TIME OPTIMAL CONTROL FOR ASYMPTOTIC TRACKING OF LINEAR SYSTEMS WITH INPUT SATURATION

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Summary

The objective of the research described in this thesis is to investigate some properties of a class of linear systems with input saturation, and to design the corresponding composite nonlinear feedback controller to achieve satisfactory certain tracking performance.

In this thesis, we have extended the composite nonlinear feedback control law from single-input-single-output systems to multi-input-multi-output systems, from continuous time systems to discrete time systems. We have explored the problem of asymptotic time optimal tracking for a class of linear systems with input saturation. A formula on the optimal settling time is given and the corresponding composite nonlinear feedback control law can serve as a proper control law to approximate the optimal settling time.

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Chapter 1

Introduction

1.1 Background and Motivation

1.1.1 Composite Nonlinear Feedback Control

Nonlinearities are widely coexisting with real physical systems. One of the commonly seen nonlinearities is the actuator saturation. For this actuator saturation, it can be described as the following mapping from input x to output y:

$$y = \begin{cases} a & \text{if} \quad x > x^+ \\ f(x) & \text{if} \quad x^+ \geq x > x^- \\ b & \text{if} \quad x \leq x^- \end{cases} \tag{1.1}$$

For linear actuator saturation with equally distribution, $a = -b$ and $f(x) = x$ and $x^+ = -x^- > 0$. It is well known that there are two main approaches to deal with actuator saturation. The first approach is to neglect the saturation in the first stage of the controller design and then add some problem-specific schemes to overcome the adverse effects caused by the saturation. The basic idea behind these schemes is to introduce additional feedbacks in such a way that the actuator stays properly within its limits. Most of these schemes lead to improved performance but poorly stability properties.

Another approach is more systematic. The control law designed should meet either the performance or stability requirement. It analyzes the closed-loop system under actuator saturation systematically and redesigns the controller in such a way that the performance is retained while stability is improved. This is the approach we will take in the thesis.

The composite nonlinear feedback control was first proposed by Chen et. al. in ([1]) and has been applied successfully to the single input single output (SISO) linear systems with actuator saturation. In this thesis, we will extend this controller design scheme to the multi-input multi-output (MIMO) linear systems and discrete time systems.

1.1.2 Asymptotic Time Optimal Tracking

Time optimal control has been the research focus of a lot of researchers and engineers since the middle of last century. In [23], Ryan first described the trajectory of a system output in his historical paper. In [16], Bushaw gave a mathematical solution to a relatively simple optimal control problem in 1953. Contiuous research has been done by a lot of researchers, including Rose (1953, [24]), LaSalle (1953, [25]), (1960, [26]), Bellman (1956, [27]), Pontryagin (1956, [28]), et. al. However, we have paid much attention to the problems about asymptotic time optimal control and tracking.

Nearly all the previous issues on time optimal control have been focused on point-point time optimal control and tracking. But in real physical systems, we have some other issues to deal with, such as fast tracking, less overshoot and less undershoot. Moreover, it is generally not necessary to have a precise point to point tracking in practical situations, such as in missile tracking. When the missile is approaching the target aircraft, it can destroy the aircraft effectively if the missile explodes in a nearby region. So it does not necessarily hit the aircraft. Instead, in some applications, it would be more preferable to consider asymptotic tracking where the tracking target is defined as a small neighborhood of the given reference. We would like to track the reference as soon as possible. So the issues concerned the optimal settling time will be of importance.

The settling time quantifies the time it takes the transient to decay below a given settling level, say ε, commonly between 1% and 10%. It is defined, as in ([22]), by

$$t_s = \inf_\delta \ \{\delta : |y(t) - 1| \leq \varepsilon, \ \text{ for all t } \in \ [\delta, \infty)\} \tag{1.2}$$

Here the step response of the system has been normalized to have a constant final value.

In this thesis, we will quantify the optimal settling time (1.2) to a specific system, a linear system with actuator saturation at input. And moreover, the corresponding controller design will be presented to approximate the optimal settling time.

1.2 Organization of Thesis

The thesis is organized as following:

In Chapter 2, we extend the composite nonlinear feedback control law from single-input-single-output systems to multi-input-multi-output systems. The state feedback control law and the output feedback control law are both given to achieve asymptotic tracking for a linear input-saturated system.

In Chapter 3, we derive the discrete time composite nonlinear feedback control law. Then a whole controller design process with identification, design and simulation on a dual-stage hard disk drive is given in Chapter 4.

In Chapter 5, we explore the problem of asymptotic time optimal tracking for a class of linear systems with actuator saturation. A formula on the optimal settling time is given and the corresponding composite nonlinear feedback control law can serve as a good control law to approximate the optimal settling time.

Then in the last chapter, concluding remarks and recommendations for future work are included.

Chapter 2

CNF Control for Linear MIMO Systems with Input Saturation

2.1 Introduction

Every physical system in our real life has nonlinearities and very little can be done to overcome them. Many practical systems are sufficiently nonlinear so that important features of their performance may be completely overlooked if they are analyzed and designed through linear techniques (see *e.g.*, Hu and Lin [18]). For example, in the computer hard disk drive (HDD) servo systems (see *e.g.*, Chen *et al.* [1]), major nonlinearities are friction, high frequency mechanical resonance and actuator saturation nonlinearities. Among all these, the actuator saturation could be the most significant nonlinearity in designing an HDD servo system. When the actuator is saturated, the performance of the control system designed will seriously deteriorate. As such, the topic of nonlinear control for saturated linear systems has attracted considerable attentions in the past (see *e.g.*, Garcia *et al.* [39], Henrion *et al.* [40], Suarez *et al.* [42], and Wredenhagen and Belanger [46] to name a few).

Most of these works are using approaches based on certain parameterized Riccati equations.

Typically, when dealing with "point-and-shoot" fast-targeting for single-input and single-output (SISO) systems with actuator saturation, one would naturally think of using the well known time optimal control (TOC) (known also as the bang-bang control), which uses maximum acceleration and maximum deceleration for a predetermined time period. Unfortunately, it is well known that the classical TOC is not robust with respect to the system uncertainties and measurement noises. It can hardly be used in any real situation. For SISO systems with input saturation, another commonly used controller for target tracking is known as the proximate time-optimal servomechanism (PTOS), which was originally proposed by Workman [9] to overcome the above mentioned drawback of the TOC design.

Inspired by a work of Lin *et al.* [8], which was introduced to improve the tracking performance under state feedback laws for a class of second order systems subject to actuator saturation, Chen *et al.* [6] have recently extended the technique to general SISO systems with measurement feedback. The work of Chen *et al.* [6] has been successfully applied to design an HDD servo system, which outperforms conventional methods by more than 30%. The extension of the results of [8] to multi-input and multi-output (MIMO) systems under state feedback was reported in a nice work by Turner *et al.* [44]. However, the extension was made under a pretty odd assumption on the system that excludes many systems including those originally considered in [8]. The restrictiveness of the assumption of [44] will be discussed later. Also, as in [8], only state feedback is considered in [44].

In this chapter, we present a design procedure of composite nonlinear feedback (CNF) control for general multivariable systems with actuator saturation. We consider both the state feedback case and the measurement feedback case without imposing any restrictive assumption on the given systems. As in the earlier works [6, 8, 44], our CNF control consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time not exceeding the actuator limits for the desired command input levels. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part.

The chapter is organized as follows. In Section 2.2, the theory of the composite nonlinear feedback control is developed. Three different cases, *i.e.*, the state feedback, the full order measurement feedback, and the reduced order measurement cases, are considered with all detailed derivations and proofs. We will address the issue on the selection of nonlinear gain parameter in this section. The application of the CNF technique to an MIMO system will be presented in Section 2.3, which shows that the proposed design method yields a very satisfactory performance. Finally, we draw some concluding remarks in Section 2.4.

2.2 Composite Nonlinear Feedback Control for MIMO Systems

We present in this section the CNF controller design for the following multivariable linear system Σ with an amplitude-constrained actuator characterized by

$$\begin{cases} \dot{x} = A\ x + B\ \mathrm{sat}(u), \quad x(0) = x_0 \\ y = C_1\ x \\ h = C_2\ x + D_2\ \mathrm{sat}(u) \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $h \in \mathbb{R}^\ell$ are respectively the state, control input, measurement output and controlled output of the given system Σ. A, B, C_1 and C_2 are appropriate dimensional constant matrices, and the saturation function is defined by

$$\mathrm{sat}(u) = \begin{pmatrix} \mathrm{sat}(u_1) \\ \mathrm{sat}(u_2) \\ \vdots \\ \mathrm{sat}(u_m) \end{pmatrix}, \tag{2.2}$$

with

$$\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\ \min(|u_i|,\ \bar{u}_i), \tag{2.3}$$

where $\bar{u}_i$ is the maximum amplitude of the i-th control channel. The objective of this chapter is to design an appropriate control law for (2.1) using the CNF approach such that the resulting controlled output will track some desired step references as fast and as smooth as possible. We will address the CNF control system design for the given system (2.1) for three different situations, namely, the state feedback case, the full order measurement feedback case, and the reduced order measurement feedback case. For tracking purpose, the following assumptions on the given system are required: i) (A, B) is stabilizable; ii) (A, C_1) is detectable;

and iii) (A, B, C_2, D_2) is right invertible and has no invariant zeros at $s = 0$. Our objective here is to design control laws that are capable of achieving fast tracking of target references under input saturation. As such, it is well understood in the literature that these assumptions are standard and necessary.

2.2.1 State Feedback Case

We first proceed to develop a composite nonlinear feedback control technique for the case when all the state variables of the plant Σ are measurable, *i.e.*, $y = x$. The design will be done in three steps, which is a natural extension of the results of Chen *et al.* [6]. We have the following step-by-step design procedure.

STEP S.1: Design a linear feedback law,

$$u_L = Fx + Gr, \tag{2.4}$$

where $r \in \mathbb{R}^m$ contains a set of step references. The state feedback gain matrix $F \in \mathbb{R}^{m\times n}$ is chosen such that the closed-loop system matrix $A+BF$ is asymptotically stable and the resulting closed-loop system transfer matrix, *i.e.*, $D_2 + (C_2 + D_2F)(sI - A - BF)^{-1}B$, has certain desired properties, *e.g.*, having a small dominating damping ratio in each channel. We note that such an F can be worked out using some well-studied methods such as the LQR, H_∞ and H_2 optimization approaches (see, *e.g.*, Anderson and Moore [38], Chen [2] and Saberi *et al.* [5]). Furthermore, G is an $m \times m$ square constant matrix and is given by

$$G := G_0' \left(G_0 G_0'\right)^{-1}, \tag{2.5}$$

with $G_0 := D_2 - (C_2 + D_2 F)(A + BF)^{-1}B$. Here we note that both G_0 and G are well defined because $A+BF$ is stable, and (A, B, C_2, D_2) is right invertible and has no invariant zeros at $s = 0$, which implies $(A + BF, B, C + D_2F, D_2)$ is right invertible and has no invariant zeros at $s = 0$ (see *e.g.*, Lemma 2.5.1 of Chen [2]).

STEP S.2: Next, we compute

$$H := \Big[I - F(A + BF)^{-1}B \Big] \, G \tag{2.6}$$

and

$$x_e := G_e \, r := -(A + BF)^{-1} BG \, r. \tag{2.7}$$

Note that the definitions of H, G_e and x_e would become transparent later in our derivation. Given a positive definite matrix $W \in \mathbb{R}^{n \times n}$, solve the following Lyapunov equation:

$$(A + BF)'P + P(A + BF) = -W, \tag{2.8}$$

for $P > 0$. Such a P exists since $A + BF$ is asymptotically stable. Then, the nonlinear feedback control law u_N is given by

$$u_N = \rho(r, y) B' P (x - x_e), \tag{2.9}$$

where

$$\rho(r, y) = \mathrm{diag}\{\rho_1, \cdots, \rho_m\} = \begin{bmatrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_m \end{bmatrix}, \tag{2.10}$$

and $\rho_i = \rho_i(r, y)$, $i = 1, 2, \cdots,\ m$, are respectively some nonpositive functions, uniformly bounded and locally Lipschitz in y, which are used to change the closed-loop system damping ratios as the outputs approach the targets. The choice of these nonlinear functions will be discussed at the end of this section.

STEP S.3: The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller:

$$u = u_L + u_N = Fx + Gr + \rho(r, y)B'P(x - x_e). \tag{2.11}$$

This completes the design of the CNF controller for the state feedback case.

For further development, we partition $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{m\times n}$ and $H \in \mathbb{R}^{m\times m}$ as follows:

$$B = [\, B_1 \quad \cdots \quad B_m \,], \quad F = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ \vdots \\ H_m \end{bmatrix}. \tag{2.12}$$

The following theorem shows that the closed-loop system comprising the given plant in (2.1) and the CNF control law of (2.11) is asymptotically stable. It also determines the magnitudes of the step functions in r that can be tracked by such a control law without exceeding the control limit.

Theorem 2.1 *Consider the given system in (2.1) with $y = x$, which satisfies the assumptions i) and iii), the linear control law of (2.4) and the composite nonlinear feedback control law of (2.11). For any $\delta \in (0, 1)$, let $c_\delta > 0$ be the largest positive scalar such that for all $x \in \boldsymbol{X}_\delta$, where*

$$\boldsymbol{X}_\delta := \Big\{ x : x'Px \le c_\delta \Big\}, \tag{2.13}$$

the following property holds,

$$|\, F_i x \,| \le (1 - \delta)\bar{u}_i, \quad i = 1, \cdots, m. \tag{2.14}$$

Then, the linear control law of (2.4) is capable of driving the system controlled output $h(t)$ *to track asymptotically a set of step references, i.e.,* r*, provided that the initial state* x_0 *and* r *satisfy:*

$$\tilde{x}_0 := (x_0 - x_e) \in \boldsymbol{X}_\delta, \quad |H_i r| \le \delta \bar{u}_i, \quad i = 1, \cdots, m. \tag{2.15}$$

Furthermore, for any nonpositive function $\rho(r, y)$*, uniformly bounded and locally Lipschitz in* y*, the composite nonlinear feedback law in (2.11) is capable of driving the system controlled output* $h(t)$ *to track asymptotically the step command input of amplitude* r*, provided that the initial state* x_0 *and* r *satisfy (2.15).*

Proof. Let us first define a new state variable $\tilde{x} = x - x_e$. It is simple to verify that the linear feedback control law of (2.4) can be rewritten as

$$u_L(t) = F\tilde{x}(t) + [I - F(A + BF)^{-1}B]Gr \tag{2.16}$$

$$= F\tilde{x}(t) + Hr, \tag{2.17}$$

and hence for all $\tilde{x} \in \boldsymbol{X}_\delta$ and, provided that $|H_i r| \le \delta \bar{u}_i$, $i = 1, \cdots, m$, the closed-loop system is linear and is given by

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + Ax_e + BHr. \tag{2.18}$$

Noting that

$$\begin{aligned} Ax_e + BHr &= \Big\{B[I - F(A + BF)^{-1}B]G - A(A + BF)^{-1}BG\Big\}r \\ &= \Big\{[I - BF(A + BF)^{-1}]BG - A(A + BF)^{-1}BG\Big\}r \\ &= \Big\{I - BF(A + BF)^{-1} - A(A + BF)^{-1}\Big\}BGr \\ &= 0, \end{aligned} \tag{2.19}$$

the closed-loop system in (2.18) can then be simplified as

$$\dot{\tilde{x}} = (A + BF)\tilde{x}. \tag{2.20}$$

Similarly, the closed-loop system comprising the given plant in (2.1) and the CNF control law of (2.11) can be expressed as

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + Bw, \tag{2.21}$$

where

$$w = \text{sat}(F\tilde{x} + Hr + u_N) - F\tilde{x} - Hr. \tag{2.22}$$

Clearly, for the given x_0 satisfying (2.15), we have $\tilde{x}_0 = (x_0 - x_e) \in \boldsymbol{X}_\delta$. We note that (2.21) is reduced to (2.20) if $\rho(r, y) = 0$.

Next, we define a Lyapunov function $V = \tilde{x}' P \tilde{x}$ and evaluate the derivative of V along the trajectories of the closed-loop system in (2.21), *i.e.*,

$$\begin{aligned} \dot{V} &= \dot{\tilde{x}}' P \tilde{x} + \tilde{x}' P \dot{\tilde{x}} \\ &= \tilde{x}'(A + BF)' P \tilde{x} + \tilde{x}' P (A + BF)\tilde{x} + 2\tilde{x}' P B w \\ &= -\tilde{x}' W \tilde{x} + 2\tilde{x}' P B w. \end{aligned} \tag{2.23}$$

Note that for all

$$\tilde{x} \in \boldsymbol{X}_\delta = \{\tilde{x} : \tilde{x}' P \tilde{x} \leq c_\delta\} \quad \Rightarrow \quad |F_i \tilde{x}| \leq (1 - \delta)\bar{u}_i, \quad i = 1, \cdots, m. \tag{2.24}$$

In the remainder of this proof, we adopt similar lines of reasoning as those of Turner *et al.* [44] by considering the following different scenarios. For simplicity, we drop the dependent variables of the nonlinear function ρ in the rest of this proof.

Case 1. All input channels are unsaturated. It is obvious that we have

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\tilde{x}'PB\rho B'P\tilde{x} \leq -\tilde{x}'W\tilde{x}. \tag{2.25}$$

Case 2. All input channels are exceeding their upper limits. In this case, we have

$$F_i\tilde{x} + H_i r + \rho_i B_i' P\tilde{x} \geq \bar{u}_i, \quad i = 1, \cdots, m. \tag{2.26}$$

For all $\tilde{x} \in \boldsymbol{X}_\delta$, which implies (2.24) holds, and r satisfying (2.15), we have

$$F_i\tilde{x} + H_i r \leq \bar{u}_i, \quad i = 1, \cdots, m, \tag{2.27}$$

and thus

$$w_i = \mathrm{sat}(F_i\tilde{x} + H_i r + \rho_i B_i' P\tilde{x}) - F_i\tilde{x} - H_i r = \bar{u}_i - F_i\tilde{x} - H_i r \geq 0 \tag{2.28}$$

and

$$\rho_i B_i' P\tilde{x} \geq \bar{u}_i - (F_i\tilde{x} + H_i r) \geq 0 \quad \Rightarrow \quad B_i' P\tilde{x} = \tilde{x}'PB_i \leq 0. \tag{2.29}$$

Hence,

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\sum_{i=1}^{m} \tilde{x}'PB_i\bar{w}_i \leq -\tilde{x}'W\tilde{x}. \tag{2.30}$$

Case 3. All input channels are exceeding their lower limits. For this case, we have

$$F_i\tilde{x} + H_i r + \rho_i B_i' P\tilde{x} \leq -\bar{u}_i, \quad i = 1, \cdots, m. \tag{2.31}$$

For all $\tilde{x} \in \boldsymbol{X}_\delta$, which implies (2.24) holds, and r satisfying (2.15), we have

$$F_i\tilde{x} + H_i r \geq -\bar{u}_i, \quad i = 1, \cdots, m, \tag{2.32}$$

and thus

$$w_i = \mathrm{sat}(F_i\tilde{x} + H_i r + \rho_i B_i' P\tilde{x}) - F_i\tilde{x} - H_i r = -u_i - F_i\tilde{x} - H_i r \leq 0 \tag{2.33}$$

and

$$\rho_i B_i' P \tilde{x} \leq -\bar{u}_i - (F_i \tilde{x} + H_i r) \leq 0 \quad \Rightarrow \quad B_i' P \tilde{x} = \tilde{x}' P B_i \geq 0. \tag{2.34}$$

Hence,

$$\dot{V} = -\tilde{x}' W \tilde{x} + 2 \sum_{i=1}^{m} \tilde{x}' P B_i w_i \leq -\tilde{x}' W \tilde{x}. \tag{2.35}$$

Case 4. Some control channels are saturated and some are unsaturated. In view of Cases 1 to 3, it is simple to note that for those unsaturated channels, we have

$$\tilde{x}' P B_i w_i = \rho_i \tilde{x}' P B_i B_i' P \tilde{x} \leq 0, \tag{2.36}$$

and those input channels whose signals exceed their upper limits, we have

$$w_i \geq 0, \quad \tilde{x}' P B_i \leq 0 \quad \Rightarrow \quad \tilde{x}' P B_i w_i \leq 0, \tag{2.37}$$

and finally for those channels whose signals exceeds their lower limits,

$$w_i \leq 0, \quad \tilde{x}' P B_i \geq 0 \quad \Rightarrow \quad \tilde{x}' P B_i w_i \leq 0. \tag{2.38}$$

Thus, for this case, again we have

$$\dot{V} = -\tilde{x}' W \tilde{x} + 2 \sum_{i=1}^{m} \tilde{x}' P B_i w_i \leq -\tilde{x}' W \tilde{x}. \tag{2.39}$$

In conclusion, we have shown that

$$\dot{V} \leq -\tilde{x} W \tilde{x}, \quad \tilde{x} \in \boldsymbol{X}_\delta, \tag{2.40}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (2.21). Noting that $W > 0$, all trajectories of (2.21) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial state x_0 and the step command input r that satisfy (2.15), we have

$$\lim_{t\to\infty} x(t) = x_e, \tag{2.41}$$

which implies

$$\lim_{t\to\infty} u(t) = F \lim_{t\to\infty} x(t) + Gr + \lim_{t\to\infty} \rho B' P[x(t) - x_e] = F x_e + Gr, \tag{2.42}$$

since $\rho(r, y)$ is uniformly bounded. Hence,

$$\begin{aligned}
\lim_{t\to\infty} h(t) &= C_2 \lim_{t\to\infty} x(t) + D_2 \lim_{t\to\infty} u(t) \\
&= C_2 x_e + D_2(F x_e + Gr) \\
&= (C_2 + D_2 F) x_e + D_2 G r \\
&= -(C_2 + D_2 F)(A + BF)^{-1} B G r + D_2 G r \\
&= [D_2 - (C_2 + D_2 F)(A + BF)^{-1} B] G r \\
&= G_0 G_0' (G_0 G_0')^{-1} r = r.
\end{aligned} \tag{2.43}$$

This completes the proof of Theorem 2.1. ◊

Lastly, assuming that the dynamic equation of the given system is transformed into the following form,

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} \mathrm{sat}(u), \tag{2.44}$$

where $\bar{B}$ is nonsingular, Turner *et al.* [44] have solved the problem under a rather strange condition, *i.e.*, A_{11} is nonsingular. It was suggested in [44] to add some small perturbations to A_{11} if it is singular. Recently, it has been pointed by Turner and

Postlethwaite [43] for the case when the system is stabilizable and B is of full rank, there exists nonsingular state transformation that would convert the given system with the form of (2.44) with A_{11} being nonsingular. Nonetheless, it is obvious from our development that such a transformation is totally unnecessary. We further note that our approach to the CNF design is much more elegant compared to that given in [44], and it carries over nicely to the measurement feedback cases in the following subsections.

