NONLINEAR SCHRÖDINGER EQUATIONS WITH VARIABLE COEFFICIENTS

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Summary

In this thesis, we focus on the cubic nonlinear Schrödinger equations (NLS) with variable coefficients. First, we consider the Cauchy problem for the vector-valued NLS with space- and time-dependent coefficients on $\mathbb{R}^N$ and $\mathbb{T}^N$. By an approximation argument we prove that for suitable initial maps, the Cauchy problem admits unique local solutions, which preserve the regularity of the initial data. Particularly, if the initial map is smooth, the solution is smooth. We also discuss the global existence in the cases $N = 1, 2$ and prove that the solutions are global when $N = 1$ or when $N = 2$ provided the L^2-norms of initial data are small enough and the coefficients satisfy certain additional conditions. We remark that the cubic nonlinearity is critical in the latter case.

Second, we study blow-up solutions to the Cauchy problem of the inhomogeneous scalar NLS with spatial dimension two. On $\mathbb{R}^2$, we make use of so-called virial identities and the ground state solution to construct a family of blow-up solutions. We also present non-existence results and investigate qualitative properties, namely, L^2-concentration and L^2-minimality, of blow-up solutions when they exist. These results are related to, and in some cases, extend the work of Merle [29] and Nawa–Tsutsumi [33]. On $\mathbb{T}^2$, we obtain an L^2-concentration in terms of the ground state solution on $\mathbb{R}^2$. It is remarkable that there is no restriction on the L^2-norms of initial data which is required in [2]. In particular, in each case, a sufficient condition for global existence of solutions is provided and the singular points of the L^2-minimal blow-up solutions can be located if the coefficients satisfy certain conditions.

Chapter 1

Introduction

1.1 Background and motivation

In the past two decades, tremendous progress has been made in the study of the nonlinear Schrödinger equation (NLS),

$$i\partial_t u + \Delta u \pm |u|^{\sigma-2}u = 0, \quad (t,x) \in [0,\infty) \times M, \tag{1.1}$$

where $\sigma > 2$ is a constant and M is the base space $\mathbb{R}^N$ or $\mathbb{T}^N$. (Here and after, the reader is referred to Section 1.2 for the explanation of general notations.) The Cauchy problem of the above equation has been used as a mathematical model in a variety of physical contexts. Although there are still many open problems, a satisfactory analysis of the wave phenomena associated with the equation could be accomplished by answering questions like existence and uniqueness of solutions, regularity properties of solutions, continuity with respect to initial data, and blow-up behavior. For blow-up solutions, some interesting qualitative properties such as L^2-concentration have been discovered; and the characterization of the L^2-minimal blow-up solutions has been exploited when the exponent of the nonlinear term is critical for blowup, i.e, $\sigma = 4/N + 2$. There are two important conserved quantities associated with solutions of the equation, known as (L^2-) mass and energy, respectively:

$$\int |u(t,x)|^2 dx = \int |u_0(x)|^2 dx, \tag{1.2}$$

$$E(u(t)) = E(u_0), \tag{1.3}$$

where

$$E(u) = \frac{1}{2}\int |\nabla u(x)|^2 dx \mp \frac{1}{\sigma}\int |u(x)|^\sigma dx. \tag{1.4}$$

These conservation laws combined with the Strichartz inequalities play a crucial role in the discussion of existence and blow-up. The reader is referred to the surveys [5, 17, 30, 37, 7, 8] and the references therein for more details.

Recently, considerable interest on Schrödinger type equations with variable coefficients has arisen among both mathematicians and physicists, and some remarkable progress on the well-posedness of the Cauchy problem has been made, see [14, 15, 16, 19, 20, 24, 40] and references therein. In the linear case, several authors have studied the equation

$$\frac{\partial u}{\partial t} - i\sum_{j,k}\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right) - \sum_j b_j(t,x)\frac{\partial u}{\partial x_j} - c(t,x)u = f(t,x), \tag{1.5}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^N$, and typically, $a_{jk}(x) \in B^\infty(\mathbb{R}^N)$, $a_{jk}(x) = a_{kj}(x)$, $b_j(t,x), c(t,x) \in C^0([0,T]; B^\infty(\mathbb{R}^N))$, and a_{jk} satisfy the uniform ellipticity condition

$$\lambda^{-1}|\xi|^2 \le \sum_{j,k} a_{jk}(x)\xi_j\xi_k \le \lambda|\xi|^2, \quad \text{for any } x,\xi \in \mathbb{R}^N, \tag{1.6}$$

for some positive constant λ. In particular, Ichinose [20] and Hara [19] provided necessary conditions on $b_j(t,x)$ for the well-posedness of the Cauchy problem in $L^2(\mathbb{R}^N)$ and $H^\infty(\mathbb{R}^N)$. Doi [15] also studied such equations on Riemannian manifolds.

Staffilani and Tataru [38] studied the Cauchy problem of the following linear Schrödinger equation with nonsmooth coefficients:

$$i\frac{\partial u}{\partial t} + \sum_{j,k}\frac{\partial}{\partial x_j}\left(a_{jk}(t,x)\frac{\partial u}{\partial x_k}\right) = 0, \quad x \in \mathbb{R}^N, \quad t \ge 0, \tag{1.7}$$

where $a_{jk}(t,x) \in [L^\infty(C^{1,1}) \cap C^{0,1}(L^\infty)](\mathbb{R}\times\mathbb{R}^N)$. When $a_{jk}(t,x)$ is a C^2 compactly supported perturbation of the identity and the Hamiltonian system associated with the Hamiltonian function

$$a(x,\xi) = \sum_{j,k=1}^{N} a_{jk}(t,x)\xi_j\xi_k$$

has empty trapping set, they used the so-called FBI transformation to construct a micro-local parametrix for the equation and consequently established Strichartz estimates.

Tsutsumi [41] considered the initial-boundary value problems for the following NLS in an external domain $\Omega \subset \mathbb{R}^3$:

$$i\frac{\partial u}{\partial t} + \sum_{j,k=1}^{3} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right) = \lambda(t,x)|u|^{\gamma-1}u + f(t,x), \quad t \geq 0. \tag{1.8}$$

When the coefficients satisfy certain conditions and $\gamma \geq 4$, he addressed the global existence of solutions with small initial values by making use of the asymptotic vanishing property of solutions to the corresponding homogeneous equation in $L^\infty(\Omega)$ and a generalized Pohozaev estimate.

Merle [29] considered the Cauchy problem of the following scalar critical NLS on $\mathbb{R}^N$:

$$\partial_t u = i\left(\Delta u + k(x)|u|^{\frac{4}{N}}u\right),$$

where $k(x)$ is a real-valued function on $\mathbb{R}^N$. He studied the existence of blow-up solutions as well as the nonexistence of L^2-minimal blow-up solutions.

Lim and Ponce [27] studied the Cauchy problem of the general quasi-linear Schrödinger equation in one space dimension

$$\begin{aligned}\partial_t u = {} & ia(u,\bar{u},\partial_x u,\partial_x\bar{u})\partial_x^2 u + ib(u,\bar{u},\partial_x u,\partial_x\bar{u})\partial_x^2\bar{u} \\ & + c(u,\bar{u},\partial_x u,\partial_x\bar{u})\partial_x u + d(u,\bar{u},\partial_x u,\partial_x\bar{u})\partial_x\bar{u} + f(u,\bar{u}), \quad x \in \mathbb{R}.\end{aligned}$$

Under certain conditions on the coefficients a, b, c, d and f, they established local existence and uniqueness results in $H^s(\mathbb{R})$ and $H^s(\mathbb{R}) \cap L^2(|x|^r dx)$ respectively.

Also of relevance is the inhomogeneous Heisenberg spin system (see, for instance, [10]) and its generalization – the inhomogeneous Schrödinger flow ([34, 35, 42, 43]):

$$\frac{\partial u}{\partial t} = \sigma(x)J(u)\tau(u) + \nabla\sigma(x)\cdot J(u)\nabla u, \quad x \in \mathcal{M}. \tag{1.9}$$

In the above, $\mathcal{M}$ is a Riemannian manifold, $u : \mathcal{M} \times [0,\infty) \to \mathcal{N}$ where $\mathcal{N}$ is a Kähler manifold with complex structure J, σ is a positive smooth real-valued function, and $\tau(u)$ is the tension field at u. In the case $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$, $\mathcal{N}$ is a Riemann surface, for instance, under a generalized Hasimoto transform ([11]), the flow (1.9) yields the focusing nonlinear Schrödinger equation with variable coefficients

$$\frac{\partial v}{\partial t} = i\left(\sigma(x)v_{xx} + 2\sigma_x v_x + \frac{\sigma(x)\kappa(x)}{2}|v|^2 v + r(t,x)v\right), \tag{1.10}$$

where κ is the Gaussian curvature of $\mathcal{N}$ and $r(t,x)v$ is the residual term.

Presently we would like to consider the Cauchy problem of the following *non-autonomous nonlinear Schrödinger equation* (NNLS henceforth):

$$\begin{cases} \dfrac{\partial u}{\partial t} = i\left\{f(t,x)\Delta u + p\nabla f(t,x)\cdot\nabla u + k(t,x)|u|^2 u\right\}, \quad t \geq 0,\ x \in M, \\ u(0,x) = u_0(x), \end{cases} \tag{1.11}$$

where p is a fixed real constant, f and k are appropriately smooth real-valued functions on $[0,\infty) \times M$ and $u \in \mathbb{C}^m$. We note that when $f(t,x) \equiv 1$ and $k(t,x) \equiv$ constant, (1.11) is just the ordinary (homogeneous) cubic NLS, which has been extensively studied, see [2, 5, 7, 8] and references therein.

We will first discuss the local existence of solutions to the Cauchy problem (1.11). Moreover, we will prove that the solutions are global when $N = 1$, and for small initial data when $N = 2$. Inspired by [12], our strategy is to approximate (1.11) by parabolic systems. To prove convergence, we will derive uniform estimates for these approximating systems by an energy method. In the consideration of global existence, to highlight the difference between the non-autonomous and the autonomous (even inhomogeneous) case, we stress that in the latter case, there are conservation laws which have no counterpart in the former case.

Then we will focus on the Cauchy problem of the *scalar cubic inhomogeneous* Schrödinger equation with *spatial dimension two*:

$$\begin{cases} \partial_t u = i\left(f(x)\Delta u + \nabla f(x) \cdot \nabla u + k(x)|u|^2 u\right) \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{1.12}$$

where $f(x)$ and $k(x)$ are real-valued functions on M ($= \mathbb{R}^2$ or $\mathbb{T}^2$) and $u_0 \in H^1(M)$. Clearly this is the special case of (1.11) with $m = 1, N = 2$ and $p = 1$. Also, this equation is the generalization of the NLS of critical nonlinearity on $\mathbb{R}^2$ and $\mathbb{T}^2$. We are interested in the singular solutions of (1.12) in the inhomogeneous case, i.e., $f(x)$ or $k(x)$ are not constant functions.

We first conduct our analysis on $\mathbb{R}^2$ and discuss some qualitative properties of blow-up solutions to the Cauchy problem (1.12) under certain conditions on $f(x)$ and $k(x)$. We obtain an L^2-concentration result and consequently a sharp condition for global existence. We make use of so-called virial identities and the ground state solution to construct a family of blow-up solutions. Then we focus on L^2-minimal blow-up solutions, locate their singular points if they exist and the coefficients satisfy appropriate conditions, and give a sufficient condition of nonexistence. Finally we investigate the blow-up solutions of (1.12) on $\mathbb{T}^2$. We describe the L^2-concentration and L^2-minimality in terms of the ground state solution and locate the singular points of the L^2-minimal blow-up solutions as well. Particularly, a sufficient condition of global existence of solutions is given.

1.2 Notations

We shall use the generic symbols C, C_j and c_j ($j \in \mathbb{Z}$) to denote positive constants depending on specified arguments, and ϵ to denote various small positive quantities. M is either the N-dimensional Euclidean space $\mathbb{R}^N$ or the N-dimensional flat torus $\mathbb{T}^N$ $(N = 1, 2, \cdots)$. $W^{k,q}$ $(0 \le k < \infty, 1 \le q \le \infty)$ denote usual Sobolev spaces on specified domains, $H^k = W^{k,2}$, $H^0 = L^2$, $H^\infty = \cap_{i=0}^\infty H^k$; B^∞ denotes the space of complex-valued smooth functions with all derivatives bounded.

We normally use $x = (x_1, \cdots, x_N)$ to denote the space variable, and t to denote the time variable. $|y - x|$ denotes the distance between two points $x, y \in M$, $B(x, r) = \{y \in M||y - x| < r\}$ and δ_x denotes the Dirac δ-function at x. If x is a variable of integration, we use dx to denote Lebesgue measure. An integral over all of M is simply denoted by $\int dx$. When referring to the function u defined on $[0, T) \times M$, we will use the shorthand $u(t)$ and $u(x)$ for $u(t, \cdot)$ and $u(\cdot, x)$, respectively.

Derivatives with respect to x_j and t are denoted by $\nabla_j = \partial/\partial x_j$ and $\partial_t = \partial/\partial t$ respectively. Sometimes, we denote $\partial_t u$ by u_t. ∇ denotes the spatial gradient, Δ is the Laplace-Beltrami operator on M. For the multi-index $\alpha = (\alpha_1, \cdots, \alpha_N)$ of length $|\alpha| := \sum_{j=1}^N \alpha_j$,

$$\nabla_\alpha = \nabla_1^{\alpha_1} \cdots \nabla_m^{\alpha_N}.$$

In this notation, the norm of the Sobolev space $H^k(M)$ is given by

$$\|u\|_{H^k}^2 = \sum_{|\alpha|=0}^{k} \int |\nabla_\alpha u(x)|^2 \, dx.$$

We say that two multi-indices satisfy $\beta \le \alpha$ if and only if $\beta_j \le \alpha_j$ for all $1 \le j \le N$, and write $\alpha - \beta = (\alpha_1 - \beta_1, \cdots, \alpha_N - \beta_N)$ when $\beta \le \alpha$.

$\mathbb{C}^m$ is the m-dimensional complex space with the standard real inner product $\langle u, v\rangle = \text{Re } (u \cdot \bar{v})$, where $\bar{v}$ is the conjugate of v. Clearly $\langle u, iu\rangle = 0$. We say two nonnegative functions $g_1(x) \sim g_2(x)$ if there exist positive constants c_1, c_2 such that $c_1 g_1(x) \le g_2(x) \le c_2 g_1(x)$ for all $x \in M$. Finally, $[s]$ denotes the integral part of the positive number s.

Chapter 2

Non-autonomous NLS: Existence and Uniqueness

In this chapter, we study the the Cauchy problem of the NNLS:

$$\begin{cases} \dfrac{\partial u}{\partial t} = i\left\{f(t,x)\Delta u + p\nabla f(t,x)\cdot\nabla u + k(t,x)|u|^2u\right\}, \quad t\geq 0,\ x\in M, \\ u(0,x) = u_0(x), \end{cases} \tag{2.1}$$

where p is a fixed real constant, f and k are appropriately smooth real-valued functions on $M\times[0,\infty)$ and $u\in\mathbb{C}^m$.

We will also be referring to the following assumptions:

(A1) There exists a positive continuous function $L(t)$ such that

$$\inf_{x\in M}|f(t,x)|\geq L(t), \quad \text{for all } 0\leq t<\infty;$$

(A2) f is C^1 with respect to t and there exists a positive continuous function $U(t)$ such that

$$\|\partial_t f(t,\cdot)\|_{L^\infty}\leq U(t), \quad \text{for all } 0\leq t<\infty;$$

(A3) $(p-1)\partial_t f\leq 0$ and there exists a positive constant c such that

$$\|f^{-p}(t,\cdot)\|_{L^\infty}\leq c \quad \text{and} \quad \sup_{x\in M}|f(t,x)|\leq c\inf_{x\in M}|f(t,x)|, \quad \text{for all } 0\leq t<\infty.$$

Our main results are as follows:

Theorem 2.1 *Let M be either $\mathbb{R}^N$ or $\mathbb{T}^N$ $(N \geq 1)$ and let $k_0 = [\frac{N}{2}]+1$. Suppose $s_0 \geq k_0 + 2$ is an integer and $f \in C^{1,s_0+1}([0,\infty) \times M)$ is a positive function satisfying* (A1)-(A2)*, $f(t,\cdot) \in W^{s_0+1,\infty}(M)$ and $k(t,\cdot) \in W^{s_0,\infty}(M)$ for all $0 \leq t < \infty$. Then, given any initial map $u_0 \in H^{s_0}(M)$, the Cauchy problem of the NNLS* (2.1) *admits a unique local solution $u \in L^\infty([0,T), H^{s_0}(M))$ where $T = T(\|u_0\|_{H^{k_0}})$. Moreover the solution is global in the sense that $u \in L^\infty_{\text{loc}}([0,\infty), H^{s_0}(M))$ when $N = 1$, or when $N = 2$ provided f satisfies* (A3) *and $\|u_0\|_{L^2}$ is small enough.*

Theorem 2.2 *Let M be either $\mathbb{R}^N$ or $\mathbb{T}^N$ $(N \geq 1)$. Suppose $f \in C^{1,\infty}([0,\infty) \times M)$ is a positive function satisfying* (A1)-(A2) *and $f(t,\cdot)$, $k(t,\cdot) \in B^\infty(M)$ for all $0 \leq t < \infty$. Then, given any initial map $u_0 \in H^\infty(M)$, the Cauchy problem of the NNLS* (2.1) *admits a unique local solution $u \in L^\infty([0,T), H^\infty(M))$ where $T = T(\|u_0\|_{H^{k_0}})$. Moreover the solution is global in the sense that $u \in L^\infty_{\text{loc}}([0,\infty), H^\infty(M))$ when $N = 1$, or when $N = 2$ provided f satisfies* (A3) *and $\|u_0\|_{L^2}$ is small enough.*

As the proof of Theorem 2.2 is similar to that of Theorem 2.1, we will only provide the proof of the latter in this chapter. We address uniqueness and local existence, and global existence, respectively, in the subsequent two sections. To minimize technicalities, we shall assume $k(t,x) \equiv 1$ in the sequel. For general coefficient $k(t,x)$, only a simple modification is needed.

2.1 Uniqueness and local existence

First of all we address the uniqueness of solution for the Cauchy problem of the NNLS (2.1).

Proposition 2.1 *Suppose that $T < \infty$ and $f \in C^{1,1}(M \times [0,T))$ is a real function satisfying* (A1)-(A2)*. Let $u \in L^\infty([0,T), H^{k_0+2}(M))$ be a solution to the Cauchy problem of the NNLS* (2.1)*. Then u is unique.*

Proof. The proof for the case $M = \mathbb{R}^N$ being almost the same, here we give the proof for $M = \mathbb{T}^N$ only. Without loss of generality, we may assume that $f > 0$. Let $u, v : [0, T) \times M \to \mathbb{C}^m$ be two solutions to (2.1) with the same initial map at $t = 0$. Then

$$\partial_t(u - v) = i\left(f\Delta(u - v) + p\nabla f \cdot \nabla(u - v) + |u|^2u - |v|^2v\right).$$

From this equation we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |u - v|^2 f^{p-1}\,dx \\
&= \int \langle \partial_t(u - v), u - v\rangle f^{p-1}\,dx + (p-1)\int |u - v|^2 f^{p-2}\partial_t f\,dx \\
&= \int \langle u + v, u - v\rangle\langle iu, u - v\rangle f^{p-1}\,dx + (p-1)\int |u - v|^2 f^{p-2}\partial_t f\,dx \\
&\le C\int |u - v|^2 f^{p-1}\,dx,
\end{aligned} \tag{2.2}$$

where the constant C only depends on

$$\sup_{t\in[0,T)}\|u\|^2_{L^\infty(M)},\ \sup_{t\in[0,T)}\|v\|^2_{L^\infty(M)},\ \inf_{t\in[0,T)}\|f\|^2_{L^\infty(M)},\ \text{and}\ \sup_{t\in[0,T)}\|\partial_t f\|^2_{L^\infty(M)}.$$

By the Gronwall's inequality and the assumption $u(\cdot, 0) = v(\cdot, 0)$, we obtain

$$\int |u - v|^2 f^{p-1}\,dx = 0 \qquad \forall t \in [0, T),$$

which implies that $u(t, x) = v(t, x)$ for all $(t, x) \in [0, T) \times M$. □

In the remainder of this section, we establish the local existence result for the Cauchy problem of the NNLS (2.1). For this, we will study the following approximating Cauchy problems parameterized by ϵ:

$$\begin{cases}
\partial_t u = (\epsilon + i)\big(\mathrm{div}(f\nabla u)\big) + i(p-1)\nabla f \cdot \nabla u + i|u|^2u, \quad t \ge 0,\ x \in M, \\
u(x, 0) = u_0(x).
\end{cases} \tag{2.3}$$

If $f \in C^{1,s_0+1}([0,\infty) \times M)$ is a positive function satisfying (A1), and $f(t, \cdot) \in W^{s_0+1,\infty}(M)$ for all $0 \le t < \infty$, then it is easy to see that (2.3) is a second-order uniformly parabolic system on $[0, T] \times M$. Thus, by the standard theory of parabolic equations, for each $0<\epsilon\le 1$, given any initial map $u_0\in C_0^\infty(M)$, the Cauchy problem (2.3) admits a unique local smooth solution ([1] Remark 10.7; [39] p.327). In fact, from the following discussions we will see that $u^\epsilon \in C([0,T_\epsilon),H^{s_0}(M))\cap L^\infty([0,T_\epsilon),H^{s_0+1}(M))$. Now, we need to establish some uniform *a priori* estimates and a uniform lower bound for T_ϵ with respect to ϵ.

