Structural Decomposition of General Singular Linear Systems and Its Applications

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Contents

Summary

This thesis presents a structural decomposition technique for singular linear systems. Such a decomposition can explicitly display the finite and infinite zero structures, system invertibility structure, invariant geometric subspaces, as well as redundant states of a given singular system. It is expected to be a powerful tool in solving singular system and control problems as its counterpart for nonsingular linear systems. To illustrate its potential applications, the structural decomposition technique is finally applied to solve disturbance decoupling problem of singular systems.

Firstly, after giving necessary background materials, we present a structural decomposition technique for single-input and single-output (SISO) singular systems. The decomposition results show that it is efficient in displaying internal structure features of a given system. And compared with its counterpart for linear nonsingular systems, the decomposition technique for SISO singular systems has more properties in revealing the redundant states.

The results for SISO singular systems give us important clues for the structural decomposition form of multi-input and multi-output (MIMO) singular systems, but the situation of multivariable case is much more difficult. To propose the structural decomposition for MIMO singular systems, a constructive algorithm is developed in decomposing the given singular state space into several distinct subspaces. The structural decomposition technique is given in equation form and compact matrix form. The decomposed subspaces also

include redundant states and states of linear combination of system input and its derivatives of different orders. Moreover, such a structural decomposition can explicitly display all its structure properties such as invariant zero structure, infinite zero structure, invertibility structure, as well as stabilizability and detectibility features. Numerical examples show that the structural decomposition is a powerful tool in revealing and understanding structure features of singular systems.

Furthermore, to give the geometric interpretations for the structurally decomposed subspaces, we define several invariant geometric subspaces for singular systems. And with these definitions, we show that the structural decomposition technique can also explicitly display the invariant geometric subspaces of the given singular system. These invariant geometric subspaces also give geometric interpretation of the structurally decomposed subspaces.

After completing the theory of the structural decomposition technique. We explore its application in solving disturbance decoupling problem of singular systems. With a sufficient condition, we show that the structural decomposition can give an easier understanding and a clearer solution for such problems. This enhances the expectation of its potential applications in solving singular system and control problems as its counterpart for nonsingular systems.

Finally, to make this thesis more complete, we include main MATLAB codes for the realization of the structural decomposition in the appendix. Such codes are essential in the applications of this technique.

List of Figures

Chapter 1

Introduction

1.1 Introduction

Linear singular systems, also commonly called generalized state space systems or descriptor systems in the literature, appear in many practical situations including engineering systems, economic systems, network analysis, and biological systems, to name a few but far from complete (see e.g., Dai [29], Kuijper [45] and Lewis [47]). To be more specific, a linear singular system generally can be expressed in the following state space form,

$$\Sigma : \begin{cases} E\,\dot{x} = A\,x + B\,u, & x(0) = x_0 \\ \quad y = C\,x + D\,u, & \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$ represents internal state variable, $y \in \mathbb{R}^p$ is the system output, $u \in \mathbb{R}^m$ is the system input and $\text{rank}(E) < n$. When the rank of matrix E is equal to n, the system Σ is called a linear nonsingular system.

Further, when $|sE - A|$ is not always equal zero, the matrix pencil (E, A) will be called regular. Unique (classical) solutions are guaranteed to exist if (E, A) is regular. Hence without loss of any generality, we assume that the matrix pencil (E, A) is always regular throughout this thesis.

In fact, many systems in the real life are singular in nature. They are usually simplified as or approximated by nonsingular models because it is still lacking of efficient tools to tackle problems related to such systems. However, a singular system model represents more practical information, and such information like interconnection relationships, will be crucial to the whole system in some critical situations. This makes it an important research topic in the last three decades and motivates us to develop an innovative technique for singular systems.

To develop an efficient tool for singular systems, structural properties are essential. From those earlier days, they have received much attention in the literature. Weierstrass [77] firstly gave a fundamental study for regular cases and Kronecker [44] extended the study to non-regular cases by introducing structural indices. Gantmacher [35] systemically described Kronecker Canonical Form and made it a popular tool in analyzing singular systems. Along this line, Kokotovic *et al.* [43] analyzed the relationship of fast subsystem and slow subsystem in Weierstrass decomposition form. While Verghese *et al.* defined a strong system equivalence using a trivial augmentation and deflation technique. Further, Misra *et al.* [59] and Liu *et al.* [53] have presented their algorithms to compute the invariant structural indices of singular systems. On the other hand, in the literature of geometric approaches, Malabre [58] presented a new way of introducing invariant subspaces for singular systems and defined their structure indices like the one presented by Morse [60] for nonsingular systems. Geerts [36] also defined and analyzed several geometric subspaces by means of a fully algebraic distributional framework. However, as a matter of fact, all of these methods for structural properties of singular systems are simply focusing either on merely structural indices or on only some special parts of state space but have not give a full image of the whole state space. The objective of this thesis is to develop an efficient technique for decomposing the whole state space into several distinct subspaces corresponding to special structural features such as invariant zero structures, infinite zero structures, redundant states, system invertibility and so on.

Most techniques for singular systems, generally speaking, are natural extensions of their counterpart for nonsingular systems. Since Kalman [41] and other people [42] [37] presented state space model in 1960's, nonsingular systems have been intensively researched and many techniques have been presented in the literature. Among these methods, there is a structural decomposition technique [70] [67] [19] which can explicitly display the zero structures, invertibility and invariant geometric subspaces of a given nonsingular system. It has been used in the literature to solve many system and control problems such as the squaring down and decoupling of linear systems (see e.g., Sannuti and Saberi [70]), linear system factorizations (see e.g., Chen et al [11], and Lin et al [51]), blocking zeros and strong stabilizability (see e.g., Chen et al [12]), zero placements (see e.g. Chen and Zheng [15]), loop transfer recovery (see e.g., Chen [10], Chen and Chen [16], and Saberi et al [68]), H_2 optimal control (see e.g., Chen et al [13, 14], and Saberi et al [69]), disturbance decoupling (see e.g., Chen [18], and Ozcetin et al [63, 64]), H_∞ optimal control (see e.g., Chen et al [11] and control with saturations (see e.g., Lin [50]). The list here is far from complete.

The applications of the structural decomposition technique for nonsingular system prove that it is a powerful tool. The main objective of this thesis is to extend this structural decomposition technique to singular systems. We will focus on developing a structural decomposition technique for singular systems to capture all structure properties, such as invariant zero structures, infinite zero structures, invertibility structures, invariant geometric subspaces, as well as redundant dynamics of a given singular system. Moreover, we will exploit its applications in solving singular system and control problems, such as disturbance decoupling, almost disturbance decoupling, H_2 optimal control, H_∞ control and model reduction, as its counterpart for nonsingular systems.

1.2 Notations

Throughout this thesis, we shall adopt the following notations:

$$\begin{aligned}
\mathbb{R} &:= \text{the set of real numbers,} \\
\mathbb{C} &:= \text{the entire complex plane,} \\
\mathbb{C}^- &:= \text{the open left-half complex plane,} \\
\mathbb{C}^+ &:= \text{the open right-half complex plane,} \\
\mathbb{C}^0 &:= \text{the imaginary axis in the complex plane,} \\
I &:= \text{an identity matrix,} \\
I_k &:= \text{an identity matrix of dimension } k \times k, \\
X' &:= \text{the transpose of } X, \\
\mathrm{rank}(X) &:= \text{the rank of } X, \\
\lambda(X) &:= \text{the set of eigenvalues of } X, \\
\mathrm{Ker}\,(X) &:= \text{the null space of } X, \\
\mathrm{Im}\,(X) &:= \text{the range space of } X, \\
\dim(\mathcal{X}) &:= \text{the dimension of a subspace } \mathcal{X}, \\
C^{-1}\{\mathcal{X}\} &:= \text{the inverse image of C, where } \mathcal{X} \text{ is a subspace and } C \text{ is a matrix ,} \\
u^{(v)} &:= \text{the } v\text{-th order derivative of a function } u(t), \\
\Sigma &:= \text{a singular system characterized by } (E, A, B, C, D) \text{ ,} \\
\Sigma_\star &:= \text{a singular system characterized by } (E_\star, A_\star, B_\star, C_\star, D_\star) \text{ ,} \\
M^\perp &:= \text{the orthogonal complement of the space spanned by the columns of a matrix } M, \\
S_\infty(M) &:= \text{a matrix with orthogonal columns spanning the right null space of a matrix } M, \\
T_\infty(M) &:= \text{a matrix with orthogonal columns spanning the right null space of } M^T.
\end{aligned}$$

1.3 Preview of Each Chapter

This thesis can naturally be divided into three parts. The first part includes Chapter 1 and Chapter 2 and gives some preliminary results and background materials. Chapter 1 gives the background and motivations of this thesis. Chapter 2 recalls some basic linear system tools on system structure such as the Jordan Canonical Form, some controllability decomposition form and the structural decomposition method for nonsingular systems. All of these techniques will play essential roles in the later chapters. Chapter 2 also provides a comprehensive study on singular systems and its properties. Some distinct features of singular systems such as impulsive mode will be presented and discussed. The initial conditions of a given singular system is discussed intensively before introducing some important tools for singular systems such as Kronecker Canonical Form and invariant structural indices. The last section of Chapter 2 lists some basic definitions such as stability, stabilizability, detectibility and so on.

The second part is the core of this thesis and consists of Chapter 3 to Chapter 5. Chapter 3 gives our research results on structural decomposition for linear single-input and single-output (SISO) singular systems. This is the first step of our research on extending the structural decomposition technique to singular systems. The results present a clear view of the technique for singular systems. Chapter 4 is the most important section of this thesis because it presents the structural decomposition technique for general multivariable singular systems. The properties of this technique show that it has a distinct feature of explicitly displaying the zero structures, invertibility, stabilizability and detectibility properties of the given systems, just as its counterpart in nonsingular systems. Chapter 5 defines the invariant geometric subspaces of singular system in state space form and presents the properties of our structural decomposition in displaying the invariant geometric subspaces.

The last part of this thesis focuses on the applications of our structural decomposition technique. In Chapter 6, we apply the structural decomposition technique to solve disturbance decoupling problem of singular systems with state feedback. It shows that the structural decomposition technique is powerful in eliminating the influence of disturbance. With a sufficient condition, we can see that the whole algorithm is based on decomposing the system into several subspaces, and we can use the state feedback algorithm to eliminate the corresponding disturbance in those subspaces. Moreover, Chapter 7 gives concluding remarks on this thesis and propose our future work in the applications of this structural decomposition in solving singular system and control problems.

Finally, in the appendix part, main MATLAB codes are given for the constructive algorithm on computing the structural decomposition form. All essential procedures are illustrated in detail. And complete source codes for those main functions are attached for references.

Chapter 2

Background Materials

2.1 Introduction

This chapter intends to recall necessary background material for the main work of this thesis, the structural decomposition of singular systems and its applications. Such preliminary materials include mathematical tools of matrix decomposition, the structural decomposition for nonsingular systems, and a brief introduction of singular systems. All of these are crucial in deriving, proving and understanding our structural decomposition technique and its properties.

Mathematical tools for decomposing matrices and matrix pairs are widely used in linear system theories. In this thesis, they are applied to constructively decompose state space into several distinct subspaces displaying internal structural features of the given linear system. Such tools include Jordan canonical form, controllability canonical form, as well as block diagonal control canonical form.

The structural decomposition for nonsingular systems has a distinct feature of explicitly displaying a given nonsingular system's internal structural properties such as invariant and infinite zero structures, system invertibility, invariant geometric subspaces and so on. This technique was first proposed by Sannuti *et al.* [70] and Saberi *et al.* [67] while Chen [19] proved all of its properties and further decompose several subspaces, and more important, gave clear geometric interpretations for the subspaces with a list of invariant geometric subspaces. Our work in this thesis is to extend this powerful technique for singular systems and apply it in solving singular systems and control problems.

At last, a brief knowledge on singular systems is recalled to make this thesis more self-contained. Moreover, such knowledge is necessary in proving our structural decomposition theorem and its properties, as well as its application in solving singular systems and control problems. The background knowledge ranges from several basic definitions, such as stabilizability, invariant zero structure and system invertibility, to very well known Kronecker canonical form and invariant structural indices.

2.2 Mathematical Tools for Linear System Decomposition

Matrix decomposition is a must-go step in structural decomposition of linear systems. This section recalls some important tools which will be used intensively in decomposing a given singular system into its structural decomposition form. Firstly, the theorem on Jordan and Real Jordan Canonical Form will be introduced, which can show the structural properties of a given matrix according to its eigenvalues. Then some Controllability Canonical Forms will be recalled for the decomposition of system matrix pair (A, B).

The following subsections give these important tools for matrices and matrix pairs.

2.2.1 Structural Decomposition of (A, B)

This section recalls two important Controllability Canonical Forms, that is, Controllability Structural Decomposition (CSD) and Block Diagonal Control Canonical Form (BDCCF). All the canonical forms are presented for a linear system characterized by a matrix pair (A, B) and display its controllability information in different ways.

Controllability canonical form is a very well-known tool in the literature. It decomposes a given system into controllable and uncontrollable parts with an invertible coordinate transform. Controllability structural decomposition form is generally called Brunovsky canonical form in the literature, and in fact it is due to Luenberger [56] in 1967 and Brunovsky [6] in 1970. Block diagonal control canonical form was presented by Chen [20], it gives a totally new and powerful canonical form and its MATLAB software realization can be found in Chen [17]. All these tools will pay key roles in the derivations of our structural decomposition technique for singular systems.

The following theorem conducts a controllability structural decomposition for a matrix pair (A, B).

Theorem 2.2.1 (CSD) Consider a pair of constant matrices (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that B is of full rank. Then, there exist nonsingular state and input transformations T_s and T_i such that $(\tilde{A}, \tilde{B}) := (T_{\rm s}^{-1} A T_{\rm s}, T_{\rm s}^{-1} B T_{\rm i})$ has the following form,

$$\left(\begin{bmatrix} A_o & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \cdots & 0 & 0 \\ \star & \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{k_m-1} \\ \star & \star & \star & \cdots & \star & \star \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix} \right), \tag{2.1}$$

where $k_i > 0$, $i = 1, \cdots, m$, A_o is of dimension $n_o := n - \sum_{i=1}^{m} k_i$ and its eigenvalues are the uncontrollable modes of the pair $(A,\, B)$. Moreover, the set of integers, $\mathcal{C}(A,B) := \{\, n_o,\, k_1,\, \cdots,\, k_m \,\}$, is referred to as the *controllability index* of (A,B). □

Proof. See Luenberger [56]. The software realization of such a canonical form can be found in Lin and Chen [52]. ∎

At last, the theorem on block diagonal control canonical form is given in the following.

Theorem 2.2.2 (BDCCF) [20] Consider a constant matrix pair (A,B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ and with (A,B) being completely controllable. Then there exist an integer $k \le m$, a set of κ integers $k_1, k_2, \cdots, k_\kappa$, and nonsingular state and input transformations T_s and T_i such that (A,B) can be transformed into the following block diagonal control canonical form,

$$\tilde{A} = T_{\rm s}^{-1} A T_{\rm s} = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_\kappa \end{bmatrix}, \tag{2.2}$$

and

$$\tilde{B} = T_{\rm s}^{-1} B T_{\rm i} = \begin{bmatrix} B_1 & \star & \star & \cdots & \star & \star \\ 0 & B_2 & \star & \cdots & \star & \star \\ 0 & 0 & B_3 & \cdots & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_\kappa & \star \end{bmatrix}, \tag{2.3}$$

where $\star$s represent some matrices of less interest, and A_i and B_i, $i = 1, 2, \cdots, \kappa$, have the following control canonical forms,

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a^i_{k_i} & -a^i_{k_i-1} & -a^i_{k_i-2} & \cdots & -a^i_1 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{2.4}$$

for some scalars $a^i_1, a^i_2, \cdots, a^i_{k_i}$. And it is obvious that $\sum_{i=1}^{\kappa} k_i = n$. □

The block diagonal control canonical form plays a key role in the derivation of our structural decomposition for singular systems. This will be introduced in detail in the Chapter 4 and 5.

2.2.2 Structural Decomposition of Linear Nonsingular Systems

Structural properties, such as invariant zero structures, are essential in understanding the internal states of linear systems, which is the first step in solving linear systems and control problems. Hence a good technique in displaying the structural properties is crucial for us to find a better solution. And after so many years' intensive research, there are a large number of techniques for nonsingular systems in the literature to reveal their internal structural features (see e.g., Lewis [47], Chen [20]). However, a better way to display the structural properties is to decompose the whole state space into several distinct subspaces each of which corresponding to special system structural properties. This has been proven to be a successful technique in solving real applications by the structural decomposition technique for nonsingular systems (see e.g. Chen *et al.* [13]).

In this section, structural decomposition for nonsingular systems is presented briefly. The decomposition can explicitly display the zero structures, invertibility and geometric subspaces of the given nonsingular system. And It has been proved to be a powerful tool in solving nonsingular system and control problems. Our structural decomposition technique for singular systems is a natural extension of this method.

The structural decomposition for nonsingular systems was first presented by Sannuti and Saberi [70] and Saberi and Sannuti [67]. Chen [19] proved the essential properties of the structural decomposition technique and moreover, and linked them for the first time with invariant geometric subspaces of geometric control theories, thus completing this theory.

Let us first consider a linear time-invariant (LTI) system Σ_* characterized by a matrix quadruple (A_*, B_*, C_*, D_*) or in the state space form,

$$\Sigma_* \ : \begin{cases} \dot{x} = A_* \ x + B_* \ u, \\ y = C_* \ x + D_* \ u, \end{cases} \tag{2.5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, the input and the output of Σ_*. Without loss of any generality, we assume that both $[\, B_*' \;\; D_*' \,]^T$ and $[\, C_* \;\; D_* \,]$ are of full col and row rank respectively. The transfer function of Σ_* is then given by

$$H_*(s) = C_*(sI - A_*)^{-1} B_* + D_*, \tag{2.6}$$

It is well-known that there exist non-singular transformations U and V such that

$$U D_* V = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.7}$$

where m_0 is the rank of matrix D_*. Without loss of generality, it is assumed that the matrix D_* has the form given on the right hand side of (2.7). One can now rewrite system Σ_* of (2.5) as,

$$\begin{cases} \dot{x} = A_* \ x + [\, B_{*,0} \;\; B_{*,1} \,] \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{bmatrix} C_{*,0} \\ C_{*,1} \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \end{cases} \tag{2.8}$$

where the matrices $B_{*,0}$, $B_{*,1}$, $C_{*,0}$ and $C_{*,1}$ have appropriate dimensions. We have the following theorem.

Theorem 2.2.3 [20] Given the linear system Σ_* of (2.5), there exist

1. Coordinate free non-negative integers $n_{\rm a}^-$, $n_{\rm a}^0$, $n_{\rm a}^+$, $n_{\rm b}$, $n_{\rm c}$, $n_{\rm d}$, $m_{\rm d} \le m - m_0$ and q_i, $i = 1, \cdots, m_{\rm d}$, and

2. Non-singular state, output and input transformations $\Gamma_{\rm s}$, $\Gamma_{\rm o}$ and $\Gamma_{\rm i}$ which take the given Σ_* into the structural decomposition form that displays explicitly both the invariant and infinite zero structures of Σ_*.

The structural decomposition can be described by the following set of equations:

$$x = \Gamma_{\rm s}\tilde{x}, \quad y = \Gamma_{\rm o}\tilde{y}, \quad u = \Gamma_{\rm i}\tilde{u}, \tag{2.9}$$

$$\tilde{x} = \begin{pmatrix} x_{\rm a} \\ x_{\rm b} \\ x_{\rm c} \\ x_{\rm d} \end{pmatrix}, \quad x_{\rm a} = \begin{pmatrix} x_{\rm a}^- \\ x_{\rm a}^0 \\ x_{\rm a}^+ \end{pmatrix}, \quad x_{\rm d} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_{\rm d}} \end{pmatrix}, \tag{2.10}$$

$$\tilde{y} = \begin{pmatrix} y_0 \\ y_{\rm d} \\ y_{\rm b} \end{pmatrix}, \; y_d = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m_{\rm d}} \end{pmatrix}, \; \tilde{u} = \begin{pmatrix} u_0 \\ u_{\rm d} \\ u_{\rm c} \end{pmatrix}, \; u_{\rm d} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_{\rm d}} \end{pmatrix}, \tag{2.11}$$

and

$$\dot{x}_{\rm a}^- = A_{\rm aa}^- x_{\rm a}^- + B_{\rm 0a}^- y_0 + L_{\rm ad}^- y_{\rm d} + L_{ab}^- y_{\rm b}, \tag{2.12}$$

$$\dot{x}_{\rm a}^0 = A_{\rm aa}^0 x_{\rm a}^0 + B_{\rm 0a}^0 y_0 + L_{\rm ad}^0 y_{\rm d} + L_{ab}^0 y_{\rm b}, \tag{2.13}$$

$$\dot{x}_{\rm a}^+ = A_{\rm aa}^+ x_{\rm a}^+ + B_{\rm 0a}^+ y_0 + L_{\rm ad}^+ y_{\rm d} + L_{ab}^+ y_{\rm b}, \tag{2.14}$$

$$\dot{x}_{\rm b} = A_{\rm bb}x_{\rm b} + B_{0\rm b}y_0 + L_{\rm bd}y_{\rm d}, \quad y_{\rm b} = C_{\rm b}x_{\rm b}, \tag{2.15}$$

$$\dot{x}_{\rm c} = A_{\rm cc}x_{\rm c} + B_{0\rm c}y_0 + L_{\rm cb}y_{\rm b} + L_{\rm cd}y_{\rm d} + B_{\rm c}\left[E_{\rm ca}^{-}x_{\rm a}^{-} + E_{\rm ca}^{0} + E_{\rm ca}^{+}x_{\rm a}^{+}\right] + B_{\rm c}u_{\rm c}, \tag{2.16}$$

$$y_0 = C_{0\rm c}x_{\rm c} + C_{0\rm a}^{-}x_{\rm a}^{-} + C_{0\rm a}^{+}x_{\rm a}^{0} + C_{0\rm a}^{+}x_{\rm a}^{+} + C_{0\rm d}x_{\rm d} + C_{0\rm b}x_{\rm b} + u_0, \tag{2.17}$$

and for each $i = 1, \cdots, m_{\rm d}$,

$$\dot{x}_i = A_{q_i}x_i + L_{i0}y_0 + L_{i\rm d}y_{\rm d} + B_{q_i}\left[u_i + E_{i\rm a}x_{\rm a} + E_{i\rm b}x_{\rm b} + E_{i\rm c}x_{\rm c} + \sum_{j=1}^{m_{\rm d}} E_{ij}x_j\right], \tag{2.18}$$

$$y_i = C_{q_i}x_i, \quad y_{\rm d} = C_{\rm d}x_{\rm d}. \tag{2.19}$$

Here the states $x_{\rm a}^{-}$, $x_{\rm a}^{0}$, $x_{\rm a}^{+}$, $x_{\rm b}$, $x_{\rm c}$ and $x_{\rm d}$ are respectively of dimensions $n_{\rm a}^{-}$, $n_{\rm a}^{0}$, $n_{\rm a}^{+}$, $n_{\rm b}$, $n_{\rm c}$ and $n_{\rm d} = \sum_{i=1}^{m_{\rm d}} q_i$, while x_i is of dimension q_i for each $i = 1, \cdots, m_{\rm d}$. The control vectors u_0, $u_{\rm d}$ and $u_{\rm c}$ are respectively of dimensions m_0, $m_{\rm d}$ and $m_{\rm c} = m - m_0 - m_{\rm d}$ while the output vectors y_0, $y_{\rm d}$ and $y_{\rm b}$ are respectively of dimensions $p_0 = m_0$, $p_{\rm d} = m_{\rm d}$ and $p_{\rm b} = p - p_0 - p_{\rm d}$. The matrices A_{q_i}, B_{q_i} and Cq_i have the following form:

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [\,1 \quad 0 \quad \cdots \quad 0\,]. \tag{2.20}$$

Assuming that x_i, $i = 1, 2, \cdots, m_{\rm d}$, are arranged such that $q_i \leq q_{i+1}$, the matrix $L_{i\rm d}$ has the particular form

$$L_{i\rm d} = [\,L_{i1} \quad L_{i2} \quad \cdots \quad L_{ii-1} \quad 0 \quad \cdots \quad 0\,]. \tag{2.21}$$

Also, the last row of each $L_{i\rm d}$ is identically zero. Moreover,

$$\lambda(A_{\rm aa}^{-}) \subset \mathbb{C}^{-}, \quad \lambda(A_{\rm aa}^{0}) \subset \mathbb{C}^{0}, \quad \lambda(A_{\rm aa}^{+}) \subset \mathbb{C}^{+}. \tag{2.22}$$

Also, the pair $(A_{\rm cc}, B_{\rm c})$ is controllable and the pair $(A_{\rm bb}, C_{\rm b})$ is observable. □

The software toolboxes that realize the continuous-time structural decomposition can be found in LAS by Chen [9] or in MATLAB by Lin [49]. The realization of this unified structural decomposition can be found in Chen [17]. ∎

We can rewrite the special coordinate basis of the quadruple (A_*, B_*, C_*, D_*) given by Theorem 2.2.3 in a more compact form,

$$\begin{aligned}\tilde{A}_* &= \Gamma_{\rm s}^{-1}(A_* - B_{*,0}C_{*,0})\Gamma_{\rm s}\\ &= \begin{bmatrix} A_{\rm aa}^- & 0 & 0 & L_{\rm ab}^- C_{\rm b} & 0 & L_{\rm ad}^- C_{\rm d} \\ 0 & A_{\rm aa}^0 & 0 & L_{\rm ab}^0 C_{\rm b} & 0 & L_{\rm ad}^0 C_{\rm d} \\ 0 & 0 & A_{\rm aa}^+ & L_{\rm ab}^+ C_{\rm b} & 0 & L_{\rm ad}^+ C_{\rm d} \\ 0 & 0 & 0 & A_{\rm bb} & 0 & L_{\rm bd} C_{\rm d} \\ B_{\rm c} E_{\rm ca}^- & B_{\rm c} E_{\rm ca}^0 & B_{\rm c} E_{\rm ca}^+ & L_{\rm cb} C_{\rm b} & A_{\rm cc} & L_{\rm cd} C_{\rm d} \\ B_{\rm d} E_{\rm da}^- & B_{\rm d} E_{\rm da}^0 & B_{\rm d} E_{\rm da}^+ & B_{\rm d} E_{\rm db} & B_{\rm d} E_{\rm dc} & A_{\rm dd} \end{bmatrix},\end{aligned} \tag{2.23}$$

$$\tilde{B}_* = \Gamma_{\rm s}^{-1}\,[\,B_{*,0} \quad B_{*,1}\,]\,\Gamma_{\rm i} = \begin{bmatrix} B_{\rm 0a}^- & 0 & 0 \\ B_{\rm 0a}^0 & 0 & 0 \\ B_{\rm 0a}^+ & 0 & 0 \\ B_{\rm 0b} & 0 & 0 \\ B_{\rm 0c} & 0 & B_{\rm c} \\ B_{\rm 0d} & B_{\rm d} & 0 \end{bmatrix}, \tag{2.24}$$

$$\tilde{C}_* = \Gamma_o^{-1} \begin{bmatrix} C_{*,0} \\ C_{*,1} \end{bmatrix} \Gamma_{\rm s} = \begin{bmatrix} C_{\rm 0a}^- & C_{\rm 0a}^0 & C_{\rm 0a}^+ & C_{\rm 0b} & C_{\rm 0c} & C_{\rm 0d} \\ 0 & 0 & 0 & 0 & 0 & C_{\rm d} \\ 0 & 0 & 0 & C_{\rm b} & 0 & 0 \end{bmatrix}, \tag{2.25}$$

$$\tilde{D}_* = \Gamma_{\rm o}^{-1} D_* \Gamma_{\rm i} = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.26}$$

We note the following intuitive points regarding the special coordinate basis:

1. The variable u_i controls the output y_i through a stack of q_i integrators (or backward shifting operators), while x_i is the state associated with those integrators (or backward shifting operators) between u_i and y_i. Moreover, (A_{q_i}, B_{q_i}) and (A_{q_i}, C_{q_i}) respectively form controllable and observable pair. This implies that all the states x_i are both controllable and observable.

2. The output $y_{\rm b}$ and the state $x_{\rm b}$ are not directly influenced by any inputs, however, they could be indirectly controlled through the output $y_{\rm d}$. Moreover, $(A_{\rm bb}, C_{\rm b})$ forms an observable pair. This implies that the state $x_{\rm b}$ is observable.

3. The state $x_{\rm c}$ is directly controlled by the input $u_{\rm c}$, but it does not directly affect any output. Moreover, $(A_{\rm cc}, B_{\rm c})$ forms a controllable pair. This implies that the state $x_{\rm c}$ is controllable.

4. The state $x_{\rm a}$ is neither directly controlled by any input nor does it directly affect any output.

In what follows, we state some important properties of the above structural decomposition for nonsingular systems.

Property 2.2.1 The given system Σ_* is observable (detectable) if and only if the pair $(A_{\rm obs}, C_{\rm obs})$ is observable (detectable), where

$$A_{\rm obs} := \begin{bmatrix} A_{\rm aa} & 0 \\ B_{\rm c}E_{\rm ca} & A_{\rm cc} \end{bmatrix}, \quad C_{\rm obs} := \begin{bmatrix} C_{\rm 0a} & C_{\rm 0c} \\ E_{\rm da} & E_{\rm dc} \end{bmatrix}, \tag{2.27}$$

and where

$$A_{\rm aa} := \begin{bmatrix} A_{\rm aa}^- & 0 & 0 \\ 0 & A_{\rm aa}^0 & 0 \\ 0 & 0 & A_{\rm aa}^+ \end{bmatrix}, \quad C_{\rm 0a} := [\, C_{\rm 0a}^- \quad C_{\rm 0a}^0 \quad C_{\rm 0a}^+ \,], \tag{2.28}$$

$$E_{\rm da} := [\, E_{\rm da}^- \quad E_{\rm da}^0 \quad E_{\rm da}^+ \,], \quad E_{\rm ca} := [\, E_{\rm ca}^- \quad E_{\rm ca}^0 \quad E_{\rm ca}^+ \,]. \tag{2.29}$$

Also, define

$$A_{\rm con} := \begin{bmatrix} A_{\rm aa} & L_{\rm ab}C_{\rm b} \\ 0 & A_{\rm bb} \end{bmatrix}, \quad B_{\rm con} := \begin{bmatrix} B_{\rm 0a} & L_{\rm ad} \\ B_{\rm 0b} & L_{\rm bd} \end{bmatrix}, \tag{2.30}$$

$$B_{\rm 0a} := \begin{bmatrix} B_{\rm 0a}^- \\ B_{\rm 0a}^0 \\ B_{\rm 0a}^+ \end{bmatrix}, \quad L_{\rm ab} := \begin{bmatrix} L_{\rm ab}^- \\ L_{\rm ab}^0 \\ L_{\rm ab}^+ \end{bmatrix}, \quad L_{\rm ad} := \begin{bmatrix} L_{\rm ad}^- \\ L_{\rm ad}^0 \\ L_{\rm ad}^+ \end{bmatrix}. \tag{2.31}$$

Similarly, Σ_* is controllable (stabilizable) if and only if the pair $(A_{\rm con}, B_{\rm con})$ is controllable (stabilizable). □

The invariant zeros of a system Σ_* characterized by (A_*, B_*, C_*, D_*) can be defined via the Smith canonical form of the Rosenbrock system matrix [66] of Σ_* defined as the polynomial matrix $P_{\Sigma_*}(s)$,

$$P_{\Sigma_*}(s) := \begin{bmatrix} sI - A_* & -B_* \\ C_* & D_* \end{bmatrix}. \tag{2.32}$$

We have the following definition for the invariant zeros (see also [57]).

Definition 2.2.1 (Invariant Zeros). A complex scalar $\alpha \in \mathbb{C}$ is said to be an invariant zero of Σ_* if

$$\text{rank}\, \{P_{\Sigma_*}(\alpha)\} < n + \text{normrank}\ \{H_*(s)\}, \tag{2.33}$$

where normrank$\{H_*(s)\}$ denotes the normal rank of $H_*(s)$, which is defined as its rank over the field of rational functions of s with real coefficients. □

The special coordinate basis of Theorem 2.2.3 shows explicitly the invariant zeros and the normal rank of Σ_*. To be more specific, we have the following properties.

Property 2.2.2

1. The normal rank of $H_*(s)$ is equal to $m_0 + m_{\rm d}$.

2. Invariant zeros of Σ_* are the eigenvalues of $A_{\rm aa}$, which are the unions of the eigenvalues of $A_{\rm aa}^-$, $A_{\rm aa}^0$ and $A_{\rm aa}^+$. Moreover, the given system Σ_* is of minimum phase if and only if $A_{\rm aa}$ has only stable eigenvalues, marginal minimum phase if and only if A_{aa} has no unstable eigenvalue but has at least one marginally stable eigenvalue, and non-minimum phase if and only if $A_{\rm aa}$ has at least one unstable eigenvalue. □

In order to display various multiplicities of invariant zeros, let $X_{\rm a}$ be a non-singular transformation matrix such that $A_{\rm aa}$ can be transformed into a Jordan canonical form, i.e.,

$$X_{\rm a}^{-1} A_{\rm aa} X_{\rm a} = J = \text{blkdiag}\left\{J_1, J_2, \cdots, J_k\right\}, \tag{2.34}$$

where J_i, $i = 1, 2, \cdots, k$, are some $n_i \times n_i$ Jordan blocks:

$$J_i = \text{diag}\left\{\alpha_i, \alpha_i, \cdots, \alpha_i\right\} + \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}. \tag{2.35}$$

For any given $\alpha \in \lambda(A_{aa})$, let there be τ_α Jordan blocks of A_{aa} associated with α. Let $n_{\alpha,1}$, $n_{\alpha,2}$, $\cdots$, n_{α,τ_α} be the dimensions of the corresponding Jordan blocks. Then we say α is an invariant zero of Σ_* with multiplicity structure $S_\alpha^\star(\Sigma_*)$ (see also [68]),

$$S_\alpha^\star(\Sigma_*) = \left\{n_{\alpha,1}, n_{\alpha,2}, \cdots, n_{\alpha,\tau_\alpha}\right\}. \tag{2.36}$$

The geometric multiplicity of α is then simply given by τ_α, and the algebraic multiplicity of α is given by $\sum_{i=1}^{\tau_\alpha} n_{\alpha,i}$. Here we should note that the invariant zeros together with their structures of Σ_* are related to the structural invariant indices list $\mathcal{I}_1(\Sigma_*)$ of Morse [60].

The special coordinate basis can also reveal the infinite zero structure of Σ_*. We note that the infinite zero structure of Σ_* can be either defined in association with root-locus theory or as Smith-McMillan zeros of the transfer function at infinity. For the sake of simplicity, we only consider the infinite zeros from the point of view of Smith-McMillan theory here.

To define the zero structure of $H_*(s)$ at infinity, one can use the familiar Smith-McMillan description of the zero structure at finite frequencies of a general not necessarily square but strictly proper transfer function matrix $H_*(s)$. Namely, a rational matrix $H_*(s)$ possesses an infinite zero of order k when $H_*(1/z)$ has a invariant zero of precisely that order at $z = 0$ (see [27], [65], [66] and [75]). The number of zeros at infinity together with their orders indeed defines an infinite zero structure. Owens [62] related the orders of the infinite zeros of the root-loci of a square system with a non-singular transfer function matrix to $\mathcal{C}^*$ structural invariant indices list $\mathcal{I}_4$ of Morse [60]. This connection reveals that even for general not necessarily strictly proper systems, the *structure at infinity is in fact the topology of inherent integrations between the input and the output variables.* The special coordinate basis of Theorem 2.2.3 explicitly shows this topology of inherent integrations. The following property pinpoints this.

Definition 2.2.2 [69] The system Σ possesses an infinite zero of order k if the associated rational matrix $C(\frac{1}{z}I - A)^{-1}B$ has a invariant zero of precisely that order at $z = 0$. If each q_i of $q_1 \geq \cdots \geq q_{m_d} \geq 1$ corresponds to an infinite zero of system Σ with order q_i, then $S_\infty(\Sigma) = \{q_1, \cdots, q_{m_d}\}$ is called the infinite zero structure of system Σ. □

Property 2.2.3 Σ_* has $m_0 =$ rank (D_*) infinite zeros of order 0. The infinite zero structure (of order greater than 0) of Σ_* is given by

$$S_\infty^\star(\Sigma_*) = \Big\{q_1, q_2, \cdots, q_{m_{\rm d}}\Big\}. \tag{2.37}$$

That is, each q_i corresponds to an infinite zero of Σ_* of order q_i. Note that for a single-input-single-output system Σ_*, we have $S_\infty^\star(\Sigma_*) = \{q_1\}$, where q_1 is the *relative degree* of Σ_*. □

The structural decomposition can also exhibit the invertibility structure of a given system Σ_*. The formal definitions of right invertibility and left invertibility of a linear system can be found in [61]. Basically, for the usual case when $[\,B_*' \;\; D_*'\,]$ and $[\,C_* \;\; D_*\,]$ are of maximal rank, the system Σ_* or equivalently $H_*(s)$ is said to be left invertible if there exists a rational matrix function, say $L_*(s)$, such that

$$L_*(s)H_*(s) = I_m. \tag{2.38}$$

Σ_* or $H_*(s)$ is said to be right invertible if there exists a rational matrix function, say $R_*(s)$, such that

$$H_*(s)R_*(s) = I_p. \tag{2.39}$$

Σ_* is invertible if it is both left and right invertible, and Σ_* is degenerate if it is neither left nor right invertible.

