PARAMETER-UNIFORM NUMERICAL METHODS FOR PROBLEMS WITH LAYER PHENOMENA: APPLICATION IN MATHEMATICAL FINANCE

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To my family

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Contents

Summary

In many fields of application, the differential equations are singularly perturbed. Usually, the exact solution of a non-trivial problem involving a singularly perturbed differential equation is unknown. Approaches for such problems are largely confined to analytical and numerical studies of solutions to these problems. In this thesis we construct numerical methods based on analytical theories for solving singularly perturbed Black-Scholes equation, which has non-smooth solutions with singularities related to interior and boundary layers.

A problem for the Black-Scholes equation that arises in financial mathematics, by a transformation of variables, is leaded to the Cauchy problem for a singularly perturbed parabolic equation with variables x, t and a perturbation parameter ε, $\varepsilon \in (0, 1]$. This problem has several singularities such as: the unbounded domain; the piecewise smooth initial function (its first order derivative in x has a discontinuity of the first kind at the point $x = 0$); an interior (moving in time) layer generated by the piecewise smooth initial function for small values of the parameter ε; etc.

In this thesis, we construct *the singularity splitting method* for grid approximation of the solution and its first order derivative of the singularly perturbed Black-Scholes equation in a finite domain including the interior layer. On a uniform mesh, using the method of additive splitting of a singularity of the interior layer type *(briefly, the singularity splitting method)*, a special difference scheme is constructed that allows us to approximate ε-uniformly both the solution of the boundary value problem and its first order derivative in x with convergence orders close to 1 and 0.5, respectively.

In order to construct adequate grid approximations for the singularity of the interior layer type, we consider a singularly perturbed boundary value problem with a piecewise smooth initial condition. Moreover, the singularity of the boundary layer is stronger than that of the interior layer, which makes it difficult to construct and study special numerical methods suitable for the adequate description of the singularity of the interior layer type. Using the method of special meshes that condense in a neighbourhood of the boundary layer and the method of additive splitting of the singularity of the interior layer type, a special finite difference scheme is designed that make it possible to approximate ε-uniformly the solution of the boundary value problem on the whole domain, its first order derivative in x on the whole domain except the discontinuity point (outside a neighbourhood of the boundary layer), and also the normalized derivative (the first order spatial derivative multiplied by the parameter ε) in a finite neighbourhood of the boundary layer.

In Chapter 1, a brief overview of several popular analytical and numerical methods for solving singularly perturbed differential equations are presented. Merits and drawbacks of various methods are also discussed. After summarized survey on methods for financial derivatives, the need of alternative parameter-uniform numerical method in financial derivatives computing is clarified.

Chapter 2 presents deduction of the dimensionless singularly perturbed Black-Scholes equation and formulation of the initial boundary value problem. A priori analysis of the singularly perturbed Black-Scholes equation with different controlled smoothness initial functions on condition of Dirichlet problem and Cauchy problem are also given.

In Chapter 3, an ε-uniform method, singularity splitting method is constructed theoretically for resolving the singularity due to the discontinuity of the first derivative of the initial condition for the singularly perturbed Black-Scholes equation. Experimental results for both solutions and derivatives from the classical finite difference method and singularity splitting method are presented. Conclusion is drawn that the additive splitting method is ε-uniformly convergent for both solutions and derivatives of the singularly perturbed Black-Scholes equation with interior layer arise from the discontinuity of the first derivative of the initial condition whereas the classical finite difference method does not.

In Chapter 4, boundary value problem in bounded domains for parabolic equations coming from the Black-Scholes equation with a discontinuous initial condition is studied. The use of a non-uniform boundary layer resolving mesh and the singularity splitting method are combined together to solve the problem. Numerical solutions and their derivatives are computed to evaluate the effectiveness of the method for problems with both interior and boundary layers.

Finally, we discuss conclusions of our research in Chapter 5.

