DEVELOPMENT OF THE FINITE AND INFINITE INTERVAL LEARNING CONTROL THEORY

JING XU

NATIONAL UNIVERSITY OF SINGPAORE
2003

DEVELOPMENT OF THE FINITE AND INFINITE INTERVAL LEARNING CONTROL THEORY

BY

JING XU

(B. ENG., M. ENG.)

A DISSERTATION SUBMITTED

IN PARTICIAL FULFILMENT OF THE

REQUIREMENTS

FOR THE DEGREE OF

DOCOTOR OF PHILOSOPHY IN ENGINEERING

NATIONAL UNIVERSITY OF SINGAPORE

2003

Acknowledgments

I would like to express my deepest gratitude to my supervisor, Prof. Xu Jian-Xin, for his valuable guidance, encouragement and patience during my entire PhD study. His wealthy knowledge and accurate foresight have impressed and benefited me very much, especially in the area of nonlinear control and learning control. Moreover, his rigorous scientific approach and endless enthusiasm to the career have influenced me significantly. Without his continuous guidance and help, I could not have accomplished this thesis and all the relevant works.

Thanks are also presented to the researchers, working in the center of intelligent control of the Electrical & Computer Engineering Department, National University of Singapore, for their encouragement and valuable advice.

I would like to take this opportunity to thank Dr. Chen Jianping, Mr. Zhang Hengwei, Dr. Pan Yajun, Ms. Yan Rui, Ms. Zheng Qing and all the other labmates in the Control & Simulation Lab for their kindly assistance in both my research work and the other personal aspects. My very special thanks go to Dr. Tan Ying from whom I have learned a lot via frequent discussions.

Finally, I am indebted to my parents, my husband Mr. Ou Ke and my younger sister Miss Wei Zeli, for their constant support and encouragement throughout all my studies. It is impossible to thank them adequately. I would like to dedicate this thesis to all my family members.

Xu Jing

June 2003

Contents

Summary

This thesis centers on the control theories of Finite Interval Learning (FIL) and Infinite Interval Learning (IIL) for nonlinear systems with deterministic uncertainties. The main contributions of this thesis lie in the following three aspects:

- **Contraction Mapping (CM) Based FIL for Systems with Nonsmooth Nonlinearities**

  Traditional Iterative Learning Control (ILC), based on CM principle, is an effective way for FIL and has been successfully applied to a variety of repeatable control problems. However, the application is limited to smooth system dynamics. Considering the wide existence of nonsmooth nonlinearities in real control systems, in this thesis CM-type FIL, i.e. ILC, has been extended to nonlinear discrete-time systems with input deadzone or backlash. Based on the scheme we proposed, only if both the control target and the dynamic system are repeatable, the unknown deadzone or backlash can be compensated automatically via learning and perfect tracking over the entire time interval can be obtained iteratively. This new methodology provides a simple way to deal with such kind of high nonlinearities.

- **Composite Energy Function (CEF) Based FIL**

  CEF-type FIL was introduced to fully consider the impact of system dynamics, based on which FIL was extended to Non-Global Lipschitz Continuous (NGLC) systems. Benefiting from CEF, we have developed several FIL and robust FIL schemes to deal with systems with norm-bounded uncertainties which may be Global Lipschitz Continuous (GLC) or NGLC. Furthermore, uniform learning convergence for all the developed algorithms can be guaranteed.

Conventional FIL schemes are only applicable to uniform trajectory tracking problems. To overcome this limitation, we have constructed a new kind of CEF-type FIL approaches to enable the learning from non-uniform tracking control tasks in the presence of time-varying and/or time-invariant parametric uncertainties. Therefore, the target trajectories of any two consecutive iterations can be completely different, which greatly widens the application areas of FIL.

To further extend the implementation of FIL, a novel Fuzzy Logic Learning Control (FLLC) scheme has been outlined in this thesis. The FLLC approach integrates two main control strategies: Fuzzy Logic Control (FLC) as the basic control part and FIL as the refinement part. The incorporation of FIL into FLC ensures the capability of improving control performance through learning iterations.

- **CEF Based IIL**

  By taking the advantage of CEF analysis method, we further extended FIL to IIL for both parametric and norm-bounded uncertainties, which includes the conventional Repetitive Control (RC) as a special case.

  In CEF-type FIL/IIL schemes, system states are assumed to be available. To facilitate the practical application, this thesis provides a kind of observer-based IIL algorithm for a class of nonlinear uncertain systems with unknown system states. Based on the state estimation and periodic updating, the proposed IIL scheme guarantees the asymptotical convergence of the output tracking in the presence of system nonlinearity and periodic time-varying parametric uncertainties. Furthermore, the observer based IIL can be applied to FIL directly.

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background

In a control system design, if all the information about the controlled process is known *a priori*, vast majority of conventional control techniques can be used. However, in most practical instances, the systems to be controlled are unknown or incompletely known. One general approach is to design a controller which is able to estimate the unknown information and a control action is further added based on the estimated information. As a result, if the estimated one converges to the true case gradually, the controller design eventually becomes same as the case when all the information is known *a priori*. Because of the capability of progressively improving the control performance , such kind of control systems are called learning control systems (Hklansky, 1966; Fu, 1970).

In this thesis, we will focus on a certain category of learning control systems where the controlled process and/or the tracking tasks are of a *repetitive* or *periodic* nature. Ultimately, high tracking performance, i.e. *perfect tracking*, is our control target. This kind of control problems is often encountered in many industrial processes, such as industrial robots on assembly processes, batch reactors, IC welding processes and wafter processes. The main idea of such class of learning control is to improve the tracking performance in an iterative manner by using the information obtained from previous iteration or period, which is similar to the learning methodology of human beings.

According to the time domain nature of a system, and the requirement from a control task, we classify this kind of learning into Finite Interval Learning (FIL) and Infinite Interval Learning (IIL).

1.1.1 Finite Interval Learning Control (FIL)

FIL refers to the learning over a fixed finite time interval $[0, T]$, during which both the controlled system and the control target are repeatable. The goal of FIL control system design is to get the control signal iteratively which ensures the system output could follow the desired trajectory perfectly over the whole time interval even in the presence of deterministic system uncertainties.

Lots of nonlinear control approaches, such as adaptive control and robust control, have been proposed to cope with the tracking problems of uncertain systems, however in most cases only bounded tracking error or asymptotic convergence can be obtained. Hence, precise tracking along entire span of trajectory is impossible. Therefore, FIL complements the existing control methods in the sense that it targets at perfect tracking in a finite time interval.

The basic idea of FIL comes from Iterative Learning Control (ILC), which is used to deal with repeated tracking control problems or repeatable disturbance rejection problems over finite time interval. In the past two decades, ILC has been developed to a typical method of FIL.

The concept of ILC was first proposed and formulated by (Arimoto *et al.*, 1984). So far, all kinds of ILC control schemes have been proposed and investigated (Bien and Chung, 1980; Hwang *et al.*, 1991; Moore, 1993; Fang and Chow, 1998; Kurek and Zaremba, 1993; Saab, 1995; Xu, 1997; Chien, 1998; Bien and Xu, 1998; Bien *et al.*, 1999; Chen and Wen, 1999; Wang, 2000; Park and Bien, 2000). Moreover, ILC has been widely applied to mechanical systems such as robotics, electrical systems such as servo motors, chemical systems such as batch reactors, as well as aerodynamic systems, etc.

Briefly speaking, the strategy of ILC is to update the control inputs iteratively to generate the required outputs. Fig. 1.1 shows the basic ILC schematic diagram. In

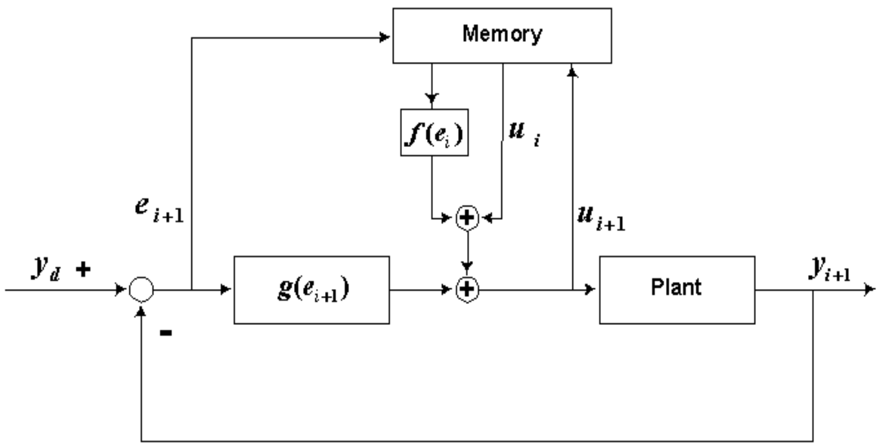

Figure 1.1: Basic structure of Iterative Learning Control.

addition to the standard feedback loop, memory components are used to record the preceding control signal $u_i(t)$ and error signal $e_i(t)$ which are incorporated into the present control $u_{i+1}(t)$. Here time $t \in [0, T]$ and $i \in \mathcal{Z}_+ \triangleq \{0, 1, \cdots\}$ denotes the iteration number. From the control point view, the memory components are used to realize the feedforward compensation. It can be clearly seen that, when $y_i = y_d$, the tracking error is zero and the control feedback part is also zero. However, to track a target trajectory and reject a persistent disturbance, a non-zero control profile will be demanded. Therefore, due to the implementation of the memory components, it is possible to achieve perfect tracking over the whole time interval $[0, T]$. The necessity of incorporating feedforward loop can be justified in terms of "Internal Model Principle" (IMP). According to the IMP (Francis and Wonham, 1975), to achieve perfect tracking, the control signal must contain a suitably reduplicated model of the target trajectory and deterministic disturbance.

Hitherto, lots of FIL schemes, including ILC, have been proposed. In the following, let us review and summarize the numerous methodologies of FIL according to three different categories – analysis methods for FIL, uncertainties addressed by FIL and tracking tasks of FIL.

Analysis Methods for FIL

Differing from many existing intelligent control methods such as fuzzy logic control and neural network control, the effectiveness of FIL schemes is guaranteed rigorously with convergence analysis. There are several theories which can be employed to analyze the convergence property of FIL.

- Analysis Methods for ILC

  Basically, the analysis methods for classical ILC schemes contain Contraction Mapping (CM) principle and two-dimensional (2-D) system theory.

  CM Principle

  Let S be a closed subset of a Banach space and let A be a mapping that maps S into S. Suppose that$\|A(\mathbf{x}) - A(\mathbf{y})\| \leq \alpha\|\mathbf{x} - \mathbf{y}\|$, where $0 \leq \alpha < 1$ and $\mathbf{x}, \mathbf{y} \in S$, then there exists a unique point $\mathbf{x}^* \in S$ such that $A(\mathbf{x}^*) = \mathbf{x}^*$ and x^* can be obtained by the method of successive approximation starting from any arbitrary initial point in S. This is the famous CM Principle (Khalil, 1990).

  CM-type method is a systematic and traditional way to analyze the learning convergence of ILC and most of the ILC works are based on it so far. However, the use of CM principle in learning control has two folds. On one hand, it achieves geometric convergence speed with very little system knowledge; on the other hand, it is hard to incorporate available system knowledge, whether parametric or structural, into the learning controller design, hence it can only handle limited classes of nonlinear uncertain systems, i.e. Global Lipschitz Continuous (GLC) systems. The reason is that in the presence of Non-Global Lipschitz Continuous (NGLC) nonlinearities, finite escape time phenomenon may occur and CM principle is no longer applicable. Consequently, further extension of CM-type ILC to more general class of nonlinear systems is very difficult.

2-D System Theory

2-D systems are those systems in which the inputs, outputs and states depend on two independent variables. Roesser first presented a two-dimensional discrete state-space model in mid 70's (Roesser, 1975).

2-D system theory has been applied to analyze ILC in both discrete-time systems (Zheng *et al.*, 1990; Kurek and Zaremba, 1993; Fang and Chow, 1998; Fang *et al.*, 2002) and continuous-time systems (Chow and Fang, 1998; Chow and Fang, 1998). The basic idea is to set up the mathematical model for the entire learning control system including the dynamics of the control system and the behavior of the learning process. Although 2-D system theory provides a useful tool to ILC design and analysis, almost all the schemes based on it are only applicable to linear time-invariant/time-varying systems.

Note that in all the ILC algorithms, the *Identical Initial Condition* (I.I.C.), i.e. $\mathbf{e}_i(0) = 0$, is essential. It means that the controlled process is required to return to the same initial configuration after each learning trial. The I.I.C. is one of the main limitations for further applications of ILC.

- EF/CEF-type FIL

Recently, FIL under the frame of Energy Function (EF)/Composite Energy Function (CEF) acquired much attention.

EF-based method evolves from Lyapunov function theory, which is a basic analysis tool in nonlinear control design, and is subsequently extended to the leaning domain of FIL. In (Ham *et al.*, 1995; Park *et al.*, 1996), the energy function with respect to iterations has been set up to facilitate the leaning design and analysis.

In CEF-type FIL schemes (Xu and Tan, 2002), a CEF which reflects the energy in both the time domain and the iteration domain is defined. Hence, the convergency of CEF guarantees not only the finiteness of system states, but also the convergence of tracking error along the learning axis. The main advantages of CEF-type FIL have been summarized in (Xu and Tan, 2002): (1) the learning convergence along the learning horizon and the system performance along time horizon can be considered concurrently; (2) because of the incorporation of system states information, the learning control approach can handle both GLC and NGLC systems.

Moreover, based on CEF analysis method, if $\mathbf{x}_d(0) = \mathbf{x}_d(T)$, the I.I.C. may be replaced by a less restricted initial condition - alignment condition, i.e. $\mathbf{x}_{i+1}(0) = \mathbf{x}_i(T)$. In (Xu, 2002), the learning convergence with the alignment condition for a certain class of systems was derived under the framework of CEF.

All in one, EF/CEF analysis method greatly widens the application areas of FIL.

System Uncertainties Addressed by FIL

The unknown system information addressed by FIL may be either the parameters only or the form together with parameters which describe a deterministic function.

Here, we classify the system uncertainties into the following two cases.

- Parametric Uncertainty

  The parametric uncertainty is only time-related and can be either constant or time-varying.

  In (Xu, 2002; Xu and Tan, 2002), the parametric uncertainties have been expressed as $\boldsymbol{\theta}(t)\boldsymbol{\xi}(\mathbf{x},t) \in \mathcal{C}(\mathcal{R}^{1\times n_1},[0,T])$ where $\boldsymbol{\theta}(t)$ is a set of unknown time-varying uncertainties and $\boldsymbol{\xi}(\mathbf{x},t) \in \mathcal{R}^{n_1}$ is a set of known functions of states. Here n_1 is an appropriate integer specifying the dimension. Note that here $\boldsymbol{\xi}(\mathbf{x},t)$ can be GLC or NGLC. It has been clearly shown in (Xu and Tan, 2002) that , if $\boldsymbol{\xi}(\mathbf{x},t)$ is NGLC, although CM-type ILC fails to work, the CEF-type FIL still can ensure the learning convergence.

- Norm-Bounded Uncertainty

  For norm-bounded uncertainty, neither its structure nor its parameters are known. The only available information is its bounding function. Obviously, constructing FIL algorithm for norm-bounded uncertainties is much more difficult.

  The norm-bounded uncertainties can be either GLC or NGLC. For GLC case, even without knowing its bounding function, CM-type ILC can handle it effectively. However, for NGLC case, CM-type ILC can not be applied any more. Therefore, how to handle NGLC norm-bounded uncertainties needs further investigation.

  From another point of view, the norm-bounded uncertainties can be classified into the following two different kinds: one is that the uncertainties will vanish as the tracking error approaches to zero; the other is that the uncertainties will not be zero even if the tracking error approaches zero. In (Qu *et al.*, 2001) EF-based FIL for systems with both parametric uncertainties and norm-bounded vanishing uncertainties has been proposed. How to deal with norm-bounded nonvanishing uncertainties is still an unknown area.

Tracking Tasks of FIL

Tracking control tasks over finite interval can be classified into uniform and non-uniform cases.

- Uniform Tracking Control Problem

  Hitherto, most FIL schemes, including both classical ILC and EF/CEF-type FIL, are only valid for uniform trajectory tracking problems, i.e. the control target must be strictly repeatable over $[0, T]$. Therefore, if any change occurs due to the variation of control objectives or task specifications, the control system has to start learning process from the very beginning.

- Non-Uniform Tracking Control Problem

  From a practical point of view, we often face non-uniform trajectory tracking tasks, i.e. the desired tracking targets are different from iteration to iteration.

  In (Saab *et al.*, 1997), D-, PD- and PID-type ILC algorithms were presented for tracking trajectories "slowly" varying in the iteration domain. In that work, the difference between two consecutive iterations is assumed to be bounded by a small constant. Due to the presence of non-parametric system uncertainties, only a bounded tracking error is guaranteed if the target trajectory keeps changing along the iteration axis.

  To partially solve non-uniform tracking problems, Direct Learning Control (DLC) and Recursive Direct Learning Control (RDLC) schemes were developed to make use of previously obtained control information to design the control input for a new trajectory (Xu *et al.*, 1996; Xu, 1997; Xu and Song, 2000). The basic idea behind these schemes is as follows. The control input of the

Table 1.1: Brief summary for the background of FIL.

Uncertainty / Analysis Method / Tracking Task	Parametric Uncertainty $\theta(t)\xi(x,t)$		Norm-Bounded Uncertainty $d(x,t)$		
	GLC	NGLC	GLC	NGLC	
				Vanishing	Non-Vanishing
Uniform Tracking Task	CM/CEF-type ILC	CEF-type ILC	CM/CEF-type ILC	EF-type ILC	Open Area
Non-Uniform Tracking Task	DLC, RDLC, etc. (Further Investagation Needed)		Open Area		

system can be partitioned into a basis function vector and a known matrix reflecting the relations between different trajectories. Based on the knowledge of the desired control inputs for the different trajectories, it is possible to identify the basis function vector. The batch processing nature of the DLC leads to some implementation difficulties such as the long computation time and singularities. Therefore, RDLC was proposed to overcome the difficulties in DLC. Although good learning results are obtained by DLC and RDLC, they are only limited to trajectories with different magnitude scales or different time scales.

Recently, high-order iterative learning update laws were also suggested for iteration-varying references or disturbances and evidenced only by simulation results (Moore and Chen, 2002).

Obviously, how to deal with non-uniform trajectories learning is still worthy of further study.

To clearly show the background of FIL, the main points are summarized in Table 1.1.

1.1.2 Infinite Interval Learning Control (IIL)

IIL represents the learning over infinite time interval $[0, \infty)$. As the continuity of the system states can be observed in most real control systems, extending the results in FIL to IIL is very constructive.

Repetitive Control (RC) is a typical method of IIL, which was first introduced by (Inoue *et al.*, 1981) for SISO plants in continuous time. RC approach is the design of a controller to track periodic reference commands and/or reject periodic disturbance with a fixed but known periodicity T. Unlike FIL, the learning process of RC is continuous, i.e. the initial state at the start of each period is equal to the final state of the preceding period.

The basic structure of RC scheme can be described in Fig. 1.2.

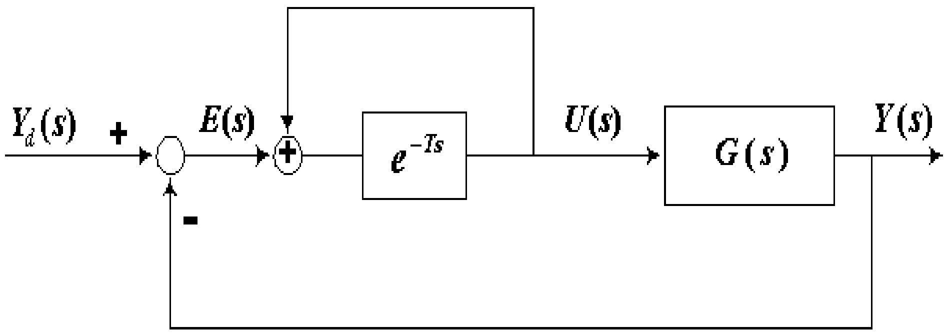

Figure 1.2: Basic structure of Repetitive Control.

From Fig. 1.2, it can be seen that the control signal is calculated using the information of the previous period. With consecutive iterations, it is expected that the RC system has the potential to substantially decrease the tracking error, and perfect tracking can be obtained eventually. The basis of RC is IMP which implies that a generator of periodic signals and a stabilizing controller, i.e. a controller that stabilizes the resulting closed-loop system, are needed to obtain the perfect tracking. As any periodic signal with period T can be generated by the free time-delay system with an appropriate initial function, a delay with positive feedback around it is used as an internal model in Fig. 1.2. Therefore, the perfect asymptotic tracking of periodic references can be achieved, provided that the closed loop system is stable.

So far lots of works have been finished about the theories and applications of RC approach. The modified RC designs were proposed in (Hara *et al.*, 1988; Sadegh, 1991) to relax the requirement for zero relative degree. The tradeoff between system stability and tracking performance has been considered in (Hara *et al.*, 1988; Srinivasan and Shaw, 1991). RC approaches for discrete time systems were discussed in (Nakano and Hara, 1986; Tomizuka, 1987; Middleton *et al.*, 1989). The stability analysis was enhanced in (Curtelin and Caron, 1993) and the robustness analysis was conducted in (Srinivasan and Shaw, 1991; Hara *et al.*, 1994; Liu and Tsao, 2001). Moreover, RC schemes have been successfully applied in a number of areas, such as robot manipulators (Hara *et al.*, 1987), disk drive systems (Sacks *et al.*, 1995), casting processes (Manayathara *et al.*, 1996) and satellite systems (Broberg and Molyet, 1992).

Next let us focus on the following three aspects.

- Plant Properties of IIL

  In most existing RC algorithms, the system internal stability and the learning convergence are guaranteed under the assumption of linearity or linearizability of the dynamic systems. Works for nonlinear RC are very limited. In (Hikita *et al.*, 1993), sliding mode control has been used to eliminate nonlinearities, thus the problem can be reduced to a linear one. In (Khalil, 1994), a robust control was first applied to bring the tracking error close to zero and then depended on the internal model servomechanism to work locally to bring the error to zero. A feedback linearizable nonlinear system has also been considered in (Alleyne and Pomykalski, 2000). Obviously, how to extend RC approach to nonlinear dynamic systems deserves further investigation.

- Analysis Methods of IIL

  Traditionally, the analysis of RC schemes is based on the small gain theorem which can be regarded as the extension of CM principle to infinite time horizon. Hence, it can be named as CM-type IIL. Recently, Lyapunov-based techniques have been applied to analyze the RC properties (Sadegh *et al.*, 1988; Dixon *et al.*, 2002) and we call them EF-type IIL. In (Sadegh *et al.*, 1988), by using the passivity properties of robot manipulators, based on EF-type RC, the system stability and the learning convergence can be obtained without requiring any assumption about the linearity of the system. Moreover, by taking advantage of EF-type RC, many other Lyapunov-based techniques can be easily fused and the stability analysis is straightforward.

- Tracking Tasks of IIL

  In all the developed RC schemes, it is required that the reference input signals and/or disturbances must be periodic. Hence, IIL for non-periodic reference signals is still an open area.

The background of IIL is summarized in Table 1.2. We can see that there is much space for us to further investigate the theories of IIL.

1.1.3 Learning for Nonsmooth Nonlinearities

In real control systems, many physical components contain nonsmooth nonlinearities, such as saturation, relay, deadzone, backlash and hysteresis. This kind of nonlinearities is especially common in actuators used in practice, such as motors,

Table 1.2: Brief summary for the background of IIL.

Plant Property / Analysis Method / Tracking Task	Linear/Linearizable System	Nonlinear System
Periodic Tracking Task	CM/EF-type IIL	EF-type IIL (Further Investigation Needed)
Non-Periodic Tracking Task	Open Area	Open Area

gear and hydraulic servo valves. The existence of these nonsmooth factors severely decreases the control accuracy or causes oscillations, even leads to system instability.

As the nonsmooth nonlinearities are usually unknown and even vary with operation conditions, conventional controllers, such as PD or PID controllers, exhibit poor performance. Therefore, the study of methods to deal with nonsmooth nonlinearities has been of interest to control engineers for some time. Next we will briefly review the works on the control of systems with deadzone or backlash.

- Systems with Deadzone

  The mathematical model of deadzone is a typical kind of nonsmooth nonlinearity, which is important not only in itself, but also to other nonsmooth nonlinearities, such as hysteresis and stiction, which can be model1ed using deadzones (Recker and Kokotović, 1991). The standard techniques to overcome a deadzone include variable structure control (Utkin, 1978) and dithering (Desoer and Sharuz, 1986). Motivated by the limitations in these approaches, such as chattering in sliding mode control, several adaptive inverse approaches were proposed (Recker and Kokotović, 1991; Tao and Kokotović, 1994; Tao and Kokotović, 1996; Tao and Kokotović, 1997), which employed an adaptive inverse for canceling the effect of an unknown nonlinearity and a fixed (or adaptive) linear control law for a known (or unknown) linear dynamics. Recently, soft computing such as fuzzy logic and neural network based control algorithms has also been applied to handle problems relevant to deadzones. Fuzzy logic based controllers were developed in (Kim *et al.*, 1994; Lewis *et al.*, 1997). Fuzzy precompensation schemes for PD controller and PID controller were proposed in (Kim *et al.*, 1993) and (Kim *et al.*, 1993) respectively. Neural network schemes (Cetinkunt and Domez, 1993; Lee and Kim, 1994; Selmić and Lewis, 2000) were also given to identify and compensate an unknown deadzone.

- Systems with Backlash

Backlash is another kind of highly practical-relevant control problem. Comparing with deadzone which is memoryless, backlash has an element of memory. Hence, overcoming backlash is more difficult.

The control of systems with backlash has been studied since 1940. Linear controllers, such as PI, PID and observer-based controllers, were first investigated. By now, many works have been finished. Approximating the inverse of the backlash has often been suggested as an effective way. The inverse compensation methods were proposed in (Tao and Kokotović, 1993; Dean *et al.*, 1995) based on online identification of backlash parameters; Switched control (Nordin and Gutman, 2000), dithered control (Desoer and Sharuz, 1986), Taylor's SIDF method (Taylor and Lu, 1995) and etc. have also been applied to avoid the harmful effect of backlash. Due to the capability of learning any nonlinear functions, neural networks have been used to identify and compensate backlash. A recurrent neural network with unsupervised learning by genetic algorithm was developed in (Shibata *et al.*, 1993). In (Seidl *et al.*, 1995), a neural network was proposed to handle gear backlash in precision position-controlled mechanisms.

In almost all the proposed learning schemes for nonsmooth nonlinearities, the algorithms are quite complicated and only bounded tracking error or asymptotic convergence can be guaranteed. Moreover, if the parameters of deadzone or backlash are time-varying, the compensation based on adaptive control fails to work. On the other hand, according to Sections 1.1.1, FIL has been widely applied due to its simplicity and effectiveness. As a complement to the existing methods, the control target of FIL is perfect tracking over the entire finite time interval. However, the implemented areas of FIL are only limited to smooth systems so far and the application to unknown nonsmooth nonlinearities is absent.

1.2 Objective of This Thesis

Although both FIL and IIL schemes have been developed for quite a long period, there are a number of problems which hinder the further applications.

- So far, the application of FIL is only limited to smooth nonlinearity. Is it possible to extend FIL to systems with unknown nonsmooth nonlinearities?
- How to deal with norm-bounded nonvanishing uncertainties which maybe NGLC?
- How to handle non-uniform trajectories learning is worthy of further investigation.
- Can we add the FIL scheme to some existing effective control methodology such that the original control approach also has learning ability?
- The implementation of IIL is very limited. How to further relax the limitation and widen its application areas?

- Many FIL and IIL approaches are under the framework of EF/CEF, in which the system states are assumed to be available. If the system states are not measurable, can we combine state estimation with the proposed CEF-type learning control approaches?

In this thesis, the major efforts are to develop theories to solve the above problems. The main contributions are summarized as follows.

- FIL for Systems with Nonsmooth Nonlinearities

  Under the framework of CM principle, FIL has been extended to discrete-time systems with unknown high nonlinearities such as input deadzone and input backlash. Based on the simple learning law, the unknown input deadzone or backlash can be compensated effectively and the perfect tracking can eventually be obtained iteratively.

- FIL for Systems with Norm-Bounded Uncertainties

  CEF-type FIL for systems with norm-bounded uncertainties has been discussed and several schemes have been proposed. A FIL approach for SISO dynamic systems with GLC norm-bounded uncertainties has been first outlined. A novel robust FIL scheme which combines robust control with CEF-type FIL has been proposed to deal with SISO dynamic systems with NGLC norm-bounded uncertainties. The basic idea of robust FIL is that the robust control is employed to guarantee that all the system states belong to a compact set, subsequently FIL is applied to improve the tracking performance gradually. Furthermore, FIL for systems with norm-bounded uncertainties under alignment condition is also considered. Finally, based on the discussion for SISO dynamic systems, the robust FIL approach has been extended to MIMO dynamic systems with norm-bounded NGLC uncertainties.

- FIL for Non-Uniform Tracking Problems

  Novel FIL algorithms have been introduced for non-uniform trajectory tracking problems in the presence of time-varying and/or time-invariant parametric uncertainties. The proposed approaches can learn from different motion patterns and are capable of generating the control profile for any new motion pattern, thus retaining the main advantages of DLC over ILC. On the other hand, the new methods require no *a priori* control knowledge, which overcomes the main limitation of DLC. Rigorous proofs based on CEF analysis method have been given to validate the proposed approaches.

  The proposed new FIL scheme includes the FIL approach for parametric uncertainties as its subset when the trajectories to be learned are identical over iterations. Obviously, the new developed approach could be applied to much broader nonlinear control systems.

- Fuzzy Logic Learning Control (FLLC)

  Although Fuzzy Logic Control (FLC) is an effective way to deal with nonlinear system uncertainties, experts have to spend a long time on re-adjusting the parameters when the tracking task changes. One way to partially tackle this problem is to offer the FLC system a learning mechanism.

  In this thesis a new modular approach - Fuzzy Logic Learning Control (FLLC) has been proposed, which integrates two complementary control approaches, FLC and FIL, and improves the tracking performance through tasks repetitions. The incorporation of the learning function into fuzzy controllers ensures exact tracking because it completely nullifies the effects of reference signal and periodic disturbances on the tacking error.

  Through rigorous proof based on EF, we show that the proposed FLLC system achieves the following novel properties: (1) the tracking error converges uniformly to zero; (2) learning control sequence converges to the desired control profile almost everywhere.

- IIL for Systems with Parametric Uncertainties

  By taking the advantage of CEF analysis method, the CEF-type FIL for systems with parametric uncertainties has been further extended to the IIL case. It has been shown that, only if the parametric uncertainty is periodic, based on the known periodicity T, the perfect tracking can be realized asymptotically. Moreover, in the proposed IIL schemes, the tracking tasks can be either periodic or non-periodic which greatly widens the application of IIL. This work can also be treated as an extension of FIL for non-uniform tracking problems.

- IIL for Systems with Norm-Bounded Uncertainties

  The CEF-type FIL for systems with norm-bounded uncertainties has been extended to infinite time interval $[0, \infty)$. To clearly show the basic idea, IIL for SISO dynamic systems with both GLC and NGLC uncertainties have been discussed in the first place, followed by implementing IIL to MIMO dynamic systems with NGLC uncertainties. It has been shown that, when the uncertainty is periodic in time t and the tracking target has a common periodicity, the learning convergence can be guaranteed even in the presence of norm-bounded uncertainties.

- Observer Based IIL for Systems with Parametric Uncertainties

  In all the CEF-type learning control schemes, the system states are assumed to be available. To facilitate the practical application, the observer based IIL algorithm, which combines the state estimation with IIL, has been proposed for systems with parametric uncertainties. Based on the state estimation, the perfect tracking can be assured as time proceeds. Moreover, if the I.I.C. or alignment condition is satisfied, the algorithm can be directly applied to CEF-type FIL for systems with parametric uncertainties.

1.3 Thesis Organization

The thesis consists of 10 chapters, organized as follows.

Chapters 2-6 cover the theories of FIL and Chapters 7-9 focus on the theories of IIL.

In Chapter 2 and Chapter 3, the CM-type FIL is extended to nonlinear systems with input deadzone and input backlash. Because of the singularity property of the systems with input deadzone or backlash, in Chapter 2-3 we consider a kind of discrete-time control system described as:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{f}(\mathbf{x}(k), k) + \mathbf{b}u(k) \\ u(k) &= W[v(k)] \\ y(k+1) &= \mathbf{c}\mathbf{x}(k+1), \end{aligned}$$

where $\mathbf{x} \in \mathcal{R}^n$ is the system state; $y \in \mathcal{R}$ is the measurable system output; $u \in \mathcal{R}$ is the plan input, but not available for control; $v \in \mathcal{R}$ is the actual system input; $k \in \mathcal{K}$ and $W[*]$ represents the input deadzone ($W = DZ$) or the input backlash ($W = BL$);

From Chapter 4 to Chapter 9, different kinds of CEF-type learning control schemes are proposed. In all these chapters, the following nonlinear systems with matched uncertainties are considered.

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) + B(\mathbf{x}, t)[\mathbf{u}(t) + \mathbf{d}(\mathbf{x}, t)], \tag{1.1}$$

where $\mathbf{x} \in \mathcal{R}^n$ is the state vector, $\mathbf{u} \in \mathcal{R}^m$ is the control input vector, $\mathbf{d}(\mathbf{x}, t)$ is the system uncertainties and t either belongs to $[0, T]$ (Chapter 4-6) or to $[0, \infty)$ (Chapter 7-9).

According to the different proposed schemes, several subsets of system (1.1) are discussed in different chapters.

Chapter 4 develops novel CEF-type FIL approaches for nonlinear systems with norm-bounded uncertainties $\mathbf{d}(\mathbf{x}, t)$ which can be GLC or NGLC. Rigorous proofs are provided therein. The control system we discussed in Chapter 4 is:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) + B_0(t)H(\mathbf{x}, t)[\mathbf{u}(t) + \mathbf{d}(\mathbf{x}, t)], \tag{1.2}$$

where $B_0(t) \in \mathcal{R}^{n\times m}$ and $H(\mathbf{x}, t) : \mathcal{R}^n \times \mathcal{R}_+ \to \mathcal{R}^{m\times m}$. We can see that in (1.1), if $n \neq m$, $B(\mathbf{x}, t) = B_0(t)H(\mathbf{x}, t)$ is needed where $H(\mathbf{x}, t)$ is square.

New FIL schemes suitable for non-uniform trajectories in the presence of parametric uncertainties are proposed in Chapter 5. The convergence analysis based on CEF is presented and the effectiveness of the new schemes is validated by simulation results. In Chapter 5, the following high-order MIMO system is considered.

$$\begin{aligned} \dot{\mathbf{x}}_j(t) &= \mathbf{x}_{j=1}(t) \;\; j = 1, \cdots, m-1 \\ \dot{\mathbf{x}}_m(t) &= \mathbf{f}(\mathbf{x}, t) + B(\mathbf{x}, t)[\mathbf{u}(t) + \mathbf{d}_1(x, t)] \end{aligned} \tag{1.3}$$

where $\mathbf{x}_k \in \mathcal{R}^n$, $k = 1, \cdots, m$; $\mathbf{x} \triangleq [\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_m^T]^T \in \mathcal{R}^{nm}$; $\mathbf{u} \in \mathcal{R}^n$; and $\mathbf{d}_1(\mathbf{x}, t)$ is the parametric uncertainty.

Note that all the approaches outlined in Chapter 5 are also valid for system (1.2) with the assumption that $\mathbf{d}(x, t)$ is a kind of parametric uncertainty. Similarly, the schemes proposed in Chapter 4 can also be applied to system (1.3) if $\mathbf{d}_1(\mathbf{x}, t)$ is a kind of norm-bounded uncertainty. Therefore, we give two different kinds of control systems to which our FIL approaches can be implemented.

Chapter 6 is devoted to a PD type FLLC approach which adds the FIL mechanism to the existing fuzzy logic controller in an additive form. Both theoretical analysis and simulation results are provided. According to the properties of FLC and FIL, we only consider the following nonlinear dynamic system.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(\mathbf{x}, t) + b(x_1, t)u \end{aligned} \tag{1.4}$$

where $\mathbf{x} = [x_1, x_2] \in \mathcal{R}^2$, $u \in \mathcal{R}$ and $f(\mathbf{x}, t)$ and $b(x_1, t)$ are nonlinear uncertain functions. Obviously, the system (1.4) is also a subset of (1.1).

CEF-type FIL approaches for systems with parametric and norm-bounded uncertainties are extended to IIL in Chapter 7 and Chapter 8 respectively. All the algorithms given in Chapter 7 are suitable either to (1.2) or to (1.3) if both $\mathbf{d}(\mathbf{x}, t)$ and $\mathbf{d}_1(\mathbf{x}, t)$ are parametric uncertainties and $t \in [0, \infty)$. In this thesis only the results for system (1.3) are given and the results for system (1.2) can be obtained directly. While in Chapter 8, we discuss more restrictive control systems:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) + B(t)[\mathbf{u}(t) + \mathbf{d}(\mathbf{x}, t)], \tag{1.5}$$

where $\mathbf{x} \in \mathcal{R}^n$, $\mathbf{u} \in \mathcal{R}^n$, $B(t) \in \mathcal{R}^{n \times n}$, and $\mathbf{d}(\mathbf{x}, t)$ is the norm-bounded uncertainty. It clearly shows that IIL for systems with norm-bounded uncertainties is the most difficult control problem.

The observer based IIL for system with parametric uncertainty is presented in Chapter 9. Considering the requirement of an observer design, we proposed the IIL algorithm for the following MIMO system.

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B[\mathbf{u}(t) + \mathbf{d}(\mathbf{x}, t)] \\ \mathbf{y} &= C\mathbf{x}, \end{aligned}$$

where $\mathbf{x} \in \mathcal{R}^n$ is not measurable; $\mathbf{y} \in \mathcal{R}^m$ is the physically accessible output vector; $\mathbf{u} \in \mathcal{R}^m$; $\mathbf{d}(\mathbf{x}, t)$ is the parametric uncertainty and A, B and C are constant matrices of appropriate dimensions.

Chapter 10 summarizes the fulfilled work and gives recommendation on the future research.

Finally, to clearly show the background and the contributions of this thesis, the main results related to the theories of FIL and IIL are summarized as Table 1.3.

Table 1.3: Summary of the Main Results Related to FIL and IIL.

	Finite Interval Learning (FIL)				
Analysis Method	CM-Type FIL		EF/CEF-Type FIL		
System Property	Smooth GLC	Nonsmooth GLC	Smooth GLC/NGLC		
Uncertainty Property	GLC	GLC	Parametric		Norm-Bounded (GLC/NGLC)
Tracking Target	Output Tracking (Uniform)	Output Tracking (Uniform)	State Tracking (Uniform)	State Tracking (Non-Uniform)	State Tracking (Uniform)
Available Control Signal	System Output	System Output	System State	System State	System State
Related Results	Many Published Works	*Chapter 2-3*	(Xu,2002), (Xu and Tan, 2002), etc.	*Chapter 5*	*Chapter 4*

	Infinite Interval Learning (IIL)			
Analysis Method	CM-Type IIL	EF/CEF-Type IIL		
System Property	Smooth Linear/Linearizable	Smooth GLC/NGLC		
Uncertainty Property	Parametric (Periodic)	Parametric (Periodic)		Norm-Bounded (GLC/NGLC)
Tracking Target	Output Tracking (Periodic)	State Tracking (Periodic/Non-Periodic)	Output Tracking (Periodic/Non-Periodic)	State Tracking (Periodic)
Available Control Signal	System Output	System State	System Output	System State
Related Results	Many Published Works	*Chapter 7*	*Chapter 9*	*Chapter 8*

Chapter 2

FIL for Systems with Input Deadzone

2.1 Introduction

FIL, such as ILC, has been widely applied to repeated tracking control and repeatable disturbance rejection in the past two decades due to its simplicity and effectiveness. However, only the smooth nonlinearity is considered hitherto and the absent from these results is the application of FIL to unknown nonsmooth nonlinearity.

The mathematical model of deadzone is a typical kind of nonsmooth nonlinearity. In practice, the parameters of the model are poorly known and even vary with operation conditions. Therefore, it is a challenge to control engineer.

The goal of this chapter is to investigate the control problem for a class of uncertain nonlinear systems with input deadzone. A possible alternative but much simpler approach making use of FIL is outlined to deal with a certain class of systems with input deadzone. It will be shown that even if the width of the deadzone is completely unknown, only by using the tacking error of previous learning cycle, the deadzone compensation can be conducted automatically. Hence, as the learning iteration approaches to infinity, the system tracking error converges to zero. Moreover, we assume that the system itself also has some nonlinear uncertainty which is totally unknown but GLC.

Many systems with deadzones can be modeled in discrete time. Furthermore, for implementation in digital controllers, a discrete-time deadzone compensator is needed. Therefore, in our work, we focus on a class of discrete-time systems.

This chapter is organized as follows. In Section 2.2 FIL for the static mapping of a deadzone is analyzed. The FIL for dynamic systems with input deadzone is presented in Section 2.3. Section 2.4 contains an illustrative example. The proposed scheme has been applied to a linear piezoelectric motor and the experimental results are given in Section 2.5. Finally, conclusion is drawn in Section 2.6.

2.2 Preliminaries

Lemma 2.1. *Let two sequences be* $\{z_i\} \subset \mathcal{R}$ *and* $\delta_i \subset \mathcal{R}$*, with* $i \in \mathcal{Z}_+$*. Assume that* $\forall i \in \mathcal{Z}_+$*, the inequality* $|z_{i+1} - a| \leq \gamma |z_i - a| + |\delta_i|$ *holds, where* $a \in \mathcal{R}$ *and* $0 < \gamma < 1$*. Then* $\lim_{i\to\infty} z_i = a$ *can be derived if* $\lim_{i\to\infty} |\delta_i| = 0$*.*

Lemma 2.2. *Separate the entire real axis* $\mathcal{R}$ *into three intervals:* $I_1 \triangleq (-\infty, a)$*,* $I_2 \triangleq [a, b]$ *and* $I_3 \triangleq (b, \infty)$*, where* $a \leq b$*. Assume* $\forall i \in \mathcal{Z}_+$*, the following relations are valid:*

$$if \quad z_i \in I_1, \qquad \gamma_1(z_i - a) - |\delta_i| \leq z_{i+1} - a \leq \gamma_1(z_i - a) + |\delta_i|; \tag{2.1}$$

$$if \quad z_i \in I_2, \qquad z_i - |\delta_i| \leq z_{i+1} \leq z_i + |\delta_i|; \tag{2.2}$$

$$if \quad z_i \in I_3, \qquad \gamma_2(z_i - b) - |\delta_i| \leq z_{i+1} - b \leq \gamma_2(z_i - b) + |\delta_i|, \tag{2.3}$$

where $0 < \gamma_1 < 1$*,* $0 < \gamma_2 < 1$*,* $\sup_{i\in\mathcal{Z}_+} |\delta_i|$ *is finite and* $\lim_{i\to\infty} |\delta_i| = 0$*. For any finite*

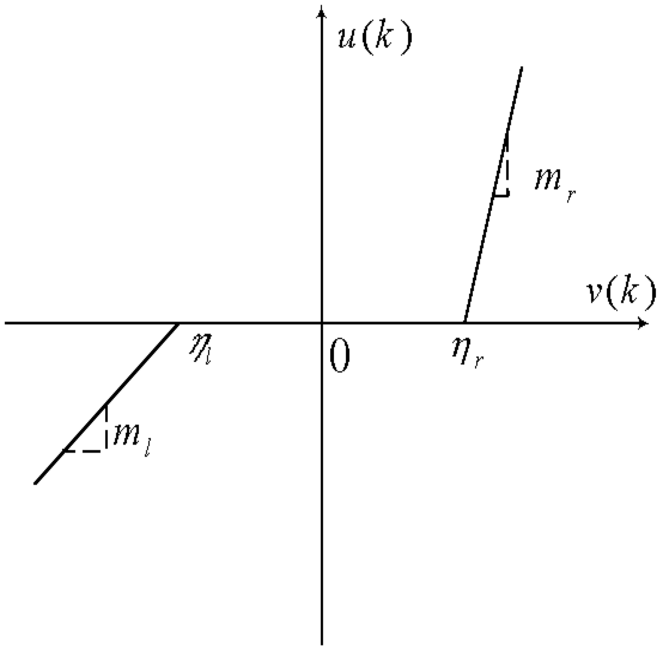

Figure 2.1: The deadzone nonlinearities $u(k) = DZ[v(k)]$.

$z_0 \in \mathcal{R}$, *under the mappings (2.1) – (2.3),* $\lim_{i\to\infty} z_i \in I_2$ *can be derived.*

The proofs for Lemma 2.1 and Lemma 2.2 are given in Appendix A.

2.3 FIL for A Pure Deadzone Component

Consider the following static mapping of a deadzone,

$$u(k) = DZ[v(k)] = \begin{cases} m_r[v(k) - \eta_r] & v(k) \in I_R \\ 0 & v(k) \in I_D \\ m_l[v(k) - \eta_l] & v(k) \in I_L \end{cases}, \tag{2.4}$$

where $v(k) \in \mathcal{R}$ is the input of the deadzone; $u(k) \in \mathcal{R}$ is the output of the deadzone; $k \in \mathcal{K} \triangleq \{0, 1, 2, \cdots, N\}$ and N is a finite integer; $I_R \triangleq (\eta_r, \infty)$, $I_D \triangleq [\eta_l, \eta_r]$ and $I_L \triangleq (-\infty, \eta_l)$; $m_l > 0$, $m_r > 0$, $\eta_l \leq 0$, $\eta_r \geq 0$ are constant parameters;

The static relationship can be described in Fig. 2.1. Note that the deadzone can be nonsymmetric.

The following assumption is first made for the deadzone (2.4).

Assumption 2.1. The upper bound of m_l and m_r is known and denoted as $B_1 \geq \max\{m_l, m_r\}$.

The control objective is to find a sequence of appropriate control signal $v_i(k)$, such that $u_i(k)$ converges to the target $u_d(k)$ iteratively.

The learning law is constructed as

$$\begin{aligned} v_i(k) &= v_{i-1}(k) + \beta\delta u_{i-1}(k), \\ 0 &< 1 - \beta B_1 < 1, \end{aligned} \tag{2.5}$$

where $\delta u_{i-1}(k) = u_d(k) - u_{i-1}(k)$ and $\beta > 0$ is the learning gain. As m_l, m_r and β are all positive, $0 < 1 - \beta B_1 < 1$ implies that $0 < \gamma_l \triangleq 1 - \beta m_l < 1$ and $0 < \gamma_r \triangleq 1 - \beta m_r < 1$.

Let $v_{-1}(k) = u_{-1}(k) = 0$, based on FIL law (2.5), the following result can be obtained.

Theorem 2.1. *For the static mapping (2.4), under Assumption 2.1, the control law (2.5) guarantees that, $\forall k \in \mathcal{K}$, $\delta u_i(k)$ converges to zero as i approaches to infinity.*

Proof:

Given any $k \in \mathcal{K}$, when $i = 0$, we have $v_0(k) = \beta u_d(k)$ from (2.5).

If $u_d(k) = 0$, according to (2.5), it can be derived that $\forall i \in \mathcal{Z}_+$, $v_i(k) = 0$, hence $u_i(k) = u_d(k) = 0$ can be guaranteed for any iteration.

Now let us discuss the case when $u_d(k) \neq 0$. Here we assume $u_d(k) > 0$. The result for $u_d(k) < 0$ can be derived analogously.

According to (2.4) and (2.5), a finite iteration number $p_k \geq 0$ can be found such that $v_{p_k}(k) = (p_k + 1)\beta u_d(k) \in I_R$ and $\forall i < p_k$ $v_i(k) = (i+1)\beta u_d(k) \in I_D$. Next we will show $\forall i \geq p_k$, $v_i(k) \in I_R$ can be derived. The induction method is used here.

1. $n = p_k$

Let $i = p_k$, $v_{p_k}(k) = (p_k + 1)\beta u_d(k) \in I_R$.

From (2.4) and considering $v_{p_k-1}(k) = p_k\beta u_d(k) \notin I_R$, we have

$$\begin{aligned} \delta u_{p_k}(k) &= u_d(k) - m_r[v_{p_k}(k) - \eta_r] \\ &= u_d(k) - m_r(p_k+1)\beta u_d(k) + m_r\eta_r \\ &= \gamma_r u_d(k) + m_r[\eta_r - p_k\beta u_d(k)] \\ &= \gamma_r u_d(k) + m_r[\eta_r - v_{p_k-1}(k)] > 0. \end{aligned}$$

2. $\forall n \geq p_k$, assume $v_n(k) \in I_R$ and $\delta u_n(k) > 0$.

According to updating law (2.5) and considering the positiveness of β and $\delta u_n(k)$, it can be derived that

$$v_{n+1}(k) = v_n(k) + \beta\delta u_n(k) > v_n(k) > \eta_r.$$

Furthermore, from (2.4) and considering $\gamma_r > 0$, we have

$$\begin{aligned} \delta u_{n+1}(k) &= \delta u_n(k) + u_n(k) - u_{n+1}(k) \\ &= \delta u_n(k) + m_r[v_n(k) - \eta_r] - m_r[v_{n+1}(k) - \eta_r] \\ &= \delta u_n(k) + m_r[v_n(k) - v_{n+1}(k)] \\ &= \gamma_r \delta u_n(k) > 0. \end{aligned} \tag{2.6}$$

Hence, $v_{n+1}(k) \in I_R$ and $\delta u_{n+1}(k) > 0$ can be derived.

3. Therefore, $\forall i \geq p_k$, $v_i(k) \in I_R$ and $\delta u_i(k) > 0$ can be guaranteed. Furthermore, (2.6) is always valid. As $0 < \gamma_r < 1$, according to (2.6), $u_i(k)$ converges to $u_d(k)$ as i approaches to infinity. ■

2.4 FIL for Dynamic Systems with Input Deadzone

Consider the following dynamic system

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{f}(\mathbf{x}(k), k) + \mathbf{b}u(k) \\ u(k) &= DZ[v(k)] \\ y(k+1) &= \mathbf{c}\mathbf{x}(k+1), \end{aligned} \tag{2.7}$$

where $\mathbf{x} \in \mathcal{R}^n$ is the system state; $y \in \mathcal{R}$ is the measurable system output; $u \in \mathcal{R}$ is the plan input, but not available for control; $v \in \mathcal{R}$ is the actual system input; $\mathbf{f} : \mathcal{R}^n \times \mathcal{K} \to \mathcal{R}^n$, $\mathbf{b} \in \mathcal{R}^n$ and $\mathbf{c} \in \mathcal{R}^{1\times n}$; $k \in \mathcal{K}$ and $DZ[*]$ is defined same as in (2.4).

The following assumptions are made for the dynamic system (2.7).

Assumption 2.2. $\mathbf{f}(\mathbf{x}(k), k)$ is GLC with respect to $\mathbf{x}(k)$, i.e. $\|\mathbf{f}(\mathbf{x}_1(k), k) - \mathbf{f}(\mathbf{x}_2(k), k)\| \le l_{\mathbf{f}}\|\mathbf{x}_1(k) - \mathbf{x}_2(k)\|$, where $l_{\mathbf{f}}$ is an unknown global Lipschitz constant.

Assumption 2.3. System (2.7) satisfies the I.I.C., i.e. $\delta\mathbf{x}_i(0) \triangleq \mathbf{x}_d(0) - \mathbf{x}_i(0) = 0$, hence $e_i(0) \triangleq y_d(0) - y_i(0) = 0$, where $i \in \mathcal{Z}_+$.

Assumption 2.4. The prior information with $\mathbf{cb} \in R$ is its sign and its bound $B_2 \ge |\mathbf{cb}|$. Without loss of generality, assume $\mathbf{cb} > 0$ in this chapter.

Remark 2.1. From the practical point of view, the I.I.C. (Assumption 2.3) is difficult to be met in practice. A possible way to solve the problem is to modify the target trajectory at the initial stage by making an appropriate interpolation (Sun and Wang, 2002), in the sequel guarantee $e_i(0) = 0$.

The ultimate control target is to find the control signal $v_i(k)$ iteratively such that $y_i(k)$ converges to the desired output $y_d(k)$ as $i \to \infty$, where $y_d(k)$ can be described as

$$y_d(k) = \mathbf{c}\mathbf{x}_d(k) = \mathbf{c}\mathbf{f}(\mathbf{x}_d(k-1), k-1) + \mathbf{c}\mathbf{b}u_d(k-1). \tag{2.8}$$

Remark 2.2. According to (2.4), when $u_d(k) \neq 0$, the unique desired input $v_d(k)$ exists. However, when $u_d(k) = 0$, $v_d(k)$ is not unique any more and could be any value belonging to I_D.

The FIL law is

$$\begin{aligned} v_i(k) &= v_{i-1}(k) + \beta e_{i-1}(k+1), \quad v_0(k) = 0 \\ 0 &< 1 - \beta B_1 B_2 < 1. \end{aligned} \tag{2.9}$$

Similarly, $0 < 1 - \beta B_1 B_2 < 1$ leads to $0 < \gamma_l' \triangleq 1 - \beta \mathbf{c}\mathbf{b} m_l < 1$ and $0 < \gamma_r' \triangleq 1 - \beta \mathbf{c}\mathbf{b} m_r < 1$.

According to (2.7) and (2.8) and considering Assumption 2.2, $\forall k \in \mathcal{K}$, we have

$$\begin{aligned} &\|\delta \mathbf{x}_i(k)\| \\ &\leq \|\mathbf{f}_d(k-1) - \mathbf{f}_i(k-1)\| + \|\mathbf{b}\||\delta u_i(k-1)| \\ &\leq l_{\mathbf{f}}\|\delta \mathbf{x}_i(k-1)\| + \|\mathbf{b}\||\delta u_i(k-1)|, \end{aligned} \tag{2.10}$$

where $\mathbf{f}_d(k-1) = \mathbf{f}(\mathbf{x}_d(k-1), k-1)$, $\mathbf{f}_i(k-1) = \mathbf{f}(\mathbf{x}_i(k-1), k-1)$ and $\delta u_i(k-1) = u_d(k-1) - u_i(k-1)$. By using (2.10) repeatedly, we obtain

$$\begin{aligned} \|\delta \mathbf{x}_i(k)\| &\leq l_{\mathbf{f}}^k \|\delta \mathbf{x}_i(0)\| + l_{\mathbf{f}}^{k-1}\|\mathbf{b}\||\delta u_i(0)| + l_{\mathbf{f}}^{k-2}\|\mathbf{b}\||\delta u_i(1)| \\ &\quad + \cdots + \|\mathbf{b}\||\delta u_i(k-1)| \\ &= \|\mathbf{b}\| \sum_{j=0}^{k-1} l_{\mathbf{f}}^{k-1-j} |\delta u_i(j)|. \end{aligned} \tag{2.11}$$

Moreover, from the above equation, the following can be derived straightforwardly.

$$\begin{aligned} \|\mathbf{e}_i(k)\| &\leq \|\mathbf{c}\|\|\delta \mathbf{x}_i(k)\| \\ &\leq \|\mathbf{c}\|\|\mathbf{b}\| \sum_{j=0}^{k-1} l_{\mathbf{f}}^{k-1-j} |\delta u_i(j)|. \end{aligned} \tag{2.12}$$

To simplify the proof of the main result, two Lemmas are given first.

Lemma 2.3. *Assume* $\lim_{i\to\infty}|\delta u_i(k)| = 0$ *where* $k = 0, \cdots, m$ *and* $0 \leq m \leq N-1$. *Under Assumptions 2.1 -2.4 and the control law (2.9), the system input* $v_i(m+1) \in I_R$ *will always be guaranteed after finite iteration if* $u_d(m+1) > 0$.

Proof:

Since $u_d(m+1) > 0$, two arbitrarily small constants ϵ_m and ρ can be found such that $u_d(m+1) \geq \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}} + \rho$, where $\Gamma_m \triangleq \|\mathbf{b}\|\sum_{j=0}^{m} l_{\mathbf{f}}^{m-j}\epsilon_m$.

As $\forall k \in \{0, \cdots, m\}$, $\lim_{i\to\infty}|\delta u_i(k)| = 0$, for any given ϵ_m, a finite constant p'_m can be found such that $\forall i \geq p'_m$, $|\delta u_i(k)| \leq \epsilon_m$.

According to (2.11), we have

$$\|\delta\mathbf{x}_i(m+1)\| \leq \|\mathbf{b}\|\sum_{j=0}^{m} l_{\mathbf{f}}^{m-j}\epsilon_m = \Gamma_m. \tag{2.13}$$

Therefore, considering Assumption 2.2, for any $i \geq p'_m$,

$$\begin{aligned} -l_{\mathbf{f}}\|\delta\mathbf{x}_i(m+1)\| &\leq \mathbf{f}_d(m+1) - \mathbf{f}_i(m+1) \leq l_{\mathbf{f}}\|\delta\mathbf{x}_i(m+1)\| \\ -l_{\mathbf{f}}\Gamma_m &\leq \mathbf{f}_d(m+1) - \mathbf{f}_i(m+1) \leq l_{\mathbf{f}}\Gamma_m. \end{aligned} \tag{2.14}$$

The rest of the proof contains two parts. Par A shows that the control law (2.9) maps $v_0(m+1)$ into I_R in finite iteration p_m. In Part B, we will prove that $\forall i \geq p_m$, the control law (2.9) maps $v_i(m+1)$ from I_R to I_R.

Part A

Substituting (2.9) into (2.7) yields

$$\mathbf{x}_i(k+1) = \mathbf{f}_i(k) + \mathbf{b}DZ[v_{i-1}(k) + \beta\mathbf{c}(\mathbf{x}_d(k+1) - \mathbf{x}_{i-1}(k+1))].$$

Considering $\mathbf{f}_i(k)$ is GLC, N is finite, $v_0(k) = 0$ and the definition of $DZ[*]$, it can be derived that for any finite iteration, the system state $x_i(k)$, the control input $v_i(k)$ and the system output $y_i(k)$ are all bounded.

As $u_d(m+1) \geq \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}} + \rho > 0$ and $\mathbf{cb} > 0$, from (2.8), it can be derived that

$$\begin{aligned} y_d(m+2) - \mathbf{cf}_d(m+1) &= \mathbf{cb}u_d(m+1) \\ &\geq \mathbf{cb}(\frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}} + \rho) \\ &= l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m + \mathbf{cb}\rho. \end{aligned} \tag{2.15}$$

$\forall i \geq p'_m$, assume $v_i(m+1) \notin I_R$. From updating law (2.9), it can be derive that

$$\begin{aligned} v_{i+1}(m+1) &= v_i(m+1) + \beta e_i(m+2) \\ &= v_i(m+1) + \beta\{y_d(m+2) - \mathbf{cf}_i(m+1) - \mathbf{cb}DZ[v_i(m+1)]\} \\ &= v_i(m+1) + \beta\{y_d(m+2) - \mathbf{cf}_d(m+1) + \mathbf{cf}_d(m+1) \\ &\quad -\mathbf{cf}_i(m+1) - \mathbf{cb}DZ[v_i(m+1)]\}. \end{aligned}$$

Considering (2.14) and (2.15), we can obtain that

$$\begin{aligned} v_{i+1}(m+1) &\geq v_i(m+1) + \beta(l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m + \mathbf{cb}\rho) - \beta l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m - \beta\mathbf{cb}DZ[v_i(m+1)] \\ &= v_i(m+1) + \beta\mathbf{cb}\rho - \beta\mathbf{cb}DZ[v_i(m+1)]. \end{aligned}$$

As $v_i(m+1) \notin I_R$, $DZ[v_i(m+1)] \leq 0$. Considering $\beta\mathbf{cb} > 0$, we have

$$v_{i+1}(m+1) \geq v_i(m+1) + \beta\mathbf{cb}\rho. \tag{2.16}$$

As p'_m is finite, $v_{p'_m}(m)$ is bounded. According to (2.16), there exists a finite iteration $p_m > p'_m$ such that $v_{p_m}(m+1) \in I_R$.

Part B

From *Part A*, $v_{p_m}(m+1) \in I_R$. Next we will prove that $\forall i \geq p_m$, if $v_i(m+1) \in I_R$, $v_{i+1}(m+1) \in I_R$ can be derived. As $u_d(m+1) > 0$, the uniqueness of $v_d(m)$ is ensured. Moreover, $u_d(m+1) \geq \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}} + \rho$ leads to $v_d(m+1) \geq \eta_r + \frac{\rho}{m_r} + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}m_r}$.