Property 2.2.4 The given system Σ_* is right invertible if and only if $x_{\rm b}$ (and hence $y_{\rm b}$) are non-existent, left invertible if and only if $x_{\rm c}$ (and hence $u_{\rm c}$) are non-existent, and invertible if and only if both $x_{\rm b}$ and $x_{\rm c}$ are non-existent. Moreover, Σ_* is degenerate if and only if both $x_{\rm b}$ and $x_{\rm c}$ are present. □

The special coordinate basis can also be modified to obtain the structural invariant indices lists $\mathcal{I}_2$ and $\mathcal{I}_3$ of Morse [60] of the given system Σ_*. In order to display $\mathcal{I}_2(\Sigma_*)$, we let $X_{\rm c}$ and X_i be non-singular matrices such that the controllable pair $(A_{\rm cc}, B_{\rm c})$ is transformed into Brunovsky canonical form (see Theorem 2.2.1), i.e.,

$$X_{\rm c}^{-1}A_{\rm cc}X_{\rm c} = \begin{bmatrix} 0 & I_{\ell_1-1} & \cdots & 0 & 0 \\ \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{\ell_{m_{\rm c}}-1} \\ \star & \star & \cdots & \star & \star \end{bmatrix}, \quad X_{\rm c}^{-1}B_{\rm c}X_i = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \tag{2.40}$$

where $\star$'s denote constant scalars or row vectors. Then we have

$$\mathcal{I}_2(\Sigma_*) = \Big\{\ell_1, \cdots, \ell_{m_{\rm c}}\Big\}, \tag{2.41}$$

which is also called the controllability index of $(A_{\rm cc}, B_{\rm c})$. Similarly, we have

$$\mathcal{I}_3(\Sigma_*) = \Big\{\mu_1, \cdots, \mu_{p_{\rm b}}\Big\}, \tag{2.42}$$

where $\{\mu_1, \cdots, \mu_{p_{\rm b}}\}$ is the controllability index of the controllable pair $(A'_{\rm bb}, C'_{\rm b})$.

By now it is clear that the special coordinate basis decomposes the state-space into several distinct parts. In fact, the state-space $\mathcal{X}$ is decomposed as

$$\mathcal{X} = \mathcal{X}_{\rm a}^- \oplus \mathcal{X}_{\rm a}^0 \oplus \mathcal{X}_{\rm a}^+ \oplus \mathcal{X}_{\rm b} \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d}. \tag{2.43}$$

Here $\mathcal{X}_{\rm a}^-$ is related to the stable invariant zeros, i.e., the eigenvalues of $A_{\rm aa}^-$ are the stable invariant zeros of Σ_*. Similarly, $\mathcal{X}_{\rm a}^0$ and $\mathcal{X}_{\rm a}^+$ are respectively related to the invariant zeros of Σ_* located in the marginally stable and unstable regions. On the other hand, $\mathcal{X}_{\rm b}$ is related to the right invertibility, i.e., the system is right invertible if and only if $\mathcal{X}_{\rm b} = \{0\}$, while $\mathcal{X}_{\rm c}$ is related to left invertibility, i.e., the system is left invertible if and only if $\mathcal{X}_{\rm c} = \{0\}$. Finally, $\mathcal{X}_{\rm d}$ is related to zeros of Σ_* at infinity.

There are interconnections between the special coordinate basis and various invariant geometric subspaces. To show these interconnections, we introduce the following geometric subspaces:

Definition 2.2.3 (Geometric Subspaces $\mathcal{V}^{\rm X}$ and $\mathcal{S}^{\rm X}$). The weakly unobservable subspaces of Σ_*, $\mathcal{V}^{\rm X}$, and the strongly controllable subspaces of Σ_*, $\mathcal{S}^{\rm X}$, are defined as follows:

1. $\mathcal{V}_{\rm X}(\Sigma_*)$ is the maximal subspace of $\mathbb{R}^n$ which is $(A_*+B_*F_*)$-invariant and contained in $\operatorname{Ker}(C_* + D_*F_*)$ such that the eigenvalues of $(A_* + B_*F_*)|\mathcal{V}_{\rm X}$ are contained in $\mathbb{C}_{\rm X} \subseteq \mathbb{C}$ for some constant matrix F_*.

2. $\mathcal{S}_{\rm X}(\Sigma_*)$ is the minimal $(A_* + K_*C_*)$-invariant subspace of $\mathbb{R}^n$ containing $\operatorname{Im}(B_* + K_*D_*)$ such that the eigenvalues of the map which is induced by $(A_* + K_*C_*)$ on the factor space $\mathbb{R}^n/\mathcal{S}_{\rm X}$ are contained in $\mathbb{C}_{\rm X} \subseteq \mathbb{C}$ for some constant matrix K_*.

Furthermore, we let $\mathcal{V}^- = \mathcal{V}_\mathrm{X}$ and $\mathcal{S}^- = \mathcal{S}_\mathrm{X}$, if $\mathbb{C}_\mathrm{X} = \mathbb{C}^- \cup \mathbb{C}^0$; $\mathcal{V}^+ = \mathcal{V}_\mathrm{X}$ and $\mathcal{S}^+ = \mathcal{S}_\mathrm{X}$, if $\mathbb{C}_\mathrm{X} = \mathbb{C}^+$; and finally $\mathcal{V}^* = \mathcal{V}_\mathrm{X}$ and $\mathcal{S}^* = \mathcal{S}_\mathrm{X}$, if $\mathbb{C}_\mathrm{X} = \mathbb{C}$. □

Various components of the state vector of the special coordinate basis have the following geometrical interpretations.

Property 2.2.5

1. $\mathcal{X}_\mathrm{a}^- \oplus \mathcal{X}_\mathrm{a}^0 \oplus \mathcal{X}_\mathrm{c}$ spans $\mathcal{V}^-(\Sigma_*)$.

2. $\mathcal{X}_\mathrm{a}^+ \oplus \mathcal{X}_\mathrm{c}$ spans $\mathcal{V}^+(\Sigma_*)$.

3. $\mathcal{X}_\mathrm{a}^- \oplus \mathcal{X}_\mathrm{a}^0 \oplus \mathcal{X}_\mathrm{a}^+ \oplus \mathcal{X}_\mathrm{c}$ spans $\mathcal{V}^*(\Sigma_*)$.

4. $\mathcal{X}_\mathrm{a}^+ \oplus \mathcal{X}_\mathrm{c} \oplus \mathcal{X}_\mathrm{d}$ spans $\mathcal{S}^-(\Sigma_*)$.

5. $\mathcal{X}_\mathrm{a}^- \oplus \mathcal{X}_\mathrm{a}^0 \oplus \mathcal{X}_\mathrm{c} \oplus \mathcal{X}_\mathrm{d}$ spans $\mathcal{S}^+(\Sigma_*)$.

6. $\mathcal{X}_\mathrm{c} \oplus \mathcal{X}_\mathrm{d}$ spans $\mathcal{S}^*(\Sigma_*)$. □

This property relates structural decomposition to invariant geometric subspaces, and thus give a clear geometric interpretation for the distinct subspaces of structural decomposition.

2.3 Linear singular systems

Linear singular system, or alternatively called generalized linear system or linear descriptor system [29] [47], is a better system model than nonsingular system since it represents more general information of a real system. Roughly speaking, most real systems in this world are singular in nature. Such systems include biological system, financial system, social system, power system and electrical system, to name just a few. According to this fact, most real systems should be characterized as singular systems. However, due to lacking of efficient tools, they just simply be treated as nonsingular systems in many cases. To propose a new powerful tool for singular systems, we present the structural decomposition in this thesis.

In general, most definitions and techniques for singular systems are natural extension of their counterpart for nonsingular systems. This will be seen clearly in the following when this section gives a brief introduction of definitions for singular systems.

Let us first look at the following example of an electrical circuit (see also [47] [29]).

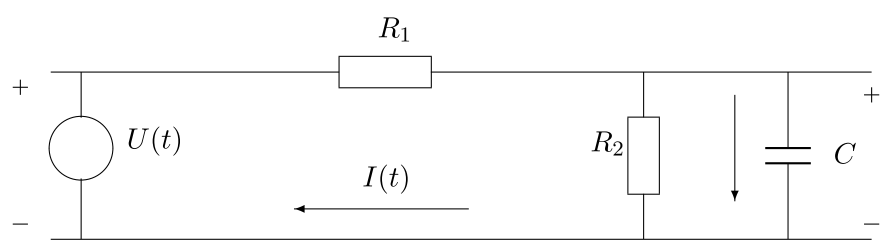

Figure 2.1: A simple electrical circuit.

Now we have at least two methods to model this circuit. First one is using nonsingular system model, and we will have

$$\dot{U}_{\mathrm{C}} = -\frac{R_1 + R_2}{R_1 R_2 C} U_{\mathrm{C}} + \frac{1}{R_1 C} U(t), \tag{2.44}$$

where $U_{\mathrm{C}}(t)$ is the voltage across the capacity.

We can see that some internal information can not be represented by (2.44), such as the relationship between $U_{\mathrm{C}}(t)$ and $I(t)$. While such information will be revealed in the following singular system description.

$$\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{I}(t) \\ \dot{U}_{\mathrm{C}}(t) \end{bmatrix} = \begin{bmatrix} R_1 & 1 \\ 1 & -\frac{1}{R_2} \end{bmatrix} \begin{bmatrix} I(t) \\ U_{\mathrm{C}}(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} U(t). \tag{2.45}$$

It is clear that the whole circuit's information has been included in the singular system of (2.45). And in general, singular systems provide more internal information of the real systems. This is the reason that the singular system has been in attention for so many years.

Generally, a singular system can be expressed in the following state space form,

$$\Sigma : \begin{cases} E\,\dot{x} = A\,x + B\,u, \qquad x(0) = x_0 \\ \quad y = C\,x + D\,u, \end{cases} \tag{2.46}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and $\text{rank}(E) < n$.

2.3.1 Impulsive Mode and Initial Conditions

Linear nonsingular system, or alteratively called nonsingular system, is a simple expression characterizing many real systems. And it has received an intensive research during last three decades. A lot of methods have been presented in literature to solve system and control problem in large variety. However, a singular system is a more natural model for most real systems in this world. It is more general than a linear nonsingular system, simply because it contains more complete information of the objects it characterized, which can be seen clearly in its state space expression of (2.46) that a linear nonsingular system is merely a special case of singular systems.

Further, singular systems have their own system features. One of these is impulsive mode.

Consider the following singular system,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u, \tag{2.47}$$

we can find its state variable as

$$\begin{cases} x_1 = \dot{u} \\ x_2 = u. \end{cases} \tag{2.48}$$

It is clear that there is an input derivative in the state variable x_1, thus it will have a impulse response factor $\delta(t)$ if the input is a function of unit step function $u(t)$.

Hence the state x_1 will have impulsive behavior at the starting point if the initial conditions are not consistent, that is, if $x_2(0) \neq 0$ or $x_1(0) \neq u(0)$. And actually this is quite like the jump behavior in nonsingular systems when their initial conditions are not consistent. Furthermore, if there is a jump in the input or even the input function is continuous, the system response may also have impulsive modes or jump behaviors. All of such behaviors are caused by the input derivatives, which is caused by the special structure properties of singular systems. And this forms a distinct feature for singular systems which is totally different from nonsingular systems.

In order to lay off those unnecessary discussion on initial conditions, and without loss of any generality, we assume the initial conditions are consistent in this thesis, just like what have happened in nonsingular systems. And if it is not consistent in some cases, we can treat them case by case.

Then the only cause of impulsive behavior is the structural property of singular system after the above general assumption on their initial conditions.

2.3.2 Restricted System Equivalence

Singular systems are also called descriptor systems, implicit systems or generalized state space systems in the literature. As one of the main research topics in system and control theories, singular system has been of attraction for more than three decades. This directly result in the large number of techniques presented for singular systems. Among these methods, one important group is about system equivalence. Because most other methods are simply based on system equivalence to begin their development.

An equivalent relationship between two systems possesses reflexivity, transitivity and invertibility. While restricted system equivalence give more rigorous conditions and can be defined as follows,

Definition 2.3.1 (see also [85] [38]) Two singular systems $\Sigma(E, A, B, C)$ and $\tilde{\Sigma}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ are restricted equivalent if there exist two invertible matrices P and Q such that

$$\tilde{E} = PEQ, \quad \tilde{A} = PAQ, \quad \tilde{B} = PB, \quad \tilde{C} = CQ. \tag{2.49}$$

Restricted equivalent singular systems have many identical properties such as structural features. And here we recall two restricted equivalence forms very often used in the literature.

Lemma 2.3.1 (see also [29]) For any singular system Σ of (2.46), if it is regular, there exist an invertible coordinate transformation,

$$x = Q\tilde{x} = Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_1 \in \mathbb{R}^{n_1}, \quad x_2 \in \mathbb{R}^{n_2}, \tag{2.50}$$

and an invertible matrix P such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CQ = [\, C_1 \quad C_2 \,], \tag{2.51}$$

and the original system Σ is decomposed into and restricted equivalent to the following system,

$$\Sigma_1 : \begin{cases} \dot{x_1} = A_1\ x_1 + B_1\ u, & x_1(0) = x_{10} \\ y_1 = C_1\ x_1 + D\ u, \end{cases} \tag{2.52}$$

$$\Sigma_1 : \begin{cases} N\ \dot{x_2} = I_{n_2}\ x_2 + B_2\ u, & x_2(0) = x_{20} \\ y_2 = C_2\ x_2, \end{cases} \tag{2.53}$$

where $y = y_1 + y_2$ and N is a nilpotent matrix with an index of h, that is, $N^h = 0$ while $N^{h-1} \neq 0$.

This decomposition gives a restricted equivalent singular system, and it also called Weierstrass decomposition or slow-fast decomposition in the literature. Such a decomposition separates a nonsingular subsystem from another singular subsystem, and thus play an important role in developing many techniques for singular systems.

For such a decomposition, we also have the following theorem,

Theorem 2.3.1 (see also [29]) Suppose $\Sigma_1(I_{n_1}, A_1, B_1, C_1, D)$ and $\Sigma_2(N, I_{n_2}, B_2, C_2, 0)$ are the two subsystems decomposed from Σ by Lemma 2.3.1 with invertible P and Q, while $\bar{\Sigma}_1(I_{\bar{n}_1}, \bar{A}_1, \bar{B}_1, \bar{C}_1, \bar{D})$ and $\bar{\Sigma}_2(\bar{N}, I_{\bar{n}_2}, \bar{B}_2, \bar{C}_2, 0)$ are the two subsystems decomposed from Σ by Lemma 2.3.1 with invertible $\bar{P}$ and $\bar{Q}$, then there exist two invertible transform matrices $U \in \mathbb{R}^{n_1 \times n_1}$ and $V \in \mathbb{R}^{n_2 \times n_2}$ such that

$$\begin{aligned} P &= \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \bar{P}, \quad Q = \bar{Q} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}, \\ A_1 &= U \bar{A}_1 U^{-1}, \quad N = V \bar{N} V^{-1}, \\ B_1 &= U \bar{B}_1, \quad C_1 = \bar{C}_1 U^{-1}, \\ B_2 &= V \bar{B}_2, \quad C_2 = \bar{C}_2 V^{-1}. \end{aligned} \tag{2.54}$$

This theorem shows that different decompositions by Lemma 2.3.1 are similar to each other.

Now we can look at another kind of system equivalence for singular systems.

Lemma 2.3.2 [29] For any singular system Σ of (2.46), if it is regular, there exist an invertible coordinate transformation,

$$x = Q\tilde{x} = Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_1 \in \mathbb{R}^q, \quad x_2 \in \mathbb{R}^{n-q}, \tag{2.55}$$

and an invertible transform matrix P such that

$$PEQ = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CQ = [\, C_1 \quad C_2 \,], \tag{2.56}$$

and the original system Σ is decomposed into and restricted equivalent to the following system,

$$\Sigma_3 : \begin{cases} \dot{x_1} = A_{11}\ x_1 + A_{12}\ x_2 + B_1\ \ u, \\ \ 0\ = A_{21}\ x_1 + A_{22}\ x_2 + B_2\ \ u, \\ \ y\ = \ C_1\ \ x_1 + \ C_2\ \ x_2 + \ D\ \ u, \end{cases} \tag{2.57}$$

This restricted equivalent decomposition shows the physical meaning of singular system clearly. The first equation is of dynamic state variables and the second equation is a algebraic one and represents the constrains among the internal state variables. Such a decomposition shows that a singular system is a combination of several interconnected subsystems.

2.3.3 Stabilizability and Detectability

Stabilizability and detectability are two essential properties of linear systems. Stabliizability gives the possibility that we can revise a linear system while remaining its stability at the same time. If it is totally stabilizable, we can design feedback controllers to improve system's performance and retain its internal stability as well. And we can not change internal states if they are uncontrollable. Similarly, we can get the information of internal state variables if the given system is detectible, otherwise we have to estimate them before designing a feedback controller.

Before defining stabilizability and detecbility, we first give the definition on controllability and observability.

Definition 2.3.2 (Controllability) [84] [26] A singular system Σ of (2.46) is said to be controllable if, for any $t_1 > 0$, $x(0) \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, there exists a control input $u(t) \in \mathbb{C}_{\mathrm{p}}^{h-1}$ such that $x(t_1) = w$. Here $\mathbb{C}_{\mathrm{p}}^{h-1}$ represents the $(h-1)$-times piecewise continuously differentiable function set.

From the definition of controllability on system states, we can see that if a state is controllable, we can use a control input to set its value as we like. This is critical in designing singular systems. Generally, a linear system is said to be controllable if and only if all its states are controllable.

The following theorem gives a general criterion on controllability.

Theorem 2.3.2 [29] Singular system Σ is controllable if and only if

$$\operatorname{rank}\,[\,sE - A \quad B\,] = n, \quad \text{and} \quad \operatorname{rank}\,[\,E \quad B\,] = n, \tag{2.58}$$

for all finite $s \in \mathbb{C}$.

This theorem is a simple rule for us to determine whether or not a given singular system is controllable. There are also many other methods on judging a singular system's controllability, but basically they are all equivalent to this one.

Now we recall a theorem on the stabilizability of singular systems.

Theorem 2.3.3 [29] Singular system Σ is stabilizable if and only if

$$\operatorname{rank}\,[\,sE - A \quad B\,] = n, \tag{2.59}$$

for all finite $s \in \mathbb{C}$.

Dual to controllability, observability is also a critical concept in system and control theories. Observability of a singular system shows how much internal state information we can get to design output feedback controllers. This is essential for the success of designing a good controller because internal state information is the basis of design. In general, people design an observer to estimate internal state information when the given system is unobservable. However, such an estimation will have error for more or less and can not perform as well as the internal states themselves.

We now give the definition on observability in following.

Definition 2.3.3 (Observability) [29] A singular system is said to be observable if its initial condition $x(0)$ can be uniquely determined by its input $u(t)$ and output $y(t)$ for $0 \leq t \leq \infty$.

The observability states that the state of observable system may be determined by observing the initial condition $x(0)$, followed by constructing the state response at any time t.

The following theorem gives matrix form judgement rule on observability.

Theorem 2.3.4 [29] A singular system Σ is observable if and only if,

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n, \quad \text{and} \quad \text{rank} \begin{bmatrix} sE - A \\ C \end{bmatrix} = n \tag{2.60}$$

for all finite $s \in \mathbb{C}$.

And the following theorem is for detectibility.

Theorem 2.3.5 [29] A singular system Σ is detectible if and only if,

$$\text{rank} \begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \tag{2.61}$$

for all finite $s \in \mathbb{C}$.

Controllability and observability, or stabilizability and detectibility are basis for our later discussions. They play essential roles in solving singular system and control problems.

2.3.4 Zero Structures

Zero structures play essential roles in understanding internal structural information of a singular system. Invariant and infinite zero structures have been widely used in solving various system and control problems. And they are proved to be efficient in representing given systems' structural features.

The definition of invariant zeros of singular systems can be done similarly as that for nonsingular systems (see e.g., Chen [19] and MacFarlane and Karcanias [57]) or in the Kronecker canonical form associated with Σ (see e.g., Malabre [58]).

Definition 2.3.4 (Invariant Zeros) *A complex scalar $\alpha \in \mathbb{C}$ is said to be an invariant zero of the singular system Σ of (2.46) if*

$$\text{rank}\{P_\Sigma(\alpha)\} < n + \text{normrank}\{H(s)\}, \tag{2.62}$$

where $\text{normrank}\{(H(s)\}$ *denotes the normal rank of $H(s) = C(sE - A)^{-1}B + D$, which is defined as its rank over the field of rational functions with real coefficients, and $P_\Sigma(s)$ is the Rosenbrock system matrix associated with Σ and is given by*

$$P_\Sigma(s) = \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}. \tag{2.63}$$

The infinite zero structure of the given system Σ can be either defined in association with the Kronecker canonical form of $P_\Sigma(s)$ or as Smith-McMillan zeros of the transfer function from $\tilde{u}$ to $\tilde{y}$, say $\tilde{H}(s)$, at infinity. The indices defined by these two methods are identical. To define the zero structure of $\tilde{H}(s)$ at infinity, one can use the familiar Smith-McMillan description of the zero structure at finite frequencies of $\tilde{H}(s)$. Namely, a rational matrix $\tilde{H}(s)$ possesses an infinite zero of order k when $\tilde{H}(1/z)$ has a finite zero of precisely that order at $z = 0$ (see [27], [65] and [75]). The number of zeros at infinity together with their orders indeed defines an infinite zero structure.

2.3.5 System Invertibility

The invertibility structure of a given singular system Σ is useful and it is an important structural property of singular systems. And the definition of system invertibility for singular systems are also similar to that for nonsingular systems [61].

Basically, for the usual case when $[\,B' \quad D'\,]$ and $[\,C \quad D\,]$ are of maximal rank, the system Σ or equivalently $H(s)$ is said to be left invertible if there exists a rational matrix function $L(s)$ such that

$$L(s)H(s) = I_m. \tag{2.64}$$

Σ is right invertible if there exists a rational matrix function $R(s)$ such that

$$H(s)R(s) = I_p. \tag{2.65}$$

Moreover, Σ is said to be invertible if it is both left and right invertible, and Σ is non-invertible if it is not invertible.

2.3.6 Kronecker Canonical Form and Invariant Indices

The Kronecker canonical form plays an important role in the structural analysis of singular systems. It exhibits the invariant zeros and infinite zeros of the system, and also shows the left and right null-space structure. We recall that two pencils $sM_1 - N_1$ and $sM_2 - N_2$ of dimension $m \times n$ are *strictly equivalent* if there exist constant nonsingular matrices P and Q such that

$$Q(sM_1 - N_1)P = sM_2 - N_2. \tag{2.66}$$

It is showed in [35] that any pencil $sM - N$ can be reduced, under strict equivalence, to a canonical quasidiagonal form, which is given by

$$S(sM - N)T = \text{blkdiag}\{R_{r_1}, \ldots, R_{r_p}, L_{l_1}, \ldots, L_{l_q}, I - sH, sI - J\}, \tag{2.67}$$

with R_k and L_k being the $k \times (k+1)$ and $(k+1) \times k$ bidiagonal pencil respectively,

$$R_k := \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}, \qquad L_k := \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}. \tag{2.68}$$

J is in Jordan canonical form, and $sI - J$ has the following $\Sigma_{i=1}^{\delta} d_i$ pencils as its diagonal blocks,

$$sI_{m_{i,j}} - J_{m_{i,j}}(\beta_i) := \begin{bmatrix} s-\beta_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\beta_i & -1 \\ & & & s-\beta_i \end{bmatrix}, \quad j = 1, \ldots, d_i, \quad i = 1, \ldots, \delta. \tag{2.69}$$

H is nilpotent and in Jordan canonical form, and $I - sH$ has the following d pencils as its diagonal blocks,

$$I_{n_j} - sJ_{n_j}(0) := \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix}, \qquad j = 1, \ldots, d. \tag{2.70}$$

Then, $\{(s-\beta_i)^{m_{i,j}}, j = 1, \ldots, d_i\}$ is finite elementary divisors at β_i, $i = 1, \ldots, \delta$. The index sets $\{r_1, \ldots, r_p\}$ and $\{l_1, \ldots, l_q\}$ are right and left minimal indices respectively. $\{(1/s)^{n_j}, j = 1, \ldots, d\}$ are the infinite elementary divisors.

The definition of structural invariants of singular systems is based on invariant indices of its system pencil. For singular systems, the right and left invertibility indices of singular system are right and left minimal indices of system pencil respectively, and the invariant and infinite zero structures of singular system are relate to finite and infinite elementary divisors of system pencil.

Several methods have been developed to compute the structural invariants of singular linear systems under algebraic setting. In the algebraic approaches, the row and column compressions of a matrix are often used in the decomposition, i.e.,

$$UH := \begin{bmatrix} H_r \\ 0 \end{bmatrix}, \qquad HV := \begin{bmatrix} H_c & 0 \end{bmatrix}, \tag{2.71}$$

where H is an arbitrary $m \times n$ matrix, U and V are unitary matrices, H_r and H_c have full row and column rank respectively. By using the row and column compression, Van Dooren *et al.* [31, 32] reduce an arbitrary pencil $sM - N$ to the form

$$U(sM - N)V = \begin{bmatrix} sM_l - N_l & * & * & * \\ 0 & sM_f - N_f & * & * \\ 0 & 0 & sM_i - N_i & * \\ 0 & 0 & 0 & sM_r - N_r \end{bmatrix}, \tag{2.72}$$

where $*$ is polynomial of s. $sM_l - N_l$ and $sM_r - N_r$ are nonsquare pencils with the informations of Kronecker indices of $sM - N$; $sM_i - N_i$ and $sM_f - N_f$ are regular pencils with the infinite and finite elementary divisors of $sM - N$ respectively. Varga [73] presents several condensed Kronecker-like forms which exhibit either the complete Kronecker structure or only a part of the Kronecker structure of the system pencil.

For the singular system $\Sigma(E, A, B, C, D)$, by using the transformations U and V such that $UEV = \text{blkdiag}\{E_{11}, 0\}$, where E_{11} is invertible, a "compressed generalized state space system" $\Sigma_\pi(E_{11}, \check{A}, \check{B}, \check{C}, \check{D})$ is introduced [59]. The transformations are conducted on the base of the compressed system,

$$Q\mathbf{P}_{\Sigma_\pi}(s)P = \begin{bmatrix} N_1 - sM_1 & * \\ 0 & N_2 - sM_2 \end{bmatrix}, \tag{2.73}$$

where

$$N_2 - sM_2 = \begin{bmatrix} N^c_{j,j} & N^c_{j,j-1} - sM^c_{j,j-1} & \dots & N^c_{j,2} - sM^c_{j,2} & N^c_{j,1} - sM^c_{j,1} \\ 0 & N^c_{j-1,j-1} & \dots & N^c_{j-1,2} - sM^c_{j-1,2} & N^c_{j-1,1} - sM^c_{j-1,1} \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & N^c_{2,2} & N^c_{2,1} - sM^c_{2,1} \\ 0 & 0 & \dots & 0 & N^c_{1,1} \end{bmatrix}. \tag{2.74}$$

and where $M_{i+1,i}$ has full row rank, $N^c_{i,i}$ has full column rank, and $N_1 - sM_1$ and $N_2 - sM_2$ have the similar structure. From the structure of the pencil, the structural invariants can be obtained.

2.4 Conclusions

In this chapter, we recall a series of crucial tools for linear systems, which include Jordan and Real Jordan Canonical Form, Controllability Canonical Form (CCF), Controllability Structural Decomposition (CSD), Block Diagonal Control Canonical Form (BDCCF) and the structural decomposition for nonsingular systems. Such tools have been used in the literature to solve many system and control problems such as the squaring down and decoupling of linear systems (see e.g., Sannuti and Saberi [70]), linear system factorizations (see e.g., Chen et al [11], and Lin et al [51]), blocking zeros and strong stabilizability (see e.g., Chen et al [12]), zero placements (see e.g. Chen and Zheng [15]), loop transfer recovery (see e.g., Chen [10], Chen and Chen [16], and Saberi et al [68]), H_2 optimal control (see e.g., Chen et al [13, 14], and Saberi et al [69]), disturbance decoupling (see e.g., Chen [18], and Ozcetin et al [63, 64]), H_∞ optimal control (see e.g., Chen et al [11] and control with saturations (see e.g., Lin [50]). The list here is far from complete.

The main objective of this thesis is to extend the structural decomposition technique to singular systems and apply it to solve singular system and control problems as its counterpart in nonsingular systems. All of these will be introduced in the following chapters.

Chapter 3

Structural Decomposition of SISO Singular Systems

3.1 Introduction

Singular systems, also commonly called generalized or descriptor systems in the literature, appear in many practical situations including engineering systems, economic systems, network analysis, and biological systems (see e.g., Dai [29], Kuijper [45] and Lewis [48]). In fact, many systems in the real life are singular in nature. They are usually simplified as or approximated by nonsingular models because there is still lacking of efficient tools to tackle problems related to such systems. The structural analysis of singular systems, using either algebraic or geometric approach, has attracted considerable attention from many researchers over the last three decades (see e.g., Van Dooren [31, 32], Geerts [36], Loiseau [55], Malabre [58], Misra *et al.* [59], Verghese [76], Zhou *et al.* [85], Chu *et al.* [21] [22] and the references cited therein). Generally speaking, almost all the research work dealing with singular systems is the natural extension of those results for nonsingular counterparts, although it is much harder in obtaining solutions associated with singular systems.

It has been extensively demonstrated and proven for nonsingular systems that the system structural properties, such as the finite zero and infinite zero structures as well as the invertibility structures, play a very important role in solving related control problems including H_2, H_∞ control and disturbance decoupling (see e.g., Chen [20] and Saberi *et al.* [69]). In this chapter, we present a structural decomposition of general single-input and single-output singular systems, which is capable of capturing and displaying all the structural properties of the given system. Our method can be regarded as a natural extension of the work of Sannuti and Saberi [70]. However, it will be seen shortly that the structural decomposition of a singular system is much more complicated than that of a nonsingular system. Such a decomposition technique is expected to be a powerful tool and play an important role in solving control problems for singular systems, such as H_2 and H_∞ control, model reduction and disturbance decoupling, to name just a few.

To extend the structural decomposition for singular systems, we first focus on single input single output case because it is easier to find a solution and can bring us hints for general singular systems.

In this chapter, we first presents the structural decomposition theorem for general single-input and single-output singular systems. Such a decomposition is a natural extension of structural decomposition for nonsingular systems. As its counterpart for nonsingular systems, it is expected to be a powerful tool in solving control problems for singular systems. And in Section 3, we will show that the decomposition technique also has a distinct feature of capturing and displaying all the structural properties, such as the finite and infinite zero structures and redundant dynamics, of the given system. All proofs for the main theorem and its properties are shown in Section 4. Further, an illustrative example will be given in Section 5 to show the constructive decomposition procedure and verify its important properties. And finally, a conclusion will be draw in Section 6.

3.2 Structural Decomposition Theorem

We first consider a linear time-invariant system Σ characterized by

$$\begin{cases} E\,\dot{x} = A\,x + B\,u, & x(0) = x_0 \\ \quad y = C\,x \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are respectively the state, input and output of the system, and E, A, B, C and D are constant matrices of appropriate dimension. The system Σ is said to be singular if $\text{rank}(E) < n$. As usual, in order to avoid any ambiguousness in solutions to the system, we assume that the given singular system Σ is regular, i.e., $\det(sE - A) \not\equiv 0$, for all $s \in \mathbb{C}$.

In this section, we will present a constructive algorithm that decomposes the state of the system x into several distinct parts, which are directly associated with the finite zero dynamics and infinite zero dynamics of the given system. It is interesting to note that our decomposition will automatically and explicitly separate the redundant dynamics of the system as well.

We present in the following the main results of the paper, i.e., the structural decomposition of the singular system (3.1).

Theorem 3.2.1 *Consider the singular system Σ of (3.1) satisfying the regularity assumption, i.e., $\det(sE - A) \not\equiv 0$ for $s \in \mathbb{C}$, and its transfer function is nontrivial, i.e., $H(s) = C(sE - A)^{-1}B \not\equiv 0$ for $s \in \mathbb{C}$. There exist*

1. *non-negative integers n_0, $n_{\rm a}$, $n_{\rm d}$, $n_{\rm e}$ and v; and*

2. *nonsingular state, input and output transformations* $\Gamma_{\rm s} \in \mathbb{R}^{n\times n}$, $\Gamma_{\rm i} \in \mathbb{R}$ *and* $\Gamma_{\rm o} \in \mathbb{R}$, *and a nonsingular constant matrix* $\Gamma_{\rm e} \in \mathbb{R}^{n\times n}$, *which together give a structural decomposition of* Σ *and display explicitly its finite and infinite zero structures.*

The structural decomposition of Σ, *or the transformed system, can be described by the following set of equations:*

$$x = \Gamma_{\rm s}\tilde{x}, \quad x_0 = \Gamma_{\rm s}\tilde{x}(0), \quad \tilde{x} = \begin{pmatrix} x_{\rm e} \\ x_{\rm z} \\ x_{\rm a} \\ x_{\rm d} \end{pmatrix}, \quad x_{\rm d} = \begin{pmatrix} x_{\rm d1} \\ x_{\rm d2} \\ \vdots \\ x_{{\rm d}n_{\rm d}} \end{pmatrix}, \quad y = \Gamma_{\rm o}\tilde{y}, \quad u = \Gamma_{\rm i}\tilde{u}, \tag{3.2}$$

where $x_{\rm e} \in \mathbb{R}^{n_{\rm e}}$, $x_{\rm z} \in \mathbb{R}^{n_0}$, $x_{\rm a} \in \mathbb{R}^{n_{\rm a}}$, $x_{\rm d} \in \mathbb{R}^{n_{\rm d}}$, *and*

Case 1: If $n_{\rm d} > 0$,

$$\left.\begin{aligned} x_e &= \tilde{u}^{(v)}, \\ x_{\rm z} &= 0, \\ \dot{x}_{\rm a} &= A_{\rm aa}x_{\rm a} + L_{\rm ad}y_{\rm d}, \\ \dot{x}_{\rm d1} &= x_{\rm d2}, \\ \dot{x}_{\rm d2} &= x_{\rm d3}, \\ &\vdots \\ \dot{x}_{{\rm d}n_{\rm d}} &= M_{\rm da}x_{\rm a} + L_{\rm dd}y_{\rm d} + \tilde{u}^{(v)}, \quad \tilde{y} = y_{\rm d} = x_{\rm d1}, \end{aligned}\right\}; \tag{3.3}$$

Case 2: If $n_{\rm d} = 0$,

$$\left.\begin{aligned} x_e &= \tilde{u}^{(v)}, \\ x_{\rm z} &= 0, \\ \dot{x}_{\rm a} &= A_{\rm aa}x_{\rm a} + B_{\rm 0a}\tilde{y}, \quad \tilde{y} = \bar{C}x_{\rm a} + \bar{D}\tilde{u}^{(v)}. \end{aligned}\right\}. \tag{3.4}$$

A constructive proof of the structural decomposition in Theorem 3.2.1 will be given later in the next section. Figure 3.1 gives a block diagram interpretation of the dynamics of the structurally decomposed system in Case 1 of Theorem 3.2.1. In the figure, a signal given

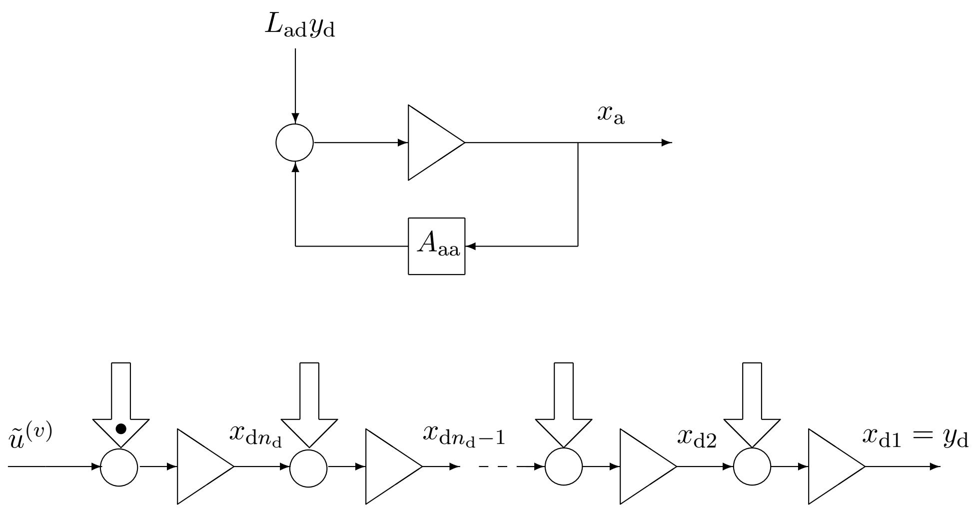

Figure 3.1: Block diagram representation of dynamics of the structurally decomposed system.

by a double-edged arrow is some linear combination of output y_{d}, whereas a signal given by the double-edged arrow with a solid dot is some linear combination of all the states.