List of Symbols

Nomenclature

a, b, c, q	coefficients of singular perturbation problem
D_ε^N	double mesh differences with respect to ε
$\overline{D}_h$	uniform mesh in closed domain $\overline{D}$
$\mathrm{erf}(\xi)$	$= 2/\sqrt{\pi}\int_0^\xi e^{-t^2}dt$, error function
E	the exercise price
E_ε^N	maximum mesh differences with respect to ε
G	open domain
$\overline{G}$	$= \bigcup_j \overline{G}^{\,j}, \quad j = 1, 2, 3$, closed domain
$\overline{G}^{\,*}$	$=\overline{G} \setminus S^{(*)}$
$\overline{G}_0^{\,*}$	$=\overline{G}_0^{\,*}(m) = \overline{G}^{\,*} \cap \{x \geq -d+m\}$
$\overline{G}_h^{NF}$	finest uniform/nonuniform mesh in bounded domain
$\overline{G}^{\,\delta}$	$=\{(x,t):\ r\big((x,t),\, S_*\big) \leq \delta\}$, the δ-neighbourhood of the set S_*
H^α	Hölder space

L	general differential operators
$L_{(j.\,k)}$	operators (constants, meshes) introduced in formula $(j.k)$
$m,\ m_0,\ m_1$	constants, $m_0 = a^{-1}\,b$, $m \in (0,\, m_0)$
M	sufficiently large positive constants independent of the parameter ε and parameters of difference schemes
N^F	finest mesh grid number in space
$N(\cdot)$	$N(d^+) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d^+} e^{-\frac{1}{2}s^2}ds$, standard normal distribution function
p_ε^N	double mesh convergence order with respect to ε
$p(x,t)$	first derivative in x
$\overline{p}^h(x,t)$	interpolant of first derivative in x
$P(x,t)$	normalized first derivative in x (diffusion flux)
$\overline{P}^h(x,t)$	interpolant of normalized first derivative in x
q_ε^N	maximum mesh convergence order with respect to ε
r	riskless interest rate in option pricing
$r\big((x,t),\,\overline{S}^{\,l}\big)$	distance from point (x,t) to set $\overline{S}^{\,l}$
S	asset price **or** boundary set
$S^{(*)}$	$=\{(0,\,0)\}$
S_0	$=\overline{S}_0^- \bigcup \overline{S}_0^+$, lower parts of the boundary S
S_0^-	$=\{(x,t):\ x \in [-d,\,0),\ t=0\}$
S_0^+	$=\{(x,t):\ x \in (0,\,d],\ t=0\}$
S^L	$=S^{\,l} \bigcup S^{\,r}$, lateral parts of the boundary S
$S^{\,l},\ S^{\,r}$	left and right parts of the boundary $S^{\,L}$
$\overline{S}_0^-,\ \overline{S}_0^+,\ \overline{S}^L$	closed set of $S_0^-,\ S_0^+,\ S^L$
S_*	$=S_0 \bigcap \overline{S}^{\,L}$
$S^{\,\gamma}$	$=\{(x,t):\ x=\gamma(t),\ (x,t) \in \overline{G}\}$, $\gamma(t) = -b\,q^{-1}\,t$, $t \geq 0$, characteristic of equation passing through the point $(0,\,0)$
T	time of expiration

Greek

α	order of space
β	fitting factor
$\mathcal{L}$	general differential operators
ε	singular perturbation parameter
$\eta(x,t)$	sufficiently smooth function to prevent interaction between boundary and interior layers
$\triangle$	Laplace operator
$\overline{\omega}_1$	uniform meshes on x domain
$\overline{\omega}^*$	$= \overline{\omega}^*(\sigma)$ piecewise uniform mesh on x domain
$\overline{\omega}_0$	uniform meshes on t domain
σ	fitting factor, **or** volatility in option pricing
τ^*	$=rT$ dimensionless variable
$\|\| \cdot \|\|$	maximum absolute error measure

subscripts

i, j, k	iteration labels

superscripts

(k)	kth derivative

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Chapter 1

Introduction

1.1 Partial Differential Equations

Differential equations are mathematical models that express the behaviors of physical systems in science and engineering. In mathematics, a differential equation is an equation in which the derivatives of a function appear as variables. Many of the fundamental laws of physics, chemistry, biology and economics can be formulated as differential equations, including the laws of Finance.