According to the updating law (2.9), we have

$$\begin{aligned}
v_{i+1}(m+1) &= v_i(m+1) + \beta e_i(m+2) \\
&= v_i(m+1) + \beta[\mathbf{c}\mathbf{f}_d(m+1) + \mathbf{c}\mathbf{b}m_r v_d(m+1) \\
&\quad -\mathbf{c}\mathbf{f}_i(m+1) - \mathbf{c}\mathbf{b}m_r v_i(m+1)] \\
&= v_i(m+1) + \beta\mathbf{c}\mathbf{b}m_r\delta v_i(m+1) + \beta\mathbf{c}[\mathbf{f}_d(m+1) \\
&\quad -\mathbf{f}_i(m+1)] \\
&\geq \beta\mathbf{c}\mathbf{b}m_r v_d(m+1) + \gamma_r' v_i(m+1) - \beta l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m \\
&\geq \beta\mathbf{c}\mathbf{b}m_r v_d(m+1) + \gamma_r'\eta_r - \beta l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m. \qquad (2.17)
\end{aligned}$$

where $\delta v_i(m+1) = v_d(m+1) - v_i(m+1)$.

Considering $v_d(m+1) \geq \eta_r + \frac{\rho}{m_r} + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{c}\mathbf{b}m_r}$, (2.17) can be rewritten as

$$\begin{aligned}
v_{i+1}(m+1) &\geq \beta\mathbf{c}\mathbf{b}m_r\left(\eta_r + \frac{\rho}{m_r} + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{c}\mathbf{b}m_r}\right) + \gamma_r'\eta_r - \beta l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m \\
&= \eta_r + \beta\mathbf{c}\mathbf{b}\rho > \eta_r
\end{aligned}$$

Therefore, $\forall v_i(m+1) \in I_R$ $(i \geq p_m)$, the control law (2.9) always maps it into I_R.

■

Remark 2.3. Analogous to Lemma 2.3, under the same assumptions, the control law (2.9) ensures that the system input $v_i(m+1) \in I_L$ after finite iterations if $u_d(m+1) < 0$.

According to Remark 2.2, if $u_d(k) = 0$, $v_d(k)$ is not unique. Next we will show that in such situation, $\lim_{i\to\infty} \delta u_i(m+1) = 0$ can be derived directly if $\lim_{i\to\infty} \delta u_i(k) = 0$ $(k = 0, \cdots, m)$ and $\lim_{i\to\infty} v_i(m+1) \in I_D$.

Lemma 2.4. *Assume* $\lim_{i\to\infty} \delta u_i(k) = 0$ *where* $k = 0, \cdots, m$. *If* $u_d(m+1) = 0$, $\lim_{i\to\infty} \delta u_i(m+1) = 0$ *and* $\lim_{i\to\infty} v_i(m+1) \in I_D$ *can be derived.*

Proof:

Similarly to Lemma 2.3, ϵ_m and p'_m can be found such that $\forall i \geq p'_m$, $|\delta u_i(k)| \leq \epsilon_m$ $(k = 0, \cdots, m)$. Moreover, (2.13) is still valid.

Let us check the system input $v_{i+1}(m+1)$ $(i \geq p'_m)$ according to the following three cases.

Case 1: $v_i(m+1) \in I_R$ $(i \geq p'_m)$

$$\begin{aligned} & v_{i+1}(m+1) - \eta_r \\ & \quad = v_i(m+1) + \beta \mathbf{c}[\mathbf{f}_d(m+1) - \mathbf{f}_i(m+1)] + \beta \mathbf{c}\mathbf{b}[u_d(m+1) - u_i(m+1)] - \eta_r \\ & \quad = v_i(m+1) + \beta \mathbf{c}[\mathbf{f}_d(m+1) - \mathbf{f}_i(m+1)] + \beta \mathbf{c}\mathbf{b} m_r[\eta_r - v_i(m+1)] - \eta_r \\ & \quad = \gamma'_r[v_i(m+1) - \eta_r] + \beta \mathbf{c}[\mathbf{f}_d(m+1) - \mathbf{f}_i(m+1)]. \end{aligned}$$

From (2.11), we have

$$\gamma'_r[v_i(m+1) - \eta_r] - \Delta_i(m) \leq v_{i+1}(m+1) - \eta_r \leq \gamma'_r[v_i(m+1) - \eta_r] + \Delta_i(m) \tag{2.18}$$

where $\Delta_i(m) \triangleq \beta l_{\mathbf{f}} \|\mathbf{c}\| \|\mathbf{b}\| \sum_{j=0}^{m} l_{\mathbf{f}}^{m-j} |\delta u_i(j)|$.

Case 2: $v_i(m+1) \in I_L$ $(i \geq p'_m)$

Analogous to *Case 1*,

$$v_{i+1}(m+1) - \eta_l \;=\; \gamma'_l[v_i(m+1) - \eta_l] + \beta \mathbf{c}[\mathbf{f}_d(m+1) - \mathbf{f}_i(m+1)],$$

and

$$\gamma'_l[v_i(m+1) - \eta_l] - \Delta_i(m) \leq v_{i+1}(m+1) - \eta_l \leq \gamma'_l[v_i(m+1) - \eta_l] + \Delta_i(m). \tag{2.19}$$

Case 3: $v_i(m+1) \in I_D$ $(i \geq p'_m)$

$$\begin{aligned} v_{i+1}(m+1) &= v_i(m+1) + \beta \mathbf{c}[\mathbf{f}_d(m+1) - \mathbf{f}_i(m+1)] + \beta \mathbf{c}\mathbf{b}[u_d(m+1) - u_i(m+1)] \\ &= v_i(m+1) + \beta \mathbf{c}[\mathbf{f}_d(m+1) - \mathbf{f}_i(m+1)]. \end{aligned}$$

Hence,

$$v_i(m+1) - \Delta_i(m) \leq v_{i+1}(m+1) \leq v_i(m+1) + \Delta_i(m). \tag{2.20}$$

Considering (2.18), (2.19) and (2.20), as $\lim_{i\to\infty} \Delta_i(m) = 0$ and $v_{p'_m}(m+1)$ is bounded, according to Lemma 2.2, it can be derived that $\lim_{i\to\infty} v_i(m+1) \in I_D$. Consequently, $\lim_{i\to\infty} u_i(m+1) = u_d(m+1) = 0$. ■

Theorem 2.2. *Under Assumptions 2.1-2.4, the learning law (2.9) guarantees that $y_i(k)$ and $u_i(k)$ converge to $y_d(k)$ and $u_d(k)$ respectively for any $k \in \mathcal{K}$. Moreover, the system input signal $v_i(k)$ converges to $v_d(k)$ if $u_d(k) \neq 0$, otherwise $v_i(k)$ converges to I_D.*

Proof:

We will prove this theorem by induction on $k \in \mathcal{K}$. The convergence property of $u_i(0)$ and $y_i(1)$ is first derived in *Part A*. Assume $u_i(k)$ and $y_i(k+1)$ converge to $u_d(k)$ and $y_d(k+1)$ respectively, where $k = 0, \cdots, n$ and $1 \leq n \leq N-1$. *Part B* shows the convergece of $u_i(n+1)$ and $y_i(n+2)$. Therefore, $\forall k \in \mathcal{K}$, as i approaches to infinity, $u_i(k)$ and $y_i(k)$ approach to $u_d(k)$ and $y_d(k)$ respectively can be guaranteed.

Part A

(1) $u_d(0) = 0$

If $u_d(0) = 0$, $y_d(1) = \mathbf{cf}(\mathbf{x}_d(0), 0)$. Assume $v_i(0) = 0$, from (2.9) we have

$$\begin{aligned} v_{i+1}(0) &= v_i(0) + \beta[y_d(1) - y_i(1)] \\ &= \beta\{y_d(1) - \mathbf{cf}_i(0) - \mathbf{cb}DZ[v_i(0)]\} \\ &= \beta[y_d(1) - \mathbf{cf}_d(0)] \\ &= 0. \end{aligned} \tag{2.21}$$

As $v_0(0) = 0$, from (2.21), $v_i(0) = 0$ can always be ensured.

Since $y_i(1) = \mathbf{cf}_i(0) + \mathbf{cb}DZ[v_i(0)]$, $x_i(0) = x_d(0)$ and $v_i(0) = 0$ lead to $y_i(1) = y_d(1)$.

Hence, $\forall i \in \mathcal{Z}_+$, $v_i(0) = 0 \in I_D$, $u_i(0) = u_d(0)$ and $y_i(1) = y_d(1)$.

(2) $u_d(0) \neq 0$

Assume $u_d(0) > 0$ which implies $v_d(0) > \eta_r$. Considering $v_0(0) = 0$ and $\mathbf{x}_0(0) = \mathbf{x}_d(0)$, we have

$$\begin{aligned} v_1(0) &= v_0(0) + \beta[y_d(1) - y_0(1)] \\ &= \beta[\mathbf{c}\mathbf{f}_d(0) + \mathbf{c}\mathbf{b}u_d(0) - \mathbf{c}\mathbf{f}_0(0) - \mathbf{c}\mathbf{f}u_0(0)] \\ &= \beta\mathbf{c}\mathbf{b}u_d(0). \end{aligned}$$

As $\forall i \in \mathcal{Z}_+$, $\mathbf{x}_i(0) = \mathbf{x}_d(0)$, if $v_i(0) \in I_D$, $v_{i+1}(0) = v_i(0) + \beta\mathbf{c}\mathbf{b}u_d(0)$ can be derived analogously. Hence, a finite iteration number p_0 can be found such that $v_{p_0-1} = (p_0 - 1)\beta\mathbf{c}\mathbf{b}u_d(0) \leq \eta_r$ and $v_{p_0} = p_0\beta\mathbf{c}\mathbf{b}u_d(0) > \eta_r$.

Moreover, $\forall i \geq p_0$, if $v_i(0) > \eta_r$, we have

$$\begin{aligned} v_{i+1}(0) &= v_i(0) + \beta[\mathbf{c}\mathbf{b}m_r(v_d(0) - \eta_r) - \mathbf{c}\mathbf{b}m_r(v_i(0) - \eta_r)] \\ &= \gamma_r' v_i(0) + \beta\mathbf{c}\mathbf{b}m_r v_d(0) \\ &> \gamma_r'\eta_r + \beta\mathbf{c}\mathbf{b}m_r\eta_r \\ &= \eta_r. \end{aligned}$$

Hence, considering $v_{p_0} > \eta_r$, for any $i \geq p_0$, $v_i(0) > \eta_r$ can be guaranteed.

According to (2.9), the following can be derived for any $i \geq p_0$.

$$\begin{aligned} \delta v_{i+1}(0) &= \delta v_i(0) - \beta e_i(1) \\ &= \delta v_i(0) - \beta\mathbf{c}[\mathbf{f}_d(0) - \mathbf{f}_i(0)] - \beta\mathbf{c}\mathbf{b}m_r\delta v_i(0) \\ &= \gamma_r'\delta v_i(0). \end{aligned} \tag{2.22}$$

As $0 < \gamma_r' < 1$, $\lim_{i\to\infty} \delta v_i(0) = 0$, hence $\lim_{i\to\infty} \delta u_i(0) = 0$.

Moreover, from (2.12), we have

$$\|\mathbf{e}_i(1)\| \leq \|\mathbf{c}\|\|\mathbf{b}\|l_{\mathbf{f}}|\delta u(0)|.$$

Hence, $\lim_{i\to\infty} \delta u_i(0) = 0$ leads to $\lim_{i\to\infty} \mathbf{e}_i(1) = 0$.

For $u_d(0) < 0$ the same result can be derived straightforwardly.

Part B

Assume $\lim_{i\to\infty} \delta u_i(k) = 0$ and $\lim_{i\to\infty} e_i(k+1) = 0$, where $k = 0, \cdots, n$ and $1 \le n \le N-1$. Let us check the convergence propery for $k = n+1$.

(1) $u_d(n+1) = 0$

If $u_d(n+1) = 0$, from Lemma 2.4, it can be derived that $\lim_{i\to\infty} u_i(n+1) = u_d(n+1)$ and $\lim_{i\to\infty} v_i(n+1) \in I_D$. Consequently, according to (2.12), $\lim_{i\to\infty} e_i(n+2) = 0$.

(2) $u_d(n+1) \neq 0$

If $u_d(n+1) < 0$ or $u_d(n+1) > 0$, according to Lemma 2.3 and Remark 2.3, a finite iteration number p_n can be found such that $v_i(n+1) \in I_L$ or $v_i(n+1) \in I_R$ respectively.

Assume $u_d(n+1) > 0$. According to updating law (2.9) and considering (2.11), for any $i \ge p_n$, we have

$$\begin{aligned}\delta v_{i+1}(n+1) &= \delta v_i(n+1) - \beta e_i(n+2) \\ &= \delta v_i(n+1) - \beta \mathbf{c}[\mathbf{f}_d(n+1) - \mathbf{f}_i(n+1)] - \beta \mathbf{c}\mathbf{b} m_r \delta v_i(n+1) \\ &\le \gamma_r' \delta v_i(n+1) + \beta l_{\mathbf{f}} \|\mathbf{c}\| \|\mathbf{b}\| \sum_{j=0}^{n} l_{\mathbf{f}}^{n-j} |\delta u_i(j)|.\end{aligned}$$

Therefore,

$$|\delta v_{i+1}(n+1)| \le \gamma_r' |\delta v_i(n+1)| + \beta l_{\mathbf{f}} \|\mathbf{c}\| \|\mathbf{b}\| \sum_{j=0}^{n} l_{\mathbf{f}}^{n-j} |\delta u_i(j)|. \qquad (2.23)$$

According to Lemma 2.1, $\lim_{i\to\infty} |\delta u_i(k)| = 0$ $(k = 0, \cdots, n)$ leads to $\lim_{i\to\infty} |\delta v_i(n+1)| = 0$. Consequently, $\lim_{i\to\infty} |\delta u_i(n+1)| = 0$ can be ensured. From (2.12), $\lim_{i\to\infty} |e_i(n+2)| = 0$ is further derived.

For $u_d(n+1) < 0$, the same result can be obtained similarly. ■

Next let us consider system (2.7) again, but $DZ[*]$ is defined as

$$u(k) = DZ[v(k)] = \begin{cases} m_l(k)[v(k) - \eta_l(k)] & v(k) \in I_L(k) \\ 0 & v(k) \in I_D(k) \\ m_r(k)[v(k) - \eta_r(k)] & v(k) \in I_R(k) \end{cases} . \tag{2.24}$$

where $\forall k \in \mathcal{K}$ $m_l(k) > 0$, $m_r(k) > 0$, $\eta_l(k) \leq 0$ and $\eta_r(k) \geq 0$. Note that in (2.24), all the parameters of the deadzone are time-varying and satisfy the following assumption.

Assumption 2.5. The upper bound of $m_l(k)$ and $m_r(k)$ is known and denoted as B_1.

Based on the same learning law (2.9), the following corollary can be obtained.

Corollary 2.3. *Under Assumption 2.2 - 2.4 and 2.5, the learning law (2.9) guarantees that $u_i(k)$ and $y_i(k)$ converge to $u_d(k)$ and $y_d(k)$ respectively for any $k \in \mathcal{K}$. Furtherfmore, the system signal $v_i(k)$ converges to $v_d(k)$ if $u_d(k) \neq 0$, otherwise $\lim_{i\to\infty} v_i(k) \in I_D$.*

The proof for Collorary 2.3 is exactly same as the proof of Theorem 2.2.

2.5 Illustrative Example

The following systems with input deadzone is considered.

$$\begin{aligned} x_1(kT_s + T_s) &= x_2(kT_s) \\ x_2(kT_s + T_s) &= 0.7x_1(kT_s) + 0.15x_2(kT_s) + DZ[v(kT_s)] \\ y(kT_s + T_s) &= x_2(kT_s + T_s), \end{aligned}$$

where the deadzone parameters are $\eta_l(k) = -0.6 - 0.12sin(kT_s)$, $\eta_r(k) = 0.9 + 0.13sin(kT_s)$, $m_l(k) = 0.8 + 0.15sin(kT_s)$ and $m_r(k) = 1.2 + 0.1sin(kT_s)$. Note that

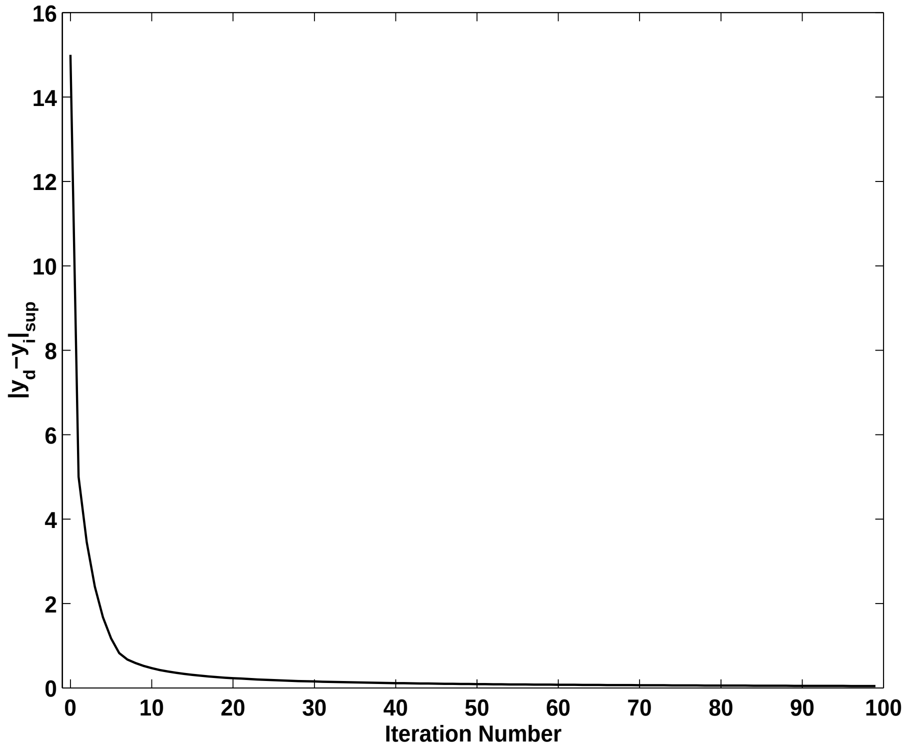

Figure 2.2: Learning convergence of $y_d - y_i$ for system with input deadzone.

m_l, m_r, η_l and η_r are all time-varying. The desired output is $y_d(k) = 10sin^3(kT_s)$, $k = 0, 1, \cdots, 6283$. To satisfy Assumption 2.3, let $x_{1,i} = x_{2,i}(0) = 0$.

Assume the known bound of m_l, m_r and **cb** are $B_1 = 1.5$ and $B_2 = 1.5$ respectively. Choose $\beta = 0.2$ to guarantee $0 < 1 - \beta B_1 B_2 < 1$. Let $T_s = 0.001s$.

By applying the control law (2.9), the simulation result is shown in Fig. 2.2. The horizon is the iteration number and the vertical is $|y_d - y_i|_{sup} \triangleq \sup_{k \in \mathcal{K}} |y_d(k) - y_i(k)|$.

Fig. 2.3 shows the control signal v_i at the $100th$ iteration.

To demonstrate how the input deadzone is learned by FIL, next we focus on the learning performance during $k = 0, 1, \cdots, 200$. In Fig. 2.4, the control signals of different iterations are given. Obviously, the system input deadzone is overcome gradually just by iterations.

From the simulation results, it can be clearly seen that although all the parameters

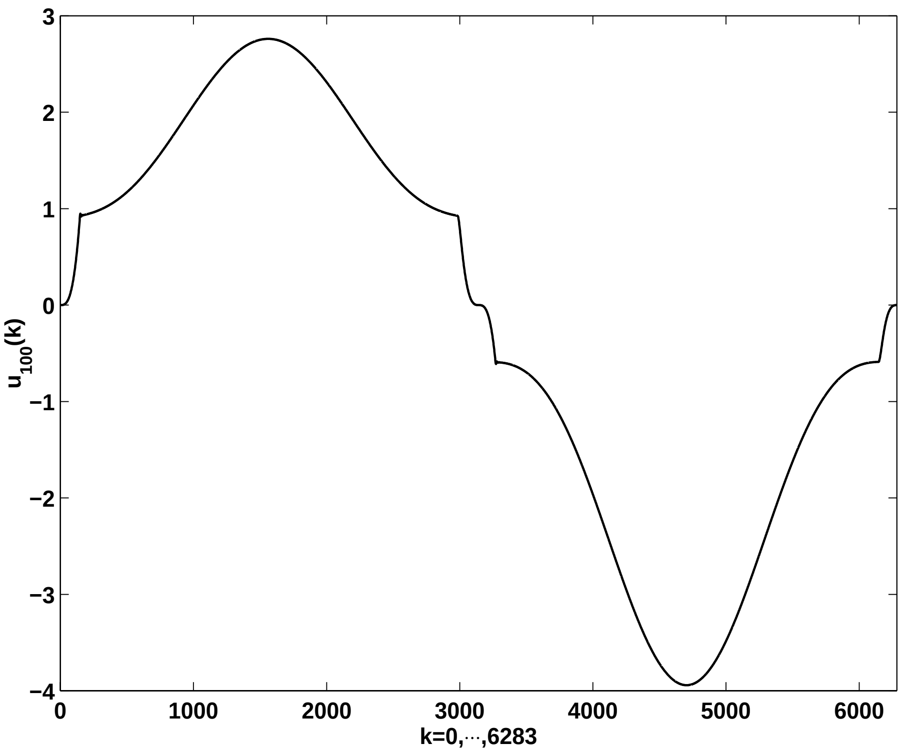

Figure 2.3: Control signal at the $100th$ iteration.

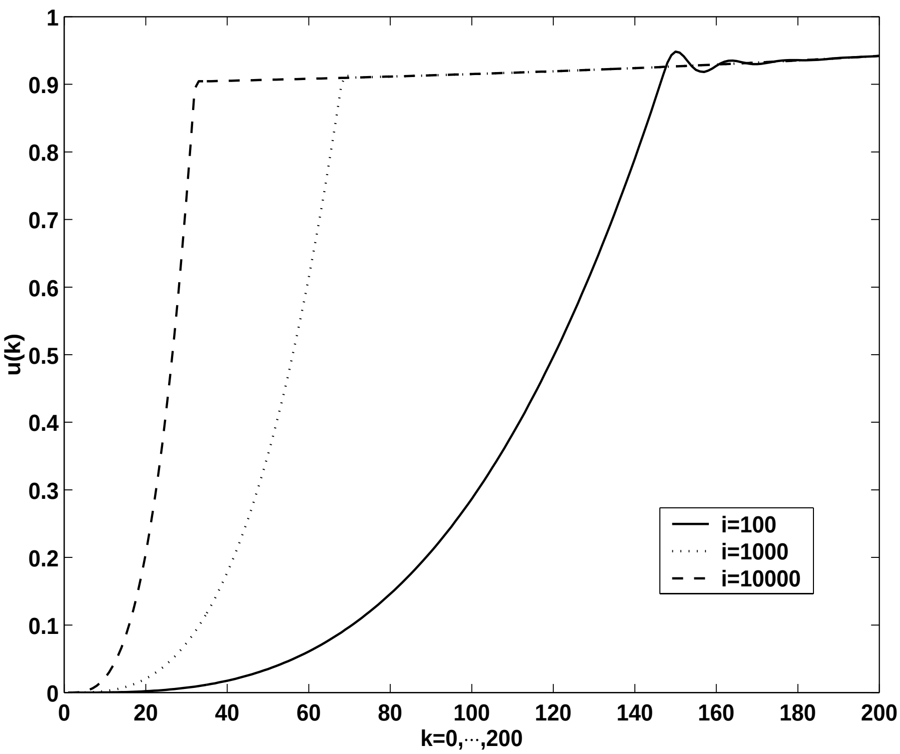

Figure 2.4: Control signal v_i in different iterations.

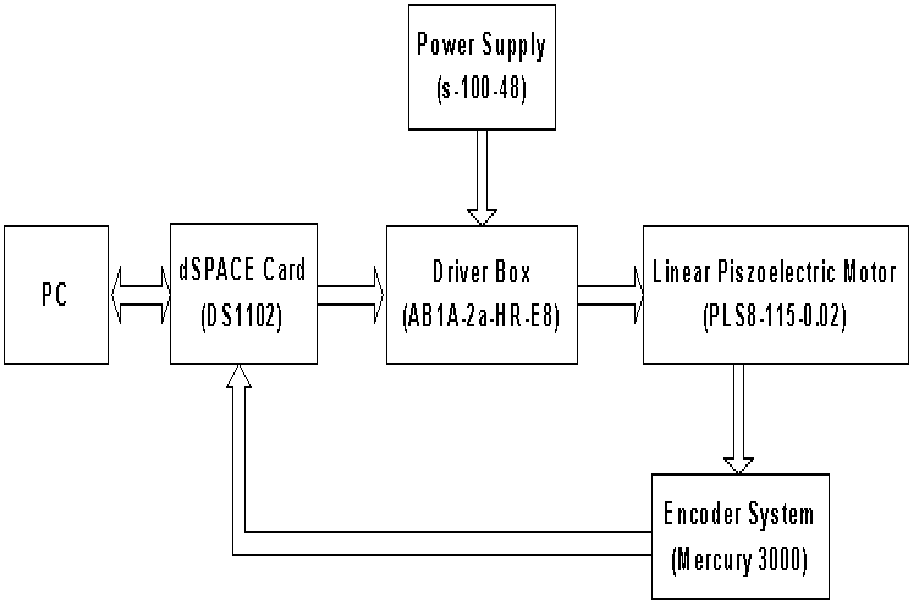

Figure 2.5: Structure of the control system.

of the deadzone are time-varying, the proposed FIL scheme still works quite well.

2.6 Experimental Results

In order to verify the effectiveness of the proposed algorithm, experiments have been carried out using a linear piezoelectric motor which has many promising applications in industries. The piezoelectric motors are characterized by low speed and high torque, which are in contrast to the high speed and low torque properties of the conventional electromagnetic motors. Moreover, piezoelectric motors are compact, light and operates quietly. They can't be affected by external magnetic or radioactive fields. However, the accurate mathematical model of piezoelectric motors are unavailable and their control characteristics are highly nonlinear. Therefore, precision control of piezoelectric motors is a challenge to control engineers.

The configuration of the whole control system is outlined in Fig. 2.5. Approximately, the driver and the motor can be modeled as:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{k_{fv}}{M}x_2(t) + \frac{k_f}{M}u(t) \\ y(t) &= x_1(t) \end{aligned} \tag{2.25}$$

where x_1 is the motion position, x_2 is the motion velocity, $M = 1kg$ is the moving mass, $k_{fv} = 144N$ is the velocity damping factor and $K_f = 6N/Volt$ is the force constant.

Choose the sampling time to be $T_s = 0.004s$. Substitute the system parameters and the discretized model of (2.25) is:

$$\begin{aligned} x_1(k+1) &= x_1(k) + 0.003x_2(k) + 6.662 \times 10^{-6}u(k) \\ x_2(k+1) &= 0.5621x_2(k) + 0.003u(k) \\ y(k+1) &= x_1(k+1). \end{aligned} \tag{2.26}$$

The dominant linear model (2.26) does not contain the nonlinear effects which are caused by frictional forces and high-order dynamics etc. Note that here the piezo-electric motor's deadzone is not only non-symmetry but also affected by the motor's position.

Although the proposed ILC algorithm can be implemented to (2.26) directly, the learning speed is very slow as (2.26) is an open-loop system and the tracking error at first iteration is very large. In practice, to improve the learning speed, a P controller may be applied first, which could be treated as a part of $\mathbf{f}(\mathbf{x}(k), k)$. Then the ILC part can be further added to the closed-loop system. Therefore, in our experiments, a simple discrete P controller, i.e., $u(k) = k_p e(k)$, is used, where $k_p = 1.5$ and $e(k) = y_d(k) - y(k)$.

Let $T = 6s$, hence $k \in \{0, 1, \cdots, 1500\}$. The system is repeatable over $[0, T]$ with a repeatability of $0.1\mu m$. The desired tracking trajectory is: $y_d(k) = [20 +$

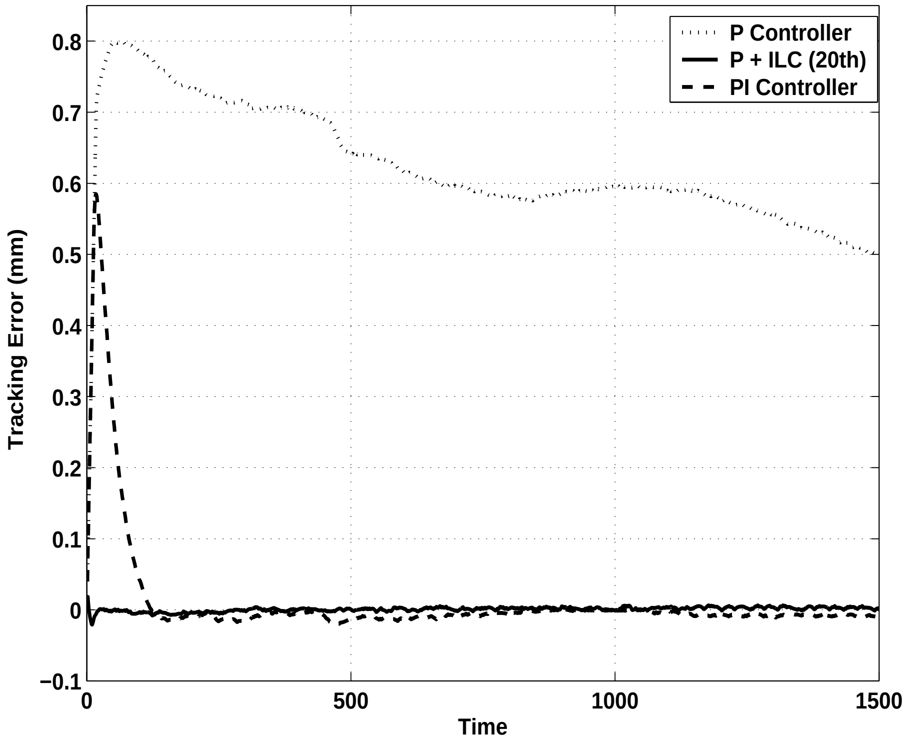

Figure 2.6: Comparison of different tracking errors.

$50\sin(0.001k)]mm$, $k \in \{0, \cdots, 1500\}$. To satisfy Assumption 2.3, the system initial condition is set to be $x_1(0) = 20mm$ and $x_2(0) = 0$, which is realized by a PI controller in the experiments. Choose $\beta = 0.6$. The tracking errors of the $1st$ and $20th$ iterations are given in Fig. 2.6. For comparison, the control performance of a discrete PI controller, i.e., $u(z) = (1.5 + 10\frac{T_s}{z-1})e(z)$, is also shown in Fig. 2.6. Obviously, the proposed ILC scheme can effectively compensate the system input deadzone and greatly reduce the tracking error. The control signals for $i = 1$, $i = 20$ and the PI controller are provided in Fig. 2.7.

To demonstrate the learning process, the maximum dynamic tracking error, i.e., $\max\limits_{k \in \mathcal{K}_1} |e(k)|$ where $\mathcal{K}_1 \triangleq \{0, \cdots, 100\}$, and the maximum steady tracking error, i.e., $\max\limits_{k \in \mathcal{K}_2} |e(k)|$ where $\mathcal{K}_2 \triangleq \{101, \cdots, 1500\}$, of each iteration are recorded and given in Fig. 2.8. We can see the convergence speed of the steady tracking error is much faster than that of the dynamic tracking error. The maximum steady tracking

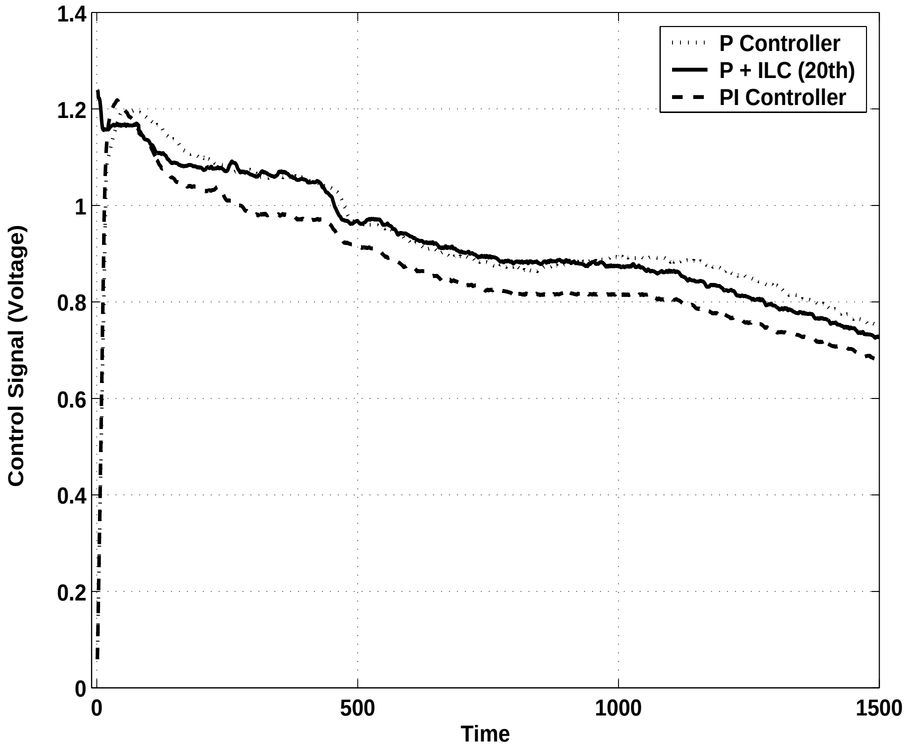

Figure 2.7: Comparison of different control signals.

error is below $0.01mm$ after only 15 iterations. The slower convergence speed of the dynamic tracking error implies the difficulty in the learning of a deadzone. However, the dynamic tracking error still can be reduced to around $0.02mm$ after 20 iterations. On the other hand, the dynamic tracking error is defined on $[0, 0.4s]$ which is very short comparing with $[0, 6s]$. Therefore, the learning performance is satisfying.

To clearly explain how the input deadzone is compensated by ILC, we focus on the time interval $k \in \{0, \cdots, 150\}$. The control signals and the tracking errors of different iterations are given in Fig. 2.9 and Fig. 2.10 respectively. Obviously, based on the iterative updating, the system input goes out of the input deadzone by iterations and the tracking error is reduced accordingly.

Here we only gave the experimental results for a tracking problem. The proposed ILC algorithm can also be applied to regulation control problems and the better control performance can be achieved. The experimental results show that the steady

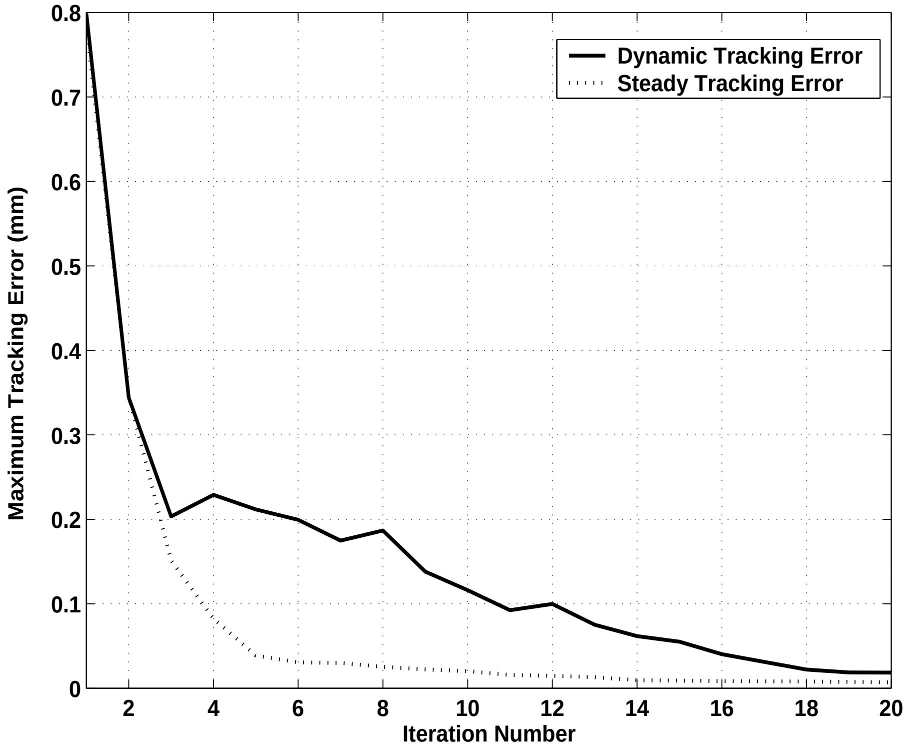

Figure 2.8: Convergence of the maximum tracking error.

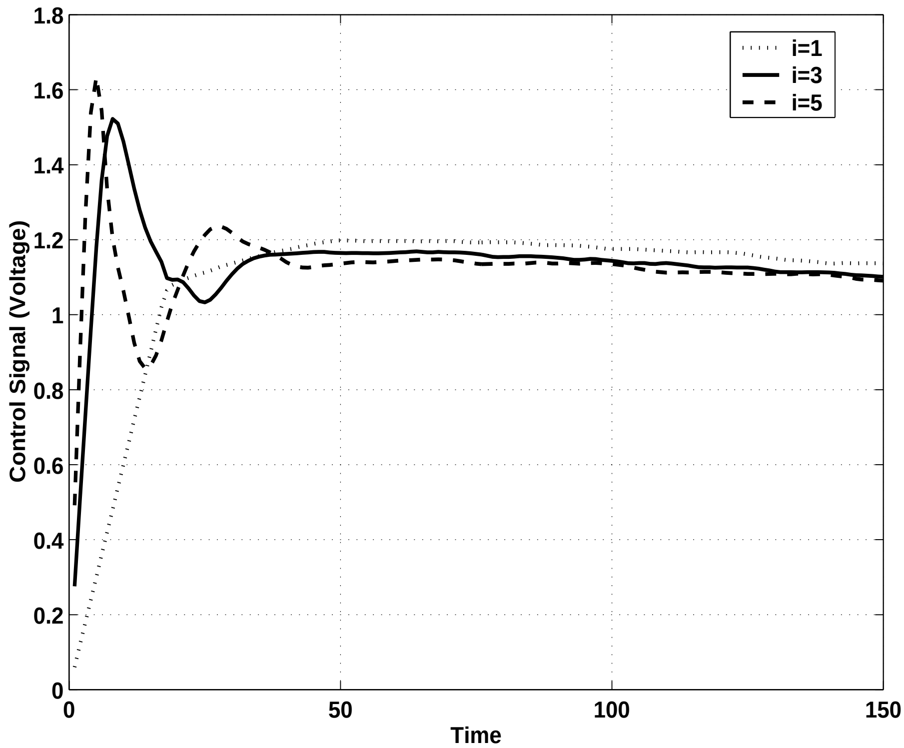

Figure 2.9: The control signals of different iterations.

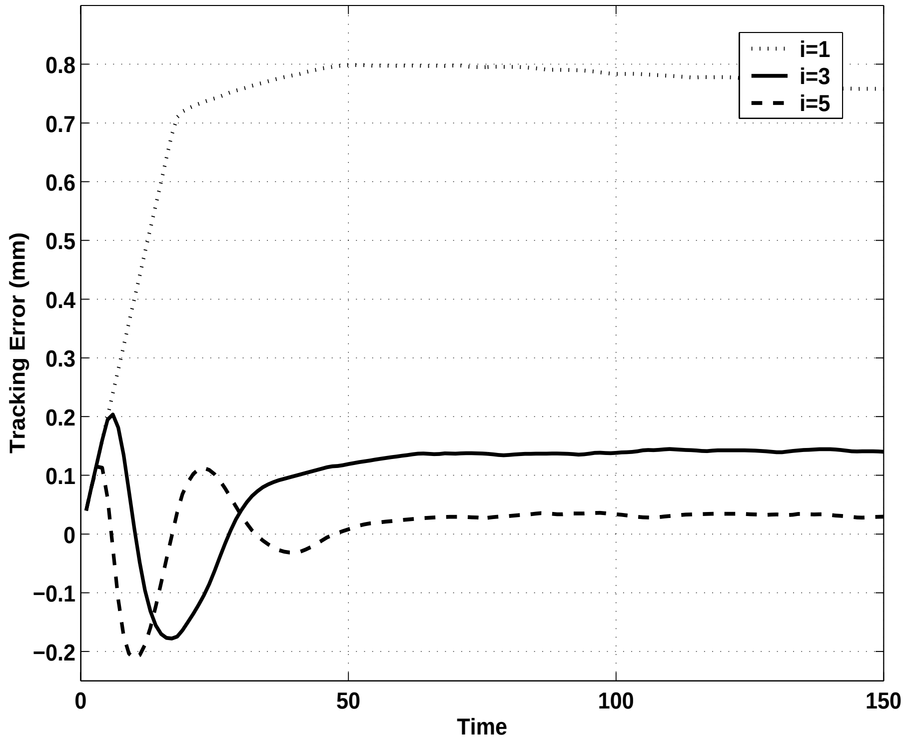

Figure 2.10: The tracking errors of different iterations.

state error can be reduced to $1\mu m$ within 20 learning iterations for a regulation control problem.

2.7 Conclusion

In this chapter, FIL is applied to a class of discrete-time systems with nonsmooth nonlinearity, i.e. a input deadzone. Therefore, if the controlled system and the control traget are repeatable, the input deadzone can be compensated by the learning iteration. Rigorous proof for the convergence property based on CM principle is given and the illustrative example and experimental results show the effectiveness of the proposed approach.

Chapter 3

FIL for Systems with Input Backlash

3.1 Introduction

In Chapter 2, FIL for systems with input deadzone has been proposed and analyzed. As a continuity of it, FIL will be extended to the dynamic systems with input backlash in this chapter. Backlash is another kind of highly practical-relevant control problem. Comparing with deadzone which is memoryless, backlash has an element of memory. Hence, overcoming backlash properly is a more difficult problem.

In this chapter, the simple FIL approach will be applied for systems with input backlash to achieve the perfect tracking over the whole finite time interval iteratively. Through rigorous proof based on CM principle, it is clearly shown that the FIL algorithm can address the unknown input backlash effective by iterative learning. The perfect tracking can be obtained as the iteration number approaches to infinity.

This chapter is organized as follows. In Section 3.2 FIL for a pure backlash component is proposed and analyzed. Based on it the FIL for dynamic systems with input backlash is presented in Section 3.3. Section 3.4 gives an illustrative example to show the effectiveness of the proposed FIL algorithm. Finally, conclusion is given in Section 3.5.

3.2 FIL for A Pure Backlash Component

Consider a class of backlash described by the following equation

$$u_i(k) = BL[v_i(k)] = \begin{cases} m_l[v_i(k) - \eta_l] & v(k) \in I_{L,i}(k-1) \\ u_i(k-1) & v(k) \in I_{D,i}(k-1) \\ m_r[v_i(k) - \eta_r] & v(k) \in I_{R,i}(k-1) \end{cases} . \tag{3.1}$$

where $m_l > 0$, $m_r > 0$, $\eta_l \leq 0$, $\eta_r \geq 0$, $k \in \mathcal{K}$, $i \in \mathcal{Z}_+$; $I_{L,i}(k-1) \triangleq (-\infty, v_{l,i}(k-1))$, $I_{D,i}(k-1) \triangleq [v_{l,i}(k-1), v_{r,i}(k-1)]$ and $I_{R,i}(k-1) \triangleq (v_{r,i}(k-1), \infty)$ with $v_{l,i}(k-1) = \frac{u_i(k-1)}{m_l} + \eta_l$ and $v_{r,i}(k-1) = \frac{u_i(k-1)}{m_r} + \eta_r$.

The characteristic of the backlash can be described as Fig. 3.1. Obviously, unlike deadzone which is memoryless, backlash has an element of memory and is, in a certain sense, dynamic.

The control objective is to find a sequence of appropriate control signal $v_i(k)$ such that $u_i(k)$ convergence to the target $u_d(k)$.

Including Assumption 2.1, the following I.I.C. is further made for the backlash (3.1).

Assumption 3.1. $\forall i \in \mathcal{Z}_+$, $\delta u_i(0) \triangleq u_d(0) - u_i(0) = 0$.

The FIL law is

$$v_i(k) = v_{i-1}(k) + \beta \delta u_{i-1}(k), \tag{3.2}$$

$$0 < 1 - \beta B_1 < 1 \tag{3.3}$$

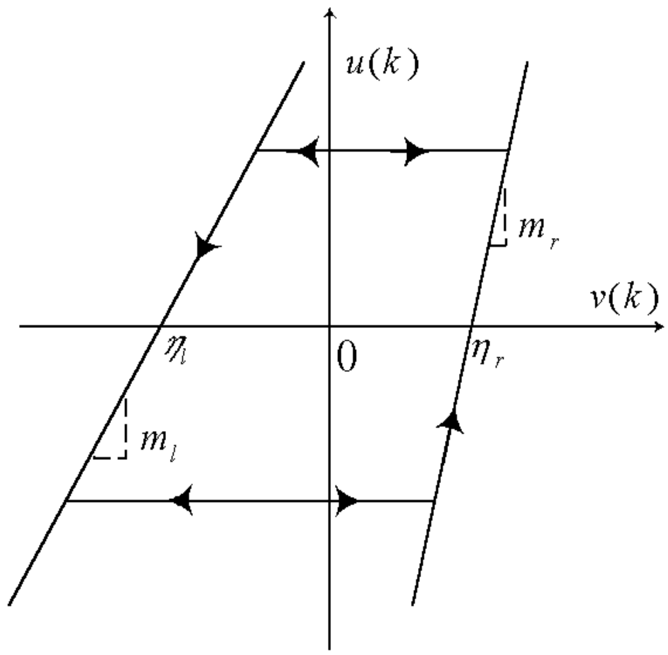

Figure 3.1: The backlash nonlinearities $u(k) = BL[v(k)]$.

where $\beta > 0$ is the learning gain and $\forall k \in \mathcal{K}$ $v_{-1}(k) = u_{-1}(k) = 0$. From (3.3), $0 < \gamma_l = 1 - \beta m_l < 1$ and $0 < \gamma_r = 1 - \beta m_r < 1$ can be ensured.

To facilitate the explanation, $\forall k \in \mathcal{K}$ and $\epsilon_k > 0$, the following notations are defined first.

$$
\begin{aligned}
I_{L,d}(k) &\triangleq (-\infty, v_{d,l}(k)), & v_{d,l}(k) &= \frac{u_d(k)}{m_l} + \eta_l \\
I_{R,d}(k) &\triangleq (v_{d,r}(k), \infty), & v_{d,r}(k) &= \frac{u_d(k)}{m_r} + \eta_r \\
I_{D,d}(k) &\triangleq [v_{d,l}(k), v_{d,r}(k)] & & \\
I'_{L,d}(k) &\triangleq (-\infty, v'_{d,l}(k)), & v'_{d,l}(k) &= v_{d,l}(k) - \frac{\epsilon_k}{m_l} \\
I'_{R,d}(k) &\triangleq (v'_{d,r}(k), \infty), & v'_{d,r}(k) &= v_{d,r}(k) + \frac{\epsilon_k}{m_r}.
\end{aligned}
$$

The main result can be summarized as the following theorem.

Theorem 3.1. *For the backlash (3.1), under Assumption 2.1 and 3.1, the control law (3.2) guarantees the convergence of $u_i(k)$ to $u_d(k)$ for any $k \in \mathcal{K}$ as $i \to \infty$.*

Proof:

1. $k = 0$

From Assumption 3.1, we have $u_i(0) = u_d(0)$ is valid for all iterations.

2. Assume that $\lim_{i\to\infty} u_i(m) = u_d(m)$, where $0 \le m \le N$. We will show $\lim_{i\to\infty} u_i(m + 1) = u_d(m + 1)$ can be derived.

(A) $u_d(m + 1) = u_d(m)$

If $u_d(m + 1) = u_d(m)$, as $\forall k \in \mathcal{K}$ $v_{-1}(k) = u_{-1}(k) = 0$, according to (3.1) and (3.2), $v_i(m) = v_i(m + 1)$ and $u_i(m) = u_i(m + 1)$ can always be ensured. Therefore, $\lim_{i\to\infty} u_i(m + 1) = u_d(m + 1)$ can be derived directly.

(B) $u_d(m + 1) \ne u_d(m)$

As $u_d(m + 1) \ne u_d(m)$, two arbitrarily small constants $\epsilon_m > 0$ and $\rho > 0$ can be found such that $|u_d(m + 1) - u_d(m)| \ge \epsilon_m + \rho$. Here we only consider $u_d(m + 1) \ge u_d(m) + \epsilon_m + \rho$. For $u_d(m + 1) \le u_d(m) - \epsilon_m - \rho$, the same result can be derived similarly.

As $\lim_{i\to\infty} u_i(m) = u_d(m)$, for any given ϵ_m, a finite constant p'_m can be found such that $\forall i \ge p'_m$, $|u_d(m) - u_i(m)| \le \epsilon_m$, hence, $v_{r,i}(m) \le v'_{d,r}(m)$.

First we will show $v_i(m+1) \in I'_{R,d}(m)$ can be realized within finite iteration $p_m \ge p'_m$.

$\forall i \ge p'_m$, assume $v_i(m + 1) \notin I'_{R,d}(m)$. According to (3.2), it can be derived that

$$\begin{aligned} v_{i+1}(m + 1) &= v_i(m + 1) + \beta[u_d(m + 1) - u_i(m + 1)] \\ &= v_i(m + 1) + \beta[u_d(m + 1) - BL[v_i(m + 1)]]. \end{aligned} \tag{3.4}$$

Next let us examine the term $-BL[v_i(m+1)]$ according to the following three cases.

Case 1: $v_i(m + 1) \in I_{L,i}(m)$

Since $v_i(m+1) \le v_{l,i}(m) \le \frac{u_d(m)+\epsilon_m}{m_l} + \eta_l$, it can be derived that

$$\begin{aligned} -BL[v_i(m+1)] &= -m_l[v_i(m+1) - \eta_l] \\ &\ge -u_d(m) - \epsilon_m. \end{aligned} \tag{3.5}$$

Case 2: $v_i(m+1) \in I_{D,i}(m)$

$$-BL[v_i(m+1)] = -u_i(m) \ge -u_d(m) - \epsilon_m. \tag{3.6}$$

Case 3: $v_i(m+1) \in I_{R,i}(m) - I'_{R,d}(m)$

$$\begin{aligned} -BL[v_i(m+1)] &= -m_r[v_i(m+1) - \eta_r] \\ &\ge -m_r[v'_{d,r}(m) - \eta_r]\} \\ &= -u_d(m) - \epsilon_m. \end{aligned} \tag{3.7}$$

According to (3.5), (3.6) and (3.7), $\forall i \ge p'_m$, if $v_i(m) \notin I'_{R,d}(m)$, $-BL[v_i(m+1)] \ge -u_d(m) - \epsilon_m$.

Considering $u_d(m+1) - u_d(m) \ge \rho + \epsilon_m$, (3.4) can be rewritten as

$$\begin{aligned} v_{i+1}(m+1) &\ge v_i(m+1) + \beta[u_d(m+1) - u_d(m) - \epsilon_m] \\ &\ge v_i(m+1) + \beta\rho. \end{aligned}$$

Hence, a finite iteration $p_m \ge p'_m$ can always be found such that $v_{p_m}(m+1) \ge v'_{d,r}(m)$, i.e. $v_{p_m}(m+1) \in I'_{R,d}(m)$.

Next we will show $\forall i \ge p_m$, $v_i(m+1) \in I'_{R,d}(m)$ is guaranteed.

As $u_d(m+1) - u_d(m) \ge \epsilon_m + \rho$, the uniqueness of $v_d(m+1)$ can be ensured and $u_d(m+1) = m_r[v_d(m+1) - \eta_r]$. Hence,

$$\begin{aligned} & u_d(m+1) \ge u_d(m) + \epsilon_m + \rho \\ \Rightarrow\ & m_r[v_d(m+1) - \eta_r] \ge m_r[v_{d,r}(m) - \eta_r] + \epsilon_m + \rho \\ \Rightarrow\ & v_d(m+1) \ge v_{d,r}(m) + \frac{\epsilon_m}{m_r} + \frac{\rho}{m_r} \\ \Rightarrow\ & v_d(m+1) \ge v'_{d,r}(m) + \frac{\rho}{m_r}. \end{aligned} \tag{3.8}$$

$\forall i \geq p_m$, if $v_i(m+1) \in I'_{R,d}(m)$, according to (3.2), we have

$$\begin{aligned} v_{i+1}(m+1) &= v_i(m+1) + \beta\{m_r[v_d(m+1) - \eta_r] - m_r[v_i(m+1) - \eta_r]\} \\ &= v_i(m+1) + \beta m_r \delta v_i(m+1) \\ &= \beta m_r v_d(m+1) + \gamma_r v_i(m+1) \end{aligned} \tag{3.9}$$

Substituting (3.8) into (3.9) yields

$$\begin{aligned} v_{i+1}(m+1) &\geq \beta m_r[v'_{d,r}(m) + \frac{\rho}{m_r}] + \gamma_r v'_{d,r}(m) \\ &= v'_{d,r}(m) + \beta\rho > v'_{d,r}(m). \end{aligned}$$

As $v_{p_m} \in I'_{R,d}(m)$, $\forall i \geq p_m$, $v_i(m+1) \in I'_{R,d}(m)$ can be derived.

Furthermore, the following can be derived for any $i \geq p_m$.

$$\begin{aligned} \delta u_{i+1}(m+1) &= \delta u_i(m+1) + u_i(m+1) - u_{i+1}(m+1) \\ &= \delta u_d(m+1) + m_r[v_i(m+1) - v_{i+1}(m+1)] \\ &= \gamma_r \delta u_i(m+1). \end{aligned} \tag{3.10}$$

As $0 < \gamma_r < 1$, from (3.10), $\lim_{i\to\infty} u_i(m+1) = u_d(m+1)$ can be obtained.

3. According to the induction method, $\forall k \in \mathcal{K}$, $\lim_{i\to\infty} u_i(k) = u_d(k)$. ■

3.3 FIL for Dynamic Systems with Input Backlash

In this part we will discuss FIL for the dynamic systems with input backlash. Consider the following dynamic system

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{f}(\mathbf{x}(k), k) + \mathbf{b}u(k) \\ u(k) &= BL[v(k)] \\ y(k+1) &= \mathbf{c}\mathbf{x}(k+1), \end{aligned} \tag{3.11}$$

where $BL[*]$ is the input backlash defined as in (3.1). It is assumed that the backlash output $u(k)$ is not accessible.

The same Assumptions 2.1-2.4 are made for the system (3.11) and the backlash (3.1). The control target is to find the control signal $v_i(k)$ iteratively such that $y_i(k)$ converges to the desired output $y_d(k)$ as $i \rightarrow \infty$. To meet the control objective the following learning law is used.

$$\begin{aligned} v_i(k) &= v_{i-1}(k) + \beta e_{i-1}(k+1). \\ 0 &< 1 - \beta B_1 B_2 < 1, \end{aligned} \tag{3.12}$$

which is same as (2.9). Note that the proposed control law is quite simple, however, it can deal with both the input deadzone and the input backlash. $0 < 1-\beta B_1 B_2 < 1$ leads to $0 < \gamma_l' \triangleq 1 - \beta \mathbf{cb} m_l < 1$ and $0 < \gamma_r' \triangleq 1 - \beta \mathbf{cb} m_r < 1$.

To facilitate the analysis, two Lemmas are given first.

Lemma 3.1. *Assume* $\lim_{i\rightarrow\infty} |\delta u_i(k)| = 0$ *where* $k = 0, \cdots, m$ *and* $0 \leq m \leq N-1$. *For system (3.11), under the learning law (3.12), the system input* $v_i(m+1) \in I'_{R,d}(m)$ *will always be guaranteed after finite iteration if* $u_d(m+1) > u_d(m)$.

Proof:

Since $u_d(m+1) > u_d(m)$, two arbitrarily small constants ϵ_m and ρ can be found such that $u_d(m+1) - u_d(m) \geq \epsilon_m + \rho + \frac{l_\mathbf{f}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}}$.

Same as the proof in Lemma 2.3 of Chapter 1, there exists a finite iteration number p_m' such that $\forall i \geq p_m'$, $|\delta u_i(k)| \leq \epsilon_m$, where $k = 0, \cdots, m$. Moreover, (2.10) - (2.14) are also valid.

The following proof contains two parts. *Part A* shows that a finite iteration p_m can be found such that $v_0(m+1)$ is mapped into $I'_{R,d}(m)$ and *Part B* proves that for any $i \geq p_m$, the control law (3.12) always maps $v_i(m+1)$ from $I'_{R,d}(m)$ to $I'_{R,d}(m)$.

Part A

Analogous to the *Part A* of Lemma 2.3, for finite iteration, the boundedness of $v_i(k)$, $u_i(k)$ and $y_i(k)$ can be guaranteed.

$\forall i \geq p'_m$, assume $v_i(m+1) \notin I'_{R,d}$. According to (3.12), we have

$$\begin{aligned} v_{i+1}(m+1) &= v_i(m+1) + \beta e_i(m+2) \\ &= v_i(m+1) + \beta \mathbf{c}[\mathbf{f}_d(m+1) - \mathbf{f}_i(m+1)] \\ &\quad +\beta \mathbf{cb} \delta u_i(m+1). \end{aligned} \tag{3.13}$$

Considering (2.14), it can be derived that

$$v_{i+1}(m+1) \geq v_i(m+1) - \beta l_{\mathbf{f}} \|\mathbf{c}\| \Gamma_m + \beta \mathbf{cb} \{u_d(m+1) - BL[v_i(m+1)]\}. \tag{3.14}$$

Now let us check the term $-BL[v_i(m+1)]$ according to the following three cases.

Case 1: $v_i(m+1) \in I_{L,i}(m)$

From (3.1) we have

$$\begin{aligned} -BL[v_i(m+1)] &= -m_l[v_i(m+1) - \eta_l] \\ &\geq -m_l[v_{l,i}(m) - \eta_l] \\ &= -u_i(m) \\ &\geq -u_d(m) - \epsilon_m. \end{aligned} \tag{3.15}$$

Case 2: $v_i(m+1) \in I_{D,i}(m)$

$-BL[v_i(m+1)]$ can be expressed as

$$-BL[v_i(m+1)] = -u_i(m) \geq -u_d(m) - \epsilon_m. \tag{3.16}$$

Case 3: $v_i(m+1) \in [I_{R,i}(m) - I'_{R,d}(m)]$

From (3.13), it can be derived that

$$
\begin{aligned}
-BL[v_i(m+1)] &= -m_r(v_i(m+1)-\eta_r) \\
&\geq -m_r(v'_{d,r}(m)-\eta_r) \\
&= -u_d(m)-\epsilon_m. \qquad (3.17)
\end{aligned}
$$

According to (3.15), (3.16) and (3.17), it can be concluded that when $v_i(m+1) \notin I'_{R,d}(m)$, $-BL[v_i(m+1)] \geq -u_d(m)-\epsilon_m$.

Considering $u_d(m+1)-u_d(m) \geq \epsilon_m + \rho + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}}$, (3.14) can be rewritten as

$$
\begin{aligned}
v_{i+1}(m+1) &\geq v_i(m+1) - \beta l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m + \beta\mathbf{cb}[u_d(m+1)-u_d(m)-\epsilon_m] \\
&\geq v_i(m+1) + \beta\mathbf{cb}\rho > v_i(m+1).
\end{aligned}
$$

Therefore, as $v_{p'_m}(m+1)$ is bounded, $v_i(m+1) \in I'_{R,d}(m)$ can be obtained in finite iteration p_m.

Part B

As $u_d(m+1)-u_d(m) \geq \epsilon_m + \rho + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}} > 0$, the uniqueness of $v_d(m+1)$ can be ensured and $u_d(m+1) = m_r[v_d(m+1)-\eta_r]$. Hence,

$$
\begin{aligned}
& u_d(m+1) \geq u_d(m) + \epsilon_m + \rho + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}} \\
\Rightarrow\ & m_r[v_d(m+1)-\eta_r] \geq m_r[v_{d,r}(m)-\eta_r] + \epsilon_m + \rho + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}} \\
\Rightarrow\ & v_d(m+1) \geq v_{d,r}(m) + \frac{\epsilon_m}{m_r} + \frac{\rho}{m_r} + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}m_r} \\
\Rightarrow\ & v_d(m+1) \geq v'_{d,r}(m) + \frac{\rho}{m_r} + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb}m_r}. \qquad (3.18)
\end{aligned}
$$

$\forall i \geq p_m$, if $v_i(m+1) \in I'_{R,d}(m)$, from (3.13), we have

$$
\begin{aligned}
v_{i+1}(m+1) &= v_i(m+1) + \beta\mathbf{c}[\mathbf{f}_d(m+1) - \mathbf{f}_i(m+1)] \\
&\quad + \beta\mathbf{cb}m_r\delta v_i(m+1) \\
&\geq v_i(m+1) - \beta l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m + \beta\mathbf{cb}m_r\delta v_i(m+1) \\
&= \beta\mathbf{cb}m_r v_d(m+1) + \gamma'_r v_i(m+1) - \beta l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m \\
&\geq \beta\mathbf{cb}m_r v_d(m+1) + \gamma'_r v'_{d,r}(m) - \beta l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m. \qquad (3.19)
\end{aligned}
$$

Substituting (3.18) into (3.19) yields

$$\begin{aligned} v_{i+1}(m+1) &\geq \beta \mathbf{cb} m_r [v'_{d,r}(m) + \frac{l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m}{\mathbf{cb} m_r} + \frac{\rho}{m_r}] + \gamma'_r v'_{d,r}(m) - \beta l_{\mathbf{f}}\|\mathbf{c}\|\Gamma_m \\ &= v'_{d,r}(m) + \beta \mathbf{cb}\rho > v'_{d,r}(m). \end{aligned}$$

As $v_{p_m} \in I'_{R,d}(m)$, $\forall i \geq p_m$, $v_i(m+1) \in I'_{R,d}(m)$ can be derived. ■

Remark 3.1. Analogous to Lemma 3.1, under the same assumptions, updating law (3.12) guarantees that the system input $v_i(m+1) \in I'_{L,d}(m)$ can be realized after finite iteration $p_m \geq p'_m$, if $u_d(m+1) < u_d(m)$.