Lemma 2.1 *Suppose $f\in C^{1,s_0+1}([0,\infty)\times M)$ is a positive function satisfying* (A1)-(A2)*, and $f(t,\cdot)\in W^{s_0+1,\infty}(M)$ for all $0\le t<\infty$. Let $u=u^\epsilon$ be a solution of* (2.3) *in $C([0,T_\epsilon),H^{s_0}(M))$. Then there exists $T=T(\|u_0\|_{H^{k_0}})>0$, which is independent of ϵ, such that for any integer $0\le l\le s_0$, there exists $C_l=C_l(m,u_0,f)$ such that*

$$\sup_{t\in[0,T]}\|u\|_{H^l}\le C_l. \tag{2.4}$$

Proof. For an integer $0\le l\le s_0$ and a multi-index α with $|\alpha|=l$, we consider the integral

$$I(t)=\int|\nabla_\alpha u|^2 f^q\,dx,$$

where $q=l+p-1$. First, we note that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla_\alpha u|^2 f^q\,dx\\
&=\frac{q}{2}\int|\nabla_\alpha u|^2 f^{q-1}\partial_t f\,dx+\int\langle\nabla_\alpha\partial_t u,\nabla_\alpha u\rangle f^q\,dx\\
&=\frac{q}{2}\int|\nabla_\alpha u|^2 f^{q-1}\partial_t f\,dx+\int\langle(\epsilon+i)\nabla_\alpha\big(\mathrm{div}(f\nabla u)\big),\nabla_\alpha u\rangle f^q\,dx\\
&\quad+(p-1)\int\langle i\nabla_\alpha(\nabla f\cdot\nabla u),\nabla_\alpha u\rangle f^q\,dx+\int\langle i\nabla_\alpha(|u|^2u),\nabla_\alpha u\rangle f^q\,dx\\
&=:A_0+A_1+A_2+A_3,
\end{aligned}\tag{2.5}$$

where A_0,A_1,A_2,A_3 denote the integral terms in the sum as given above. We will compute these terms separately.

First, by the assumptions on f, it is clear that

$$\begin{aligned}
A_0&=\frac{q}{2}\int|\nabla_\alpha u|^2 f^{q-1}\partial_t f\,dx\\
&\le C(f)\int|\nabla_\alpha u|^2\,dx\le C(f)\sum_{|\alpha|=l}\int|\nabla_\alpha u|^2\,dx.
\end{aligned}\tag{2.6}$$

For the second term A_1, integration by parts yields

$$\begin{aligned}
A_1 &= \int \langle (\epsilon+i)\nabla_\alpha\big(\mathrm{div}(f\nabla u)\big), \nabla_\alpha u\rangle f^q\,dx \\
&= -\sum_{j=1}^N \int \langle (\epsilon+i)\nabla_\alpha(f\nabla_j u), \nabla_\alpha\nabla_j u\rangle f^q\,dx \\
&\quad -q\sum_{j=1}^N \int \langle (\epsilon+i)\nabla_\alpha(f\nabla_j u), \nabla_\alpha u\rangle f^{q-1}\nabla_j f\,dx \\
&= -\sum_{j=1}^N \int \langle (\epsilon+i)\nabla_\alpha\nabla_j u, \nabla_\alpha\nabla_j u\rangle f^{q+1}\,dx \\
&\quad -\sum_{j=1}^N \sum_{|\beta|=1,\beta\le\alpha} \int \langle (\epsilon+i)\nabla_\beta f\nabla_{\alpha-\beta}\nabla_j u, \nabla_\alpha\nabla_j u\rangle f^q\,dx \\
&\quad -\sum_{j=1}^N \sum_{|\beta|\ge 2,\beta\le\alpha} \int \langle (\epsilon+i)\nabla_\beta f\nabla_{\alpha-\beta}\nabla_j u, \nabla_\alpha\nabla_j u\rangle f^q\,dx \\
&\quad -q\sum_{j=1}^N \int \langle (\epsilon+i)\nabla_\alpha\nabla_j u, \nabla_\alpha u\rangle f^q\nabla_j f\,dx \\
&\quad -q\sum_{j=1}^N \sum_{|\beta|\ge 1,\beta\le\alpha} \int \langle (\epsilon+i)\nabla_\beta f\nabla_{\alpha-\beta}\nabla_j u), \nabla_\alpha u\rangle f^{q-1}\nabla_j f\,dx \\
&=: A_{11}+A_{12}+A_{13}+A_{14}+A_{15}. \qquad (2.7)
\end{aligned}$$

By direct computation, we have, term by term,

$$\begin{aligned}
A_{11} &= -\sum_{j=1}^N \int \langle (\epsilon+i)\nabla_\alpha\nabla_j u, \nabla_\alpha\nabla_j u\rangle f^{q+1}\,dx \\
&= -\epsilon\sum_{j=1}^N \int |\nabla_\alpha\nabla_j u|^2 f^{q+1}\,dx \quad\le\quad 0; \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
A_{12} &= -\sum_{j=1}^{N}\sum_{|\beta|=1,\beta\le\alpha}\int\langle(\epsilon+i)\nabla_\beta f\nabla_{\alpha-\beta}\nabla_j u,\nabla_\alpha\nabla_j u\rangle f^q\,dx\\
&= \sum_{j=1}^{N}\sum_{|\beta|=1,\beta\le\alpha}\left\{\frac{\epsilon}{2}\int|\nabla_{\alpha-\beta}\nabla_j u|^2\nabla_\beta\left(f^q\nabla_\beta f\right)\,dx\right.\\
&\quad\left.+\int\langle i\nabla_\alpha\nabla_j u,\nabla_{\alpha-\beta}\nabla_j u\rangle f^q\nabla_\beta f\,dx\right\}\\
&\le C(f)\|u\|_{H^l}^2+\sum_{j=1}^{N}\sum_{|\beta|=1,\beta\le\alpha}\int\langle i\nabla_\alpha\nabla_j u,\nabla_{\alpha-\beta}\nabla_j u\rangle f^q\nabla_\beta f\,dx; \qquad (2.9)
\end{aligned}$$

$$\begin{aligned}
A_{13} &= -\sum_{j=1}^{N}\sum_{|\beta|\ge2,\beta\le\alpha}\int\langle(\epsilon+i)\nabla_\beta f\nabla_{\alpha-\beta}\nabla_j u,\nabla_\alpha\nabla_j u\rangle f^q\,dx\\
&= \sum_{j=1}^{N}\sum_{|\beta|\ge2,\beta\le\alpha}\left\{\int\langle(\epsilon+i)\nabla_\beta\nabla_j f\nabla_{\alpha-\beta}\nabla_j u,\nabla_\alpha u\rangle f^q\,dx\right.\\
&\quad+\int\langle(\epsilon+i)\nabla_\beta f\nabla_{\alpha-\beta}\Delta u,\nabla_\alpha u\rangle f^q\,dx\\
&\quad\left.+q\int\langle(\epsilon+i)\nabla_\beta f\nabla_{\alpha-\beta}\nabla_j u,\nabla_\alpha u\rangle f^{q-1}\nabla_j f\,dx\right\}\\
&\le C(f)\|u\|_{H^l}^2, \qquad (2.10)
\end{aligned}$$

where the last inequality follows from the Hölder's inequality;

$$\begin{aligned}
A_{14} &= -q\sum_{j=1}^{N}\int\langle(\epsilon+i)\nabla_\alpha\nabla_j u,\nabla_\alpha u\rangle f^q\nabla_j f\,dx\\
&= \sum_{j=1}^{N}\left\{\frac{q\epsilon}{2}\int|\nabla_\alpha u|^2\nabla_j\left(f^q\nabla_j f\right)\,dx-q\int\langle i\nabla_\alpha\nabla_j u,\nabla_\alpha u\rangle f^q\nabla_j f\,dx\right\}\\
&\le C(f)\|u\|_{H^l}^2-q\sum_{j=1}^{N}\int\langle i\nabla_\alpha\nabla_j u,\nabla_\alpha u\rangle f^q\nabla_j f\,dx; \qquad (2.11)
\end{aligned}$$

and

$$\begin{aligned}
A_{15} &= -q\sum_{j=1}^{N}\sum_{|\beta|\ge1,\beta\le\alpha}\int\langle(\epsilon+i)\nabla_\beta f\nabla_{\alpha-\beta}\nabla_j u),\nabla_\alpha u\rangle f^{q-1}\nabla_j f\,dx\\
&\le C(f)\|u\|_{H^l}^2. \qquad (2.12)
\end{aligned}$$

Substituting (2.8)–(2.12) into (2.7), we obtain

$$\begin{aligned} A_1 \;\le\; & \sum_{j=1}^{N} \sum_{|\beta|=1,\beta\le\alpha} \int \langle i\nabla_\alpha \nabla_j u, \nabla_{\alpha-\beta} u\rangle f^q \nabla_\beta f\, dx \\ & -q \sum_{j=1}^{N} \int \langle i\nabla_\alpha \nabla_j u, \nabla_\alpha u\rangle f^q \nabla_j f\, dx + C(f)\|u\|_{H^l}^2. \qquad (2.13) \end{aligned}$$

For the term A_2, a direct computation leads to

$$\begin{aligned} A_2 &= (p-1)\int \langle i\nabla_\alpha(\nabla f\cdot\nabla u), \nabla_\alpha u\rangle f^q\, dx \\ &= (p-1)\sum_{j=1}^{N}\int \langle i\nabla_j f \nabla_\alpha \nabla_j u, \nabla_\alpha u\rangle f^q\, dx \\ &\quad +(p-1)\sum_{|\beta|\ge 2,\beta\le\alpha}\int \langle i\nabla_\beta\nabla_j f\nabla_{\alpha-\beta}\nabla_j u, \nabla_\alpha u\rangle f^q\, dx \\ &\le (p-1)\sum_{j=1}^{N}\int \langle i\nabla_\alpha\nabla_j u, \nabla_\alpha u\rangle f^q \nabla_j f\, dx + C(f)\|u\|_{H^l}^2. \qquad (2.14) \end{aligned}$$

Hence, it follows from (2.13) and (2.14) that

$$\sum_{|\alpha|=l}\{A_1 + A_2\} \le C(f)\|u\|_{H^l}^2. \qquad (2.15)$$

Note that here we have used the facts that $q = l + p - 1$ and

$$\begin{aligned} & \sum_{|\alpha|=l}\sum_{j=1}^{N}\sum_{|\beta|=1,\beta\le\alpha}\int \langle i\nabla_\alpha\nabla_j u, \nabla_{\alpha-\beta}\nabla_j u\rangle f^q \nabla_\beta f\, dx \\ & = l\sum_{|\alpha|=l}\sum_{j=1}^{N}\int \langle i\nabla_\alpha\nabla_j u, \nabla_\alpha u\rangle f^q\nabla_j f\, dx. \qquad (2.16) \end{aligned}$$

To complete the proof of the Lemma, we need an estimate on the term A_3. We will do so by making use of a technique of [13]. First, we note that

$$\begin{aligned} A_3 &= \int \langle i\nabla_\alpha(|u|^2 u), \nabla_\alpha u\rangle f^\beta\, dx \\ &\leq C(f)\Big\{\int |\nabla_\alpha u|^2 |u|^2\, dx + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|,|\gamma|\geq 1}} \int |\nabla_\alpha u|\cdot|\nabla_\beta u|\cdot|\nabla_\gamma u|\cdot|u|\, dx \\ &\quad + \sum_{\substack{\beta+\gamma+\theta=\alpha \\ |\beta|,|\gamma|,|\theta|\geq 1}} \int |\nabla_\alpha u|\cdot|\nabla_\beta u|\cdot|\nabla_\gamma u|\cdot|\nabla_\theta u|\, dx\Big\} \\ &=: C(f)\{A_{31}+A_{32}+A_{33}\}. \end{aligned} \tag{2.17}$$

The remainder of the proof comprises two cases:

Case I: $l \leq k_0$. From the Sobolev imbedding theorem we have

$$\|u\|_{L^\infty} \leq C(N)\|u\|_{H^{k_0}}, \tag{2.18}$$

where $C(N)$ is the Sobolev constant which depends only on the dimension N. Thus,

$$\begin{aligned} A_{31} &= \int |\nabla_\alpha u|^2 |u|^2\, dx \\ &\leq \|u\|_{L^\infty}^2 \int |\nabla_\alpha u|^2\, dx \leq C(N)\|u\|_{H^{k_0}}^4. \end{aligned} \tag{2.19}$$

Now we assume $l \geq 2$ and note that the term A_{32} does not appear unless this is the case. Given $1 \leq s_1, s_2 \leq l-1 \leq k_0-1$, $s_1+s_2=l$, since

$$0 \leq \frac{1}{2} + \frac{s_j-1}{N} - \frac{k_0-1}{N} < \frac{1}{2}, \quad j=1,\ 2,$$

and

$$\sum_{j=1}^{2}\Big(\frac{1}{2} + \frac{s_j-1}{N} - \frac{k_0-1}{N}\Big) = 1 + \frac{l}{N} - \frac{2k_0}{N} < \frac{1}{2},$$

we can choose p_{s_j}'s such that

$$\frac{1}{2} + \frac{s_j-1}{N} > \frac{1}{p_{s_j}} > \frac{1}{2} + \frac{s_j-1}{N} - \frac{k_0-1}{N}, \quad j=1,\ 2,$$

and

$$\frac{1}{p_{s_1}} + \frac{1}{p_{s_2}} \;=\; \frac{1}{2}.$$

Then, by the Gagliardo-Nirenberg inequality [3], there exist $\frac{s_j-1}{k_0-1} \le r_j < 1$ such that for any multi-index μ with $|\mu| = s_j$,

$$\begin{aligned} \left(\int |\nabla_\mu u|^{p_{s_j}}\, dx\right)^{1/p_{s_j}} &\le C\left(\|u\|_{H^{k_0}}\right)^{r_j}\left(\|\nabla u\|_{L^2}\right)^{1-r_j} \\ &\le C\|u\|_{H^{k_0}}, \quad j = 1,\ 2, \end{aligned} \tag{2.20}$$

where the constants $C = C(N, s_j, p_{s_j}, k_0, r_j)$ are independent of u, f and ϵ.

Now let $s_1 = |\beta|$ and $s_2 = |\gamma|$. Then by Hölder's inequality and the above argument,

$$\begin{aligned} A_{32} &= \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|,|\gamma|\ge 1}} \int |\nabla_\alpha u|\cdot|\nabla_\beta u|\cdot|\nabla_\gamma u|\cdot|u|\, dx \\ &\le \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|,|\gamma|\ge 1}} \|u\|_{L^\infty}\left(\int |\nabla_\alpha u|^2\, dx\right)^{1/2} \\ &\quad \cdot\left(\int |\nabla_\beta u|^{p_{s_1}}\, dx\right)^{1/p_{s_1}}\left(\int |\nabla_\gamma u|^{p_{s_2}}\, dx\right)^{1/p_{s_2}} \\ &\le C(N)\|u\|_{H^{k_0}}^4. \end{aligned} \tag{2.21}$$

Similarly, for the term A_{33}, let $s_1 = |\beta|$, $s_2 = |\gamma|$ and $s_3 = |\theta|$. Then given $1 \le s_1, s_2, s_3 \le l-1 \le k_0 - 1$, $s_1 + s_2 + s_3 = l$, we can choose positive p_{s_j}'s such that

$$\frac{1}{p_{s_j}} \;>\; \frac{1}{2} + \frac{s_j - 1}{N} - \frac{k_0 - 1}{N}, \qquad j = 1,\ 2,\ 3,$$

and

$$\frac{1}{p_{s_1}} + \frac{1}{p_{s_2}} + \frac{1}{p_{s_3}} \;=\; \frac{1}{2}.$$

By the same argument as above, we have

$$\begin{aligned} A_{33} &= \sum_{\substack{\beta+\gamma+\theta=\alpha \\ |\beta|,|\gamma|,|\theta|\geq 1}} \int |\nabla^\alpha u| \cdot |\nabla^\beta u| \cdot |\nabla^\gamma u| \cdot |\nabla^\theta u| \, dx \\ &\leq \sum_{\substack{\beta+\gamma+\theta=\alpha \\ |\beta|,|\gamma|,|\theta|\geq 1}} \left(\int |\nabla_\alpha u|^2 \, dx \right)^{1/2} \left(\int |\nabla_\beta u|^{p_{s_1}} \, dx \right)^{1/p_{s_1}} \\ &\qquad \cdot \left(\int |\nabla_\gamma u|^{p_{s_2}} \, dx \right)^{1/p_{s_2}} \left(\int |\nabla_\theta u|^{p_{s_3}} \, dx \right)^{1/p_{s_3}} \\ &\leq C(N) \|u\|^4_{H^{k_0}}. \end{aligned} \tag{2.22}$$

Combining the estimates (2.19), (2.21) and (2.22), we conclude that

$$A_3 \ \leq \ C(f)\{A_{31} + A_{32} + A_{33}\} \ \leq \ C(N, f)\|u\|^4_{H^{k_0}} \tag{2.23}$$

for the case $l \leq k_0$. Consequently, taking summation over $|\alpha| = l$ and $l = 0, 1, \cdots, k_0$ in (2.5) and using the estimates (2.13), (2.15) and (2.23), we obtain

$$\frac{d}{dt} \left(\sum_{l=0}^{k_0} \sum_{|\alpha|=l} \int |\nabla^l u|^2 f^{l+p-1} \, dx \right) \ \leq \ C(N, f) \left(\|u\|^4_{H^{k_0}} + \|u\|^2_{H^{k_0}} \right). \tag{2.24}$$

Finally, by the hypothesis of the Lemma, all the constants depending on f in this proof are finite when $t < \infty$. Therefore, the ordinary differential inequality (2.24) implies that for any constant $K > \|u_0\|^2_{H^{k_0}}$, we can find $T^* = T^*(K)$ such that

$$\|u\|^2_{H^{k_0}} \ \leq \ K \tag{2.25}$$

for all $t \in [0, T^*]$. This completes the proof of the Lemma in Case I.

Case II: $l \geq k_0 + 1$. We argue inductively on l. Suppose that there is a constant $C_{l-1} = C_{l-1}(N, K, u_0, f)$ such that

$$\|u\|^2_{H^{l-1}} \ \leq \ C_{l-1} \qquad \text{for all } t \in [0, T^*]. \tag{2.26}$$

From (2.18) and (2.25) we can see that

$$\begin{aligned} A_{31} &= \int |\nabla^\alpha u|^2 |u|^2 \, dx \\ &\leq \|u\|^2_{L^\infty} \int |\nabla^\alpha u|^2 \, dx \ \leq \ C(N) K \|u\|^2_{H^l}. \end{aligned} \tag{2.27}$$

To estimate the terms A_{32} and A_{33}, we proceed similarly as in Case I.