The structural decomposition technique decomposes the state space $\mathcal{X}$ into several distinct subspaces. Such subspaces are associated with special structural properties of the given singular system. The next section will give more details on this.

3.3 Properties of Structural Decomposition

As mentioned earlier, the structural decomposition of Theorem 3.2.1 has distinct feature of revealing the structural properties of the given singular system Σ. In what follows, we will study how the system properties of Σ such as the stabilizability, detectability, invertibility, as well as finite zero and infinite zero structures, can be obtained from our decomposition.

We have the following property.

Property 3.3.1 (Stabilizability and Detectability) *The given system Σ of (3.1) is stabilizable if and only if the pair $(A_{\rm con}, B_{\rm con})$ is stabilizable. Σ is detectable if and only if the pair $(A_{\rm obs}, C_{\rm obs})$ is detectable. Here $A_{\rm con} := A_{\rm aa}$ and $A_{\rm obs} := A_{\rm aa}$. Moreover, $B_{\rm con} := L_{\rm ad}$ and $C_{\rm obs} := M_{\rm da}$ in Case 1 while $B_{\rm con} := B_{0\rm a}$ and $C_{\rm obs} := \bar{C}$ in Case 2.*

The definition of invariant zeros of singular systems has already been given in Chapter 2, which can be done similarly to that for nonsingular systems (see e.g., Chen [19] and MacFarlane and Karcanias [57]) or in the Kronecker canonical form associated with Σ (see e.g., Malabre [58]).

The following property shows that the invariant zeros of Σ can be obtained in the structural decomposition in a trivial matter.

Property 3.3.2 (Invariant Zeros) *The invariant zeros of Σ are the eigenvalues of $A_{\rm aa}$.*

The infinite zero structure of Σ can be either defined in association with the Kronecker canonical form of $P_\Sigma(s)$ or as Smith-McMillan zeros of the transfer function $H(s)$ at infinity. This has been shown in Chapter 2 in detail.

Property 3.3.3 (Infinite Zero Structure) *The infinite zero structure of the singular system Σ is given by $\{n_{\rm d} - v\}$, i.e., Σ has an infinite zero of order or relative degree $n_{\rm d} - v$. However, Σ has an infinite elementary divisor of order $n_{\rm d}$ in its corresponding Kronecker canonical form.*

Again, the rigorous proofs to all these properties are given in the next section.

3.4 Proofs of Main Results

We are ready to give proofs to the main results of our paper, i.e., the structural decomposition of Theorem 3.2.1 and its properties.

3.4.1 Proof of Theorem 3.2.1

The following is a step-by-step constructive procedure for the structural decomposition of Σ.

Step 1 (Preliminary Decomposition): It follows from Lemma 2.3.1 or Dai [29] that there exist two nonsingular matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CQ = [\, C_1 \quad C_2 \,], \tag{3.5}$$

where A_1, B_1, B_2, C_1 and C_2 are matrices with appropriate dimensions, and N is a nilpotent matrix with an appropriate nilpotent index, say h, i.e., $N^{h-1} \neq 0$ and $N^h = 0$. Equivalently, Σ can be decomposed into the following two subsystems:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = A_1 \, x_1 + B_1 \, u, \quad x_1(0) = x_{10} \\ y_1 = C_1 \, x_1 \end{cases} \tag{3.6}$$

and

$$\Sigma_2 : \begin{cases} N \, \dot{x}_2 = \quad x_2 + B_2 \, u, \quad x_2(0) = x_{20} \\ y_2 = C_2 \, x_2 \end{cases} \tag{3.7}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$, and $y = y_1 + y_2$.

Step 2 (Decomposition of Σ_2): If $B_2 = 0$, we have $x_0 = x_2$, $n_0 = n_2$, $x_\mathrm{e} = \emptyset$, $n_\mathrm{e} = 0$ and $v = 0$. For this case, the following procedure does not apply. We go directly to Step 3.

For the case when $B_2 \neq 0$, it follows from Brunovsky [6] and Luenberger [56] (see also Chen [20]) that there exists a nonsingular transformation T_2 and $\alpha \neq 0$ such that

$$x_2 = T_2 \begin{pmatrix} x_{\rm v} \\ x_{\rm z} \end{pmatrix}, \quad x_{\rm z} \in \mathbb{R}^{n_0}, \quad x_{\rm v} \in \mathbb{R}^{v_{\rm d}}, \quad x_{\rm v} = \begin{pmatrix} x_{\rm v1} \\ \vdots \\ x_{{\rm v}v_{\rm d}} \end{pmatrix}, \tag{3.8}$$

and

$$T_2^{-1} N T_2 = \begin{bmatrix} J_{\rm c0} & N_{\rm c\bar{c}} \\ 0 & J_{n_0} \end{bmatrix}, \quad T_2^{-1} B_2 = \begin{bmatrix} B_{\rm 2c} \\ 0 \end{bmatrix}, \quad C_2 T_2 = [\, C_{\rm 2c} \quad C_{\rm 2\bar{c}} \,], \tag{3.9}$$

where $(J_{\rm c0}, B_{\rm 2c})$ is a completely controllable pair. Since N has all its eigenvalues at 0 and $B_{\rm 2c}$ is a column vector, $(J_{\rm c0}, B_{\rm 2c})$ can actually be written as,

$$J_{\rm c0} = \begin{bmatrix} 0 & I_{v_{\rm d}-1} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_{\rm 2c} = \begin{bmatrix} 0 \\ -1/\alpha \end{bmatrix}. \tag{3.10}$$

Also note that J_{n_0} has all its eigenvalues at 0. As such, it is simple to verify that Σ_2 is decomposed into the following two subsystems:

$$J_{n_0} \dot{x}_{\rm z} = x_{\rm z} \quad \Longrightarrow \quad x_{\rm z} = 0, \tag{3.11}$$

and $J_{\rm c0} \dot{x}_{\rm v} + N_{\rm c\bar{c}} \dot{x}_{\rm z} = x_{\rm v} + B_{\rm 2c} u$, which is equivalent to $J_{\rm c0} \dot{x}_{\rm v} = x_{\rm v} + B_{\rm 2c} u$ or

$$u = \alpha x_{{\rm v}v_{\rm d}}, \quad \dot{x}_{{\rm v}v_{\rm d}} = x_{{\rm v}v_{\rm d}-1}, \quad \cdots, \quad \dot{x}_{\rm v2} = x_{\rm v1}, \tag{3.12}$$

which implies

$$x_{\rm e} := x_{\rm v1} = \frac{1}{\alpha} u^{(v)} \quad \text{and} \quad n_{\rm e} = 1, \tag{3.13}$$

and where $v = max(0, v_{\rm d} - 1)$. The output y_2 can then be expressed as

$$y_2 = C_{\rm 2c} x_{\rm v} + C_{\rm 2\bar{c}} x_{\rm z} = C_{\rm 2c} x_{\rm v}. \tag{3.14}$$

Step 3 (Decomposition of the Finite and Infinite Zero Structure): Observing the results in (3.6), (3.7), (3.11), (3.12), (3.13) and (3.14), we can obtain the following trivial system,

$$\left.\begin{aligned} \dot{x}_1 &= A_1 x_1 + \alpha B_1 x_{\mathrm{v}v_{\mathrm{d}}}, \\ x_{\mathrm{v}1} &= x_{\mathrm{e}} = \tfrac{1}{\alpha} u^{(v)}, \\ \dot{x}_{\mathrm{v}2} &= \tfrac{1}{\alpha} u^{(v)}, \\ &\vdots \\ \dot{x}_{\mathrm{v}v_{\mathrm{d}}} &= x_{\mathrm{v}v_{\mathrm{d}}-1}, \\ x_{\mathrm{z}} &= 0, \\ y &= C_1 x_1 + C_{2\mathrm{c}} x_{\mathrm{v}}, \end{aligned}\right\} \tag{3.15}$$

which is equivalent to Σ if the following initial conditions are satisfied,

$$\begin{pmatrix} x_1(0) \\ x_{\mathrm{v}}(0) \\ x_{\mathrm{z}}(0) \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & T_2^{-1} \end{bmatrix} Q^{-1} x_0. \tag{3.16}$$

Furthermore, it can be seen that the impulsive modes of Σ are also reserved since these impulsive modes are introduced by the derivatives of the input.

Next, let us partition

$$C_{2c} = [\, c_{\mathrm{v}1} \quad c_{\mathrm{v}2} \quad \cdots \quad c_{\mathrm{v}v_{\mathrm{d}}} \,]. \tag{3.17}$$

Thus, the nonsingular system (3.15) can be rewritten as,

$$\begin{cases} \dot{\bar{x}} = \bar{A}\ \bar{x} + \bar{B}\ \bar{u} \\ y = \bar{C}\ \bar{x} + \bar{D}\ \bar{u} \end{cases} \tag{3.18}$$

where

$$\bar{x} = \begin{pmatrix} x_1 \\ x_{\mathrm{v}2} \\ \vdots \\ x_{\mathrm{v}v_{\mathrm{d}}-1} \\ x_{\mathrm{v}v_{\mathrm{d}}} \end{pmatrix}, \quad \bar{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 & \alpha B_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{3.19}$$

and

$$\bar{u} = \frac{1}{\alpha} u^{(v)}, \quad \bar{C} = [\, C_1 \quad c_{\mathrm{v}2} \quad \cdots \quad c_{\mathrm{v}v_{\mathrm{d}}-1} \quad c_{\mathrm{v}v_{\mathrm{d}}} \,], \quad \bar{D} = c_{\mathrm{v}1}. \tag{3.20}$$

Note that $H(s)$ is nontrivial. We have the following two distinct cases.

1°: $\bar{D}=0$ and it is corresponding to Case 1 of Theorem 3.2.1. It follows from the result of Sannuti and Saberi [70] that there exist nonsingular transformations $\bar{\Gamma}_{\rm s}$ and $\Gamma_{\rm o}$ such that when we apply the following changes of coordinates,

$$\bar{x}=\bar{\Gamma}_{\rm s}\check{x}=\bar{\Gamma}_{\rm s}\begin{pmatrix} x_{\rm a} \\ x_{\rm d} \end{pmatrix}, \quad y=\Gamma_{\rm o}\tilde{y}, \tag{3.21}$$

to the system in (3.18), and in view of (3.13), we have

$$\dot{\check{x}}=\begin{bmatrix} A_{\rm aa} & L_{\rm ad}C_{\rm d} \\ B_{\rm d}M_{\rm da} & A_{\rm dd} \end{bmatrix}\check{x}+\begin{bmatrix} 0 \\ B_{\rm d} \end{bmatrix}\alpha^{-1}u^{(v)} \tag{3.22}$$

and

$$\tilde{y}=[\,0 \quad C_{\rm d}\,]\,\check{x}, \tag{3.23}$$

where $A_{\rm dd}$, $B_{\rm d}$ and $C_{\rm d}$ have the form as given in (3.31). Let

$$u=\Gamma_{\rm i}\tilde{u}=\alpha\tilde{u} \quad \Longrightarrow \quad \alpha^{-1}u^{(v)}=\tilde{u}^{(v)}. \tag{3.24}$$

Furthermore, noting the coordinate transforms in (3.5), (3.8), (3.20) and (3.21), we have,

$$\begin{aligned}\tilde{x}=\begin{pmatrix} x_{\rm e} \\ x_{\rm z} \\ x_{\rm a} \\ x_{\rm d} \end{pmatrix} &= \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_{\rm s}^{-1} \end{bmatrix}\begin{pmatrix} x_{\rm e} \\ x_{\rm z} \\ \bar{x} \end{pmatrix}=\begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_{\rm s}^{-1} \end{bmatrix}T\begin{pmatrix} x_1 \\ x_{\rm v} \\ x_{\rm z} \end{pmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_{\rm s}^{-1} \end{bmatrix}T\begin{bmatrix} I & 0 \\ 0 & T_2^{-1} \end{bmatrix}Q^{-1}x=\Gamma_{\rm s}^{-1}x, \end{aligned} \tag{3.25}$$

where T is an $n\times n$ permutation matrix.

And from the results in (3.16) and (3.25), we can get the following initial condition for the decomposed system,

$$\tilde{x}(0)=\Gamma_{\rm s}^{-1}x_0. \tag{3.26}$$

2°: $\bar{D}\neq 0$ and it is corresponding to Case 2 of Theorem 3.2.1. In this case, it is simple to obtain $x_{\rm d}=\emptyset$, $n_{\rm d}=0$, $x_{\rm a}=\bar{x}$, $n_{\rm a}=n_1+v$ and

$$\dot{x}_{\rm a}=(\bar{A}-\bar{B}\bar{D}^{-1}\bar{C})x_{\rm a}+\bar{B}\bar{D}^{-1}y=A_{\rm aa}x_{\rm a}+B_{\rm 0a}y \tag{3.27}$$

and

$$y = \bar{C}x_{\rm a} + \bar{D}\alpha^{-1}u^{(v)} = \bar{C}x_{\rm a} + \bar{D}\tilde{u}^{(v)}, \tag{3.28}$$

if we let $u = \Gamma_{\rm i}\tilde{u} = \alpha\tilde{u}$.

Moreover, similar to that in 1°, there exist,

$$\Gamma_{\rm s}^{-1} = T \begin{bmatrix} I & 0 \\ 0 & T_2^{-1} \end{bmatrix} Q^{-1}, \quad \Gamma_{\rm o} = 1, \quad \tilde{x}(0) = \Gamma_{\rm s}^{-1}x_0. \tag{3.29}$$

This complete the algorithm for the structural decomposition of Σ.

Actually, we can rewrite the structural decomposition of Σ in a compact matrix form, which will be handy in proving the properties of the structural decomposition. For simplicity, we will only focus on Case 1 of Theorem 3.2.1, i.e., $n_{\rm d} > 0$. The compact form for Case 1 of Theorem 3.2.1 is given by

$$\left.\begin{aligned}
\tilde{E} = \Gamma_{\rm e}^{-1}E\Gamma_{\rm s} &= \begin{bmatrix} 0 & E_{\rm e0} & 0 & 0 \\ 0 & J_{n_0} & 0 & 0 \\ 0 & E_{\rm a0} & I_{n_{\rm a}} & 0 \\ 0 & E_{\rm d0} & 0 & I_{n_{\rm d}} \end{bmatrix} \\
\tilde{A} = \Gamma_{\rm e}^{-1}A\Gamma_{\rm s} &= \begin{bmatrix} 0 & A_{\rm e0} & N_{\rm ea} & N_{\rm ed} \\ 0 & I_{n_0} & 0 & 0 \\ 0 & A_{\rm a0} & A_{\rm aa} & L_{\rm ad}C_{\rm d} \\ B_{\rm d} & A_{\rm d0} & B_{\rm d}M_{\rm da} & A_{\rm dd} \end{bmatrix}, \\
\tilde{B} = \Gamma_{\rm e}^{-1}B\Gamma_{\rm i} &= \begin{bmatrix} B_{\rm e} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\tilde{C} = \Gamma_{\rm o}^{-1}C\Gamma_{\rm s} &= [\,0 \quad C_0 \quad 0 \quad C_{\rm d}\,],
\end{aligned}\right\} \tag{3.30}$$

where J_{n_0} is in a Jordan canonical form with all its diagonal elements being equal to 0, and $N_{\rm ea}$, $N_{\rm ed}$ and are sub-matrices with appropriate dimensions, and $B_{\rm e} \neq 0$. Furthermore, matrices $A_{\rm dd}$, $B_{\rm d}$, $C_{\rm d}$ are in the following forms:

$$A_{\rm dd} = \begin{bmatrix} 0 & I_{n_{\rm d}-1} \\ \star & 0 \end{bmatrix}, \quad B_{\rm d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{\rm d} = [\,1 \quad 0 \quad \cdots \quad 0\,]. \tag{3.31}$$

3.4.2 Proof of Property 3.3.1

It follows from Dai [29] that the singular system Σ of (3.1) is stabilizable if and only if

$$\operatorname{rank}\,[\, sE - A \quad B\,] = n, \tag{3.32}$$

for all $s \in \mathbb{C}^0 \cup \mathbb{C}^+$. Let us again focus on Case 1 of Theorem 3.2.1. In the structural decomposition form

$$\begin{aligned}
\operatorname{rank}\,[\, sE - A \quad B\,] &= \operatorname{rank}\,[\, s\tilde{E} - \tilde{A} \quad \tilde{B}\,] \\
&= \operatorname{rank}\begin{bmatrix} 0 & sE_{\mathrm{e}0} - A_{\mathrm{e}0} & -N_{\mathrm{ea}} & -N_{\mathrm{ed}} & B_{\mathrm{e}} \\ 0 & sJ_{n_0} - I_{n_0} & 0 & 0 & 0 \\ 0 & sE_{\mathrm{a}0} - A_{\mathrm{a}0} & sI_{n_\mathrm{a}} - A_{\mathrm{aa}} & -L_{\mathrm{ad}}C_{\mathrm{d}} & 0 \\ -B_{\mathrm{d}} & sE_{\mathrm{d}0} - A_{\mathrm{d}0} & -B_{\mathrm{d}}M_{\mathrm{da}} & sI_{n_\mathrm{d}} - A_{\mathrm{dd}} & 0 \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & 0 & B_{\mathrm{e}} \\ 0 & sJ_{n_0} - I_{n_0} & 0 & 0 & 0 \\ 0 & 0 & sI_{n_\mathrm{a}} - A_{\mathrm{aa}} & -L_{\mathrm{ad}}C_{\mathrm{d}} & 0 \\ -B_{\mathrm{d}} & 0 & 0 & sI_{n_\mathrm{d}} - A_{\mathrm{dd}} & 0 \end{bmatrix}.
\end{aligned}$$

Noting that $B_{\mathrm{e}} \neq 0$ and the special structures of J_{n_0}, A_{dd}, C_{d} and B_{d}, it is straightforward to show that Σ is stabilizable if and only if $(A_{\mathrm{aa}}, L_{\mathrm{ad}})$ is stabilizable. Results for Case 2 of Theorem 3.2.1 can be shown in a similar way.

Similarly, the proof for the detectability can be done in a dual fashion. This completes the proof of Property 3.3.1.

3.4.3 Proof of Property 3.3.2

Again, we prove this property for Case 1 of Theorem 3.2.1. Observing that for $\alpha \in \mathbb{C}$, we have

$$\operatorname{rank}\left\{P_{\Sigma}(\alpha)\right\} = \operatorname{rank}\left\{P_{\tilde{\Sigma}}(\alpha)\right\}$$

$$
\begin{aligned}
&= \operatorname{rank}\begin{bmatrix} 0 & A_{\mathrm{e0}}-\alpha E_{\mathrm{e0}} & N_{\mathrm{ea}} & N_{\mathrm{ed}} & B_{\mathrm{e}} \\ 0 & I_{n_0}-\alpha J_{n_0} & 0 & 0 & 0 \\ 0 & A_{\mathrm{a0}}-\alpha E_{\mathrm{a0}} & A_{\mathrm{aa}}-\alpha I_{n_{\mathrm{a}}} & L_{\mathrm{ad}}C_{\mathrm{d}} & 0 \\ B_d & A_{\mathrm{d0}}-\alpha E_{\mathrm{d0}} & B_{\mathrm{d}}M_{\mathrm{da}} & A_{\mathrm{dd}}-\alpha I_{n_{\mathrm{d}}} & 0 \\ 0 & C_0 & 0 & C_{\mathrm{d}} & 0 \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & 0 & B_{\mathrm{e}} \\ 0 & I_{n_0}-\alpha J_{n_0} & 0 & 0 & 0 \\ 0 & 0 & A_{\mathrm{aa}}-\alpha I_{n_{\mathrm{a}}} & 0 & 0 \\ B_{\mathrm{d}} & 0 & 0 & A_{\mathrm{dd}}-\alpha I_{n_{\mathrm{d}}} & 0 \\ 0 & 0 & 0 & C_{\mathrm{d}} & 0 \end{bmatrix} \\
&= n_{\mathrm{e}}+n_0+n_{\mathrm{d}}+1+\operatorname{rank}\left\{A_{\mathrm{aa}}-\alpha I_{n_{\mathrm{a}}}\right\}. \qquad (3.33)
\end{aligned}
$$

Obviously, the rank of P_Σ drops if and only if $\alpha \in \lambda(A_{\mathrm{aa}})$. Hence, the invariant zeros of $\tilde{\Sigma}$ are given by the eigenvalues of A_{aa}. In fact, the eigenstructure of A_{aa} defines the finite zero structure of Σ. This completes the proof of Property 3.3.2.

3.4.4 Proof of Property 3.3.3

It is well known that the infinite zero structure or relative degree of Σ is nothing more than the number of integrators that are inherent in between the system input u and the system output y. As all transformations involved in our structural decomposition are nonsingular, the number of inherent integrators remains unchanged under such transformations. It follows from the constructive proof of Theorem 3.2.1 (see also Figure 3.1) that there are n_{d} integrators in between $\tilde{u}^{(v)}$ and $\tilde{y}$, where $v = max(0, v_{\mathrm{d}} - 1)$. Thus, the number of inherent integrators in between u and y is $n_{\mathrm{d}} - v$. Hence, the result of Property 3.3.3 follows.

3.5 An Illustrative Example

In this section, an example is presented to illustrate the structural decomposition procedure and its properties. We consider a singular system of (3.1) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{3.34}$$

and

$$C = [\,2 \quad 0 \quad -2 \quad 1 \quad -1\,], \quad D = 0. \tag{3.35}$$

Step 1 (Preliminary Decomposition). The given system is already in the forms of (3.5), i.e., we have

$$\Sigma_1 : \begin{cases} \dot{x}_1 = \quad x_1 + u \\ y_1 = 2\, x_1 \end{cases} \tag{3.36}$$

and

$$\Sigma_2 : \begin{cases} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u \\ y_2 = [\,0 \quad -2 \quad 1 \quad -1\,] \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \end{cases} \tag{3.37}$$

with $n_1 = 1$ and $n_2 = 4$.

Step 2 (Decomposition of Σ_2). Using the toolbox of Lin and Chen [52], we obtain a nonsingular transformation

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{3.38}$$

which transform Σ_2 to the following canonical form

$$T_2^{-1} N T_2 = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \quad T_2^{-1} B_2 = \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \end{array}\right], \quad C_2 T_2 = \left[\begin{array}{ccc|c} 0 & -1 & -2 & 1 \end{array}\right], \tag{3.39}$$

with $v_{\mathrm{d}} = 3$ and $n_0 = 1$. Thus, $n_{\mathrm{e}} = 1$ and $v = 2$.

Step 3 (Decomposition of the Finite and Infinite Zero Structures). Following from the results of (3.15) to (3.18), we obtain an auxiliary nonsingular system

$$\begin{cases} \dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,\bar{u} \\ y = \bar{C}\,\bar{x} + \bar{D}\,\bar{u} \end{cases} \tag{3.40}$$

with

$$\bar{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{C} = [\,2 \quad -1 \quad -2\,], \quad \bar{D} = 0. \tag{3.41}$$

Again, using the toolbox of Lin and Chen [52], we obtain

$$\bar{\Gamma}_{\mathrm{s}} = \begin{bmatrix} -0.3333 & -0.7071 & 0 \\ 0.6667 & 0 & 1 \\ -0.6667 & -0.7071 & 0 \end{bmatrix}, \quad \bar{\Gamma}_{\mathrm{o}} = -1 \quad \text{and} \quad \bar{\Gamma}_{\mathrm{i}} = 1, \tag{3.42}$$

which transform the nonsingular system (3.40) into the so-call special coordinate basis,

$$\bar{\Gamma}_{\mathrm{s}}^{-1} \bar{A} \bar{\Gamma}_{\mathrm{s}} = \begin{bmatrix} -1 & 0 & -3 \\ 0 & 0 & 1.4142 \\ 0.6667 & 0 & 2 \end{bmatrix}, \quad \bar{\Gamma}_{\mathrm{s}}^{-1} \bar{B} \bar{\Gamma}_{\mathrm{i}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{\Gamma}_{\mathrm{o}}^{-1} \bar{C} \bar{\Gamma}_{\mathrm{s}} = [\,0 \quad 0 \quad 1\,] \tag{3.43}$$

with $n_\mathrm{a} = 2$ and $n_\mathrm{d} = 1$. Finally, putting all the sub-transformations together, we obtain

$$\Gamma_\mathrm{e} = \begin{bmatrix} 1 & 0 & -0.3333 & -0.7071 & 0 \\ 0 & 0 & 0.6667 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.6667 & -0.7071 & 0 \end{bmatrix}, \quad \Gamma_\mathrm{s} = \begin{bmatrix} 0 & 0 & -0.3333 & -0.7071 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.6667 & -0.7071 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.6667 & 0 & 1 \end{bmatrix}, \tag{3.44}$$

and $\Gamma_\mathrm{o} = -1$, $\Gamma_\mathrm{i} = 1$.

The transformed system is then given by

$$x_\mathrm{e} = \ddot{u}, \quad x_\mathrm{z} = 0, \tag{3.45}$$

$$\dot{x}_\mathrm{a} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x_\mathrm{a} + \begin{bmatrix} -3 \\ 1.4142 \end{bmatrix} y_\mathrm{d}, \tag{3.46}$$

$$\dot{x}_\mathrm{d1} = [\,0.6667 \quad 0\,]\, x_\mathrm{a} + 2y_\mathrm{d} + \ddot{u}, \quad \tilde{y} = y_\mathrm{d} = x_\mathrm{d1}, \quad \tilde{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1.4142 \\ 1 \end{bmatrix}, \tag{3.47}$$

or in the following compact form

$$\tilde{E} = \Gamma_\mathrm{e}^{-1} E \Gamma_\mathrm{s} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 1 & 0 & 0 \\ 0 & 4.2426 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{C} = \Gamma_\mathrm{o}^{-1} C \Gamma_\mathrm{s} = [\,0 \quad -1 \quad 0 \quad 0 \quad 1\,], \tag{3.48}$$

and

$$\tilde{A} = \Gamma_\mathrm{e}^{-1} A \Gamma_\mathrm{s} = \begin{bmatrix} 0 & 0 & -0.6667 & -0.7071 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1.4142 \\ 1 & 0 & 0.6667 & 0 & 2 \end{bmatrix}, \quad \tilde{B} = \Gamma_\mathrm{e}^{-1} B \Gamma_\mathrm{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.49}$$

It is simple to see now from the above decomposition that there are two invariant zeros are $s_1 = -1$ and $s_2 = 0$, and the infinite zero structure or relative degree of Σ from $\ddot{u}$ to y is equal to 1. Thus, Σ has a relative degree of -1 from u to y. These results can be easily verified from the transfer function of Σ,

$$H(s) = C(sE - A)^{-1}B = \frac{s(s+1)}{s-1}. \tag{3.50}$$

Finally, we note that it can be shown that there is an infinite elementary divisor of order $n_{\mathrm{d}} = 1$ in the the Kronecker canonical form associated with Σ.

3.6 Conclusions

We have presented in this chapter the structural decomposition technique for general single-input and single-output singular systems, which has a distinct feature of explicitly capturing and displaying the structural properties, such as the finite and infinite zero structures, of the given system. As its counterpart in nonsingular systems, the technique is expected to play an important role in solving many control problems related to singular systems. This will actually be the subject of the following chapters. The next chapter will give the structural decomposition technique for multi-input and multi-output singular systems.

Chapter 4

Structural Decomposition of Multivariable Singular Linear Systems

4.1 Introduction

Last chapter describes our work on the structural decomposition technique for single input single output singular linear systems. The technique and its essential properties show that it can explicitly display the given system's internal structural features with decomposed distinct subspaces. In this chapter, we will further extend this technique for multivariable singular linear systems.

The internal structural features of multi-input and multi-output (MIMO) singular systems are much more complex than those of single-input and single-output (SISO) singular systems. This largely dues to the increasing number of input and output. And compared to a SISO singular system, a MIMO singular system has more internal structural features such as system invertibility and so on. However, although the structural decomposition technique for a MIMO singular system may be more complex and difficult to be derived, it also has the similar and even more properties in revealing the internal structural features. As we have described before, such a decomposition has a distinct feature of capturing and displaying all the structural properties, such as the finite and infinite zero structures, invertibility structures, redundant dynamics, and the invariant geometric subspaces of the given system. As its counterpart for nonsingular systems, we believe that the technique is a powerful tool in solving control problems for singular systems, including H_2 and H_∞ control, model reduction, disturbance decoupling problems, to name a few.

In this chapter, we will give some existing research results in Section 2. Such preliminary results include the invariant structure indices for singular systems and some other efforts in revealing internal structural properties of a given MIMO singular system. And Section 3 will give our main result of this chapter, that is, the main theorem of the structural decomposition for MIMO singular systems and its essential properties. Moreover, in Section 4, a constructive decomposition algorithm is presented to prove our main theorem of structural decomposition. After that, the essential properties of this structural decomposition technique will be proved in Section 5. To give a clear understanding of the structural decomposition technique and its properties, an illustrative example will be given in Section 6. And finally in Section 7, a concluding remark will be drawn.

4.2 Preliminary Results

As we have pointed out in the last chapter, since it is a better model for most real systems, the singular system has attracted many researchers in the last three decades. And just like the nonsingular system, its structural features are crucial in solving systems and control problems. To find a good technique for revealing such structural features, many efforts have been presented in the literature (see e.g., Dooren [31, 32], Geerts [36], Loiseau [55], Malabre [58], Misra *et al.* [59], Verghese [76], Zhou *et al.* [85], and the references cited therein). Generally speaking, almost all the research works dealing with singular systems are the natural extensions of their nonsingular system counterparts, although these extensions are usually non-trivial.

In this section, we try to give some important preliminary results which will be essential in our derivation in the following sections. Such results include essential knowledge on strictly equivalence based on Kronecker canonical form.

To be more specific, we consider a linear time-invariant system Σ characterized by

$$\Sigma : \begin{cases} E\,\dot{x} = A\,x + B\,u, \quad x(0) = x_0, \\ \quad y = C\,x + D\,u, \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state, input and output of the system, and E, A, B, C and D are constant matrices of appropriate dimensions. Traditionally, the Kronecker canonical form, a classical form of matrix pencils under strictly equivalent transformation, has been used extensively in the structural analysis of singular systems. Malabre [58] presents a geometric approach and introduces structural invariants of singular systems. In that paper, some definitions are shown to be consistent with other ones directly deduced from matrix pencil tools. It extends many geometric and structural results (see e.g., Wonham [83]) from the nonsingular systems to singular systems.

The Kronecker canonical form exhibits the finite- and infinite-zero structures (i.e., invariant indices) of the system, and shows the left and right null-space structures. And first we recall the following lemma on strictly equivalent.

Lemma 4.2.1 (Strictly Equivalence) Two singular systems Σ_1 and Σ_2 are strictly equivalent if they have same Kronecker canonical forms.

The result of this lemma is obvious in the literature [74] [76]. And it is clear that two strictly equivalent singular systems have the same structural features because they have the same Kronecker canonical form of their system matrices.

4.3 The Structural Decomposition Theorem

It has been extensively demonstrated and proven for nonsingular systems that the system structural properties, such as the finite and infinite zero structures and the invertibility structures, play a very important role in solving various control problems including H_2, H_∞ control and disturbance decoupling (see e.g., [20] and [69]). The structural properties of singular systems and their applications to the control problems of singular systems are however less emphasized in the literature. In their recent work, He and Chen [39] have developed a technique that gives a structural decomposition for single-input and single-output (SISO) singular systems. The technique is capable of revealing all the structural properties, including the finite and infinite zero structures. In this section, we present a structural decomposition technique for general multivariable singular systems. Again, such a technique can be used to capture and display the structural properties of general singular systems. Our work generalizes the result of He and Chen [39]. It can also be regarded as a natural extension and counterpart of the work of Sannuti and Saberi [70] for nonsingular systems. However, it will be seen shortly that the structural decomposition of a general multivariable singular system is much more involved.

Here we first summarize the structural decomposition of general multivariable singular systems in the following main theorem. All its properties and its connection to the concept of geometric spaces will also be given. The constructive algorithm for the structural decomposition and proofs of all these properties will be separately given in Section 4 and Section 5 for clarity of presentation.

We have the following theorem.

Theorem 4.3.1 *Consider the general multivariable singular system* Σ *of (4.1). Then,*

1. *there exist non-negative integers* $n_{\rm z}$, $n_{\rm e}$, $n_{\rm a}$, $n_{\rm b}$, $n_{\rm c}$, $n_{\rm d}$, $m_{\rm d}$, m_0, $m_{\rm c}$, $p_{\rm b}$, *and positive integers* p_i, $i = 1, 2, \cdots, n_{\rm e}$, *if* $n_{\rm e} > 0$, *and* q_i, $i = 1, 2, \cdots, m_{\rm d}$, *if* $m_{\rm d} > 0$; *and*

2. *there exist nonsingular state and output transformations* $\Gamma_{\rm s} \in \mathbb{R}^{n\times n}$ *and* $\Gamma_{\rm o} \in \mathbb{R}^{p\times p}$, *and a nonsingular transformation* $\Gamma_{\rm e} \in \mathbb{R}^{n\times n}$, *as well as an* $m \times m$ *input transformation* $\Gamma_{\rm i}(s)$, *whose inverse has all its elements being some polynomials of* s *(i.e., its inverse contains various differentiation operators), which together give a structural decomposition of* Σ *and display explicitly its structural properties.*

The structural decomposition of Σ *can be described by the following set of equations:*

$$x = \Gamma_{\rm s}\tilde{x}, \quad y = \Gamma_{\rm o}\tilde{y}, \quad u = \Gamma_{\rm i}(s)\tilde{u}, \tag{4.2}$$

and

$$\tilde{x} = \begin{pmatrix} x_{\rm z} \\ x_{\rm e} \\ x_{\rm a} \\ x_{\rm b} \\ x_{\rm c} \\ x_{\rm d} \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} y_0 \\ y_{\rm b} \\ y_{\rm d} \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_0 \\ u_{\rm c} \\ u_{\rm d} \end{pmatrix}, \tag{4.3}$$

$$x_{\rm d} = \begin{pmatrix} x_{\rm d1} \\ x_{\rm d2} \\ \vdots \\ x_{{\rm d}m_{\rm d}} \end{pmatrix}, \quad y_{\rm d} = \begin{pmatrix} y_{\rm d1} \\ y_{\rm d2} \\ \vdots \\ y_{{\rm d}m_{\rm d}} \end{pmatrix}, \quad u_{\rm d} = \begin{pmatrix} u_{\rm d1} \\ u_{\rm d2} \\ \vdots \\ u_{{\rm d}m_{\rm d}} \end{pmatrix}, \tag{4.4}$$

and

$$x_{\rm z} = 0, \tag{4.5}$$

$$x_{\rm e} = B_{\rm e0} u_0 + B_{\rm ec} u_{\rm c} + B_{\rm ed} u_{\rm d}, \tag{4.6}$$

$$\dot{x}_{\rm a} = A_{\rm aa} x_{\rm a} + B_{\rm 0a} y_0 + L_{\rm ad} y_{\rm d} + L_{\rm ab} y_{\rm b}, \tag{4.7}$$

$$\dot{x}_{\rm b} = A_{\rm bb} x_{\rm b} + B_{\rm 0b} y_0 + L_{\rm bd} y_{\rm d}, \quad y_{\rm b} = C_{\rm b} x_{\rm b}, \tag{4.8}$$

$$\dot{x}_{\rm c} = A_{\rm cc} x_{\rm c} + B_{\rm 0c} y_0 + L_{\rm cd} y_{\rm d} + L_{\rm cb} y_{\rm b} + B_{\rm c} M_{\rm ca} x_{\rm a} + B_{\rm c} u_{\rm c}, \tag{4.9}$$

$$y_0 = C_{\rm 0a} x_{\rm a} + C_{\rm 0b} x_{\rm b} + C_{\rm 0c} x_{\rm c} + C_{\rm 0d} x_{\rm d} + u_0, \tag{4.10}$$

and for each $i = 1, 2, \cdots, m_{\rm d}$,

$$\dot{x}_{{\rm d}i} = A_{q_i} x_{{\rm d}i} + L_{i0} y_0 + L_{i{\rm d}} y_{\rm d} + B_{q_i} \left[u_{{\rm d}i} + M_{i{\rm a}} x_{\rm a} + M_{i{\rm b}} x_{\rm b} + M_{i{\rm c}} x_{\rm c} + \sum_{j=1}^{m_{\rm d}} M_{ij} x_{{\rm d}j} \right], \tag{4.11}$$

$$y_{{\rm d}i} = C_{q_i} x_i, \quad y_{\rm d} = C_{\rm d} x_{\rm d}, \tag{4.12}$$

for some appropriate dimensional constant submatrices. Here the states $x_{\rm z}$, $x_{\rm e}$, $x_{\rm a}$, $x_{\rm b}$, $x_{\rm c}$ *and* $x_{\rm d}$ *are respectively of dimensions* $n_{\rm z}$, $n_{\rm e}$, $n_{\rm a}$, $n_{\rm b}$, $n_{\rm c}$ *and* $n_{\rm d} = \sum_{i=1}^{m_{\rm d}} q_i$, *while* $x_{{\rm d}i}$ *is of dimension* q_i *for each* $i = 1, 2, \cdots, m_{\rm d}$. *The control vectors* u_0, $u_{\rm d}$ *and* $u_{\rm c}$ *are respectively of dimensions* m_0, $m_{\rm d}$ *and* $m_{\rm c} = m - m_0 - m_{\rm d}$ *while the output vectors* y_0, $y_{\rm d}$ *and* $y_{\rm b}$ *are respectively of dimensions* m_0, $m_{\rm d}$ *and* $p_{\rm b} = p - m_0 - m_{\rm d}$. *The pair* $(A_{\rm bb}, C_{\rm b})$ *is completely observable, the pair* $(A_{\rm cc}, B_{\rm c})$ *is completely controllable, and the triple* $(A_{q_i}, B_{q_i}, C_{q_i})$ *has the following form:*

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [\,1 \quad 0 \quad \cdots \quad 0\,]. \tag{4.13}$$

Assuming that $x_i, i = 1, 2, \cdots, m_{\rm d}$, *are arranged such that* $q_i \le q_{i+1}$, *the matrix* $L_{i{\rm d}}$ *will be in the following particular form:*

$$L_{i{\rm d}} = [\, L_{i1} \quad L_{i2} \quad \cdots \quad L_{ii-1} \quad 0 \quad \cdots \quad 0\,], \tag{4.14}$$

with its last row of $L_{i{\rm d}}$ *being all zeros. Finally, the initial condition of the transformed system* $\tilde{x}_0 = \Gamma_{\rm s}^{-1} x_0$. □

A constructive proof of the structural decomposition in Theorem 4.3.1 will be given later in the next section. Figure 4.1 gives a block diagram interpretation of the dynamics of the structurally decomposed system. In the figure, a signal given by a double-edged arrow is some linear combination of output $y_{\rm d}$, whereas a signal given by the double-edged arrow with a solid dot is some linear combination of all the states.