According to (3.1), if $u_d(k+1) = u_d(k)$, the control signal $v_d(k+1)$ is not unique. Next we will show that when $u_d(m+1) = u_d(m)$, $\lim_{i\to\infty} \delta u_i(m+1) = 0$ can be ensured if $\lim_{i\to\infty} \delta u_i(k) = 0$ $(k = 0, \cdots, m)$.

Lemma 3.2. *Assume* $\lim_{i\to\infty} \delta u_i(k) = 0$ *where* $k = 0, \ldots, m$. *If* $u_d(m+1) = u_d(m)$, $\lim_{i\to\infty} \delta u_i(m+1) = 0$ *and* $\lim_{i\to\infty} v_i(m+1) \in I_{D,d}(m)$ *can be derived.*

Proof:

Analogous to the proof of Lemma 2.4, $\forall i \geq p'_m$, the following relationships can be derived:

$$\begin{aligned} &if \quad v_i(m+1) \in I_{R,i}(m), && \gamma'_r[v_i(m+1) - v_{r,i}(m)] - \Delta_i(m) \\ & && \leq v_{i+1}(m+1) - v_{r,i}(m) \\ & && \leq \gamma'_r[v_i(m+1) - v_{r,i}(m)] + \Delta_i(m); \qquad (3.20) \\ &if \quad v_i(m+1) \in I_{L,i}(m), && \gamma'_l[v_i(m+1) - v_{l,i}(m)] - \Delta_i(m) \\ & && \leq v_{i+1}(m+1) - v_{l,i}(m) \\ & && \leq \gamma'_l[v_i(m+1) - v_{l,i}(m)] + \Delta_i(m); \qquad (3.21) \\ &if \quad v_i(m+1) \in I_{D,i}(m), && v_i(m+1) - \Delta_i(m) \\ & && \leq v_{i+1}(m+1) \leq v_i(m+1) + \Delta_i(m). \qquad (3.22) \end{aligned}$$

According to (3.20), (3.21) and (3.22) and considering $\lim_{i\to\infty} \Delta_i(m) = 0$, from Lemma 2.2, we have, $\lim_{i\to\infty} v_i(m) \in \lim_{i\to\infty} I_{D,i}(m) = I_{D,d}(m)$. Hence, $\lim_{i\to\infty} u_i(m) = \lim_{i\to\infty} u_i(m-1) = u_d(m-1) = u_d(m)$. ■

The main result for the control law (3.12) is summarized as the following theorem.

Theorem 3.2. *For system (3.11), under Assumptions 2.1-2.4, $\forall k \in \mathcal{K}$, the learning law (3.12) guarantees that $u_i(k)$ and $y_i(k)$ converges to $u_d(k)$ and $y_d(k)$ respectively as i approaches to infinity. The control signal $v_i(k)$ converges to $v_d(k)$ if $u_d(k) \neq u_d(k-1)$, otherwise $\lim_{i\to\infty} v_i(k) \in I_{D,d}(k-1)$.*

Proof:

Analogous to Theorem 2.2, *Part A* shows the convergence of $u_i(0)$ and $y_i(1)$. Assume the convergence of $u_i(k)$ and $y_i(k+1)$, where $k = 0, \cdots, n$ and $1 \leq n \leq N-1$, the convergence of $u_i(n+1)$ and $y_i(n+2)$ is proven in *Part B*. From the induction method, it can be derived that, for all $k \in \mathcal{K}$, $u_i(k)$ and $y_i(k)$ converge to $u_d(k)$ and $y_d(k)$.

Part A

As $x_i(0) = x_d(0)$, we assume that $x_i(-1) = x_d(-1)$ and $u_i(-1) = u_d(-1)$ for all $i \in \mathcal{Z}_+$.

(1) $u_d(0) = u_d(-1)$

If $u_d(0) = u_d(-1)$, Lemma 3.2 leads to $\delta u_i(0) = 0$ and $\lim_{i\to\infty} v_i(0) \in I_{D,d}(0)$. As $x_i(0) = x_d(0)$, according to the system dynamic (3.11), $\lim_{i\to\infty} e_i(1) = 0$ can be derived.

(2) $u_d(0) \neq u_d(-1)$

If $u_d(0) \neq u_d(-1)$, according to Lemma 3.1 and Remark 3.1, a finite constant p_{-1} can be found such that $\forall i \geq p_{-1}$ $v_i(0) \in I'_{L,d}(-1)$ or $v_i(0) \in I'_{R,d}(-1)$.

Assume $u_d(0) - u_d(-1) > 0$. Analogous to Theorem 2.2, (2.22) can be derived for any $i \geq p_{-1}$ which leads to $\lim_{i\to\infty} v_i(0) = v_d(0)$ and $\lim_{i\to\infty} u_i(0) = u_d(0)$. As the relationship (2.12) is still valid, $\lim_{i\to\infty} y_i(1) = y_d(1)$ can be guaranteed.

For $u_d(0) - u_d(-1) < 0$, the same result can be derived straightforwardly.

Part B

Assume $\lim_{i\to\infty} \delta u_i(k) = 0$ and $\lim_{i\to\infty} e_i(k+1) = 0$, where $k = 0, \cdots, n$ and $1 \leq n \leq N-1$. Let us examine the property for $k = n + 1$.

(1) $u_d(n + 1) = u_d(n)$

From Lemma 3.2, $\lim_{i\to\infty} \delta u_i(n + 1) = 0$ and $\lim_{i\to\infty} v_i(n + 1) \in I_{D,d}(n)$. According to (2.12), $\lim_{i\to\infty} e_i(n + 2) = 0$.

(2) $u_d(n + 1) \neq u_d(n)$

According to Lemma 3.1 and Remark 3.1, there exists a finite constant p_n such that $\forall i \geq p_n$, $v_i(n + 1) \in I'_{R,d}(n)$ or $v_i(n + 1) \in I'_{L,d}(n)$ respectively.

Assume $u_d(n + 1) - u_d(n) > 0$. Analogous to the proof Theorem 2.2, (2.23) can be obtained for any $i \geq p_n$. Therefore, $\lim_{i\to\infty} \delta v_i(n + 1) = 0$, $\lim_{i\to\infty} \delta u_i(n + 1) = 0$ and $\lim_{i\to\infty} e_i(n + 2) = 0$ can be ensured.

For $u_d(n + 1) - u_d(n) < 0$, same result can be obtained. ■

3.4 Illustrative Example

To illustrate the effectiveness of our FIL method, the following system is considered.

$$\begin{aligned} x_1(kT_s+T_s) &= x_2(kT_s) \\ x_2(kT_s+T_s) &= 0.4sin[x_1(kT_s)] + 0.15x_2(kT_s) + BL[v(kT_s)] \\ y(kT_s+T_s) &= x_2(kT_s+T_s), \end{aligned}$$

where the backlash parameters are $\eta_l = -1.3$, $\eta_r = 1.5$, $m_l = 1.1$ and $m_r = 1.0$. The desired output is $y_d(k) = 10sin^3(kT_s)$, $k = \{0, 1, \cdots, 6283\}$. To satisfy Assumption 2.3, let $x_{2,i}(0) = y_d(0) = 0$ and $x_{1,i} = 0$.

Assume the known bound of m_l, m_r and **cb** are $B_1 = 1.2$ and $B_2 = 1.2$ respectively. Choose $\beta = 0.6$ to guarantee $0 < 1 - \beta B_1 B_2 < 1$. Let $T_s = 0.001s$.

By applying the control law (3.12), the simulation result is shown in Fig. 3.2. The horizon is the iteration number and the vertical is $|y_d - y_i|_{sup}$.

Fig. 3.3 shows the control signal v_i at the $100th$ iteration.

3.5 Conclusion

In this chapter, FIL is further extended to dynamic systems with input backlash. It has been shown that the learning convergence can be guaranteed by the simple proposed FIL control law. The illustrative example verifies the effectiveness of the developed FIL scheme.

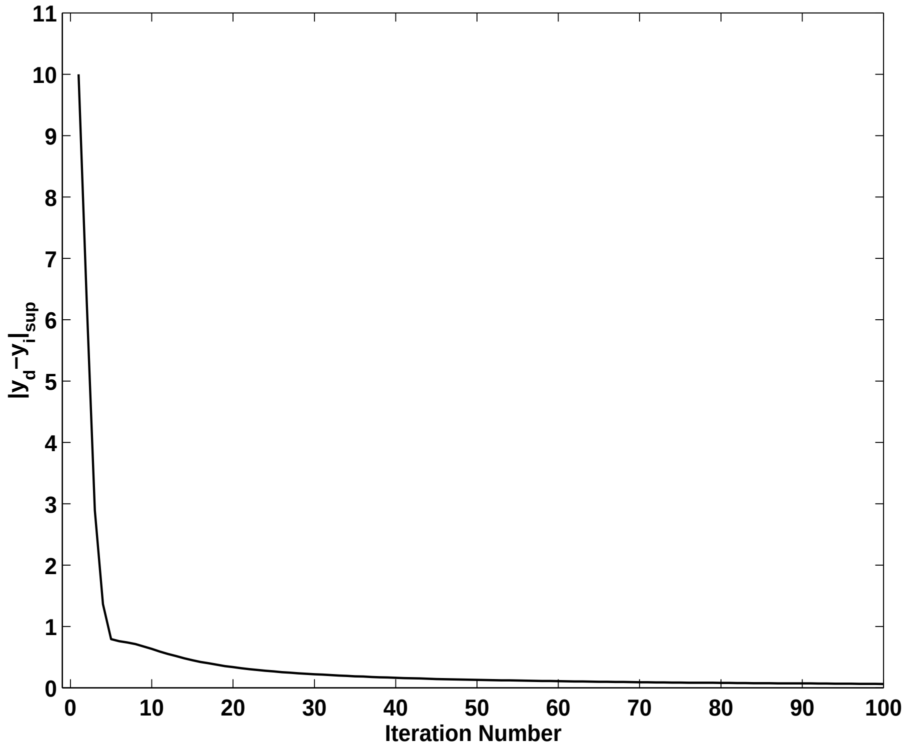

Figure 3.2: Learning convergence of $y_d - y_i$ for system with input backlash.

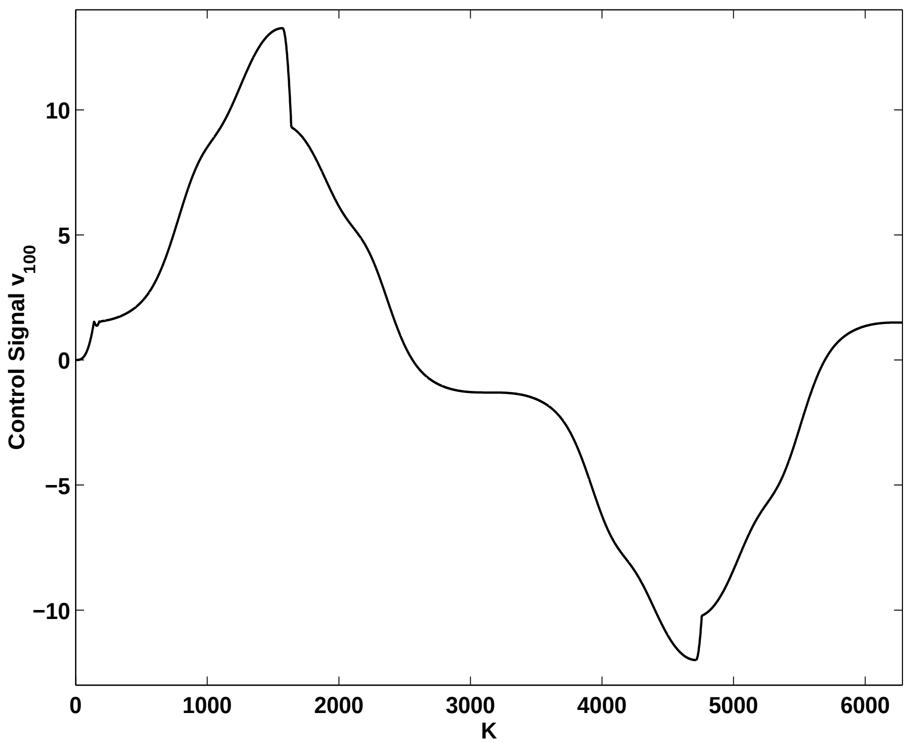

Figure 3.3: Control signal at the $100th$ iteration.

Chapter 4

FIL for Systems with Norm-bounded Uncertainties

4.1 Introduction

Traditional FIL approaches are based on the CM principle and their applications are limited to GLC systems. Recently, CEF-type FIL was proposed (Xu and Tan, 2002), hence much broader classes of nonlinearities can be easily addressed. In terms of CEF, we can evaluate the tracking performance along time axis by a Lyapunov function, meanwhile evaluate the learning performance along learning axis by a $\mathcal{L}^2$ functional.

CEF is a general concept and can be implemented to systems with both parametric and norm-bounded uncertainties. However, in (Xu and Tan, 2002), the research was focused only on FIL for systems with parametric uncertainties. Therefore, there exists in a more challenging problem: can we learn and deal with norm-bounded uncertainties?

In this chapter, CEF-type FIL will be extended to address norm-bounded uncertainties. FIL algorithms for SISO systems with both GLC and NGLC uncertainties are proposed in Section 4.2. The robust FIL is further applied to MIMO dynamics in Section 4.3. Simulation results are given in Section 4.6 to show the effectiveness of all the developed FIL schemes.

4.2 FIL for SISO Systems with Norm-bounded Uncertainties

To clearly explain the basic idea, FIL for the following SISO dynamic system is considered first.

$$\dot{x} = u + d(x,t), \quad t \in [0,T] \tag{4.1}$$

where $x \in \mathcal{R}$ is the measurable system state, $u \in \mathcal{R}$ is the control input and $d(x,t) : \mathcal{R} \times \mathcal{R}_+ \to \mathcal{R}$ is the lumped uncertainty.

Both the system dynamics (4.1) and the tracking task $x_d(t) \in \mathcal{C}^1[0,T]$ are assumed to be repeatable over $[0,T]$. Moreover, as part of the repeatability, the following I.I.C. is made.

Assumption 4.1. $\forall i \in \mathcal{Z}_+$, $x_i(0) = x_d(0)$.

The ultimate control objective is to find a suitable control profile iteratively so as to track the following given target trajectory $x_d(t)$

$$\dot{x}_d(t) = u_d(t) + d(x_d,t) \quad t \in [0,T], \tag{4.2}$$

where u_d is the desired control input.

From the system dynamics (4.1) and the control target (4.2), we have

$$\delta u_i = (\dot{x}_d - d_d) - (\dot{x}_i - d_i) = \dot{e}_i + d_i - d_d \tag{4.3}$$

where $\delta u_i = u_d - u_i$, $d_d = d(x_d, t)$, $d_i = d(x_i, t)$ and $e_i = x_d - x_i$.

In the rest part of this section, two FIL schemes will be developed according to the different properties of d_i. If d_i is GLC, the boundedness property of the dynamic system over a finite time interval $[0, T]$ can be guaranteed. Hence, a simple FIL control algorithm is constructed. When d_i is NGLC, robust control is incorporated to ensure the finiteness of the system state, which leads to a new robust FIL approach.

4.2.1 FIL for Systems with GLC Uncertainties

The following assumption is first made for the system uncertainty d.

Assumption 4.2. The system uncertainty $d(x, t)$ is GLC, i.e. $|d(x_1, t) - d(x_2, t)| \leq l_d|x_1 - x_2|$, where the Lipschitz constant l_d is completely unknown.

The learning law is designed as

$$\begin{aligned} u_i &= proj[u_{i-1}] + \beta e_i, \qquad u_{-1}(t) = 0 \\ proj[\cdot] &\triangleq \begin{cases} \cdot & |\cdot| \leq u^* \\ sign(\cdot)u^* & |\cdot| > u^* \end{cases}, \end{aligned} \tag{4.4}$$

where $\beta > 0$ is the learning gain and u^* is a projection bound which is sufficiently large such that $u^* \geq \sup_{t\in[0,T]} |u_d(t)|$. In practice, u^* is either a physical process limitation or a virtual saturation bound which can be arbitrarily large but finite.

The main result for the proposed FIL control law (4.4) is summarized as the following theorem.

Theorem 4.1. *For system (4.1), under Assumptions 4.1 and 4.2, the control law (4.4) guarantees that the tracking error e_i converges to 0 uniformly and the control signal u_i converges to u_d almost everywhere as i approaches to infinity.*

Proof:

To facilitate the derivation and analysis of the learning properties, define the following time-weighted CEF for the ith iteration:

$$E_i(t) = \frac{1}{2}e^{-\lambda t}e_i^2 + \frac{1}{2\beta}\int_0^t e^{-\lambda\tau}\delta u_i^2 d\tau, \tag{4.5}$$

where λ is a finite positive constant.

The proof consists of three parts which address respectively the boundedness of the system internal signals, the monotone decrease of the CEF along the learning axis i, and the uniform convergence of the tracking error.

(I) Boundedness Property

Substituting the control law (4.4) into the system dynamics (4.1) yields

$$\dot{x}_i = d_i + proj[u_{i-1}] + \beta(x_d - x_i). \tag{4.6}$$

Since $d_i - \beta x_i$ is still GLC, $x_d(t)$ is bounded, and $proj[u_{i-1}]$ is also bounded by u^*, considering the I.I.C., we can immediately derive the boundedness of x_i for any i. In the sequel the RHS of (4.6) is bounded, i.e. $\dot{x}_i$ is bounded. Further from (4.4) the boundedness of u_i is straightforward.

(II) Difference of $E_i(t)$

The difference of $E_i(t)$ is

$$\begin{aligned} \Delta E_i(t) &\triangleq E_i(t) - E_{i-1}(t) \\ &= \frac{1}{2}e^{-\lambda t}e_i^2 + \frac{1}{2\beta}\int_0^t e^{-\lambda t}(\delta u_i^2 - \delta u_{i-1}^2)d\tau - \frac{1}{2}e^{-\lambda t}e_{i-1}^2. \end{aligned} \tag{4.7}$$

The first term on the RHS of (4.7), with the I.I.C., can be rewritten as

$$\frac{1}{2}e^{-\lambda t}e_i^2 = -\frac{\lambda}{2}\int_0^t e^{-\lambda\tau}e_i^2 d\tau + \int_0^t e^{-\lambda\tau}e_i\dot{e}_i d\tau. \tag{4.8}$$

We can easily verify the property $(a-b)^2 \geq [a - proj[b]]^2$, for any quantities b and $|a| \leq a^*$, where a^* is the bound of the projector. Hence, the second term on the

RHS of (4.7) can be expressed as

$$\begin{aligned}
&\int_0^t e^{-\lambda\tau}(\delta u_i^2 - \delta u_{i-1}^2)d\tau \\
&\leq \int_0^t e^{-\lambda\tau}[\delta u_i^2 - (u_d - proj[u_{i-1}])^2]d\tau \\
&= \int_0^t e^{-\lambda\tau}[-2\delta u_i(u_i - proj[u_{i-1}]) - (u_i - proj[u_{i-1}])^2]d\tau. \qquad (4.9)
\end{aligned}$$

Substitute (4.3) and (4.4) into (4.9), we have

$$\begin{aligned}
&\frac{1}{2\beta}\int_0^t e^{-\lambda\tau}(\delta u_i^2 - \delta u_{i-1}^2)d\tau \\
&\leq -\int_0^t e^{-\lambda\tau}\dot{e}_i e_i d\tau + \int_0^t e^{-\lambda\tau}|d_d - d_i||e_i|d\tau - \frac{\beta}{2}\int_0^t e^{-\lambda\tau}e_i^2 d\tau \\
&\leq -\int_0^t e^{-\lambda\tau}\dot{e}_i e_i d\tau + (l_d - \frac{\beta}{2})\int_0^t e^{-\lambda\tau}e_i^2 d\tau. \qquad (4.10)
\end{aligned}$$

Substituting (4.8) and (4.10) into (4.7) yields

$$\Delta E_i(t) \;\leq\; -(\frac{\lambda}{2} + \frac{\beta}{2} - l_d)\int_0^t e^{-\lambda\tau}e_i^2 d\tau - \frac{1}{2}e^{-\lambda t}e_{i-1}^2.$$

There exists a sufficiently large λ such that $\lambda > 2l_d - \beta$ to ensure that

$$\Delta E_i(t) \leq -\frac{1}{2}e^{-\lambda t}e_{i-1}^2(t) \leq -\frac{1}{2}e^{-\lambda T}e_{i-1}^2(t) \leq 0, \qquad (4.11)$$

which implies the monotonically decreasing property of $E_i(t)$.

(III) Uniform Convergence

By using (4.11) repeatedly, we have

$$E_i(t) \leq E_0(t) - \frac{1}{2}e^{-\lambda T}\sum_{j=0}^{i-1} e_j^2(t). \qquad (4.12)$$

According to (4.12), from the boundedness of $E_0(t)$ and the positiveness of $E_i(t)$, we can derive that $\lim_{i\to\infty} e_i(t) = 0$ pointwisely. Therefore, $\lim_{i\to\infty}|d_i - d_d| \leq \lim_{i\to\infty} l_d|e_i| = 0$. Using (4.3) and the boundedness of $\dot{e}_i$ we further derive

$$\begin{aligned}
\lim_{i\to\infty} E_i(t) &= \lim_{i\to\infty}\frac{1}{2}e^{-\lambda t}e_i^2 + \lim_{i\to\infty}\frac{1}{2\beta}\int_0^t e^{-\lambda\tau}\delta u_i^2 d\tau \\
&= \lim_{i\to\infty}\frac{1}{2\beta}\int_0^t e^{-\lambda\tau}(\dot{e}_i + d_i - d_d)^2 d\tau \\
&= \lim_{i\to\infty}\frac{1}{2\beta}\int_0^{e_i} e^{-\lambda\tau}\dot{e}_i de_i.
\end{aligned}$$

Since $\dot{e}_i$ is bounded, $\lim_{i\to\infty} e_i = 0$ leads to $\lim_{i\to\infty} E_i(t) = 0$ pointwisely. Therefore we first acquire the convergence properties: $e_i(t) \to 0$ pointwisely and $u_i \to u_d$ almost everywhere as $i \to \infty$. On the other hand, as $\dot{x}$ is bounded and $x_d \in \mathcal{C}^1[0,T]$, we can derive the boundedness of $\dot{e}_i(t)$, which assures the uniform continuity of $e_i(t)$ in the interval $[0,T]$. According to Barbalat Lemma (Khalil, 1992), $e_i(t) \to 0$ uniformly as $i \to \infty$ can be derived. ■

4.2.2 FIL for Systems with NGLC Uncertainties

System (4.1) is considered again, however, a different assumption is made for the lumped uncertainty $d(x,t)$.

Assumption 4.3. System uncertainty $d(x,t)$ is only local Lipschitz continuous, nevertheless, it is bounded by a known smooth bounding function $\eta(x,t)$, i.e. $|d(x,t)| \leq \eta(x,t)$.

We will show that even if with NGLC uncertainty, only if the system repeats, the learning convergence can also be guaranteed.

The underlying idea is as follows. Since the system is only Local Lipschitz, robust control is employed to ensure that the system state is bounded by a compact set. Consequently, the dynamic system is Lipschitz continuous on the compact set. Thus, by adding the FIL, the perfect tracking can be obtained iteratively. This leads to a new FIL strategy – robust FIL.

The robust FIL scheme can be described as

$$
\begin{aligned}
u_i &= proj[u_{i-1}] + u_{r,i} && (4.13)\\
u_{r,i} &= (\rho_i \kappa_i + 1)e_i && (4.14)\\
\rho_i &= \sqrt{\dot{x}_d^2 + \varepsilon} + \eta_i \\
\kappa_i &= \frac{\sqrt{e_i^2 + 3\varepsilon^2} + 8\varepsilon}{(\sqrt{e_i^2 + 3\varepsilon^2} + \varepsilon)^2},
\end{aligned}
$$

where ε is a positive constant and $\eta_i = \eta(x_i, t)$. Both ρ_i and κ_i are smooth functions of e_i and t.

Remark 4.1. In the robust controller design, there is a tradeoff between the value of ϵ and the control performance. The smaller the ϵ is, the smaller the tracking error is. However, if the ϵ is too small, the control signal will become chattering which is not practical in real control systems. Hence, the perfect tracking can not be obtained only by Robust Control.

In our scheme, a larger ϵ can be chosen to guarantee the smoothness of the control signal in the first iteration. Then based on the FIL, the tracking error can be reduced iteratively. Eventually, the perfect tracking and a smooth control signal can be ensured.

The main result of the proposed learning algorithm is given in the following theorem.

Theorem 4.2. *For system (4.1), under Assumptions 4.1 and 4.3, the control law (4.13) and (4.14) ensure that* e_i *converges to* 0 *uniformly and the control signal* u_i *converges to* u_d *almost everywhere as* $i \to \infty$.

Proof:

To analyze the convergence property of the proposed robust FIL, the following time-weighted CEF is used.

$$E_i(t) = e^{-\lambda t} e_i^2 + \int_0^t e^{-\lambda \tau} \delta u_i^2 d\tau. \tag{4.15}$$

Analogous to Theorem 4.1, the proof also contains three parts.

(I) Boundedness Property

Define a Lyapunov function $V_i = \frac{1}{2}e_i^2$. Note the following fact provided that $|e_i| \geq \varepsilon$,

$$\begin{aligned} 1 - \kappa_i|e_i| &= \frac{e_i^2 + 3\varepsilon^2 + \varepsilon^2 + 2\varepsilon\sqrt{e_i^2 + 3\varepsilon^2} - \sqrt{e_i^2 + 3\varepsilon^2}|e_i| - 8\varepsilon|e_i|}{(\sqrt{e_i^2 + 3\varepsilon^2} + \varepsilon)^2} \\ &\leq \frac{e_i^2 + 4\varepsilon^2 + 2\varepsilon\sqrt{e_i^2 + 3e_i^2} - \sqrt{e_i^2}|e_i| - 8\varepsilon|e_i|}{(\sqrt{e_i^2 + 3\varepsilon^2} + \varepsilon)^2} \\ &\leq \frac{4\varepsilon^2 + 4\varepsilon|e_i| - 8\varepsilon|e_i|}{(\sqrt{e_i^2 + 3\varepsilon^2} + \varepsilon)^2} \\ &\leq \frac{4\varepsilon(\varepsilon - |e_i|)}{(\sqrt{e_i^2 + 3\varepsilon^2} + \varepsilon)^2} \\ &< 0. \end{aligned} \tag{4.16}$$

Consequently it can be derived that, if $|e_i| \geq \epsilon$,

$$\begin{aligned} \dot{V}_i &= e_i\dot{e}_i \\ &= e_i(\dot{x}_d - d_i - u_i) \\ &= e_i\{\dot{x}_d - d_i - proj[u_{i-1}] - [(\sqrt{\dot{x}_d^2 + \epsilon} + \eta_i)\kappa_i + 1]e_i\} \\ &\leq |e_i||\dot{x}_d| + |e_i|\eta_i + |e_i|u^* - |\dot{x}_d|\kappa_i e_i^2 - \eta_i\kappa_i e_i^2 - e_i^2 \\ &\leq |e_i|u^* - e_i^2 + (1 - \kappa_i|e_i|)(|\dot{x}_d| + \eta_i)|e_i| \\ &\leq |e_i|u^* - e_i^2 \\ &= -|e_i|(|e_i| - u^*). \end{aligned}$$

Therefore, $|e_i|$ is Globally Uniformly Bounded (GUB) by $max\{\varepsilon, u^*\}$. Hence $x_i \in \mathcal{X}$ where $\mathcal{X}$ is a compact set.

Since x_i is bounded and d_i is local *Lipschitz*, there exists a *Lipschitz* constant $l_{d'} \triangleq \sup_{(x_i,t)\in\mathcal{X}\times[0,T]} \left|\frac{\partial d_i}{\partial x_i}\right| < \infty$, such that

$$|d_i - d_d| \leq l_{d'}|e_i|. \tag{4.17}$$

Moreover, according to the control law (4.13) and (4.14) the boundedness of x_i guarantees the finiteness of $u_{r,i}$ and u_i. Therefore, $\dot{x}_i$ and $\dot{e}_i$ are also finite on $\mathcal{X}$.

From the definition of κ_i and ρ_i, it can be derived that there exists a finite constant $c_1 \triangleq \sup_{(x_i,t)\in\mathcal{X}\times[0,T]} \rho_i\kappa_i + 1$ and a finite constant $c_2 \triangleq \sup_{(x_i,t)\in\mathcal{X}\times[0,T]} \frac{d(\rho_i\kappa_i)}{dt}$.

(II) Difference of $E_i(t)$

From (4.15), the difference of $E_i(t)$ is

$$\Delta E_i = e^{-\lambda t}e_i^2 + \int_0^t e^{-\lambda t}(\delta u_i^2 - \delta u_{i-1}^2)d\tau - e^{-\lambda t}e_{i-1}^2. \tag{4.18}$$

Obviously, both (4.8) and (4.9) are still valid. Furthermore, (4.9) can be rewritten as

$$\int_0^t e^{-\lambda\tau}(\delta u_i^2 - \delta u_{i-1}^2)d\tau \le \int_0^t e^{-\lambda\tau}[-2\delta u_i u_{r,i} - u_{r,i}^2]d\tau. \tag{4.19}$$

Substituting (4.3) and (4.14) into (4.19) and dropping the $u_{r,i}^2$ term, we have

$$\begin{aligned}
&\int_0^t e^{-\lambda\tau}(\delta u_i^2 - \delta u_{i-1}^2)d\tau \\
&\le -2\int_0^t e^{-\lambda\tau}(d_i - d_d)(\rho_i\kappa_i + 1)e_i d\tau - 2\int_0^t e^{-\lambda\tau}(\rho_i\kappa_i + 1)e_i\dot{e}_i d\tau \\
&\le 2\int_0^t e^{-\lambda\tau}l_{d'}|x_d - x_i|c_1|e_i|d\tau - \int_0^t e^{-\lambda\tau}\rho_i\kappa_i d(e_i^2) - 2\int_0^t e^{-\lambda\tau}e_i\dot{e}_i d\tau \\
&\le 2l_{d'}c_1\int_0^t e^{-\lambda\tau}e_i^2 d\tau - e^{-\lambda\tau}\rho_i\kappa_i e_i^2 + \int_0^t e^{-\lambda\tau}e_i^2 d(\rho_i\kappa_i) - \lambda\int_0^t e^{-\lambda\tau}e_i^2\rho_i\kappa_i d\tau \\
&\quad -2\int_0^t e^{-\lambda\tau}e_i\dot{e}_i d\tau \\
&\le 2l_{d'}c_1\int_0^t e^{-\lambda\tau}e_i^2 d\tau + \int_0^t e^{-\lambda\tau}e_i^2 d(\rho_i\kappa_i) - 2\int_0^t e^{-\lambda\tau}e_i\dot{e}_i d\tau \\
&\le (2l_{d'}c_1 + c_2)\int_0^t e^{-\lambda\tau}e_i^2 d\tau - 2\int_0^t e^{-\lambda\tau}e_i\dot{e}_i d\tau.
\end{aligned} \tag{4.20}$$

Substituting (4.8) and (4.20) into (4.18) and considering (4.17), yield

$$\begin{aligned}
\Delta E_i(t) &\le -\lambda\int_0^t e^{-\lambda\tau}e_i^2 d\tau + (2l_{d'}c_1 + c_2)\int_0^t e^{-\lambda\tau}e_i^2 d\tau - e^{-\lambda t}e_{i-1}^2 \\
&= -(\lambda - 2l_{d'}c_1 - c_2)\int_0^t e^{-\lambda\tau}e_i^2 d\tau - e^{-\lambda t}e_{i-1}^2.
\end{aligned}$$

There exists a sufficiently large λ such that $\lambda > 2l_{d'}c_1 + c_2$ to ensure that

$$\Delta E_i(t) \le -e^{-\lambda t}e_{i-1}^2(t) \le -e^{-\lambda T}e_{i-1}^2(t).$$

(III) Uniform Convergence

Analogous to the Part *(III)* in Theorem 4.1, it can be proven that $e_i(t)$ converges to 0 uniformly and u_i converges to u_d almost everywhere. ■

4.3 FIL for Norm-bounded Uncertainties under Alignment Condition

I.I.C. is an essential requirement for FIL, however it is difficult to be satisfied in many practical engineering systems. By taking advantage of the concept of CEF, the I.I.C. may be relaxed to alignment condition for systems with parametric uncertainties (Xu, 2002). The alignment condition is $x_{i+1}(0) = x_i(T)$, which can be easily perceived: restart from wherever stopped at. Under the alignment condition we need not do extra work to bring the system back to a specific place after every iteration. In this section, we will explore the possibility of replacing I.I.C. by alignment condition in FIL for systems with norm-bounded uncertainties.

Consider (4.1) again and the following assumption is further made for $x_d(t)$.

Assumption 4.4. For the desired trajectory $x_d(t) \in C^1[0,T]$, $x_d(0) = x_d(T)$ is guaranteed.

The control target is same as in Section 4.2 and we also discuss the problem according to the property of $d(x,t)$.

4.3.1 FIL for GLC Systems under Alignment Condition

Assume the system uncertainty $d(x,t)$ satisfies the following assumption.

Assumption 4.5. The system uncertainty $d(x,t)$ is GLC and the Lipschitz constant l_d or its bound is known *a priori*.

By using the same learning law (4.4), the following theorem can be obtained.

Theorem 4.3. *For system (4.1), under Assumptions 4.4 - 4.5 and alignment condition, if* $\beta \geq 2(l_d+1)$ *the learning law (4.4) ensures that the tracking error* e_i *converges to 0 uniformly and the control signal* u_i *converges to* u_d *almost everywhere.*

Proof:

The following CEF is defined.

$$E_i(t) = \frac{1}{2}e_i^2 + \frac{1}{2\beta}\int_0^t \delta u_i^2(\tau)d\tau. \tag{4.21}$$

Same as Part *(I)* of Theorem 4.1, the boundedness of x_i, $\dot{x}_i$ and u_i can be ensured for any $i \in \mathcal{Z}_+$. Next let us check the difference of $E_i(t)$.

$$\begin{aligned}\Delta E_i(t) &= \frac{1}{2}e_i^2(t) - \frac{1}{2}e_{i-1}^2(t) + \frac{1}{2\beta}\int_0^t (\delta u_i^2 - \delta u_{i-1}^2)d\tau \\ &= \frac{1}{2}e_i^2(0) + \int_0^t e_i\dot{e}_i d\tau - \frac{1}{2}e_{i-1}^2(t) + \frac{1}{2\beta}\int_0^t (\delta u_i^2 - \delta u_{i-1}^2)d\tau. \end{aligned}\tag{4.22}$$

According to (4.10) and letting $\lambda = 0$, the last term on the RHS of (4.22) can be expressed as

$$\begin{aligned}&\frac{1}{2\beta}\int_0^t (\delta u_i^2 - \delta u_{i-1}^2)d\tau \\ &\leq -\int_0^t e_i\dot{e}_i d\tau + (l_d - \frac{\beta}{2})\int_0^t e_i^2 d\tau. \end{aligned}\tag{4.23}$$

Substituting (4.23) into (4.22) yields

$$\Delta E_i(t) \leq \frac{1}{2}e_i(0)^2 - \frac{1}{2}e_{i-1}(t)^2 + (l_d - \frac{\beta}{2})\int_0^t e_i^2 d\tau.$$

From the alignment condition, it can be derived that $e_i(0) = e_{i-1}(T)$. Hence, choosing $t = T$ and considering $\beta \geq 2(l_d+1)$, we can obtain

$$\Delta E_i(T) \leq -\int_0^T e_i^2 d\tau. \tag{4.24}$$

Analogous to the Part *III* of Theorem 4.1, as $E_0(T)$ is bounded and $E_i(t)$ is positive, (4.24) leads to $\lim_{i\to\infty}\int_0^T e_i^2 d\tau = 0$. Furthermore, the uniform continuity of e_i implies the uniform convergence of e_i, i.e. $\lim_{i\to\infty}|e_i|_{sup} = 0$. Hence, according to the definition of $E_i(t)$, u_i converges to u_d almost everywhere can also be derived. ■

4.3.2 FIL for NGLC Systems under Alignment Condition

Including Assumption 4.3, the following assumption is further made for the system uncertainties $d(x,t)$.

Assumption 4.6. $|d(x_1,t) - d(x_2,t)| \le \eta'(x,t)|x_1 - x_2|$ where $\eta'(x,t)$ is a known bounding function.

Remark 4.2. Assumption 4.6 implies that the variation of the norm-bounded uncertainty is within an acceptable range.

The new FIL scheme for systems with NGLC uncertainties under alignment condition is constructed as

$$u_i(t) = w_i(t) + v_i(t) \tag{4.25}$$

$$w_i(t) = proj[w_{i-1}(t)] + \beta e_i(t) \tag{4.26}$$

$$v_i(t) = (\rho_i \kappa_i + 1)e_i(t) + \eta' e_i(t), \tag{4.27}$$

where $\eta' = \eta'(x,t)$ and ρ_i and κ_i are same defined as in (4.14).

The main result for control laws (4.25) - (4.27) is summarized in the following theorem.

Theorem 4.4. *For system (4.1), under Assumptions 4.3, 4.4 and 4.6 and the alignment condition, the control laws (4.25) - (4.27) guarantee that e_i converges to 0 uniformly and u_i converges to u_d almost everywhere.*

Proof:

The following CEF is used in the proof.

$$E_i(t) = \frac{1}{2}e_i^2(t) + \frac{1}{2\beta}\int_0^t (u_d - w_i)^2 d\tau$$

(I) Boundedness Property

According to the Part I of Theorem 4.2, if $|e_i| \geq \epsilon$, $1 - \kappa_i|e_i| < 0$. Therefore, by defining the same V_i, we have

$$\begin{aligned}
\dot{V}_i &= e_i(\dot{x}_d - d_i - proj[w_{i-1}] - \beta e_i - v_i) \\
&\leq |e_i|w^* - (1+\beta)e_i^2 + (1 - \kappa_i|e_i|)(|\dot{x}_d + \eta_i)|e_i| \\
&\leq |e_i|w^* - (1+\beta)e_i^2 \\
&= -|e_i|[(1+\beta)|e_i| - w^*], \qquad (4.28)
\end{aligned}$$

where w^* is the projection bound of w_i. Hence, $|e_i|$ is GUB by $\max\{\epsilon, w^*/(1+\beta)\}$ and x belongs to a compact set $\mathcal{X}$. Moreover, the boundedness of x_i leads to the finiteness of w_i, v_i, u_i, $\dot{x}_i$ and $\dot{e}_i$.

(II) Difference of CEF

Analogous to (4.22), the difference of CEF is

$$\begin{aligned}
\Delta E_i(t) &= \frac{1}{2}e_i^2(0) + \int_0^t e_i\dot{e}_i d\tau - \frac{1}{2}e_{i-1}^2(t) \\
&\quad + \frac{1}{2\beta}\int_0^t [(u_d - w_i)^2 - (u_d - w_{i-1})^2] d\tau. \qquad (4.29)
\end{aligned}$$

The second term on the RHS of (4.29) can be rewritten as

$$\begin{aligned}
&\int_0^t e_i\dot{e}_i d\tau \\
&= \int_0^t e_i(d_d - d_i + u_d - w_i - v_i) d\tau \\
&\leq \int_0^t \eta_i'|e_i|^2 d\tau + \int_0^t e_i(u_d - w_i) - \int_0^t (\rho_i\kappa_i + 1)e_i^2 d\tau - \int_0^t \eta_i' e_i^2 d\tau \\
&\leq \int_0^t e_i(u_d - w_i) d\tau - \int_0^t e_i^2 d\tau. \qquad (4.30)
\end{aligned}$$

From (4.9), by letting $\lambda = 0$ and $u_i = w_i$, the last term on the RHS of (4.29) can be described as

$$\begin{aligned} &\frac{1}{2\beta}\int_0^t [(u_d - w_i)^2 - (u_d - w_{i-1})^2] d\tau \\ &\leq -\frac{1}{2\beta}\int_0^t (u_d - w_i)(w_i - proj[w_{i-1}]) d\tau \\ &= \int_0^t -(u_d - w_i) e_i d\tau. \end{aligned} \tag{4.31}$$

Substituting (4.30) and (4.31) into (4.32) and considering the alignment condition, it can be derived that

$$\Delta E_i(T) \leq -\int_0^T e_i^2 d\tau. \tag{4.32}$$

(III) Uniform Convergence

Analogous to Theorem 4.3, based on the results of Part *I* and Part *II*, it can be derived that as i approaches to infinity, the tracking error e_i converges to 0 uniformly and the control signal u_i converges to u_d almost everywhere. ■

4.4 Robust FIL for MIMO Systems with NGLC Uncertainties

In this section, the robust FIL will be extended to the following MIMO nonlinear system.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + B_0(t) H(\mathbf{x}, t)[\mathbf{u}(t) + \mathbf{d}(\mathbf{x}, t)] \tag{4.33}$$

where $\mathbf{x} \in \mathcal{R}^n$ is the measurable state vector; $\mathbf{u} \in \mathcal{R}^m$ is the control input vector; $\mathbf{f}(\mathbf{x}, t) : \mathcal{R}^n \times \mathcal{R}_+ \to \mathcal{R}^n$ is known; $B_0(t) \in \mathcal{R}^{n\times m}$ and $H(\mathbf{x}, t) : \mathcal{R}^n \times \mathcal{R}_+ \to \mathcal{R}^{m\times m}$ are known functions with full rank; $\mathbf{d}(\mathbf{x}, t) : \mathcal{R}^n \times \mathcal{R}_+ \to \mathcal{R}^m$ is system uncertainties.

To facilitate the analysis of the control performance, an extended tracking error $\boldsymbol{\sigma}(\mathbf{x}_i, t) : \mathcal{R}^n \times \mathcal{R}_+ \to \mathcal{R}^m$, which is linear to $\mathbf{x}_i$, is defined at the ith iteration.

The control objective is: for a given desired trajectory $\mathbf{x}_d \in \mathcal{C}^1[0,T]$, $\boldsymbol{\sigma}_i \triangleq \boldsymbol{\sigma}(\mathbf{x}_i, t) \to \mathbf{0}$ and $\mathbf{x}_i \to \mathbf{x}_d$ uniformly can be obtained as the iteration number i approaches to infinity.

Differentiating $\boldsymbol{\sigma}_i$ with respect to time t and considering system dynamics (4.33),

$$\dot{\boldsymbol{\sigma}}_i = G_0\dot{\mathbf{x}}_i + \mathbf{h}_i = G_0\mathbf{f}_i + \mathbf{h}_i + \boldsymbol{\alpha}_i(\mathbf{u}_i + \mathbf{d}_i) \tag{4.34}$$

where $G_0 \triangleq G_0(t) = \frac{\partial \boldsymbol{\sigma}_i}{\partial \mathbf{x}_i}$, $\mathbf{h}_i \triangleq \mathbf{h}(\mathbf{x}_i, t) = \frac{\partial \boldsymbol{\sigma}_i}{\partial t}$, $\mathbf{f}_i \triangleq \mathbf{f}(\mathbf{x}_i, t)$, $\boldsymbol{\alpha}_i \triangleq G_0B_0H(\mathbf{x}_i, t) = G_0B_0H_i$ and $\mathbf{d}_i \triangleq \mathbf{d}(\mathbf{x}_i, t)$.

The dynamic system (4.33) and the extended tracking error $\boldsymbol{\sigma}_i$ satisfy the following assumptions.

Assumption 4.7. The known functions $\mathbf{f}_i$, $\mathbf{h}_i$ and H_i are all GLC, i.e. $\forall \mathbf{p} \in \{\mathbf{f}_i, \mathbf{h}_i, H_i\}$, $\|\mathbf{p}_d - \mathbf{p}_i\| \le l_{\mathbf{p}}\|\mathbf{x}_d - \mathbf{x}_i\|$. The uncertainty $\mathbf{d}_i$ is only locally Lipschitz continuous but bounded by a known function, i.e. $\|\mathbf{d}_i\| \le \eta_i$.

Assumption 4.8. Both H_i and G_0B_0 are invertible. Moreover, if $\mathbf{x}_i$ belongs to a compact set $\mathcal{X}$, the boundedness of H_i^{-1}, $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_i^{-1}$ can be guaranteed. $Q \triangleq B_0(G_0B_0)^{-1}$ and $\frac{dQ}{dt}$ are assumed to be finite over $[0,T]$.

Assumption 4.9. The dynamic system (4.33) will repeat itself under the I.I.C., i.e. $\mathbf{x}_i(0) = \mathbf{x}_d(0)$ and $\boldsymbol{\sigma}_i(0) = \mathbf{0}$ $\forall i \in \mathcal{Z}_+$.

From (4.34), it can be derived that

$$\mathbf{u}_i = -\boldsymbol{\alpha}_i^{-1}G_0\mathbf{f}_i - \boldsymbol{\alpha}_i^{-1}\mathbf{h}_i - \mathbf{d}_i + \boldsymbol{\alpha}_i^{-1}\dot{\boldsymbol{\sigma}}_i. \tag{4.35}$$

Let $\dot{\boldsymbol{\sigma}}_i = \mathbf{0}$ and the desired control signal $\mathbf{u}_d$ can be expressed as

$$\mathbf{u}_d = -\boldsymbol{\alpha}_d^{-1}G_0\mathbf{f}_d - \boldsymbol{\alpha}_d^{-1}\mathbf{h}_d - \mathbf{d}_d. \tag{4.36}$$

Substituting (4.35) and (4.36) into (4.33), we have

$$\dot{\mathbf{x}}_i = \mathbf{f}_i - QG_0\mathbf{f}_i - Q\mathbf{h}_i + Q\dot{\boldsymbol{\sigma}}_i \tag{4.37}$$

$$\dot{\mathbf{x}}_d = \mathbf{f}_d - QG_0\mathbf{f}_d - Q\mathbf{h}_d. \tag{4.38}$$

The robust FIL scheme is constructed as

$$\mathbf{u}_i = proj[\mathbf{u}_{i-1}] + \mathbf{u}_{r,i} \tag{4.39}$$

$$\begin{aligned} \mathbf{u}_{r,i} &= -(\rho_i \kappa_i + 1)\boldsymbol{\alpha}_i^T \boldsymbol{\sigma}_i \\ \rho_i &= \sqrt{(\boldsymbol{\alpha}_i^{-1} G_0 \mathbf{f}_i + \boldsymbol{\alpha}_i^{-1}\mathbf{h}_i)^T(\boldsymbol{\alpha}_i^{-1} G_0 \mathbf{f}_i + \boldsymbol{\alpha}_i^{-1}\mathbf{h}_i) + \varepsilon} + \eta_i \\ &\triangleq \sqrt{\mathbf{v}^T\mathbf{v} + \varepsilon} + \eta_i \\ \kappa_i &= \frac{\sqrt{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 3\varepsilon^2} + 8\varepsilon}{(\sqrt{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 3\varepsilon^2} + \varepsilon)^2} \end{aligned} \tag{4.40}$$

where $\varepsilon > 0$. Both ρ_i and κ_i are the smooth functions of t and $\boldsymbol{\sigma}_i$.

For a given matrix $A \in R^{n\times m}$, the operator $proj[\cdot]$ is defined as

$$\begin{aligned} proj[A] &= \{proj[a_{ij}]\}_{n\times m} \\ proj[a_{ij}] &= \begin{cases} a_{ij} & |a_{ij}| \le a_{ij}^* \\ a_{ij}^* \cdot sign(a_{ij}) & |a_{ij}| > a_{ij}^* \end{cases}, \end{aligned}$$

with a_{ij}^* the known bound.

The following CEF is defined to analyze the convergence property of the proposed learning scheme.

$$E_i(t) = e^{-\lambda t}\|\boldsymbol{\sigma}_i\|^2 + \int_0^t e^{-\lambda\tau}\|\delta\mathbf{u}_i\|^2 d\tau \tag{4.41}$$

where $\delta\mathbf{u}_i = \mathbf{u}_d - \mathbf{u}_i$.

First, two lemmas will be given, which reveal the boundedness relationships among quantities $\boldsymbol{\sigma}_i$, $\mathbf{x}_i$ and $\mathbf{u}_i$.

Lemma 4.1. *For the system (4.33), under Assumptions 4.7-4.9, the control laws (4.39) and (4.40) guarantee that $\boldsymbol{\sigma}_i$ is bounded for any $i \in \mathcal{Z}_+$.*

Proof:

Define a Lyapunov function $V_i = \frac{1}{2}\|\boldsymbol{\sigma}_i\|^2$. If $\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\| \geq \varepsilon$,

$$\begin{aligned}
&1 - \kappa_i\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\| \\
&= \frac{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 4\varepsilon^2 + 2\varepsilon\sqrt{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 3\varepsilon^2} - \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|\sqrt{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 3\varepsilon^2} - 8\varepsilon\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|}{(\sqrt{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 3\varepsilon^2} + \varepsilon)^2} \\
&\leq \frac{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 4\varepsilon^2 + 2\varepsilon\sqrt{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 3\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2} - \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|\sqrt{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2} - 8\varepsilon\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|}{(\sqrt{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 3\varepsilon^2} + \varepsilon)^2} \\
&= \frac{4\varepsilon(\varepsilon - \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|)}{(\sqrt{\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + 3\varepsilon^2} + \varepsilon)^2} \\
&\leq 0.
\end{aligned} \tag{4.42}$$

Consequently, when $\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\| \geq \varepsilon$,

$$\begin{aligned}
\dot{V}_i &= \boldsymbol{\sigma}_i^T\dot{\boldsymbol{\sigma}}_i \\
&= \boldsymbol{\sigma}_i^T[G_0\mathbf{f}_i + \mathbf{h}_i + \boldsymbol{\alpha}_i(\mathbf{u}_i + \mathbf{d}_i)] \\
&\leq \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|\|\mathbf{v}_i\| + \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|\eta_i - \kappa_i\rho_i\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 - \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|u^* \\
&\leq \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|\|\mathbf{v}_i\| + \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|\eta_i - \kappa_i(\|\mathbf{v}_i\| + \eta_i)\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma})i\|^2 - \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|u^* \\
&= (1 - \kappa_i\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|)(\|\mathbf{v}_i\| + \eta_i)\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\| - \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|^2 + \|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|u^* \\
&\leq -\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\|(\|\boldsymbol{\alpha}_i^T\boldsymbol{\sigma}_i\| - u^*).
\end{aligned}$$

$\|\boldsymbol{\sigma}_i\|$ is globally uniformly bounded by $max\{\varepsilon, u^*\}/\sqrt{\lambda_{min}(\boldsymbol{\alpha}_i\boldsymbol{\alpha}_i^T)}$. ■

Lemma 4.2. *For the system (4.33), under Assumptions 4.7-4.9, the control laws (4.39) and (4.40) ensure that* $\mathbf{x}_i$, $\mathbf{u}_{r,i}$, $\dot{\boldsymbol{\sigma}}_i$ *and* $\dot{\mathbf{x}}_i$ *are all bounded for any* $i \in \mathcal{Z}_+$. *Moreover, we have*

$$\|\mathbf{x}_d - \mathbf{x}_i\| \leq b_Q\|\boldsymbol{\sigma}_i\| + b_2\int_0^t \|\boldsymbol{\sigma}_i\|d\tau \tag{4.43}$$

where b_Q *and* b_2 *are finite positive constants defined in Appendix B.*

The proof of Lemma 4.2 can be found in Appendix B.

According to (4.35) and (4.36), we can obtain

$$\delta\mathbf{u}_i = -\boldsymbol{\alpha}_i^{-1}\dot{\boldsymbol{\sigma}}_i - \boldsymbol{\gamma}_i \tag{4.44}$$

where $\boldsymbol{\gamma}_i = \boldsymbol{\alpha}_d^{-1}G_0\mathbf{f}_d + \boldsymbol{\alpha}_d^{-1}\mathbf{h}_d + \mathbf{d}_d - \boldsymbol{\alpha}_i^{-1}G_0\mathbf{f}_i - \boldsymbol{\alpha}_i^{-1}\mathbf{h}_i - \mathbf{d}_i$. Under *Assumptions* 4.7-4.9, we have

$$\|\boldsymbol{\gamma}_i\| \le b_3\|\mathbf{x}_d - \mathbf{x}_i\|, \tag{4.45}$$

where b_3 is a finite constant. The finiteness of b_s can be derived as follows.

$$\begin{aligned}
\boldsymbol{\gamma}_i &= \boldsymbol{\alpha}_d^{-1}G_0\mathbf{f}_d + \boldsymbol{\alpha}_d^{-1}\mathbf{h}_d + \mathbf{d}_d - \boldsymbol{\alpha}_i^{-1}G_0\mathbf{f}_i - \boldsymbol{\alpha}_i^{-1}\mathbf{h}_i - \mathbf{d}_i \\
&= (\boldsymbol{\alpha}_d^{-1} - \boldsymbol{\alpha}_i^{-1})G_0\mathbf{f}_d + \boldsymbol{\alpha}_i^{-1}G_0(\mathbf{f}_d - \mathbf{f}_i) + (\boldsymbol{\alpha}_d^{-1} - \boldsymbol{\alpha}_i^{-1})\mathbf{h}_d + \boldsymbol{\alpha}_i^{-1}(\mathbf{h}_d - \mathbf{h}_i) \\
&\quad + \mathbf{d}_d - \mathbf{d}_i \\
&= H_d^{-1}(H_i - H_d)H_i^{-1}G_0\mathbf{f}_d + \boldsymbol{\alpha}_i^{-1}G_0(\mathbf{f}_d - \mathbf{f}_i) + H_d^{-1}(H_i - H_d)H_i^{-1}\mathbf{h}_d \\
&\quad + \boldsymbol{\alpha}_i^{-1}(\mathbf{h}_d - \mathbf{h}_i) + \mathbf{d}_d - \mathbf{d}_i.
\end{aligned}$$

$\forall p \in \{H^{-1}, G_0, \boldsymbol{\alpha}^{-1}, \mathbf{f}_d, \mathbf{h}_d\}$, define $b_p \triangleq \sup_{\mathcal{X}\times[0,T]} \|p\|$. Then,

$$\|\boldsymbol{\gamma}_i\| \le b_3\|\mathbf{x}_d - \mathbf{x}_i\|,$$

where $b_3 = b_{H^{-1}}^2 b_{G_0} b_{\mathbf{f}_d} l_H + b_{\boldsymbol{\alpha}^{-1}} b_{G_0} b_{H^{-1}} l_{\mathbf{f}} + b_{H^{-1}}^2 b_{\mathbf{h}_d} l_H + b_{\boldsymbol{\alpha}^{-1}} l_{\mathbf{h}} + l_{\mathbf{d}}$. According to Assumptions 4.7 -4.8, the finiteness of b_3 can be guaranteed.

Now the main result for the robust FIL is given in the following theorem.

Theorem 4.5. *Consider the nonlinear system (4.33) satisfying Assumptions 4.7-4.9. Under the control laws (4.39) and (4.40),* $\boldsymbol{\sigma}_i(t)$ *and* $\mathbf{x}_i(t)$ *uniformly converge to* $\mathbf{0}$ *and* $\mathbf{x}_d(t)$ *respectively. Furthermore, the control signal* $\mathbf{u}_i(t)$ *converges to* $u_d(t)$ *almost everywhere.*

Proof:

(I) Boundedness Property

Lemma 4.1 and Lemma 4.2 clearly show that the finiteness of $\mathbf{x}_i$, $\boldsymbol{\sigma}_i$, $\mathbf{u}_i$, $\dot{x}_i$ and $\dot{\boldsymbol{\sigma}}_i$, for any $i \in \mathcal{Z}_+$.

(II) Difference of $E_i(t)$

$$\Delta E_i(t) = e^{-\lambda t}\|\boldsymbol{\sigma}_i\|^2 + \int_0^t e^{-\lambda t}(\|\delta \mathbf{u}_i\|^2 - \|\delta \mathbf{u}_{i-1}\|^2)d\tau - e^{-\lambda t}\|\boldsymbol{\sigma}_{i-1}\|^2. \quad (4.46)$$

The first term on the RHS of (4.46) can be rewritten as

$$e^{-\lambda t}\|\boldsymbol{\sigma}_i\|^2 = -\lambda \int_0^t e^{-\lambda\tau}\|\boldsymbol{\sigma}_i\|^2 d\tau + \int_0^t 2e^{-\lambda\tau}\boldsymbol{\sigma}_i\dot{\boldsymbol{\sigma}}_i d\tau. \quad (4.47)$$

The second term on the RHS of (4.46) can be expressed as

$$\begin{aligned}
&\int_0^t e^{-\lambda\tau}(\|\delta \mathbf{u}_i\|^2 - \|\delta \mathbf{u}_{i-1}\|^2)d\tau \\
&\leq \int_0^t e^{-\lambda\tau}(\|\delta \mathbf{u}_i\|^2 - \|\mathbf{u}_d - proj[\mathbf{u}_{i-1}]\|^2)d\tau \\
&\leq \int_0^t e^{-\lambda\tau}(-2\mathbf{u}_{r,i}^T\delta \mathbf{u}_i - \|\mathbf{u}_{r,i}\|^2)d\tau. \quad (4.48)
\end{aligned}$$

According to Lemma 4.2 and the definition of $\rho_i\kappa_i$, there exist a finite constant $b_4 \triangleq \sup_{t\in[0,T]} \rho_i\kappa_i + 1$ and a finite constant $b_5 \triangleq \sup_{t\in[0,T]} \frac{d(\rho_i\kappa_i)}{dt}$.

Substituting (4.40) and (4.44) into (4.48), we have

$$\begin{aligned}
&\int_0^t e^{-\lambda\tau}(\|\delta \mathbf{u}_i\|^2 - \|\delta \mathbf{u}_{i-1}\|^2)d\tau \\
&\leq -2\int_0^t e^{-\lambda\tau}(\rho_i\kappa_i + 1)\boldsymbol{\sigma}_i^T\dot{\boldsymbol{\sigma}}_i d\tau - 2\int_0^t e^{-\lambda\tau}(\rho_i\kappa_i + 1)\boldsymbol{\sigma}_i^T\boldsymbol{\alpha}_i\boldsymbol{\gamma}_i d\tau \\
&\leq -2\int_0^t e^{-\lambda\tau}\boldsymbol{\sigma}_i^T\dot{\boldsymbol{\sigma}}_i d\tau - \rho_i\kappa_i\|\boldsymbol{\sigma}_i\|^2 e^{-\lambda t} + \int_0^t e^{-\lambda\tau}\|\boldsymbol{\sigma}_i\|^2 d(\rho_i\kappa_i) \\
&\quad +2b_4 b_\alpha b_3 \int_0^t e^{-\lambda\tau}\|\boldsymbol{\sigma}_i\|\|\mathbf{x}_d - \mathbf{x}_i\| d\tau \\
&\leq -2\int_0^t e^{-\lambda\tau}\boldsymbol{\sigma}_i^T\dot{\boldsymbol{\sigma}}_i d\tau + b_5\int_0^t e^{-\lambda\tau}\|\boldsymbol{\sigma}_i\|^2 d\tau \\
&\quad +2b_4 b_\alpha b_3(b_Q + b_2 T)\int_0^t e^{-\lambda\tau}\|\boldsymbol{\sigma}_i\|^2 d\tau. \quad (4.49)
\end{aligned}$$

Substituting (4.47) and (4.49) into (4.46) and considering (B.3), it can be derived that

$$\Delta E_i \leq -(\lambda - b_6)\int_0^t e^{-\lambda\tau}\|\boldsymbol{\sigma}_i\|^2 d\tau - e^{-\lambda t}\|\boldsymbol{\sigma}_{i-1}\|^2 \quad (4.50)$$

where $b_6 = b_5 + 2b_4 b_\alpha b_3(b_Q + b_2 T)$. There exists a sufficiently large λ such that $\lambda > b_6$ to ensure that

$$\Delta E_i(t) \leq -e^{-\lambda t}\|\boldsymbol{\sigma}_{i-1}\|^2 \leq -e^{-\lambda T}\|\boldsymbol{\sigma}_{i-1}\|^2. \tag{4.51}$$

(III) Uniform Convergence

By using (4.51) repeatedly, the following can be obtained.

$$E_i(t) \leq E_0(t) - e^{-\lambda T}\sum_{j=0}^{i-1}\|\boldsymbol{\sigma}_i\|^2. \tag{4.52}$$

Since both $\mathbf{x}_0(t)$ and $\mathbf{u}_0(t)$ are bounded, $E_0(t)$ is bounded. From the positiveness of $E_i(t)$ and (4.52), we can derive that $\lim_{i\to\infty}\|\boldsymbol{\sigma}_i\| = 0$ pointwisely. Moreover, the boundedness of $\dot{\boldsymbol{\sigma}}_i$ implies the uniform continuity of $\boldsymbol{\sigma}_i$ which leads to the uniform convergence of $\boldsymbol{\sigma}_i$. According to Lemma 4.2, $\mathbf{x}_i(t)$ uniformly converges to $\mathbf{x}_d(t)$ can also be derived.