Suppose $l > 2$, $1 \le s_2 \le s_1 \le l-1$ and $s_1 + s_2 = l$. Then $s_1 \ge \frac{l}{2} > 1$ and $s_2 \le \frac{l}{2} < l-1$. As

$$\frac{1}{2} + \frac{s_1 - 1}{N} - \frac{l-1}{N} \;=\; \frac{1}{2} - \frac{s_2}{N} \;<\; \frac{1}{2},$$

$$\frac{1}{2} + \frac{s_2 - 1}{N} - \frac{l-2}{N} \;=\; \frac{1}{2} - \frac{s_1 - 1}{N} \;<\; \frac{1}{2}$$

and

$$\sum_{j=1}^{2}\left(\frac{1}{2} + \frac{s_j - 1}{N} - \frac{l-j}{N}\right) \;=\; 1 - \frac{l-1}{N} \;<\; \frac{1}{2},$$

we can choose positive p_{s_j}'s such that

$$\frac{1}{2} + \frac{s_j - 1}{N} \;>\; \frac{1}{p_{s_j}} \;>\; \frac{1}{2} + \frac{s_1 - 1}{N} - \frac{l-j}{N}, \qquad j = 1,\ 2,$$

and

$$\frac{1}{p_{s_1}} + \frac{1}{p_{s_2}} \;=\; \frac{1}{2}.$$

Then, by the Gagliardo-Nirenberg inequality, for any multi-index μ with $|\mu| = s_j$ we have

$$\left(\int |\nabla_\mu u|^{p_{s_j}}\, dx\right)^{1/p_{s_j}} \le C \left(\|u\|_{H^{l+1-j}}\right)^{r_j} \left(\|\nabla u\|_{L^2}\right)^{(1-r_j)}, \quad j = 1,\ 2, \tag{2.28}$$

where r_j satisfies

$$\frac{s_j - 1}{l-j} \le r_j \;<\; 1,$$

and

$$\frac{1}{p_{s_j}} - \frac{s_1 - 1}{N} \;=\; r_j\left(\frac{1}{2} - \frac{l-j}{N}\right) \;+\; \frac{1}{2}(1 - r_j).$$

We emphasize that the constants C in (2.28) are independent of u, f and ϵ. Without loss of generality, assume $|\beta| \ge |\gamma|$. Denote $s_1 = |\beta|$, $s_2 = |\gamma|$. Using the Hölder inequality and the above inequalities, by virtue of the assumption (2.26) we have

$$
\begin{aligned}
A_{32} &= \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|,|\gamma|\geq 1}} \int |\nabla_\alpha u|\cdot|\nabla_\beta u|\cdot|\nabla_\gamma u|\cdot|u|\,dx \\
&\leq \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|,|\gamma|\geq 1}} \|u\|_{L^\infty}\left(\int |\nabla_\alpha u|^2\,dx\right)^{1/2} \\
&\quad \cdot\left(\int |\nabla_\beta u|^{p_{s_1}}\,dx\right)^{1/p_{s_1}}\left(\int |\nabla_\gamma u|^{p_{s_2}}\,dx\right)^{1/p_{s_2}} \\
&\leq C(N,K,u_0,f,C_{l-1})\|u\|^2_{H^l}.
\end{aligned}
\tag{2.29}
$$

For the case $k_0+1\leq l\leq 2$, which necessarily arises from $l=2$, $N=1$ and $s_2=s_1=1$, which is not covered above, we note that, as in (2.21), we have

$$
\begin{aligned}
A_{32} &\leq \int |\nabla_\alpha u|\cdot|\nabla u|^2\cdot|u|\,dx \\
&\leq \|u\|_{L^\infty}\|\nabla u\|_{L^\infty}\left(\int |\nabla_\alpha u|^2\,dx\right)^{1/2}\left(\int |\nabla u|^2\,dx\right)^{1/2} \\
&\leq C(N)\|u\|^2_{H^1}\|\nabla u\|_{H^1}\left(\int |\nabla^2 u|^2\,dx\right)^{\frac{1}{2}} \\
&\leq C(N,K,u_0,f)\|u\|^2_{H^2}.
\end{aligned}
\tag{2.30}
$$

Combining (2.29) and (2.30), we conclude that for $l\geq k_0+1$,

$$A_{32}\leq C(N,K,u_0,f,C_{l-1})\|u\|^2_{H^l}. \tag{2.31}$$

Now we turn to the term A_{33}. For positive integers s_1, s_2 and s_3 with $s_1\geq s_2\geq s_3$, $1\leq s_1,s_2,s_3\leq l-2$ and $s_1+s_2+s_3=l$, it is easy to see

$$\left(\frac{1}{2}+\frac{s_1-1}{N}-\frac{l-1}{N}\right)+\left(\frac{1}{2}+\frac{s_2-1}{N}-\frac{l-2}{N}\right)+\left(\frac{1}{2}+\frac{s_3-1}{N}-\frac{l-2}{N}\right) < \frac{1}{2}.$$

Therefore, by choosing p_{s_j}'s suitably and using the same argument as in (2.31), we may employ the interpolation inequality to get

$$A_{33} \leq C(N,K,u_0,f,C_{l-1})\|u\|^2_{H^l}. \tag{2.32}$$

Hence by (2.27), (2.31) and (2.32),

$$A_3 \leq C(f)\{A_{31}+A_{32}+A_{33}\} \leq C(N,K,u_0,f,C_{l-1})\|u\|^2_{H^l} \tag{2.33}$$

for $l \geq k_0 + 1$. Consequently, substituting (2.15) and (2.33) into (2.5), summing over $|\alpha| = l$, and using the assumption (2.26), we have, for any integer $l \geq k_0 + 1$,

$$\begin{aligned} & \frac{d}{dt}\left(\sum_{|\alpha|=l} \int |\nabla_\alpha u|^2 f^{l+p-1}\, dx\right) \\ \leq \ & C(N, K, u_0, f, C_{l-1}) \left(\sum_{|\alpha|=l} \int |\nabla_\alpha u|^2\, dx + 1\right) \end{aligned} \tag{2.34}$$

for all $t \in [0, T^*]$. In view of the hypothesis (A1), the Gronwall's inequality tells us that

$$\sum_{|\alpha|=l} \int |\nabla_\alpha u|^2\, dx \ \leq \ C(N, K, u_0, f, C_{l-1}) \tag{2.35}$$

for all $t \in [0, T^*]$. Combining this estimate with the assumption (2.26), one obtain constants $C_l = C_l(m, K, u_0, f)$ such that for $l \geq k_0 + 1$

$$\|u\|^2_{H^l} \ \leq \ C_l, \quad \forall t \in [0, T^*]. \tag{2.36}$$

By fixing $K > \|u_0\|^2_{H^{k_0}}$ and letting $T = T^*(K)$, the proof of the Lemma is now complete. □

Remark 2.1 We emphasize that, in the above estimates, the dependence on u_0 is only on the Sobolev norm of u_0. In particular T depends only on $\|u_0\|_{H^{k_0}}$.

Now we are in the position to establish the local existence result. We first consider smooth initial maps $u_0 \in C_0^\infty(M)$. From Lemma 2.1 we know that there exist $T > 0$ and a positive constant $C_{s_0}(N, u_0, f)$ such that u^ϵ is defined on $[0, T] \times M$ and

$$\sup_{t \in [0,T]} \|u^\epsilon\|^2_{H^{s_0}} \leq C_{s_0}(N, u_0, f) \tag{2.37}$$

uniformly for the parameter ϵ. Therefore we can select a sequence $\{\epsilon_j\}$, $\epsilon_j \to 0$, such that $u^{\epsilon_j} \to u$ [weakly*] in $L^\infty([0, T], H^{s_0}(M))$. Obviously u is a solution of the Cauchy problem (2.1).

For general initial maps $u_0 \in H^{s_0}(M)$, one can use a sequence of smooth maps $\{u_{0,j} \in C_0^\infty(M)\}$ to approximate u_0 in H^{s_0}. From the argument above and Remark 2.1, the Cauchy problem (2.3) admits local smooth solutions u_j^ϵ on $[0,T] \times M$ with initial maps $u_{0,j}$ respectively and

$$\sup_{t\in[0,T]} \|u_j^\epsilon\|_{H^{s_0}}^2 \leq C_{s_0}(N, u_0, f) \qquad \forall j, \quad \forall 0 < \epsilon \leq 1. \tag{2.38}$$

Therefore, after relabelling if necessary, there exists a subsequence $\{u_j^\epsilon\}$ such that

$$u_j^\epsilon \longrightarrow u^\epsilon \quad \text{[weakly*]} \quad \text{in} \quad L^\infty([0,T]; H^{s_0}(M)). \tag{2.39}$$

It is easy to see that the limit u^ϵ is a classical solution to (2.3) with the initial map u_0 and the estimate (2.37) holds true for any $\epsilon \in (0,1]$. Then the same limitting procedure as in previous paragraph gives a local solution u of the Cauchy problem (2.1).

2.2 Global existence

In the previous section, we have established that, given an initial map $u_0 \in H^{s_0}(M)$, the Cauchy problem of the NNLS (2.1) admits a unique, local solution. In this section, we will show that this solution can be extended to all times when $N = 1$ and also when $N = 2$ for suitable initial maps u_0. As we have explained earlier, the main difference between the non-autonomous and the autonomous (even inhomogeneous) case is the absence of conservation laws in the former case. Thus, to establish global existence in the non-autonomous case, we will need to establish some *a priori* estimates on the Sobolev norms of the solutions. These estimates will play the role of the conservation laws in arguments used in [42] (see also [12, 35]).

Lemma 2.2 *Let M be either $\mathbb{R}^N$ or $\mathbb{T}^N$, $N = 1, 2$. Suppose that $f(t,x) > 0$ is a $C^{1,1}$ function satisfying* (A1)-(A2)*, and $f(t,\cdot) \in L^\infty(M)$ for all $t \in [0,T)$. If u is* *a solution to* (2.1) *such that* $u(t,\cdot) \in H^{k_0+2}(M)$ *for all* $t \in [0,T)$*, then*

$$\sup_{t\in[0,T)} \int |u|^2 f^{p-1}\, dx \;\le\; C(N,u_0,f,T). \tag{2.40}$$

Moreover, when $N = 1$*,*

$$\sup_{t\in[0,T)} \int |\nabla u|^2 f^{p}\, dx \;\le\; C(N,u_0,f,T). \tag{2.41}$$

If f *satisfies* (A3) *and* $\|u_0\|_{L^2}$ *is small enough,* (2.41) *also holds when* $N = 2$*.*

Proof. It is easy to see that

$$\begin{aligned} \frac{d}{dt}\int |u|^2 f^{p-1}\, dx &= (p-1)\int |u|^2 f^{p-2}\partial_t f\, dx \\ &\le C(f)\int |u|^2 f^{p-1}\, dx, \end{aligned}$$

from which (2.40) follows from the Gronwall's inequality. In particular, if f satisfies $(p-1)\partial_t f \le 0$ (part of the condition (A3)), then for any $t > 0$

$$\int |u(t,x)|^2 f^{p-1}(t,x)\, dx \le \int |u_0(x)|^2 f^{p-1}(x,0)\, dx. \tag{2.42}$$

We also note (*cf* the second conservation law in the inhomogeneous case [42]) that

$$\begin{aligned} &\frac{d}{dt}\left\{\int |\nabla u|^2 f^p\, dx - \frac{1}{2}\int |u|^4 f^{p-1}\, dx\right\} \\ &= p\int |\nabla u|^2 f^{p-1}\partial_t f\, dx - \frac{p-1}{2}\int |u|^4 f^{p-2}\partial_t f\, dx. \end{aligned}$$

By the Gagliardo-Nirenberg inequality and the estimate (2.40),

$$\begin{aligned} &\int |u|^4 f^{p-2}\partial_t f\, dx \\ &\le C(f)\int |u|^4\, dx \\ &\le C(N,f)\left(\int |\nabla u|^2\, dx + \int |u|^2\, dx\right)^{N/2}\left(\int |u|^2\, dx\right)^{(4-N)/2} \\ &\le C(N,f)\left(\int |\nabla u|^2 f^p\, dx + \int |u|^2 f^{p-1}\, dx\right)^{N/2}\left(\int |u|^2 f^{p-1}\, dx\right)^{(4-N)/2} \\ &\le C(N,u_0,f,T)\left(\int |\nabla u|^2 f^p\, dx\right)^{N/2} + C(N,u_0,f,T). \end{aligned} \tag{2.43}$$

Thus, as $N \leq 2$,

$$\begin{aligned}
&\int |\nabla u(t,\cdot)|^2 f(t,\cdot)^p \, dx - \frac{1}{2}\int |u(t,\cdot)|^4 f(t,\cdot)^{p-1}\, dx \\
&\leq \int |\nabla u_0|^2 f(\cdot,0)^p \, dx - \frac{1}{2}\int |u_0|^4 f(\cdot,0)^{p-1}\, dx \\
&\quad + C(N,u_0,f,T)\int_0^t \int |\nabla u(\cdot,s)|^2 f^p \, dxds + C(N,u_0,f,T)t. \qquad (2.44)
\end{aligned}$$

When $N = 1$, (2.41) follows immediately from (2.44) and the Gagliardo-Nirenberg inequality (see also (2.43)). For the case $N = 2$, by the assumption $(p-1)\partial_t f \leq 0$ (A3), we have the uniform estimate (2.42). Therefore, applying again the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
&\int |u(t,\cdot)|^4 f(t,\cdot)^{p-1}\, dx \\
\leq\ & \|f(t,\cdot)^{p-1}\|_{L^\infty} \int |u(t,\cdot)|^4 \, dx \\
\leq\ & \|f(t,\cdot)^{p-1}\|_{L^\infty} \|f(t,\cdot)^{1-p}\|_{L^\infty} \|f(t,\cdot)^{-p}\|_{L^\infty} \left(\int |\nabla u(t,\cdot)|^2 f(t,\cdot)^p \, dx\right) \\
& \cdot \left(\int |u(t,\cdot)|^2 f(t,\cdot)^{p-1}\, dx\right) + C(u_0,f,T) \\
\leq\ & C\left(\int |\nabla u(t,\cdot)|^2 f(t,\cdot)^p \, dx\right)\left(\int |u(\cdot,0)|^2 f(\cdot,0)^{p-1}\, dx\right) \\
& + C(u_0,f,T), \qquad (2.45)
\end{aligned}$$

where the positive constant C, by the assumption (A3), is independent of t. We can see easily that on the left hand side of (2.44), the second term can be absorbed by the first term as long as $\|u_0\|_{L^2}$ is small enough. In this case,

$$\begin{aligned}
\int |\nabla u(t,\cdot)|^2 f^p \, dx \ \leq\ & C(N,u_0,f,T)\int_0^t \int |\nabla u(\cdot,s)|^2 f^p \, dxds \\
& + C(N,u_0,f,T)t + C(N,u_0,f,T), \qquad (2.46)
\end{aligned}$$

and the desired estimate (2.41) follows from the Gronwall's inequality. □

In order to establish global existence in the case $N = 2$, we will need to derive some *a priori* estimates for the H^2-norm of the solution u (see Remark 2.1). To do so, we refer to the following result due to Brezis and Gallouet ([6], Lemma 2 with slight modification):

Lemma 2.3 *Let M be either $\mathbb{R}^2$ or $\mathbb{T}^2$. Then*

$$\|v\|_{L^\infty(M)} \;\le\; C(M)\left(1+\sqrt{\log(1+\|v\|_{H^2(M)})}\right)$$

for every $v \in H^2(M)$ with $\|v\|_{H^1(M)} \le 1$.

Now we are ready to complete the proof of Theorem 2.1:

Proof of Theorem 2.1. Let u be the solution of the Cauchy problem (2.1) existing on the maximal time interval $[0, T)$ such that $u(t, \cdot) \in H^{s_0}(M)$ for all $t \in [0, T)$. Suppose $T < \infty$. We will derive contradictions in the one- and two-spatial-dimensional cases separately.

Case A: $N = 1$. From Lemma 2.2, we know that there exists a positive constant $C(u_0, f, T)$ such that

$$\sup_{t\in[0,T)} \|u\|_{H^1} \le C(u_0, f, T). \tag{2.47}$$

Then, for $0 < \delta < T$, by the local existence result of Section 3, the NNLS (2.1) for u_δ satisfying the initial data

$$u_\delta(x, T-\delta) \;=\; u(x, T-\delta)$$

has a solution u_δ on the time interval $[T-\delta, T-\delta+\eta)$ for some $\eta > 0$. Since we have uniform bounds (independent of δ) on $\|u\|_{H^1}$ if $T < \infty$ as given in (2.47), by Remark 2.1, it follows that η is independent of δ. Thus, if we choose δ sufficiently small, we have

$$T - \delta + \eta \;>\; T.$$

However, by Proposition 2.1, u_δ and u coincide on $M \times [T-\delta, T)$, and therefore u_δ extends u beyond the maximal time interval of existence. This is a contradiction.

Case B: $N = 2$. Lemma 2.2 shows that if $\|u\|_{L^2}$ is small enough, then there exists a positive constant $C(u_0, f, T)$, such that

$$\sup_{t\in[0,T)} \|u\|_{H^1} \;\le\; C(u_0, f, T). \tag{2.48}$$

Similar to the proof of Lemma 2.1 (let $\epsilon = 0$ in that argument), we have

$$\begin{aligned} &\frac{d}{dt}\left(\sum_{l=0}^{2}\sum_{|\alpha|=l}\int |\nabla_\alpha u|^2 f^{l+p-1}\,dx\right) \\ \leq\ & C(f)\left(\|u\|_{H^2}^2 + \||u|^2u\|_{H^2}\cdot\|u\|_{H^2}\right). \end{aligned} \tag{2.49}$$

In view of the Gagliardo-Nirenberg inequality, it is easy to verify that

$$\||u|^2u\|_{H^2} \ \leq\ C\|u\|_{L^\infty}^2\|u\|_{H^2}. \tag{2.50}$$

From Lemma 2.3 and the estimate (2.48) we deduce that

$$\|u\|_{L^\infty} \ \leq\ C\left(1+\sqrt{\log(1+\|u\|_{H^2})}\right).$$

Hence with the assumptions (A1)-(A3), the inequality (2.49) leads to

$$\|u(t)\|_{H^2}^2 \leq C + C\int_0^t \|u(s)\|_{H^2}^2\left(1+\log(1+\|u(s)\|_{H^2})\right)ds, \quad \forall t\in[0,T), \tag{2.51}$$

where $C = C(u_0, f, T)$. Denoting the RHS of (2.51) by $G(t)$, we have

$$G'(t) \ =\ C\|u(t)\|_{H^2}^2\left(1+\log(1+\|u(t)\|_{H^2})\right) \ \leq\ CG(t)\left(1+\log(1+G(t))\right).$$

Consequently

$$\frac{d}{dt}\log(1+\log(1+G(t))) \ \leq\ C, \quad \forall t\in[0,T),$$

and we find an estimate for $\|u\|_{H^2}$ of the form

$$\|u(t)\|_{H^2} \ \leq\ \exp(c_1\exp(c_2 t)), \qquad \forall t\in[0,T),$$

where c_1 and c_2 are constants. Thus $\|u\|_{H^2}$ remains bounded on every finite time interval. A contradiction can now be derived as in Case A and the proof of Theorem 1 is complete. □

Finally, we remark that there has been a lot of interest in the Ginzburg-Landau equation (see [9, 21, 22] and references therein)

$$\operatorname{div}(a(x)\nabla u) + (1-|u|^2)u = 0 \quad \text{in } \mathbb{R}^2$$

for a complex order parameter with a variable coefficient arising in a macroscopic description of superconductivity associated with the inhomogeneous Ginzburg-Landau functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 a(x)dx + \frac{1}{4}\int_{\mathbb{R}^2}(1-|u|^2)^2 dx.$$

We point out that, for suitable $a(x)$, our method can be used to address the global existence of the following Schrödinger flow corresponding to this functional:

$$\begin{cases} \dfrac{\partial u}{\partial t} = i\left(\mathrm{div}(a(x)\nabla u) + (1-|u|^2)u\right), \quad (t,x) \in [0,\infty)\times\mathbb{R}^2 \text{ or } [0,\infty)\times\mathbb{T}^2, \\ u(x,0) = u_0(x). \end{cases}$$

Chapter 3

Inhomogeneous NLS: Blow-up Analysis

In this chapter, we study the Cauchy problem of the inhomogeneous Schrödinger equation with spatial dimension two:

$$\begin{cases} \partial_t u = i\left(f(x)\Delta u + \nabla f(x)\cdot\nabla u + k(x)|u|^2 u\right), \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{3.1}$$

where u takes values in $\mathbb{C}$, $f(x)$ and $k(x)$ are positive real-valued functions on M ($=\mathbb{R}^2$ or $\mathbb{T}^2$) and $u_0 \in H^1(M)$. As observed in Chapter 1, this equation is the special case of the NNLS when $m = 1, N = 2$ and $p = 1$; and the nonlinearity is critical for blowup. First of all, we recall the following existence and uniqueness result for the above problem established in Chapter 2.

