FUZZY MODELING FOR MULTICRITERIA DECISION MAKING UNDER UNCERTAINTY

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TABLE OF CONTENTS

SUMMARY

Multiple criteria decision making (MCDM) refers to the problem of selecting or ranking a finite set of alternatives with usually noncommensurate and conflicting criteria. MCDM methods have been developed and applied in many areas. Obviously, uncertainty always exists in the human world. Fuzzy set theory is a perfect means for modeling imprecision, vagueness, and subjectiveness of information. With the application of fuzzy set theory, the fuzzy MCDM methods are effective and flexible to deal with complex and ill-defined problems.

Two fuzzy MCDM methods are developed in this thesis. The first one is fuzzy extension of ELECTRE. In this method, fuzzy ranking measurement and fuzzy preference measurement are proposed to construct fuzzy outranking relations between alternatives. With reference to the decision maker (DM)'s preference attitude, we establish the concordance sets and discordance sets. Then the concordance index and discordance index are used to express the strengths and weaknesses of alternatives. Finally, the performance index is obtained by the net concordance index and net discordance index. The sensitivity analysis of the threshold of the DM's preference attitude can allow comprehension of the problem and provide a flexible solution.

Another method we proposed is fuzzy MCDM based on the risk and confidence analysis. Towards uncertain information, the DM may show different risk attitudes. The optimist tends to solve the problem in a favorable way, while the pessimist tends to solve the problem in an unfavorable way. In assessing uncertainty, the DM may have different confidence attitudes. More confidence means that he prefers the values with higher possibility. In this method, risk attitude and confidence attitude are incorporated into the decision process for expressing the DM's subjective judgment and assessment. Linguistic terms of risk attitude towards interval numbers are defined by triangular fuzzy numbers. Based on the α-cut concept, refined triangular fuzzy numbers are defined to express confidence towards uncertainty. By two imagined ideal solutions of alternatives: the positive ideal solution and the negative ideal solution, we measure the alternatives' performances under confidence levels. These values are aggregated by confidence vectors into the overall performance. This method is effective in treating the DM's subjectiveness and imprecision in the decision process. The sensitivity analysis on both risk and confidence attitudes provides deep insights of the problem.

NOMENCLATURE

R	Set of real numbers
R^+	Set of positive real number
$\widetilde{A}$	Fuzzy set and fuzzy number
$\mu_{\widetilde{A}}(x)$	Membership function of x in $\widetilde{A}$
$Supp(\widetilde{A})$	Support set of $\widetilde{A}$
$\widetilde{A}^{\alpha}$	α -cut of $\widetilde{A}$
$a_l(\alpha)$	Lower value of interval of confidence at level α
$a_u(\alpha)$	Upper value of interval of confidence at level α
(a_1, a_2, a_3)	A triangular fuzzy number
$T(x)$	Set of linguistic term
$\forall x$	Universal quantifier (for all x)
$\exists x$	Existential quantifier (there exists an x)
$<$	Strict total order relation
$\leq$	Non-strict total order relation
$\cup$	Union
$\cap$	Intersection
$\varnothing$	Empty subset

LIST OF FIGURES

LIST OF TABLES

Chapter 1

Introduction

1.1 Background

Making decisions is a part of our lives. Most decision problems are made based on multiple criteria. For example, in a personal context, one chooses a job based on its salary, location, promotion opportunity, reputation and so on. In a business context, a car manufacturer needs to design a model which maximizes fuel efficiency, maximizes riding comfort, and minimizes production cost and so on. In these problems, a decision maker needs to have relevant criteria or objectives. These criteria or objectives usually conflict with one another and the measurement units of these criteria or objectives are usually incommensurable. Solutions of these problems are either to design the best alternative or to select or rank the predefined alternatives.

Multicriteria decision making (MCDM) is one of the most well known branches of decision making and has been one of the fast growing problem areas during the last two decades. From a practical viewpoint, two main theoretical streams can be distinguished. First, by assuming continuous solution spaces, multiple objective decision making (MODM) models solve problems given a set of objectives and a set of well defined constraints. MODM problems are usually called multiple objective optimization problems. The second stream focuses on problems with discrete decision spaces. That is to solve problems by ranking, selecting or prioritizing given a finite number of courses of action (alternatives). This stream is often called multiple attribute decision making. Methods and applications of these two streams in the case of a single decision maker have been thoroughly reviewed and classified (Hwang and Yoon, 1981; Hwang and Masud, 1979). In this thesis, our research scope focuses on the second stream. The more general term MCDM is used here.

The basic characteristics of MCDM are alternatives and criteria. They are explained as follows.

Alternatives

A finite number of alternatives need to be screened, prioritized, selected and ranked. The alternatives may be referred to as “candidates” or “actions”, among others.

Multiple Criteria

Each MCDM problem is associated with multiple criteria. Criteria represent the different dimensions from which the alternatives can be viewed.

In the case where the number of criteria is large, the criteria may be arranged in a hierarchical structure for a clear representation of problems. Each major criterion may be associated with several sub-criteria and each sub-criterion may be associated with several sub-sub-criteria and so on. Although some MCDM problems may have a hierarchical structure, most of them assume a single level of criteria. A desirable list of criteria should: (1) be complete and exhaustive. All important performance criteria relevant to the final decision should be represented; (2) be mutually exclusive. This permits listed criteria as independent entities among which appropriate trade-offs may later be made. And this helps prevent undesirable "double-counting" in the worth sense; (3) be restricted to performance criteria of the highest degree of importance. The purpose is to provide a sound basis from which lower level criteria may subsequently be derived.

Conflict among Criteria

Criteria usually conflict with one another since different criteria represent different dimensions of the alternatives. For instance, cost may conflict with profit etc.

Incommensurable Units

Criteria usually have different units of measurement. For instance, in buying a car, the criteria "cost" and "mileage" may be measured in terms of dollars and thousands of miles, respectively. Normalization methods can be used for commensuration among criteria. Some methods that are often used include vector normalization and linear scale transformation.

Decision Weights

Most MCDM problems require that the criteria be assigned weights to express their corresponding importance. Normally, these weights add up to one. Besides the weights being assigned by a decision maker directly, other main methods include: (1) eigenvector method (Saaty, 1977), (2) weighted least square method (Chu et al, 1979), (3) entropy method (Shannon, 1947), and (4) LINMAP (Srinivasan and Shocker, 1973) (Hwang, C.L. and Yoon, K., 1981).

Decision Matrix

MCDM problems can be concisely expressed in a matrix format. Suppose that there are m alternatives and n criteria in a decision-making problem. A decision matrix D is a $m \times n$ matrix. It is also assumed that the decision maker has determined the weights of relative importance of the decision criteria. This information is expressed as follows:

$$D = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix},$$

$$W = (w_1, \quad \ldots, \quad w_j, \quad \ldots, \quad w_n),$$

where x_{ij} is the rating of alternative A_i with respect to criterion C_j, represented by a matrix referred to as the decision matrix. w_j is the weight of criterion C_j, represented by a vector referred to as the weighting vector.

1.2 Motivations

In the real world, an exact description of real situations may be virtually impossible. In MCDM problems, uncertainties mainly come from four sources: (1) unquantifiable information, (2) incomplete information, (3) nonobtainable information, (4) partial ignorance. Classical MCDM methods do not handle problems with such imprecise information. The application of fuzzy set theory to MCDM problems provides an effective way of dealing with the subjectiveness and vagueness of the decision processes for the general MCDM problem. Research on fuzzy MCDM methods and its applications have been explored in many monographs and papers (Bellman and Zadeh, 1970; Carlsson

1982; Zimmermann, 1987; Dubois and Prade, 1994; Herrera and Verdegay, 1997; Chen and Hwang, 1992). In these fuzzy MCDM approaches, the majority of the methods require cumbersome computations. This leads to difficulties in solving problems with many alternatives and criteria. The complex computation in the ranking of fuzzy numbers often leads to unreliable, even counter-intuitive results. Human subjective attitude towards uncertainty is seldom studied to provide human-oriented solutions in the fuzzy decision problems.

1.3 Methodology

Zadeh (1965) proposed fuzzy set theory as the means for representing, quantifying, and measuring the inherent uncertainty in the real world. Fuzziness is a type of imprecision which may be associated with sets in which there are no sharp transition from membership to nonmembership. It presents a mathematical way to deal with vagueness, impreciseness and subjectiveness in complex and ill-defined decision problems.

Triangular Fuzzy Number

For many practical applications and fuzzy mathematics problems, triangular fuzzy numbers are simple in operating and approximating. In the triangular fuzzy number $\widetilde{A} = (a_1, a_2, a_3)$, a_1 , a_2 and a_3 represents lower, modal and upper value of presumption to uncertainty. In the inverse, multiplication, and division operations, the outcome does not necessarily give a real triangular fuzzy number. But using an approximation of triangular fuzzy numbers is enough to reflect the facts without much divergence. When the DM considers the uncertain ratings of the alternatives and the weights of the criteria, the triangular fuzzy number approach is usually used. Linguistic terms also can be simply expressed by triangular fuzzy numbers.

Linguistic Variable

The linguistic approach is intended to be used in situations in which the problem is too complex or too ill-defined to be amenable to quantitative characterization. It deals with the pervasive fuzziness and imprecision of human judgment, perception and modes of reasoning. A linguistic variable can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms.

1.4 Contributions

The objective of this research is to develop fuzzy MCDM methods. This thesis proposes two novel approaches.

The first proposed method is a fuzzy extension of ELECTRE. In this method, we first propose a fuzzy ranking measurement to construct the relations between two alternatives. Preference measurement is then used to represent pairwise preferences between two alternatives with reference to the whole set of alternatives. Based on the DM's preference attitudes, we establish the concordance and discordance sets. The corresponding concordance and discordance indices are used to express the strengths of outranking relations. The net concordance and discordance indices are combined to obtain the performance of alternatives. In this procedure, the preference attitude is incorporated in the outranking process to provide a more flexible way to evaluate and analyze alternatives.

The second method that we (Wang and Poh, 2003a, 2003b, 2003c, 2003d, and 2003e) proposed is a fuzzy MCDM method based on risk and confidence analysis. In this method, the risk attitude and confidence attitude are defined by linguistic terms. The triangular fuzzy numbers are proposed to incorporate the DM's risk attitudes towards an interval of uncertainty. In order to deal with the DM's confidence in the fuzzy assessments, based on the α-cut concept, we proposed refined triangular fuzzy numbers to assess the confidence level towards uncertainty. Confidence vectors are obtained from the membership functions of confidence attitudes. By using confidence vectors, the alternatives' performances on confidence levels are aggregated as the final performance to evaluate the alternatives. This method incorporates the DM's subjective judgment and assessments towards uncertainty into the decision process. Thus, by considering human adaptability and dynamics of preference, the proposed method is effective in solving complex and ill-defined MCDM problems.

1.5 Origination of The Thesis

The next chapter presents a state-of-the-art survey of crisp MCDM methods, an overview of the fuzzy set theory and operations, as well as the fuzzy MCDM methods. Then in chapters three and four we present the proposed fuzzy extension of ELECTRE method and an example, respectively. In chapters five and six we introduce the proposed fuzzy

MCDM method based on risk attitude and confidence attitude and an example, respectively. Finally, chapter seven concludes our work in this thesis.

Chapter 2

Literature Survey

In this Chapter, we first present an overview of crisp MCDM methods. Then we give an introduction of fuzzy set theory and operations. Finally, by the application of fuzzy set theory, we introduce the fuzzy MCDM methods.

2.1 Crisp MCDM Methods

An MCDM method is a procedure to process alternatives' values in order to arrive at a choice. There are three basic steps in MCDM methods to evaluate the alternatives. First of all, we formulate the problem by determining the relevant criteria and alternatives. Secondly, we attach numerical measures to the relative importance of the criteria as the weights and to the impacts of the alternatives on criteria as the ratings. Finally, we process the numerical values of the ratings of alternatives and weights of criteria to evaluate alternatives and determine a ranking order.

There are two major approaches in information processing: noncompensatory and compensatory models. Each category includes the relevant MCDM methods. Noncompensatory models do not permit tradeoffs among criteria. An unfavorable value in one criterion cannot be offset by a favorable value in some criteria. The comparisons are made on a criterion-by-criterion basis. The models in this category are dominance, maximin, maximax, conjunctive constraint method, disjunctive constraint method, and lexicographic method. Compensatory models make tradeoffs among criteria. These models include the weighted sum model (WSM), the weighted product model (WPM), the analytic hierarchy process (AHP), TOPSIS, ELECTRE, LINMAP, nonmetric MDS, permutation method, linear assignment method.

The weighted sum model (WSM) is the earliest and widely used method. The weighted product model (WPM) can be considered as a modification of the WSM, and has been proposed for overcoming some of the weaknesses in WSM. The AHP proposed by Saaty (1980) is a later development and has recently become increasingly popular. A revised AHP suggested by Belton and Gear (1983) appears to be more consistent than the original approach. Other widely used methods are the TOPSIS and ELECTRE. Next, we give an overview of some of the popular methods, namely WSM, WPM, AHP, TOPSIS, and ELECTRE.

2.1.1 The Weighted Sum Method

The WSM is probably the best known and highly used method of decision making. Suppose there are m alternatives and n criteria in a decision-making problem. An alternative's performance is defined as (Fishburn, 1967):

$$p_i = \sum_{j=1}^{n} x_{ij} w_j \,, \; i = 1,2,...,m \,, \tag{2.1}$$

where x_{ij} is the rating of the i th alternative in terms of the j th decision criterion, and w_j is the weight of the j th criterion. The best alternative is the one which has the maximum value (in the maximization case). The WSM method can be applied without difficulty in single-dimensional cases where all units of measurement are identical. Because of the additive utility assumption, a conceptual violation occurs when the WSM is used to solve multidimensional problems in which the units are different.

2.1.2 The Weighted Product Method

The WPM uses multiplication to rank alternatives. Each alternative is compared with others by multiplying a number of ratios, one for each criterion. Each ratio is raised to the power of the relative weight of the corresponding criterion. Generally, in order to compare two alternatives A_k and A_l , the following formula (Miller and Starr, 1969) is used:

$$Q\left(\frac{A_k}{A_l}\right)=\prod_{j=1}^{n}\left(\frac{x_{kj}}{x_{lj}}\right)^{w_j}, \tag{2.2}$$

where x_{ij} is the rating of the i th alternative in terms of the j th decision criterion, and w_j is the weight of the j th criterion. If the above ratio is greater than or equal to one, then (in the maximization case) the conclusion is that alternative A_k is better than alternative A_l. Obviously, the best alternative is the one which is better than or at least as good as all other alternatives. The WPM is sometimes called dimensionless analysis because its structure eliminates any units of measurement. Thus, the WPM can be used in single- and multidimensional decision problems.

2.1.3 The AHP Method

The Analytic Hierarchy Process (AHP) approach deals with the construction of a matrix (where there are m alternatives and n criteria). In this matrix the element a_{ij} represents the relative performance of the ith alternative in terms of the jth criterion. The vector $A_i = (a_{i1}, a_{i2}, ..., a_{in})$ for the ith alternative ($i = 1,2,...,m$) is the eigenvector of an $n \times n$ reciprocal matrix which is determined through a sequence of pairwise comparisons (Saaty, 1980). In the original AHP, $\sum_{j=1}^{n} w_{ij} = 1$.

According to the AHP, an alternative's performance is defined as:

$$p_i = \sum_{j=1}^{n} a_{ij} w_j \ , \ i = 1,2,...,m . \tag{2.3}$$

The AHP uses relative values instead of actual ones. Therefore, the AHP can be used in single- and multidimensional decision problems.

The RAHP (Belton and Gear, 1983) is a revised version of the original AHP model. The shortcoming of the AHP is that it is sometimes possible to yield unjustifiable ranking reversals. The reason for the ranking inconsistency is that the relative performance measures of all alternatives in terms of each criterion are summed to one. Instead of having the relative values sum to one, they propose that each relative value be divided by the maximum value in the corresponding vector of relative values. That is known as the ideal-model of AHP.

2.1.4 The ELECTRE Method

The ELECTRE (Elimination and Choice Translating Reality; English translation from the French original) method was originally introduced by Benayoun et al. (1966). It focuses on the concept of outranking relation by using pairwise comparisons among alternatives under each criterion separately. The outranking relationship of the two alternatives A_k and A_l, denoted as $A_k \rightarrow A_l$, describes that even though A_k does not dominate A_l quantitatively, the DM accepts the risk of regarding A_k as almost surely better than A_l (Roy, 1973).

The ELECTRE method begins with pairwise comparisons of alternatives under each criterion. It elicits the so-called concordance index, named as the amount of evidence to support the conclusion that A_k outranks or dominates A_l, as well as the discordance index, the counterpart of the concordance index. This method yields binary outranking relations between the alternatives. It gives a clear view of alternatives by eliminating less favorable ones and is convenient in solving problems with a large number of alternatives and a few criteria. There are many variants of the ELECTRE method. The original version of the ELECTRE method is illustrated in the following steps.

Suppose there are m alternatives and n criteria. The decision matrix element x_{ij} is the rating of the i th alternative in terms of the j th criterion, and w_j is the weight of the j th criterion.

***Step* 1: Normalizing the Decision Matrix**

The vector normalization method is used here. This procedure transforms the various criteria scales into comparable scales.

The normalized matrix is defined as follows:

$$D^n = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix}, \tag{2.4}$$

where

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, \; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n.$$

***Step* 2: Weighting the Normalized Decision Matrix**

This matrix is obtained by multiplying each column of matrix R with its associated weight. These weights are determined by the DM. Therefore, the weighted normalized decision matrix V is equal to

$$V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix}, \tag{2.5}$$

where

$$v_{ij} = w_j r_{ij}, \; \sum_{j=1}^{n} w_j = 1, \; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n.$$

***Step* 3: Determine the Concordance and Discordance Sets**

For two alternatives A_k and A_l ($1 \leq k, l \leq m$), the set of decision criteria J={ $j \mid j = 1,2,...,n$} is divided into two distinct subsets. The concordance set C_{kl} of A_k and A_l is composed of criteria in which A_k is preferred to A_l. In other words,

$$C_{kl} = \{j \mid v_{kj} \geq v_{lj}\}. \tag{2.6}$$

The complementary subset is called the discordance set, described as:

$$D_{kl} = \{j \mid v_{kj} < v_{lj}\} = J - C_{kl}. \tag{2.7}$$

***Step* 4: Construct the Concordance and Discordance Matrices**

The relative value of the concordance set is measured by means of the concordance index. The concordance index is equal to the sum of the weights associated with those criteria which are contained in the concordance set. Therefore, the concordance index C_{kl} between A_k and A_l is defined as:

$$c_{kl} = \sum_{j \in C_{kl}} w_j. \tag{2.8}$$

The concordance index reflects the relative importance of A_k with respect to A_l. Obviously, $0 \leq c_{kl} \leq 1$. The concordance matrix C is defined as follows:

$$C = \begin{bmatrix} - & c_{12} & \cdots & c_{1m} \\ c_{21} & - & \cdots & c_{2m} \\ \vdots & & & \vdots \\ c_{m1} & c_{m2} & \cdots & - \end{bmatrix}.$$

The elements of matrix C are not defined when $k = l$. In general, this matrix is not symmetric.

The discordance matrix expresses the degree that A_k is worse than A_l. Therefore a second index, called the discordance index, is defined as:

$$d_{kl} = \frac{\max_{j \in D_{kl}} \left| v_{kj} - v_{lj} \right|}{\max_{j \in J} \left| v_{kj} - v_{lj} \right|}. \tag{2.9}$$

It is clear that $0 \le d_{kl} \le 1$. The discordance matrix D is defined as follows:

$$D = \begin{bmatrix} - & d_{12} & \cdots & d_{1m} \\ d_{21} & - & \cdots & d_{2m} \\ \vdots & & & \vdots \\ d_{m1} & d_{m2} & \cdots & - \end{bmatrix}.$$

In general, matrix D is not symmetric.

***Step* 5: Determine the Concordance and Discordance Dominance Matrices**

This matrix can be calculated with the aid of a threshold value for the concordance index. A_k will only have a chance of dominating A_l, if its corresponding concordance index c_{kl} exceeds at least a certain threshold value $\bar{c}$. That is:

$c_{kl} \ge \bar{c}$.