In what follows, we illustrate the essential features of the structural decomposition of general singular systems given in Theorem 4.3.1.

1. The state $x_{\rm z}$ is purely static and identically zero for all time t. It can neither be controlled at all by the system input nor be affected by other states.

2. The state $x_{\rm e}$ is again static and contains a linear combination of the input variables of the system and their derivatives of appropriate orders. It contains the impulse modes of Σ, if any, as impulse modes are caused by the derivatives of the system input.

3. The state $x_{\rm a}$ is neither directly controlled by the system input nor does it directly affect the system output.

4. The output $y_{\rm b}$ and the state $x_{\rm b}$ are not directly influenced by any input, although they could be indirectly controlled through the output $y_{\rm d}$. Moreover, $(A_{\rm bb}, C_{\rm b})$ forms an observable pair. This implies that the state $x_{\rm b}$ is observable.

5. The state $x_{\rm c}$ is directly controlled by the input $u_{\rm c}$, but it does not directly affect any output. $(A_{\rm cc}, B_{\rm c})$ forms a controllable pair. This implies that the state $x_{\rm c}$ is controllable.

6. The variables $u_{{\rm d}i}$ controls the output $y_{{\rm d}i}$ through a stack of q_i integrators. Furthermore, all the states $x_{{\rm d}i}$ are both controllable and observable.

We mentioned earlier that the structural decomposition of Theorem 4.3.1 has the distinct feature of revealing the structural properties of the given singular system Σ. We are now

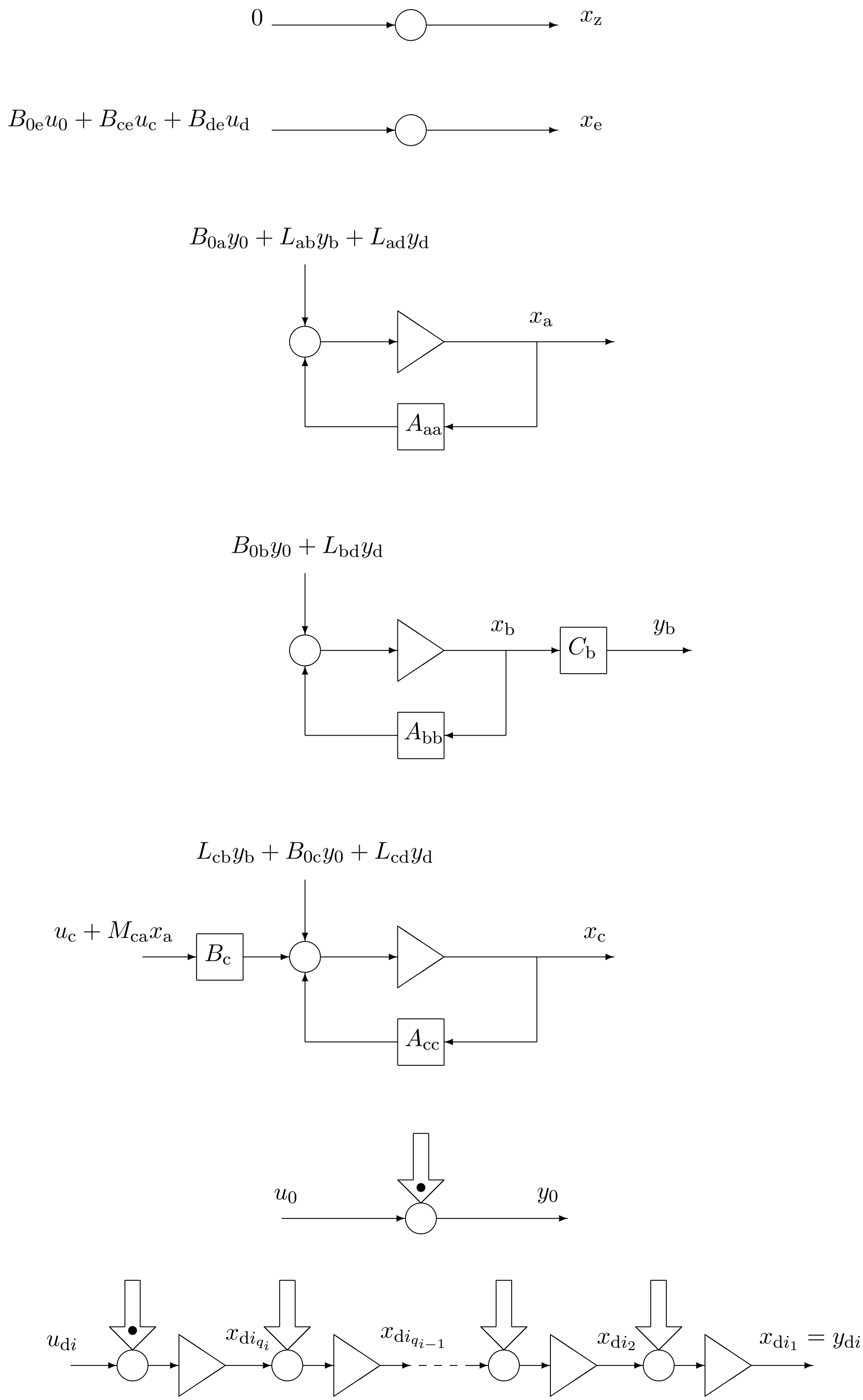

Figure 4.1: Block diagram representation of dynamics of the decomposed system.

ready to study how the system properties of Σ, such as the stabilizability, detectability, finite zero and infinite zero structures, and invariant geometric subspaces, can be obtained from the decomposition.

The definitions of stability, stabilizability and detectability of singular systems have been recalled in Chapter 2. And the following property gives our structural decomposition technique's function in reflecting the given system's stabilizability and detectability.

Property 4.3.1 (Stabilizability and Detectability) *The given system Σ of (4.1) is stabilizable if and only if $(A_{\rm con}, B_{\rm con})$ is stabilizable, and is detectable if and only if $(A_{\rm obs}, C_{\rm obs})$ is detectable, where*

$$A_{\rm con} := \begin{bmatrix} A_{\rm aa} & L_{\rm ab}C_{\rm b} \\ 0 & A_{\rm bb} \end{bmatrix}, \quad B_{\rm con} := \begin{bmatrix} B_{\rm 0a} & L_{\rm ad} \\ B_{\rm 0b} & L_{\rm bd} \end{bmatrix}, \tag{4.15}$$

and

$$A_{\rm obs} := \begin{bmatrix} A_{\rm aa} & 0 \\ B_{\rm c}M_{\rm ca} & A_{\rm cc} \end{bmatrix}, \quad C_{\rm obs} := \begin{bmatrix} C_{\rm 0a} & C_{\rm 0c} \\ M_{\rm da} & M_{\rm dc} \end{bmatrix}. \tag{4.16}$$

Again, the definition of invariant zeros of singular systems has already been given in detail in Chapter 2. The following property shows that the invariant zeros of Σ can be obtained in the structural decomposition in a trivial manner.

Property 4.3.2 (Invariant Zeros, Normal Rank) *The invariant zeros of the given singular system Σ are the eigenvalues of $A_{\rm aa}$. The normal rank of Σ is equal to $m_0 + m_{\rm d}$.*

In fact, in many applications, it is handy and necessary to further separate the state variable associated with the invariant zero dynamics, i.e., $x_{\rm a}$, into a stable part, an unstable part and the part associated with invariant zeros on the imaginary axis. It is simple to note that there exists a nonsingular transformation, say $T_{\rm a}$, such that

$$x_{\rm a} = T_{\rm a} \begin{pmatrix} x_{\rm a}^- \\ x_{\rm a}^0 \\ x_{\rm a}^+ \end{pmatrix}, \quad T_{\rm a}^{-1} A_{\rm aa} T_{\rm a} = \begin{bmatrix} A_{\rm aa}^- & 0 & 0 \\ 0 & A_{\rm aa}^0 & 0 \\ 0 & 0 & A_{\rm aa}^+ \end{bmatrix}, \tag{4.17}$$

where $\lambda(A_{\rm aa}^-) \subset \mathbb{C}^-$, i.e., the stable invariant zeros, $\lambda(A_{\rm aa}^0) \subset \mathbb{C}^0$, i.e., the invariant zeros on the imaginary axis, and $\lambda(A_{\rm aa}^+) \subset \mathbb{C}^-$, the unstable invariant zeros.

The infinite zero structure of the given system Σ can be defined as the structure associated with the corresponding block in the Kronecker canonical form of its system matrix $P_\Sigma(s)$. It can also be defined using the well-known Smith-McMillan form.

Property 4.3.3 (Infinite Zero Structure) *Σ has m_0 infinite zeros of order 0. The infinite zero structure (of order greater than 0) of Σ is given by*

$$S_\infty^\star(\Sigma) = \{q_1, q_2, \cdots, q_{m_{\rm d}}\}, \tag{4.18}$$

i.e., for each $i = 1, 2, \cdots, m_{\rm d}$, Σ has an infinite zero of order q_i, respectively.

Our structural decomposition can also exhibit the invertibility structure of a given singular system Σ. As its counterpart in nonsingular systems, the following property shows that the invertibility of a singular system is associated with the decomposed subspaces $x_{\rm b}$ and $x_{\rm c}$.

Property 4.3.4 (Invertibility Structure) *The singular system Σ is right invertible if and only if $x_{\rm b}$ and hence $y_{\rm b}$ are non-existent, left invertible if and only if $x_{\rm c}$ and hence $u_{\rm c}$ is non-existent, and invertible if and only if both $x_{\rm b}$ and $x_{\rm c}$ are non-existent.*

There are also interconnections between the structural decomposition form and invariant geometric subspaces. The definitions of various invariant geometric subspaces for singular systems and their relation with the structurally decomposed subspaces will be given in detail in the next chapter.

Finally, we can conclude that the structural decomposition separates the state-space into several distinct parts. In fact, the state-space $\mathcal{X}$ is decomposed as

$$\mathcal{X} = \mathcal{X}_{\rm z} \oplus \mathcal{X}_{\rm e} \oplus \mathcal{X}_{\rm a}^{-} \oplus \mathcal{X}_{\rm a}^{0} \oplus \mathcal{X}_{\rm a}^{+} \oplus \mathcal{X}_{\rm b} \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d}. \tag{4.19}$$

Here $\mathcal{X}_{\rm z}$ is related to the state that is identically zero and $\mathcal{X}_{\rm e}$ is related to the state that is a linear combination of the system input. $\mathcal{X}_{\rm a}^{-}$ is related to the stable invariant zeros, i.e., the eigenvalues of $A_{\rm aa}^{-}$ are the stable invariant zeros of Σ. Similarly, $\mathcal{X}_{\rm a}^{0}$ and $\mathcal{X}_{\rm a}^{+}$ are respectively related to the invariant zeros of Σ located in the marginally stable and unstable regions. On the other hand, $\mathcal{X}_{\rm b}$ is related to the right invertibility, i.e., the system is right invertible if and only if $\mathcal{X}_{\rm b} = \{0\}$, while $\mathcal{X}_{\rm c}$ is related to left invertibility, i.e., the system is left invertible if and only if $\mathcal{X}_{\rm c} = \{0\}$. Finally, $\mathcal{X}_{\rm d}$ is related to zeros of Σ at infinity.

4.4 A Constructive Algorithm for the Structural Decomposition

We now present a constructive proof for the main results of the previous section, i.e., Theorem 4.3.1. The following is a step-by-step algorithm for the structural decomposition of general multivariable singular systems.

STEP 1 (PRELIMINARY DECOMPOSITION): *This step is to separate the given singular system into a nonsingular subsystem and a singular subsystem with a special structure.* It follows from Campbell [8] (see also Dai [29]) that there exist two nonsingular matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CQ = [\, C_1 \quad C_2 \,], \tag{4.20}$$

where A_1, B_1, B_2, C_1 and C_2 are matrices with appropriate dimensions, and N is a nilpotent matrix with an appropriate nilpotent index, say h, i.e., $N^{h-1} \neq 0$ and $N^h = 0$. Equivalently, Σ can be decomposed into the following two subsystems:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = A_1 \; x_1 + B_1 \; u \\ y_1 = C_1 \; x_1 + \; D \;\; u \end{cases} \tag{4.21}$$

and

$$\Sigma_2 : \begin{cases} N \; \dot{x}_2 = \quad\quad x_2 + B_2 \; u \\ \quad y_2 = C_2 \; x_2 \end{cases} \tag{4.22}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$, $y = y_1 + y_2$.

To conduct the above decomposition, one can follow such procedures in the following (see also Dai [29]).

First, since (E, A) pair is regular, thus there exists a numerous α such that $|\alpha E+A| \neq 0$. Now we can construct the following pencil with this α,

$$\hat{E} = (\alpha E + A)^{-1}E, \quad \hat{A} = (\alpha E + A)^{-1}A, \tag{4.23}$$

and it is clear that $\hat{A} = I - \alpha\hat{E}$.

Secondly, it is obvious that there exists an invertible transform matrix T such that

$$T^{-1}\hat{E}T = \begin{bmatrix} \hat{E}_1 & 0 \\ 0 & \hat{E}_2 \end{bmatrix}, \tag{4.24}$$

where $\hat{E}_1 \in \mathbb{R}^{n_1\times n_1}$ is nonsingular and $\hat{E}_2 \in \mathbb{R}^{n_2\times n_2}$ is nilpotent.

Moreover, since $\hat{E}_2$ is nilpotent and hence we have a nonsingular $(I - \alpha\hat{E}_2)$ to construct the following two transformation matrices,

$$P = \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I - \alpha\hat{E}_2)^{-1} \end{bmatrix} T^{-1}(\alpha E + A)^{-1}, \quad Q = T. \tag{4.25}$$

Remember that $\hat{A} = I - \alpha\hat{E}$, we can verify the decomposition as follows,

$$\begin{aligned} PEQ &= \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I-\alpha\hat{E}_2)^{-1} \end{bmatrix} T^{-1}(\alpha E + A)^{-1} E T = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \\ PAQ &= \begin{bmatrix} \hat{E}_1^{-1} & 0 \\ 0 & (I-\alpha\hat{E}_2)^{-1} \end{bmatrix} T^{-1}(\alpha E + A)^{-1} A T = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \end{aligned} \tag{4.26}$$

where $N = (I - \alpha\hat{E}_2)^{-1}\hat{E}_2$ is nilpotent and $A_1 = \hat{E}_1^{-1}(I - \alpha\hat{E}_2)$.

The above algorithm is simple in decomposing a given singular system into its canonical form.

STEP 2 (DECOMPOSITION OF $x_{\rm z}$ AND $x_{\rm e}$): *The key idea is to separate the controllable and uncontrollable parts of the pair (N, B_2) in Σ_2. It will be simple to observe that some of the state variables of Σ_2 are identically zero and some are the derivatives of the system inputs.* It follows from Chen [20] (see e.g., Theorems 2.3.1 and 2.3.2) that there exist following nonsingular coordinate transformations

$$x_2 = T_{\rm s}\hat{x}_2, \quad u = T_{\rm i}\hat{u}, \tag{4.27}$$

such that

$$\hat{x}_2 = \begin{pmatrix} x_{\rm v} \\ x_{\rm z} \end{pmatrix}, \quad x_{\rm v} = \begin{pmatrix} x_{\rm v1} \\ x_{\rm v2} \\ \vdots \\ x_{{\rm v}n_{\rm e}} \end{pmatrix}, \quad x_{\rm z} \in \mathbb{R}^{n_{\rm z}}, \quad \hat{u} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n_{\rm e}} \\ \hat{u}_* \end{pmatrix}, \tag{4.28}$$

where

$$x_{{\rm v}i} \in \mathbb{R}^{p_i}, \quad x_{{\rm v}i} = \begin{pmatrix} x_{{\rm v}i,1} \\ x_{{\rm v}i,2} \\ \vdots \\ x_{{\rm v}i,p_i} \end{pmatrix}, \quad i = 1, 2, \cdots, n_{\rm e}, \quad p_1 \le p_2 \le \cdots \le p_{n_{\rm e}}, \tag{4.29}$$

and

$$\hat{N} = T_{\rm s}^{-1} N T_{\rm s} = \begin{bmatrix} J_{\rm v} & N_{\rm zv} \\ 0 & J_{n_{\rm z}} \end{bmatrix} = \begin{bmatrix} J_{\rm v1} & 0 & \cdots & 0 & N_{1\rm z} \\ 0 & J_{\rm v2} & \cdots & 0 & N_{2\rm z} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{{\rm v}n_{\rm e}} & N_{n_{\rm e}\rm z} \\ 0 & 0 & \cdots & 0 & J_{n_{\rm z}} \end{bmatrix}, \tag{4.30}$$

$$\hat{B}_2 = T_{\rm s}^{-1} B_2 T_{\rm i} = \begin{bmatrix} B_{\rm v} \\ 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n_{\rm e}} & B_{1\rm z} \\ 0 & B_{22} & \cdots & B_{2n_{\rm e}} & B_{2\rm z} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{n_{\rm e}n_{\rm e}} & B_{n_{\rm e}\rm z} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{4.31}$$

and where $(J_{\rm v}, B_{\rm v})$ is completely controllable. Furthermore, N being nilpotent implies that $J_{{\rm v}i}$ and $J_{n_{\rm z}}$ have all their eigenvalues at 0, and $J_{{\rm v}i}$ and B_{ij} have the following control special forms,

$$J_{{\rm v}i} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{ii} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad B_{i\rm z} = \begin{bmatrix} b_{i{\rm z},1} \\ \vdots \\ b_{i{\rm z},p_i-1} \\ 0 \end{bmatrix}, \quad B_{ij} = \begin{bmatrix} b_{ij,1} \\ \vdots \\ b_{ij,p_i-1} \\ 0 \end{bmatrix}. \tag{4.32}$$

As such, by the transformation of (4.27), Σ_2 is decomposed into the following subsystems:

$$J_{n_{\rm z}} \dot{x}_{\rm z} = x_{\rm z} \quad \Longrightarrow \quad x_{\rm z} = 0, \tag{4.33}$$

and for $i = 1, 2, \cdots, n_{\rm e}$,

$$J_{{\rm v}i} \dot{x}_{{\rm v}i} + N_{i\rm z} \dot{x}_{\rm z} = x_{{\rm v}i} + B_{ii} \hat{u}_i + \sum_{j=i+1}^{n_{\rm e}} B_{ij} \hat{u}_j + B_{i\rm z} \hat{u}_*, \tag{4.34}$$

which is equivalent to

$$J_{{\rm v}i} \dot{x}_{{\rm v}i} = x_{{\rm v}i} + B_{ii} \hat{u}_i + \sum_{j=i+1}^{n_{\rm e}} B_{ij} \hat{u}_j + B_{i\rm z} \hat{u}_*. \tag{4.35}$$

Owing the special structure of $J_{\mathrm{v}i}$, we have for $i = 1, 2, \cdots, n_{\mathrm{e}}$,

$$\left.\begin{aligned}
\dot{x}_{\mathrm{v}i,2} &= x_{\mathrm{v}i,1} + \sum_{j=i+1}^{n_{\mathrm{e}}} b_{ij,1}\hat{u}_j + b_{i\mathrm{z},1}\hat{u}_* \\
\dot{x}_{\mathrm{v}i,3} &= x_{\mathrm{v}i,2} + \sum_{j=i+1}^{n_{\mathrm{e}}} b_{ij,2}\hat{u}_j + b_{i\mathrm{z},2}\hat{u}_* \\
&\ \,\vdots \\
\dot{x}_{\mathrm{v}i,p_i} &= x_{\mathrm{v}i,p_i-1} + \sum_{j=i+1}^{n_{\mathrm{e}}} b_{ij,p_i-1}\hat{u}_j + b_{i\mathrm{z},p_i-1}\hat{u}_*
\end{aligned}\right\} \tag{4.36}$$

$$\hat{u}_i = -x_{\mathrm{v}i,p_i} \tag{4.37}$$

and

$$x_{\mathrm{v}i,1} = -\hat{u}_i^{(p_i-1)} - \sum_{k=0}^{p_i-2}\sum_{j=i+1}^{n_{\mathrm{e}}} b_{ij,k+1}\hat{u}_j^{(k)} - \sum_{k=0}^{p_i-2} b_{i\mathrm{z},k+1}\hat{u}_*^{(k)}. \tag{4.38}$$

Let us define a new input variable

$$\check{u}_i = x_{\mathrm{v}i,1} = \psi_i(s)\hat{u}, \tag{4.39}$$

for an appropriate vector $\psi_i(s)$ whose elements are polynomials in s. Then, we can rewrite (4.36) as follows

$$\left.\begin{aligned}
\dot{x}_{\mathrm{v}i,2} &= -\sum_{j=i+1\ \&\ p_j>1}^{n_{\mathrm{e}}} b_{ij,1}\hat{x}_{\mathrm{v}j,p_j} + b_{i\mathrm{z},1}\hat{u}_* + \check{u}_i + \sum_{j=i+1\ \&\ p_j=1}^{n_{\mathrm{e}}} b_{ij,1}\check{u}_j \\
\dot{x}_{\mathrm{v}i,3} &= x_{\mathrm{v}i,2} - \sum_{j=i+1\ \&\ p_j>1}^{n_{\mathrm{e}}} b_{ij,2}\hat{x}_{\mathrm{v}j,p_j} + b_{i\mathrm{z},2}\hat{u}_* + \sum_{j=i+1\ \&\ p_j=1}^{n_{\mathrm{e}}} b_{ij,2}\check{u}_j \\
&\ \,\vdots \\
\dot{x}_{\mathrm{v}i,p_i} &= x_{\mathrm{v}i,p_i-1} - \sum_{j=i+1\ \&\ p_j>1}^{n_{\mathrm{e}}} b_{ij,p_i-1}\hat{x}_{\mathrm{v}j,p_j} + b_{i\mathrm{z},p_i-1}\hat{u}_* + \sum_{j=i+1\ \&\ p_j=1}^{n_{\mathrm{e}}} b_{ij,p_i-1}\check{u}_j
\end{aligned}\right\} \tag{4.40}$$

Next, define

$$x_{\mathrm{e}} = \check{u}_{\mathrm{e}} = \begin{pmatrix} \check{u}_1 \\ \check{u}_2 \\ \vdots \\ \check{u}_{n_{\mathrm{e}}} \end{pmatrix} = \begin{pmatrix} x_{\mathrm{v}1,1} \\ x_{\mathrm{v}2,1} \\ \vdots \\ x_{\mathrm{v}n_{\mathrm{e}},1} \end{pmatrix}. \tag{4.41}$$

It is now straightforward to verify that the transformed system of Σ_2 as given in

(4.22) can be rearranged into the following form

$$\begin{cases} x_{\rm z} = 0 \\ x_{\rm e} = \check{u}_{\rm e} \\ \dot{\check{x}}_2 = \check{A}_2 \check{x}_2 + \check{B}_{2\rm e} \check{u}_{\rm e} + \check{B}_{2*} \hat{u}_* \\ y_2 = \check{C}_2 \check{x}_2 + \check{D}_{2\rm e} \hat{u}_{\rm e} \end{cases} \tag{4.42}$$

where $\check{x}_2$ consists of all the state variables of $x_{\rm v}$ that are not contained in $x_{\rm e}$, and $\check{A}_2$, $\check{B}_{2\rm e}$, $\check{B}_{2*}$, $\check{C}_2$ and $\check{D}_{2\rm e}$ are constant matrices with appropriate dimensions. Furthermore, Σ_1 of (4.21) can be rewritten as follows

$$\begin{cases} \dot{x}_1 = A_1 x_1 + \check{A}_{12} \check{x}_2 + \check{B}_{1\rm e} \check{u}_{\rm e} + \check{B}_{1*} \check{u}_* \\ y_1 = C_1 x_1 + \check{C}_{12} \check{x}_2 + \check{D}_{1\rm e} \check{u}_{\rm e} + \check{D}_{1*} \check{u}_* \end{cases} \tag{4.43}$$

for some appropriate dimensional constant matrices $\check{A}_{12}$, $\check{B}_{1\rm e}$, $\check{B}_{1*}$, $\check{C}_{12}$, $\check{D}_{1\rm e}$ and $\check{D}_{1*}$.

STEP 3 (FORMATION OF A NONSINGULAR SYSTEM AND FINAL DECOMPOSITION): *The key idea is to form a nonsingular system from the subsystems (4.42) and (4.43), and then apply the result of nonsingular systems to obtain a structural decomposition for the original system given in (4.1).* Following (4.42) and (4.43), we obtain a nonsingular system

$$\bar{\Sigma} : \begin{cases} \dot{\bar{x}} = \bar{A}\ \bar{x} + \bar{B}\ \bar{u} \\ y = \bar{C}\ \bar{x} + \bar{D}\ \bar{u} \end{cases} \tag{4.44}$$

where

$$\bar{x} = \begin{pmatrix} x_1 \\ \check{x}_2 \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} \check{u}_{\rm e} \\ \hat{u}_* \end{pmatrix}, \tag{4.45}$$

$$\bar{A} = \begin{bmatrix} A_1 & \check{A}_{12} \\ 0 & \check{A}_2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \check{B}_{1\rm e} & \check{B}_{1*} \\ \check{B}_{2\rm e} & \check{B}_{2*} \end{bmatrix} \tag{4.46}$$

and

$$\bar{C} = [\, C_1 \quad \check{C}_2 + \check{C}_{12} \,], \quad \bar{D} = [\, \check{D}_{1\rm e} + \check{D}_{2\rm e} \quad \check{D}_{1*} \,]. \tag{4.47}$$

It then follows from the result of Sannuti and Saberi [70] and Saberi and Sannuti [67] that there exist nonsingular transformations $\bar{\Gamma}_{\rm s} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, where $\bar{n} = n - n_{\rm e} - n_{\rm z}$,

$\bar{\Gamma}_{\rm o} \in \mathbb{R}^{p\times p}$ and $\bar{\Gamma}_{\rm i} \in \mathbb{R}^{m\times m}$ such that when they are applied to $\bar{\Sigma}$, i.e.,

$$\bar{x} = \bar{\Gamma}_{\rm s} \begin{pmatrix} x_{\rm a} \\ x_{\rm b} \\ x_{\rm c} \\ x_{\rm d} \end{pmatrix}, \quad y = \bar{\Gamma}_{\rm o}\tilde{y} = \bar{\Gamma}_{\rm o} \begin{pmatrix} y_0 \\ y_{\rm b} \\ y_{\rm d} \end{pmatrix}, \quad \bar{u} = \bar{\Gamma}_{\rm i}\tilde{u} = \bar{\Gamma}_{\rm i} \begin{pmatrix} u_0 \\ u_{\rm c} \\ u_{\rm d} \end{pmatrix}, \tag{4.48}$$

where $x_{\rm a} \in \mathbb{R}^{n_{\rm a}}$, $x_{\rm b} \in \mathbb{R}^{n_{\rm b}}$, $x_{\rm c} \in \mathbb{R}^{n_{\rm c}}$, $x_{\rm d} \in \mathbb{R}^{n_{\rm d}}$, $u_0 \in \mathbb{R}^{n_0}$, $u_{\rm c} \in \mathbb{R}^{m_{\rm c}}$, $u_{\rm d} \in \mathbb{R}^{m_{\rm d}}$, $y_0 \in \mathbb{R}^{n_0}$, $y_{\rm b} \in \mathbb{R}^{p_{\rm b}}$, $y_{\rm d} \in \mathbb{R}^{m_{\rm d}}$,

$$x_{\rm d} = \begin{pmatrix} x_{\rm d1} \\ x_{\rm d2} \\ \vdots \\ x_{{\rm d}m_{\rm d}} \end{pmatrix}, \quad y_{\rm d} = \begin{pmatrix} y_{\rm d1} \\ y_{\rm d2} \\ \vdots \\ y_{{\rm d}m_{\rm d}} \end{pmatrix}, \quad u_{\rm d} = \begin{pmatrix} u_{\rm d1} \\ u_{\rm d2} \\ \vdots \\ u_{{\rm d}m_{\rm d}} \end{pmatrix}, \tag{4.49}$$

we have

$$\dot{x}_{\rm a} = A_{\rm aa}x_{\rm a} + B_{\rm 0a}y_0 + L_{\rm ad}y_{\rm d} + L_{\rm ab}y_{\rm b}, \tag{4.50}$$

$$\dot{x}_{\rm b} = A_{\rm bb}x_{\rm b} + B_{\rm 0b}y_0 + L_{\rm bd}y_{\rm d}, \quad y_{\rm b} = C_{\rm b}x_{\rm b}, \tag{4.51}$$

$$\dot{x}_{\rm c} = A_{\rm cc}x_{\rm c} + B_{\rm 0c}y_0 + L_{\rm cd}y_{\rm d} + L_{\rm cb}y_{\rm b} + B_{\rm c}\left[u_{\rm c} + M_{\rm ca}x_{\rm a}\right], \tag{4.52}$$

$$y_0 = C_{\rm 0a}x_{\rm a} + C_{\rm 0b}x_{\rm b} + C_{\rm 0c}x_{\rm c} + C_{\rm 0d}x_{\rm d} + u_0, \tag{4.53}$$

and

$$\dot{x}_{{\rm d}i} = A_{q_i}x_{{\rm d}i} + L_{i0}y_0 + L_{i{\rm d}}y_{\rm d} + B_{q_i}\left[u_{{\rm d}i} + M_{i{\rm a}}x_{\rm a} + M_{i{\rm b}}x_{\rm b} + M_{i{\rm c}}x_{\rm c} + \sum_{j=1}^{m_{\rm d}} M_{ij}x_{{\rm d}j}\right], \tag{4.54}$$

$$y_{{\rm d}i} = C_{q_i}x_{{\rm d}i}, \quad y_{\rm d} = C_{\rm d}x_{\rm d}, \tag{4.55}$$

with $(A_{q_i}, B_{q_i}, C_{q_i})$ having the following special form as given in (4.13).

This completes the proof of Theorem 4.3.1. ∎

For future use, we can rewrite the structural decomposition of Theorem 4.3.1 in its compact form. This compact form will be handy in developing many applications of the theory.

It will be frequently used later in the next section to prove the structural properties of the system. The following corollary gives the compact matrix form of the structural decomposition.

Corollary 4.4.1 (Compact Form) *The structural decomposition in Theorem 4.3.1 can also be given as the following compact matrix transformation form,*

$$\tilde{E} = \Gamma_{\mathrm{e}}^{-1} E \Gamma_{\mathrm{s}} = \tilde{E}_{\mathrm{cmp}} = \begin{bmatrix} J_{n_{\mathrm{z}}} & 0 & 0 & 0 & 0 & 0 \\ E_{\mathrm{ez}} & 0 & 0 & 0 & 0 & 0 \\ E_{\mathrm{az}} & 0 & I_{n_{\mathrm{a}}} & 0 & 0 & 0 \\ E_{\mathrm{bz}} & 0 & 0 & I_{n_{\mathrm{b}}} & 0 & 0 \\ E_{\mathrm{cz}} & 0 & 0 & 0 & I_{n_{\mathrm{c}}} & 0 \\ E_{\mathrm{dz}} & 0 & 0 & 0 & 0 & I_{n_{\mathrm{d}}} \end{bmatrix},$$

$$\begin{aligned} \tilde{A} &= \Gamma_{\mathrm{e}}^{-1} A \Gamma_{\mathrm{s}} = \tilde{A}_{\mathrm{cmp}} + \tilde{A}_{\mathrm{k}} \\ &= \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_{\mathrm{e}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{\mathrm{aa}} & L_{\mathrm{ab}} C_{\mathrm{b}} & 0 & L_{\mathrm{ad}} C_{\mathrm{d}} \\ 0 & 0 & 0 & A_{\mathrm{bb}} & 0 & L_{\mathrm{bd}} C_{\mathrm{d}} \\ 0 & 0 & B_{\mathrm{c}} M_{\mathrm{ca}} & L_{\mathrm{cb}} C_{\mathrm{b}} & A_{\mathrm{cc}} & L_{\mathrm{cd}} C_{\mathrm{d}} \\ 0 & 0 & B_{\mathrm{d}} M_{\mathrm{da}} & B_{\mathrm{d}} M_{\mathrm{db}} & B_{\mathrm{d}} M_{\mathrm{dc}} & A_{\mathrm{dd}} \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ B_{0\mathrm{a}} \\ B_{0\mathrm{b}} \\ B_{0\mathrm{c}} \\ B_{0\mathrm{d}} \end{bmatrix} \begin{bmatrix} 0 & 0 & C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \end{bmatrix} + \tilde{A}_{\mathrm{k}}, \end{aligned}$$

$$\tilde{B} = \Gamma_{\rm e}^{-1} B \Gamma_{\rm i}(s) = \tilde{B}_{\rm cmp} + \tilde{B}_{\rm k}(s) = \begin{bmatrix} 0 & 0 & 0 \\ B_{\rm 0e} & B_{\rm de} & B_{\rm ce} \\ B_{\rm 0a} & 0 & 0 \\ B_{\rm 0b} & 0 & 0 \\ B_{\rm 0c} & 0 & B_{\rm c} \\ B_{\rm 0d} & B_{\rm d} & 0 \end{bmatrix} + \tilde{B}_{\rm k}(s),$$

$$\tilde{C} = \Gamma_{\rm o}^{-1} C \Gamma_{\rm s} = \tilde{C}_{\rm cmp} + \tilde{C}_{\rm k} = \begin{bmatrix} C_{\rm 0z} & 0 & C_{\rm 0a} & C_{\rm 0b} & C_{\rm 0c} & C_{\rm 0d} \\ C_{\rm dz} & 0 & 0 & 0 & 0 & C_{\rm d} \\ C_{\rm bz} & 0 & 0 & C_{\rm b} & 0 & 0 \end{bmatrix} + \tilde{C}_{\rm k}, \tag{4.56}$$

$$\tilde{D} = \Gamma_{\rm o}^{-1} D \Gamma_{\rm i}(s) = \tilde{D}_{\rm cmp} + \tilde{D}_{\rm k}(s) = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \tilde{D}_{\rm k}(s), \tag{4.57}$$

where

$$\tilde{A}_{\rm k}\tilde{x} + \tilde{B}_{\rm k}(s)\tilde{u} = 0, \qquad \tilde{C}_{\rm k}\tilde{x} + \tilde{D}_{\rm k}(s)\tilde{u} = 0. \tag{4.58}$$

Proof: Although the constructive decomposition process has been given in the proof of Theorem 4.3.1, the process is in equation form and thus not direct when we want to compute it with a computer program. The following algorithm proves this corollary and gives a different form of decomposition in the view of compact matrix computation.