Theorem 3.1 *Let $s_0 \geq 4$ be an integer. Suppose $f \in C^{s_0+1}(M) \cap W^{s_0+1,\infty}(M)$ and $k \in C^{s_0}(M) \cap W^{s_0,\infty}(M)$ are real functions and $\inf_{x\in M} f(x) > 0$. Then, given $u_0 \in H^{s_0}(M)$, the Cauchy problem* (3.1) *admits a unique local smooth solution $u \in L^\infty([0,T), H^{s_0}(M))$. Moreover the solution is global in the sense that $u \in L^\infty_{\mathrm{loc}}([0,\infty), H^{s_0}(M))$ provided $\|u_0\|_{L^2}$ is small enough.*

From this result, a natural question arises: How small does the L^2-norm of the initial data have to be to guarantee global existence? The answer will be provided in this chapter in Corollaries 3.1 and 3.3. Furthermore, we will show that under appropriate conditions on f and k, we will have

$$\lim_{t\uparrow T} \|u(t)\|_{H^1} = \infty$$

for some $0 < T < \infty$. Such a solution is called a *blow-up* solution and T is called the blow-up time. In the rest of this chapter, we suppose the existence and uniqueness of the solution and only focus on the behavior of blow-up properties.

We also note that, as in the homogeneous case, one can easily check that solutions of (3.1) obey conservation of mass and energy as follows:

$$\int |u(t,x)|^2\, dx = \int |u_0(x)|^2\, dx, \tag{3.2}$$

$$E_{f,k}(u(t)) = E_{f,k}(u_0), \tag{3.3}$$

where

$$E_{f,k}(u) = \frac{1}{2}\int f(x)|\nabla u(x)|^2\, dx - \frac{1}{4}\int k(x)|u(x)|^4\, dx.$$

3.1 Blow-up analysis on $\mathbb{R}^2$

In this section, we investigate the blow-up phenomenon of the Cauchy problem (3.1) on the plane $\mathbb{R}^2$. We will study some qualitative properties, namely, L^2-concentration and L^2-minimality, of blow-up solutions. For this, we will be referring to the following conditions:

(H1) $0 < L \equiv \inf_{x\in\mathbb{R}^2} f(x) \le f(x) \le \sup_{x\in\mathbb{R}^2} f(x) < \infty,\ \forall x \in \mathbb{R}^2;$

(H2) $|x\cdot\nabla f(x)| + |\nabla f(x)| \le C,\ \forall x \in \mathbb{R}^2,$ for some $C > 0$;

(H3) there is x_0 such that $f(x_0) = L$.

(H1)′ $0 < \inf_{x\in\mathbb{R}^2} k(x) \le k(x) \le \sup_{x\in\mathbb{R}^2} k(x) \equiv K < \infty,\ \forall x \in \mathbb{R}^2;$

(H2)′ $|x\cdot\nabla k(x)| + |\nabla k(x)| \le C,\ \forall x \in \mathbb{R}^2,$ for some $C > 0$;

(H3)′ there is x_0 satisfying (H3) such that $k(x_0) = K$.

As in the homogeneous case, the blow-up solutions of the inhomogeneous equation can be described in terms of the unique radially symmetric positive solution $Q_{L,K}$ of

$$L\Delta Q + K|Q|^2 Q = Q, \quad \text{in } \mathbb{R}^2,$$

called the ground state solution (see [36] for existence and [25] for uniqueness). Our main results are as follows:

Theorem 3.2 (L^2-concentration) *Assume that* $f(x)$ *and* $k(x)$ *satisfy* (H1)-(H2) *and* (H1)′-(H2)′ *respectively. Let* $u(t)$ *be a blow-up solution of the Cauchy problem* (3.1) *and* T *its blow-up time. Then*

(i) *there is* $x(t) \in \mathbb{R}^2$ *such that* $\forall R > 0$

$$\liminf_{t \uparrow T} \int_{|x-x(t)|<R} |u(t,x)|^2 dx \geq \|Q_{L,K}\|_{L^2}^2; \tag{3.4}$$

(ii) *there is no sequence* $\{t_n\}$ *such that* $t_n \uparrow T$ *and* $u(t_n)$ *converges in* $L^2(\mathbb{R}^2)$ *as* $n \to \infty$.

Theorem 3.2 implies that blow-up solutions have a lower L^2-bound, namely, $\|u(t)\|_{L^2} \geq \|Q_{L,K}\|_{L^2}$. Therefore, as a consequence of the conservation of mass, we have a sufficient condition for the global existence of solutions. This result is sharp in the sense described in Theorems 3.4 and 3.5.

Corollary 3.1 *Assume that* $f(x)$ *and* $k(x)$ *satisfy* (H1)-(H2) *and* (H1)′-(H2)′ *respectively, then the solution* $u(t)$ *is globally defined in time provided* $\|u_0\|_{L^2} < \|Q_{L,K}\|_{L^2}$.

Theorem 3.3 (L^2-concentration: Radial case) *Let* $f(x)$ *and* $k(x)$ *be radial with respect to* x_0 *i.e.,* $f(x) = f(|x - x_0|)$ *and* $k(x) = k(|x - x_0|)$*, and satisfy* (H1)-H(2) *and* (H1)′-(H2)′ *respectively. Let* $u(t)$ *be a blow-up solution with radial (w.r.t.*

x_0) *initial data* u_0, *and* T *its blow-up time. Assume in addition that there exists* $\rho_0 > 0$ *such that for* $|x - x_0| < \rho_0$,

$$(x - x_0) \cdot \nabla k(x) \le 0 \le (x - x_0) \cdot \nabla f(x). \tag{3.5}$$

Then the following are equivalent:

(A) $|u(t,x)|^2 \to \|u_0\|_{L^2}^2 \delta_{x_0}$ *in the distribution sense as* $t \uparrow T$;

(B) $|x - x_0| u_0 \in L^2(\mathbb{R}^2)$ *and* $\lim_{t \uparrow T} \| |x - x_0| u(t) \|_{L^2} = 0.$

Theorem 3.4 (Existence) *Suppose* $f(x)$ *and* $k(x)$ *satisfy* (H1)-H(3) *and* (H1)$'$-(H3)$'$ *respectively. Assume in addition that*

$$\operatorname{curl}(\frac{x - x_0}{f(x)}) = 0 \quad (\textit{integrability condition}), \tag{3.6}$$

$$(x - x_0) \cdot \nabla f(x) \ge 0 \quad \textit{for all } x \in \mathbb{R}^2 \ \textit{or} \tag{3.7}$$

$$(x - x_0) \cdot \nabla f(x) > 0 \quad \textit{for } 0 < |x - x_0| < \rho_0 \quad \textit{for some } \rho_0 > 0, \tag{3.8}$$

and

$$(x - x_0) \cdot \nabla k(x) \le 0 \quad \textit{for all } x \in \mathbb{R}^2 \ \textit{or} \tag{3.9}$$

$$(x - x_0) \cdot \nabla k(x) < 0 \quad \textit{for } 0 < |x - x_0| < \rho_0 \quad \textit{for some } \rho_0 > 0. \tag{3.10}$$

Then there exists $\epsilon_0 > 0$ *such that* $\forall \epsilon \in (0, \epsilon_0)$, *there is* $\phi_\epsilon \in H^1$ *such that*

(a) $\|\phi_\epsilon\|_{L^2} = \|Q_{L,K}\|_{L^2} + \epsilon$,

(b) u_ϵ *blows up in finite time where* u_ϵ *is the solution of* (3.1) *with initial data* ϕ_ϵ.

Moreover, $\epsilon_0 = \infty$ *if* $f(x)$ *and* $k(x)$ *satisfy* (3.7) *and* (3.9) *respectively.*

Remark 3.1 Let $b(x) = (b_1(x), b_2(x))$ be a smooth map from $\mathbb{R}^2$ into $\mathbb{R}^2$. If

$$\operatorname{curl}(b(x)) = 0, \quad \text{i.e.,} \ \frac{\partial b_1}{\partial x_2} = \frac{\partial b_2}{\partial x_1},$$

then there exist a function $a(x)$ with $\nabla a(x) = b(x)$. In particular, the integrability condition (3.6) implies that there exists ψ such that $\nabla\psi(x - x_0) = (x - x_0)/f(x)$. It is also easy to check that if f is radial with respect to x_0, then (3.6) is fulfilled automatically. Also, the assumption (3.8) can be weakened to

$$\begin{cases} (x - x_0) \cdot \nabla f(x) \geq 0 \quad \text{for } |x - x_0| < \rho_0 \\ (x - x_0) \cdot \nabla f(x) > 0 \quad \text{on } S, \end{cases}$$

where S is a closed curve (hypersurface for higher dimension) contained in $\{|x - x_0| < \rho_0\}$ with x_0 in its interior, and (3.10) can be weakened similarly.

Theorem 3.5 (L^2-minimal blow-up solutions) *Assume $\|u_0\|_{L^2} = \|Q_{L,K}\|_{L^2}$ and $u(t)$ is the solution of* (3.1). *Let $f(x)$, $k(x)$ satisfy* (H1)-H(2) *and* (H1)$'$-(H2)$'$ *respectively. Suppose there are $\gamma_0 > 0$, $R_0 > 0$ such that*

$$f(x) \geq L + \gamma_0 \text{ for } |x| > R_0, \text{ and } \mathcal{M} = \{x; f(x) = L\} \text{ is finite} \tag{3.11}$$

$$\text{or} \quad k(x) \leq K - \gamma_0 \text{ for } |x| > R_0, \text{ and } \mathcal{M}' = \{x; k(x) = K\} \text{ is finite.} \tag{3.12}$$

(i) *If $u(t)$ blows up in finite time T, then there exists $y_0 \in \mathcal{M} \cap \mathcal{M}'$ such that*

$$|u(t, x)|^2 \to \|Q_{L,K}\|_{L^2}^2 \delta_{y_0}, \text{ in the distribution sense as } t \uparrow T,$$

$$|x - y_0| u_0 \in L^2(\mathbb{R}^2) \quad \text{and} \quad \lim_{t\uparrow T} \||x - y_0| u(t, x)\|_{L^2} = 0.$$

(ii) *Assume in addition that for each $y_0 \in \mathcal{M} \cap \mathcal{M}'$, there are $\rho_0 > 0$, $\alpha_0 \in (0, 1)$, $c_0 > 0$ such that for $|x - y_0| < \rho_0$*

$$(x - y_0) \cdot \nabla f(x) \geq c_0 |x - y_0|^{1+\alpha_0} \quad \text{or} \quad (x - y_0) \cdot \nabla k(x) \leq -c_0 |x - y_0|^{1+\alpha_0}, \tag{3.13}$$

then $u(t)$ does not blow up in finite time.

As a direct consequence of the above theorem, we have:

Corollary 3.2 *Under the same assumption as in Theorem* 3.5. *If $\mathcal{M} \cap \mathcal{M}' = \emptyset$, then there is no blow-up solution to* (3.1) *with $\|u_0\|_{L^2} = \|Q_{L,K}\|_{L^2}$.*

Remark 3.2 Note that, in contrast with Theorem 3.3, in Theorem 3.5, the initial data u_0, the functions $f(x)$ and $k(x)$ are not assumed to be radial with respect to y_0. For the general initial data u_0 with $\|u_0\|_{L^2} > \|Q_{L,K}\|_{L^2}$, it is not known whether the concentration point of the blow-up solution is a critical point of either $f(x)$ or $k(x)$.

Remark 3.3 Our arguments are also essentially valid for the general setting on $\mathbb{R}^N$ for the inhomogeneous NLS

$$\partial_t u = i\left(f(x)\Delta u + \nabla f(x)\cdot\nabla u + k(x)|u|^{\frac{4}{N}}u\right).$$

Note that the above equation is in general not conformally equivalent to the equation

$$\partial_t u = i\left(\Delta u + \tilde{k}(x)|u|^{\frac{4}{N}}u\right),$$

which was studied by Merle [29].

To minimize technicalities, we shall assume $k(x) \equiv 1$ in the sequel. The proofs for the non-constant function $k(x)$ follow essentially the same arguments with some modifications. Notationally, we write $Q_L = Q_{L,1}$ and $E_f = E_{f,1}$; when no confusion arises, we sometimes denote E_f simply as E. We note that solutions of (3.1) satisfy $E_L(u) \leq E(u)$.

3.1.1 Preliminaries

In this section, we collect a few basic results which will be used in the subsequent sections.

Lemma 3.1 *Let $u(t)$ be a solution of (2.1), and let $\phi, \tilde{\psi} \in C^4(\mathbb{R}^2)$ be functions with compact support (up to constants) that satisfy*

$$\nabla\tilde{\psi}(x - x_0) = \frac{\nabla\phi(x - x_0)}{f(x)}.$$

Then,

$$\frac{d}{dt}\int \tilde{\psi}(x-x_0)|u(t,x)|^2\,dx = 2\ \mathrm{Im}\int \nabla\phi(x-x_0)\cdot\nabla u(t,x)\bar{u}(t,x)\,dx,$$

and

$$\begin{aligned}
&\frac{d^2}{dt^2}\int \tilde{\psi}(x-x_0)|u(t,x)|^2\,dx\\
&= -\int \Delta\phi(x-x_0)|u(t,x)|^4\,dx\\
&\quad +4\ \mathrm{Re}\int (\nabla^2\phi(x-x_0)\cdot\nabla u(t,x))\cdot\nabla\bar{u}(t,x)f(x)\,dx \qquad (3.14)\\
&\quad -\ 2\int \nabla\phi(x-x_0)\cdot\nabla f(x)|\nabla u(t,x)|^2\,dx\\
&\quad -\int (\Delta^2\phi(x-x_0)+\nabla\Delta\phi(x-x_0)\cdot\nabla f(x))|u(t,x)|^2\,dx.
\end{aligned}$$

Proof. By a straightforward computation,

$$\begin{aligned}
\frac{d}{dt}\int \tilde{\psi}(x-x_0)|u(t,x)|^2\,dx &= 2\ \mathrm{Re}\int \tilde{\psi}(x-x_0)u_t(t,x)\bar{u}(t,x)\,dx\\
&= -\ 2\ \mathrm{Im}\int \tilde{\psi}(x-x_0)\mathrm{div}\,(f(x)\nabla u(t,x))\,\bar{u}(t,x)\,dx\\
&= 2\ \mathrm{Im}\int \nabla\phi(x-x_0)\cdot\nabla u(t,x)\bar{u}(t,x)\,dx.
\end{aligned}$$

Differentiating again,

$$\begin{aligned}
&2\,\frac{d}{dt}\ \mathrm{Im}\int \nabla\phi(x-x_0)\cdot\nabla u(x)\bar{u}(x)\,dx\\
&=\ 2\ \mathrm{Im}\int \nabla\phi(x-x_0)\cdot\nabla u_t(x)\bar{u}(x)\,dx + 2\ \mathrm{Im}\int \nabla\phi(x-x_0)\cdot\nabla u(x)\bar{u}_t(x)\,dx\\
&=\ 4\ \mathrm{Im}\int \nabla\phi(x-x_0)\cdot\nabla u(x)\bar{u}_t(x)\,dx - 2\ \mathrm{Im}\int \Delta\phi(x-x_0)u_t(x)\bar{u}(x)\,dx\\
&= -\ 4\ \mathrm{Re}\int \nabla\phi(x-x_0)\cdot\nabla u(x)|u(x)|^2\bar{u}(x)\,dx\\
&\quad -4\ \mathrm{Re}\int \nabla\phi(x-x_0)\cdot\nabla u(x)\nabla\left(f(x)\nabla\bar{u}(x)\right)\,dx\\
&\quad -\ 2\ \mathrm{Re}\int \Delta\phi(x-x_0)\nabla(f(x)\nabla u(x))\bar{u}(x)\,dx - 2\ \mathrm{Re}\int \Delta\phi(x-x_0)|u(x)|^4\,dx\\
&= -\int \Delta\phi(x-x_0)|u(x)|^4\,dx + 4\ \mathrm{Re}\int (\nabla^2\phi(x-x_0)\cdot\nabla u(x))\cdot\nabla\bar{u}(x)f(x)\,dx\\
&\quad +\ 2\int \nabla\phi(x-x_0)\cdot\nabla(|\nabla u(x)|^2)f(x)\,dx +\ 2\int \Delta\phi(x-x_0)f(x)|\nabla u(x)|^2\,dx\\
&\quad +\int \nabla\Delta\phi(x-x_0)\cdot\nabla(|u(x)|^2)f(x)\,dx,
\end{aligned}$$

from which (3.14) follows upon integration by parts. $\square$

Lemma 3.2 *Let* $u(t,x)$ *be the solution of* (3.1) *for* $t \in [0,T)$. *Suppose* $|x-x_0|u_0 \in L^2$, $\psi(x) \geq 0$ *and* $\nabla\psi(x-x_0) = \dfrac{x-x_0}{f(x)}$. *Then, for* $t \in [0,T)$,

$$\begin{aligned} &\frac{d^2}{dt^2}\int \psi(x-x_0)|u(t,x)|^2\,dx \\ &= \ 8E(u_0) - 2\int (x-x_0)\cdot\nabla f(x)|\nabla u(t,x)|^2\,dx. \end{aligned} \tag{3.15}$$

Proof. As for the previous Lemma, the proof is a straightforward computation. We remark that $\psi(x-x_0) \sim |x-x_0|^2$ and the regularity (at least H^2) of u guarantees that the left hand side of (3.15) makes sense. $\square$

Lemma 3.3 *Let* $\eta(x) \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ *and* $\Omega = \mathrm{supp}(\eta)$. *Then there is a constant* $c(N) > 0$ *such that, for all* $v \in H^1(\mathbb{R}^N)$,

$$\begin{aligned} \int |v(x)|^{\frac{4}{N}+2}\eta^2(x)\,dx \ &\leq \ c(N)\Big(\int_\Omega v^2(x)\,dx\Big)^{2/N} \\ &\cdot\Big\{\int_\Omega \eta^2(x)|\nabla v(x)|^2\,dx + \|\nabla\eta\|^2_{L^\infty}\int_\Omega v^2(x)\,dx\Big\}. \end{aligned}$$

Proof. See [32] (Appendix A) or [28]. $\square$

Lemma 3.4 ([44]) *For any* $v \in H^1$,

$$\frac{L}{2}\left[1 - \left(\frac{\|v\|_{L^2}}{\|Q_L\|_{L^2}}\right)^2\right]\|\nabla v\|^2_{L^2} \leq E_L(v).$$

Lemma 3.5 *There are positive constants* c_1 *and* c_2 *such that*

$$|\nabla Q_{L,K}(x)| \leq c_1\exp(-c_2|x|), \quad \forall x \in \mathbb{R}^2.$$

Proof. It suffices to prove the result for $Q = Q_1$. We follow the idea of [4]. From [36], we know that Q decreases exponentially. Thus, for $r = |x|$ large enough, say $r \geq r_0 > 0$, $|Q|^2Q - Q < 0$. Hence

$$-\frac{d^2Q}{dr^2} \leq -\frac{d^2Q}{dr^2}Q - \frac{1}{r}\frac{dQ}{dr} = |Q|^2Q - Q < 0.$$

It follows that, for $r > r_0 + 1$, one has

$$0 \geq \frac{dQ}{dr} \geq \int_{r-1}^{r}\frac{dQ}{dr}(s)\,ds = Q(r) - Q(r-1) \geq -C\exp(-C(r-1)),$$

which implies the result. $\square$

3.1.2 L^2-concentration

For the homogeneous NLS, blow-up phenomena have been observed in L^2 as well as in H^1. Particularly, concentration occurs in L^2 for blow-up solutions. This phenomenon persists in the inhomogeneous case and the corresponding results are stated in Theorems 3.2 and 3.3. We investigate this in detail and give the proofs of these results in this section.

General case First of all, we introduce a crucial lemma which is essentially due to Merle [29] (Propositions 2.4, 2.5 and Corollary 2.7) (see also [45] for the second part). For the reader's convenience, we shall give a sketch of the proof here.