This threshold value can be determined, for example, as the average concordance index:

$$\bar{c} = \frac{1}{m(m-1)} \sum_{\substack{k=1 \\ k \ne l}}^{m} \sum_{\substack{l=1 \\ l \ne k}}^{m} c_{kl} . \tag{2.10}$$

Based on the threshold value, the elements of the concordance dominance matrix F are determined as follows:

$f_{kl} = 1, \quad \text{if } c_{kl} \ge \bar{c};$

$f_{kl} = 0, \quad \text{if } c_{kl} < \bar{c}.$

Similarly, the discordance dominance matrix G is defined by using a threshold value $\overline{d}$, which is defined as :

$$\overline{d} = \frac{1}{m(m-1)} \sum_{\substack{k=1 \\ k \neq l}}^{m} \sum_{\substack{l=1 \\ l \neq k}}^{m} d_{kl} , \tag{2.11}$$

where

$g_{kl} = 1, \quad \text{if } d_{kl} \leq \overline{d}$;

$g_{kl} = 0, \quad \text{if } d_{kl} > \overline{d}$.

***Step* 6: Determine the Aggregate Dominance Matrix**

The elements of the aggregate dominance matrix E are defined as follows:

$$e_{kl} = f_{kl} \times g_{kl} . \tag{2.12}$$

***Step* 7: Eliminate the Less Favorable Alternatives**

The aggregate dominance matrix E gives the partial-preference ordering of the alternatives. If $e_{kl} = 1$, then A_k is preferred to A_l for both the concordance and discordance criteria, but A_k still has the chance of being dominated by the other alternatives. Hence the condition that A_k is not dominated by the ELECTRE procedure is:

$e_{kl} = 1$, for at least one l, $l = 1,2,...,m, k \neq l$;

$e_{ik} = 0$, for all $i, i = 1,2,...,m, i \neq k, i \neq l$.

This condition appears difficult to apply, but the dominated alternatives can be easily identified in the E matrix. If any column of the E matrix has at least one element of 1, then this column is 'ELECTREcally' dominated by the corresponding row(s). Hence we simply eliminate any column(s) which has an element of 1.

2.1.5 The TOPSIS Method

TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) was developed by Hwang and Yoon (1980) as an alternative to the ELECTRE method. The basic concept of this method is that the selected best alternative should have the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution in a geometrical (i.e., Euclidean) sense. The TOPSIS assumes that each criterion has a tendency toward monotonically increasing or decreasing utility. Therefore, it is easy to locate the ideal and negative-ideal solutions. The Euclidean distance approach is used to evaluate the relative closeness of alternatives to the ideal solution. Thus, the preference order of alternatives can be derived by comparing these relative distances.

Suppose there are m alternatives and n criteria. The decision matrix element x_{ij} is the rating of the i th alternative in terms of the j th criterion, and w_j is the weight of the j th criterion.

***Step* 1: Normalizing the Decision Matrix**

The TOPSIS converts the various criteria dimensions into nondimensional criteria, as in the ELECTRE method. An element r_{ij} of the normalized decision matrix R is calculated as follows:

$$D^n = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix}, \tag{2.13}$$

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.$$

***Step* 2: Construct the Weighted Normalized Decision Matrix**

A set of weight $W = (w_{1,} w_2, \cdots, w_n)$, $\sum_{j=1}^{n} w_j = 1$, specified by the decision maker, is used in conjunction with the previous normalized decision matrix to determine the weighted normalized matrix V defined as:

$$V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix}, \tag{2.14}$$

where

$$v_{ij} = w_j r_{ij}, \ \sum_{j=1}^{n} w_j = 1, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.$$

***Step* 3: Determine the Positive Ideal and the Negative Ideal Solutions**

The positive ideal A^* and the negative ideal A^- solutions are defined as follows:

$$A^* = \{(\max_{1 \le i \le m} v_{ij} \mid j \in J), (\min_{1 \le i \le m} v_{ij} \mid j \in J^{'})\}$$

$$= \{ v_1^*, \ldots, v_j^*, \ldots, v_n^* \}, \tag{2.15}$$

$$A^- = \{(\min_{1 \le i \le m} v_{ij} \mid j \in J), (\max_{1 \le i \le m} v_{ij} \mid j \in J^{'})\}$$

$$= \{ v_1^-, ..., v_j^-, ..., v_n^- \}, \tag{2.16}$$

where

$J = \{ j = 1,2,...,n \mid j$ is associated with benefit criteria$\}$,

and $J' = \{ j = 1,2,...,n \mid j$ is associated with cost criteria$\}$.

It is clear that these two created alternatives A^* and A^- indicate the most preferable alternative (positive ideal solution) and the least preferable alternative (negative ideal solution), respectively.

***Step* 4: Calculate the Separation Measure**

In this step the concept of the *n*-dimensional Euclidean distance is used to measure the separation distances of each alternative to the positive ideal solution and negative ideal solution.

The separation of each alternative from the positive ideal solution is defined as:

$$S_{i^*} = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^*)^2}, \quad i = 1,2,...,m. \tag{2.17}$$

Similarly, the separation of each alternative from the negative ideal one is defined as:

$$S_{i^-} = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2}, \quad i = 1,2,...,m. \tag{2.18}$$

***Step* 5: Calculate the Relative Closeness to the Ideal Solution**

The alternative with a lower value of S_{i^*} and a higher value of S_{i^-} is preferred. The relative closeness of A_i with respect to A^* is defined as:

$$C_{i^*} = \frac{S_{i^-}}{S_{i^*} + S_{i^-}}, \quad i = 1,2,...,m. \tag{2.19}$$

It is clear that $C_{i^*} = 1$ if $A_i = A^*$ and $C_{i^*} = 0$ if $A_i = A^-$. An alternative A_i is closer to A^* as C_{i^*} approaches 1.

***Step* 6: Rank the Preference Order**

The best alternative can be decided according to the preference rank order of C_{i^*}. Therefore, the best alternative is the one which has the shortest distance to the positive ideal solution. The way the alternatives are processed in the previous steps reveals that if an alternative has the shortest distance from the positive ideal solution, then this alternative is guaranteed to have the longest distance from the negative ideal solution.

2.2 Fuzzy Set Theory and Operations

Very often in MCDM problems data are imprecise and vague. Also, the DM may encounter difficulty in quantifying linguistic statements that can be used in decision making. Fuzzy set theory, proposed by Zadeh (1965), has been effectively used in representing and measuring uncertainty. It is desired to develop decision making methods in the fuzzy environment. In this section, we will present basic concepts and definitions of fuzzy set theory and operations from mathematical aspects. In many fuzzy MCDM methods, the final performances of alternatives are expressed in terms of fuzzy numbers. Thus, the fuzzy ranking methods need to be introduced here also. The application of fuzzy set theory to MCDM problems will be introduced in section 2.3.

2.2.1 Basic Concepts and Definitions

Definition 2.1: If X is a universe of discourse denoted generically by x, then a fuzzy set $\widetilde{A}$ in the universe of discourse X is characterized by a membership function $\mu_{\widetilde{A}}(x)$ which associates with each element x in X a real number in the interval $[0, \ 1]$. $\mu_{\widetilde{A}}(x)$ is called the membership function of x in $\widetilde{A}$.

Definition 2.2: A crisp set is a collection of elements or objects $x \in X$ that can be finite, countable, or over countable. Each single element can either belong to or not belong to a set A, $A \subseteq X$.

Definition 2.3: The support of a fuzzy set $\widetilde{A}$ ($Supp(\widetilde{A})$) in the universe of discourse *X* is the crisp set of all $x \in X$, such that $\mu_{\widetilde{A}}(x) > 0$.

Definition 2.4: A fuzzy set $\widetilde{A}$ in the universe of discourse *X* is called a normal fuzzy set means that $\exists x \in X$, such that $\mu_{\widetilde{A}}(x) = 1$.

Definition 2.5: A fuzzy set $\widetilde{A}$ in the universe of discourse X is convex means that $\mu_{\widetilde{A}}(x_2) \geq \min\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_3)\}$, for all $x_1, x_3 \in X$, and any $x_2 \in [x_1, x_3]$.

Definition 2.6: A fuzzy number is a fuzzy set in the universe of discourse *X* that is both convex and normal. Figure 2.1 shows a fuzzy number in the universe of discourse *X*.

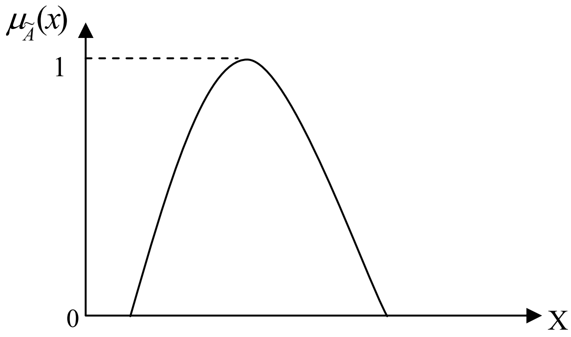

Figure 2.1 A fuzzy number $\widetilde{A}$

Definition 2.7: A fuzzy number $\widetilde{A}$ is positive (negative) if its membership function is such that $\mu_{\widetilde{A}}(x) = 0$, $\forall x \leq 0$ ($\forall x \geq 0$).

Definition 2.8: If $\widetilde{A}$ is a fuzzy set in the universe of discourse *X*, then the α-cut set of $\widetilde{A}$ is defined as $\widetilde{A}^{\alpha} = \{x \in X \mid \mu_{\widetilde{A}}(x) > \alpha\}$, $0 \leq \alpha \leq 1$.

For any fuzzy number $\widetilde{A}$, $\widetilde{A}^{\alpha}$ is a non-empty closed, bounded interval for $0 \leq \alpha \leq 1$. It can be denoted as $\widetilde{A}^{\alpha} = [a_l(\alpha), \quad a_u(\alpha)]$, where $a_l(\alpha)$ and $a_u(\alpha)$ represent the lower boundary and upper boundary of the interval, respectively. $a_l(\alpha)$ is an increasing function of α with $a_l(1) \leq a_u(1)$, while $a_u(\alpha)$ is a decreasing function of α with $a_l(1) \leq a_u(1)$. Figure 2.2 shows a fuzzy $\widetilde{A}$ with α-cuts, where $\widetilde{A}^{\alpha_1} = [a_l(\alpha_1), \quad a_u(\alpha_1)]$ and $\widetilde{A}^{\alpha_2} = [a_l(\alpha_2), \quad a_u(\alpha_2)]$. It is obvious when $\alpha_2 \geq \alpha_1$, $[a_l(\alpha_2), a_u(a_2)] \subset [a_l(\alpha_1), \quad a_u(\alpha_1)]$.

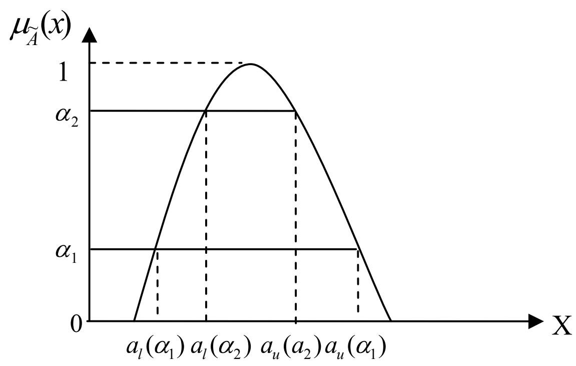

Figure 2.2 A fuzzy number $\widetilde{A}$ with α-cuts

Definition 2.9: A triangular fuzzy number $\widetilde{A}$ is defined by a triplet (a_1, a_2, a_3) shown in Figure 2.3. The membership function is defined as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a_1, \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2, \\ \dfrac{x - a_3}{a_2 - a_3}, & a_2 \le x \le a_3, \\ 0, & x > a_3. \end{cases} \tag{2.20}$$

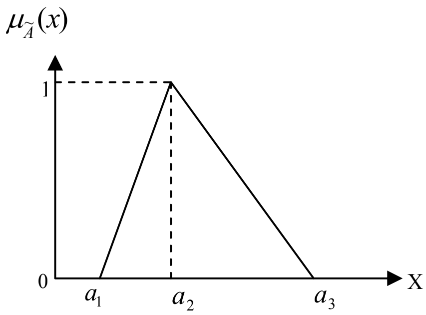

Figure 2.3 A triangular fuzzy number $\widetilde{A} = (a_1, a_2, a_3)$

Definition 2.10: If $\widetilde{A}$ is a triangular fuzzy number, and $a_l(\alpha) > 0$, for $0 \le \alpha \le 1$, then $\widetilde{A}$ is called a positive triangular fuzzy number.

Let $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$ be two positive triangular fuzzy numbers. The basic arithmetic operators are defined as:

a. Negation: $-\widetilde{A} = (-a_3, -a_2, -a_1)$.

b. Inverse: $\widetilde{A}^{-1} = (1/a_3, 1/a_2, 1/a_1)$.

c. Addition: $\widetilde{A} + \widetilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

d. Subtraction: $\widetilde{A} - \widetilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$.

e. Multiplication: $\widetilde{A}\widetilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3)$.

f. Division: $\widetilde{A}/\widetilde{B} = (a_1/b_3, a_2/b_2, a_3/b_1)$.

g. Scalar multiplication:

$$\forall k > 0, k \in R,\ kA = (ka_1, ka_2, ka_3);\ \forall k < 0,\ k \in R,\ kA = (ka_3, ka_2, ka_1). \quad (2.21)$$

Definition 2.11: If $\widetilde{A}$ is a triangular fuzzy number and $a_l(\alpha) > 0$, $a_u(\alpha) \le 1$ for $0 \le \alpha \le 1$, then $\widetilde{A}$ is called a normalized positive triangular fuzzy number.

Definition 2.12: A matrix $\widetilde{D}$ is called a fuzzy matrix, if at least an element in $\widetilde{D}$ is a fuzzy number.

Definition 2.13: Let $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$ be two positive triangular fuzzy numbers, then the vertex method is defined to calculate the distance between them:

$$d(\widetilde{A}, \widetilde{B}) = \{[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2]/3\}^{1/2}. \quad (2.22)$$

Definition 2.14: Let $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. The fuzzy number $\widetilde{A}$ is closer to fuzzy number $\widetilde{B}$ as $d(\widetilde{A}, \widetilde{B})$ approaches 0.

Definition 2.15: Let $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. If $\widetilde{A} = \widetilde{B}$, then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

2.2.2 Ranking of Fuzzy Numbers

In many fuzzy MCDM methods, the final performances of alternatives are represented in terms of fuzzy numbers. In order to choose the best alternatives, we need a method for building a crisp ranking order from fuzzy numbers. The problem of ranking fuzzy numbers appears often in literature (McCahon and Lee, 1988; Zhu and Lee, 1991). Each method of ranking has its advantages over others in certain situations. It is hard to determine which method is the best one. The important factors in deciding which ranking method is the most appropriate one for a given situation include the complexity, flexibility, accuracy, ease of interpretation of the fuzzy numbers which are used.

A widely used method for comparing fuzzy numbers was introduced by Bass and Kwakernaak (1977). The concept of dominance measure was introduced by Tong and Bonissone (1981) and it was proved to be equivalent to Bass and Kwakernaak's ranking measure. The method proposed by Zhu and Lee (1991) is less complex and still effective. It allows the DM to implement it without difficulty and with ease of interpretation. This is adopted in fuzzy MCDM by Triantaphyllou (1996).

The procedure of Zhu and Lee's method for ranking fuzzy numbers is to compare the membership function as follows:

For fuzzy numbers $\tilde{A}$ and $\tilde{B}$, we define:

$$e_{\tilde{A}\tilde{B}} = \max_{x \geq y}\{\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))\} . \qquad (2.23)$$

Then $\tilde{A} > \tilde{B}$ if and only if $e_{\tilde{A}\tilde{B}} = 1$ and $e_{\tilde{B}\tilde{A}} < Q$, where $Q \in [0,1)$. Values such as 0.7, 0.8, or 0.9 might be appropriate for Q, and the value of Q should be set by the DM or can be varied for sensitivity analysis.

2.3 Fuzzy MCDM Methods

Fuzzy MCDM methods are proposed to solve problems involving fuzzy data. Bellman and Zadeh (1970) first introduced fuzzy set theory to decision making problems. Bass and Kwakernaak (1977) proposed a fuzzy MCDM method that is regarded as classical work. A systematic review of fuzzy MCDM has been conducted by Zimmermann (1987) and Chen and Hwang (1992). Zimmermann treated the fuzzy MCDM method as a two-phase process. The first phase is to aggregate the fuzzy ratings of the alternatives as the fuzzy final ratings. The second phase is to obtain the ranking order of the alternatives by fuzzy ranking methods.

Next we will present the widely used fuzzy MCDM method that is based on traditional MCDM methods presented in section 2.1. These are the WSM, the WPM, the AHP, and the TOPSIS method. Fuzzy ELECTRE methods are based mainly on the fuzzy outranking relations. We will discuss fuzzy ELECTRE methods and propose a new approach in chapter 3. In these fuzzy MCDM methods, the values which the DM assigns to the alternatives in terms of the decision criteria are fuzzy. These fuzzy numbers are often assigned as triangular fuzzy numbers. The procedure is based on the corresponding crisp MCDM method.

2.3.1 The Fuzzy Weighted Sum Method

Suppose there are m alternatives and n criteria in a decision-making problem. The rating of the ith alternative in terms of the jth criterion is a fuzzy number denoted as $\widetilde{x}_{ij}$. Analogously, it is assumed that the DM uses fuzzy numbers in order to express the weights of the criteria, denoted as $\widetilde{w}_j$. Now the overall fuzzy utility is defined as:

$$\widetilde{p}_i = \sum_{j=1}^{n} \widetilde{x}_{ij} \widetilde{w}_j , \ i = 1,2,...,m . \tag{2.24}$$

The next procedure is to use a fuzzy ranking method to determine the ranking order of these fuzzy numbers. The fuzzy ranking method (2.23) can be effectively used here. The best alternative is the one with the maximum value.

2.3.2 The Fuzzy Weighted Product Method

For the fuzzy version of the weighted product model, the corresponding formula is defined as:

$$\widetilde{Q}\left(\frac{A_k}{A_l}\right) = \prod_{j=1}^{n}\left(\frac{\widetilde{x}_{kj}}{\widetilde{x}_{lj}}\right)^{\widetilde{w}_j}, \tag{2.25}$$

where the $\widetilde{x}_{kj}$, $\widetilde{x}_{lj}$ are the respective ratings of the alternatives in terms of criteria, and $\widetilde{w}_j$ is the weight of criterion j. These are all expressed as fuzzy numbers. Alternative A_k dominates alternative A_l if and only if the numerator in (2.25) is greater than the denominator.

2.3.3 The Fuzzy AHP Method

The extension of the crisp AHP to fuzzy environment has been developed (Buckley, 1985; Boender et al., 1989; Laarhoven and Pedrycz, 1983). In the fuzzy version of the AHP method, triangular fuzzy numbers were used in pairwise comparisons to compute the weights of importance of the decision criteria. The fuzzy performance values of the alternatives in terms of each decision criterion were computed by using triangular fuzzy numbers also. The most widely used procedure is proposed by Buckley (1985) and is well-known for its simplicity. In this method, the rating of the i th alternative in terms of the j th criterion is a fuzzy number denoted as: $\widetilde{x}_{ij}$. First the aggregated fuzzy rating can be calculated as:

$$\widetilde{z}_i = [\widetilde{x}_{i1} \times \ldots \times \widetilde{x}_{in}]^{1/n},\ i = 1,2,...,m. \tag{2.26}$$

Next, the geometric mean method is used for obtaining the fuzzy weight as follows:

$$\tilde{w}_i = \frac{\tilde{z}_i}{\sum_{i=1}^{m} \tilde{z}_i}, \; i = 1,2,...,m. \tag{2.27}$$

Finally, the overall fuzzy utility (performance) can be obtained as:

$$\tilde{p}_i = \sum_{j=1}^{n} \tilde{w}_j \tilde{x}_{ij}, \; i = 1,2,...,m. \tag{2.28}$$

2.3.4 The Fuzzy TOPSIS Method

One approach of fuzzy TOPSIS is to use fuzzy numbers in the procedure of crisp TOPSIS in section 2.1.5. The fuzzy positive ideal solution and the fuzzy negative ideal solution are determined by fuzzy ranking methods. Finally, the fuzzy closeness index to ideal solutions determines the ranking order of the alternatives. A fuzzy TOPSIS method was proposed by Chen (2000). One merit of this method is that the fuzzy ranking procedure is avoided. In this method, the rating of the i th alternative in terms of the j th criterion is a fuzzy number denoted as $\tilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3})$, and the weight of the criteria is denoted as $\tilde{w}_j = (\tilde{w}_{j1}, \tilde{w}_{j2}, \tilde{w}_{j3})$. Suppose there are m alternatives and n criteria. The linear scale is used in normalization instead of the vector method in the fuzzy version. It is defined as:

$$\tilde{R} = [\tilde{r}_{ij}]_{m \times n}, \tag{2.29}$$

where

$$\tilde{r}_{ij} = \left(\frac{x_{ij1}}{c_j^*}, \frac{x_{ij2}}{c_j^*}, \frac{x_{ij3}}{c_j^*}\right), \text{ if } j \in B;$$

$$c_j^* = \max_i x_{ij3}, \text{ if } j \in B;$$

$\tilde{r}_{ij} = (\frac{c_j^-}{x_{ij3}}, \frac{c_j^-}{x_{ij2}}, \frac{c_j^-}{x_{ij1}})$, if $j \in C$;

$c_j^- = \min_i x_{ij1}$, if $j \in C$,

with B and C being the set of benefit criteria and cost criteria, respectively.