2.2.2 Full Order Measurement Feedback Case

The assumption that all the state variables of the given system Σ are measurable is, in general, not practical. For example, in HDD servo systems (see Chen *et al.* [1]), the velocity of the actuator is usually hard to be measured. As such, in this subsection and the next subsection, we proceed to develop CNF design using only measurement information. We first deal with the full order measurement feedback case, in which the dynamical order of the controller is exactly the same as that of the given plant. The following is a step-by-step procedure for the CNF design using full order measurement feedback.

STEP F.1: We first construct a linear full order measurement feedback control law,

$$\begin{cases} \dot{x}_v = (A + KC_1)x_v - Ky + B\ \mathrm{sat}(u_L) \\ u_L = F(x_v - x_e) + Hr, \end{cases} \tag{2.45}$$

where r is the set of step reference signals and x_v is the state of the controller. As usual, K, F are gain matrices and are chosen such that $(A + KC_1)$ and

$(A+BF)$ are asymptotically stable and the resulting closed loop system has desired properties. Finally, G, H and x_e are as defined in (2.5)–(2.7).

STEP F.2: Given a positive definite matrix $W_P \in \mathbb{R}^{n\times n}$, solve the Lyapunov equation

$$(A+BF)'P + P(A+BF) = -W_P, \tag{2.46}$$

for $P > 0$. As in the state feedback case, the linear control law of (2.45) obtained in the above step is to be combined with a nonlinear control law to form the following CNF controller:

$$\begin{cases} \dot{x}_v = (A+KC_1)x_v - Ky + B\,\mathrm{sat}(u) \\ u = F(x_v - x_e) + Hr + \rho(r,y)B'P(x_v - x_e), \end{cases} \tag{2.47}$$

where $\rho(r,y)$ is as given in (2.10) with all its diagonal elements being respectively a nonpositive function, locally Lipschitz in y, which are to be chosen to improve the performance of the closed-loop system.

It turns out that, for the measurement feedback case, the choice of $\rho_i(r,y)$, $i = 1,\dots m$, the nonpositive scalar functions, are not totally free. They are subject to certain constraints. We have the following result.

Theorem 2.2 *Consider the given system in (2.1), which satisfies the standard assumptions i) to iii), the full order linear measurement feedback control law of (2.45) and the composite nonlinear measurement feedback control law of (2.47). Given a positive define matrix* $W_Q \in \mathbb{R}^{n\times n}$ *with*

$$W_Q > F'B'PW_P^{-1}PBF, \tag{2.48}$$

Let $Q > 0$ be the solution to the Lyapunov equation,

$$(A + KC_1)'Q + Q(A + KC_1) = -W_Q. \tag{2.49}$$

Note that such a Q exists as $A + KC_1$ is asymptotically stable. For any $\delta \in (0, 1)$, let $c_\delta > 0$ be the largest positive scalar such that for all $x \in \boldsymbol{X}_{\mathrm{F}\delta}$, where

$$\boldsymbol{X}_{\mathrm{F}\delta} := \left\{ \begin{pmatrix} x \\ x_v \end{pmatrix} : \begin{pmatrix} x \\ x_v \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} x \\ x_v \end{pmatrix} \le c_\delta \right\}, \tag{2.50}$$

the following property holds

$$\left| [\, F_i \quad F_i \,] \begin{pmatrix} x \\ x_v \end{pmatrix} \right| \le (1 - \delta)\bar{u}_i, \quad i = 1, \cdots, m. \tag{2.51}$$

Then, the linear measurement feedback control law in (2.47) will drive the system's controlled output $h(t)$ to track asymptotically a set of step references, i.e., r, from an initial state x_0, provided that x_0, $x_{v0} = x_v(0)$ and r satisfy:

$$\begin{pmatrix} x_0 - x_e \\ x_{v0} - x_0 \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta} \quad \text{and} \quad |H_i\, r| \le \delta \bar{u}_i, \quad i = 1, \cdots, m. \tag{2.52}$$

Furthermore, there exist positive scalars $\rho_i^* > 0$, $i = 1, \ldots m$, such that for any nonpositive functions $\rho_i(r, y)$, $i = 1, \cdots, m$, locally Lipschitz in y and $|\rho_i(r, y)| \le \rho_i^*$, $i = 1, \cdots, m$, the CNF control law of (2.47) will drive the system controlled output $h(t)$ to track asymptotically the reference r from an initial x_0, provided that x_0, x_{v0} and r satisfy (2.52).

Proof. For simplicity, we again drop r and y in $\rho(r, y)$ throughout the proof of this theorem. Let $\tilde{x} = x - x_e$ and $\tilde{x}_v = x_v - x$. The linear feedback control law of (2.45) can be written as

$$\dot{\tilde{x}}_v = (A + KC_1)\tilde{x}_v, \quad u_L = [\, F \quad F \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr. \tag{2.53}$$

Hence, for all

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta} \quad \Rightarrow \quad \left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right| \le (1-\delta)\bar{u}_i, \;\; i = 1, \cdots, m, \tag{2.54}$$

and for any r satisfying

$$|H_i\, r| \le \delta \bar{u}_i, \;\; i = 1, \cdots, m, \tag{2.55}$$

each channel of u_L, say $u_{L,i}$, has the following property

$$u_{L,i} = \left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + H_i\, r \right| \le \left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right| + |H_i\, r| \le \bar{u}_i. \tag{2.56}$$

Thus, for all $\tilde{x}$ and $\tilde{x}_v$ satisfying the condition as given in (2.54), the closed-loop system comprising the given plant and the linear control law of (2.45) can be rewritten as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_v \end{pmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}. \tag{2.57}$$

Similarly, the closed-loop system with the CNF control law of (2.47) can be expressed as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_v \end{pmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w, \tag{2.58}$$

where

$$w = \mathrm{sat}\left[[\, F \quad F \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr + \rho\, [\, B'P \quad B'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right] - [\, F \quad F \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} - Hr. \tag{2.59}$$

Clearly, for x_0 and x_{v0} satisfying (2.52), we have

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_{v0} \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta}, \tag{2.60}$$

where $\tilde{x}_0 = \tilde{x}(0)$ and $\tilde{x}_{v0} = \tilde{x}_v(0)$. We note that (2.57) and (2.58) are identical when $\rho = 0$. Again, the results of Theorem 2.2 for both the linear and the nonlinear feedback case can be proved in one shot.

Next, we define a Lyapunov function:

$$V = \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}, \tag{2.61}$$

and evaluate the derivative of V along the trajectories of the closed-loop system in (2.58), *i.e.*,

$$\dot{V} = \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}' \begin{bmatrix} -W_P & PBF \\ F'B'P & -W_Q \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + 2\tilde{x}'PBw. \tag{2.62}$$

Note that for all

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta} \quad \Rightarrow \quad \left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right| \leq (1-\delta)\bar{u}_i, \;\; i = 1, \cdots, m. \tag{2.63}$$

Again, as done in the full state feedback case, let us find the above derivative of V for four different cases.

Case 1. All input channels are unsaturated. For this case, we have

$$\left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + H_i r + \rho_i \, [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right| \leq \bar{u}_i, \;\; i = 1, \cdots, m, \tag{2.64}$$

which implies

$$w_i = \rho_i \, [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}, \tag{2.65}$$

and

$$\begin{aligned} \dot{V} &= \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}' \begin{bmatrix} -W_P & PB(F+\rho B'P) \\ (F+\rho B'P)'B'P & -W_Q \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + 2\tilde{x}'PB\rho B'P\tilde{x} \\ &\leq \begin{pmatrix} \hat{x} \\ \tilde{x}_v \end{pmatrix}' \begin{bmatrix} -W_P & 0 \\ 0 & -\tilde{W}_Q \end{bmatrix} \begin{pmatrix} \hat{x} \\ \tilde{x}_v \end{pmatrix}, \end{aligned} \tag{2.66}$$

where

$$\hat{x} = \tilde{x} - W_P^{-1}PB(F+\rho B'P)\tilde{x}_v \tag{2.67}$$

and

$$\tilde{W}_Q = W_Q - (F+\rho B'P)'B'PW_P^{-1}PB(F+\rho B'P). \tag{2.68}$$

Noting (2.48), *i.e.*, $W_Q > F'B'PW_P^{-1}PBF$, and ρ_i is locally Lipschitz, it is clear that there exist positive scalars $\rho_{i,1}^* > 0$, $i = 1, \cdots, m$, such that for any scalar function satisfying $|\rho_i| \le \rho_{i,1}^*$, $i = 1, \cdots, m$, we have $\tilde{W}_Q > 0$ and hence $\dot{V} \le 0$.

Case 2. All input channels are exceeding their upper limits. In such a situation, we have for all $i = 1, \cdots, m$,

$$[\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + H_i r + \rho_i [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \ge \bar{u}_i. \tag{2.69}$$

For all the trajectories inside $\boldsymbol{X}_{\mathrm{F}\delta}$,

$$\left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + H_i r \right| \le \bar{u}_i, \tag{2.70}$$

we have for $i = 1, \cdots, m$,

$$0 \le w_i \le \rho_i [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}. \tag{2.71}$$

Next, let us express

$$w_i = q_i \rho_i [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}, \tag{2.72}$$

for some appropriate positive continuous function matrix $q_i(t)$ bounded by 1 for all t. In this case, the derivative of V becomes

$$\begin{aligned} \dot{V} &= \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}' \begin{bmatrix} -W_P & PB(F + q\rho B'P) \\ (F + q\rho B'P)'B'P & -W_Q \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + 2\tilde{x}'PBq\rho B'P\tilde{x} \\ &\le \begin{pmatrix} \hat{x}_+ \\ \tilde{x}_v \end{pmatrix}' \begin{bmatrix} -W_P & 0 \\ 0 & -\tilde{W}_{Q+} \end{bmatrix} \begin{pmatrix} \hat{x}_+ \\ \tilde{x}_v \end{pmatrix}, \end{aligned} \tag{2.73}$$

where

$$q = \operatorname{diag}\Big\{ q_1, \cdots, q_m \Big\}, \tag{2.74}$$

$$\hat{x}_+ = \tilde{x} - W_P^{-1} PB(F + q\rho B'P)\tilde{x}_v \tag{2.75}$$

and

$$\tilde{W}_{Q_+} = W_Q - (F + q\rho B'P)'B'PW_P^{-1}PB(F + q\rho B'P). \tag{2.76}$$

Again, noting (2.48), *i.e.*, $W_Q > F'B'PW_P^{-1}PBF$, and ρ_i is locally Lipschitz, it is clear that there exist positive scalars $\rho_{i,2}^* > 0$, $i = 1, \cdots, m$, such that for any scalar function satisfying $|\rho_i| \le \rho_{i,2}^*$, $i = 1, \cdots, m$, we have $\tilde{W}_{Q_+} > 0$ and hence $\dot{V} \le 0$.

Case 3. All input channels are exceeding their lower limits. In this case, we have for $i = 1, \cdots, m$,

$$[\,F_i \quad F_i\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + H_i\,r + \rho_i\,[\,B_i'P \quad B_i'P\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} \le -\bar{u}_i. \tag{2.77}$$

For all the trajectories inside $\boldsymbol{X}_{\mathrm{F}\delta}$,

$$\left|[\,F_i \quad F_i\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + H_i r\right| \le \bar{u}_i, \tag{2.78}$$

we have for $i = 1, \cdots, m$,

$$\rho_i\,[\,B_i'P \quad B_i'P\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} \le w_i \le 0. \tag{2.79}$$

Next, let us express

$$w_i = q_i\rho_i\,[\,B_i'P \quad B_i'P\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix}, \tag{2.80}$$

for some appropriate positive continuous function matrix $q_i(t)$ bounded by 1 for all t. Following the similar arguments as in the previous case, we can show that there exist positive scalars $\rho_{i,3}^* > 0$, $i = 1, \cdots, m$, such that for any scalar function satisfying $|\rho_i| \le \rho_{i,3}^*$, $i = 1, \cdots, m$, the corresponding $\dot{V} \le 0$.

Case 4. Some control channels are saturated and some are unsaturated. Following the similar arguments as those in Cases 1 to 3, we can express that for $i = 1, \cdots, m$,

$$w_i = q_i \rho_i \,[\, B_i' P \quad B_i' P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}, \tag{2.81}$$

for some appropriate positive continuous function matrix $q_i(t)$ bounded by 1 for all t, and show that there exist positive scalars $\rho_{i,4}^* > 0$, $i = 1, \cdots, m$, such that for any scalar function satisfying $|\rho_i| \le \rho_{i,4}^*$, $i = 1, \cdots, m$, the corresponding $\dot{V} \le 0$.

Finally, we let $\rho_i^* = \min\{\rho_{i,1}^*, \rho_{i,2}^*, \rho_{i,3}^*, \rho_{i,4}^*\}$. Then, we have for any scalar functions ρ_i satisfying $|\rho_i| < \rho_i^*$, $i = 1, \cdots, m$,

$$\dot{V} \le 0, \quad \forall \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta}. \tag{2.82}$$

Thus, $\boldsymbol{X}_{\mathrm{F}\delta}$ is an invariant set of the closed-loop system in (2.58), and all trajectories starting from $\boldsymbol{X}_{\mathrm{F}\delta}$ will remain inside and asymptotically converge to the origin. This, in turn, indicates that, for the initial state of the given system x_0, the initial state of the controller x_{v0}, and step command input r that satisfy (2.52),

$$\lim_{t\to\infty} \tilde{x}_v(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} x(t) = x_e, \tag{2.83}$$

and then it follows from (2.43) that the controlled output $h(t)$ converges asymptotically to the target reference r. This completes the proof of Theorem 2.2. $\diamondsuit$

2.2.3 Reduced Order Measurement Feedback Case

For the given system in (2.1), it is clear that there are p state variables of the system, which are measurable if C_1 is of maximal rank. Thus, in general, it is not necessary to estimate these measurable state variables in measurement feedback laws. As such, we will proceed in this subsection to design a dynamic controller that has a dynamical order less than that of the given plant. For simplicity of presentation, we assume that C_1 is already in the form

$$C_1 = [\, I_p \quad 0 \,]. \tag{2.84}$$

Then, the system in (2.1) can be rewritten as

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathrm{sat}(u) \\ \quad y \quad = [\ I_p \quad 0\ ] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \quad h \quad = \quad C_2 \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \end{cases} \tag{2.85}$$

where the original state x is partitioned into two parts, x_1 and x_2 with $y \equiv x_1$. Thus, we will only need to estimate x_2 in the reduced order measurement feedback design. Next, we let F be chosen such that i) $A + BF$ is asymptotically stable, and ii) $(C_2 + D_2F)(sI - A - BF)^{-1}B + D_2$ has desired properties, and let K_{R} be chosen such that $A_{22} + K_{\mathrm{R}}A_{12}$ is asymptotically stable. Here we note that it can be shown that (A_{22}, A_{12}) is detectable if and only if (A, C_1) is detectable. Thus, there exists a stabilizing K_{R}. Again, such F and K_{R} can be designed using an appropriate control technique. We then partition F in conformity with x_1 and x_2:

$$F = [\, F_1 \quad F_2 \,]. \tag{2.86}$$

We further partition F_2 as follows:

$$F_2 = \begin{bmatrix} F_{2,1} \\ \vdots \\ F_{2,m} \end{bmatrix}. \tag{2.87}$$

Also, let G, H and x_e be as given in (2.5)–(2.7). The reduced order CNF controller is given by

$$\dot{x}_v = (A_{22}+K_{\mathrm{R}}A_{12})x_v + (B_2+K_{\mathrm{R}}B_1)\,\mathrm{sat}(u) + \Big[A_{21}+K_{\mathrm{R}}A_{11}-(A_{22}+K_{\mathrm{R}}A_{12})K_{\mathrm{R}}\Big]y \quad (2.88)$$

and

$$u = F\left[\begin{pmatrix} y \\ x_v - K_{\mathrm{R}}y \end{pmatrix} - x_e\right] + Hr + \rho(r,y)B'P\left[\begin{pmatrix} y \\ x_v - K_{\mathrm{R}}y \end{pmatrix} - x_e\right], \quad (2.89)$$

where $\rho(r, y)$ is as given in (2.10).

Next, given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, let $P > 0$ be the solution to the Lyapunov equation

$$(A + BF)'P + P(A + BF) = -W_P. \quad (2.90)$$

Given another positive definite matrix $W_R \in \mathbb{R}^{(n-p)\times(n-p)}$ with

$$W_{\mathrm{R}} > F_2'B'PW_P^{-1}PBF_2, \quad (2.91)$$

let $Q_{\mathrm{R}} > 0$ be the solution to the Lyapunov equation

$$(A_{22} + K_{\mathrm{R}}A_{12})'Q_{\mathrm{R}} + Q_{\mathrm{R}}(A_{22} + K_{\mathrm{R}}A_{12}) = -W_{\mathrm{R}}. \quad (2.92)$$

Note that such P and Q_{R} exist as $A + BF$ and $A_{22} + K_{\mathrm{R}}A_{12}$ are asymptotically stable. For any $\delta \in (0, 1)$, let c_δ be the largest positive scalar such that for all

$$\begin{pmatrix} x \\ x_v \end{pmatrix} \in \boldsymbol{X}_{\mathrm{R}\delta} := \left\{\begin{pmatrix} x \\ x_v \end{pmatrix} : \begin{pmatrix} x \\ x_v \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q_{\mathrm{R}} \end{bmatrix} \begin{pmatrix} x \\ x_v \end{pmatrix} \le c_\delta \right\} \quad (2.93)$$

the following property holds:

$$\left| [\, F_i \quad F_{2,i} \,] \begin{pmatrix} x \\ x_v \end{pmatrix} \right| \le \bar{u}_i(1 - \delta), \quad i = 1, \cdots, m. \quad (2.94)$$

We have the following theorem.

Theorem 2.3 *Consider the given system in (2.1), which satisfies the usual assumptions i) to iii). Then, there exist positive scalars* $\rho_i^* > 0$, $i = 1, \cdots, m$, *such that for any nonpositive function* $\rho_i(r, y)$, $i = 1, \cdots, m$, *locally Lipschitz in* y_i *and* $|\rho_i(r, y)| \le \rho_i^*$, *the reduced order CNF law given by (2.88) and (2.89) will drive the system controlled output* $h(t)$ *to asymptotically track the reference* r *from an initial state* x_0, *provided that* x_0, x_{v0} *and* r *satisfy*

$$\begin{pmatrix} x_0 - x_e \\ x_{v0} - x_{20} - K_{\mathrm{R}} x_{10} \end{pmatrix} \in \boldsymbol{X}_{\mathrm{R}\delta}, \quad |H_i r| \le \delta \bar{u}_i, \quad i = 1, \cdots, m. \tag{2.95}$$

Proof. Let $\tilde{x} = x - x_e$ and $\tilde{x}_v = x_v - x_2 - K_{\mathrm{R}} x_1$. Then, the closed-loop system comprising the given plant in (2.1) and the reduced order CNF control law of (2.88) and (2.89) can be expressed as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_v \end{pmatrix} = \begin{bmatrix} A + BF & BF_2 \\ 0 & A_{22} + K_{\mathrm{R}} A_{12} \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \tag{2.96}$$

where

$$w = \mathrm{sat}\left\{ [\, F \quad F_2 \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr + \rho(r, y) B' P \left[\tilde{x} + \begin{pmatrix} 0 \\ \tilde{x}_v \end{pmatrix} \right] \right\} - [\, F \quad F_2 \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} - Hr. \tag{2.97}$$

The rest of the proof follows along similar lines to the reasoning given in the full order measurement feedback case.