Next from (4.45)

$$\lim_{i\to\infty}\|\boldsymbol{\gamma}_i\| \leq \lim_{i\to\infty} b_3\|\mathbf{x}_d - \mathbf{x}_i\| = 0.$$

Thus using (4.44) and the boundedness of $\dot{\boldsymbol{\sigma}}_i$ we further derive

$$\begin{aligned}
\lim_{i\to\infty} E_i(t) &= \lim_{i\to\infty} e^{-\lambda t}\|\boldsymbol{\sigma}_i\|^2 + \lim_{i\to\infty}\int_0^t e^{-\lambda\tau}\|\delta\mathbf{u}_i\|^2 d\tau \\
&= \lim_{i\to\infty}\int_0^t e^{-\lambda\tau}\|\boldsymbol{\alpha}_i\|^2\dot{\boldsymbol{\sigma}}_i^T\dot{\boldsymbol{\sigma}}_i d\tau \\
&\leq \lim_{i\to\infty} b_{\boldsymbol{\alpha}}\int_0^t e^{-\lambda\tau}\dot{\boldsymbol{\sigma}}_i d\boldsymbol{\sigma}_i \\
&= 0.
\end{aligned}$$

Hence, $\mathbf{u}_i$ converges to $\mathbf{u}_d$ almost everywhere as $i \to \infty$. ■

4.5 Illustrative Examples

Case 1. FIL for SISO Dynamic Systems

Consider system (4.1) with the target trajectory $x_d = 1.5sin^3t$, $t \in [0, 2\pi]$.

(1) $d(x,t) = 3xsint$ and $x_i(0) = x_d(0)$

Obviously, $d(x,t)$ is GLC. Choose $\beta = 10$ and $u^* = 10$. Applying the control law (4.4), the simulation result is shown in Fig. 4.1. The horizontal axis denotes the iteration number i, and the vertical axis denotes the sup-norm $|e_i|_{sup} \triangleq \sup_{t\in[0,2\pi]} |e_i(t)|$.

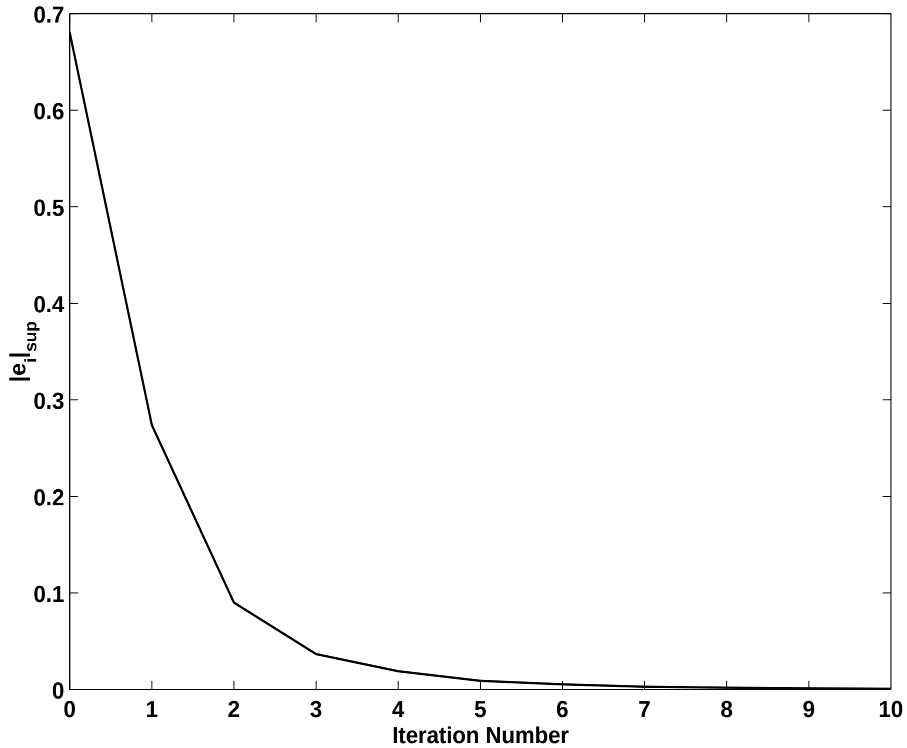

Figure 4.1: Learning convergence for SISO system with GLC uncertainty $t \in [0, T]$.

(2) $d(x,t) = 3x^2sint + 5x^2$ and $x_i(0) = x_d(0)$

$d(x,t)$ is NGLC. Assume the known bounding function $\eta(x,t) = 10x^2$. Choose $\epsilon = 0.25$ and $u^* = 20$. Applying the robust learning laws (4.13) and (4.14), Fig. 4.2 demonstrates the learning convergence.

Case 2. FIL Under Alignment Condition

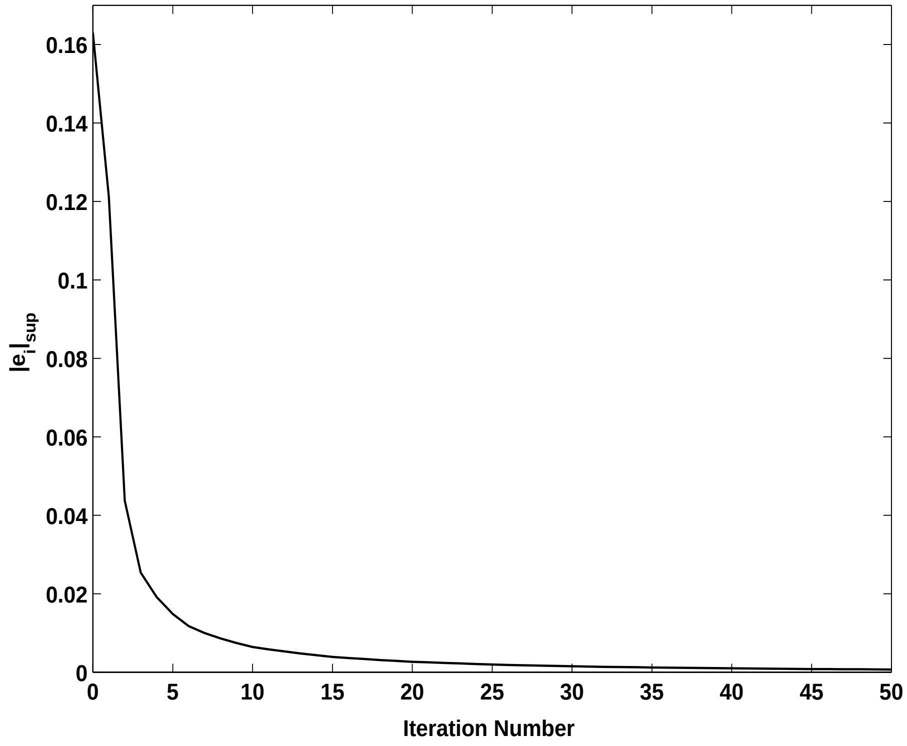

Figure 4.2: Learning convergence for SISO system with NGLC uncertainty $t \in [0, T]$.

The same desired trajectory $x_d = 1.5sin^3t$, $t \in [0, 2\pi]$ is used, which obviously satisfies Assumption 4.4.

(1) $d(x, t) = 3xsint$ and $x_0(0) = 0.1 \neq x_d(0)$

Assume the known bound of l_d is 4. Choose $\beta = 10$ and $u^* = 10$. Under the alignment condition, $x_{i+1}(0) = x_i(T)$, the simulation result is shown in Fig. 4.3. Comparing with Fig. 4.1, it can be clearly seen that, under the alignment condition, the learning becomes more difficult, however, the convergence still can be guaranteed.

(2) $d(x, t) = 3x^2sint + 5x^2$ and $x_0(0) = -0.2 \neq x_d(0)$

Assume the know bounding functions are $\eta(x, t) = 10x^2$ and $\eta'(x, t) = 20x$. Choose $\epsilon = 0.25$, $\beta = 10$ and $u^* = 20$. The learning convergence of the tracking error under the proposed FIL control laws (4.25) - (4.27) is given in Fig. 4.4.

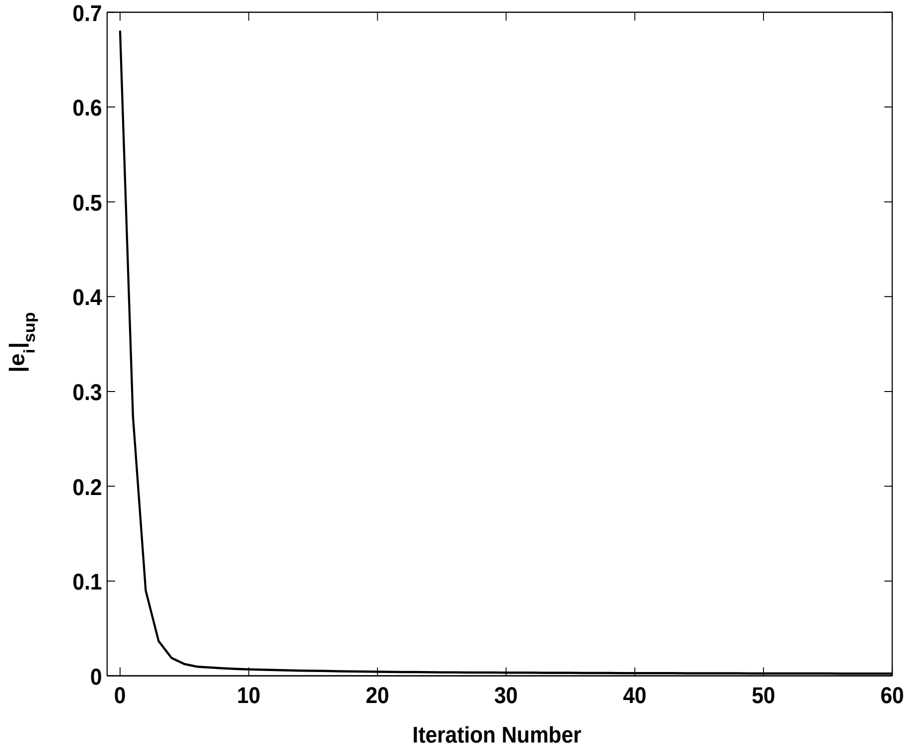

Figure 4.3: Learning convergence for GLC system under alignment condition $t \in [0, T]$.

Case 3. Robust FIL for MIMO Dynamic Systems

Consider the following deterministic system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1 sinx_2 + (t^2+1)(1+sin^3x_1)(u+5x_1^2 sint + 3x_2^2), \end{aligned} \tag{4.53}$$

which is repeatable over $[0 \;\; 2\pi]$. In this case, $\mathbf{f} = [x_2 \;\; 2x_1 sinx_2]^T$, $B_0 = [0 \;\; t^2+1]^T$ and $H = 1 + sin^3 x_1$ are known functions. $d = 5x_1^2 sint + 3x_2^2$ is NGLC with the known bounding function $\eta = (3x_1 + 2x_2)^2$.

The desired trajectory to be followed is

$$x_{1,d} = sin^3 t \quad x_{2,d} = \dot{x}_{1,d}, \quad t \in [0, 2\pi]. \tag{4.54}$$

The extended tracking error is chosen as $\sigma_i = (x_{1,d} - x_{1,i}) + 3(x_{2,d} - x_{2,i})$. Let $\varepsilon = 0.3$ and $u^* = 10$. Apply the robust FIL laws (4.39) and (4.40). The simulation result is

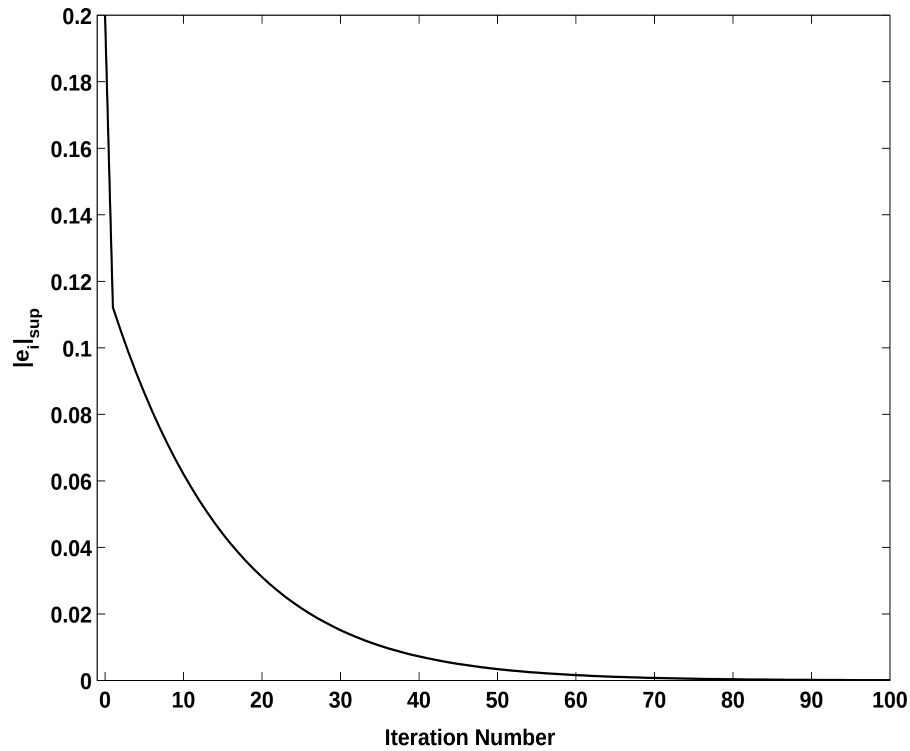

Figure 4.4: Learning convergence for NGLC system under alignment condition $t \in [0, T]$.

shown in Fig. 4.5.

4.6 Conclusion

In this chapter CEF-type FIL is extended to handle systems with norm-bounded uncertainties which may be GLC or NGLC. The possibility of replacing I.I.C. by alignment condition has been discussed. Rigorous proofs based on CEF for all FIL methodologies are given. Illustrative examples clearly show the effectiveness of the proposed FIL schemes.

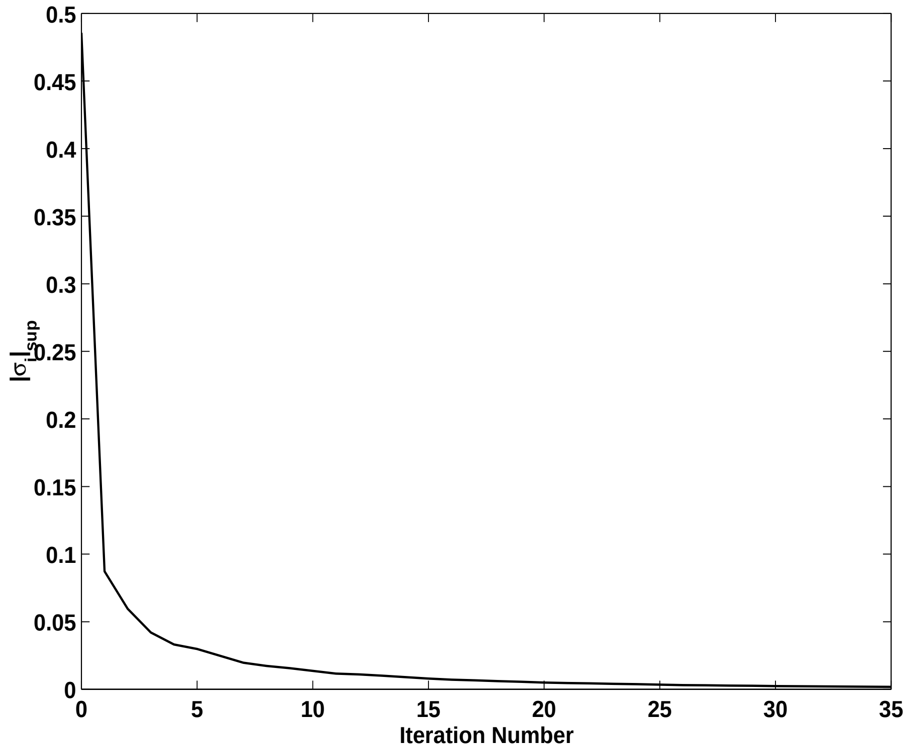

Figure 4.5: Learning convergence for MIMO system with NGLC uncertainty $t \in [0, T]$.

Chapter 5

FIL for Non-Uniform Tracking Tasks in the Presence of Parametric Uncertainties

5.1 Introduction

In most of the works on FIL, it is required that the target trajectory must be invariant in all iterations. If there is a change in the target trajectory due to the variation of control objectives or task specifications, no matter how small it might be, the control system will have to start the learning process from the very beginning and the previously learned control input profiles can no longer be used.

Can a control system learn consecutively from different tracking control tasks? To answer this question, we need to make the learnability of FIL clear. A typical FIL, in the time domain, is under a simple closed-loop or even open-loop control. The novel learning functionality comes from the extra updating activity in the iteration domain. Indeed, the iterative learning mechanism, whether derived from CM approach or EF/CEF approach, is in essence a pointwise integration along the learning axis. This pointwise integration imposes certain conditions on what we can learn – learn an invariant set in the iteration domain. If we define a time axis and an iteration axis, whatever to be learned must be a constant along the iteration axis, as far as the pointwise integration is employed. A simple example of a pointwise integrator that characterizes the FIL mechanism is

$$w_i(t) = w_{i-1}(t) + f_{i-1}(t), \qquad t \in [0,\ T], \qquad i \in \mathcal{Z}_+$$

where $w_i(t)$ is to learn some unknown function $\eta(t)$, and $f_{i-1}(t)$ is a correcting term. Clearly, for each t, $w_i(t)$ is a discrete integrator in the iteration domain with the objective to approach $\eta(t)$ which is an invariant set in the iteration domain.

Next question is, what is this invariant set $\eta(t)$ that is learnable? Although there is no definite conclusion made hitherto in this aspect, we can summarize from the numerous publications in traditional CM-based FIL, that the target trajectory must be invariant, i.e. repeatable, in the iteration domain. This limitation arises because of the existence of the non-parametric uncertainties in the system nonlinear dynamics. Suppose we are going to compensate or cancel a lumped nonlinear unknown function, $\eta(x)$, of the system state, $x(t)$. In the ideal case, we wish to capture the nonlinear uncertain function with the argument being the desired system state, $x_d(t)$.If however the target trajectory varies in the iteration domain, i.e. $x_{d,i}(t)$ is i-dependent, the unknown function will vary accordingly as $\eta(x_{d,i})$. As a consequence, CM-type FIL is not able to work because the function to be learned, $\eta(x_{d,i})$, is no longer an invariant set in the iteration domain.

When the system uncertainties can be represented as parametric types, $\theta(t)$, which are invariant in the iteration domain, it is possible for us to conduct learning even if the target trajectory varies from iteration to iteration. The reason is simple: now we need only to learn unknown parameters, which can be time-varying, but iteration-invariant.

In this chapter we present a novel FIL method that can fulfill the challenging objective. The new learning control law consists of a feedback term and a learning term. The learning term is updated, by a learning mechanism, pointwisely in the time axis and iteratively in the iteration axis. To facilitate the learning control design and convergence analysis, a CEF is employed, which consists of a Lyapunov function to evaluate the tracking performance in the time axis, and a functional to evaluate the learning performance in the iteration axis.

In practice, often we know that some of the system parameters, though unknown, are unlikely time-varying, such as the inertia of a robotic link and the stiffness of a flexible link. In such circumstance, it would be far-fetched to treat them as time-varying ones. If a parameter is invariant in both the time and the iteration axis, the pointwise integration mechanism can be simplified into a conventional integrator working consecutively in the iteration axis. However, for time-varying uncertainties, an integrator along the time axis such as adaptive control fails to work.

This chapter is organized as follow. The dynamic system and the tracking control task are formulated in Section 5.2. Section 5.3 presents a new FIL method for systems with time-varying parametric uncertainties. Based on it, the FIL scheme is extended to systems with both time-varying and time-invariant uncertainties. Section 5.5 applies the proposed learning control approaches to a one-link robotic arm and gives the simulation results.

5.2 Problem Formulation

To clearly explain the main idea, here we only consider the following simple nonlinear dynamic system

$$\dot{x} = \boldsymbol{\theta}^o(t)\boldsymbol{\xi}^o(x,t) + b(t)u$$
$$x(0) = x_0 \quad t \in [0,T], \tag{5.1}$$

where $x \in R$ is the measurable system state, $u \in R$ is the system control input, $b(t) \in \mathcal{C}^1([0,T])$ is the perturbed gain of the system input, $\boldsymbol{\theta}^o(t) \in \mathcal{C}(R^{1\times n_1},[0,T])$ is a vector of unknown time-varying parameters, and $\boldsymbol{\xi}^o(x,t) \in R^{n_1}$ is a known vector-valued function. The elements of $\boldsymbol{\xi}^o(x,t)$ are assumed to be local Lipschitz continuous with respect to x. Here n_1 is an appropriate integer specifying the dimension.

The following assumption is made for the system input gain $b(t)$.

Assumption 5.1. The prior information with regards to $b(t)$ is that the control direction is known and invariant, that is, $b(t)$ is either positive or negative and non-singular for all $t \in [0,T]$.

Without loss of generality, assume that $b > 0\ \forall t \in [0,T]$.

Since the target trajectories could be different from iteration to iteration, the target trajectory in the i-th iteration is denoted as $x_{d,i}(t) \in \mathcal{C}^1[0,T]$.

Define the tracking error $e_i = x_{d,i} - x_i$. The error dynamics at the i-th iteration is

$$\begin{aligned}
\dot{e}_i &= \dot{x}_{d,i} - \dot{x}_i \\
&= \dot{x}_{d,i} - \boldsymbol{\theta}^o\boldsymbol{\xi}_i^o - bu_i \\
&= b(b^{-1}\dot{x}_{d,i} - b^{-1}\boldsymbol{\theta}^o\boldsymbol{\xi}_i^o - u_i) \qquad (5.2)\\
e_i(0) &= x_{d,i}(0) - x_i(0),
\end{aligned}$$

where $\boldsymbol{\theta}^o = \boldsymbol{\theta}^o(t)$, $\boldsymbol{\xi}_i^o = \boldsymbol{\xi}^o(x_i, t)$ and $b = b(t)$. The control objective is to track the trajectories by determining a sequence of control input u_i, such that the tracking error converges to zero as the iteration number i approaches infinity.

As is common in FIL field, the following I.I.C. is assumed.

Assumption 5.2. $\forall i \in \mathcal{Z}_+$, $e_i(0) = 0$ is satisfied.

5.3 FIL Configuration and Convergence Analysis

The proposed learning control law at the i-th iteration is

$$u_i = ke_i + \hat{\boldsymbol{\theta}}_i \boldsymbol{\xi}_i, \tag{5.3}$$

where $k > 0$ is the feedback gain, $\hat{\boldsymbol{\theta}}_i \in R^{1\times(n_1+2)}$ is to learn the time-varying parametric uncertainty consisting of $\boldsymbol{\theta} = [b^{-1}, \quad -b^{-1}\boldsymbol{\theta}^o, \quad b^{-2}\dot{b}] \in R^{1\times(n_1+2)}$ and $\boldsymbol{\xi}_i = [\dot{x}_{d,i}, \quad \boldsymbol{\xi}_i^{oT}, \quad -\frac{1}{2}e_i]^T \in R^{(n_1+2)\times 1}$ is the known vector-valued function.

The updating law for $\hat{\boldsymbol{\theta}}_i$ is

$$\hat{\boldsymbol{\theta}}_i = \hat{\boldsymbol{\theta}}_{i-1} + \beta\boldsymbol{\xi}_i^T e_i, \quad \hat{\boldsymbol{\theta}}_{-1}(t) = 0 \quad \forall t \in [0, T] \tag{5.4}$$

where $\beta > 0$ is the learning gain.

The convergence property of the proposed learning controller is derived in the following theorem.

Theorem 5.1. *For system (5.1), under the Assumptions 5.1-5.2, the learning control law (5.3) and the updating law (5.4) guarantee that the tracking error converges to zero pointwisely over $[0, T]$ when the iteration number i approaches to infinity.*

Proof:

To evaluate the learning property, define the CEF at the i-th iteration as

$$E_i(t) = \frac{1}{2}b^{-1}e_i^2 + \frac{1}{2\beta}\int_0^t \boldsymbol{\phi}_i\boldsymbol{\phi}_i^T d\tau, \tag{5.5}$$

where $\boldsymbol{\phi}_i \stackrel{\triangle}{=} \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_i$. Note that $\frac{1}{2}b^{-1}e_i^2$ is a quadratic type Lyapunov function used to evaluate the tracking performance in the time axis. The functional, the second term on the RHS of (5.5), is essentially an $\mathcal{L}^2$-norm reflecting the parametric learning error.

The proof consists of two parts. Part A derives the difference of the CEF, and Part B proves the pointwise convergence of the tracking error.

Part A: Difference of CEF

Consider the difference of $E_i(t)$ at the i-th iteration.

$$\begin{aligned}
\Delta E_i(t) &= E_i(t) - E_{i-1}(t) \\
&= \frac{1}{2}b^{-1}e_i^2 + \frac{1}{2\beta}\int_0^t (\boldsymbol{\phi}_i\boldsymbol{\phi}_i^T - \boldsymbol{\phi}_{i-1}\boldsymbol{\phi}_{i-1}^T)d\tau - \frac{1}{2}b^{-1}e_{i-1}^2.
\end{aligned} \tag{5.6}$$

Let us examine the first term on the RHS of (5.6). According to the I.I.C. (Assumption 5.2), the error dynamics (5.2) and the control law (5.3), the following can be derived.

$$\begin{aligned}
\frac{1}{2}b^{-1}e_i^2 &= \int_0^t b^{-1}e_i\dot{e}_i d\tau - \frac{1}{2}\int_0^t b^{-2}\dot{b}e_i^2 d\tau + \frac{1}{2}b^{-1}(0)e_i^2(0) \\
&= \int_0^t e_i(b^{-1}\dot{x}_{d,i} - b^{-1}\boldsymbol{\theta}^o\boldsymbol{\xi}_i^o - u_i)d\tau - \frac{1}{2}\int_0^t b^{-2}\dot{b}e_i^2 d\tau \\
&= \int_0^t e_i(\boldsymbol{\theta}\boldsymbol{\xi}_i - u_i)d\tau \\
&= -k\int_0^t e_i^2 d\tau + \int_0^t \boldsymbol{\phi}_i\boldsymbol{\xi}_i e_i d\tau \\
&= -k\int_0^t e_i^2 d\tau + \int_0^t \varsigma_i d\tau,
\end{aligned} \tag{5.7}$$

where $\varsigma_i = \boldsymbol{\phi}_i\boldsymbol{\xi}_i e_i$.

According to the updating law (5.4), the following can be obtained.

$$\begin{aligned}
&\frac{1}{2\beta}(\boldsymbol{\phi}_i\boldsymbol{\phi}_i^T - \boldsymbol{\phi}_{i-1}\boldsymbol{\phi}_{i-1}^T)\\
&= \frac{1}{2\beta}(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\theta}}_{i-1})(\hat{\boldsymbol{\theta}}_i + \hat{\boldsymbol{\theta}}_{i-1} - 2\boldsymbol{\theta})^T\\
&= -\frac{1}{\beta}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_i)(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\theta}}_{i-1})^T - \frac{1}{2\beta}(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\theta}}_{i-1})(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\theta}}_{i-1})^T\\
&= -\boldsymbol{\phi}_i\boldsymbol{\xi}_i e_i - \frac{\beta}{2}\|\boldsymbol{\xi}_i\|^2 e_i^2\\
&= -\varsigma_i - \frac{\beta}{2}\|\boldsymbol{\xi}_i\|^2 e_i^2. \qquad (5.8)
\end{aligned}$$

Substituting (5.7) and (5.8) into (5.6) yields

$$\begin{aligned}
\Delta E_i(t) &= -k\int_0^t e_i^2 d\tau - \frac{\beta}{2}\int_0^t \|\boldsymbol{\xi}_i\|^2 e_i^2 d\tau - \frac{1}{2}b^{-1}e_{i-1}^2\\
&\leq -\frac{1}{2}b^{-1}e_{i-1}^2. \qquad (5.9)
\end{aligned}$$

Part B: Convergence of Tracking Error

According to (5.9), it can be derived that the finiteness of $E_i(t)$ is ensured for any iteration provided $E_0(t)$ is finite. In the following we will show the finiteness of $E_0(t)$. From the definition of $E_i(t)$ in (5.5), we have

$$E_0(t) = \frac{1}{2}b^{-1}e_0^2 + \frac{1}{2\beta}\int_0^t \boldsymbol{\phi}_0\boldsymbol{\phi}_0^T d\tau.$$

Hence, the derivative of $E_0(t)$ is

$$\dot{E}_0(t) = b^{-1}e_0\dot{e}_0 - \frac{1}{2}b^{-2}\dot{b}e_0^2 + \frac{1}{2\beta}\boldsymbol{\phi}_0\boldsymbol{\phi}_0^T.$$

From (5.7) it can be derived that

$$b^{-1}e_0\dot{e}_0 - \frac{1}{2}b^{-2}\dot{b}e_0^2 = -ke_0^2 + \varsigma_0.$$

From (5.8) and the fact $\hat{\boldsymbol{\theta}}_{-1} = 0$, we have

$$\begin{aligned}
\frac{1}{2\beta}\boldsymbol{\phi}_0\boldsymbol{\phi}_0^T &= \frac{1}{2\beta}(\boldsymbol{\phi}_0\boldsymbol{\phi}_0^T - \boldsymbol{\phi}_{-1}\boldsymbol{\phi}_{-1}^T) + \frac{1}{2\beta}\boldsymbol{\phi}_{-1}\boldsymbol{\phi}_{-1}^T\\
&= -\varsigma_0 - \frac{\beta}{2}\|\boldsymbol{\xi}_0\|^2 e_0^2 + \frac{1}{2\beta}\boldsymbol{\theta}\boldsymbol{\theta}^T. \qquad (5.10)
\end{aligned}$$

Consequently,

$$\dot{E}_0(t) = -ke_0^2 - \frac{\beta}{2}\|\boldsymbol{\xi}_0\|^2 e_0^2 + \frac{1}{2\beta}\boldsymbol{\theta}\boldsymbol{\theta}^T \leq \frac{1}{2\beta}\boldsymbol{\theta}\boldsymbol{\theta}^T.$$

Because $\boldsymbol{\theta}$ is continuous, it is bounded over the time interval $[0, T]$. Therefore there exists a constant

$$L = \max_{t\in[0,T]}(\frac{1}{2\beta}\boldsymbol{\theta}\boldsymbol{\theta}^T) < \infty.$$

Considering $e_0(0) = 0$, we have

$$\begin{aligned} E_0(t) &\leq |E_0(0)| + |\int_0^t \dot{E}_0(\tau)d\tau| \\ &\leq \int_0^t |\dot{E}_0(\tau)|d\tau \\ &\leq \int_0^t L d\tau \leq LT < \infty. \end{aligned}$$

The finiteness of $E_0(t)$ implies that $E_i(t)$ is finite, hence both $x_i(t)$ and $\int_0^t \|\hat{\boldsymbol{\theta}}_i\|^2 d\tau$ are bounded for all $i \in \mathcal{Z}_+$.

Using (5.9) repeatedly we have

$$\begin{aligned} E_i(t) &\leq E_0(t) - \frac{1}{2}b^{-1}(t)\sum_{j=0}^{i-1} e_j^2(t). \\ \lim_{i\to\infty} E_i(t) &\leq E_0(t) - \frac{1}{2}b^{-1}(t)\lim_{i\to\infty}\sum_{j=0}^{i-1} e_j^2(t) \\ &\leq E_0(t) - \frac{1}{2}b_{max}^{-1}\lim_{i\to\infty}\sum_{j=0}^{i-1} e_j^2(t), \end{aligned}$$

where $b_{max} = \max_{t\in[0,T]} b(t) < \infty$.

Since $E_0(t)$ is finite and $E_i(t)$ is positive, $\sum_{j=0}^{\infty} e_j^2(t)$ converges. From the convergence theorem of the sum of series, $\lim_{i\to\infty} e_i^2(t) = 0$, $\forall t \in [0, T]$, is guaranteed. Hence $e_i(t)$ converges to zero pointwisely as i approaches infinity.

Since $\boldsymbol{\xi}_i$ is continuous with respect to x_i, the boundedness of x_i leads to the boundedness of $\boldsymbol{\xi}_i$. Therefore, according to the control law (5.3) and considering the boundedness of $\int_0^t \|\hat{\boldsymbol{\theta}}_i\|^2 d\tau$, the control signal u_i is bounded in $\mathcal{L}^2$-norm. ■

Remark 5.1. By substituting the updating law (5.4) into the control law (5.3), we can reach the following FIL law including u_{i-1}

$$u_i(t) = u_{i-1}(t) + f(e_{i-1}, \hat{\boldsymbol{\theta}}_{i-1}, \boldsymbol{\xi}_{i-1}, \boldsymbol{\xi}_i) + g(e_i, \boldsymbol{\xi}_i),$$

where $g = ke_i + \beta \boldsymbol{\xi}_i^T e_i$, and $f = -ke_{i-1} - \beta \boldsymbol{\xi}_{i-1}^T e_{i-1} + \hat{\boldsymbol{\theta}}_{i-1}(\boldsymbol{\xi}_i - \boldsymbol{\xi}_{i-1})^T$.

It can be interpreted that the new FIL is updated consecutively between u_i and u_{i-1}, but with a nonlinear feedback term and a general nonlinear correcting term. On the contrary, the traditional ILC updating law $u_i(t) = u_{i-1}(t) + \beta e_{i-1}(t)$, though simple and linear, could not capture the nonlinear structure characteristics of the system.

Remark 5.2. It is known mathematically that the pointwise convergence does not guarantee the convergent sequence to have a fixed upperbound, for instance

$$e_i(t) = i^2 t e^{-it} \tag{5.11}$$

If possible, the uniform convergence should be targeted.

Note that in the above learning control design, we do not need the system knowledge regarding the parameter bounds. Without knowing those bounds, robust control methods cannot be applied. On the other hand, in many control problems, the upper and lower bounds of unknown system parameters are known *a priori*. In such circumstance, the updating law (5.4) can be modified as

$$\hat{\boldsymbol{\theta}}_i = proj[\hat{\boldsymbol{\theta}}_{i-1}] + \beta \boldsymbol{\xi}_i^T e_i. \tag{5.12}$$

Here the question is, by incorporating the additional system bounding information in the learning control, can we improve the control performance? In the following we show that the control law (5.3) and the updating law (5.12) lead to the *uniform* convergence of the tracking error, instead of the *pointwise* convergence.

Corollary 5.2. *For system (5.1), under the Assumptions 5.1and 5.2, the learning control law (5.3) and the updating law (5.12) guarantee the uniform convergence of the tracking error sequence over* $[0, T]$*, when the iteration approaches to infinity.*

Proof:

Define the same CEF in (5.5), the relations (5.6) and (5.7) can be derived straightforward. Let us look at the relation (5.8), which may be affected by the introduction of the projection operator. Using the updating law (5.12), and comparing with (5.8), we have

$$\begin{aligned}
&\frac{1}{2\beta}(\boldsymbol{\phi}_i\boldsymbol{\phi}_i^T - \boldsymbol{\phi}_{i-1}\boldsymbol{\phi}_{i-1}^T)\\
&\leq \frac{1}{2\beta}\{\boldsymbol{\phi}_i\boldsymbol{\phi}_i^T - [\boldsymbol{\theta} - proj[\hat{\boldsymbol{\theta}}_{i-1}]][\boldsymbol{\theta} - proj[\hat{\boldsymbol{\theta}}_{i-1}]]^T\}\\
&= \frac{1}{2\beta}(\hat{\boldsymbol{\theta}}_i - proj[\hat{\boldsymbol{\theta}}_{i-1}])(\hat{\boldsymbol{\theta}}_i + proj[\hat{\boldsymbol{\theta}}_{i-1}] - 2\boldsymbol{\theta})\\
&= -\varsigma_i - \frac{\beta}{2}\|\boldsymbol{\xi}_i\|^2 e_i^2,
\end{aligned} \tag{5.13}$$

which turns out to be the same as (5.8). Consequently, substituting (5.7) and (5.13) into (5.6) yields the same result as (5.9)

$$\Delta E_i(t) \leq -k\int_0^t e_i^2 d\tau - \frac{\beta}{2}\int_0^t \|\boldsymbol{\xi}_i\|^2 e_i^2 d\tau - \frac{1}{2}b^{-1}e_{i-1}^2 \leq 0. \tag{5.14}$$

In the sequel, the pointwise convergence of e_i can be obtained according to Theorem 5.1.

According to the system dynamics (5.1), the control law (5.3) and the updating law (5.12), the boundedness of x_i ensures the finiteness of $\hat{\boldsymbol{\theta}}_i$, $u_i(t)$ and $\dot{x}_i(t)$. The boundedness of $\dot{x}_i(t)$ implies the uniform continuity of $x_i(t)$, thereafter the uniform continuity of the tracking error e_i, as $x_{d,i} \in \mathcal{C}^1[0, T]$. Therefore

$$\lim_{i\to\infty} |e_i(t)| = 0 \Rightarrow \lim_{i\to\infty} |e_i|_{sup} = 0. \tag{5.15}$$

■

Remark 5.3. In our work, b is related to time only. If b is a function of system states, but factorable into $b_1(t)b_2(x,t)$ where $b_2(x,t)$ is known, our approach still applies. If $b_2(x,t)$ is unknown, the learning control problem is still open.

5.4 FIL with Mixed Updating Laws

Often we have some prior knowledge about the system parametric uncertainties, for instance we may know that some unknown parameters are time-invariant, whereas the rest are time-varying. This is a non-trivial case, as the more we know, the better we should be able to improve the control performance. It would be far-fetched if we still apply the difference updating to those constant parameters, and the traditional integrator based adaptation is more suitable (Moore, 1989; French and Rogers, 2000). Indeed, differential updating mechanism may generate a smoother profile comparing with the difference type.

Instead of Assumption 5.1, the following assumption is made for system (5.1).

Assumption 5.3. $\boldsymbol{\theta}^o\boldsymbol{\xi}^o$ can be separated into $\boldsymbol{\theta}^o\boldsymbol{\xi}^o = \boldsymbol{\theta}_1^o(t)\boldsymbol{\xi}_1^o(x,t) + \boldsymbol{\theta}_2^o\boldsymbol{\xi}_2^o(x,t)$, where $\boldsymbol{\theta}_1^o(t) \in \mathcal{C}\left(R^{1\times n_1},[0,T]\right)$ is an unknown time-varying parameter vector, $\boldsymbol{\theta}_2^o \in R^{1\times n_2}$ is an unknown time-invariant parameter vector and both $\boldsymbol{\xi}_1^o(x,t) \in R^{n_1}$ and $\boldsymbol{\xi}_2^0(x,t) \in R^{n_2}$ are known continuous vector-valued functions, and are local Lipschitz continuous with respect to x. n_1 and n_2 are integers specifying dimensions. In addition, the system input gain b is an unknown constant and the only prior knowledge is that its sign is known.

In fact, if b is time-varying, according to the definition of the vector $\boldsymbol{\theta}$ in the preceding section, all parameters to be learned are time-varying .

Now the error dynamics can be expressed as

$$\begin{aligned} \dot{e}_i &= b[b^{-1}\dot{x}_{d,i} - b^{-1}\boldsymbol{\theta}_1^o(t)\boldsymbol{\xi}_1^o(x_i,t) - b^{-1}\boldsymbol{\theta}_2^o\boldsymbol{\xi}_2^o(x_i,t) - u_i] \\ &= b(\boldsymbol{\theta}_1\boldsymbol{\xi}_1 + \boldsymbol{\theta}_2\boldsymbol{\xi}_2 - u_i) \end{aligned} \tag{5.16}$$

where $\boldsymbol{\theta}_1 = -b^{-1}\boldsymbol{\theta}_1^o(t) \in R^{1\times n_1}$, $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_{1,i}^o(x_i,t) \in R^{n_1}$, $\boldsymbol{\theta}_2 = [b^{-1}, \quad -b^{-1}\boldsymbol{\theta}_2^o] \in R^{1\times(n_2+1)}$ and $\boldsymbol{\xi}_{2,i} = [\dot{x}_{d,i}, \quad \boldsymbol{\xi}_2^o(x_i,t)^T]^T \in R^{(n_2+1)}$. $\boldsymbol{\theta}_1$ presents all the time-varying parametric uncertainties, while $\boldsymbol{\theta}_2$ represents all the time-invariant parametric uncertainties.

The learning control law is constructed as

$$u_i = ke_i + \hat{\boldsymbol{\theta}}_{1,i}\boldsymbol{\xi}_{1,i} + \hat{\boldsymbol{\theta}}_{2,i}\boldsymbol{\xi}_{2,i}, \tag{5.17}$$

where $k > 0$ is the feedback gain, $\hat{\boldsymbol{\theta}}_{1,i} \in R^{1\times n_1}$ is to learn $\boldsymbol{\theta}_1$, and $\hat{\boldsymbol{\theta}}_{2,i} \in R^{1\times(n_2+1)}$ is to learn $\boldsymbol{\theta}_2$.

For the time-varying uncertainty $\boldsymbol{\theta}_1$, the preceding difference type updating law is used

$$\hat{\boldsymbol{\theta}}_{1,i} = \hat{\boldsymbol{\theta}}_{1,i-1} + \beta_1\boldsymbol{\xi}_{1,i}^T e_i, \quad \hat{\boldsymbol{\theta}}_{1,-1}(t) = 0 \quad \forall t \in [0,T]. \tag{5.18}$$

For the constant part $\boldsymbol{\theta}_2$, the traditional differential type updating law is employed

$$\dot{\hat{\boldsymbol{\theta}}}_{2,i} = \beta_2\boldsymbol{\xi}_{2,i}^T e_i, \quad \hat{\boldsymbol{\theta}}_{2,i}(0) = \hat{\boldsymbol{\theta}}_{2,i-1}(T), \quad \hat{\boldsymbol{\theta}}_{2,0}(0) = 0. \tag{5.19}$$

Both $\beta_1 > 0$ and $\beta_2 > 0$ are the learning gains.

Remark 5.4. Note the difference in the initial conditions of the difference type updating law (5.18) and differential type updating law (5.19). In fact, since $e_i(0) = 0$, $\hat{\boldsymbol{\theta}}_{1,i}(0) = 0$ for all iterations, namely, the difference type updating mechanism has an *resetting action* along the iteration axis. On the contrary, the differential type updating mechanism has a *consecutive initial condition* along the iteration axis, that is, the end value of preceding iteration becomes the initial value of the present iteration. The reason that accounts for the difference is, that a constant parameter will hold the same value at $t = 0$ and $t = T$, whereas a time-varying parameter may not. If $\boldsymbol{\theta}_1(0) \neq \boldsymbol{\theta}_1(T)$, it would be meaningless to apply the consecutive initial condition. The consecutive initial condition is applicable to the difference type updating mechanism, only if we have additional knowledge that $\boldsymbol{\theta}_1(0) = \boldsymbol{\theta}_1(T)$.

The main result of the above learning control approach is summarized in the following theorem. Here we also assume $b > 0$.

Theorem 5.3. *For system (5.16), under the Assumptions 5.2 and 5.3, the learning control law (5.17) and the updating laws (5.18) and (5.19) guarantee that the tracking error converges to zero in $\mathcal{L}^2$-norm over $[0, T]$ as iteration number i approaches to infinity.*

Proof:

Define a CEF as

$$E_i(t) = \frac{1}{2}b^{-1}e_i^2 + \frac{1}{2\beta_1}\int_0^t \boldsymbol{\phi}_{1,i}\boldsymbol{\phi}_{1,i}^T d\tau + \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}\boldsymbol{\phi}_{2,i}^T, \tag{5.20}$$

where $\boldsymbol{\phi}_{1,i} \triangleq \boldsymbol{\theta}_1 - \hat{\boldsymbol{\theta}}_{1,i}$ and $\boldsymbol{\phi}_{2,i} \triangleq \boldsymbol{\theta}_2 - \hat{\boldsymbol{\theta}}_{2,i}$.

Because of the involvement of the mixed difference-differential updating, the proof becomes more complicated, and consists of three parts. Part A derives the difference of the CEF; Part B proves the convergence of the tracking error; Part C examines the boundedness property of the system state and the control signal.

Part A: Difference of CEF

The difference of $E_i(t)$ is

$$\begin{aligned}\Delta E_i(t) &= \frac{1}{2}b^{-1}e_i^2 + \frac{1}{2\beta_1}\int_0^t (\boldsymbol{\phi}_{1,i}\boldsymbol{\phi}_{1,i}^T - \boldsymbol{\phi}_{1,i-1}\boldsymbol{\phi}_{1,i-1}^T)d\tau \\ &\quad + \frac{1}{2\beta_2}(\boldsymbol{\phi}_{2,i}\boldsymbol{\phi}_{2,i}^T - \boldsymbol{\phi}_{2,i-1}\boldsymbol{\phi}_{2,i-1}^T) - \frac{1}{2}b^{-1}e_{i-1}^2.\end{aligned} \tag{5.21}$$

Let us examine the terms on the RHS of (5.21) separately.

According to the I.I.C., the error dynamics (5.16) and the control law (5.17), the first term on the RHS of (5.21) can be expressed as

$$\begin{aligned}\frac{1}{2}b^{-1}e_i^2 &= b^{-1}\int_0^t e_i\dot{e}_i d\tau + \frac{1}{2}b^{-1}(0)e_i^2(0)\\ &= \int_0^t e_i(\boldsymbol{\theta}_1\boldsymbol{\xi}_{1,i} + \boldsymbol{\theta}_2\boldsymbol{\xi}_{2,i} - u_i)d\tau\\ &= -k\int_0^t e_i^2 d\tau + \int_0^t \boldsymbol{\phi}_{1,i}\boldsymbol{\xi}_{1,i}e_i d\tau + \int_0^t \boldsymbol{\phi}_{2,i}\boldsymbol{\xi}_{2,i}e_i d\tau\\ &= -k\int_0^t e_i^2 d\tau + \int_0^t \varsigma_{1,i}d\tau + \int_0^t \varsigma_{2,i}d\tau, \qquad (5.22)\end{aligned}$$

where $\varsigma_{1,i} = \boldsymbol{\phi}_{1,i}\boldsymbol{\xi}_{1,i}e_i$ and $\varsigma_{2,i} = \boldsymbol{\phi}_{2,i}\boldsymbol{\xi}_{2,i}e_i$.

Analogous to the derivation of (5.8), from the updating law (5.18) the second term on the RHS of (5.21) is

$$\begin{aligned}&\frac{1}{2\beta_1}(\boldsymbol{\phi}_{1,i}\boldsymbol{\phi}_{1,i}^T - \boldsymbol{\phi}_{1,i-1}\boldsymbol{\phi}_{1,i-1}^T)\\ &= -\frac{1}{\beta_1}\boldsymbol{\phi}_{1,i}(\hat{\boldsymbol{\theta}}_{1,i} - \hat{\boldsymbol{\theta}}_{1,i-1})^T - \frac{1}{2\beta_1}(\hat{\boldsymbol{\theta}}_{1,i} - \hat{\boldsymbol{\theta}}_{1,i-1})(\hat{\boldsymbol{\theta}}_{1,i} - \hat{\boldsymbol{\theta}}_{1,i-1})^T\\ &= -\boldsymbol{\phi}_{1,i}\boldsymbol{\xi}_{1,i}e_i - \frac{\beta_1}{2}\|\boldsymbol{\xi}_{1,i}\|^2 e_i^2\\ &= -\varsigma_{1,i} - \frac{\beta_1}{2}\|\boldsymbol{\xi}_{1,i}\|^2 e_i^2. \qquad (5.23)\end{aligned}$$

From the updating law (5.19), the third term on RHS of (5.21) is

$$\begin{aligned}&\frac{1}{2\beta_2}(\boldsymbol{\phi}_{2,i}\boldsymbol{\phi}_{2,i}^T - \boldsymbol{\phi}_{2,i-1}\boldsymbol{\phi}_{2,i-1}^T)\\ &= \frac{1}{\beta_2}\int_0^t \boldsymbol{\phi}_{2,i}\dot{\boldsymbol{\phi}}_{2,i}^T d\tau + \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}(0)\boldsymbol{\phi}_{2,i}^T(0) - \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i-1}\boldsymbol{\phi}_{2,i-1}^T\\ &= -\int_0^t \boldsymbol{\phi}_{2,i}\boldsymbol{\xi}_{2,i}e_i d\tau + \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}(0)\boldsymbol{\phi}_{2,i}^T(0) - \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i-1}\boldsymbol{\phi}_{2,i-1}^T\\ &= -\int_0^t \varsigma_{2,i}d\tau + \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}(0)\boldsymbol{\phi}_{2,i}^T(0) - \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i-1}\boldsymbol{\phi}_{2,i-1}^T. \qquad (5.24)\end{aligned}$$

Substituting (5.22), (5.23) and (5.24) back into (5.21) yields

$$\begin{aligned}\Delta E_i(t) &= -k\int_0^t e_i^2 d\tau - \frac{\beta_1}{2}\int_0^t \|\boldsymbol{\xi}_{1,i}\|^2 e_i^2 d\tau + \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}(0)\boldsymbol{\phi}_{2,i}^T(0)\\ &\quad -\frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i-1}(t)\boldsymbol{\phi}_{2,i-1}^T(t) - \frac{1}{2}b^{-1}e_{i-1}^2(t). \qquad (5.25)\end{aligned}$$

Considering the consecutive initial condition $\hat{\boldsymbol{\theta}}_{2,i}(0) = \hat{\boldsymbol{\theta}}_{2,i-1}(T)$, at the time instant $t = T$, we have $\boldsymbol{\phi}_{2,i}(0) = \boldsymbol{\phi}_{2,i-1}(T)$, thus

$$\begin{aligned}\Delta E_i(T) &= -k\int_0^T e_i^2 d\tau - \frac{\beta_1}{2}\int_0^T \|\boldsymbol{\xi}_{1,i}\|^2 e_i^2 d\tau - \frac{1}{2}b^{-1}e_{i-1}^2(T) \\ &\leq -k\int_0^T e_i^2 d\tau \leq 0. \end{aligned} \tag{5.26}$$

Part B: Convergence of the Tracking Error

According to (5.26), it can be derived that the finiteness of $E_i(T)$ is ensured for any iteration provided $E_0(T)$ is finite. In the following we will show the finiteness of $E_0(t)$.

$$E_0(t) = \frac{1}{2}b^{-1}e_0^2 + \frac{1}{2\beta_1}\int_0^t \boldsymbol{\phi}_{1,0}\boldsymbol{\phi}_{1,0}^T d\tau + \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,0}\boldsymbol{\phi}_{2,0}^T.$$

The derivative of $E_0(t)$ is

$$\dot{E}_0(t) = b^{-1}e_0\dot{e}_0 + \frac{1}{2\beta_1}\boldsymbol{\phi}_{1,0}\boldsymbol{\phi}_{1,0}^T + \frac{1}{\beta_2}\dot{\boldsymbol{\phi}}_{2,0}\boldsymbol{\phi}_{2,0}^T.$$

From (5.22), it can be derived that

$$b^{-1}e_0\dot{e}_0 = -ke_0^2 + \varsigma_{1,0} + \varsigma_{2,0}.$$

Analogous to the derivation in (5.10), from (5.23) we have

$$\frac{1}{2\beta_1}\boldsymbol{\phi}_{1,0}\boldsymbol{\phi}_{1,0}^T = -\varsigma_{1,0} - \frac{\beta_1}{2}\|\boldsymbol{\xi}_{1,0}\|^2 e_0^2 + \frac{1}{2\beta_1}\boldsymbol{\theta}_1\boldsymbol{\theta}_1^T.$$

According to the updating law (5.19) and using $\dot{\boldsymbol{\theta}}_2 = 0$, it can be derived that

$$\frac{1}{\beta_2}\dot{\boldsymbol{\phi}}_{2,0}\boldsymbol{\phi}_{2,0}^T = -\boldsymbol{\phi}_{2,0}\boldsymbol{\xi}_{2,0}e_0 = -\varsigma_{2,0}.$$

Therefore,

$$\dot{E}_0(t) = -ke_0^2 - \frac{\beta_1}{2}\|\boldsymbol{\xi}_{1,0}\|^2 e_0^2 + \frac{1}{2\beta_1}\boldsymbol{\theta}_1\boldsymbol{\theta}_1^T \leq \frac{1}{2\beta_1}\boldsymbol{\theta}_1\boldsymbol{\theta}_1^T.$$

Since $\boldsymbol{\theta}_1$ is continuous, it is bounded over the time interval $[0, T]$. There exists a constant

$$L = \max_{t\in[0,T]}\left(\frac{1}{2\beta_1}\boldsymbol{\theta}_1\boldsymbol{\theta}_1^T\right) < \infty.$$

Considering $e_0(0) = 0$, $\hat{\boldsymbol{\theta}}_{2,0}(0) = 0$ and the boundedness of $\boldsymbol{\theta}_2$, the following can be derived.

$$\begin{aligned} E_0(t) &\leq |E_0(0)| + |\int_0^t \dot{E}_0(\tau)d\tau| \\ &\leq \frac{1}{2\beta_2}\boldsymbol{\theta}_2\boldsymbol{\theta}_2^T + \int_0^t |\dot{E}_0(\tau)|d\tau \\ &\leq \frac{1}{2\beta_2}\boldsymbol{\theta}_2\boldsymbol{\theta}_2^T + \int_0^t L d\tau \\ &\leq \frac{1}{2\beta_2}\boldsymbol{\theta}_2\boldsymbol{\theta}_2^T + LT < \infty. \end{aligned}$$

The finiteness of $E_0(t)$ implies $E_0(T)$ is bounded, hence $E_i(T)$ is finite for all $i \in \mathcal{Z}_+$.

According to (5.26), we obtain

$$\begin{aligned} E_i(T) &\leq E_0(T) - k\sum_{j=1}^{i}\int_0^T e_j^2 d\tau, \\ \lim_{i\to\infty} E_i(T) &\leq E_0(T) - k\lim_{i\to\infty}\sum_{j=1}^{i}\int_0^T e_j^2 d\tau. \end{aligned}$$

The finiteness of $E_0(T)$ and the positiveness of $E_i(T)$ lead to $\lim_{i\to\infty}\int_0^T e_i^2 d\tau = 0$. Hence e_i converges to zero in $\mathcal{L}^2$-norm.

Part C: Boundedness Property

Finally, let us check the boundedness property of the system state x_i and the control signal u_i. Note that, up to now we only prove the boundedness of $E_i(T)$, from which we need to further derive the boundedness of $E_i(t)$ for any $t \in [0, T]$.

According to the definition of $E_i(t)$ and the finiteness of $E_i(T)$, the boundedness of $\int_0^T \boldsymbol{\phi}_{1,i}\boldsymbol{\phi}_{1,i}^T d\tau$ and $\boldsymbol{\phi}_{2,i}(T)\boldsymbol{\phi}_{2,i}^T(T)$ is guaranteed for all iterations. Therefore, $\forall i \in \mathcal{Z}_+$, there exist finite constants M_1 and M_2 satisfying

$$\int_0^t \boldsymbol{\phi}_{1,i}\boldsymbol{\phi}_{1,i}^T d\tau \leq \int_0^T \boldsymbol{\phi}_{1,i}\boldsymbol{\phi}_{1,i}^T d\tau \leq M_1 < \infty$$
$$\boldsymbol{\phi}_{2,i+1}(0)\boldsymbol{\phi}_{2,i+1}^T(0) = \boldsymbol{\phi}_{2,i}(T)\boldsymbol{\phi}_{2,i}^T(T) \leq M_2 < \infty.$$

Hence, from (5.20), we have

$$E_i(t) \leq \frac{1}{2}b^{-1}e_i^2(t) + M_1 + \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}(t)\boldsymbol{\phi}_{2,i}^T(t). \quad (5.27)$$

On the other hand, from (5.25), we have

$$\begin{aligned}\Delta E_{i+1}(t) &\leq \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i+1}(0)\boldsymbol{\phi}_{2,i+1}^T(0) - \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}(t)\boldsymbol{\phi}_{2,i}^T(t) - \frac{1}{2}b^{-1}e_i^2(t) \\ &\leq M_2 - \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}(t)\boldsymbol{\phi}_{2,i}^T(t) - \frac{1}{2}b^{-1}e_i^2(t). \quad (5.28)\end{aligned}$$

Adding (5.27) and (5.28) leads to

$$\begin{aligned}E_{i+1}(t) &= E_i(t) + \Delta E_{i+1}(t) \\ &\leq \frac{1}{2}b^{-1}e_i^2 + M_1 + \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}\boldsymbol{\phi}_{2,i}^T + M_2 - \frac{1}{2\beta_2}\boldsymbol{\phi}_{2,i}\boldsymbol{\phi}_{2,i}^T - \frac{1}{2}b^{-1}e_i^2 \\ &= M_1 + M_2. \quad (5.29)\end{aligned}$$

As we have shown that $E_0(t)$ is bounded, hence $E_i(t)$ is finite for all $i \in \mathcal{Z}_+$, which implies the boundedness of x_i, $\int_0^t \|\hat{\boldsymbol{\theta}}_{1,i}\|^2 d\tau$ and $\hat{\boldsymbol{\theta}}_{2,i}(t)$. Because $\boldsymbol{\xi}_{1,i}$ and $\boldsymbol{\xi}_{2,i}$ are local Lipschitz continuous with respect to x_i, the boundedness of x_i leads to the boundedness of $\boldsymbol{\xi}_{1,i}$ and $\boldsymbol{\xi}_{2,i}$. Hence, from learning control law (5.17), it can be derived that u_i is bounded in $\mathcal{L}^2$-norm. ■

Remark 5.5. Analogous to Corollary 5.2, if the bound of $\boldsymbol{\theta}_{1,i}$ is known *a priori*, the updating law (5.18) can be modified as

$$\hat{\boldsymbol{\theta}}_{1,i} = proj(\hat{\boldsymbol{\theta}}_{1,i-1}) + \beta_1\boldsymbol{\xi}_{1,i}e_i.$$

Consequently, the boundedness of $\hat{\boldsymbol{\theta}}_{1,i}$ can be ensured, which leads to the boundedness of u_i and $\dot{x}_i$. The finiteness of $\dot{x}_i$ implies the uniform continuity of x_i. Hence, the uniform convergence of the tracking error is guaranteed.

Remark 5.6. To clearly explain the basic idea, only a first-order system is consider in this chapter. However, the proposed FIL approaches can be easily extended to the following class of systems,

$$\begin{aligned}\dot{x}_j &= x_{j+1}, \quad j = 1, \cdots, n-1 \\ \dot{x}_n &= \boldsymbol{\theta}^o(t)\boldsymbol{\xi}^o(\mathbf{x}, t) + b(t)u,\end{aligned}$$

where $\mathbf{x} = [x_1, \cdots, x_n]^T \in R^n$.

Define the extended tracking error $\sigma = \sum_{j=1}^{n} c_j e_j(t)$ $(c_n = 1)$, where $e_j(t) = x_d^{(j-1)}(t) - x_j(t)$ and c_j $(j = 1, \cdots, n)$ are coefficients of a Hurwitz polynomial. The derivative of $\sigma(t)$ with respect to time t is

$$\begin{aligned}\dot{\sigma}(t) &= \sum_{j=1}^{n-1} c_j e_{j+1} + \dot{x}_d^{(n)} - \boldsymbol{\theta}^o \boldsymbol{\xi}^o(\mathbf{x}, t) - bu \\ &= b[b^{-1} \sum_{j=1}^{n-1} c_j e_{j+1} + b^{-1} \dot{x}_d^{(n)} - b^{-1} \boldsymbol{\theta}^o \boldsymbol{\xi}^o(\mathbf{x}, t) - u],\end{aligned}$$

which has a similar form as equation (5.2). Therefore, the proposed FIL algorithm can be applied directly and the convergence of σ is guaranteed which leads to the convergence of $\mathbf{x}(t)$ to $\mathbf{x}_d(t)$.

5.5 Illustrative Examples

In this section, the following one-link robotic manipulator is considered

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2+I} \end{bmatrix} [u - gl\cos x_1 + \eta_1],$$

where x_1 is the joint angle, x_2 is the angular velocity, m is the mass, l is the length, I is the moment of inertia, u is the joint input and $\eta_1 = 5x_1^2 \sin^3(5t)$ is a disturbance. Let x_2 be the control target and the desired trajectories for the i-th iteration is $x_{d,i}$.