The weighted normalized fuzzy decision matrix is defined as:

$$V = [\tilde{v}_{ij}]_{m \times n}, \tag{2.30}$$

where

$\tilde{v}_{ij} = \tilde{r}_{ij} \tilde{w}_{ij}$.

Then the fuzzy positive ideal solution and fuzzy negative ideal solution are defined as:

$A^* = (\tilde{v}_1^*, \ldots \tilde{v}_n^*)$,

$$A^- = (\tilde{v}_1^-, \ldots \tilde{v}_n^-), \tag{2.31}$$

where

$\tilde{v}_j^* = (1,1,1)$ and $\tilde{v}_j^- = (0,0,0)$ for all j.

The distance of each alternative from A^* and A^- are calculated as:

$d_i^* = \sum_{j=1}^{n} d(\tilde{v}_{ij}, \tilde{v}_j^*)$, $i = 1,2,...,m$;

$$d_i^- = \sum_{j=1}^{n} d(\tilde{v}_{ij}, \tilde{v}_j^-),\ i = 1,2,...,m, \tag{2.32}$$

where $d(\cdot,\cdot)$ is the distance measurement between two fuzzy numbers by the vertex method.

Finally, the closeness of each alternative is defined as:

$$CC_i = \frac{d_i^-}{d_i^* + d_i^-},\ i = 1,2,...,m\ . \tag{2.33}$$

Using the closeness index, the ranking order of alternatives can be determined.

2.4 Summary

Classical MCDM methods are introduced in this chapter. An overview of fuzzy set theory and operations is presented here and these provide tools to deal with uncertainty in MCDM problems. The fuzzy MCDM methods follow in the third section. In chapters 3 and 4, we will propose a fuzzy extension of the ELECTRE method with an illustrating example. In chapters 5 and 6, we will propose a fuzzy MCDM method based on risk and confidence analysis, also with an illustrating example.

Chapter 3

Fuzzy Extension of ELECTRE

In this chapter, we propose an approach to extend the ELECTRE method into fuzzy environment. A fuzzy outranking method is proposed to determine the relations between alternatives.

3.1 Introduction

The ELECTRE method and its family including ELECTRE I, IS, II, III, and IV are decision aids popular in Europe. This method was originally proposed in the mid sixties last century (Benayoun, Roy and Sussman, 1966; Roy, 1968). Since then it has been developed greatly (Nijkamp and Delft, 1977; Voogd, 1983). Based on the concept of outranking relations, the ELECTRE method uses a concordance-discordance analysis to solve multicriteria decision problems.

Many fuzzy relations have been introduced to model individual preferences. Preference modeling is an important aid in the decision process (Roy, 1990, 1996; Vincke, 1990; Fodor and Roubens, 1994). Zadeh (1971) first introduced the concept of fuzzy relations. The types of relation include fuzzy preference relation (Orlovsky, 1978) and fuzzy outranking relation (Roy, 1977; Siskos et al., 1984). Roy and Siskos et al. used outranking relations effectively by introducing fuzzy concordance relations and fuzzy discordance relations. A fuzzy concordance relation is an aggregation of fuzzy partial relations, each is being considered as a model for a unique criterion. The fuzzy discordance relation takes into consideration the importance of the differences between the performances of alternatives for each criterion. Both Roy and Siskos used crisp data as criteria.

Here we propose a new method that combines both fuzzy outranking and fuzzy criteria to provide a more flexible way for comparing and evaluating alternatives. A novel fuzzy outranking measurement is also proposed here. Specifically, in our method, the ratings of alternatives and weights of criteria are given in triangular fuzzy numbers to express the DM's assessments. Fuzzy ranking measurement is proposed to construct the relations between two alternatives. Preference measurement is used to represent pairwise preference between two alternatives with reference to the whole set of alternatives. By considering the DM's preference attitude, we establish the concordance and discordance sets. Then, concordance and discordance indices are used to express the strength of outranking relations. Finally, the net concordance and net discordance indexes are combined to evaluate the performance of alternatives. Sensitivity analysis of the threshold of the DM's preference attitudes can allow deep comprehension of the problems.

Next, in section 3.2, we introduce the measurements between fuzzy numbers and propose a new measurement method. Based on fuzzy measurement, we propose our fuzzy ELECTRE approach in section 3.3.

3.2 The Proposed Method

3.2.1 Fuzzy Outranking Measurement

For any two given alternatives A_k and A_l, the outranking relation principle is based on the fact that even though A_k and A_l do not dominate each other, the DM accepts the risk of regarding A_k is at least as good as A_l, given the available information. The problem of uncertainty results in a fuzzy outranking relation that makes the comparison more realistic and accurate.

Here we propose a method of ranking measurement between two fuzzy numbers. We define a fuzzy outranking function in $A \times A$ as a function $f : A \times A \rightarrow R$ in which the different $f(k,l)$ values indicate the degree of outranking associated with the pair of alternatives (k,l). A corresponding preference measurement will reflect the credibility of an existing preference of A_k over A_l. Specifically, the ranking measurement evaluates the average comparison of fuzzy interval numbers under α-cuts and integrates these values to produce the ranking relations. In our method, preference measurements are proposed to express pairwise preference relations between two fuzzy numbers with reference to the whole fuzzy numbers. By comparing with indices which represent the DM's preference attitudes, we establish the concordance and discordance sets. This method can utilize all information included in the fuzzy numbers and determine the outranking relations between two fuzzy numbers effectively. The outranking relation between two fuzzy numbers is defined as:

Definition 3.1: The ranking measurement between $\widetilde{A}_i$ and $\widetilde{A}_j$ ($i, j = 1,2,\ldots,m$) is a mapping of this relation into the real line R as defined below:

$$r(\widetilde{A}_i, \widetilde{A}_j) = \int_{\alpha=0}^{1} r(\widetilde{A}_i^{\alpha}, \widetilde{A}_j^{\alpha}) d\alpha = \frac{1}{2}\int_{\alpha=0}^{1} (a_{il}(\alpha) - a_{jl}(\alpha) + a_{iu}(\alpha) - a_{ju}(\alpha)) d\alpha \qquad (3.1)$$

Definition 3.2: The preference measurement between $\widetilde{A}_i$ and $\widetilde{A}_j$ ($i, j = 1,2,\ldots,m$) is a mapping of this relation into the interval $[-1,1]$ as defined below:

$$p(\widetilde{A}_i, \widetilde{A}_j) = \frac{1}{(\beta_2 - \beta_1)} \int_{\alpha=0}^{1} r(\widetilde{A}_i^{\alpha}, \widetilde{A}_j^{\alpha}) d\alpha\,,$$

$$= \frac{1}{2(\beta_2 - \beta_1)} \int_{\alpha=0}^{1} (a_{il}(\alpha) - a_{jl}(\alpha) + a_{iu}(\alpha) - a_{ju}(\alpha)) d\alpha\,, \qquad (3.2)$$

where

$Supp(\widetilde{A}_1) \cup Supp(\widetilde{A}_2) \cup \ldots \cup Supp(\widetilde{A}_m) = [\beta_1, \beta_2]$.

Given the DM's preference attitude index λ ($\lambda \in [0,1]$), the interval $[0,1]$ represents a range from the most strict attitude to the most weak attitude on preference. We have preference relations between $\widetilde{A}_i$ and $\widetilde{A}_j$ as:

(1) if $p(\widetilde{A}_i, \widetilde{A}_j) > \lambda$, then $\widetilde{A}_i \succ \widetilde{A}_j$;

(2) if $| p(\widetilde{A}_i, \widetilde{A}_j) | \le \lambda$, then $\widetilde{A}_i \sim \widetilde{A}_j$;

(3) if $p(\widetilde{A}_i, \widetilde{A}_j) < -\lambda$, then $\widetilde{A}_i \prec \widetilde{A}_j$.

Let $\widetilde{A}_i = (a_{i1}, a_{i2}, a_{i3})$, $\widetilde{A}_j = (a_{j1}, a_{j2}, a_{j3})$ ($i, j = 1,2,\ldots,m$) be two positive triangular fuzzy numbers, we calculate the ranking measurement as:

$$r(\widetilde{A}_i, \widetilde{A}_j) = \int_{\alpha=0}^{1} r(\widetilde{A}_i^{\alpha}, \widetilde{A}_j^{\alpha}) d\alpha$$

$$= \frac{1}{2} \int_{\alpha=0}^{1} (a_{il}(\alpha) - a_{jl}(\alpha) + a_{iu}(\alpha) - a_{ju}(\alpha)) d\alpha$$

$$= \frac{1}{2} \int_{\alpha=0}^{1} [a_{i1} - a_{j1} + a_{i3} - a_{j3} + \alpha(2a_{i2} + a_{j1} + a_{j3} - 2a_{j2} - a_{i1} - a_{i3})] d\alpha$$

$$= \frac{a_{i1} + 2a_{i2} + a_{i3} - a_{j1} - 2a_{j2} - a_{j3}}{4}. \qquad (3.3)$$

Similarly, the preference measurement is calculated as:

$$p(\widetilde{A}_i, \widetilde{A}_j) = \frac{1}{(\beta_2 - \beta_1)} \int_{\alpha=0}^{1} r(\widetilde{A}_i^{\alpha}, \widetilde{A}_j^{\alpha}) d\alpha$$

$$= \frac{a_{i1} + 2a_{i2} + a_{i3} - a_{j1} - 2a_{j2} - a_{j3}}{4(\max(a_{13}, a_{23}, \ldots, a_{m3}) - \min(a_{11}, a_{21}, \ldots, a_{m1}))}. \qquad (3.4)$$

3.2.2 Proposed Fuzzy ELECTRE

In this section, we introduce the proposed method. The method consists of six steps as follows.

***Step* 1: Problem Formulation**

A fuzzy MCDM problem can be concisely expressed in the matrix format as:

$$\tilde{D} = \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \dots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \dots & \tilde{x}_{2n} \\ \dots & \dots & \dots & \dots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \dots & \tilde{x}_{mn} \end{bmatrix}, \tag{3.5}$$

$$\tilde{W} = (\tilde{w}_1, \quad \tilde{w}_2, \quad \dots, \quad \tilde{w}_n), \tag{3.6}$$

where $\tilde{x}_{ij}$ and $\tilde{w}_j$ ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$) are positive triangular fuzzy numbers. $\tilde{x}_{ij}$ is the rating of alternative A_i with respect to criterion C_j, and it forms a fuzzy matrix referred to as a decision matrix. $\tilde{w}_j$ is the weight of criterion C_j, and it forms a fuzzy vector referred to as a weighting vector.

***Step* 2: Normalize the Decision Matrix**

This procedure transforms the various attribute scales into comparable scales. Linear scale normalization is used for its simplicity.

$$\tilde{D}^n = \begin{bmatrix} \tilde{r}_{11} & \tilde{r}_{12} & \dots & \tilde{r}_{1n} \\ \tilde{r}_{21} & \tilde{r}_{22} & \dots & \tilde{r}_{2n} \\ \dots & \dots & \dots & \dots \\ \tilde{r}_{m1} & \tilde{r}_{m2} & \dots & \tilde{r}_{mn} \end{bmatrix}, \tag{3.7}$$

where

$$\tilde{r}_{ij} = \begin{cases} (\frac{x_{ij1}}{M}, \frac{x_{ij2}}{M}, \frac{x_{ij3}}{M}), & M = \max_i x_{ij3}, \quad j \in B. \\ (\frac{N}{x_{ij3}}, \frac{N}{x_{ij2}}, \frac{N}{x_{ij1}}), & N = \min_i x_{ij1}, \quad j \in C. \end{cases}$$

Here B and C represent benefit criteria and cost criteria, respectively. A maximum value among the alternatives is expected for benefit criteria. While a minimum value among the alternatives is expected for cost criteria.

***Step* 3: Calculate the Weighted Normalized Decision Matrix**

The weighted normalized decision matrix is defined by multiplying each column of matrix with its associated weight as:

$$\widetilde{V} = \begin{bmatrix} \widetilde{v}_{11} & \widetilde{v}_{12} & \cdots & \widetilde{v}_{1n} \\ \widetilde{v}_{21} & \widetilde{v}_{22} & \cdots & \widetilde{v}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{v}_{m1} & \widetilde{v}_{m2} & \cdots & \widetilde{v}_{mn} \end{bmatrix}, \tag{3.8}$$

where

$$\widetilde{v}_{ij} = \widetilde{w}_j \widetilde{x}_{ij} = (w_{j1}x_{ij1}, w_{j2}x_{ij2}, w_{j3}x_{ij3}).$$

***Step* 4: Determine the Concordance and Discordance Sets**

For each pair of alternatives A_k and A_l ($k,l = 1,2,\ldots,m$ and $k \neq l$), when the DM prefers A_k to A_l, the set of decision criteria $J = (j \mid j = 1,2,\ldots,n\}$ is divided into concordance sets C_{kl} and discordance sets D_{kl} with corresponding definitions:

$$C_{kl} = \{j \mid p(\widetilde{v}_{kj}, \widetilde{v}_{lj}) > \lambda\},\ \lambda \in [0,1]; \tag{3.9}$$

$$D_{kl} = \{j \mid p(\widetilde{v}_{kj}, \widetilde{v}_{lj}) < -\lambda\},\ \lambda \in [0,1], \tag{3.10}$$

where

$p(\cdot,\cdot)$ is the preference measurement between two fuzzy numbers.

If $| p(\tilde{v}_{kj}, \tilde{v}_{lj}) | \leq \lambda$, the DM is indifferent between alternatives A_k and A_l. Therefore, the relevant criteria neither belong to concordance set nor discordance set.

***Step* 5: Calculate the Concordance and Discordance Indices**

The concordance index measures the strength of confidence by evaluating the criteria weights in the concordance set, while the discordance index measures the strength of disagreement by evaluating the ratings of the alternatives in the discordance set. The concordance index is defined as:

$$\tilde{C}_{kl} = \sum_{j \in C_{kl}} \tilde{w}_j \,. \tag{3.11}$$

Correspondingly, the discordance index is defined as:

$$D_{kl} = \frac{\sum_{j \in D_{kl}} | r(\tilde{v}_{kj}, \tilde{v}_{lj}) |}{\sum_{j \in J} | r(\tilde{v}_{kj}, \tilde{v}_{lj}) |}, \tag{3.12}$$

where

$$r(\tilde{v}_{kj}, \tilde{v}_{lj}) = \frac{1}{2} \int_{\alpha=0}^{1} (v_{kjl}(\alpha) - v_{ljl}(\alpha) + v_{kju}(\alpha) - v_{lju}(\alpha)) d\alpha \,.$$

Note that the information contained in the discordance index differs significantly from that contained in the concordance index, making the information content of $\tilde{C}_{kl}$ and D_{kl} complementary. Differences among weights are represented by means of the concordance matrix, whereas differences among rating values are represented by means of the discordance matrix.

***Step* 6: Determine the Outranking Relations**

One traditional method uses the average values of concordance indices and discordance indices as thresholds to establish the outranking relations between two alternatives. These thresholds are rather arbitrary and have great impact on the final outranking. Moreover, this method leads to cumbersome computing in fuzzy environment. Van Delft and Nijkamp (1977) introduced the net dominance relationships for the complementary analysis of the ELECTRE method. Similarly, we extend it to the fuzzy number situation. The net concordance index C_k, which measures the strength of the total dominance of alternative A_k that exceeds the strength to which other alternatives dominate A_k, is defined as:

$$C_k = r(\sum_{\substack{n=1 \\ n\neq k}}^{m} \widetilde{C}_{kn}, \sum_{\substack{n=1 \\ n\neq k}}^{m} \widetilde{C}_{nk}), \tag{3.13}$$

where

$r(\cdot,\cdot)$ is the ranking measurement between two fuzzy numbers as defined in (3.1).

Similarly, the net discordance index D_k, which measures the relative weakness of alternative A_k compared to other alternatives, is defined as:

$$D_k = \sum_{\substack{n=1 \\ n\neq k}}^{m} D_{kn} - \sum_{\substack{n=1 \\ n\neq k}}^{m} D_{nk}. \tag{3.14}$$

Obviously alternative A_k has a higher preference with a higher value of C_k and a lower value of D_k.

Step 7: Determine the Performance Index

Finally, the net concordance and net discordance indices are combined to evaluate the performance of alternatives. According to the performance index we can obtain the ranking order and choose the best one. We define the final performance index as:

$$E_k = C_k - D_k . \tag{3.15}$$

In summary, the procedure of proposed fuzzy extension of ELECTRE is given as follows:

Step 1: Formulate the problem as expressed in (3.5) and (3.6).

Step 2: Normalize the decision matrix as expressed in (3.7).

Step 3: Calculate the weighted normalized decision matrix by (3.8).

Step 4: Determine the Concordance and Discordance Sets by (3.9) and (3.10).

Step 5: Calculate the Concordance and Discordance Indices by (3.11) and (3.12).

Step 6: Determine the Outranking Relations by (3.13) and (3.14).

Step 7: Determine the Performance Index by (3.15) and rank the order of the alternatives.

In the following chapter, a numerical example is given to illustrate the computation process.

Chapter 4

A Numerical Example of Fuzzy ELECTRE

In this Chapter, we illustrate our fuzzy ELECTRE method with an example.

4.1 A Step-by-step Approach

Here we have four alternatives with four benefit criteria that need to be evaluated and ranked. The procedure is as follows.

Step 1: Problem Formulation

The decision matrix and the weighting vector of the problem are given in Table 4.1.

Table 4.1 Decision matrix and weighting vector

	C1	C2	C3	C4
	(0.20, 0.21,0.25)	(0.25,0.28, 0.30)	(0.30, 0.40, 0.53)	(0.10, 0.12, 0.14)
A1	(8.00, 9.00, 9.00)	(2.00, 6.00, 7.00)	(5.00, 6.00, 8.00)	(2.00, 3.00, 9.00)
A2	(3.00, 4.00, 9.00)	(6.00, 6.00, 8.00)	(1.00, 4.00, 5.00)	(4.00, 5.00, 6.00)
A3	(1.00, 6.00, 9.00)	(3.00, 7.00, 8.00)	(3.00, 7.00, 8.00)	(5.00, 7.00, 8.00)
A4	(4.00, 5.00, 6.00)	(4.00, 4.00, 5.00)	(4.00, 8.00, 9.00)	(7.00, 7.00, 8.00)

Step 2: Normalize the Decision Matrix

We normalize the decision matrix by (3.7) and the resulting matrix is shown in Table 4.2.

Table 4.2 Normalized decision matrix

	C1	C2	C3	C4
A1	(0.889, 1.000, 1.000)	(0.250, 0.750, 0.875)	(0.556, 0.667, 0.889)	(0.222, 0.333, 1.000)
A2	(0.333, 0.444, 1.000)	(0.750, 0.750, 1.000)	(0.111, 0.444, 0.556)	(0.444, 0.556, 0.667)
A3	(0.111, 0.667, 1.000)	(0.375, 0.875, 1.000)	(0.333, 0.778, 0.889)	(0.556, 0.778, 0.889)
A4	(0.444, 0.556, 0.667)	(0.500, 0.500, 0.625)	(0.444, 0.889, 1.000)	(0.778, 0.778, 0.889)

***Step* 3: Weighting the Normalized Matrix**

We construct the weighted normalized matrix by (3.8) in Table 4.3.

Table 4.3 Weighted normalized decision matrix

	C1	C2	C3	C4
A1	(0.178, 0.210, 0.250)	(0.063, 0.210, 0.263)	(0.167, 0.267, 0.471)	(0.022, 0.040, 0.140)
A2	(0.067, 0.093, 0.250)	(0.188, 0.210, 0.300)	(0.033, 0.178, 0.294)	(0.044, 0.067, 0.093)
A3	(0.022, 0.140, 0.250)	(0.094, 0.245, 0.300)	(0.100, 0.311, 0.471)	(0.056, 0.093, 0.124)
A4	(0.089, 0.117, 0.167)	(0.125, 0.140, 0.188)	(0.133, 0.356, 0.530)	(0.078, 0.093, 0.124)

***Step* 4: Determine the Concordance and Discordance Sets**

The preference measurements between two alternatives (row alternative preference measurement to column alternative) are calculated with respect to each criterion by (3.9) and (3.10) in Tables 4.4, 4.5, 4.6, and 4.7. According to the DM's preference attitude λ=0.2, the outranking relations are determined in Tables 4.8, 4.9, 4.10, and 4.11, in which 1 represents that the row alternative outranks the column alternative, 0 represents indifference between the two alternatives, and -1 represents the row alternative is outranked by the column alternative. The concordance and discordance sets of the criteria are determined from these outranking relations.