2.2.4 Selecting the Nonlinear Gain $\rho(r, y)$

The freedom to choose the function $\rho(r, y)$ is used to tune the control laws so as to improve the performance of the closed-loop system as the controlled output h

approaches the set point. Since the main purpose of adding the nonlinear part to the CNF controllers is to speed up the settling time, or equivalently to contribute a significant value to the control input when the tracking error, $r - h$, is small. The nonlinear part, in general, will be in action when the control signal is far away from its saturation level, and thus it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising the given plant in (2.1) and the three different types of control law can be expressed as

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + \rho(r, y)BB'P\tilde{x}. \tag{2.98}$$

We note that the additional term $\rho(r, y)$ does not affect the stability of the estimators. It is now clear that eigenvalues of the closed-loop system in (2.98) can be changed by the function $\rho(r, y)$. There are different types of nonlinear gains that have been suggested in the literature (see *e.g.*, [6, 8, 44]). Assuming that h is available, we follow the work of [6] to propose the following nonlinear gains,

$$\rho_i(r_i, h_i) = -\beta_i \left| e^{-\alpha_i |h_i(t) - r_i|} - e^{-\alpha_i |h_i(0) - r_i|} \right|, \quad i = 1, \cdots, m, \tag{2.99}$$

which starts from 0 and gradually increases to a final gain of $-\beta_i \left| 1 - e^{-\alpha_i |h_i(0) - r_i|} \right|$ as h_i approaches to the target reference r_i. α_i is used to determine the speed of change in ρ_i. Thus, one could properly select scalar gains α_i and β_i, $i = 1, \cdots, m$, to yield a desired performance. We further note that for the case when (A, B, C_2, D_2) is a SISO system, Chen *et al.* [6] have recently shown a nice interconnection on the mechanism of the nonlinear gain ρ with the classical root-locus theory. They have also shown that W can actually be connected to the zero placement for an auxiliary system. Unfortunately, these nice properties generally do not carry over to the MIMO systems. As a rule of thumb, we might follow the idea of the so-called sequential loop closing method in multivariable control system design (see *e.g.*, Stephanopoulos [41]) to select nonlinear gain β_i for each individual input channel.

2.3 An MIMO System Design

To illustrate the concept of the CNF control, we present in this section a roll-yaw autopilot system for the Extended Medium Range Air-to-Air Technology (EMRAAT) airframe. We will compare the performance of the CNF design with a corresponding LQR design. The airframe is a generic, non-axisymmetrical airframe and as such, lends itself to highly g coordinated bank-to-turn maneuvers. The linearized roll-yaw state space model for the EMRAAT airframe for the flight conditions of Mach = 2.5, Velocity = 2420 ft/sec, Dynamic Pressure = 1720 lbs/ft^2, and Angle of Attack = 10°, is given by

$$\dot{x} = \begin{bmatrix} -0.501 & -0.985 & 0.174 & 0 & 0.109 & 0.007 \\ 16.83 & -0.575 & 0.0123 & 0 & -132.8 & 27.19 \\ -3227 & 0.321 & -2.10 & 0 & -1620 & -1240 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -179 & 0 \\ 0 & 0 & 0 & 0 & 0 & -179 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 179 & 0 \\ 0 & 179 \end{bmatrix} \mathrm{sat}(u), \tag{2.100}$$

where

$$x = \begin{pmatrix} \beta \\ \alpha \\ p \\ \int p \\ \delta_r \\ \delta_a \end{pmatrix}, \quad u = \begin{pmatrix} \delta_{rc} \\ \delta_{ac} \end{pmatrix}. \tag{2.101}$$

and where β is sideslip, α is yaw rate, p is roll rate, $\int p$ is roll angle, δ_r is rudder position, δ_a is aileron position, and δ_{rc} and δ_{ac} are respectively the controls applied to the rudder and aileron. The measurement of the system is given by

$$y = \begin{pmatrix} \beta \\ \alpha \\ p \\ \int p \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x. \tag{2.102}$$

This air-to-air missile system is taken from the work of Wilson *et al.* [45], in which the authors had designed an autopilot system based on a Lyapunov-constrained eigenstructure assignment approach. We note that in [45], they did not consider any input saturation in their formulation. The same system was adopted by Turner *et al.* [44] for illustration of their work, although they had added a small perturbation in the $(4,4)$ entry in the system matrix A into order to make A_{11} nonsingular. However, in [44], the authors had assumed that all the state variables of the system are measurable and assumed that both input channels are bounded by $\pm 20°$. The controlled output of the system is defined as the the sideslip and the the roll angle, *i.e.*,

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \beta \\ \int p \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u. \tag{2.103}$$

To demonstrate our results, we choose a command reference:

$$r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 80 \end{pmatrix}. \tag{2.104}$$

Our aim is to design appropriate CNF controllers with full state feedback, full order measurement feedback and reduced order measurement feedback, which would control the controlled output of the system to track the command reference as fast as possible and as smooth as possible. Following the procedures given in the previous section and with appropriate selections of design parameters, we have obtained the following CNF control laws. We note that the linear parts of the control laws are carried out using the standard LQR design.

1. CNF controller using full state feedback:

$$u = Fx + Gr + \rho(r, y)F_n(x - x_e), \tag{2.105}$$

where

$$F = \begin{bmatrix} -2.573875 & 0.124261 & 0.037199 & 1.891459 & -0.351318 & -0.186503 \\ -0.039226 & -0.131115 & 0.037657 & 1.192637 & -0.186503 & -0.235628 \end{bmatrix},$$

$$G = \begin{bmatrix} 1.675090 & -1.891459 \\ -2.656604 & -1.192637 \end{bmatrix},$$

$$F_n = \begin{bmatrix} 2.573875 & -0.124261 & -0.037199 & -1.891459 & 0.351318 & 0.186503 \\ 0.039226 & 0.131115 & -0.037657 & -1.192637 & 0.186503 & 0.235628 \end{bmatrix},$$

$$x_e = [\,8 \quad 3.66 \quad 0 \quad 80 \quad -2.5755 \quad -19.4536\,]'$$

and

$$\rho(r, y) = \text{diag}\Big\{\rho_1(r_1, h_1),\ \rho_2(r_2, h_2)\Big\},$$

and where

$$\rho_1(r_1, h_1) = -200\left|e^{-0.005|h_1(t)-r_1|} - e^{-0.005|h_1(0)-r_1|}\right|,$$

and

$$\rho_2(r_2, h_2) = -200\left|e^{-0.025|h_2(t)-r_2|} - e^{-0.025|h_2(0)-r_2|}\right|.$$

2. CNF controller using full order measurement feedback:

$$\begin{cases} \dot{x}_v = (A + KC_1)x_v - Ky + B\ \text{sat}(u) \\ u\ = F(x_v - x_e) + Hr + \rho(r, y)F_n(x_v - x_e), \end{cases} \tag{2.106}$$

where F, F_n, x_e, $\rho(r,y)$ are as given in the state feedback case, and

$$K = \begin{bmatrix} -29.6237 & 0.7142 & -0.1485 & 0 \\ -46.2737 & 119.3702 & -0.6416 & 0 \\ 3495.4107 & 18.1069 & 105.4275 & 0 \\ 0 & 0 & -1 & -60 \\ -20.7195 & 131.5970 & 2.0269 & 0 \\ 56.8973 & -169.5411 & 13.2893 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} -0.321939 & 0 \\ -2.181703 & 0 \end{bmatrix}.$$

3. CNF controller using reduced order measurement feedback:

$$\dot{x}_v = A_{\rm cmp} x_v + K_{\rm cmp} y + B_{\rm cmp}\ {\rm sat}(u) \tag{2.107}$$

and

$$u = F\left[\begin{pmatrix} y \\ x_v - K_{\rm R} y \end{pmatrix} - x_e\right] + Hr + \rho(r,y)F_n\left[\begin{pmatrix} y \\ x_v - K_{\rm R} y \end{pmatrix} - x_e\right], \tag{2.108}$$

where

$$A_{\rm cmp} = \begin{bmatrix} -15 & 0 \\ 0 & -20 \end{bmatrix}, \quad K_{\rm cmp} = \begin{bmatrix} 52.553834 & -14.061997 & -0.287191 & 0 \\ 347.215285 & 23.940526 & -1.796177 & 0 \end{bmatrix},$$

$$B_{\rm cmp} = \begin{bmatrix} 179 & 0 \\ 0 & 179 \end{bmatrix}, \quad K_{\rm R} = \begin{bmatrix} 0.000578 & -0.974320 & -0.021364 & 0 \\ -0.000730 & 1.234094 & -0.101165 & 0 \end{bmatrix},$$

and F, H, x_e, $\rho(r,y)$ and F_n are the same as those given in the previous two cases.

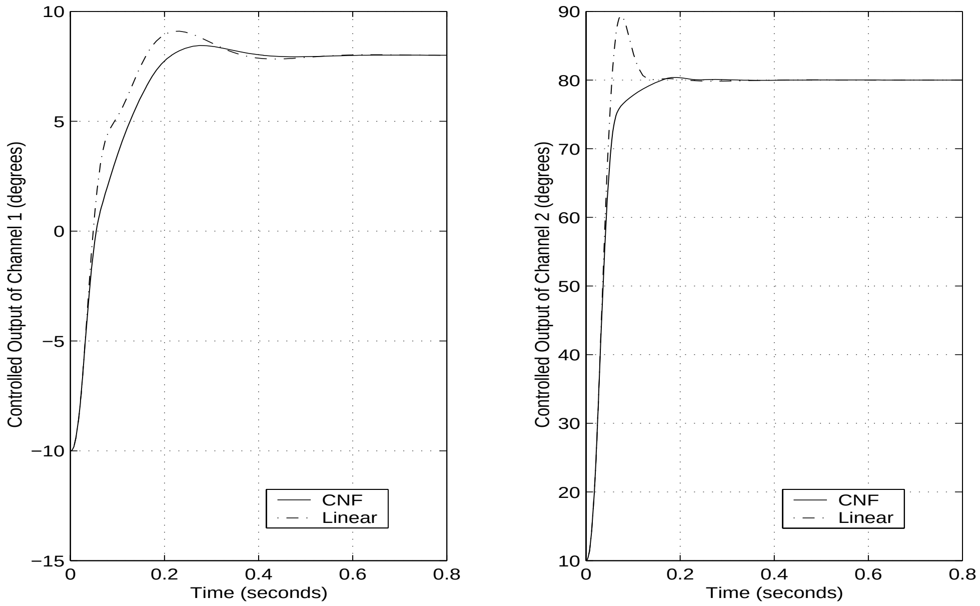
Using SIMULINK in MATLAB, we obtain a set of simulation results in Figures 2.1–2.3, which are done under the following initial condition,

$$x_0 = [\,-10 \quad 0 \quad 0 \quad 10 \quad 0 \quad 0\,]', \tag{2.109}$$

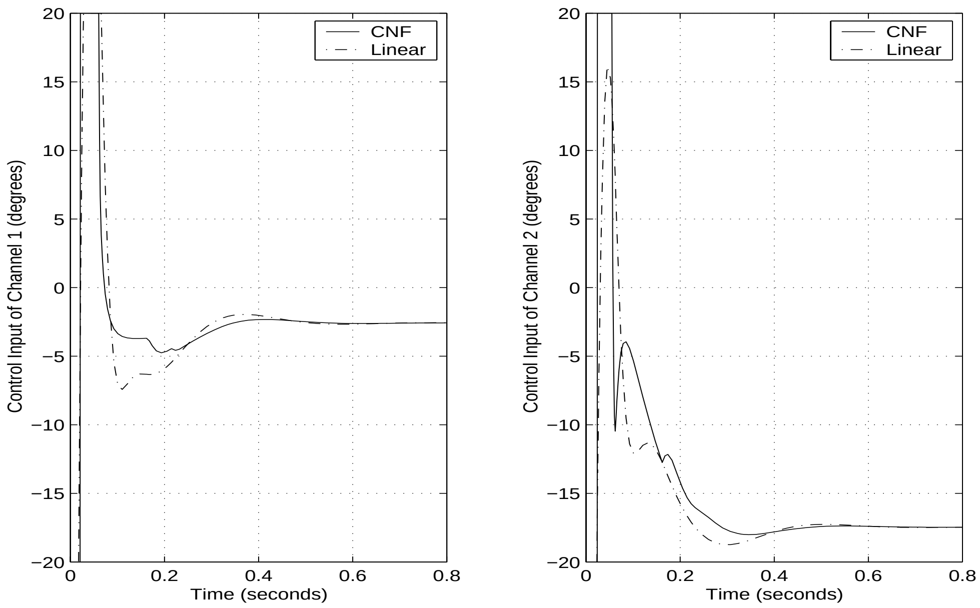
together with initial conditions for both full and reduced order controllers being set to zero. The results clearly show that the control laws with the nonlinear components, *i.e.*, the CNF controllers, outperform their linear counterparts a great deal. It is interesting to note that the results for the CNF state feedback case and the CNF reduced order measurement feedback case are almost identical, and have very minimal overshoots in their controlled output responses. The controlled output responses in the CNF full order measurement feedback case are, however, having a small overshoot in the second channel.

2.4 Conclusions

We have proposed a nonlinear tracking control technique, *i.e.*, the so-called composite nonlinear feedback (CNF) control design, which consists of two parts, a linear component and a nonlinear component. The former is usually chosen to give fast rising time while the latter is added to smooth out the transient peaks or overshoots when the controlled output is approaching the target reference. The technique is applicable to general multivariable system with some standard assumptions and a natural extension of some recent work in the field. It is successfully demonstrated by a practical example on an air-to-air missile system. Finally, we note that unlike the SISO case, the mechanism of the nonlinear gains in the CNF design for MIMO systems is still not clearly captured. It requires more investigations and research.

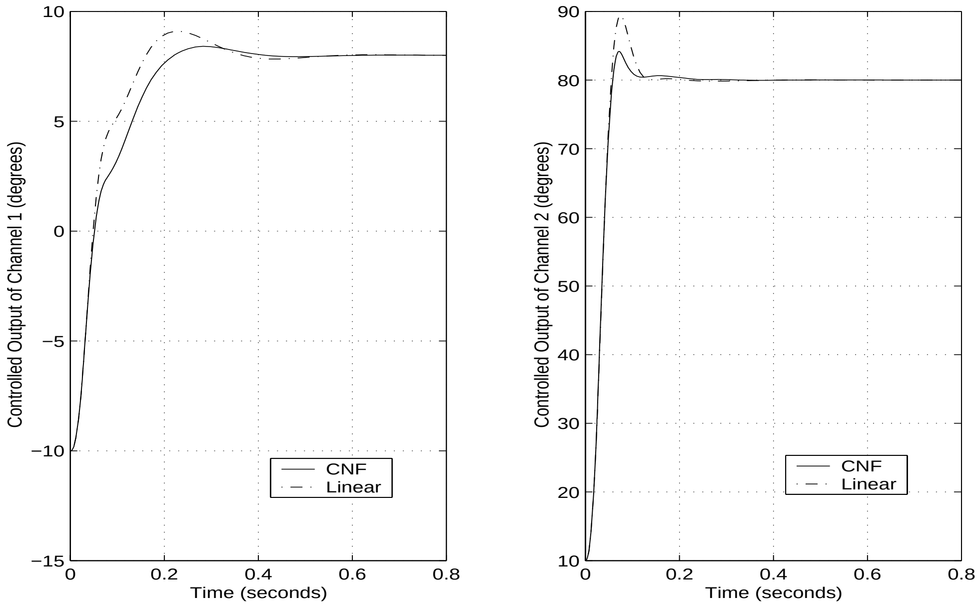


(a) Controlled output

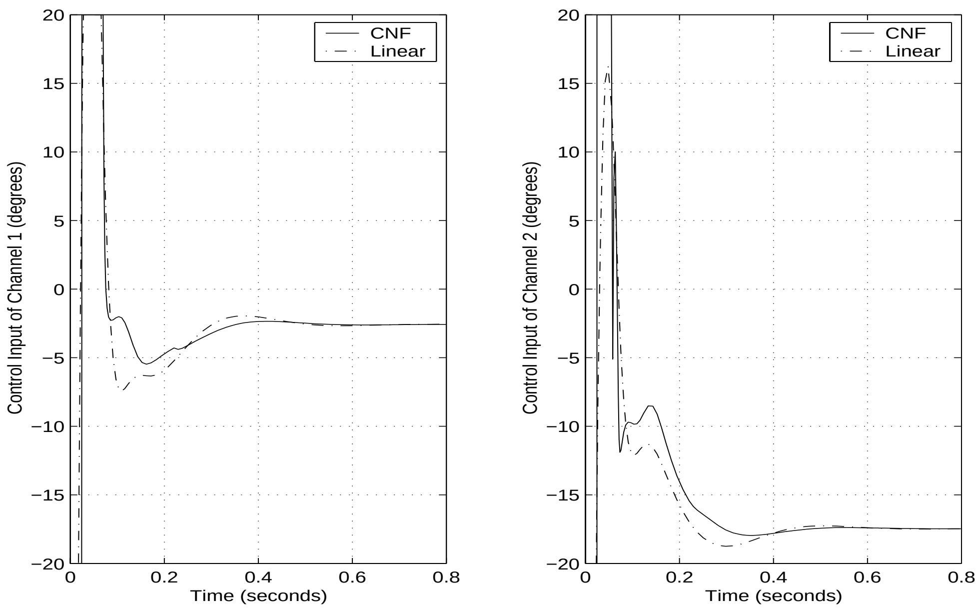


(b) Control input

Figure 2.1: Input and output responses under state feedback.

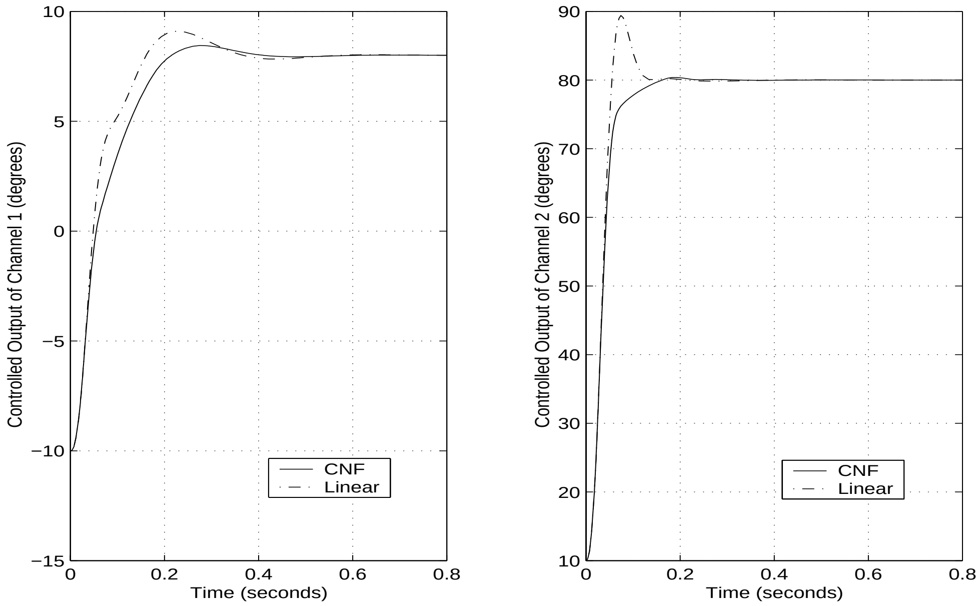


(a) Controlled output

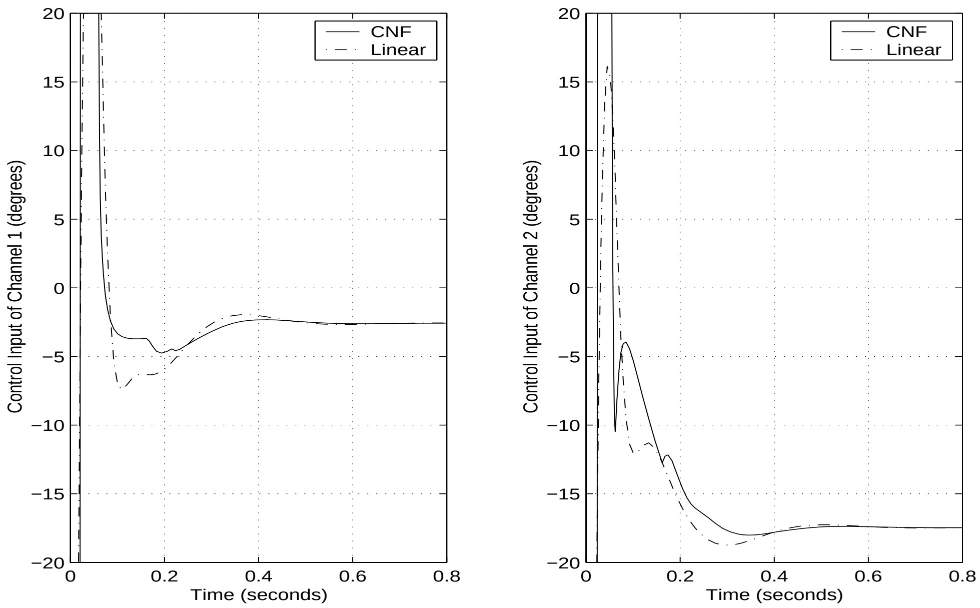


(b) Control input

Figure 2.2: Input and output responses under full order measurement feedback.