A partial differential equation (PDE) is a differential equation involving functions and their derivatives of more than one single independent variable while an ordinary differential equation (ODE) is a differential equation involving one function and its derivatives. Partial differential equations are used to formulate and solve problems that involve unknown functions of several variables, such as the propagation of sound or heat, electrostatics, fluid flow, elasticity, or more generally any process that is distributed in space, or distributed in space and time. Very different physical problems may have identical mathematical formulations. Mathematical theory is often a useful connection between diverse fields.

The analysis and solution of partial differential equation is a difficult subject. A basic problem is that of determining whether the differential equations have solutions. Closely related questions of interest are: under what conditions do solutions exist, are there multiple solutions and if so which solutions are meaningful to the problem being solved and which are auxiliary mathematical solutions. Most of these issues are the concern of professional mathematicians. Engineers and scientists would be interested in simply finding solutions to the equations.

Generally, analytical methods and numerical methods are used to solve a PDE. Analytical methods are concerned with obtaining exact or approximate solutions or with establishing their qualitative properties by some theoretical considerations. Analytical methods produce, when possible, exact analytical solutions in the form of general mathematical expressions. Solutions of differential equations will give expressions for functions. While numerical methods on the other hand produce approximate solutions in the form of discrete values or numbers. Finding exact solutions to higher-order algebraic equations will not, in general, be a feasible task and numerical methods must be employed to find approximate solutions instead.

The analytical solution of some partial differential equations in certain conditions, are much more difficult to get the analytical solutions, for example, the differential equations governing the behavior of an inviscid gas, the Euler equations, have been known to scientists for centuries, but the exact solutions of these equations available today are only valid for very simple physical situations. Also the Black-Scholes equation with a linear complementarity involving a differential operator and a constraint on the value of the option which governs the American option does not admit an analytical solution. Therefore scientists require numerical methods. Mostly, scientists use both analytical and numerical methods to analyze problems.

In many fields of application, the PDE and ODE are singularly perturbed. Indeed, it is feature of the equations that explains theoretically the physical phenomenon of boundary layers. Typical examples of the problems are presented by singularly perturbed equations which have a small parameter ε, the singular perturbation parameter, effecting the higher derivatives. These problems arise frequently in many practical applications such as fluid mechanics, chemical reactions, control theory, and finance.

A brief review of the derivation and some basic definitions in singular perturbation phenomena will be given in the following sections.

1.2 Derivation of Singularly Perturbed Problems

The first formulations of singularly perturbed differential equations modeling fluid motion near boundaries were performed by Prandtl (1905). A general description of various phenomena of practical problems which are modeled by singularly perturbed equations was originally given by Friedrichs (1955).

The fundamental mathematical problem with singular perturbation phenomenon is a singular perturbation problem. In singular perturbation problems the coefficient of the highest derivative in the differential equation is multiplied by a small parameter, called the singular perturbation parameter ε. For example, in Convection-Diffusion problems, singular perturbation phenomenon arise when the small parameter ε multiplies the Laplace operator $\triangle$. Singular perturbation phenomena also emerge in other equations, such as in Momentum Conservation laws, in Prandtl equations, in Burger's equation and in Black-Scholes equation, etc. We give a briefly introduction of the derivation of the singularly perturbed differential equation with Navier-Stokes equations.