Throughout simulations the following two functions are chosen as target trajectories:

Class 1 $x_{d,i} = \kappa_i \sin^3(0.5t)$, i is odd,

Class 2 $x_{d,i} = \kappa_i 0.05 e^{-t}(2\pi t^3 - t^4)(2\pi - t)$, i is even,

where $t \in [0, 2\pi]$, and κ_i is generated randomly from the interval of $[-1, 0) \cup (0, 1]$ for each iteration i. The desired trajectories for the first four iterations are shown in

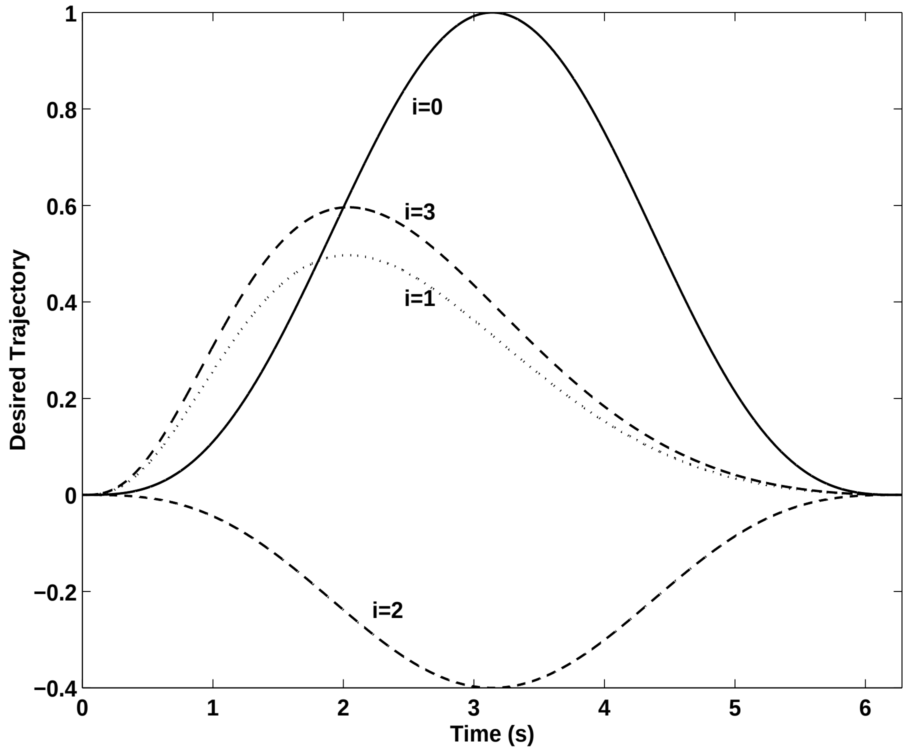

Figure 5.1: Desired trajectories for the first four learning iterations.

Fig. 5.1. Obviously, there is little similarity in the target trajectories between any two consecutive iterations, except for the fixed interval $T = 2\pi$. Define $b = \frac{1}{ml^2+I}$ and the extended tracking error $\sigma = 3e_1 + e_2$. Then the dynamics of the extended tracking error at the i-th iteration is

$$\begin{aligned} \dot{\sigma}_i &= 3\dot{e}_{1,i} + \dot{e}_{2,i} \\ &= 3e_{2,i} + \dot{x}_{d,i} - b(u_i - gl\cos x_{1,i} + \eta_{1,i}) \\ &= b(3b^{-1}e_{2,i} + b^{-1}\dot{x}_{d,i} - u_i + gl\cos x_{1,i} - \eta_{1,i}). \end{aligned} \tag{5.30}$$

Case 1: b is time-varying.

The system parameters are chosen as: $m = (3 + 0.1 sint)kg$, $l = 1m$ and $I = 0.5kg{\cdot}m^2$. b is assumed to be unknown and the only available information is that $b(t)$ is positive. The system initial condition is: $x_{1,i}(0) = 0$ and $x_{2,i}(0) = 0$. In this case, $\boldsymbol{\theta} = [3b^{-1},\ \ b^{-1},\ \ gl,\ \ -5\sin^3(5t),\ \ b^{-2}\dot{b}]$ and $\boldsymbol{\xi} = [e_{2,i},\ \ \dot{x}_{d,i},\ \ \cos x_{1,i},\ \ x_{1,i}^2,\ \ -\frac{1}{2}\sigma_i]^T$.

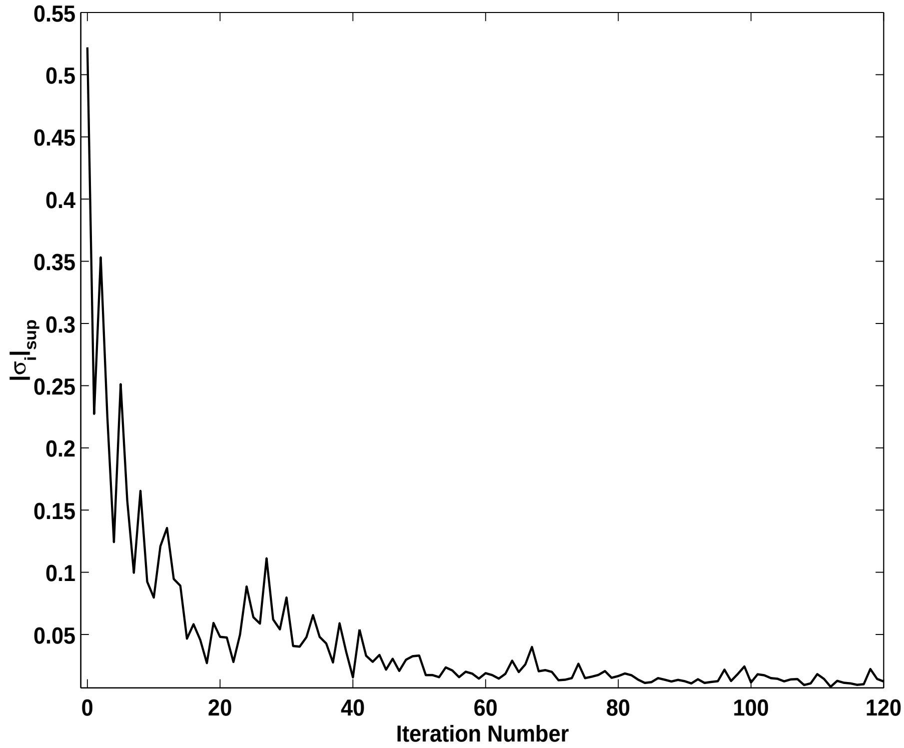

Figure 5.2: Convergence of the extended tracking error σ_i in *Case 1*.

Choose $k = 5$ and $\beta = 20$. Applying control law (5.3) and (5.4), the learning convergence is shown in Fig. 5.2. The horizon is the iteration number and the vertical is the sup-norm $|\sigma_i|_{sup}$.

Case 2: b is time-invariant.

The system parameters are: $m = 3kg$, $l = 1m$ and $I = 0.5kg \cdot m^2$. b is an unknown positive constant. The system initial condition is the same as in *Case 1*.

The system uncertainty can be expressed as $\boldsymbol{\theta}_1(t)\boldsymbol{\xi}_{1,i}+\boldsymbol{\theta}_2\boldsymbol{\xi}_{2,i}$, where $\boldsymbol{\theta}_1 = [-5\sin^3(5t)]$, $\boldsymbol{\theta}_2 = [3b^{-1}, \quad b^{-1}, \quad gl]$, $\boldsymbol{\xi}_{1,i} = [x_{1,i}^2]$ and $\boldsymbol{\xi}_{2,i} = [e_{2,i}, \quad \dot{x}_{d,i}, \quad \cos x_{1,i}]^T$. Choose $\beta_1 = \beta_2 = 20$. Applying the control laws (5.17), (5.18) and (5.19), the learning convergence is shown in Fig. 5.3. The tracking error reduces to 1% of $|\sigma_0|_{sup}$ after a number of iterations.

The simulation results in both *Case 1* and *Case 2* demonstrate clearly the ability of

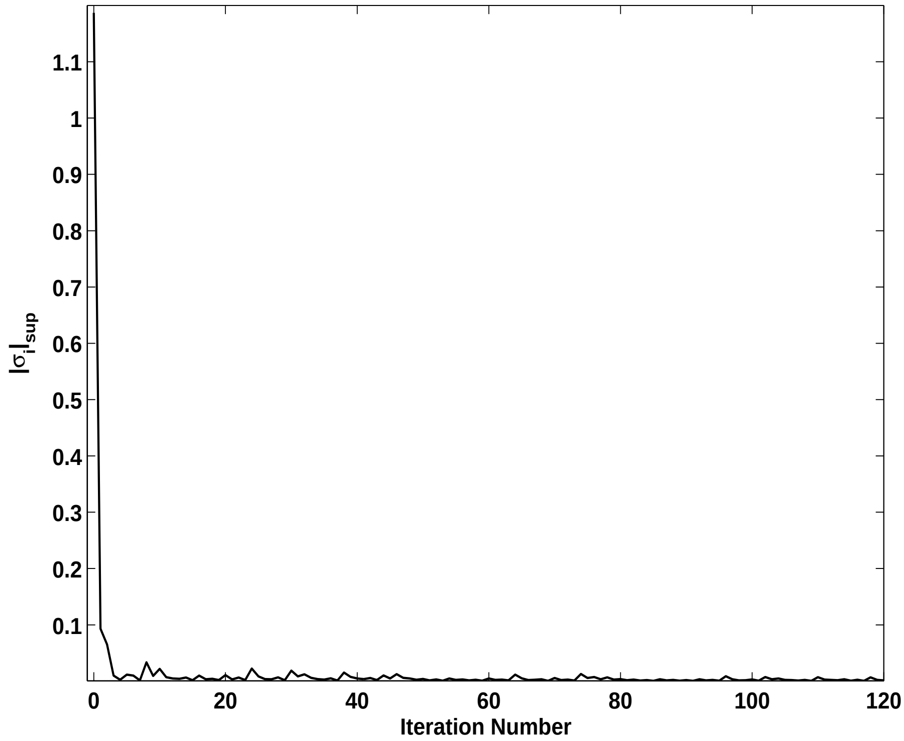

Figure 5.3: Convergence of the extended tracking error σ_i in *Case 2*.

the learnability from different motion patterns.

Case 3: Learning for identical trajectory.

For comparison purpose, here the tracking control is conducted for a fixed target trajectory $x_d = \sin(0.5t)$ $t \in [0, 2\pi]$. Applying the same control design as in *Case 2*, Fig. 5.4 gives the simulation result. Since the identical trajectory tracking task is a special case of non-identical trajectory tracking problems, obviously it is much easier to learn.

Case 4: Comparison with the differential-type updating.

If we are not sure whether a parametric uncertainty is time-varying or time-invariant, the safe way is to treat it as time-varying. In the following, we show that the differential updating law alone fails to work for time-varying uncertainties.

Again consider *Case 2*, but treat the term $-5\sin^3(5t)$ as time-invariant. Hence

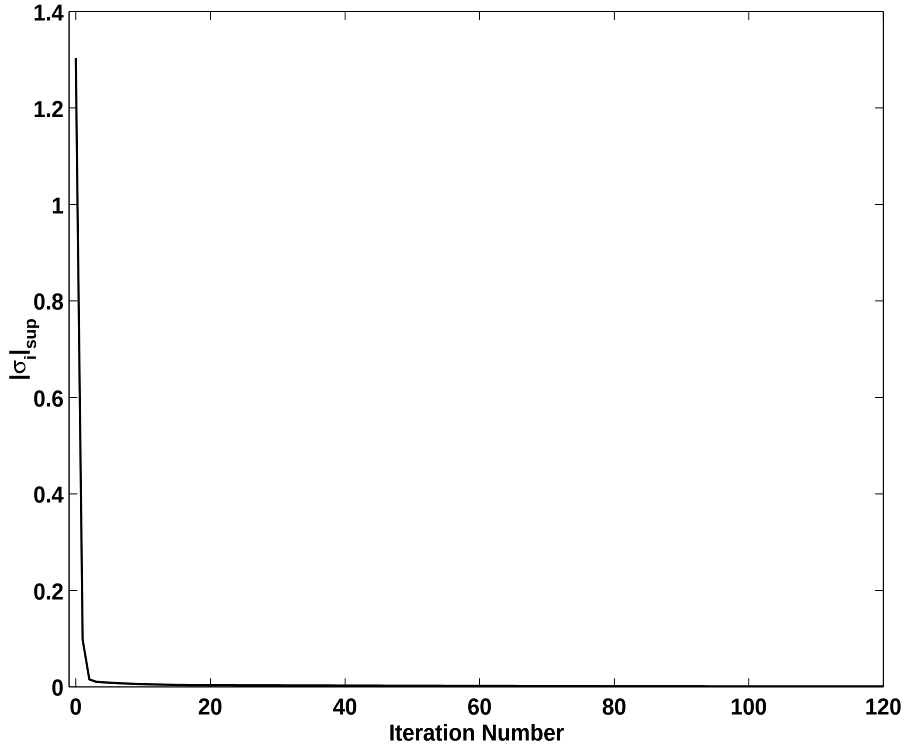

Figure 5.4: Convergence of the extended tracking error σ_i in *Case 3*.

only the differential updating law is applied to all parameters. Choose controller parameters k and β_2 to be same as in *Case 2*, the simulation result is shown in Fig. 5.5.

From Fig. 5.5 we can see that the tracking error retains at a rather high level in comparison with the previous case, due to the lack of the learnability of a differential-type updating mechanism to time-varying parameters.

Case 5: Comparison with the traditional ILC approach

The following traditional D-type ILC is applied,

$$u_i = u_{i-1} + \beta\dot{\sigma}_{i-1}.$$

Use the same model and parameters as in *Case 2* and let $\beta = 0.1$, the simulation result is given in Fig. 5.6. It is clearly shown that if the desired trajectories are non-uniform, the learning convergence cannot be guaranteed by the traditional ILC

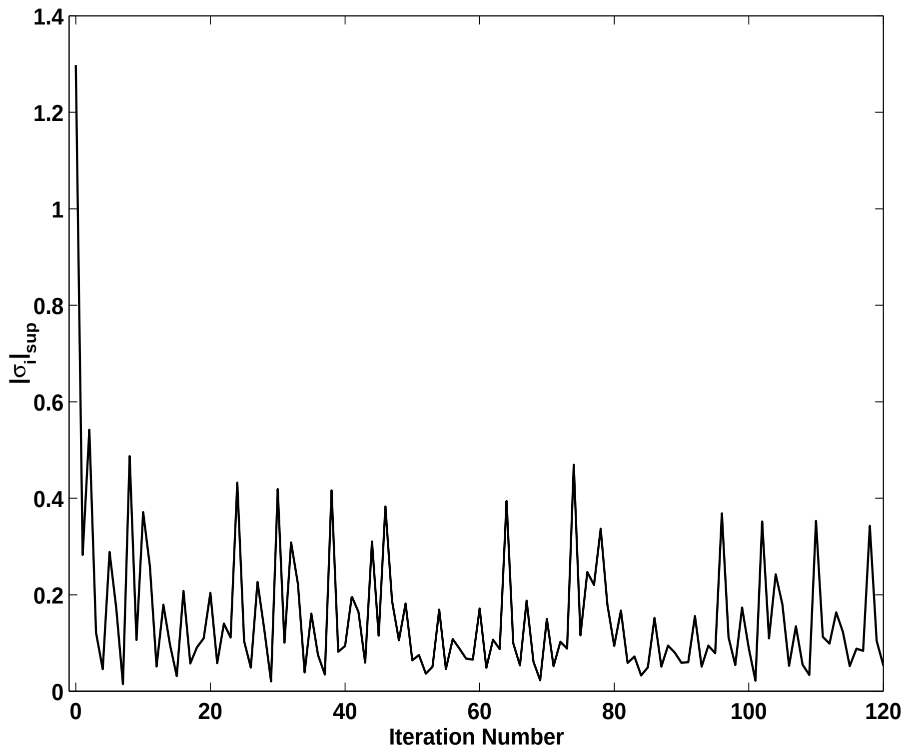

Figure 5.5: Extended tracking error σ_i in *Case 4*.

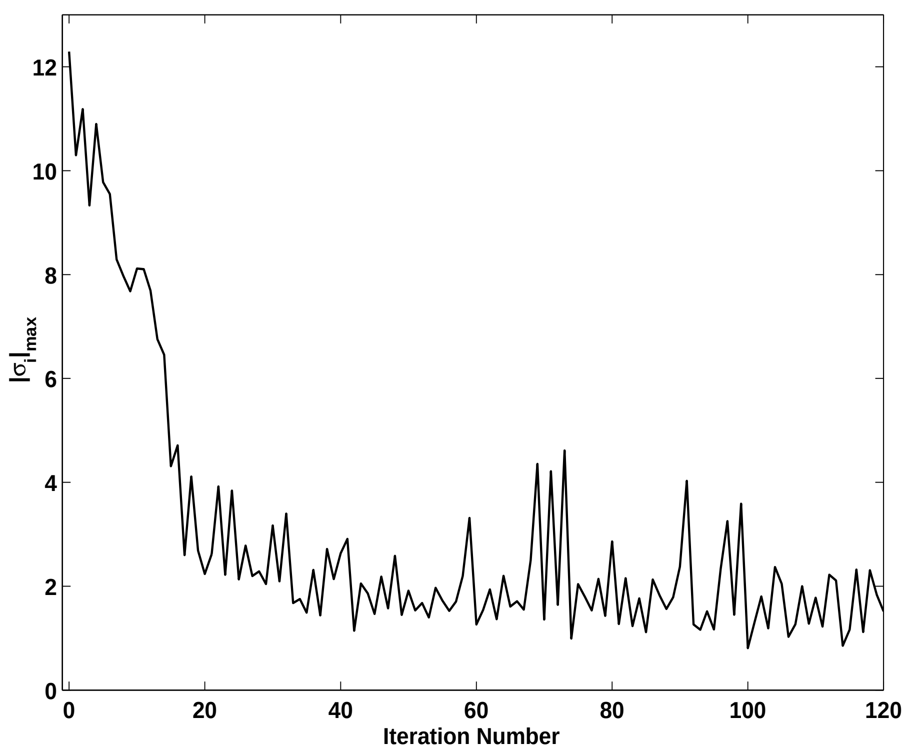

Figure 5.6: Extended tracking error σ_i in *Case 5*.

any more.

5.6 Conclusion

A novel FIL control method has been developed in this chapter. The new method is able to learn from different tracking tasks, that is, possessing the learnability along learning axis for non-identical trajectories. Through detailed discussions and rigorous analysis, we show that the system learnability comes from pointwise integration iteratively, and that the learnable part must be invariant along the iteration axis, which in our case is the time-varying but iteration-invariant parametric uncertainties. By introducing the CEF, it is convenient to derive the convergence property of the tracking error, and boundedness property of the system signal. Simulation results demonstrate the effectiveness of the proposed FIL method.

Chapter 6

Fuzzy Logic Learning Control

6.1 Introduction

Fuzzy Logic Control (FLC) was originally advocated by Zadeh (Zadeh, 1973) and Mamdani (Mamdani and Assilian, 1974) as a means of collecting human knowledge and experience to deal with uncertainties in the control process. In recent years, Fuzzy Logic Controllers have been widely used for industrial processes owing to their heuristic nature associated with simplicity and effectiveness especially for nonlinear uncertain systems. When a control task is given, a FLC is customized suitable for the task by experienced experts or skilled operators who "learn" to develop the FLC wherever the control task repeats.

The effectiveness of a FLC is mainly because of its structured nonlinearity. Many FLCs are essentially fuzzy PD-, fuzzy PI- or fuzzy PID-type controllers associated with nonlinear gains (Ying *et al.*, 1990; Ying, 1999; Lee, 1990; Malki *et al.*, 1994; Xu, 1998). Because of the nonlinear property of control gains, this kind of FLCs possesses the potential to improve and achieve better system performance. For instance, the farther the system error or change of error is off the equilibrium point, the higher the control gain is. Thus the closed-loop system will respond faster to the set-point change and recover faster from the load disturbance comparing to the conventional PID control.

Generally speaking, the nonlinear structure property of a heuristically designed FLC cater well to the characteristics of the industrial process under control. However when a new control task is given, it is always imperative to re-adjust the FLC so as to produce reasonable responses. It will naturally take experts or operators long time and great efforts to re-adjust the FLC suitable for the new task through trial and error. A simple and feasible idea is to retain the well established FLC nonlinear structure and only tune the FLC parameters such as the input-output scaling coefficients. FLC auto-tuning methods (Xu, 1998; Xu, 2000) have been proposed which work effectively and can satisfy the specified gain margin and phase margin. The main limitation of FLC auto-tuning is that the auto-tuning schemes are only applicable to simple control tasks such as set-point control or step-type load disturbance rejection. It would be a challenging work for a FLC to perform complicated tracking control tasks.

One way to partially address the trajectory tracking problem is to offer the FLC system a learning mechanism. Instead of letting experts learn to adjust, it is better to let FLC incorporate adaptive or learning functions to adjust itself to best meet the control task, which would be much more efficient and more accurate. Applying neural network into the FLC (Lin and Lee, 1991; Ichikawa *et al.*, 1992; Ng and Trivedi, 1998; Behera and Anand, 1999; Chien, 2000) is one such possible approach. However, a neural controller tends to be over complicated due to its large number of nodes and weights. On the other hand, a simple neural network may not achieve sufficient tracking precision. As a kind of input-to-output mapping approaches, most neural controllers will reconstruct the whole control system, which is neither practical from control engineering point of view, nor advisable from the FLC point of view where the "good" nonlinear structure is to be retained.

In this chapter we propose a new modular approach - Fuzzy Logic Learning Control (FLLC), which integrates two complementary control approaches, FLC and FIL, and improves the tracking performance through tasks repetitions.

In the configuration, FLLC consists of two control modules in an additive form: a simple fuzzy logic controller, and a learning mechanism which updates the current control profile from the previous control sequence. Such a construction does not alter the existing FLC which is heuristic and proved effective from expert's experience. From the control point of view, FLC provides feedback and the learning mechanism realizes feedforward compensation. Now if the control environment is repeatable or more or less repeatable over a finite duration, the proposed FLLC can provide a simple and effective way to possess such an internal model.

In this chapter we limit our discussion to a simple PD-type FLC. The proposed FLLC method based on the Fuzzy PD focuses on learning for the repeatable control tasks. The nonrepeatable factors such as random disturbance are assumed to be very small, consequently negligible. Through rigorous proof based on EF, we show that the FLLC system achieves the following novel properties: (1) the tracking error sequence converges uniformly to zero; (2) learning control sequence converges to the desired control profile almost everywhere.

The chapter is organized as follows. In Section 6.2, problem formulation and control objective are introduced. The structure and properties of a PD-type FLC are derived in Section 6.3. In Section 6.4, FLLC with learning updating is introduced with rigorous convergence analysis. Simulation work is presented in Section 6.5 to demonstrate the effectiveness of the proposed scheme. Finally, Section 6.6 gives the conclusion.

6.2 Problem Formulation

In this chapter, we consider the second order nonlinear dynamical system described by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(\mathbf{x}, t) + b(x_1, t)u \end{cases} \tag{6.1}$$

where $y(t) = x_1(t)$, $\mathbf{x} = [x_1, x_2] \in R^2$ is the physically measurable state vector, and u is the control input. $f(\mathbf{x}, t)$ and $b(x_1, t)$ are nonlinear uncertain functions.

For this system we make the following assumptions:

Assumption 6.1. $f(\mathbf{x}, t)$ is bounded by a known function $f_{max}(\mathbf{x}, t)$, and $0 < b_{min} \leq b(x_1, t) \leq b_{max}$ where b_{min} and b_{max} are known constants.

Assumption 6.2. $\forall q \in \{f, b\}$, $q(\mathbf{x}, t) \in \mathcal{C}(\mathcal{R}^2 \times [0, T])$ and $q(\mathbf{x}, t)$ satisfies the *Lipschitz* condition, $\|q(\mathbf{x}_1, t) - q(\mathbf{x}_2, t)\| \leq l_q \|\mathbf{x}_1 - \mathbf{x}_2\|$, $\forall t \in [0, T]$ and $\forall \mathbf{x}_1, \mathbf{x}_2 \in R^2$. Here the positive constant $l_q < \infty$.

Given a finite initial state $\mathbf{x}_i(0)$ and a finite time interval $[0, T]$ where i denotes the iteration sequence, the control objective is to design a FLC combined with FIL approach such that, as $i \to \infty$, the system state $\mathbf{x}_i$ of the nonlinear uncertain system (6.1) tracks the desired trajectory $\mathbf{x}_d = [x_{d,1}, x_{d,2}] \in R^2$ which is generated by the following dynamics over $[0, T]$

$$\begin{cases} \dot{x}_{d,1} = x_{d,2} \\ \dot{x}_{d,2} = \alpha(\mathbf{x}_d, t) + r(t) \end{cases} \tag{6.2}$$

where $\alpha(\mathbf{x}_d, t) \in \mathcal{C}(\mathcal{R}^2 \times [0, T])$ is a known function and $r(t) \in \mathcal{C}([0, T])$ is a reference input. As part of the repeatability condition, the I.I.C., i.e. $\mathbf{x}_i(0) = \mathbf{x}_d(0)$, is available for all trials.

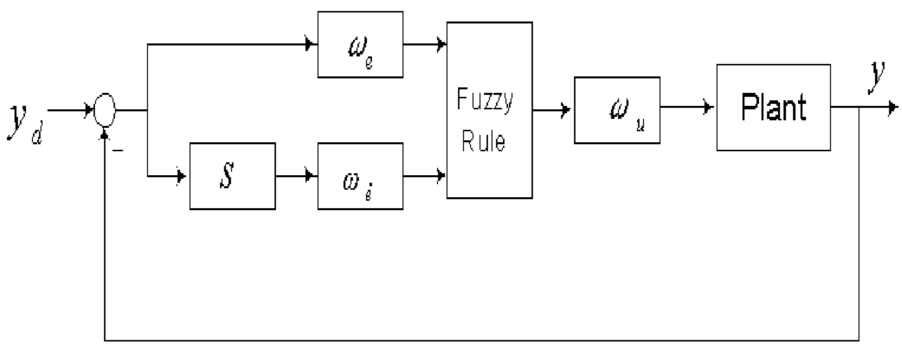

Figure 6.1: Overall structure of the FLC closed-loop system.

6.3 Properties of A Fuzzy PD Controller

For a large class of FLCs, fuzzy input variables are the error e and the change of error $\dot{e}$. The fuzzy rule table is then established on the phase plane $(e, \dot{e})$. In essence, these fuzzy controllers are fuzzy PD-, fuzzy PI- or fuzzy PID-type controllers associated with nonlinear gains. Because of the nonlinear property of control gains, FLCs possess the potential to improve and achieve better system performance. Due to the existence of nonlinearity, it is usually difficult to conduct theoretical analysis and find out appropriate design methods.

Consider a typical class of fuzzy PD controllers (Ying, 1993) and the control system is shown in Fig. 6.1. The inputs of the fuzzy rule base are the normalized error $(\omega_e e)$ and the normalized change of error $(\omega_{\dot{e}} \dot{e})$ where ω_e and $\omega_{\dot{e}}$ are weighting factors. The error and the change of error are defined as

$$\begin{cases} e(t) = y_d(t) - y(t) \\ \dot{e}(t) = \frac{dy_d(t)}{dt} - \frac{dy(t)}{dt}. \end{cases}$$

The membership functions used to fuzzify the inputs are triangular in shape shown in Fig. 6.2 and, consequently, there are four simple fuzzy control rules (Table 6.1) used in the FLC. The reasons to choose this type of FLC are (1) theoretical analysis

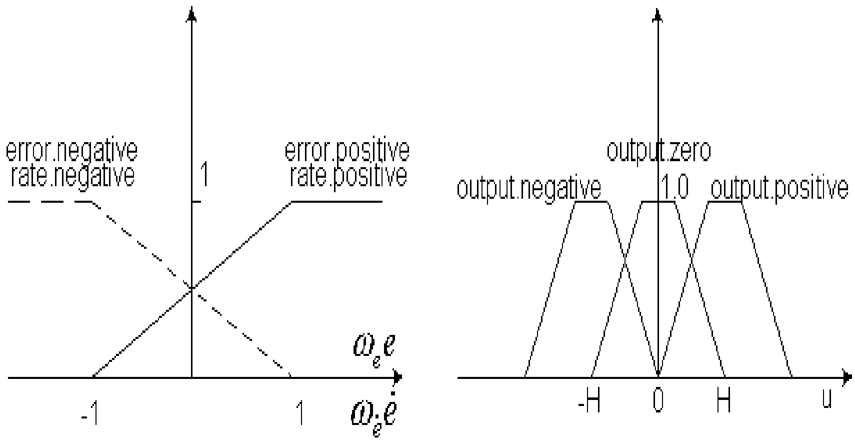

Figure 6.2: The membership functions of inputs ($\omega_e e$, $\omega_{\dot{e}}\dot{e}$) and output.

Table 6.1: Fuzzy control rules. N: negative; P: positive; Z: zero.

Rule 1	If error is N and change of error is N, control action is N
Rule 2	If error is N and change of error is P, control action is Z
Rule 3	If error is P and change of error is N, control action is Z
Rule 4	If error is P and change of error is P, control action is P

is possible owing to the known structural knowledge; (2) the nonlinearity of the simplest fuzzy PD controller is the strongest in the case of linear distributed rules (Buckley and Ying, 1989); (3) it is highly desirable to make the FLCs as simple as possible and leave the performance refining task to learning control, i.e. maximize the automated learning and minimize the heuristic learning efforts in deriving FLC rules.

The fuzzy output variables have trapezoidal shape membership functions and the lengths of their upper and lower bases are $2A$ and $2H$ (Fig. 6.2), respectively. Zadeh's AND (MIN) and Lukasiewicz's OR are used in the fuzzy inference and the most general inference method, the Mamdamni's minimum inference method (Xu *et al.*, 1998), is considered in the discussion. By using the center of gravity (COG) defuzzification method, (Ying, 1993) has discussed the control property when

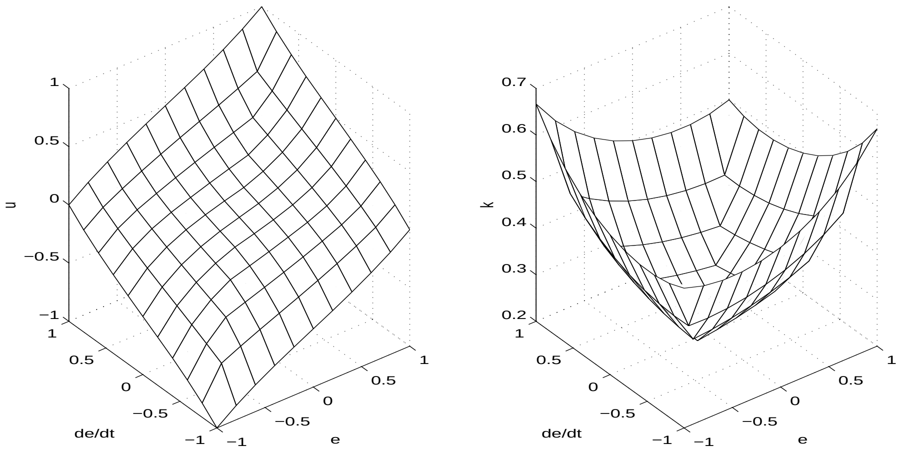

Figure 6.3: Control surface u (left) and nonlinear control gain k (right) produced by FLC (Unsaturated region).

$A \leq 0.5H$, and the overall control output can be obtained (inside the unsaturated region of the universe of discourse)

$$
\begin{aligned}
u_f &= k(e,\dot{e})(\omega_e e + \omega_{\dot{e}}\dot{e}) \qquad (6.3)\\
k(e,\dot{e}) &= \frac{0.5H\omega_u[(1+\theta)+0.5(1-\theta)|\omega_e e - \omega_{\dot{e}}\dot{e}|]}{(3+\theta)-[(1+\theta)max(\omega_e|e|,\omega_{\dot{e}}|\dot{e}|)+0.5(1-\theta)((\omega_e e)^2+(\omega_{\dot{e}}\dot{e})^2)]}
\end{aligned}
$$

where $\theta = \frac{A}{H}$ and $k(e,\dot{e})$ is the nonlinear part of the FLC output.

Let $H = 1$, $\theta = 0.5$ and $\omega_e = \omega_{\dot{e}} = \omega_u = 1$, the control surface of the FLC and the surface of $k(e,\dot{e})$ of the unsaturated region are shown in Fig. 6.3.

In most cases, we find that the two-dimensional rule table has the skew-symmetry property (Choi *et al.*, 1999). The unsaturated phase plane is divided into two semi-planes by means of a switching line σ. Within the semi-planes positive and negative control outputs are produced respectively. While outside the unsaturated region, the output of FLCs will be partially or fully saturated. In general we can choose ω_e and $\omega_{\dot{e}}$ to ensure that the control task can be fulfilled by the FLC in the unsaturated region. From (6.3), the PD-type FLC can be expressed as

$$u_f = k(e,\dot{e})\sigma$$

where $\sigma = \omega_e e + \omega_{\dot{e}}\dot{e}$ is the switching line, $0 < k_{min} \le k(e,\dot{e}) \le k_{max}$ and $|u_f|$ is bounded by u_F.

Remark 6.1. Note that $k(e,\dot{e})$ is a bounded function of the arguments e and $\dot{e}$. Thus, $u_f = 0$ whenever the system is at its equilibrium $e = \dot{e} = 0$. However, from (6.1) we can see that the desired control input at the equilibrium is

$$u_d = b^{-1}(x_{d,1},t)[\dot{x}_{d,2} - f(\mathbf{x}_d,t)]$$

which may not be zero $\forall t \in [0,T]$. This shows the essential problem of all kinds of feedback control inclusive of FLCs due to the lack of "internal model". Learning control, as one of the most effective feedforward methods, complements FLCs.

6.4 Fuzzy Logic Learning Control

The proposed FLLC is given below

$$u_i = proj[u_{i-1}] + u_{f,i}, \qquad u_{-1}(t) = 0 \tag{6.4}$$

$$u_{f,i} = k(e_i,\dot{e}_i)\sigma_i \tag{6.5}$$

$$\sigma_i = \omega_e e_i + \omega_{\dot{e}}\dot{e}_i \tag{6.6}$$

where i denotes the iteration sequence, u_i is the system input and $0 < k_{min} < k(e_i,\dot{e}_i) \le k_{max}$. Moreover, it is assumed that the original FLC based on heuristic knowledge should ensure the system stability, though may still yield a large tracking error with respect to the tracking task specified by (6.2).

To evaluate the learning performance, the following time-weighted $\mathcal{L}^2$ norm of $u_i - u_d$ is used

$$J_i(t) = \int_0^t e^{-\lambda\tau}[u_i(\tau) - u_d(\tau)]^2 d\tau. \tag{6.7}$$

The difference of $J_i(t)$ between two successive trials can be derived as

$$\begin{aligned}
\Delta J_i(t) &\triangleq J_i(t) - J_{i-1}(t) \\
&= \int_0^t e^{-\lambda\tau}(u_i - u_d)^2 d\tau - \int_0^t e^{-\lambda\tau}(u_{i-1} - u_d)^2 d\tau \\
&\leq \int_0^t e^{-\lambda\tau}(u_i - u_d)^2 d\tau - \int_0^t e^{-\lambda\tau}(proj[u_{i-1}] - u_d)^2 d\tau \\
&= \int_0^t e^{-\lambda\tau}[u_i - proj[u_{i-1}]][u_i + proj[u_{i-1}] - 2u_d] d\tau \\
&= \int_0^t e^{-\lambda\tau}\{u_{f,i}^2 + 2u_{f,i}[proj[u_{i-1}] - u_d]\} d\tau. \qquad (6.8)
\end{aligned}$$

First we derive the expressions of $proj[u_{i-1}] - u_d$ and $u_i - u_d$. From (6.6) we can obtain

$$\sigma_i = \omega_e e_i + \omega_{\dot{e}} \dot{e}_i = \omega_e (x_{d,1} - x_{1,i}) + \omega_{\dot{e}} (x_{d,2} - x_{2,i}). \qquad (6.9)$$

Differentiating (6.9) with respect to t yields

$$\dot{\sigma}_i = \omega_e (\dot{x}_{d,1} - \dot{x}_{1,i}) + \omega_{\dot{e}} (\dot{x}_{d,2} - \dot{x}_{2,i}). \qquad (6.10)$$

Substituting (6.1) and (6.2) into (6.10) gives

$$\begin{aligned}
\dot{\sigma}_i &= \omega_e (x_{d,2} - x_{2,i}) + \omega_{\dot{e}} (\alpha(\mathbf{x}_d, t) + r(t)) - h_i - l_i u_i \\
&= g_i - h_i - l_i u_i
\end{aligned}$$

where $h_i \triangleq \omega_{\dot{e}} f(\mathbf{x}_i, t)$, $l_i \triangleq \omega_{\dot{e}} b(x_{1,i}, t)$, $g_i \triangleq \omega_e (x_{d,2} - x_{2,i}) + g_d$, $g_d \triangleq \omega_{\dot{e}} \alpha(\mathbf{x}_d, t) + \omega_{\dot{e}} r(t)$.

Then

$$u_i = -l_i^{-1} \dot{\sigma}_i + l_i^{-1} g_i - l_i^{-1} h_i. \qquad (6.11)$$

Let $\dot{\sigma}_i = 0$, consequently $\sigma_i(t) = \sigma_i(0) = 0$. According to (6.11), the desired control is

$$u_d = l_d^{-1} g_d - l_d^{-1} h_d$$

where $h_d \stackrel{\triangle}{=} \omega_{\dot{e}} f(\mathbf{x}_d, t)$, $l_d \stackrel{\triangle}{=} \omega_{\dot{e}} b(x_{d,1}, t)$.

It can be derived that

$$u_i - u_d = -l_i^{-1}\dot{\sigma}_i - \gamma_i \quad (6.12)$$

$$proj[u_{i-1}] - u_d = u_i - u_{f,i} - u_d = -u_{f,i} - l_i^{-1}\dot{\sigma}_i - \gamma_i \quad (6.13)$$

where

$$\gamma_i = (l_d^{-1} g_d - l_i^{-1} g_i) - (l_d^{-1} h_d - l_i^{-1} h_i). \quad (6.14)$$

Here γ_i is the equivalent system uncertainties. From (6.14) we know

$$\gamma_i = (l_d^{-1} g_d - l_i^{-1} g_i) - (l_d^{-1} h_d - l_i^{-1} h_i).$$

It can be derived that

$$\begin{aligned} |\gamma_i| &\leq |l_d^{-1} g_d - l_i^{-1} g_d + l_i^{-1} g_d - l_i^{-1} g_i| + |l_d^{-1} h_d - l_i^{-1} h_d + l_i^{-1} h_d - l_i^{-1} h_i| \\ &\leq l_i^{-1} l_d^{-1} |l_d - l_i| \cdot |g_d| + l_i^{-1} |g_d - g_i| + l_i^{-1} l_d^{-1} |l_d - l_i| \cdot |h_d| + l_i^{-1} |h_d - h_i|. \end{aligned}$$

Since $g_d - g_i = \omega_e (x_{d,2} - x_{i,2})$, we have

$$|g_d - g_i| \leq \omega_e \|\mathbf{x}_d - \mathbf{x}_i\|.$$

Under Assumption 6.1, b_i^{-1} is bounded by b_{min}^{-1}, so l_i^{-1} and l_d^{-1} are also bounded by $(\omega_{\dot{e}} b_{min})^{-1}$. Since h_d and g_d are both bounded, we denote that $\bar{h}_d = \sup_{t \in [0,\ T]} h_d(t)$ and $\bar{g}_d = \sup_{t \in [0,\ T]} g_d(t)$. Using the *Lipschitz* condition described in Assumption 6.2 we can obtain

$$|\gamma_i| \leq c\|\mathbf{x}_d - \mathbf{x}_i\| \quad (6.15)$$

where

$$c = \omega_{\dot{e}} b_{min}^{-1} (\omega_{\dot{e}} l_b \omega_{\dot{e}} b_{min}^{-1} \bar{g}_d + \omega_{\dot{e}} + \omega_{\dot{e}} b_{min}^{-1} \omega_{\dot{e}} l_b \bar{h}_d + \omega_{\dot{e}} l_f)$$

which is a finite positive constant.

To facilitate FLLC analysis, we give three propositions which reveal the bound relationships among the quantities σ_i, $\mathbf{x}_i$, and γ_i.

Proposition 6.1. *For system (6.1), given the desired trajectory (6.2) and the FLLC laws (6.4) and (6.5), the following stands*

$$\dot{\mathbf{x}}_d - \dot{\mathbf{x}}_i = A(\mathbf{x}_d - \mathbf{x}_i) + \mathbf{p}\dot{\sigma}_i, \tag{6.16}$$

$$\|\mathbf{x}_d - \mathbf{x}_i\| \leq \omega_{\dot{e}}^{-1}\|A\| \int_0^t |\sigma_i(\tau)| e^{\|A\|(t-\tau)} d\tau + \omega_{\dot{e}}^{-1}|\sigma_i| \tag{6.17}$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & -\omega_{\dot{e}}^{-1}\omega_e \end{bmatrix}$ *and* $\mathbf{p} = \begin{bmatrix} 0 & \omega_{\dot{e}}^{-1} \end{bmatrix}^T$.

Proof:

Combining (6.1) and (6.2) yields

$$\dot{x}_{d,1} - \dot{x}_{1,i} = x_{d,2} - x_{2,i}. \tag{6.18}$$

Rearranging (6.10) gives

$$\dot{x}_{d,2} - \dot{x}_{2,i} = -\omega_{\dot{e}}^{-1}\omega_e(x_{d,2} - x_{2,i}) + \omega_{\dot{e}}^{-1}\dot{\sigma}_i. \tag{6.19}$$

Combining (6.18) and (6.19) gives (6.16). Integrating both sides of (6.16) and noticing $\sigma_i(0) = 0$ and $\mathbf{x}_i(0) = \mathbf{x}_d(0)$ obtain

$$\mathbf{x}_d - \mathbf{x}_i = A \int_0^t (\mathbf{x}_d - \mathbf{x}_i)\, d\tau + \mathbf{p}\sigma_i.$$

Taking the norm of the above and since $\|\mathbf{p}\| = \omega_{\dot{e}}^{-1}$, the following stands

$$\|\mathbf{x}_d - \mathbf{x}_i\| \leq \|A\| \int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau + \omega_{\dot{e}}^{-1}|\sigma_i|.$$

Applying *Bellman-Gronwell Lemma I* (Ioannou and Sun, 1996), we can obtain (6.17). ■

Proposition 6.2. *For system (6.1), given the desired trajectory (6.2) and the FLLC laws (6.4) and (6.5), the following stands*

$$\int_0^t e^{-\lambda\tau}|\sigma_i(\tau)| \cdot |\gamma_i(\tau)| d\tau \leq \left(c\omega_{\dot{e}}^{-1} + c\omega_{\dot{e}}^{-1}\|A\|Te^{\|A\|T}\right) \int_0^t e^{-\lambda\tau}\sigma_i^2(\tau) d\tau. \tag{6.20}$$

Proof:

It can be obtained from (6.15) and (6.17) that

$$|\gamma_i| \leq c\omega_{\dot{e}}^{-1}\|A\| \int_0^t |\sigma_i(\tau)| e^{\|A\|(t-\tau)} d\tau + c\omega_{\dot{e}}^{-1}|\sigma_i(t)|. \tag{6.21}$$

Since $0 \leq \nu \leq \tau \leq t \leq T$, then $0 \leq \tau - \nu \leq \tau \leq T$ and $-\frac{\lambda}{2}\tau \leq -\frac{\lambda}{2}\nu$. Using *Hölder inequality* (Ioannou and Sun, 1996), it can be obtained from (6.21) that

$$\begin{aligned}
&\int_0^t e^{-\lambda\tau}|\sigma_i(\tau)| \cdot |\gamma_i(\tau)| d\tau \\
&\leq \int_0^t \left[\int_0^\tau c\omega_{\dot{e}}^{-1}\|A\| e^{\|A\|(\tau-\nu)} e^{-\lambda\tau}|\sigma_i(\tau)| \cdot |\sigma_i(\nu)| d\nu\right] d\tau + \int_0^t c\omega_{\dot{e}}^{-1} e^{-\lambda\tau}\sigma_i^2(\tau) d\tau \\
&\leq c\omega_{\dot{e}}^{-1}\|A\| e^{\|A\|T} \int_0^t \left[\int_0^\tau e^{-\lambda\tau}|\sigma_i(\tau)| \cdot |\sigma_i(\nu)| d\nu\right] d\tau + \int_0^t c\omega_{\dot{e}}^{-1} e^{-\lambda\tau}\sigma_i^2(\tau) d\tau \\
&= c\omega_{\dot{e}}^{-1}\|A\| e^{\|A\|T} \int_0^t e^{-\frac{\lambda}{2}\tau}|\sigma_i(\tau)| \left[\int_0^t e^{-\frac{\lambda}{2}\tau}|\sigma_i(\nu)| d\nu\right] d\tau + \int_0^t c\omega_{\dot{e}}^{-1} e^{-\lambda\tau}\sigma_i^2(\tau) d\tau \\
&\leq c\omega_{\dot{e}}^{-1}\|A\| e^{\|A\|T} \int_0^t e^{-\frac{\lambda}{2}\tau}|\sigma_i(\tau)| \left[\int_0^t e^{-\frac{\lambda}{2}\nu}|\sigma_i(\nu)| d\nu\right] d\tau + \int_0^t c\omega_{\dot{e}}^{-1} e^{-\lambda\tau}\sigma_i^2(\tau) d\tau \\
&= c\omega_{\dot{e}}^{-1}\|A\| e^{\|A\|T} \left[\int_0^t e^{-\frac{\lambda}{2}\tau}|\sigma_i(\tau)| d\tau\right]^2 + \int_0^t c\omega_{\dot{e}}^{-1} e^{-\lambda\tau}\sigma_i^2(\tau) d\tau \\
&\leq c\omega_{\dot{e}}^{-1}\|A\| e^{\|A\|T} \left[\int_0^t e^{-\lambda\tau}\sigma_i^2(\tau) d\tau\right] \left[\int_0^t 1^2 d\tau\right] + \int_0^t c\omega_{\dot{e}}^{-1} e^{-\lambda\tau}\sigma_i^2(\tau) d\tau \\
&\leq \left(c\omega_{\dot{e}}^{-1} + c\omega_{\dot{e}}^{-1}\|A\| T e^{\|A\|T}\right) \int_0^t e^{-\lambda\tau}\sigma_i^2(\tau) d\tau.
\end{aligned}$$

■

Proposition 6.3. *For system (6.1), given the desired trajectory (6.2) and under the control laws (6.4) and (6.5), the following stands*

$$\|\mathbf{x}_i - \mathbf{x}_d\| \leq b_{max} e^{lT} T^{\frac{1}{2}} J_i^{\frac{1}{2}}(T), \tag{6.22}$$

$$|\sigma_i| \leq b_{max}(\omega_e^2 + \omega_{\dot{e}}^2)^{\frac{1}{2}} e^{lT} T^{\frac{1}{2}} J_i^{\frac{1}{2}}(T). \tag{6.23}$$

where $l \triangleq max(\lambda, \|A\| + b_{max}c)$.

Proof:

From (6.12), it can be obtained that

$$\dot{\sigma}_i = l_i u_d - l_i u_i - l_i \gamma_i.$$

Substituting the above into (6.16) yields

$$\dot{\mathbf{x}}_d - \dot{\mathbf{x}}_i = A(\mathbf{x}_d - \mathbf{x}_i) + \mathbf{p}(l_i u_d - l_i u_i - l_i \gamma_i)$$

Since $\mathbf{x}_i(0) = \mathbf{x}_d(0)$, $\|\mathbf{p}\| = \omega_{\dot{e}}^{-1}$ and from Assumption 6.1 $l_i \leq \omega_{\dot{e}} b_{max}$, it can be obtained from the above that

$$\begin{aligned} \|\mathbf{x}_d - \mathbf{x}_i\| \;\leq\; & \|A\| \int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau + b_{max} \int_0^t |u_i - u_d| d\tau \\ & + b_{max} \int_0^t |\gamma_i| d\tau \end{aligned} \tag{6.24}$$

Substituting (6.15) into (6.24) yields

$$\|\mathbf{x}_d - \mathbf{x}_i\| \leq (\|A\| + b_{max} c) \int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau + b_{max} \int_0^t |u_{l,i} - u_d| d\tau.$$

It can be obtained by the *Hölder inequality* and *Bellman-Gronwall Lemma II* (Ioannou and Sun, 1996) that

$$\begin{aligned} \|\mathbf{x}_d - \mathbf{x}_i\| &\leq l_1 \int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau + b_{max} \int_0^t |u_{l,i} - u_d| d\tau \\ &\leq \int_0^t b_{max} e^{l(t-\tau)} |u_{l,i} - u_d| d\tau \\ &\leq b_{max} e^{lT} \int_0^T e^{-l\tau} |u_{l,i} - u_d| d\tau \\ &\leq b_{max} e^{lT} \left[\int_0^T e^{-2l\tau} (u_{l,i} - u_d)^2 d\tau \right]^{\frac{1}{2}} \left[\int_0^T 1^2 d\tau \right]^{\frac{1}{2}} \\ &\leq b_{max} e^{lT} T^{\frac{1}{2}} \left[\int_0^T e^{-\lambda\tau} (u_{l,i} - u_d)^2 d\tau \right]^{\frac{1}{2}} \\ &= b_{max} e^{lT} T^{\frac{1}{2}} J_i^{\frac{1}{2}}(T) \end{aligned} \tag{6.25}$$

where $l_1 = \|A\| + b_{max} c$ and $l \stackrel{\triangle}{=} max(\lambda, l_1)$. From (6.5) we have

$$\sigma_i = \begin{bmatrix} \omega_e & \omega_{\dot{e}} \end{bmatrix}^T (\mathbf{x}_d - \mathbf{x}_i) \tag{6.26}$$

$$|\sigma_i| \leq (\omega_e^2 + \omega_{\dot{e}}^2)^{\frac{1}{2}} \|\mathbf{x}_d - \mathbf{x}_i\| \tag{6.27}$$

Hence from (6.25) and the above, we can obtain (6.23) which completes the proof.

■

In the following, we first give the convergence property of the proposed FLLC when the FLC part works within the unsaturated region.

Theorem 6.1. *Consider the nonlinear system (6.1) satisfying assumptions Assumption 6.1 and Assumption 6.2, together with the desired trajectory* $\mathbf{x}_d$ *defined in (6.2). Under the control laws (6.4) and (6.5), as* $i \to \infty$, u_i *converges to* u_d *almost everywhere,* σ_i *converges uniformly to 0 and* $\mathbf{x}_i$ *converges uniformly to* $\mathbf{x}_d$*.*

Proof:

Substituting (6.13) into (6.8) gives

$$\Delta J_i(t) \leq \int_0^t e^{-\lambda\tau}(-u_{f,i}^2 - 2u_{f,i}l_i^{-1}\dot{\sigma}_i - 2u_{f,i}\gamma_i)d\tau.$$

Then

$$\begin{aligned}\Delta J_i(t) &\leq \int_0^t e^{-\lambda\tau}(-2k_i l_i^{-1}\sigma_i\dot{\sigma}_i - 2k_i\sigma_i\gamma_i)d\tau \\ &= -\int_0^t 2e^{-\lambda\tau}k_i l_i^{-1}\sigma_i\dot{\sigma}_i d\tau - \int_0^t 2e^{-\lambda\tau}k_i\sigma_i\gamma_i d\tau.\end{aligned}$$

Since $b(x_1,t)$ is bounded, $l_i \in [\omega_{\dot{e}}b_{min}, \omega_{\dot{e}}b_{max}]$,

$$\begin{aligned}\Delta J_i(t) &\leq -k_{min}(\omega_{\dot{e}}b_{max})^{-1}\int_0^{\sigma_i^2(t)} e^{-\lambda t}d\sigma_i^2 + 2k_{max}\int_0^t e^{-\lambda\tau}|\sigma_i||\gamma_i|d\tau \\ &\leq -k_{min}(\omega_{\dot{e}}b_{max})^{-1}e^{-\lambda t}\sigma_i^2 - \lambda k_{min}(\omega_{\dot{e}}b_{max})^{-1}\int_0^t e^{-\lambda\tau}\sigma_i^2 d\tau \\ &\quad +2k_{max}\int_0^t e^{-\lambda\tau}|\sigma_i||\gamma_i|d\tau.\end{aligned}$$

Using Proposition 6.2, we can derive

$$\begin{aligned}\Delta J_i(t) &\leq -k_{min}(\omega_{\dot{e}}b_{max})^{-1}e^{-\lambda t}\sigma_i^2 - \lambda k_{min}(\omega_{\dot{e}}b_{max})^{-1}\int_0^t e^{-\lambda\tau}\sigma_i^2 d\tau \\ &\quad +2k_{max}(c\omega_{\dot{e}}^{-1} + c\omega_{\dot{e}}^{-1}\|A\|T_f e^{\|A\|T})\int_0^t e^{-\lambda\tau}\sigma_i^2(\tau)d\tau \\ &= -k_{min}(\omega_{\dot{e}}b_{max})^{-1}e^{-\lambda t}\sigma_i^2 \\ &\quad -k_{min}(\omega_{\dot{e}}b_{max})^{-1}\int_0^t [\lambda - 2k_{max}k_{min}^{-1}b_{max}(c + c\|A\|Te^{\|A\|T})]e^{-\lambda\tau}\sigma_i^2(\tau)d\tau.\end{aligned}$$

Since $2k_{max}k_{min}^{-1}b_{max}(c + c\|A\|Te^{\|A\|T})$ is a finite positive constant, there exists a sufficiently large λ such that $\lambda \geq 2k_{max}k_{min}^{-1}b_{max}(c+c\|A\|Te^{\|A\|T}) + k_{min}^{-1}(\omega_{\dot{e}}b_{max})$ to ensure

$$\Delta J_i(t) \leq -k_{min}(\omega_{\dot{e}}b_{max})^{-1}e^{-\lambda t}\sigma_i^2 - \int_0^t e^{-\lambda\tau}\sigma_i^2 d\tau. \tag{6.28}$$

According to (6.7), $J_i(t) \geq 0$, then from (6.28) we have

$$0 \leq J_i(t) \leq J_{i-1}(t) \leq \cdots \leq J_1(t).$$

From (6.28), taking the summation over $j = 1$ to i obtains

$$J_i(t) - J_1(t) \leq -k_{min}(\omega_{\dot{e}}b_{max})^{-1}e^{-\lambda t}\sum_{j=1}^{i}\sigma_j^2(t)$$

As $J_i \geq 0$, we have from the above that

$$\lim_{i\to\infty}\sum_{j=1}^{i}\sigma_j^2(t) \quad \leq \quad {k_{min}}^{-1}(\omega_{\dot{e}}b_{max})e^{\lambda t}J_1(t)$$

which concludes that

$$\lim_{i\to\infty}\sigma_i(t) = 0, \forall t \in [0, T].$$

As $\lim_{i\to\infty}\sigma_i(t) = 0$, from (6.5) and (6.17), $\lim_{i\to\infty}u_{f,i} = 0$ and $\lim_{i\to\infty}\mathbf{x}_i = \mathbf{x}_d$. According to (6.15), $\lim_{i\to\infty}\gamma_i = 0$.

From (6.12) and (6.7), it can be obtained

$$\begin{aligned}
\lim_{i\to\infty}J_i(t) &= \lim_{i\to\infty}\int_0^t e^{-\lambda\tau}[u_i(\tau) - u_d(\tau)]^2 d\tau \\
&= \lim_{i\to\infty}\int_0^t e^{-\lambda\tau}(\omega_{\dot{e}}b_i)^{-2}\dot{\sigma}_i^2 d\tau \\
&= \lim_{i\to\infty}\int_0^{\sigma(t)} e^{-\lambda\tau}(\omega_{\dot{e}}b_i)^{-2}\dot{\sigma}_i d\sigma_i.
\end{aligned} \tag{6.29}$$

From (6.5) we can obtain

$$\begin{aligned}
\dot{\sigma}_i &= \omega_e\dot{e}_i + \omega_{\dot{e}}\ddot{e}_i \\
&= \omega_e(\dot{x}_{d,1} - \dot{x}_{1,i}) + \omega_{\dot{e}}(\dot{x}_{d,2} - \dot{x}_{2,i}) \\
&= \omega_e(x_{d,2} - x_{2,i}) + \omega_{\dot{e}}(\dot{x}_{d,2} - f_i - bu_i).
\end{aligned} \tag{6.30}$$

As u_i is bounded, considering Proposition 6.3, $\dot{\sigma}_i$ is bounded. Since $e^{-\lambda\tau}(\omega_{\dot{e}} b_i)^{-2}\dot{\sigma}_i$ is bounded and $\lim_{i\to\infty} \sigma_i(t) = 0$, we can obtain

$$\lim_{i\to\infty} J_i(t) = 0, \forall t \in [0, T] \tag{6.31}$$

and u_i converges to u_d almost everywhere.

From Proposition 6.3 and (6.31), both $\mathbf{x}_i$ and σ_i are bounded. We have

$$\lim_{i\to\infty} \sup_{t\in[0,\ T]} |\sigma_i| = 0, \quad \lim_{i\to\infty} \sup_{t\in[0,\ T]} \|\mathbf{x}_d - \mathbf{x}_i\| = 0,$$

σ_i and $\mathbf{x}_i$ are uniformly convergent.

From (6.5) and $e_i(0) = 0$, by solving the differential equation (6.1) with the FLLC, we can reach that e_i and $\dot{e}_i$ uniformly converge to zero as $i \to \infty$. ■

Now let us consider the circumstances where FLC may enter its saturated or semi-saturated regions. Note that $k(e, \dot{e})$ is undefined or is zero where $(\omega_e e > 1) \cap (\omega_{\dot{e}} \dot{e} < -1)$ or $(\omega_e e < -1) \cap (\omega_{\dot{e}} \dot{e} > 1)$ because of the null control action in these two regions. Nevertheless, we can still prove that the FLC part will re-enter and remain in the unsaturated region after finite iterations. Consequently, FLLC will converge uniformly as $i \to \infty$.

Theorem 6.2. *In the presence of FLC saturation, consider the nonlinear system (6.1) satisfying Assumption 6.1 and Assumption 6.2, together with the desired trajectory $\mathbf{x}_d$ defined in (6.2). Under the control laws (6.4) and (6.5), as $i \to \infty$, u_i converges to u_d almost everywhere, σ_i converges uniformly to 0 and $\mathbf{x}_i$ converges uniformly to $\mathbf{x}_d$.*

Proof:

Rewrite the system (6.1) as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t)u$$

where $\mathbf{f}(\mathbf{x}, t) = [x_2 \;\; f(\mathbf{x}, t)]^T$ and $\mathbf{b}(\mathbf{x}, t) = [0 \;\; b(x_1, t)]^T$. The desired system is

$$\dot{\mathbf{x}}_d = \mathbf{f}(\mathbf{x}_d, t) + \mathbf{b}(\mathbf{x}_d, t)u_d.$$

Then

$$\begin{aligned} \dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}_d - \dot{\mathbf{x}} \\ &= \mathbf{f}(\mathbf{x}_d, t) - \mathbf{f}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}_d, t)u_d - \mathbf{b}(\mathbf{x}, t)u \end{aligned}$$

where $\mathbf{e} = [e, \dot{e}]^T \triangleq [e_1, e_2]^T$.

From Proposition 6.3 we can obtain

$$\begin{aligned} \|\mathbf{e}\| &= \|\mathbf{x}_d - \mathbf{x}\| \\ &\leq b_{max} e^{lT} T^{\frac{1}{2}} J_i^{\frac{1}{2}}(T) \end{aligned}$$

where $J_i^{\frac{1}{2}}(T)$ is bounded according to (6.4) and (6.7).

From (6.1), it can be derived

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= f(\mathbf{x}_d, t) - f(\mathbf{x}, t) + b(x_{d,1}, t)u_d - b(x_1, t)u. \end{aligned}$$

Then

$$\begin{aligned} |\dot{e}_1| &= |e_2| \\ &\leq \|\mathbf{e}(t)\| \\ &\leq b_{max} e^{lT} T^{\frac{1}{2}} J_i^{\frac{1}{2}}(T) && (6.32) \\ |\dot{e}_2| &\leq L_{\mathbf{f}} \|\mathbf{e}(t)\| + 2b_{max}u_{max} \\ &\leq 2L_{\mathbf{f}} b_{max} e^{lT} T^{\frac{1}{2}} J_i^{\frac{1}{2}}(T) + 2b_{max}u_{max} && (6.33) \end{aligned}$$

where $u_{max} = u_M + u_F$ is the bound of the system input and $L_{\mathbf{f}}$ is the Lipschitz constant of $\mathbf{f}(\mathbf{x}, t)$. The relations (6.32) and (6.33) show that both e_1 and e_2, in the worst case, have finite divergent speed.