(a) Controlled output



(b) Control input

Figure 2.3: Input and output responses under reduced order measurement feedback.

Chapter 3

Discrete-Time CNF Control for Linear MIMO Systems with Input Saturation

3.1 Introduction

In the previous chapter, we have presented the continuous time composite nonlinear feedback controller design, and the illustrative example applied to the air to air missile system is proved to be very successful. We want to explore deeper the applicability of the theory on the discrete time controller design. Much of the scheme in this chapter will follow last chapter. We will try to use our discrete time composite nonlinear feedback control to design a controller for a dual stage hard disk drive (HDD) system in Chapter 4.

3.2 Discrete-Time Composite Nonlinear Feedback Control for MIMO Systems

We present in this section the CNF controller design for the following multivariable linear system Σ with an amplitude-constrained actuator characterized by

$$\begin{cases} x(k+1) &= Ax(k) + B\,\mathrm{sat}(u(k)), \quad x(0)=x_0 \\ y(k) &= C_1 x(k) \\ h(k) &= C_2 x(k) + D_2\,\mathrm{sat}(u(k)) \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $h \in \mathbb{R}^\ell$ are respectively the state, control input, measurement output and controlled output of the given system Σ. A, B, C_1 and C_2 are appropriate dimensional constant matrices, and the saturation function is defined by

$$\mathrm{sat}(u) = \begin{pmatrix} \mathrm{sat}(u_1) \\ \mathrm{sat}(u_2) \\ \vdots \\ \mathrm{sat}(u_m) \end{pmatrix}, \tag{3.2}$$

with

$$\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\,\min(|u_i|,\ \bar{u}_i), \tag{3.3}$$

where $\bar{u}_i$ is the maximum amplitude of the i-th control channel. The objective of this chapter is to design an appropriate control law for (3.1) using the CNF approach such that the resulting controlled output will track some desired step references as fast and as smooth as possible. We will address the CNF control system design for the given system (3.1) for three different situations, namely, the state feedback case, the full order measurement feedback case, and the reduced order measurement feedback case. For tracking purpose, the following assumptions on the given system are required:

1. (A, B) is stabilizable.

2. (A, C_1) is detectable.

3. (A, B, C_2, D_2) is right invertible and has no invariant zeros at $z = 1$.

Our objective here is to design control laws that are capable of achieving fast tracking of target references under input saturation. As such, it is well understood in the literature that these assumptions are standard and necessary.

3.2.1 State Feedback Case

We first proceed to develop a composite nonlinear feedback control technique for the case when all the state variables of the plant Σ are measurable, *i.e.*, $y = x$. The design will be done in three steps, which is a natural extension of the results of Chen *et al.* [6]. We have the following step-by-step design procedure.

STEP S.1: Design a linear feedback law,

$$u_L(k) = Fx(k) + Gr, \tag{3.4}$$

where $r \in \mathbb{R}^m$ contains a set of step references. The state feedback gain matrix $F \in \mathbb{R}^{m\times n}$ is chosen such that the closed-loop system matrix $A+BF$ is asymptotically stable and the resulting closed-loop system transfer matrix, *i.e.*, $D_2 + (C_2 + D_2F)(zI - A - BF)^{-1}B$, has certain desired properties, *e.g.*, having a small dominating damping ratio in each channel. We note that such an F can be worked out using some well-studied methods such as the LQR,

H_∞ and H_2 optimization approaches (see, *e.g.*, Anderson and Moore [38], Chen [2] and Saberi *et al.* [5]). Furthermore, G is an $m \times m$ square constant matrix and is given by

$$G := G_0' \left(G_0 G_0'\right)^{-1}, \tag{3.5}$$

with $G_0 := D_2 - (C_2 + D_2 F)(A + BF)^{-1} B$. Here we note that both G_0 and G are well defined because $A+BF$ is stable, and (A, B, C_2, D_2) is right invertible and has no invariant zeros at $z = 1$, which implies $(A+BF, B, C_2+D_2F, D_2)$ is right invertible and has no invariant zeros at $z = 1$ (see *e.g.*, Lemma 2.5.1 of Chen [2]).

Step s.2: Next, we compute

$$H := \left[I - F(A + BF)^{-1} B\right] G \tag{3.6}$$

and

$$x_{\mathrm{R}} := G_{\mathrm{R}}\, r := -(A + BF)^{-1} BG\, r. \tag{3.7}$$

Note that the definitions of H, G_{R} and x_{R} would become transparent later in our derivation. Given a positive definite matrix $W \in \mathbb{R}^{n \times n}$, solve the following Lyapunov equation:

$$P = (A + BF)' P (A + BF) + W, \tag{3.8}$$

for $P > 0$. Such a P exists since $A + BF$ is asymptotically stable. Then, the nonlinear feedback control law $u_N(k)$ is given by

$$u_N(k) = \rho(r, y) B' P (A + BF)(x(k) - x_{\mathrm{R}}), \tag{3.9}$$

where

$$\rho(r,y) = \text{diag}\{\rho_1, \cdots, \rho_m\} = \begin{bmatrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_m \end{bmatrix}, \tag{3.10}$$

and $\rho_i = \rho_i(r,y)$, $i = 1, 2, \cdots, m$, are respectively some nonpositive functions, uniformly bounded and locally Lipschitz in y, which are used to change the closed-loop system damping ratios as the outputs approach the targets. The choice of these nonlinear functions will be discussed at the end of this section.

STEP S.3: The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller:

$$u(k) = u_L(k) + u_N(k) = Fx(k) + Gr + \rho(r,y)B'P(A+BF)(x(k) - x_{\rm R}). \tag{3.11}$$

This completes the design of the CNF controller for the state feedback case.

For further development, we partition $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{m\times n}$ and $H \in \mathbb{R}^{m\times m}$ as follows:

$$B = [\, B_1 \quad \cdots \quad B_m \,], \quad F = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ \vdots \\ H_m \end{bmatrix}. \tag{3.12}$$

The following theorem shows that the closed-loop system comprising the given plant in (3.1) and the CNF control law of (3.11) is asymptotically stable. It also determines the magnitudes of the step functions in r that can be tracked by such a control law without exceeding the control limit.

Theorem 3.1 *Consider the given system in (3.1) with $y = x$, which satisfies the assumptions i) and iii), the linear control law of (3.4) and the composite nonlinear feedback control law of (3.11). For any $\nabla \in (0,1)$, let $c_\nabla > 0$ be the largest positive* *scalar such that for all* $x(k) \in \boldsymbol{X}_\nabla$, *where*

$$\boldsymbol{X}_\nabla := \Big\{ x : x'(k) P x(k) \le c_\nabla \Big\}, \tag{3.13}$$

the following property holds,

$$|\, F_i\, x(k) \,| \le (1 - \nabla) \bar{u}_i, \quad i = 1, \cdots, m. \tag{3.14}$$

Then, the linear control law of (3.4) is capable of driving the system controlled output $h(t)$ *to track asymptotically a set of step references, i.e.,* r, *provided that the initial state* x_0 *and* r *satisfy:*

$$\tilde{x}_0 := (x_0 - x_{\mathrm{R}}) \in \boldsymbol{X}_\nabla, \quad |H_i\, r| \le \nabla \bar{u}_i, \quad i = 1, \cdots, m. \tag{3.15}$$

Furthermore, for any nonpositive function $\rho(r, y)$, *uniformly bounded and locally Lipschitz in* y, *the composite nonlinear feedback law in (3.11) is capable of driving the system controlled output* $h(t)$ *to track asymptotically the step command input of amplitude* r, *provided that the initial state* x_0 *and* r *satisfy (3.15).*

Proof. Let us first define a new state variable $\tilde{x}(k) = x(k) - x_{\mathrm{R}}$. It is simple to verify that the linear feedback control law of (3.4) can be rewritten as

$$u_L(k) = F\tilde{x}(k) + [I - F(A + BF)^{-1} B] G r \tag{3.16}$$

$$= F\tilde{x}(k) + Hr, \tag{3.17}$$

and hence for all $\tilde{x}(k) \in \boldsymbol{X}_\nabla$ and, provided that $|H_i\, r| \le \nabla \bar{u}_i$, $i = 1, \cdots, m$, the closed-loop system is linear and is given by

$$\tilde{x}(k+1) = (A + BF)\tilde{x}(k) + A x_{\mathrm{R}} + BHr. \tag{3.18}$$

Noting that

$$\begin{aligned} Ax_{\mathrm{R}} + BHr &= \Big\{B[I - F(A+BF)^{-1}B]G - A(A+BF)^{-1}BG\Big\}r \\ &= \Big\{[I - BF(A+BF)^{-1}]BG - A(A+BF)^{-1}BG\Big\}r \\ &= \Big\{I - BF(A+BF)^{-1} - A(A+BF)^{-1}\Big\}BGr \\ &= 0, \end{aligned} \tag{3.19}$$

the closed-loop system in (3.18) can then be simplified as

$$\tilde{x}(k+1) = (A+BF)\tilde{x}(k). \tag{3.20}$$

Similarly, the closed-loop system comprising the given plant in (3.1) and the CNF control law of (3.11) can be expressed as

$$\tilde{x}(k+1) = (A+BF)\tilde{x}(k) + Bw(k), \tag{3.21}$$

where

$$w(k) = \mathrm{sat}(F\tilde{x}(k) + Hr + u_N(k)) - F\tilde{x}(k) - Hr. \tag{3.22}$$

Clearly, for the given x_0 satisfying (3.15), we have $\tilde{x}_0 = (x_0 - x_{\mathrm{R}}) \in \boldsymbol{X}_{\nabla}$. We note that (3.21) is reduced to (3.20) if $\rho(r,y) = 0$.

Next, we define a Lyapunov function $V(k) = \tilde{x}'(k)P\tilde{x}(k)$ and evaluate the increment of $V(k)$ along the trajectories of the closed-loop system in (3.21), *i.e.*,

$$\begin{aligned} \nabla V(k+1) &= \tilde{x}'(k+1)P\tilde{x}(k+1) - \tilde{x}'(k)P\tilde{x}(k) \qquad (3.23) \\ &= \tilde{x}'(k)(A+BF)'P(A+BF)\tilde{x}(k) - \tilde{x}'(k)P\tilde{x}(k) \\ &\quad + 2\tilde{x}'(k)(A+BF)'PBw(k) + w'(k)B'PBw(k) \\ &= -\tilde{x}'(k)W\tilde{x}(k) + 2\tilde{x}'(k)(A+BF)'PBw(k) + w'(k)B'PBw(k) \end{aligned}$$

Note that for all

$$\tilde{x}(k) \in \boldsymbol{X}_{\nabla} = \{\tilde{x}(k) : \tilde{x}'(k)P\tilde{x}(k) \leq c_{\nabla}\} \quad \Rightarrow \quad |F_i\, \tilde{x}(k)| \leq (1-\nabla)\bar{u}_i, \quad i = 1, \cdots, m. \tag{3.24}$$

In the remainder of this proof, we adopt similar lines of reasoning as those of Turner *et al.* [44] by considering the following different scenarios. For simplicity, we drop the dependent variables of the nonlinear function ρ in the rest of this proof.

Case 1. All input channels are unsaturated. It is obvious that we have

$$w(k) = u_N(k) = \rho B'P(A + BF)\tilde{x}(k) \tag{3.25}$$

$$\begin{aligned} \nabla V(k+1) &= -\tilde{x}'(k)W\tilde{x}(k) + 2\tilde{x}'(k)(A+BF)'PB\rho B'P(A+BF)\tilde{x}(k) \\ &\quad + \tilde{x}(k)(A+BF)'PB\rho B'PB\rho B'P(A+BF)\tilde{x} \\ &= -\tilde{x}'(k)W\tilde{x}(k) + \tilde{x}'(k)(A+BF)'PB\rho(2I + B'PB\rho)B'P(A+BF)\tilde{x}'(k) \end{aligned}$$

So if

$$2I + B'PB\rho > 0 \tag{3.26}$$

and because $\rho < 0$

$$\nabla V(k+1) < -\tilde{x}'(k)W\tilde{x}(k) < 0 \tag{3.27}$$

Case 2. All input channels are exceeding their upper limits. In this case,define

$$u_{Ni}(k) = \rho_i B_i' P(A + BF)\tilde{x}(k) \tag{3.28}$$

$$F_i\tilde{x}(k) + H_i r + u_{Ni}(k) \geq \bar{u}_i, \;\; i = 1, \cdots, m. \tag{3.29}$$

$$u_{Ni}(k) \geq \bar{u}_i - F_i\tilde{x}(k) - H_i r, \;\; i = 1, \cdots, m. \tag{3.30}$$

and

$$w_i(k) = \bar{u}_i - (F_i\tilde{x}(k) + H_i r) \tag{3.31}$$

For all $\tilde{x}(k) \in \boldsymbol{X}_\nabla$, which implies (3.24) holds, and r satisfying (3.15), we have

$$F_i\tilde{x}(k) + H_i r \le \bar{u}_i, \quad i = 1, \cdots, m, \tag{3.32}$$

Hence,

$$0 < w_i(k) < u_{Ni}(k) \tag{3.33}$$

$$\begin{aligned}
\nabla V(k+1) &= -\tilde{x}'(k)W\tilde{x}(k) + 2\tilde{x}'(k)(A+BF)'PBw(k) + w'(k)B'PBw(k) \\
&= -\tilde{x}'(k)W\tilde{x}(k) + w'(k)[2B'P(A+BF)\tilde{x}]w(k) + w'(k)B'PBw(k) \\
&= -\tilde{x}'(k)W\tilde{x}(k) + \sum_{i=1}^{m} w_i(k)[2\rho_i^{-1}u_{Ni}(k)] + w'(k)B'PBw(k) \\
&< -\tilde{x}'(k)W\tilde{x}(k) + \sum_{i=1}^{m} w_i(k)[2\rho_i^{-1}w_i(k)] + w'(k)B'PBw(k) \\
&= -\tilde{x}'(k)W\tilde{x}(k) + w'(k)[2\rho^{-1}]w(k) + w'(k)B'PBw(k) \\
&= -\tilde{x}'(k)W\tilde{x}(k) + \sum_{i=1}^{m} w'(k)[2\rho^{-1} + B'PB]w(k) \\
&< 0.
\end{aligned} \tag{3.34}$$

$$\nabla V(k+1) = -\tilde{x}'(k)W\tilde{x}(k) + 2\sum_{i=1}^{m} \tilde{x}'(k)PB_i\bar{w}_i(k) \le -\tilde{x}'(k)W\tilde{x}(k). \tag{3.35}$$

Case 3. All input channels are exceeding their lower limits. For this case, we have

$$F_i\tilde{x}(k) + H_i r + \rho_i B_i' P\tilde{x}(k) \le -\bar{u}_i, \quad i = 1, \cdots, m. \tag{3.36}$$

Similarly, it can be shown that

$$\nabla V(k+1) = -\tilde{x}'(k)W\tilde{x}(k) \le 0. \tag{3.37}$$

Case 4. Some control channels are saturated and some are unsaturated. In view of Cases 1 to 3, the increment is just a combination of the above three cases.

For those understaurated channels, we have

$$w_i(k) = u_{Ni}(k) = \rho_i B' P (A + BF)\tilde{x}(k) \tag{3.38}$$

And

$$w_i(k)[2\rho_i^{-1}]u_{Ni}(k) = w_i(k)[2\rho_i^{-1}]w_i(k) \tag{3.39}$$

For those saturated channels, we have

$$w_i = \bar{u}_i - (F_i \tilde{x}(k) + H_i r) \tag{3.40}$$

or

$$w_i = -\bar{u}_i - (F_i \tilde{x}(k) + H_i r) \tag{3.41}$$

And

$$w_i(k)[2\rho_i^{-1}]u_{Ni}(k) < w_i(k)[2\rho_i^{-1}]w_i(k) \tag{3.42}$$

Thus, for this case, again we have

$$\sum_{i=1}^{m} w_i(k)[2\rho_i^{-1}u_{Ni}(k)] \leq \sum_{i=1}^{m} w_i(k)[2\rho_i^{-1}w_i(k)] \tag{3.43}$$

and hence

$$\nabla V(k+1) = -\tilde{x}'(k) W \tilde{x}(k) \leq 0. \tag{3.44}$$

In conclusion, we have shown that

$$\nabla V(k+1) \leq -\tilde{x}(k) W \tilde{x}(k), \quad \tilde{x}(k) \in \boldsymbol{X}_{\nabla}, \tag{3.45}$$

which implies that $\boldsymbol{X}_\nabla$ is an invariant set of the closed-loop system in (3.21). Noting that $W > 0$, all trajectories of (3.21) starting from inside $\boldsymbol{X}_\nabla$ will converge to the origin. This, in turn, indicates that, for all initial state x_0 and the step command input r that satisfy (3.15), we have

$$\lim_{k\to\infty} x(k) = x_e, \tag{3.46}$$

which implies

$$\lim_{k\to\infty} u(k) = F \lim_{k\to\infty} x(k) + Gr + \lim_{k\to\infty} \rho B' P[x(k) - x_{\rm R}] = F x_{\rm R} + Gr, \tag{3.47}$$

since $\rho(r, y)$ is uniformly bounded and

$$2I + \rho B' P B > 0, \tag{3.48}$$

Hence,

$$\begin{aligned} \lim_{k\to\infty} h(k) &= C_2 \lim_{k\to\infty} x(k) + D_2 \lim_{k\to\infty} u(k) \\ &= C_2 x_{\rm R} + D_2(F x_{\rm R} + Gr) \\ &= (C_2 + D_2 F) x_{\rm R} + D_2 G r \\ &= -(C_2 + D_2 F)(A + BF)^{-1} B G r + D_2 G r \\ &= [D_2 - (C_2 + D_2 F)(A + BF)^{-1} B] G r \\ &= G_0 G_0' (G_0 G_0')^{-1} r = r. \end{aligned} \tag{3.49}$$

This completes the proof of Theorem 3.1. ♢

Note that this conclusion follows from the fact that $u_N(k)$ decays to zero as $t \to \infty$, which is the reason for the uniform boundedness condition. This assumption is not required if either $D = 0$ or if only stability and not asymptotic tracking is required. Obviously this note applies also to full-order and reduced-order measurement feedback cases and we will not repeat it later.

3.2.2 Full Order Measurement Feedback Case

The assumption that all the state variables of the given system Σ are measurable is, in general, not practical. For example, in HDD servo systems (see Chen *et al.* [1]), the velocity of the actuator is usually hard to be measured. As such, in this subsection and the next subsection, we proceed to develop CNF design using only measurement information. We first deal with the full order measurement feedback case, in which the dynamical order of the controller is exactly the same as that of the given plant. The following is a step-by-step procedure for the CNF design using full order measurement feedback.

STEP F.1: We first construct a linear full order measurement feedback control law,

$$\begin{cases} x_{\rm R}(k+1) = (A + KC_1)x_{\rm R}(k) - Ky(k) + B\ \text{sat}(u_L(k)) \\ \quad\quad u_L(k) = F(x_{\rm R}(k) - x_{\rm R}) + Hr, \end{cases} \tag{3.50}$$

where r is the set of step reference signals and $x_{\rm R}(k)$ is the state of the controller. As usual, K, F are gain matrices and are chosen such that $(A + KC_1)$ and $(A + BF)$ are asymptotically stable and the resulting closed loop system having desired properties. Finally, G, H and $x_{\rm R}$ are as defined in (3.5)–(3.7).

STEP F.2: Given a positive definite matrix $W_{\mathrm{P}} \in \mathbb{R}^{n\times n}$, solve the Lyapunov equation

$$P = (A + BF)' P (A + BF) + W_{\mathrm{P}}, \tag{3.51}$$

for $P > 0$. As in the state feedback case, the linear control law of (3.50) obtained in the above step is to be combined with a nonlinear control law to form the following CNF controller:

$$\begin{cases} x_{\mathrm{R}}(k+1) = (A + KC_1)x_{\mathrm{R}}(k) - Ky(k) + B\ \mathrm{sat}(u(k)) \\ \quad u(k) \quad\ = F(x_{\mathrm{R}}(k) - x_{\mathrm{R}}) + Hr + q\rho(r,y)B'P(A+BF)(x_{\mathrm{R}}(k) - x_{\mathrm{R}}), \end{cases} \tag{3.52}$$

where $\rho(r, y)$ is given in (3.10) with all its diagonal elements being respectively a nonpositive function, locally Lipschitz in y, and q is a diagonal matrix with each diagonal element positive but less that one, which are to be chosen to improve the performance of the closed-loop system.

It turns out that, for the measurement feedback case, the choice of $\rho_i(r, y)$, $i = 1, \ldots m$, the nonpositive scalar functions, are not totally free. They are subject to certain constraints. We have the following result.

