VALUE EFFICIENCY IN DATA ENVELOPMENT ANALYSIS: WEIGHTED GLOBAL MEASURE

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Summary

This research has focused on a number of important issues related to the efficiency measurement in the presence of additional slack after Farrell efficiency is achieved. Firstly, we define our measure of efficiency and then investigate its properties and demonstrate its characteristics theoretically. In addition, we provide an effective method to capture the internal value information in the production systems which is usually omitted in the traditional efficiency measures. Furthermore, we show how the effect of weights factors on the efficiency and efficient frontier in our model. Finally, we compare our measure with other measures theoretically as well as empirically and find that there are some differences between our measure and others. We believe that the use of this measure is practical, in the sense that it requires little detailed information on the part of the analyst, and consistent, in the sense that – if a factor is deemed important enough to include in the analysis then its importance should be reflected in its contribution to the benefit of DMU activity.

In addition, the ability to rank or differentiate the efficient units is of both theoretically and practically importance. One concern about these super-efficiency models is that they may not always be possible to determine their optimal value when the super-efficiency models are applied under other alternate returns to scale (RTS) conditions other than constant returns to scales (CRS). Another concern is that these super-efficiency measures cannot capture certain inherent relationships among the inputs and the outputs which can be known or predetermined beforehand. In this study, we discuss the use of the weighted super-efficiency measure which is derived from the weighted global efficiency measure. This super-efficiency measure is useful to differentiate efficient units and motivate appropriate behavior.

Furthermore, we have studied various approaches for incorporating undesirable factors in the DEA models under the assumption of variables return to scales. A new efficiency measure is oriented to both desirable factors and undesirable factors simultaneously on the basis of classification invariance so that the weighted global DEA model allows the expansion of desirable outputs and the contraction of undesirable outputs and all inputs with different proportions. The new approach can also be applied to situations when some inputs need to be increased to improve the performance.

Finally, we have discussed the use of the weighted global efficiency measure in the production systems without inputs or outputs. And we have also developed a new super-efficiency measure which can be used to discriminate the relative performance among the efficient DMU.

KEYWORDS: *Data Envelopment Analysis*; *Value efficiency analysis*; *Weighted global efficiency; Super-efficiency; Undesirable factors; DEA model without inputs/outputs.*

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Abbreviations

AR I	Assurance Region Type I
AR II	Assurance Region Type II
CRS	Constant Returns to Scales
DEA	Data Envelopment Analysis
DM	Decision Maker
DMU	Decision Making Unit
DRS	Decreasing Returns to Scale
ERGTE	Extended Russell Graph Measure of Efficiency
FTE	Farrell Measure of Technical Efficiency
IRS	Increasing Returns to Scale
LP	Linear Programming
MCDM	Multiple Criteria Decision Making
MOLP	Multiple Objectives Linear Programming
MPS	Most Preferred Solution
MPSS	Most Productive Scale Size
PPS	Production Possibility Set
RGTE	Russell Graph Measure of Technical Efficiency
RTE	Russell Measure of Technical Efficiency
SBM	Slack-based Model
SFA	Stochastic Frontier Analysis
STE	Slack-based Technical Efficiency
UDMU	Unobserved Decision Making Unit
VRS	Variable Returns to Scale
WGTE	Weighted Global Measure of Technical Efficiency
WR	Weighted Restriction
WRTE	Weighted Russell Measure of Technical Efficiency

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Chapter 1

Introduction

1.1 Data Envelopment Analysis

Data Envelopment Analysis (DEA) is a creative model for efficiency evaluation based on mathematical programming theory. It uses the optimization method of mathematical programming to generalize Farrell (1957)'s single-output/single-input technical efficiency measure to multiple-input/multiple-output cases. DEA does so by constructing a single "virtual" output to a single "virtual" input relative efficiency measure. It is an extension of optimization techniques to solve resource allocation problems, and offers an alternative to classical statistical methods in extracting information from sample observation. DEA can be used to evaluate the relative efficiency of managerial performance among organizations with multiple inputs and multiple outputs. In DEA applications, these organizations are generally called Decision Making Units (DMUs), and they can either be industrial companies, government facilities, or service systems.

DEA was first proposed by Charnes et al. (1978), and has experienced extensive extension in both theoretical development and empirical application. According to a recent bibliography compiled by Seiford (1996), there have been more than 700 papers published in major international journals on DEA since 1978. There are a number of models developed in DEA theory. Most of them are deterministic in nature. The milestone models include the CCR model (Charnes et al., 1978), the BCC model (Banker et al., 1984), and the Additive model (Charnes et al., 1985). DEA has been widely used in both public and private sectors, and in both business and non-profit organizations. The DEA study fields include education (public schools and universities), health care (hospital, clinics, and physicians), banking, armed forces (recruiting, aircraft maintance), auditing, sports, market research, mining, agriculture, siting and spatial studies, retail outlets, transportation (ferries, highway maintenance), and public housing (Seiford, 1996). DEA has indeed established itself as an important analytical tool whose acceptance in no longer in doubt.

1.2 Motivations and Objectives

The purpose of DEA is to empirically estimate the so-called efficient frontier based on the set of available DMUs. DEA provides the user with information about the efficient and inefficient units, as well as the efficiency scores and reference sets for inefficient units. The results of the DEA analysis, especially the efficiency scores, are used in practical applications as performance indicators of DMUs. As noted by Roll et al. (1991), the classical engineering approach to input-output analysis was to reduce the analysis of a multi-input, multi-output situation to that of examining the ratio of a single composite measure of output to a single composite measure of input. This requires that the user should specify a set of weights so that all inputs and all outputs are effectively measured in the same units. This allows it to be readily applied to non-engineering environments such as not-for-profit or public sector organizations. In DEA, the relative efficiency of a unit is still assessed by calculating the ratio of its weighted sum of outputs to its weighted sum of inputs. However, the weights attached to inputs and outputs are not specified a priori. Instead, they are selected by the program to show each unit in its most favorable light. As a result, the weights chosen by DEA in assessing one unit's efficiency may be completely different from the weights selected for another unit.

There is a modest literature on how the flexibility of choice of weights on inputs and outputs in DEA might be restricted. However, in general the large DEA research effort—

particularly in the applied studies—has hitherto paid relatively little attention to the analysis of the weights used to assess the efficiency of units. This omission is surprising bearing in mind the fundamental role weights play in determining the measured efficiency of the unit.

The flexibility in choosing weights in DEA implies that (a) no a priori values or limits are set for the various weights and (b) the weights assigned to the different inputs and outputs will typically be different from one DMU to another. Traditionally, in most of the DEA literature, flexibility has been considered to be one of the major advantages of DEA when comparing it with other techniques to measure efficiency. If the weights are not constrained in any way, a DMU evaluated as inefficient by DEA cannot claim that its inefficiency arises because the set of weights selected for its inputs and outputs. The DMU must be *a fortiori* overall inefficient. However, such complete weight flexibility in DEA often leads to inappropriate estimates of efficiency. DMUs can attribute low enough weights to certain inputs and outputs so as to effectively ignore them. Therefore, the desire to incorporate restrictions on the weights attached to the input/outputs of DMUs is one of these areas of development in DEA and also is the main research topic in my research. Nowadays, weights restrictions and value judgments cover a considerable part of the DEA research literature without, however, showing any signs of saturation. The intention of incorporating value judgments is to incorporate prior views or information regarding the assessment of efficiency of DMUs. This prior information can be incorporated in a multitude of different ways having different implications on the assessed relative efficiency of DMUs.

In addition, in recent years, a substantial amount of scholarly efforts has been devoted to the development of so-called super-efficiency measures for differentiating some of the efficient DMUs that have identical efficiency scores equal to one in the basic models. The ability to rank or differentiate the efficient DMUs is of both theoretically and practically importance. Theoretically, the inability to differentiate the efficient units creates a considerable number of observations typically characterized as efficient, unless the sum of the number of inputs and outputs is small relative to the number of observations. Specialized units may be rated as efficient due to a single input or output, even though that input or output may be seen as relatively important. Thus this poses analytical difficulties to any post-DEA statistical inference analysis. In practice, further differentiation among efficient DMUs is also desirable and even necessary in many cases. One classical example of the application of the super-efficiency DEA model is the work by Lovell et al. (1994). In the Farrell tradition, ranking efficient units on the frontier was first researched by Andersen and Petersen (1993). Since then, other scholarly efforts attributed to this topic include the works by Doyle and Green (1993, 1994), Stewart (1994), Wilson (1995), Charnes et al. (1996), Tofallis (1996), Zhu (1996), Seiford and Zhu (1998, 1999), Tone (2002), Xue and Harker (2002) among others. However, one concern about these super-efficiency measures is that they may not always be possible to determine their value when the super-efficiency models are applied under other alternate returns to scale (RTS) conditions other than constant returns to scales (CRS). In other words, the mathematical program defining the super-efficiency measures may not have a feasible solution. This has been a concern in the literature since the introduction of the Farrell-based super-efficiency measure and was first noticed in Thrall (1996). Another concern is that these super-efficiency measures cannot capture certain inherent relationships among the inputs and the outputs.

Furthermore, in DEA literature, a substantial amount of scholarly efforts has been devoted to address those production systems in which both desirable (good) and undesirable (bad) output and input factors may be present. Consider a paper mill production where paper is produced with undesirable outputs of pollutants such as biochemical oxygen demand, suspended solids, particulates and sulfur oxides. If inefficiency exists in the production, the undesirable pollutants should be reduced to improve the inefficiency, i.e., the undesirable and desirable outputs should be treated differently when we evaluate the production performance of paper mills. However, in the standard DEA models, decreases in outputs are not allowed and only inputs are allowed to decrease. (Similarly, increases in inputs are not allowed and only outputs are allowed to increase.) If one treats the undesirable outputs as inputs, the resulting DEA model does not reflect the true production process. Similarly situations when some inputs need to be increased to improve the performance are also likely to occur.

Finally, we discuss the efficiency evaluation in some complex production systems where input data (or output data) are unavailable, thus making performance evaluation based only on the output data (or input data). Although from an economic point of view it is difficult to accept a DEA model without inputs or outputs, the BCC model without inputs has been widely used in performance evaluation in many fields, e. g. Lovell (1995), Ozcan and Mccue (1996), and Lovell and Pastor (1995), (1997). In addition, Lovell and Pastor (1999) made a detailed analysis on some radial DEA models without inputs or without outputs from the theoretical perspective. Therefore, research on the DEA models without inputs/outputs is both theoretically and practically importance.

1.3 Organization of the Thesis

The rest of this thesis is organized as follows: Chapter 2 provides a review of the evolution and development on the use of weights restrictions and value judgements in data envelopment analysis. In Chapter 3 we first make a comparative research on the technical efficiency measures and then develop a new methodology to measure technical efficiency, which satisfies two essential objectives: the introduction of a nonradial measure for measuring efficiency in the full input-output orientation and the introduction of a weighting scheme for inputs and outputs. In Chapter 4 we propose a weighted measure of super-efficiency which can be useful to differentiate efficient units. In Chapter 5, by using the classification invariance property, we apply our new measure to evaluate the specific production systems with undesirable factors (desirable or undesirable). In Chapter 6 we discuss the application of our new weighted global efficiency in DEA models without inputs/outputs and demonstrate some desirable characteristics theoretically and empirically. Finally, some concluding remarks and a summary of the works that we have done in this research are provided in Chapter 7.

Chapter 2

Literature Survey on Value Efficiency in DEA

2.1 Introduction

In this chapter, we first try to give a detailed description of the general structure of value efficiency problem. Then we review the evolution of the methodology of weights factors and related fields. In the second section, we introduce a basic DEA model. In the third section, alternative types of weights restrictions are presented as they arose from the application of DEA to real problems. The final section summaries our findings and provides some research directions for this research.

2.2 General Structure of Value Efficiency Problem

The DEA model was first developed by Charnes et al (1978) based on the seminal work of Farrell (1957). It requires comprehensive data on inputs and outputs for a set of homogenous decision making units. Using mathematical programming techniques, the model compares the efficiency of a chosen DMU with all possible linear combinations of other DMUs. Mathematically, assume that we have n DMUs each consuming m inputs and producing s outputs. Suppose that DMU$_0$ (x_0, y_0) is the unit under evaluation, $x_0 = (x_{01}, \cdots, x_{0m}) \in \Re_+^m$ is the vector of m inputs consumed and $y_0 = (y_{01}, \cdots, y_{0s}) \in \Re_+^s$ is the vector of s nonnegative outputs produced by this unit, where $\Re_+^m$ and $\Re_+^s$ represent vector set consisting of m and s nonnegative elements respectively. Let X and Y be the input and output matrices respectively, consisting of nonnegative elements and containing the observed input and output measures for all

DMUs.

$$Maximize\ \ h_0 = \frac{\sum_{r=1}^{s} \mu_r y_{0r}}{\sum_{i=1}^{m} v_i x_{0i}} \tag{2.1.a}$$

subject to

$$\frac{\sum_{r=1}^{s} \mu_r y_{jr}}{\sum_{i=1}^{m} v_i x_{ji}} \le 1, \quad j = 1, \cdots, n \tag{2.1.b}$$

$$\mu_r, v_i \ge 0, \quad i = 1, \cdots, m, \quad r = 1, \cdots, s \tag{2.1.c}$$

where:

y_{jr} is the amount of the rth output produced by DMU_j;

μ_r is the weight given to rth output;

x_{ji} is the amount of the ith input consumed by DMU_j;

v_i is the weight given to ith input.

Using the above formulation it is clear that DEA can be viewed as an extension of the simple ratio output-input analysis (Ganley and Cubbin, 1992; Boussofiane et al. 1991; Sherman, 1984). The efficiency of the DMU is defined as the ratio of a weighted sum of inputs to a weighted sum of inputs. However, instead of using an exogenously specified set of weights μ_r and v_i, the technique searches for the set of weights which maximize the assessed efficiency of DMU_0 (the DMU that is being evaluated) subject to the restrictions that it must be compared with all other DMUs using the same set of weights, and that none of the other DMUs can have an efficiency score higher than one. If, subject to these constraints, it is possible to find a set of weights for which the efficiency ratio of DMU_0 is equal to one, DMU_0 will be regarded as efficient; otherwise it will be regarded as inefficient. The program is computed separately for each DMU, generating n sets of optimal weights which, in general, will vary from DMU to DMU. Thus DEA is a non-parametric approach to measuring the relative efficiency of organizations. As Banker, Charnes and Cooper (1984) explained, DEA does not prescribe an underlying functional form for the efficiency frontier or specific values for the weights in an *a priori* manner. The technique can therefore be said to be "empirically based", in contrast to parametric and statistical approaches used for measuring efficiency.

However, DEA calculations in above methods are traditionally value-free. The underlying assumption is that no output or input is more important than another, although, in the real world there generally exist some outputs or inputs which are less important than other outputs or inputs in the production systems. In DEA models, a DMU which, for example, is a superior producer of a less important output is diagnosed as efficient even if it performs poorly with respect to all other outputs. Hence, in the original DEA models, the efficiency scores are not necessarily good performance indicators. Here, we use Figure 2.1 to clarify our point. The example consists of five DMUs, each producing two

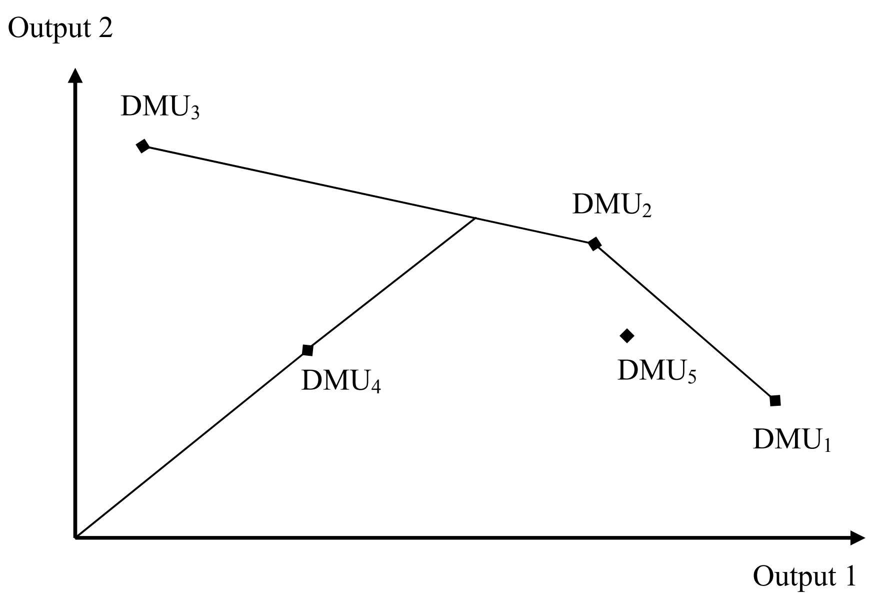

Figure 2.1 Classical DEA

outputs and all consuming the same amount of one input. We can see that DMU_1, DMU_2 and DMU_3 are efficient while DMU_4 and DMU_5 are inefficient. Thus DMU_1, DMU_2 and DMU_3 all receive an efficiency score of 1. Let us assume that for some reasons the Decision Maker (DM) considers output 1 to be much more important than output 2. In this case DMU_1 would be far more preferred to DMU_3. The DM might even prefer DMU_5 to DMU_3, even though the former is inefficient.

2.3 Review of Value Efficiency Problem

2.3.1 Taxonomy and Annotation

This Section reviews and summarizes some existing researches, both theoretical and empirical, on the value efficiency evaluation through imposing restrictions on weights. We identify four types of approach, and introduce a terminology to distinguish these approaches:

- Direct restrictions on the weights;
- Adjusting the observed input-output levels to capture value judgements;
- Restricting the virtual inputs and outputs.

These approaches are now outlined in turn. We will restrict the discussion to incorporating value judgements in the basic DEA model proposed by Charnes et al. (1978).

a) Direct Restrictions on the Weights

Following the DEA model (2.1), the following linear programming model illustrates some of the direct restriction on DEA weights typically found.

$$Maximize \sum_{r=1}^{s} \mu_r y_{0r} \tag{2.2.a}$$

subject to

$$\sum_{i=1}^{m} v_i x_{0i} = 1, \quad r = 1,\cdots,s \tag{2.2.b}$$

$$\sum_{r=1}^{s} \mu_r y_{jr} - \sum_{i=1}^{m} v_i x_{ji} \le 0, \quad i = 1,\cdots,m \tag{2.2.c}$$

$$\kappa_i v_i + \kappa_{i+1} v_{i+1} \le v_{i+2} \tag{2.2.d}$$

$$\alpha_i \le \frac{v_i}{v_{i+1}} \le \beta_i \tag{2.2.e}$$

$$\gamma_i v_i \ge \mu_r \tag{2.2.f}$$

$$v_i \le v_i \le v_i' \tag{2.2.g}$$

$$u_r \le \mu_r \le u_r' \tag{2.2.h}$$

$$v_i \ge \varepsilon, \ \ \mu_r \ge \varepsilon \tag{2.2.i}$$

$v = (v_1,\cdots,v_m)^T \in \Re_+^m$ and $\mu = (\mu_1,\cdots,\mu_2) \in \Re_+^s$ are the weight vectors of m inputs and s outputs, respectively, and are the variables of the model. The variables $(\kappa_i, \alpha_i, \beta_i, \gamma_i, u_i, u_i', v_r, v')$ are user-specified constants to reflect value judgements regarding the relative importance of the input or output factors. Constraints of type (2.2.d) and (2.2.e) can involve output rather than input weights. The five types of weights restrictions, (2.2.d) to (2.2.h), can essentially be divided into three categories and these are discussed in more detail next.

The first type of restrictions is illustrated by (2.2.d) and (2.2.e), and is introduced to incorporate into the analysis the relative ordering or values of the inputs/outputs. We call it Assurance regions of type I (ARI). Thompson et al. (1990) termed restrictions (2.2.d) and (2.2.e) as “type I Assurance Regions” (ARI). Form (2.2.d) is similar to the type used in Thompson et al. (1986) and Kornbluth (1991). The use of restriction form (2.2.e) is more prevalent, reflecting marginal rates of substitution, although the upper bound or alternatively the lower bound is often omitted. Clearly, the bound values for ARI are dependent on the scaling of the inputs and outputs, that is, they are sensitive to the units of measure of the related factors. Charnes et al. (1990) and Thompson et al. (1990) noted that when imposing ARI, there will always be at least one efficient DMU. Moreover, whether the output or input orientation is used, a DEA model incorporating ARI produces the same relative efficiency scores. This type of weights restriction is mainly based on the implementation of the economic notion of marginal rates of substitution in the context of the Charnes et al. (1978) and Charnes et al. (1985) definition. The setting of bounds for ARI in practical applications has been based either solely on expert opinion (Beasley 1990, Kornbluth 1991), or expert opinion in conjunction with price/cost information (Thompson et al. 1990, 1992).

The second type of restriction is depicted by (2.2.f). Thompson et al. (1990) termed relationships between the input and output weights as “type II Assurance Regions” (ARII). The linking of input and output weights is required in many DEA applications as it is the combination rather than the individual values of the variables that the efficiency measure should reflect. This is, clearly, the case for using ARII in Thanassoulis et al. (1995). It can be shown that ARII may render (2.2) infeasible. Moreover, a DEA model incorporating ARII does produce the same relative efficiency scores when switching from an input to an output orientation or vice versa. Similar to ARI, ARII is dependent on the scaling of the inputs and outputs. Methods for developing suitable ARII have not received much attention in the literature other than Thompson et al. (1994) in assessing world-wide major oil companies and Thanassoulis et al. (1995) in assessing output quality in health care. Thompson et al. (1994) relied on market prices obtained by corporately industry reports, whilst the Thanassoulis et al. (1995) approach is described in more detail next. For the purpose of Thanassoulis et al.’s assessment of the perinatal care units in England to recognize environmental impacts on mortality, they used a standardized survival rate – namely survival rate of babies at risk – to reflect the quality of perinatal care medical outcomes. This variable was incorporated in the DEA model as two variables: “Babies at risk” is an input and “survivals” an output. Evidently, the weight on survivals ought to be linked to that of babies at risk as otherwise a unit could exploit its high number of survivals or low number of babies at risk to improve its efficiency rating irrespective of its actual survival rate. To ensure that the efficiency estimates obtained reflect the actual survival rate, when either survivals or babies at risk are given any weight, the authors suggest equal weights for the two related variables.

The third type is absolute weights restrictions. These restrictions are illustrated by (2.2.g) and (2.2.h) and are mainly introduced to prevent the inputs or outputs from being over emphasized or ignored in the analysis. The value of the restriction is context dependent. For example, it may represent either the maximum or minimum cost of the associated factor. The bounds used in the restrictions are dependent on the normalization constant. There is a strong interdependence between the bounds on different weights. For example, setting an upper bound on one input weight imposes a lower bound on the total virtual input of the remaining variables and this in turn has implications for the values that the remaining input weights can take. It should be noted that when absolute weights restrictions are used in a DEA model, switching from an input to an output orientation produces different relative efficiency scores, and hence the bounds need to be set in light of the model orientation used. Finally, absolute weights restrictions may render model (2.2) infeasible. The key difficulty in using any one of the three types of weight restrictions outlined above is the estimation of the appropriate values for the constants in the restrictions, compatible with the value judgements to be reflected in the efficiency assessments. A number of methods have been developed to aid the estimation of such constants as is now outlined.

b) Adjusting Input-output Levels to Capture Value Judgements

Two approaches can be found where transformed input-output data is used to simulate weights restrictions. They are those of Charnes et al. (1990) and Golany (1988); both methods derive the transformations with reference to the dual of (2.2).