First, observing (4.20) to (4.41) in the constructive decomposition procedure, we can combine the Step 1 and Step 2 with coordinate transform matrices $\Gamma_{\rm e1}$, $\Gamma_{\rm s1}$ and $\Gamma_{\rm i1}$. These transform matrices decompose the original singular system Σ as,

$$x = \Gamma_{\rm s1}\hat{x} = \Gamma_{\rm s1} \begin{pmatrix} x_{\rm z} \\ x_{\rm e} \\ x_{\rm f} \end{pmatrix}, \qquad u = \Gamma_{\rm i1}\hat{u} = \Gamma_{\rm i1} \begin{pmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_{n_{\rm e}} \\ \hat{u}_\star \end{pmatrix} = \Gamma_{\rm i1} \begin{pmatrix} \hat{u}_{\rm e} \\ \hat{u}_\star \end{pmatrix}, \tag{4.59}$$

and Σ is transformed into the following $\hat{\Sigma}$,

$$\begin{aligned}
\hat{E} &= \Gamma_{\mathrm{e1}} E \Gamma_{\mathrm{s1}} = U \begin{bmatrix} I_{n_1} & 0 \\ 0 & T_{\mathrm{s}}^{-1} \end{bmatrix} PEQ \begin{bmatrix} I_{n_1} & 0 \\ 0 & T_{\mathrm{s}} \end{bmatrix} V = \begin{bmatrix} J_{n_{\mathrm{z}}} & 0 & 0 \\ E_{\mathrm{ez}} & 0 & 0 \\ E_{\mathrm{fz}} & 0 & I_{n_{\mathrm{f}}} \end{bmatrix}, \\
\hat{A} &= \Gamma_{\mathrm{e1}} A \Gamma_{\mathrm{s1}} = U \begin{bmatrix} I_{n_1} & 0 \\ 0 & T_{\mathrm{s}}^{-1} \end{bmatrix} PAQ \begin{bmatrix} I_{n_1} & 0 \\ 0 & T_{\mathrm{s}} \end{bmatrix} V = \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 \\ 0 & 0 & A_{\mathrm{g}} \\ 0 & A_{\mathrm{e}} & A_{\mathrm{f}} \end{bmatrix}, \\
\hat{B} &= \Gamma_{\mathrm{e1}} B \Gamma_{\mathrm{i1}} = U \begin{bmatrix} I_{n_1} & 0 \\ 0 & T_{\mathrm{s}}^{-1} \end{bmatrix} PBT_{\mathrm{i}} = \begin{bmatrix} 0 & 0 \\ I_{n_{\mathrm{e}}} & 0 \\ 0 & B_{\mathrm{f}} \end{bmatrix}, \\
\hat{C} &= C\Gamma_{\mathrm{s1}} = CQ \begin{bmatrix} I_{n_1} & 0 \\ 0 & T_{\mathrm{s}} \end{bmatrix} V = [\, C_{\mathrm{z}} \quad C_{\mathrm{e}} \quad C_{\mathrm{f}} \,], \\
\hat{D} &= D\Gamma_{\mathrm{i1}} = DT_{\mathrm{i}} = [\, D_{\mathrm{e}} \quad D_{\mathrm{f}} \,],
\end{aligned} \tag{4.60}$$

where U and V are permutation matrices. And it is obvious that the original singular system $\Sigma : (E, A, B, C, D)$ is equivalent to $\hat{\Sigma} : (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$.

Next, noting (4.39) and (4.41), we have the following input transformation,

$$\check{u} = \begin{pmatrix} \check{u}_{\mathrm{e}} \\ \hat{u}_{\star} \end{pmatrix} = \Gamma_{\mathrm{is}}(s)\hat{u} = \begin{bmatrix} \Psi_1(s) & \Psi_2(s) \\ 0 & I_{m-n_{\mathrm{e}}} \end{bmatrix} \begin{pmatrix} \hat{u}_{\mathrm{e}} \\ \hat{u}_{\star} \end{pmatrix}, \tag{4.61}$$

and

$$x_{\mathrm{e}} = \check{u}_{\mathrm{e}}. \tag{4.62}$$

Now if we apply the input transformation $\check{u} = \Gamma_{\mathrm{is}}(s)\hat{u}$ to the transformed system $\hat{\Sigma}$, we will have,

$$\begin{aligned}
\hat{E} &= \check{E} = \begin{bmatrix} J_{n_{\mathrm{z}}} & 0 & 0 \\ E_{\mathrm{ez}} & 0 & 0 \\ E_{\mathrm{fz}} & 0 & I_{n_{\mathrm{f}}} \end{bmatrix}, \\
\hat{A} &= \check{A} + \check{A}_{\mathrm{k}} = \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 \\ 0 & I_{n_{\mathrm{e}}} & 0 \\ 0 & 0 & A_{\mathrm{f}} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I_{n_{\mathrm{e}}} & A_{\mathrm{g}} \\ 0 & A_{\mathrm{e}} & 0 \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} I_{n_z} & 0 & 0 \\ 0 & 0 & A_g \\ 0 & A_e & A_f \end{bmatrix},$$

$$\hat{B}\Gamma_{is}^{-1} = \check{B} + \check{B}_k(s) = \begin{bmatrix} 0 & 0 \\ I_{n_e} & 0 \\ A_e & B_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \Psi_1^\star(s) - I_{n_e} & \Psi_2^\star(s) \\ -A_e & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ \Psi_1^\star(s) & \Psi_2^\star(s) \\ 0 & B_f \end{bmatrix}$$

$$\hat{C} = \check{C} + \check{C}_k = [\, C_z \quad 0 \quad C_f - D_e A_g \,] + [\, 0 \quad C_e \quad D_e A_g \,]$$

$$= [\, C_z \quad C_e \quad C_f \,],$$

$$\hat{D}\Gamma_{is}^{-1}(s) = \check{D} + \check{D}_k(s)$$

$$= [\, C_e \quad D_f \,] + [\, D_e \Psi_1^\star(s) - C_e \quad D_e \Psi_2^\star(s) \,]$$

$$= [\, D_e \Psi_1^\star(s) \quad D_e \Psi_2^\star(s) + D_f \,]. \tag{4.63}$$

Here

$$\Gamma_{is}^{-1}(s) = \begin{bmatrix} \Psi_1(s) & \Psi_2(s) \\ 0 & I_{m-n_e} \end{bmatrix}^{-1} = \begin{bmatrix} \Psi_1^\star(s) & \Psi_2^\star(s) \\ 0 & I_{m-n_e} \end{bmatrix}. \tag{4.64}$$

Now the system $\hat{\Sigma}$ has been further transformed into the following system $\check{\Sigma}$,

$$\check{\Sigma} : \begin{cases} \check{E}\dot{\check{x}} = \check{A}\check{x} + \check{B}\check{u} + \check{A}_k\check{x} + \check{B}_k(s)\check{u}, \\ y = \check{C}\check{x} + \check{D}\check{u} + \check{C}_k\check{x} + \check{D}_k(s)\check{u}, \end{cases} \tag{4.65}$$

where $\check{x} = \hat{x}$.

And noting (4.33), (4.59) and (4.60), we have the following equation in system $\hat{\Sigma}$,

$$A_g x_f + \hat{u}_e = 0. \tag{4.66}$$

Then after the input transformation of (4.64), (4.66) becomes,

$$A_g x_f + [\, \Psi_1^\star(s) \quad \Psi_2^\star(s) \,]\, \check{u} = 0. \tag{4.67}$$

Now with (4.67), and check (4.63), we have

$$\check{A}_{\mathrm{k}}\check{x} + \check{B}_{\mathrm{k}}(s)\check{u} = 0, \qquad \check{C}_{\mathrm{k}}\check{x} + \check{D}_{\mathrm{k}}(s)\check{u} = 0. \tag{4.68}$$

And thus the singular system $\hat{\Sigma} : (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$ is equivalent to the following transformed system $\check{\Sigma} : (\check{E}, \check{A}, \check{B}, \check{C}, \check{D})$,

At last, according to the decomposition procedure (4.44) to (4.48) in Section 4.4, we have

$$\begin{aligned}
\tilde{E} &= \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 \\ 0 & I_{n_{\mathrm{e}}} & 0 \\ 0 & 0 & \bar{\Gamma}_{\mathrm{s}}^{-1} \end{bmatrix} \check{E} \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 \\ 0 & I_{n_{\mathrm{e}}} & 0 \\ 0 & 0 & \bar{\Gamma}_{\mathrm{s}} \end{bmatrix} = \tilde{E}_{\mathrm{cmp}} \\
&= \begin{bmatrix} J_{n_{\mathrm{z}}} & 0 & 0 & 0 & 0 & 0 \\ E_{\mathrm{ez}} & 0 & 0 & 0 & 0 & 0 \\ E_{\mathrm{az}} & 0 & I_{n_{\mathrm{a}}} & 0 & 0 & 0 \\ E_{\mathrm{bz}} & 0 & 0 & I_{n_{\mathrm{b}}} & 0 & 0 \\ E_{\mathrm{cz}} & 0 & 0 & 0 & I_{n_{\mathrm{c}}} & 0 \\ E_{\mathrm{dz}} & 0 & 0 & 0 & 0 & I_{n_{\mathrm{d}}} \end{bmatrix}, \\
\tilde{A} &= \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 \\ 0 & I_{n_{\mathrm{e}}} & 0 \\ 0 & 0 & \bar{\Gamma}_{\mathrm{s}}^{-1} \end{bmatrix} (\check{A} + \check{A}_{\mathrm{k}}) \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 \\ 0 & I_{n_{\mathrm{e}}} & 0 \\ 0 & 0 & \bar{\Gamma}_{\mathrm{s}} \end{bmatrix} \\
&= \tilde{A}_{\mathrm{cmp}} + \tilde{A}_{\mathrm{k}} \\
&= \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_{\mathrm{e}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{\mathrm{aa}} & L_{\mathrm{ab}}C_{\mathrm{b}} & 0 & L_{\mathrm{ad}}C_{\mathrm{d}} \\ 0 & 0 & 0 & A_{\mathrm{bb}} & 0 & L_{\mathrm{bd}}C_{\mathrm{d}} \\ 0 & 0 & B_{\mathrm{c}}M_{\mathrm{ca}} & L_{\mathrm{cb}}C_{\mathrm{b}} & A_{\mathrm{cc}} & L_{\mathrm{cd}}C_{\mathrm{d}} \\ 0 & 0 & B_{\mathrm{d}}M_{\mathrm{da}} & B_{\mathrm{d}}M_{\mathrm{db}} & B_{\mathrm{d}}M_{\mathrm{dc}} & A_{\mathrm{dd}} \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
& + \begin{bmatrix} 0 \\ 0 \\ B_{0\mathrm{a}} \\ B_{0\mathrm{b}} \\ B_{0\mathrm{c}} \\ B_{0\mathrm{d}} \end{bmatrix} [0 \quad 0 \quad C_{0\mathrm{a}} \quad C_{0\mathrm{b}} \quad C_{0\mathrm{c}} \quad C_{0\mathrm{d}}] + \tilde{A}_{\mathrm{k}}, \\
\tilde{B} &= \begin{bmatrix} I_{n_\mathrm{z}} & 0 & 0 \\ 0 & I_{n_\mathrm{e}} & 0 \\ 0 & 0 & \bar{\Gamma}_\mathrm{s}^{-1} \end{bmatrix} (\check{B} + \check{B}_\mathrm{k})\bar{\Gamma}_\mathrm{i} \\
&= \tilde{B}_\mathrm{cmp} + \tilde{B}_\mathrm{k}(s) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ B_{0\mathrm{e}} & B_\mathrm{de} & B_\mathrm{ce} \\ B_{0\mathrm{a}} & 0 & 0 \\ B_{0\mathrm{b}} & 0 & 0 \\ B_{0\mathrm{c}} & 0 & B_\mathrm{c} \\ B_{0\mathrm{d}} & B_\mathrm{d} & 0 \end{bmatrix} + \tilde{B}_\mathrm{k}(s), \\
\tilde{C} &= \bar{\Gamma}_\mathrm{o}^{-1}(\check{C} + \check{C}_\mathrm{k}) \begin{bmatrix} I_{n_\mathrm{z}} & 0 & 0 \\ 0 & I_{n_\mathrm{e}} & 0 \\ 0 & 0 & \bar{\Gamma}_\mathrm{s} \end{bmatrix} \\
&= \tilde{C}_\mathrm{cmp} + \tilde{C}_\mathrm{k} \\
&= \begin{bmatrix} 0 & 0 & C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \\ 0 & 0 & 0 & 0 & 0 & C_\mathrm{d} \\ 0 & 0 & 0 & C_\mathrm{b} & 0 & 0 \end{bmatrix} + \tilde{C}_\mathrm{k}, \\
\tilde{D} &= \bar{\Gamma}_\mathrm{o}^{-1}(\check{D} + \check{D}_\mathrm{k})\bar{\Gamma}_\mathrm{i} \\
&= \tilde{D}_\mathrm{cmp} + \tilde{D}_\mathrm{k} \\
&= \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \tilde{D}_\mathrm{k}(s),
\end{aligned} \tag{4.69}
$$

where

$$\tilde{A}_{\rm k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I_{n_{\rm e}} & A_{\rm g}\bar{\Gamma}_{\rm s} \\ 0 & \bar{\Gamma}_{\rm s}^{-1}A_{\rm e} & 0 \end{bmatrix},$$

$$\tilde{B}_{\rm k}(s) = \begin{bmatrix} 0 & 0 \\ \Psi_1(s) - I_{n_{\rm e}} & \Psi_2(s) \\ -\bar{\Gamma}_{\rm s}^{-1}A_{\rm e} & 0 \end{bmatrix} \bar{\Gamma}_{\rm i},$$

$$\tilde{C}_{\rm k} = [\,0 \quad \bar{\Gamma}_{\rm o}^{-1}C_{\rm e} \quad \bar{\Gamma}_{\rm o}^{-1}D_{\rm e}A_{\rm g}\bar{\Gamma}_{\rm s}\,],$$

$$\tilde{D}_{\rm k}(s) = [\,\bar{\Gamma}_{\rm o}^{-1}(-C_{\rm e} + D_{\rm e}\Psi_1(s)) \quad \bar{\Gamma}_{\rm o}^{-1}D_{\rm e}\Psi_2(s)\,]\,\bar{\Gamma}_{\rm i}, \tag{4.70}$$

This completes the proof of Corollary 4.4.1. ∎

We also have the following corollary,

Corollary 4.4.2 (Strictly Equivalence) *The structurally decomposed system in equation form, or in compact form, is strictly equivalent to the original singular system* Σ.

Proof: Actually, from the computation algorithm for the compact form of the structural decomposition, we can easily get that the structural decomposition is nothing more than an invertible transformation on the original system's system matrix, that is,

$$P_{\tilde{\Sigma}}(s) = \begin{bmatrix} \tilde{A}_{\rm cmp} - s\tilde{E}_{\rm cmp} & \tilde{B}_{\rm cmp} \\ \tilde{C}_{\rm cmp} & \tilde{D}_{\rm cmp} \end{bmatrix} = \Gamma_{\rm p} P_{\Sigma}(s) \Gamma_{\rm q} = \Gamma_{\rm p} \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix} \Gamma_{\rm q}, \tag{4.71}$$

and

$$\Gamma_{\rm p} = \begin{bmatrix} I_{n_{\rm z}} & 0 & 0 & 0 \\ 0 & I_{n_{\rm e}} & 0 & 0 \\ 0 & 0 & \bar{\Gamma}_{\rm s}^{-1} & 0 \\ 0 & 0 & 0 & \bar{\Gamma}_{\rm o}^{-1} \end{bmatrix} \begin{bmatrix} I_{n_{\rm z}} & 0 & 0 & 0 \\ 0 & I_{n_{\rm e}} & 0 & 0 \\ 0 & 0 & I_{n_{\rm f}} & 0 \\ 0 & -D_{\rm e} & 0 & I_{\rm p} \end{bmatrix} U \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & T_{\rm s}^{-1} & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I_p \end{bmatrix}$$

$$= \begin{bmatrix} I_{n_z} & 0 & 0 & 0 \\ 0 & I_{n_e} & 0 & 0 \\ 0 & 0 & \bar{\Gamma}_s^{-1} & 0 \\ 0 & -\bar{\Gamma}_o^{-1} D_e & 0 & \bar{\Gamma}_o^{-1} \end{bmatrix} U \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & T_s^{-1} & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I_p \end{bmatrix},$$

$$\Gamma_q = \begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & T_s & 0 \\ 0 & 0 & T_i \end{bmatrix} V \begin{bmatrix} I_{n_z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_e} & 0 \\ 0 & 0 & I_{n_f} & 0 & 0 \\ 0 & I_{n_e} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{m-n_e} \end{bmatrix} \cdot$$

$$\begin{bmatrix} I_{n_z} & 0 & 0 & 0 & 0 \\ 0 & I_{n_e} & -A_g & 0 & 0 \\ 0 & 0 & I_{n_f} & 0 & 0 \\ 0 & 0 & 0 & I_{n_e} & 0 \\ 0 & 0 & 0 & 0 & I_{m-n_e} \end{bmatrix} \begin{bmatrix} I_{n_z} & 0 & 0 & 0 & 0 \\ 0 & I_{n_e} & 0 & -I_{n_e} & 0 \\ 0 & 0 & I_{n_f} & 0 & 0 \\ 0 & 0 & 0 & I_{n_e} & 0 \\ 0 & 0 & 0 & 0 & I_{m-n_e} \end{bmatrix} \begin{bmatrix} I_{n_z} & 0 & 0 & 0 \\ 0 & I_{n_e} & 0 & 0 \\ 0 & 0 & \bar{\Gamma}_s & 0 \\ 0 & 0 & 0 & \bar{\Gamma}_i \end{bmatrix}$$

$$= \begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & T_s & 0 \\ 0 & 0 & T_i \end{bmatrix} V \begin{bmatrix} I_{n_z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_e} & 0 \\ 0 & 0 & I_{n_f} & 0 & 0 \\ 0 & I_{n_e} & -A_g & -I_{n_e} & 0 \\ 0 & 0 & 0 & 0 & I_{m-n_e} \end{bmatrix} \begin{bmatrix} I_{n_z} & 0 & 0 & 0 \\ 0 & I_{n_e} & 0 & 0 \\ 0 & 0 & \bar{\Gamma}_s & 0 \\ 0 & 0 & 0 & \bar{\Gamma}_i \end{bmatrix}.$$

Here $n_f = n - n_z - n_e$ and Γ_p and Γ_q can be clearly computed from the above algorithm in proving Corollary 4.4.1.

And thus according to the definition of strictly equivalence, the decomposed system $\tilde{\Sigma}$: $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly equivalent to Σ. So its Kronecker canonical form will remain unchanged and all of its structural properties are reserved. ∎

4.5 Proofs of Properties of Structural Decomposition

We present in this section the proofs of all the properties of the structural decomposition given in the previous sections. The following lemmas are essential and instrumental to our proofs of the structural properties of singular systems.

Lemma 4.5.1 *Consider a singular system Σ characterized by (E, A, B, C, D) or in the state space form of (4.1). Then, for any state feedback gain $F \in \mathbb{R}^{m\times n}$ satisfying $\det(sE - A - BF) \not\equiv 0$, $\Sigma_{\rm F}$ as characterized by $(E, A + BF, B, C + DF, D)$ has the following properties:*

1. *$\Sigma_{\rm F}$ is stabilizable if and only if Σ is stabilizable;*
2. *the normal rank of $\Sigma_{\rm F}$ is equal to that of Σ;*
3. *the invariant zero structure of $\Sigma_{\rm F}$ is the same as that of Σ;*
4. *the infinite zero structure of $\Sigma_{\rm F}$ is the same as that of Σ;*
5. *$\Sigma_{\rm F}$ is (left or right or non) invertible if and only if Σ is (left or right or non) invertible;*

Proof. Item 1 is obvious. In view of the following reductions,

$$\begin{aligned} H_{\rm F}(s) :&= (C + DF)(sE - A - BF)^{-1}B + D \\ &= (C + DF)(sE - A)^{-1} \left[I - BF(sE - A)^{-1} \right]^{-1} B + D \\ &= (C + DF)(sE - A)^{-1}B \left[I - F(sE - A)^{-1}B \right]^{-1} + D \\ &= \left[C(sE - A)^{-1}B + D \right] \left[I - F(sE - A)^{-1}B \right]^{-1} \\ &= H(s) \left[I - F(sE - A)^{-1}B \right]^{-1}, \end{aligned} \tag{4.72}$$

Item 2 follows. Next, noting that

$$\begin{bmatrix} A + BF - sE & B \\ C + DF & D \end{bmatrix} = \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}, \tag{4.73}$$

and the fact that the invariances of $P_\Sigma(s)$ are strictly equivalent under nonsingular constant transformations, the results of Items 3, 4 and 5 follow.

This completes the proof of Lemma 4.5.1. ∎

Lemma 4.5.2 *Consider a singular system* Σ *characterized by* (E, A, B, C, D) *or in the state space form of (4.1). Then, for a constant output injection gain* $K \in \mathbb{R}^{n\times p}$ *satisfying* $\det(sE - A - KC) \not\equiv 0$, $\Sigma_{\rm K}$ *as characterized by* $(E, A + KC, B + KD, C, D)$ *has the following properties:*

1. $\Sigma_{\rm K}$ *is stabilizable if and only if* Σ *is stabilizable;*
2. *the normal rank of* $\Sigma_{\rm K}$ *is equal to that of* Σ*;*
3. *the invariant zero structure of* $\Sigma_{\rm K}$ *is the same as that of* Σ*;*
4. *the infinite zero structure of* $\Sigma_{\rm K}$ *is the same as that of* Σ*;*
5. $\Sigma_{\rm K}$ *is (left or right or non) invertible if and only if* Σ *is (left or right or non) invertible.*

Proof: It is a dual version of Lemma 4.5.1. ∎

We now proceed to prove the properties of the structural decomposition. Noting the properties in Corollary 4.4.2, and without loss of generality, we assume throughout the rest of this section that the given system Σ has been transformed into the structural decomposition of Theorem 4.3.1 or into the compact form of (4.57), i.e.

$$E = \begin{bmatrix} J_{n_z} & 0 & 0 & 0 & 0 & 0 \\ E_{ez} & 0 & 0 & 0 & 0 & 0 \\ E_{az} & 0 & I_{n_a} & 0 & 0 & 0 \\ E_{bz} & 0 & 0 & I_{n_b} & 0 & 0 \\ E_{cz} & 0 & 0 & 0 & I_{n_c} & 0 \\ E_{dz} & 0 & 0 & 0 & 0 & I_{n_d} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ B_{0e} & B_{de} & B_{ce} \\ B_{0a} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \tag{4.74}$$

$$A = \begin{bmatrix} I_{n_z} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_e} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{aa} & L_{ab}C_b & 0 & L_{ad}C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd}C_d \\ 0 & 0 & B_cM_{ca} & L_{cb}C_b & A_{cc} & L_{cd}C_d \\ 0 & 0 & B_dM_{da} & B_dM_{db} & B_dM_{dc} & A_{dd} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B_{0a} \\ B_{0b} \\ B_{0c} \\ B_{0d} \end{bmatrix} \begin{bmatrix} 0 & 0 & C_{0a} & C_{0b} & C_{0c} & C_{0d} \end{bmatrix} \tag{4.75}$$

and

$$C = \begin{bmatrix} 0 & 0 & C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.76}$$

We further note that A_{dd}^*, B_d and C_d have the following forms:

$$A_{dd}^* = \text{blkdiag}\{A_{q_1}, \cdots, A_{q_{m_d}}\}, \tag{4.77}$$

and

$$B_d = \text{blkdiag}\{B_{q_1}, \cdots, B_{q_{m_d}}\}, \quad C_d = \text{blkdiag}\{C_{q_1}, \cdots, C_{q_{m_d}}\}, \tag{4.78}$$

where A_{q_i}, B_{q_i} and C_{q_i}, $i = 1, 2, \cdots, m_d$, are as defined in (4.13).

The following proofs of the properties of the structural decomposition for singular systems follow pretty closely to those given in Chen [19] for nonsingular systems.

4.5.1 Proof of Property 4.3.1

It follows from Dai [29] that the singular system Σ of (4.1) is stabilizable if and only if

$$\text{rank}\,[\,sE-A \quad B\,]=n, \tag{4.79}$$

for all $s\in\mathbb{C}^0\cup\mathbb{C}^+$. Let us define a state feedback gain matrix

$$F=-\begin{bmatrix} 0 & 0 & C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \\ 0 & 0 & M_{\mathrm{da}} & M_{\mathrm{db}} & M_{\mathrm{dc}} & M_{\mathrm{dd}} \\ 0 & 0 & M_{\mathrm{ca}} & 0 & 0 & 0 \end{bmatrix}, \tag{4.80}$$

which gives

$$A+BF=\begin{bmatrix} I_{n_\mathrm{z}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_\mathrm{e}} & N_{\mathrm{ea}} & N_{\mathrm{eb}} & N_{\mathrm{ec}} & N_{\mathrm{ed}} \\ 0 & 0 & A_{\mathrm{aa}} & L_{\mathrm{ab}}C_{\mathrm{b}} & 0 & L_{\mathrm{ad}}C_{\mathrm{d}} \\ 0 & 0 & 0 & A_{\mathrm{bb}} & 0 & L_{\mathrm{bd}}C_{\mathrm{d}} \\ 0 & 0 & 0 & L_{\mathrm{cb}}C_{\mathrm{b}} & A_{\mathrm{cc}} & L_{\mathrm{cd}}C_{\mathrm{d}} \\ 0 & 0 & 0 & 0 & 0 & A^*_{\mathrm{dd}} \end{bmatrix}, \tag{4.81}$$

where N_{ea}, N_{eb}, N_{ec} and N_{ed} are constant matrices with appropriate dimensions.

Noting that $(A_{\mathrm{cc}},B_{\mathrm{c}})$ is completely controllable, we have for any $s\in\mathbb{C}^0\cup\mathbb{C}^+$,

$$\begin{aligned}&\text{rank}\,[\,sE-A-BF \quad B\,]\\ &=\text{rank}\begin{bmatrix} sJ_{n_\mathrm{z}}-I_{n_\mathrm{z}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{n_\mathrm{e}} & -N_{\mathrm{ea}} & -N_{\mathrm{eb}} & -N_{\mathrm{ec}} & -N_{\mathrm{ed}} & B_{0\mathrm{e}} & B_{\mathrm{de}} & B_{\mathrm{ce}} \\ sE_{\mathrm{az}} & 0 & sI_{n_\mathrm{a}}-A_{\mathrm{aa}} & -L_{\mathrm{ab}}C_{\mathrm{b}} & 0 & -L_{\mathrm{ad}}C_{\mathrm{d}} & B_{0\mathrm{a}} & 0 & 0 \\ sE_{\mathrm{bz}} & 0 & 0 & sI_{n_\mathrm{b}}-A_{\mathrm{bb}} & 0 & -L_{\mathrm{bd}}C_{\mathrm{d}} & B_{0\mathrm{b}} & 0 & 0 \\ sE_{\mathrm{cz}} & 0 & 0 & -L_{\mathrm{cb}}C_{\mathrm{b}} & sI_{n_\mathrm{c}}-A_{\mathrm{cc}} & -L_{\mathrm{cd}}C_{\mathrm{d}} & B_{0\mathrm{c}} & 0 & B_{\mathrm{c}} \\ sE_{\mathrm{dz}} & 0 & 0 & 0 & 0 & sI_{n_\mathrm{d}}-A^*_{\mathrm{dd}} & B_{0\mathrm{d}} & B_{\mathrm{d}} & 0 \end{bmatrix}\end{aligned}$$

$$= \text{rank} \begin{bmatrix} sJ_{n_z} - I_{n_z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{n_e} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & sI_{n_a} - A_{aa} & -L_{ab}C_b & 0 & -L_{ad}C_d & B_{0a} & 0 & 0 \\ 0 & 0 & 0 & sI_{n_b} - A_{bb} & 0 & -L_{bd}C_d & B_{0b} & 0 & 0 \\ 0 & 0 & 0 & 0 & sI_{n_c} - A_{cc} & 0 & 0 & 0 & B_c \\ 0 & 0 & 0 & 0 & 0 & sI_{n_d} - A_{dd}^{**} & B_{0d} & B_d & 0 \end{bmatrix}$$

$$= \text{rank} \begin{bmatrix} sJ_{n_z} - I_{n_z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{n_e} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & sI - A_{con} & 0 & B_{con1}C_d & B_{con0} & 0 & 0 \\ 0 & 0 & 0 & sI_{n_c} - A_{cc} & 0 & 0 & 0 & B_c \\ 0 & 0 & 0 & 0 & sI_{n_d} - A_{dd}^{**} & B_{0d} & B_d & 0 \end{bmatrix}, \qquad (4.82)$$

where $A_{dd}^{**} = A_{dd}^{*} - L_{dd}C_d$ and

$$A_{con} = \begin{bmatrix} A_{aa} & L_{ab}C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{con} = [\, B_{con0} \quad B_{con1} \,] = \begin{bmatrix} B_{0a} & L_{ad} \\ B_{0b} & L_{bd} \end{bmatrix}. \qquad (4.83)$$

Also, noting the special structure of J_{n_z} and the properties of (A_{dd}^{*}, C_d, B_d), it is simple to verify that $[\, sE - A - BF \quad B \,]$ is of maximal rank if and only if $[\, sI - A_{con} \quad B_{con} \,]$ is of maximal rank. By Lemma 4.5.1, we have that Σ is stabilizable if and only if (A_{con}, B_{con}) is stabilizable.

Similarly, the property of detectability of the system can be proved in an output injection way. This completes the proof of Property 4.3.1. ∎

4.5.2 Proof of Property 4.3.2

To prove this property, we first define a state feedback gain matrix F as in (4.80) and an output injection gain matrix K as follows:

$$K = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_{0\mathrm{a}} & L_{\mathrm{ad}} & L_{\mathrm{ab}} \\ B_{0\mathrm{b}} & L_{\mathrm{bd}} & 0 \\ B_{0\mathrm{c}} & L_{\mathrm{cd}} & L_{\mathrm{cb}} \\ B_{0\mathrm{d}} & L_{\mathrm{dd}} & 0 \end{bmatrix}, \tag{4.84}$$

and noting the results in (4.81), we thus have,

$$E_\star = E = \begin{bmatrix} J_{n_\mathrm{z}} & 0 & 0 & 0 & 0 & 0 \\ E_{\mathrm{ez}} & 0 & 0 & 0 & 0 & 0 \\ E_{\mathrm{az}} & 0 & I_{n_\mathrm{a}} & 0 & 0 & 0 \\ E_{\mathrm{bz}} & 0 & 0 & I_{n_\mathrm{b}} & 0 & 0 \\ E_{\mathrm{cz}} & 0 & 0 & 0 & I_{n_\mathrm{c}} & 0 \\ E_{\mathrm{dz}} & 0 & 0 & 0 & 0 & I_{n_\mathrm{d}} \end{bmatrix}, \tag{4.85}$$

$$A_\star = A + BF + KC + KDF = \begin{bmatrix} I_{n_\mathrm{z}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_\mathrm{e}} & N_{\mathrm{ea}} & N_{\mathrm{eb}} & N_{\mathrm{ec}} & N_{\mathrm{ed}} \\ 0 & 0 & A_{\mathrm{aa}} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{\mathrm{bb}} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{\mathrm{cc}} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{\mathrm{dd}}^{**} \end{bmatrix}, \tag{4.86}$$

$$B_\star = B + KD = \begin{bmatrix} 0 & 0 & 0 \\ B_{0\mathrm{e}} & B_{\mathrm{de}} & B_{\mathrm{ce}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{\mathrm{c}} \\ 0 & B_{\mathrm{d}} & 0 \end{bmatrix}, \tag{4.87}$$

and

$$C_\star = C + DF = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_\mathrm{d} \\ 0 & 0 & 0 & C_\mathrm{b} & 0 & 0 \end{bmatrix}, \quad D_\star = D = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.88}$$

Next, it can be shown by some matrix manipulations that the transfer function of the singular system $\Sigma_\star$ characterized by $(E_\star, A_\star, B_\star, C_\star, D_\star)$ is given by

$$H_\star(s) = C_\star(sE_\star - A_\star)^{-1}B_\star + D_\star = \begin{bmatrix} I_{\mathrm{m}_0} & 0 & 0 \\ 0 & C_\mathrm{d}(sI_{n_\mathrm{d}} - A_\mathrm{dd}^*)^{-1}B_\mathrm{d} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.89}$$

with

$$C_\mathrm{d}(sI_{n_\mathrm{d}} - A_\mathrm{dd}^*)^{-1}B_\mathrm{d} = \begin{bmatrix} \frac{1}{s^{q_1}} & & \\ & \ddots & \\ & & \frac{1}{s^{q_{m_\mathrm{d}}}} \end{bmatrix}. \tag{4.90}$$

By Lemma 4.5.1 and Lemma 4.5.2, we have

$$\text{normrank}\,\{H(s)\} = \text{normrank}\,\{H_\star(s)\} = m_0 + m_\mathrm{d}. \tag{4.91}$$

Next, noting the special structure of the triple $(A_\mathrm{dd}^*, B_\mathrm{d}, C_\mathrm{d})$, and the properties of $(A_\mathrm{bb}, C_\mathrm{b})$ and $(A_\mathrm{cc}, B_\mathrm{c})$, we have, for a complex scalar α,

$$\begin{aligned} &\text{rank}\,\{P_{\Sigma_\star}(\alpha)\} \\ &\quad = \text{rank} \begin{bmatrix} A_\star - \alpha E_\star & B_\star \\ C_\star & D_\star \end{bmatrix} \\ &\quad = \text{rank} \begin{bmatrix} I_{n_\mathrm{z}} - \alpha J_{n_\mathrm{z}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha E_\mathrm{az} & A_\mathrm{aa} - \alpha I_{n_\mathrm{a}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha E_\mathrm{ez} & I_{n_\mathrm{e}} & N_\mathrm{ea} & N_\mathrm{eb} & N_\mathrm{ec} & N_\mathrm{ed} & B_\mathrm{0e} & B_\mathrm{de} & B_\mathrm{ce} \\ -\alpha E_\mathrm{bz} & 0 & 0 & A_\mathrm{bb} - \alpha I_{n_\mathrm{b}} & 0 & 0 & 0 & 0 & 0 \\ -\alpha E_\mathrm{cz} & 0 & 0 & 0 & A_\mathrm{cc} - \alpha I_{n_\mathrm{c}} & 0 & 0 & 0 & B_\mathrm{c} \\ -\alpha E_\mathrm{dz} & 0 & 0 & 0 & 0 & A_\mathrm{dd} - \alpha I_{n_\mathrm{d}} & 0 & B_\mathrm{d} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{m_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_\mathrm{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_\mathrm{b} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$
= \text{rank} \begin{bmatrix} I_{n_z} - \alpha J_{n_z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_e} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{aa} - \alpha I_{n_a} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{bb} - \alpha I_{n_b} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{cc} - \alpha I_{n_c} & 0 & 0 & 0 & B_c \\ 0 & 0 & 0 & 0 & 0 & A_{dd} - \alpha I_{n_d} & 0 & B_d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{m_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_d & 0 & 0 & 0 \\ 0 & 0 & 0 & C_b & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
= n_z + n_e + \text{rank}\{A_{aa} - \alpha I_{n_a}\} + n_b + n_c + n_d + m_0 + m_d. \tag{4.92}
$$

It is clear that the rank of $P_\star(\alpha)$ drops below $n + m_0 + m_d$ if and only if $\alpha \in \lambda(A_{aa})$. Hence, by Lemmas 4.5.1–4.5.2, the invariant zeros of Σ are given by the eigenvalues of A_{aa}. This completes the proof of Property 4.3.2. ∎

4.5.3 Proof of Property 4.3.3

Observing (4.89) and (4.90), we can easily see that $\Sigma_\star$, or equivalently by Lemmas 4.5.1–4.5.2 the given singular system Σ, has m_0 infinite zeros of order 0 and has m_d infinite zeros of orders q_i respectively, where $i = 1, 2, \cdots, m_d$. ∎

4.5.4 Proof of Property 4.3.4

Following from the results of Lemmas 4.5.1–4.5.2, we have that Σ or $H(s)$ is (left or right or non) invertible if and only if $\Sigma_\star$ or $H_\star(s)$ is (left or right or non) invertible. Then, the results of Property 4.3.4 follow explicitly from the properties of $H_\star(s)$ in (4.89). ∎

4.6 An Illustrative Example

We now present a numerical example to illustrate the structural decomposition technique and the properties. We consider a singular system of (4.1) characterized by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}, \quad A = I_7, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \tag{4.93}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 2 & 1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.94}$$

Step 1 (PRELIMINARY DECOMPOSITION). The given system is already in the form of (4.20), i.e., we have

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_1 + \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} u \\ y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u \end{cases} \tag{4.95}$$

and

$$
\Sigma_2 : \left\{
\begin{aligned}
\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}
\begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{pmatrix}
&= \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}
+ \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} u \\
y_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 & 1 & -2 \end{bmatrix}
\begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}
\end{aligned}
\right. \tag{4.96}
$$

with $n_1 = 1$ and $n_2 = 6$.