Theorem 3.2 *Consider the given system in (3.1), which satisfies the standard assumptions i) to iii), the full order linear measurement feedback control law of (3.50) and the composite nonlinear measurement feedback control law of (3.52). Given a positive define matrix $W_Q \in \mathbb{R}^{n\times n}$ with*

$$W_Q \geq \mu\, I \tag{3.53}$$

where

$$\mu = \max\{||(F + q\rho B'P(A + BF))'B'PB(F + q\rho B'P(A + BF))||\} \tag{3.54}$$

where $q = diag\{q_1, q_2, \cdots q_n\}$ *with each element,* $q_i \in [0, 1], i = 1, 2, \cdots n$, *whose definition will become transparent later, and* ρ *is a diagnal matrix with each element being non-positive function satisfying*

$$2I + B'PB\rho > 0 \tag{3.55}$$

Let $Q > 0$ *be the solution to the Lyapunov equation,*

$$Q = (A + KC_1)'Q(A + KC_1) + W_Q. \tag{3.56}$$

Note that such a Q *exists as* $A + KC_1$ *is asymptotically stable. For any* $\nabla \in (0, 1)$, *let* $c_\nabla > 0$ *be the largest positive scalar such that for all* $x(k) \in \boldsymbol{X}_{\mathrm{F}\nabla}$, *where*

$$\boldsymbol{X}_{\mathrm{F}\nabla} := \left\{ \begin{pmatrix} x(k) \\ x_v(k) \end{pmatrix} : \begin{pmatrix} x(k) \\ x_v(k) \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} x(k) \\ x_v(k) \end{pmatrix} \le c_\nabla \right\}, \tag{3.57}$$

the following property holds

$$\left| [\, F_i \quad F_i \,] \begin{pmatrix} x(k) \\ x_{\mathrm{R}}(k) \end{pmatrix} \right| \le (1 - \nabla)\bar{u}_i, \;\; i = 1, \cdots, m. \tag{3.58}$$

Then, the linear measurement feedback control law in (3.52) will drive the system's controlled output $h(t)$ *to track asymptotically a set of step references, i.e.,* r, *from an initial state* x_0, *provided that* x_0, $x_{\mathrm{R}0} = x_{\mathrm{R}}(0)$ *and* r *satisfy:*

$$\begin{pmatrix} x_0 - x_{\mathrm{R}} \\ x_{\mathrm{R}0} - x_0 \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\nabla} \quad \text{and} \quad |H_i r| \le \nabla \bar{u}_i, \;\; i = 1, \cdots, m. \tag{3.59}$$

Furthermore, there exist positive scalars $\rho_i^* > 0$, $i = 1, \ldots m$, *such that for any nonpositive functions* $\rho_i(r, y)$, $i = 1, \cdots, m$, *locally Lipschitz in* y *and* $|\rho_i(r, y)| \le \rho_i^*$,

$i = 1, \cdots, m$, the CNF control law of (3.52) will drive the system controlled output $h(t)$ to track asymptotically the reference r from an initial x_0, provided that x_0, $x_{\rm R0}$ and r satisfy (3.59).

Proof. For simplicity, we again drop r and y in $\rho(r, y)$ throughout the proof of this theorem. Let $\tilde{x} = x - x_{\rm R}$ and $\tilde{x}_{\rm R} = x_{\rm R} - x$. The linear feedback control law of (3.50) can be written as

$$\tilde{x}_{\rm R}(k+1) = (A + KC_1)\tilde{x}_{\rm R}(k), \quad u_L(k) = [\,F \quad F\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix} + Hr. \tag{3.60}$$

Hence, for all

$$\begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix} \in \boldsymbol{X}_{\rm F\nabla} \quad \Rightarrow \quad \left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix} \right| \le (1 - \nabla)\bar{u}_i, \ \ i = 1, \cdots, m, \tag{3.61}$$

and for any r satisfying

$$|H_i r| \le \nabla \bar{u}_i, \ \ i = 1, \cdots, m, \tag{3.62}$$

each channel of u_L, say $u_{L,i}$, has the following property

$$u_{L,i}(k) = \left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix} + H_i r \right| \le \left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix} \right| + |H_i r| \le \bar{u}_i. \tag{3.63}$$

Thus, for all $\tilde{x}(k)$ and $\tilde{x}_{\rm R}(k)$ satisfying the condition as given in (3.61), the closed-loop system comprising the given plant and the linear control law of (3.50) can be rewritten as

$$\begin{pmatrix} \tilde{x}(k+1) \\ \tilde{x}_{\rm R}(k+1) \end{pmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix}. \tag{3.64}$$

Similarly, the closed-loop system with the CNF control law of (3.52) can be expressed as

$$\begin{pmatrix} \tilde{x}(k+1) \\ \tilde{x}_{\rm R}(k+1) \end{pmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w(k), \tag{3.65}$$

where

$$\begin{aligned} w(k) &= \text{sat}\left[[\,F \quad F\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} + Hr + \rho\,[\,B'P \quad B'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \right] \\ &- [\,F \quad F\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} - Hr. \end{aligned} \tag{3.66}$$

If some channels are undersaturated, i.e.,

$$\left[[\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} + H_i r + \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \right] \leq \bar{u}_i \tag{3.67}$$

then

$$w_i(k) = \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \tag{3.68}$$

else if some channels are saturated from above, i.e.,

$$\left[[\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} + H_i r + \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \right] \geq \bar{u}_i \tag{3.69}$$

then

$$\begin{aligned} w_i(k) &= \bar{u}_i - [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \\ &- H_i r \leq \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \end{aligned} \tag{3.70}$$

Define a diagonal matrix $q = \text{diag}\{q_1, q_2, \cdots q_n\}$ with each element, $q_i \in [0,1], i = 1, 2, \cdots n$, then $w_i(k)$ can be expressed as following:

$$w_i(k) = q_i \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \tag{3.71}$$

similarly if some channels are saturated from below, or some channels are saturated and some undersaturated, we can still define $w(k)$ as following:

$$w_i(k) = q_i \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \tag{3.72}$$

or

$$w(k) = q\rho \,[\, B'P \quad B'P \,]\, (A + BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \tag{3.73}$$

So we use (3.72) or (3.73) to cover all cases.

Clearly, for x_0 and $x_{\mathrm{R}0}$ satisfying (3.59), we have

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_{\mathrm{R}0} \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\nabla}, \tag{3.74}$$

where $\tilde{x}_0 = \tilde{x}(0)$ and $\tilde{x}_{\mathrm{R}0} = \tilde{x}_{\mathrm{R}}(0)$. We note that (3.64) and (3.65) are identical when $\rho = 0$. Again, the results of Theorem 3.2 for both the linear and the nonlinear feedback case can be proved in one shot.

Next, we define a Lyapunov function:

$$V(k) = \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix}, \tag{3.75}$$

and evaluate the increment of $V(k)$ along the trajectories of the closed-loop system in (3.65), *i.e.*,

$$\begin{aligned} \nabla V(k+1) &= \begin{pmatrix} \tilde{x}(k+1) \\ \tilde{x}_{\mathrm{R}}(k+1) \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} \tilde{x}(k+1) \\ \tilde{x}_{\mathrm{R}}(k+1) \end{pmatrix} - \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix}' \begin{bmatrix} -W_{\mathrm{P}} & (A+BF)'PBF \\ (BF)'P(A+BF) & -W_Q + (BF)'PBF \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} \\ &+ w' \,[\, B'P(A+BF) \;\; B'PBF \,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix} + \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix}' \begin{bmatrix} (A+BF)'PB \\ (BF)'PB \end{bmatrix} w(k) \\ &+ w'(k)B'PBw(k) \end{aligned} \tag{3.76}$$

So substitute (3.73) into $\nabla V(k+1)$, we have

$$\nabla V(k+1) = \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix}' \begin{bmatrix} -W_{\mathrm{P}} & (A+BF)'PBF \\ (BF)'P(A+BF) & -W_Q + (BF)'PBF \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{R}}(k) \end{pmatrix}$$

$$
\begin{aligned}
&+\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}'\begin{bmatrix}(A+BF)'PB\\ (A+BF)'PB\end{bmatrix}q\rho[B'P(A+BF)\ \ B'PBF]\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}\\
&+\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}'\begin{bmatrix}(A+BF)'PB\\ (BF)'PB\end{bmatrix}q\rho[B'P(A+BF)\ \ B'P(A+BF)]\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}\\
&+\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}\begin{bmatrix}(A+BF)'PB\\ (A+BF)'PB\end{bmatrix}q\rho B'PB\rho[B'P(A+BF)\ \ B'P(A+BF)]\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}
\end{aligned}
$$

Define $T = B'P(A+BF)$, then

$$
\begin{aligned}
\nabla V(k+1) &= \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}'\begin{bmatrix}-W_{\mathrm{P}} & T'F\\ F'T & -W_Q+(BF)'PBF\end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}\\
&+ \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}'\begin{bmatrix}0 & T'q\rho B'PBF\\ T'q\rho T & T'q\rho B'PBF\end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}\\
&+ \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}'\begin{bmatrix}0 & T'q\rho T\\ F'B'PBq\rho T & F'B'PBq\rho T\end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}\\
&+ \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}'\begin{bmatrix}0 & T'q\rho B'PBq\rho T\\ T'\rho B'PBq\rho T & T'q\rho B'PBq\rho T\end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}\\
&+ \tilde{x}(k)'T'q\rho[2I+B'PBq\rho]T\tilde{x}(k)\\
&= -\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}'\begin{bmatrix}W_{\mathrm{P}} & -W_d\\ -W_d' & W_{qm}\end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{R}}(k)\end{pmatrix}\\
&+ \tilde{x}(k)'T'q\rho[2I+B'PBq\rho]T\tilde{x}(k) \qquad (3.77)
\end{aligned}
$$

where

$$
\begin{aligned}
W_d &= T'F+T'q\rho T+T'q\rho B'PBF+T'q\rho B'PBq\rho T\\
&= T'(I+q\rho B'PB)(F+q\rho T) \qquad (3.78)
\end{aligned}
$$

$$
W_{qm} = -W_Q + F'B'PBF + T'q\rho B'PBF + F'B'PBq\rho T + T'q\rho B'PBq\rho T
$$

Define four matrices

$$
M_1 = \begin{bmatrix} I & 0\\ -W_d'W_{\mathrm{P}}{}^{-1} & I\end{bmatrix} \qquad (3.79)
$$

$$
M_2 = \begin{bmatrix} I & 0\\ W_d'W_{\mathrm{P}}{}^{-1} & I\end{bmatrix} \qquad (3.80)
$$

and

$$N_1 = \begin{bmatrix} I & W_{\text{P}}{}^{-1}W_d \\ 0 & I \end{bmatrix} \tag{3.81}$$

$$N_2 = \begin{bmatrix} I & -W_{\text{P}}{}^{-1}W_d \\ 0 & I \end{bmatrix} \tag{3.82}$$

Note the properties: $M_1M_2 = I$, $N_1N_2 = I$ and $N_1' = M_2$, $N_2' = M_1$. So in (3.77), we have

$$\begin{aligned} \nabla V(k+1) &= -\begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\text{R}}(k) \end{pmatrix}' M_1(M_2 \begin{bmatrix} W_{\text{P}} & -W_d \\ -W_d' & W_{qm} \end{bmatrix} N_1)N_2 \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\text{R}}(k) \end{pmatrix} \\ &+ \tilde{x}(k)'T'q\rho[2I + B'PBq\rho]T\tilde{x}(k) \end{aligned} \tag{3.83}$$

Define

$$\hat{x}_m(k) = N_2 \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\text{R}}(k) \end{pmatrix} \tag{3.84}$$

We have

$$\begin{aligned} \nabla V(k+1) &= -\hat{x}_m(k)' \begin{bmatrix} W_{\text{P}} & 0 \\ 0 & W_{qq} \end{bmatrix} \hat{x}_m(k) \\ &+ \tilde{x}(k)'T'q\rho[2I + B'PBq\rho]T\tilde{x}(k) \end{aligned} \tag{3.85}$$

where

$$\hat{x}_m(k) = N_2 \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\text{R}}(k) \end{pmatrix} = \begin{pmatrix} \tilde{x}(k) - W_{\text{P}}^{-1}W_d\tilde{x}_{\text{R}}(k) \\ \tilde{x}_{\text{R}}(k) \end{pmatrix} \tag{3.86}$$

and from (3.53), we can get $W_{\text{Q}} \geq (F + q\rho T)'B'PB(F + q\rho T)$

$$\begin{aligned} W_{qq} &= W_{qm} - W_d'W_{\text{P}}^{-1}W_d \\ &= -W_Q + F'B'PBF + T'q\rho B'PBF + F'B'PBq\rho T + T'q\rho B'PBq\rho T - W_d'W_{\text{P}}{}^{-1}W_d \end{aligned}$$

$$= -[W_Q + W_d' W_P^{-1} W_d - (F + q\rho T)' B' P B (F + q\rho T)]$$

$$\leq -[W_Q - (F + q\rho T)' B' P B (F + q\rho T)]$$

$$\leq 0 \tag{3.87}$$

So

$$-\hat{x}_m(k)' \begin{bmatrix} W_{\mathrm{P}} & 0 \\ 0 & W_{qq} \end{bmatrix} \hat{x}_m(k) \leq 0 \tag{3.88}$$

Since in (3.55), $2I + B'PB\rho > 0$ and $||q|| \leq 1$, then

$$\tilde{x}(k)' T' q\rho [2I + B'PBq\rho] T \tilde{x}(k) \leq 0 \tag{3.89}$$

Combing (3.88) and (3.89), we have $\nabla V(k+1) \leq 0$

Thus, $\boldsymbol{X}_{\mathrm{F}\nabla}$ is an invariant set of the closed-loop system in (3.65), and all trajectories starting from $\boldsymbol{X}_{\mathrm{F}\nabla}$ will remain inside and asymptotically converge to the origin. This, in turn, indicates that, for the initial state of the given system x_0, the initial state of the controller $x_{\mathrm{R}0}$, and step command input r that satisfy (3.59),

$$\lim_{k\to\infty} \tilde{x}_{\mathrm{R}}(k) = 0 \quad \text{and} \quad \lim_{k\to\infty} x(k) = x_{\mathrm{R}}, \tag{3.90}$$

and then it follows from (3.49) that the controlled output $h(t)$ converges asymptotically to the target reference r. This completes the proof of Theorem 3.2. ♢

3.2.3 Reduced Order Measurement Feedback Case

For the given system in (3.1), it is clear that there are p state variables of the system, which are measurable if C_1 is of maximal rank. Thus, in general, it is not necessary to estimate these measurable state variables in measurement feedback laws. As such, we will proceed in this subsection to design a dynamic controller that has a dynamical order less than that of the given plant. For simplicity of presentation, we assume that C_1 is already in the form

$$C_1 = [\, I_p \quad 0 \,]\,. \tag{3.91}$$

Then, the system in (3.1) can be rewritten as

$$\begin{cases} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathrm{sat}(u(k)) \\ \qquad y(k) \qquad = [\ I_p \quad 0\ ] \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \\ \qquad h(k) \qquad = \qquad C_2 \qquad \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}, \quad x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \end{cases} \tag{3.92}$$

where the original state x is partitioned into two parts, x_1 and x_2 with $y \equiv x_1$. Thus, we will only need to estimate x_2 in the reduced order measurement feedback design. Next, we let F be chosen such that i) $A + BF$ is asymptotically stable, and ii) $(C_2 + D_2F)(sI - A - BF)^{-1}B + D_2$ has desired properties, and let K_R be chosen such that $A_{22} + K_RA_{12}$ is asymptotically stable. Here we note that it can be shown that (A_{22}, A_{12}) is detectable if and only if (A, C_1) is detectable. Thus, there exists a stabilizing K_R. Again, such F and K_R can be designed using an appropriate control technique. We then partition F in conformity with x_1 and x_2:

$$F = [\, F_1 \quad F_2 \,]\,. \tag{3.93}$$

We further partition F_2 as follows:

$$F_2 = \begin{bmatrix} F_{2,1} \\ \vdots \\ F_{2,m} \end{bmatrix}. \tag{3.94}$$

Also, let G, H and $x_{\rm R}$ be as given in (3.5)–(3.7). The reduced order CNF controller is given by

$$x_{\rm R}(k+1) = (A_{22}+K_R A_{12})x_{\rm R}(k) + (B_2+K_R B_1)\,\mathrm{sat}(u(k)) + \Big[A_{21}+K_R A_{11}-(A_{22}+K_R A_{12})K_R\Big]y \tag{3.95}$$

and

$$u(k) = F\left[\begin{pmatrix} y(k) \\ x_{\rm R}(k)-K_R y(k)\end{pmatrix} - x_{\rm R}(k)\right] + Hr + \rho(r,y)B'P\left[\begin{pmatrix} y \\ x_{\rm R}(k)-K_R y(k)\end{pmatrix} - x_{\rm R}(k)\right], \tag{3.96}$$

where $\rho(r, y)$ is as given in (3.10).

Next, given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, let $P > 0$ be the solution to the Lyapunov equation

$$P = (A + BF)'P(A + BF) + W_{\rm P}. \tag{3.97}$$

Given another positive definite matrix $W_R \in \mathbb{R}^{(n-p)\times(n-p)}$ with

$$W_R \geq \mu\, I \tag{3.98}$$

where again

$$\mu = \max\{||(F + q\rho B'P(A + BF))'B'PB(F + q\rho B'P(A + BF))||\} \tag{3.99}$$

let $Q_{\rm R} > 0$ be the solution to the Lyapunov equation

$$(A_{22} + K_R A_{12})'Q_{\rm R} + Q_{\rm R}(A_{22} + K_R A_{12}) = -W_R. \tag{3.100}$$

Note that such P and $Q_{\rm R}$ exist as $A + BF$ and $A_{22} + K_R A_{12}$ are asymptotically stable. For any $\nabla \in (0, 1)$, let c_∇ be the largest positive scalar such that for all

$$\begin{pmatrix} x(k) \\ x_{\rm R}(k)\end{pmatrix} \in \boldsymbol{X}_{\rm R\nabla} := \left\{\begin{pmatrix} x(k) \\ x_{\rm R}(k)\end{pmatrix} : \begin{pmatrix} x(k) \\ x_{\rm R}(k)\end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q_{\rm R}\end{bmatrix}\begin{pmatrix} x(k) \\ x_{\rm R}(k)\end{pmatrix} \leq c_\nabla\right\} \tag{3.101}$$

the following property holds:

$$\left| [\, F_i \quad F_{2,i} \,] \begin{pmatrix} x(k) \\ x_{\rm R}(k) \end{pmatrix} \right| \le \bar{u}_i(1-\nabla), \quad i = 1, \cdots, m. \tag{3.102}$$

We have the following theorem.

Theorem 3.3 *Consider the given system in (3.1), which satisfies the usual assumptions i) to iii). Then, there exist positive scalars* $\rho_i^* > 0$, $i = 1, \cdots, m$, *such that for any nonpositive function* $\rho_i(r, y)$, $i = 1, \cdots, m$, *locally Lipschitz in* y_i *and* $|\rho_i(r, y)| \le \rho_i^*$, *the reduced order CNF law given by (3.95) and (3.96) will drive the system controlled output* $h(t)$ *to asymptotically track the reference* r *from an initial state* x_0, *provided that* x_0, $x_{\rm R0}$ *and* r *satisfy*

$$\begin{pmatrix} x_0 - x_{\rm R} \\ x_{\rm R0} - x_{20} - K_R x_{10} \end{pmatrix} \in \boldsymbol{X}_{{\rm R}\nabla}, \quad |H_i r| \le \nabla \bar{u}_i, \quad i = 1, \cdots, m. \tag{3.103}$$

Proof. Let $\tilde{x}(k) = x(k) - x_{\rm R}$ and $\tilde{x}_{\rm R}(k) = x_{\rm R}(k) - x_2(k) - K_R x_1(k)$. Then, the closed-loop system comprising the given plant in (3.1) and the reduced order CNF control law of (3.95) and (3.96) can be expressed as

$$\begin{pmatrix} \tilde{x}(k+1) \\ \tilde{x}_{\rm R}(k+1) \end{pmatrix} = \begin{bmatrix} A + BF & BF_2 \\ 0 & A_{22} + K_R A_{12} \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \tag{3.104}$$

where

$$\begin{aligned} w \;=\; & \operatorname{sat}\left\{ [\, F \quad F_2 \,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix} + Hr + \rho(r,y) B' P \left[\tilde{x}(k) + \begin{pmatrix} 0 \\ \tilde{x}_{\rm R}(k) \end{pmatrix} \right] \right\} \\ & - \; [\, F \quad F_2 \,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm R}(k) \end{pmatrix} - Hr. \end{aligned} \tag{3.105}$$

The rest of the proof follows along similar lines to the reasoning given in the full order measurement feedback case.

Chapter 4

A Dual Stage HDD Servo CNF Controller

4.1 Introduction

In recent years, a lot of researchers and engineers have paid much attention to hard disk drive controller design techniques. Gradually, to achieve smaller-size hard disks with increasingly larger capacities, dual stage hard disk drive has been drawn the attention of the public.