The principal governing equations of fluid dynamics are: the continuity equation, the momentum equation and the energy equation. These are the mathematical statements of the fundamental physical principles: the conservation of mass, momentum, and energy. Based on these principles, fluid and gas dynamics can be described by the Navier-Stokes equations. In two dimensions these comprise the systems of four nonlinear partial differential equations

$$\begin{aligned}
\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y} &= 0,\\
\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+p)+\frac{\partial}{\partial y}(\rho v u)-\mu\Big(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}\Big) &= 0,\\
\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho u v)+\frac{\partial}{\partial y}(\rho v^2+p)-\mu\Big(\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}\Big) &= 0,\\
\frac{\partial}{\partial t}(\rho e)+\frac{\partial}{\partial x}\Big(\rho u(e+\frac{p}{\rho})\Big)+\frac{\partial}{\partial y}\Big(\rho v(e+\frac{p}{\rho})\Big)-\mu\Big(\frac{\partial}{\partial x}(u\tau_{xx}+v\tau_{xy}) & \\
+\frac{\partial}{\partial y}(u\tau_{yx}+v\tau_{yy})\Big)-k\Big(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\Big) &= 0.
\end{aligned}$$

For the four dependent variables (ρ, u, v, e), where ρ is the density of the material(fluid or gas), u and v are the components of its velocity, and e is the internal energy. The variables T and p in the system can be expressed in terms of these variables using the definition of the internal energy $e = C_v T + \frac{1}{2}(u^2+v^2)$ where C_v is the specific heat and the equation of state $p = p(\rho, T)$ for the material, which expresses the pressure p as a function of the density ρ and the temperature T(For example, $p = \rho RT$ for a perfect gas). The components $\tau_{xx}, \tau_{xy}, \tau_{yx}, \tau_{yy}$ of the viscous stress tensor τ are expressed in terms of the rate of change in space of the velocities by the relations

$$\tau_{xx}=\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y},\quad \tau_{yy}=-\frac{2}{3}\frac{\partial u}{\partial x}+\frac{4}{3}\frac{\partial v}{\partial y},\quad \tau_{yx}=\tau_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}.$$

For complete physical definiteness of solutions for the system of equations, boundary and initial conditions must be prescribed. In a viscous fluid, the components of the velocity on walls are equal to zero. The singularly perturbed nature of these equations becomes apparent when the magnitude of the convective terms is much larger than that of the diffusion terms, that is when the magnitude of the terms involving first order derivatives is much larger than that of the terms involving second derivatives. In specific situations, and with appropriate scaling of the variables, this is equivalent to the condition that the corresponding value of the scaled coefficients μ and k have magnitudes that are much smaller than unity (The scaled coefficient μ is $1/Re$, where Re is the Reynolds number and scaled coefficient k is $\frac{1}{Pr}$, where Pr is the Prandtl number). It is precisely this situation, which is referred to as a singularly perturbed system of differential equations and the small coefficients are called the singular perturbation parameters.

A robust numerical method is considered in [20] for the Prandtl problem of laminar flow of an incompressible fluid past semi-infinite plate. The Prandtl boundary layer equations are an essential simplification of the Navier-Stokes equations. The solution of the Prandtl problem retains singularities of the solution of the Navier-Stokes equations. It is shown by numerical experiments that the numerical method for the Prandtl problem that is constructed on the basis of the condensing mesh technique converges ε-uniformly where $\varepsilon = Re^{-1}$. A technique for experimental studying of the rate of ε-uniform convergence is given in [20]. A similar technique is used also for the numerical investigation of the difference scheme constructed in the present work.

A distinctive feature of the singularly perturbed equations is that their solutions and (or) the solution derivatives have intrinsic narrow zones (boundary and interior layers) of large variations in which they jump from one stable state to another or to prescribed boundary values. In physics, for example, this happens in viscous gas flows in the zones near the boundary layers where the viscous flow jumps from the boundary values prescribed by the condition of adhesion to the inviscid flow or in the zones near the shock wave where the flow jumps from a subsonic to supersonic state. In chemical reactions the rapid transition from one state to another is typical for solution process. In finance, the value of a call option at and before the expire time is typical for derivatives process.