As the initial state $\mathbf{e}_i(0) = \mathbf{0}$ is available for all the trials and its divergent speed is limited, at least one non-infinitesimal time interval $[0, T_1]$ exists such that FLC works in the unsaturated region $(|\omega_e e_1| < 1) \cap (|\omega_{\dot{e}} e_2| < 1)$, i.e.

$$T_1 = Min(t_1, \tau_1)$$

where

$$\begin{aligned} t_1 &= \frac{1-0}{\omega_e |\dot{e}_1|_{max}} \\ \tau_1 &= \frac{1-0}{\omega_{\dot{e}} |\dot{e}_2|_{max}}. \end{aligned}$$

According to Theorem 6.1, $\forall t \in [0, T_1]$, $e_{1,i}(t)$ and $e_{2,i}(t)$ uniformly converge to zero as $i \to \infty$. It means there always exists a non-infinitesimal quantity ϵ_1 and a finite integer N_1, such that $|\omega_e e_{1,i}(t)| < \epsilon_1 << 1$ and $|\omega_{\dot{e}} e_{2,i}(t)| < \epsilon_1 << 1$ ($\forall t \in [0, T_1]$) when iteration number $i > N_1$.

Analogously, there exits another time interval $[T_1, T_2]$ such that FLC works in the unsaturated region for any iteration $i > N_1$.

$$T_2 = T_1 + Min(t_2, \tau_2)$$

where

$$\begin{aligned} t_2 &= \frac{1-\epsilon_1}{\omega_e |\dot{e}_1|_{max}} \\ \tau_2 &= \frac{1-\epsilon_1}{\omega_{\dot{e}} |\dot{e}_2|_{max}}. \end{aligned}$$

Applying Theorem 6.1 again, we can obtain $\forall t \in [0, T_2]$, $e_{1,i}(t)$ and $e_{2,i}(t)$ uniformly converge to zero as $i \to \infty$. In other words, there always exists a non-infinitesimal quantity ϵ_2 and a finite integer N_2, such that $|\omega_e e_{1,i}(t)| < \epsilon_2 << 1$ and $|\omega_{\dot{e}} e_{2,i}(t)| < \epsilon_2 << 1$ ($\forall t \in [0, T_2]$) when iteration time $i > N_2$.

Since $[0, T]$ is a finite interval, by repeating the above procedure for finite times K, a time interval $[T_{K-1}, T_K]$ ($T_K \geq T$) can be found in which the FLC works in the unsaturated region.

$$T_K = T_{K-1} + Min(t_K, \tau_K)$$

where

$$\begin{aligned} t_K &= \frac{1-\epsilon_{K-1}}{\omega_e |\dot{e}_1|_{max}} \\ \tau_K &= \frac{1-\epsilon_{K-1}}{\omega_{\dot{e}} |\dot{e}_2|_{max}}. \end{aligned}$$

Eventually the FLC will work in the unsaturated region over the whole cycle $[0, T]$, because each interval $[T_j, T_{j+1}]$ $(j = 1, \cdots, K)$ is a non-infinitesimal interval. According to Theorem 6.1, $\forall t \in [0, T]$, $e_{1,i}(t)$ and $e_{2,i}(t)$ uniformally converge to zero as $i \to \infty$.

From the above derivation, it can be clearly seen that even if FLC works in the saturated region during some iterations, the FLC part will re-enter the unsaturated region and remain in it. As $i \to \infty$, we can still derive that u_i converges to u_d almost everywhere, σ_i converges uniformly to 0 and $\mathbf{x}_i$ converges uniformly to $\mathbf{x}_d$. ■

6.5 Illustrative Examples

In this section, the FLLC will be applied to a simple nonlinear mass-spring-damper mechanical system (Wang *et al.*, 1996) as shown in Fig. 6.4. The behavior of this system can be described by

$$\begin{aligned} M\ddot{x} + g(x, \dot{x}) + f(x) &= \phi(\dot{x})u \qquad (6.34) \\ g(x, \dot{x}) &= D(c_1 x + c_2 \dot{x} + c_3 \dot{x}^3) \\ f(x) &= c_4 x + c_5 x^3 \\ \phi(\dot{x}) &= 1 + c_6 \dot{x} + c_7 \dot{x}^3 + c_8 sin \dot{x} \end{aligned}$$

where M is the mass and u is the force. $f(x)$, $g(x, \dot{x})$ and $\phi(\dot{x})$ describe the spring, the damper and the input nonlinearity and uncertainty respectively. The control

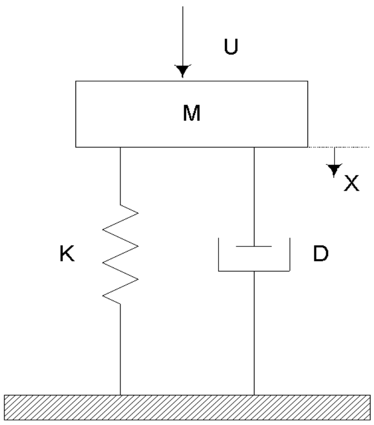

Figure 6.4: Mass-spring-damper system.

task is to track the desired trajectory

$$x_d = 1.728 \times sin^3(0.7t) \quad t \in [0, 9].$$

Case 1

The system parameters are set to be: $M = 1.0$, $D = 1.0$, $c_1 = 0.01$, $c_2 = 0.1$, $c_3 = 0$, $c_4 = 0.01$, $c_5 = 0$, $c_6 = 0.01$, $c_7 = 0$, $c_8 = -0.01$. The plant (6.34) can be rewritten as

$$\ddot{x} = -0.1\dot{x} - 0.02x + (1 - 0.01sin\dot{x} + 0.01\dot{x})u. \tag{6.35}$$

Consider the FLC described in Section 4.3. Since there is no systematic way to fine tune the three FLC parameters (ω_e, $\omega_{\dot{e}}$, ω_u), for demonstration purpose six sets of parameters are randomly chosen within the range of $[4, 8]$. The FIL is further added to the FLC to improve the tracking performance. To demonstrate the effectiveness of the proposed FLLC, the maximum tracking error of each iteration (e_{max}) is recorded and shown in Table 6.2. We can see that the incorporation of FIL can dramatically reduce the tracking error even if only one iteration is performed. After a number of learning iterations, the maximum tracking error can be reduced to less than 0.001 regardless of the FLC parameters settings.

Case 2

Table 6.2: Comparison of FLLC with different FLC parameters (*Case 1).*

ω_e	$\omega_{\dot{e}}$	ω_u	FLC Error	$e_{max}(i=1)$	$e_{max}(i=2)$	$e_{max}\ (i=3)$	$e_{max}\ (i=4)$	i ($e_{max} < 10^{-3}$)
5	5	4	0.1678	0.0423	0.0161	0.0073	0.0038	12
6	6	5	0.1035	0.0175	0.0041	0.0019	0.0010	5
7	7	6	0.0791	0.0097	0.0017	0.0007	0.0006	3
8	8	7	0.0624	0.0057	0.0008	0.0003	0.0002	2
4	4	8	0.1152	0.0186	0.0040	0.0017	0.0008	4
8	8	4	0.1028	0.0174	0.0054	0.0022	0.0011	5

The system parameters are chosen to be: $M = 1.0$, $D = 1.0$, $c_1 = 0.01$, $c_2 = 0.1$, $c_3 = 0.15$, $c_4 = 0.01$, $c_5 = 0.1$, $c_6 = 0.01$, $c_7 = 0$, $c_8 = -0.6$. The plant (6.34) can be rewritten as

$$\ddot{x} = -0.1\dot{x} - 0.02x - 0.15\dot{x}^3 - 0.1x^3 + (1 - 0.6sin\dot{x} + 0.01\dot{x})u.$$

Applying FLLC with the same parameters as in *Case 1*, the tracking control results are summarized in Table 6.3.

The FLLC can work equally well in the presence of stronger nonlinearities.

Case 3

From Table 6.2 and Table 6.3, we can observe that the larger the (ω_e, $\omega_{\dot{e}}$, ω_u), the smaller the FLC tracking error. However, it is not advisable to reduce the tracking error only through increasing the FLC gains. Due to the discrete-time control nature, the FLC gains are limited by the system sampling period. Again consider the plant given in *Case 1*, but with a larger sampling period of $10ms$. Choosing $\omega_e = 7$, $\omega_{\dot{e}} = 7$, $\omega_u = 5$ and applying FLLC, Fig. 6.5 shows the control signal and the tracking error after six iterations.

Table 6.3: Comparison of FLLC with different FLC parameters (*Case 2*).

ω_e	$\omega_{\dot{e}}$	ω_u	FLC Error	$e_{max}(i=1)$	$e_{max}(i=2)$	e_{max} $(i=3)$	e_{max} $(i=4)$	i ($e_{max} < 10^{-3}$)
5	5	4	0.2061	0.0700	0.0395	0.0241	0.0145	33
6	6	5	0.1400	0.0429	0.0208	0.0106	0.0054	12
7	7	6	0.1077	0.0251	0.0109	0.0048	0.0024	8
8	8	7	0.0854	0.0139	0.0062	0.0023	0.0014	6
4	4	8	0.1528	0.0305	0.0215	0.0111	0.0067	16
8	8	4	0.1399	0.0418	0.0178	0.0080	0.0046	11

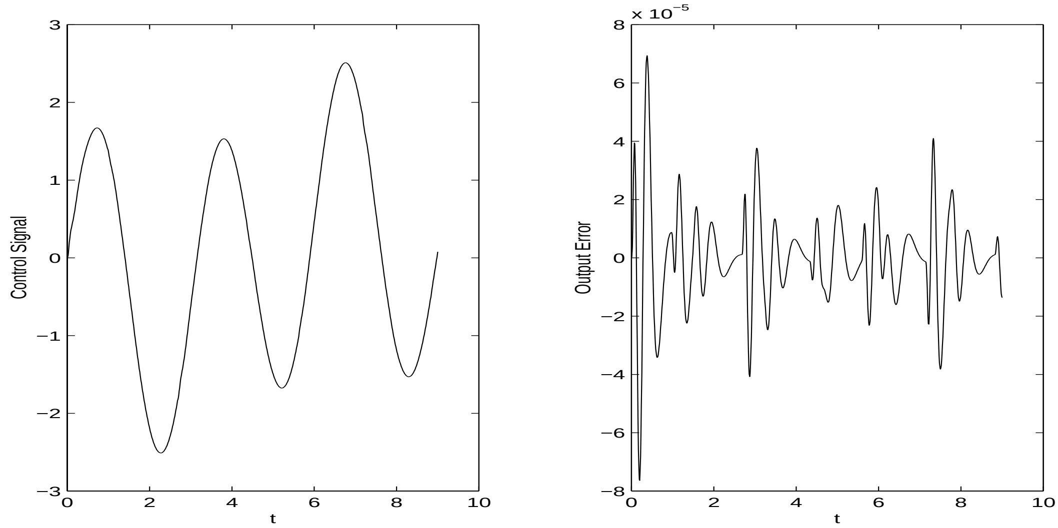

Figure 6.5: Control signal and output error of FLLC with low gain.

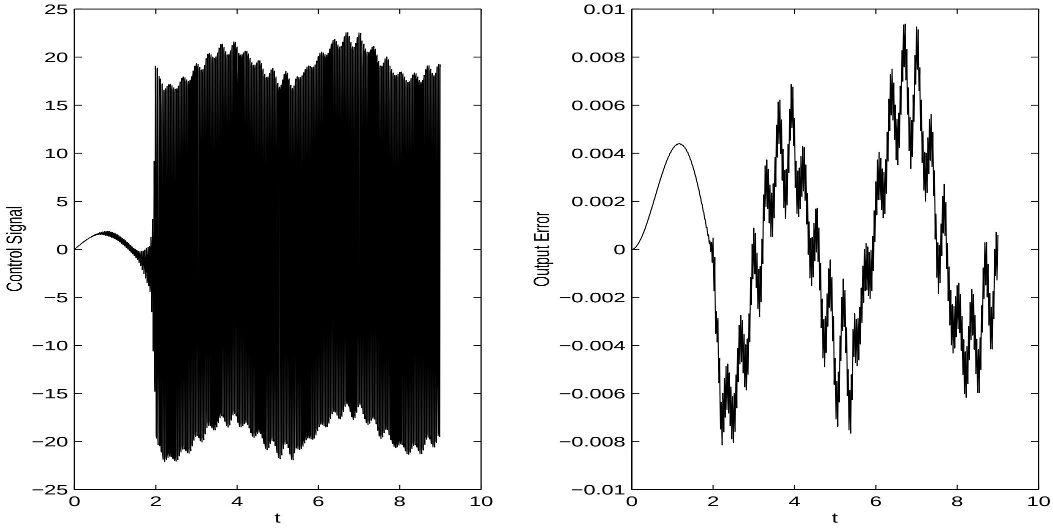

Figure 6.6: Control signal and output error of FLC with high gain.

For comparison purpose, choosing higher FLC gains $\omega_e = 15$, $\omega_{\dot{e}} = 15$, $\omega_u = 20$ and only applying FLC, Fig. 6.6 shows the control signal and the tracking error.

Due to large (ω_e, $\omega_{\dot{e}}$, ω_u), control chattering phenomenon occurs (Fig. 6.6), yet the tracking error is about 100 times larger than that of FLLC. Obviously, in practice it is difficult for such a simple FLC to obtain accurate tracking performance. FLLC, on the other hand, can obtain much better tracking performance and much smoother control profiles with only a few iterations.

6.6 Conclusion

In this chapter, a novel control scheme - Fuzzy Logic Learning Control (FLLC) is proposed for repeatable tracking control tasks. The new FLLC is constructed in an add-on fashion: FIL mechanism is added to the existing FLC without changing the FLC structure and settings. Both theoretical analysis and simulations show that the FLLC method possesses the capability of improving control performance through learning iterations. Through rigorous proof, we reach the conclusion that, by means of the proposed FLLC, the tracking error uniformly converges to zero, the system states converge to the desired trajectory and the learning control profile converges to the desired one almost everywhere.

Chapter 7

IIL for Systems with Parametric Uncertainties

7.1 Introduction

In Chapters 2– 6, theories of FIL have been discussed and several FIL schemes have been proposed. From this chapter, the learning over finite time interval $[0, T]$ is extended to $[0, \infty)$.

Adaptive control is a systematical design method for nonlinear systems with time-invariant parametric uncertainties. Based on a parametric adaptation mechanism, the asymptotic tracking convergence can be guaranteed in the presence of constant parametric uncertainties. However, it is difficult to extend the traditional adaptive control into nonlinear systems with time-varying parametric uncertainties. RC is an effective way to handle systems with periodic time-varying uncertainties. However, most of the RC schemes are only applicable to linear/linearizable systems and the requirement for the periodicity of the control target is essential.

If the parametric uncertainties are periodic, can we find a novel learning algorithm by taking the advantage of the repeatability property?

Based on the concept of CEF, a new IIL approach is developed in this chapter for systems with time-varying but periodic parametric uncertainties. Through rigorous proof, it will be shown that, only if the periodicity of the time-varying parametric uncertainty is known *a priori*, the learning convergence can be obtained, even if the desired tracking trajectory is non-periodic.

Comparing with CM-type IIL, i.e. RC, the new IIL can be applied to systems with NGLC nonlinearities and the control target can be periodic or non-periodic. Hence, the developed IIL scheme greatly widens the application area of learning control.

This chapter is organized as follows. In Section 7.2, a CEF-type IIL scheme for a class of SISO system with parametric uncertainty is first analyzed. Based on it, the IIL approach is further extended to high-order MIMO systems in Section 7.3. Illustrative examples are given in Section 7.4. Finally, Section 7.5 draws the conclusion.

7.2 IIL for SISO Systems with Parametric Uncertainties

Consider the following simple dynamic system:

$$\dot{x} = u(t) + \theta(t)\xi(x,t), \qquad x(0) = x_0 \tag{7.1}$$

where $\theta(t) \in \mathcal{C}^0(\mathcal{R}, [0,\infty))$ is an unknown time-varying parametric uncertainty and $\xi(x,t) : \mathcal{R} \times \mathcal{R}_+ \to \mathcal{R}$ is a known function which may be GLC or NGLC.

It is assumed that the system uncertainty $\theta(t)$ satisfies the following assumption.

Assumption 7.1. The system uncertainty $\theta(t)$ is periodic with a known period of

T, i.e. $\theta(t) = \theta(t-T)$.

The ultimate control objective is to find an appropriate control signal $u(t)$, such that the system state x converges to the desired trajectory $x_d \in \mathcal{C}^1(\mathcal{R}, [0, \infty))$ in $\mathcal{L}_T^2$ norm, where the $\mathcal{L}_T^2$ norm of a function $h(t)$, $t \in [t-T, t)$, is defined as $\int_{t-T}^{t} |h(\tau)|^2 d\tau$. Note that the desired trajectory x_d can be non-periodic.

The CEF-type IIL is constructed as follows.

$$u = ke + \dot{x}_d - \hat{\theta}(t)\xi \tag{7.2}$$

$$\hat{\theta}(t) = \begin{cases} -\gamma_0(t)\xi e(t) & t \in [0, T) \\ \hat{\theta}(t-T) - \gamma\xi e(t) & t \in [T, \infty) \end{cases}, \tag{7.3}$$

where $\xi = \xi(x, t)$, $\gamma > 0$ is a constant learning gain for $t \geq T$ and $\gamma_0(t)$ is a continuous and strictly increasing function satisfying $\gamma_0(0) = 0$ and $\gamma_0(T) = \gamma$. The special design for the learning gain of the first period, $\gamma_0(t)$, is to ensure the continuity of $\hat{\theta}(t)$ in the neighborhoods centered around $t = iT$ where $i \in \mathcal{Z}_+$.

Substituting the control law (7.2) into the dynamics (7.1) yields the error dynamics

$$\begin{aligned} \dot{e} &= \dot{x}_d - \theta\xi - u \\ &= -ke - \phi\xi \\ e(0) &= x_d(0) - x_0, \end{aligned} \tag{7.4}$$

where $\phi = \theta - \hat{\theta}$.

The main result for the proposed IIL approach is summarized in the following theorem.

Theorem 7.1. *For system (7.1), under Assumption 7.1, the control laws (7.2) and (7.3) ensure that the tracking error $e(t)$ converges to 0 in the sense of $\mathcal{L}_T^2$ norm.*

Proof:

The following CEF will be adopted to analyze the learning convergence:

$$E(t) = \frac{1}{2}e^2(t) + \frac{1}{2\gamma}\int_{t-T}^{t} \phi^2(\tau)d\tau. \tag{7.5}$$

The proof contains three parts to address the difference of the defined CEF, the learning convergence and the boundedness property of the controlled system respectively.

(I) Difference of CEF

Let us first derive the difference of the CEF over one period for any $t \geq T$.

$$\begin{aligned} \Delta E(t) &\overset{\triangle}{=} E(t) - E(t-T) \\ &= \frac{1}{2}e^2(t) - \frac{1}{2}e^2(t-T) + \frac{1}{2\gamma}\int_{t-T}^{t}[\phi^2(\tau) - \phi^2(\tau - T)]d\tau. \end{aligned} \tag{7.6}$$

Looking into the first two term on the RHS of the ΔE and using the error dynamics (7.4), we have

$$\begin{aligned} \frac{1}{2}e^2(t) - \frac{1}{2}e^2(t-T) &= \int_{t-T}^{t} e\dot{e}d\tau \\ &= \int_{t-T}^{t}(-ke^2 - \phi\xi e)d\tau. \end{aligned} \tag{7.7}$$

Using the algebraic relationship $(a-b)^2 - (a-c)^2 = (c-b)[2(a-b) + (b-c)]$ and the periodicity $\theta(t) = \theta(t-T)$, by substituting the parameter updating law (7.3), the third term on the RHS of (7.6) can be expressed as

$$\begin{aligned} &\frac{1}{2\gamma}\int_{t-T}^{t}[\phi^2(\tau) - \phi^2(\tau - T)]d\tau \\ &= \frac{1}{2\gamma}\int_{t-T}^{t}[\hat{\theta}(\tau - T) - \hat{\theta}(\tau)]\{2[\theta(\tau) - \hat{\theta}(\tau)] + \hat{\theta}(\tau) - \hat{\theta}(\tau - T)\}d\tau \\ &= \int_{t-T}^{t}[\phi(\tau)\xi(\tau)e(\tau) - \frac{\gamma}{2}\xi^2(\tau)e^2(\tau)]d\tau. \end{aligned} \tag{7.8}$$

Substituting (7.7) and (7.8) into (7.6) yields

$$\Delta E(t) = -k\int_{t-T}^{t} e^2 d\tau - \frac{\gamma}{2}\int_{t-T}^{t}\xi^2 e^2 d\tau \leq -k\int_{t-T}^{t} e^2 d\tau \leq 0. \tag{7.9}$$

(II) Convergence Property

Applying (7.9) repeatedly for any $t \in [nT, (n+1)T]$, and denoting $t_0 = t - nT$, we have

$$E(t) = E(t_0) + \sum_{i=0}^{n-1} \Delta E(t - iT)$$

and

$$\lim_{t\to\infty} E(t) < E(t_0) - \lim_{n\to\infty} k \sum_{i=0}^{n-1} \int_{t-(i+1)T}^{t-iT} e^2 d\tau. \tag{7.10}$$

Considering the positiveness of $E(t)$, if $E(t_0)$ is bounded, the tracking error $e(t)$ converges to zero asymptotically in $\mathcal{L}_T^2$ norm, i.e.

$$\lim_{t\to\infty} \int_{t-T}^{t} e^2 d\tau = 0.$$

(III) Boundedness Property

Now we will prove the finiteness of $E(t_0)$. The finiteness property is necessary, as $\xi(x,t)$ may be a local Lipschitz continuous function and finite escape time phenomenon may occur.

From the system dynamics (7.1) and the proposed control laws (7.2) and (7.3), it can be derived that the RHS of (7.1) is continuous with respect to all the arguments. According to the existence theorem of differential equation (Yoshizawa, 1996), there exists a solution in an interval $[0, T_1) \subset [0, T)$, where T_1 is not infinitesimal. Therefore, the boundedness of $E(t)$ over $[0, T_1]$ can be guaranteed and we need only focus on the interval (T_1, T).

For any $t \in [T_1, T)$, the derivative of $E(t)$ is

$$\dot{E} = e\dot{e} + \frac{1}{2\gamma}\phi^2 = -ke^2 - \phi\xi e + \frac{1}{2\gamma}\phi^2. \tag{7.11}$$

Since $\gamma_0(t)$ is strictly increasing in $[0, T)$, $\frac{1}{\gamma} \leq \frac{1}{\gamma_0(t)}$ is ensured in the time interval

$[T_1, T)$. Therefore, by substituting the learning law (7.3), we have

$$\begin{aligned}\frac{1}{2\gamma}\phi^2 &\leq \frac{1}{2\gamma_0(t)}\phi^2 \\ &= \frac{1}{2\gamma_0(t)}\theta^2 - \frac{1}{\gamma_0(t)}\hat{\theta}\phi - \frac{1}{2\gamma_0(t)}\hat{\theta}^2 \\ &\leq \frac{1}{2\gamma_0(t)}\theta^2 - \phi\xi e. \end{aligned} \tag{7.12}$$

Substituting (7.12) into (7.11) yields

$$\dot{E} \leq -ke^2 + \frac{1}{2\gamma_0(t)}\theta^2. \tag{7.13}$$

The boundedness of θ leads to the boundedness of $\dot{E}$. As $E(T_1)$ is finite, $\forall t \in (T_1, T)$ the boundedness of $E(t)$ is obvious.

From (7.10), as $E(t_0)$ is bounded, $E(t)$ is finite for any $t \in [0, \infty)$, which implies the boundedness of $e(t)$ and the $\mathcal{L}_T^2$ boundedness of $\hat{\theta}(t)$. Hence, the boundedness of $x(t)$ and the $\mathcal{L}_T^2$ boundedness of the control input $u(t)$ can be derived. ∎

7.3 IIL for MIMO Systems with Parametric Uncertainties

7.3.1 Problem Formulation

Consider a high-order MIMO nonlinear dynamic system described by

$$\begin{aligned}\dot{\mathbf{x}}_j &= \mathbf{x}_{j+1}, \quad j = 1, \cdots, m-1 \\ \dot{\mathbf{x}}_m &= \mathbf{f}(\mathbf{x}, t) + B(\mathbf{x}, t)[\mathbf{u}(t) + \mathbf{d}_1(\mathbf{x}, t)] \end{aligned} \tag{7.14}$$

where $\mathbf{x}_j \in \mathcal{R}^n$, $j = 1, \cdots, m$; $\mathbf{x} \triangleq [\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_m^T]^T \in \mathcal{R}^{nm}$ is the physically measurable state vector of the system; $\mathbf{u} \in \mathcal{R}^n$ is the control input vector of the system; $B(\mathbf{x}, t) : \mathcal{R}^{nm} \times \mathcal{R}_+ \to \mathcal{R}^{n \times n}$ is a known function with full rank; $\mathbf{f}(\mathbf{x}, t)$:

$\mathcal{R}^n \times \mathcal{R}_+ \rightarrow \mathcal{R}^n$ is a known mapping; and $\mathbf{d}_1(\mathbf{x}, t) : \mathcal{R}^{nm} \times \mathcal{R}_+ \rightarrow \mathcal{R}^n$ is the system uncertainties.

The desired trajectory for $\mathbf{x}_1$ is denoted as $\mathbf{x}_{1d}$ which is defined on the $[0, \infty)$. $\mathbf{x}_{1d}$ is differential with respect to t up to the mth order and all its higher-order derivatives

$$\mathbf{x}_{1d}^{(j)} \triangleq \mathbf{x}_{(j+1)d}, \quad j = 0, \cdots, m$$

are available over $t \in [0, \infty)$.

For the mth order dynamic system (7.14), an extended tracking error is defined as

$$\boldsymbol{\sigma}(t) = \sum_{j=1}^{m} c_j \mathbf{e}_j(t), \quad c_m = 1$$

where $\mathbf{e}_j(t) \triangleq \mathbf{x}_{jd}(t) - \mathbf{x}_j(t)$ and c_j $(j = 1, \cdots, m)$ are coefficients of a Hurwitz polynomial.

Taking derivative of $\boldsymbol{\sigma}(t)$ with respect to time t yields

$$\dot{\boldsymbol{\sigma}}(t) = \sum_{j=1}^{m} c_j \dot{\mathbf{x}}_{(j+1)d} - \sum_{j=1}^{m-1} c_j \mathbf{x}_{j+1} - \mathbf{f} - B(\mathbf{u} + \mathbf{d}_1) \tag{7.15}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$, $B = B(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(t)$ and $\mathbf{d}_1 = \mathbf{d}_1(\mathbf{x}, t)$.

The following assumption is made first.

Assumption 7.2. There exist a C^1 Lyapunov function $V : \mathcal{R}^n \rightarrow \mathcal{R}_+$ and functions γ_1, γ_2 and γ_3, where γ_1, γ_2 belong to class-KR and γ_3 belongs to class-K, such that for a vector $\boldsymbol{\zeta} \in \mathcal{R}^n$

$$\begin{aligned} 0 \leq \gamma_1(||\boldsymbol{\zeta}||) \leq V(\boldsymbol{\zeta}(t)) \leq \gamma_2(||\boldsymbol{\zeta}||) \\ \frac{\partial V^T}{\partial \boldsymbol{\zeta}} \mathbf{g}(\boldsymbol{\zeta}, t) \leq -\gamma_3(\|\boldsymbol{\zeta}\|). \end{aligned} \tag{7.16}$$

According to Assumption 7.2, the extended error dynamics (7.15) can be rewritten as

$$\dot{\boldsymbol{\sigma}}(t) = \mathbf{g}(\boldsymbol{\sigma}, t) - B[\mathbf{u} + \mathbf{d} + B^{-1}\mathbf{g}(\boldsymbol{\sigma}, t)] \tag{7.17}$$

where $\mathbf{d} \triangleq \mathbf{d}_1 + B^{-1}[\mathbf{f} - \sum_{j=1}^{m} c_j \dot{\mathbf{x}}_{(j+1)d} + \sum_{j=1}^{m-1} c_j \mathbf{x}_{j+1}]$ are the system uncertainties satisfying following assumption.

Assumption 7.3. The system uncertainties $\mathbf{d}$ can be represented as

$$\mathbf{d} \;=\; \Theta(t)\boldsymbol{\xi}(\mathbf{x}, \mathbf{x}_d, t), \quad \Theta \in \mathcal{R}^{n\times n_1} \quad \boldsymbol{\xi} \in \mathcal{R}^{n_1}$$

where n_1 is an appropriate number of dimension. $\Theta(t)$ is an unknown continuous time-varying parameter matrix with a known period T, i.e. $\Theta(t-T) = \Theta(t)$, and $\boldsymbol{\xi}$ is a known vector function.

Remark 7.1. Although the second term in $\mathbf{d}$, $B^{-1}[\mathbf{f} - \sum_{j=1}^{m} c_j \dot{\mathbf{x}}_{(j+1)d} + \sum_{j=1}^{m-1} c_j \mathbf{x}_{j+1}]$, is known, it is treated by learning control. In this way the learning capability can be maximized.

The control objective is to track the desired trajectories by determining the control input $\mathbf{u} \in \mathcal{R}^n$, such that the tracking error converges to zero in $\mathcal{L}_T^2$ norm as time t approaches to infinitely.

7.3.2 IIL Configuration and Convergence Analysis

The proposed CEF-type IIL algorithm is

$$\mathbf{u}(t) = -\hat{\Theta}(t)\boldsymbol{\xi} - B^{-1}\mathbf{g}(\boldsymbol{\sigma}, t), \tag{7.18}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}, \mathbf{x}_d, t)$. Here $\hat{\Theta}$ is to learn Θ and updated pointwisely as

$$\begin{aligned} \hat{\Theta}(t) &= \begin{cases} -\gamma_0(t)\boldsymbol{\alpha}(\mathbf{x}, t)\boldsymbol{\xi}^T & t \in [0, T) \\ \hat{\Theta}(t-T) - \gamma\boldsymbol{\alpha}(\mathbf{x}, t)\boldsymbol{\xi}^T & t \in [T, \infty) \end{cases} \\ \boldsymbol{\alpha}^T(\mathbf{x}, t) &\triangleq \frac{\partial V^T}{\partial \boldsymbol{\sigma}} B, \end{aligned} \tag{7.19}$$

where $\gamma_0(t)$ and γ are defined same as in (7.3). The convergence of the proposed control scheme is given by the following theorem.

Theorem 7.2. *For system (7.14), under the Assumptions 7.2 and 7.3, the learning control law (7.18) and the updating law (7.19) guarantee the convergence of the tracking error in* $\mathcal{L}_T^2$ *norm.*

Proof:

Define a non-negative CEF as:

$$E(t) \;=\; V(\boldsymbol{\sigma}(t)) + \frac{1}{2\gamma}\int_{t-T}^{t} trace[\Phi^T(\tau)\Phi(\tau)]d\tau$$

where $V(\boldsymbol{\sigma}(t))$ is a Lyapunov function which satisfies Assumption 7.2 and $\Phi(t) = \Theta(t) - \hat{\Theta}(t)$.

The proof consists of three parts. Part *I* derives the difference of the CEF, Part *II* eximines the boundedness property of the controlled system and Part *III* proves the convergence of the tracking error.

(I) Difference of the CEF

$\forall t \geq T$, the difference of $E(t)$ over one period is

$$\begin{aligned} \Delta E(t) \;=\; & V(\boldsymbol{\sigma}(t)) + \frac{1}{2\gamma}\int_{t-T}^{t}\{trace[\Phi^T(\tau)\Phi(\tau)] - trace[\Phi^T(\tau - T)\Phi(\tau - T)]\}d\tau \\ & -V(\boldsymbol{\sigma}(t-T)). \end{aligned} \tag{7.20}$$

According to Assumption 7.2, control law (7.18) and updating law (7.19), the following can be derived.

$$\begin{aligned} V(\boldsymbol{\sigma}(t)) - V(\boldsymbol{\sigma}(t-T)) &= \int_{t-T}^{t}\frac{\partial V^T}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}}d\tau \\ &= \int_{t-T}^{t}\frac{\partial V^T}{\partial \boldsymbol{\sigma}}\mathbf{g}(\boldsymbol{\sigma},\tau)d\tau - \int_{t-T}^{t}\frac{\partial V^T}{\partial \boldsymbol{\sigma}}B(\mathbf{x},\tau)\Phi(\tau)\boldsymbol{\xi}d\tau \\ &\leq -\int_{t-T}^{t}\gamma_3(\|\boldsymbol{\sigma}\|)d\tau - \int_{t-T}^{t}\boldsymbol{\alpha}^T(\mathbf{x},\tau)\Phi(\tau)\boldsymbol{\xi}d\tau \\ &\triangleq -\int_{t-T}^{t}\gamma_3(\|\boldsymbol{\sigma}\|) - \int_{t-T}^{t}\varsigma(\tau)d\tau. \end{aligned} \tag{7.21}$$

Similarly, the following relationship is valid.

$$\dot{V}(\boldsymbol{\sigma}(t)) \leq -\gamma_3(\|\boldsymbol{\sigma}\|) - \varsigma(t). \tag{7.22}$$

According to updating law (7.19) and considering $\Theta(t-T)=\Theta(t)$, it can be derived that

$$
\begin{aligned}
&\frac{1}{2\gamma}\{trace[\Phi^T(\tau)\Phi(\tau)] - trace[\Phi^T(\tau-T)\Phi(\tau-T)]\}\\
&= \frac{1}{\gamma}\ trace\{[\hat{\Theta}(t-T)-\hat{\Theta}(t)]^T\Phi(t)\}\\
&\quad -trace\{[\hat{\Theta}(t-T)-\hat{\Theta}(t)]^T[\hat{\Theta}(t-T)-\hat{\Theta}(t)]\}\\
&= \boldsymbol{\alpha}^T\Phi\boldsymbol{\xi} - \frac{\gamma}{2}\|\boldsymbol{\alpha}(\mathbf{x},t)\|^2\|\boldsymbol{\xi}\|^2\\
&= \varsigma(t) - \frac{\beta}{2}\|\boldsymbol{\alpha}(\mathbf{x},t)\|^2\|\boldsymbol{\xi}\|^2. \qquad (7.23)
\end{aligned}
$$

The following properties of *trace* have been used in the above derivations. For $A_1, A_2, A_4, W \in \mathcal{R}^{n\times n_1}$, $\mathbf{w}_1 \in \mathcal{R}^{n_1}$, and $\mathbf{w}_2 \in \mathcal{R}^n$,

$$
\begin{aligned}
P1^\circ \quad : \quad & trace[(A_1-A_2)^T(A_1-A_2)] - trace[(A_1-A_4)^T(A_1-A_4)]\\
& = 2trace[(A_4-A_2)^T 2(A_1-A_2)] - trace[(A_4-A2)^T(A4-A2)]\\
P2^\circ \quad : \quad & trace(\mathbf{w}_1\mathbf{w}_2^T W) = \mathbf{w}_2^T W\mathbf{w}_1.
\end{aligned}
$$

Substituting (7.21) and (7.23) into (7.20) yields

$$
\begin{aligned}
\Delta E(t) &\le -\int_{t-T}^{t}\gamma_3(\|\boldsymbol{\sigma}\|)d\tau - \int_{t-T}^{t}\frac{\gamma}{2}\|\boldsymbol{\alpha}(\mathbf{x},\tau)\|^2\|\boldsymbol{\xi}\|^2 d\tau\\
&\le 0. \qquad (7.24)
\end{aligned}
$$

(II) Boundedness Property

From (7.24), it can be derived that the finiteness of $E(t)$ is ensured for any learning iteration provided that $E(t)$ is finite over $[0,T)$. Analogous to the Part *III* of Theorem 7.1, a non-infinitesimal interval $[0,T_1]$ can be found such that $E(t)$ is bounded and $\forall t\in[T_1,T)$, the derivative of CEF $E(t)$ is

$$
\begin{aligned}
\dot{E}(t) &= \frac{\partial V^T}{\partial\boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} + \frac{1}{2\gamma}trace[\Phi^T(t)\Phi(t)]\\
&\le \frac{\partial V^T}{\partial\boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} + \frac{1}{2\gamma_0(t)}trace[\Phi^T(t)\Phi(t)]. \qquad (7.25)
\end{aligned}
$$

The second term on the RHS of (7.25) can be rewritten as

$$\begin{aligned}
&\frac{1}{2\gamma_0(t)} trace[\Phi^T(t)\Phi(t)] \\
&= \frac{1}{2\gamma_0(t)} trace(\Theta^T\Theta) - \frac{1}{\gamma_0(t)} trace(\hat{\Theta}\Phi) - \frac{1}{2\gamma_0(t)} trace(\hat{\Theta}^T\hat{\Theta}) \\
&\leq \frac{1}{2\gamma_0(t)} trace(\Theta^T\Theta) + \varsigma(t). \qquad (7.26)
\end{aligned}$$

Substituting (7.22) and (7.26) into (7.25), for any $t \in [T_1, T)$, we have,

$$\begin{aligned}
\dot{E}(t) &\leq -\gamma_3(\|\boldsymbol{\sigma}\|) + \frac{1}{2\gamma_0(t)} trace(\Theta^T\Theta) \\
&\leq \frac{1}{2\gamma_0(t)} trace(\Theta^T\Theta). \qquad (7.27)
\end{aligned}$$

Therefore, the boundedness of $\Theta(t)$ leads to the boundedness of $\dot{E}(t)$ over $[T_1, T)$. Considering the finiteness of $E(T_1)$, for any $t \in [0, T)$, the boundedness of $E(t)$ can be guaranteed.

Moreover, the boundedness of $E(t)$ implies that $\mathbf{x}_j$ $(j = 1, \cdots, m)$ is bounded and $\hat{\boldsymbol{\theta}}$ is $\mathcal{L}_T^2$ bounded over $[0, \infty)$. Hence, the $\mathcal{L}_T^2$ boundedness of the control signal $\mathbf{u}(t)$ can be ensured.

(III) Convergence Property

Simiarly to the Part *II* of Theorem 7.1, from (7.24), for any $t \in [nT, (n+1)T]$, we have

$$\lim_{t\to\infty} E(t) \leq E(t_0) - \lim_{n\to\infty} \sum_{i=0}^{n-1} \int_{t-(i+1)T}^{t-iT} \gamma_3(\|\boldsymbol{\sigma}\|) d\tau. \qquad (7.28)$$

Since $E(t)$ is positive and $E(t_0)$ is finite, $\lim_{t\to\infty} \int_{t-T}^{t} \gamma_3(\|\boldsymbol{\sigma}\|) d\tau = 0$ which leads to $\lim_{t\to\infty} \int_{t-T}^{t} \|\boldsymbol{\sigma}\| d\tau = 0$ and $\lim_{t\to\infty} \int_{t-T}^{t} \|\mathbf{e}_j\| d\tau = 0$. ■

Remark 7.2. If the bound for each element of Θ is known *a priori*, $\forall t \geq T$, the learning law (7.19) can be modifed as

$$\hat{\Theta}(t) = proj[\hat{\Theta}(t-T)] - \gamma \boldsymbol{\alpha}(\mathbf{x}, t)\boldsymbol{\xi}^T.$$

According to control laws (7.18)-(7.19) and system dynamics (7.14), the boundedness of $\mathbf{x}$ ensures the finiteness of $\hat{\Theta}(t)$, $\mathbf{u}(t)$ and $\dot{\mathbf{x}}(t)$. Consequently, the boundedness of $\dot{\mathbf{x}}(t)$ implies the uniform continuity of $\mathbf{x}(t)$.

Since $\mathbf{x}_j(t)$ is uniformly continuous,

$$\lim_{t\to\infty} \|\mathbf{e}_j(t)\| = 0 \Rightarrow \lim_{t\to\infty} \|\mathbf{e}_j(t)\|_{sup} = 0. \tag{7.29}$$

Therefore, as t approaches infinity, $\mathbf{x}_j$ uniformly converges to $\mathbf{x}_{jd}$ and the tracking error $\mathbf{e}_j$ uniformly converges to 0.

7.4 Illustrative Examples

Case 1: IIL for SISO Dynamic System

Consider the following system

$$\dot{x} = \theta(t)x^2 + u \quad\quad x(0) = 0.2 \tag{7.30}$$

where $\theta = |2\sin(\frac{1}{3}\pi t)|$. Obviously, the learning period is $T = 3$. The desired tracking trajectory is $x_d = 2\sin 2t$ which has no common period with $\theta(t)$. Choose $k = 1$, $\gamma = 1$ and $\gamma_0(t) = \frac{\gamma t}{T}$ and apply the proposed IIL laws (7.2) and (7.3). We use $|e|_{sup}$ to record the maximum absolute tracking error during the i-th period and Fig. 7.1 shows the maximum tracking error over each period. The effectiveness can be seen clearly.

Case 2: IIL for MIMO Dynamic System

The following high-order MIMO dynamic system is considered

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f + b(u + d_1) \\ x_1(0) &= 0.2; \quad x_2(0) = 0.3, \end{aligned} \tag{7.31}$$

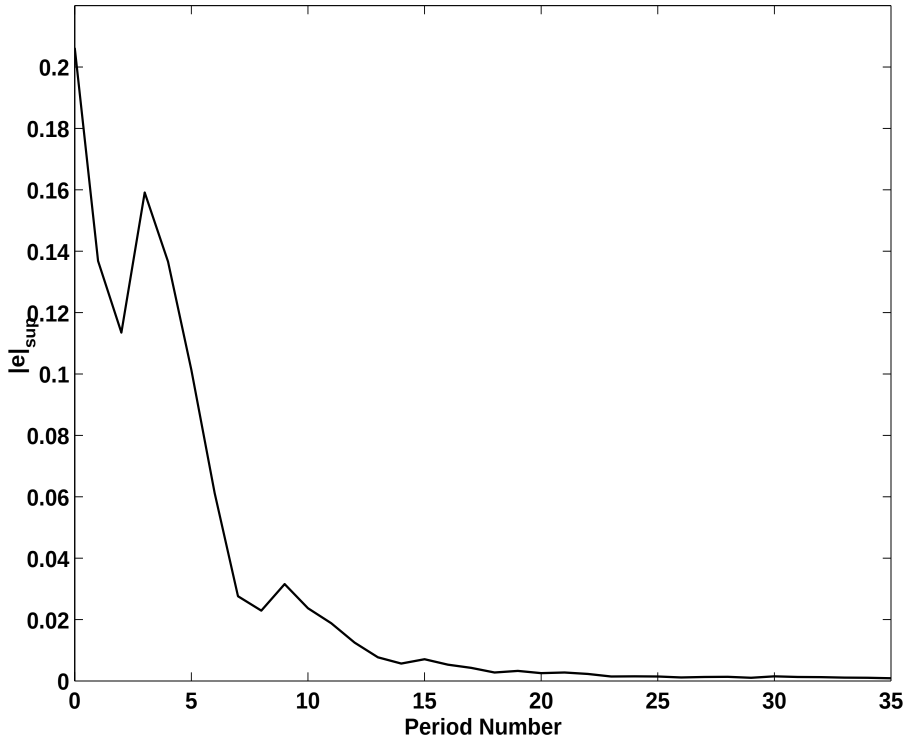

Figure 7.1: Learning convergence for SISO Dynamic System $t \in [0, \infty)$.

where $b = e^{sinx_1}$ and $f = 9(1 - cost)$ are known functions. System uncertainties $d_1 = 4sin^3tcosx_1$ is unknown.

The desired tracking trajectory is $x_{1d} = \sin^3(\frac{\pi}{4}t)$.

Define the extended tracking error as

$$\sigma = x_{2d} - x_2 + 5(x_{1d} - x_1).$$

Then

$$\begin{aligned} \dot{\sigma} &= \dot{x}_{2d} - \dot{x}_2 + 5(\dot{x}_{1d} - \dot{x}_1) \\ &= g - b[u + d + b^{-1}g] \end{aligned}$$

where the system unknown part $d = d_1 + b^{-1}(f + 5x_2 - 5x_{2d} - \dot{x}_{2d})$ can be factorized as $\Theta\boldsymbol{\xi}$. Here $\Theta = [4sin^3t \quad 9(1 - cost) \quad 5]$ and $\boldsymbol{\xi} = [cosx_1 \quad e^{-sinx_1} \quad (x_2 - x_{2d} - \dot{x_{2d}}/5)e^{-sinx_1}]^T$. Hence the learning period is $T = 2\pi$.

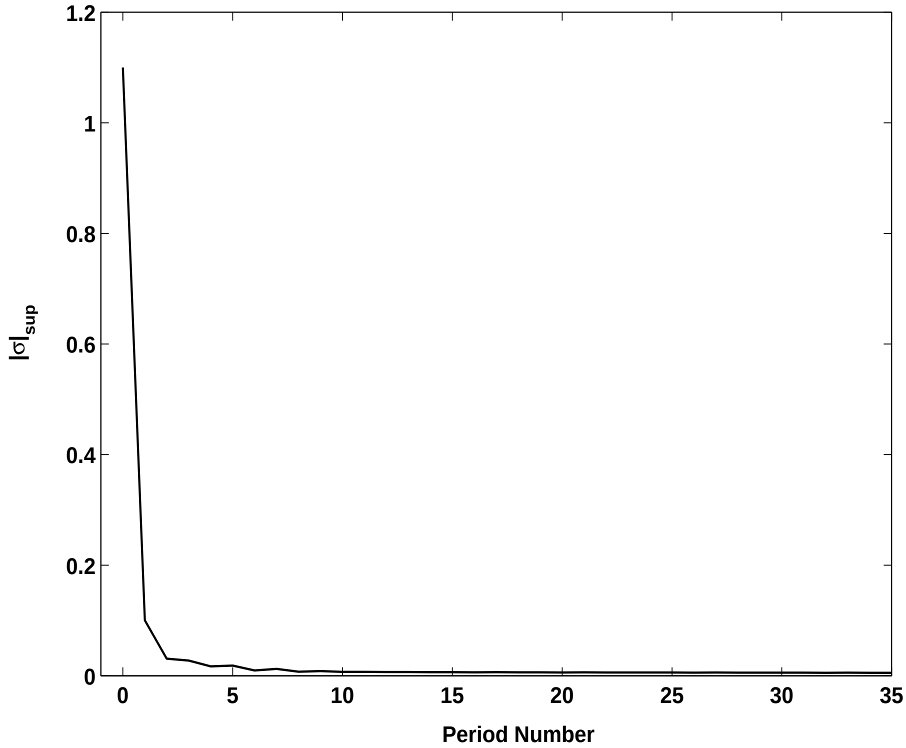

Figure 7.2: Learning convergence for MIMO dynamic system $t \in [0, \infty)$.

Construct g as $g = -6\sigma$ and choose V to be $V = 5\sigma^2$. Let $\gamma = 8$ and $\gamma_0(t) = \frac{\gamma t}{T}$. The maximum extended tracking error σ is recorded for each period and Fig. 7.2 shows the convergence property.

7.5 Conclusion

In this chapter, a new CEF-type IIL control approach is developed to deal with time-varying parametric uncertainties. The novel scheme is applicable to the unknown parameters which maybe time-varying and the only prior knowledge needed is the periodicity. Validity of the proposed approach is confirmed through theoretical analysis and numerical simulations.

Chapter 8

IIL for Systems with Norm-bounded Uncertainties

8.1 Introduction

FIL for systems with parametric uncertainties has been extended to IIL case in Chapter 7. Can we further extended FIL for systems with norm-bounded uncertainties to IIL? It is a challenge problem, as norm-bounded uncertainties are state-related and much more complicated than parametric uncertainties.

In this chapter, we will focus on IIL for systems with norm-bounded uncertainties. Analogous to Chapter 4, the system nonlinearities under consideration are classified into two categories: GLC and NGLC. For both GLC and NGLC systems, when unknown, a CEF, which consists of a quadratic term of the tracking error and a $\mathcal{L}_T^2$ term of the learning error, can be found to unify the theoretical analysis and controller design.

The chapter is organized as follows. In order to clearly demonstrate the underlying idea of the new nonlinear IIL, the simplest system dynamics – first-order and SISO system is discussed in Section 8.2. Based on it, IIL for MIMO dynamic systems with NGLC norm-bounded uncertainties are described in Section 8.3. Simulation results are given in Section 8.4.

8.2 IIL for SISO Systems with Norm-bounded Uncertainties

Consider the following dynamic system

$$\dot{x} = u + d(x,t), \qquad t \in [0,\infty) \tag{8.1}$$

which is similarly to the SISO dynamic system (4.1) in Chapter 4 except that the finite time interval $[0,T]$ is extended to $[0,\infty)$. Hence, the following assumption is further needed.

Assumption 8.1. The system uncertainty $d(x,t)$ and the tracking target x_d are both periodic with respect to t. A common periodicity T can be found such that $d(x,t-T) = d(x,t)$ and $x_d(t-T) = x_d(t)$.

Note that according to (8.1), Assumption 8.1 implies that $u_d(t-T) = u_d(t)$.

The control target is to find an appropriate control signal $u(t)$, such that the system state x uniformly converges to the periodic tracking task $x_d \in \mathcal{C}^1[0,\infty)$

$$\dot{x}_d(t) = u_d(t) + d(x_d,t). \tag{8.2}$$

8.2.1 IIL for Systems with GLC Uncertainties

First the system uncertainty $d(x,t)$ is supposed to satisfy the following assumption.

Assumption 8.2. The system uncertainty $d(x,t)$ is GLC, i.e. $|d(x_1,t)-d(x_2,t)| \leq l_d|x_1-x_2|$. The Lipschitz constant l_d or its bound is known.

The proposed IIL law is

$$u(t) = \begin{cases} \gamma_0(t)e(t) & t \in [0,T) \\ proj[u(t-T)] + \gamma e(t) & t \in [T,\infty) \end{cases}, \tag{8.3}$$

where $e(t) = x_d(t) - x(t)$, $\gamma > 0$ is the learning gain for $t \in [T,\infty)$ and $\gamma_0(t)$ is the learning gain for the first period $[0,T)$. Note that $\gamma_0(t)$ is chosen to be same as in (7.3) which can guarantee the continuity of the control signal $u(t)$ at $t = iT$ where $i \in Z_+$.

From (8.1) and (8.2), we have

$$\delta u(t) = \dot{e}(t) + d - d_d, \tag{8.4}$$

where $\delta u = u_d - u$, $d = d(x,t)$ and $d_d = d(x_d,t)$.

The following theorem gives a sufficient condition for the learning convergence over $[0,\infty)$.

Theorem 8.1. *For system (8.1), under Assumptions 8.1 and 8.2, the control law (8.3) ensures that the tracking error* $e(t)$ *converges to* 0 *uniformly over* $[0,\infty)$ *if* $\gamma \geq 2(l_d+1)$.

Proof:

CEF for IIL is defined as

$$E(t) = \frac{1}{2}e^2(t) + \frac{1}{2\gamma}\int_{t-T}^{t} \delta u^2(\tau)d\tau. \tag{8.5}$$

Note that the time-weighted CEF (4.5) is not suitable for infinite horizon problems.

The proof contains three parts. The system boundedness property is analyzed in Part *I*; Part *II* derives the difference of the CEF; and the uniform convergence of the tracking error is shown in Part *III*.

(I) Boundedness Property

For the first period $t \in [0, T)$, according to (8.3), we have

$$\dot{x}(t) = d + \gamma_0(t)[x_d(t) - x(t)]. \tag{8.6}$$

As both $\gamma_0(t)$ and $x_d(t)$ are bounded and d is GLC, the boundedness of x, u and $\dot{x}$ over $[0, T)$ can be derived straightforwardly.

$\forall t \geq T$, the closed-loop system can be written as

$$\dot{x}(t) = d + proj[u(t-T)] + \gamma e(t). \tag{8.7}$$

Choose a Lyapunov function $V(t) = \frac{1}{2}e^2(t)$, we have

$$\begin{aligned} \dot{V} &= e[\dot{x}_d - d - proj[u(t-T)] - \gamma e] \\ &= e[\dot{x}_d - d_d + (d_d - d) - proj[u(t-T)] - \gamma e]. \end{aligned}$$

Since $x_d \in \mathcal{C}^1[0, \infty)$ and d is GLC with respect to the argument x, $\dot{x}_d - d_d$ is globally uniformly bounded (GUB) and $|d_d - d| \leq l_d|e|$.

Therefore, considering $\gamma \geq 2(l_d + 1)$, we can obtain

$$\begin{aligned} \dot{V} &= e[\dot{x}_d - d_d + (d_d - d) - proj[u(t-T)] - \gamma e] \\ &\leq |e(\dot{x}_d - d_d)| + l_d e^2 + |e|u^* - 2(l_d + 1)e^2 \\ &= [|\dot{x}_d - d_d| + u^* - (l_d + 2)|e|]|e|. \end{aligned} \tag{8.8}$$

Hence, $e(t)$ is GUB by $(|\dot{x}_d - d_d| + u^*)/(l_d + 2)$. Consequently $x(t)$ and $d(x, t)$ are GUB, which implies that $\dot{x}$ is bounded. The boundedness of $\dot{x}$ warrants the uniform continuity of the differentiable state variable $x(t)$.

(II) Difference of $E(t)$

For any $t \geq T$, the difference of $E(t)$ is

$$\begin{aligned}
\Delta E(t) &\triangleq E(t) - E(t-T) \\
&= \frac{1}{2}e^2(t) + \frac{1}{2\gamma}\int_{t-T}^{t}[\delta u^2(\tau) - \delta u^2(\tau - T)]d\tau - \frac{1}{2}e^2(t-T) \\
&= \int_{t-T}^{t} e(\tau)\dot{e}(\tau)d\tau + \frac{1}{2\gamma}\int_{t-T}^{t}[\delta u^2(\tau) - \delta u^2(\tau - T)]d\tau. \qquad (8.9)
\end{aligned}$$

Considering $u_d(t-T) = u_d(t)$, similarly to (4.9), we have

$$\begin{aligned}
&\int_{t-T}^{t}[\delta u^2(\tau) - \delta u^2(\tau - T)]d\tau \\
&\leq \int_{t-T}^{t}\{\delta u^2(\tau) - [u_d(\tau - T) - proj[u(\tau - T)]]^2\}d\tau \\
&= \int_{t-T}^{t}\{-2\delta u(\tau)[u(\tau) - proj[u(\tau - T)]] \\
&\quad -[u(\tau) - proj[u(\tau - T)]]^2\}d\tau. \qquad (8.10)
\end{aligned}$$

Substitute (8.3) and (8.4) into (8.10) yields

$$\begin{aligned}
&\int_{t-T}^{t}[\delta u^2(\tau) - \delta u^2(\tau - T)]d\tau \\
&\leq -2\gamma\int_{t-T}^{t} e(\tau)\dot{e}(\tau)d\tau + 2\gamma l_d\int_{t-T}^{t} e^2(\tau)d\tau - \gamma^2\int_{t-T}^{t} e^2(\tau)d\tau \\
&= -2\gamma\int_{t-T}^{t} e(\tau)\dot{e}(\tau)d\tau - (\gamma^2 - 2\gamma l_d)\int_{0}^{t} e^2(\tau)d\tau. \qquad (8.11)
\end{aligned}$$

Substituting (8.11) into (8.9) and considering $\gamma \geq 2(l_d + 1)$, we can obtain

$$\Delta E(t) \leq -(\frac{\gamma}{2} - l_d)\int_{t-T}^{t} e^2(\tau)d\tau \leq -\int_{t-T}^{t} e^2(\tau)d\tau \leq 0. \qquad (8.12)$$

(III) Uniform Convergence

Applying (8.12) repeatedly for any $t \in [nT, (n+1)T]$, and denoting $t_0 = t - nT$ we have

$$\begin{aligned}
E(t) &= E(t_0) + \sum_{i=0}^{n-1}\Delta E(t - iT) \\
&\leq E(t_0) - \sum_{i=0}^{n-1}\int_{t-(i+1)T}^{t-iT} e^2(\tau)d\tau. \qquad (8.13)
\end{aligned}$$

Let $n \to \infty$ which is equivalent to $t \to \infty$, we obtain

$$\lim_{t\to\infty} E(t) \le E(t_0) - \lim_{n\to\infty} \sum_{i=0}^{n-1} \int_{t-(i+1)T}^{t-iT} e^2(\tau)d\tau. \tag{8.14}$$

Since $E(t)$ is positive and $E(t_0)$ is finite, $\lim_{t\to\infty} \int_{t-T}^{t} e^2(\tau)d\tau = 0$. Analogous to the Part *III* of Theorem 4.1, as $\dot{e}(t)$ is bounded, $e(t)$ converges to 0 uniformly and $u(t)$ converges to $u_d(t)$ almost everywhere when t approaches to infinity. ■

8.2.2 IIL for Systems with NGLC Uncertainties

Similar to the FIL for GLC uncertainties, the robust FIL for NGLC systems can also be extended to the periodic tasks defined in the infinite interval $[0, \infty)$. Including Assumption 8.1, the following assumptions are further made for $d(x, t)$.

Assumption 8.3. The system uncertainty $d(x, t)$ is NGLC with a known smooth bounding function $\eta_1(x, t)$.

Assumption 8.4. $d(x_1, t) - d(x_2, t) = \frac{\partial d(\xi)}{\partial x}(x_1 - x_2)$, where $\xi = x_2 + \tau(x_1 - x_2)$ and $\tau \in [0, 1]$. We assume $|\frac{\partial d(\xi)}{\partial x}| \le \eta_2(x, t)$ and $\eta_2(x, t)$ is a known bounding function.

Remark 8.1. Assumption 8.4 implies that the variation of the non-parametric uncertainty is within an acceptable range.

To deal with the NGLC uncertainties, the robust control is incorporated to ensure the system boundedness. Hence, the robust IIL scheme is constructed as

$$u(t) = w(t) + v(t) \tag{8.15}$$

$$w(t) = \begin{cases} \gamma_0(t)e(t) & t \in [0, T) \\ proj[w(t-T)] + \gamma e(t) & t \in [T, \infty) \end{cases} \tag{8.16}$$

$$v(x, t) = (\rho\kappa + 1)e(t) + \eta_2 e(t) \tag{8.17}$$

$$\rho = \sqrt{\dot{x}_d^2 + \varepsilon} + \eta_1$$

$$\kappa = \frac{\sqrt{e^2(t) + 3\varepsilon^2} + 8\varepsilon}{(\sqrt{e^2(t) + 3\varepsilon^2} + \varepsilon)^2},$$

where $\eta_1 = \eta_1(x,t)$, $\eta_2 = \eta_2(x,t)$ and $\varepsilon > 0$ is a constant. The main result is summarized in the following theorem.

Theorem 8.2. *For system (8.1) under Assumption 8.1 and Assumptions 8.3 - 8.4, the learning control laws (8.15), (8.16) and (8.17) guarantee the tracking error $e(t)$ converges to 0 uniformly as t approaches to infinity.*

Proof:

The CEF is defined as

$$E(t) = \frac{1}{2}e^2(t) + \frac{1}{2\gamma}\int_{t-T}^{t}[u_d(\tau) - w(\tau)]^2 d\tau.$$

(I) Boundedness Property

Define a Lyapunov function $V = \frac{1}{2}e^2$. Analogous to (4.16), when $|e| \geq \varepsilon$, $1-\kappa|e| < 0$ can be derived. Consequently $\forall t \geq T$, we have

$$\begin{aligned}
\dot{V} &= e\dot{e} \\
&= e(\dot{x}_d - d - u) \\
&= e[\dot{x}_d - d - proj[w(t-T)] - \gamma e - v] \\
&\leq |e|w^* - (1+\gamma)e^2 + (1-\kappa|e|)(|\dot{x}_d| + \eta_1)|e| \\
&\leq |e|w^* - (1+\gamma)e^2 = -|e|[(1+\gamma)|e| - w^*].
\end{aligned}$$

$|e|$ is uniformly bounded by $max\{\varepsilon, w^*/(1+\gamma)\}$ and $x \in \mathcal{X}$.

For the first period, $|e|$ is uniformly bounded by ϵ and $x \in \mathcal{X}$ can be derived analogously.

According to the control laws (8.15), (8.16) and (8.17), the boundedness of x leads to the finiteness of w, v and u. Therefore, $\dot{x}$ and $\dot{e}$ are also finite on $\mathcal{X}$. Moreover, the boundedness of $\dot{x}$ leads to the uniform continuity of the system states $x(t)$.