Table 4.4 Preference measurements with respect to C1

	A1	A2	A3	A4
A1	-	0.378	0.324	0.394
A2	-0.378	-	-0.054	0.016
A3	-0.324	0.054	-	0.070
A4	-0.394	-0.016	-0.070	-

Table 4.5 Preference measurements with respect to C2

	A1	A2	A3	A4
A1	-	-0.171	-0.146	0.161
A2	0.171	-	0.025	0.332
A3	0.146	-0.025	-	0.307
A4	-0.161	-0.332	-0.307	-

Table 4.6 Preference measurements with respect to C3

	A1	A2	A3	A4
A1	-	0.246	-0.011	-0.102
A2	-0.246	-	-0.257	-0.348
A3	0.011	0.257	-	-0.091
A4	0.102	0.348	0.091	-

Table 4.7 Preference measurements with respect to C4

	A1	A2	A3	A4
A1	-	-0.061	-0.264	-0.311
A2	0.061	-	-0.203	-0.250
A3	0.264	0.203	-	-0.047
A4	0.311	0.250	0.047	-

Table 4.8 Outranking relations with respect to C1 when $\lambda=0.2$

	A1	A2	A3	A4
A1	-	1	1	1
A2	-1	-	0	0
A3	-1	0	-	0
A4	-1	0	0	-

Table 4.9 Outranking relations with respect to C2 when $\lambda=0.2$

	A1	A2	A3	A4
A1	-	0	0	0
A2	0	-	0	1
A3	0	0	-	1
A4	0	-1	-1	-

Table 4.10 Outranking relations with respect to C3 when $\lambda=0.2$

	A1	A2	A3	A4
A1	-	1	0	0
A2	-1	-	-1	-1
A3	0	1	-	0
A4	0	1	0	-

Table 4.11 Outranking relations with respect to C4 when $\lambda=0.2$

	A1	A2	A3	A4
A1	-	0	-1	-1
A2	0	-	-1	-1
A3	1	1	-	0
A4	1	1	0	-

***Step* 5: Calculate the Concordance and Discordance Indices**

The concordance and discordance indices are calculated by (3.11) and (3.12) respectively, and the results when $\lambda=0.2$ are shown in Tables 4.12 and 4.13.

Table 4.12 Concordance indices when $\lambda=0.2$

	A1	A2	A3	A4
A1	-	(0.50, 0.61, 0.78)	(0.20, 0.21, 0.25)	(0.20, 0.21, 0.25)
A2	(0.00, 0.00, 0.00)	-	(0.00, 0.00, 0.00)	(0.25, 0.28, 0.30)
A3	(0.10, 0.12, 0.14)	(0.40, 0.52, 0.67)	-	(0.25, 0.28, 0.30)
A4	(0.10, 0.12, 0.14)	(0.40, 0.52, 0.67)	(0.00, 0.00, 0.00)	-

Table 4.13 Discordance indices when $\lambda=0.2$

	A1	A2	A3	A4
A1	-	0	0.214	0.170
A2	0.813	-	0.893	0.711
A3	0.509	0	-	0
A4	0.417	0.277	0.522	-

***Step* 6: Determine the Outranking Relation**

The net concordance indices and the net discordance indices are calculated by (3.13) and (3.14), and the results when λ=0.2 are shown in Table 4.14.

Table 4.14 Net concordance indices (NCI) and net discordance indices (NDI) when λ=0.2

	NCI	NDI
A1	0.820	-1.354
A2	-1.403	2.140
A3	0.708	-1.120
A4	-0.125	0.335

***Step* 7: Determine the Performance Index**

Calculate the performance indices by (3.15) in Table 4.15 when λ=0.2.

Table 4.15 Performance indices (PI) when λ=0.2

	A1	A2	A3	A4
PI	2.174	-3.542	1.828	-0.460

Repeating the same steps, the performance indices with respect to the DM's preference attitudes taken as 0, 0.1, …, 1 are calculated, and the results are shown in Table 4.16 and Figure 4.1.

Table 4.16 Performance indices with respect to λ values

λ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.438	1.032	2.174	1.624	0.000	0.000	0.000	0.000	0.000	0.000	0.000
A2	-2.705	-3.106	-3.542	-1.014	0.000	0.000	0.000	0.000	0.000	0.000	0.000
A3	2.164	2.344	1.828	0.073	0.000	0.000	0.000	0.000	0.000	0.000	0.000
A4	0.102	-0.271	-0.460	-0.683	0.000	0.000	0.000	0.000	0.000	0.000	0.000

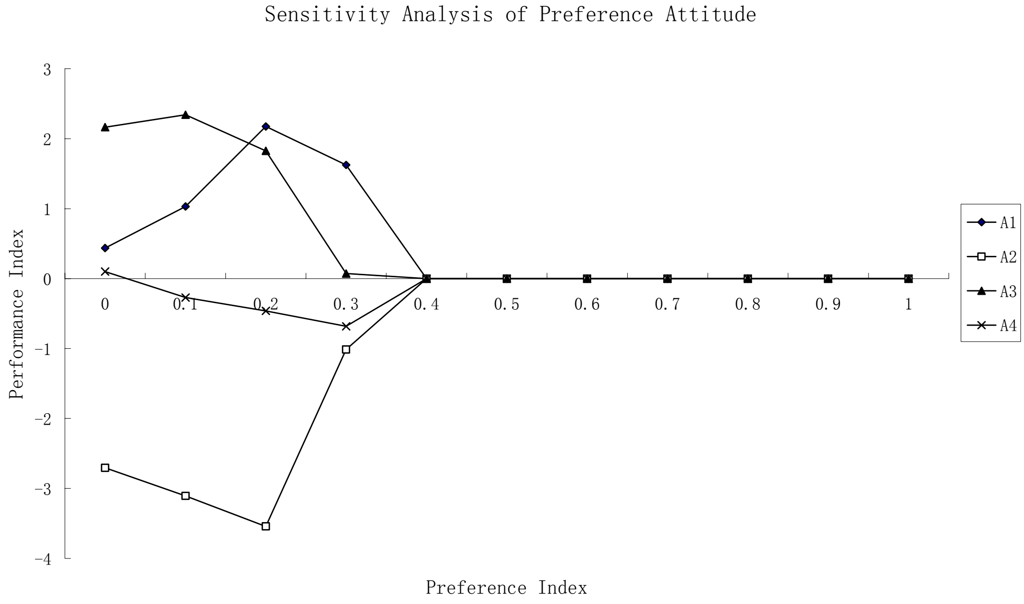

Figure 4.1 Sensitivity analysis with the DM's preference attitudes

The results in Figure 4.1 show that when the DM's preference threshold is approximately below 0.2, the ranking order is almost A3, A1, A4, and A2 from best to worst. Beyond 0.2, the four lines begin to converge to 0 gradually. When the preference threshold reaches 0.4, we cannot distinguish the four alternatives. Therefore, we can choose A3 as the best alternative.

4.2 Summary

In chapters 3 and 4, the fuzzy extension of the ELECTRE method is proposed to solve problems in the fuzzy environment by incorporating the DM's preference attitudes. Fuzzy ranking measurement and preference measurement are proposed to determine ranking relations between fuzzy numbers.

The ELECTRE method is regarded as one of the best MCDM methods because of its simple logic, full utilization of information and refined computational procedure. Our proposed fuzzy ELECTRE method provides an efficient way to treat the imprecision and subjectiveness that may arise in the decision process, and it is flexible in solving complex problems.

In the next two chapters, we will propose a fuzzy MCDM method based on risk and confidence analysis, followed by an example.

Chapter 5

Fuzzy MCDM Based on Risk and Confidence Analysis

In this chapter, we propose a fuzzy MCDM method based on risk and confidence analysis. First we propose the methods to model the DM's risk attitude and confidence attitude towards uncertainty by the linguistic approach. Then we present our detailed fuzzy MCDM model.

5.1 Introduction

To deal with uncertainty in decision analysis, the human-related, subjective judgment and interpretation of "uncertainty" is needed (Zimmermann, 2002). Indubitably, the value of fuzzy MCDM methods will be improved if the human adaptability, intransitivity, and dynamic adjustment of preferences can be considered in the decision process (Liang, 1999). The DM's subjective preference and judgment are intuitively involved in the process of decision analysis. Incorporating the optimism index into fuzzy MCDM is first proposed by Zeleny (1982). Some other MCDM methods (Cheng and Mon, 1994; Cheng, 1996; Deng, 1999; Yeh and Deng, 1997) also utilize the DM's confidence interval and optimism index to evaluate the alternatives.

We (2003a, 2003b, 2003c, 2003d and 2003e) proposed fuzzy MCDM based on risk and confidence analysis. This method introduces the modeling of confidence attitude and risk attitude towards uncertainty to support normative fuzzy MCDM. In this approach, the DM's subjective preference, judgment and assessment are incorporated into decision process. Thus, it provides an effective way to solve complex, ill-defined and human-oriented MCDM problems.

This method uses fuzzy numbers and the linguistic approach to establish risk and confidence analysis into the multiple criteria decision model. The linguistic approach is first introduced in section 5.2, and in section 5.3, we will introduce the linguistic modeling of risk and confidence attitude and the proposed fuzzy MCDM model based on risk and confidence analysis.

5.2 Modeling of Linguistic Approach

Fuzzy set theory is useful in processing linguistic information. The linguistic approach is an effective way of expressing the DM's subjectiveness under different decision situations. It is used in situations in which the problem is too complex or too ill-defined. By using a vector-valued objective function, it provides a language for an approximate linguistic characterization of the trade-offs between its components. The central concept of the linguistic approach is the linguistic variable. A linguistic variable can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms. By means of linguistic variables, the membership functions of fuzzy number are processed accordingly. Linguistic terms have been intuitively used in expressing the subjectiveness and imprecision of the DM's assessments (Zadeh, 1975; Deng and Yeh, 1998; Liang, 1999). The basic definitions are as follows.

Definition 5.1: A linguistic variable is characterized by a quintuple (x , $T(x)$, U , G , $\widetilde{M}$) in which x is the name of the variable; $T(x)$ (or simply (T)) denotes the term set of x, that is, the set of names of linguistic values of x, with each value being a fuzzy variable denoted generically by X and ranging over a universe of discourse U that is associated with the base variable u; G is a syntactic rule for generating the name, X, of values of x; and M is a semantic rule for associating with each X its meaning, $\widetilde{M}(x)$, which is a fuzzy subset of U. Any X, generated by G, is called a term. Often the name of the variable and the generic name of the elements of the variable are denoted by the same symbol. The same holds for X and $\widetilde{M}$.

Definition 5.2: A linguistic variable x is called termed if $T(x)$ and the meaning $\widetilde{M}(x)$ can be regarded as algorithms that generate the terms of the term set and associate meanings with them.

Definition 5.3: A linguistic hedge or a modifier is an operation that changes the meaning of a term or more generally, of a fuzzy set. If $\widetilde{A}$ is a fuzzy set, its modifier m generates the term $\widetilde{B} = m(\widetilde{A})$.

Mathematical models frequently used for modifiers are as follows:

a. Concentration:

$$\mu_{con(\widetilde{A})}(x) = (\mu_{\widetilde{A}}(x))^2 .$$

b. Dilation:

$$\mu_{dil(\widetilde{A})}(x) = (\mu_{\widetilde{A}}(x))^{1/2} .$$

c. Contrast intensification:

$$\mu_{\mathrm{int}(\tilde{A})}(x) = \begin{cases} 2(\mu_{\tilde{A}}(x))^2, & \mu_{\tilde{A}}(x) \in [0,0.5], \\ 1-2(1-\mu_{\tilde{A}}(x))^2, & otherwise. \end{cases}$$

5.3 The Proposed Method

Interval information is common in uncertain situations (Moore, R.E. 1979; Neumaier, A., 1990; Alefeld, G., Mayer, G., 1996). An interval number is based on a two-value judgment: the minimum possible value and the maximum possible value. In our proposed method, we use the interval number to represent the uncertain rating of alternatives and weights of criteria in the MCDM problem.

For the DM's risk attitude towards uncertainty, the optimist tends to feel that the uncertainty will be resolved in a favorable manner and the pessimist tends to feel that the uncertainty will be resolved in an unfavorable manner (Yager, 2000). In the case of risk attitude to interval assessments, optimism (absolute) means a higher preference to superior value, while pessimism (absolute) means a higher preference to inferior value. Next, another kind of subjectiveness we deal with is the DM's confidence in the fuzzy assessments. More confidence means that the DM will give a higher preference to the values with a higher possibility and a lower preference to the values with a lower possibility. For the confidence attitude to a triangular fuzzy number, more confidence means assessment towards uncertainty is closer to the modal value. Naturally the DM's risk attitudes and confidence attitudes are vague in complex and ill-defined situations. Linguistic terms are intuitively used to express these attitudes.

5.3.1 Modeling of Risk Attitudes

Interval arithmetic is introduced in detail by Moore (1979) and Neumaier (1990). We define an interval number and its arithmetic operations as follows:

Definition 5.4: An interval number $\overline{A}$ is defined by a closed interval $[a^{\inf}, a^{\sup}]$.

Let $\overline{A} = [a^{\inf}, a^{\sup}]$ and $\overline{B} = [b^{\inf}, b^{\sup}]$ be two positive interval numbers ($a^{\inf} > 0$ and $b^{\inf} > 0$). The basic arithmetic operators are defined as:

a. Negation: $-\overline{A} = [-a^{\sup}, -a^{\inf}]$.

b. Inversion: $\overline{A}^{-1} = [1/a^{\sup}, 1/a^{\inf}]$.

c. Addition: $\overline{A} + \overline{B} = [a^{\inf} + b^{\inf}, a^{\sup} + b^{\sup}]$.

d. Subtraction: $\overline{A} - \overline{B} = [a^{\inf} - b^{\sup}, a^{\sup} - b^{\inf}]$.

e. Multiplication: $\overline{A}\overline{B} = [a^{\inf} b^{\inf}, a^{\sup} b^{\sup}]$.

f. Division: $\overline{A}/\overline{B} = [a^{\inf}/b^{\sup}, a^{\sup}/b^{\inf}]$. (5.1)

For the risk attitude towards the interval number expressed by a superior value and an inferior value, optimism (absolute) means a higher preference to the superior value, while pessimism (absolute) means a higher preference to the inferior value. A linguistic variable "risk attitude" is defined as a mathematical model. Here linguistic terms we use are absolutely optimism (*AO*), very optimism (*VO*), optimism (*O*), fairly optimism (*FO*), neutral (*N*), fairly pessimism (*FP*), pessimism (*P*), very pessimism (*VP*), and absolutely pessimism (*AP*) to represent the decision maker's qualitative assessments. The number nine is based on Miller's theory (1956) that seven plus or minus two represents the great amount of information that the DM can express on the basis of a subjective judgment.

Definition 5.5: *T* (Risk Attitude) = {*AO*, *VO*, *O*, *FO*, *N*, *FP*, *P*, *VP*, *AP*}.

Fuzzy numbers are intuitively easy and effective in expressing the DM's qualitative assessments (Liang, 1999; Yeh and Deng, 2000; Chen, 2000; Cheng, 2002). Here we propose triangular fuzzy numbers to express linguistic terms of risk attitudes to the interval uncertainty. With the reference of the inferior value and superior value as the lower value and upper value of the support boundary, respectively, the modal values are taken in an average distribution with respect to the optimism (pessimism) attitudes accordingly. Thus, we define the triangular fuzzy numbers to represent optimism (pessimism) attitude towards risk as:

Definition 5.6: To express the decision attitude to an interval $\overline{A} = [a^{\inf}, a^{\sup}]$, a triangular fuzzy number is defined as:

$$\widetilde{A} = (a_1,\ a_2,\ a_3)\ , \tag{5.2}$$

where

$a_1 = a^{\inf}$, $a_3 = a^{\sup}$, and $a_2 = a^{\inf} + (a^{\sup} - a^{\inf})(x-1)/8$, $x = 1,\ 2,\ \ldots,\ 9$ represent the linguistic terms *AP*, *VP*, *P*, *FP*, *N*, *FO*, *O*, *VO*, and *AO*, respectively.

By this method, we incorporate the DM's risk attitudes into interval assessments. The linguistic terms of risk attitudes expressed by triangular fuzzy numbers are presented in Table 5.1 and Figure 5.1.

Table 5.1 Linguistic terms of risk attitude

Linguistic term	Triangular Fuzzy number
Absolutely optimism (*AO*)	$(a^{\inf}, a^{\sup}, a^{\sup})$
Very optimism (*VO*)	$(a^{\inf}, (a^{\inf} + 7a^{\sup})/8, a^{\sup})$
optimism (*O*)	$(a^{\inf}, (a^{\inf} + 3a^{\sup})/4, a^{\sup})$
Fairly optimism (*FO*)	$(a^{\inf}, (3a^{\inf} + 5a^{\sup})/8, a^{\sup})$
Neutral (*N*)	$(a^{\inf}, (a^{\inf} + a^{\sup})/2, a^{\sup})$
Fairly pessimism (*FP*)	$(a^{\inf}, (5a^{\inf} + 3a^{\sup})/8, a^{\sup})$
pessimism (*P*)	$(a^{\inf}, (3a^{\inf} + a^{\sup})/4, a^{\sup})$
Very pessimism (*VP*)	$(a^{\inf}, (7a^{\inf} + a^{\sup})/8, a^{\sup})$
Absolutely pessimism (*AP*)	$(a^{\inf}, a^{\inf}, a^{\sup})$

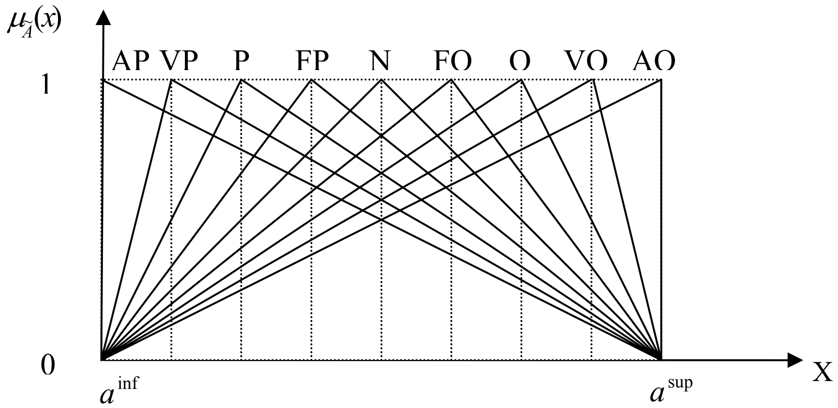

Figure 5.1 Linguistic terms of risk attitude

5.3.2 Modeling of Confidence Attitudes

In assessing the uncertainty of the fuzzy numbers, we need to analyze the confidence of the DM to the truth or reliability. The interval of confidence (A. Kaufmann and M.M. Gupta, 1985) is a way to incorporate the confidence attitude into fuzzy numbers. Some MCDM methods (Cheng and Mon, 1994; Cheng, 1996; Deng, 1999; Yeh and Deng, 1997) have used confidence interval concepts to evaluate the alternatives. However, this confidence interval cannot fully incorporate the DM's confidence towards the uncertainty. A fuzzy number on confidence is more effective on this matter.

We propose a method to express the DM's confidence on fuzzy numbers. More confidence means that the DM's assessment is closer to the most likely value. In the case of a triangular fuzzy number, this means that the DM's assessment is closer to the modal value. Therefore, we define a modified triangular fuzzy number based on the α-cut concept to incorporate the DM's confidence assessment to the uncertainty as:

Definition 5.7: Assuming that confidence to the triangular fuzzy number $\widetilde{A} = (a_1,\ a_2,\ a_3)$ is at level α, the refined fuzzy number on confidence level is defined as:

$$\widetilde{A}^{\alpha} = (a_1(\alpha), a_2, a_3(\alpha)) = (a_1 + \alpha(a_2 - a_1), a_2, a_3 - \alpha(a_3 - a_2)),\ \alpha \in [0,1]. \tag{5.3}$$

Figure 5.2 shows a triangular fuzzy number and its corresponding α-cut triangular fuzzy number.