Lemma 3.6 ([29]) *Suppose $f(x)$ satisfies* (H1)-(H2) *and let $\{u_n\}$ be such that*

$$\|u_n\|_{L^2} \leq c_1 \text{ and } E_L(u_n) \leq c_2;$$
$$\lambda_n = \|\nabla u_n\|_{L^2}/\|\nabla Q_L\|_{L^2} \to \infty \text{ as } n \to \infty.$$

Then there exist $x_n \in \mathbb{R}^2$ such that for all $R > 0$,

$$\liminf_{n\to\infty} \frac{\|u_n\|_{L^2(B(x_n,R))}}{\|Q_{f(x_n)}\|_{L^2}} \geq 1.$$

Moreover if $\|u_n\|_{L^2} \to \|Q_L\|_{L^2}$ as $n \to \infty$, then there exist $\tilde{x}_n \in \mathbb{R}^2, \theta_n \in \mathbb{R}$ such that

$$\frac{1}{\lambda_n} e^{i\theta_n} u_n\left(\frac{\cdot + \tilde{x}_n}{\lambda_n}\right) \to Q_L(\cdot) \text{ in } H^1 \text{ as } n \to \infty.$$

Sketch of the proof. We follow the idea of [29] and [18]. First of all, we introduce the following non-vanishing result.

Lemma 3.7 *Assume that $v_n \in H^1$ such that*

$$\int |v_n(x)|^2 dx \leq c_1, \quad \int |\nabla v_n(x)|^2 dx \leq c_2, \quad \int |v_n(x)|^4 \geq c_3.$$

Then there exist a constant $c_4 = c_4(c_1, c_2, c_3) > 0$ *and a sequence* $\{x_n \in \mathbb{R}^2\}$ *such that*

$$\int_{|x-x_n|<1} |v_n(x)|^2 dx > c_4. \tag{3.16}$$

Proof. Clearly there exists $\{x_n \in \mathbb{R}^2\}$ such that for all n,

$$\int_{S_n} |v_n(x)|^4 dx \geq c_5 \int_{S_n} (|\nabla v_n(x)|^2 + |v_n(x)|^2) dx,$$

where S_n is the unit square of center x_n and $c_5 = c_3/(2c_1 + 2c_2)$, for if not, we would obtain $c_3 \leq c_5(c_1 + c_2) \leq c_3/2$ which is a contradiction. Therefore it follows from the Sobolev inequality that

$$\left(\int_{S_n} |v_n(x)|^4 dx\right)^{1/2} \leq c \int_{S_n} (|\nabla v_n(x)|^2 + |v_n(x)|^2) dx,$$

which implies

$$\int_{S_n} |v_n(x)|^4 dx \geq c_6 > 0, \tag{3.17}$$

where c, c_6 are independent of n.

To see (3.16), assume by contradiction that there is a subsequence $\{v_n\}$ (relabelled) such that

$$\int_{S_n} |v_n(x)|^2 dx \to 0 \quad \text{as } n \to \infty,$$

which implies

$$v_n(x_n + \cdot) \to 0 \quad \text{weakly in } L^2(S_0) \text{ as } n \to \infty, \tag{3.18}$$

where S_0 is the unit square centered at the origin. Moreover we can assume that

$$v_n(x_n + \cdot) \to v \quad \text{weakly in } H^1(S_0) \text{ as } n \to \infty,$$

for some $v \in H^1(S_0)$. Then

$$v_n(x_n + \cdot) \to v \quad \text{strongly in } L^4(S_0) \text{ as } n \to \infty. \tag{3.19}$$

Thus it follows from (3.18) and (3.19)

$$\int_{S_n} |v_n(x)|^4 dx \to 0 \quad \text{as } n \to \infty,$$

which is a contradiction to (3.17). The lemma is proved. □

The proof of the second part of Lemma 3.6 can be found in [45], hence is omitted here. In the following, we verify the first part of Lemma 3.6 by making use of the concentration compactness principle. For simplicity, we will only prove a weak version of the result. The proof of the strong version is essentially the same and makes use of the observations that

$$\|Q_{f(x)}\|_{L^2}^2 = \frac{f(x)}{L}\|Q_L\|_{L^2}^2,$$
$$\lim_{n\to\infty} \sup_{|x-y|<R} |f(\frac{x}{\lambda_n}) - f(\frac{y}{\lambda_n})| = 0, \quad \forall R > 0,$$

where $\lambda_n = \|\nabla u_n\|_{L^2} \to \infty$ as $n \to \infty$.

Lemma 3.8 *Suppose $f(x)$ satisfies* (H1)-(H2). *Let $\{u_n\}$ be such that $\|u_n\|_{L^2}^2 \leq C_1$, $E_L(u_n) \leq C_2$, and $\|\nabla u_n\|_{L^2} \to \infty$ as $n \to \infty$. Then there exists $\{x_n\}$ such that for all $R > 0$,*

$$\liminf_{n\to\infty} \frac{\|u_n\|_{L^2(B(x_n,R))}}{\|Q_L\|_{L^2}} \geq 1.$$

Proof. We argue by contradiction. Suppose there are $R_0 > 0, \gamma_0 > 0$ and a subsequence $\{u_n\}$ (relabelled) such that

$$\sup_{x\in\mathbb{R}^2} \left(\int_{|x-y|<R_0} |u_n(y)|^2 dy \right) \leq \|Q_L\|_{L^2}^2 - \gamma_0.$$

Consider the scaling

$$U_n(x) = \lambda_n^{-1} u_n(\lambda_n^{-1} x),$$

where $\lambda_n = \|\nabla u_n\|_{L^2}$. It is easy to verify that

$$\|U_n\|_{L^2}^2 = \|u_n\|_{L^2}^2 \leq C_1, \qquad \|\nabla U_n\|_{L^2} = 1,$$
$$\liminf_{n\to\infty} E_L(U_n) = \liminf_{n\to+\infty} \frac{E_L(u_n)}{\lambda_n^2} \leq 0,$$
$$\sup_{x\in\mathbb{R}^2} \left(\int_{|x-y|<R} |U_n(y)|^2 dy \right) \leq \|Q_L\|_{L^2}^2 - \gamma_0, \quad \forall 0 < R \leq \lambda_n R_0.$$

Therefore, extracting a subsequence (still labelled by U_n), we have

$$\int |U_n(x)|^4 dx \geq 2L \int |\nabla U_n(x)|^2 dx - L = L, \quad \text{for large } n, \tag{3.20}$$

$$\liminf_{n\to\infty} \left(\sup_{x\in\mathbb{R}^2} \int_{|x-y|<R} |U_n(y)|^2 dy \right) \leq \|Q_L\|_{L^2}^2 - \gamma_0, \quad \forall R > 0. \tag{3.21}$$

Applying the concentration compactness principle, by Lemma 3.7 (and its proof), we have the dichotomy

$$U_n = U_n^1 + \tilde{U}_n^1$$

such that, for a sequence $\{x_n^1 \in \mathbb{R}^2\}$ and some $\psi_1 \in H^1$,

$$U_n^1(x_n^1 + \cdot) \to \psi_1 \quad \text{weakly in } H^1, \text{ locally (strongly) in } L^4 \text{ and } L^2 \quad \text{as } n \to \infty,$$

$$\int_{|x-x_n^1|<1} |U_n^1(x)|^4 dx \geq C, \quad \text{and} \quad \int_{|x-x_n^1|<1} |U_n^1(x)|^2 dx \geq \gamma_1,$$

where C and γ_1 are positive constants depending only on C_1 and L.

On one hand, from (3.21) we have

$$\liminf_{n\to\infty} \int_{|x-x_n^1|<R} |U_n^1(x)|^2 dx \leq \|Q_L\|_{L^2}^2 - \gamma_0, \quad \forall R > 0.$$

By usual techniques of concentration compactness method, we have a suitable choice of U_n^1 such that

$$\|U_n^1\|_{L^2}^2 + \|\tilde{U}_n^1\|_{L^2}^2 - \|U_n\|_{L^2}^2 \to 0 \quad \text{as } n \to \infty, \tag{3.22}$$

$$\gamma_1 \leq \|\psi_1\|_{L^2}^2 = \lim_{n\to\infty} \|U_n^1\|_{L^2}^2 \leq \|Q_L\|_{L^2}^2 - \gamma_0. \tag{3.23}$$

On the other hand,

$$\begin{aligned} E_L(\psi_1) + \liminf_{n\to\infty} E_L(\tilde{U}_n^1) &\leq \liminf_{n\to\infty} (E_L(U_n^1) + E_L(\tilde{U}_n^1)) \\ &\leq \liminf_{n\to\infty} E_L(U_n) \leq 0 \end{aligned}$$

Therefore, by Lemma 3.4 and (3.23),

$$\liminf_{n\to\infty} E_L(\tilde{U}_n^1) \leq -E_L(\psi_1) < 0.$$

Thus, extracting a subsequence (still labelled by $\tilde{U}_n^1$), we have

$$\|\tilde{U}_n^1\|_{L^2}^2 \to C_1^1 \le C_1 - \gamma_1 \quad \text{and} \quad \liminf_{n\to\infty} E_L(\tilde{U}_n^1) < 0.$$

Redefine the sequences

$$\lambda_n = \|\nabla \tilde{U}_n^1\|_{L^2} \quad \text{and} \quad U_n(x) = \lambda_n^{-1}\tilde{U}_n^1(\lambda_n^{-1}x).$$

Then, extracting a subsequence if necessary, we have

$$\|U_n\|_{L^2}^2 \to C_1^1 \le C_1 - \gamma_1, \qquad \liminf_{n\to\infty} E_L(U_n) < 0,$$
$$\liminf_{n\to\infty}\left(\sup_{x\in\mathbb{R}^2}\int_{|x-y|<R}|U_n(y)|^2dy\right) \le \|Q_L\|_{L^2}^2 - \gamma_0, \quad \forall R > 0.$$

Iterating the same procedure we can get

$$U_n = U_n^2 + \tilde{U}_n^2$$

where, for some $\{x_n^2 \in \mathbb{R}^2\}$,

$$\int_{|x-x_n^2|<1}|U_n^2(x)|^2dx \ge \gamma_1.$$

Define p as the number such that $-p\gamma_1 + C_1 < \|Q_L\|_{L^2}^2$. Applying the same procedure at most p times, we can find $j \le p$ and a function U_n^j such that (extracting a subsequence if necessary) for large n

$$E_L(\tilde{U}_n^j) < 0 \quad \text{and} \quad \|\tilde{U}_n^j\|_{L^2}^2 < \|Q_L\|_{L^2}^2.$$

This contradicts Lemma 3.4 and completes the proof. □

With Lemma 3.6 and the conservation of mass and energy in hand, we can prove Theorem 3.2 easily.

Proof of Theorem 3.2. The proof makes use of the observation that $E_L(u) \le E(u)$. Part (i) follows directly from Lemma 3.6 and the conservation of mass and energy. Part (ii) is essentially the same with that of Proposition 1 in [31]. □

Radial case Throughout the remainder of this subsection, we assume that $f(x)$ and the initial data u_0 are radial (w.r.t. x_0), $f(x)$ satisfies (H1)-(H2) and (3.5) holds.

Let $u(t)$ be a blow-up solution with blow-up time T. By the uniqueness of the solution, it is easy to see that $u(t)$ is also radial. We first establish some useful estimates by making use of Lemma 3.1.

Lemma 3.9 *Suppose that property* (A) *in Theorem 3.3 holds. Then for any $R > 0$, there exists a constant $c(R) > 0$ such that*

$$\int_0^T (T-t) \int_{|x-x_0|\geq 2R} |\nabla u(t,x)|^2 dx dt \leq c(R), \tag{3.24}$$

$$\int_0^T (T-t) \int_{|x-x_0|\geq 2R} |u(t,x)|^4 dx dt \leq c(R). \tag{3.25}$$

Moreover, $c(R)$ can be chosen to be decreasing in R.

Proof. As in [33], we choose an auxiliary function φ with certain smoothness on $\mathbb{R}$ such that

$$\varphi(r) = \begin{cases} r, & 0 \leq r < 1, \\ r-(r-1)^3, & 1 \leq r < 1+1/\sqrt{3}, \\ \text{smooth with } \varphi' \leq 0, & 1+1/\sqrt{3} \leq r < 2, \\ 0, & r \geq 2. \end{cases}$$

Define, for $R > 0$,

$$\varphi_R(r) = R\varphi(r/R)$$

and

$$\phi_R(x) = \int_0^{|x|} \varphi_R(s) ds.$$

It is easy to see that

$$\Delta\phi_R(x) \equiv 2 \quad \text{for } |x| \leq R$$

$$|\nabla\Delta\phi_R(x-x_0)\cdot\nabla f(x)| + |\nabla(2-\Delta\phi_R(x-x_0))^{1/2}| \leq \frac{C}{R} \quad \text{for all } x \in \mathbb{R}^2$$

$$|\Delta^2\phi_R(x)| \leq \frac{C}{R^2} \quad \text{for all } x \in \mathbb{R}^2.$$

Since $f(x)$ is radial, we can find nonnegative radial functions ψ_R such that for all $x \in \mathbb{R}^2$,

$$\psi_R(x-x_0) \sim \phi_R(x-x_0) \quad \text{and} \quad \nabla\psi_R(x-x_0) = \frac{\nabla\phi_R(x-x_0)}{f(x)}.$$

Taking $\tilde{\psi}(x) = \psi_R(x-x_0)$ in Lemma 3.1, and using the identity

$$(\nabla^2\phi_R(r)\cdot\nabla u)\cdot\nabla\bar{u} = \varphi_R'(r)|u_r|^2 = \varphi_R'(r)|\nabla u|^2,$$

we obtain for any $t \in [0,T)$,

$$\begin{aligned}
&\int \psi_R(x-x_0)|u(t,x)|^2 dx \\
&= \bigg\{ \int \psi_R(x-x_0)|u_0(x)|^2 dx \\
&\quad + 2t \text{ Im } \int \nabla\phi_R(x-x_0)\cdot\nabla u_0(x)\bar{u}_0(x)dx + 4E(u_0)t^2 \bigg\} \\
&\quad + \int_0^t (t-s)\bigg\{ \int (2-\Delta\phi_R(x-x_0))|u(s,x)|^4 dx \\
&\quad + \int 4(\varphi_R'(|x-x_0|)-1)f(x)|\nabla u(s,x)|^2 dx \bigg\} ds \\
&\quad - 2\int_0^t (t-s)\int \nabla\phi_R(x-x_0)\cdot\nabla f(x)|\nabla u(s,x)|^2 dxds \\
&\quad - \int_0^t (t-s)\int (\Delta^2\phi_R(x-x_0) + \nabla\Delta\phi_R(x-x_0)\cdot\nabla f(x))|u(s,x)|^2 dxds \\
&=: I + II + III + IV.
\end{aligned} \tag{3.26}$$

First, we note that

$$|IV| \leq Ct^2\left(\frac{1}{R} + \frac{1}{R^2}\right), \quad \forall t \in [0,T). \tag{3.27}$$

Next, we observe that it suffices to prove this lemma for small R, say, $R < \rho_0/2$. In view of the fact that

$$\varphi_R'(|x-x_0|) - 1 \leq 0 \leq 2 - \Delta\phi_R(x-x_0)$$

and that these functions are supported in $\Omega_R = \{|x - x_0| \geq R\}$, it follows from Lemma 3.3 that

$$\begin{aligned}
&\int (2 - \Delta\phi_R(x - x_0))|u(t, x)|^4 dx \\
&\leq C\|u(t)\|^2_{L^2(\Omega_R)} \int (2 - \Delta\phi_R(x - x_0))|\nabla u(t, x)|^2 dx \\
&\quad + C\|u(t)\|^4_{L^2(\Omega_R)} \|\nabla(2 - \Delta\phi_R(x - x_0))^{1/2}\|^2_{L^\infty}. \qquad (3.28)
\end{aligned}$$

By (3.28) and (3.26), for any $0 \leq t_0 < t < T$,

$$\begin{aligned}
&\int_{t_0}^t (t - s) \int_{\Omega_R} \big\{4(1 - \varphi_R'(|x - x_0|))f(x) \\
&\quad - C\|u(s)\|^2_{L^2(\Omega_R)}(2 - \Delta\phi_R(x - x_0))\big\}|\nabla u(s, x)|^2 dx ds \\
&\leq \int \psi_R(x - x_0)|u(t_0, x)|^2 dx + 4E(u_0)(t - t_0)^2 \\
&\quad + 2(t - t_0) \text{ Im } \int \nabla\phi_R(x - x_0) \cdot \nabla u(t_0, x)\bar{u}(t_0, x) dx \\
&\quad - 2\int_{t_0}^t (t - s) \int \nabla\phi_R(x - x_0) \cdot \nabla f(x)|\nabla u(s, x)|^2 dx ds \\
&\quad - \int_{t_0}^t (t - s) \int (\Delta^2\phi_R(x - x_0) + \nabla\Delta\phi_R(x - x_0) \cdot \nabla f(x))|u(s, x)|^2 dx ds \\
&\quad + C\|u(t)\|^4_{L^2(\Omega_R)} \|\nabla(2 - \Delta\phi_R(x - x_0))^{1/2}\|^2_{L^\infty}(t - t_0)^2. \qquad (3.29)
\end{aligned}$$

As $\text{supp}(\phi_R) = \{|x - x_0| \leq 2R\}$ and $2R < \rho_0$ and, by (3.5),

$$\nabla\phi_R(x - x_0) \cdot \nabla f(x) \geq 0, \quad \forall x \in \mathbb{R}^2,$$

we have, for all $t \in [0, T)$,

$$\int_{t_0}^t (t - s) \int \nabla\phi_R(x - x_0) \cdot \nabla f(x)|\nabla u(s, x)|^2 dx ds \ \geq \ 0.$$

Moreover, analogous to (3.27) we have

$$\begin{aligned}
&\left| \int_{t_0}^t (t - s) \int (\Delta^2\phi_R(x - x_0) + \nabla\Delta\phi_R(x - x_0) \cdot \nabla f(x))|u(s, x)|^2 dx ds \right| \\
&\leq \ C(t - t_0)^2 \left(\frac{1}{R} + \frac{1}{R^2} \right).
\end{aligned}$$

By the property (A), for any $\epsilon > 0$, there exists $0 \le t_* = t_*(R, \epsilon) < T$ such that

$$\int_{|x-x_0|>R} |u(t,x)|^2 dx < \epsilon, \quad \forall t_* \le t < T.$$

It is obvious that

$$\int |\nabla u(t,x)|^2 dx \le C(t_*) \quad \text{for } 0 \le t \le t_*;$$

also, it is easy to check that

$$\inf_{|x-x_0|\ge R} \frac{(1-\varphi'(|x-x_0|))f(x)}{2-\Delta\phi_R(x-x_0)} \ge C,$$
$$2-\Delta\phi_R(x-x_0) \equiv 2, \quad \text{for } |x-x_0| \ge 2R.$$

Therefore choosing $\epsilon > 0$ small enough and $t_0 = t_*$ in (3.29), we obtain (3.24) for $R < \rho_0/2$ as $t \to T$. As the left hand side of (3.24) is decreasing in R, the inequality holds for all $R > 0$. The proof of (3.25) is similar. □

Lemma 3.10 *Suppose in addition that property* (A) *holds. Then*

$$\int_0^T (T-t) \int_{|x-x_0|<\rho_0} (x-x_0)\cdot\nabla f(x)|\nabla u(t,x)|^2 dx dt \le C \tag{3.30}$$

for some positive constant C which may depend on ρ_0.