(II) Difference of CEF

For any $t \geq T$, the difference of $E(t)$ is

$$\begin{aligned} \Delta E(t) &= \int_{t-T}^{t} e(\tau)\dot{e}(\tau)d\tau + \frac{1}{2\gamma}\int_{t-T}^{t}\{[u_d(\tau) - w(\tau)]^2 \\ &\quad -[u_d(\tau - T) - w(\tau - T)]^2\}d\tau. \end{aligned} \tag{8.18}$$

The first term on the RHS of (8.18) is

$$\begin{aligned} &\int_{t-T}^{t} e(\tau)\dot{e}(\tau)d\tau \\ &= \int_{t-T}^{t} e(\tau)(d_d - d + u_d - u)d\tau \\ &\leq \int_{t-T}^{t} |e||d_d - d|d\tau + \int_{t-T}^{t} e(u_d - w - v)d\tau \\ &\leq \int_{t-T}^{t} \eta_2 e^2 d\tau + \int_{t-T}^{t} e(u_d - w)d\tau - \int_{t-T}^{t} (\rho\kappa + 1)e^2 d\tau - \int_{t-T}^{t} \eta_2 e^2 d\tau \\ &= \int_{t-T}^{t} e(u_d - w)d\tau - \int_{t-T}^{t} (\rho\kappa + 1)e^2 d\tau. \end{aligned} \tag{8.19}$$

According to (8.10) and (8.16) and letting $u(t) = w(t)$, we obtain

$$\begin{aligned} &\frac{1}{2\gamma}\int_{t-T}^{t}[(u_d(\tau) - w(\tau))^2 - (u_d(\tau - T) - w(\tau - T))^2]d\tau \\ &\leq \int_{t-T}^{t} -[u_d(\tau) - w(\tau)]e(\tau)d\tau. \end{aligned} \tag{8.20}$$

Substituting (8.19) and (8.20) into (8.18) yields

$$\Delta E(t) \leq -\int_{t-T}^{t} e^2(\tau)d\tau \leq 0. \tag{8.21}$$

(III) Convergence Property

Similarly to Theorem 8.1, $\lim_{t\to\infty}\int_{t-T}^{t} e^2(\tau)d\tau = 0$ can be guaranteed. Considering the boundedness of $\dot{e}(t)$, we can derive that $e(t)$ converges to 0 uniformly and $u(t)$ converges to $u_d(t)$ almost everywhere as t approaches infinity. ■

8.3 IIL for MIMO Systems with NGLC Uncertainties

In this section, the robust IIL for NGLC systems will be extended to the following MIMO dynamic systems.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + B(t)[\mathbf{u}(t) + \mathbf{d}(\mathbf{x}, t)], \tag{8.22}$$

where $\mathbf{x} \in \mathcal{R}^n$ is the measurable system state vector, $\mathbf{u} \in \mathcal{R}^n$ is the system control input, $B(t) \in \mathcal{R}^{n\times n}$, and $\mathbf{d}(\mathbf{x}, t) : \mathcal{R}^n \times \mathcal{R}_+ \rightarrow \mathcal{R}^n$ is the system uncertainty.

The following assumptions are made for the system dynamic (8.22).

Assumption 8.5. $\mathbf{f}(\mathbf{x}, t)$ is GLC and the Lipschitz constant $l_{\mathbf{f}}$ or its bound is known. $B(t)$ is invertible and bounded, i.e. $\bar{b}_B \triangleq \sup_{t\in\mathcal{R}_+} \|B\| < \infty$ and $\underline{b}_B \triangleq \min_{t\in\mathcal{R}_+} \|B\| \neq 0$. The system uncertainty $\mathbf{d}(\mathbf{x}, t)$ is NGLC with the known bounding function $\eta_1(x, t)$, i.e. $\|\mathbf{d}(\mathbf{x}, t)\| \leq \eta_1(\mathbf{x}, t)$. Moreover, it is assumed that $\|\mathbf{d}(\mathbf{x}_1, t) - \mathbf{d}(\mathbf{x}_2, t)\| \leq \eta_2(x, t)\|B^T(t)(\mathbf{x}_1 - \mathbf{x}_2)\|$, where $\eta_2(\mathbf{x}, t)$ is known.

Assumption 8.6. $\mathbf{f}(\mathbf{x}, t)$, $B(t)$, $\mathbf{d}(\mathbf{x}, t)$ and $\mathbf{x}_d(t)$ are all periodic in time t with a known common periodicity T.

The control objective is to track the periodic desired trajectory $\mathbf{x}_d$:

$$\dot{\mathbf{x}}_d = \mathbf{f}(\mathbf{x}_d, t) + B(t)[\mathbf{u}_d(t) + \mathbf{d}(\mathbf{x}_d, t)], \tag{8.23}$$

by determining the control input $\mathbf{u}(t)$, such that the tracking error converges to zero uniformly.

The IIL control law is described as

$$\mathbf{u}(t) = \mathbf{w}(t) + \mathbf{v}(t) \tag{8.24}$$

$$\mathbf{w}(t) = \begin{cases} \gamma_0(t)B^T\mathbf{e}(t) & t \in [0, T) \\ proj[\mathbf{w}(t-T)] + \gamma B^T\mathbf{e}(t) & t \in [T, \infty) \end{cases} \tag{8.25}$$

$$\mathbf{v}(t) = [\rho(\mathbf{x}, t)\kappa(\mathbf{x}, t) + 1]B^T\mathbf{e}(t) + \eta_2(\mathbf{x}, t)B^T\mathbf{e}(t) \tag{8.26}$$

$$\rho(\mathbf{x}, t) = \sqrt{(B^{-1}\dot{\mathbf{x}}_d)^T(B^{-1}\dot{\mathbf{x}}_d) + \epsilon} + \eta_1(\mathbf{x}, t)$$

$$\kappa(\mathbf{x}, t) = \frac{\sqrt{\|B\mathbf{e}(t)\|^2 + 3\epsilon^2} + 8\epsilon}{(\sqrt{\|B\mathbf{e}(t)\|^2 + 3\epsilon^2} + \epsilon)^2}.$$

Theorem 8.3. *For system (8.22), under Assumptions 8.5 and 8.6, the control laws (8.24), (8.25) and (8.26) guarantee the system states* $\mathbf{x}(t)$ *converges to the desired states* $\mathbf{x}_d(t)$ *uniformly, if* $\gamma \geq \frac{2l_{\mathbf{f}}}{\underline{b}_B^2}$.

Proof:

The CEF is defined as

$$E(t) = \frac{1}{2}\|\mathbf{e}(t)\|^2 + \frac{1}{2\gamma}\int_{t-T}^{t} \|\mathbf{u}_d(\tau) - \mathbf{w}(\tau)\|^2 d\tau.$$

(I) Boundedness Property

$\forall t \in [T, \infty)$, define a Lyapunov function $V(t) = \frac{1}{2}\mathbf{e}^T\mathbf{e}$. Similarly to (4.42), when

$\|B(t)\mathbf{e}(t)\| \geq \epsilon$, $1 - \kappa\|B\mathbf{e}\| \leq 0$. Therefore, we have

$$\begin{aligned}
\dot{V} &= \mathbf{e}^T\dot{\mathbf{e}} \\
&= \mathbf{e}^T[\dot{\mathbf{x}}_d - \mathbf{f}(\mathbf{x},t) - B\mathbf{w} - B\mathbf{v} - B\mathbf{d}] \\
&= \mathbf{e}^T[\dot{\mathbf{x}}_d - \mathbf{f}(\mathbf{x}_d,t) + \mathbf{f}(\mathbf{x}_d,t) - \mathbf{f}(\mathbf{x},t) - Bproj[\mathbf{w}(t-T)] - \gamma BB^T\mathbf{e} \\
&\quad -\rho\kappa BB^T\mathbf{e} - BB^T\mathbf{e} - \eta_2 BB^T\mathbf{e} - B\mathbf{d}] \\
&\leq \mathbf{e}^T\dot{\mathbf{x}}_d - \mathbf{e}^T\mathbf{f}(\mathbf{x}_d,t) + l_{\mathbf{f}}\|\mathbf{e}\|^2 + \|B^T\mathbf{e}\|w^* - (\gamma+1)\|B^T\mathbf{e}\|^2 \\
&\quad -\rho\kappa\|B^T\mathbf{e}\|^2 - \mathbf{e}^T B\mathbf{d} \\
&\leq \|B^T\mathbf{e}\|\|B^{-1}\dot{\mathbf{x}}_d\| - \mathbf{e}^T\mathbf{f}(\mathbf{x}_d,t) + l_{\mathbf{f}}\|\mathbf{e}\|^2 + \|B^T\mathbf{e}\|w^* - (\gamma+1)\|B^T\mathbf{e}\|^2 \\
&\quad -(\|B^{-1}\dot{\mathbf{x}}_d\| + \eta_1)\kappa\|B^T\mathbf{e}\|^2 + \|B^T\mathbf{e}\|\eta_1 \\
&= -\mathbf{e}^T\mathbf{f}(\mathbf{x}_d,t) + l_{\mathbf{f}}\|\mathbf{e}\|^2 + \|B^T\mathbf{e}\|w^* - (\gamma+1)\|B^T\mathbf{e}\|^2 \\
&\quad +(1 - \kappa\|B^T\mathbf{e}\|)(\|B^{-1}\dot{\mathbf{x}}_d\| + \eta_1)\|B^T\mathbf{e}\| \\
&\leq \|\mathbf{e}\|[\|\mathbf{f}(\mathbf{x}_d,t)\| + L_{\mathbf{f}}\|\mathbf{e}\| + \bar{B}w^* - (\gamma+1)\underline{b}_B^2\|\mathbf{e}\|] \\
&= \|\mathbf{e}\|\{\|\mathbf{f}(\mathbf{x}_d,t) + \bar{b}_B w^* - [(\gamma+1)\underline{b}_B^2 - l_{\mathbf{f}}]\|\mathbf{e}\|\} \\
&\leq \|\mathbf{e}\|[\|\mathbf{f}(\mathbf{x}_d,t) + \bar{b}_B w^* - (\underline{b}_B^2 + l_{\mathbf{f}})\|\mathbf{e}\|].
\end{aligned}$$

Therefore, $\|\mathbf{e}\|$ is GUB by $max\{\epsilon/\underline{b}_B, [\mathbf{f}(\mathbf{x}_d,t) + \bar{b}_B w^*]/(\underline{b}_B^2 + l_{\mathbf{f}})\}$ which leads to the boundedness of $\mathbf{x}(t)$ and $\mathbf{d}(t)$. Moreover, $\dot{\mathbf{x}}$ is also finite, hence $\mathbf{x}(t)$ is uniformly continuous.

For $t \in [0,T]$, the boundedness of $\mathbf{e}$ and $\mathbf{w}$ can also be derived analogously, which leads to the finiteness of $E(t)$ $(t \in [0,T])$.

(II) Difference of $E(t)$

$\forall t \geq T$, the difference of $E(t)$ over one period is

$$\begin{aligned}
\Delta E(t) &= \int_{t-T}^{t} \mathbf{e}^T\dot{\mathbf{e}}d\tau + \int_{t-T}^{t} [\|\mathbf{u}_d(\tau) - \mathbf{w}(\tau)\|^2 \\
&\quad -\|\mathbf{u}_d(\tau-T) - \mathbf{w}(\tau-T)\|^2]d\tau. \qquad (8.27)
\end{aligned}$$

The first term on the RHS of (8.27) is

$$\begin{aligned}
&\int_{t-T}^{t} \mathbf{e}^T \dot{\mathbf{e}} d\tau \\
&= \int_{t-T}^{t} \mathbf{e}^T [\mathbf{f}(\mathbf{x}_d, \tau) - \mathbf{f}(\mathbf{x}, \tau) + B(\mathbf{u}_d - \mathbf{u}) + B(\mathbf{d}_d - \mathbf{d})] d\tau \\
&\leq \int_{t-T}^{t} l_{\mathbf{f}} \|\mathbf{e}\|^2 d\tau + \int_{t-T}^{t} \mathbf{e}^T B(\mathbf{u}_d - \mathbf{w}) d\tau \\
&\quad - \int_{t-T}^{t} \mathbf{e}^T B(\rho\kappa + 1) B^T \mathbf{e} d\tau - \int_{t-T}^{t} \eta_2 \|B^T \mathbf{e}\|^2 d\tau + \int_{t-T}^{t} \eta_2 \|B^T \mathbf{e}\|^2 d\tau \\
&\leq \int_{t-T}^{t} l_{\mathbf{f}} \|\mathbf{e}\|^2 d\tau + \int_{t-T}^{t} \mathbf{e}^T B(\mathbf{u}_d - \mathbf{w}) d\tau - \int_{t-T}^{t} \underline{b}_B^2 \|\mathbf{e}\|^2 d\tau.
\end{aligned}$$

The second term on the RHS of (8.27) is

$$\begin{aligned}
&\frac{1}{2\gamma} \int_{t-T}^{t} [\|\mathbf{u}_d(\tau) - \mathbf{w}(\tau)\|^2 - \|\mathbf{u}_d(\tau - T) - \mathbf{w}(\tau - T)\|^2] d\tau \\
&\leq \frac{1}{2\gamma} \int_{t-T}^{t} \{-2[\mathbf{w}(\tau) - proj[\mathbf{w}(\tau - T)]]^T [\mathbf{u}_d(\tau) - \mathbf{w}(\tau)] \\
&\quad - \|\mathbf{w}(\tau) - proj[\mathbf{w}(\tau - T)]\|^2\} d\tau \\
&= \int_{t-T}^{t} -\mathbf{e}^T B[\mathbf{u}_d(\tau) - \mathbf{w}(\tau)] d\tau - \frac{\gamma}{2} \int_{t-T}^{t} \|B^T \mathbf{e}\|^2 d\tau \\
&\leq \int_{t-T}^{t} -\mathbf{e}^T B[\mathbf{u}_d(\tau) - \mathbf{w}(\tau)] d\tau - \frac{\gamma \underline{b}_B^2}{2} \int_{t-T}^{t} \|\mathbf{e}\|^2 d\tau.
\end{aligned}$$

Therefore,

$$\Delta E(t) \leq \int_{t-T}^{t} (l_{\mathbf{f}} - \underline{b}_B^2 - \frac{\gamma \underline{b}_B^2}{2}) \|\mathbf{e}\|^2 d\tau.$$

Considering $\gamma \geq \frac{2l_{\mathbf{f}}}{\underline{b}_B^2}$, we have

$$\Delta E(t) \leq -\underline{b}_B^2 \int_{t-T}^{t} \|\mathbf{e}\|^2 d\tau \leq 0. \tag{8.28}$$

(III) Convergence Property

Similarly to the Part *III* of Theorem 8.1, according to (8.28) and considering the system boundedness property, it can be derived that $\mathbf{e}(t)$ converges to $\mathbf{0}$ uniformly and $\mathbf{u}(t)$ converges to $\mathbf{u}_d(t)$ almost everywhere as t approaches to infinity. ■

8.4 Illustrative Examples

Case 1. IIL for SISO Dynamic Systems

Consider system (8.1) with the target trajectory $x_d = 1.5sin^3t$ where $t \in [0, \infty)$.

(a) $d(x,t) = 3xsin(2t)$ and $x(0) = 0.6 \neq x_d(0)$.

The learning period should be $T = 2\pi$. Suppose the known bound of l_d is 4. Choose $u^* = 10$, $\gamma = 10$ and $\gamma_0 = \begin{cases} 6\gamma t/T & t \in [0, T/6] \\ \gamma & t \in [T/6, T) \end{cases}$. Applying control law (8.3), the maximum error for each period is recorded in Fig. 8.1. The effectiveness is obvious.

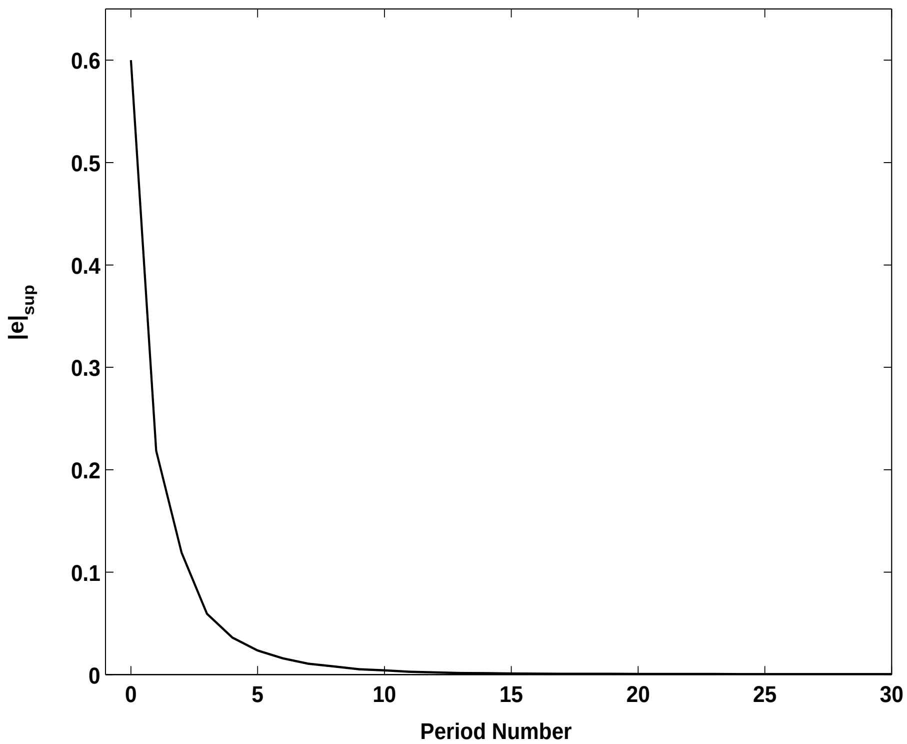

Figure 8.1: Learning convergence for SISO system with GLC uncertainty $t \in [0, \infty)$.

(b) $d(x,t) = 3x^2sint + 5x^2$, $t \in [0, \infty)$ and $x(0) = 0.3 \neq x_d(0)$.

$d(x,t)$ is NGLC with the known bounding functions $\eta_1 = 10x^2$ and $\eta_2 = 18x$. The learning period is $T = 2\pi$. Choose $\epsilon = 0.3$ and $w^* = 20$. γ and $\gamma_0(t)$ are same as in Part (a). Applying control law (8.15), (8.16) and (8.17), the maximum error for each learning period is shown in Fig. 8.2.

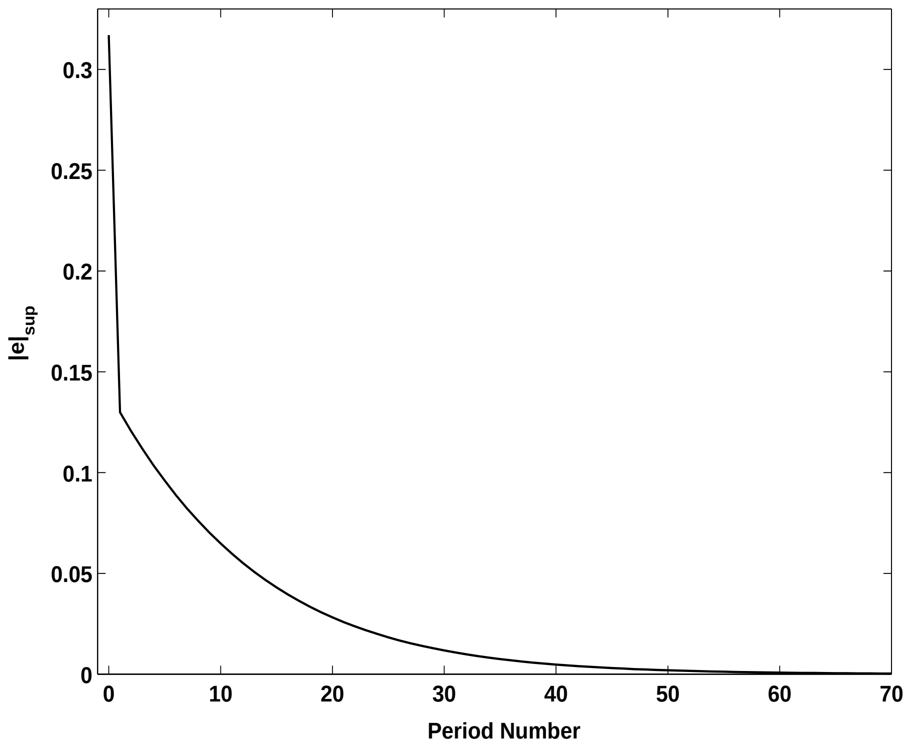

Figure 8.2: Learning convergence for SISO system with NGLC uncertainty $t \in [0, \infty)$.

Case2. IIL for MIMO Dynamic Systems

Consider system (8.22) and let $\mathbf{f} = [x_2 \;\; 2x_1 sinx_2]^T$, $B(t) = [1 \;\; 0; 0 \;\; 1 + 0.5sint]$ and $\mathbf{d} = [x_1^2 + cost \;\; x_1^2 sint + 2x_2^2]$. Assume $l_{\mathbf{f}} = 3$, $\underline{b}_B = 1$ and the known bounding functions are: $\eta_1 = 2(x_1^2 + x_2^2)$ and $\eta_2 = 4\sqrt{x_1^2 + x_2^2}$.

The desired trajectory to be followed is

$$x_{1,d} = sin^3 t \quad x_{2,d} = \dot{x}_{1,d} \quad t \in [0, \infty). \tag{8.29}$$

The learning updating period is 2π.

Let $\varepsilon = 0.3$ and $w^* = 10$. γ and $\gamma_0(t)$ are same as in *Case 1* which guarantees that

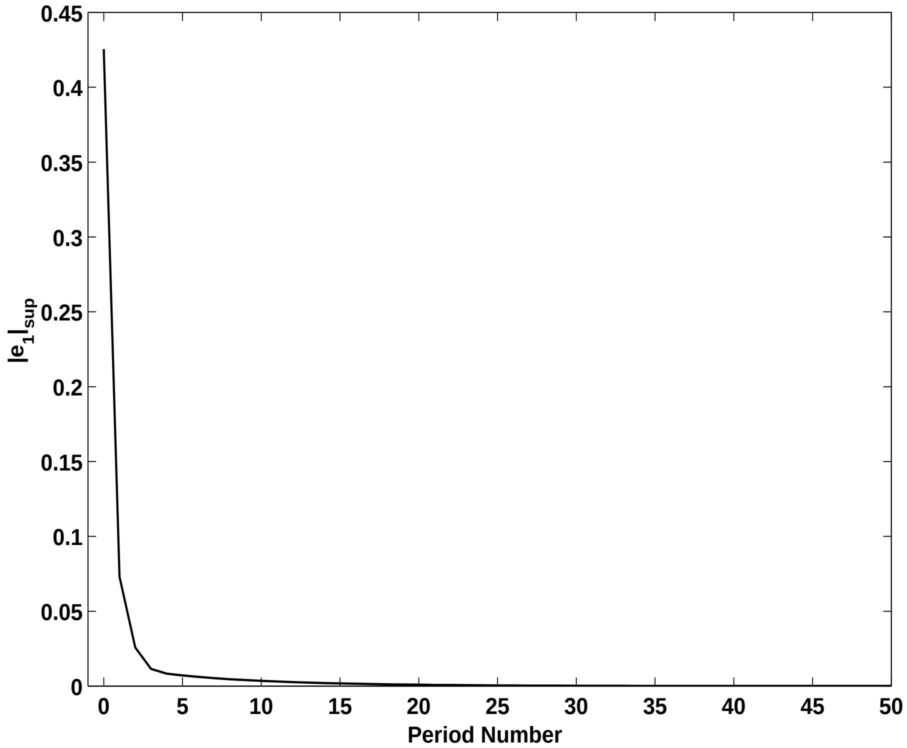

Figure 8.3: Convergence of e_1 for MIMO system with NGLC uncertainties $t \in [0, \infty)$.

$\gamma = \frac{2l_f}{\underline{b}_B^2}$. By applying the control laws (8.24) - (8.26), the convergence properties of $e_1 = x_{1,d} - x_1$ and $e_2 = x_{2,d} - x_2$ are clearly shown in Fig. 8.3 and Fig. 8.4 respectively.

8.5 Conclusion

In this chapter IIL schemes are extended to address systems with norm-bounded uncertainties. It is clearly shown that the proposed IIL approaches work effective no matter the uncertainties are GLC or NGLC. Rigorous proofs based on CEF are given and simulation examples demonstrate the validity.

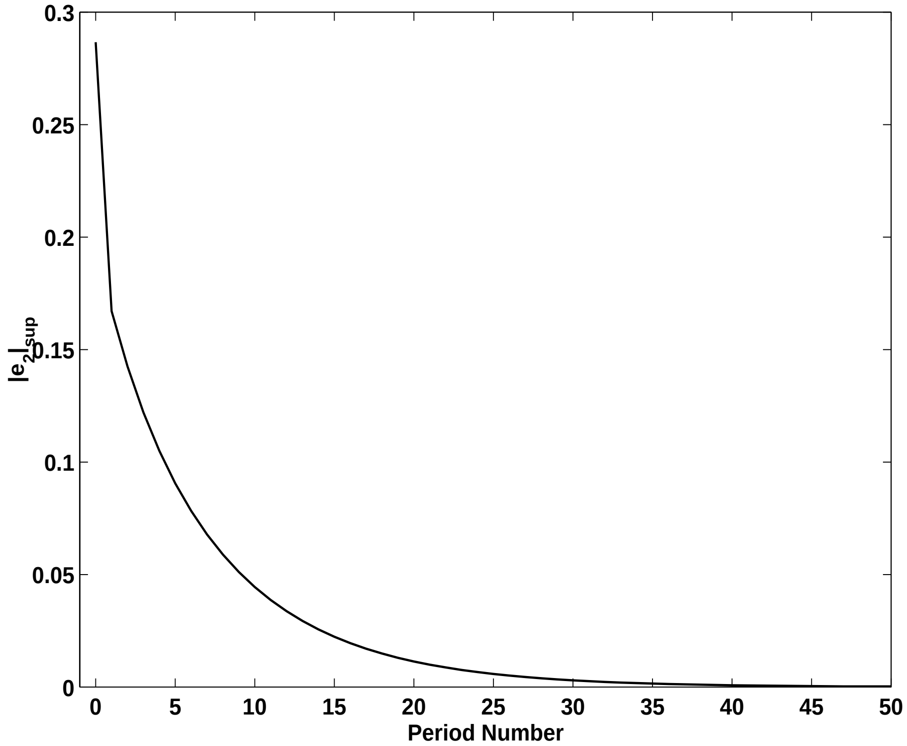

Figure 8.4: Convergence of e_2 for MIMO system with NGLC uncertainties $t \in [0, \infty)$.

Chapter 9

Observer Based IIL for Systems with Parametric Uncertainties

9.1 Introduction

CEF suggests a new avenue, which shortens the gap between FIL and IIL, removes the limitations such as I.I.C., GLC and zero relative degree, and enables both FIL and IIL for GLC and NGLC systems with parametric or non-parametric uncertainties. Note that In all the CEF based learning schemes, the system states are assumed to be available. Hence we will consider a new challenging problem: can FIL and IIL deal with output tracking tasks where the system state information is not available?

In this chapter, we combine the state estimation with IIL control to address periodic parametric uncertainties. The difficulty in this kind of problems lies in the presence of the product terms in the system dynamics, which consist of the unknown time-varying parameters and the state functions which are also unknown due to the lack of the state information. In such circumstance, the product of unknown parameters and state-dependent functions cannot be treated simply as parametric type uncertainties. Many otherwise effective observers, such as the Luenberger observer, adaptive observer, robust observer (sliding mode observer), are difficult to apply. In our work IIL is combined with a specific observer (Yang and Wilde, 1988; Darouach *et al.*, 1994), which is able to nullify the influence from input disturbances without any extra robust feedback, provided that the system linear nominal part is observable. In addition to the time-varying parametric uncertainties, we will consider two classes of nonlinearities: the GLC function of state variables, and the NGLC function of output variables.

Comparing with the RC which also uses only output information, the observer based IIL control applies to more general nonlinear uncertain systems, and to more general control tasks such as tracking non-periodic target trajectories.

The chapter is organized as follow. Section 9.2 focuses on the observer based IIL control for systems with state-dependent GLC nonlinearities. The observer based IIL control for system with output-dependent NGLC nonlinearities is discussed in Section 9.3. Illustrative examples are given in Section 9.4.

9.2 Problem Formulation

Considering the following uncertain nonlinear system

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B[\mathbf{u}(t) + \Theta(t)\boldsymbol{\xi}(\mathbf{z}, t)] \\ \mathbf{y} &= C\mathbf{x}, \end{aligned} \tag{9.1}$$

where $\mathbf{x} \in \mathcal{R}^n$ is the system state vector; $\mathbf{y} \in \mathcal{R}^m$ is the physically accessible output vector; $\mathbf{u} \in \mathcal{R}^m$ is the system input vector; $\Theta(t) \in \mathcal{C}^0(\mathcal{R}^{m\times n_1}, [0, \infty))$ represents the time-varying parametric uncertainty; and $\boldsymbol{\xi}(\mathbf{z}, t) \in \mathcal{R}^{n_1}$ is a known vector-valued function with $\mathbf{z}$ being either the state $\mathbf{x}$ or output $\mathbf{y}$. A, B and C are known constant matrices of appropriate dimensions.

The system (9.1) satisfies the following assumptions.

Assumption 9.1. For system (9.1), (A, C) is observable and $rank(CB) = m$. The invariant zeros of (A, B, C) lie in the left-half complex plane.

Assumption 9.2. $\Theta(t)$ is periodic with a known period of T, i.e. $\Theta(t) = \Theta(t - T)$.

Clearly, from Assumption 9.2, $\|\Theta(t)\|$ is bounded over $[0, \infty)$: $\theta_m \triangleq \|\Theta(t)\|_{sup} < \infty$. Regarding the system nonlinearities, we have

Assumption 9.3. When $\mathbf{z} = \mathbf{x}$, $\boldsymbol{\xi}(\mathbf{x}, t)$ is GLC, i.e. $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{R}^n$, $\|\boldsymbol{\xi}(\mathbf{x}_1, t) - \boldsymbol{\xi}(\mathbf{x}_2, t)\| \leq l\|\mathbf{x}_1 - \mathbf{x}_2\|$. When $\mathbf{z} = \mathbf{y}$, $\boldsymbol{\xi}(\mathbf{y}, t)$ is only local Lipschitz continuous.

The ultimate control objective is to find an appropriate control signal $\mathbf{u}(t)$, such that the system output $\mathbf{y}$ converges to the target $\mathbf{y}_d \in \mathcal{C}^1\{\mathcal{R}^m, [0, \infty)\}$ in $\mathcal{L}_T^2$ norm as t approaches to infinity. Note that the desired trajectory $\mathbf{y}_d$ can be non-periodic.

The following observer is used to obtain the estimated system states $\hat{\mathbf{x}}$ (Fang and Wilde, 1988; Darouach *et al.*, 1994).

$$\begin{aligned} \hat{\mathbf{x}} &= \mathbf{v} - D\mathbf{y} \\ \dot{\mathbf{v}} &= (FA - LC)\mathbf{v} + [L(I_m + CD) - FAD]\mathbf{y}, \end{aligned} \tag{9.2}$$

where $\mathbf{v} \in \mathcal{R}^n$, $D = -B(CB)^{-1} \in \mathcal{R}^{n \times m}$ and $F = I_n + DC \in \mathcal{R}^{n \times n}$. Here we can arbitrarily choose $\mathbf{v}(0) = 0$. By defining the estimation error $\delta\mathbf{x} = \mathbf{x} - \hat{\mathbf{x}}$, it can be easily derived

$$\delta\dot{\mathbf{x}} = (FA - LC)\delta\mathbf{x}, \tag{9.3}$$

i.e. the observer is independent of the input uncertainty. Under Assumption 9.1, according to the Theorem 2 in (Darouach *et al.*, 1994), (FA, C) is detectable. Hence, there exists a matrix $L \in \mathcal{R}^{n \times m}$ such that $FA - LC$ is asymptotically stable. Given a positive definite matrix $Q \in \mathcal{R}^{n\times n}$, there exists a unique positive definite matrix $P \in \mathcal{R}^{n\times n}$ satisfying the following Lyapunov equation

$$(FA - LC)^T P + P(FA - LC) = -Q. \tag{9.4}$$

Therefore, $-\mathbf{w}^T Q \mathbf{w} \le -\lambda \|\mathbf{w}\|^2$ holds for any $\mathbf{w} \in \mathcal{R}^n$, where λ is the minimum eigenvalue of the matrix Q.

9.3 Observer Based IIL for GLC System

9.3.1 Observer Based IIL With Known θ_m And l

First, we assume that the bound of the time-varying uncertainty θ_m, and the Lipschitz constant l are known *a priori*.

Define $\mathbf{e} = \mathbf{y}_d - \mathbf{y}$, the observer based IIL control scheme is constructed as

$$\mathbf{u} = -\hat{\Theta}\hat{\boldsymbol{\xi}} + (CB)^{-1}(\dot{\mathbf{y}}_d - CA\hat{\mathbf{x}} + K\mathbf{e}), \tag{9.5}$$

$$\hat{\Theta}(t) = \begin{cases} -\Gamma_0(t)(CB)^T\mathbf{e}\hat{\boldsymbol{\xi}}^T & t \in [0,T) \\ \hat{\Theta}(t-T) - \Gamma(CB)^T\mathbf{e}\hat{\boldsymbol{\xi}}^T & t \in [T,\infty) \end{cases}, \tag{9.6}$$

where $\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}(\hat{\mathbf{x}}, t)$; $K \in \mathcal{R}^{m\times m}$ is a positive definite matrix with the minimum eigenvalue γ; $\Gamma \in \mathcal{R}^{m\times m}$ is a diagonal, positive learning gain matrix for $t \ge T$; $\Gamma_0(t) \in \mathcal{C}\{\mathcal{R}^{m\times m}, [0,T)\}$ is a diagonal, positive learning gain matrix for the first period $[0,T)$ satisfying $\Gamma_0(0) = 0$, $\Gamma_0(T) = \Gamma$, and each element of $\Gamma_0(t)$ is chosen to be strictly increasing. The purpose of choosing such a $\Gamma_0(t)$ is to ensure the continuity of $\hat{\Theta}(t)$ at the instants $t = jT$ $j \in \mathcal{Z}_+$, when the algebraic updating law (9.6) is used.

The error feedback gain K is chosen to satisfy that $\gamma \ge \frac{(\|CA\|+\|CB\|\theta_m l)^2}{\lambda} + 1$.

According to (9.1) and (9.5), the output tracking error dynamics is

$$\begin{aligned}\dot{\mathbf{e}} &= \dot{\mathbf{y}}_d - C\dot{\mathbf{x}} \\ &= \dot{\mathbf{y}}_d - CA\mathbf{x} - CB[\mathbf{u} + \Theta\boldsymbol{\xi}] \\ &= -K\mathbf{e} - CA\delta\mathbf{x} - CB[\Theta\boldsymbol{\xi} - \hat{\Theta}\hat{\boldsymbol{\xi}}] \\ &\overset{\triangle}{=} -K\mathbf{e} + \mathbf{g} - CB\Phi\hat{\boldsymbol{\xi}}, \end{aligned} \tag{9.7}$$

where $\boldsymbol{\xi} \overset{\triangle}{=} \boldsymbol{\xi}(\mathbf{x}, t)$, $\Phi = \Theta - \hat{\Theta}$ and $\mathbf{g} = -CA\delta\mathbf{x} - CB\Theta(\boldsymbol{\xi} - \hat{\boldsymbol{\xi}})$.

It should be noted that the controller (9.5) and the observer (9.2) work concurrently, and the observer (9.2) will not be able to work if the input uncertainties in (9.1) grow unbounded. Further, although the original system (9.1) is GLC, the closed-loop system with state estimation is no longer GLC, a finite escape time may exist. Consequently the separation principle does not hold, even if the estimation error dynamics (9.3) appears to be independent of the input uncertainties. The following theorem exhibits the convergence and boundedness of the closed-loop control system with state estimation.

Theorem 9.1. *The system (9.1), under the learning laws (9.5) and (9.6), achieves the convergence of* $\mathbf{e}$ *and* $\delta\mathbf{x}$ *in the sense of* $\mathcal{L}_T^2$ *norm.*

Proof:

To evaluate the convergence property, we define the following CEF

$$E(t) = \delta\mathbf{x}^T P\delta\mathbf{x} + \frac{1}{2}\|\mathbf{e}\|^2 + \frac{1}{2}\int_{t-T}^{t} trace[\Phi^T(\tau)\Gamma^{-1}\Phi(\tau)]d\tau. \tag{9.8}$$

The proof consists of three parts. *Part I* derives the difference of the CEF; *Part II* proves the convergence of the tracking error; and *Part III* examines the boundedness property of the system.

(I) Difference of CEF

For any $t \geq T$, the difference of the CEF over one period is

$$\begin{aligned}\Delta E(t) &\stackrel{\triangle}{=} E(t) - E(t-T)\\ &= \delta\mathbf{x}^T(t)P\delta\mathbf{x}(t) - \delta\mathbf{x}^T(t-T)P\delta\mathbf{x}(t-T) + \frac{1}{2}\|\mathbf{e}(t)\|^2 - \frac{1}{2}\|\mathbf{e}(t-T)\|^2\\ &\quad +\frac{1}{2}\int_{t-T}^{t}\{trace[\Phi^T(\tau)\Gamma^{-1}\Phi(\tau)]\\ &\quad -trace[\Phi^T(\tau-T)\Gamma^{-1}\Phi(\tau-T)]\}d\tau. \end{aligned} \tag{9.9}$$

The first two terms on the RHS of (9.9) can be rewritten as

$$\begin{aligned}&\delta\mathbf{x}^T(t)P\delta\mathbf{x}(t) - \delta\mathbf{x}^T(t-T)P\delta\mathbf{x}(t-T)\\ &\quad = \int_{t-T}^{t}[\delta\dot{\mathbf{x}}^T(\tau)P\delta\mathbf{x}(\tau) + \delta\mathbf{x}^T(\tau)P\delta\dot{\mathbf{x}}(\tau)]d\tau. \end{aligned} \tag{9.10}$$

From (9.3), we have

$$\begin{aligned}&\delta\dot{\mathbf{x}}^T(t)P\delta\mathbf{x}(t) + \delta\mathbf{x}^T(t)P\delta\dot{\mathbf{x}}(t)\\ &= \delta\mathbf{x}^T(t)[(FA-LC)^TP + P(FA-LC)]\delta\mathbf{x}(t)\\ &\leq -\lambda\|\delta\mathbf{x}\|^2. \end{aligned} \tag{9.11}$$

Therefore

$$\delta\mathbf{x}^T(t)P\delta\mathbf{x}^T(t) - \delta\mathbf{x}^T(t-T)P\delta\mathbf{x}(t-T) \leq -\lambda\int_{t-T}^{t}\|\delta\mathbf{x}\|^2 d\tau. \tag{9.12}$$

Similarly, the third and the fourth terms on the RHS of (9.9) can be rewritten as

$$\frac{1}{2}\|\mathbf{e}(t)\|^2 - \frac{1}{2}\|\mathbf{e}(t-T)\|^2 = \int_{t-T}^{t}\mathbf{e}^T(\tau)\dot{\mathbf{e}}(\tau)d\tau. \tag{9.13}$$

According to (9.7), it can be derived that

$$\begin{aligned}&\mathbf{e}^T(t)\dot{\mathbf{e}}(t)\\ &\quad = -\mathbf{e}^T(t)K\mathbf{e}(t) + \mathbf{e}^T\mathbf{g} - \mathbf{e}^TCB\Phi\hat{\boldsymbol{\xi}}\\ &\quad \leq -\gamma\|\mathbf{e}\|^2 + \|\mathbf{e}\|\|\mathbf{g}\| - \mathbf{e}^TCB\Phi\hat{\boldsymbol{\xi}}. \end{aligned} \tag{9.14}$$

Let us evaluate the bound of $\|\mathbf{e}\|\|\mathbf{g}\|$. According to Assumptions 9.2 and 9.3,

$$\begin{aligned}\|\mathbf{e}\|\|\mathbf{g}\| &\leq \|\mathbf{e}\|[\|CA\delta\mathbf{x}\| + \|CB\Theta\|\|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|]\\ &\leq (\|CA\| + \|CB\|\theta_m l)\|\mathbf{e}\|\|\delta\mathbf{x}\|. \end{aligned}$$

Let us further seek the upper-bound of the cross term $\|\mathbf{e}\|\|\delta\mathbf{x}\|$ in quadratic form using Young's inequality $ab \le ca^2 + \frac{1}{4c}b^2$ with $c > 0$. Set $c = \dfrac{1}{\lambda}$, $a = (\|CA\| + \|CB\|\theta_m l)\|\mathbf{e}\|$, and $b = \|\delta\mathbf{x}\|$, it is straightforward to derive

$$\begin{aligned} \|\mathbf{e}\|\|\mathbf{g}\| &\le \frac{(\|CA\| + \|CB\|\theta_m l)^2}{\lambda}\|\mathbf{e}\|^2 + \frac{\lambda}{4}\|\delta\mathbf{x}\|^2 &(9.15)\\ &\le (\gamma - 1)\|\mathbf{e}\|^2 + \frac{\lambda}{4}\|\delta\mathbf{x}\|^2. &(9.16)\end{aligned}$$

In the sequel

$$\mathbf{e}^T(t)\dot{\mathbf{e}}(t) \le -\|\mathbf{e}\|^2 + \frac{\lambda}{4}\|\delta\mathbf{x}\|^2 - \mathbf{e}^T CB\Phi\hat{\boldsymbol{\xi}}, \qquad (9.17)$$

and

$$\begin{aligned}&\frac{1}{2}[\|\mathbf{e}(t)\|^2 - \|\mathbf{e}(t-T)\|^2]\\ &\le -\int_{t-T}^{t}\|\mathbf{e}\|^2 d\tau + \frac{\lambda}{4}\int_{t-T}^{t}\|\delta\mathbf{x}\|^2 d\tau - \int_{t-T}^{t}\mathbf{e}^T CB\Phi\hat{\boldsymbol{\xi}} d\tau. \qquad (9.18)\end{aligned}$$

Regarding the last term on the RHS of (9.9), using the learning law (9.6) we first derive

$$\begin{aligned}&trace\{[\Phi^T(t)\Gamma^{-1}\Phi(t)] - trace[\Phi^T(t-T)\Gamma^{-1}\Phi(t-T)]\}\\ &= trace\{[\hat{\Theta}(t-T) - \hat{\Theta}(t)]^T\Gamma^{-1}[2\Phi(t) - (\hat{\Theta}(t-T) - \hat{\Theta}(t))]\}\\ &\le trace\{2\hat{\boldsymbol{\xi}}[(CB)^T\mathbf{e}]^T\Phi\}\\ &= 2\mathbf{e}^T CB\Phi\hat{\boldsymbol{\xi}}. \qquad (9.19)\end{aligned}$$

Since the relation (9.19) holds for any t, integrating both sides leads to

$$\begin{aligned}&\frac{1}{2}\int_{t-T}^{t} trace\{[\Phi^T(\tau)\Gamma^{-1}\Phi(\tau)] - trace[\Phi^T(\tau-T)\Gamma^{-1}\Phi(\tau-T)]\}d\tau\\ &\le \int_{t-T}^{t}\mathbf{e}^T CB\Phi\hat{\boldsymbol{\xi}} d\tau. \qquad (9.20)\end{aligned}$$

Substituting (9.12), (9.18) and (9.20) into (9.9) yields

$$\Delta E(t) \le -\int_{t-T}^{t}\|\mathbf{e}\|^2 d\tau - \frac{3\lambda}{4}\int_{t-T}^{t}\|\delta\mathbf{x}\|^2 d\tau. \qquad (9.21)$$

(II) Convergence Analysis

For $t \in [iT, (i+1)T)$, denote $t = iT + t_0$ where $t_0 \in [0, T)$ and $i = 1, 2, \cdots$. Obviously, when $t \to \infty$, $i \to \infty$. Applying (9.21) repeatedly, we have

$$\begin{aligned} E(t) &= E(t_0) + \sum_{j=1}^{i} \Delta E(jT + t_0) \\ &\leq E(t_0) - \sum_{j=1}^{i} \int_{(j-1)T+t_0}^{jT+t_0} \|\mathbf{e}\|^2 d\tau - \frac{3\lambda}{4} \sum_{j=1}^{i} \int_{(j-1)T+t_0}^{jT+t_0} \|\delta\mathbf{x}\|^2 d\tau. \end{aligned} \quad (9.22)$$

The above relationship holds for any t, thus

$$\begin{aligned} \lim_{t\to\infty} E(t) &\leq E(t_0) - \lim_{i\to\infty} \sum_{j=1}^{i} \int_{(j-1)T+t_0}^{jT+t_0} \|\mathbf{e}\|^2 d\tau \\ &\quad - \lim_{i\to\infty} \frac{3\lambda}{4} \sum_{j=1}^{i} \int_{(j-1)T+t_0}^{jT+t_0} \|\delta\mathbf{x}\|^2 d\tau. \end{aligned} \quad (9.23)$$

As $E(t)$ is positive, if $E(t_0)$ is finite, it can be derived both $\lim_{i\to\infty} \sum_{j=1}^{i} \int_{(j-1)T+t_0}^{jT+t_0} \|\mathbf{e}\|^2 d\tau$ and $\lim_{i\to\infty} \sum_{j=1}^{i} \int_{(j-1)T+t_0}^{jT+t_0} \|\delta\mathbf{x}\|^2 d\tau$ converge. According to the convergence theorem of the sum of series, $\lim_{t\to\infty} \int_{t-T}^{t} \|\mathbf{e}\|^2 d\tau = 0$ and $\lim_{t\to\infty} \int_{t-T}^{t} \|\delta\mathbf{x}\|^2 d\tau = 0$. Therefore, as t approaches infinity, $\hat{\mathbf{x}}$ converges to $\mathbf{x}$ and $\mathbf{y}$ converges to $\mathbf{y}_d$ asymptotically in $\mathcal{L}_T^2$ norm.

Now let us check the finiteness property of $E(t)$ for the first period $t \in [0, T)$. From the system dynamics (9.1) and the proposed control law (9.5) and (9.6), it can be derived that the RHS of (9.1) is continuous with respect to all the arguments. According to the existence theorem of differential equation, there exists a solution in an interval $[0, T_1) \subset [0, T)$, where T_1 is not infinitesimal. Therefore, the boundedness of $E(t)$ over $[0, T_1)$ can be guaranteed and we need only focus on the interval $[T_1, T)$.

For any $t \in [T_1, T)$, the derivative of $E(t)$ is

$$\dot{E}(t) = (\delta\dot{\mathbf{x}}^T P \delta\mathbf{x} + \delta\mathbf{x}^T P \delta\dot{\mathbf{x}}) + \mathbf{e}^T \dot{\mathbf{e}} + \frac{1}{2} trace(\Phi^T \Gamma^{-1} \Phi). \quad (9.24)$$

Note that in above equation, the first and second terms on the RHS have been derived and given in (9.11) and (9.17) respectively. Let us concentrate on the third term on the RHS of (9.24). Since $\Gamma_0(t)$ is diagonal and each diagonal element is strictly increasing in $[0, T)$, $trace(\Gamma^{-1}) \leq trace(\Gamma_0^{-1})$ is ensured in the time interval $[T_1, T)$. Therefore, by substituting the learning law (9.6),

$$\begin{aligned}
&\frac{1}{2} trace(\Phi^T \Gamma^{-1} \Phi) \\
&\leq \frac{1}{2} trace(\Phi^T \Gamma_0^{-1} \Phi) \\
&= \frac{1}{2} trace(\Theta^T \Gamma_0^{-1} \Theta) - trace(\hat{\Theta}^T \Gamma_0^{-1} \Phi) - \frac{1}{2} trace(\hat{\Theta}^T \Gamma_0^{-1} \hat{\Theta}) \\
&\leq \frac{1}{2} trace(\Theta^T \Gamma_0^{-1} \Theta) + \mathbf{e}^T C B \Phi \hat{\boldsymbol{\xi}}.
\end{aligned} \tag{9.25}$$

Substituting (9.11), (9.17) and (9.25) into (9.24) yields

$$\dot{E} \leq -\|\mathbf{e}\|^2 - \frac{3\lambda}{4} \|\delta \mathbf{x}\|^2 + \frac{1}{2} trace(\Theta^T \Gamma_0^{-1} \Theta) \leq \frac{1}{2} trace(\Theta^T \Gamma_0^{-1} \Theta). \tag{9.26}$$

The boundedness of Θ leads to the boundedness of $\dot{E}(t)$. As $E(T_1)$ is bounded, $\forall t \in [T_1, T)$ the finiteness of $E(t)$ is obvious.

(III) Boundedness Property

According to preceding derivations, $E(t)$ is bounded for any $t \in [0, \infty)$, which leads to the boundedness of $\mathbf{y}(t)$ and $\delta \mathbf{x}(t)$. According to the structure of the observer (9.2), a stable $FA - LC$ and bounded $\mathbf{y}$ ensures the boundedness of $\mathbf{v}$, in the sequel the finiteness of $\hat{\mathbf{x}}$. As both $\delta \mathbf{x}$ and $\hat{\mathbf{x}}$ are finite, the system states $\mathbf{x}$ are bounded. On the other hand, the boundedness of $E(t)$ implies the $\mathcal{L}_T^2$ boundedness of $\hat{\Theta}$. Therefore, according to the control law (9.5), $\mathbf{u}$ is bounded in the $\mathcal{L}_T^2$ norm. ■

9.3.2 Observer Based IIL With Unknown θ_m and l

In this section, θ_m and l are supposed to be finite but completely unknown. The proposed IIL control law is

$$\mathbf{u} = -\hat{\Theta}(t)\hat{\boldsymbol{\xi}} + (CB)^{-1}(\dot{\mathbf{y}}_d - CA\hat{\mathbf{x}} + K\mathbf{e} + \hat{s}\mathbf{e}) \tag{9.27}$$

$$\hat{\Theta}(t) = \begin{cases} -\Gamma_0(t)(CB)^T\mathbf{e}\hat{\boldsymbol{\xi}}^T & t \in [0, T) \\ \hat{\Theta}(t-T) - \Gamma(CB)^T\mathbf{e}\hat{\boldsymbol{\xi}}^T & t \in [T, \infty) \end{cases} \tag{9.28}$$

$$\dot{\hat{s}} = \|\mathbf{e}\|^2 \qquad \hat{s}(0) = 0, \tag{9.29}$$

where $\hat{\Theta}$ is to approximate Θ and $\hat{s}$ is to estimate a constant $s = \dfrac{(\|CA\| + \|CB\|\theta_m l)^2}{\lambda}$ where $\theta_m l$ is unknown.

Similarly to (9.7), according to the control law (9.27), (9.28) and (9.29), the output tracking error dynamics is

$$\dot{\mathbf{e}} = -K\mathbf{e} + \mathbf{g} - CB\Phi\hat{\boldsymbol{\xi}} - \hat{s}\mathbf{e}. \tag{9.30}$$

Theorem 9.2. *The control law (9.27), the algebraic learning law (9.28) and the adaptation law (9.29) ensure the convergence of the state estimation and the output tracking in $\mathcal{L}_T^2$ norm.*

Proof:

The following CEF is used to evaluate the convergence property.

$$E(t) = \delta\mathbf{x}^T P \delta\mathbf{x} + \frac{1}{2}\|\mathbf{e}\|^2 + \frac{1}{2}\int_{t-T}^{t} trace[\Phi^T(\tau)\Gamma^{-1}\Phi(\tau)]d\tau + \frac{1}{2}\tilde{s}^2, \tag{9.31}$$

where $\tilde{s}(t) = s - \hat{s}(t)$.

For any $t \geq T$, the difference of the CEF over one period is

$$\begin{aligned}
\Delta E(t) &\triangleq E(t) - E(t-T) \\
&= [\delta\mathbf{x}^T(t)P\delta\mathbf{x}(t) - \delta\mathbf{x}^T(t-T)P\delta\mathbf{x}(t-T)] + \frac{1}{2}[\|\mathbf{e}(t)\|^2 - \|\mathbf{e}(t-T)\|^2] \\
&\quad + \frac{1}{2}\int_{t-T}^{t}\{trace[\Phi^T(\tau)\Gamma^{-1}\Phi(\tau)] - trace[\Phi^T(\tau-T)\Gamma^{-1}\Phi(\tau-T)]\}d\tau \\
&\quad + \frac{1}{2}[\tilde{s}^2(t) - \tilde{s}^2(t-T)]. \qquad (9.32)
\end{aligned}$$

There are four terms on the RHS of (9.32): the first is concerned with the state estimation error; the second is concerned with the output tracking error; and the third and fourth are concerned with the parametric estimation errors respectively.

For the first term, (9.11) and (9.12) are still valid. For the second term, (9.13) can be derived. Therefore, from (9.30), we can obtain

$$\begin{aligned}
&\mathbf{e}^T(t)\dot{\mathbf{e}}(t) \\
&= -\mathbf{e}^T(t)K\mathbf{e}(t) + \mathbf{e}^T\mathbf{g} - \mathbf{e}^TCB\Phi\hat{\boldsymbol{\xi}} - \hat{s}\|\mathbf{e}\|^2 \\
&\leq -\gamma\|\mathbf{e}\|^2 + \|\mathbf{e}\|\|\mathbf{g}\| - \mathbf{e}^TCB\Phi\hat{\boldsymbol{\xi}} - \hat{s}\|\mathbf{e}\|^2.
\end{aligned}$$

On the other hand, (9.16) can be rewritten as

$$\begin{aligned}
\|\mathbf{e}\|\|\mathbf{g}\| &\leq \frac{(\|CA\| + \|CB\|\theta_m l)^2}{\lambda}\|\mathbf{e}\|^2 + \frac{\lambda}{4}\|\delta\mathbf{x}\|^2 \\
&= s\|\mathbf{e}\|^2 + \frac{\lambda}{4}\|\delta\mathbf{x}\|^2.
\end{aligned}$$

Hence,

$$\mathbf{e}^T(t)\dot{\mathbf{e}}(t) \leq -\gamma\|\mathbf{e}\|^2 + \frac{\lambda}{4}\|\delta\mathbf{x}\|^2 - \mathbf{e}^TCB\tilde{\Theta}\hat{\boldsymbol{\xi}} + \tilde{s}\|\mathbf{e}\|^2, \qquad (9.33)$$

and

$$\begin{aligned}
&\frac{1}{2}[\|\mathbf{e}(t)\|^2 - \|\mathbf{e}(t-T)\|^2] \\
&\leq -\gamma\int_{t-T}^{t}\|\mathbf{e}\|^2 d\tau + \frac{\lambda}{4}\int_{t-T}^{t}\|\delta\mathbf{x}\|^2 d\tau - \int_{t-T}^{t}\mathbf{e}^TCB\tilde{\Theta}\hat{\boldsymbol{\xi}}d\tau \\
&\quad + \int_{t-T}^{t}\tilde{s}\|\mathbf{e}\|^2 d\tau. \qquad (9.34)
\end{aligned}$$

Regarding the third term on the RHS of (9.32), (9.19) and (9.20) are valid.

Now let us check the last term on the RHS of (9.32). First, it can be rewritten as

$$\frac{1}{2}[\tilde{s}^2(t) - \tilde{s}^2(t-T)] = \int_{t-T}^{t} \tilde{s}(\tau)\dot{\tilde{s}}(\tau)d\tau. \tag{9.35}$$

Since s is a constant, according to the adaptation law (9.29), we have

$$\tilde{s}\dot{\tilde{s}} = -\tilde{s}\dot{\hat{s}} = -\tilde{s}\|\mathbf{e}\|^2. \tag{9.36}$$

Therefore, from (9.35) and (9.36), it can be obtained that

$$\frac{1}{2}[\tilde{s}^2(t) - \tilde{s}^2(t-T)] = -\int_{t-T}^{t} \tilde{s}\|\mathbf{e}\|^2 d\tau. \tag{9.37}$$

Substitution of (9.12), (9.20), (9.34) and (9.37) into (9.32), $\forall t \geq T$, yields a negative difference of CEF

$$\Delta E(t) \leq -\gamma \int_{t-T}^{t} \|\mathbf{e}\|^2 d\tau - \frac{3\lambda}{4}\int_{t-T}^{t} \|\delta\mathbf{x}\|^2 d\tau. \tag{9.38}$$

Analogous to the *Part II* and *Part III* in the Theorem 9.1, the convergence and the boundedness properties can be guaranteed. ■

9.4 IIL for NGLC Systems

Now let us consider $\mathbf{z} = \mathbf{y}$ and $\boldsymbol{\xi}(\mathbf{y}, t)$ is local Lipschitz continuous. The same observer (9.2) is used for state estimation, and the IIL control law is constructed as

$$\mathbf{u} = -\hat{\Theta}\boldsymbol{\xi} + (CB)^{-1}(\dot{\mathbf{y}}_d - CA\hat{\mathbf{x}} + K\mathbf{e} + \hat{s}\mathbf{e}) \tag{9.39}$$

$$\hat{\Theta}(t) = \begin{cases} -\Gamma_0(t)(CB)^T\mathbf{e}\boldsymbol{\xi}^T & t \in [0, T) \\ \hat{\Theta}(t-T) - \Gamma(CB)^T\mathbf{e}\boldsymbol{\xi}^T & t \in [T, \infty) \end{cases} \tag{9.40}$$

$$\dot{\hat{s}} = \|\mathbf{e}\|^2 \qquad \hat{s}(0) = 0, \tag{9.41}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{y}, t)$. The proposed IIL scheme is analogous to the preceding learning control laws (9.27), (9.28) and (9.29), except for the replacement of the nonlinear term $\boldsymbol{\xi}(\hat{\mathbf{x}}, t)$ by $\boldsymbol{\xi}(\mathbf{y}, t)$. As a consequence, the error dynamics is

$$\dot{\mathbf{e}} = -K\mathbf{e} + \mathbf{g}' - CB\Phi\boldsymbol{\xi} - \hat{s}\mathbf{e}. \tag{9.42}$$

where $\mathbf{g}' = -CA\delta\mathbf{x}$.

The convergence of the proposed control scheme is summarized in the following theorem.

Theorem 9.3. *The control laws (9.39), (9.40) and (9.41) ensure that both the state estimation and the output tracking, i.e.* $\delta\mathbf{x}$ *and* $\mathbf{e}$*, converge in* $\mathcal{L}_T^2$ *norm.*

Proof:

The proof is much the same as Theorem 9.2, the only difference lies in between the two functions $\mathbf{g}$ and $\mathbf{g}'$. In Theorem 9.2, $\mathbf{g}$ is expressed as $-CA\delta\mathbf{x} - CB\Theta[\boldsymbol{\xi}(\mathbf{x}, t) - \boldsymbol{\xi}(\hat{\mathbf{x}}, t)]$ and is upper bounded by

$$\|\mathbf{g}\| \leq (\|CA\| + \|CB\theta_m l\|)\|\delta\mathbf{x}\|,$$

which leads to $s = \dfrac{(\|CA\| + \|CB\|\theta_m l)^2}{\lambda}$. On the other hand, in (9.42), $\mathbf{g}' = -CA\delta\mathbf{x}$, which is upper bounded by

$$\|\mathbf{g}'\| \leq \|CA\|\|\delta\mathbf{x}\|.$$

Therefore, simply let $s = \dfrac{\|CA\|^2}{\lambda}$ in the CEF (9.31), all the derivations and conclusions in Theorem 9.2 hold with regards to the learning convergence property and boundedness property. ∎

Remark 9.1. If the bound for each element of Θ is known *a priori*, $\forall t \geq T$, the learning laws (9.6), (9.28) and (9.40) can be modified as

$$\hat{\Theta}(t) = proj[\hat{\Theta}(t - T)] - \Gamma(CB)^T\mathbf{e}\hat{\boldsymbol{\xi}}^T,$$

and

$$\hat{\Theta}(t) = proj[\hat{\Theta}(t-T)] - \Gamma(CB)^T \mathbf{e}\boldsymbol{\xi}^T.$$

Consequently, the boundedness of $\hat{\Theta}$ can be ensured, which lead to the boundedness of $\mathbf{u}$, $\dot{\mathbf{x}}$, $\dot{\mathbf{y}}$ and $\dot{\hat{\mathbf{x}}}$. The finiteness of $\dot{\mathbf{y}}$ and $\delta\dot{\mathbf{x}}$ implies the uniform continuity of $\mathbf{y}$ and $\delta\mathbf{x}$. Hence, the uniform convergence of the tracking error $\mathbf{e}$ and the state estimation error $\delta\mathbf{x}$ is guaranteed.

9.5 Illustrative Examples

Consider the circuit model (Fig. 9.1). The system parameters are: resistors $R_1 = 1\Omega$ and $R_2 = 1\Omega$, inductors $L_1 = 0.36H$ and $L_2 = 0.5H$, and the mutual inductor $M = 0.15H$. i_1 and i_2 are the loop currents, u is an input voltage, and η represents the input perturbation. Defining $x_1 = i_1$ and $x_2 = i_2$, the circuit can be formulated as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1 L_2}{L_1 L_2 - M^2} & \frac{R_2 M}{L_1 L_2 - M^2} \\ \frac{R_1 M}{L_1 L_2 - M^2} & -\frac{R_2 L_1}{L_1 L_2 - M^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{L_2 - M}{L_1 L_2 - M^2} \\ \frac{L_1 - M}{L_1 L_2 - M^2} \end{bmatrix} [u(t) + \eta].$$

The physically measurable output is $y = x_1$. The target trajectory is $y_d = \sin^3(\pi t)$.

Case 1: η is GLC and θ_m and l are known.