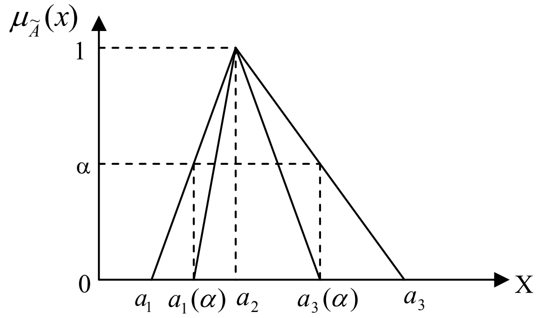

Figure 5.2 A triangular fuzzy number $\widetilde{A}$ and its α-cut triangular fuzzy number

Some main operations for positive triangular fuzzy numbers on confidence level α are as follows:

$\forall a_1(\alpha),\ b_1(\alpha) \in R^+$, $\widetilde{A}^\alpha = (a_1(\alpha), a_2, a_3(\alpha))$, $\widetilde{B}^\alpha = (b_1(\alpha), b_2, b_3(\alpha))$, and $\alpha \in [0,1]$.

a. Addition: $\widetilde{A}^\alpha + \widetilde{B}^\alpha = (a_1(\alpha) + b_1(\alpha), a_2 + b_2, a_3(\alpha) + b_3(\alpha))$;

b. Subtraction: $\widetilde{A}^\alpha - \widetilde{B}^\alpha = (a_1(\alpha) - b_3(\alpha), a_2 - b_2, a_3(\alpha) - b_1(\alpha))$;

c. Multiplication: $\widetilde{A}^\alpha \widetilde{B}^\alpha = (a_1(\alpha) b_1(\alpha), a_2 b_2, a_3(\alpha) b_3(\alpha))$;

d. Division: $\widetilde{A}^\alpha / \widetilde{B}^\alpha = (a_1(\alpha)/b_3(\alpha), a_2/b_2, a_3(\alpha)/b_1(\alpha))$. (5.4)

The DM's confidence attitudes are often vague in complex and ill-defined situations. Like risk attitudes, an effective way is to use linguistic terms to express the DM's subjective attitudes under different situations. For the linguistic variable "confidence attitude", we use linguistic terms as absolutely confidence (*AC*), very confidence (*VC*), confidence (*C*), fairly confidence (*FC*), neutral (*N*), fairly non-confidence (*FNC*), non-confidence (*NC*), very non-confidence (*VNC*), and absolutely non-confidence (*ANC*) to represent the DM's qualitative assessments. We define these linguistic terms as:

Definition 5.8: *T* (Confidence Attitude) = {*AC*, *VC*, *C*, *FC*, *N*, *FNC*, *NC*, *VNC*, *ANC*}.

Using the confidence level α in the interval [0, 1], we define the membership function of the linguistic terms of "confidence attitude" to express the DM's subjective confidence. Obviously, the membership degree of confidence will increase linearly when α increases from 0 to 1. Thus, we can use a linear function to represent it and other confidence terms can be defined by the concentration, dilation and contrast intensification operations, accordingly.

Definition 5.9: The linguistic terms and their corresponding membership functions are defined in Table 5.2 and shown in Figure 5.3.

Table 5.2 Linguistic terms of confidence attitude

Linguistic term	Membership function
Absolutely confidence (*AC*)	$\mu_{AC}(\alpha)=\begin{cases}1, & \alpha=1\\ 0, & otherwise\end{cases}$, $\alpha\in[0,1]$.
Very confidence (*VC*)	$\mu_{VC}(\alpha)=(\mu_C(\alpha))^2=\alpha^2$, $\alpha\in[0,1]$.
Confidence (*C*)	$\mu_C(\alpha)=\alpha$, $\alpha\in[0,1]$.
Fairly confidence (*FC*)	$\mu_{FC}(\alpha)=(\mu_C(\alpha))^{0.5}=\sqrt{\alpha}$, $\alpha\in[0,1]$.
Neutral (*N*)	$\mu_U(\alpha)=1$, $\alpha\in[0,1]$.
Fairly non-confidence (*FNC*)	$\mu_{FNC}(\alpha)=(1-\mu_C(\alpha))^{0.5}=\sqrt{1-\alpha}$, $\alpha\in[0,1]$.
Non-confidence (*NC*)	$\mu_{NC}(\alpha)=1-\mu_C(\alpha)=1-\alpha$, $\alpha\in[0,1]$.
Very non-confidence (*VNC*)	$\mu_{VNC}(\alpha)=(1-\mu_C(\alpha))^2=(1-\alpha)^2$, $\alpha\in[0,1]$.
Absolutely non-confidence (*ANC*)	$\mu_{ANC}(\alpha)=\begin{cases}1, & \alpha=0\\ 0, & otherwise\end{cases}$, $\alpha\in[0,1]$.

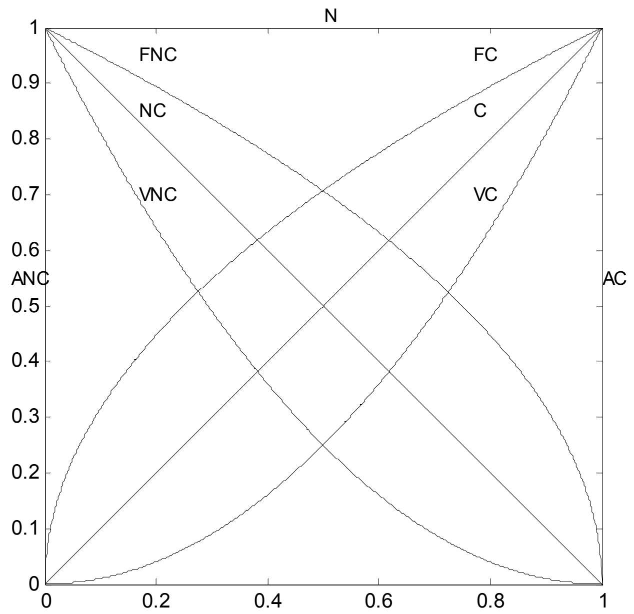

Figure 5.3 Linguistic terms of confidence attitude

Based on the definition above, there are four basic properties of the linguistic terms of confidence attitude:

a. $(very)^n$ confidence $\rightarrow$ absolutely confidence as $n \rightarrow \infty$;

b. $(very)^n$ non-confidence $\rightarrow$ absolutely non-confidence as $n \rightarrow \infty$;

c. $(fairly)^n$ confidence $\rightarrow$ neutral as $n \rightarrow \infty$;

d. $(fairly)^n$ non confidence $\rightarrow$ neutral as $n \rightarrow \infty$.

We still need to compare, evaluate and aggregate the performance of the alternatives on the confidence levels. Therefore, a vector method is proposed here. According to the membership function of the linguistic term, the confidence membership value is determined with respect to the confidence level. We define the confidence vector as:

Definition 5.10: Assume that there is a total of l ($l \geq 2$) confidence levels. The confidence vector is defined as:

$$C_{LT} = (c_1, \quad \ldots \quad c_k, \quad \ldots \quad c_l), \tag{5.5}$$

where

$c_k = \mu_{LT}(\alpha)$, $\alpha = {k-1}/{l-1}$, $k = 1,2,...,l,\ l \geq 2$, and LT represents linguistic terms *AC*, *VC*, *C*, *FC*, *N*, *FNC*, *NC*, *VNC*, and *ANC*, respectively.

The selection of l is rather arbitrary. The larger the l, the more calculation is needed, but a closer to real confidence membership function is achieved. We need a normalized scale for comparable calculation. Therefore, a normalized format is defined as:

Definition 5.11: The normalized confidence vector is defined as:

$$C_{LT}^* = (c_1^*, \quad \ldots \quad c_k^*, \quad \ldots \quad c_l^*), \tag{5.6}$$

where

$c_k^* = c_k \Big/ \sum_{k=1}^{l} c_k$, and symbol c_k has the same meaning as equation (5.5).

For the linguistic terms defined in Table 5.2, the corresponding confidence vectors are obtained as follows:

(1) Absolutely Confidence Vector:

$$C_{AC}^* = (0, \quad \ldots, \quad 0, \quad \ldots, \quad 1). \tag{5.7}$$

(2) Very Confidence Vector:

$$C_{VC}^{*}=\left(0,\quad \ldots,\quad \frac{\left(\frac{k-1}{l-1}\right)^{2}}{\sum_{k=1}^{l}\left(\frac{k-1}{l-1}\right)^{2}},\quad \ldots,\quad 1\right). \tag{5.8}$$

(3) Confidence Vector:

$$C_{C}^{*}=\left(0,\quad \ldots,\quad \frac{\frac{k-1}{l-1}}{\sum_{k=1}^{l}\left(\frac{k-1}{l-1}\right)},\quad \ldots,\quad \frac{1}{\sum_{k=1}^{l}\left(\frac{k-1}{l-1}\right)}\right). \tag{5.9}$$

(4) Fairly Confidence Vector:

$$C_{FC}^{*}=\left(0,\quad \ldots,\quad \frac{\sqrt{\frac{k-1}{l-1}}}{\sum_{k=1}^{l}\left(\sqrt{\frac{k-1}{l-1}}\right)},\quad \ldots,\quad \frac{1}{\sum_{k=1}^{l}\left(\sqrt{\frac{k-1}{l-1}}\right)}\right). \tag{5.10}$$

(5) Neutral vector:

$$C_{N}^{*}=\left(\frac{1}{l},\quad \ldots,\quad \frac{1}{l},\quad \ldots,\quad \frac{1}{l}\right). \tag{5.11}$$

(6) Fairly Non-Confidence Vector:

$$C_{FNC}^{*}=\left(\frac{1}{\sum_{k=1}^{l}\sqrt{1-\frac{k-1}{l-1}}},\quad \ldots,\quad \frac{\sqrt{1-\frac{k-1}{l-1}}}{\sum_{k=1}^{l}\sqrt{1-\frac{k-1}{l-1}}},\quad \ldots,\quad 0\right). \tag{5.12}$$

(7) Non-Confidence Vector:

$$C_{NC}^{*}=\left(\frac{1}{\sum_{k=1}^{l}(1-{k-1}/{l-1})},\quad \ldots,\quad \frac{1-{k-1}/{l-1}}{\sum_{k=1}^{l}(1-{k-1}/{l-1})},\quad \ldots,\quad 0\right). \tag{5.13}$$

(8) Very Non-Confidence Vector:

$$C_{VNC}^{*}=\left(\frac{1}{\sum_{k=1}^{l}(1-{k-1}/{l-1})^{2}},\quad \ldots,\quad \frac{(1-{k-1}/{l-1})^{2}}{\sum_{k=1}^{l}(1-{k-1}/{l-1})^{2}},\quad \ldots,\quad 0\right). \tag{5.14}$$

(9) Absolutely Non-Confidence Vector:

$$C_{AC}^{*}=(1,\quad \ldots,\quad 0,\quad \ldots,\quad 0). \tag{5.15}$$

5.3.3 Proposed Fuzzy MCDM based on Risk and Confidence Analysis

Fuzzy MCDM models are typically based on a two-phase approach (Zimmermann, 1987; Chen and Hwang, 1992; Munda et al., 1995; Ribeiro, 1996). The first phase is to aggregate the performance of the ratings of alternatives under criteria. Usually triangular fuzzy numbers are used to express the DM's assessments on the alternatives' performance in terms of each criterion. After the criteria are weighted, the fuzzy utilities represented by fuzzy numbers are aggregated by fuzzy arithmetic (Kaufmanns and Gupta, 1991). The second phase is to rank alternatives with respect to the aggregated performances. This involves the ranking of the alternatives based on the comparison of their corresponding fuzzy utilities.

In the second phase of fuzzy MCDM analysis, ranking of fuzzy numbers is a hard task. Though many methods have been proposed, the computation is complex and unreliable. This is because the comparison process may (a) involve considerable computations, (b) produce inconsistent outcomes by different fuzzy ranking methods, and (c) generate counter-intuitive ranking outcomes for similar fuzzy utilities (Bortolan and Degani, 1985; Zimmermann, 1987; Chen and Hwang, 1992; Chen and Klien, 1997). In our method, with reference to the imaged ideal alternative solutions, fuzzy numbers are aggregated into crisp performance in the second phase. Thus it makes the computation efficient and avoids the complicated and unreliable fuzzy number ranking.

We propose this approach to solve the fuzzy MCDM problems by incorporating the DM's risk attitude and confidence attitude. Interval numbers are used to assess the ratings of alternatives and the weights of criteria. The decision matrix is transformed into a performance matrix representing a weighted interval assessment. Risk attitudes are incorporated by triangular fuzzy numbers. Based on the α-cut concept, the fuzzy numbers are incorporated with confidence levels. According to the concept of ideal solutions, we define the fuzzy ideal solutions as: fuzzy positive ideal solution and fuzzy negative ideal solution. Then we measure the degree of separation of fuzzy numbers by the vertex method. The degree of separation transforms fuzzy performance into a crisp performance under confidence levels. According to the confidence attitudes, we obtain confidence vectors with respect to the membership functions. Finally, by aggregating performance values under confidence levels, the overall performance is obtained to evaluate the alternatives. We give the procedure as follows:

***Step* 1: Problem Formulation**

For conciseness, fuzzy MCDM can be expressed in the matrix format as:

$$\overline{D} = \begin{bmatrix} \overline{x}_{11} & \overline{x}_{12} & \cdots & \overline{x}_{1n} \\ \overline{x}_{21} & \overline{x}_{22} & \cdots & \overline{x}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{x}_{m1} & \overline{x}_{m2} & \cdots & \overline{x}_{mn} \end{bmatrix}, \tag{5.16}$$

$$\overline{W} = (\overline{w}_1, \ \ \overline{w}_2, \ \ \ldots, \ \ \overline{w}_n), \tag{5.17}$$

where

$\overline{x}_{ij}$ and $\overline{w}_j$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) are positive interval numbers. $\overline{x}_{ij}$ is the rating of alternative A_i with respect to criterion C_j, and it forms a matrix referred to as the decision matrix. $\overline{w}_j$ is the weight of criterion C_j, and it forms a vector referred to as the weighting vector.

***Step* 2: Construct the Performance Matrix**

Considering the importance of each criterion, we construct the fuzzy performance matrix by multiplying the weighting vector by the decision matrix, using the interval multiplication arithmetic operation.

$$\overline{P} = \begin{bmatrix} \overline{p}_{11} & \overline{p}_{12} & \cdots & \overline{p}_{1n} \\ \overline{p}_{21} & \overline{p}_{22} & \cdots & \overline{p}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{p}_{m1} & \overline{p}_{m2} & \cdots & \overline{p}_{mn} \end{bmatrix} = [(p_{ij}^{\inf}, p_{ij}^{\sup})], \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n, \tag{5.18}$$

where $p_{ij}^{\inf} = w_j^{\inf} x_{ij}^{\inf}$ and $p_{ij}^{\sup} = w_j^{\sup} x_{ij}^{\sup}$.

This process transforms the fuzzy decision matrix into a weighted fuzzy decision matrix, referred to as the performance matrix.

***Step* 3: Incorporate the Risk Attitude**

The DM may show different optimistic or pessimistic preference towards risk in different situations. To incorporate this decision attitude into the fuzzy MCDM, we introduce triangular fuzzy numbers to express the linguistic terms of risk attitude. Thus we construct the performance matrix with decision attitude as follows:

$$\widetilde{P} = \begin{bmatrix} \widetilde{p}_{11} & \widetilde{p}_{12} & \cdots & \widetilde{p}_{1n} \\ \widetilde{p}_{21} & \widetilde{p}_{22} & \cdots & \widetilde{p}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{p}_{m1} & \widetilde{p}_{m2} & \cdots & \widetilde{p}_{mn} \end{bmatrix} = [(p_{ij1}, p_{ij2}, p_{ij3})], \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n, \qquad (5.19)$$

where

$p_{ij1} = p_{ij}^{\inf}$, $p_{ij2} = p_{ij}^{\inf} + (d-1)(p_{ij}^{\sup} - p_{ij}^{\inf})/8$, $p_{ij3} = p_{ij}^{\sup}$ and $d = 1, 2, \ldots, 9$ represents decision attitudes *AP*, *VP*, *P*, *FP*, *N*, *FO*, *O*, *VO*, and *AO*, respectively.

***Step* 4: Incorporate the Confidence Attitude**

For the uncertainty of triangular fuzzy numbers, the DM's may have different confidence preference in different situations. Based on the α-cut concept, we introduce refined triangular fuzzy numbers to express the DM's degree of confidence to the fuzzy assessments. Thus we construct the performance matrix on confidence as follows:

$$\widetilde{P}^{\alpha} = \begin{bmatrix} \widetilde{p}_{11}^{\alpha} & \widetilde{p}_{12}^{\alpha} & \cdots & \widetilde{p}_{1n}^{\alpha} \\ \widetilde{p}_{21}^{\alpha} & \widetilde{p}_{22}^{\alpha} & \cdots & \widetilde{p}_{2n}^{\alpha} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{p}_{m1}^{\alpha} & \widetilde{p}_{m2}^{\alpha} & \cdots & \widetilde{p}_{mn}^{\alpha} \end{bmatrix} = [(p_{ij1}(\alpha), p_{ij2}, p_{ij3}(\alpha))], i = 1,2, \ldots, m, j = 1,2, \ldots, n, \qquad (5.20)$$

where

$p_{ij1}(\alpha) = p_{ij1} + \alpha(p_{ij2} - p_{ij1})$, $p_{ij3}(\alpha) = p_{ij3} - \alpha(p_{ij3} - p_{ij2})$, and $\alpha \in [0,1]$.

The values of α express confidence levels in assessment of the uncertainty of triangular fuzzy numbers. A larger value means a higher confidence toward uncertainty.

Step 5: Normalization

Generally criteria are incommensurate. The normalization process aims at obtaining comparable scales. Two main methods, namely vector normalization and linear scale normalization, are usually used in MCDM (Hwang and Yoon, 1981). Vector normalization cannot guarantee a criterion scale with an equal length. Linear scale normalization uses the ways in which the relative outcomes are equal. Moreover, linear scale normalization is often used for its simplicity. Thus, we will use linear scale normalization here.

We normalize the fuzzy numbers in the performance matrix on confidence level α as follows:

$$\tilde{P}^{\alpha}{}_{n} = \begin{bmatrix} \tilde{p}^{\alpha}_{11n} & \tilde{p}^{\alpha}_{12n} & \cdots & \tilde{p}^{\alpha}_{1nn} \\ \tilde{p}^{\alpha}_{21n} & \tilde{p}^{\alpha}_{22n} & \cdots & \tilde{p}^{\alpha}_{2nn} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{p}^{\alpha}_{m1n} & \tilde{p}^{\alpha}_{m2n} & \cdots & \tilde{p}^{\alpha}_{mnn} \end{bmatrix}, \; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n, \tag{5.21}$$

where $\tilde{p}^{\alpha}_{ij\,n} = \begin{cases} \left(\dfrac{p_{ij1}(\alpha)}{M}, \dfrac{p_{ij2}}{M}, \dfrac{p_{ij3}(\alpha)}{M}\right), & M = \max\limits_{i} p_{ij3}(\alpha), \quad j \in B. \\ \left(\dfrac{N}{p_{ij3}(\alpha)}, \dfrac{N}{p_{ij2}}, \dfrac{N}{p_{ij1}(\alpha)}\right), & N = \min\limits_{i} p_{ij1}(\alpha), \quad j \in C. \end{cases}$

Here B and C represent benefit criteria and cost criteria, respectively. For benefit criteria, the DM wants to have a maximum value among the alternatives. For cost criteria, the DM wants to have a minimum value among the alternatives. This method preserves the ranges of normalized triangular fuzzy numbers to be [0, 1].

***Step* 6: Determine the Positive Ideal and Negative Ideal Solutions**

The ideal solution in decision analysis means the desired decision outcome in a given decision situation. The positive (negative) ideal solution consists of the best (or worst) criteria values attainable from all the alternatives if each criterion takes monotonically increasing or decreasing values. The most preferred alternative should have the shortest distance from the positive ideal solution and the longest distance from the negative ideal solution (Hwang and Yoon, 1981; Zeleny, 1982). This concept has been widely used in developing various methodologies for solving practical decision problems (Shipley, DeKorvin and Obid, 1991; Yeh and Deng, 1997, 1999) due to: (a) its simplicity and comprehensibility in concept, (b) its computation efficiency, and (c) its ability to measure the relative performance of the alternatives in a simple mathematical form.

In line with this concept, in the normalized fuzzy performance matrix where its element is the normalized positive triangular fuzzy number, we can define the fuzzy positive ideal solution ($\widetilde{A}^*$) and fuzzy negative ideal solution ($\widetilde{A}^-$). These two ideal alternatives are used as references to measure the alternatives' performance. We determine the positive ideal solution and the negative ideal solution as follows:

$$\widetilde{A}^* = (\widetilde{p}_1^*, \widetilde{p}_2^*, \ldots, \widetilde{p}_n^*),$$

$$\widetilde{A}^- = (\widetilde{p}_1^-, \widetilde{p}_2^-, \ldots, \widetilde{p}_n^-), \tag{5.22}$$

where

$\widetilde{p}_j^* = (1, 1, 1),$

$\widetilde{p}_j^- = (0, 0, 0), \ j = 1, 2, \ldots, n.$

***Step* 7: Measure the Separations**

The distance of each alternative to the ideal solutions is measured by the vertex method (Chen, 2000). The vertex method measures distance between two triangular fuzzy numbers. It avoids the complexity of ranking fuzzy numbers.