The first method is Charnes et al. (1990) "Cone-ratio" approach. In this method, an artificial data set is generated which produces the same relative efficiency scores as imposing ARI of the DEA model form (2.2.e). The primal cone ratio DEA model is as follows:

$$Maximize\ u^T (BY_0) \tag{2.3.a}$$

subject to

$$v^T (AX_0) = 1 \tag{2.3.b}$$

$$-v^T (AX) + u^T (BY) \leq 0 \tag{2.3.c}$$

$$u \geq 0,\ \ v \geq 0 \tag{2.3.d}$$

where the matrices A and B are defined in relation to the matrices D and F above, that is they are equivalent alternative forms, with $A^T = (D^T D)^{-1} D^T$ and $B^T = (F^T F)^{-1} F^T$, which is shown in Charnes et al. (1990).

Approaches are suggested such that the cones used in (2.3) can favor either specific inputs/outputs or individual DMUs. In the Charnes et al. (1990) bank application of the cone-ratio theory, cones that favored individual model banks were defined. For example, let us suppose that DMU_a and DMU_b, are considered as model banks. Suppose further that the optimal unrestricted DEA weights of DMU_a are $v_1 = a_1, v_2 = a_2$ and of the DMU_b, $v_1 = b_1$, $v_2 = b_2$. It can be deduced that these cones imply that the banks are being assessed under the marginal rates of substitution, as determined by the sets of optimal DEA weights for the model DMUs *a* and *b*. That is, $b_1 / b_2 \leq v_1/v_2 \leq a_1 / a_2$. This gives the following matrix:

$$D = \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \tag{2.4}$$

and from the stated matrix transformations we obtain

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \tag{2.5}$$

which can then be applied to the observed data to generate an artificial data set.

The second method is the method proposed by Golany (1988) method is the second method. Golany sought to incorporate ordinal relationships of the form $v_1 \geq v_2 \geq v_3 \geq \varepsilon$ among the DEA weights. Without allowing the weights to take a zero value, the relative efficiency scores obtained are the same as those obtained by transforming the input-output data to generate an artificial data set, by accumulating the related factors. However, Ali et al. (1993) pointed out that the data transformations proposed by Golany (1988) only provide suitable solutions for strict, not weak, ordinal relationships between DEA weights due to the weights being strictly positive. In addition, they noted that the weights themselves can be accumulated, rather than the data, to obtain the same relative efficiency scores as under the original weights restrictions.

These artificial data sets have several advantages. Firstly, they allow the use of DEA software which does not otherwise offer weights restrictions facilities. Secondly, they allow zero or even negative observed data levels to be used. However, their disadvantage is that the data must be transformed and then once results are obtained it must be transformed back to the original form in order to interpret the results. This can prove more cumbersome than the direct application of weights restrictions to the original data where the software allows it.

c) Restricting Virtual Inputs and Outputs

Wong and Beasley (1990) explored the use of such restrictions in DEA. Rather than restricting the actual DEA weights, the proportion of the total virtual output of DMU_0 devoted to output, i.e. the "importance" attached to output by DMU_0, can be restricted to range between $[\xi_r, \zeta_r]$, with ξ_r and ζ_r being determined by expert opinion, see Beasley (1990) for details. Thus, the restriction on the rth virtual output takes the form

$$\xi_r \le \frac{u_r y_{0r}}{u^T y_0} \le \zeta_r \tag{2.6}$$

where $u^T y = 1$ represents the total virtual output of DMU_0. The total virtual input or output is included in the constraint (2.6) as a standardization mechanism that would facilitate the assignment of values to ξ_r, ζ_r . A similar restriction can be set on the virtual inputs. Implementing this type of restriction is not straightforward, due to the fact that the implied restrictions on the DEA weights are DMU specific. Hence, several modifications have been suggested by Wong and Beasley (1990).

The efficiency ratings obtained with restrictions applied on the virtual inputs/outputs are sensitive to the orientation of the model (input/output). Restrictions on the virtual input/output weights have received relatively little attention in the DEA literature. More research is necessary to explore the *pros* and *cons* of setting restrictions on the virtual inputs and outputs. Heretofore, there has been no attempt to compare methods for setting restrictions on the actual DEA weights with those restricting virtual inputs and/or outputs.

This section has illustrated the rich variety of approaches to the use of weights restrictions in DEA. It is clear, however, that no overall approach to setting weights restrictions in DEA has been identified. Moreover, different approaches are likely to prove more appropriate in different contexts. For example, in a single input multi-output case, the approach by Dyson and Thanassoulis (1988) may prove suitable, while in a case with strong expert identification of good DMUs, the approach by Charnes et al. (1990) may prove more appropriate.

2.3.2 Results and Discussion

The growing expansion of the weights restrictions methodology since its original development by Thompson et al. (1986) and Dyson and Thanassoulis (1988) gives encouraging signs regarding the contribution of the method in assessing performance. Taking account of the evolutionary stages of the method, it can be said that:

- Weights restrictions are based on mathematical modifications of the Charnes et al. (1978) model that seek to encapsulate value judgements in the assessment of performance.
- Weights restrictions do not seek to eliminate the fundamental tenet of the original DEA model, which the assessment of productive efficiency should allow DMUs freedom on the value attached to the input/output variables.
- There is no all purpose method for translating value judgements into restrictions on DEA weights.
- The mathematical and managerial implications of the introduction of value judgements in DEA models have yet to be explored in full.

The interpretation of the efficiency rating as a measure of the radial contraction of inputs or radial expansion of outputs feasible under efficient operation breaks down under weights restrictions. The targets yielded by DEA models incorporating weights restrictions are not necessarily radial projections of the inefficient DMU onto the efficient frontier of the production possibility set.

It is possible to think of developing systematic methods to capture progressively the internal value in DEA assessments with the help of other methodologies. Moreover, it is also possible to incorporate the intrinsic values of production systems or the preferences of decision makers by setting weight restrictions in the function of the programming. This is in contrast to weights restrictions which are set in the constraints.

Chapter 3

Value Efficiency: Weighted Global Measure

3.1 Introduction

Theoretical consideration of technical efficiency has existed in the economic literature since Koopmans (1951) defined technical efficiency as a feasible input/output vector where it is technologically impossible to increase any output (and/or reduce any input) without simultaneously reducing another output (and/or increasing any other input). Debreu (1951) and later Farrell (1957) developed indices of technical efficiency measured as the maximum radial reduction in all inputs consistent with equivalent production of observed output. After all inputs have been radially reduced, however, there may still exist additional slack in the use of some but not all inputs. As a result, a Farrell efficient producer may not be Koopmans efficient. (Färe and Lovell (1978), and Lovell (1993) provided useful discussions.). Interest in this early theoretical work on technical inefficiency was renewed in the late 1970s with the development of DEA, a Farrell-based mathematical programming approach to frontier estimation pioneered in Charnes et al. (1978) and extended in Banker et al. (1984) and Färe et al. (1985, 1994). (For more details about the strengths and weaknesses of DEA, see Cooper et al. (2000) and Thanassoulis (2001))

Potential problems arise with the DEA measure of inefficiency because it is not based on the conceptual notions of Koopmans. DMUs may be identified as efficient even though additional slack exists in some but not all of the inputs. As a result, the DEA measure may not capture all of the existing inefficiency. To solve this problem, Färe and Lovell (1978) introduced the non-proportional Russell measure which identifies a producing unit as technically efficient if it does have any slack in any inputs. Rather than determing the maximum radial in all inputs holding output constant, the Russell measure minimizes the average weighted arithmetic mean of proportional reductions in all individual inputs. Thus, a Koopmans inefficient producing unit can be effectively identified. One problem with the Russell measure is the implicit assumption that all inputs equally affect the level of potential production. As will be shown, this can lead to distorted efficiency measurement. The main purpose of this chapter is to introduce the weighted global measure of technical efficiency that not only allows non-proportionate reduction in both input and output space but also introduces a weighting scheme for inputs and outputs which takes account of the characteristics of the DMUs.

3.2 Comparative Research on Efficiency Measures

Assume that we have n DMUs each consuming m inputs and producing s outputs. Suppose that DMU_0 (x_0, y_0) is the unit under evaluation, $x_0 \in \Re_+^m$ is the vector of m inputs consumed and $y_0 \in \Re_+^s$ is the vector of s outputs produced by this unit. Let $X \in \Re_+^{m\times n}$ and $Y \in \Re_+^{s\times n}$ be the input and output matrices respectively, consisting of nonnegative elements and containing the observed input and output measures for DMUs.

3.2.1 The Radial Efficiency Measures

Farrell provided the first comprehensive measure efficiency as one minus the maximum equal-proportional reduction in all inputs that maintains observed output. Following Banker et al. (1984), the Farrell input measure of technical efficiency (FTE) (assuming variable returns to scale) for DMU_0 (x_0, y_0) can be calculated as

$$FTE = \theta \tag{3.1}$$

Following Banker et al. (1984), the Farrell measure (assuming variable returns to scale) for DMU$_0$ (x_0, y_0) can be calculated as:

$$Minimize\ \ \theta \tag{3.2.a}$$

subject to

$$X\lambda \le \theta x_0, \quad 0 \le \theta \le 1 \tag{3.2.b}$$

$$Y\lambda \ge y_0 \tag{3.2.c}$$

$$1^T \lambda = 1, \quad \lambda \ge 0 \tag{3.2.d}$$

The Farrell measure is radial; for a given DMU, it determines the maximal amount by which input vector can be proportionally reduced while maintaining production of output vector. Note that the Farrell measure does not require comparison of a given input vector to an input vector that belongs to the identified efficient subset. A potential problem arises with the Farrell measure because inputs are radially reduced. Even after this reduction is achieved, there may still exist slack in the outputs and some but not all of the inputs. This is evident from the inequality constraints (3.2.b) and (3.2.c) in the use of either LP or DEA model. As a result, a Farrell efficient DMU may be technically inefficient in the Koopmans sense. This problem is illustrated in Figure 3.1.

Referring to the diagram, assume that four DMUs A, B, C and D employ one input to produce one output. Based on the above postulates, the efficient frontier, the geometrical illustration of DEA efficiency, is identified as convex combinations of the observed production possibilities, which consists of line segments AB and BC. While D is technically and scale inefficient, A and C are technically efficient but scale inefficient. Only decision making unit B is technically and scale efficient. In this case, inclusion of the convexity constraint (3.2.d) leads to increasing returns to scale along AB and decreasing returns along BC. The variable returns to scale (VRS) model

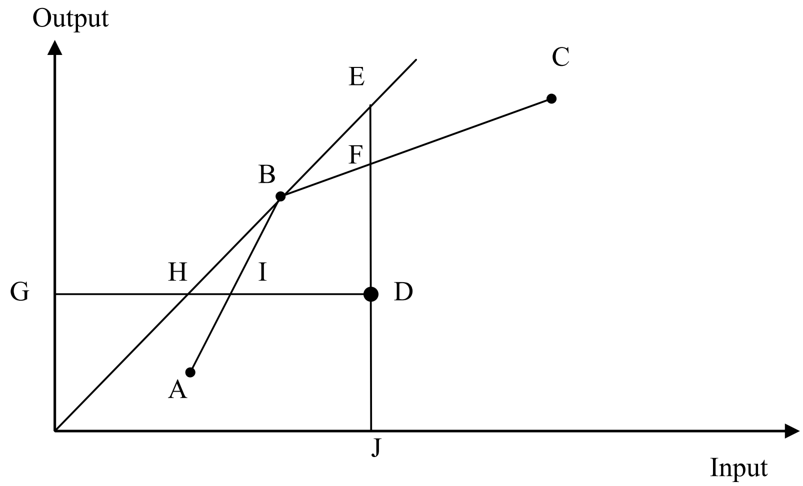

Figure 3. 1 Measurement of Radial Efficiency

presented above measures the technical efficiency of DMU_D to be $\frac{IG}{DG}$, where the composite reference production possibility is labeled I. This measure can be considered to be pure technical efficiency measure since it allows variable returns to scale. By assuming constant returns to scale in production, the technical efficiency of DMU_D would be measured to be $\frac{HG}{DG} \leq \frac{IG}{DG}$. It has been shown by banker et al. (1984), Färe at al. (1985) and Banker and Thrall (1992) that the measure of inefficiency obtained from the solution of the constant returns to scale DEA model consists of not only technical but also scale inefficiency.

3.2.2 The Extended Radial Efficiency Measures

A potential problem arises with the Farrell measure because Farrell efficiency of a DMU is determined either by maximizing outputs subject to given inputs level or minimizing inputs subject to given output levels. Thus, the difference in both efficiency measures is inevitable. Following the above analysis for DMU_D in Figure 3.1, the technical efficiency in output-oriented model is $\frac{JD}{JE}$ and the inefficiency is $\frac{DE}{JE}$. Therefore, we have no reason to be sure that the following equality will be satisfied: $\frac{DG}{GH} = \frac{JD}{JE}$. Therefore, two models can address this problem: Cooper et al. (1996) proposed a new efficiency measure which can consider both input and output inefficiency simultaneously and is the optimal function value of the following mathematical programming problem:

$$Minimize\ \frac{\theta}{\delta} \tag{3.3.a}$$

subject to

$$X\lambda \leq \theta x_0\,, \quad 0 \leq \theta \leq 1 \tag{3.3.b}$$

$$Y\lambda \geq \delta y_0\,, \quad \delta \geq 1 \tag{3.3.c}$$

$$1^T \lambda = 1, \quad \lambda \geq 0 \tag{3.3.d}$$

For a DMU to be efficient, two following conditions must be satisfied: (i) $\theta^* / \delta^* = 1$ and (ii) all the slacks must be zero in any optimal solution. The measure is rather different from and stronger than that of Banker et al. (1984) because of considering slacks in alternative optimal solutions.

Alternatively, Joro et al. (1998) treat both output and input rows as objective rows. This leads to the following formulation:

$$Maximize\ Z = \delta + \varepsilon\left(1^T s^+ + 1^T s^-\right) \tag{3.4.a}$$

subject to

$$X\lambda + s^- = (1-\delta) x_0 \tag{3.4.b}$$

$$Y\lambda - s^+ = (1+\delta) y_0 \tag{3.4.c}$$

$$1^T \lambda = 1, \quad \lambda \geq 0 \tag{3.4.d}$$

$$s^+ \geq 0\,, \ s^- \geq 0\,, \ \varepsilon \geq 0 \tag{3.4.e}$$

3.2.3 The Nonradial Efficiency Measures

A potential problem arises with both the radial and extended radial efficiency measures because inputs are radially reduced (See Färe and Lovell (1978) for a further discussion). Even after this reduction is achieved, there may still exist slack in the outputs and some but not all of the inputs. This is evident from the inequality constraints (3.2.b) and (3.2.c) in the programming model. As a result, a Farrell efficient DMU may be technically inefficient in the Koopmans sense. This problem is shown in Figure 3.2, where two inputs x_1 and x_2 are to produce the same level of output y_0. Using the original DEA model results in a piecewise linear y_0 that consists of relevant segments AB, BD and DE. DMU_C is the only Farrell inefficient DMU with excess usage in both inputs. DMUs A, B, D and E are identified as Farrell efficient. Note, however, that DMUs A and E are not Koopmans efficient since additional input slack exists in x_1 and x_2 respectively.

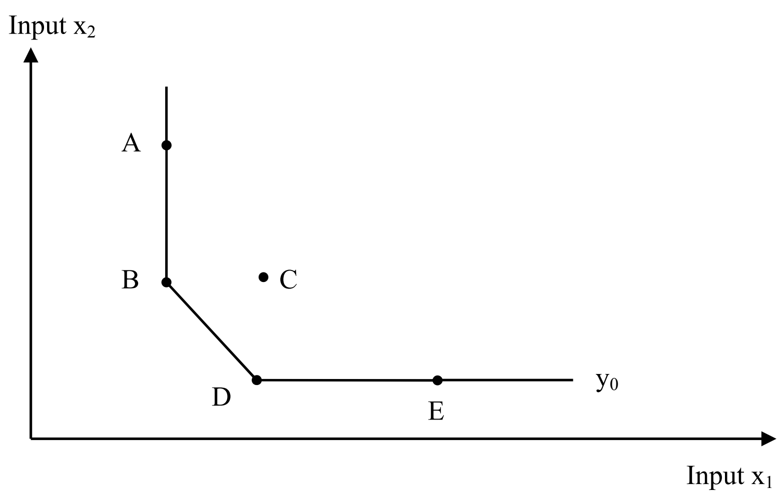

Figure 3. 2 Radial and Koopmans Efficiency

The potential problem of the Farrell measure arises when there exists slack in some but not all of the inputs after radial efficiency is achieved. Färe and Lovell (1978) not only recognized this problem but provided a solution by introducing the nonradial Russell measure of efficiency. The Russell measure of technical (RTE) efficiency is defined as

$$RTE = (1^T \theta)/m \tag{3.5}$$

where $\theta = (\theta_1, \cdots, \theta_m)^T$ represents the scalar for the ith inputs. The Russell measure for each DMU can be calculated as the solution to the following linear program:

$$Minimize \;\; RTE = (1^T \theta)/m \tag{3.6.a}$$

subject to

$$X\lambda \leq \theta_i x_0 \,, \quad 0 \leq \theta_i \leq 1 \tag{3.6.b}$$

$$Y\lambda \geq y_0 \tag{3.6.c}$$

$$1^T \lambda = 1, \quad \lambda \geq 0 \tag{3.6.d}$$

The advantage of the Russell measure over the Farrell measure can be inferred from Figure 3.2 as discussed above, DMU_A is Farrell efficient, achieving $RTE = 1$. This results because DMU is compared to itself in the solution of the DEA model (6). The Russell measure, on the other hand, allows non-radial contraction of inputs and hence, compares A to B. Solution to (3.6) results in a Russell measure $RTE = 0.83$. This solution is obtained from $\theta_1 = 1$ and $\theta_2 = 2/3$, i. e. DMU_A is efficient in the use of x_1 but inefficient in the use of x_2 relative to DMU_B. The Russell measure of efficiency for each DMU can be inferred from Figure 3.2, using the Russell measure, DMU_B and DMU_D are efficient, while DMU_A, DMU_C and DMU_E are not.

Correspondingly, the following three The Russell Graph Measure of technical efficiency was defined as a combination of the Input and Output Russell Measure of technical efficiency. For a given DMU, the value of this measure can be obtained from the following formulation:

$$Minimize \;\; RGTE = \frac{1}{m+s}\left(1^T \theta + \frac{1}{(1^T \delta)}\right) \tag{3.7.a}$$

subject to

$$X\lambda \le \theta x_0\,,\quad 0 \le \theta \le 1 \tag{3.7.b}$$

$$Y\lambda \ge \delta\, y_0\,,\quad \delta \ge 1 \tag{3.7.c}$$

$$1^T \lambda = 1,\quad \lambda \ge 0 \tag{3.7.d}$$

where constraints $0 \le \theta = (\theta_1,\cdots,\theta_m)^T \le 1$ and $\delta = (\delta_1,\cdots,\delta_s)^T \ge 1$ are the requirements for dominance. In addition, the convexity constraint $1^T \lambda = 1$ would be included if T were not assumed to satisfy constant returns to scale.

Although RGTE is well defined and it also satisfies the four basic properties listed by Cooper and Pastor, there are some difficulties with this measure. Firstly, it must be computed from a nonlinear programming problem whose solution is not easily obtained. Secondly, it is not readily understood because, as Cooper et al. 1998 note, RGTE is a weighted average of arithmetic and harmonic means. Therefore, based on this measure, Pastor et al. (1999) propose the Enhanced Russell Graph Measure as an alternative to this measure which, although closely related, avoids the mentioned difficulties. Instead of combining the input and output Russell measures in an additive way, as in (3.8), they define the following measure as the ratio between them:

$$ERGTE = \left(1^T \theta / m\right) / \left(1^T \delta / s\right) \tag{3.8}$$

In above definition, they separately average the input and the output efficiency and then combine these two efficiency component in a ratio form. The result is the following model:

$$Minimize\quad ERGTE = \left(1^T \theta / m\right) / \left(1^T \delta / s\right) \tag{3.9.a}$$

subject to

$$X\lambda \le \theta\, x_0\,,\quad 0 \le \theta \le 1 \tag{3.9.b}$$

$$Y\lambda \ge \delta\, y_0\,,\quad \delta \ge 1 \tag{3.9.c}$$

$$1^T \lambda = 1,\quad \lambda \ge 0 \tag{3.9.d}$$

In attempt to define inefficiency based on the slacks, Russell (1985, 1988), Pastor (1996), Lovell and Pastor (1995), Torgersen et al. (1996), Cooper and Pastor (1997), Cooper and Tone (1997), Thrall (1997), Tone (2001) and others have proposed several formulae for finding a scalar measure. Here we discuss the efficiency measure proposed by Tone (2001) and the possible relationship between such measures with other nonradial efficiency measures. Consider an expression for describing a certain DMU_0 (x_0, y_0) as

$$x_0 = X\lambda + s^- \tag{3.10}$$

$$y_0 = Y\lambda - s^+ \tag{3.11}$$

where $s^- = (s_1^-, \cdots, s_m^+)^T \in \Re_m^+$ and $s^+ = (s_1^+, \cdots, s_s^+)^T \in \Re_s^+$ denote the input excess and output shortfall of this expression, respectively, which are called slacks. Using s^- and s^+, Tone (2001) defines the slack-based technical efficiency as follows:

$$STE = \frac{1 - \frac{1}{m}\left[1^T\left(s^- / x_0\right)\right]}{1 + \frac{1}{s}\left[1^T\left(s^+ / y_0\right)\right]} \tag{3.12}$$

The efficiency of the DMU can be obtained by solving the following fractional program:

$$Minimize\ \ STE = \frac{1 - \frac{1}{m}\left[1^T\left(s^- / x_0\right)\right]}{1 + \frac{1}{s}\left[1^T\left(s^+ / y_0\right)\right]} \tag{3.13.a}$$

subject to

$$x_0 = X\lambda + s^- \tag{3.13.b}$$

$$y_0 = Y\lambda + s^+ \tag{3.13.c}$$

$$1^T \lambda = 1, \quad \lambda \geq 0 \tag{3.13.d}$$

$$s^- \geq 0,\ \ s^+ \geq 0 \tag{3.13.e}$$

The above fractional program can also be obtained by transformation of (3.9). Thus, the slack-based efficiency measure in (3.13) is equivalent to that in (3.9).