Step 2 (DECOMPOSITION OF $x_{\rm z}$ AND $x_{\rm e}$). Using the toolbox of Lin and Chen [52], we obtain two nonsingular transformations

$$
T_{\rm s} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad
T_{\rm i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \tag{4.97}
$$

which transform Σ_2 to the following canonical form

$$
T_{\rm s}^{-1} N T_{\rm s} = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad
T_{\rm s}^{-1} B_2 T_{\rm i} = \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right],
$$

$$C_2 T_\mathrm{s} = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]. \tag{4.98}$$

As such, Σ_2 is decomposed into:

$$x_\mathrm{z} = \hat{x}_7 = 0, \tag{4.99}$$

and

$$\left.\begin{aligned} \hat{u}_1 &= -x_{\mathrm{v1}_2}, \\ \dot{x}_{\mathrm{v1}_2} &= x_{\mathrm{v1}_1} + \hat{u}_2 + \hat{u}_3, \end{aligned}\right\} \tag{4.100}$$

$$\left.\begin{aligned} \hat{u}_2 &= -x_{\mathrm{v2}_3}, \\ \dot{x}_{\mathrm{v2}_3} &= x_{\mathrm{v2}_2} + \hat{u}_3, \\ \dot{x}_{\mathrm{v2}_2} &= x_{\mathrm{v2}_1} + \hat{u}_3, \end{aligned}\right\} \tag{4.101}$$

Thus $n_\mathrm{z} = 1$, $n_\mathrm{e} = 2$, $p_1 = 2$, $p_2 = 3$, $v_\mathrm{d} = p_2 - 1 = 2$ and

$$\begin{aligned} x_\mathrm{e1} &= -\dot{\hat{u}}_1 - \hat{u}_2 - \hat{u}_3 = -\dot{u}_1 + u_1 - u_2 - u_3, \\ x_\mathrm{e2} &= -\ddot{\hat{u}}_2 - \dot{\hat{u}}_3 - \hat{u}_3 = -\ddot{u}_2 + \dot{u}_1 - \dot{u}_3 + u_1 - u_3. \end{aligned} \tag{4.102}$$

Step 3 (FORMATION OF A NONSINGULAR SYSTEM AND FINAL DECOMPOSITION).

According to the input transformation in (4.98), and noting the results in (4.100) and (4.101), we can rewrite Σ_1 as:

$$\dot{x}_1 = x_1 + \hat{u}_1 + \hat{u}_2 = x_1 - x_{\mathrm{v1}_1} - x_{\mathrm{v2}_1}. \tag{4.103}$$

Now, combining the results of (4.100), (4.101) and (4.103), we obtain an auxiliary nonsingular system

$$\begin{cases} \dot{\bar{x}} = \bar{A}\ \bar{x} + \bar{B}\ \bar{u} \\ y = \bar{C}\ \bar{x} + \bar{D}\ \bar{u} \end{cases} \tag{4.104}$$

with

$$\bar{x} = \begin{pmatrix} x_1 \\ x_{\mathrm{v1}_2} \\ x_{\mathrm{v2}_2} \\ x_{\mathrm{v2}_3} \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} x_\mathrm{e1} \\ x_\mathrm{e2} \\ \hat{u}_3 \end{pmatrix}, \tag{4.105}$$

and

$$\bar{A} = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \bar{D} = 0. \tag{4.106}$$

Again, using the toolbox of Lin and Chen [52], we obtain

$$\bar{\Gamma}_{\rm s} = \begin{bmatrix} 0.7746 & 0 & 0 & 0 \\ -0.2582 & -0.3162 & 0 & 0.6 \\ -0.5164 & 0.3162 & 1 & 0.4 \\ 0.2582 & 0.3162 & 0 & 0.4 \end{bmatrix}, \quad \bar{\Gamma}_{\rm i} = \begin{bmatrix} 0 & 0.2 & -0.8944 \\ 1 & 0 & 0 \\ 0 & 0.4 & 0.4472 \end{bmatrix}, \quad \bar{\Gamma}_{\rm o} = I_2, \tag{4.107}$$

$n_{\rm a} = 1$, $n_{\rm b} = 0$, $n_{\rm c} = 1$, $n_{\rm d} = 2$,

$$\bar{\Gamma}_{\rm s}^{-1} \bar{A} \bar{\Gamma}_{\rm s} = \begin{bmatrix} 1 & 0 & 0 & -1.2910 \\ -1.4697 & 1 & 1.8974 & 2.3190 \\ 1.2910 & -0.3162 & -1 & -1.4000 \\ -0.7746 & 0 & 1 & 0 \end{bmatrix}, \tag{4.108}$$

and

$$\bar{\Gamma}_{\rm s}^{-1} \bar{B} \bar{\Gamma}_{\rm i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1.4142 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{\Gamma}_{\rm o}^{-1} \bar{C} \bar{\Gamma}_{\rm s} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.109}$$

Finally, the structural decomposition of the given singular system is given by

$$x_{\rm z} = 0, \tag{4.110}$$

$$x_{\rm e} = \begin{bmatrix} 0.8944 \\ 0 \end{bmatrix} u_{\rm c} + \begin{bmatrix} 0 & -0.2 \\ -1 & 0 \end{bmatrix} u_{\rm d}, \quad u_{\rm d} = \begin{pmatrix} u_{\rm d1} \\ u_{\rm d2} \end{pmatrix}, \tag{4.111}$$

$$\dot{x}_{\rm a} = x_{\rm a} + [\,0 \quad -1.2910\,]\, y_{\rm d}, \tag{4.112}$$

$$\dot{x}_{\rm c} = x_{\rm c} - 1.4697 x_{\rm a} + [\,1.8974 \quad 2.3190\,]\, y_{\rm d} + 1.4142 u_{\rm c}, \tag{4.113}$$

and

$$\begin{pmatrix} \dot{x}_{\rm d1} \\ \dot{x}_{\rm d2} \end{pmatrix} = \begin{bmatrix} 1.2910 \\ -0.7746 \end{bmatrix} x_{\rm a} + \begin{bmatrix} -0.3162 \\ 0 \end{bmatrix} x_{\rm c} + \begin{bmatrix} -1 & -1.4 \\ 1 & 0 \end{bmatrix} y_{\rm d} + u_{\rm d}, \quad y_{\rm d} = \begin{pmatrix} x_{\rm d1} \\ x_{\rm d2} \end{pmatrix}. \tag{4.114}$$

It is simple to see now from the above decomposition that the given system is right invertible with one invariant zero at $s = 1$ and two infinite zeros of order 1. The given system has one state variable, which is identically zero, and two state variables, which are nothing but the linear combination of the system inputs and their derivatives. These state variables are actually redundant in the system dynamics. All of these properties can be verified clearly from the following Kronecker canonical form of its system matrix,

$$\Gamma_{\rm u} \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix} \Gamma_{\rm v} = \begin{bmatrix} 1-s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -s & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (4.115)$$

with

$$\Gamma_{\rm u} = \begin{bmatrix} 1 & 1 & -1 & 0 & -2 & -1 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 1 & -4 & -1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{bmatrix},$$

$$\Gamma_{\mathrm{v}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 2 & 0 & 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -3 & 2 & 0 & 1 & -1 & 3 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \end{bmatrix}. \tag{4.116}$$

Finally, for completeness, we give below all the necessary transformation matrices:

$$\Gamma_{\mathrm{e}} = \begin{bmatrix} 0 & 1 & 1 & 0.7746 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.2582 & -0.3162 & 0 & 0.6 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.5164 & 0.3162 & 1 & 0.4 \\ 0 & 0 & 0 & 0.2582 & 0.3162 & 0 & 0.4 \\ 1 & 0 & 2 & -0.2582 & -0.3162 & 0 & 0.6 \\ 1 & 0 & 0 & 0.2582 & 0.3162 & 0 & 0.4 \end{bmatrix}, \tag{4.117}$$

$$\Gamma_{\mathrm{s}} = \begin{bmatrix} 0 & 0 & 0 & 0.7746 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -0.2582 & -0.3162 & 0 & 0.6 \\ 0 & 0 & 1 & -0.2582 & -0.3162 & 0 & 0.6 \\ 0 & 0 & 0 & -0.5164 & 0.3162 & 1 & 0.4 \\ 1 & 1 & 0 & 0.2582 & 0.3162 & 0 & 0.4 \\ 1 & 0 & 0 & 0.5164 & 0.3162 & 1 & 0.4 \end{bmatrix}, \tag{4.118}$$

and

$$\Gamma_{\mathrm{i}}(s) = \begin{bmatrix} s+1 & -s^2 & -s-1 \\ -s-2 & -1 & -1 \\ 0.8944s - 0.4472 & 0.8944 & 1.3416 \end{bmatrix}^{-1}, \quad \Gamma_{\mathrm{o}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.119}$$

Finally, we note that the s-dependent input transformation $\Gamma_{\mathrm{i}}(s)$ simply implies that

$$\begin{pmatrix} u_{\mathrm{c}} \\ u_{\mathrm{d1}} \\ u_{\mathrm{d2}} \end{pmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{u} + \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0.8944 & 0 & 0 \end{bmatrix} \dot{u} + \begin{bmatrix} 1 & 0 & -1 \\ -2 & -1 & -1 \\ -0.4472 & 0.8944 & 1.3416 \end{bmatrix} u. \tag{4.120}$$

And the structurally decomposed system is strictly equivalent to the original system. This can be verified by the following transformations.

$$\begin{aligned}
&\begin{bmatrix} \tilde{A} - s\tilde{E} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \\
= \;& \Gamma_{\mathrm{p}} \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix} \Gamma_{\mathrm{q}} \\
= \;& \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2000 & 0.8944 \\
0 & 0 & 1 & 0.0000 & -0.0000 & 0 & 0 & -1.0000 & 0 & 0 \\
0 & 0 & 0 & 1.0000 - s & 0.0000 & 0 & -1.2910 & 0 & 0 & -0.0000 \\
0 & 0 & 0 & -1.4698 & 1.0000 - s & 1.8975 & 2.3192 & 0.0000 & 0 & 1.4143 \\
0 & 0 & 0 & 1.2910 & -0.3162 & -1.0000 - s & -1.4000 & 1.0000 & 0 & 0.0000 \\
0 & 0 & 0 & -0.7746 & 0 & 1.0000 & -s & -0.0000 & 1.0000 & 0 \\
1 & 0 & 0 & -0.0000 & 0.0000 & 1.0000 & 0.0000 & -0.0000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0
\end{bmatrix},
\end{aligned}$$

where

$$
\Gamma_p = \begin{bmatrix}
0 & 0 & 0 & 0 & -1.0000 & 0 & 1.0000 & 0 & 0 \\
0 & 0 & 1.0000 & 0 & 1.0000 & 0 & -1.0000 & 0 & 0 \\
0.0000 & -1.0000 & -0.0000 & 0 & 1.0000 & 1.0000 & -1.0000 & 0 & 0 \\
1.2910 & 1.2910 & -1.2910 & 0 & -2.5820 & -1.2910 & 2.5820 & 0 & 0 \\
-1.0542 & -3.5842 & 1.0542 & 0 & 5.2709 & 2.3192 & -3.3734 & 0 & 0 \\
1.0000 & 1.0000 & -2.0000 & 1.0000 & -4.0000 & -1.0000 & 3.0000 & 0 & 0 \\
-0.0000 & 2.0000 & 0.0000 & 0 & 0.0000 & -1.0000 & 1.0000 & 0 & 0 \\
-0.0000 & 1.0000 & 0.0000 & 0 & -1.0000 & -1.0000 & 1.0000 & 1.0000 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000
\end{bmatrix}, \tag{4.121}
$$

and

$$
\Gamma_q = \begin{bmatrix}
0 & 0 & 0 & 0.7746 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2000 & -0.8944 \\
1 & 0 & 0 & -0.2582 & -0.3162 & 0 & 0.6000 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.2582 & -0.3162 & 0 & 0.6000 & 1.0000 & 0 & 0 \\
0 & 0 & 0 & -0.5164 & 0.3162 & 1.0000 & 0.4000 & 0 & 0 & 0 \\
1 & 0 & 0 & 0.2582 & 0.3162 & 0 & 0.4000 & 0 & 0.2000 & -0.8944 \\
1 & 0 & 0 & -0.5164 & 0.3162 & 1.0000 & 0.4000 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0.2582 & 0.3162 & 0 & -0.6000 & 0 & -0.2000 & 0.8944 \\
0 & 0 & 1.0000 & -0.2582 & -0.3162 & 0 & -0.4000 & -1.0000 & 0 & 0 \\
0 & 1.0000 & 0 & 0.2582 & 0.3162 & 0 & -0.6000 & 0 & 0.2000 & 1.3416
\end{bmatrix}. \tag{4.122}
$$

4.7 Conclusions

We have presented in this chapter a structural decomposition technique for general multivariable singular systems, which has a distinct feature of explicitly capturing and displaying all structural properties, such as the finite and infinite zero structures, invertibility structure, as well as redundant states of a given singular system. And all the properties of the structural decomposition can be verified by the Kronecker canonical form of its system matrix. As its counterpart in nonsingular systems, the technique is expected to play an important role in solving many control problems related to singular systems. This will actually be the subject of our future research. And Chapter 6 will give an example of the potential applications of the structural decomposition technique.

Chapter 5

Geometric Subspaces of Singular Systems

5.1 Introduction

Geometric approach for linear systems firstly appeared in the literature when Basile and Marro [3], [4], [5] and Wonham and Morse [81], [82] presented the notations of (A, B)-invariance and (C, A)-invariance in the late 1960's. From then on, more geometric invariant subspaces have been proposed, such as (C, A, B)-pair introduced by Schumacher [71] and almost invariant subspaces by Willems [78], [79], [80]. Geometric invariant subspaces and pairs play core roles in geometric approach since they can be used to characterize structural properties of linear systems such as controllability, observability, system invertibility and so on. In general, the essence of geometric approach is to characterize a system or control problem as a verifiable property of some constructible state subspaces. Then a specific solution will be derived from a set of such state subspaces and their generations.

For singular systems, geometric approach is also a powerful tool and has been widely used in the literature [58], [45]. And this chapter is to give geometric interpretations of the structural decomposition technique through those invariant geometric subspaces.

The structural decomposition technique for singular systems is a natural extension of its counterpart for linear nonsingular systems [70] [67]. Its initial ideas are structure properties and system equivalence, which have been intensively researched in the literature. Campbell [8] presented an effective structural decomposition method and got the corresponding equivalent system, while Verghese *et al.* [76] defined a strong system equivalence using a trivial augmentation and deflation technique. On the other hand, structural invariants also received intensive research in literature. Further, Misra *et al.* [59] and Liu *et al.* [53] have presented their algorithms to compute the invariant structural indices of singular systems. More recently, He and Chen [39] and He *et al.* [40] have developed a structural decomposition method for single-input single-output and multivariable singular systems respectively. Such a structural decomposition can not only give the invariant structural indices but also explicitly display the structural features, such as the finite and infinite zero dynamics, invertibility structures and redundant dynamics of the given systems. And it is expected to be a powerful tool in solving system and control problems as its counterpart in nonsingular linear system [13].

In this chapter, we first give the definitions of several geometric subspaces for singular systems. Then the geometric subspaces will be alternatively expressed in matrix form by some theorems and lemmas. And finally, the relations between structurally decomposed subspaces and those geometric subspaces will be presented and proved.

5.2 Geometric Subspaces of Singular Systems

The definitions of many geometric subspaces have already been presented in the literature by Malabre [58] in geometric form, and Geerts [36] in the so called algebraic distributional framework. However, to our best knowledge, there are still some problems in their definitions in algebraic frame. Thus we first try to give the definitions of geometric subspaces in an algebraic framework in this section.

The following definitions are natural extensions from those for nonsingular systems (see e.g., Trentelman *et al.* [72]).

Weakly unobservable subspace is a superset of unobservable subspace, which is generally denoted as $\langle \text{Ker } C \mid A \rangle$. It is associated with system zeros and thus become a very useful tool of geometric approach.

Definition 5.2.1 (Weakly Unobservable Subspace $\mathcal{V}^\star$) For a singular system Σ, an initial point $x_0 \in \mathcal{X}$ is called weakly unobservable if there exists an input function u such that the corresponding output $y_u(t_1, x_0) = 0$ for all $t_1 \geqslant 0$. The weakly unobservable subspace $\mathcal{V}^\star$ of Σ is the set of all its weakly unobservable points.

Based on the above definition, strongly controllable subspace is contained in the controllable subspace $\langle\ A \mid \text{Im } B \rangle$. And it is also widely used in geometric approach.

Definition 5.2.2 (Strongly Controllable Subspace $\mathcal{S}^\star$) For a singular system Σ, an initial point $x_0 \in \mathcal{X}$ is called strongly controllable if there exists an impulsive input function u such that $x_u(t_1, x_0) = 0$ for all $t_1 \geq 0$. The strongly controllable subspace $\mathcal{S}^\star$ of Σ is the set of all its strongly controllable points.

The following two definitions are generally extension of weakly unobservable subspace and strongly controllable subspace.

Definition 5.2.3 (Controllable Weakly Unobservable Subspace $\mathcal{R}^\star$) For a singular system Σ, an initial point $x_0 \in \mathcal{X}$ is called controllable weakly unobservable if there exists an input function u and $T > 0$ such that the corresponding output $y_u(t_1, x_0) = 0$ for all $t_1 \in [0, T]$ and $x_u(T, x_0) = 0$. The controllable weakly unobservable subspace $\mathcal{R}^\star$ of Σ is the set of all its weakly unobservable points.

Definition 5.2.4 (Distributionally Weakly Unobservable Subspace $\mathcal{W}^\star$) For a singular system Σ, an initial point $x_0 \in \mathcal{X}$ is called distributionally weakly unobservable if there exists an distributionally impulsive input function u such that the corresponding output $y_u(t_1, x_0) = 0$ for all $t_1 \geqslant 0$. The distributionally weakly unobservable subspace $\mathcal{W}^\star$ of Σ is the set of all such points.

After defining these subspaces, we give their relationship in the following. Since they are quite obvious, we do not need to prove them here.

Remark 5.2.1 *It can be seen clearly that $\mathcal{V}^\star$ is dual to $\mathcal{S}^\star$ from their definitions.*

Further, we have the following lemma on the relationship of the above four geometric subspaces.

Lemma 5.2.1 *The relationship of geometric subspaces can be characterized as:*

$$\mathcal{R}^\star = \mathcal{V}^\star \cap \mathcal{S}^\star, \quad \mathcal{W}^\star = \mathcal{V}^\star \cup \mathcal{S}^\star. \tag{5.1}$$

5.3 Geometric Expression of the Subspaces

The definitions in Section 5.2 are direct but not in geometric form. This section will express the definition in geometric form and prove their equivalence.

We first give the geometric description of weakly unobservable subspace in the following theorem.

Theorem 5.3.1 *$\mathcal{V}^\star(\Sigma)$ is the largest subspace $\mathcal{V}$ of $\mathcal{X}$ for which there exists a constant matrix F such that*

$$(A+BF)\mathcal{V} \subset E\mathcal{V} \quad \text{and} \quad (C+DF)\mathcal{V} = 0. \tag{5.2}$$

Or alternatively, $\mathcal{V}^\star(\Sigma)$ is the largest subspace $\mathcal{V}$ of $\mathcal{X}$ such that

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subset \begin{bmatrix} E\mathcal{V} \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} B \\ D \end{bmatrix}. \tag{5.3}$$

This theorem gives a direct way in finding the weakly unobservable subspace for a given singular system Σ. To prove it, we first need the following lemma.

Lemma 5.3.1 *If $x_0 \in \mathcal{V}^\star(\Sigma)$ and let u be an input function such that the corresponding output function satisfies $y_u(x_0,t) = 0$ for all $t \geq 0$. Then the associated state satisfies $x_u(x_0,t) \in \mathcal{V}^\star(\Sigma)$ for all $t \geq 0$.*

The proof of this lemma is similar to its counterpart in nonsingular systems which is found in the literature (see e.g. Trentelman *et al.* [72]).

Proof of Theorem 5.3.1:

Let $x_0 \in \mathcal{V}^\star(\sigma)$ and let u be such that $y_u(x_0, t) = 0$ for all $t \geq 0$. Since $\mathcal{V}^\star(\sigma)$ is a linear subspace of $\mathcal{X}$, we have

$$\dot{x}(0^+) := \lim_{t \to 0} \frac{x_u(x_0, t) - x_0}{t} \in \mathcal{V}^\star(\Sigma). \tag{5.4}$$

Now because $E\dot{x}(0^+) = Ax_0 + Bu_0$ and $Cx_0 + Du_0 = 0$, we see that for any given $x_0 \in \mathcal{V}^\star(\sigma)$, there exists a vector $u_0 \in \mathcal{U}$ such that $Ax_0 + Bu_0 \in E\mathcal{V}^\star(\Sigma)$ and $Cx_0 + Du_0 = 0$. Equivalently, the subspace $\mathcal{V} = \mathcal{V}^\star(\Sigma)$ satisfies

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subset \begin{bmatrix} E\mathcal{V} \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} B \\ D \end{bmatrix}. \tag{5.5}$$

And now let $\mathcal{V}$ be any subspace of $\mathcal{X}$ with the property (5.5). Choose a basis $x_1, \cdots, x_n$ for $\mathcal{X}$ such that $x_1, \cdots, x_r$ is a basis for $\mathcal{V}$ with $r \leq n$. And by (5.5), there are vectors $u_i \in \mathcal{U}$ such that for $i = 1, 2, \cdots, r$ we have $Ax_i + Bu_i \in E\mathcal{V}$ and $Cx_i + Du_i = 0$. Let $F : \mathcal{X} \to \mathcal{U}$ be any linear map such that $Fx_i = u_i$ for $i = 1, 2, \cdots, r$. Then we have $(A + BF)x_i \in E\mathcal{V}$ and $(C + DF)x_i = 0$. Since $x_1, \cdots, x_r$ is a basis of $\mathcal{V}$, we conclude that there exists a map $F : \mathcal{X} \to \mathcal{U}$ such that

$$(A + BF)\mathcal{V} \subset E\mathcal{V} \quad \text{and} \quad (C + DF)\mathcal{V} = 0. \tag{5.6}$$

Now we can use (5.5) and (5.6) to prove Theorem 5.3.1. First, we have already shown that $\mathcal{V} = \mathcal{V}^\star(\Sigma)$ satisfies (5.5). Now let $\mathcal{V}$ be an arbitrary subspace satisfies (5.5), according to the above, there exists an F such that (5.6) holds. Let $x_0 \in \mathcal{V}$ and apply the feedback control $u(t) = Fx(t)$. The resulting trajectory $x_u(x_0, t)$ then remains in $\mathcal{V}$ for all $t \geq 0$. Hence,

$$y_u(x_0, t) = (C + DF)x_u(x_0, t) = 0 \tag{5.7}$$

for all $t \geq 0$. This means that $x_0 \in \mathcal{V}^\star(\Sigma)$ and thus $\mathcal{V} \subset \mathcal{V}^\star(\Sigma)$.

On the other hand, we also already showed that there exists an F such that (5.6) holds with $\mathcal{V} = \mathcal{V}^\star(\Sigma)$. Now let $\mathcal{V}$ be any subspace such that (5.6) holds for some F. It can be seen immediately that $\mathcal{V}$ satisfies (5.5). According to what we have proved above, this implies that $\mathcal{V} \subset \mathcal{V}^\star(\Sigma)$.

In this way, we complete the proof for Theorem 5.3.1.

Similarly, for the strongly controllable subspace $\mathcal{S}^\star$ of the given singular system Σ, we have the following theorem,

Theorem 5.3.2 *$\mathcal{S}^\star(\Sigma)$ is the smallest subspace $\mathcal{S}$ of $\mathcal{X}$ for which there exists a constant matrix K such that*

$$(A+KC)\mathcal{S} \subset E\mathcal{S} \quad \text{and} \quad \mathrm{Im}(B+KD) \subset \mathcal{S}. \tag{5.8}$$

Again, its proof is similar to that for nonsingular systems.

The following lemmas show that the weakly unobservable subspace and strongly controllable subspace are invariant according to state feedback and output injection.

Lemma 5.3.2 *The weakly unobservable subspace $\mathcal{V}^\star(\Sigma)$ of a given singular system Σ is invariant under state feedback and output injection.*

Proof: Firstly, from Theorem 5.3.1, it is obviously invariant under any state feedback laws.

Next, for any subspace $\mathcal{V}$ that satisfies the following conditions:

$$(A+BF)\mathcal{V} \subset E\mathcal{V}, \quad \text{and} \quad (C+DF)\mathcal{V} = 0, \tag{5.9}$$

we have an output injection matrix K and

$$(A+(B+KD)F+KC)\mathcal{V}$$

$$
\begin{aligned}
&= (A+BF+K(C+DF))\mathcal{V} \\
&= (A+BF)\mathcal{V}+K(C+DF)\mathcal{V} \\
&= (A+BF)\mathcal{V} \subset E\mathcal{V}
\end{aligned} \tag{5.10}
$$

So $\mathcal{V}^\star$ is also invariant under any output injection laws. This complete the proof.

Lemma 5.3.3 *The strongly controllable subspace $\mathcal{S}^\star(\Sigma)$ of a given singular system Σ is invariant under state feedback and output injection.*

Similarly, the proof of this lemma can be derived accordingly.

5.4 Geometric Interpretation of Structural Decomposition

As its counterpart for nonsingular systems, our structural decomposition for singular systems also has the following geometrical interpretations.

Theorem 5.4.1 *Suppose that the state-space $\mathcal{X}$ is structurally decomposed into the following distinct subspaces,*

$$
\mathcal{X} = \mathcal{X}_{\mathrm{z}} \oplus \mathcal{X}_{\mathrm{e}} \oplus \mathcal{X}_{\mathrm{a}} \oplus \mathcal{X}_{\mathrm{b}} \oplus \mathcal{X}_{\mathrm{c}} \oplus \mathcal{X}_{\mathrm{d}}, \tag{5.11}
$$

we have

$$
\mathcal{V}^\star(\Sigma) = \mathrm{Im}\left\{\begin{bmatrix} 0 & 0 \\ N_{\mathrm{ea}} & N_{\mathrm{ec}} \\ I_{n_{\mathrm{a}}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\mathrm{c}}} \\ 0 & 0 \end{bmatrix}\right\}, \tag{5.12}
$$

and

$$\mathcal{S}^\star(\Sigma) = \mathrm{Im}\left\{\begin{bmatrix} 0 & 0 \\ N_{\mathrm{ec}} & N_{\mathrm{ed}} \\ 0 & 0 \\ 0 & 0 \\ I_{n_\mathrm{c}} & 0 \\ 0 & I_{n_\mathrm{d}} \end{bmatrix}\right\}, \tag{5.13}$$

where

$$\begin{aligned} N_{\mathrm{ea}} &= B_{0\mathrm{e}}C_{0\mathrm{a}} + B_{\mathrm{de}}M_{\mathrm{da}} + B_{\mathrm{ce}}M_{\mathrm{ca}}, \\ N_{\mathrm{ec}} &= B_{0\mathrm{e}}C_{0\mathrm{c}} + B_{\mathrm{de}}M_{\mathrm{dc}}, \\ N_{\mathrm{ed}} &= B_{0\mathrm{e}}C_{0\mathrm{d}} + B_{\mathrm{de}}M_{\mathrm{dd}}. \end{aligned} \tag{5.14}$$

Proof:

First, without loss of any generality, we assume throughout the rest of this section that the given system Σ has been transformed into the structural decomposition or into the compact form as follows,

$$E = \begin{bmatrix} J_{n_\mathrm{z}} & 0 & 0 & 0 & 0 & 0 \\ E_{\mathrm{ez}} & 0 & 0 & 0 & 0 & 0 \\ E_{\mathrm{az}} & 0 & I_{n_\mathrm{a}} & 0 & 0 & 0 \\ E_{\mathrm{bz}} & 0 & 0 & I_{n_\mathrm{b}} & 0 & 0 \\ E_{\mathrm{cz}} & 0 & 0 & 0 & I_{n_\mathrm{c}} & 0 \\ E_{\mathrm{dz}} & 0 & 0 & 0 & 0 & I_{n_\mathrm{d}} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ B_{0\mathrm{e}} & B_{\mathrm{de}} & B_{\mathrm{ce}} \\ B_{0\mathrm{a}} & 0 & 0 \\ B_{0\mathrm{b}} & 0 & 0 \\ B_{0\mathrm{c}} & 0 & B_{\mathrm{c}} \\ B_{0\mathrm{d}} & B_{\mathrm{d}} & 0 \end{bmatrix}, \tag{5.15}$$

$$A = \begin{bmatrix} I_{n_\mathrm{z}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_\mathrm{e}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{\mathrm{aa}} & L_{\mathrm{ab}}C_{\mathrm{b}} & 0 & L_{\mathrm{ad}}C_{\mathrm{d}} \\ 0 & 0 & 0 & A_{\mathrm{bb}} & 0 & L_{\mathrm{bd}}C_{\mathrm{d}} \\ 0 & 0 & B_{\mathrm{c}}M_{\mathrm{ca}} & L_{\mathrm{cb}}C_{\mathrm{b}} & A_{\mathrm{cc}} & L_{\mathrm{cd}}C_{\mathrm{d}} \\ 0 & 0 & B_{\mathrm{d}}M_{\mathrm{da}} & B_{\mathrm{d}}M_{\mathrm{db}} & B_{\mathrm{d}}M_{\mathrm{dc}} & A_{\mathrm{dd}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B_{0\mathrm{a}} \\ B_{0\mathrm{b}} \\ B_{0\mathrm{c}} \\ B_{0\mathrm{d}} \end{bmatrix} \begin{bmatrix} 0 & 0 & C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \end{bmatrix} \tag{5.16}$$

and

$$C = \begin{bmatrix} C_{0\mathrm{z}} & 0 & C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \\ C_{\mathrm{dz}} & 0 & 0 & 0 & 0 & C_{\mathrm{d}} \\ C_{\mathrm{bz}} & 0 & 0 & C_{\mathrm{b}} & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.17}$$

Now we begin to prove the invariant geometric subspace $\mathcal{V}^\star(\Sigma)$, i.e.,

$$\mathcal{V}^\star(\Sigma) = \mathrm{Im}\left\{\begin{bmatrix} 0 & 0 \\ N_{\mathrm{ea}} & N_{\mathrm{ec}} \\ I_{n_\mathrm{a}} & 0 \\ 0 & 0 \\ 0 & I_{n_\mathrm{c}} \\ 0 & 0 \end{bmatrix}\right\}. \tag{5.18}$$

Firstly, we need to prove that

$$\mathrm{Im}\left\{\begin{bmatrix} 0 & 0 \\ N_{\mathrm{ea}} & N_{\mathrm{ec}} \\ I_{n_\mathrm{a}} & 0 \\ 0 & 0 \\ 0 & I_{n_\mathrm{c}} \\ 0 & 0 \end{bmatrix}\right\} \subseteq \mathcal{V}^\star(\Sigma). \tag{5.19}$$

It follows from Lemma 5.3.2 that $\mathcal{V}^\star(\Sigma)$ is invariant under output injection laws. Thus first we can choose an output injection gain matrix K as

$$K = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_{0\mathrm{a}} & L_{\mathrm{ad}} & L_{\mathrm{ab}} \\ B_{0\mathrm{b}} & L_{\mathrm{bd}} & 0 \\ B_{0\mathrm{c}} & L_{\mathrm{cd}} & L_{\mathrm{cb}} \\ B_{0\mathrm{d}} & L_{\mathrm{dd}} & 0 \end{bmatrix}, \tag{5.20}$$

then we have

$$\hat{A} = A + KC = \begin{bmatrix} I_{n_z} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_e} & 0 & 0 & 0 & 0 \\ W_{az} & 0 & A_{aa} & 0 & 0 & 0 \\ W_{bz} & 0 & 0 & A_{bb} & 0 & 0 \\ W_{cz} & 0 & B_c M_{ca} & 0 & A_{cc} & 0 \\ W_{dz} & 0 & B_d M_{da} & B_d M_{db} & B_d M_{dc} & A^{\star}_{dd} + B_d M_{dd} \end{bmatrix}, \tag{5.21}$$

where W_{az}, W_{bz}, W_{cz} and W_{dz} are matrix blocks with less interest and

$$\hat{B} = B + KD = \begin{bmatrix} 0 & 0 & 0 \\ B_{0e} & B_{de} & B_{ce} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_c \\ 0 & B_d & 0 \end{bmatrix}. \tag{5.22}$$

Let $\hat{\Sigma}$ be a system characterized by $(\hat{A}, \hat{B}, C, D)$. Then it is sufficient to show the property of $\mathcal{V}^{\star}(\Sigma)$ by showing that

$$\mathcal{V}^{\star}(\hat{\Sigma}) = \mathrm{Im} \left\{ \begin{bmatrix} 0 & 0 \\ N_{ea} & N_{ec} \\ I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} \right\}. \tag{5.23}$$

First, let us choose a state feedback gain matrix F as

$$F = -\begin{bmatrix} 0 & 0 & C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & M_{da} & M_{db} & M_{dc} & M_{dd} \\ 0 & 0 & M_{ca} & 0 & 0 & 0 \end{bmatrix}, \tag{5.24}$$

then we have

$$\hat{A} + \hat{B}F = \begin{bmatrix} I_{n_z} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_e} & -N_{ea} & -N_{eb} & -N_{ec} & -N_{ed} \\ W_{az} & 0 & A_{aa} & 0 & 0 & 0 \\ W_{bz} & 0 & 0 & A_{bb} & 0 & 0 \\ W_{cz} & 0 & 0 & 0 & A_{cc} & 0 \\ W_{dz} & 0 & 0 & 0 & 0 & A^{\star}_{dd} \end{bmatrix}, \tag{5.25}$$

where

$$\begin{aligned} N_{ea} &= B_{0e}C_{0a} + B_{de}M_{da} + B_{ce}M_{ca}, \\ N_{eb} &= B_{0e}C_{0b} + B_{de}M_{db}, \\ N_{ec} &= B_{0e}C_{0c} + B_{de}M_{dc}, \\ N_{ed} &= B_{0e}C_{0d} + B_{de}M_{dd}. \end{aligned} \tag{5.26}$$

and

$$C + DF = \begin{bmatrix} C_{0z} & 0 & 0 & 0 & 0 & 0 \\ C_{dz} & 0 & 0 & 0 & 0 & C_{d} \\ C_{bz} & 0 & 0 & C_{b} & 0 & 0 \end{bmatrix}. \tag{5.27}$$

Now it is simple to see that for any

$$\zeta \in \operatorname{Im} \left\{ \begin{bmatrix} 0 & 0 \\ N_{ea} & N_{ec} \\ I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} \right\} \subset \mathcal{X}, \tag{5.28}$$

we have

$$\zeta = \begin{pmatrix} 0 \\ \zeta_{\mathrm{e}} \\ \zeta_{\mathrm{a}} \\ 0 \\ \zeta_{\mathrm{c}} \\ 0 \end{pmatrix}. \tag{5.29}$$

And since $\zeta \in \mathcal{X}$, it should satisfy

$$\hat{A}\zeta + \hat{B}F\zeta = E\dot{\zeta}, \tag{5.30}$$

which implies that

$$\zeta_{\mathrm{e}} = N_{\mathrm{ea}}\zeta_{\mathrm{a}} + N_{\mathrm{ec}}\zeta_{\mathrm{c}}, \tag{5.31}$$

and thus,

$$(\hat{A} + \hat{B}F) \begin{pmatrix} 0 \\ \zeta_{\mathrm{e}} \\ \zeta_{\mathrm{a}} \\ 0 \\ \zeta_{\mathrm{c}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A_{\mathrm{aa}}\zeta_{\mathrm{a}} \\ 0 \\ A_{\mathrm{cc}}\zeta_{\mathrm{c}} \\ 0 \end{pmatrix} \in E\mathrm{Im}\left\{ \begin{bmatrix} 0 & 0 \\ N_{\mathrm{ea}} & N_{\mathrm{ec}} \\ I_{n_{\mathrm{a}}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\mathrm{c}}} \\ 0 & 0 \end{bmatrix} \right\} = \mathrm{Im}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_{n_{\mathrm{a}}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\mathrm{c}}} \\ 0 & 0 \end{bmatrix} \right\}. \tag{5.32}$$

Moreover, we have

$$(C + DF) \begin{pmatrix} 0 \\ \zeta_{\mathrm{e}} \\ \zeta_{\mathrm{a}} \\ 0 \\ \zeta_{\mathrm{c}} \\ 0 \end{pmatrix} = 0. \tag{5.33}$$

Thus, by Theorem 5.3.1, we have

$$\mathrm{Im}\left\{\begin{bmatrix} 0 & 0 \\ N_{\mathrm{ea}} & N_{\mathrm{ec}} \\ I_{n_{\mathrm{a}}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\mathrm{c}}} \\ 0 & 0 \end{bmatrix}\right\} \subseteq \mathcal{V}^\star(\hat{\Sigma}). \tag{5.34}$$

Conversely, for any $\zeta \in \mathcal{V}^\star(\hat{\Sigma})$, by Theorem 5.3.1, there exists a gain matrix $\hat{F} \in \mathbb{R}^{m\times n}$ such that

$$(\hat{A} + \hat{B}\hat{F})\zeta \in E\mathcal{V}^\star(\hat{\Sigma}), \tag{5.35}$$

and

$$(C + D\hat{F})\zeta = 0. \tag{5.36}$$

Noting that $E\mathcal{V}^\star(\hat{\Sigma}) \subset \mathcal{V}^\star(\hat{\Sigma})$, we can get from (5.35) and (5.36) that, for any $\zeta \in \mathcal{V}^\star(\hat{\Sigma})$,

$$(C + D\hat{F})(\hat{A} + \hat{B}\hat{F})^k\zeta = 0, \quad k = 0, 1, \cdots, n-1. \tag{5.37}$$

Thus, (5.37) and (5.34) imply

$$(C + D\hat{F})(\hat{A} + \hat{B}\hat{F})^k \begin{bmatrix} 0 & 0 \\ N_{\mathrm{ea}} & N_{\mathrm{ec}} \\ I_{n_{\mathrm{a}}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\mathrm{c}}} \\ 0 & 0 \end{bmatrix} = 0, \quad k = 0, 1, \cdots, n-1. \tag{5.38}$$

Now if we partition $\hat{F}$ as follows:

$$\hat{F} = \begin{bmatrix} F_{\mathrm{z0}} & F_{\mathrm{e0}} & F_{\mathrm{a0}} - C_{\mathrm{0a}} & F_{\mathrm{b0}} - C_{\mathrm{0b}} & F_{\mathrm{c0}} - C_{\mathrm{0c}} & F_{\mathrm{d0}} - C_{\mathrm{0d}} \\ F_{\mathrm{zd}} & F_{\mathrm{ed}} & F_{\mathrm{ad}} - M_{\mathrm{da}} & F_{\mathrm{bd}} - M_{\mathrm{db}} & F_{\mathrm{cd}} - M_{\mathrm{dc}} & F_{\mathrm{dd}} - M_{\mathrm{dd}} \\ F_{\mathrm{zc}} & F_{\mathrm{ec}} & F_{\mathrm{ac}} - M_{\mathrm{ca}} & F_{\mathrm{bc}} & F_{\mathrm{cc}} & F_{\mathrm{dc}} \end{bmatrix}, \tag{5.39}$$

we will have

$$C + D\hat{F} = \begin{bmatrix} \star & F_{\mathrm{e0}} & F_{\mathrm{a0}} & F_{\mathrm{b0}} & F_{\mathrm{c0}} & F_{\mathrm{d0}} \\ C_{\mathrm{dz}} & 0 & 0 & 0 & 0 & C_{\mathrm{d}} \\ C_{\mathrm{bz}} & 0 & 0 & C_{\mathrm{b}} & 0 & 0 \end{bmatrix}, \tag{5.40}$$

and

$$\hat{A} + \hat{B}\hat{F} = \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star \\ \star & 0 & A_{\mathrm{aa}} & 0 & 0 & 0 \\ \star & 0 & 0 & A_{\mathrm{bb}} & 0 & 0 \\ \star & B_{\mathrm{c}}F_{\mathrm{ec}} & B_{\mathrm{c}}F_{\mathrm{ac}} & B_{\mathrm{c}}F_{\mathrm{bc}} & A_{\mathrm{cc}} + B_{\mathrm{c}}F_{\mathrm{cc}} & B_{\mathrm{c}}F_{\mathrm{dc}} \\ \star & B_{\mathrm{d}}F_{\mathrm{ed}} & B_{\mathrm{d}}F_{\mathrm{ad}} & B_{\mathrm{d}}F_{\mathrm{bd}} & B_{\mathrm{d}}F_{\mathrm{cd}} & A_{\mathrm{dd}}^{\star\star} \end{bmatrix}, \tag{5.41}$$

where $\star$s are some matrices of not much interest and $A_{\mathrm{dd}}^{\star\star} = A_{\mathrm{dd}}^{\star} + B_{\mathrm{d}}F_{\mathrm{dd}}$.