The distance between each alternative and the positive ideal solution is calculated as:

$$d_i^{\alpha^*} = \sum_{j=1}^{n} d(\widetilde{p}_{ij\,n}^{\alpha}, \widetilde{p}_j^*), \ i = 1, 2, \ldots, m, \tag{5.23}$$

where

$$d(\widetilde{p}_{ij\,n}^{\alpha}, \widetilde{p}_j^*) = \{[(p_{ij1_n}^{\alpha} - 1)^2 + (p_{ij2_n}^{\alpha} - 1)^2 + (p_{ij3_n}^{\alpha} - 1)^2]/3\}^{1/2};$$

The distance between each alternative and the negative ideal solution is calculated as:

$$d_i^{\alpha^-} = \sum_{j=1}^{n} d(\widetilde{p}_{ij\,n}^{\alpha}, \widetilde{p}_j^-), \ i = 1, 2, \ldots, m, \tag{5.24}$$

where

$$d(\widetilde{p}_{ij\,n}^{\alpha}, \widetilde{p}_j^-) = \{[(p_{ij1_n}^{\alpha} - 0)^2 + (p_{ij2_n}^{\alpha} - 0)^2 + (p_{ij3_n}^{\alpha} - 0)^2]/3\}^{1/2}.$$

The smaller the value of $d_i^{\alpha^*}$ and $d_i^{\alpha^-}$, the higher the degree of similarity between each alternative and the positive ideal solution and the negative ideal solution, respectively.

***Step* 8: Determine the Performance on Confidence Level**

A preferred alternative should have a higher degree of similarity to the positive ideal solution, and at the same time have a lower degree of similarity to the negative ideal solution (Hwang and Yoon, 1981; Zeleny, 1982; Shipley, deKorvin, and Obid, 1991; Yeh and Deng, 1997, 1999). We prefer the alternative with a lower distance to the positive ideal solution (d_i^*) and a higher distance to the negative ideal solution (d_i^-). Therefore, an overall performance index for each alternative on confidence level α with respect to the positive ideal solution and the negative ideal solution is defined as:

$$p_i^\alpha = \frac{1}{2n}[d_i^{\alpha^-} + (n - d_i^{\alpha^*})], \; i = 1, 2, \ldots, m, \tag{5.25}$$

where n is the number of criteria.

Obviously, the nearer p_i^α is to 1 means the better the performance of alternative A_i ($i = 1$, 2, …, m).

The alternatives usually have different performance values on different confidence levels. Assuming that we take a total of l confidence levels that are equally distributed in the interval [0, 1], we need to obtain all the performance values of alternatives on these levels. Referring to (5.5), we define a performance vector with respect to the confidence levels as:

$$P_i = (p_{i1}, \; \ldots, \; p_{ik} \; \ldots \; p_{il}), \; i = 1, 2, \ldots, m, \tag{5.26}$$

where $p_{ik} = p_i^\alpha$, $\alpha = {k-1}/{l-1}$, $k = 1, 2, \ldots, l$ ($l \geq 2$).

***Step* 9: Determine the Performance on Confidence Attitude**

We use linguistic terms as absolutely confidence (*AC*), very confidence (*VC*), confidence (*C*), fairly confidence (*FC*), neutral (*N*), fairly non-confidence (*FNC*), non-confidence (*NC*), very non-confidence (*VNC*), and absolutely non-confidence (*ANC*) to represent the DM's confidence attitude. According to the membership functions (Table 5.2), we determine the confidence vectors from (5.7) to (5.15). The performance of the alternatives with respect to confidence attitude is obtained as:

$$P_i^{LT} = P_i (C_{LT}^*)^T = \sum_{k=1}^{l} p_{ik} c_k \text{ , } i = 1, 2, \ldots, m, \tag{5.27}$$

where

LT represents linguistic terms as *AC*, *VC*, *C*, *FC*, *N*, *FNC*, *NC*, *VNC*, and *ANC*, respectively.

In summarizing the discussion above, we present the steps for the approach developed as follows:

Step 1**:** Formulate the problem in the decision matrix and weighting vector as expressed in (5.16) and (5.17).

Step 2: Construct the fuzzy performance matrix expressed in (5.18) by multiplying the weighting vector by the decision matrix.

Step 3: Obtain the DM's risk attitude in definition 5.5 and construct the performance matrix with risk attitude in (5.19).

Step 4: Construct the performance matrix on confidence level as expressed in (5.20).

Step 5: Normalize the performance matrix by (5.21) to get comparable scales.

Step 6: Determine the positive ideal solution and the negative ideal solution by (5.22). The positive ideal solution and the negative ideal solution are used as references to measure the alternatives' performance.

Step 7: Measure the separations of the alternatives to the ideals solutions by (5.23) and (5.24).

Step 8: Determine the performance on confidence level by (5.25). Take a total of l confidence levels as denoted in (5.5) and calculate the performance vector with respect to confidence levels as expressed in (5.26).

Step 9: According to the DM's confidence attitudes in definition 5.8, determine the confidence vectors and calculate the alternatives' performance by (5.27). The DM ranks, selects or prioritizes the alternatives according to their performance index values.

In the following chapter, we will give a numerical example to illustrate the computation process.

Chapter 6

A Numerical Example of Fuzzy MCDM Based on Risk and Confidence Analysis

In this chapter, we give a numerical example to illustrate the computation process of fuzzy MCDM based on risk and confidence analysis.

6.1 A Step-by-step Approach

We illustrate our method by a MCDM problem with four alternatives under four benefit criteria. In the following, we consider absolutely optimism (*AO*) attitude towards risk in the solving process.

Step **1: Problem Formulation**

The decision matrix and the weighting vector of the problem are given in Table 6.1.

Table 6.1 Decision matrix and weighting vector

	C1	C2	C3	C4
	[0.10 0.30]	[0.20 0.40]	[0.30 0.50]	[0.05 0.15]
A1	[2.00 6.00]	[3.00 7.00]	[3.00 8.00]	[4.00 9.00]
A2	[2.00 7.00]	[3.00 7.00]	[1.00 5.00]	[4.00 8.00]
A3	[5.00 9.00]	[1.00 8.00]	[2.00 7.00]	[4.00 9.00]
A4	[1.00 5.00]	[3.00 6.00]	[5.00 9.00]	[7.00 9.00]

***Step* 2: Construct the Performance Matrix**

The performance matrix is constructed in Table 6.2.

Table 6.2 Performance matrix

	C1	C2	C3	C4
A1	[0.20 1.80]	[0.60 2.80]	[0.90 4.00]	[0.20 1.35]
A2	[0.20 2.10]	[0.60 2.80]	[0.30 2.50]	[0.20 1.20]
A3	[0.50 2.70]	[0.20 3.20]	[0.60 3.50]	[0.20 1.35]
A4	[0.10 1.50]	[0.60 2.40]	[1.50 4.50]	[0.35 1.35]

***Step* 3: Incorporate the Risk Attitude**

The performance matrix is incorporated with absolutely optimism attitude in Table 6.3.

Table 6.3 Performance matrix under *AO* attitude

	C1	C2	C3	C4
A1	(0.20, 1.80, 1.80)	(0.60, 2.80, 2.80)	(0.90, 4.00, 4.00)	(0.20, 1.35, 1.35)
A2	(0.20, 2.10, 2.10)	(0.60, 2.80, 2.80)	(0.30, 2.50, 2.50)	(0.20, 1.20, 1.20)
A3	(0.50, 2.70, 2.70)	(0.20, 3.20, 3.20)	(0.60, 3.50, 3.50)	(0.20, 1.35, 1.35)
A4	(0.10, 1.50, 1.50)	(0.60, 2.40, 2.40)	(1.50, 4.50, 4.50)	(0.35, 1.35, 1.35)

***Step* 4: Incorporate the Confidence Attitude**

Taking a total of 11 ($\alpha = 0, 0.1, \ldots, 1$) confidence levels, we construct the performance matrix on confidence. The performance matrix under *AO* on 0.5 confidence level is presented in Table 6.4.

Table 6.4 Performance matrix under *AO* attitude when α=0.5

	C1	C2	C3	C4
A1	(1.00, 1.80, 1.80)	(1.70, 2.80, 2.80)	(2.45, 4.00, 4.00)	(0.78, 1.35, 1.35)
A2	(1.15, 2.10, 2.10)	(1.70, 2.80, 2.80)	(1.40, 2.50, 2.50)	(0.70, 1.20, 1.20)
A3	(1.60, 2.70, 2.70)	(1.70, 3.20, 3.20)	(2.05, 3.50, 3.50)	(0.78, 1.35, 1.35)
A4	(0.80, 1.50, 1.50)	(1.50, 2.40, 2.40)	(3.00, 4.50, 4.50)	(0.85, 1.35, 1.35)

***Step* 5: Normalization**

The normalized performance matrix under *AO* on 0.5 confidence level is presented in Table 6.5.

Table 6.5 Normalized performance matrix under *AO* attitude when α=0.5

	C1	C2	C3	C4
A1	(0.370, 0.667, 0.667)	(0.531, 0.875, 0.875)	(0.544, 0.889, 0.889)	(0.574, 1.000, 1.000)
A2	(0.426, 0.778, 0.778)	(0.531, 0.875, 0.875)	(0.311, 0.556, 0.556)	(0.519, 0.889, 0.889)
A3	(0.593, 1.000, 1.000)	(0.531, 1.000, 1.000)	(0.456, 0.778, 0.778)	(0.574, 1.000, 1.000)
A4	(0.296, 0.556, 0.556)	(0.469, 0.750, 0.750)	(0.667, 1.000, 1.000)	(0.630, 1.000, 1.000)

***Step* 6: Determine the Positive Ideal and Negative Ideal Solutions**

The alternatives' separation distance to the positive ideal solution and the negative ideal solution are calculated in Table 6.6.

Table 6.6 Separation distance under *AO* when α=0.5

	C1		C2		C3		C4		Overall	
	P	N	P	N	P	N	P	N	P	N
A1	0.454	0.585	0.289	0.778	0.278	0.791	0.246	0.881	1.268	3.034
A2	0.378	0.681	0.289	0.778	0.538	0.488	0.292	0.785	1.498	2.732
A3	0.235	0.885	0.271	0.872	0.363	0.687	0.246	0.881	1.115	3.326
A4	0.545	0.485	0.368	0.670	0.193	0.903	0.214	0.894	1.320	2.951

***Step* 7: Measure the Separations**

The performance indices under 11 confidence levels are calculated in Table 6.7 and shown in Figure 6.1. We can clearly observe how the alternatives' performance values vary with the confidence level in the figure.

Table 6.7 Performance index under *AO* with 11 confidence levels

	Confidence level										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.598	0.621	0.644	0.669	0.695	0.721	0.748	0.775	0.803	0.831	0.858
A2	0.544	0.564	0.586	0.608	0.631	0.654	0.678	0.703	0.727	0.751	0.774
A3	0.633	0.659	0.687	0.716	0.745	0.776	0.809	0.841	0.875	0.910	0.944
A4	0.595	0.615	0.636	0.658	0.681	0.704	0.728	0.752	0.777	0.801	0.826

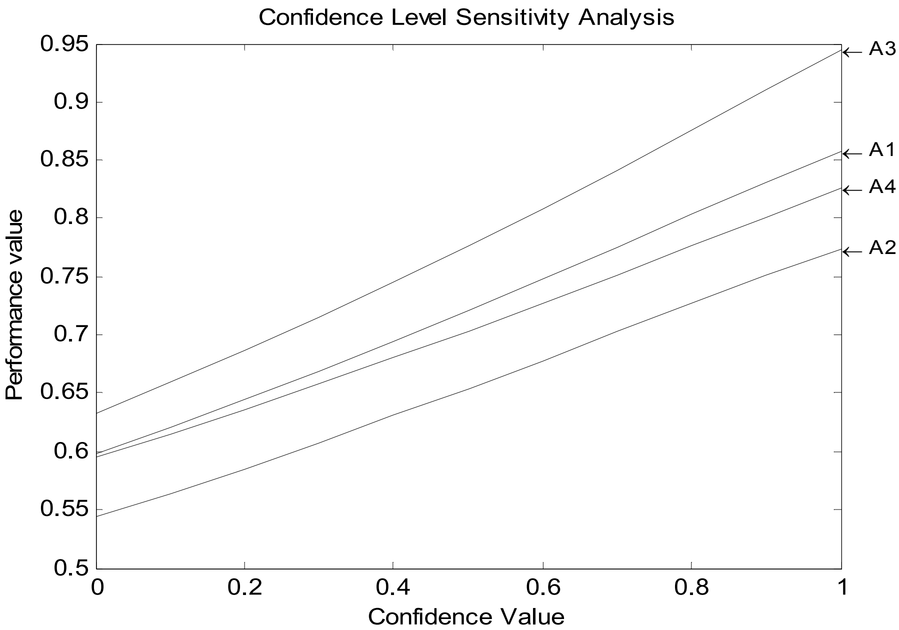

Figure 6.1 Performance value under *AO* with respect to confidence levels

Step 8: Determine the Performance on Confidence Level

According to the DM’s confidence attitudes, we take a total of 11 levels to calculate the confidence vectors. The entries in the vectors are presented in Table 6.8.

Table 6.8 Confidence vector under 11 confidence levels

	Confidence vector										
	1	2	3	4	5	6	7	8	9	10	11
AC	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000
VC	0.000	0.003	0.010	0.023	0.042	0.065	0.094	0.127	0.166	0.210	0.260
C	0.000	0.018	0.036	0.055	0.073	0.090	0.109	0.127	0.145	0.164	0.182
FC	0.000	0.044	0.063	0.077	0.089	0.100	0.109	0.118	0.126	0.134	0.141
N	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091
FNC	0.141	0.134	0.126	0.118	0.109	0.100	0.089	0.077	0.063	0.044	0.000
NC	0.182	0.164	0.145	0.127	0.109	0.090	0.073	0.055	0.036	0.018	0.000
VNC	0.260	0.210	0.166	0.127	0.094	0.065	0.042	0.023	0.010	0.003	0.000
ANC	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

***Step* 9: Determine the Performance on Confidence Attitude**

Calculate the alternatives' performance under *AO* from absolute confident to absolute non confident attitudes in Table 6.9.

Table 6.9 Performance index under *AO* with respect to confidence attitudes

	A1		A2		A3		A4	
	P	Order	P	Order	P	Order	P	Order
AC	0.8576	2	0.7743	4	0.9444	1	0.8264	3
VC	0.8001	2	0.7242	4	0.8727	1	0.7744	3
C	0.7685	2	0.6960	4	0.8355	1	0.7456	3
FC	0.7549	2	0.6840	4	0.8188	1	0.7338	3
N	0.7165	2	0.6498	4	0.7736	1	0.6995	3
FNC	0.6845	2	0.6215	4	0.7349	1	0.6714	3
NC	0.6646	2	0.6036	4	0.7117	1	0.6534	3
VNC	0.6501	2	0.5908	4	0.6939	1	0.6413	3
ANC	0.5977	2	0.5438	4	0.6332	1	0.5946	3

Finally, we analyze the results in Table 6.9 and make a ranking order. It is clear that A3 is the best alternative under absolute optimism attitude with respect to all confidence attitudes, and the other alternatives ranking order are A1, A4 and A2. Repeating the same steps, we can evaluate and analyze the alternatives' performances under other risk attitudes with respect to confidence attitudes. The data and figures are given as follows.

Performance under *VO* Attitude

Table 6.10 Performance index under *VO* with 11 confidence levels

	Confidence level										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.576	0.598	0.622	0.646	0.673	0.700	0.729	0.759	0.791	0.822	0.854
A2	0.522	0.542	0.563	0.585	0.609	0.634	0.659	0.686	0.714	0.742	0.770
A3	0.610	0.635	0.662	0.691	0.723	0.753	0.786	0.822	0.859	0.897	0.936
A4	0.575	0.596	0.617	0.640	0.664	0.689	0.715	0.742	0.770	0.799	0.829

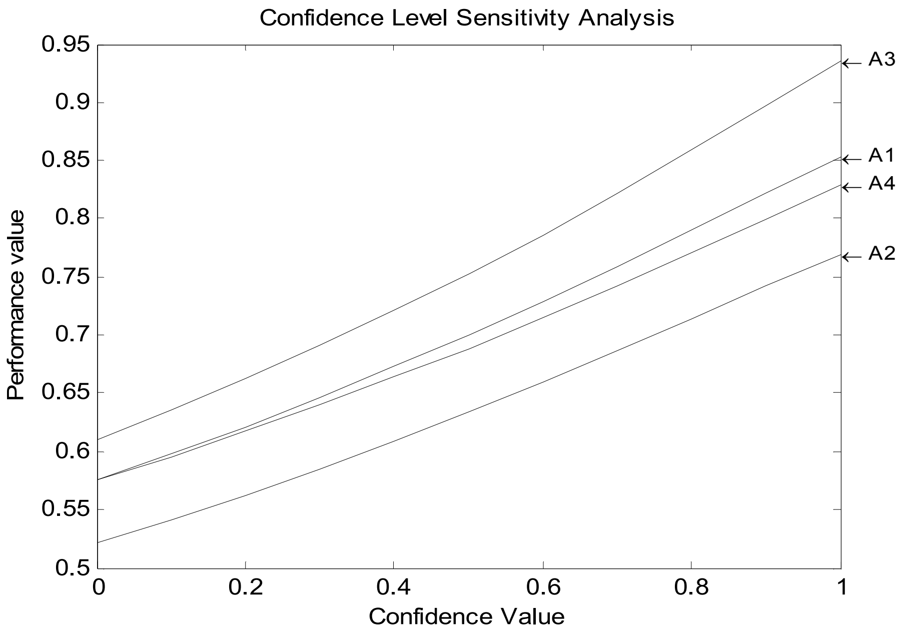

Figure 6.2 Performance value under *VO* with respect to confidence levels

Table 6.11 Performance index under *VO* with respect to confidence attitudes

	A1		A2		A3		A4	
	P	Order	P	Order	P	Order	P	Order
AC	0.8535	2	0.7697	4	0.9363	1	0.8291	3
VC	0.7883	2	0.7119	4	0.8571	1	0.7688	3
C	0.7546	2	0.6817	4	0.8182	1	0.7376	3
FC	0.7396	2	0.6683	4	0.8001	1	0.7242	3
N	0.6993	2	0.6323	4	0.7534	1	0.6873	3
FNC	0.6645	2	0.6013	4	0.7124	1	0.6559	3
NC	0.6439	2	0.5828	4	0.6887	1	0.6369	3
VNC	0.6283	2	0.5689	4	0.6702	1	0.6233	3
ANC	0.5756	2	0.5217	4	0.6101	1	0.5751	3

Performance under *O* Attitude

Table 6.12 Performance index under *O* with 11 confidence levels

	Confidence level										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.552	0.574	0.597	0.622	0.648	0.677	0.708	0.740	0.775	0.812	0.849
A2	0.499	0.518	0.538	0.561	0.585	0.610	0.638	0.667	0.698	0.731	0.765
A3	0.585	0.609	0.635	0.622	0.693	0.725	0.760	0.798	0.838	0.881	0.927
A4	0.554	0.575	0.597	0.620	0.645	0.672	0.700	0.730	0.763	0.797	0.833