3.2.4 The Weighted Nonradial Efficiency Measures

The problem with the nonradial measures is shown in Figure 3.3, where the true (but

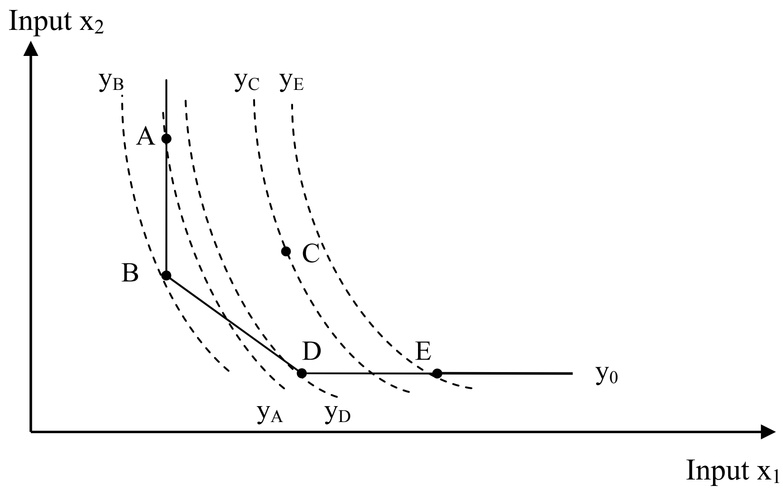

Figure 3.3 Theoretical problem of the Russell measure

unknown) isoquants are superimposed on the piecewise linear isoquants. The true isoquants were generated from the production function y_0. As seen from Figure 3.3, while all DMUs produce the same level of output, the efficient amount of output differs for all DMUs. DMU_B, which is Koopmans efficient, can produce the least amount of output given its input usage. DMU_E could have produced the most amount of output given its high level of input 1. This is interesting because the Russell measure identifies DMU_A and DMU_E to be equally efficient and more efficient than DMU_C. As shown, DMU_C is more inefficient than DMU_A but more efficient than DMU_B. Consequently, both the radial and nonradial measures fail to rank the DMUs properly. The failure of the nonradial measure can be attributed to the invalid assumption of equal weights when different inputs impact output differently in the production process. Thanassoulis and Dyson (1992) extended the Russell measure to allow unequal factor weights. Their model, however, is motivated by the preferred target input and output levels; each DMU can assign weights based on its preferences. Ruggiero and Bretschneider (1998) extended the important model of Thanassoulis and Dyson to accommodate the excess slack inherent in the Farrell measure without assuming equal factor weights. Weights are not chosen to achieve preferred target level but rather to recognize the possibility of differential factor weights in the production process. The resulting Weighted Russell measure, can be defined as

$$WRTE = w^T \theta \tag{3.14}$$

where $w = (w_1, \cdots, w_m)^T$ denotes the weight of input and satisfies $1^T w = 1$. One important qualification of using the Weighted Russell measure is the necessity of determing the factor weights w prior to measurement. One means of inferring the weights is a first-stage regression analysis. Alternatively, one could employ an LP model to constraint the 'residuals' to be one side. Allowing variable return to scale, the Weighted Russell measure of technical efficiency (WRTE) for each DMU can be calculated as the solution to the following linear programming:

$$Minimize\ WRTE = w^T \theta \tag{3.15.a}$$

subject to

$$X\lambda \le \theta x_0, \quad 0 \le \theta \le 1 \tag{3.15.b}$$

$$Y\lambda \ge y_0 \tag{3.15.c}$$

$$1^T \lambda = 1, \quad \lambda \ge 0 \tag{3.15.d}$$

Unlike the Farrell measure, this measure compares inefficient production possibilities to those production possibilities identified from the linear programming model to be relatively efficient. However, similar to the Russell measure, the weighted Russell measure does not make allowance for output slack.

Prior to this research, Briec (1997) has introduced a new efficiency measure which not only measures efficiency in full input-output space but also introduces a weighting scheme for inputs and outputs. This defines an orientation account of particularities of the market and characterizing the criteria of management chosen by the producer. The efficiency measure can be formulated as the optimal function value of the following programming:

$$Maximize\ \ Z = \delta + \varepsilon\left(1^T s^+ + 1^T s^-\right) \tag{3.16.a}$$

subject to

$$X\lambda + s^- = (I - A\delta)x_0 \tag{3.16.b}$$

$$Y\lambda - s^+ = (I + B\delta)y_0 \tag{3.16.c}$$

$$1^T \lambda = 1, \quad \lambda \geq 0 \tag{3.16.d}$$

$$s^+ \geq 0\,,\ \ s^- \geq 0\,,\ \ \varepsilon \geq 0 \tag{3.16.e}$$

Basing upon the relationship between the proportional distance and radial efficiency measures, the above linear programming is identical to the DEA linear program. Hence, the new measure presented in Briec's research generalizes the DEA method introduced by Charnes et al. (1978).

3.3 The Weighted Global Measure of Efficiency

Assume that we have n DMUs each consuming m inputs and producing s outputs. Suppose that DMU$_0$ (x_0, y_0) is the unit under evaluation, $x_0 \in \Re_+^m$ is the vector of m inputs consumed and $y_0 \in \Re_+^s$ is the vector of s outputs produced by this unit. Let $X \in \Re_+^{m\times n}$ and $Y \in \Re_+^{s\times n}$ be the input and output matrices respectively, consisting of nonnegative elements and containing the observed input and output measures for DMUs. We also assume that there are no duplicated units in the data set. Then, following returns to variables, the production possibility set P is defined as

$$\mathrm{P} = \left\{(x_0, y_0) \middle| x_0 \geq \lambda X, y_0 \leq \lambda Y, 1^T \lambda = 1, \lambda \geq 0\right\} \tag{3.17}$$

Suppose that $v = (v_1, \cdots, v_m)^T \in \Re_+^m$ and $u = (u_1, \cdots, u_s)^T \in \Re_+^s$ are the weight vectors of m inputs and s outputs, respectively. We also assume that the input and output vectors satisfy $1^T u = 1$ and $1^T v = 1$. Then the weighted global efficiency score of DMU_0 (x_0, y_0) can be defined as the optimal solution of the following program:

$$Minimize\ \Phi = \frac{v^T \theta}{u^T \delta} \tag{3.18.a}$$

subject to

$$X\lambda = \theta x, \quad 0 \leq \theta \leq 1 \tag{3.18.b}$$

$$Y\lambda = \delta y, \quad \delta \geq 1 \tag{3.18.c}$$

$$1^T \lambda = 1, \quad \lambda \geq 0 \tag{3.18.d}$$

where $\theta = (\theta_1, \cdots \theta_m)^T$ and $\delta = (\delta_1, \cdots, \delta_s)^T$ represent the scalar vector for the inputs and outputs respectively. Let an optimal solution of (3.18) be $(\Phi^*, \theta^*, \delta^*, \lambda^*)$. It can be interpreted as ratio between the weighted efficiency of inputs and the weighted efficiency of outputs, which is a more straightforward interpretation than other measures listed before. Moreover, Φ may be decomposed into an input component of weighted efficiency and an output one to better explain the efficiency of the DMU being evaluated. Based on the optimal solution, we define a DMU_0 (x_0, y_0) as being *weighted global efficient* as follows:

DEFINITION 3.1 *A DMU_0 (x_0, y_0) is weighted global efficient if and only if $\Phi^* = 1$, that is, $\delta^* = 1$ and $\theta^* = 1$.*

If the DMU_0 (x_0, y_0) is not weighted global Efficient, it is called *weighted global Inefficient*. Thus we also have

***THEOREM* 3.1** *DMU₀* $\left(x_0, y_0\right)$ *is weighted global efficient if and only if the optimal value of objective function* Φ^* *is equal to unity.*

PROOF. Suppose that DMU_0 $\left(x_0, y_0\right)$ is efficient, if $(\Phi^*, \theta^*, \delta^*, \lambda^*)$ is the optimal solution of (3.18) whose objective function value Φ^* is unity, from (3.18.a), we have

$$v^T\theta^* = u^T\delta^* \tag{3.19}$$

Since $0 \le \theta^* \le 1$ and $\delta^* \ge 1$, then the only condition to satisfy (3) is $\theta^* = \delta^* = 1$. Therefore, according to definition, DMU_0 $\left(x_0, y_0\right)$ is efficient. This completes the proof.

□

The above theorem shows that model (3.18) can not only determine efficient DMUs but also determine inefficiency of DMU as well as show how to improve the inefficient DMUs relative to those efficient ones. On the other hand, by means of the following change of variables:

$$\theta = \frac{x_0 - s^-}{x_0} = 1 - \frac{s^-}{x_0} \tag{3.20}$$

$$\delta = \frac{y_0 + s^+}{y_0} = 1 + \frac{s^+}{y_0} \tag{3.21}$$

It is easy to reexpress formulation (3.18) in terms of total slacks. The result is this new problem which provides an alternative expression of the weighted global measure connecting Φ with the usual GEMs:

$$\textit{Minimize } \Phi = \frac{1 - v^T\left(s^- / x_0\right)}{1 + u^T\left(s^+ / y_0\right)} \tag{3.22.a}$$

subject to

$$x_0 = X\lambda + s^- \tag{3.22.b}$$

$$y_0 = Y\lambda + s^+ \tag{3.22.c}$$

$$1^T\lambda = 1, \quad \lambda \ge 0 \tag{3.22.d}$$

$$s^- \geq 0, \ \ s^+ \geq 0 \tag{3.22.e}$$

In a similar fashion, if we set all the inputs and outputs are equally important, then the above formulation is identical to the slack-based model proposed by Tone (2001).

REMARK Färe and Lovell (1978) were the first ones who proposed a set of desirable properties that an ideal efficiency measure should satisfy, although these were enunciated for the particular case of an input oriented measure. Recently, Cooper and Pastor (1995) listed similar requirements for the DEA context and suggested some others. Next, we study the properties which the proposed weighted global measure satisfies. The following is true for the weighted global measure of efficiency Φ :

PROPOSITION 3.2 *The weighted global efficiency score Φ^* is units invariant, i. e. it is independent of the units in which the inputs and outputs are measured provided these units are the same for every DMU.*

PROOF. This propostion holds, since both the objective function and constraints are units invariant. This completes the proof. □

PROPOSITION 3.3 *The weighted global measure of efficiency is strongly monotonic in inputs and in outputs.*

PROOF. Firstly, we are going to rate two units differing only in one input. Consider an observation DMU$_0$ with vector of inputs $(x_{10},\cdots,x_{m0})$ and outputs $(y_{10},\cdots,y_{s0})$, and another observation, DMU$_a$ with the same values for all inputs and outputs but input k, which has the value $x_{ka}=x_{k0}+a, \quad a>0$. We have to show that the optimal value of Φ_a^* for the second observation, DMU$_a$, is smaller than Φ_0^*, the optimal value for the first unit. Throughout this proof let us use the fractional problems (P$_0$) and (P$_a$) to evaluate DMU$_0$ and DMU$_a$, respectively, and let $\left(\Phi_0^*,\theta_0^*,\delta_0^*,\lambda_0^*\right)$ and $\left(\Phi_a^*,\theta_a^*,\delta_a^*,\lambda_a^*\right)$ be the corresponding optimal solution. Let $\left(\lambda_{1a}^*,\cdots,\lambda_{na}^*,\theta_{1a}^*,\cdots,\theta_{ma}^*,\delta_{1a}^*,\cdots,\delta_{sa}^*\right)$ be a solution of

(P_a). Then, we can see that $\left(\lambda^*_{1a},\cdots,\lambda^*_{na},\theta'^*_{1a},\cdots,\theta'^*_{ma},\delta^*_{1a},\cdots,\delta^*_{sa}\right)$, where $\theta^*_{ra}=\theta'^*_{r0}$, $r \neq p$, and $\theta^*_{pa}=\theta'^*_{p0}-a/x_{p0}$, is a feasible solution for (P_0) verifying the above requirement. Then it is easy to check that Φ'^*_a is greater than or equal to Φ^*_a and thus greater than Φ^*_0. So, we can conclude $\Phi^*_a \geq \Phi^*_0$.

Following the notation above, let us now consider DMU_a equal to DMU_0 except for the output p, taking the value $y_{ka}=y_{k0}+a$, $a>0$, for DMU_a. Now, we have to prove $\Phi^*_a \geq \Phi^*_0$. Let us start the proof by showing that any solution of the problem (P_a) gives a feasible solution of the problem (P_0) with a smaller value of the objective function. Let $\left(\lambda^*_{1a},\cdots,\lambda^*_{na},\theta^*_{1a},\cdots,\theta^*_{ma},\delta^*_{1a},\cdots,\delta^*_{sa}\right)$ be a solution of (P_a). Then, we can see that $\left(\lambda^*_{1a},\cdots,\lambda^*_{na},\theta^*_{1a},\cdots,\theta^*_{ma},\delta'^*_{1a},\cdots,\delta'^*_{sa}\right)$, where $\delta^*_{ra}=\delta'^*_{r0}$, $r \neq p$, and $\delta^*_{pa}=\delta'^*_{p0}+a/y_0$, is a feasible solution for (P_0) verifying the above requirement. In particular, if the starting solution of (P_a) is an optimum, we find a solution of problem (P) with as associated value of the objective function less than Φ^*_a, so we conclude $\Phi^*_a \geq \Phi^*_0$. This completes the proof. □

PROPOSITION 3.4 *Let* $(\varphi x_0, \psi y_0)$ *with* $0 \leq \varphi \leq 1$ *and* $\psi \geq 1$ *be a DMU with the reduced inputs and enlarged outputs than* $\left(x_0, y_0\right)$*. Then, the weighted global efficiency score of* $(\varphi x_0, \psi y_0)$ *is not less than that of* $\left(x_0, y_0\right)$*.*

PROOF. Supposing that $(\Phi^*, \theta^*, \delta^*, \lambda^*)$ is the optimal solution of (3.18) when DMU_0 $\left(x_0, y_0\right)$ is under evaluation. If $0 \leq \varphi \leq 1$ and $\psi \geq 1$, then $\left(\theta^*/\varphi, \delta^*/\psi, \lambda^*\right)$ is a feasible solution of (3.18) when $(\varphi x_0, \psi y_0)$ is being evaluated, because the constraints for the inputs and outputs are clearly satisfied and $\theta^* \leq \theta^*/\varphi$ as well as $\delta^*/\psi \leq \delta^* \leq 1$. Therefore, we have

$$\Phi'^{*} = \frac{u^T\left[\theta^{*}/\varphi\right]}{v^T\left[\delta^{*}/\psi\right]} \geq \frac{u^T\theta^{*}}{v^T\delta^{*}} = \Phi^{*} \tag{3.23}$$

Thus, the *weighted global efficiency score* Φ'^{*} of $(\varphi x_0, \psi y_0)$ is not less than Φ^{*} of (x_0, y_0). This completes the proof. □

REMARK. One important qualification of using the weighted efficiency measure is the necessity of determining the factor weights prior to measurement. One method of estimation for this multi-output and multi-input production function is based on the multivariate technique of canonical correlation analysis. For more details see Vinod (1968). This method creates two variables, U and V, consisting of linear combinations of outputs and inputs respectively (specified in log form):

$$V = v_1 lnx_1 + \cdots + v_m lnx_m \tag{3.24}$$

and

$$U = u_1 lny_1 + \cdots + u_s lny_s \tag{3.25}$$

The optimal weights $v = (v_1, \cdots, v_m)^T$ and $u = (u_1, \cdots, u_s)^T$ can be obtained by maximizing the correlation between U and V:

$$\rho^{*} = \underset{u,v}{Maximize}\, Corr\,(U, V) \tag{3.26}$$

leading to estimates $u^{*} = (u_1^{*}, \cdots, u_m^{*})^T$ and $v = (v_1^{*}, \cdots, v_s^{*})^T$. If the weights are precisely recognized, this new measure can adjust inputs and outputs at the same time. Because the proposed measure requires specification of weights, it should be considered semi-parametric.

REMARK. In order to preserve the linearity and convexity of DEA through our model, the fractional program (3.18) can be transformed into a linear programming problem using the Charnes-Cooper transformation as follows:

$$t > 0\,,\ \theta = \frac{\theta'}{t}\,,\ \delta = \frac{\delta'}{t}\,,\ \lambda = \frac{\lambda'}{t}\,,\ \Phi = \Phi' \qquad (3.27)$$

Then, (3.18) becomes the following linear program in t, θ', δ' and λ':

$$\text{Minimize } \Phi' = \mathrm{u}^{\mathrm{T}}\theta' \qquad (3.28.a)$$

subject to

$$\mathrm{X}\lambda' = \theta'\mathrm{x}_0\,, \quad 0 \le \theta \le \mathrm{t} \qquad (3.28.b)$$

$$\mathrm{Y}\lambda' = \delta'\mathrm{y}_0\,, \quad \delta' \ge \mathrm{t} \qquad (3.28.c)$$

$$\mathrm{v}^{\mathrm{T}}\delta' = 1 \qquad (3.28.d)$$

$$\mathbf{1}^{\mathrm{T}}\lambda' = \mathrm{t}\,, \quad \lambda' \ge 0 \qquad (3.28.f)$$

Suppose that the optimal solution of (3.28) is $\left(\Phi'^{*}, \theta'^{*}, \delta'^{*}, \lambda'^{*}, t^{*}\right)$, using (3.27), we can obtain an optimal solution of (3.18) as expressed by $\left(\Phi^{*}, \theta^{*}, \delta^{*}, \lambda^{*}\right)$.

3.4 Tracing Out the Efficient Frontier

The intent of frontier estimation is to deduce empirically the production function in the form of an efficient frontier. That is, rather than knowing how to convert functionally inputs and outputs, these methods take the inputs and outputs as given, map out the best performers, and produce a relative notion of the efficiency of each. The problem with the existing methods is that they each measure efficiency in a conceptually suspect, albeit computationally effective, way. If the DMUs are plotted in their input/output space, then an efficient frontier that provides a tight envelope around all of the DMUs can be determined. The main function of this envelope is to get as close as possible to each DMU without passing by any others. A simple example of an efficient frontier (using variable returns to scale) is shown in Figure 3.4. Each DMU along the frontier is considered efficient while those falling below the frontier, (e.g., DMU_5) are considered inefficient.

The method of determining the efficiency score for DMU_5 varies according to the technique employed. Of the two classic methods, the input-oriented or output-oriented methods, the efficiency score is determined, in effect, by determining the projection directly along the horizontal axis (holding outputs constant), or along the vertical axis (holding inputs constant). The method developed in this section determines the shortest projection from an inefficient DMU_5 to the frontier, in both the input and output space.

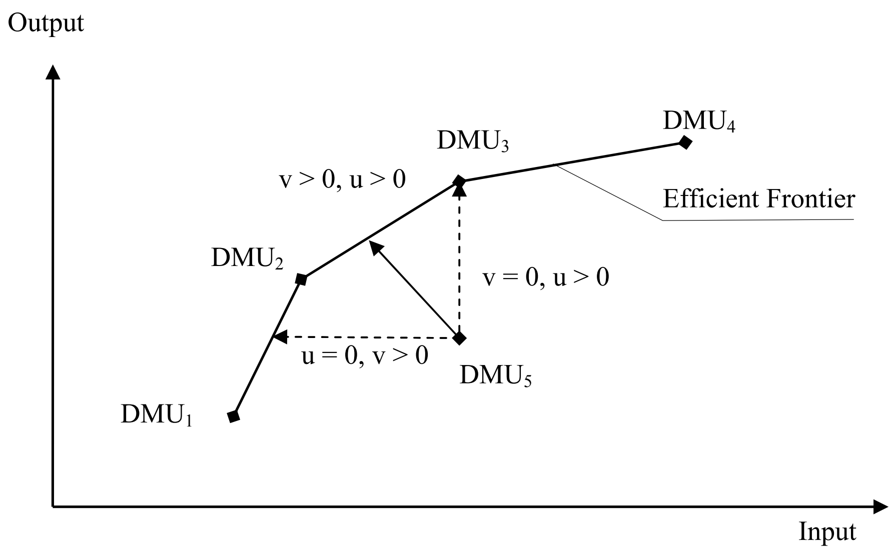

Figure 3.4 Efficient frontier and shortest projection

This projection is more meaningful than either the input- or output-oriented projection as it permits the simultaneous movement of inputs and outputs.

***DEFINITION* 3.2** *Supposing that the optimal solution of (3.18) is* $(\Phi^*, \theta^*, \delta^*, \lambda^*)$, *$DMU_k$* $(\widetilde{x}_k, \widetilde{y}_k)$ *can be expressed as follows:*

$$\widetilde{x}_k = \theta^* x_0 = X\lambda^*, \quad 0 \le \theta^* \le 1 \tag{3.29}$$

$$\widetilde{y}_k = \delta^* y_0 = Y\lambda^*, \quad \delta^* \ge 1 \tag{3.30}$$

$$1^T \widetilde{\lambda}^* = 1, \quad \lambda^* \ge 0 \tag{3.31}$$

then DMU_k $(\widetilde{x}_k, \widetilde{y}_k)$ is defined as the projection of DMU_0 (x_0, y_0) onto the efficient *frontier.*

These relationships suggest that the efficiency of DMU_0 (x_0, y_0) can be improved if the input values are reduced nonproportionally by the ratio θ^* while the output values are augmented by the ratio δ^*. Thus, we have a method for improving an inefficient DMU that accords with Definition 3. 2. In the following theorem, we will show that the improved activity $(\tilde{x}_k, \tilde{y}_k)$ projects DMU_0 onto the reference set Θ and any nonnegative combination of DMUs in Θ is weighted global efficient.