Then, using (5.38) with a positive integer k together with (5.40), we have

$$(C + D\hat{F})(\hat{A} + \hat{B}\hat{F})^k = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & 0 & 0 & \star & 0 & C_{\mathrm{d}}(A_{\mathrm{dd}}^{\star\star})^k \\ \star & 0 & 0 & C_{\mathrm{b}}(A_{\mathrm{bb}})^k & 0 & 0 \end{bmatrix} = 0. \tag{5.42}$$

Now, for any

$$\zeta = \begin{pmatrix} \zeta_{\mathrm{z}} \\ \zeta_{\mathrm{e}} \\ \zeta_{\mathrm{a}} \\ \zeta_{\mathrm{b}} \\ \zeta_{\mathrm{c}} \\ \zeta_{\mathrm{d}} \end{pmatrix} \in \mathcal{V}^{\star}(\hat{\Sigma}), \tag{5.43}$$

there exists

$$\begin{pmatrix} \zeta_{\mathrm{z}} \\ \zeta_{\mathrm{e}} \\ \zeta_{\mathrm{a}} \\ \zeta_{\mathrm{b}} \\ \zeta_{\mathrm{c}} \\ \zeta_{\mathrm{d}} \end{pmatrix} \in \mathrm{Im} \begin{pmatrix} 0 \\ x_{\mathrm{e}} \\ x_{\mathrm{a}} \\ x_{\mathrm{b}} \\ x_{\mathrm{c}} \\ x_{\mathrm{d}} \end{pmatrix}, \tag{5.44}$$

because $\mathcal{V}^\star(\hat{\Sigma})$ and $\mathcal{X}$ are in the same coordinate and $\mathcal{V}^\star(\hat{\Sigma}) \subset \mathcal{X}$, we thus have $\zeta_{\rm z} = 0$.

Furthermore, for any

$$\zeta = \begin{pmatrix} 0 \\ \zeta_{\rm e} \\ \zeta_{\rm a} \\ \zeta_{\rm b} \\ \zeta_{\rm c} \\ \zeta_{\rm d} \end{pmatrix} \in \mathcal{V}^\star(\hat{\Sigma}), \tag{5.45}$$

it follows from (5.38) and (5.42) that

$$C_{\rm b}(A_{\rm bb})^k \zeta_{\rm b} = 0, \quad k = 0, 1, \cdots, n-1, \tag{5.46}$$

which implies $\zeta_{\rm b} = 0$ because $(A_{\rm bb}, C_{\rm b})$ is completely observable, and

$$C_{\rm d}(A_{\rm dd}^{\star\star})^k \zeta_{\rm d} + \star \cdot \zeta_{\rm b} = C_{\rm d}(A_{\rm dd}^{\star\star})^k \zeta_{\rm d} = 0, \quad k = 0, 1, \cdots, n-1, \tag{5.47}$$

which implies $\zeta_{\rm d} = 0$ because $(A_{\rm dd}^{\star\star}, C_{\rm d})$ is also completely observable.

Thus, for any $\zeta \in \mathcal{V}^\star(\hat{\Sigma})$, there exists

$$\zeta \in \operatorname{Im} \left\{ \begin{bmatrix} 0 & 0 \\ N_{\rm ea} & N_{\rm ec} \\ I_{n_{\rm a}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\rm c}} \\ 0 & 0 \end{bmatrix} \right\}, \tag{5.48}$$

and we finally have

$$\mathcal{V}^\star(\hat{\Sigma}) \subseteq \operatorname{Im} \left\{ \begin{bmatrix} 0 & 0 \\ N_{\rm ea} & N_{\rm ec} \\ I_{n_{\rm a}} & 0 \\ 0 & 0 \\ 0 & I_{n_{\rm c}} \\ 0 & 0 \end{bmatrix} \right\}. \tag{5.49}$$

Obviously, (5.34) and (5.49) show the result. Similarly, one can follow the same procedure as in the above to show the property about strongly controllable subspace.

5.5 Conclusions

In this chapter, we introduced and defined important geometric subspaces for singular systems. Moreover, the internal relationship between these geometric subspaces and our structural decomposition was presented and proved. Such relationship shows that the structural decomposition technique can explicitly display the invariant geometric subspaces of a given singular system.

Chapter 6

Disturbance Decoupling of Singular Systems via State Feedback

6.1 Introduction

Among the problems of singular system and control, disturbance decoupling plays a crucial role in robust control of singular systems. Let's consider the following time-invariant continuous singular system,

$$\Sigma : \begin{cases} E\,\dot{x} = A\,x + B\,u + G\,w, \quad x(0) = x_0, \\ y = x, \\ h = C\,x, \end{cases} \tag{6.1}$$

where $\text{rank}(E) < n$ and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state, input and output of the system while $w \in \mathbb{R}^q$ is the system disturbance, which may represent noise or errors from measuring and modelling. And E, A, B, C and G are constant matrices of appropriate dimensions. As usual, in order to avoid any ambiguity in the solutions to the system, we assume throughout this paper that the given singular system Σ is regular, i.e., $\det(sE - A) \not\equiv 0$, for all $s \in \mathbb{C}$.

Now the disturbance decoupling problem of singular systems by state feedback is to find a constant state feedback matrix F such that the closed loop system is internally stable and the influence of disturbance to system output is eliminated. The problem of disturbance decoupling for singular systems was first formulated and solved by Fletcher and Aasaraai [34] but with an extra assumption that the output is independent of input disturbance in the sense that there is a set of admissible initial conditions such that the system's response is zero. After that, with geometric concepts of sliding and coasting subspaces, Banaszuk *et al.* [2] gave necessary and sufficient conditions for solving the disturbance decoupling problem of implicit discrete-time systems while Lebret [46] presented structural equivalent characterization to solve the same problem. Then Ailon [1] considered the standard disturbance decoupling problem for singular systems and formulated a solution in state space form. And more recently, Chu and Mehrmann [23] developed a numerically stable solution for the problem and Liu and Ho [54] designed a constructive method for the disturbance decoupling problem of linear time-varying singular systems. All of these researchers' work has improved the understanding for the disturbance decoupling problem and their solutions are effective under the accordingly conditions, however, to our best knowledge, such solutions are either cumbersome or too technical to be applied in the real controls.

This chapter intends to present a easy and clear solution for the disturbance decoupling problem of singular systems by state feedback. The solution is based on structural decomposition technique and a sufficient condition is proposed to guarantee the feasibility of such a solution. It should be pointed out that the solution presented in this chapter is far from complete in solving the disturbance decoupling problem since it needs a sufficient condition and the necessary condition is skipped. However, the objective of this chapter is to illustrate how the structural decomposition technique can be applied in solving singular system and control problems. Furthermore, the full solution including both necessary and sufficient conditions is on researching and the result will be completed soon.

Without loss of any generality, we also assume throughout this chapter that the matrix B in Σ is of full collum rank and C is of full row rank.

The organization of this chapter is as follows. Section 6.1 gives background and formulation of the disturbance decoupling problem for singular systems. To make this chapter more self-contained, some essential preliminary materials are included in Section 6.2. In Section 6.3, our main results are presented by a constructive algorithm for solving the disturbance decoupling problem and a sufficient condition is also proposed. And a numerical example is given in Section 6.4 to show the detail of the application of the structural decomposition. Finally, a concluding remark is drawn in Section 6.5.

6.2 Preliminary Materials

In this section, we briefly introduce necessary background materials for solving disturbance decoupling of singular systems by state feedback. First, the necessary and sufficient conditions for the existence of a state feedback matrix to get a regular and impulse free closed-loop system are remembered. To make this chapter more self-contained, we also include the necessary and sufficient conditions for the existence of a state feedback for disturbance decoupling. And further more, several essential definitions are also introduced in this chapter.

We first recall the following lemma on regularity by state feedback.

Lemma 6.2.1 (See [29]) For the system in (6.1), there exists a state feedback matrix

$F \in \mathbb{R}^{m\times n}$ such that $(E, A+BF)$ is regular if and only if

$$\operatorname{rank}\,[\, sE - A \quad B \,] = n. \tag{6.2}$$

□

Here is another lemma on both regularity and impulse free through state feedback.

Lemma 6.2.2 (See [7] [73]) For the system in (6.1), there exists a state feedback matrix $F \in \mathbb{R}^{m\times n}$ such that $(E, A+BF)$ is regular and impulse free if and only if

$$\operatorname{rank}\,[\, E \quad AS_\infty(E) \quad B \,] = n. \tag{6.3}$$

□

Here $S_\infty(M)$ denotes a matrix with orthogonal columns spanning the right null space of a given matrix M.

For the disturbance decoupling of singular systems by state feedback, many researchers have presented their solutions. Recently, Chu and Mehrmann [23] proposed their solution for this problem and presented the necessary and sufficient conditions. To give their solution here, we need some background materials as following.

Lemma 6.2.3 (See [30]) There exist orthogonal matrices U and V such that a matrix pencil (E, A) can be decomposed into the following form:

$$U(sE-A)V = \begin{bmatrix} sE_{11}-A_{11} & sE_{12}-A_{12} & sE_{13}-A_{13} & sE_{14}-A_{14} \\ 0 & sE_{22}-A_{22} & sE_{23}-A_{23} & sE_{24}-A_{24} \\ 0 & 0 & sE_{33}-A_{33} & sE_{34}-A_{34} \\ 0 & 0 & 0 & sE_{44}-A_{44} \end{bmatrix}. \tag{6.4}$$

Here $E_{11}, A_{11} \in \mathbb{R}^{n_1\times l_1}$, $E_{22}, A_{22} \in \mathbb{R}^{n_2\times l_2}$, and furthermore E_{22}, A_{22}, E_{33} and A_{33} are quare matrices. And $sE_{11}-A_{11}$ and $sE_{44}-A_{44}$ contain all left and right singular Kronecker blocks of $(sE - A)$ respectively. Moreover, $sE_{22} - A_{22}$ and $sE_{33} - A_{33}$ are regular and contain the finite and infinite structures of $(sE - A)$ respectively. □

Basing on the above lemma, we have the following definitions.

Definition 6.2.1 (see [30]) Given a matrix pencil (E, A) and orthogonal matrices U and V with $U(sE - A)V$ in the form of (6.4), we can define,

1. The left reducing subspace $V_{f-l}[E, A]$ corresponding to the finite spectrum of (E, A), is the space spanned by the leading $n_1 + n_2$ columns of U^T.

2. The right reducing subspace $V_{f-r}[E, A]$ corresponding to the finite spectrum of (E, A), is the space spanned by the leading $l_1 + l_2$ columns of V. □

Then some necessary computation can be conducted with the definitions introduced above.

$$
\begin{aligned}
\Pi &:= T_\infty(\begin{bmatrix} B & G \\ 0 & 0 \end{bmatrix}), \quad \Psi := T_\infty(G), \\
\Gamma_1 &:= \begin{bmatrix} 0 & \Psi^T E \\ 0 & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} \Psi^T B & \Psi^T A \\ 0 & 0 \end{bmatrix}, \\
\Lambda_r &:= V_{f-r}[\Pi^T \begin{bmatrix} E \\ 0 \end{bmatrix}, \Pi^T \begin{bmatrix} A \\ C \end{bmatrix}], \\
\Lambda_l &:= V_{f-l}[\Pi^T \begin{bmatrix} E \\ 0 \end{bmatrix}, \Pi^T \begin{bmatrix} A \\ C \end{bmatrix}], \\
\Lambda_t &:= [\Pi^T \quad \Pi] \begin{bmatrix} I & 0 \\ 0 & \Lambda_l \end{bmatrix}, \\
\Lambda_1 &:= \Lambda_t^T \begin{bmatrix} E \\ 0 \end{bmatrix} \Lambda_r, \quad \Lambda_2 := \Lambda_t^T \begin{bmatrix} A \\ C \end{bmatrix} \Lambda_r, \\
\Lambda_3 &:= \Lambda_t^T \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \Lambda_4 := (V_{f-l}^\perp[\Gamma_1, \Gamma_2])^T \Gamma_1 V_{f-r}^\perp[\Gamma_1, \Gamma_2],
\end{aligned} \tag{6.5}
$$

where $T_\infty(M)$ denotes a matrix with orthogonal columns spanning the right null space of M^T.

Furthermore, the following computations are also necessary,

$$\begin{aligned}
\tau &:= \dim(V_{f-l}[\begin{bmatrix} 0 & \Psi^T E \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \Psi^T B & \Psi^T A \\ 0 & C \end{bmatrix}]), \\
\mu &:= \dim(V_{f-r}^{\perp}[\begin{bmatrix} \Psi^T B & \Psi^T E \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Psi^T A \\ 0 & C \end{bmatrix}]), \\
\eta &:= \dim(V_{f-l}[\begin{bmatrix} \Psi^T B & \Psi^T E \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Psi^T A \\ 0 & C \end{bmatrix}]).
\end{aligned} \tag{6.6}$$

Now we can remember the following lemma on the necessary and sufficient conditions for the disturbance decoupling problem of singular systems by state feedback.

Lemma 6.2.4 (see [23]) For the singular system in (6.1) with disturbance, there exists a state feedback matrix F such that the disturbance is decoupled if and only if the following conditions are satisfied,

1. $\operatorname{rank}[\,E \quad AS_\infty(E) \quad B\,] = n$;
2. $\tau + \mu \le n - p$;
3. $\operatorname{rank}(\Lambda_1) + \operatorname{rank}(\Lambda_4) = \operatorname{rank}(E)$;
4. $\operatorname{rank}[\,\Lambda_1 \quad \Lambda_2 S_\infty(\Lambda_1) \quad \Lambda_3\,] = p + \tau + \eta$. □

Now we have finished introducing the necessary and sufficient conditions for disturbance decoupling of singular systems by state feedback. Chu and Mehrmann [23] have also presented an algorithm for computing the state feedback matrix F, however, the process is too complex and is not transparent in physical meaning. In the next section, we will find that our structural decomposition technique can be used to find the state feedback matrix F in a clear way. The detailed algorithm will be presented in the following section and a numerical example will be given in the section after that.

6.3 A Constructive Solution for the Disturbance Decoupling of Singular Systems

In this section, we try to give a novel solution for the disturbance decoupling of singular systems. Such a solution uses the structural decomposition technique and need a special sufficient condition. Although this kind of solution is far from complete due to the sufficient condition may be more rigorous than that in Lemma 6.2.4 and the necessary condition is skipped. However, such a solution is a good example in illustrating how the structural decomposition technique can be used in solving such problems. And what is more, the full necessary and sufficient conditions for the disturbance decoupling of singular systems by the structural decomposition technique still remains in our research topics and the results will be completed soon.

To illustrate how the structural decomposition technique can be applied in solving linear system and control problems, we present a constructive algorithm in solving the disturbance decoupling problem step by step. And during this process, a sufficient condition is proposed to guarantee the feasibility of such a solution.

Step 1:

For the singular system Σ with disturbance, according to the structural decomposition technique in Chapter 4, there exist transform matrices $\Gamma_{\rm e}$ and $\Gamma_{\rm s}$ such that Σ can be coordinately transformed into the following form,

$$
\begin{aligned}
\hat{E} &= \Gamma_{\mathrm{e1}}^{-1} E \Gamma_{\mathrm{s1}} = \begin{bmatrix} J_{n_{\mathrm{z}}} & 0 & 0 \\ E_{\mathrm{ez}} & 0 & 0 \\ E_{\mathrm{fz}} & 0 & I_{n_{\mathrm{f}}} \end{bmatrix}, \quad \hat{A} = \Gamma_{\mathrm{e1}}^{-1} A \Gamma_{\mathrm{s1}} = \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 \\ 0 & 0 & A_{\mathrm{g}} \\ 0 & A_{\mathrm{e}} & A_{\star} \end{bmatrix}, \\
\hat{B} &= \Gamma_{\mathrm{e1}}^{-1} B = \begin{bmatrix} 0 & 0 \\ I_{n_{\mathrm{e}}} & 0 \\ 0 & B_{\star} \end{bmatrix}, \quad \hat{G} = \Gamma_{\mathrm{e1}}^{-1} G = \begin{bmatrix} G_{\mathrm{z}} \\ G_{\mathrm{e}} \\ G_{\mathrm{f}} \end{bmatrix}, \\
\hat{C} &= C \Gamma_{\mathrm{s1}} = [\, C_{\mathrm{z}} \quad C_{\mathrm{e}} \quad C_{\star} \,],
\end{aligned} \tag{6.7}
$$

where E_{ez}, E_{fz}, A_{g}, A_{e}, $A_{\star}$, $B_{\star}$, G_{z}, G_{e}, G_{f}, C_{z}, C_{e} and $C_{\star}$ are matrices with appropriate dimensions. And the transformation matrices can be computed by the following manipulations.

$$
\Gamma_{\mathrm{e1}} = U \begin{bmatrix} I & 0 \\ 0 & T_{\mathrm{s}}^{-1} \end{bmatrix} P, \quad \Gamma_{\mathrm{s1}} = Q \begin{bmatrix} I & 0 \\ 0 & T_{\mathrm{s}} \end{bmatrix} V, \tag{6.8}
$$

where the transformation matrices can be found in the constructive decomposition procedure in Chapter 4.

Step 2:

Now for the transformed system in (6.7), we can find a constant state feedback matrix F as in the following,

$$
F = \begin{bmatrix} 0 & I_{n_{\mathrm{e}}} & -A_{\mathrm{g}} \\ 0 & -F_{\mathrm{e}} & F_{\star} \end{bmatrix}, \tag{6.9}
$$

where $B_{\star} F_{\mathrm{e}} = A_{\mathrm{e}}$. And it can be proved that such a F_{e} is existent because B is full collum rank and thus $B_{\star}$ is full collum rank, and furthermore, from the decomposition process in Chapter 4, we can get that A_{e} is also full collum rank and every collum of A_{e} has only one non-zero element.

Then the feedback matrix F further change the system in (6.7) into,

$$
\hat{E} = \begin{bmatrix} J_{n_{\mathrm{z}}} & 0 & 0 \\ E_{\mathrm{ez}} & 0 & 0 \\ E_{\star} & 0 & I_{n_{\mathrm{f}}} \end{bmatrix}, \quad \hat{A} + \hat{B}F = \begin{bmatrix} I_{n_{\mathrm{z}}} & 0 & 0 \\ 0 & I_{n_{\mathrm{e}}} & 0 \\ 0 & 0 & A_{\star} + B_{\star} F_{\star} \end{bmatrix},
$$

$$
\begin{aligned}
\hat{B} &= \Gamma_{\rm e}^{-1}B = \begin{bmatrix} 0 & 0 \\ I_{n_{\rm e}} & 0 \\ 0 & B_\star \end{bmatrix}, \quad \hat{G} = \Gamma_{\rm e}^{-1}G = \begin{bmatrix} G_{\rm z} \\ G_{\rm e} \\ G_{\rm f} \end{bmatrix}, \\
\hat{C} &= C\Gamma_{\rm s} = [\, C_{\rm z} \quad C_{\rm e} \quad C_{\rm f} \,].
\end{aligned} \tag{6.10}
$$

Step 3:

Now if we set

$$
H_\star = sI - A_\star - B_\star F_\star, \quad H_{\rm k} = \begin{bmatrix} sJ - I & 0 \\ sE_{\rm ez} & -I_{n_{\rm e}} \end{bmatrix}, \tag{6.11}
$$

and

$$
C_{\rm k} = [\, C_{\rm z} \quad C_{\rm e} \,], \qquad G_{\rm k} = \begin{bmatrix} G_{\rm z} \\ G_{\rm e} \end{bmatrix}, \tag{6.12}
$$

then according to the formulas on inverse of block matrix, it is simple to compute that the transfer function from disturbance to output is

$$
T_{\rm wh} = C_{\rm k}H_{\rm k}^{-1}G_{\rm k} - C_\star H_\star^{-1} [\, sE_\star \quad 0 \,] H_{\rm k}^{-1}G_{\rm k} + C_\star H_\star^{-1} G_\star. \tag{6.13}
$$

Step 4:

Now it can be seen clearly from (6.11) and (6.12) that

$$
\begin{aligned}
C_{\rm k}H_{\rm k}^{-1}G_{\rm k} &= [\, C_{\rm z} \quad C_{\rm e} \,] \begin{bmatrix} (sJ-I)^{-1} & 0 \\ sE_{\rm ez}(sJ-I)^{-1} & -I \end{bmatrix} \begin{bmatrix} G_{\rm z} \\ G_{\rm e} \end{bmatrix} \\
&= C_{\rm z}(sJ-I)^{-1}G_{\rm z} + C_{\rm e}sE_{\rm ez}(sJ-I)^{-1}G_{\rm z} - C_{\rm e}G_{\rm e}, \\
C_\star H_\star^{-1} [\, sE_\star \quad 0 \,] H_{\rm k}^{-1}G_{\rm k} &= C_\star H_\star^{-1} [\, sE_\star \quad 0 \,] \begin{bmatrix} G_{\rm z} \\ G_{\rm e} \end{bmatrix} \\
&= C_\star(sI - A_\star - B_\star F_\star)^{-1} sE_\star (sJ-I)^{-1}G_{\rm z}, \\
C_\star H_\star^{-1} G_\star &= C_\star(sI - A_\star - B_\star F_\star)^{-1}G_\star.
\end{aligned} \tag{6.14}
$$

Thus we have,

$$
\begin{aligned}
T_{\rm wh} &= C_{\rm k}H_{\rm k}^{-1}G_{\rm k} - C_\star H_\star^{-1} [\, sE_\star \quad 0 \,] H_{\rm k}^{-1}G_{\rm k} + C_\star H_\star^{-1} G_\star \\
&= [C_{\rm e}sE_{\rm ez}(sJ-I)^{-1}G_{\rm z}] + [C_{\rm z}(sJ-I)^{-1}G_{\rm z} - C_{\rm e}G_{\rm e}
\end{aligned}
$$

$$
\begin{aligned}
& - \quad C_\star(sI - A_\star - B_\star F_\star)^{-1} sE_\star(sJ - I)^{-1}G_{\mathrm{z}}] + [C_\star(sI - A_\star - B_\star F_\star)^{-1}G_\star] \\
& = \quad T_\mu + T_\star + C_\star H_\star^{-1} G_\star. \qquad (6.15)
\end{aligned}
$$

where

$$
\begin{aligned}
T_\mu &= C_{\mathrm{e}} s E_{\mathrm{ez}} (sJ - I)^{-1} G_{\mathrm{z}}, \\
T_\star &= C_{\mathrm{z}}(sJ - I)^{-1}G_{\mathrm{z}} - C_{\mathrm{e}}G_{\mathrm{e}} - C_\star(sI - A_\star - B_\star F_\star)^{-1} sE_\star(sJ - I)^{-1}G_{\mathrm{z}}. \qquad (6.16)
\end{aligned}
$$

Step 5:

Now it is clear that all entities of T_μ are strictly non-proper polynomials while all entities of $C_\star H_\star^{-1} G_\star$ are strictly proper polynomials, and all entities of $T_\star$ are polynomials with orders different from those in T_μ and $C_\star H_\star^{-1} G_\star$.

Thus a sufficient condition for $T_{\mathrm{wh}} = 0$ is,

$$
T_\mu = 0, \quad T_\star = 0, \quad C_\star H_\star^{-1} G_\star = 0. \qquad (6.17)
$$

And if we want the closed-loop system to be internally stable, the non-proper entities should not exist in final system. Furthermore, for the nonsingular system $C_\star H_\star^{-1} G_\star$, according to the structural decomposition technique presented in Chapter 2, we can find $\Gamma_{\star \mathrm{s}}$, $\Gamma_{\star \mathrm{i}}$ and $\Gamma_{\star \mathrm{o}}$ such that it is decomposed in the structural decomposition form.

And if we set,

$$
\Gamma_{\star \mathrm{s}}^{-1} G_\star = \begin{bmatrix} G_{\star \mathrm{a}^-} \\ G_{\star \mathrm{a}^0} \\ G_{\star \mathrm{a}^+} \\ G_{\star \mathrm{b}} \\ G_{\star \mathrm{c}} \\ G_{\star \mathrm{d}} \end{bmatrix}, \qquad (6.18)
$$

then the sufficient condition will be,

Lemma 6.3.1 *For the given singular system Σ with disturbance, its disturbance can be decoupled if the following conditions are satisfied,*

1. $(A_\star, B_\star)$ *is stabilizable;*
2. $(A_\star, C_\star)$ *is detectable;*
3. $E_{\mathrm{ez}} = 0$, $E_\star = 0$ *and* $C_{\mathrm{z}}(sJ - I)^{-1}G_{\mathrm{z}} = C_{\mathrm{e}}G_{\mathrm{e}}$*;*
4. $G_{\star \mathrm{a}^+} = 0$, $G_{\star \mathrm{a}^0} = 0$, $G_{\star \mathrm{b}} = 0$ *and* $G_{\star \mathrm{d}} = 0$.

This Lemma is obvious according to the above decomposition procedures and the results for nonsingular systems [20].

Step 6:

This last step is to finish constructing the feedback matrix F, the only part left in (6.9) is $F_\star$, which is the feedback matrix for nonsingular system $(A_\star, B_\star, C_\star, G_\star)$. There have been many research results in this in the literature [24] [33] [28] and it is easy to design such a $F_\star$.

6.4 An Example

In this section, we give a numerical example to show the detail process of applying the structural decomposition technique in solving disturbance decoupling problems.

Let us consider the following singular system with disturbance,

$$E = \begin{bmatrix} -0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0 & 0 & 1 \\ -0.5 & 1.5 & 0.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad C = [\,-0.5 \quad 1.5 \quad -0.5\,]. \tag{6.19}$$

Now according to the structural decomposition technique and the constructive decomposition algorithm presented in Chapter 4, we can find the such two invertible transformation matrices,

$$\Gamma_{\rm e1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \Gamma_{\rm s1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \tag{6.20}$$

such that the given system in (6.19) can be decomposed into,

$$\begin{aligned} \hat{E} &= \Gamma_{\rm e1}^{-1} E \Gamma_{\rm s1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \hat{A} &= \Gamma_{\rm e1}^{-1} A \Gamma_{\rm s1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \\ \hat{B} &= \Gamma_{\rm e1}^{-1} B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{G} = \Gamma_{\rm e1}^{-1} G = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\ \hat{C} &= C \Gamma_{\rm s1} = [\,-1 \quad 1 \quad 1\,]. \end{aligned} \tag{6.21}$$

Now we can check the sufficient condition in Lemma 6.3.1. It can be easily seen that

$$\begin{aligned} A_\star &= 1, \;\; B_\star = 1, \;\; C_\star = 1, \;\; G_\star = 0, \\ C_{\rm z} &= -1, \;\; G_{\rm z} = 1, \;\; C_{\rm e} = 1, \;\; G_{\rm e} = 1. \end{aligned} \tag{6.22}$$

Hence the sufficient condition in Lemma 6.3.1 is satisfied. Thus we can find the following state feedback matrix according to (6.9),

$$F = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & -2 \end{bmatrix}. \tag{6.23}$$

Furthermore, we can compute $T_{\rm wh}$ by the following manipulations.

$$\begin{aligned} T_{\rm wh} &= \hat{C}\,[\,s\hat{E}-\hat{A}-\hat{B}F\,]\,\hat{G} \\ &= [\,-1 \quad 1 \quad 1\,]\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= [\,-1 \quad 1 \quad 1\,]\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{s+1} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= 0. \end{aligned} \tag{6.24}$$

It is clear that with such a F, the influence of the disturbance is eliminated and the closed-loop system is also internally stable.

This simple example illustrates in detail the process of applying structural decomposition technique in solving disturbance decoupling problem of singular systems. It is obvious that such a technique may be powerful in solving disturbance decoupling problems.

6.5 Conclusions

In this chapter, we give an example on how to apply the structural decomposition technique in solving singular system and control problems. The structural decomposition technique for disturbance decoupling of singular systems has been illustrated and a sufficient condition is given. Although it is still a great distance to a complete solution, the sufficient condition and the state feedback matrix show in detail the procedures in how to analyze the problem using the structural decomposition technique. The complete solution by the structural decomposition technique is still on going and it will further be used to solve many other problems such as all most disturbance decoupling, H_2 and H_∞ optimal control of singular systems, to name just a few.

Chapter 7

Conclusions and Future Work

This thesis presents a structural decomposition technique for linear singular systems and its applications in solving system and control problems. The main focus is the structural decomposition theorem and the constructive algorithm for the decomposition.

Firstly, the structural decomposition for single-input single-output (SISO) linear singular systems is proposed. The results show that there are two situations, one is with x_{d} and the other is with y_0. However, both situations can explicitly display the internal structural properties of the given linear singular system. Such structural properties include invariant zero structure, infinite zero structure, stabilizability and detectability. To illustrate the decomposition process, a numerical example is given and all properties are verified after the decomposition.

The results of structural decomposition for SISO linear singular system provide a clearer view for that of multi-input multi-output (MIMO) linear singular system. But the MIMO case is much more complex largely because it has more system inputs. To prove our structural decomposition theorem for MIMO linear singular systems, a constructive algorithm is deduced. This algorithm gives details of decomposition process and its MATLAB codes have been attached in the appendix for reference. As its counterpart for linear nonsingular systems, the structural decomposition technique we presented for MIMO linear singular systems can explicitly display all structure properties such like invariant zero structure, infinite zero structure, invertibility structure, stabilizability and detectability of the given linear singular system. Furthermore, it can also clearly reveals the redundant states of the given system, such redundant states are either static and equal to zero or are linear combinations of system inputs and their derivatives in different orders.

Moreover, we define several invariant geometric subspaces and use them to find the relationship between these subspaces and the structurally decomposed subspaces. It has been showed that our structural decomposition technique can also explicitly display the invariant geometric subspaces of the given linear singular system.

To show the potential applications of the structural decomposition technique in solving linear singular system and control problems, we use it to solve disturbance decoupling problem of linear singular systems. A sufficient condition is presented after decomposing the given system into several subspaces. Although the solution is not complete because it has no necessary condition, it shows in detail how the structural decomposition technique can be used in solving practical questions.

The structural decomposition technique has been widely used in solving linear nonsingular system and control problems. Our future work includes applying this technique to solve linear singular system and control problems, such as H_2 optimal control, H_∞ control, almost disturbance decoupling and etc. Moreover, its application for nonlinear systems will also be studied in the future.

Bibliography

[1] A. Ailon, "A solution to the disturbance decoupling problem in singular systems via analogy with state space systems," *Automatica*, Vol. 29, pp.1541-1545, 1993.

[2] A. Banaszuk, M. Kociecki and K. A. Przyluski, "The disturbance decoupling problem for implicit linear descrete-time systems," *SIAM Journal on Control Optimization*, Vol. 28, pp.1270-1293, 1990.

[3] G. Basile and G. Marro, "Luoghi Caratteristici dello spazio degli stati relativi al controllo dei sistemi lineari," *L'Elettrotecnica*, Vol. 55, No. 12, pp.1-7, 1968.

[4] G. Basile and G. Marro, "Controlled and conditioned invariant subspaces in linear system theory," *Technical Report Number: AM-68-7, Div. of Appl. Mech., College of Engineering, University of California, Berkeley*, June-July, 1968.

[5] G. Basile and G. Marro, "Controlled and conditioned invariant subspaces in linear system theory," *Journal of Optimization Theory and Applications*, Vol 3, No. 5, pp.306-315, 1969.

[6] P. Brunovsky, "A classification of linear controllable systems," *Kybernetika (Praha)*, Vol. 3, pp.173-187, 1970.

[7] A. Bunsegerstner, V. Mehrmann and N. K. Nichols, "Regularization of descriptor systems by derivative and proportional state feedback," *SIAM Journal on Matrix Analysis and Applications*, Vol. 13, pp.46-67, 1992.

[8] S. L. Campbell, *Singular Systems of Differential Equations II*, Pitman, New York, 1982.

[9] B. M. Chen, *Software Manual for the Special Coordinate Basis of Multivariable Linear Systems*, Washington Sate University Technical Report Number: ECE 0094, Pullman, Washington, 1988.

[10] B. M. Chen, *Theory of Loop Transfer Recovery for Multivariable Linear Systems*, Ph.D. Dissertation, Washington State University, 1991.

[11] B.M. Chen, A. Saberi and U. Ly, "Exact computation of infimum in H_∞-optimization via output feedback," *IEEE Transactions on Automatic Control*, Vol. 37, pp.70-78, 1992.

[12] B. M. Chen, A. Saberi and P. Sannuti, "On blocking zeros and strong stabilizability of linear multivariable systems," *Automatica*, Vol. 28, No. 5, pp.1051-1055, 1992.

[13] B. M. Chen, A. Saberi, P. Sannuti and Y. Shamash, "Construction and parameterization of all static and dynamic -optimal state feedback solutions, optimal fixed modes and fixed decoupling zeros," *IEEE Transactions on Automatic Control*, Vol. 38, pp.248-261, 1993.

[14] B. M. Chen, A. Saberi, Y. Shamash and P. Sannuti, "Construction and parameterization of all static and dynamic H_2-optimal state feedback solutions for discrete time systems," *Automatica*, Vol. 30, No. 10, pp.1617-1624, 1994.

[15] B. M. Chen and D. Z. Zheng, "Simultaneous finite- and infinite-zero assignments of linear systems," *Automatica*, Vol. 31, pp.643-648, 1995.