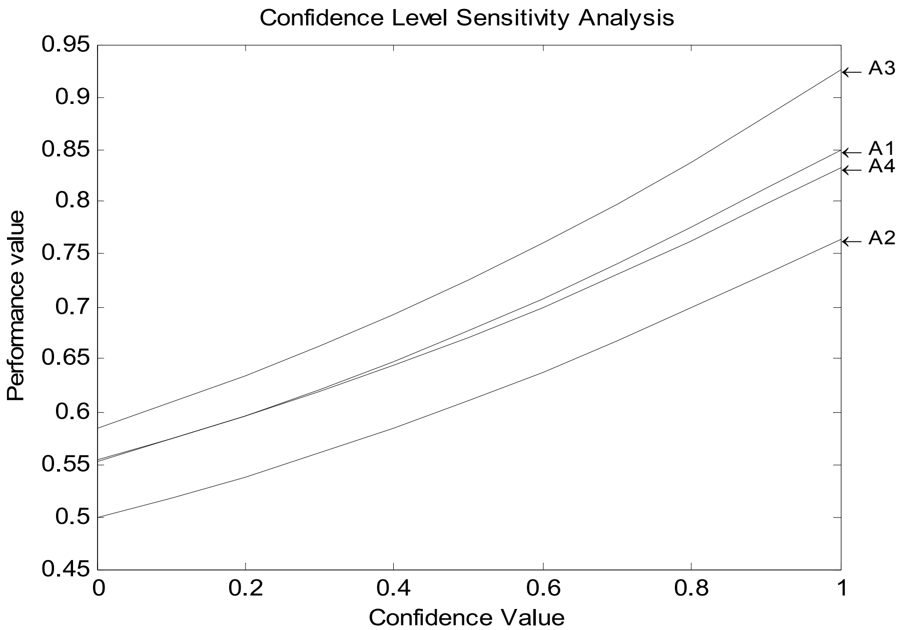

Figure 6.3 Performance value under *O* with respect to confidence levels

Table 6.13 Performance index under *O* with respect to confidence attitudes

	A1		A2		A3		A4	
	P	Order	P	Order	P	Order	P	Order
AC	0.8489	2	0.7645	4	0.9265	1	0.8327	3
VC	0.7744	2	0.6977	4	0.8383	1	0.7624	3
C	0.7386	2	0.6655	4	0.7975	1	0.7285	3
FC	0.7220	2	0.6507	4	0.7781	1	0.7132	3
N	0.6797	2	0.6128	4	0.7301	1	0.6735	3
FNC	0.6422	2	0.5792	4	0.6866	1	0.6386	3
NC	0.6209	2	0.5601	4	0.6626	1	0.6185	3
VNC	0.6044	2	0.5454	4	0.6435	1	0.6036	3
ANC	0.5522	3	0.4987	4	0.5850	1	0.5543	2

Performance under *FO* Attitude

Table 6.14 Performance index under *FO* with 11 confidence levels

	Confidence level										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.528	0.548	0.570	0.594	0.621	0.650	0.682	0.718	0.757	0.799	0.844
A2	0.475	0.493	0.512	0.534	0.558	0.584	0.613	0.645	0.680	0.718	0.759
A3	0.558	0.580	0.604	0.631	0.660	0.693	0.729	0.769	0.813	0.861	0.914
A4	0.533	0.552	0.574	0.598	0.623	0.651	0.682	0.716	0.753	0.794	0.838

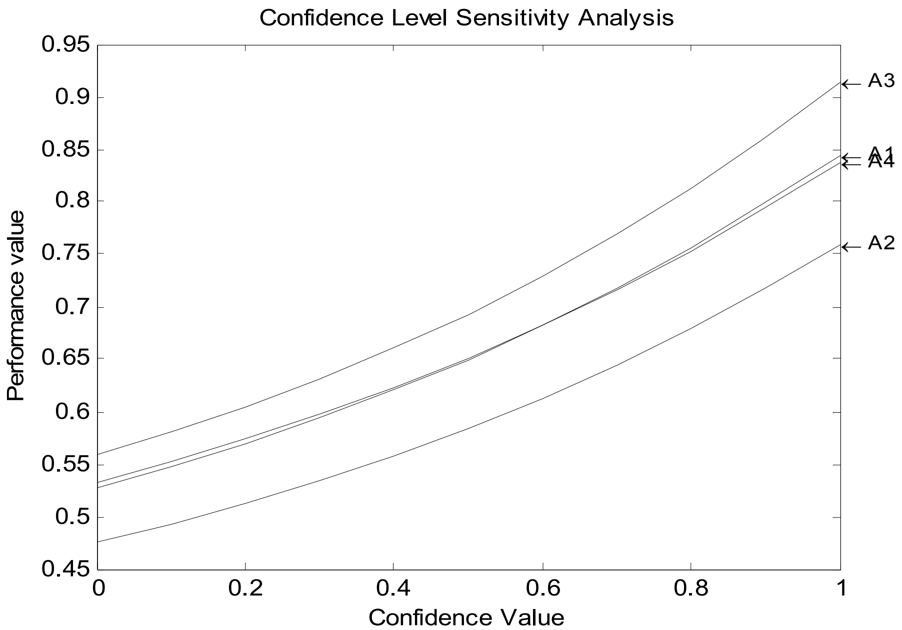

Figure 6.4 Performance value under *FO* with respect to confidence levels

Table 6.15 Performance index under *FO* with respect to confidence attitudes

	A1		A2		A3		A4	
	P	Order	P	Order	P	Order	P	Order
AC	0.8437	2	0.7586	4	0.9143	1	0.8378	3
VC	0.7581	2	0.6812	4	0.8155	1	0.7550	3
C	0.7200	2	0.6470	4	0.7728	1	0.7181	3
FC	0.7018	2	0.6306	4	0.7522	1	0.7001	3
N	0.6578	3	0.5913	4	0.7031	1	0.6582	2
FNC	0.6174	3	0.5550	4	0.6575	1	0.6195	2
NC	0.5956	3	0.5355	4	0.6334	1	0.5984	2
VNC	0.5785	3	0.5203	4	0.6141	1	0.5822	2
ANC	0.5277	3	0.4751	4	0.5584	1	0.5326	2

Performance under *N* Attitude

Table 6.16 Performance index under *N* with 11 confidence levels

	Confidence level										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.503	0.521	0.541	0.564	0.590	0.619	0.653	0.690	0.734	0.783	0.838
A2	0.451	0.467	0.485	0.506	0.528	0.554	0.584	0.618	0.657	0.701	0.752
A3	0.531	0.550	0.572	0.596	0.624	0.656	0.692	0.733	0.781	0.836	0.899
A4	0.510	0.529	0.550	0.573	0.599	0.628	0.661	0.699	0.741	0.790	0.846

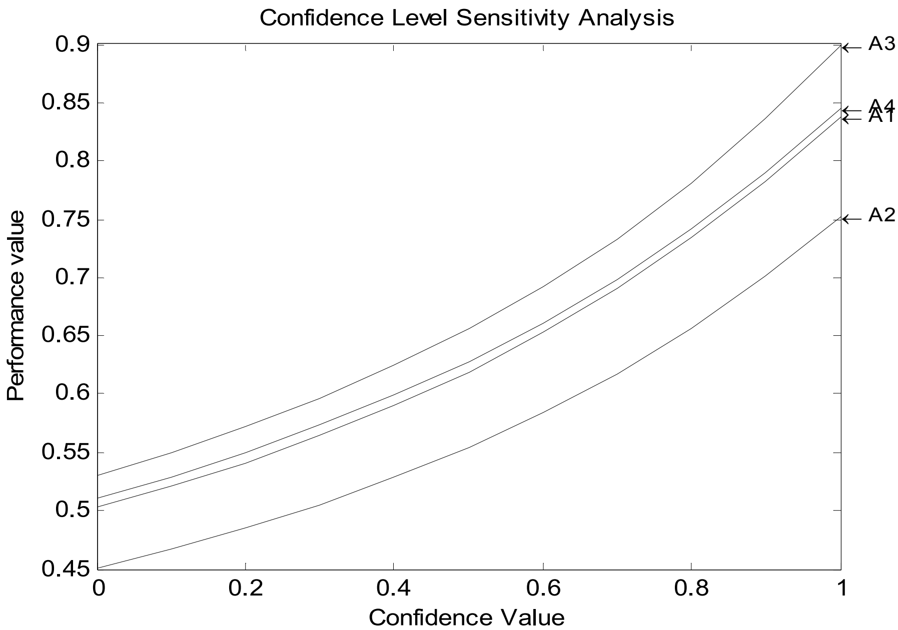

Figure 6.5 Performance value under *N* with respect to confidence levels

Table 6.17 Performance index under *N* with respect to confidence attitudes

	A1		A2		A3		A4	
	P	Order	P	Order	P	Order	P	Order
AC	0.8384	3	0.7522	4	0.8988	1	0.8456	2
VC	0.7389	3	0.6621	4	0.7877	1	0.7468	2
C	0.6984	3	0.6256	4	0.7433	1	0.7064	2
FC	0.6786	3	0.6079	4	0.7215	1	0.6868	2
N	0.6333	3	0.5674	4	0.6722	1	0.6415	2
FNC	0.5900	3	0.5287	4	0.6248	1	0.5984	2
NC	0.5681	3	0.5091	4	0.6011	1	0.5765	2
VNC	0.5508	3	0.4938	4	0.5822	1	0.5592	2
ANC	0.5026	3	0.4513	4	0.5307	1	0.5103	2

Performance under *FP* Attitude

Table 6.18 Performance index under *FP* with 11 confidence levels

	Confidence level										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.478	0.493	0.512	0.532	0.557	0.585	0.618	0.658	0.705	0.756	0.815
A2	0.428	0.441	0.457	0.475	0.496	0.521	0.551	0.586	0.628	0.674	0.728
A3	0.503	0.519	0.538	0.559	0.584	0.613	0.648	0.690	0.740	0.796	0.861
A4	0.488	0.505	0.525	0.547	0.573	0.602	0.637	0.678	0.727	0.779	0.842

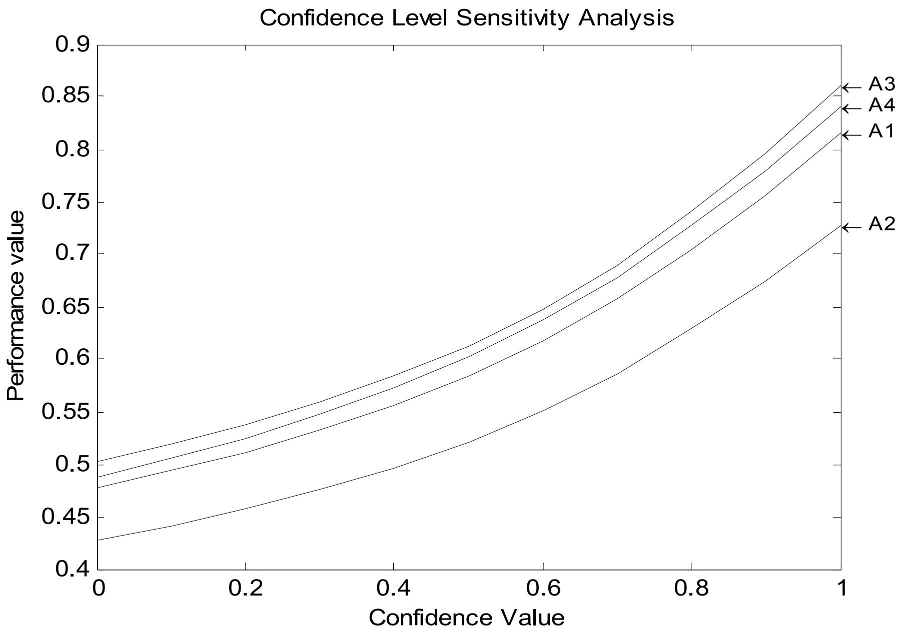

Figure 6.6 Performance value under *FP* with respect to confidence levels

Table 6.19 Performance index under *FP* with respect to confidence attitudes

	A1		A2		A3		A4	
	P	Order	P	Order	P	Order	P	Order
AC	0.8150	3	0.7276	4	0.8608	1	0.8416	2
VC	0.7101	3	0.6333	4	0.7472	1	0.7324	2
C	0.6689	3	0.5965	4	0.7034	1	0.6895	2
FC	0.6486	3	0.5785	4	0.6818	1	0.6682	2
N	0.6036	3	0.5386	4	0.6345	1	0.6211	2
FNC	0.5597	3	0.4996	4	0.5880	1	0.5752	2
NC	0.5383	3	0.4807	4	0.5655	1	0.5528	2
VNC	0.5214	3	0.4660	4	0.5480	1	0.5348	2
ANC	0.4775	3	0.4277	4	0.5028	1	0.4878	2

Performance under *P* Attitude

Table 6.20 Performance index under *P* with 11 confidence levels

	Confidence level										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.453	0.466	0.481	0.499	0.520	0.546	0.577	0.615	0.658	0.713	0.782
A2	0.405	0.416	0.428	0.443	0.462	0.484	0.512	0.545	0.582	0.630	0.693
A3	0.475	0.488	0.502	0.519	0.540	0.565	0.596	0.634	0.678	0.735	0.807
A4	0.465	0.481	0.498	0.519	0.543	0.573	0.608	0.650	0.698	0.759	0.835

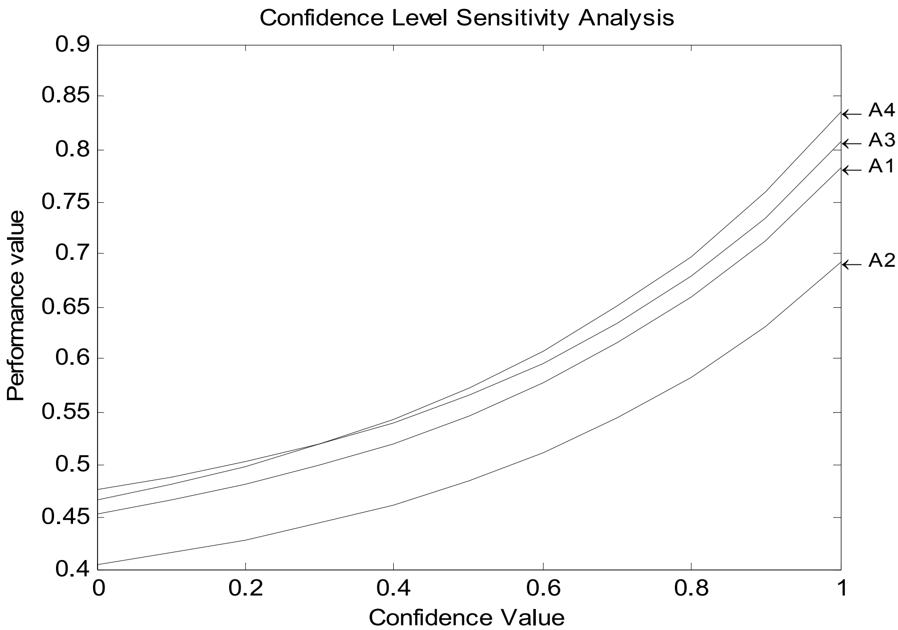

Figure 6.7 Performance value under *P* with respect to confidence levels

Table 6.21 Performance index under *P* with respect to confidence attitudes

	A1		A2		A3		A4	
	P	Order	P	Order	P	Order	P	Order
AC	0.7821	3	0.6927	4	0.8069	2	0.8354	1
VC	0.6704	3	0.5936	4	0.6919	2	0.7113	1
C	0.6301	3	0.5582	4	0.6509	2	0.6666	1
FC	0.6103	3	0.5409	4	0.6310	2	0.6443	1
N	0.5678	3	0.5039	4	0.5886	2	0.5966	1
FNC	0.5253	3	0.4668	4	0.5459	2	0.5488	1
NC	0.5055	3	0.4497	4	0.5262	2	0.5267	1
VNC	0.4905	3	0.4369	4	0.5118	1	0.5091	2
ANC	0.4527	3	0.4048	4	0.4751	1	0.4654	2

Performance under *VP* Attitude

Table 6.22 Performance index under *VP* with 11 confidence levels

	Confidence level										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.429	0.438	0.450	0.463	0.480	0.502	0.528	0.558	0.598	0.653	0.732
A2	0.383	0.390	0.399	0.411	0.425	0.443	0.464	0.489	0.522	0.569	0.639
A3	0.448	0.456	0.466	0.478	0.493	0.511	0.534	0.561	0.597	0.649	0.724
A4	0.444	0.456	0.471	0.490	0.512	0.539	0.572	0.611	0.660	0.728	0.824

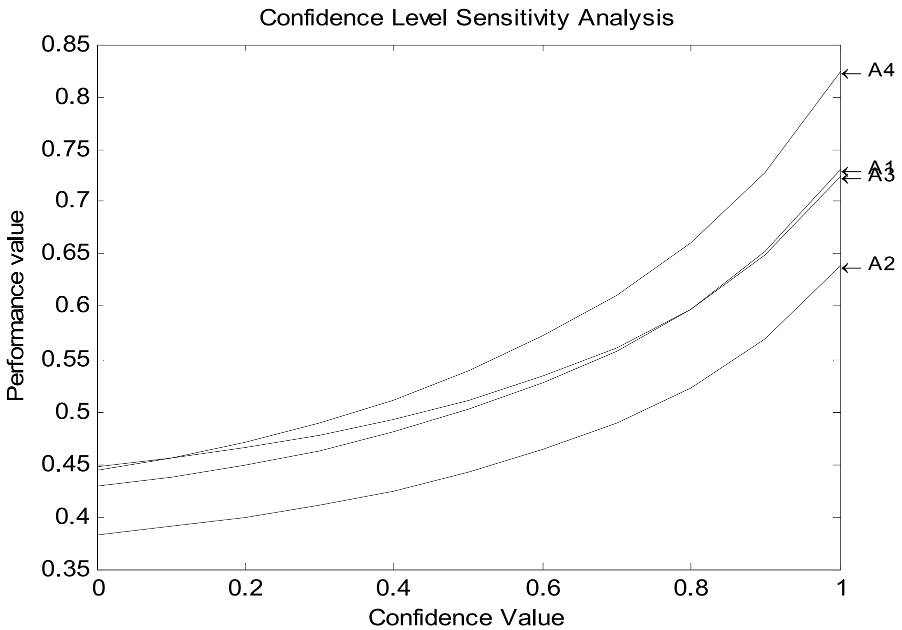

Figure 6.8 Performance value under *VP* with respect to confidence levels

Table 6.23 Performance index under *VP* with respect to confidence attitudes

	A1		A2		A3		A4	
	P	Order	P	Order	P	Order	P	Order
AC	0.7316	2	0.6388	4	0.7235	3	0.8244	1
VC	0.6169	2	0.5398	4	0.6164	3	0.6828	1
C	0.5797	3	0.5081	4	0.5819	2	0.6372	1
FC	0.5619	3	0.4932	4	0.5661	2	0.6143	1
N	0.5247	3	0.4620	4	0.5325	2	0.5677	1
FNC	0.4864	3	0.4299	4	0.4982	2	0.5190	1
NC	0.4697	3	0.4159	4	0.4831	2	0.4981	1
VNC	0.4581	3	0.4067	4	0.4737	2	0.4820	1
ANC	0.4287	3	0.3828	4	0.4483	1	0.4435	2

Performance under *AP* attitude

Table 6.24 Performance index under *AP* with 11 confidence levels

	Confidence level										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A1	0.406	0.412	0.419	0.427	0.439	0.454	0.468	0.487	0.516	0.562	0.643
A2	0.362	0.366	0.371	0.377	0.386	0.397	0.407	0.420	0.441	0.476	0.543
A3	0.423	0.426	0.431	0.436	0.443	0.452	0.460	0.471	0.489	0.518	0.576
A4	0.422	0.432	0.444	0.459	0.478	0.502	0.527	0.561	0.609	0.681	0.800

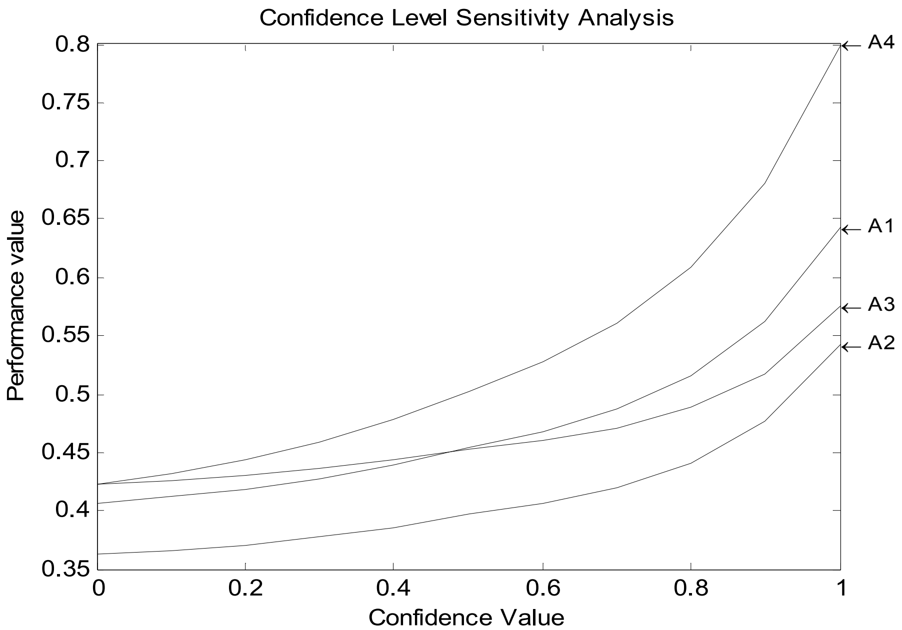

Figure 6.9 Performance value under *AP* with respect to confidence levels

Table 6.25 Performance index under *AP* with respect to confidence attitudes

	A1		A2		A3		A4	
	P	Order	P	Order	P	Order	P	Order
AC	0.6429	2	0.5429	4	0.5762	3	0.8000	1
VC	0.5403	2	0.4619	4	0.5068	3	0.6425	1
C	0.5109	2	0.4390	4	0.4863	3	0.5978	1
FC	0.4980	2	0.4296	4	0.4789	3	0.5757	1
N	0.4708	2	0.4091	4	0.4612	3	0.5324	1
FNC	0.4424	3	0.3880	4	0.4440	2	0.4855	1
NC	0.4307	3	0.3792	4	0.4362	2	0.4671	1
VNC	0.4245	3	0.3755	4	0.4344	2	0.4538	1
ANC	0.4058	3	0.3621	4	0.4229	1	0.4224	2

Performance of the Alternatives under Risk and Confidence Attitudes

For a clearer representation, we show the alternatives' performance results under risk and confidence attitude simultaneously. The data and figures are as follows. Performance of A1 under different risk and confidence attitudes are given in Table 6.26 and shown in Figure 6.10.