THEOREM 3.5 *Compared with all the other DMUs under evaluation, the projection DMU_k $(\tilde{x}_k, \tilde{y}_k)$ is weighted global efficient.*

PROOF. Supposing that the DMU_k $(\tilde{x}_k, \tilde{y}_k)$ is the projection of DMU_0 (x_0, y_0) onto the efficient frontier, we use the following (14) to evaluate the efficiency of DMU_k relative to all the other DMUs under evaluation:

$$Minimize\ \tilde{\Phi} = \frac{v^T \tilde{\theta}}{u^T \tilde{\delta}} \tag{3.32.a}$$

subject to

$$X\tilde{\lambda} + \tilde{x}_k \tilde{\lambda}_k = \tilde{\theta}\tilde{x}_k, \quad 0 \le \tilde{\theta} \le 1 \tag{3.32.b}$$

$$Y\tilde{\lambda} + \tilde{y}_k \tilde{\lambda}_k = \tilde{\delta}\tilde{y}_k, \quad \tilde{\delta} \ge 1 \tag{3.32.c}$$

$$1^T \tilde{\lambda} + \tilde{\lambda}_k = 1, \quad \tilde{\lambda} \ge 0, \quad \tilde{\lambda}_k \ge 0 \tag{3.32.d}$$

Suppose that the optimal solution of (3.32) is $(\tilde{\theta}^*, \tilde{\delta}^*, \tilde{\lambda}^*, \tilde{\lambda}_k^*)$, from the constraints of (3.32), we have

$$X\tilde{\lambda}^* + \tilde{x}_k \tilde{\lambda}_k^* = X(\tilde{\lambda}^* + \tilde{\lambda}^* \tilde{\lambda}_k^*) = \tilde{\theta}^* \tilde{x}_k = \tilde{\theta}^* \theta^* x_o \tag{3.33}$$

$$Y\tilde{\lambda}^* + \tilde{y}_k \tilde{\lambda}_k^* = Y(\tilde{\lambda}^* + \tilde{\lambda}^* \tilde{\lambda}_k^*) = \tilde{\delta}^* \tilde{y}_k = \tilde{\delta}^* \delta^* y_0 \tag{3.34}$$

$$1^T \tilde{\lambda}^* + \tilde{\lambda}_k^* = 1^T \left(\tilde{\lambda}^* + \tilde{\lambda}^* \tilde{\lambda}_k^*\right) = 1 \tag{3.35}$$

Thus $(\widetilde{\theta}^{*}\theta^{*},\widetilde{\delta}^{*}\delta^{*},\widetilde{\lambda}^{*}+\widetilde{\lambda}^{*}\lambda_{k}^{*})$ must be a feasible solution of (3.18), we also have

$$\frac{v^{T}[\widetilde{\theta}^{*}\theta^{*}]}{u^{T}[\widetilde{\delta}^{*}\delta^{*}]}\geq\frac{v^{T}\theta^{*}}{u^{T}\delta^{*}} \tag{3.36}$$

Since $0\leq\widetilde{\theta}^{*}\leq 1$ and $\widetilde{\delta}^{*}\geq 1$, then the only condition for (3.36) to be satisfied is $\widetilde{\theta}^{*}=1$ and $\widetilde{\delta}^{*}=1$, that is $\widetilde{\Phi}^{*}=1$. Therefore, DMU_k $(\widetilde{x}_k,\widetilde{y}_k)$ is weighted global efficient relative to all the DMUs under evaluation. This completes the proof. □

From the above definition and theorem, the weighted global measure can be expressed at point $DMU_0(x_0, y_0)$ as a particular shortage function in the direction of $(\theta^{*}x_0,\delta^{*}y_0)$. A similar viewpoint is developed by Chambers et al. (1995), Briec (1997) and Ruggiero and Bretschneider (1998). They introduce a function they term "input directional distance," which is similarly related to the input distance function defined by Malmquist (1953). Now we focus on the particular relationship between the weighted global projection and the factor weights. Let us analyze the following several categories of cases:

Assume that $u^{T}\delta=1$, the weighted global efficiency is identical to the Farrell weighted measure of efficiency defined by Ruggiero and Bretschneider (1998). Moreover, if assume that input weighting factors satisfy $v_1=v_2=1/2$, the weighted global measure coincides with the nonradial measurement of technical efficiency defined by Färe and Lovell (1978). Furthermore, if we set $\theta_1=\theta_2$, then we can obtain the radial efficiency measure. If $v_1<v_2$ the second input is reduced more than the first one, and the weighted global projection is more oriented to the direction of the X_1-axis. If $v_1>v_2$, the first input is reduced more than the second one, and the weighted global projection is more oriented to the direction of the X_2-axis. Figure 3.5 illustrates the above differences. Assume that $v^{T}\theta=1$, the weighted global efficiency coincides with the weighted Russell efficiency. If $u_1=u_2$, all outputs are equiproportionately increased. If $u_1<u_2$, the second output is

increased more than the first one, and the weighted global projection is more oriented to the direction of the Y_2-axis. If $u_1 > u_2$, then the first output is increased more than the second one, and the weighted global projection is more oriented to the direction of the Y_1-axis.

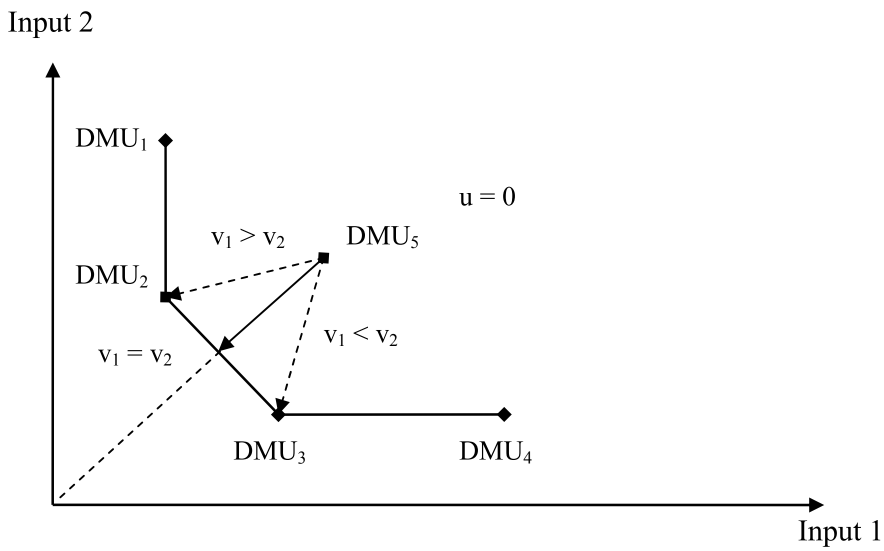

Figure 3.5 Effect of factor weights on the Input-oriented projection

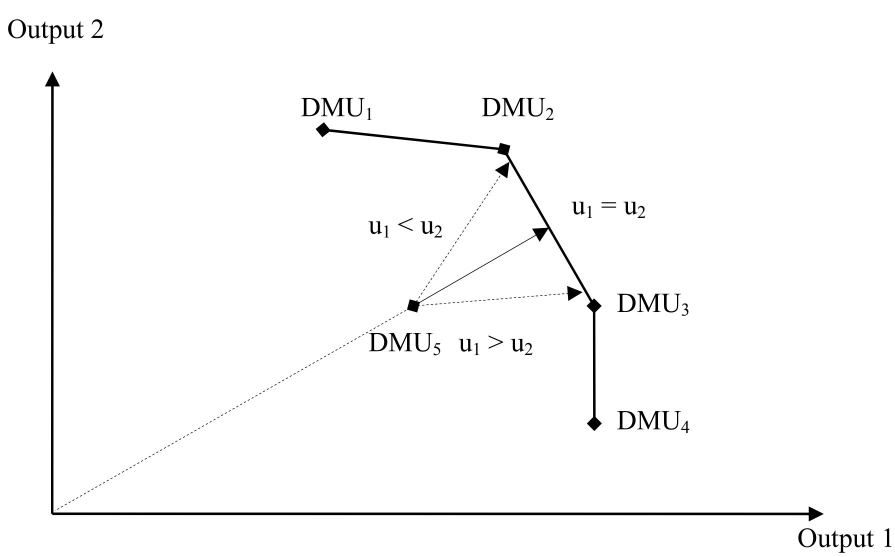

Figure 3.6 Effect of factor weights on the output-oriented projection

3.5 Comparison with Other Efficiency Measures

In this section, we first compare our model with a Farrell-based efficiency measure proposed by Banker et al. (1984) which is regarded as one of the basic DEA models and point out remarkable differences among them.

***THEOREM* 3.6.** *The weighted global efficiency score* Φ^* *of DMU₀ in (5) is less than the weighted Russell efficiency score* η^* *in (19).*

PROOF. Suppose that the optimal solution of (19) is $(\eta^*, \lambda^*, s^{-*}, s^{+*})$, let

$$\theta^* = \eta^* - s^{-*}/x_0 \tag{3.36}$$

$$\delta^* = 1 + s^{+*}/y_0 \tag{3.37}$$

Obviously, $(\eta^*, \lambda^*, s^{-*}, s^{+*})$ makes $(\theta^*, \delta^*, \lambda^*)$ be the optimal solution of (3.18), then

$$\Phi^* = \frac{v^T\theta^*}{u^T\delta^*} = \frac{v^T[\eta^* - s^{-*}/x_0]}{u^T[1 + s^{+*}/y_0]} = \frac{v^T\eta^* - v^T s^{-*}/x_0}{1 + u^T s^{+*}/y_0} \le v^T\eta^* \tag{3.38}$$

Since $s^{-*} \ge 0$ and $s^{+*} \ge 0$, then we have $\Phi^* \le \eta^*$. This completes the proof. □

***THEOREM* 3.7.** *A DMU₀* (x_0, y_0) *is weighted global efficient, if and only if it is weighted Russell efficient.*

PROOF. Suppose that DMU₀ (x_0, y_0) is weighted Russell inefficient. Then, we have either $\eta^* \le 1$ or ($\eta^* = 1$ and $(s^{-*}, s^{+*}) \ne (0,0)$). From (23), in both cases, we have $\Phi^* \le 1$ for a feasible solution of (5). Hence, DMU₀ (x_0, y_0) is weighted global inefficient.

On the other hand, suppose that DMU₀ (x_0, y_0) is weighted global inefficient. Then, it holds $(\theta^*, \delta^*) \ne (1,1)$. By the statement (16) and (18), (η^*, $s^{-*} = \eta^* x_0 - \theta^* x_0$, $s^{+*} = \delta^* y_0 - y_0$) is a feasible solution for (4), provided $s^{-*} = \eta^* x_0 - \theta^* x_0 = 0$, that is $\theta^* = \eta^*$. There are two cases.

Case 1 ($\theta^* = \eta^* = 1$) then $\delta^* > 1$, $s^{+*} = \delta^* y_0 - y_0 \geq 0$. In this case, an optimal solution for (4) is weighted Russell inefficient.

Case 2 ($\theta^* = \eta^* \neq 1$), then an optimal solution for (4) is also weighted Russell inefficient. On the other hand, provided $s^{-*} = \eta^* x_0 - \theta^* x_0 \geq 0$, that is $\theta^* \leq \eta^*$. In this case, an optimal solution for (4) is also weighted Russell inefficient.

Therefore, weighted Russell inefficiency is equivalent to weighted global inefficiency. Since the definition of efficiency and inefficiency are mutually exclusive, we have proved the theorem. This completes the proof. □

3.6 An Illustrative Example

To facilitate comparison, five measures were used to measure the efficiency of the DMUs using the observed data reported in Table 3.1. In calculating the weighted global measure, the true factor weights and the average factor weights were used in the linear programs respectively. All the results by applying model (3.18) are shown in the Table 3.2.

Table 3. 1 Example data

DMU	1	2	3	4	5	6	7	8	9	10
Input 1	5	2	1	2	1	3	4	2	4	5
Input 2	3	5	3	4	1	2	3	11	5	3
Output 1	5	3.5	4	5	1	5	6	7	9	3
Output 2	4	4	3.5	4.5	2	1.5	6.5	8	4	6.5

Using standard efficiency models, we can show that four DMUs (DMU 1, 2, 4 and 10) are weighted global inefficient. Results for the weighted global measures of efficiency which consider the average weightings of inputs and outputs are displayed in Panel 1 of Table 3.2. For each efficient DMU the efficiency score and the position in the ranking based on these scores are displayed. In addition, Panel 1 also shows the values of projections and reference units. The projection of DMU_0 onto the efficient frontier can be expressed as a linear combination of other DMUs under evaluation. In the other panel the weights factors are considered. It turns out that this measure yields not only different super-efficiency scores and thus different rankings but also different values of projection and different reference units. This is not surprising because the factors are equally important in the production process in Panel 1 while biased relative weights used in Panel 2. Finally, we compared our measures with other measures in Table 3.3.

Other application of such global measure can be found in performance measurement of Chinese investment funds. For more details, see Chen and Poh (2003).

Table 3.2 Results for (3.18) with averagely weighted and weighted reference units

DMU	Scores	Ranks	Projection Points				Reference Units
			Input 1	Input 2	Output 1	Output 2	
Panel 1: Average weighted							
1	0.6372	8	4	3	4	6	7
2	0.5957	10	1.33	3	4.22	3.83	3(.89), 7(.11)
4	0.9439	7	2	3.5	5	4.5	3, 7, 8, 9
10	0.6000	9	4	3	4	6	7
Panel 2: Weighted ($(Y_1^{0.530} Y_2^{0.470} = X_1^{0.322} X_2^{0.678})$							
1	0.6683	8	4	3	4	6	7
2	0.5757	10	1.51	3	4.34	4	3(.83), 7(.17)
4	0.9239	7	2	3.5	5	4.5	3, 7, 8, 9
10	0.6115	9	4	3	4	6	7

Table 3.3 Results from different efficiency measures

Type	1	2	3	4	5	6	7	8	9	10
BCC	0.789	0.672	1	0.948	1	1	1	1	1	1
RTE	0.717	0.667	1	0.944	1	1	1	1	1	0.900
STE	0.637	0.615	1	0.944	1	1	1	1	1	0.600
Ours	0.637	0.596	1	0.944	1	1	1	1	1	0.600

3.7 Conclusion

In this study, we propose a weighted efficiency measure which focuses on inputs minimization and output maximization simultaneously. Firstly, we define our measure of efficiency and then investigate its properties and demonstrate its characteristics theoretically. In addition, we provide one effective method to capture the internal value information in the production systems which is usually omitted in the traditional efficiency measures. Furthermore, we show how the effect of weights factors on the efficiency and efficient frontier in our model. Finally, we compare our measure with other measures theoretically as well as empirically and find that there are some differences between our measure and others.

Chapter 4

Value Super-Efficiency for Ranking Efficient Units

4.1 Introduction

Since the original publication, DEA has become a popular method for analyzing the efficiency of various organizational units, e. g. see Charnes et al. (1994). In recent years, a substantial amount of scholarly efforts has been devoted to the development of so-called super-efficiency measures for differentiating some of the efficient DMUs that have identical efficiency scores equal to one in the basic models. The ability to rank or differentiate the efficient DMUs is of both theoretically and practically importance. Theoretically, the inability to differentiate the efficient units creates a considerable number of observations typically characterized as efficient, unless the sum of the number of inputs and outputs is small relative to the number of observations. Specialized units may be rated as efficient due to a single input or output, even though that input or output may be seen as relatively important. Thus this poses analytical difficulties to any post-DEA statistical inference analysis. In practice, further differentiation among efficient DMUs is also desirable and even necessary in many cases. One classical example of the application of the super-efficiency DEA model is the work by Lovell et al. (1994).

In the Farrell tradition, ranking efficient units on the frontier was first researched by Andersen and Petersen (1993). Since then, other scholarly efforts attributed to this topic include the works by Doyle and Green (1993, 1994), Stewart (1994), Wilson (1995), Charnes et al. (1996), Tofallis (1996), Zhu (1996), Seiford and Zhu (1998, 1999), Tone

(2002), Xue and Harker (2002) among others. However, one concern about these super-efficiency measures is that they may not always be possible to determine their value when the super-efficiency models are applied under other alternate returns to scale (RTS) conditions other than constant returns to scales (CRS). In other words, the mathematical program defining the super-efficiency measures may not have a feasible solution. This has been a concern in the literature since the introduction of the Farrell-based super-efficiency measure and was first noticed in Thrall (1996). Another concern is that these super-efficiency measures cannot capture certain inherent relationships among the inputs and the outputs.

In this chapter we propose a weighted global measure of super-efficiency based on the weighted global measure of efficiency proposed in the Chapter 3. This super-efficiency measure differs from traditional radial measures of super-efficiency due to several aspects. Firstly, this measure considers both inputs minimization and output maximization simultaneously while traditional measures are usually determined either by maximizing outputs subject to given input levels or minimizing inputs subject to given output levels. Secondly, unlike the slacked-based measure of super-efficiency proposed by Tone (2002), our super-efficiency measure can deal with input and output slacks directly, as well as account for all sources of inefficiency, including radial and nonradial inefficiency in inputs and outputs. Finally, this measure presents a weighted global measure which recognizes the possibility of differential factor weights in production process and can be applied to situation where the relative worth of a subset (or subsets) of inputs and/or outputs is known or predetermined beforehand.

This chapter is organized as follows: Section 4.2 proposes a weighted global measure of super-efficiency and investigates its desirable characteristics as well as its computational feasibility. In Section 4.3, we explore the super efficient frontier and discuss the differences between our super projection and traditional radial projections onto the super efficient frontier. Furthermore, we compare our model with other super-efficiency models and demonstrate the rationalities of our model in Section 4.4. Finally, an illustrative example is illustrated in Section 4.5. Section 4.6 contains some concluding remarks.

4.2 A New Value Super-efficiency Measure

Following the above research, we suppose that DMU_0 (x_0, y_0) is an efficient unit under evaluation, $x_0 \in \Re_+^m$ is the vector of m inputs consumed and $y_0 \in \Re_+^s$ is the vector of s outputs produced by this unit. Similarly, let $X \in \Re_+^{m\times(n-1)}$ and $Y \in \Re_+^{m\times(n-1)}$ be the input and output matrices respectively, consisting of nonnegative elements and excluding the observed input vector and output vector of DMU_0. The production possibility set $P/(x_0, y_0)$ can be redefined as

$$P/(x_0, y_0) = \{(x_0, y_0) | x_0 \le \lambda X, y_0 \ge \lambda Y, B\lambda \le b, \lambda \ge 0\} \tag{4.1}$$

Following the above section, the *weighted global super-efficiency score* Φ for DMU_0 can be defined as follows:

$$\Phi = \frac{v^T \theta}{u^T \delta} \tag{4.2}$$

where Φ is the optimal solution of the following problem:

$$Minimize \ \Phi = \frac{v^T \theta}{u^T \delta} \tag{4.3.a}$$

subject to

$$X\lambda = \theta x_0 \tag{4.3.b}$$

$$Y\lambda = \delta y_0 \tag{4.3.c}$$

$$B\lambda \le b, \quad \lambda \ge 0 \tag{4.3.d}$$

where $\delta = (\delta_1, \cdots \delta_s)^T$ and $\theta = (\theta_1, \cdots, \theta_m)^T$ represent the scalar vectors for the outputs and the inputs respectively. Suppose that $(\Phi^*, \delta^*, \theta^*, \lambda^*)$ is the optimal solution of (4.3), We have the following propositions:

PROPOSITION 4.1 *The weighted global super-efficiency score Φ^* is units invariant, i. e. it is independent of the units in which the inputs and outputs are measured provided these units are the same for every DMU.*

PROOF. This propostion holds, since both the objective function and constraints are units invariant. This completes the proof. □

PROPOSITION 4.2 *Let $(\varphi x_0, \psi y_0)$ with $0 \le \varphi \le 1$ and $\psi \ge 1$ be a DMU with the reduced inputs and enlarged outputs than (x_0, y_0). Then, the weighted global super-efficiency score of $(\varphi x_0, \psi y_0)$ is not less than that of (x_0, y_0).*

PROOF. Supposing that $(\Phi^*, \theta^*, \delta^*, \lambda^*)$ is the optimal solution of (4.3) when DMU$_0$ (x_0, y_0) is under evaluation. If $0 \le \varphi \le 1$ and $\psi \ge 1$, then $(\theta^* / \varphi, \delta^* / \psi, \lambda^*)$ is a feasible solution of (4.3) when $(\varphi x_0, \psi y_0)$ is being evaluated, because the constraints for the inputs and outputs are clearly satisfied and $\theta^* \le \theta^* / \varphi$ as well as $\delta^* / \psi \le \delta^* \le 1$. Therefore, we have

$$\Phi'^* = \frac{u^T \theta^* / \varphi}{v^T \delta^* / \psi} \ge \frac{u^T \theta^*}{v^T \delta^*} = \Phi^* \tag{4.4}$$

Thus, the *weighted global super-efficiency score* Φ'^* of $(\varphi x_0, \psi y_0)$ is not less than Φ^* of (x_0, y_0). This completes the proof. □

In order to preserve the linearity and convexity of DEA through our model, the fractional program (4.3) can be transformed into a linear programming problem using the Charnes-Cooper transformation as follows:

$$t > 0\,,\ \theta = \frac{\theta'}{t}\,,\ \delta = \frac{\delta'}{t}\,,\ \lambda = \frac{\lambda'}{t}\,,\ \Phi = \Phi' \tag{4.5}$$

Then, (4.3) becomes the following linear program in t, θ', δ' and λ':

$$Minimize\ \Phi' = v^T\theta' \tag{4.6.a}$$

subject to

$$\mathrm{X}\lambda' = \theta' x_0 \tag{4.6.b}$$

$$\mathrm{Y}\lambda' = \delta' y_0 \tag{4.6.c}$$

$$u^T\delta' = 1 \tag{4.6.d}$$

$$B\lambda' \le bt\,,\quad \lambda' \ge 0 \tag{4.6.f}$$

Let an optimal solution of (4.6) be $(\Phi'^*, \theta'^*, \delta'^*, \lambda'^*, t^*)$. By using (4.5), we can obtain an optimal solution of (4.3) as expressed by $(\Phi^*, \theta^*, \delta^*, \lambda^*)$.

4.3 Exploring the Super-efficient Frontier

Traditionally, when we apply DEA models and DMU under evaluation results as inefficient, we obtain single efficient reference point which Pareto dominates the DMU under evaluation. This point can also be interpreted as target point on the frontier such that inefficient DMU should adopt its output mix to became efficient. Here we also apply such concept to characterize the super-efficient frontier. In Anderson and Peterson model, the radially projected current position is generally not the same as the most preferred future position. Accordingly, when setting target point, the radial projection of an efficient DMU to the super-efficient frontier is too restrictive a technique because the current values of outputs (or inputs) are projected onto the super-efficient frontier by decreasing (or increasing) them in the same proportion. In our models, the super-efficiency of the efficient DMU is determined by adjusting each input to its given level unproportionally at the same time adjusting each output to its given level unproportionally. Variables are aggravated until the boundary of the super-efficient frontier is achieved. Below, in Figure 4.1, we have illustrated such difference.

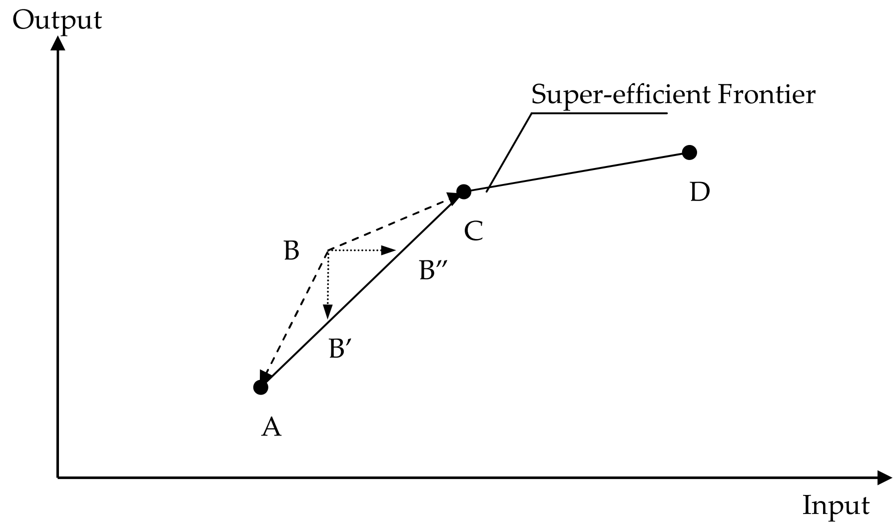

Figure 4. 1 Super-efficient Frontier and Projection

Suppose that four efficient DMUs, namely DMU_A, DMU_B and DMU_C and DMU_D, are composed of the original efficient frontier. Now the DMU_B is under evaluation and thus excluded from the efficient frontier, then we may define the resulting efficient frontier ACD as the super-efficient frontier. The traditional projection, either B' or B", is inefficient relative to all the DMUs under evaluation because it is possible that a composition of other DMUs shows more efficient than the projection whereas our super projection is targeted at the efficient points on the original efficient frontier and thus still efficient relative to the rest DMUs under evaluation.