The input perturbation $\eta(\mathbf{x}, t) = x_2 \sin^3 t + 0.8 \sin^2 t \sin x_1$ is state-related and GLC. It can be factorized as $\boldsymbol{\theta\xi}$ where $\boldsymbol{\theta} = [\sin^3 t \quad 0.8 \sin^2 t]$ and $\boldsymbol{\xi} = [x_2 \quad \sin x_1]^T$. Note that the period of $\boldsymbol{\theta}$ is $T = 2\pi$ which has no common period with $y_d(t)$. The initial conditions are set as: $x_1(0) = 0.3$, $x_2(0) = 0.2$, $z_1(0) = 0$ and $z_2(0) = 0$.

Let $L = [3.5 \quad 4]^T$, the eigenvalues of $FA - LC$ are -3.50 and -2.86 respectively. Assume the known bound $\theta_m = 1.5$ and $l = 1$. Choose $K = 5$. The learning gains are chosen to be $\Gamma = 50$ and $\Gamma_0 = 50t/T$.

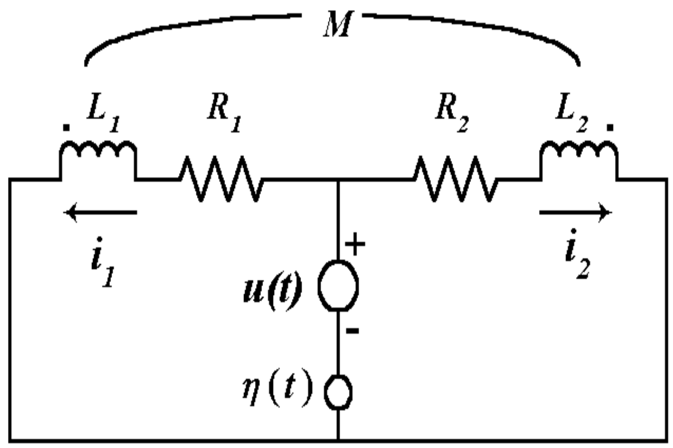

Figure 9.1: The circuit network.

Applying the learning control law (9.5) and (9.6), the simulation results are shown in Fig. 9.2 and Fig. 9.3. The horizon is the number of periods and the vertical quantities are $|y_d - y|_{sup}$ and $|x_k - \hat{x}_k|_{sup}$ $(k = 1, 2)$ respectively. It can be seen, that observer converges very quickly. The rapid learning convergence can also been observed.

Case 2. η is GLC and θ_m is unknown.

Choose the same L, $\Gamma_0(t)$ and Γ. Let $K = 1$. Based on the control law (9.27), (9.28) and (9.29), Fig. 9.4 and Fig. 9.5 show the achieved results. From the simulation results, it can be seen that both the estimated state error and the output tracking error have been reduced greatly after a number of periods.

Case 3: η is NGLC.

Assume $\eta = 0.2y\sin^3 t + 0.1y^2 \sin t$, which is output-dependent and NGLC. Here $\boldsymbol{\theta} = [0.2\sin^3 t \;\; 0.1sint]$ and $\boldsymbol{\xi} = [y \;\; y^2]^T$. The learning control design is the same as *Case 2.* Based on learning control law (9.39), (9.40) and (9.41), Fig. 9.6 and Fig. 9.7 show the achieved results. From the simulation results, it can be seen that although the system nonlinearities are NGLC, the learning convergence of both the estimated state error and the output tracking error still can be guaranteed.

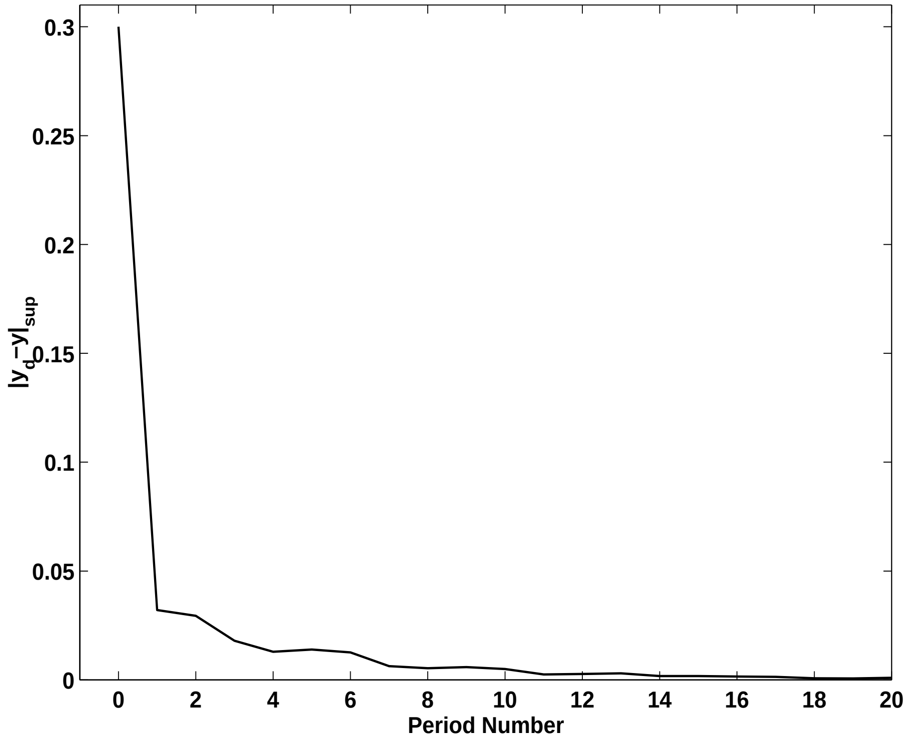

Figure 9.2: Convergence profile of $y_d - y$ (*Case 1*).

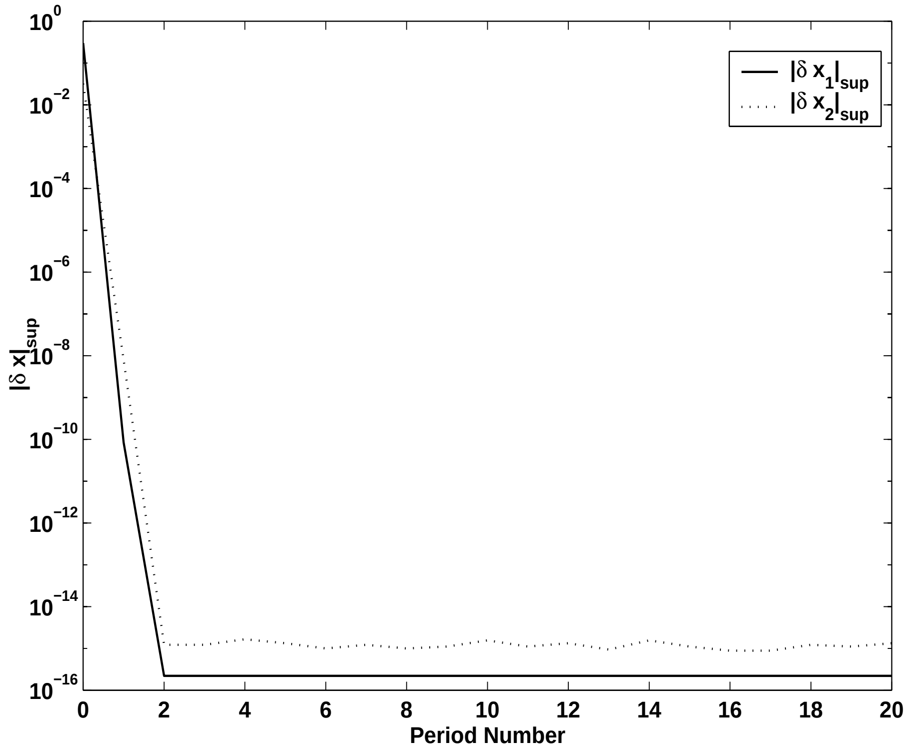

Figure 9.3: Convergence profile of $\mathbf{x} - \hat{\mathbf{x}}$ (*Case 1*).

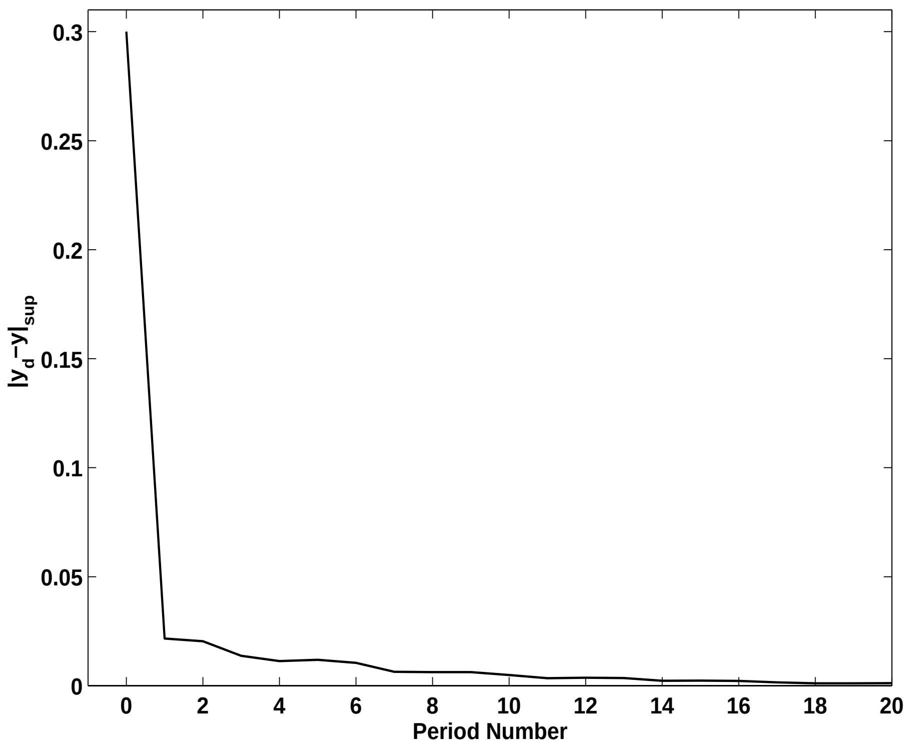

Figure 9.4: Convergence profile of $y_d - y$ (*Case 2*).

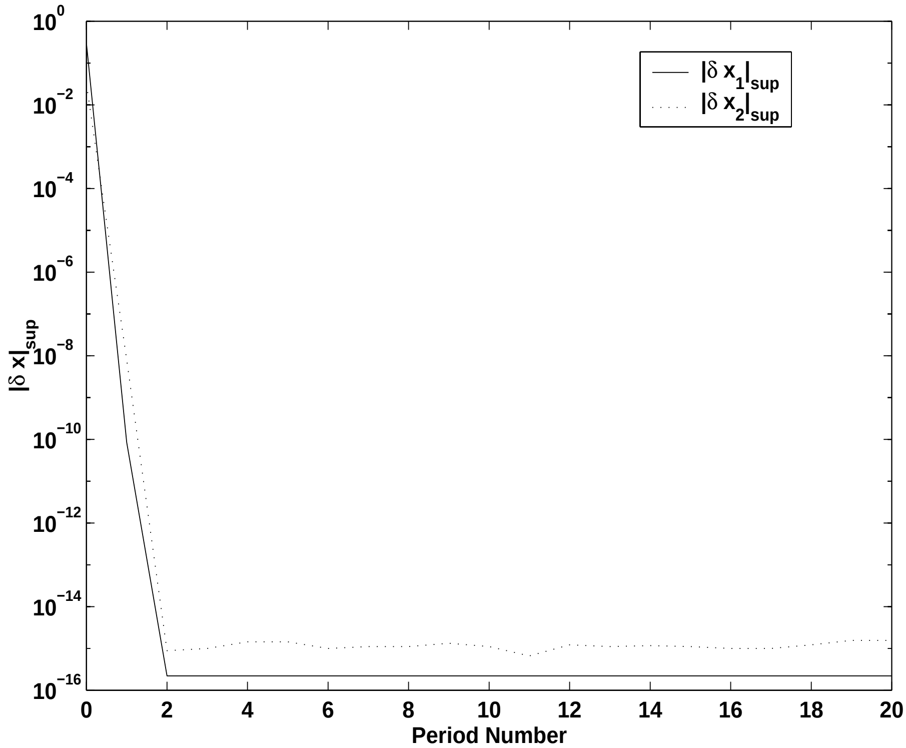

Figure 9.5: Convergence profile of $\mathbf{x} - \hat{\mathbf{x}}$ (*Case 2*).

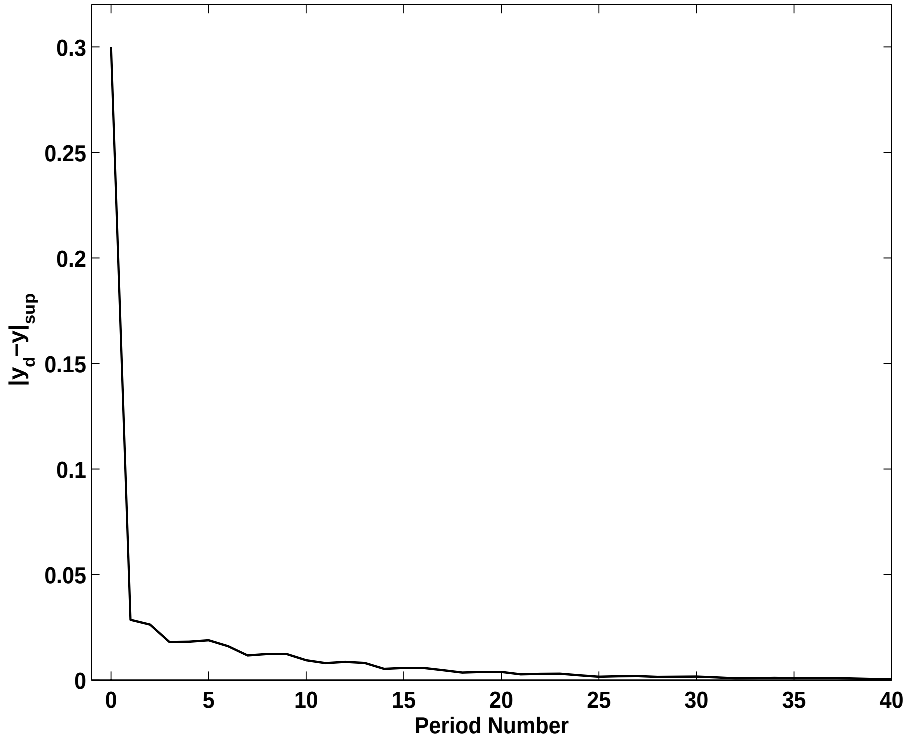

Figure 9.6: Convergence profile of $y_d - y$ (*Case 3*).

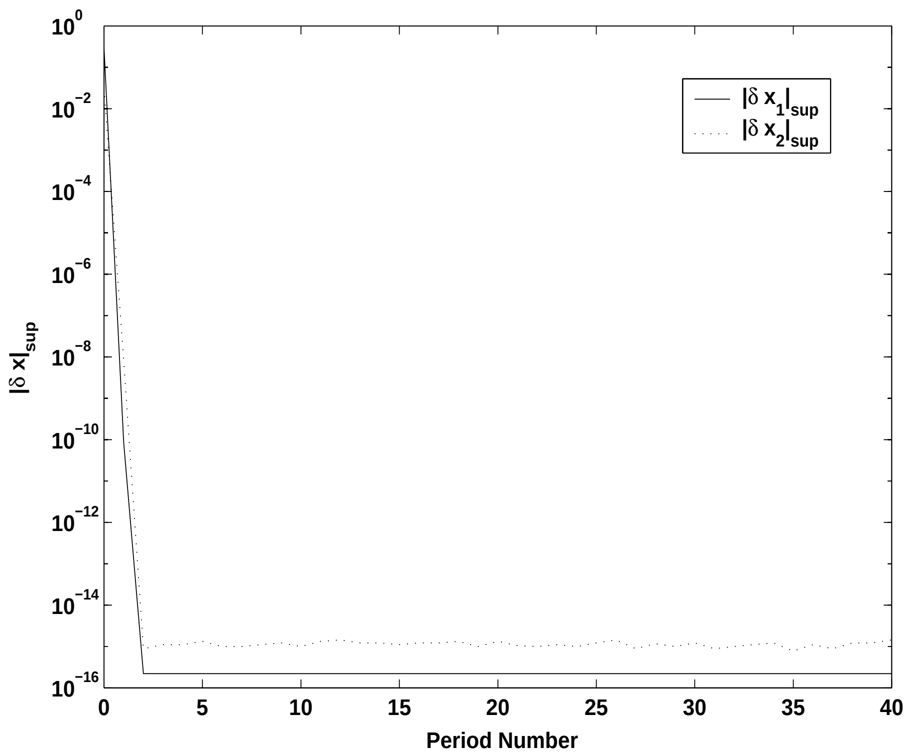

Figure 9.7: Convergence profile of $\mathbf{x} - \hat{\mathbf{x}}$ (*Case 3*).

9.6 Conclusion

This chapter has developed a new IIL methodology for systems with time-varying parametric uncertainties, global and non-global Lipschitzian nonlinearities. Based on the state estimation and periodic updating, the proposed IIL scheme guarantees the asymptotical convergence of the output tracking in $\mathcal{L}_T^2$ norm and the boundedness of the system states. Simulation results clearly demonstrate the effectiveness of the observer based IIL approach.

Chapter 10

Conclusion

10.1 Conclusion

This thesis was centered on the control theories of FIL and IIL for nonlinear systems with deterministic uncertainties.

- **Theories of FIL**

  - **CM-type FIL**

    In Chapter 2, CM-type FIL was extended to discrete-time systems with input deadzone which is a typical kind of non-smooth nonlinearities. It has been shown that although the parameters of the input deadzone are completely unknown, only if the control environment and the tracking target are repeatable, the proposed simple FIL can automatically compensate the input deadzone by iteration. It is assumed that, including the unknown input deadzone, the dynamic system may also have some unknown but GLC uncertainties. Moreover, the parameters of the input deadzone can be constant or time-varying. Via rigorous proof based on

CM principle, it is clearly shown that, in the presence of all the uncertainties, the perfect tracking can be obtained as the iteration approaches to infinity.

Chapter 3 was a continuity of Chapter 2. In this chapter, CM-type FIL was further applied to handle discrete-time systems with input backlash which is also a class of practice-relevant high nonlinearity. However, backlash is much more complicated due to its property of memory. Analogous to Chapter 2, based on CM principle, it has been proved that, in the presence of unknown backlash and unknown but GLC system dynamics, the developed FIL can cancel the harmful effect of all the uncertainties and guarantee the perfect tracking iteratively.

– **CEF-type FIL**

In Chapter 4, CEF-type FIL schemes were presented for continuous-time systems with norm-bounded uncertainties. It has been shown that, for GLC norm-bounded uncertainties, which may be handled by CM-type FIL, CEF-type FIL can also work effectively. On the other hand, for NGLC norm-bounded uncertainties, which can not be addressed by CM-type FIL, the proposed robust FIL, combining robust control with CEF-type FIL, still can guarantee the learning convergence. Moreover, benefiting from the concept of CEF, the I.I.C. for systems with norm-bounded uncertainties may be replaced by the alignment condition.

In Chapter 5, we explored the possibility for FIL to learn from nonuniform tracking trajectories for continuous-time systems with time-varying parametric uncertainties. Based on CEF, two novel FIL approaches were introduced for tracking non-uniform trajectories in the presence of time-varying and both time-varying and time-invariant parametric uncertainties respectively. In the proposed algorithms, the time-varying parametric uncertainties are handled by CEF-type FIL, while the known system dynamics related to the different tracking targets are canceled by the control signal. It has been proven that the tracking error uniformly converges to zero as the iteration time approaches infinity.

A new FIL control approach - FLLC was outlined for repeatable tracking control tasks in Chapter 6. FLLC integrates two main control strategies: FLC as the basic control part and FIL as the refinement part. The new FLLC is constructed by simply adding a FIL mechanism to a PD-type fuzzy logic controller in additive form. Through rigorous proof based on EF, it has been shown that the tracking error of the proposed FLLC system converges uniformly to zero iteratively.

- **Theories of IIL**

By taking the advantage of the concept and the analysis method of CEF, most of the theories of FIL can be extended to IIL.

In Chapter 7, FIL for systems with time-varying parametric uncertainties (Xu and Tan, 2002) was first extended to IIL case. Only if the parametric uncertainties are periodic, the proposed IIL scheme can guarantee the perfect tracking as time approach infinity, no matter the tracking trajectory is periodic or not. Therefore, this chapter can also be treated as an extension of Chapter 5. Moreover, both the GLC requirement in CM-type FIL and the I.I.C. in almost all FIL can be removed, which greatly widens the application of learning control.

As a counterpart of Chapter 3, IIL for systems with norm-bounded uncertainties was discussed in Chapter 8. Only if the norm-bounded uncertainties are periodic in time and the desired trajectory has a common period, FIL for systems with norm-bounded uncertainties can be extended to IIL case. It has been shown that, even with norm-bounded uncertainties, no matter they are

GLC or NGLC, the perfect tracking can be realized asymptotically

To facilitate the implementation of CEF-type learning approaches, observer based IIL for systems with time-varying parametric uncertainties was proposed in Chapter 9. Based on the state estimation, the learning convergence still can be guaranteed even if the system states are not available. Moreover, if the I.I.C. is satisfied, observer based IIL can be applied to FIL directly.

10.2 Recommendation for Future Research

Based on the prior research, the following points deserve further investigation.

- CM-type FIL has been extended to deal with systems with GLC uncertainties and input deadzone or input backlash. Is it possible to further apply it to the other non-smooth nonlinearities, such as hysteresis? As we have seen, in Chapter 2 and Chapter 3, the I.I.C. and GLC are essential as the proposed FIL schemes are based on CM principle. If the CEF based design and analysis method can be applied, the GLC can be removed accordingly. Moreover, based on CEF, the FIL for non-smooth nonlinearities may be extended to IIL case. How to apply CEF concept to systems with non-smooth nonlinearities is worthy of further study.

- FIL/IIL for systems with either parametric uncertainties or norm-bounded uncertainties have been discussed. If both parametric and norm-bounded uncertainties exist, we need to find an appropriate way to integrate the methods proposed in this thesis.

- In Chapter 5, FIL from different tracking targets for systems with parametric uncertainties was discussed. Extending it to systems with norm-bounded uncertainties is a quite interesting future work. Similarly, how to apply observer-based FIL/IIL proposed in Chapter 9 to systems with norm-bounded uncertainties could also be a future study.

- In the IIL, the learning updating is based on the known common period T. If T is unknown or not accurately known, what will the effect be and how to eliminate the harmful effect? If all the uncertainties are periodic, however, a common period T can not be found, how to construct the learning approaches?

- In this thesis, several CEF-type FIL/IIL have been proposed for continuous-time systems. How to implement CEF based learning to discrete-time uncertain systems will be the future work.

- Through rigorous proof, it has been clearly shown that the learning convergence can be guaranteed in all the proposed FIL/IIL schemes. However, it is only a steady state property. In classical control many indices are used to specify transient performance such as setting time, overshoot and oscillatory response. Is it possible for us to quantify similar performance indices to describe the transient properties along learning axis? How to conduct quantitative evaluation and design in learning domain is a meaningful work.

- Including CM principle, 2-D theory and CEF theory, is it possible to find some other analysis method which could extend FIL/IIL to more general systems? Lyapunov functional is a good candidate. Lots of future studies needed to answer this question.

All in one, there are still many open problems in FIL/IIL for further investigation and study.

Bibliography

Alleyne, A. and Pomykalski, M. (2000). Control of a class of nonlinear systems subject to periodic exogenous signals. *IEEE Transactions on Control Systems Technology* **8**(2), 279–287.

Arimoto, S., Kawamura, S. and Miyazaki, F. (1984). Bettering operation of robots by learning. *Journal of Robotic Systems* **1**, 123–140.

Behera, L. and Anand, K. K. (1999). Guaranteed tracking and regulatory performance of nonlinear dynamic systems using fuzzy neural networks. *IEE Proceedings-Control Theory and Applications* **146**(5), 484–491.

Bien, Z. and Chung, D. H. (1980). Optimal control of delay systems with final function condition. *International Journal of Control* **32**(3), 539–560.

Bien, Z. and Xu, J. X. (1998). *Iterative Learning Control – Analysis, Design, Integration and Applications*. Kluwer Academic Publishers. Boston, USA. ISBN 0–7923–8213–7, (Hardcover).

Bien, Z., Park, K. H. and Wang, D. H. (1999). Study on the robustness of PID type iterative learning control with current feedback for uncertain linear system. *International Journal of System Science* **30**, 49–59.

Broberg, H. L. and Molyet, R. G. (1992). Correction of period errors in a weather satellite servo using repetitive control. In: *1st IEEE Conference on Control Application*. pp. 682–683.

Buckley, J. J. and Ying, H. (1989). Fuzzy controllers theory: Limit theorems for linear fuzzy control rules. *Automatica* **25**, 469–472.

Cetinkunt, S. and Domez, A. (1993). CMAC learning controller for servo control of high precision machine tools. In: *Proceedings of the American Control Conference*. pp. 1976–1980.

Chen, Y. Q. and Wen, C. Y. (1999). *Iterative Learning Control – convergence, robustness and applications*. Vol. 248 of *Lecture Notes in Control and Information Science*. Springer-Verlag. London, UK.

Chien, C. J. (1998). Discrete iterative learning control for a class of nonlinear time–varying systems. *IEEE Transactions on Automatic Control* **43**(5), 748–752.

Chien, C. J. (2000). A sampled-data iterative learning control using fuzzy network design. *International Journal of Control* **73**(10), 902–913.

Choi, B.-J., Kwak, S.-W. and Kim, B. K. (1999). Design of a single-input fuzzy logic controller and its properties. *Fuzzy Sets and Systems* **106**, 299–308.

Chow, T. W. S. and Fang, Y. (1998). An iterative learning control method for continuous -time systems based on 2-D system theory. *IEEE Transaction on Circuits and Systems I: Fundamental Theory and Applications* **45**(4), 683–689.

Chow, T. W. S. and Fang, Y. (1998). An iterative learning control method for continuous-time systems based on 2-D system theory. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* **45**(6), 683–689.

Curtelin, G., Saari, H. and Caron, B. (1993). Repetitive control of continuous systems: comparative study and application. *IEEE Transaction on Systems, Man and Cybernetics* **5**(2), 231–234.

Darouach, M., Zasadzinski, M. and Xu, S. J. (1994). Full-order observer for linear systems with unknown inputs. *IEEE Transaction on Automatic Control* **39**(3), 606–609.

Dean, S. R. H., Surgenor, B. W. and Iordanou, H. N. (1995). Experimental evaluaiton of a backlash inverter as applied to a servomotor with gear train. In: *Proceedings of the Fourth IEEE Conference Control Application*. pp. 580–585.

Desoer, C. and Sharuz, S. (1986). Stability of dithered non-linear systems with backlash or hysteresis. *International Journal of Control* **43**(4), 1045–1060.

Dixon, W. E., Zergeroglu, E., Dawson, D. M. and Costic, B. T. (2002). Repetitive learning control : A Lyapunov-based approach. *IEEE Trans. on Systems, Man and Cybernetics - Part B* **32**(4), 538–545.

Fang, Y. and Wilde, R. W. (1988). Observers for linear systems with unknown inputs. *IEEE Transaction on Automatic Control* **33**, 677–681.

Fang, Y. and Chow, T. W. S. (1998). Iterative learning control of linear discrete-time multivariable systems. *Automatica* **34**(11), 1459–1462.

Fang, Y., Yeng, C. S. and Feng, G. G. (2002). Convergence analysis of iterative learning control with uncertain initial conditions. In: *Proceedings of the 4th World Congress on Intelligent Control and Automation*. pp. 960–963.

Francis, B. and Wonham, W. (1975). The internal model principle for linear multivariable regulators. *Journal of Applied Mathematics and Optimization* **2**(2), 175–194.

French, M. and Rogers,E. (2000). Non-linear iterative learning by an adaptive Lyapunov technique. *International Journal of Control* **73**(10), 840–850.

Fu, K. S. (1970). Learning control systems - review and outlook. *IEEE Transaction on Automatic Control* **15**(4), 210–221.

Ham, C., Qu, Z. H. and Kaloust, J. H. (1995). Nonlinear learning control for a class of nonlinear systems based on Lyapunov's direct method. In: *Proc. of American Control Conference*. Vol. 11. Seattle, Whashinton, USA. pp. 3024–3028.

Hara, S., Omata, T. and Nakano, M. (1987). Nonlinear repetitive control with application to trajectory of manipulators. *Journal of Robotic System* **4**(5), 631–652.

Hara, S., Yamamoto, Y., Omata, T. and Nakano, M. (1988). Repetitive control system: A new type servo system for periodic exogenous signals. *IEEE Transaction on Automatic Control* **37**(7), 659–668.

Hara, S., Trannitad, P. and Chen, Y. (1994). Robust stabilization for repetitive control systems. In: *Proceedings of 1st Asian Control Conference*. pp. 541–544.

Hikita, H., Yamashita, M. and Kubota, Y. (1993). Repetitive control for a class of nonlinear systems. *JSME International Journal* **36**(4), 430–434.

Hklansky, J. (1966). Learning systems for automatic control. *IEEE Transaction on Automatic Control* **11**(1), pp.6–19.

Hwang, D. H., Bien, Z. and Oh, S. R. (1991). Iterative learning control method for discrete-time dynamic systems. In: *IEE Proceedings-D*. **138**(2), pp. 139–144.

Ichikawa, R., Nishimura, K., Kunugi, M. and Shimada, K. (1992). Auto-tuning method of fuzzy membership functions using neutral network learning algorithm. In: *Proceedings of Second Conference on Fuzzy Logic and Neural Networks*. pp. 345–348.

Inoue, T., Nakano, M., Kubo, T., Matsumoto, S. and Baba, H. (1981). High accuracy control of a proton synchrotron magnet power supply. In: *Proceedings of 8th World Congress of IFAC*. pp. 3137–3142.

Ioannou, P. A. and Sun, J. (1996). *Robust adaptive control*. Prenticd-Hall International. Englewood Cliffs, New Jersey.

Khalil, H. K. (1990). *Nonlinear Systems (Second Edition)*. John Wiley and Sons, Inc.. New York.

Khalil, H. K. (1992). *Nonlinear Systems*. Vol. 8 of *Second Prentice-Hall International Editions*. Macmillan Publishing Company. New York.

Khalil, H. K. (1994). Robust servomechanism output feedback controllers for feedback linearizable systems. *Automatica* **30**(10), 1587–1599.

Kim, J. H., Park, J. H. and Lee, S. W. (1993). Fuzzy precompensation of PD controllers for systems with deadzones. *Journal of Intelligent and Fuzzy Systems* **1**(2), 125–133.

Kim, J. H., Park, J. H. and Lee, S. W. (1993). Fuzzy presompensated PID controller. In: *Proceedings of Second IEEE Conf. on Control Applications*. pp. 183–188.

Kim, J. H., Park, J. H., Lee, S. W. and Cheng, E. K. P. (1994). A two layered fuzzy logic controller for systems with deadzones. *IEEE Trans. Ind. Electron.* **41**, 155–162.

Kurek, J. E. and Zaremba, M. B. (1993). Iterative learning control synthesis based on 2-D system theory. *IEEE Transaction on Automatic Control* **38**, 121–125.

Lee, C. C. (1990). Fuzzy logic in control system: fuzzy logic controller. *IEEE Transactions on Systems, Man and Cybernetics* **20**(2), 408–435.

Lee, S. W. and Kim, J. H. (1994). Control of systems with deadzones using neural network based learning control. In: *Proc. IEEE Int. Conf. Neural Networks.* pp. 2535–2538.

Lewis, F. L., Liu,K. and Wang,L,-X. (1997). Adaptive fuzzy logic compensation of actuator deadzones. *Journal of Robotic Systems* **14**(6), 501–511.

Lin, C. T. and Lee, C. S. G. (1991). Neural network-based fuzzy logic control and decision system. *IEEE Transactions on Computer* **40**(12), 1320–1336.

Liu, J. W. and Tsao, T.-C. (2001). Robust performance repetitive control systems. *ASME J. Dynam. Syst., Measur., Contr.* **123**(3), 330–337.

Malki, H. A., Li, H. and Chen, G. (1994). New design and stability analysis of fuzzy proportional-derivative control systems. *IEEE Transactions on Fuzzy Systems* **2**(4), 245–254.

Mamdani, E. H. and Assilian, S. (1974). Applications of fuzzy algorithms for control of simple dynamic plant. In: *Proceedings Institute of Electrical Engineering.* Vol. 121. pp. 1585–1588.

Manayathara, T. J., Tsao, T.-C. and Bentsman, J. (1996). Rejection of unknown periodic load disturbances in continuous steel casting process using learning repetitive control approach. *IEEE Trans. on Control Systems Technology* **4**(3), 259–265.

Middleton, R. H., Goodwin, G. C. and Longman, R. W. (1989). A method for improving the dynamic accuracy of a robot performing a repetitive task. *Int. J. Robotics Res.* **8**, 67–74.

Moore, K. L. (1989). *Design techniques for transient response control.* PhD thesis, Texas A& M. USA.

Moore, K. L. (1993). *Iterative learning control for deterministic systems*. Vol. 22 of *Advances in Industrial Control*. Springer-Verlag. London.

Moore, K. L. and Chen, Y. Q. (2002). On monotonic convergence of high order iterative learning update laws. In: *Proceedings of 15th Triennial World Congress*.

Nakano, M. and Hara, S. (1986). *Microprocessor based repetitive control*. D. Reidel publishing Co.

Ng, K. C. and Trivedi, M. M (1998). A neuro-fuzzy controller for mobile robot navigation and multirobot. *IEEE Transactions on Systems, Man and Cybernetics* **28**(6), 829–843.

Nordin, M. and Gutman, P.-O. (2000). Nonlinear speed control of elastic systems with backlash. In: *Proceedings of the 39th IEEE Conference on Decision and Control*. pp. ThP13–4.

Park, K. H., Kuc, T.-Y. and Lee, J. S (1996). Adaptive learning control of uncertain robotic systems. *International Journal of Control* **65**(5), 725–744.

Park, K. H. and Bien, Z. (2000). A generalized iterative learning controller against initial state error. *International Journal of Control* **73**(10), 871–881.

Qu, Z. H., Ham, C. and Kaloust, J. (2001). Nonlinear learning control for a class of nonlinear systems. *Automatica* **37**, 419–428.

Recker, D. and Kokotović, P. V. (1991). Adaptive nonlinear control of systems contaning a deadzone. In: *Proceedings of the 30th Conference on Decision and Control*. pp. 2111–2115.

Roesser, R. P. (1975). A discrete state-space model for linear processing. *IEEE Transaction on Automatic Control* **20**, 1–10.

Saab, S. S. (1995). A discrete-time learning control algorithm for a class of linear time-invariant systems. *IEEE Transaction on Automatic Control* **40**(6), 1138–1141.

Saab, S. S., Vogt, W. G. and Mickle, M. H. (1997). Learning control algorithms for tracking slowly varying trajectories. *IEEE Transactions on Systems, Man and Cybernetics* **27**(4), 657–670.

Sacks, A. H., Bodson, M. and Messner, W. (1995). Advanced methods for repeatable runout compensation. *IEEE Transactions on Magnetics* **31**(2), 1031–1036.

Sadegh, N., Horowitz,R., Kao, W. W. and Tomizuka, M. (1988). A unified approach to design of adaptive and repetitive controllers for robotic manipulators. In: *Proceedings of USA-japan Symposium on Flezible Automation*. Vol. 1. pp. 223–231.

Sadegh, N. (1991). Synthesis and stability analysis of repetitive controllers. In: *Proceedings of American Control Conference*. pp. 2634–2639.

Seidl, D. R., Lam, S.-L., Putman, J. A. and Lorenz, D. R. (1995). Neural network compensation of gear backlash hysteresis in position-controlled mechanisms. *IEEE Transaction on Industry Application* **31**(6), 1475–1483.

Selmić, R. R. and Lewis, F. L. (2000). Deadzone compensation in motion control systems using neural networks. *IEEE Transaction on Automatic Control* **45**(4), 602–613.

Shibata, T., Fukuda, T. and Tanie, K. (1993). Nonlinear backlash compensation using recurrent neural network - unsupervised learning by genetic algorithm. In: *Proceedings of 1993 International Joint Conference on Neural Networks*. pp. 742–745.

Srinivasan, K. and Shaw, F.-R. (1991). Analysis and design of repetitive control systems using regeneration spectrum. *ASME J. Dynam. Syst., Measur., Contr.* **113**(2), 216–222.

Sun, M. X. and Wang, D. W. (2002). Iterative learning control with initial rectifying action. *Automatica* **38**, 1177–1182.

Tao, G., and Kokotović, P. V. (1993). Adaptive control of systems with backlash. *Automatica* **29**(2), 323–335.

Tao, G. and Kokotović, P. V. (1994). Adaptive control of plants with unknown dead-zones. *IEEE Transaction on Automatic Control* **39**(1), 59–68.

Tao, G. and Kokotović, P. V. (1996). *Adaptive control of systems with actuator and sensor nonlinearities*. John Wiley and Sons, Inc.. New York.

Tao, G. and Kokotović, P. V. (1997). Adaptive control of systems with unknown non-smooth non-linearities. *International Journal of Adaptive Control and Signal Processing* **11**, 81–100.

Taylor, J. and Lu, J. (1995). Robust nonlinear control system synthesis method for electro-mechanical pointing systems with flexible modes. *Journal of Systems Engineering* pp. 192–204.

Tomizuka, M. (1987). Zero phase error tracking algorithm for digital control. *ASME J. of Dyn. syst., Meas., and Contr.* **109**(1), 65–68.

Utkin, V. (1978). *Sliding modes and their application in variable structure systems*. Mir. Moscow.

Wang, D. H. (2000). On D-type and P-type ILC designs and anticipatory approach. *International Journal of Control* **73**(10), 890–901.

Wang, H. O., Tanaka,K. and Griffin, M. F. (1996). An approach to fuzzy control of nonlinear systems: stability and the design issues. *IEEE Transactions on Fuzzy Systems* **4**(1), 14–23.

Xu, J. X., Jia, Q. W and Lee, T. H. (1996). Analysis of adaptive backstepping control schemes for PM synchronous motors. In: *In Proceedings of 35th IEEE Conference on Decision & Control*. Kobe, Japan. pp. 3516–3521.

Xu, J. X. (1997). Analysis of iterative learning control for a class of nonlinear discrete-time systems. *Automatica* **33**, 1905–1907.

Xu, J. X. (1997). Direct learning of control effors for trajectories with different magnitude scales. *Automatica* **33**(12), 2191–2195.

Xu, J. X., Pok, Y.-M., Liu, C. and Hang, C.-C. (1998). Tuning and analysis of a fuzzy PI controller based on gain and phase margins. *IEEE Transactions on Systems, Man and Cybernetics* **28**(5), 685–691.

Xu, J. X. and Qu, Z. H. (1998). Robust iterative learning control for a class of nonlinear systems. *Automatica* **34**, 983–988.

Xu, J. X. and Viswanathan, B. (2000). Adaptive robust iterative learning control with deadzone scheme. *Automatica* **36**, 91–99.

Xu, J. X. and Song, Y. B. (2000). Multi-scale direct learning of control of linear time-varying high-order systems. *Automatica* **36**, 61–68.

Xu, J. X. (2002). The frontiers of iterative learning control – part II. *Journal of Systems, Control and Information.*

Xu, J. X. and Tan,Y. (2002). A composite energy function based learning control approach for nonlinear systems with time-varying parametric uncertainties. *IEEE Transactions on Automatic Control* **47**(11), 1940–1945.

Ying, H., Siler, W. and Buckley, J. J. (1990). Fuzzy control theory: a nonlinear case. *Automatica* **26**, 513–520.

Ying, H. (1993). The simplest fuzzy controllers using different inference methods are different nonlinear proportional-integral controllers with variable gains. *Automatica* **29**(6), 1579–1589.

Ying, H. (1999). Analytical structure of the typical fuzzy controllers employing trapezoidal input fuzzy sets and nonlinear control rules. *Information Sciences* **116**, 177–203.

Yoshizawa, T. (1976). Stability theory by Liapunov's second method. *Mathematical Society of Japan.*

Zadeh, L. A. (1973). Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Transactions on Systems, Man and Cybernetics* **3**(1), 28–44.

Zheng, G., Carroll, R. and Xie, J. (1990). Two-dimensional model and algorithm analysis for a class of iterative learning control systems. *International Journal of Control* **52**(4), 833–862.

Appendix A

Appendix for Chapter 2

A.1 Proof of Lemma 2.1

Proof:

Define a new sequence $\bar{\delta} \triangleq \{\bar{\delta}_0, \bar{\delta}_1, \cdots, \bar{\delta}_i\}$, where $\bar{\delta}_n = sup\{|\delta_n|, |\delta_{n+1}|, \cdots, |\delta_i|\}$. Obviously, $\bar{\delta}_n \geq \bar{\delta}_{n+1} \geq 0$ and $\bar{\delta}_n \geq |\delta_n|$. As $\lim_{i\to\infty} |\delta_i| = 0$, $\lim_{i\to\infty} \bar{\delta}_i = 0$ can be derived.

By using $|z_{i+1}| \leq \gamma|z_i| + |\delta_i|$ repeatedly, the following equation can be derived.

$$\begin{aligned} |z_i| &\leq \gamma^i|z_0| + \gamma^{i-1}|\delta_0| + \gamma^{i-2}|\delta_1| + \cdots + \gamma|\delta_{i-2}| + |\delta_{i-1}| \\ &\leq \gamma^i|z_0| + \gamma^{i-1}\bar{\delta}_0 + \gamma^{i-2}\bar{\delta}_1 + \cdots + \gamma\bar{\delta}_{i-2} + \bar{\delta}_{i-1}. \end{aligned} \tag{A.1}$$

If i is even, (A.1) can be rewritten as

$$\begin{aligned} |z_i| &\leq \gamma^i|z_0| + \gamma^{i-1}\bar{\delta}_0 + \cdots + \gamma^{\frac{i}{2}}\bar{\delta}_{\frac{i}{2}-1} + \gamma^{\frac{i}{2}-1}\bar{\delta}_{\frac{i}{2}} + \cdots + \gamma\bar{\delta}_{i-2} + \bar{\delta}_{i-1} \\ &\leq \gamma^{\frac{i}{2}}(|z_0| + \bar{\delta}_0 + \cdots + \bar{\delta}_{\frac{i}{2}-1}) + \bar{\delta}_{\frac{i}{2}}(\gamma^{\frac{i}{2}-1} + \cdots + \gamma + 1) \\ &\leq \gamma^{\frac{i}{2}}(|z_0| + \frac{i}{2}\bar{\delta}_0) + \bar{\delta}_{\frac{i}{2}}\frac{1-\gamma^{\frac{i}{2}}}{1-\gamma}. \end{aligned}$$

Therefore,

$$\lim_{i\to\infty} |z_i| \leq \lim_{i\to\infty} \gamma^{\frac{i}{2}}(|z_0| + \frac{i}{2}\bar{\delta}_0) + \lim_{i\to\infty} \bar{\delta}_{\frac{i}{2}}\frac{1-\gamma^{\frac{i}{2}}}{1-\gamma} = 0.$$

Similarly, when i is odd, (A.1) can be expressed as

$$
\begin{aligned}
|z_i| &\leq \gamma^i|z_0| + \gamma^{i-1}\bar{\delta}_0 + \cdots + \gamma^{\frac{i+1}{2}}\bar{\delta}_{\frac{i-1}{2}-1} + \gamma^{\frac{i-1}{2}}\bar{\delta}_{\frac{i-1}{2}} + \cdots + \gamma\bar{\delta}_{i-2} + \bar{\delta}_{i-1} \\
&\leq \gamma^{\frac{i+1}{2}}(|z_0| + \bar{\delta}_0 + \cdots + \bar{\delta}_{\frac{i-1}{2}-1}) + \bar{\delta}_{\frac{i-1}{2}}(\gamma^{\frac{i-1}{2}} + \cdots + \gamma + 1) \\
&\leq \gamma^{\frac{i+1}{2}}(|z_0| + \frac{i-1}{2}\bar{\delta}_0) + \bar{\delta}_{\frac{i-1}{2}}\frac{1-\gamma^{\frac{i+1}{2}}}{1-\gamma}.
\end{aligned}
$$

Therefore,

$$
\lim_{i\to\infty}|z_i| \leq \lim_{i\to\infty}\gamma^{\frac{i+1}{2}}(|z_0| + \frac{i-1}{2}\bar{\delta}_0) + \lim_{i\to\infty}\bar{\delta}_{\frac{i-1}{2}}\frac{1-\gamma^{\frac{i+1}{2}}}{1-\gamma} = 0.
$$

■

A.2 Proof of Lemma 2.2

Proof:

Define the same sequence $\bar{\delta}$ as in the proof of Lemma 2.1. The mapping (2.3) can be rewritten as

$$
\begin{aligned}
&if \quad z_i \in I_1, \quad \gamma_1(z_i - a) - \bar{\delta}_i \leq (z_{i+1} - a) \leq \gamma_1(z_i - a) + \bar{\delta}_i; && \text{(A.2)} \\
&if \quad z_i \in I_2, \quad z_i - \bar{\delta}_i \leq z_{i+1} \leq z_i + \bar{\delta}_i; && \text{(A.3)} \\
&if \quad z_i \in I_3, \quad \gamma_2(z_i - b) - \bar{\delta}_i \leq (z_{i+1} - b) \leq \gamma_2(z_i - b) + \bar{\delta}_i. && \text{(A.4)}
\end{aligned}
$$

For any finite $n \in \mathcal{Z}_+$, if $\bar{\delta}_n = 0$, $\forall i \geq n$, $\bar{\delta}_i = 0$ can be derived. Hence, $\forall i \geq n$, the relations (A.2)- (A.4) can be rewritten as

$$
\begin{aligned}
&if \quad z_i \in I_1, \quad \gamma_1(z_i - a) \leq (z_{i+1} - a) \leq \gamma_1(z_i - a); && \text{(A.5)} \\
&if \quad z_i \in I_2, \quad z_{i+1} = z_i; && \text{(A.6)} \\
&if \quad z_i \in I_3, \quad \gamma_2(z_i - b) \leq (z_{i+1} - b) \leq \gamma_2(z_i - b). && \text{(A.7)}
\end{aligned}
$$

Obviously, as $i \to \infty$, $z_i \in I_2$.

Next let us check the convergence property if for any finite i, $\bar{\delta}_i > 0$. The proof contains three parts. *Part A* shows $\forall n \in \mathcal{Z}_+$, if z_n is bounded, a finite constant q_n can be found such that $z_{n+q_n} \in I'_n \triangleq [a'_n, b'_n]$ where $a'_n = a - \frac{\bar{\delta}_n}{\min\{\gamma_1, 1-\gamma_1\}}$, $b'_n = b + \frac{\bar{\delta}_n}{\min\{\gamma_2, 1-\gamma_2\}}$. *Part B* proves that $z_i \in I'_n$ is guaranteed for any $i \geq n + q_n$. The convergence property of z_i is given in *Part C*.

Part A

For any finite $n \in \mathcal{Z}_+$, assume $z_n \notin I'_n$ which implies $z_n > b'_n$ or $z_n < a'_n$.

Suppose $z_n > b'_n$. According to Lemma 2.1 and (A.4), as z_n is bounded, $\bar{\delta}_n > 0$ and $\lim_{i\to\infty} \bar{\delta}_i = 0$, a finite iteration number q_n can be found such that $z_{n+q_n-1} > b'_n$ and $z_{n+q_n} \leq b'_n$.

On the other hand, as $z_{n+q_n-1} > b'_n$, $z_{n+q_n-1} \geq b + \frac{\bar{\delta}_n}{\gamma_2}$ can be derived. Therefore, from (A.4), we have

$$\begin{aligned} z_{n+q_n} - b &\geq \gamma_2(z_{n+q_n-1} - b) - \bar{\delta}_{n+q_n-1} \\ &\geq \gamma_2 \frac{\bar{\delta}_n}{\gamma_2} - \bar{\delta}_n \\ &= 0. \end{aligned}$$

Hence, $z_{n+q_n} \in [b, b'_n] \subset I'_n$.

Similarly, for $z_n < a'_n$, a finite constant q_n can also be found such that $z_{n+q_n} \in [a'_n, a] \subset I'_n$.

Hence, there exists a finite q_n such that $z_{n+q_n} \in I'_n$ can be realized.

Part B

As $z_{n+q_n} \in I'_n$, the property of z_{n+q_n+1} can be analyzed in the following three cases.

Case 1. $z_{n+q_n} \in I_2$

According to (A.3) and considering $0 < \gamma_1 < 1$ and $0 < \gamma_2 < 1$, it can be derived that

$$a'_n < a - \bar{\delta}_n \le a - \bar{\delta}_{n+q_n} \le z_{n+q_n+1} \le b + \bar{\delta}_{n+q_n} \le b + \bar{\delta}_n < b'_n. \tag{A.8}$$

Obviously, $z_{n+q_n+1} \in I'_n$.

Similarly, for any $i \ge n + q_n$, if $z_i \in I_2$, $z_{i+1} \in I'_n$ can be derived.

Case 2. $z_{n+q_n} \in (b, b'_n]$

$0 < z_{n+q_n} - b \le \frac{\bar{\delta}_n}{\min\{\gamma_2, 1-\gamma_2\}}$ can be derived directly. Therefore, from (A.4), we have

$$\begin{aligned} & -\bar{\delta}_{n+q_n} \le z_{n+q_n+1} - b \le \gamma_2 \frac{\bar{\delta}_n}{\min\{\gamma_2, 1-\gamma_2\}} + \bar{\delta}_{n+q_n} \\ \Rightarrow \quad & b - \bar{\delta}_n \le z_{n+q_n+1} \le b + \gamma_2 \frac{\bar{\delta}_n}{\min\{\gamma_2, 1-\gamma_2\}} + \bar{\delta}_n \\ \Rightarrow \quad & a'_n \le z_{n+q_n+1} \le b + \gamma_2 \frac{\bar{\delta}_n}{\min\{\gamma_2, 1-\gamma_2\}} + \bar{\delta}_n. \end{aligned} \tag{A.9}$$

If $0 < \gamma_2 \le 0.5$, $\min\{\gamma_2, 1-\gamma_2\} = \gamma_2$ and $\frac{1}{\gamma_2} \ge 2$, which leads to $b'_n = b + \frac{\bar{\delta}_n}{\gamma_2} \ge b + 2\bar{\delta}_n$. Hence, (A.9) can be rewritten as

$$a'_n \le z_{n+q_n+1} \le b + \gamma_2 \frac{\bar{\delta}_n}{\gamma_2} + \bar{\delta}_n = b + 2\bar{\delta}_n \le b'_n. \tag{A.10}$$

If $0.5 < \gamma_2 < 1$, $\min\{\gamma_2, 1-\gamma_2\} = 1-\gamma_2$, which implies that $b'_n = b + \frac{\bar{\delta}_n}{1-\gamma_2}$. Therefore, (A.9) can be expressed as

$$a'_n \le z_{n+q_n+1} \le b + \gamma_2 \frac{\bar{\delta}_n}{1-\gamma_2} + \bar{\delta}_n = b + \frac{\bar{\delta}_n}{1-\gamma_2} = b'_n. \tag{A.11}$$

According to (A.10) and (A.11), $z_{n+q_n+1} \in I'_n$ is guaranteed.

$\forall i \ge n + q_n$, only if $z_i \in (b, b'_n]$, the above proof is still valid, hence, $z_{i+1} \in I'_n$ can be derived.

Case 3. $z_{n+q_n} \in [a'_n, a)$

Analogous to the proof in Case 2, it can be derived that, $\forall i \ge n + q_n$, if $z_i \in [a'_n, a)$, $z_{i+1} \in I'_n$.

According to the results of Case 1, Case 2 and Case 3, we can conclude that $z_i \in I'_n$ can always be ensured for any $i \geq n + q_n$.

Part C

Considering the finiteness of z_0 and $\bar{\delta}_0$, from the results of *Part A* and *Part B*, it can be derived that, a finite q_0 can be found such that, $\forall i \geq q_0$, $z_i \in I'_0$. Consequently, $\forall i \in \mathcal{Z}_+$, the boundedness z_i can be guaranteed.

For every $\epsilon > 0$, as $\lim_{i\to\infty} \bar{\delta}_i = 0$, there exists a finite N' such that for any $i \geq N'$, $\bar{\delta}_i \leq \gamma\epsilon$ where $\gamma = \min\{\min\{\gamma_1, 1-\gamma_1\}, \min\{\gamma_2, 1-\gamma_2\}\}$. According to *Part A* and *Part B*, a finite $N = N' + q_{N'}$ can be found such that $\forall i \geq N$, the following equation is valid.

$$a - \frac{\bar{\delta}_N}{\min\{\gamma_1, 1-\gamma_1\}} \leq z_i \leq b + \frac{\bar{\delta}_N}{\min\{\gamma_2, 1-\gamma_2\}}.$$

Considering $\bar{\delta}_i \leq \gamma\epsilon$, we have

$$a - \epsilon \leq a - \frac{\gamma\epsilon}{\min\{\gamma_1, 1-\gamma_1\}} \leq z_i \leq b + \frac{\gamma\epsilon}{\min\{\gamma_2, 1-\gamma_2\}} \leq b + \epsilon.$$

Hence, for every $\epsilon > 0$, a finite N can be found such that $\forall i \geq N$, $z_i \in [a-\epsilon, b+\epsilon]$. According to the definition of *limitation*, $\lim_{i\to\infty} z_i \in I_2$ can be derived. ■

Appendix B

Appendix for Chapter 4

B.1 Proof of Lemma 4.2

Proof:

From (4.37) and (4.38), it can be obtained

$$\dot{\mathbf{x}}_d - \dot{\mathbf{x}}_i = \boldsymbol{\phi}_i - Q\dot{\boldsymbol{\sigma}}_i \tag{B.1}$$

where

$$\begin{aligned}\boldsymbol{\phi}_i &= \mathbf{f}_d - \mathbf{f}_i - Q(\mathbf{f}_d - \mathbf{f}_i) - Q(\mathbf{h}_d - \mathbf{h}_i) \\ &\le l_{\mathbf{f}}\|\mathbf{x}_d - \mathbf{x}_i\| + b_Q l_{\mathbf{f}}\|\mathbf{x}_d - \mathbf{x}_i\| + b_Q l_{\mathbf{h}}\|\mathbf{x}_d - \mathbf{x}_i\| \\ &= b_1\|\mathbf{x}_d - \mathbf{x}_i\| \end{aligned} \tag{B.2}$$

where $b_Q = \sup_{t\in[0,T]} |Q(t)|$ and $c_1 = l_{\mathbf{f}} + b_Q l_{\mathbf{f}} + b_Q l_{\mathbf{h}}$. As $\mathbf{x}_i(0) = \mathbf{x}_d(0)$ and $\boldsymbol{\sigma}_i(0) = \mathbf{0}$, integrating both sides of equation (B.1), we can obtain that

$$\begin{aligned}\|\mathbf{x}_d - \mathbf{x}_i\| &\le \int_0^t \|\boldsymbol{\phi}_i\| d\tau - \int_0^t Q d\boldsymbol{\sigma}_i \\ &\le b_1\int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau + \|Q\boldsymbol{\sigma}_i\| + \int_0^t \|\boldsymbol{\sigma}_i\|\|\frac{dQ}{d\tau}\| d\tau \\ &\le b_1\int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau + b_Q\|\boldsymbol{\sigma}_i\| + b_{\frac{dQ}{dt}}\int_0^t \|\boldsymbol{\sigma}_i\| d\tau,\end{aligned}$$

where $b_{\frac{dQ}{dt}} = \sup_{t\in[0,T]} |\frac{dQ}{dt}|$.

Applying *Gronwall-Bellman Lemma* we have

$$\begin{aligned}
&\|\mathbf{x}_d - \mathbf{x}_i\| \\
&\leq b_Q\|\boldsymbol{\sigma}_i\| + b_{\frac{dQ}{dt}}\int_0^t \|\boldsymbol{\sigma}_i\|d\tau + b_1 b_Q\int_0^t \|\boldsymbol{\sigma}_i\|e^{c_1(t-\tau)}d\tau \\
&\quad + b_1 b_{\frac{dQ}{dt}}\int_0^t (\int_0^\tau \|\boldsymbol{\sigma}_i\|ds)e^{b_1(t-\tau)}d\tau \\
&\leq b_Q\|\boldsymbol{\sigma}_i\| + (b_{\frac{dQ}{dt}} + b_1 b_Q e^{b_1 T})\int_0^t \|\boldsymbol{\sigma}_i\|d\tau + b_1 b_{\frac{dQ}{dt}}\int_0^t e^{b_1(t-\tau)}d\tau\int_0^t \|\boldsymbol{\sigma}_i\|d\tau \\
&\leq b_Q\|\boldsymbol{\sigma}_i\| + (b_{\frac{dQ}{dt}} + b_1 b_Q e^{b_1 T})\int_0^t \|\boldsymbol{\sigma}_i\|d\tau + b_1 b_{\frac{dQ}{dt}} T e^{b_1 T}\int_0^t \|\boldsymbol{\sigma}_i\|d\tau \\
&= b_Q\|\boldsymbol{\sigma}_i\| + b_2\int_0^t \|\boldsymbol{\sigma}_i\|d\tau
\end{aligned}$$

where $b_2 = b_{\frac{dQ}{dt}} + b_1 b_Q e^{b_1 T} + b_1 b_{\frac{dQ}{dt}} T e^{b_1 T}$. Therefore, the boundedness of $\boldsymbol{\sigma}_i$ leads to the finiteness of $\mathbf{x}_i$ since $\mathbf{x}_d$ is bounded, i.e., $\mathbf{x}_i \in \mathcal{X}$.

Since $\mathbf{x}_i$ is bounded and $\mathbf{d}_i$ is local *Lipschitz*, there exists a *Lipschitz* constant

$$l_{\mathbf{d}} \triangleq \sup_{(x_i,t)\in\mathcal{X}\times[0,T]} \frac{\partial \mathbf{d}_i}{\partial \mathbf{x}_i}| < \infty \quad \forall i \in \mathcal{Z}_+$$

such that

$$\|\mathbf{d}_i - \mathbf{d}_d\| \leq l_{\mathbf{d}}\|\mathbf{x}_i - \mathbf{x}_d\|. \tag{B.3}$$

Moreover, from the control law, the boundedness of $\mathbf{x}_i$ guarantees the finiteness of $\mathbf{u}_{r,i}$ and $\mathbf{u}_i$. Consequently, $\dot{\mathbf{x}}_i$ and $\dot{\boldsymbol{\sigma}}_i$ are also finite. ■

Appendix C

Author's Publications

The author has contributed to the following papers:

Published (Accepted) Journal Papers

1. Jian-Xin Xu and Jing Xu, "Observer-based Control for a Class of Nonlinear Systems with Parametric Uncertainties", accepted by *IEEE Transactions on Automatic Control*.

2. Jian-Xin Xu and Jing Xu, "On Iterative Learning from Different Tracking Tasks in The Presence of Time-Varying Uncertainties", accepted by *IEEE Transactions on Systems, Man and Cybernetics* as a regular paper.

3. Jian-Xin Xu and Jing Xu, "A New Fuzzy Logic Control Approach for Repetitive Trajectory Tracking Problems", *Fuzzy Sets and Systems*, Vol. 133, No. 1, January 2003, pp. 57-75.

4. Jian-Xin Xu, Jing Xu and Badrinath V, "Recursive Direct Learning of Control Efforts for Trajectories with Different Magnitude Scales", *Asian Journal of Control*, Vol. 4, No. 1, March 2002, pp. 49-59.

5. Jian-Xin Xu, Jing Xu and Wen-Jun cao, "A PD Type Fuzzy Logic Learning Control for Repeatable Tracking Control Problems", *Acta Automatica Sinica*, Vol. 27, No. 4, July, 2001, pp. 434-446.

Published Conference Papers

1. Jian-Xin Xu and Jing Xu, "Memory Based Nonlinear Internal Model: What Can a Control System Learn", *In Proceeding of Asian Control Conference 2002*, September, 2002, Singapore.

2. Jian-Xin Xu and Jing Xu, "Iterative Learning Control for Non-Uniform Trajectory Tracking Problems", *In Proceeding of IFAC 2002*, July, 2002, Spain.

3. Jian-Xin Xu, Jing Xu and Ying Tan, "Robust Learning Control for Nonlinear Uncertain Systems Based on Composite Energy Function", *In Proceeding of IFAC 2002*, July, 2002, Spain.

4. Jian-Xin Xu, Jing Xu, "A New Fuzzy Logic Control Approach for Repetitive Trajectory Tracking Problems", *In Proceeding of IEEE 2001 American Control Conference*, June, 2001, Arlington, VA, USA, pp. 3878 - 3883.

Submitted Journal Papers

1. Jian-Xin Xu, Tong H. Lee and Jing Xu, "Iterative Learning Control for Systems With Input Deadzone: Theory and an Application", submitted to *IEEE Transactions On Automatic Control.*