One limitation in the conventional hard disk drives to achieve higher data capacity is the bandwidth. That is, the voice coil motor (VCM) as an actuator has a lot of flexible resonances in high frequencies over 2kHz, which limits the increase of bandwidth. A possible solution to this kind of problems is to introduce an additional micro-actuator on top of the conventional VCM actuator to provide a faster and finer response. Dual stage actuator refers to the fact that there is a small actuator mounted on a large conventional VCM actuator. This small actuator or micro-actuator will be used only to follow a small data track. The following figure (4.1) shows the configuration of a dual stage hard disk drive:

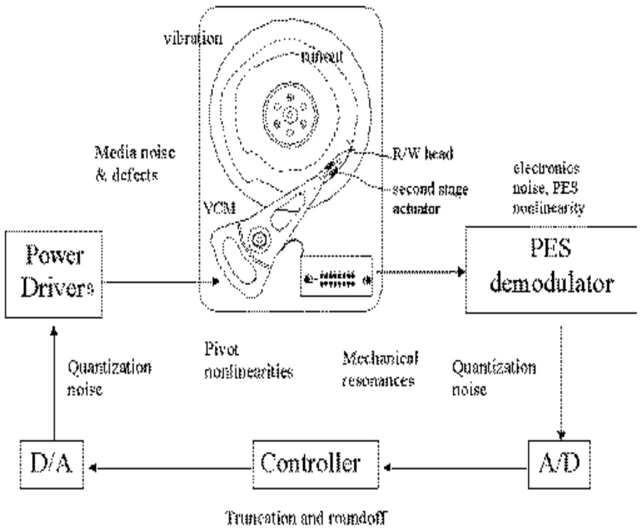


Figure 4.1: A hard disk drive servo mechanism with dual stage actuator

4.1.1 Modeling and Identification of Dual Stage HDD

The following four graphs (4.2), (4.3),(4.4) and (4.5) are the instruments we have used in the modeling and identification process.

A detailed VCM model of an HDD can be as high as 40th order, see ([31]). Factors needed to be considered include: nonlinearities in power amplifier saturation, in voice coil hysteresis, saturation, in actuator material damping, in torque factor, in friction and in striction.

Using non-linear least squares as the algorithm for identification, we will get the following graph (4.6) :

Figure 4.2: Modeling system configuration

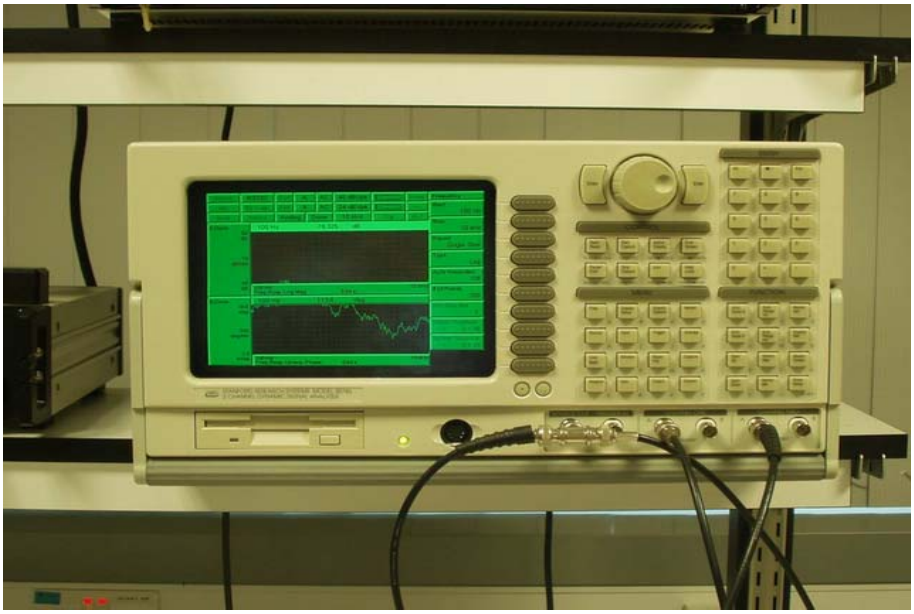


Figure 4.3: Signal analyzer

And the transfer function of VCM is (4.1):

$$G_v(s) = \frac{num_1 + num_2 + num_3}{den_1 + den_2 + den_3} \tag{4.1}$$

where

$$\begin{aligned} num_1 &= 1.531e012s^{11} - 5.246e016s^{10} + 4.801e021s^9 - 1.687e026s^8 \\ num_2 &= 5.36e030s^7 - 1.841e035s^6 + 2.577e039s^5 - 8.584e043s^4 \end{aligned}$$

Figure 4.4: Polytech meter

Figure 4.5: Laser meter

$$num_3 = 5.276e047s^3 - 1.738e052s^2 + 3.574e055s - 1.205e060 \quad (4.2)$$

and

$$\begin{aligned} den_1 &= s^{14} + 5454s^{13} + 4.271e009s^{12} + 1.807e013s^{11} + 6.909e018s^{10} \\ den_2 &= 2.14e022s^9 + 5.265e027s^8 1.097e031s^7 + 1.929e036s^6 \\ den_3 &= 2.421e039s^5 + 3.209e044s^4 + 1.841e047s^3 + 1.883e052s^2 \end{aligned} \quad (4.3)$$

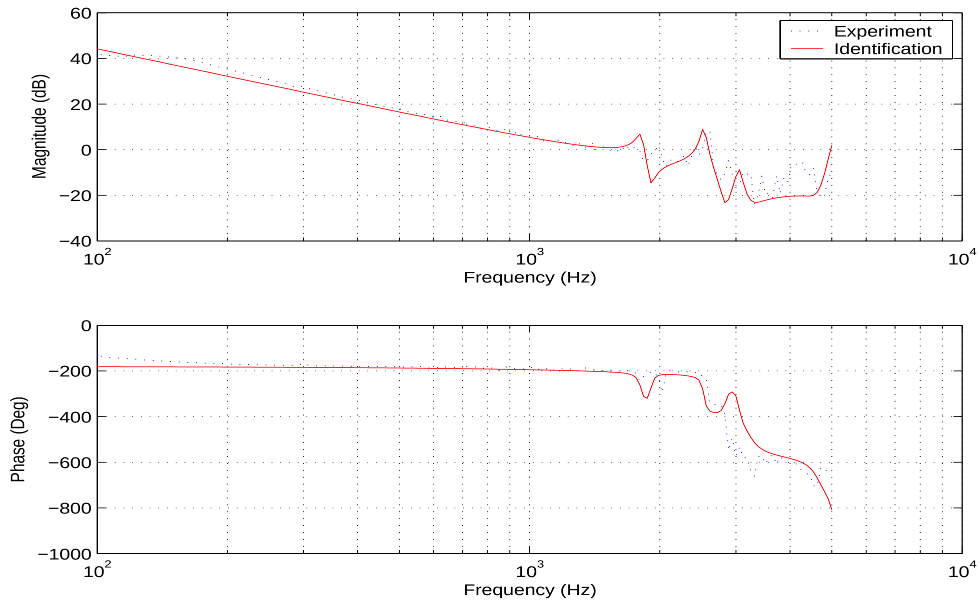


Figure 4.6: Nonlinear least square approach to identify VCM model

The transfer function of micro-actuator is (4.4):

$$G_m(s) = \frac{num_4 + num_5 + num_6}{den_4 + den_5 + den_6} \tag{4.4}$$

where

$$\begin{aligned} num_4 &= 1.531e012s^{11} - 5.246e016s^{10} + 4.801e021s^9 - 1.687e026s^8 \\ num_5 &= 5.36e030s^7 - 1.841e035s^6 + 2.577e039s^5 - 8.584e043s^4 \\ num_6 &= 5.276e047s^3 - 1.738e052s^2 + 3.574e055s - 1.205e060 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} den_4 &= s^{14} + 5454s^{13} + 4.271e009s^{12} + 1.807e013s^{11} + 6.909e018s^10 \\ den_5 &= 2.14e022s^9 + 5.265e027s^8 + 1.097e031s^7 + 1.929e036s^6 \\ den_6 &= 2.421e039s^5 + 3.209e044s^4 + 1.841e047s^3 + 1.883e052s^2 \end{aligned} \tag{4.6}$$

Unfortunately, the above identified model is a fourteenth order system, which is too complicated to be used to design a controller. Moreover, there is no need to use such a complicated model to identify the system. What the model we need is that the model can grasp the main characteristic of the system, not to re-draw the system. So we will use a model with fourth order to identify the system.

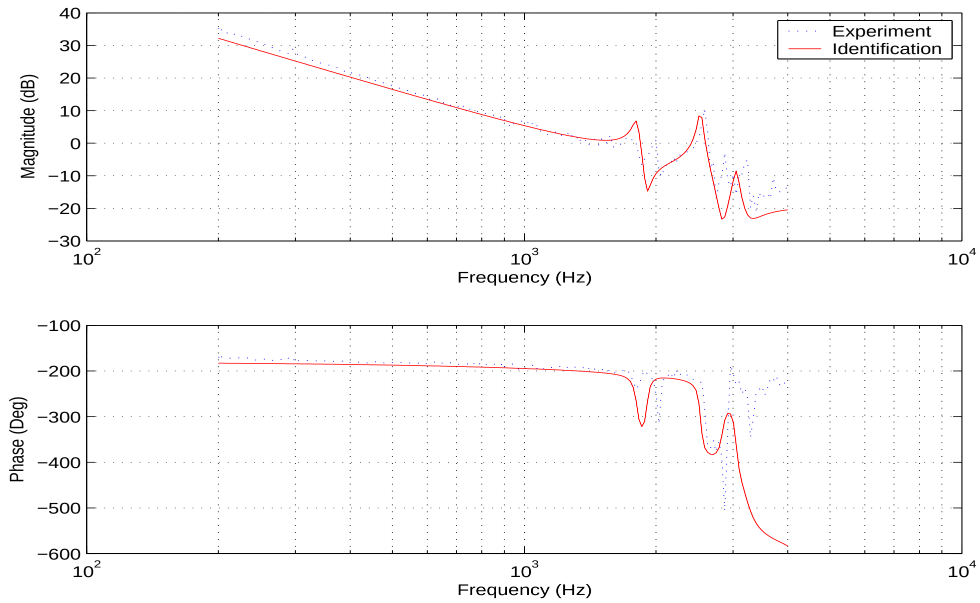


Figure 4.7: Nonlinear least square approach to identify micro-actuator model

Refered to ([4]), we use the data from actual systems and the corresponding algorithms by Eykhoff ([30]) and identify a fourth order model for the VCM actuator,

$$G_v(s) = \frac{6.4017e07}{s^2} \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2} \tag{4.7}$$

where $\xi = 0.085$, $w_n = 1.1309e04\text{rad/sec}$ and a fourth order model for the micro-actuator,

$$G_m(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \tag{4.8}$$

where

$$\begin{aligned} b_0 &= 1.1593e09 \\ b_1 &= 1.708e12 \\ b_2 &= 2.512e18 \\ a_3 &= 4256 \end{aligned}$$

$$\begin{aligned} a_2 &= 3.506e09 \\ a_1 &= 7.496e12 \\ a_0 &= 2.512e18 \end{aligned} \tag{4.9}$$

The only measured output, y is

$$y = y_v + y_m \tag{4.10}$$

where y_v and y_m are outputs of VCM and micro-actuator respectively. The units are set in volts and μm respectively for the input and output in the models. These models compose the model of the dual stage actuator HDD servo system, which will be used throughout the rest of the chapter.

Then we will convert these two models into state space representation and transform them into discrete time system. The sample time for VCM is

$$T_{s1} = 1e05Hz \tag{4.11}$$

and the sample time for micro-actuator is

$$T_{s2} = 1e05Hz \tag{4.12}$$

4.1.2 CNF Controller Design Preparation

Transform the models into the discrete time state space representation, we will have two models, one is for VCM and the other is for micro-actuator.

The VCM model is represented by A_v, B_v, C_v, D_v:

$$A_v = \begin{bmatrix} 3.9680 & -5.9180 & 3.9300 & -0.9810 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}; B_v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.13}$$

$$C_v = 1.0e-004 \left[\begin{array}{cccc} 0.4026 & 0.0041 & 0.4006 & 0.0010 \end{array} \right]; D_v = \; 5.0520e-006 \quad (4.14)$$

And the micro-actuator is represented by A_m, B_m, C_m, D_m:

$$A_m = \left[\begin{array}{llll} 3.6170 & -5.2240 & 3.5410 & -0.9583 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \end{array} \right]; B_m = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right] \quad (4.15)$$

$$C_m = \left[\begin{array}{cccc} 0.0776 & -0.0655 & -0.0640 & 0.0759 \end{array} \right]; D_m = \; 0 \quad (4.16)$$

Simplified as a second order system in Chapter 5, which will be discussed later, now the hard disk drive will be presented as a fourth order system and has been decoupled as two parts, VCM part and micro-actuator. So it is easier to design two independent controller to control each part. However, as the output is a combination of outputs from VCM and micro-actuator (4.17),

$$y = y_v + y_m \quad (4.17)$$

So the problem is the coordination between y_v and y_m. But the internal structures of VCM and micro-actuator are quite different, the outputs to the reference signal are quite different too. Moreover, during the low frequency, VCM model is dominant, while, during the high frequency, the micro-actuator is dominant. How to design the controller and coordinate the two models will need some tricks and non-standard design strategies.

However, if we apply the CNF control law, the process is systematic and simple. The CNF controller is basically stable, however, the performance can not be guaranteed if we cannot select the nonlinear part properly. So we use the approaches by try and error in the simulation.

4.1.3 Controller Design

In ([32]), a typical design specification for dual stage HDD servo is summarized in the following table (4.1.3):

Table 4.1: Specifications for a dual stage HDD controller design

Items	Objective
Open-loop bandwidth f_0:	more than 200 Hz
Disturbance attenuation:	more than 40 dB below 100 Hz
Phase margin (PM):	more than 40 degree
Gain margin (GM):	more than 6 dB
Rise time:	less than 0.2 ms
Overshoot:	less than 20%
Sensitivity transfer function peak:	less than 10 dB

Combine the two subsystems together, we will get system parameters as following:

$$\left\{ \begin{array}{lcl} x_c(k+1) & = & A_c x_c(k) + B_c u_c(k) \\ y_c(k) & = & y_v(k) + y_m(k) = C_c(k) x_c(k) \end{array} \right. \tag{4.18}$$

where

$$A_c = \begin{bmatrix} A_v & 0 \\ 0 & A_m \end{bmatrix}, B_c = \begin{bmatrix} B_v & 0 \\ 0 & B_m \end{bmatrix}, C_c = \begin{bmatrix} C_v & C_m \end{bmatrix} \tag{4.19}$$

In ([17]), they proposed two auxiliary states , which were integrals of tracking errors of secondary-stage actuator and VCM, respectively,

$$x_{ci}(k+1) = x_{ci}(k) + r(k) - y_c(k) \tag{4.20}$$

$$x_{vi}(k+1) = x_{vi}(k) + r(k) - y_v(k) \tag{4.21}$$

Then it follows that the generalized system is

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + B_r r(k) \\ y(k) &= C(k)x(k) \end{cases} \tag{4.22}$$

where $x = [x_c^T \;\; x_{ci}^T \;\; x_{vi}^T]$, $C = [C_c \;\; 0 \;\; 0]$ and

$$A = \begin{bmatrix} A_c & 0 & 0 \\ -C_c & 1 & 0 \\ [0 \quad -C_v] & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} B_c \\ 0 \\ 0 \end{bmatrix}, B_r = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{4.23}$$

We will design our CNF controller through two phases. In step 1, we can use LQR to design linear controller. In step 2, we can add the nonlinear part and adjust the nonlinear parameter $\rho(y, r)$ to improve the dynamics of the system.

(1). LQR Design.

Introduce the following quadratic performance index,

$$J = \sum_{0}^{\infty} z(k)^T R_1 z(k) + u(k)^T R_2 u(k) \tag{4.24}$$

where $R_1 \geq 0$, $R_2 > 0$ are weight matrices and $z(k) = [x_{ci} \quad x_{vi}]^T$. Then our design problem can be formulated as : find a proper controller for the generalized system such that the closed-loop systems is stable and the performance index J is minimized.

We just assume all the states are available and design a state feedback controller. The control law is

$$u(k) = Fx(k) \tag{4.25}$$

where F is obtained by

$$F = -(R_2 + B^T P B)^{-1} B^T P A \tag{4.26}$$

and $P > 0$ is the solution of the following Riccati equation:

$$A^T PA - P + R_1 - A^T PB(R_2 + B^T PB)^{-1} B^T PA = 0 \tag{4.27}$$

(2). CNF Controller.

The CNF controller consists of two parts. One is linear part, which we can obtain in the above step, and the other one is nonlinear part, which we will try to construct in the following.

$$u = u_L(k) + u_N(k) \tag{4.28}$$

$$= Fx(k) + Gr(k) + \rho(y,r)B^T P(A + BF)(x(k) - x_e(k)) \tag{4.29}$$

where F can be chosen from (4.26) and G is given by

$$G = G_0'(G_0 G_0')^{-1} \tag{4.30}$$

$$G_0 = -C(A + BF)^{-1} B \tag{4.31}$$

4.1.4 Simulation

We use the following Simulink block (4.8), where the parameters are given as following:

$$F_{c1} = \begin{bmatrix} -1.1508 & 2.8011 & -2.3559 & 0.6756 & -0.0046 \\ -0.0027 & 0.0061 & -0.0046 & 0.0008 & -0.6260 \end{bmatrix} \tag{4.32}$$

$$F_{c2} = \begin{bmatrix} 0.0085 & -0.0059 & 0.0014 & 0.0079 & -0.0003 \\ 1.5629 & -1.4208 & 0.4673 & 0.0004 & 0.0101 \end{bmatrix} \tag{4.33}$$

And

$$F_c = [F_{c1}; F_{c2}]$$

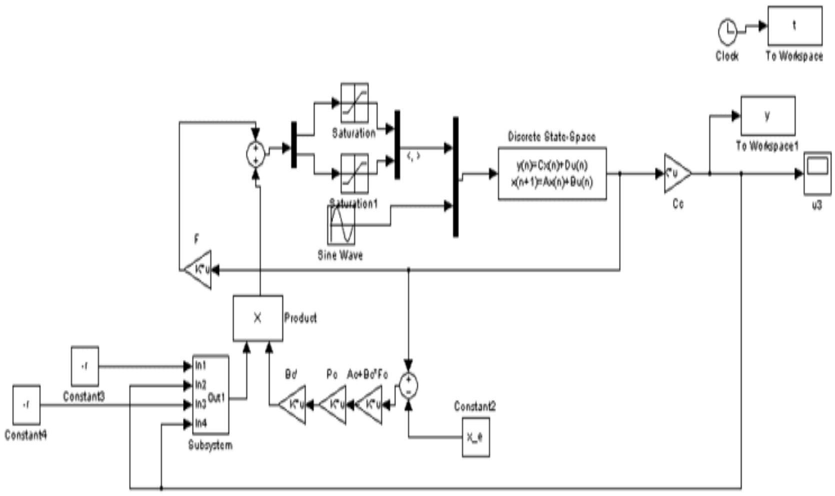


Figure 4.8: Simulink block for the simulation

$$Gc = \begin{bmatrix} 0.0757 \\ 6.4663 \end{bmatrix} \tag{4.34}$$

The nonlinear part is

$$\rho_1(r, y) = -\alpha_1 * (e^{(-|(1-\lambda_1 * u(2)/u(1)|} - e^{-1}) \tag{4.35}$$

$$\rho_2(r, y) = -\alpha_2 * (e^{-|1-\lambda_2 * u(4)/u(3)|} - e^{-1}) \tag{4.36}$$

where

$$\alpha_1 = 0.001;\ \alpha_2 = 0.003;\ \lambda_1 = 0.8;\ \lambda_2 = 0.2 \tag{4.37}$$

The simulation result is the Figure (4.9). In Figure (4.9), a fast and stable tracking is achieved.

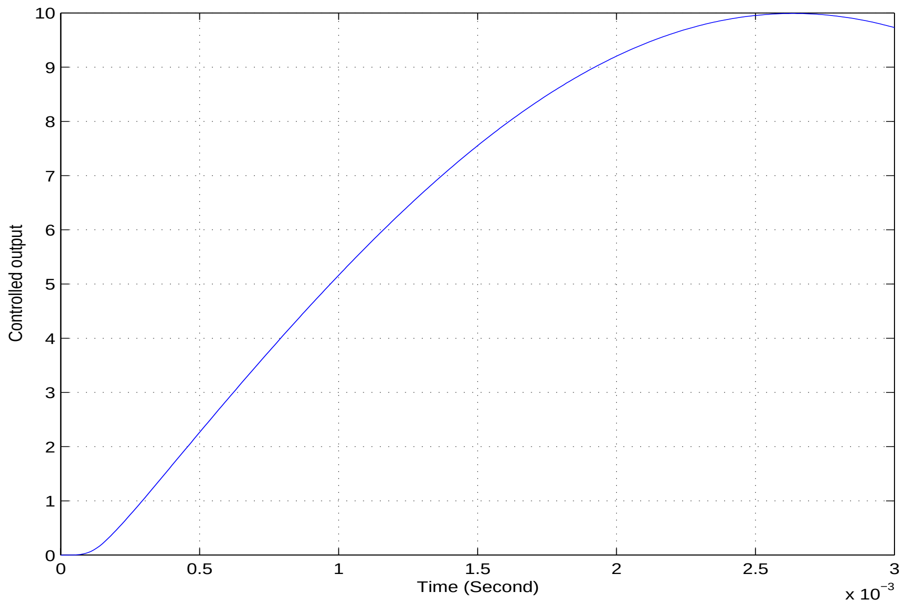


Figure 4.9: Output of a dual stage hard disk drive via CNF controller

4.2 Conclusions

In this chapter, we have successfully designed a CNF discrete-time controller which can control the dual stage hard disk drive and achieve a very good performance.