We may define the super projection of the efficient DMU onto the super-efficient frontier of (4.3) as follows:

DEFINITION 4.1 *Supposing that the optimal solution of (4.3) is* $\left(\Phi^*,\theta^*,\delta^*,\lambda^*\right)$*, the* DMU_k $\left(\widetilde{x}_k,\widetilde{y}_k\right)$ *can be expressed as follows:*

$$\widetilde{x}_k = \theta^* x_0 = \mathrm{X}\lambda^* \tag{4.7}$$

$$\widetilde{y}_k = \delta^* y_0 = \mathrm{Y}\,\lambda^* \tag{4.8}$$

then DMU$_k$ $(\widetilde{x}_k, \widetilde{y}_k)$ is defined as the super projection of DMU$_0$ onto the super-efficient frontier.

PROPOSITION 4.3 *Compared with all the other DMUs (excluding DMU$_0$) under evaluation, the super projection DMU$_k$ $(\widetilde{x}_k, \widetilde{y}_k)$ is weighted global efficient.*

PROOF. Supposing that the DMU$_k$ $(\widetilde{x}_k, \widetilde{y}_k)$ is the super projection of DMU$_0$ onto the super efficient frontier, we use the following (4.9) to evaluate the weighted efficiency of DMU$_k$ relative to all the other DMUs under evaluation:

$$Minimize\ \ \widetilde{\Phi} = \frac{u^T \widetilde{\theta}}{v^T \widetilde{\delta}} \tag{4.9.a}$$

subject to

$$\mathrm{X}\,\widetilde{\lambda} + \widetilde{x}_k \widetilde{\lambda}_k = \widetilde{\theta}\widetilde{x}_k\,, \quad 0 \le \widetilde{\theta} \le 1 \tag{4.9.b}$$

$$\mathrm{Y}\,\widetilde{\lambda} + \widetilde{y}_k \widetilde{\lambda}_k = \widetilde{\delta}\widetilde{y}_k\,, \quad \widetilde{\delta} \ge 1 \tag{4.9.c}$$

$$B\widetilde{\lambda} + \widetilde{\lambda}_k \le b\,, \quad \widetilde{\lambda} \ge 0\,, \quad \widetilde{\lambda}_k \ge 0 \tag{4.9.d}$$

We suppose that the optimal solution of (4.9) is $\left(\widetilde{\theta}^*, \widetilde{\delta}^*, \widetilde{\lambda}^*, \widetilde{\lambda}_k^*\right)$, from the constraints of (4.9), we have

$$\mathrm{X}\,\widetilde{\lambda}^* + \widetilde{x}_k \widetilde{\lambda}_k^* = \mathrm{X}(\widetilde{\lambda}^* + \widetilde{\lambda}^* \widetilde{\lambda}_k^*) = \widetilde{\theta}^* \widetilde{x}_k = \widetilde{\theta}^* \theta^* x_o \tag{4.10}$$

$$\mathrm{Y}\,\widetilde{\lambda}^* + \widetilde{y}_k \widetilde{\lambda}_k^* = \mathrm{Y}(\widetilde{\lambda}^* + \widetilde{\lambda}^* \widetilde{\lambda}_k^*) = \widetilde{\delta}^* \widetilde{y}_k = \widetilde{\delta}^* \delta^* y_0 \tag{4.11}$$

$$B\widetilde{\lambda}^* + \widetilde{\lambda}_k^* = B\left(\widetilde{\lambda}^* + \widetilde{\lambda}^* \widetilde{\lambda}_k^*\right) \le b \tag{4.12}$$

Thus $(\widetilde{\theta}^* \theta^*, \widetilde{\delta}^* \delta^*, \widetilde{\lambda}^* + \widetilde{\lambda}^* \lambda_k^*)$ must be a feasible solution of (4.3), we also have

$$\frac{u^T \widetilde{\theta}^* \theta^*}{v^T \widetilde{\delta}^* \delta^*} \ge \frac{u^T \theta^*}{v^T \delta^*} \tag{4.13}$$

Since $0 \le \widetilde{\theta}^* \le 1$ and $\widetilde{\delta}^* \ge 1$, then the only condition for (4.13) to be satisfied is

$\tilde{\theta}^* = 1$ and $\tilde{\delta}^* = 1$, that is $\tilde{\Phi}^* = 1$. Therefore, DMU$_k$ $(\tilde{x}_k, \tilde{y}_k)$ is weighted global efficient. This completes the proof. □

4.4 Comparison with Other Super-efficiency Measures

In this section, we compare our model with a Farrell-based super-efficiency measure proposed by Anderson and Peterson (1993) as well as a slack-based super-efficiency measure proposed by Tone (2002), and point out remarkable differences among them. For more details about the comparison between the Farrell-based super-efficiency and the Slack-based super-efficiency, see Tone (2002).

4.4.1 Anderson and Petersen Farrell-based model

This model can be described, in the input-oriented general model, as follows:

$$Minimize\ \eta \tag{4.14.a}$$

subject to

$$\mathrm{X}\lambda + s^- = \eta x_0 \tag{4.14.b}$$

$$\mathrm{Y}\lambda - s^+ = y_0 \tag{4.14.c}$$

$$B\lambda \le b, \quad \lambda \ge 0 \tag{4.14.d}$$

$$s^- \ge 0, \ s^+ \ge 0 \tag{4.14.e}$$

where s^- and s^+ represent input and output slack vectors, respectively. Let an optimal solution of (4.14) be $(\eta^*, \lambda^*, s^{-*}, s^{+*})$. For an efficient DMU$_0$ (x_0, y_0), Farrell-based efficiency score η^* is not less than unity, and this value indicates Farrell-based "super-efficiency". However, the subproblems for some "extreme points" (Thrall 1996) may become infeasible when the super-efficiency models are applied under variable returns to scale (VRS). Previous researches on this topic in the DEA literature have basically concluded that in case of infeasibility in the super-efficiency DEA models, the ranking of the whole set is impossible and, consequently, it is suggested that the use of the super-efficiency DEA models should be restricted under alternate RTS assumption. Regarding this measure we also have the following proposition:

***PROPOSITION* 4.4** *The Anderson and Peterson model returns the same super-efficiency score* η^* *for any DMU represented by* $(x_0 - \alpha s^{-*}/\eta^*)$ *for the range* $0 \le \alpha \le 1$.

This contradicts our common understanding that a reduction of input values usually increases super-efficiency. This irrationality is caused by the fact that this model deals only with the radial measure and neglects the existence of input slacks as represented by s^{-*}.Furthermore, if we set $v^T\delta = 1$, $\theta = \eta$, then (4.3) can be equivalent to (4.14). Thus, we also have the following relationships between (4.3) and (4.14).

***LEMMA* 4.5** *Let us define*

$$\tau^* = minimize\left\{\frac{(\eta^* - 1)x_0}{s^{-*}} \middle| s^{-*} \ge 0\right\} = 0, \quad if \ \ s^{-*} = 0 \tag{4.15}$$

Then, $\left(\theta^* = \eta^* - \tau^* s^{-*}/x_0, \delta^* = 1 + s^{+*}/y_0, \lambda^* = \lambda^*\right)$ *is a feasible solution for (4. 3).*

PROOF. From (4.15), we have $\theta \le 1$ and $\delta \ge 1$, hence the solution $\left(\theta^*, \delta^*, \lambda^*\right)$ satisfies the constraints of the weighted super-efficiency model. □

***THEOREM* 4.6** *The weighted global super-efficiency score* Φ^* *of* DMU_0 *in (4.3) is less than the Farrell-based super-efficiency score* η^* *in (4.14).*

PROOF. Suppose that the optimal solution of (4.14) is $\left(\eta^*, \lambda^*, s^{-*}, s^{+*}\right)$, let

$$\theta^* = \eta^* - s^{-*}/x_0 \tag{4.16}$$

$$\delta^* = 1 + s^{+*}/y_0 \tag{4.17}$$

Obviously, $\left(\eta^*, \lambda^*, s^{-*}, s^{+*}\right)$ makes $\left(\theta^*, \delta^*, \lambda^*\right)$ be the optimal solution of (4.3), then

$$\Phi^* = \frac{u^T\theta^*}{v^T\delta^*} = \frac{u^T[\eta^* - s^{-*}/x_0]}{v^T[1+s^{+*}/y_0]} = \frac{\eta^* - u^T s^{-*}/x_0}{1+v^T s^{+*}/y_0} \le \eta^* \quad (4.18)$$

Since $s^{-*} \ge 0$ and $s^{+*} \ge 0$, then we have $\Phi^* \le \eta^*$. This completes the proof. □

4.4.2 Tone's Slack-based Super-efficiency Model

In Tone's slack-based super-efficiency model, we assume that the DMU$_0$ (x_0, y_0) is SBM-efficient. Let $(\bar{x}_0, \bar{y}_0)$ be the projection of DMU$_0$ (x_0, y_0) in the production possibility set, the super-efficiency of DMU$_0$ (x_0, y_0) can be defined as the objective function value κ^* of the following program:

$$Minimize\ \ \kappa = \frac{s}{m} \times \frac{1^T(\bar{x}_0/x_0)}{1^T(\bar{y}_0/y_0)} \quad (4.19.a)$$

subject to

$$\mathrm{X}\lambda \le \bar{x}_0 \quad (4.19.b)$$

$$\mathrm{Y}\lambda \ge y_0 \quad (4.19.c)$$

$$\bar{x}_0 \ge x_0,\ \ 0 < \bar{y}_0 \le y_0 \quad (4.19.d)$$

$$B\lambda \le b,\quad \lambda \ge 0 \quad (4.19.e)$$

If we set $\bar{x}_0 = \alpha x_0$, $\alpha = (\alpha_1, \cdots, \alpha_m)^T$ and $\bar{y}_0 = \beta y_0$, $\beta = (\beta_1, \cdots, \beta_s)^T$, then Tone's slack-based super-efficiency model can be converted to be

$$Minimize\ \ \kappa = \frac{s}{m} \times \frac{1^T\alpha}{1^T\beta} \quad (4.20.a)$$

subject to

$$\mathrm{X}\lambda + s^- = \alpha x_0,\quad \alpha \ge 1 \quad (4.20.b)$$

$$\mathrm{Y}\lambda - s^+ = \beta y_0,\quad 0 \le \beta \le 1 \quad (4.20.c)$$

$$B\lambda \le b,\quad \lambda \ge 0 \quad (4.20.d)$$

$$s^{-} \geq 0,\ \ s^{+} \geq 0 \tag{4.20.e}$$

Let an optimal solution of (4.20) be $\left(\kappa^{*}, \alpha^{*}, \beta^{*}, s^{-*}, s^{+*}\right)$. Comparing (4.3) and (4.20), we have

THEOREM 4.7 *The weighted global super-efficiency score* Φ^{*} *in (4.3) is less than the slack-based super-efficiency score* κ^{*} *in (4.20).*

PROOF. Suppose that the optimal solution of (4.20) is $\left(\kappa^{*}, \alpha^{*}, \beta^{*}, s^{-*}, s^{+*}\right)$, let

$$\theta^{*} = \alpha^{*} - s^{-*}/x_0 \tag{4.21}$$

$$\delta^{*} = \beta^{*} + s^{+*}/y_0 \tag{4.22}$$

Obviously, $\left(\kappa^{*}, \alpha^{*}, \beta^{*}, s^{-*}, s^{+*}\right)$ makes $\left(\Phi^{*}, \theta^{*}, \delta^{*}, \lambda^{*}\right)$ be the optimal solution of (4.3), then

$$\Phi^{*} = \frac{s}{m} \cdot \frac{1^{T}\theta^{*}}{1^{T}\delta^{*}} = \frac{s}{m} \cdot \frac{[1^{T}\alpha^{*} - 1^{T}s^{-*}/x_0]}{[1^{T}\beta^{*} + 1^{T}s^{+*}/y_0]} \leq \frac{s}{m} \cdot \frac{1^{T}\alpha^{*}}{1^{T}\beta^{*}} = \kappa^{*} \tag{4.23}$$

Since $s^{-*} \geq 0$ and $s^{+*} \geq 0$, then we have $\Phi^{*} \leq \kappa^{*}$. This completes the proof.. □

4.5 An Illustrative Example

We will now demonstrate how the efficient DMUs are ranked in different models: our Weighted Super-efficiency model, the Anderson and Peterson model and Tone Slack-based Super-efficiency model. Following variables return to scales, the constraint $B\lambda \leq b$ becomes $1^{T}\lambda = 1$. Let us consider the example in Table 4. 1 with ten DMUs, two outputs and two inputs. Using standard efficiency models, we can show that four DMUs (DMU 3, 4, 5 and 7) are efficient. Results for the weighted global measures of super-efficiency which consider the average weightings of inputs and outputs are displayed in Panel 1 of Table 4.2. For each efficient DMU the super-efficiency score and the position in the ranking based on these scores are displayed. In addition, Panel 1 also

Table 4. 1 Example data

DMU	1	2	3	4	5	6	7	8	9	10
Input 1	4	12	8	6	2	3	3	8	4	6
Input 2	6	8	2	6	8	9	7	12	10	5
Output 1	2	3	2	4	2	1	2	3	1	2
Output 2	1	1	2	2	3	2	4	1	3	2

shows the values of projections and reference units. The projection of DMU_0 onto the efficient frontier can be expressed as a linear combination of other DMUs under evaluation. The other Panel contains the super-efficiency scores and rank numbers obtained when the weighting factors are considered by assuming that the production function is represented as $Y^{0.3}Y^{0.7} = X^{0.4}X^{0.6}$. It turns out that this measure yields not only different super-efficiency scores and thus different rankings but also different values of projection and different reference units. This is not surprising because the factors are equally important in the production process in Panel 1 while biased relative weights used in Panel 2. For each efficient DMU the super-efficiency score and the position in the ranking based on these scores are displayed. In addition, Panel 1 also shows the values of projections and reference units. The projection of DMU_0 onto the efficient frontier can be expressed as a linear combination of other DMUs under evaluation. The other Panel contains the super-efficiency scores and rank numbers obtained when the weighting

Table 4. 2 Result from the weighted global model of super-efficiency

DMU	Scores	Ranks	Projection Points				Reference Units
			Input 1	Input 2	Output 1	Output 2	
Panel 1: Average weighted							
3	1.250	1	6	6	4	2	4
4	0.667	4	3	7	2	4	7
5	1.108	2	3	7	2	4	7
7	1.034	3	2	8	2	3	5
Panel 2: Value-added							
3	1.324	1	3	7	2	4	7
4	0.581	4	3	7	2	4	7
5	0.912	3	3	7	2	4	7
7	1.154	2	2	8	2	3	5

expressed as a linear combination of other DMUs under evaluation. The other Panel contains the super-efficiency scores and rank numbers obtained when the weighting factors are considered by assuming that the production function is represented as $Y^{0.3}Y^{0.7} = X^{0.4}X^{0.6}$. It turns out that this measure yields not only different super-efficiency scores and thus different rankings but also different values of projection and different reference units. This is not surprising because the factors are equally important in the production process in Panel 1 while biased relative weights used in Panel 2.

We will next examine efficient DMUs and their projections in the input-oriented Anderson and Peterson model and Tone's Slack-based Super-efficiency model. The results including super-efficiency scores, ranks and projections from applying these two models are shown in Panel 1 and Panel 2 of Table 4.3, respectively. As expected, the super-efficiency scores both in Andersen and Peterson model and Tone model are greater than the weighted super-efficiency scores and thus lead to different rankings among the efficient DMUs. Moreover, when using Andersen and Peterson model, the mathematical programming defining the super-efficiency scores for DMU_4 and DMU_7 can not have a feasible solution. This represents the potential drawback of the traditional Farrell-based super-efficiency. At the same time, the super efficiencies are undervalued because there still exist relatively large input slacks against their projections which are composed of a positive combination of their reference units. This means that their super-efficiency scores are evaluated by referring to points far apart from the efficient portions of the production possibility set. On the other hand, although the slack-based super-efficiency score κ^* drops from the Farrell- based super-efficiency score η^*, it increases from the weighted super-efficiency score Φ^* due to still not completely incorporating all the input/output slacks.

Table 4.3 Results from Andersen and Peterson's model and Tone's model

DMU	Scores	Ranks	Projection Points (Slacks)				Reference Units
			Input 1	Input 2	Output 1	Output 2	
Panel 1: The Anderson and Peterson's Model							
3	2.500	1	6 (14)	5 (0)	2 (0)	2 (0)	10
4	Inf.	*					
5	1.500	2	3 (0)	7 (5)	2 (0)	4 (1)	7
7	Inf.	*					
Panel 2: The Tone's Slack-based Model							
3	1.750	1	8(2)	5	2	2	10
4	1.290	2	6	6	2. 2	2	1, 2, 3, 5
5	1.250	3	3	7	2	3 (1)	7
7	1.171	4	3	7	2	2.8 (1.2)	3, 5

4.6 Conclusion

The ability to rank or differentiate the efficient units is of both theoretically and practically importance. As illustrated by the application work by Lovell et al. (1994): "The primary benefit of this approach is the ability to make finer distinctions between efficient DMUs and to produce a logarithmic MDEA distribution of relative performance scores that are approximately normally distributed". Thus the super-efficiency DEA model has the extraordinary potential to overcome the analytical difficulties arising in the post-DEA analysis. However, one concern about these super-efficiency models is that they may not always be possible to determine their optimal value when the super-efficiency models are applied under other alternate returns to scale (RTS) conditions other than constant returns to scales (CRS). Another concern is that these super-efficiency measures cannot capture certain inherent relationships among the inputs and the outputs which can be known or predetermined beforehand.

In this study, we propose a weighted super-efficiency measure which focuses on inputs minimization and output maximization simultaneously. This super-efficiency measure is useful to differentiate efficient units and motivate appropriate behavior. Firstly, we define our measure of super-efficiency and then investigated its properties and demonstrate its characteristics theoretically. In addition, we shows how to calculate the measure in a linear program setting when it is actually applicable in the sense that the measure exists, i. e. the defining programs have a feasible solution. Finally, we compare our measure with other super-efficiency measures theoretically as well as empirically and find that there are some differences between our measure and others.

Chapter 5

Modeling Undesirable Factors in Value Efficiency

5.1 Introduction

DEA was originally developed by Charnes et al. (1978) and extended by Banker et al. (1984) as a method for evaluating the relative efficiency of Decision Making Units (DMUs) that essentially perform the same task using similar multiple inputs to produce similar multiple outputs. Since the original publication, DEA has become a popular method for analyzing the efficiency of various organizational units, e. g. see Charnes et al. (1994). In recent years, a substantial amount of scholarly efforts has been devoted to address those production systems in which both desirable (good) and undesirable (bad) output and input factors may be present. Consider a paper mill production where paper is produced with undesirable outputs of pollutants such as biochemical oxygen demand, suspended solids, particulates and sulfur oxides. If inefficiency exists in the production, the undesirable pollutants should be reduced to improve the inefficiency, i.e., the undesirable and desirable outputs should be treated differently when we evaluate the production performance of paper mills. However, in the standard DEA models, decreases in outputs are not allowed and only inputs are allowed to decrease. (Similarly, increases in inputs are not allowed and only outputs are allowed to increase.) If one treats the undesirable outputs as inputs, the resulting DEA model does not reflect the true production process. Similarly situations when some inputs need to be increased to improve the performance are also likely to occur. For example, in order to improve the performance of a waste treatment process, the amount of waste (undesirable input) to be treated should be increased rather than decreased as assumed in the standard DEA models.

As so far, there are at least five methods for dealing with undesirable outputs in the DEA framework. The first method is just simply to ignore the undesirable outputs. The second is to treat the undesirable ones as outputs and to adjust the distance measurement in order to restrict the expansion of the undesirable outputs (see the weak disposability model in Färe et al., 1989). The third is to treat the undesirable outputs as inputs. However, all these three methods do not reflect the true production process. The fourth is to treat the undesirable outputs in the non-linear DEA model developed by Färe et al. (1989) and used to model the paper production systems where the desirable outputs are increased and the undesirable outputs are decreased. The fifth is to apply a monotone decreasing transformation to the undesirable outputs and then to use the adapted variables as outputs, e. g. Seiford and Zhu (2002) applied a linear monotone decreasing transformation. Since the use of linear transformation preserves the convexity relations, it is a good choice for DEA models. However, DEA calculations in above methods are traditionally value-free. The underlying assumption is that no output or input is more important than another, although, in the real world there generally exist some undesirable outputs or inputs which are less important than other outputs or inputs in the production systems. In DEA models, a DMU which, for example, is a superior producer of a less important undesirable output is diagnosed as efficient even if it performs poorly with respect to all other outputs. Hence, in the original DEA models, the efficiency scores are not necessarily good performance indicators. Here, we use Figure 5.1 to clarify our point. The example consists of five DMUs, each producing two outputs (one desirable and another undesirable) and all consuming the same amount of one input. We can see that DMU_1, DMU_2 and DMU_3 are efficient while DMU_4 and DMU_5 are inefficient. Thus DMU_1, DMU_2 and DMU_3 all

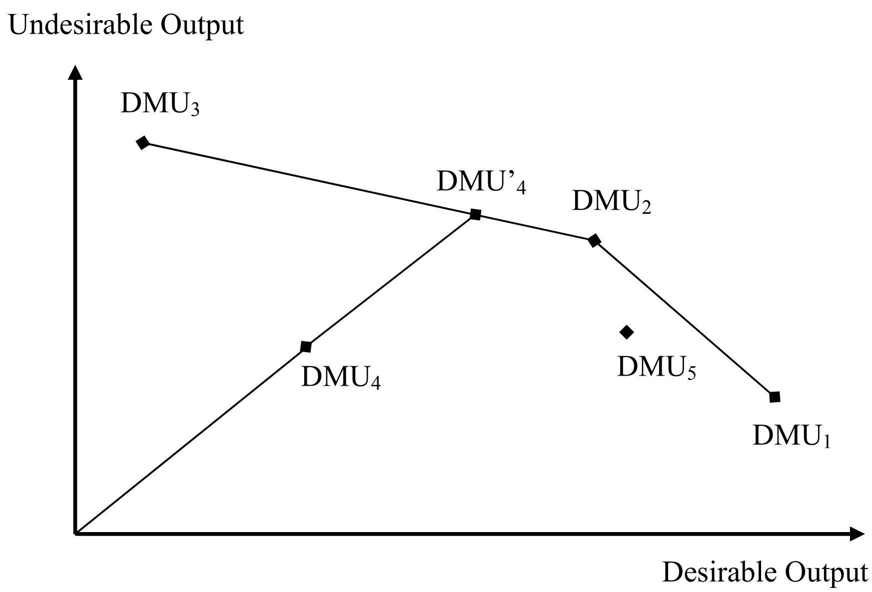

Figure 5.1 Classical DEA

receive an efficiency score of 1. Let us assume that for some reasons the Decision Maker (DM) considers the desirable output to be much more important than the undesirable output. In this case DMU_1 would be far more preferred to DMU_3. The DM might even prefer DMU_5 to DMU_3, even though the former is inefficient.

In this chapter we will first briefly illustrate the last two methods for treating the undesirable outputs in DEA framework. Then based on the linear monotone decreasing transformation, we treat both desirable and undesirable outputs differently in the weighted global DEA framework in which both radial inefficiency and nonradial inefficiency are incorporated. Furthermore, factor weights for both inputs and outputs (desirable and undesirable) are also incorporated in such DEA framework.