Proof. Choosing $R = \rho_0$ in (3.26), we have

$$\begin{aligned} & 2\int_0^t (t-s) \int_{|x-x_0|<\rho_0} (x-x_0)\cdot\nabla f(x)|\nabla u(t,x)|^2 dx ds \\ \le\ & I + II + IV \\ & -2\int_0^t (t-s) \int_{|x-x_0|\ge\rho_0} \nabla\phi_{\rho_0}(x-x_0)\cdot\nabla f(x)|\nabla u(t,x)|^2 dx ds, \end{aligned}$$

which implies (3.30) in view of Lemma 3.9. □

Proof of Theorem 3.3. First of all, (B) implies (A) by the conservation of mass and the inequality

$$\int_{|x-x_0|>R} |u(t,x)|^2 dx \le \frac{1}{R^2} \int |x-x_0|^2 |u(t,x)|^2 dx, \quad \forall R > 0.$$

Now we prove that (A) implies (B). By our construction of the auxiliary functions, it is easy to see that

$$|\nabla\phi_R(x-x_0)|^2 = \varphi_R^2(|x-x_0|) \le 2\phi_R(x-x_0) \le C\psi_R(x-x_0).$$

Thus by (3.26), (3.27), Lemmas 3.9 and 3.10, Hölder's inequality, and the fact that $|\nabla\phi_R(x-x_0)\cdot\nabla f(x)| \le C$ (which follows from (H2)), we obtain, for $R \ge \rho_0$,

$$\begin{aligned}
&\int \psi_R(x-x_0)|u(t,x)|^2dx \\
&\ge \int \psi_R(x-x_0)|u_0(x)|^2dx - 2T\left(C\int \psi_R(x-x_0)|u_0(x)|^2dx\right)^{1/2}\|\nabla u_0\|_{L^2} \\
&\quad - 4\,|E(u_0)|T^2 + II + IV \\
&\quad - 2\int_0^T (T-s)\int_{|x-x_0|<\rho_0} \nabla\phi_R(x-x_0)\cdot\nabla f(x)|\nabla u(s,x)|^2dxds \\
&\quad - 2\int_0^T (T-s)\int_{|x-x_0|\ge\rho_0} \nabla\phi_R(x-x_0)\cdot\nabla f(x)|\nabla u(s,x)|^2dxds \\
&\ge C_1\int \psi_R(x-x_0)|u_0(x)|^2dx - C_2,
\end{aligned} \tag{3.31}$$

where the constants C_1, C_2 are independent of R.

If $|x-x_0|u_0 \notin L^2$, then (3.31) leads to

$$\lim_{R\to\infty}\liminf_{t\uparrow T}\int \psi_R(x-x_0)|u(t,x)|^2dx = \infty.$$

but this contradicts

$$\lim_{t\uparrow T}\int \psi_R(x-x_0)|u(t,x)|^2dx = 0$$

which is a consequence of property (A). Hence $|x-x_0|u_0 \in L^2$.

Let $\psi(x) \ge 0$ be such that

$$\nabla\psi(x-x_0) = \frac{x-x_0}{f(x)}.$$

Then, by Lemma 3.2, we have

$$\begin{aligned}
&\int \psi(x-x_0)|u(t,x)|^2dx \\
&= \int \psi(x-x_0)|u_0(x)|^2dx + 2t\ \mathrm{Im}\int (x-x_0)\cdot\nabla u_0(x)\bar{u}_0(x)dx \\
&\quad + 4E(u_0)t^2 - 2\int_0^t (t-s)\int (x-x_0)\cdot\nabla f(x)|\nabla u(s,x)|^2dxds.
\end{aligned} \tag{3.32}$$

Furthermore, we note that ψ is radial and ψ, ψ_R may be chosen such that

$$\psi(0) = \psi_R(0) = 0,$$

which implies

$$\lim_{R\to\infty} \psi_R(x - x_0) = \psi(x - x_0), \quad \forall x \in \mathbb{R}^2.$$

Subtracting (3.26) from (3.32), we have

$$\begin{aligned}
&\int (\psi(x - x_0) - \psi_R(x - x_0))|u(t, x)|^2 dx \\
&= \int (\psi(x - x_0) - \psi_R(x - x_0))|u_0(x)|^2 dx \\
&\quad +2t \text{ Im} \int ((x - x_0) - \nabla\phi_R(x - x_0)) \cdot \nabla u_0(x)\bar{u}_0(x)dx \\
&\quad -2 \int_0^t (t - s) \int ((x - x_0) - \nabla\phi_R(x - x_0)) \cdot \nabla f(x)|\nabla u(s, x)|^2 dxds \\
&\quad -II - IV. \qquad (3.33)
\end{aligned}$$

It is clear that

$$\lim_{R\to\infty} |(x - x_0) - \nabla\phi_R(x - x_0)| = 0. \qquad (3.34)$$

By Lemma 3.9 and the Lebesgue dominated convergence theorem, for all $t \in [0, T)$,

$$\lim_{R\to\infty} II = 0$$

$$\lim_{R\to\infty} \int_0^t (t - s) \int ((x - x_0) - \nabla\phi_R(x - x_0)) \cdot \nabla f(x)|\nabla u(s, x)|^2 dxds = 0.$$

Hence, by (3.33), (3.34) and (3.27),

$$\lim_{R\to\infty} \left\{ \sup_{0\le t<T} \int (\psi(x - x_0) - \psi_R(x - x_0))|u(t, x)|^2 dx \right\} = 0.$$

Therefore for any $\epsilon > 0$, there exists $R > 0$ such that

$$\int \psi(x - x_0)|u(t, x)|^2 dx < \int \psi_R(x - x_0)|u(t, x)|^2 dx + \epsilon, \quad t \in [0, T),$$

and the desired limiting behavior follows. $\square$

3.1.3 Existence

In this subsection, we construct blow-up solutions under appropriate assumptions on the function $f(x)$ and the initial data and prove Theorem 3.4. Throughout this subsection, we assume $f(x)$ satisfies (H1)-(H3) and that (3.6) holds. Hence we can find a nonnegative real function $\psi(x)$ such that

$$\psi(x-x_0) \sim |x-x_0|^2 \quad \text{and} \quad \nabla\psi(x-x_0) = \frac{x-x_0}{f(x)}.$$

The proof of Theorem 3.4 is now presented in the following two cases:

Case of global minimum We first consider the case where x_0 is a global minimum of f, i.e., (3.7) holds. The proof of Theorem 3.4 in this case is direct and elementary but useful.

First, we assume that u_0 satisfies that $|x-x_0|u_0 \in L^2$ and $E(u_0) < 0$. Suppose $u(t,x)$, the solution of the Cauchy problem (3.1), is defined for all time. Consider

$$y(t) := \int \psi(x-x_0)|u(t,x)|^2 dx \ \geq \ 0.$$

By Lemma 3.2 and (3.7), we have, for all $t > 0$,

$$\begin{aligned} y(t) &= y(0) + ty'(0) + 4t^2 E(u_0) \\ &\quad - 2\int_0^t (t-s) \int (x-x_0)\cdot\nabla f(x)|\nabla u(s,x)|^2 dx ds, \\ &\leq y(0) + ty'(0) + 4t^2 E(u_0). \end{aligned}$$

Since $E(u_0) < 0$, the right hand side of the above inequality is negative provided t is large enough, which is a contradiction. Hence $u(t,x)$ blows up in finite time.

Now, for all $\epsilon > 0$ and $\lambda > 0$, define

$$\omega_{\epsilon,\lambda}(x) = (1+\epsilon)\lambda^{-1} Q_L(\lambda^{-1}(x-x_0)). \tag{3.35}$$

Then we have

$$\|\omega_{\epsilon,\lambda}\|_{L^2} = (1+\epsilon)\|Q_L\|_{L^2}$$

and

$$E(\omega_{\epsilon,\lambda}) = E_L(\omega_{\epsilon,\lambda}) + \frac{1}{2}\int (f(x) - f(x_0)) |\nabla \omega_{\epsilon,\lambda}(x)|^2 dx.$$

By a scaling argument,

$$\begin{aligned} E_L(\omega_{\epsilon,\lambda}) &= (1+\epsilon)^2 \frac{1}{\lambda^2} E_L(Q_L) \\ &\quad +((1+\epsilon)^2 - (1+\epsilon)^4)\frac{1}{\lambda^2}\int |Q_L(x)|^4 dx. \end{aligned}$$

Since $E_L(Q_L) = 0$, there exists $c(\epsilon) > 0$ such that

$$E_L(\omega_{\epsilon,\lambda}) \le -\frac{c(\epsilon)}{\lambda^2} \quad \forall \lambda > 0. \tag{3.36}$$

By Lemma 3.5 and the assumption (H2), we have

$$\begin{aligned} \left|\frac{1}{2}\int (f(x) - f(x_0)) |\nabla \omega_{\epsilon,\lambda}(x)|^2 dx\right| &\le C\int |x - x_0||\nabla \omega_{\epsilon,\lambda}(x)|^2 dx \\ &\le C(1+\epsilon)^2 \int \frac{|x|}{\lambda^4} e^{-c_2\frac{|x|}{\lambda}} dx \\ &\le \frac{C}{\lambda}(1+\epsilon)^2. \end{aligned} \tag{3.37}$$

Thus, it follows from (3.36) and (3.37) that for $\epsilon > 0$, $E(\omega_{\epsilon,\lambda}) < 0$ for λ small enough, say $0 < \lambda \le \lambda(\epsilon)$. Consequently $\phi_\epsilon = \omega_{\epsilon,\lambda(\epsilon)}$ satisfies the conclusions of Theorem 3.4. In particular $\epsilon_0 = \infty$.

Case of local minimum Now, we consider the case where x_0 is a local minimum, i.e., (3.8) holds.

By the argument for the global minimum case, we have the following:

Lemma 3.11 $\forall \epsilon \in (0,1)$, *for all* $A(\epsilon) > 0$, *there is a* $\phi_\epsilon \in H^2$ *such that*

(a) $\|\phi_\epsilon\|_{L^2} = \|Q_L\|_{L^2} + \epsilon$,

(b) $E(\phi_\epsilon) = -A(\epsilon)$,

(c) $\int |x - x_0|^2 |\phi_\epsilon(x)|^2 dx \le c$, *where* c *is independent of* ϵ *and* $A(\epsilon)$,

(d) $\left|\int (x - x_0)\cdot \nabla\phi_\epsilon(x)\bar{\phi}_\epsilon(x) dx\right| \le c$, *where* c *is independent of* ϵ *and* $A(\epsilon)$,

(e) $\phi_\epsilon(x)$ *is real for all* x,

(f) *as* $\epsilon \to 0$, $\|\nabla\phi_\epsilon\|_{L^2} \to \infty$, *and* $|\phi_\epsilon(x)|^2 \to \|Q_L\|_{L^2}^2 \delta_{x_0}$ *in the distribution sense.*

The rest of this subsection will be devoted to proving the following claim from which Theorem 3.4, in the local minimum case, follows.

Claim For $A(\epsilon)$ sufficiently large and ϵ sufficiently small, the solution $u_\epsilon(t)$ with Cauchy data ϕ_ϵ blows up in finite time.

We argue by contradiction. Suppose that, as $\epsilon \to 0$, $A(\epsilon) \to \infty$ and $u_\epsilon(t)$ is globally defined in time. First, we make the following observation:

Lemma 3.12 *Let $t_\epsilon \to \infty$ as $\epsilon \to 0$. Then*

$$\|\nabla u_\epsilon(t_\epsilon)\|_{L^2} \to \infty \quad \textit{as } \epsilon \to 0.$$

Proof. Suppose there exists a sequence $\{\epsilon_n\}$ such that

$$\|\nabla u_{\epsilon_n}(t_{\epsilon_n})\|_{L^2} \leq C \quad \text{as } \epsilon_n \to 0.$$

Then by the Gagliardo-Nirenberg inequality and the conservation of energy,

$$|E(\phi_{\epsilon_n})| = |E(u_{\epsilon_n}(t_{\epsilon_n}))| \leq \frac{L}{2}\|\nabla u_{\epsilon_n}(t_{\epsilon_n})\|_{L^2}^2 + \frac{1}{4}\|u_{\epsilon_n}(t_{\epsilon_n})\|_{L^4}^4 \leq C,$$

which contradicts the fact that

$$|E(\phi_{\epsilon_n})| = A(\epsilon_n) \to \infty \quad \text{as } n \to \infty.$$

□

The proof of the Claim follows in three steps:

Proposition 3.1 (Concentration properties of $u_\epsilon(t)$) *For all $\epsilon' > 0$, there exists $\epsilon_0 > 0$ such that $\forall \epsilon \in (0, \epsilon_0)$ and $\forall t \geq 0$,*

$$\left| \int_{|x-x_0|\leq\epsilon'} |u_\epsilon(t,x)|^2 dx - \int Q_L^2(x) dx \right| < \epsilon', \tag{3.38}$$

and

$$\left| \int_{|x-x_0|\geq\epsilon'} |u_\epsilon(t,x)|^2 dx \right| < \epsilon'. \tag{3.39}$$

Proof. The proof is based on the fact that x_0 is a local minimum, and Lemmas 3.6 and 3.12. First, by the assumption (H1) we can find $\gamma > 0$ such that

$$\|Q_{f(x)}\|_{L^2}^2 = \|Q_L\|_{L^2}^2 \frac{f(x)}{L} \geq 2\gamma, \quad \forall x \in \mathbb{R}^2. \tag{3.40}$$

For each $\epsilon > 0$ and $0 < r \leq \rho_0/4$, let

$$T_{\epsilon,r} = \sup\{t \in \mathbb{R}; \|u_\epsilon(t)\|_{L^2(B(x_0,r))}^2 \geq \|Q_L\|_{L^2}^2 - \gamma\}. \tag{3.41}$$

By Lemma 3.11, it is easy to see that $T_{\epsilon,r} > 0$ for $0 < \epsilon \leq \epsilon_0$ where $\epsilon_0 > 0$ is some constant possibly dependent on r and γ. In fact, $T_{\epsilon,r} = \infty$ provided ϵ is small enough.

Indeed, suppose, on the contrary, that for a sequence $\epsilon_n \to 0$, $T_{\epsilon_n,r} < \infty$. Let $u_n(x) = u_{\epsilon_n}(T_{\epsilon_n,r}, x)$. Then, by Lemmas 3.11 and 3.12, u_n satisfies the assumptions of Lemma 3.6. Therefore there exists $\{x_n\}$ such that

$$\forall R > 0, \quad \liminf_{n\to\infty} \left(\|u_n\|_{L^2(B(x_n,R))}\|Q_{f(x_n)}\|_{L^2}^{-1}\right) \geq 1. \tag{3.42}$$

Now, for sufficiently large n, we have

$$|x_n - x_0| \leq 2r < \rho_0; \tag{3.43}$$

for, if not, by (3.40)–(3.42), we have

$$\begin{aligned}
\liminf_{n\to\infty} \|u_n\|_{L^2}^2 &\geq \liminf_{n\to\infty} \|u_n\|_{L^2(B(x_0,r))}^2 + \liminf_{n\to\infty} \|u_n\|_{L^2(\mathbb{R}^2 - B(x_0,r))}^2 \\
&\geq \|Q_L\|_{L^2}^2 - \gamma + \liminf_{n\to\infty} \|u_n\|_{L^2(B(x_n,r))}^2 \\
&\geq \|Q_L\|_{L^2}^2 + \gamma
\end{aligned}$$

which is a contradiction to Lemma 3.11. Furthermore

$$x_n \longrightarrow x_0 \quad \text{as } n \to \infty. \tag{3.44}$$

To see this, from (3.42) we have $\forall \tilde{\epsilon} > 0$ and for n sufficiently large

$$\begin{aligned}
\|\phi_{\epsilon_n}\|_{L^2}^2 = \|u_n\|_{L^2}^2 &\geq (1 - \tilde{\epsilon})\|Q_{f(x_n)}\|_{L^2}^2 \\
&= (1 - \tilde{\epsilon})\|Q_L\|_{L^2}^2 \frac{f(x_n)}{L}.
\end{aligned}$$

Since $\|\phi_{\epsilon_n}\|_{L^2}^2 \to \|Q_L\|_{L^2}^2$ as $n \to \infty$, and $\tilde{\epsilon}$ is arbitrary, we have

$$\limsup_{n\to\infty} f(x_n) \leq L = \inf_{x\in\mathbb{R}^2} f(x),$$

from which (3.44) follows in view of (3.43) and (3.8). As a result of (3.44), (3.42) implies that

$$\begin{aligned}
\liminf_{n\to\infty} \|u_{\epsilon_n}(T_{\epsilon_n,r})\|_{L^2(B(x_0,r))}^2 &\geq \liminf_{n\to\infty} \|u_{\epsilon_n}(T_{\epsilon_n,r})\|_{L^2(B(x_n,\frac{r}{2}))}^2 \\
&= \liminf_{n\to\infty} \|u_n\|_{L^2(B(x_n,\frac{r}{2}))}^2 \\
&\geq \|Q_L\|_{L^2}^2,
\end{aligned}$$

which is a contradiction to the finiteness of $T_{\epsilon,r}$. Therefore there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $T_{\epsilon,r} = \infty$.

Now we are in the position to conclude the proof of the proposition. Suppose there exist $\epsilon' > 0$, $\epsilon_n \to 0$ and t_{ϵ_n} such that

$$\left| \int_{|x-x_0|<\epsilon'} |u_n(x)|^2 dx - \int |Q_L(x)|^2 dx \right| \geq \epsilon',$$

where $u_n(x) = u_{\epsilon_n}(t_{\epsilon_n}, x)$. By the conservation of mass and Lemma 3.11, this is equivalent to

$$\int_{|x-x_0|<\epsilon'} |u_n(x)|^2 dx \leq \int |Q_L(x)|^2 dx - \epsilon'. \tag{3.45}$$

Choosing $r = \min\{\epsilon', \rho_0/4\}$ and $\gamma = \min\{\epsilon'/2, \|Q_L\|_{L^2}^2/2\}$ in (3.41), since $T_{\epsilon,r} = \infty$ for $0 < \epsilon \leq \epsilon_0(r, \gamma)$ we get

$$\int_{|x-x_0|<\epsilon'} |u_n(x)|^2 dx \geq \|u_n\|_{L^2(B(x_0,r))}^2 \geq \|Q_L\|_{L^2}^2 - \frac{\epsilon'}{2},$$

which is a contradiction to (3.45). □

Proposition 3.2 (Energy estimates away from the concentration point) *For $0 < \beta < R \leq \rho_0$, there exists $\epsilon_0 > 0$ such that $\forall \epsilon \in (0, \epsilon_0)$ and $\forall t \geq 0$,*

$$\begin{aligned}
&\int_0^t (t-s) \int_{|x-x_0|\geq R} |\nabla u_\epsilon(s,x)|^2 dxds \\
\leq\ & c_1 + (c_2 + c_3 E(u_\epsilon))t^2 + c(\epsilon) \int_0^t (t-s) \int_{\beta\leq|x-x_0|\leq R} |\nabla u_\epsilon(s,x)|^2 dxds,
\end{aligned} \tag{3.46}$$

where c_j are independent of ϵ, $c(\epsilon) > 0$ and $c(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof. The proof follows from Lemma 3.1 by choosing suitable functions $\tilde{\psi}$ and ϕ. Indeed, consider a radial function $\phi \in C^4(\mathbb{R}^2) \cap W^{4,\infty}(\mathbb{R}^2)$ satisfying

$$\begin{aligned}
2\phi(x) &= |x|^2 \quad \text{for} \quad |x| \leq \beta, \\
2\phi(x) &< |x|^2 \quad \text{for} \quad |x| > \beta, \\
\phi(x) &\equiv c \qquad \text{for} \quad |x| \geq R, \\
\nabla\phi(x)\cdot x &\geq 0 \qquad \text{for all} \quad x, \\
\Delta\phi(x) &\leq 2 \qquad \text{for all} \quad x,
\end{aligned}$$

and, for $\beta \leq |x| \leq R$ and $\forall v \in \mathbb{C}^2$,

$$\left(|v|^2 - (\nabla^2\phi \cdot v)\cdot \bar{v}\right) \geq 0. \tag{3.47}$$

The existence of such a ϕ can be proved easily (see Section 3.1.2), hence the proof is omitted. In view of (3.6) and the fact that $\phi(x - x_0)$ is radial, it is easy to see that the integrability condition $\mathrm{curl}(\dfrac{\nabla\phi(x-x_0)}{f(x)}) = 0$ holds. Thus we can find a nonnegative function $\tilde{\psi}(x)$ such that

$$\tilde{\psi}(x - x_0) \sim |x - x_0|^2 \quad \text{locally, and} \quad \nabla\tilde{\psi}(x - x_0) = \frac{\nabla\phi(x - x_0)}{f(x)}.$$

We note, in particular, that $\tilde{\psi}(x)$ is a positive constant for $|x| \geq R$.