Table 6.26 Performance index of A1 under risk and confidence attitudes

A1	AO	VO	O	FO	N	FP	P	VP	AP
AC	0.8576	0.8535	0.8489	0.8437	0.8384	0.8150	0.7821	0.7316	0.6429
VC	0.8001	0.7883	0.7744	0.7581	0.7389	0.7101	0.6704	0.6169	0.5403
C	0.7685	0.7546	0.7386	0.7200	0.6984	0.6689	0.6301	0.5797	0.5109
FC	0.7549	0.7396	0.7220	0.7018	0.6786	0.6486	0.6103	0.5619	0.4980
N	0.7165	0.6993	0.6797	0.6578	0.6333	0.6036	0.5678	0.5247	0.4708
FNC	0.6845	0.6645	0.6422	0.6174	0.5900	0.5597	0.5253	0.4864	0.4424
NC	0.6646	0.6439	0.6209	0.5956	0.5681	0.5383	0.5055	0.4697	0.4307
VNC	0.6501	0.6283	0.6044	0.5785	0.5508	0.5214	0.4905	0.4581	0.4245
ANC	0.5977	0.5756	0.5522	0.5277	0.5026	0.4775	0.4527	0.4287	0.4058

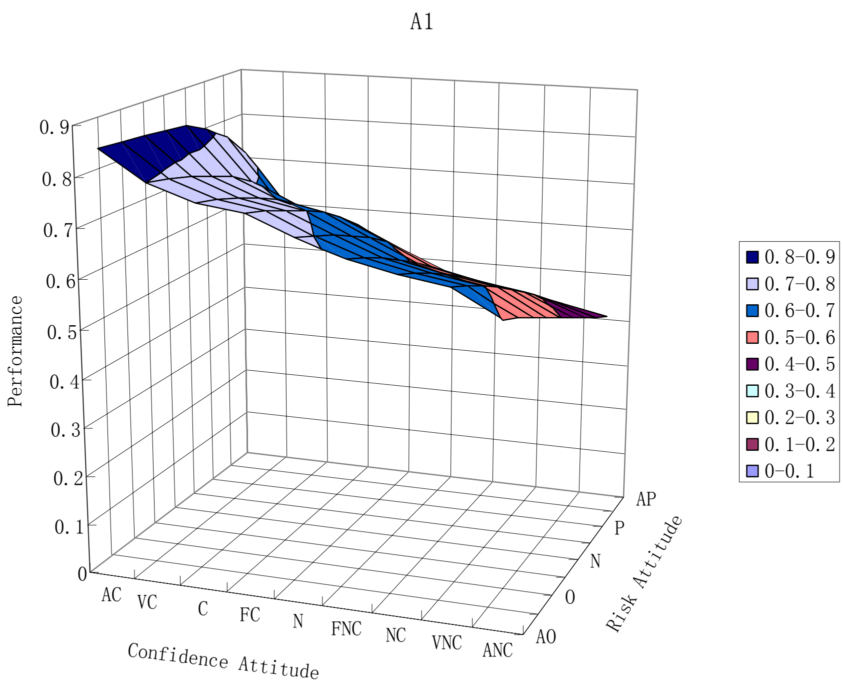

Figure 6.10 Performance index of A1 under risk and confidence attitudes

The performance of A2 under different risk and confidence attitudes are given in Table 6.27 and shown in Figure 6.11.

Table 6.27 Performance index of A2 under risk and confidence attitudes

A2	AO	VO	O	FO	N	FP	P	VP	AP
AC	0.7743	0.7697	0.7645	0.7586	0.7522	0.7276	0.6927	0.6388	0.5429
VC	0.7242	0.7119	0.6977	0.6812	0.6621	0.6333	0.5936	0.5398	0.4619
C	0.6960	0.6817	0.6655	0.6470	0.6256	0.5965	0.5582	0.5081	0.4390
FC	0.6840	0.6683	0.6507	0.6306	0.6079	0.5785	0.5409	0.4932	0.4296
N	0.6498	0.6323	0.6128	0.5913	0.5674	0.5386	0.5039	0.4620	0.4091
FNC	0.6215	0.6013	0.5792	0.5550	0.5287	0.4996	0.4668	0.4299	0.3880
NC	0.6036	0.5828	0.5601	0.5355	0.5091	0.4807	0.4497	0.4159	0.3792
VNC	0.5908	0.5689	0.5454	0.5203	0.4938	0.4660	0.4369	0.4067	0.3755
ANC	0.5438	0.5217	0.4987	0.4751	0.4513	0.4277	0.4048	0.3828	0.3621

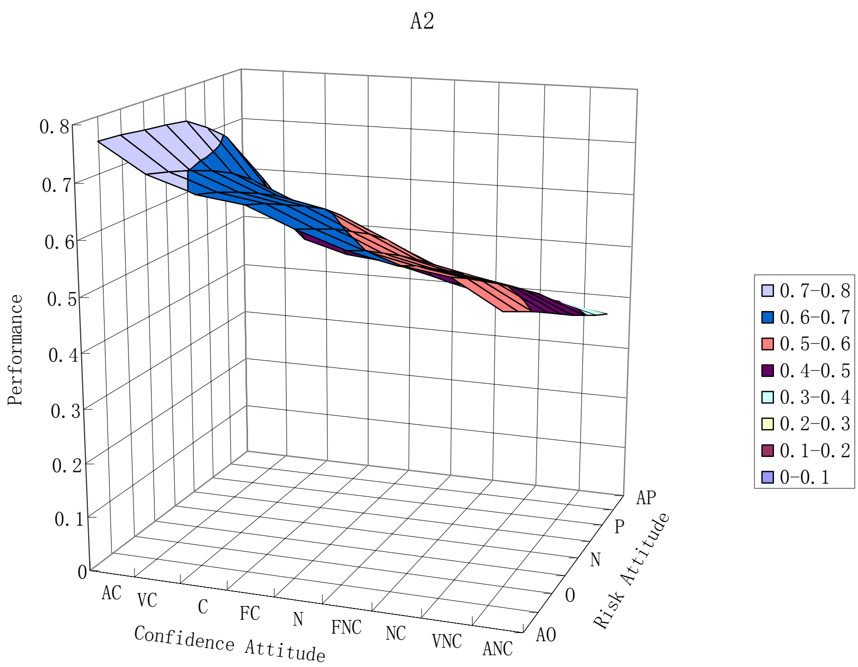

Figure 6.11 Performance index of A2 under risk and confidence attitudes

The performance of A3 under different risk and confidence attitudes are given in Table 6.28 and shown in Figure 6.12.

Table 6.28 Performance index of A3 under risk and confidence attitudes

A3	AO	VO	O	FO	N	FP	P	VP	AP
AC	0.9444	0.9363	0.9265	0.9143	0.8988	0.8608	0.8069	0.7235	0.5762
VC	0.8727	0.8571	0.8383	0.8155	0.7877	0.7472	0.6919	0.6164	0.5068
C	0.8355	0.8182	0.7975	0.7728	0.7433	0.7034	0.6509	0.5819	0.4863
FC	0.8188	0.8001	0.7781	0.7522	0.7215	0.6818	0.6310	0.5661	0.4789
N	0.7736	0.7534	0.7301	0.7031	0.6722	0.6345	0.5886	0.5325	0.4612
FNC	0.7349	0.7124	0.6866	0.6575	0.6248	0.5880	0.5459	0.4982	0.4440
NC	0.7117	0.6887	0.6626	0.6334	0.6011	0.5655	0.5262	0.4831	0.4362
VNC	0.6939	0.6702	0.6435	0.6141	0.5822	0.5480	0.5118	0.4737	0.4344
ANC	0.6332	0.6101	0.5850	0.5584	0.5307	0.5028	0.4751	0.4483	0.4229

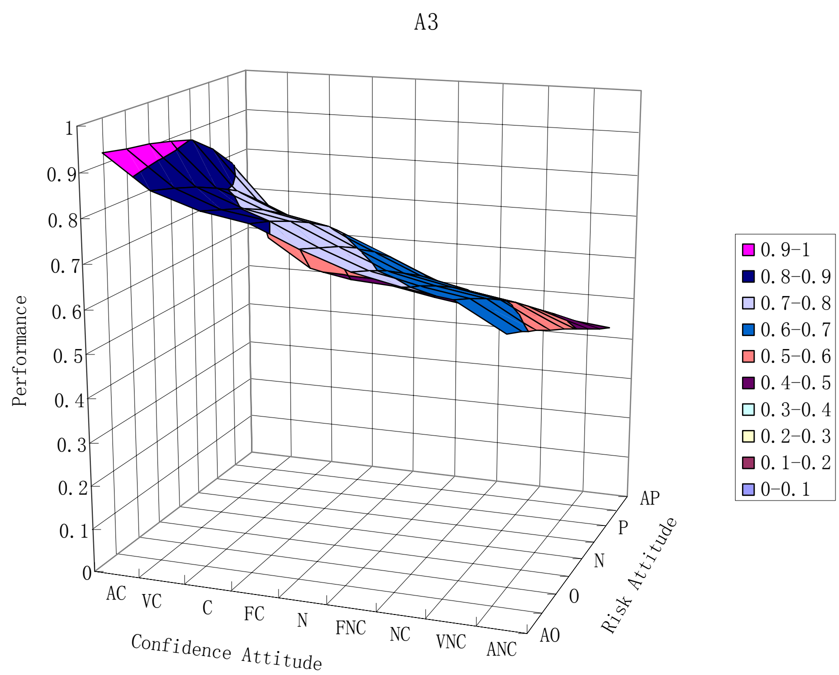

Figure 6.12 Performance index of A3 under risk and confidence attitudes

The performance of A4 under different risk and confidence attitudes are given in Table 6.29 and shown in Figure 6.13.

Table 6.29 Performance index of A4 under risk and confidence attitudes

A4	AO	VO	O	FO	N	FP	P	VP	AP
AC	0.8264	0.8291	0.8327	0.8378	0.8456	0.8416	0.8354	0.8244	0.8000
VC	0.7744	0.7688	0.7624	0.7550	0.7468	0.7324	0.7113	0.6828	0.6425
C	0.7456	0.7376	0.7285	0.7181	0.7064	0.6895	0.6666	0.6372	0.5978
FC	0.7338	0.7242	0.7132	0.7001	0.6868	0.6682	0.6443	0.6143	0.5757
N	0.6995	0.6873	0.6735	0.6582	0.6415	0.6211	0.5966	0.5677	0.5324
FNC	0.6714	0.6559	0.6386	0.6195	0.5984	0.5752	0.5488	0.5190	0.4855
NC	0.6534	0.6369	0.6185	0.5984	0.5765	0.5528	0.5267	0.4981	0.4671
VNC	0.6413	0.6233	0.6036	0.5822	0.5592	0.5348	0.5091	0.4820	0.4538
ANC	0.5946	0.5751	0.5543	0.5326	0.5103	0.4878	0.4654	0.4435	0.4224

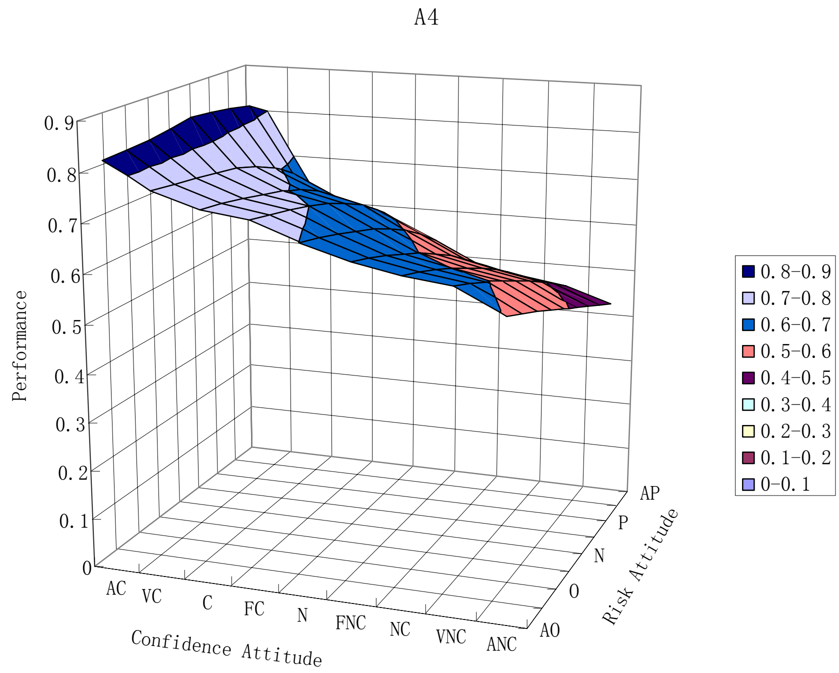

Figure 6.13 Performance index of A4 under risk and confidence attitudes

Ranking Order of the Alternatives

The ranking orders of A1 under different risk and confidence attitudes are given in Table 6.30.

Table 6.30 Ranking order of A1 under risk and confidence attitudes

A1	AO	VO	O	FO	N	FP	P	VP	AP
AC	2	2	2	2	3	3	3	2	2
VC	2	2	2	2	3	3	3	2	2
C	2	2	2	2	3	3	3	3	2
FC	2	2	2	2	3	3	3	3	2
N	2	2	2	3	3	3	3	3	2
FNC	2	2	2	3	3	3	3	3	3
NC	2	2	2	3	3	3	3	3	3
VNC	2	2	2	3	3	3	3	3	3
ANC	2	2	3	3	3	3	3	3	3

The ranking orders of A2 under different risk and confidence attitudes are given in Table 6.31.

Table 6.31 Ranking order of A2 under risk and confidence attitudes

A2	AO	VO	O	FO	N	FP	P	VP	AP
AC	4	4	4	4	4	4	4	4	4
VC	4	4	4	4	4	4	4	4	4
C	4	4	4	4	4	4	4	4	4
FC	4	4	4	4	4	4	4	4	4
N	4	4	4	4	4	4	4	4	4
FNC	4	4	4	4	4	4	4	4	4
NC	4	4	4	4	4	4	4	4	4
VNC	4	4	4	4	4	4	4	4	4
ANC	4	4	4	4	4	4	4	4	4

The ranking orders of A3 under different risk and confidence attitudes are given in Table 6.32.

Table 6.32 Ranking order of A3 under risk and confidence attitudes

A3	AO	VO	O	FO	N	FP	P	VP	AP
AC	1	1	1	1	1	1	2	3	3
VC	1	1	1	1	1	1	2	3	3
C	1	1	1	1	1	1	2	2	3
FC	1	1	1	1	1	1	2	2	3
N	1	1	1	1	1	1	2	2	3
FNC	1	1	1	1	1	1	2	2	2
NC	1	1	1	1	1	1	2	2	2
VNC	1	1	1	1	1	1	1	2	2
ANC	1	1	1	1	1	1	1	1	1

The ranking orders of A4 under different risk and confidence attitudes are given in Table 6.33.

Table 6.33 Ranking order of A4 under risk and confidence attitudes

A4	AO	VO	O	FO	N	FP	P	VP	AP
AC	3	3	3	3	2	2	1	1	1
VC	3	3	3	3	2	2	1	1	1
C	3	3	3	3	2	2	1	1	1
FC	3	3	3	3	2	2	1	1	1
N	3	3	3	2	2	2	1	1	1
FNC	3	3	3	2	2	2	1	1	1
NC	3	3	3	2	2	2	1	1	1
VNC	3	3	3	2	2	2	2	1	1
ANC	3	3	2	2	2	2	2	2	2

Finally, the DM prioritizes and selects the alternatives.

6.2 Summary

Multicriteria decision problems generally involve uncertain and imprecise data. To consider the DM's risk and confidence attitude towards intervals of uncertainty, we propose a fuzzy MCDM approach based on attitude and confidence analysis. Triangular fuzzy numbers are constructed to incorporate the DM's optimism (pessimism) attitude towards risk. The DM's confidence attitudes on the assessments of uncertainty are incorporated based on the α-cut concept. By incorporating the DM's subjectiveness towards uncertainty, this approach is effective in expressing human adaptability, intransitivity, and dynamic adjustment of preferences in the decision process. A numerical example is given to demonstrate its effectiveness in solving fuzzy MCDM problems.

Chapter 7

Conclusion and Future Work

This chapter concludes the thesis with a summary of the accomplishments and future work.

7.1 Conclusion

MCDM refers to making decisions in the presence of multiple criteria. The application of fuzzy set theory to MCDM methods can provide an effective way to solve problems involving uncertainty. An effective way to express the vagueness, impreciseness, and subjectiveness of uncertain information is to use fuzzy numbers. Fuzzy numbers usually express the uncertain numerical value for the ratings of the alternatives and weights of the criteria in MCDM. The linguistic approach relies on a systematic use of words to characterize the values of variables and the relations between variables. It is used in situations in which the problem under analysis is too complex or too ill-defined to be amenable to quantitative characterization.

In this thesis, we developed two approaches to solve the MCDM problems in the fuzzy environment.

In the fuzzy extension of ELECTRE, we propose a method to establish fuzzy outranking relations between alternatives. With reference to the DM's preference attitude, the concordance and discordance sets, as well as the concordance and discordance indices, are obtained to express the strength of agreement and disagreement in outranking relations among alternatives. The net concordance index and net discordance index are constructed to represent the strength and weakness of one alternative's domination over other alternatives. Finally, the performance index is obtained based on the net concordance index and the net discordance index. This fuzzy ELECTRE method provides a more flexible way to solve problems based on the DM's preference attitudes.

In the second proposed method, we introduced the concept of confidence attitude and risk attitude towards uncertainty in supporting normative decision making. A fuzzy MCDM is proposed by incorporating the DM's subjectiveness and imprecision into the decision process. The linguistic term of risk attitude is expressed as a triangular fuzzy number toward the interval of uncertainty. The optimism attitude towards risk prefers the uncertainty to be solved in a favorable way, while the pessimism attitude towards risk prefers the uncertainty to be solved in an unfavorable way. Based on the α-cut concept, a refined triangular fuzzy number is defined to incorporate the DM's confidence towards uncertainty. Higher confidence means a higher preference towards values with a higher possibility. The basic confidence attitude is established linearly with respect to the confidence levels. The other linguistic terms are established by modifier or hedge operations accordingly. Confidence vectors are established on the membership functions of the confidence attitudes. By making use of confidence vectors, the alternatives' performances on confidence levels are aggregated to obtain the overall performance of alternatives. Sensitivity analysis can help gain a deep insight and understanding of the problem. Therefore, it provides an effective way to solve complex, ill-defined and human-oriented MCDM problems.

7.2 Future Work

The triangular fuzzy number and linguistic terms are effective and flexible in fuzzy decision modeling. The systematic establishment and assignment of fuzzy numbers require a theoretical approach. We need more study on the triangular fuzzy number, the trapezoidal fuzzy number and other types of fuzzy number, as well as fuzzy operations and measures in decision analysis.

For the fuzzy ELECTRE method, possibility and necessity measures may be considered as ways to establish the outranking relations. For the fuzzy MCDM method based on risk and confidence attitudes, we may further consider other preference attitudes for supporting normative decision making.

We may also extend our work by considering multiple decision makers in our fuzzy decision models. Moving away from a single decision maker's setting introduces a great deal of complexity into the decision analysis process, as it no longer considers only one individual's preference structure. The analysis must be extended to account for the conflicts among different interest groups who have different objectives or criteria. By the application of fuzzy set theory and other theories such as utility theory, game theory, and social choice theory, appropriate methods can be proposed to solve the problem under different situations.

Currently, there is no single method that is good for solving all the different types of decision problems. Thus we need to establish the rules to choose a right method to solve the problems. Expert decision support systems can assist the DM in implementing MCDM methods for the appropriate problem.

In summary, this thesis presents an overview of MCDM methods and fuzzy MCDM methods, and develops two fuzzy MCDM methods. More research and application of such methods will be done in the future.

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