5.2 Traditional DEA Models with Undesirable Outputs

Assume that we have n DMUs each consuming m inputs to produce s desirable outputs and r undesirable outputs. We also suppose that DMU_0 $\left(x_0, y_0^g, y_0^b\right)$ is the unit under evaluation, $x_0 \in \Re_+^m$ is the vector of m inputs consumed this unit while

$y_0^g \in \Re_+^s$ and $y_0^b \in \Re_+^r$ are the vectors of s desirable outputs and r undesirable outputs produced by this unit, respectively. Let $X \in \Re_+^{m\times n}$ and $Y \in \Re_+^{(s+r)\times n}$ be the input and output matrices respectively, consisting of nonnegative elements and containing the observed input and output measures for all DMUs. The DEA data domain is expressed as

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y^g \\ Y^b \\ -X \end{bmatrix} \tag{5.1}$$

where Y^g and Y^b represent the desirable (good) and undesirable (bad) output matrices, respectively. Obviously, we wish to increase the desirable outputs Y^g and to decrease the undesirable outputs Y^b to improve the performance. However, in the standard BCC model, both Y^g and Y^b are supposed to increase to improve the performance. In order to improve the desirable output and to decrease the undesirable outputs, Färe et al. (1989) modify the BCC model into the following nonlinear programming problem:

$$Maxmize\ \ \alpha + \varepsilon\left(1^T s^- + 1^T s^{+g} + 1^T s^{+b}\right) \tag{5.2.a}$$

subject to

$$X\lambda + s^- = x_0 \tag{5.2.b}$$

$$Y^g\lambda - s^{+g} = \alpha y_0^g \tag{5.2.c}$$

$$Y^b\lambda - s^{+b} = \frac{1}{\alpha} y_0^b \tag{5.2.d}$$

$$1^T\lambda = 1, \quad \lambda \geq 0 \tag{5.2.e}$$

$$s^- \geq 0, \quad s^{+g} \geq 0, \quad s^{+b} \geq 0, \ \varepsilon \geq 0 \tag{5.2.f}$$

Similarly, based upon classification invariance, Seiford and Zhu (2002) show that an alternative to model (5.2) can be developed to preserve the linearity and convexity in DEA. They multiply each undesirable output by "−1" and then find a proper translation vector w to let all negative undesirable outputs be positive. The data domain of (5.1) now becomes

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y^g \\ \overline{Y}^b \\ -X \end{bmatrix} \tag{5.3}$$

where $\overline{Y}^b = -Y^b + w > 0$. Based on (5.3), undesirable factors can be formulated in the following programming problem:

$$Maxmize\ \ \beta + \varepsilon\left(1^T s^- + 1^T s^{+g} + 1^T s^{+b}\right) \tag{5.4.a}$$

subject to

$$X\lambda + s^- = x_0 \tag{5.4.b}$$

$$Y^g \lambda - s^{+g} = \beta y_0^g \tag{5.4.c}$$

$$\overline{Y}^b \lambda - s^{+b} = \beta \overline{y}_0^b \tag{5.4.d}$$

$$1^T \lambda = 1, \quad \lambda \geq 0 \tag{5.4.e}$$

$$s^- \geq 0, \quad s^{+g} \geq 0, \quad s^{+b} \geq 0, \ \varepsilon \geq 0 \tag{5.4.f}$$

Note that (5.4) expands desirable outputs and contracts undesirable outputs as in the non-linear DEA model (5.4). Therefore, under the context of the BCC model, this research provides an alternative method in dealing with desirable and undesirable factors in DEA

5.3 Modeling Undesirable Factors in Weighted Global Framework

Following Seiford and Zhu, we propose a new model that its objective function is linear and also both desirable and undesirable outputs are treated differently and simultaneously. Besides we will prove that by this model efficient DMUs can be determined. Supposing that $V = (V_1, \cdots, V_m)^T \in \Re_+^m$ is the weight vector of m inputs while $U_g = (U_1, \cdots, U_S)^T \in \Re_+^s$ and $U_b = (U_1, \cdots, U_r)^T \in \Re_+^r$ are the weight vectors of s

desirable outputs and k undesirable outputs respectively. We also assume that the input and output vectors satisfy $1^T V = 1$ and $1^T U_g + 1^T U_b = 1$, respectively. The weighted global efficiency score of DMU$_0$ $\left(x_0, y_0^g, \bar{y}_0^b\right)$ can be formulated as the objective function value of the following programming problem:

$$Maximize\ \Phi = \frac{U_g^T \delta^g + U_b^T \delta^b}{V^T \theta} \tag{5.5.a}$$

subject to

$$X\lambda = \theta x_0, \quad 0 \le \theta \le 1 \tag{5.5.b}$$

$$Y^g \lambda = \delta^g y_0^g, \quad \delta^g \ge 1 \tag{5.5.c}$$

$$\bar{Y}^b \lambda = \delta^b \bar{y}_0^b, \quad \delta^b \ge 1 \tag{5.5.d}$$

$$1^T \lambda = 1, \quad \lambda \ge 0 \tag{5.5.e}$$

where $\delta^g = \left(\delta_1^g, \cdots, \delta_s^g\right)^T$ and $\delta^b = \left(\delta_1^b, \cdots, \delta_r^b\right)^T$ represent the scalar vectors for the desirable outputs and the undesirable outputs, respectively, whereas $\theta = \left(\theta_1, \cdots, \theta_m\right)^T$ represents the scalar vector for the inputs.

REMARK: One important qualification of using the weighted efficiency measure is the necessity of determining the factor weights prior to measurement. One method of estimation for this multi-output and multi-input production function is based on the multivariate technique of canonical correlation analysis. For more details see Vinod (1968), Ruggiero and Bretscheider (1998) and Chen and Poh (2003).

REMARK: In order to preserve the linearity and convexity of DEA through our model, (5) can be transformed into a linear program using the Charnes-Cooper transformation in a similar way to the CCR model (see Charnes et al. 1978). Let

$$t > 0,\ \theta = \frac{\theta'}{t},\ \delta^g = \frac{\delta'^g}{t},\ \delta^b = \frac{\delta'^b}{t}, \lambda = \frac{\lambda'}{t}, \Phi = \Phi' \tag{5.6}$$

Then, (5) becomes the following linear programming in t, θ', δ'^g, δ'^b and λ':

$$Maximize\ \Phi' = U_g^T \delta'^g + U_b^T \delta'^b \tag{5.7.a}$$

subject to

$$V^T \theta' = 1 \tag{5.7.b}$$

$$X\lambda' = \theta' x_0\,, \quad 0 \le \theta' \le t \tag{5.7.c}$$

$$Y^g \lambda' = \delta'^g y_0^g\,, \quad \delta'^g \ge t \tag{5.7.d}$$

$$\overline{Y}^b \lambda' = \delta'^b \overline{y}_0^b\,, \quad \delta'^b \ge t \tag{5.7.e}$$

$$1^T \lambda' = t\,, \quad \lambda' \ge 0, \quad t \ge 0 \tag{5.7.f}$$

Let the optimal solution of (5.7) be $\left(\Phi'^*, \theta'^*, \delta'^{g*}, \delta'^{b*}, \lambda'^*, t^*\right)$. By using (5.6), we can obtain the optimal solution of (5.5) as expressed by $\left(\Phi^*, \theta^*, \delta^{g*}, \delta^{b*}, \lambda^*\right)$. Based on this optimal solution, we determine a DMU as being *weighted global efficient* as follows:

DEFINITION* 5.1** *(Weighted global efficiency).** *If the optimal solution* $\left(\Phi^*, \theta^*, \delta^{g*}, \delta^{b*}, \lambda^*\right)$ *of (5) satisfies* $\Phi^* = 1$*, then DMU₀* $\left(x_0, y_0^g, \overline{y}_0^b\right)$ *is weighted global efficient. Otherwise, the DMU₀* $\left(x_0, y_0^g, \overline{y}_0^b\right)$ *is weighted global inefficient.*

Note that (5.5) not only expands desirable outputs and contracts undesirable outputs as in the non-linear DEA model (5.2) and BCC model (5.4), but also contracts inputs simultaneously. The following theorem ensures that the optimized undesirable output $y_0^b (= w - \delta^{b*} \overline{y}_0^b)$ can not be negative:

***THEOREM* 5.1** *Given a translation vector* w*, suppose the scalar vector* δ^{b*} *is the optimal value of (5.5), we have* $\delta^{b*} \overline{y}_0^b \le w$.

***PROOF*.** Note that all outputs now are non-negative. Let δ^{b*} be an optimal solution associated with β^*. Since $1^T \lambda^* = 1$, therefore $\delta^{b*} \overline{y}_0^b \le \overline{y}^{b*}$, where $\overline{y}^{b*}$ is composed from (translated) maximum values among all bad outputs. Note that $\overline{y}_0^b = -\overline{y}_0^{b*} + w$, where $\bar{y}^{b*}$ is composed from (original) minimum values among all bad outputs. Thus, $\delta^{b*}\bar{y}_0^b \le w$. This completes the proof. □

Since $0 \le \theta^* \le 1$, $\delta^{g*} \ge 1$ and $\delta^{b*} \ge 1$, so it is clear that the optimal value of objective function is non-negative and not less than unity. We always have

***THEOREM* 5.2** *DMU₀* $\left(x_0, y_0^g, \bar{y}_0^b\right)$ *is weighted global efficient if and only if the optimal value of objective function* Φ^* *is equal to unity.*

PROOF. Suppose that DMU_0 is weighted global efficient, if $\left(\theta^*, \delta^{g*}, \delta^{b*}, \lambda^*\right)$ is the optimal solution of (5.5) whose objective function value Φ^* is unity, from (5.5.a), we have

$$V^T\theta^* = U_g^T\delta^{g*} + U_b^T\delta^{b*} \tag{5.8}$$

Since $0 \le \theta^* \le 1$, $\delta^{g*} \ge 1$ and $\delta^{b*} \ge 1$, then the only condition to satisfy (5.8) is $\theta^* = \delta^{g*} = \delta^{b*} = 1$. Therefore, according to *DEFINITION 1*, DMU_0 $\left(x_0, y_0^g, \bar{y}_0^b\right)$ is weighted global efficient. This completes the proof. □

The above theorem not only shows that model (5.5) can discriminate the efficient DMUs from the inefficient ones but also show how to improve those inefficient DMUs relative to the combinations of the efficient DMUs.

***THEOREM* 5.3** *For an inefficient* DMU_0 *with* $\left(x_0, y_0^g, \bar{y}_0^b\right)$ *input and output combination, figurative* DMU_F *with* $(\theta^* x_0,\ \delta^{g*} y_0^g,\ \delta^{b*}\bar{y}_0^b)$ *input and output (including desirable and undesirable) combination is weighted global efficient.*

PROOF. Supposing that the DMU_F $\left(\theta^* x_0,\ \delta^{g*} y_0^g,\ \delta^{b*}\bar{y}_0^b\right)$ is the figurative DMU of DMU_0 $\left(x_0, y_0^g, \bar{y}_0^b\right)$, we use the following (5.9) to evaluate the efficiency of DMU_F relative to all the DMUs under evaluation:

$$Maximize\ \widetilde{\Phi} = \frac{U_g^T \widetilde{\delta}^g + U_b^T \widetilde{\delta}^b}{V^T \widetilde{\theta}} \tag{5.9.a}$$

subject to

$$X\widetilde{\lambda} + x_F \widetilde{\lambda}_F = \widetilde{\theta} x_F, \quad 0 \le \widetilde{\theta} \le 1 \tag{5.9.b}$$

$$Y^g \widetilde{\lambda} + y_F^g \widetilde{\lambda}_F = \widetilde{\delta}^g y_F^g, \quad \widetilde{\delta}^g \ge 1 \tag{5.9.c}$$

$$\overline{Y}^b \widetilde{\lambda} + \overline{y}_F^b \widetilde{\lambda}_F = \widetilde{\delta}^b \overline{y}_F^b, \quad \widetilde{\delta}^b \ge 1 \tag{5.9.d}$$

$$1^T \widetilde{\lambda} + \widetilde{\lambda}_F = 1, \quad \widetilde{\lambda} \ge 0, \quad \widetilde{\lambda}_F \ge 0 \tag{5.9.e}$$

Supposing that the optimal solution of (5.9) is $\left(\widetilde{\Phi}^*, \widetilde{\theta}^*, \widetilde{\delta}^{g*}, \widetilde{\delta}^{b*}, \widetilde{\lambda}^*, \widetilde{\lambda}_F^*\right)$, from the constraints of (5.9), we can have

$$X\widetilde{\lambda}^* + x_F \widetilde{\lambda}_F^* = X(\widetilde{\lambda}^* + \widetilde{\lambda}^* \widetilde{\lambda}_F^*) = \widetilde{\theta}^* x_F = \widetilde{\theta}^* \theta^* x_0 \tag{5.10}$$

$$Y\widetilde{\lambda}^* + y_F^g \widetilde{\lambda}_F^* = Y(\widetilde{\lambda}^* + \widetilde{\lambda}^* \widetilde{\lambda}_F^*) = \widetilde{\delta}^{g*} y_F^g = \widetilde{\delta}^{g*} \delta^{g*} y_0^g \tag{5.11}$$

$$\overline{Y}^b \widetilde{\lambda}^* + \overline{y}_F^b \widetilde{\lambda}_F^* = \overline{Y}^b (\widetilde{\lambda}^* + \widetilde{\lambda}^* \widetilde{\lambda}_F^*) = \widetilde{\delta}^{b*} \overline{y}_F^b = \widetilde{\delta}^{b*} \delta^{b*} \overline{y}_0^b \tag{5.12}$$

$$1^T \widetilde{\lambda}^* + \widetilde{\lambda}_F^* = 1^T \left(\widetilde{\lambda}^* + \widetilde{\lambda}^* \widetilde{\lambda}_F^*\right) = 1 \tag{5.13}$$

Then $(\widetilde{\theta}^* \theta^*, \widetilde{\delta}^{g*} \delta^{g*}, \widetilde{\delta}^{b*} \delta^{b*}, \widetilde{\lambda}^* + \widetilde{\lambda}^* \lambda_F^*)$ must be a feasible solution of (5.5), we also have

$$\frac{U_g^T [\widetilde{\delta}^{g*} \delta^{g*}] + U_b^T [\widetilde{\delta}^{b*} \delta^{b*}]}{V^T [\widetilde{\theta}^* \theta^*]} \ge \frac{U_g^T \delta^{g*} + U_b^T \delta^{b*}}{V^T \theta^*} \tag{5.14}$$

Since $0 \le \widetilde{\theta}^* \theta^* \le 1$, $\widetilde{\delta}^{g*} \delta^{g*} \ge 1$ and $\widetilde{\delta}^{b*} \delta^{b*} \ge 1$, the only condition for (5.14) to be satisfied is $\widetilde{\theta}^* = 1$, $\widetilde{\delta}^{g*} = 1$ and $\widetilde{\delta}^{b*} = 1$, that is $\widetilde{\Phi}^* = 1$. Therefore, the DMU_F $\left(x_F, y_F^g, y_F^b\right)$ is weighted efficient relative to other DMUs under evaluation. This completes the proof. □

***THEOREM* 5.4** *If* $\left(x_0, y_0^g, \overline{y}_0^b\right)$ *is weighted global efficient, for* $\alpha > 0$, *then* $\left(\alpha x_0, \alpha y_0^g, \alpha \overline{y}_0^b\right)$ *is also weighted global efficient.*

PROOF. It is obvious. □

THEOREM 5.5 *Let* $(\varphi x_0, \psi y_0^g, \varphi \bar{y}_0^b)$ *with* $\varphi \le 1$, $\psi \ge 1$ *and* $\varphi \ge 1$ *be a DMU with the reduced inputs and enlarged outputs(enlarged desirable outputs and reduced undesirable outputs) than* $(x_0, y_0^g, \bar{y}_0^b)$. *Then, the weighted global efficiency score of* $(\varphi x_0, \psi y_0^g, \varphi \bar{y}_0^b)$ *is not greater than that of* $(x_0, y_0^g, \bar{y}_0^b)$.

PROOF. Suppose that $(\Phi^*, \theta^*, \delta^{g*}, \delta^{b*}, \lambda^*)$ is the optimal solution of (5.5) for DMU_0 $(x_0, y_0^g, \bar{y}_0^b)$. If $\varphi \le 1$, $\psi \ge 1$ and $\varphi \ge 1$, then $(\theta^*/\varphi, \delta^{g*}/\psi, \delta^{b*}/\varphi, \lambda^*)$ is a feasible solution of (5.5) when $(\varphi x_0, \psi y_0^g, \varphi \bar{y}_0^b)$ is being evaluated, because the constraints for the inputs and outputs are clearly satisfied. Since $0 \le \theta^* \le \theta^*/\varphi$, $1 \le \delta^{g*}/\psi \le \delta^{g*}$ and $1 \le \delta^{b*}/\varphi \le \delta^{b*}$, therefore, we have

$$\Phi'^* = \frac{V_g^T \delta^{g*}/\psi + V_b^T \delta^{b*}/\varphi}{U^T \theta^*/\varphi} \le \frac{V_g^T \delta^{g*} + V_b^T \delta^{b*}}{U^T \theta^*} = \Phi^* \tag{5.15}$$

Thus, the weighted efficiency score of $(\varphi x_0, \psi y_0^g, \varphi \bar{y}_0^b)$ is not greater than that of $(x_0, y_0^g, \bar{y}_0^b)$. This completes the proof. □

Comparing our model with Seiford and Zhou's model, we have the following theorems:

THEOREM 5.6 *The weighted global efficiency score* Φ^* *of* DMU_0 $(x_0, y_0^g, \bar{y}_0^b)$ *in (5.5) is not less than the BCC efficiency score* β^* *in (5.4).*

PROOF. Suppose that the optimal solution of (5.5) is $(\beta^*, \lambda^*, s^{-*}, s^{+g*}, s^{+b*})$, let

$$\theta^* = 1 - s^{-*}/x_0 \tag{5.16}$$

$$\delta^{g*} = \beta^* + s^{+g*}/y_0^g \tag{5.17}$$

$$\delta^{b*} = \beta^* + s^{+b*}/\bar{y}_0^b \tag{5.18}$$

Obviously, $(\beta^*, \lambda^*, s^{-*}, s^{+g*}, s^{+b*})$ make $(\theta^*, \delta^{g*}, \delta^{b*}, \lambda^*)$ be the optimal solution of (5.5), then

$$\Phi^* = \frac{U_g^T \delta^{g*} + U_b^T \delta^{b*}}{V^T \theta^*} = \frac{U_g^T [\beta^* + s^{+g*}/y_0^g] + U_b^T [\beta^* + s^{+b*}/\bar{y}_0^b]}{V^T [1 - s^{-*}/x_0]} \tag{5.19}$$

that is,

$$\Phi^* = \frac{\beta^* + U_g^T [s^{+g*}/y_0^g] + U_b^T [s^{+b*}/\bar{y}_0^b]}{1 - U^T [s^{-*}/x_0]} \tag{5.20}$$

Since $s^{-*} \geq 0$, $s^{+g*} \geq 0$ and $s^{+b*} \geq 0$, we have $\Phi^* \geq \beta^*$. This completes the proof. □

***THEOREM* 5.7** *The weighted global efficiency for DMU₀* $(x_0, y_0^g, \bar{y}_0^b)$ *in (5.5) is equivalent to the BCC efficiency for that in (5.4).*

PROOF. Suppose that DMU_0 $(x_0, y_0^g, \bar{y}_0^b)$ is BCC inefficient. Then, we have either $\beta^* \geq 1$ or ($\beta^* = 1$ and $(s^{-*}, s^{+g*}, s^{+b*}) \neq (0,0,0)$). From (5.20), in both cases, we have $\Phi^* \geq 1$ for a feasible solution of (5.5). Hence, DMU_0 $(x_0, y_0^g, \bar{y}_0^b)$ is weighted global inefficient. On the other hand, suppose that DMU_0 $(x_0, y_0^g, \bar{y}_0^b)$ is weighted global inefficient in (5.5). Then, it holds $(\theta^*, \delta^{g*}, \delta^{b*}) \neq (1,1,1)$. According to statements (5.16), (5.17) and (5.18), $(\beta^*, s^{-*} = x_0 - \theta^* x_0, s^{+g*} = \delta^{g*} y_0^g - \beta^* y_0^g, s^{+b*} = \delta^{b*} \bar{y}_0^b - \beta^* \bar{y}_0^b)$ is a feasible solution for (5.4). Provided $s^{-*} = x_0 - \theta^* x_0 = 0$, that is $\theta^* = 1$, there are two cases:

Case 1 ($\delta^{g*} = 1, \delta^{b*} \geq 1$) then $s^{+g*} = y_0^g - \beta^* y_0^g = 0$, $s^{+b*} = \delta^{b*} \bar{y}_0^b - \beta^* \bar{y}_0^b \geq 0$. In this case, DMU_0 $(x_0, y_0^g, \bar{y}_0^b)$ in (5.4) is BCC inefficient.

Case 2 ($\delta^{g*} \geq 1, \delta^{b*} = 1$) then $s^{+b*} = \bar{y}_0^b - \beta^* \bar{y}_0^b = 0$, $s^{+g*} = \delta^{g*} y_0^g - \beta^* y_0^g \geq 0$. In this case, DMU_0 $(x_0, y_0^g, \bar{y}_0^b)$ in (5.4) is also BCC inefficient.

Finally, provided $s^{-*} = x_0 - \theta^* x_0 \geq 0$, that is $0 \leq \theta^* \leq 1$. In this case, DMU_0 $(x_0, y_0^g, \bar{y}_0^b)$ in (5.4) is also BCC inefficient. Therefore, the BCC inefficiency is equivalent to the weighted global inefficiency. Since the definitions of efficiency and inefficiency are mutually exclusive, we have proved the theorem. This completes the proof. □

The above discussions can also be applied to situation when some inputs need to be increased rather than decreased to improve the performance. In this case, we rewrite data domain as

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y^g \\ -X^c \\ -X^d \end{bmatrix} \tag{5.21}$$

where X^c and X^d represent inputs to be increased and decreased, respectively. Next multiply X^d by "−1" and then find a proper translation vector Z to let all negative X^d be positive. The data domain of (5.21) becomes

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y^g \\ -X^c \\ -\overline{X}^d \end{bmatrix} \tag{5.22}$$

Based upon (5.22), we suppose that $U = [U_1, \cdots, U_S] \in \Re_+^s$ is the weight vectors of s outputs while $V_C = [V_1, \cdots, V_m] \in \Re_+^m$ and $V_d = [V_1, \cdots, V_r] \in \Re_+^r$ are the weight vector of m desirable inputs and r undesirable inputs respectively. We also assume that the weight factors for both inputs and outputs satisfy $\mathbf{1}^T V_c + \mathbf{1}^T V_d = 1$ and $\mathbf{1}^T U = 1$, respectively.

$$\text{Minimize } \Phi = \frac{V_c^T \theta^c + V_d^T \theta^d}{U^T \delta} \tag{5.23.a}$$

subject to

$$X^c \lambda = \theta^c x_0^c, \quad 0 \le \theta^c \le 1 \tag{5.23.b}$$

$$\overline{X}^d \lambda = \theta^d \overline{x}_0^d, \quad 0 \le \theta^d \le 1 \tag{5.23.c}$$

$$Y\lambda = \delta y_0, \quad \delta \ge 1 \tag{5.23.d}$$

$$\mathbf{1}^T \lambda = 1, \quad \lambda \ge 0 \tag{5.23.e}$$

where $\theta^c = [\theta_1^c, \cdots, \theta_m^c]^T \in \Re_+^m$ and $\theta^d = [\theta_1^d, \cdots, \theta_r^d]^T \in \Re_+^r$ represent the scalar vectors for desirable inputs and undesirable inputs, respectively, whereas $\delta = [\delta_1, \cdots, \delta_s]^T \in \Re_s^+$ represents the scalar vector for outputs.