By Lemmas 3.1 and 3.11, integrating w.r.t. x and t, we have $\forall\epsilon > 0$ and $\forall t > 0$,

$$\begin{aligned}
0 \;&\leq \int \tilde{\psi}(x - x_0)|u_\epsilon(t,x)|^2 dx \\
&= \int \tilde{\psi}(x - x_0)|\phi_\epsilon(x)|^2 dx + 2t \;\mathrm{Im} \int \nabla\phi(x - x_0)\cdot\nabla\phi_\epsilon(x)\bar{\phi}_\epsilon(x) dx \\
&\quad + \int_0^t (t-s)\Big\{ -\int \Delta\phi(x - x_0)|u_\epsilon(s,x)|^4 dx \\
&\quad + 4 \;\mathrm{Re} \int \left(\nabla^2\phi(x - x_0)\cdot\nabla u_\epsilon(s,x)\right)\cdot\nabla\bar{u}_\epsilon(s,x) f(x) dx \\
&\quad - 2\int \nabla\phi(x - x_0)\cdot\nabla f(x)|\nabla u_\epsilon(s,x)|^2 dx \\
&\quad - \int (\Delta^2\phi(x - x_0) + \nabla\Delta\phi(x - x_0)\cdot\nabla f(x))|u_\epsilon(s,x)|^2 dx \Big\} ds.
\end{aligned} \tag{3.48}$$

By the conservation of mass, Lemma 3.11 and the properties of $\tilde{\psi}$ and ϕ, there exist positive constants c_1, c_2 independent of ϵ such that

$$\begin{aligned} & - c_1 - c_2 t^2 \\ \leq & \int_0^t (t-s)\Big\{ - \int \Delta\phi(x-x_0)|u_\epsilon(s,x)|^4 dx \\ & + 4 \text{ Re} \int (\nabla^2\phi(x-x_0)\cdot\nabla u_\epsilon(s,x))\cdot\nabla\bar{u}_\epsilon(s,x) f(x)dx \\ & - 2\int \nabla\phi(x-x_0)\cdot\nabla f(x)|\nabla u_\epsilon(s,x)|^2 dx\Big\} ds \\ = & \int_0^t (t-s)\Big\{ 8E(u_\epsilon) + \int_{|x-x_0|\geq\beta} (2-\Delta\phi(x-x_0))|u_\epsilon(s,x)|^4 dx \\ & + 4 \text{ Re} \int_{|x-x_0|\geq\beta} (\nabla^2\phi(x-x_0)\cdot\nabla u_\epsilon(s,x))\cdot\nabla\bar{u}_\epsilon(s,x) f(x)dx \\ & - 4\int_{|x-x_0|\geq\beta} |\nabla u_\epsilon(s,x)|^2 f(x) dx \\ & - 2\int \nabla\phi(x-x_0)\cdot\nabla f(x)|\nabla u_\epsilon(s,x)|^2 dx\Big\} ds. \end{aligned} \tag{3.49}$$

By (3.47) and the fact $\nabla\phi(x-x_0)\cdot\nabla f(x) \geq 0$, this implies that

$$\begin{aligned} & 4\int_0^t (t-s)\int_{|x-x_0|\geq R} |\nabla u_\epsilon(s,x)|^2 f(x) dx ds \\ \leq & \; c_1 + (c_2 + 4E(u_\epsilon))t^2 \\ & + \int_0^t (t-s)\int_{|x-x_0|\geq\beta} (2-\Delta\phi(x-x_0))|u_\epsilon(s,x)|^4 dx ds. \end{aligned} \tag{3.50}$$

Since $\Omega = \text{supp}(2-\Delta\phi(x-x_0)) = \mathbb{R}^2 \setminus B(x_0,\beta)$, it follows from Lemma 3.3 that

$$\begin{aligned} & \int (2-\Delta\phi(x-x_0))|u(t,x)|^4 dx \\ \leq & \; C\|u(t)\|_{L^2(\Omega)}^2 \int (2-\Delta\phi(x-x_0))|\nabla u(t,x)|^2 dx \\ & + C\|u(t)\|_{L^2(\Omega)}^4 \|\nabla(2-\Delta\phi(x-x_0))^{1/2}\|_{L^\infty}^2. \end{aligned} \tag{3.51}$$

The following observation, which follows from Proposition 3.1, then completes the proof:

$$\sup_{s\geq 0}\int_{|x-x_0|\geq R} |u_\epsilon(s,x)|^2 dx \to 0 \quad \text{as } \epsilon \to 0.$$

□

Conclusion of proof of claim and proof of Theorem 3.4. By (3.8), the fact

$$(x - x_0) \cdot \nabla f(x) \geq C > 0 \quad \text{for} \quad 0 < \beta \leq |x - x_0| \leq R_0 < \rho_0,$$

and the hypothesis (H2), we note that

$$|(x - x_0) \cdot \nabla f(x)| \leq C.$$

Consequently, Proposition 3.2 implies that for small $\epsilon > 0$

$$\begin{aligned} &\int_0^t (t-s) \int_{|x-x_0|\geq R_0} (x - x_0) \cdot \nabla f(x) |\nabla u_\epsilon(s,x)|^2 dx ds \\ \leq\ & c_1 + (c_2 + c_3 E(u_\epsilon)) t^2 \\ &+ c(\epsilon) \int_0^t (t-s) \int_{\beta \leq |x-x_0| \leq R_0} (x - x_0) \cdot \nabla f(x) |\nabla u_\epsilon(s,x)|^2 dx ds, \end{aligned}$$

where c_j are positive constants independent of ϵ, and $c(\epsilon) > 0$ with $c(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, by Lemma 3.2, for all $t > 0$ and $0 < \epsilon \leq \epsilon_0$ (small enough), we arrive at

$$\begin{aligned} 0 \ \leq\ & \int \psi(x - x_0) |u_\epsilon(t,x)|^2 dx \\ =\ & \int \psi(x - x_0) |\phi_\epsilon(x)|^2 dx + 4E(u_\epsilon) t^2 \\ & + 2t \ \mathrm{Im} \int (x - x_0) \cdot \nabla \phi_\epsilon(x) \bar{\phi}_\epsilon(x) dx \\ & - 2 \int_0^t (t-s) \int_{|x-x_0|<R_0} (x - x_0) \cdot \nabla f(x) |\nabla u_\epsilon(s,x)|^2 dx ds \\ & - 2 \int_0^t (t-s) \int_{|x-x_0|\geq R_0} (x - x_0) \cdot \nabla f(x) |\nabla u_\epsilon(s,x)|^2 dx ds \\ \leq\ & c_1 + c_2 t - c_3 A(\epsilon) t^2, \end{aligned}$$

where c_j are positive constants independent of ϵ. It is obvious that this inequality is a contradiction to the assumption that $A(\epsilon) \to \infty$ as $\epsilon \to 0$. The claim is thus established, and the proof of Theorem 3.4 is complete. □

3.1.4 L^2-minimality

We have seen in Corollary 3.1 that if $\|u_0\|_{L^2} < \|Q_L\|_{L^2}$, then the solution $u(t)$ of the Cauchy problem (3.1) is globally defined. On the other hand, in previous section, with suitable assumptions on $f(x)$ and sufficiently small ϵ, we have constructed a family of initial data such that $\|\phi_\epsilon\|_{L^2} = \|Q_L\|_{L^2} + \epsilon$ and the corresponding solutions blow up in finite time. If the solution $u(t)$ of (3.1) blows up in finite time with $\|u_0\|_{L^2} = \|Q_L\|_{L^2}$, it is called an L^2-minimal blow-up solution. In this subsection, we focus on such solutions and prove Theorem 3.5. First, we give some preliminary results.

Lemma 3.13 ([28]) *Let $u_n \in H^1(\mathbb{R}^2)$, $c_0 > 0$ and $R_0 > 0$ be such that $E_L(u_n) \leq c_0$, $\|u_n\|_{L^2}^2 \leq \|Q_L\|^2$, $\|\nabla u_n\|_{L^2} \to \infty$ as $n \to \infty$, and*

$$\int_{|x|>R_0} |u_n(x)|^2\,dx \leq C,$$

where C is independent of n. Then, there exists a positive constant $\tilde{C}$ depending only on R_0 and c_0 such that, for all n,

$$\int_{|x|>4R_0} |\nabla u_n(x)|^2\,dx \leq \tilde{C}.$$

Lemma 3.14 *For all $t \in [0,T)$,*

$$\int (f(x)-L)|\nabla u(t,x)|^2 dx \leq 2E(u_0).$$

Proof. The result follows from Lemma 3.4 together with the observation

$$E(u_0) = E(u(t)) = E_L(u(t)) + \int (f(x)-L)|\nabla u(t,x)|^2 dx.$$

□

Lemma 3.15 *Let $u(t)$ be the solution of the Cauchy problem (3.1) with $\|u_0\|_{L^2} = \|Q_L\|_{L^2}$ and $|u(t,x)|^2 \to \|Q_L\|_{L^2}^2 \delta_{x_0}$ in the distribution sense as $t \uparrow T$. Then there exist $x(t) \in \mathbb{R}^2$ and $\theta(t) \in \mathbb{R}$ such that $x(t) \to x_0$ and*

$$\frac{1}{\lambda(t)} e^{i\theta(t)} u\left(t, x(t) + \frac{\cdot - x(t)}{\lambda(t)}\right) \to Q_L(\cdot) \text{ in } H^1 \text{ as } t \to T, \tag{3.52}$$

where $\lambda(t) = \|\nabla u(t)\|_{L^2} / \|\nabla Q_L\|_{L^2} \to \infty$ as $t \to T$.

Proof. This follows from Lemma 3.6. □

Proof of Theorem 3.5 (i). The proof comprises the next three propositions.

Proposition 3.3 (Concentration) *There exists $x(t) \in \mathbb{R}^2$ such that*

$$|u(t, x + x(t))|^2 \to \|Q_L\|^2_{L^2}\delta_0, \text{ in the distribution sense as } t \uparrow T. \tag{3.53}$$

Furthermore, for each $r > 0$, there exists a $c(r)$ such that for all $t \in [0, T)$,

$$\int_{|x-x(t)|>r} |\nabla u(t, x)|^2 dx \le c(r). \tag{3.54}$$

Proof. By Lemma 3.6, there exists $x(t)$ such that for all $R > 0$

$$\liminf_{t\uparrow T} \|u(t)\|_{L^2(B(x(t),R))} \ge \|Q_L\|_{L^2}.$$

In view of the assumption that $\|u(t)\|_{L^2} = \|u_0\|_{L^2} = \|Q_L\|_{L^2}$, we get the concentration result (3.53).

For $r > 0$, by (3.53), there exists $t_r < T$ such that for all $t \in [t_r, T)$,

$$\int_{|x|>r/4} |u(t, x + x(t))|^2 dx \le C,$$

where C is given in Lemma 3.13. Furthermore, we have a constant $\tilde{C}(r) > 0$ such that for all $t \in [t_r, T)$,

$$\int_{|x|>r} |\nabla u(t, x + x(t))|^2 dx \le \tilde{C}(r).$$

The estimate (3.54) now follows from the observation that $\forall t \in [0, t_r]$,

$$\int_{|x|>r} |\nabla u(t, x + x(t))|^2 dx \le \int |\nabla u(t, x)|^2 dx \le C.$$

□

Proposition 3.4 (Location of concentration point) *There is a $y_0 \in \mathcal{M}$ such that*

$$x(t) \to y_0 \quad \text{as } t \to T.$$

Proof. Suppose $\mathcal{M} = \{x_j\}_{j=1}^p$. We first note that

$$d(t) = \min_{j=1,\cdots,p}\{|x(t) - x_j|\} \to 0 \quad \text{as } t \to T. \tag{3.55}$$

Indeed, suppose, by contradiction, that there are $t_n \to T$ as $n \to \infty$ and $\gamma > 0$ such that

$$d(t_n) \geq \gamma \quad \text{and} \quad \min_{j\neq k}|x_j - x_k| \geq 2\gamma.$$

Denote

$$D = \mathbb{R}^2 \setminus \Sigma_{i=1}^p B(x_i, \gamma/2).$$

By the assumption $f(x) \geq L + \gamma_0$ for $|x| > R_0$ in Theorem 3.5, there is $\gamma_1 > 0$ such that

$$f(x) - L \geq \gamma_1, \quad \forall x \in D.$$

Clearly $B(x(t_n), \gamma/2) \in D$ for all n. Thus by Lemma 3.14,

$$\int_{|x-x(t_n)|\leq\gamma/2} |\nabla u(t_n, x)|^2 dx \leq \int_D |\nabla u(t_n, x)|^2 dx \leq c(\gamma),$$

for all n. Choosing $r = \gamma/2$ in (3.54), we get a contradiction to the fact

$$\|\nabla u(t_n, x)\|_{L^2} \to \infty \quad \text{as } n \to \infty.$$

Therefore $d(t) \to 0$ as $t \to T$.

By the concentration (3.53) and the conservation of mass $\|u(t)\|_{L^2} = \|u_0\|_{L^2} = \|Q_L\|_{L^2}$, we can see that there is one point $y_0 \in \mathcal{M}$ such $x(t) \to y_0$ as $t \to T$.

Indeed, following the idea in [29], let $\rho = \frac{1}{4}\min_{j\neq l}\{|x_j - x_l|; x_j, x_l \in \mathcal{M}\}$ and $\phi \in C^\infty(\mathbb{R}^2)$ be a cut-off function such that

$$\begin{aligned} \phi(x) &\equiv 1 \quad \text{for } |x| < \rho, \\ \phi(x) &\equiv 0 \quad \text{for } |x| \geq 2\rho. \end{aligned}$$

By (3.54), (3.55) and the computations in Lemma 3.1, we have

$$\begin{aligned} &\left|\frac{d}{dt}\int \phi(x-x_j)|u(t,x)|^2dx\right| \\ \le\ & 2\left|\mathrm{Im}\int f(x)\nabla\phi(x-x_j)\cdot\nabla u(t,x)\bar{u}(t,x)dx\right| \\ \le\ & c\left(\int_{\rho<|x-x_j|\le 2\rho}|\nabla u(t,x)|^2dx\right)^{1/2} \ \le\ c. \end{aligned}$$

Therefore there exists $e_j \ge 0$ such that

$$\int_{|x-x_j|<\rho} \le \int \phi(x-x_j)|u(t,x)|^2dx \to e_j \quad \text{as } t \to T,$$

which obviously implies the desired result in view of the initial mass. This concludes the proof. □

We remark that by the $k(x)$-version of Lemma 3.6, it can be shown that

$$\liminf_{t\uparrow T} k(x(t)) \ge K,$$

hence $y_0 \in \mathcal{M}'$ (the assumption (3.12) is not needed).

The proof of Theorem 3.5 (i) concludes with the following proposition whose proof is similar to those of Lemmas 4.6 and 4.7 in [29] and is hence omitted here.

Proposition 3.5 *Assume $y_0 \in \mathcal{M}$ such that $x(t) \to y_0$ as $t \to T$. Then we have*

$$|x-y_0||u_0(x)| \in L^2,$$

and

$$\int |x-y_0|^2|u(t,x)|^2dx \to 0 \quad \textit{as } t \to T.$$

Now, we turn to the nonexistence of L^2-minimal solutions. Let x_0 be such that $f(x_0) = L$, and suppose that there exits $c_0 > 0$ such that

$$(x-x_0)\cdot\nabla f(x) \ge c_0|x-x|^{1+\alpha_0} \quad \text{for } x \text{ near } x_0,$$

where $\alpha_0 \in (0,1)$. This implies in particular that

$$f(x) - L \ge c_0|x-x_0|^{1+\alpha_0} \quad \text{for } |x-x_0| \le \rho_0, \tag{3.56}$$

for some constant $\rho_0 > 0$. More generally, we can claim the following result:

Proposition 3.6 *Assume that* $f(x)$ *satisfies* (3.56) *and* $\|u_0\|_{L^2} = \|Q_L\|_{L^2}$. *Then there is no blow-up solution* $u(t)$ *of* (3.1) *such that* $|u(t,x)|^2 \to \|Q_L\|_{L^2}^2 \delta_{x_0}$ *in the distribution sense as* $t \uparrow T$, *for any* $T < \infty$.

Proof. We argue by contradiction. Suppose $u(t)$ is such a blow-up solution. We claim that

$$\int (f(x) - L)|\nabla u(t,x)|^2 dx \to \infty \quad \text{as } t \to T,$$

which will be a contradiction to Lemma 3.14. The proof of the claim is based on the profile of the L^2-minimal blow-up solutions described in Lemma 3.15.

For $\lambda > 0$ and $0 < t < T$, denote

$$D_\lambda(t) = \{x \in \mathbb{R}^2 |\ |x - x(t)| \le \frac{\rho_0}{2}\lambda,\ (x - x(t)) \cdot (x_0 - x(t)) \le 0\}$$

It is easy to see that

$$|x - x_0| \ge |x - x(t)| \quad \text{for all } x \in D_\lambda(t).$$

For t close to T, $|x(t) - x_0| < \rho_0/2$, $D_1(t) \subset B(x_0, \rho_0)$, and from Lemma 3.15 we have

$$\begin{aligned}
&\int_{|x-x_0|\le\rho_0} (f(x) - L)|\nabla u(t,x)|^2 dx \\
\ge &\int_{|x-x_0|\le\rho_0} c_0|x - x_0|^{1+\alpha_0}|\nabla u(t,x)|^2 dx \\
\ge &\int_{D_1(t)} c_0|x - x(t)|^{1+\alpha_0}|\nabla u(t, x - x(t) + x(t))|^2 dx \\
\ge &\int_{D_{\lambda(t)}(t)} c_0\left|\frac{x - x(t)}{\lambda(t)}\right|^{1+\alpha_0}\left|\nabla u(t, \frac{x - x(t)}{\lambda(t)} + x(t))\right|^2 dx \\
\ge &\ c\lambda(t)^{1-\alpha_0}\int_{D_2(t)\setminus D_1(t)} \frac{1}{\lambda(t)^2}\left|\nabla u(t, \frac{x - x(t)}{\lambda(t)} + x(t))\right|^2 dx \\
\ge &\ c\lambda(t)^{1-\alpha_0}\int_{D_2(t)\setminus D_1(t)} |\nabla Q_L(x)|^2 dx \\
\ge &\ c\lambda(t)^{1-\alpha_0},
\end{aligned}$$

where $\lambda(t) = \|\nabla u(t)\|_{L^2} \to \infty$ as $t \to T$. This establishes the claim, and the proof of the proposition is complete. □

Proof of Theorem 3.5 (ii). The desired result is a corollary of Proposition 3.6 and Theorem 3.5 (i). □

3.2 Blow-up analysis on $\mathbb{T}^2$

In this section, we focus on the space-periodic blow-up solutions of the Cauchy problem (3.1) with spatial dimension two, i.e., on $\mathbb{T}^2$. We will be referring to the following condition:

(H) $f(x)$, $k(x) \in C^1(\mathbb{T}^2)$ are positive functions with $L = \min_{x\in\mathbb{T}^2} f(x)$ and $K = \max_{x\in\mathbb{T}^2} k(x)$.

It is interesting that the L^2-concentration and L^2-minimality still can be described in terms of the ground state solution $Q_{L,K}$ of

$$L\Delta Q + K|Q|^2Q = Q, \quad \text{in } \mathbb{R}^2.$$

In the sequel, all the omitted underlying domains are supposed to be $\mathbb{T}^2$, except that the L^2-norm of $Q_{L,K}$ is taken over $\mathbb{R}^2$. Our main results are as follows:

Theorem 3.6 (L^2-concentration) *Assume that $f(x)$, $k(x)$ satisfy* (H). *Let $u(t)$ be a blow-up solution of the Cauchy problem* (3.1) *and T its blow-up time. Then*

(i) *there is $x(t) \in \mathbb{T}^2$ such that for all (small) $R > 0$*

$$\liminf_{t\uparrow T} \int_{B(x(t),R)} |u(t,x)|^2 dx \geq \|Q_{L,K}\|^2_{L^2}; \tag{3.57}$$

(ii) *there is no sequence $\{t_n\}$ such that $t_n \uparrow T$ and $u(t_n)$ converges in $L^2(\mathbb{T}^2)$ as $n \to \infty$.*

Theorem 3.6 implies that blow-up solutions have a lower L^2-bound, namely, $\|u(t)\|_{L^2} \geq \|Q_{L,K}\|_{L^2}$. Therefore, as a consequence of the conservation of mass, we have a sufficient condition for the global existence of solutions.