Chapter 5

Asymptotic Time Optimal Tracking

5.1 Introduction and Problem Statement

It is well known (see e.g., Hu and Lin [18]) that every physical system in our real life has nonlinearities and very little can be done to overcome them. Many practical systems are sufficiently nonlinear so that important features of their performance may be completely overlooked if they are analysed and designed through linear techniques. For example, as pointed out in Chen et al. [1],[6] that the actuator saturation in a hard disk drive has seriously limited the performance of its overall servo system. Traditionally, the most popular nonlinear control technique used in the design of servo systems, especially the hard disk drive servo systems, is the so-called proximate time-optimal servomechanism (PTOS) proposed by Workman [9], which achieves near time-optimal performance for a large class of motion control systems characterised a double integrator, e.g., hard disk drives and spring-mass mechanical systems. The PTOS was actually modified from the well-known time-optimal control or bang-bang control. However, it is made to yield a minimum variance with smooth switching from the track seeking to track following modes via a mode switching controller. It was shown in Workman [9] that by properly adjusting the controller parameters, the settling time for tracking a step reference in the resulting servo system with the PTOS controller can be made as close as possible to the optimal time achieved by the bang-bang control.

We note that the time-optimal control or bang-bang control indeed yields the best performance in point-to-point tracking, although such a technique cannot be used in practical situations. It is well known that the resulting system is very sensitive to the uncertainties and noises. Moreover, it is generally not necessary to have a precise point to point tracking in practical situations. Instead, it would be more preferable to consider asymptotic tracking in which the tracking target is defined as a small neighbourhood of a given setpoint. We believe that such a consideration is very practical. For example, in a hard disk drive servo system (see e.g., [1, 6]), it is a common practice to activate its read/write head to read or write data once it enters $\pm 5\%$ of the data track-width of the target setpoint.

Interestingly, it has been recently demonstrated by an example in [1, 6] that the time-optimal control or bang-bang control, and consequently the PTOS, do not necessarily yield the best performance in asymptotic tracking situations. There are control laws that would yield a better performance than that of the time-optimal control. This is actually the motivation for the work in this chapter. Our goals or contributions are two-fold: 1) to derive the optimal settling for asymptotic tracking;

and 2) to find a control law that achieves this optimal performance.

To be more specific, we consider a class of second order linear systems Σ characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ a \end{bmatrix} \text{sat}(u), \quad y = [\,1 \quad 0\,]\, x, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x(0) = x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}, \tag{5.1}$$

where x is the state, y is the measurement output, a is a constant and $\text{sat}(u)$ is control input to the system with

$$\text{sat}(u) = \text{sign}(u) \times \min\{u_{\max}, |u|\}. \tag{5.2}$$

As pointed out earlier, there are a large class of real life problems, such as hard disk drives and spring-mass mechanical systems, can be modeled as a double-integrator system characterized by (5.1). The problem we consider and solve in this chapter is the following:

Definition 5.1 *Consider the system of (5.1) with actuator nonlinearities. Let r be a reference target and δ be a positive scalar and $\delta \in [0, 1]$. Let*

$$u = \phi(y, r, \delta) \tag{5.3}$$

be an internally stabilizing controller for the system, i.e., the closed-loop system comprising of the given system Σ of (5.1) and the control law of (5.3) is asymptotically stable. Let $t_s(x_0, r, \delta, \phi)$ be the corresponding settling time for the resulting system output $y(t, \phi)$ to enter the δ-neighbourhood of the target reference, i.e, $t_s(x_0, r, \delta, \phi)$ is the smallest scalar such that for all $t \geq t_s(x_0, r, \delta, \phi)$,

$$|y(t, \phi) - r| \leq \delta \cdot |r| \quad \textit{and} \quad \lim_{t \to \infty} y(t, \phi) = r. \tag{5.4}$$

Finally, we let $t_s^*(x_0, r, \delta)$ be the optimal settling time over all the internally stabilizing controllers, i.e.,

$$t_s^*(x_0, r, \delta) := \inf\Big\{ \ t_s(x_0, r, \delta, \phi) \ \Big| \ \phi(y, r, \delta) \ \textit{internally stabilizes } \Sigma \ \Big\}. \tag{5.5}$$

The asymptotic time-optimal tracking (ATOT) control problem is to find a stabilizing measurement feedback control law $\phi^*(y, r, \delta)$ such that $t_s(x_0, r, \delta, \phi^*) = t_s^*(x_0, r, \delta)$.

The detailed derivations for the optimal asymptotic tracking performance t_s^* and the optimal controller ϕ^* are given respectively in Sections 5.2 and 5.3.

5.2 Optimal Settling Time

We derive in this section the optimal settling time $t_s^*(x_0, r, \delta)$ for the asymptotic time-optimal tracking problem defined in Definition 5.1. We will focus on the case when the target reference r is a step function, i.e., r is a constant. First, we note that x_1 in (5.1) usually represents the displacement of its corresponding physical system, while x_2 represents its velocity. For simplicity of presentation, we assume that the initial velocity of the system is zero, i.e., $x_{20} = 0$. Without loss of generality, we can also assume that the initial displacement is zero $x_{10} = 0$. If $x_{10} \neq 0$, we can re-define a new target reference $r_{\text{new}} = r - x_{10}$. Then, the problem of tracking r with nonzero initial condition is equivalent to that of tracking r_{new} with zero initial condition. Similarly, for simplicity, we assume $a = 1$ and $u_{\max} = 1$ in (5.1). This can be done by a proper scaling on u and r. We have the first main result of the chapter.

Theorem 5.1 *Consider the given system Σ of (5.1) with $a = 1$, $u_{\max} = 1$ and $x_0 = 0$. Given a step target reference r (for simplicity, we assume $r \geq 0$) and a positive scalar $\delta \in [0, 1]$, the optimal settling time for Σ under all possible stabilizing control laws (see, e.g., Definition 5.1) is given by:*

$$t_s^*(r,\delta) = \begin{cases} 2(\sqrt{r(1+\delta)} - \sqrt{r\delta}), & 0 \leq \delta < \frac{1}{3}, \\ \sqrt{2r(1-\delta)}, & \frac{1}{3} \leq \delta \leq 1. \end{cases} \tag{5.6}$$

Note that x_0 is dropped from the above expression as x_0 is assumed to be zero.

Proof. Since the system is a double integrator system, if we figure out x_2 versus time t (see figure (5.1)), then the ouput $y = x_1 = \int_0^{T_t} x_2(\tau)d\tau$, where $T_t \geq 0$ is the desired time instant, is simply the net area (with $\pm$ signs) enclosed by $t = 0$, $t = T_t$, $x_2(t)$ and the time axis $x_2 = 0$.

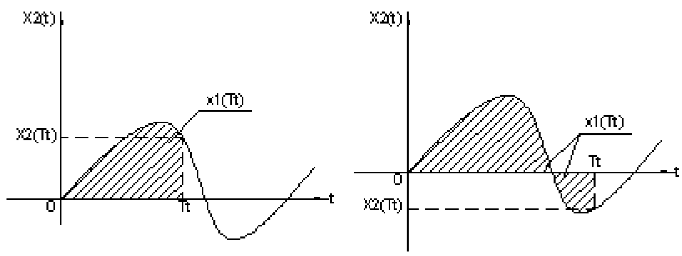


Figure 5.1: Plot of $x_2(t)$ versus t

Let us construct $\triangle OAB$ as shown in the figure (5.2) where $OA = AB$ and the slope of OA is equal to $\max(u) = u_{max} = +1$ while the slope of AB is equal to $\min(u) = u_{min} = -u_{max} = -1$.

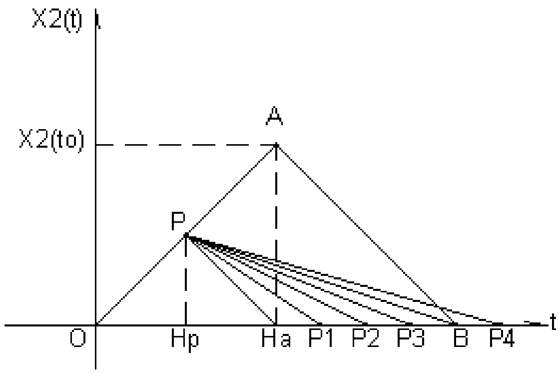


Figure 5.2: Case 1: $1/3 \le \delta \le 1$

For the case of $\frac{1}{3} \le \delta \le 1$, we first apply $u(t) = u_{max} = +1$ from $t = 0$ to $t = t_A = 2\sqrt{\frac{1}{3}r}$ and then apply $u(t) = u_{min} = -u_{max} = -1$ till $t = t_B = 4\sqrt{\frac{1}{3}r}$, as shown in figure (5.2). Since $u_{max} = 1$, $x_1 = \int_0^t au(\tau)d\tau \le \int_0^t au_{max}(\tau)d\tau = \int_0^t ad\tau = \frac{1}{2}at^2 = \frac{1}{2}t^2$, or $t \ge \sqrt{2x_1}$ for $x_1 \ge 0$, t_s^* is the time at which x_1 arrives at $(1-\delta)r$ along OA, which is $\sqrt{2r(1-\delta)}$. At $t = t_B = 4\sqrt{\frac{1}{3}r}$, the output $x_1 = \frac{4}{3}r \le (1+\delta)r$ as $\frac{1}{3} \le \delta \le 1$, so the ouput is within the region of $[(1-\delta)r,\ (1+\delta)r]$. After that, if we remove any control, $x_2 = 0$ and x_1 keeps unchanged, *i.e.*, the ouput is always within the region of $[(1-\delta)r,\ (1+\delta)r]$. This justifies our calculation of t_s^* for the case of $\frac{1}{3} \le \delta \le 1$.

For the case of $0 \le \delta < \frac{1}{3}$, we first apply $u(t) = +1$ from $t = 0$ to $t = t_A = \sqrt{(1+\delta)r}$ where the time coordinate t_A correponds to A, and then apply $u(t) = -1$ till $t = t_B = 2\sqrt{(1+\delta)r}$ where, again, the time coordinate t_B correponds to B, as shown in figure (5.3). In this case, t_s^* is the time at which x_1 arrives at $(1-\delta)r$ along OAB, which is, after some simple calculations, exactly $2(\sqrt{(1+\delta)r} - \sqrt{\delta r})$. We shall prove that there exists no shorter settling time. First we claim that $t_s^* > t_A$.

$x_1 = \int_0^{t_A} au(\tau)d\tau \leq \int_0^{t_A} au_{max}(\tau)d\tau = \int_0^{t_A} ad\tau = \frac{1}{2}at_A^2 = \frac{1}{2}t_A^2 = \frac{1}{2}(1+\delta)r < (1-\delta)r$ for $0 \leq \delta < \frac{1}{3}$. Therefore, at t_A, x_1 will not arrive at $(1-\delta)r$ and hence $t_s^* > t_A$.

Suppose we have another settling time t_s' which satisfies $t_s' < t_s^*$, then, if we indicate the point corresponding to t_s as P, there are only three possible cases for the location of the point corresponding to t_s', namely P_a, P_o or P_b, see figure (5.3), where H_a, H_p' and H_p are projection points corresponding to A, P_o (or P_a and P_b) and P respectively. Now we shall prove that all these cases are impossible. To this sequel, we will first introduce a proposition. This proposition shows that the trajectories leaving or enetring some point $x_2(t_0)$ can only take the slope between $-a$ and $+a$, which complies with $\frac{d}{dt}x_2(t) = au(t)$.

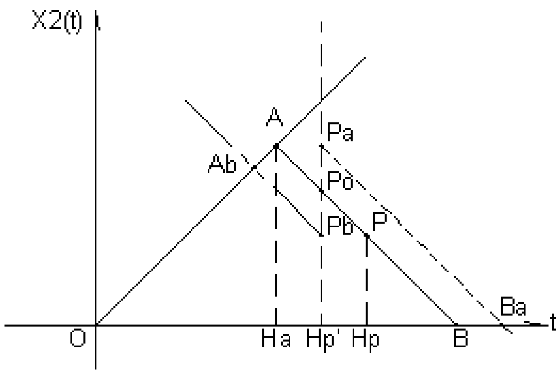


Figure 5.3: Case 2: $0 \leq \delta < 1/3$

Property 5.1 *Suppose $x_2(t_0)$ is located at some point A, then the trajectories leaving ($t > t_0$) or entering ($t > t_0$) A will be confined to the slanted shade area shown in the figure (5.4).*

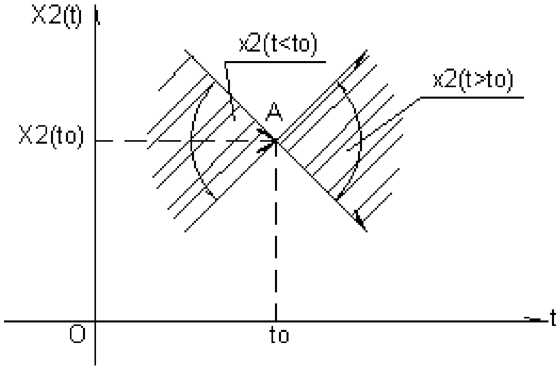


Figure 5.4: The Trajectories leaving or entering $x_2(t_0)$

Proof of the Proposition.

First we assume $t > t_0$. $x_2(t) = \int_{t_0}^{t} au(\tau)d\tau$, but $-1 \leq u(\tau) \leq +1$, which means $\int_{t_0}^{t} -ad\tau \leq x_2(t) = \int_{t_0}^{t} au(\tau)d\tau \leq \int_{t_0}^{t} ad\tau$ or, $x_2(t_0) - a(t-t_0) \leq x_2(t) \leq x_2(t_0) + a(t-t_0)$. Hence the result for the trajectories leaving A. For the case of $t < t_0$, we have $\int_{t_0}^{t} ad\tau \leq x_2(t) = \int_{t_0}^{t} au(\tau)d\tau \leq \int_{t_0}^{t} -ad\tau$ or, $x_2(t_0) + a(t-t_0) \leq x_2(t) \leq x_2(t_0) - a(t-t_0)$. Hence the result for the trajectories entering A. Now we go on with our proof of the theorem. Suppose that $x_2(t_s')$ stays at P_a, we draw a line P_aB_a parallel to PB. According to the above proposition, trajectories leaving P_a will be on or above the line P_aB_a, which implies that the area of $\triangle H_p'P_aB_a$ is the infimum for all possible $x_2(t),\ t \geq t_s'$. Since at t_s', the area is already $1-\delta$, the area or the output x_1 will definitely exceed $1+\delta$ as the area of $\triangle H_p'P_aB_a$ is larger than that of $\triangle H_pPB$, which contradicts the definition of settling time, see Definition 1.1.

Suppose now that $x_2(t_s')$ stays at P_b, we draw a line P_bA_b parallel to BP. Again, according to the above proposition, trajectories entering P_b will be on or below the line P_bA_b, which implies that the area of the polygon $OA_bP_bH'_pO$ is the supremum for all possible $x_2(t),\ 0 \le t \le t'_s$. Since at t_s, the area is already $(1-\delta)r$, we see that the area of $OA_bP_bH'_pO$ or the output $x_1(t'_s)$ will be smaller than $(1-\delta)r$, which, again, contradicts the definition of settling time in Definition 1.1.

For the last case that $x_2(t'_s)$ stays at P_o, using the same arguement as the case of $x_2(t'_s)$ staying at P_b shown above, we can claim too, that there doesn't exist such a t'_s which satisfies $t'_s < t_s$. In summary, t^*_s is indeed the desired optimal settling time for the system.

Therefore, we have

$$t^*_s = \begin{cases} 2(\sqrt{r(1+\delta)} - \sqrt{r\delta}), & 0 \le \delta < \frac{1}{3}, \\ \sqrt{2r(1-\delta)}, & \frac{1}{3} \le \delta \le 1. \end{cases} \tag{5.7}$$

This completes our proof of the theorem.

When $r = 1$, the relationship between t_s and δ can be easily plotted in Figure (5.5):

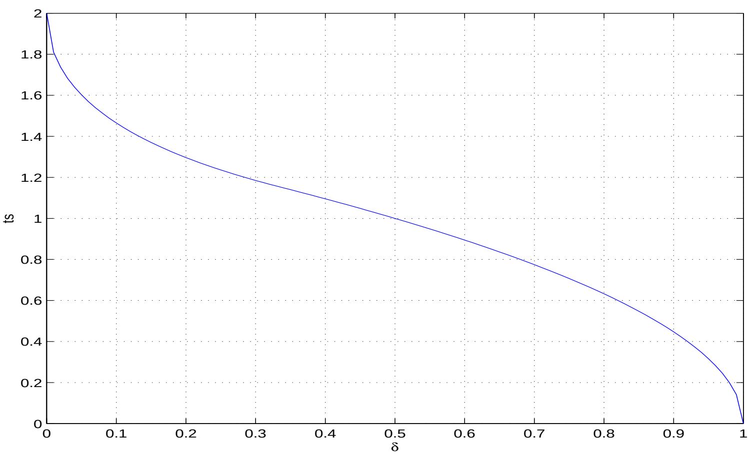


Figure 5.5: The Relationship between δ and t_s, $r = 1$

Assuming $\delta = 0.01$, the corresponding optimal settling time is $t_s^* = 1.8100$, which will be used in the illustrative example in Section 5.4.

Furthermore, assuming $\delta = \frac{1}{3}$, the corresponding optimal settling time is $t_s = 1.1547$, which is the joint point for the two different cases of δ. The point with the star is the switching point corresponding to $\delta = \frac{1}{3}$.

Remarks

1. As shown in the proof for the case of $\frac{1}{3} \le \delta \le 1$, we let the output stay at $x_1 = \frac{4}{3}r \in [(1-\delta)r,\ (1+\delta)r]$ while $x_2 = 0$. As a matter of fact, we can set it to be any $x_1 \in [2(1-\delta)r,\ (1+\delta)r]$, which can be realized by let $x_2(t)$ go along the lines PH_a, PP_1, PP_2, PP_3, PB, $PP_4 \cdots$ as shown in figure (5.2), corresponding to the decreasing amplitude of control input gradually. Obviously, we have infinitely many choices. In the following remarks, we simplify $t_s^*(r, \delta, x_{10}, u_-, u_+)$ as t_s^*.

2. For the general case when $a > 0$, $r > 0$, $x_{20} = 0$, $x_{10} < (1-\delta)r$ where $\delta \ge 0$ is desired tracking bound, and $\max(u) = u_+ > 0$, $\min(u) = -u_- < 0$ where u_+ doesn't necessarily equal u_-, by introducing new tracking area of $[(1-\delta)r - x_{10},\ (1+\delta)r - x_{10}]$ and hence artificially set a new zero initial condition for x_1, we have the following formula:

$$t_s^* = \begin{cases} \sqrt{\frac{2[(1+\delta)r - x_{10}]u_-}{au_+(u_+ + u_-)}} + \sqrt{\frac{2[(1+\delta)r - x_{10}]u_+}{au_-(u_+ + u_-)}} - 2\sqrt{\frac{r\delta}{au_-}}, & 0 \le \delta < m_1, \\ \sqrt{\frac{2[(1-\delta)r - x_{10}]}{au_+}}, & m_1 \le \delta \le 1. \end{cases} \tag{5.8}$$

where $m_1 = \frac{u_+(r - x_{10})}{(u_+ + 2u_-)r}$.

By applying max(u) first and then min(u), we obtain the desired control input.

3. For the case when $a > 0$, $r > 0$, $x_{20} = 0$, $(1-\delta)r \leq x_{10} \leq (1+\delta)r$ where $\delta \geq 0$ is desired tracking bound, obviously $t_s^* = 0$.

4. Fot the case when $a > 0$, $r > 0$, $x_{20} = 0$, $x_{10} > (1+\delta)r$ where $\delta \geq 0$ is desired tracking bound, the settling time shall be the infimum of the time instant at which the system output reaches $(1+\delta)r$. The formula for t_s^* can be revised as follows.

$$t_s^* = \begin{cases} \sqrt{\frac{-2[(1-\delta)r-x_{10}]u_+}{au_-(u_-+u_+)}} + \sqrt{\frac{-2[(1-\delta)r-x_{10}]u_-}{au_+(u_-+u_+)}} - 2\sqrt{\frac{r\delta}{au_+}}, & 0 \leq \delta < m_2, \\ \sqrt{\frac{-2[(1+\delta)r-x_{10}]}{au_-}}, & m_2 \leq \delta \leq 1. \end{cases} \tag{5.9}$$

where $m_2 = \frac{-u_-(r-x_{10})}{(u_-+2u_+)r}$

By applying min(u) first and then max(u), we obtain the desired control input.

5. For the case when $a > 0$, $r < 0$, $x_{20} = 0$, $x_{10} > (1-\delta)r$ where $\delta \geq 0$, by introducing new tracking area of $[(1+\delta)r - x_{10},\ (1-\delta)r - x_{10}]$ and hence artificially set a new zero initial condition for x_1, we apply the following formula (5.10) to get the optimal settling time.

$$t_s^* = \begin{cases} \sqrt{\frac{-2[(1+\delta)r-x_{10}]u_+}{au_-(u_-+u_+)}} + \sqrt{\frac{-2[(1+\delta)r-x_{10}]u_-}{au_+(u_-+u_+)}} - 2\sqrt{\frac{-r\delta}{au_+}}, & 0 \leq \delta < m_3, \\ \sqrt{\frac{-2[(1-\delta)r-x_{10}]}{au_-}}, & m_3 \leq \delta \leq 1. \end{cases} \tag{5.10}$$

where $m_3 = \frac{u_-(r-x_{10})}{(u_-+2u_+)r}$.