5.4 A Numerical Example

We will now demonstrate how the undesirable factors are addressed differently in efficiency evaluation using our models and others. Let us consider the example in Table 5.1 with ten DMUs, two inputs and three outputs (two desirable outputs and one undesirable output). Here in this example, we do not consider the different factor weights in the inputs and outputs, that is, we regard all the inputs and outputs are equally important in the production system.

Table 5. 1 Example Data

DMU	1	2	3	4	5	6	7	8	9	10
Input 1	5	2	1	2	4	3	4	4	4	5
Input 2	3	5	3	4	4	2	3	11	5	3
Desirable output 1	5	3.5	4	5	1	5	6	7	9	3
Desirable output 2	4	4	3.5	3.5	2	1.5	6.5	3	4	6
Undesirable output	4	3	4	4.5	4	1	4	5	4	4
Adjusted	2	3	2	1.5	2	5	2	1	2	2

The results obtained by applying different models are displayed in Table 5.2. For each measure the efficiency scores and the positions in the ranking based on these scores are displayed. Column 2 shows the optimal value to the model $(1)^a$ when the undesirable output is not included. When we ignore undesirable factor, four DMUs were deemed as efficient. Column 3 contains the results obtained from $(1)^b$ where the undesirable output is treated as input. It turns out that this model yields not only different efficiency scores but also a different ranking. The efficiency scores resulting from Seiford and Zhou model based upon classification invariance $\overline{Y}^b = -Y^b + 6.0$ of undesirable output are displayed in the Column 4. The results are not greatly different from the those resulting from model

Table 5.2 Efficiency scores and rankings for the ten DMUs

DMU	Model (1)[a]	Model (1)[b]	β^* in Model (4)	Φ^* in Model(5)
1	1.226 (8)	1.226 (8)	1.226 (8)	1.674 (8)
2	1.125 (7)	1.000 (1-5)	1.000 (1-5)	1.000 (1-5)
3	1.000 (1-4)	1.000 (1-5)	1.000 (1-5)	1.000 (1-5)
4	1.094 (6)	1.094 (7)	1.094 (7)	1.381 (6)
5	3.250 (10)	3.250 (10)	1.844 (10)	4.450 (10)
6	1.000 (1-4)	1.000 (1-5)	1.000 (1-5)	1.000 (1-5)
7	1.000 (1-4)	1.000 (1-5)	1.000 (1-5)	1.000 (1-5)
8	1.286 (9)	1.286 (9)	1.286 (9)	2.971 (9)
9	1.000 (1-4)	1.000 (1-5)	1.000 (1-5)	1.000 (1-5)
10	1.083 (5)	1.083 (6)	1.054 (6)	1.571 (7)
Mean	1.306	1.294	1.150	1.705

(1)[a] The undesirable output is ignored in the BCC model.
(1)[b] The undesirable output is treated as inputs in the BCC model.

(1)[b] where the undesirable output is treated as input. The last column shows the efficiency scores and the positions in the ranking based on these efficiency scores. Comparing the results from the Seiford and Zhu model with those from our model, we find that efficiency in both models is equivalent and our efficiency scores of the inefficient DMUs increase greatly from those obtained from Seiford and Zhou model due to considering both radial inefficiency and nonradial inefficiency. The most notable examples are that DMU_5 lost over 100% efficiency and the ranking of DMU_4 and DMU_{10} is totally reversed.

5.5 Conclusion

In this paper, we have studied various approaches for incorporating undesirable factors in the DEA models under the assumption of variables return to scales. A new efficiency measure is oriented to both desirable factors and undesirable factors simultaneously on the basis of classification invariance so that the weighted global DEA model allows the expansion of desirable outputs and the contraction of undesirable outputs and all inputs with different proportions. The new approach can also be applied to situations when some inputs need to be increased to improve the performance.

Chapter 6

Value Efficiency in DEA Models without Inputs/Outputs

6.1 Introduction

DEA measures the relative efficiency of comparable entities called Decision Making Units (DMUs) essentially performing the same task using similar multiple inputs to produce similar multiple outputs (Charnes et al. 1978). The purpose of DEA is to empirically estimate the so-call efficient frontier based on the set of available DMUs. DEA provides the user with information about the efficient and inefficient units, as well as the efficiency scores and reference sets for inefficient units. The results of the DEA analysis, especially the efficiency scores, are used in practical applications as performance indicators of DMUs. However, in some complex production systems, input data (or output data) is unavailable, thus making performance evaluation be based only on the output data (or input data). Adolphson *et al*. (1991) firstly noted that it is possible to use DEA models without inputs or outputs for such broader perspective and justified that the presence of the convexity constraint in the BCC model (Banker *et al.* 1984) provides the technical grounds of the model change. Actually prior to this research, Thompson *et al.* (1986) adopted an input-oriented CCR model with unique constant output to determinate the optimal location of a superconducting supercollider in the state of Texas in a case study. Although from an economic point of view it is difficult to accept a DEA model without inputs or outputs, the BCC model without inputs has been widely used in performance evaluation in many fields e. g. Lovell (1995), Ozcan and Mccue (1996), and Lovell and Pastor (1997a), (1997b). These applications provide solid empirical support of the methodology suggested in this paper. In addition, Lovell and

Pastor (1999) make a detailed analysis on some radial DEA models without inputs or without outputs from the theoretical perspective. Therefore, research on the DEA models without inputs/outputs is both theoretically and practically importance.

The reminder of this chapter is organized as follows. Section 6.2 briefly introduces the standard output-oriented BCC model without inputs. Basing on this model, in Section 6.3, we introduce the weighted global efficiency measure and discuss a set of desirable properties that the new measure satisfies; we also compare our new measure with the traditional BCC measure and discuss the importance of the global projection based on this efficiency measure. Section 6.4 discusses the validity of super-efficiency in the BCC model without outputs and then proposes a new procedure to discriminate super-efficiency scores among the efficient DMUs. In Section 6.5 we include an example to illustrate the performance of the measure. Finally, Section 6.6 offers some concluding remarks.

6.2 The Traditional BCC Model without Inputs/Outputs

We want to deal with n DMUs with the output matrix $Y = (y_{ji}) \in \Re_+^{n\times m}$, ($j = 1,\cdots,n$; $i = 1,\cdots,m$), where j indexes the DMUs under evaluation and i indexes the outputs of each DMU under evaluation. We define the production possibility set **P** as follows:

$$\mathrm{P} = \left\{ y_{0,i} \middle| y_{0,i} \le \sum_{j=1}^{n} \lambda_j y_{ji}, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \ge 0, j = 1,\cdots,n; i = 1,\cdots,m \right\} \tag{6.1}$$

where $y_{0,i}$ is the *ith* output of the j_0th DMU, named $\mathrm{DMU}_{0,}$ which we want to evaluate. Hence the formulation of the data envelopment analysis problem for an output-oriented BCC model without inputs is

$$Maximize\ \delta + \varepsilon \sum_{i=1}^{m} S_i^+ \tag{6.2.a}$$

subject to

$$\sum_{j=1}^{n} \lambda_j y_{ji} - S_i^+ = \delta y_{0,i}\ , \quad i = 1,\cdots,m \tag{6.2.b}$$

$$\sum_{j=1}^{n} \lambda_j = 1\,, \quad \lambda_j \geq 0, \quad j = 1,\cdots,n \tag{6.2.c}$$

$$S_i^+ \geq 0\,, \quad \varepsilon > 0 \tag{6.2.d}$$

where S_i^+ is slack in the ith output, ε is an arbitrarily small positive number, and λ_j is an intensity variable. This is a simplified version of the output-oriented BCC model (Banker et al. 1984). Actually, it corresponds to the BCC model with unique constant inputs or without inputs. Lovell and Pastor (1999) have demonstrated the equivalence of these two models as follows:

***PROPOSITION* 6.1** *An output-oriented (input-oriented) BCC model with a single constant input (output) is equivalent to an output-oriented (input-oriented) BCC model without inputs (outputs).*

PROOF. We can always assume that the constant input is at least 1, since a rescaling of any variable does not affect the optimal efficiency score obtained by means of any radial DEA model. Therefore the restriction associated with a single constant input is $\sum_{j=1}^{n} \lambda_j \leq 1$. The presence of the convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$ converts the proceeding restriction into a redundant restriction and so it can be deleted. This completes the proof. □

For observed DMU_0, the envelopment problem seeks the maximum equiproportionate expansion in all outputs that is feasible without violating best practice as defined by the m+1 function constraints in the problem. The solution to the maximization problem provides a comprehensive performance measure for DMU_0, provided that the output slacks are small. The optimal value of the objective satisfies $\delta^* \geq 1$. The optimal solution ($\delta^* = 1$ and $S_i^{+*} = 0$) suggests that the DMU under evaluation is BCC efficient, since it is impossible to expand all the outputs equiproportionately with a same scalar δ to the level which does not exceed the best practice in the observed DMUs. The optimal solution $\delta^* > 1$ means that the DMU under evaluation is BCC inefficient, since it is possible to expand all the outputs simultaneously with the same proportion to $\delta^* y_{0,i}$. Thus the larger the value of δ^*, the weaker the performance.

6.3 Measuring Efficiency in DEA Model without Inputs/Outputs

In this section, we discuss the weighted global efficiency issues by focusing our attention on an output-oriented DEA model without inputs. A similar discussion is also valid for an input-oriented DEA model without outputs.

6.3.1 Definition of the New Measure

First, we change the constraints of (6. 2) as follows:

$$\sum_{j=1}^{n} \lambda_j y_{ji} = \delta y_{0,i} + S_i^+ = (\delta + \frac{S_i^+}{y_{0,i}}) y_{0,i} \tag{6.3}$$

Let $\delta_i = \delta + \dfrac{S_i^+}{y_{0,i}}$, we define the efficiency score of DMU_0 as the optimal objective function value $\bar{\delta}^*$ of the following programming:

$$Maximize\ \bar{\delta} = [\delta_1, \cdots, \delta_m]^T \tag{6.4.a}$$

subject to

$$\sum_{j=1}^{n} \lambda_j y_{ji} = \delta_i y_{0,i}, \quad \delta_i \geq 1, \quad i = 1, \cdots, m \tag{6.4.b}$$

$$\sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \cdots, n \tag{6.4.c}$$

The objective function of above formulation is to increase each output with different scalar. Whenever performance is evaluated on the basis of more than one outputs, conflicts are bound to arise. For a production system, an increase in one output leads to a decrease in another output because the overall resources for producing these two outputs are constant. Does its performance improve or decline? The answer depends on the two magnitudes in question, and on how the two outputs are weighted. Therefore, if we suppose that w_i is the weight for the ith output as well as satisfies $\sum_{i=1}^{m} w_i = 1$, then (6.4) can be converted to the following programming:

$$Maximize\ \bar{\delta} = \sum_{i=1}^{m} w_i \delta_i \tag{6.5.a}$$

subject to

$$\sum_{j=1}^{n} \lambda_j y_{ji} = \delta_i y_{0,i}, \quad \delta_i \geq 1, \quad i = 1, \cdots, m \tag{6.5.b}$$

$$\sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \cdots, n \tag{6.5.c}$$

This is our new formulation of an output-oriented weighted global DEA model. Compared with the BCC efficiency measure, the new efficiency measure not only considers both radial and nonradial inefficiency, but also incorporates the weight factors for the outputs. Thus, we also refer to it as a Weighted Global Efficiency Measure (WGEM). One important qualification of using the weighted global measure is the necessity of determining the factor weights w_i prior to measurement. Here we do not discuss how to determine the factor weights, for more detail see Ruggiero and Bretschneider (1998). Let the optimal solution of (6. 5) be $\left(\bar{\delta}^*, \delta_i^*, \lambda_j^*\right)$, we determine a DMU as being weighted global efficient as follows:

***DEFINITION 6.1** (Weighted global efficiency). If the optimal solution $\left(\bar{\delta}^*, \delta_i^*, \lambda_j^*\right)$ of (6.5) satisfies $\bar{\delta}^* = \delta_i^* = 1$, then DMU_0 is weighted global efficient. Otherwise, the DMU_0 is weighted global inefficient.*

It is evident for an increasing value function $\bar{\delta}$ that $\bar{\delta} > 1$ if and only if the DMU_0 is weighted global inefficient. The greater the efficiency scores, the poorer the performances.

6.3.2 Properties of the Weighted Global Efficiency Measure

Färe and Lovell (1978) were the first ones who proposed a set of desirable properties that an ideal efficiency measure should satisfy, although these properties were enunciated for the particular case of an input oriented measure. Recently, Cooper and Pastor (1995) listed similar requirements for the DEA context and suggested some others. Next, we

discuss some properties which the proposed weighted global measure satisfies. The following propositions are true for the weighted global measure of efficiency:

***PROPOSITION 6. 2** The weighted global efficiency score $\bar{\delta}^*$ is units invariant, i. e. it is independent of the units in which outputs are measured provided these units are the same for every DMU.*

PROOF. This proposition holds, since both the objective function and constraints are units invariant. □

***PROPOSITION 6. 3** If the weighted global efficiency score of DMU_0 satisfies $\bar{\delta}^* = 1$, then DMU_0 is Koopmans efficient.*

PROOF. This property is also a consequence of the definition of weighted global efficiency, since observation DMU_0 is Koopmans efficient if, and only if, all slacks are zero. □

PROPOSITION 6.4 *The weighted global measure of efficiency is strictly monotonic in outputs.*

PROOF. Consider an observation DMU_0 with outputs $(y_{0,1},\cdots,y_{0,m})$, and another observation DMU_a with the same values for all outputs but the rth output, which has value $y_{a,r} = y_{0,r} + \Delta_r$, $\Delta_r > 0$. We need to show that the weighted global efficiency score $\bar{\delta}^{a*}$ for the second observation DMU_a is greater than the weighted global efficiency score $\bar{\delta}^{0*}$ for the first observation DMU_0. Denote by $(y_{a,1},\cdots,y_{a,m})$ the outputs of the second observation, where $y_{a,i} = y_{0,i}$ for all $i \neq r$ and $y_{a,r} = y_{0,r} + \Delta_r$. Throughout this proof let us use the model (6. 5) to evaluate DMU_0 and DMU_a and thus obtain the optimal solutions $(\bar{\delta}^{0*},\delta_i^{0*},\delta_r^{0*},\lambda_j^{0*})$ and $(\bar{\delta}^{a*},\delta_i^{a*},\delta_r^{a*},\lambda_j^{a*})$ for DMU_0 and DMU_a, respectively. If the both observations have the same projection onto the efficient frontier it follows that $\delta_i^{a*} = \delta_i^{0*}$, $i \neq r$, and $\delta_r^{a*} = \delta_r^{0*} - \Delta_r / y_{a,r}$. $(\bar{\delta}^{0*},\delta_i^{0*},\delta_r^{0*},\lambda_j^{0*})$ is a feasible solution for (P_a) and thus verifying the above requirement. This completes the proof. □

PROPOSITION 6.5 *Let* $(\varphi_1 y_{0,1},\cdots,\varphi_m y_{0,m})$ *with* $\varphi_i \geq 1$ $(i = 1,\cdots,m)$ *be a DMU with the enlarged outputs relative to* DMU_0 $(y_{0,1},\cdots,y_{0,m})$*. Then, the weighted global efficiency score of* $(\varphi_1 y_{0,1},\cdots,\varphi_m y_{0,m})$ *is not greater than that of* $(y_{0,1},\cdots,y_{0,m})$*.*

PROOF. Suppose that $(\bar{\delta}^*,\delta_i^*,\lambda_j^*)$ is the optimal solution of (6.5) for DMU_0 $(x_0, y_0^g, \bar{y}_0^b)$. If $\varphi_i \geq 1(i = 1,\cdots,m)$, then $(\delta_i^*/\varphi_i,\lambda_j^*)$ is a feasible solution of (6.5) when $(\varphi_1 y_{0,1},\cdots,\varphi_m y_{0,m})$ is being evaluated, because the constraints for the inputs and outputs are clearly satisfied. Since $1 \leq \delta_i^* / \varphi_i \leq \delta_i^*$, therefore, we have

$$\bar{\delta}'^* = \sum_{i=1}^{m} w_i [\delta_i^*/\varphi_i] \leq \sum_{i=1}^{m} w_i\delta_i^* = \bar{\delta}^* \tag{6.6}$$

where $\bar{\delta}'^*$ represents the weighted global efficiency score of $(\varphi_1 y_{0,1},\cdots,\varphi_m y_{0,m})$. Therefore, the weighted efficiency score of $(\varphi_1 y_{0,1},\cdots,\varphi_m y_{0,m})$ is not greater than that of $(y_{0,1},\cdots,y_{0,m})$. This completes the proof. □

6.3.3 Comparing Weighted Global Efficiency with BCC Efficiency

The relationship between the BCC efficiency and the weighted global efficiency is demonstrated by the following propositions.

PROPOSITION 6.6 *The weighted global efficiency of DMU$_0$ in (6. 5) is equivalent to the BCC efficiency of DMU$_0$ in (6. 2).*

PROOF. (i) Suppose that $(\bar{\delta}^*,\delta_i^*,\lambda_j^*)$ is the optimal solution of (6.5). If DMU$_0$ is weighted global efficient, according to DEFINITION 1, we have $\bar{\delta}^*=\delta_i^*=1$. Then the constraints of (6.5) must satisfy:

$$\sum_{j=1}^{n}\lambda_j^* y_{ji}=\delta_i^* y_{0,i}\,,\quad i=1,\cdots,m \tag{6.7}$$

$$\sum_{j=1}^{n}\lambda_j^*=1,\quad \lambda_j^*\geq 0\,,\quad j=1,\cdots,n \tag{6.8}$$

If we set $\delta^*=1$ and $S_i^{+*}=0$, obviously, $(\delta^*,\lambda_j^*,S_i^{+*})$ is also a feasible solution of (6.2). If S_i^{+*} and δ^* can be further expanded, that is, $S_i^{+*}\neq 0$ and $\delta^*\neq 1$, this will contradict condition that λ_j^* must satisfy (6.7) and (6.8). On the other hand, since $S_i^{+*}\geq 0$ and $\delta^*\geq 1$, S_i^{+*} and δ^* cannot become small. Therefore, $(\delta^*,\lambda_j^*,S_i^{+*})$ is the optimal solution of (6.2), as well as satisfies $S_i^{+*}=0$ and $\delta^*=1$. That means that DMU$_0$ is also BCC efficient.

(ii) Suppose that DMU$_0$ is BCC efficient, then the optimal solution $(\delta^*,\lambda_j^*,S_i^{+*})$ of (6.2) must satisfy $S_i^{+*}=0$ and $\delta^*=1$. Thus the solution $(\bar{\delta}^*,\delta_i^*,\lambda_j^*)$ must not only satisfy constraints (6.7) and (6.8). Just proved as above, $\left(\bar{\delta}^*, \delta_i^*, \lambda_j^*\right)$ is not only a feasible solution of (6.5) but also the only optimal solution of (6.5). According to DEFINITION 1, we know that the corresponding DMU_0 is also weighted global efficient. Therefore, the weighted global efficiency of DMU_0 in (6.5) is equivalent to the BCC efficiency of DMU_0 in (6.2). This completes the proof. □

***PROPOSITION* 6.7** *The weighted global efficiency score* $\bar{\delta}^*$ *of* DMU_0 *in (6. 5) is not less than the BCC efficiency score* δ^* *of* DMU_0 *in (6. 2).*

PROOF. Suppose that the optimal solution of (6.2) is $\left(\delta^*, \lambda_j^*, S_i^*\right)$ ($i = 1, \cdots, m$, $j = 1, \cdots, n$). According to (6. 3), we have

$$\delta_i^* = \delta^* + \frac{S_i^{+*}}{y_{0,i}} \tag{6.9}$$

It can be observed from (6. 9) that, $\left(\delta^*, \lambda_j^*, S_i^*\right)$ makes $\left(\bar{\delta}^*, \delta_i^*, \lambda_j^*\right)$ the optimal solution of (6. 5). Then we have

$$\bar{\delta}^* = \sum_{i=1}^{m} w_i \left(\delta^* + \frac{S_i^{+*}}{y_{0,i}} \right) = \delta^* + \sum_{i=1}^{m} \left(\frac{w_i S_i^{+*}}{y_{0,i}} \right) \tag{6.10}$$

Since $S_i^{+*} \geq 0$, from (6.10), we have $\bar{\delta}^* \geq \delta^*$. Therefore, the weighted global efficiency score $\bar{\delta}^*$ of DMU_0 in (6.5) is not less than the BCC efficiency score δ^* of that in (6.2). This completes the proof. □

6.3.4 Global Projection Analysis onto Efficient Frontier

The intent of frontier estimation is to deduce empirically the production function in the form of an efficient frontier. That is, rather than knowing how to convert functionally inputs and outputs, these methods take the inputs and outputs as given, map out the best performers, and produce a relative notion of the efficiency of each. The problem with the existing methods is that they each measure efficiency in a conceptually suspect, albeit computationally effective, way. In the traditional BCC model without inputs, the radial projection of an efficient DMU onto the efficient frontier is a too restrictive technique because the current output values are projected onto the efficient frontier by increasing them in the same proportion. Whereas in our model, the efficiency of an inefficient DMU is determined by increasing each output to its given level which mostly depends on the weight factors assigned to different outputs. Thus, this projection is more meaningful than the radial projection as it incorporates the effects of weight factors assigned to outputs. Here, we focus on the differences between the global projection obtained in our model and radial projection obtained in traditional BCC model by evaluating a simple production system set with only two outputs and without inputs, see Figure 6.1.