[16] B. M. Chen and Y.-L. Chen, "Loop transfer recovery design via new observer based and CSS architecture based controllers," *International Journal of Robust & Nonlinear Control*, Vol. 5, pp.649-669, 1995.

[17] B. M. Chen, *Linear Systems and Control Toolbox*, Technical Report, Department of Electrical and Computer Engineering, National University of Singapore, 1997.

[18] B. M. Chen, “Solvability conditions for the disturbance decoupling problems with static measurement feedback,” *International Journal of Control*, Vol. 68, No. 1, pp.51-60, 1997.

[19] B. M. Chen, “On properties of the special coordinate basis of linear systems,” *International Journal of Control*, Vol. 71, pp.981-1003, 1998.

[20] B. M. Chen, *Robust and H_∞ Control*, Springer, London, 2000.

[21] D. Chu, H. C. Chan and D. W. C. Ho, ”Regularization of singular systems by derivative and proportional output feedback,” *SIAM Journal on Matrix Analysis and Applications*, Vol. 19, No. 1, pp.21-38, 1998.

[22] D. Chu, V. Mehrmann and N. K. Nichols, ”Minimum norm regularization of descriptor systems by mixed output feedback,” *Linear Algebra and Its Applications*, Vol. 296, pp.39-77, 1999.

[23] D. Chu and V. Mehrmann, ”Disturbance decoupling for descriptor systems by state feedback,” *SIAM Journal on Control Optimization*, Vol. 38, No. 6, pp.1830-1858, 2000.

[24] D. Chu and V. Mehrmann, ”Disturbance decoupling for linear time-invariant systems: a matrix pencil approach,” *IEEE Transactions on Automatic Control*, Vol. 46, No. 5, pp.802-808, 2001

[25] Delin Chu, Xinmin. Liu and Roger C. E. Tan, “On the numerical computation of a structural decomposition in systems and control ,” *IEEE Transactions on Automatic Control*. Vol. 47, pp.1786- 1799, Nov 2002.

[26] J. D. Cobb, “COntrollability, observability and duality in singular systems,” *IEEE Transactions on Automatic Control*. Vol. 29, No. 12, pp.1076-1082, 1984.

[27] C. Commault and J. M. Dion, “Structure at infinity of linear multivariable systems: a geometric approach,” *IEEE Transactions on Automatic Control*, Vol. 27, pp.693-696, 1982.

[28] C. Commault, J. M. Dion and V. Hovelaque, "A geometric approach for structured systems: Application to disturbance decoupling," *Automatica* Vol. 33, No. 3, pp.403-409, 1997

[29] L. Dai, *Singular Control System*, Springer-Verlag, Berlin, 1989.

[30] J. Demmel and B. Kagstrom, "The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: Robust software with error bounds and applications. Part I: Theory and algorithms," *ACM Transactions on Mathematical Software*, Vol. 19, pp.160-174, 1993.

[31] P. V. Dooren, "The generalized eigenstructure problem in linear system theory," *IEEE Transactions on Automatic Control*, Vol. 26, pp.111-129, 1981.

[32] P. V. Dooren, "The eigenstructure of an arbitrary polynomial matrix: Computational aspects," *Linear Algebra and its Applications*, Vol. 50, pp.545-579, 1983.

[33] C. Dorea and B. Milani, "Disturbance decoupling in a class of linear systems," *IEEE Transactions on Automatic Control* Vol. 42, No. 10, pp.1427-1431, 1997

[34] L. R. Fletcher and A. Asaraai, "On disturbance decoupling in descriptor systems," *SIAM Journal on Control Optimization*, Vol. 27, pp.1319-1332, 1989.

[35] F. R. Gantmacher, *Theory of Matrices*, Chelsea, New York, 1959.

[36] T. Geerts, "Invariant subspaces and invertibility properties for singular systems: The general case," *Linear Algebra and Its Applications*, Vol. 183, pp.61-68, 1993.

[37] E. G. Gilbert, "Controllability and observability in multivariable systems," *J. SIAM Control*, Ser. A, 2, pp.128-151, 1963.

[38] G. E. Hayton, P. Fretwell and A. C. Pugh, "Fundamental equivalence of generalized state space systems," *IEEE Transactions on Automatic Control*, Vol. 31, No. 5, pp.431-439, 1986.

[39] M. He and B. M. Chen, "Structural decomposition of linear singular systems: The single-input and single-output case," *Systems and Control Letters*, Vol. 47, pp.327-334, 2002.

[40] M. He, B. M. Chen and Z. Lin, "Structural decomposition of general linear multivariable singular systems," *Submitted for publication*.

[41] R. E. Kalman, "On the general theory of control systems," *Proceedings of the First IFAC Congress*, Vol. 1, pp.481-491, Moscow, 1960.

[42] R. E. Kalman, Y. C. Ho and K. S. Navendra, "Controllability of linear dynamical systems," *Contributions to the Theory of Differential Equations*, Vol. 1, pp.189-213, Miley-Interscience, New York, 1963.

[43] P. V. Kokotovic, Jr. R. E. O'Malley and P. Sannuti, "Singular perturbations and order reduction in control theory - An overview," *Automatica*, Vol. 12, pp.123-132, 1976.

[44] L. Kronecker, "Algebraishe reduction der schaaren bilinearer formen," *S. B. Akad. Berlin*, pp.763-776, 1890.

[45] M. Kuijper, *First Order Representation of Linear Systems*, Birkhauser, Boston, 1994.

[46] G. Lebret, "Structural solution of the disturbance decoupling problem for implicit descrete-time systems," *Proceedings of the Second International Symposium on Implicit and Robust Systems* , Warsaw, 1991, pp.127-130.

[47] F. L. Lewis, "A survey of linear singular systems," *Circuits, Systems, and Signal Processing*, Vol. 5, No. 1, pp.3-36, 1986.

[48] F. L. Lewis and K. Ozcaldiran, "Geometric structures and feedback in singular systems," *IEEE Transactions on Automatic Control*, Vol. 34, No. 4, pp.450-455, 1989.

[49] Z. Lin, *The Implementation of Special Coordinate Basis for Linear Systems in MATLAB*, Washington Sate University Technical Report Number: ECE 0100, Pullman, Washington, 1989.

[50] Z. Lin, *Global and Semi-global Control Problems for Linear Systems Subject to Input Saturation and Minimum-Phase Input-Output Linearizable Systems*, Ph.D. Dissertation, Washington State University, 1994.

[51] Z. Lin, B. M. Chen, A. Saberi and Y. Shamash, "Input-output factorization of discrete-time transfer matrices," *IEEE Transactions on Circuits and Systems — I: Fundamental Theory and Applications*, Vol. 43, No. 11, pp.941-945, 1996.

[52] Z. Lin Z and B. M. Chen, *Linear Systems and Control Toolbox*, Technical Report, Department of Electrical and Computer Engineering, University of Virginia, Charlottesville, Virginia, USA, 2000.

[53] X. Liu, B. M. Chen and Z. Lin, "Computation of structural invariants of singular linear systems," *Submitted for publication*

[54] X. Liu and D. W. C. Ho, "Disturbance decoupling of linear time-varying singular systems," *IEEE Transactions on Automatic Control*, Vol. 47, No. 2, pp.335-341, 2002.

[55] J. J. Loiseau, "Some geometric considerations about the Kronecker normal form," *International Journal of Control*, Vol. 42, pp.1411-1431, 1985.

[56] D. G. Luenberger, "Canonical forms for linear multivariable systems," *IEEE Transactions on Automatic Control*, Vol. 12, pp.290-293, 1967.

[57] A. G. J. MacFarlane and N. Karcanias, "Poles and zeros of linear multivariable systems: A survey of the algebraic, geometric and complex variable theory," *International Journal of Control*, Vol. 24, pp.33-74, 1976.

[58] M. Malabre, "Generalized linear systems: Geometric and structural approaches," *Linear Algebra and its Applications*, Vol. 122-123, pp.591-621, 1989.

[59] P. Misra, P. V. Dooren and A. Varga, "Computation of structural invariants of generalized state-space systems," *Automatica*, Vol. 30, pp.1921-1936, 1994.

[60] A. S. Morse, "Structural invariants of linear multivariable systems," *SIAM Journal on Control and Optimization*, Vol. 11 , pp.446-465, 1973.

[61] P. Moylan, "Stable inversion of linear systems," *IEEE Transactions on Automatic Control*, Vol. 22, pp.74-78, 1977.

[62] D. H. Owens, "Invariant zeros of multivariable systems: A geometric analysis," *International Journal of Control*, Vol. 28, pp.187-198, 1978.

[63] H. K. Ozcetin, A. Saberi and P. Sannuti, "Design for H_∞ almost disturbance decoupling problem with internal stability via state or measurement feedback – singular perturbation approach," *International Journal of Control*, Vol. 55, No. 4, pp.901-944, 1993.

[64] H. K. Ozcetin, A. Saberi and Y. Shamash, "H_∞-almost disturbance decoupling for non-strictly proper systems–A singular perturbation approach," *Control–Theory & Advanced Technology*, Vol. 9, pp.203-245, 1993.

[65] A. C. Pugh and P. A. Ratcliffe, "On the zeros and poles of a rational matrix," *International Journal of Control*, Vol. 30, pp.213-227, 1979.

[66] H. H. Rosenbrock, *State-space and Multivariable Theory*, John-Wiley, New York, 1970.

[67] A. Saberi and P. Sannuti, "Squaring down of non-strictly proper systems," *International Journal of Control*, Vol. 51, pp.621-629, 1990.

[68] A. Saberi, B. M. Chen and P. Sannuti, *Loop Transfer Recovery: Analysis and Design*, Springer-Verlag, London, 1993.

[69] A. Saberi, P. Sannuti and B. M. Chen, *H_2 Optimal Control*, Prentice Hall, New York, 1995.

[70] P. Sannuti and A. Saberi, “A special coordinate coordinate basis of multivariable linear systems — Finite and infinite zero structure, squaring down and decoupling,” *International Journal of Control*, Vol. 45, pp.1655-1704, 1987.

[71] J. M. Schumacher, ”Compensator synthesis using (C,A,B)-pairs,” *IEEE Transactions on Automatic Control*, Vol. 25, No. 6, pp.1133-1138, 1980.

[72] H. L. Trentelman, A. A. Stoorvogel and M. Hautus, *Control Theory for Linear Systems*, Springer, London, 2001.

[73] A. Varga, ”On stabilization methods of descriptor systems,” *Systems and Control Letters*, Vol. 24, pp.133-138, 1995.

[74] A. Varga, ”Computation of Kronecker-like forms of a system pencil: applications, algorithms and software,” *Proceedings of the 1996 IEEE International Symposium on Computer-Aided Control System Design*, pp.77-82, 1996.

[75] G. Verghese, *Infinite Frequency Behavior in Generalized Dynamical Systems,* Ph.D. Dissertation, Stanford University, 1978.

[76] G. Verghese, B. C. Levy and T. Kailath, ”A generalized state-space for singular systems,” *IEEE Transactions on Automatic Control*, Vol. 26, No. 4, pp.811-831, 1981.

[77] K. Weierstrass, “Zur Theorie der Bilinearen und Quadratischen Formen,” *Monatsh. Akad. Wiss. Berlin*, pp.310-338, 1867.

[78] J. C. Willems, ”Almost A(mod B)-invariant subspaces,” *Asterisque 75-76, Analyse des Systemes,*, pp.239-248, 1978.

[79] J. C. Willems, ”Almost invariant subspaces: an approach to high gain feedback design-Part 1: almost controlled invariant subspaces,” *IEEE Transactions on Automatic Control*, Vol. 26, No. 1, pp.235-252, 1981.

[80] J. C. Willems, "Almost invariant subspaces: an approach to high gain feedback design-Part 2: almost conditionally invariant subspaces," *IEEE Transactions on Automatic Control*, Vol. 27, No. 5, pp.1071-1084, 1982.

[81] W. M. Wonham and A. S. Morse, "Decoupling and pole assignment in linear multivariable systems: a geometric approach," *Technical Report Number: PM-66, NASA Electronics Research Centre*, Cambridge, Mass., October, 1968.

[82] W. M. Wonham and A. S. Morse, "Decoupling and pole assignment in linear multivariable systems: a geometric approach," *SIAM Journal on Control*, Vol 8, No. 1, pp.1-18, 1970.

[83] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, Springer-Verlag, New York, 1979.

[84] E. L. Yip and R. F. Sincovec, "Solvability, controllability and observability of continuous descriptor systems," *IEEE Transactions on Automatic Control*, Vol. 26, No. 3, pp.702-707, 1981.

[85] Z. Zhou, M. A. Shayman and T. J. Tarn, "Singular systems: A new approach in the time domain," *IEEE Transactions on Automatic Control*, Vol. 32, No. 1, pp.42-50, 1987.

Appendix A

MATLAB Codes for Realization of the Structural Decomposition

In this appendix, we list realization codes for several main functions. This software package enhances the power of the structural decomposition technique as a useful tool in solving singular systems and control problems.

To illustrate it in a clearer way, we list the functions according to their mutual relationship, that is, a main function will be presented first and followed by its subfunctions. For those important functions but not essential in this thesis, we only give their algorithms here while their full codes can be found in Lin and Chen [52].

1. StructuralDecomposition.m

   This is the main function, that is, the structural decomposition function for general multivariable singular systems. The function transforms the given singular system (E, A, B, C, D) into its structural decomposition in compact form $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, which can explicitly display all the structural properties, such as the finite and infinite zero structures, invertibility structures and even redundant dynamics of the given system. All other procedures are sub-programs of this main function.

```
function [Ge,Gs,invGi,Go,Ev,Av,Bv,Cv,Dv,nz,ne,na,nb,nc,nd]
          =StructuralDecomposition(E,A,B,C,D,tol)

%----Structural Decomposition for singular systems-----
%----[Ge,Gs,invGi,Go,Ev,Av,Bv,Cv,Dv,nz,ne,na,nb,nc,nd]
%--------------=Structural-Decomposition(E,A,B,C,D,tol)
%    decomposes the system (E,A,B,C,D) into its
%    structurally decomposed form (Ev,Av,Bv,Cv,Dv).
%
%    Inputs:
%       E, A, B, C, D : state space matrices of a given system.
%
%    Outputs:
%        Ev, Av, Bv, Cv, Dv : state space matrices in an
%                             structurally decomposed form.
%        Ge, Gs, Go, invGi : an invertable transform matrix and
%                            state, output and input transformations
%                            respectively.
%        nz, ne, na,nb nc, nd : dimensions of Xz, Xe, Xa, Xb, Xc, Xd
%                               respectively.
%
%
%     Minghua He, NUS, Kent Ridge, Singapore, Sept. 24, 2002.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

if nargin<6
    tol=1e-5;
end

%----define the dimensions

n=size(A,1); m=size(B,2); p=size(C,1);

%----decompose it to sys_hat and separate the redundant states

[E_hat,A_hat,B_hat,C_hat,D_hat,Ge_1,Gs_1,Gi_1,
nz,ne,nf,Gi_es,Psi_1,Psi_2]=RedundantSeparation(E,A,B,C,D,tol);

%----seperate A_e, A_g, and C_e, D_e ...

Ag=A_hat(nz+1:nz+ne,nz+ne+1:n); Ae=A_hat(nz+ne+1:n,nz+1:nz+ne);
Ce=C_hat(:,nz+1:nz+ne); De=D_hat(:,1:ne);

Ak=zeros(n,n); if ne>0,
    Ak(nz+1:nz+ne,nz+1:nz+ne)=-eye(ne);
    Ak(nz+1:nz+ne,nz+ne+1:n)=Ag;
    Ak(nz+ne+1:n,nz+1:nz+ne)=Ae;
end

Bk=zeros(n,m); if ne>0,
    Bk(nz+1:nz+ne,1:ne)=2*eye(ne);
    Bk(nz+ne+1:n,1:ne)=-Ae;
end
```

```
Ck=zeros(p,n); if ne>0,
    Ck(:,nz+1:nz+ne)=Ce;
    Ck(:,nz+ne+1:n)=De*Ag;
end

Dk=zeros(p,m); if ne>0,
    Dk(:,1:ne)=De-Ce;
end

%----construct an nonsingular system----
A_chk=A_hat-Ak; A_bar=A_chk(nz+ne+1:n,nz+ne+1:n);

B_chk=B_hat-Bk; B_bar=B_chk(nz+ne+1:n,:);

C_chk=C_hat-Ck; C_bar=C_chk(:,nz+ne+1:n);

D_chk=D_hat; D_bar=D_hat-Dk;

[As,Bs,Cs,Ds,G1,G2,G3,qv,rv,dims]=scb(A_bar,B_bar,C_bar,D_bar,tol);

na=dims(1)+dims(2)+dims(3); nb=dims(4); nc=dims(5); nd=dims(6);

Gs_bar=G1; Gi_bar=G3; Go_bar=G2;

Gs_2=eye(n); Gs_2(nz+ne+1:n,nz+ne+1:n)=Gs_bar;

opq=eye(ne); for i=nz+1:nz+ne,
    for j=1:ne,
        Tmp(i,j)=Psi_1(i-nz,j)+opq(i-nz,j);
    end
    for j=ne+1:m,
        Tmp(i,j)=Psi_2(i-nz,j-ne);
    end
end

for i=1:nz,
    for j=1:m,
        Tmp(i,j)=0;
    end
end

for i=nz+ne+1:n,
    for j=1:ne,
        Tmp(i,j)=-Ae(i-nz-ne,j);
    end
    for j=ne+1:m,
        Tmp(i,j)=0;
    end
end

Bk=Tmp;

Dd1=De*Psi_1; Dd2=De*Psi_2;

for i=1:p,
    for j=1:ne,
        Bmp(i,j)=Dd1(i,j)-Ce(i,j);
    end
    for j=ne+1:m,
```

```
            Bmp(i,j)=Dd2(i,j-ne);
        end
    end

    Dk=Bmp;

    Ak_v=inv(Gs_2)*Ak*Gs_2; Bk_v=inv(Gs_2)*Bk*Gi_bar;
    Ck_v=inv(Go_bar)*Ck*Gs_2; Dk_v=inv(Go_bar)*Dk*Gi_bar;

    Ge=Ge_1*Gs_2; Gs=Gs_1*Gs_2; Gi=Gi_1*Gi_es*Gi_bar;
    invGi=inv(Gi_bar)*inv(Gi_es)*inv(Gi_1); Go=Go_bar;

    Ev=inv(Ge)*E*Gs; Av=inv(Gs_2)*A_chk*Gs_2;
    Bv=inv(Gs_2)*B_chk*Gi_bar; Cv=inv(Go_bar)*C_chk*Gs_2;
    Dv=inv(Go_bar)*D_chk*Gi_bar;
```

2. RedundantSeparation.m

This is an essential function which separates two kinds of redundant states from the original system state. One kind of such redundant states are static and identical zero all the time, whereas the other redundant states are linear combination of appropriate order of system input's derivatives. Such states are associated with the so called impulse modes, which are introduced by the derivatives of the system input. The main algorithm for this function is the following transformations.

$$x = \Gamma_{\mathrm{s}1}\hat{x} = \Gamma_{\mathrm{s}1}\begin{pmatrix} x_{\mathrm{e}} \\ x_{\mathrm{z}} \\ x_{\mathrm{f}} \end{pmatrix}, \qquad u = \Gamma_{\mathrm{i}1}\hat{u} = \Gamma_{\mathrm{i}1}\begin{pmatrix} \hat{u}_{\mathrm{e}} \\ \hat{u}_{\star} \end{pmatrix}, \tag{A.1}$$

and

$$\hat{E} = \Gamma_{\mathrm{e}1}E\Gamma_{\mathrm{s}1} = \begin{bmatrix} J_{n_{\mathrm{z}}} & 0 & 0 \\ E_{\mathrm{ez}} & 0 & 0 \\ E_{\mathrm{fz}} & 0 & I \end{bmatrix}, \tag{A.2}$$

$$\hat{A} = \Gamma_{\mathrm{e}1}A\Gamma_{\mathrm{s}1} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & A_{\mathrm{g}} \\ 0 & A_{\mathrm{e}} & A_{\mathrm{f}} \end{bmatrix}, \tag{A.3}$$

$$\hat{B} = \Gamma_{\mathrm{e}1}B\Gamma_{\mathrm{i}1} = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & B_{\mathrm{f}} \end{bmatrix}, \tag{A.4}$$

$$\hat{C} \;=\; C\Gamma_{\mathrm{s1}} = [\,C_{\mathrm{e}} \quad C_{\mathrm{z}} \quad C_{\mathrm{f}}\,], \tag{A.5}$$

$$\hat{D} \;=\; D\Gamma_{\mathrm{i1}} = [\,D_{\mathrm{e}} \quad D_{\mathrm{f}}\,]. \tag{A.6}$$

Here the states $x_{\mathrm{z}} = 0$ and x_{e} are redundant states in the structurally decomposed form.

```
function
[E_hat,A_hat,B_hat,C_hat,D_hat,Ge_1,Gs_1,Gi_1,
nz,ne,nf,Gi_es,Psi_1,Psi_2]=RedundantSeparation(E,A,B,C,D,tol)

%-----decompose system to x-> (x_z,x_e,x_f)'
%

%-----------------------------------------------------
if nargin<6
    tol=1e-5;
end

%---determine the dimensions of state,input and ouput

n=size(A,1); m=size(B,2); p=size(C,1);

%---perform the Weitress decomposition

[P,Q,n1,n2,A1,B1,C1,N,B2,C2]=Weierstrass(E,A,B,C);

%---find Ts and Ti and transform (N,B2) into
%---Controllability Canonical Form

Ts=eye(n2); Ti=eye(m);

%------trans (N,B2) into controllable and non-controllable subspaces
%  [Nbar,Bbar,Cbar,T,k] = ctrbf(N,B2,C2)
%    Abar = T * A * T' ,  Bbar = T * B ,  Cbar = C * T'
%    and the transformed system has the form

%              | Anc    0 |            | 0 |
%       Abar =  ----------   ,  Bbar =  ---   ,  Cbar = [Cnc| Cc].
%              | A21   Ac |            |Bc |
%    The number of controllable states is SUM(K)

[Nbar,Bbar,Cbar,T,k] = ctrbf(N,B2,C2); sumk=sum(k); Ts=T*Ts;

%---transform x_z to jordan form

if sumk~=n2,
    nz=n2-sumk;
    Nz=Nbar(1:nz,1:nz);
```

```
    [Nt,Ttt,xindex]=r_jordan(Nz,tol);
    Tt=eye(n2);
    Tt(1:nz,1:nz)=inv(Ttt);
    Ts=Tt*Ts;
else
    nz=0;
end

%---transform controllable part to control canonical form

if sumk~=0,
    Nc=Nbar(nz+1:n2,nz+1:n2);
    Bc=Bbar(nz+1:n2,:);
    [Nc,Bc,Ts_c,Ti_c,ks]=bdccf(Nc,Bc,tol);

    Tt=eye(n2);
    Tt(nz+1:n2,nz+1:n2)=inv(Ts_c);

    Ts=Tt*Ts;
    Ti=Ti*Ti_c;

    ne=length(ks);

    V=eye(m); dog=0;
    for i=1:ne,
        dog=dog+ks(i);
        for j=i+1:m,
            if abs(Bc(dog,j))>tol,
                Vt=eye(m);
                Vt(i,j)=-Bc(dog,j);
                V=V*Vt;
                Bc=Bc*V;
            end
        end
    end
    Ti=Ti*V;

else
    nz=n2;
    ne=0;
end

%----construct transform matrix Gie(s)

syms s;

for i=1:n2-nz,
    for j=1:m,
        Bc(i,j)=round(Bc(i,j)*10000)/10000;
    end
end

row_pos=1; for i=1:ne,
    for j=1:m,
        tmp=0;
        for k=0:ks(i)-1,
            dog=Bc(row_pos+k,j);
            tmp=tmp-(dog)*s^(k);
        end
```

```
        invGie(i,j)=tmp;
    end
    row_pos=row_pos+ks(i);
end

for i=ne+1:m,
    for j=(ne+1):m,
        invGie(i,j)=1;
    end
end

Gi_es=inv(invGie);

Psi_1=Gi_es(1:ne,1:ne); Psi_2=Gi_es(1:ne,ne+1:m);

%----construct the transform matrices---

Ge_1=eye(n); Ge_1(n1+1:n,n1+1:n)=Ts; Ge_1=Ge_1*P;

Gs_1=eye(n); Gs_1(n1+1:n,n1+1:n)=inv(Ts); Gs_1=Q*Gs_1;

Gi_1=Ti;


%---separate xe,xz and xf------------

%----------------combine the dynamics in x_1 and x_2 to x_f


%--Permutation Step-1:

%            x_e
%      x ->  x_z
%            x_f

V=eye(n); pos=1; ttt=0; for i=1:ne,
    V(i,i)=0;
    V(i,n1+nz+pos)=1;
    if n1+nz+pos>ne+nz,
        ttt=ttt+1;
        V(n1+nz+pos,ttt)=1;
        V(n1+nz+pos,n1+nz+pos)=0;
    end
    pos=pos+ks(1,i);
end

for i=1:nz,
    V(ne+i,ne+i)=0;
    V(ne+i,n1+i)=1;
    if n1+i>ne+nz,
        ttt=ttt+1;
        V(n1+i,ttt)=1;
        V(n1+i,n1+i)=0;
    end
end

%--\vt{x}=Vx

disp('---Permutation Step-1:  set sequence of x_e,x_z,x_f--')
```

```
Ge_1=V*Ge_1; Gs_1=Gs_1*inv(V);

EE=Ge_1*E*Gs_1; AA=Ge_1*A*Gs_1; BB=Ge_1*B*Gi_1; CC=C*Gs_1; DD=D;

%---rearrange the sequence to x_z->x_e->x_f

V=zeros(n,n); V(1:nz,ne+1:ne+nz)=eye(nz);
V(nz+1:nz+ne,1:ne)=eye(ne); V(nz+ne+1:n,nz+ne+1:n)=eye(n-nz-ne);

Ge_1=V*Ge_1; Gs_1=Gs_1*inv(V);

EE=Ge_1*E*Gs_1; AA=Ge_1*A*Gs_1; BB=Ge_1*B*Gi_1; CC=C*Gs_1; DD=D;

%--Permutation Step-2:

%            | 0 * 0 |         | 0 * * |        | I 0 |
%      E ->  | 0 J 0 |    A -> | 0 I 0 |   B -> | 0 0 |
%            | 0 * I |         | * * * |        | * * |


U=eye(n); for i=ne+nz+1:n,
    Tt=eye(n);
    for j=1:n,
        if EE(j,i)>1-tol,
            if j~=i,
                Tt(j,i)=1;
                Tt(i,j)=1;
                Tt(i,i)=0;
                Tt(j,j)=0;
            end
        end
    end
    EE=Tt*EE;
    U=Tt*U;
end


disp('---Permutation Step-2: unify matrix E------')

Ge_1=U*Ge_1;

EE=Ge_1*E*Gs_1; AA=Ge_1*A*Gs_1; BB=Ge_1*B*Gi_1; CC=C*Gs_1; DD=D;

%--Permutation Step-3:

%            | 0 * 0 |         | 0 * * |        | I 0 |
%      E ->  | 0 J 0 |    A -> | 0 I 0 |   B -> | 0 0 |
%            | 0 * I |         | * * * |        | 0 * |

%--substitute u_e in x_f by x_vh and unify B

U=eye(n); for i=1:ne,
    for j=1:n,
        if j~=(nz+i),
            if abs(BB(j,i))>tol,
                Ut=eye(n);
                Ut(j,nz+i)=-BB(j,i);
                U=Ut*U;
```

```
                BB=U*BB;
            end
        end
    end
end

disp('---Permutation Step-3: unify matrix BB------')

Ge_1=U*Ge_1;

Ge_1=inv(Ge_1);

E_hat=inv(Ge_1)*E*Gs_1; A_hat=inv(Ge_1)*A*Gs_1;
B_hat=inv(Ge_1)*B*Gi_1; C_hat=C*Gs_1; D_hat=D;

nf=n-nz-ne;
```

3. Weierstrass.m

This one is to perform a fast-slow decomposition (see e.g., [29] for more details) for the given singular system. With two constant transform matrices P and Q, it transforms the given singular system into two subsystems, one is nonsingular and the other is singular. The decomposition can be characterized as the following transformations,

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix},$$
$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CQ = [\, C_1 \quad C_2 \,], \tag{A.7}$$

where N is a nilpotent matrix.

```
function [P,Q,n1,n2,A1,B1,C1,N,B2,C2]=Weierstrass(E,A,B,C,tol)

%-------perform the Weitress decomposition for a singular system-----
%
%        | I 0  |  | A1 0 |
%        | 0 N2 |, | 0  I |  =  P (E,A) Q
%
%-------He Minghua, Sept.5, 2002, Kent Ridge, Singapore
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%---set the toleratio
if nargin<5,
    tol=1e-5;
end
```

```
%---determine the dimensions of state,input and ouput
n=size(A,1);

%---find P,Q and make a Weierstrass decomposition

%-----Step 1: find an nonsingular alpha*E+A

alpha=0; Ealp=eig(E); Aalp=eig(A); for i=1:n,
    alpha=alpha+Ealp(i)+Aalp(i);
end alpha=alpha/(2*n);

%-----Step 2: compute E_hat

E_hat=inv(alpha*E+A)*E;

%-----Step 3: decompose E_hat and get E_1 and E_2

[J,T,xindex]=r_jordan(E_hat,tol) T=inv(T);

SizeJordan=size(xindex);

n1=0; n2=0;

start=1; pos=0; s2=1; for i=1:SizeJordan(1),
    for j=1:SizeJordan(2),
        pos=pos+xindex(i,j);
        if xindex(i,j)~=0 & abs(J(pos,pos))<tol,
            n2=n2+xindex(i,j);
            H=J(start:start+xindex(i,j)-1,start:start+xindex(i,j)-1);
            E_2(s2:s2+xindex(i,j)-1,s2:s2+xindex(i,j)-1)=H;
            s2=s2+xindex(i,j);
            start=start+xindex(i,j);
        end
    end
end

n1=n-n2;

E_1=J(n2+1:n,n2+1:n);

%------construct P and Q
n=n n1=n1 n2=n2

U=zeros(n,n); U(1:n1,n2+1:n)=eye(n1); U(n1+1:n,1:n2)=eye(n2);

T=U*T;

Q=inv(T);

P=zeros(n); P(1:n1,1:n1)=inv(E_1);
P(n1+1:n,n1+1:n)=inv(eye(n2)-alpha*E_2); P=P*T*inv(alpha*E+A);

EE=P*E*Q; AA=P*A*Q; BB=P*B; CC=C*Q;

%---Seperate the two sub systems----

A1=AA(1:n1,1:n1); B1=BB(1:n1,:); C1=CC(:,1:n1);
```

```
N=EE(n1+1:n,n1+1:n); B2=BB(n1+1:n,:); C2=CC(:,n1+1:n);
%---the end of code-----------------------
```

4. Kronecker.m

The function transform the given system's system matrix $P_\Sigma(s)$ to its Kronecker Canonical Form with two constant transform matrices M and N, that is,

$$\begin{aligned}\tilde{P}_\Sigma(s) &= MP_\Sigma(s)N \\ &= M\begin{bmatrix} A-sE & B \\ C & D\end{bmatrix}N \\ &= \text{blkdiag}\{sI-J, R_{r_1},\ldots,R_{r_p},L_{l_1},\ldots,L_{l_q},I-sH\}.\end{aligned}$$

Here every block of the diagonal entries in $\tilde{P}_\Sigma(s)$ is associated a series of distinct structure indices.

5. CCF.m

The function transform a matrix pair (A, B) into its control canonical form as follows,

$$T^{-1}AT = \begin{bmatrix} A_{\rm c} & A_{{\rm c}\bar{\rm c}} \\ 0 & A_{\bar{\rm c}}\end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_{\rm c} \\ 0\end{bmatrix}, \tag{A.8}$$

where $(A_{\rm c}, B_{\rm c})$ is completely controllable while $(A_{\bar{\rm c}}, 0)$ is totally uncontrollable.

6. BDC.m

This function decomposes a complete controllable pair $(A_{\rm c}, B_{\rm c})$ into a special block controllability canonical form [20], in which every submatrix block corresponds to a distinct input channel. The decomposition process can be described as follows,

$$\begin{aligned} R^{-1}A_{\rm c}R &= \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & J_k\end{bmatrix}, \\ R^{-1}B_{\rm c} &= \begin{bmatrix} B_1 & B_{12} & \cdots & B_{1k} & B_{1l} \\ 0 & B_2 & \cdots & B_{2k} & B_{2l} \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & B_k & B_{kl}\end{bmatrix}, \end{aligned} \tag{A.9}$$

where J_i, $i = 1, 2, \cdots, k$ are Jordan blocks with zero eigenvalue and

$$B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad B_{ij} = \begin{bmatrix} \star \\ \star \\ \vdots \\ 0 \end{bmatrix}. \tag{A.10}$$

7. StrictEquivalence.m

This function is used to find two invertible transform matrices $\Gamma_{\rm p}$ and $\Gamma_{\rm q}$. It can be showed that the structural decomposition for singular systems is nothing more than an invertible transform on the given singular system's system matrix. This can be illustrated as the following equation,

$$\begin{aligned} P_{\tilde{\Sigma}}(s) &= \begin{bmatrix} \tilde{A} - s\tilde{E} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \\ &= \Gamma_{\rm p} P_{\Sigma}(s) \Gamma_{\rm q} \\ &= \Gamma_{\rm p} \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix} \Gamma_{\rm q}, \end{aligned} \tag{A.11}$$

where $P_{\tilde{\Sigma}}(s)$ is the system matrix of structurally decomposed system.

8. CancelledParts.m

This procedure computes the cancelled parts $\tilde{A}_{\rm k}$, $\tilde{B}_{\rm k}(s)$, $\tilde{C}_{\rm k}$ and $\tilde{D}_{\rm k}(s)$ in (4.57). The algorithm for computation is in the following,

$$\begin{aligned} \tilde{A}_{\rm k} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{n_{\rm e}} & -A_{\rm g}\bar{\Gamma}_{\rm s} \\ 0 & -\bar{\Gamma}_{\rm s}^{-1}A_{\rm e} & 0 \end{bmatrix}, \\ \tilde{B}_{\rm k}(s) &= \begin{bmatrix} 0 & 0 \\ -\Psi_1(s) - I_{n_{\rm e}} & -\Psi_2(s) \\ \bar{\Gamma}_{\rm s}^{-1}A_{\rm e} & 0 \end{bmatrix} \bar{\Gamma}_{\rm i}, \\ \tilde{C}_{\rm k} &= [\, 0 \quad -\bar{\Gamma_{\rm o}}^{-1}C_{\rm e} \quad -\bar{\Gamma_{\rm o}}^{-1}D_{\rm e}A_{\rm g}\bar{\Gamma}_{\rm s} \,], \\ \tilde{D}_{\rm k}(s) &= [\, \bar{\Gamma_{\rm o}}^{-1}(C_{\rm e} - D_{\rm e}\Psi_1(s)) \quad -\bar{\Gamma_{\rm o}}^{-1}D_{\rm e}\Psi_2(s) \,]\, \bar{\Gamma}_{\rm i}, \end{aligned}$$

where $\bar{\Gamma}_{\rm s}$, $\bar{\Gamma}_{\rm i}$ and $\bar{\Gamma}_{\rm o}$ are invertible transform matrices from the following SCB.m for decomposing a nonsingular system into its structural decomposition form.

9. SCB.m

   This is the function of structural decomposition for linear nonsingular system. The function was developed by Lin and Chen [52], and it decomposes a given linear system (A, B, C, D) and explicitly displays its structural properties. The function Structural-Decomposition.m is its natural extension to singular systems.

10. Jordan.m

    This function transforms a real matrix A to its Jordan canonical form. It is more reliable and accurate than the one provided by MATLAB.

The functions introduced here are only some main procedures, for the codes of other functions, there are more details in Lin and Chen [52].

Appendix B

Author's Publications

[1] M. He and B. M. Chen, "Structural decomposition of linear singular systems: The single-input and single-output case," *Systems and Control Letters*, Vol. 183, pp. 61-68, 2002.

[2] M. He, B. M. Chen and Z. Lin, "Structural decomposition of general linear multivariable singular systems", *submitted to publish on the International Journal of Control*.

[3] M. He, B. M. Chen and C. C. Ko, "Chinese character recognition using natural stroke sequence", Proceedings of the Sixth International Conference on Control, Automation, Robotics and Vision (CD-ROM), TM1.4, 5 pages, Singapore, Dec. 2000.

[4] M. He and B. M. Chen, "Structural decomposition of single-input single-output linear singular systems", Proceedings of the Forth Asian Control Conference (CD-ROM), TM5-5, 6 pages, Singapore, Sept. 2002.

[5] M. He, B. M. Chen and Z. Lin, Structural decomposition and its properties of general multivariable linear singular systems, Proceedings of the 2003 American

Control Conference, Denver, Colorado, USA, Page 4494-4499, June 2003.

[6] M. He and B. M. Chen, "Computation of the structural decomposition of general linear singular systems", finished and on submitting