By applying $\min(u)$ first and then $\max(u)$, we obtain the desired control input.

6. For the case when $a > 0$, $r < 0$, $x_{20} = 0$, $(1+\delta)r \leq x_{10} \leq (1-\delta)r$ where $\delta \geq 0$ is desired tracking bound, obviously $t_s^* = 0$.

7. For the case when $a > 0$, $r < 0$, $x_{20} = 0$, $x_{10} < (1+\delta)r$ where $\delta \geq 0$, we apply the following formula (5.11) to get the optimal settling time. Again, the settling time shall be the infimum of the time instant at which the system output reaches $(1+\delta)r$.

$$t_s^* = \begin{cases} \sqrt{\frac{2[(1-\delta)r - x_{10}]u_-}{au_+(u_+ + u_-)}} + \sqrt{\frac{2[(1-\delta)r - x_{10}]u_+}{au_-(u_+ + u_-)}} - 2\sqrt{\frac{-r\delta}{au_-}}, & 0 \leq \delta < m_4, \\ \sqrt{\frac{2[(1+\delta)r - x_{10}]}{au_+}}, & m_4 \leq \delta \leq 1. \end{cases} \tag{5.11}$$

where $m_4 = \frac{-u_+(r - x_{10})}{(u_+ + 2u_-)r}$.

By applying $\max(u)$ first and then $\min(u)$, we obtain the desired control input.

8. For the case when $a < 0$, $r < 0$, $x_{20} = 0$, $x_{10} > (1-\delta)r$ where $\delta \geq 0$, we have the following formula (5.8). By applying $\max(u)$ first and then $\min(u)$, we obtain the desired control input.

9. For the case when $a < 0$, $r < 0$, $x_{20} = 0$, $(1+\delta)r \leq x_{10} \leq (1-\delta)r$ where $\delta \geq 0$ is desired tracking bound, obviously $t_s^* = 0$.

10. Fot the case when $a < 0$, $r < 0$, $x_{20} = 0$, $x_{10} < (1+\delta)r$ where $\delta \geq 0$, the settling time shall be the infimum of the time instant at which the system

output reaches $(1+\delta)r$. The formula for t_s^* is exactly the same as formula (5.9). By applying $\min(u)$ first and then $\max(u)$, we obtain the desired control input.

11. For the case when $a < 0$, $r > 0$, $x_{20} = 0$, $x_{10} < (1-\delta)r$ where $\delta \geq 0$, we apply formula (5.10) to get the optimal settling time. By applying $\min(u)$ first and then $\max(u)$, we obtain the desired control input.

12. For the case when $a < 0$, $r > 0$, $x_{20} = 0$, $(1-\delta)r \leq x_{10} \leq (1+\delta)r$ where $\delta \geq 0$ is desired tracking bound, obviously $t_s^* = 0$.

13. For the case when $a > 0$, $r > 0$, $x_{20} = 0$, $x_{10} > (1+\delta)r$ where $\delta \geq 0$, we apply formula (5.11) to get the optimal settling time. Again, the settling time shall be the infimum of the time instant at which the system output reaches $(1+\delta)r$. By applying $\max(u)$ first and then $\min(u)$, we obtain the desired control input.

14. We have given the formulae for all the possible cases when $x_{20} = 0$. When $x_{20} \neq 0$, things become more complicated as there are too many different combinations of conditions regarding a, x_{10}, r, and $\max(u) = u_+ > 0$, $\min(u) = -u_- < 0$. However, for each specified case, using almost the same reasoning as the proof of Theorem 5.1, we can obtain the corresponding results.

5.3 Asymptotic Time-Optimal Tracking Controller Design

We now proceed to design a controller that would achieve the optimal settling time as given in Theorem 5.1. We have already shown in the proof of Theorem 5.1 that by applying $u = +1$ from $t = 0$ to $t = t_A = \sqrt{(1+\delta)r}$ and then apply $u = -1$ till $t = t_B = 2\sqrt{(1+\delta)r}$ for the case of $0 \le \delta \le \frac{1}{3}$, we end up with $x_1(t_B) = (1+\delta)r$ and $x_2(t_B) = 0$. For the case of $\frac{1}{3} < \delta \le 1$, we apply $u = +1$ from $t = 0$ to $t = t_A = 2\sqrt{\frac{1}{2}r}$ and then apply $u = -1$ till $t = t_B = 4\sqrt{\frac{1}{2}r}$ and end up with $x_1(t_B) = (1-\delta)r$ and $x_2(t_B) = 0$. The next step to drive the system output to the target r is a trival design problem. There are many available methods which can reach this goal, which further drives x_1 to r and x_2 to 0 asymptotically without making x_1 exceeding the tracking region of $[(1-\delta)r,\ (1+\delta)r]$. A simple choice is to use time-optimal control. It drives the system output to the target monotonically and hence will never exceed the tracking bound while at the same time x_2 reaches 0. We can use u_{max} and u_{min} for the time-optimal control design or even we can use smaller control signals, say αu_{max} and αu_{min} where $0 < \alpha < 1$, as saturation levels, which only makes the time to the target longer.

However, this controller can not be used in practical situations as it is basically the same as time-optimal controller. We will turn to other methods although we may only obtain sub-optimal ATOT controllers.

Again, we try to apply the composite nonlinear feedback (CNF) control to achieve a fast settling time. Rewrite (5.1) in the following form:

$$\begin{cases} \dot{x} &= Ax + B\text{sat}(u) \\ y &= Cx \end{cases} \tag{5.12}$$

where A, B, C are the corresponding matrices in (5.1). We present the control algorithm step by step.

Step 1: Linear Part

$$u_L = Fx + Gr \tag{5.13}$$

where F and G are chosen such that (1) $(A + BF)$ is an asymptotically stable matrix, (2) The closed system $C(sI - A - BF)^{-1}B$ has certain properties,such as having a small damping ratio, (3) G is a scalar given by $G = -[C(A + BF)^{-1}B]^{-1}$ and r is the command input.

Step 2: Nonlinear Part

$$u_N = \rho B^T P(x - x_e) \tag{5.14}$$

where ρ is a nonpositive, Lipschitz continuous function and P is the solution of the following Lyapunov equation,

$$(A + BF)^T P + P(A + BF) = -W \tag{5.15}$$

W is a positive definite matrix, $x_e = -(A + BF)^{-1}BGr$ and $H := [1-F(A + BF)^{-1}B]G$.

For any $\delta \in (0, 1)$, let c_δ be the largest positive scalar satisfying the following condition:

$$|Fx| \;\leq\; (1 - \delta)\bar{u}, \forall x \in X_\delta := \{x' P x \leq c_\delta\} \tag{5.16}$$

The following two conditions should be guaranteed in the CNF control design.

$$\hat{x}_0 = x_0 - x_e \in X_\delta \tag{5.17}$$

$$|Hr| \leq \delta\bar{u} \tag{5.18}$$

Step 3: Composite Control

$$\begin{aligned} u &= \phi_{\text{CNF}}(y, r, \delta, \varepsilon) = u_L + u_N \\ &= Fx + Gr + \rho B^T P(x - x_e) \end{aligned} \tag{5.19}$$

The following theorem (5.2) is adopted from Chen et al. [1], which tells us the CNF controller can achieve the asymptotic tracking for an SISO linear system.

Theorem 5.2 *The control law (5.19) is capable of driving the controlled output y, to track asymptotically a step command input r, provided that conditions (5.17) and (5.18) are satisified.*

There are many choices for ρ, only if ρ is a non-positive function, locally Lipschitz. In Lin et al. ([8]), it gave some ideas how to choose the nonlinear part for a second order SISO system, such that the damping ratio tended, asymptotically, to infinity. In this chapter, we choose ρ in (5.19) as following, which is a non-positive function, locally Lipschitz in y,

$$\rho = \varepsilon(e^{-r} - e^{-|r-y|}), \quad \varepsilon > 0 \tag{5.20}$$

The transient performance of this system can be improved dramatically: a faster rise time, a shorter settling time, with less overshoot, which is inherently the advantage for CNF control over the linear feedback control.

Then we shall note that the above CNF controller (5.19) is parameterized by another additional tuning parameter ε, which is to be adjusted to achieve the optimal settling time. In Section (5.4), the simulation will show how this parameter affects the settling time. And Figure (5.9) shows the trend there is one point, where $\varepsilon = \varepsilon^*$, and $t_s = t_s^*$, although there is no rigorous proof. However, it is easy to tune only one parameter to approximate the optimal settliing time by simulation.

Moreover, we can provide some guidelines to choose the parameters to achieve a faster tracking,

1. Choose F such that the eigenvalues of $(A + BF)$ such that conditions (5.17), (5.18) are satisfied. Furthermore, if there is no nonlinear part in the control law, there should exist overshoot outside the desired region.

2. Choose proper ε, and the nonlinear part will gradually change the damping ratio such that the settling time can approximate the optimal value t_s^*.
   First randomly choose an ε, if the overshoot is beyond the scope you expect, then choose a smaller one ε accordingly. If the output reaches the destination increasingly at infnity, choose a bigger one. But for the ε you have chosen, there should have overshoot in order to get a faster settling time.When the overshoot is bounded in the region, tune this parameter ε gradually and slightly around this value.

Because we add one dynamic term in the control signal, the system will move the eigenvalues away from the imaginary axis, due to the nonlinear part, which will enhance the robustness of the system. And the only part we need to change is the coefficient term ε in ρ after we choose the feedback gain F.

5.4 Simulation

We now illustrate the results in the following example. We will use the model in (5.1) with $a = 1$, $\delta = 0.01$ and $r = 1$. And we will present our results with the comparison to time optimal control.

The parameters we chose are:

$$W = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, F = [-50 \quad -10], \quad \varepsilon = 133.5 \tag{5.21}$$

Figure (5.6) gives the controlled output y under the TOC (The dot dash line) and ATOT (The solid line) approaches.

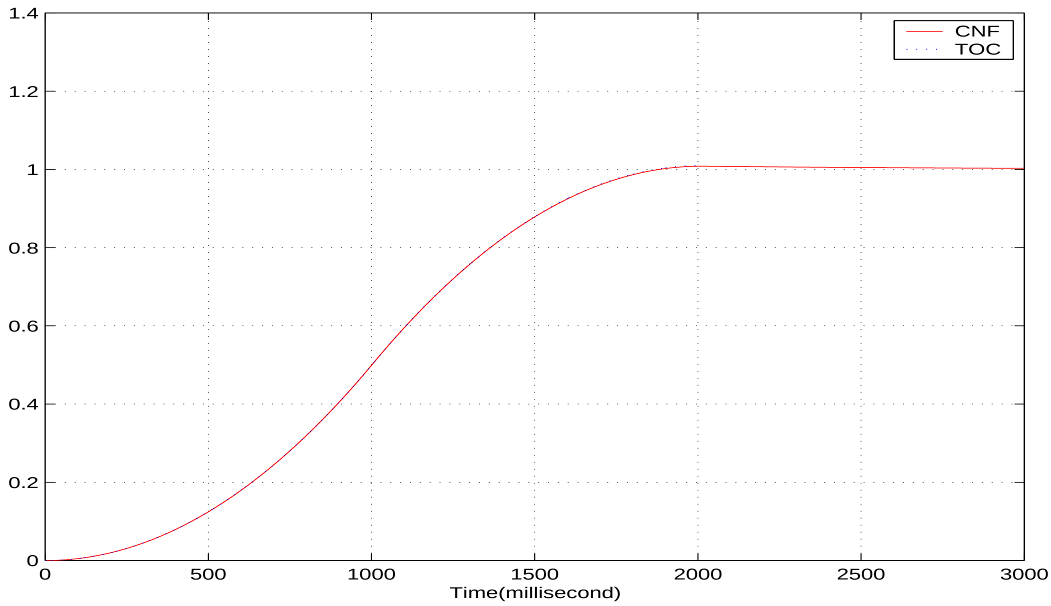


Figure 5.6: Controlled output for the whole process

The settling time under ATOT is $t_s = 1.8110$, which is very close to the optimal value $t_s = 1.8100$. While the settling time with TOC is 1.8586. We can see there exists much difference.

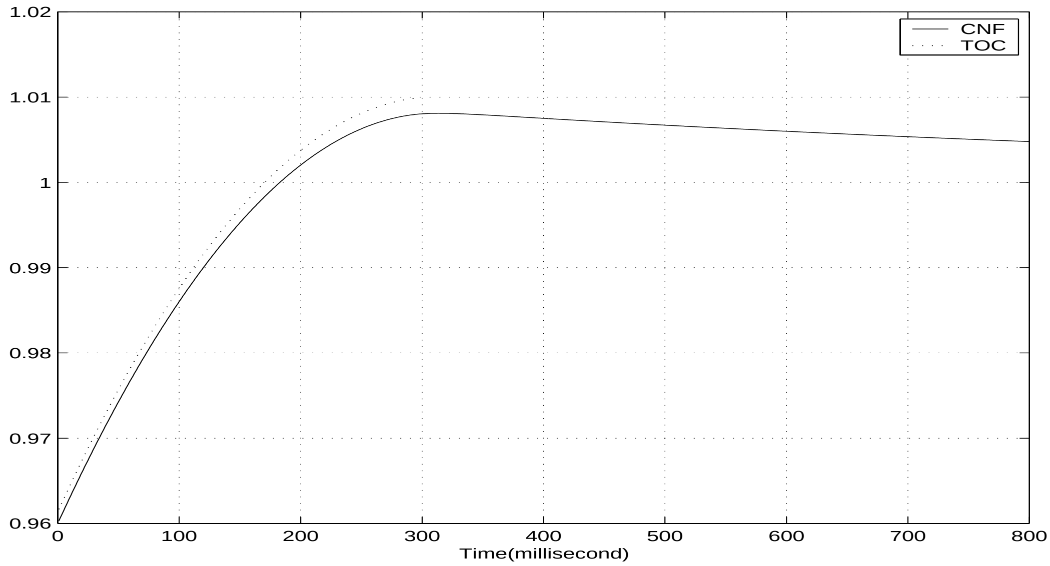


Figure 5.7: Controlled output for a selected period

In Figure (5.7), we can conclude that the ATOT is faster that the TOC under the same definition of settling time. Although the time we can spare is very short, this small improvement will be very useful in some actual physical system, such as the hard disk drive servo system. Furthermore, the controller of ATOT is robust and can reject noise as well. It shows the advantage over the TOC.

Figure (5.8) gives the controlled signal, which is continuous and will decay when the output converges to the desired position. Both linear part and nonlinear part contribute different weight to the CNF control law at different stages of the control.

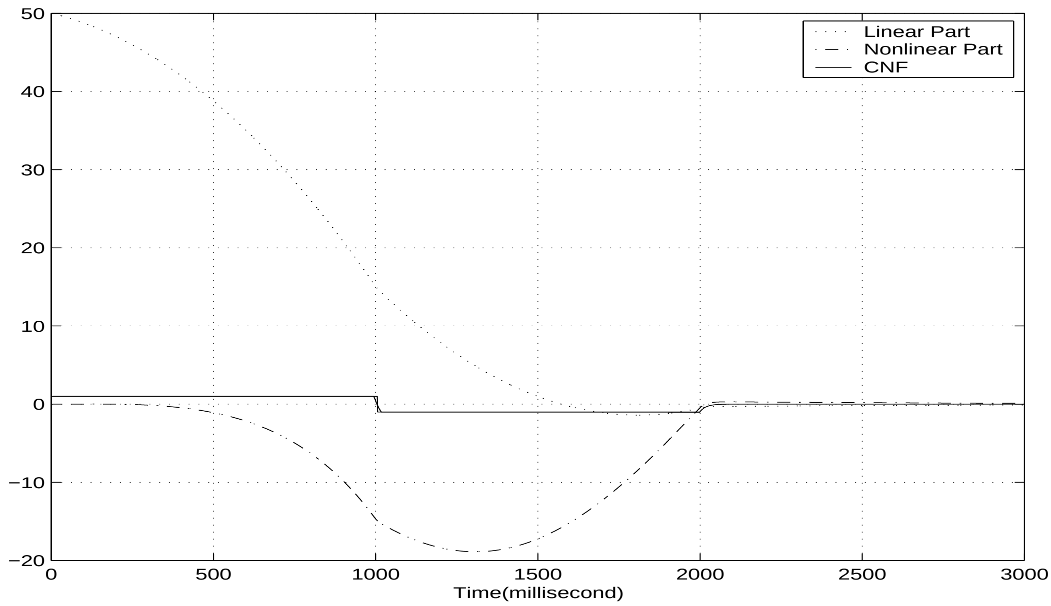


Figure 5.8: The control singal

Moreover, we present a figure in (5.9) about the relationship between different values of ε and settling time. It reveals that near the optimal value of ε, the settling time does not change much. Once it is away from the nearby region, it grows up very quickly.

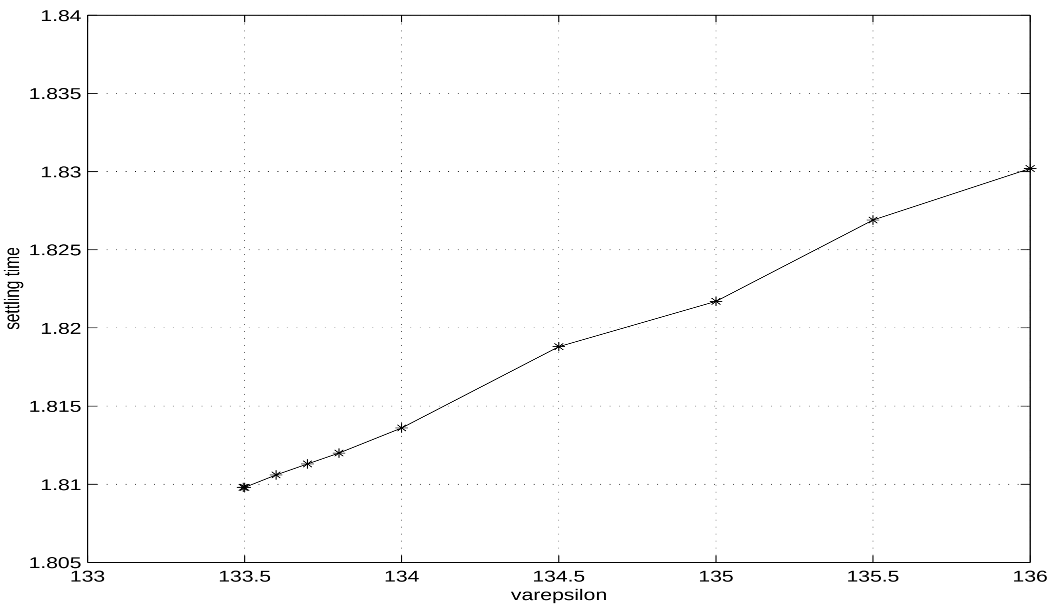


Figure 5.9: Relationship between ε and settling time

5.5 Conclusion

In this chapter, we propose the ATOT problem and present the formula of the optimal settling time under ATOT control. And the composite nonlinear feedback control can serve as a solution to approximate the optimal settling time. Further research will be focused on the rigorous findings on the optimal settling time for CNF controllers and higher order systems.

Chapter 6

Conclusions and Future Directions

6.1 Summary of Results

In this thesis, we have addressed some issues in nonlinear control. We have considered a class of linear systems with input saturation and we have a systematic controller design process. In these areas of research, so far we have achieved the following:

• The work is mainly to complete and extend the composite nonlinear feedback controller design techniques.

We have extended the composite nonlinear feedback control to the multi-input multi-outpu systems and discrete time systems. We have sucessfully constructed a state feedback controller, a full order measurement feedback controller and a reduced order measurement feedback controller.

We have showed the efficiency of the CNF controller design scheme by two examples. In the air to air missile system, we even compared our results with those of linear controller and others' controllers. The composite nonlinear feedback control has demonstrated advantage over them. In a dual-stage hard disk drive system, we have given the process for identification, modelling, and simulation in a CNF controller design.

• We have deduced the optimal setting time for a second order system with input saturation. We present a formula which gives the explicit optimal setting time according to the system parameters. This optimal settling time is a good indication to which level we can get at the first stage of the controller design. The proposed composite nonlinear feedback control has only one tuning paramenter, which is adjusted to approximate the optimal settling time.

6.2 Prospect of Research

Despite the work we have finished and the results we have obtained, there are still a lot of work we can do for the future directions.

◦ Extend the CNF control to more general systems and find other ways to determine the invariant set. We can try to find a more general way to select the nonlinear parameters in the CNF controller. And we can explore the possibility to track other kinds of signal, other than the constant setpoint, asymptotically. Furthermore, we may consider the problem of disturbance rejection based on CNF control.

◦ Extend the formula on the optimal settling time to more general systems. Due to some constraints we currently have, we can only apply this formula to a second order system. Acutally, we can try to extend it to a higher order system, even to a general linear system with no constraints on the system matrix, input matrix and output matrix.

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Details of Papers

Journal Paper

Yingjie He, Ben M. Chen and Chao Wu, "Composite Nonlinear Control with State and Measurement Feedback for General Multivariable Systems with Input Saturation," *International Journal of Control*, submitted, 2003

Conference Paper

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