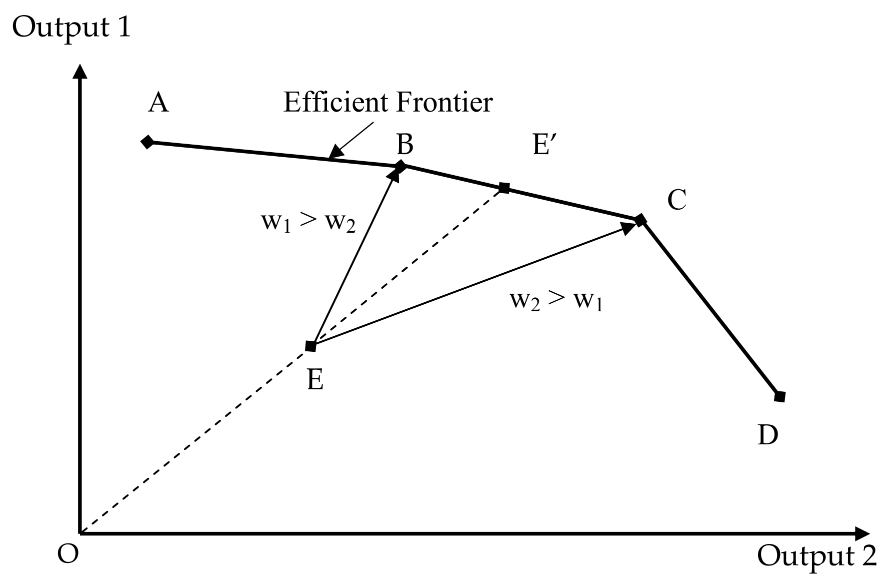

Figure 6.1 Illustration of different projections onto efficient frontier

In Figure 6.1, an inefficient DMU E can be projected onto the efficient frontier which consists of four efficient DMUs, i. e. A, B, C, and D. Assume that $\theta_1 = \theta_2$, the weighted global efficiency in (6.5) coincides with the BCC efficiency in (6.2), that is, these two models have the same projections onto the efficient frontier. The outputs are equiproportionately increased. Assume that $\theta_1 \neq \theta_2$, if $w_1 \geq w_2$, the first output is increased more than the second one and the global projection is more oriented to the direction of the Output 1-axis. Thus the inefficient DMU E can be projected onto the DMU B. Compared with the radial projection (E'), the global projection has the higher value in output 1 and lower value in output 2. Conversely, if $w_1 \leq w_2$, the second output is increased more than the first one and the global projection is more oriented to the direction of the Output 2-axis. Correspondingly, the projection on the efficient frontier will be $\mathrm{DMU_C}$ (point C).

Therefore, we may define the global projection of the inefficient DMU onto the efficient frontier of the model (6. 5) as follows:

DEFINITION 6.2. *Suppose that the optimal solution of (5) is* $\left(\bar{\delta}^*, \delta_i^*, \lambda_j^*\right)$*, let*

$$\tilde{y}_{ki} = \delta_i^* y_{0,i} = \sum_{j=1}^{n} \lambda_j^* y_{ji}, \quad \delta_i^* \geq 1, \quad i = 1, \cdots, m \tag{6.11}$$

$$\sum_{j=1}^{n} \lambda_j^* = 1, \quad \lambda_j^* \geq 0, \quad j = 1, \cdots, n \tag{6.12}$$

then DMU_k $\left(\tilde{y}_{k1}, \cdots, \tilde{y}_{km}\right)$ *is defined as the weighted global projection of* DMU_0 *onto the efficient frontier.*

These relationships suggest that the efficiency of $\mathrm{DMU_0}$ can be improved if the outputs are augmented unequalproportionally. Thus we have a method for improving an inefficient DMU that accords with DEFINITION 6. 2 in the following proposition.

PROPOSITION 6.8 *The global projection of* DMU_0 *onto the efficient frontier is weighted global efficient compared with all the other DMUs under evaluation.*

PROOF. Supposing that $\mathrm{DMU_k}$ is the global projection of $\mathrm{DMU_0}$ onto the efficient frontier, we use the following (6.13) to evaluate the weighted global efficiency of $\mathrm{DMU_k}$ relative to all the DMUs under evaluation:

$$Maximize\ \tilde{\tilde{\delta}} = \sum_{i=1}^{m} w_i \tilde{\delta}_i \tag{6.13.a}$$

subject to

$$\sum_{j=1}^{n} \tilde{\lambda}_j y_{ji} + \tilde{\lambda}_{n+1} \tilde{y}_{ki} = \tilde{\delta}_i \tilde{y}_{ki}, \quad i = 1, \cdots, m \tag{6.13.b}$$

$$\sum_{j=1}^{n} \tilde{\lambda}_j + \tilde{\lambda}_{n+1} = 1 \tag{6.13.c}$$

$$\tilde{\delta}_i \geq 1 \tag{6.13.d}$$

$$\forall \tilde{\lambda}_j \geq 0,\ \tilde{\lambda}_{n+1} \geq 0;\ j = 1, \cdots, n \tag{6.13.e}$$

where $\tilde{y}_{ki}$ is the ith output of DMU_k and $\tilde{\lambda}_{n+1}$ is an intensity variable for DMU_k. If the optimal solution of (6.13) is $\left(\tilde{\tilde{\delta}}^*, \tilde{\delta}_i^*, \tilde{\lambda}_j^*, \tilde{\lambda}_{n+1}^*\right)$, then, by incorporating (6.11) and (6.12), the constraints of (6.13) can be converted to

$$\sum_{j=1}^{n} \tilde{\lambda}_j^* y_{ji} + \tilde{\lambda}_{n+1}^* \tilde{y}_{ki} = \sum_{j=1}^{n} (\tilde{\lambda}_j^* + \tilde{\lambda}_{n+1}^* \lambda_j^*) y_{ji} = \tilde{\delta}_i^* \tilde{y}_{ki} = \tilde{\delta}_i^* \delta_i^* y_{0,i} \tag{6.14}$$

$$\sum_{j=1}^{n} \tilde{\lambda}_j^* + \tilde{\lambda}_{n+1}^* = \sum_{j=1}^{n} (\tilde{\lambda}_j^* + \tilde{\lambda}_{n+1}^* \tilde{\lambda}_j^*) = 1 \tag{6.15}$$

we know $(\tilde{\lambda}_j^* + \tilde{\lambda}_{n+1}^* \tilde{\lambda}_j^*, \tilde{\delta}_i^* \delta_i^*)$ is also a feasible solution of (6.5), thus we have

$$\sum_{i=1}^{m} w_i \tilde{\delta}_i^* \delta_i^* \leq \sum_{i=1}^{m} w_i \delta_i^* \tag{6.16}$$

Since $\tilde{\delta}_i^* \geq 1$ and thus $\tilde{\delta}_i^* \delta_i^* \geq \delta^*$, the only condition to satisfy (6.16) is $\tilde{\delta}_i^* = 1$. According to DEFINITION 1& 2, the projection of DMU_0 onto the efficient frontier is efficient compared with all the DMUs under evaluation. This completes the proof. □

6.4 Super-efficiency Based on Weighted Global Measure

In this section, we continue to discuss the super-efficiency issues under the assumption that DMU$_0$ is weighted global efficient, i.e. $\bar{\delta}^* = 1$. Let $\mathrm{Y} = (y_{ji}) \in \Re_+^{m\times(n-1)}, j \neq j_0$ be output matrices, consisting of nonnegative elements and excluding the observed output vector of DMU$_0$. The production possibility set $\mathbf{P}/\mathrm{y}_{0,\mathrm{i}}$ can be redefined as follows:

$$\mathrm{P}/\, y_{0,i} = \left\{ y_{0,i} \left| y_{0,i} \leq \sum_{\substack{j=1\\ j\neq j_0}}^{n} \lambda_j y_{ji}, \sum_{\substack{j=1\\ j\neq j_0}}^{n} \lambda_j = 1, \lambda_j \geq 0, j=1,\cdots,n; i=1,\cdots,m \right. \right\} \tag{6.17}$$

The corresponding BCC super-efficiency score δ^s of DMU$_0$ can be obtained in the following programming:

$$\textit{Maximize}\ \delta^s \tag{6.18.a}$$

subject to

$$\sum_{\substack{j=1\\ j\neq j_0}}^{n} \lambda_j y_{ji} = \delta^s y_{0,i}, \quad i=1,\cdots,m \tag{6.18.b}$$

$$\sum_{\substack{j=1\\ j\neq j_0}}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j=1,\cdots,n \tag{6.18.c}$$

Following the above sections, suppose that w_i is the weight for the ith output and satisfies $\sum_{i=1}^{m} w_i = 1$, then the *weighted global super-efficiency score* $\bar{\delta}^s$ for DMU$_0$ can be defined as follows:

$$\textit{Maximize}\ \bar{\delta}^s = \sum_{i=1}^{m} w_i \delta_i^s \tag{6.19.a}$$

subject to

$$\sum_{\substack{j=1\\ j\neq j_0}}^{n} \lambda_j y_{ji} = \delta_i^s y_{0,i}, \quad 0 \leq \delta_i^s \leq 1, \quad i=1,\cdots,m \tag{6.19.b}$$

$$\sum_{\substack{j=1 \\ j \neq j_0}}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \cdots, n \tag{6.19.c}$$

Compared (6.19) with (6.18), we have the following proposition:

***PROPOSITION* 6.9** The *weighted global super-efficiency score* $\bar{\delta}^{s*}$ *of* DMU_0 *in (6.19) is not less than the BCC super-efficiency score* δ^{s*} *in (6.18).*

PROOF. The proof is analogous to the proof of PROPOSITION 6. 7. □

The effects of weighting factors on outputs in super global projection is similar to that in global projection. Now we focus on the difference between the global projection and the super global projection. Unlike global projection, the super projection is to project an efficient DMU onto the efficient frontier which is composed of all the DMU excluding the one under evaluation. Thus the efficient DMU is more efficient than its projection onto the efficient frontier because the outputs of the efficient DMU should be decreased nonproportionally to reach the efficient frontier. Below, in Figure 6.2, we have illustrated such difference.

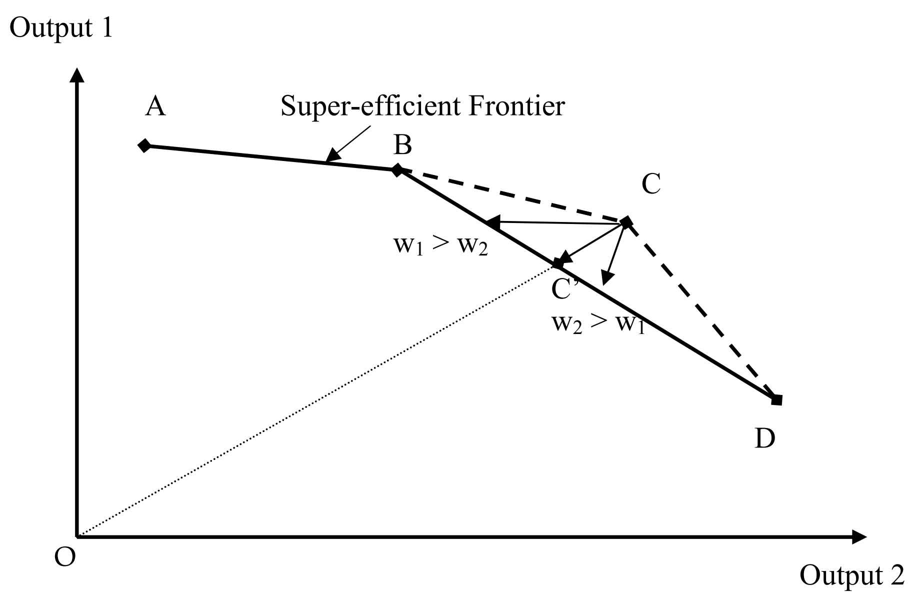

Figure 2. Super-efficient frontier and super global projection

6.5 A Numerical Example

In this section, we compare the BCC model without inputs with our model using an example in reality. We restrict our comparisons only within the above two models. This example in Table 6.1 consists of nine DMUs with three outputs as listed below: y_1 = Profitability Ratio (%); y_2 = Return on Asset (%); y_3 = Return on Equity (%).

Table 6.1 A numerical example

DMU	1	2	3	4	5	6	7	8	9
y_1	4.21	7.73	8.68	9.60	7.22	8.34	3.09	12.54	2.57
y_2	15.44	15.44	5.39	15.74	4.17	5.00	9.34	8.13	4.63
y_3	6.07	25.06	4.39	22.54	5.33	2.84	18.37	5.76	9.10

Table 6.2 exhibits the traditional BCC efficiency scores, their ranks, along with the projected point $(\delta^* y_{\bullet i})$, output slacks (S_i^{+*}) and the reference set $\{y \mid \lambda_j^* > 0\}$. From Table 6.2, we can see that three out of nine DMUs, i. e. DMU_2, DMU_4, and DMU_8, are BCC efficient relative to all the DMUs. The rest are regarded as BCC inefficient DMUs. We also apply (6.5) to these DMUs and obtain their weighted global efficiency scores and corresponding ranks, along with the projected point $(\delta_i^* y_{\bullet i})$ and the reference set $\{y \mid \lambda_j^* > 0\}$, all are shown in Table 6.3. Next, we compare results obtained for BCC

Table 6.2 Results of applying (2)

DMU	Scores	Ranks	Projected Points (Slacks)			Reference Set
			$\delta^* y_{\bullet 1}$	$\delta^* y_{\bullet 2}$	$\delta^* y_{\bullet 3}$	
1	2.366	8	9.96	14.81	14.36(6.13)	#4, #8
2	1.000	1-3	7.73	15.44	25.06	#2
3	1.434	5	12.45(0.64)	7.73	6.30	#4, #8
4	1.000	1-3	9.60	15.74	22.54	#4
5	1.662	7	12.00	6.93(2.61)	8.86	#4, #8
6	1.504	6	12.54(0.61)	7.52(1.49)	4.27	#8
7	1.364	4	4.22(3.51)	12.74(2.70)	25.06	#2
8	1.000	1-3	12.54	8.13	5.76	#8
9	2.754	9	7.08(0.65)	12.75(2.69)	25.06	#2

efficiency measure with those obtained for weighted global efficiency measure. Obviously, both measure agree to in the classification of the efficient DMUs. BCC efficient DMU_2, DMU_4, and DMU_8 remained at the efficient status under the weighted global efficiency evaluations, as claimed by PROPOSITION 6.6. Moreover, as expected by PROPOSITION 6.7, the weighted global efficiency score $\bar{\delta}^*$ is not less than the BCC efficiency score δ^*. The most notable examples are that the weighted global efficiency scores of DMU_3, DMU_5 and DMU_6 in (6.5) more than double the BCC efficiency scores in (6.2) respectively since the later ones are radial measures and take no account of slacks while the global efficiency scores contain the effects of slacks. We will now examine this difference in the case of DMU_3 in more detail. The BCC model for DMU_3 gives the solution $\delta^* y_{3\bullet} = \lambda_4^* y_{4\bullet} + \lambda_8^* y_{8\bullet} + S^{+*}$. Thus, the projected point $\delta^* y_{3\bullet}$ has slacks $S^{+*} = (0.64, 0, 0)^T$ against the referent $\lambda_4^* y_{4\bullet} + \lambda_8^* y_{8\bullet} =$ $(13.09, 7.73, 6.30)^T$ which is on the efficient frontiers. For comparison, the weighted global efficiency model for DMU_3 gives a different solution $\delta_{3i}^* y_{3i} = \lambda_2^* y_{2i} + \lambda_4^* y_{4i}$. Thus, the projected point $\delta_i^* y_{3i}$ has slacks against the referent $\lambda_2^* y_{4i} + \lambda_4^* y_{4i} = (8.68, 15.59, 23.78)^T$.

Table 6.3 Results of applying (6.5)

DMU	Global Efficiency Scores				Ranks	Projected Points			Reference Set
	δ_1^*	δ_2^*	δ_3^*	$\bar{\delta}^*$		$\delta_1^* y_{\bullet 1}$	$\delta_2^* y_{\bullet 2}$	$\delta_3^* y_{\bullet 3}$	
1	2.280	2.514	3.713	2.836	5	9.60	15.74	22.54	#4
2	1.000	1.000	1.000	1.000	1-3	7.73	15.44	25.06	#2
3	1.000	2.893	5.417	3.103	6	8.68	15.59	23.78	#2, #4
4	1.000	1.000	1.000	1.000	1-3	9.60	15.74	22.54	#4
5	1.071	3.703	4.702	3.158	7	7.73	15.44	25.06	#2
6	1.000	3.108	8.534	4.214	9	8.34	15.54	24.24	#2, #4
7	3.107	1.685	1.227	2.006	4	9.60	15.74	22.54	#4
8	1.000	1.000	1.000	1.000	1-3	12.54	8.13	5.76	#8
9	3.735	3.340	2.477	3.184	8	9.60	15.74	22.54	#4

Finally, in order to discriminate the relative efficiency among efficient DMUs, i. e. DMU_2, DMU_4 and DMU_8, we apply (6.18) to evaluate the super efficiency of these efficient DMUs relative to all the DMUs excluding the DMU under evaluation. Results for the weighted global measure of super-efficiency which consider the average weightings of outputs are displayed in Table 6.4. For each efficient DMU the super-efficiency score and the position in the ranking based on these scores are displayed. The super efficiency scores for these three global efficient DMUs, as well as rank among them, are shown in Table 6.4. From Table 6.4, we can see that DMU_4 is closer to the efficient frontier (denoted by a combination of DMU_2 and DMU_8) than DMU_2, thus its super-efficiency score is greater than that of DMU_2. Therefore, the ranking order for these three global efficient DMUs is $DMU_8 > DMU_2 > DMU_4$. For comparison, we also apply the output-oriented Anderson and Peterson model to address the same problem. The resulting super-efficiency scores and ranks are shown in Table 6.4. As expected, the super-efficiency scores in Andersen and Peterson model is less than the weighted global super-efficiency scores.

Table 6.4 Illustration of the two super-efficiency measures

DMU	Weighted global super-efficiency					BCC Super-efficiency	
	δ_1^{s*}	δ_2^{s*}	δ_3^{s*}	$\bar{\delta}^{s*}$	Ranks	δ^{s*}	Ranks
2	1.000	0.900	0.851	0.917	2	0.751	3
4	0.871	0.920	1.000	0.930	3	0.730	2
8	0.680	0.811	1.000	0.830	1	0.674	1

6.6 Conclusion

In this chapter, we have discussed the weighted global efficiency measure by formulating a new nonradial model without inputs and demonstrated theoretically and empirically that the weighted global efficiency score obtained from our model is not less than the traditional BCC efficiency score because our efficiency measure considers not only radial inefficiency but also the nonradial inefficiency. On the other hand, the weighted global efficiency is also equivalent to BCC efficiency obtained from the traditional DEA model without inputs. Next, we have discussed how to improve the efficiencies of those inefficient DMUS to be efficient by analyzing their global projection onto the efficient frontier. Finally, we have developed a new super-efficiency measure which can be used to discriminate the relative performance among the efficient DMU.

Chapter 7

Conclusion

7.1 Summary of the Research

In this study, we propose a weighted efficiency measure which focuses on inputs minimization and output maximization simultaneously. Firstly, we define our measure of efficiency and then investigate its properties and demonstrate its characteristics theoretically. In addition, we provide an effective method to capture the internal value information in the production systems which is usually omitted in the traditional efficiency measures. Furthermore, we show how the effect of weights factors on the efficiency and efficient frontier in our model. Finally, we compare our measure with other measures theoretically as well as empirically and find that there are some differences between our measure and others. We believe that the use of this measure is practical, in the sense that it requires little detailed information on the part of the analyst, and consistent, in the sense that – if a factor is deemed important enough to include in the analysis then its importance should be reflected in its contribution to the benefit of DMU activity.

In addition, the ability to rank or differentiate the efficient units is of both theoretically and practically importance. One concern about these super-efficiency models is that they may not always be possible to determine their optimal value when the super-efficiency models are applied under other alternate returns to scale (RTS) conditions other than constant returns to scales (CRS). Another concern is that these super-efficiency measures cannot capture certain inherent relationships among the inputs and the outputs which can be known or predetermined beforehand. In this study, we discuss the use of the weighted super-efficiency measure which is derived from the weighted global efficiency measure. This super-efficiency measure is useful to differentiate efficient units and motivate appropriate behavior. First of all, we define our measure of super-efficiency and then investigated its properties and demonstrate its characteristics theoretically. Second, we shows how to calculate the measure in a linear program setting when it is actually applicable in the sense that the measure exists, i. e. the defining programs have a feasible solution. Finally, we compare our measure with other super-efficiency measures theoretically as well as empirically and find that there are some differences between our measure and others.

Furthermore, we have studied various approaches for incorporating undesirable factors in the DEA models under the assumption of variables return to scales. A new efficiency measure is oriented to both desirable factors and undesirable factors simultaneously on the basis of classification invariance so that the weighted global DEA model allows the expansion of desirable outputs and the contraction of undesirable outputs and all inputs with different proportions. The new approach can also be applied to situations when some inputs need to be increased to improve the performance.

Finally, we have discussed the use of the weighted global efficiency measure in the production systems without inputs or outputs. In this chapter of research, firstly, we have demonstrated theoretically and empirically that the weighted global efficiency score obtained from our model is not less than the traditional BCC efficiency score because our efficiency measure considers not only radial inefficiency but also the nonradial inefficiency. On the other hand, the weighted global efficiency is also equivalent to BCC efficiency obtained from the traditional DEA model without inputs. Secondly, we have discussed how to improve the efficiencies of those inefficient DMUS to be efficient by analyzing their global projection onto the efficient frontier. Finally, we have developed a new super-efficiency measure which can be used to discriminate the relative performance among the efficient DMU.

7.2 Contributions

This research has focused on a number of important issues related to the efficiency measurement in the presence of additional slack after Farrell efficiency is achieved. Most analyses of efficiency provided evidence on the Farrell measure of inefficiency but provide little if any discussion on remaining slack. First, as shown in the research, the Farrell measure may be a poor measure of producer performance in the presence of slack. As a solution to this problem, the non-radial global measure is introduced. One contribution of this research is to show that the weighted global measure may not perform better than the traditional measures if inputs and outputs do not have equal factor weights in the production process. The second contribution of this research is the development of an alternative programming model that not only simultaneously allows non-radial reduction in inputs and augmentation in outputs, but only incorporates unequal factor weights. This new measure of technical inefficiency, called the Weighted Global measure, combines econometric production function estimation as part of a first stage and incorporates the resulting weights into a linear programming model. Comparison analysis suggest that the weighted global measure outperforms the existing measures in cases where inputs and outputs are not equally productive and provides similar measures in all other cases. This new technique places more structure on the production correspondence by combining econometric and linear programming techniques. The weighted global measure is useful for future empirically analyses where excess slack exists after radial efficiency is achieved.

7.3 Future Research

Our future research will focus on how to capture the judgments and preferences of decision makers using a general framework, not only of static production systems but also of dynamic and stochastic framework. The combination of our current works and future works will be regarded as systematic research on value efficiency. On the other hand, we also investigate the value efficiency in the field of finance and investment, in which very little research is conducted by the scholars in data envelopment analysis domain. Thus the empirical application of value efficiency will extend the application of DEA.

Another area in my further development is the relationship of value efficiency to the mission and objectives of the organization. The specification of inputs and outputs defines the nature and scope of the organization as a system of DMUs, and indeed an essential criterion of homogeneity for admittance of a DMU to a DEA study is the acceptance of the set of inputs and outputs. However, the definition of this set is insufficient to reflect the values and priorities of the organization. The weights associated with the inputs and outputs represent the rates of substitution or relative values of variables and it is thus the weights that need to relate to the mission and objectives of the organization. Moreover, free weights can lead to an inversion of the value system as in the study of perinatal care units, Thanassoulis et al. (1995), where the weight placed on a "very satisfied mother" was lower than that placed on a "satisfied mother" in terms of the perceived quality of treatment. This could be a fruitful area of development in my coming research.

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