Corollary 3.3 *Assume that $f(x)$, $k(x)$ satisfy* (H), *then the solution $u(t)$ is globally defined in time provided $\|u_0\|_{L^2} < \|Q_{L,K}\|_{L^2}$.*

Theorem 3.7 (L^2-minimal blow-up solutions) *Assume* $\|u_0\|_{L^2} = \|Q_{L,K}\|_{L^2}$ *and* $u(t)$ *is the solution of* (3.1). *Let* $f(x)$, $k(x)$ *satisfy* (H). *Then*

(i) *there exist* $\theta(x,t) \in \mathbb{R}, x(t) \in \mathbb{T}^2$ *such that*

$$\frac{1}{\lambda(t)} e^{i\theta(t,\frac{\cdot}{\lambda(t)})} \varphi(\frac{\cdot}{\lambda(t)}) u(t, \frac{\cdot}{\lambda(t)} + x(t)) \to Q_{L,K}(\cdot) \quad \textit{strongly in } H^1(\mathbb{R}^2) \quad \textit{as } t \uparrow T,$$

where $\lambda(t) = \|\nabla u(t)\|_{L^2}/\|\nabla Q_{L,K}\|_{L^2}$ *and* $\varphi(x)$ *is a cut-off function on* $\mathbb{R}^2$ *which is identically equal to* 1 *for* x *close to* 0;

(ii) *suppose moreover*

$$\mathcal{M} = \{x; f(x) = L\} \textit{ is finite} \tag{3.58}$$

$$\textit{or} \quad \mathcal{M}' = \{x; k(x) = K\} \textit{ is finite}, \tag{3.59}$$

then there exists $y_0 \in \mathcal{M} \cap \mathcal{M}'$ *such that*

$$|u(t,x)|^2 \to \|Q_{L,K}\|^2_{L^2} \delta_{y_0}, \textit{ in the distribution sense as } t \uparrow T,$$

As a direct consequence of the above theorem, we have:

Corollary 3.4 *Under the same assumption as in Theorem* 3.7. *If* $\mathcal{M} \cap \mathcal{M}' = \emptyset$, *then there is no blow-up solution to* (3.1) *with* $\|u_0\|_{L^2} = \|Q_{L,K}\|_{L^2}$.

Remark 3.4 Our arguments are also essentially valid for the general setting on $\mathbb{T}^N$ for the inhomogeneous NLS

$$\partial_t u = i \left(f(x)\Delta u + \nabla f(x) \cdot \nabla u + k(x)|u|^{\frac{4}{N}} u \right). \tag{3.60}$$

Also, the following lemma will be used in our argument.

Lemma 3.16 ([26]) *Let* $\{f_n\}$ *be a bounded family in* $L^q(\mathbb{R}^N)$ $(0 < q < \infty)$ *such that* $f_n \to f$ *a.e. in* $\mathbb{R}^N$. *Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} ||f_n(x)|^q - |f(x)|^q - |f_n(x) - f(x)|^q| dx = 0.$$

To minimize technicalities, we shall assume $k(x) \equiv 1$ in the sequel. The proofs for the non-constant function $k(x)$ follow essentially the same arguments with some modifications. $\mathbb{T}^2$ is represented by the unit square $[-1/2, 1/2]^2$ with the proper identifications. Thus the functions on $\mathbb{T}^2$ can be viewed as space-periodic functions on $\mathbb{R}^2$. Also, we shall use the same convention as in the last section (see the paragraph before Subsection 3.1.1). Particularly, we still have that $E_L(u) \le E(u)$.

We first establish some useful results.

Lemma 3.17 (Non-vanishing) *Let $\Omega_n = [-\lambda_n/2, \lambda_n/2]^2$ be the square of size $\lambda_n \in \mathbb{Z}^+$. Assume that $v_n \in H^1(\Omega_n)$ such that*

$$\int_{\Omega_n} |v_n(x)|^2 dx \le c_1, \quad \int_{\Omega_n} |\nabla v_n(x)|^2 dx \le c_2, \quad \int_{\Omega_n} |v_n(x)|^4 \ge c_3.$$

Then there exist a constant $c_4 = c_4(c_1, c_2, c_3) > 0$ and a sequence $\{x_n \in \lambda_n\}$ such that

$$\int_{|x-x_n|<1} |v_n(x)|^2 dx > c_4. \tag{3.61}$$

Proof. Clearly there exists $\{x_n \in \Omega_n\}$ such that for all n,

$$\int_{S_n} |v_n(x)|^4 dx \ge c_5 \int_{S_n} (|\nabla v_n(x)|^2 + |v_n(x)|^2) dx,$$

where S_n is the unit square of center x_n and $c_5 = c_3/(2c_1 + 2c_2)$, for if not, we would obtain $c_3 \le c_5(c_1 + c_2) \le c_3/2$ which is a contradiction. Therefore it follows from the Sobolev inequality that

$$\left(\int_{S_n} |v_n(x)|^4 dx\right)^{1/2} \le c \int_{S_n} (|\nabla v_n(x)|^2 + |v_n(x)|^2) dx,$$

which implies

$$\int_{S_n} |v_n(x)|^4 dx \ge c_6 > 0, \tag{3.62}$$

where c, c_6 are independent of n.

To see (3.61), assume by contradiction that there is a subsequence $\{v_n\}$ (relabelled) such that

$$\int_{S_n} |v_n(x)|^2 dx \to 0 \quad \text{as } n \to \infty,$$

which implies

$$v_n(x_n + \cdot) \to 0 \quad \text{weakly in } L^2(S_0) \text{ as } n \to \infty, \tag{3.63}$$

where S_0 is the unit square centered at the origin. Moreover we can assume that

$$v_n(x_n + \cdot) \to v \quad \text{weakly in } H^1(S_0) \text{ as } n \to \infty,$$

for some $v \in H^1(S_0)$. Then

$$v_n(x_n + \cdot) \to v \quad \text{strongly in } L^4(S_0) \text{ as } n \to \infty. \tag{3.64}$$

Thus it follows from (3.63) and (3.64) that

$$\int_{S_n} |v_n(x)|^4 dx \to 0 \quad \text{as } n \to \infty,$$

which is a contradiction to (3.62). The lemma is proved. □

Lemma 3.18 *Suppose $f \in C^1(\mathbb{T}^2)$ with $L = \min_{x\in\mathbb{T}^2} f(x)$. Let $\{u_n\}$ be such that $\|u_n\|_{L^2}^2 \le C_1$, $E_L(u_n) \le C_2$, and $\|\nabla u_n\|_{L^2} \to \infty$ as $n \to \infty$. Then there exists $\{x_n \in \mathbb{T}^2\}$ such that for all (small) $R > 0$,*

$$\liminf_{n\to\infty} \frac{\|u_n\|_{L^2(B(x_n,R))}}{\|Q_L\|_{L^2}} \ge 1.$$

Proof. We argue by contradiction. Suppose there are $R_0 > 0, \gamma_0 > 0$ and a subsequence $\{u_n\}$ (relabelled) such that

$$\sup_{x\in\mathbb{T}^2} \left(\int_{B(x,R_0)} |u_n(y)|^2 dy \right) \le \|Q_L\|_{L^2}^2 - \gamma_0.$$

Consider the scaling

$$U_n(x) = \lambda_n^{-1} u_n(\lambda_n^{-1} x),$$

where $\lambda_n = [\|\nabla u_n\|_{L^2}] \sim \|\nabla u_n\|_{L^2}$. It is easy to verify that $U_n \in H^1(\mathbb{R}^2/(\lambda_n\mathbb{Z}^2))$ and

$$\|U_n\|^2_{L^2(\Omega_n)} = \|u_n\|^2_{L^2} \le C_1, \qquad \lim_{n\to\infty} \|\nabla U_n\|_{L^2(\Omega_n)} = 1,$$
$$\tilde{E}_L(U_n) := \frac{L}{2}\int_{\Omega_n} |\nabla U_n(x)|^2 dx - \frac{1}{4}\int_{\Omega_n} |U_n(x)|^4 dx = \frac{E_L(u_n)}{\lambda_n^2},$$
$$\sup_{x\in\Omega_n}\left(\int_{|x-y|<R} |U_n(y)|^2 dy\right) \le \|Q_L\|^2_{L^2} - \gamma_0, \quad \forall 0 < R \le \lambda_n R_0,$$

where Ω_n is the square of size λ_n. Therefore, extracting a subsequence (still labelled by U_n), we have

$$\int_{\Omega_n} |U_n(x)|^4 dx \ge 2L\int_{\Omega_n} |\nabla U_n(x)|^2 dx - \frac{L}{2} \ge L, \quad \text{for large } n, \tag{3.65}$$
$$\liminf_{n\to\infty}\left(\sup_{x\in\Omega_n}\int_{|x-y|<R} |U_n(y)|^2 dy\right) \le \|Q_L\|^2_{L^2} - \gamma_0, \quad \forall R > 0. \tag{3.66}$$

By Lemma 3.17, there exists a sequence $\{x_n^1 \in \Omega_n\}$ such that

$$\int_{|x-x_n^1|<1} |U_n(x)|^2 dx > \gamma_1,$$

where γ_1 is a positive constant depending only on C_1 and L. Moreover, we can decompose U_n as follows:

$$U_n(x_n^1 + \cdot) = U_n^1(\cdot) + \tilde{U}_n^1(\cdot)$$

where

(i) $U_n^1(x) = 0$ if $|x| \ge 2R_n$, $\tilde{U}_n^1(x) = 0$ if $|x| \le R_n$, with $R_n \to \infty$ and $R_n/\lambda_n \to 0$ as $n \to \infty$;

(ii) $\int_{R_n\le|x|\le 2R_n} |U_n(x_n^1+x)|^2 + |\nabla U_n(x_n^1+x)|^2 + |U_n(x_n^1+x)|^4 \to 0$ as $n\to\infty$;

(iii) $\int_{\mathbb{R}^2} |U_n^1(x)|^2 dx + \int_{\Omega_n} |\tilde{U}_n^1(x)|^2 dx - \int_{\Omega_n} |U_n(x)|^2 dx \to 0$ as $n\to\infty$;

(iv) $\int_{\mathbb{R}^2} |\nabla U_n^1(x)|^2 dx + \int_{\Omega_n} |\nabla\tilde{U}_n^1(x)|^2 dx - \int_{\Omega_n} |\nabla U_n(x)|^2 dx \to 0$ as $n\to\infty$;

(v) $\int_{\mathbb{R}^2} |U_n^1(x)|^4 dx + \int_{\Omega_n} |\tilde{U}_n^1(x)|^4 dx - \int_{\Omega_n} |U_n(x)|^4 dx \to 0$ as $n\to\infty$.

Therefore, there exists $\psi_1 \in H^1(\mathbb{R}^2)$, such that, after extraction of a subsequence, as $n \to \infty$,

$$U_n^1(x_n^1 + \cdot) \to \psi_1 \quad \text{weakly in } H^1(\mathbb{R}^2), \text{ locally (strongly) in } L^4 \text{ and } L^2.$$

Consequently, we have

$$\lim_{n\to\infty} \left\{ \|\nabla U_n^1\|^2_{L^2(\mathbb{R}^2)} - \|\nabla \psi_1\|^2_{L^2(\mathbb{R}^2)} - \|\nabla U_n^1 - \nabla \psi_1\|^2_{L^2(\mathbb{R}^2)} \right\} = 0, \tag{3.67}$$

and by Lemma 3.16,

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} ||U_n^1|^q - |\psi_1|^q - |U_n^1 - \psi_1|^q| dx = 0, \quad (q = 2, 4). \tag{3.68}$$

Thus we have

$$\lim_{n\to\infty} \left\{ E_L(U_n^1) - E_L(\psi_1) - E_L(U_n^1 - \psi_1) \right\} = 0. \tag{3.69}$$

Since $R_n/\lambda_n \to 0$ as $n \to \infty$, by (3.66) we have, for large n,

$$\|U_n^1(x)\|^2_{L^2(\mathbb{R}^2)} \leq \|Q_L\|^2_{L^2} - \frac{3\gamma_0}{4},$$

which, by virtue of (3.68), implies

$$\|U_n^1 - \psi_1\|^2_{L^2(\mathbb{R}^2)} \leq \|Q_L\|^2_{L^2} - \frac{\gamma_0}{2} \quad \text{and} \quad \gamma_1 \leq \|\psi_1\|^2_{L^2} \leq \|Q_L\|^2_{L^2} - \frac{\gamma_0}{2}. \tag{3.70}$$

Therefore, by Lemma 3.4 and (3.70),

$$\begin{aligned} E_L(\psi_1) + \liminf_{n\to\infty} \tilde{E}_L(\tilde{U}_n^1) &\leq \liminf_{n\to\infty} (E_L(U_n^1) + \tilde{E}_L(\tilde{U}_n^1)) \\ &\leq \liminf_{n\to\infty} \tilde{E}_L(U_n) \leq 0 \end{aligned} \tag{3.71}$$

Hence, by Lemma 3.4 and (3.70) again,

$$\liminf_{n\to\infty} \tilde{E}_L(\tilde{U}_n^1) \leq -E_L(\psi_1) < 0.$$

Thus, extracting a subsequence (still labelled by $\tilde{U}_n^1$), we have

$$\|\tilde{U}_n^1\|^2_{L^2} \to C_1^1 \leq C_1 - \gamma_1 \quad \text{and} \quad \liminf_{n\to\infty} E_L(\tilde{U}_n^1) < 0.$$

Define

$$\tilde{u}_n(x) = \lambda_n \tilde{U}_n^1(\lambda_n x), \quad x \in \mathbb{R}^2/\mathbb{Z}^2.$$

Redefine the sequences

$$\lambda_n = [\|\nabla \tilde{u}_n\|_{L^2}] \quad \text{and} \quad U_n(x) = \lambda_n^{-1} \tilde{u}_n(\lambda_n^{-1} x).$$

If $\lambda_n < \infty$ for all n, then it is easy to see that $\liminf_{n\to\infty} \tilde{E}_L(\tilde{U}_n^1) = 0$, hence by (3.71) we have $E_L(\psi_1) \leq 0$ which contradicts Lemma 3.4. Thus the Lemma is proved. If $\lambda_n \to \infty$ as $n \to \infty$, then, extracting a subsequence if necessary, we have

$$\lim_{n\to\infty} \|U_n\|_{L^2(\Omega_n)}^2 = C_1^1 \leq C_1 - \gamma_1, \qquad \liminf_{n\to\infty} \tilde{E}_L(U_n) < 0,$$
$$\liminf_{n\to\infty} \left(\sup_{x\in\Omega_n} \int_{|x-y|<R} |U_n(y)|^2 dy \right) \leq \|Q_L\|_{L^2}^2 - \gamma_0, \quad \forall R > 0.$$

Therefore, we can iterate the same procedure. Since $-p\gamma_1 + C_1 < 0$ for some finite integer p, applying the same procedure at most p times, we can reach a contradiction. The proof is complete. □

As in the case in $\mathbb{R}^2$, the above Lemma can be strengthened by making use of the observation that

$$\|Q_{f(x)}\|_{L^2}^2 = \frac{f(x)}{L} \|Q_L\|_{L^2}^2,$$
$$\lim_{n\to\infty} \sup_{|x-y|<R} |f(\frac{x}{\lambda_n}) - f(\frac{y}{\lambda_n})| = 0, \quad \forall R > 0,$$

where $\lambda_n = \|\nabla u_n\|_{L^2} \to \infty$ as $n \to \infty$. Namely, we have

Lemma 3.19 *Suppose $f \in C^1(\mathbb{T}^2)$ with $L = \min_{x\in\mathbb{T}^2} f(x)$. Let $\{u_n\}$ be such that $\|u_n\|_{L^2}^2 \leq C_1$, $E_L(u_n) \leq C_2$, and $\|\nabla u_n\|_{L^2} \to \infty$ as $n \to \infty$. Then there exists $\{x_n \in \mathbb{T}^2\}$ such that for all (small) $R > 0$,*

$$\liminf_{n\to\infty} \frac{\|u_n\|_{L^2(B(x_n,R))}}{\|Q_{f(x_n)}\|_{L^2}} \geq 1.$$

Now we are in the position to prove Theorems 3.6 and 3.7.

Proof of Theorem 3.6. The proof makes use of the observation that $E_L(u) \le E(u)$. Part (i) follows directly from Lemma 3.18 and the conservation of mass and energy. Part (ii) is essentially the same with that of Proposition 1 in [31]. □

Proof of Theorem 3.7. (i) We view u as a space-periodic function on $\mathbb{R}^2$. Let $\varphi(x)$ be a cut-off function as defined in Theorem 3.7. Define

$$\tilde{u}(x,t) = \varphi(x)u(t, x + x(t)) = |\tilde{u}(t,x)|e^{-i\theta(t,x)},$$

where $\{x(t)\}$ is from Theorem 3.6. It is easy to see that $\tilde{u} \in H^1(\mathbb{R}^2)$ and $\|\tilde{u}(t)\|_{L^2} \le \|u(t)\|_{L^2(\mathbb{R}^2)} = \|Q_L\|_{L^2}$. Furthermore, by Theorem 3.6, Lemma 3.13 and Lemma 3.4, we have

$$\|\nabla\tilde{u}(t)\|_{L^2(\mathbb{R}^2)}/\|\nabla u(t)\| \to 1 \quad \text{as } t \uparrow T$$

and

$$0 \le E_L(|\tilde{u}|) \le E_L(\tilde{u}) \le C$$

which implies that $\|\tilde{u}\nabla\theta\|_{L^2(\mathbb{R}^2)} \le C$ and

$$\|\nabla|\tilde{u}(t)|\|_{L^2(\mathbb{R}^2)}/\|\nabla u(t)\| \to 1 \quad \text{as } t \uparrow T.$$

Now, define

$$\phi_k(x) = \frac{1}{\lambda_k}\tilde{u}(t_k, \frac{x}{\lambda_k}) = \frac{1}{\lambda_k}\left|\tilde{u}(t_k, \frac{x}{\lambda_k})\right| e^{-i\theta(t_k, \frac{x}{\lambda_k})},$$

where $t_k \uparrow T$ as $k \to \infty$ and $\lambda_k = \|\nabla u(t_k)\|_{L^2(\mathbb{R}^2)}/\|\nabla Q_L\|_{L^2}$. Clearly

$$\|\phi_k\|_{L^2(\mathbb{R}^2)} \uparrow \|Q_L\|_{L^2}, \quad \|\nabla|\phi_k|\|_{L^2(\mathbb{R}^2)} \to \|\nabla Q_L\|_{L^2}$$

and

$$0 \le E_L(|\phi_k|) = \frac{E_L(|\tilde{u}(t_k)|)}{\lambda_k^2} \le \frac{E_L(\tilde{u}(t_k))}{\lambda_k^2} \to 0 \ \text{ as } k \to \infty.$$

Therefore there exist $\psi \in H^1(\mathbb{R}^2)$ such that $|\phi_k| \to \psi$ weakly in $H^1(\mathbb{R}^2)$. By a similar argument as in the proof of Lemma 3.18 (see (3.67)-(3.69)), we have

$$E_L(|\phi_k|) - E_L(\psi) - E_L(|\phi_k| - \psi) \to 0 \quad \text{as } k \to \infty,$$

which implies $E_L(\psi) \leq 0$, hence $E_L(\psi) = 0$ since $\|\psi\|_{L^2(\mathbb{R}^2)} \leq \|Q_L\|_{L^2}$. Thus, $\|\psi\|_{L^2(\mathbb{R}^2)} = \|Q_L\|_{L^2}$ and $|\phi_k| \to \psi$ strongly in $L^2(\mathbb{R}^2)$. By Gagliardo-Nirenberg inequality, $|\phi_k| \to \psi$ strongly in $L^4(\mathbb{R}^2)$, hence strongly in $H^1(\mathbb{R}^2)$ since $E_L(\psi) = 0$. In view of the variational characterization of Q_L, we then can claim that $\psi(x) = Q_L(x + x_0)$ for some $x_0 \in \mathbb{R}^2$. After redefining $x(t)$, we can set $x_0 = 0$. Finally, the desired result follows from $|\phi_k(x)| = \phi_k(x)e^{i\theta(t_k, \frac{x}{\lambda_k})}$.

(ii) By Lemma 3.19, there exists $x(t) \in \mathbb{T}^2$ such that for all small $R > 0$,

$$\liminf_{t \uparrow T} \frac{\|u(t)\|^2_{L^2 B(x(t),R)}}{\|Q_{f(x(t))}\|^2_{L^2}} \geq 1.$$

Since $\|Q_{f(x(t))}\|^2_{L^2} = \frac{f(x(t))}{L}\|Q_L\|^2_{L^2}$, by the conservation of mass and the assumption that $\|u_0\|_{L^2} = \|Q_L\|_{L^2}$, we obtain $\limsup_{t\uparrow T} f(x(t)) \leq L$. The desired result is then easy to be verified by a similar argument as in the proof of Proposition 3.4 and the remark followed. $\square$

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