Usually the solution, the approximation solution or the initial condition of a singular perturbed problem has a singular component, called singular function. Some singular functions are typical for singular perturbation problems: the exponential function, power function, logarithmic functions and singular functions with interior critical points, etc. Properties of some typical singular functions for singularly perturbed problems is discussed in [50] and [20].

There are a variety of physical processes in which boundary and interior layers in the solution may arise for certain parameter ranges. The primary objective in singular perturbation analysis of such problems is to develop asymptotic approximations to the true solution that are uniformly valid with respect to the perturbation parameter. Some examples of such perturbation problems are boundary layers in viscous fluid flow and concentration or thermal layers in mass and heat transfer problems.

Various analytical and numerical methods has been proposed during the years for singularly perturbed problem. We survey some of them in next section.

1.3 Basic Approaches for Singularly Perturbed Problems

The exact solution of a non-trivial problem involving a singularly perturbed differential equation is usually unknown. Approaches for such problems are largely confined to analytical and numerical studies of solutions to these problems.

1.3.1 Analytical Methods

The basic idea of the analytical methods is to find the approximate solution of the differential equation with the absolute error bounded uniformly to $M\varepsilon^k$ for some M and k independent of ε. The theoretical background is to find appropriate coordinate transformation or layer-resolving grids through analyzing the solution derivatives to eliminate the singularities and to study the limit solutions derived from the exact solution by letting the parameter ε approach zero.

The most popular analytical methods are known as multivariable asymptotic expansions, matched asymptotic expansions and expansions with differential inequalities.

The fundamental idea of the multivariable asymptotic method is that the solution to a singularly perturbed problem is sought as an additive composite function of the slow variable x and the fast variables $\tau_j = \tau_j(x, \varepsilon)$ which, in the case of a singular layer, is found as a combination of two power series in ε referred to as inner $u_1(x, \varepsilon)$ and outer $u_0(x, \varepsilon)$ expansions. The most general foundation for the asymptotic studies of singular perturbed equations was made by Tikhonov [83, 84]. In 1983, Nipp gave an extension of Tikhonov's theorems to planar case. Detailed descriptions of the analytical methods of asymptotic expansions were showed in [86, 87, 33, 61]. This scheme for finding solutions is readily generalized to multipoint and multiscale expansions. However, this method is suitable for the problems whose reduced problems' solutions are known and smooth. Even for the problem presented in the monograph of Chang and Howes [16], the method demonstrates difficulties in spite of the fact that the solution of the reduced problem is a constant.

For the matched asymptotic expansions method, a solution is found as a combination of some separate expansions with individual coordinate scaling considered only at suitable subdomains. The scales are chosen in a way that the different expansions are valid at the intersection of the respective subdomain. Andrei Camyshev, Andrei Kolyshkin and Inta Volodko [2] used the method and got good results for analyzing rapidly changing unsteady laminar flows. One of the difficulties is the matching procedure limitations.

In the methods of expansion via differential inequalities the asymptotic solution is located and estimated with the aid of inequality techniques developed by Nagumo [60] and others. The asymptotic solution is chosen by means of a shooting technique in terms of its values on the boundary of the existence interval. This is the most general approach allowing one to obtain uniformly many new asymptotic expansions as well as those which have been obtained by other methods.

1.3.2 Numerical Methods

The singularly perturbed problems can also be solved numerically using the finite difference methods and finite element methods. The main idea of these methods is to adjust approximation of equations or specifying layer-resolving coordinate transformation or constructing layer-resolving algorithms to get the uniform convergence and eliminate the singularities.

The difficulty with standard numerical methods which employ uniform meshes is a lack of robustness with respect to the perturbation parameter ε. Since the layer contract as ε becomes smaller, the mesh needs to be refined substantially to capture the dynamics within the diminishing layer.

There are fitted operator techniques, fitted grids techniques, finite element methods and methods of Layer-Damping transformations. The motivation for contriving the numerical schemes for singularly perturbed equations with fitted operator techniques was proposed by Allen and Southwell [1], and was justified by Il'in [32]. The methods rely on a simulation of differential equations by special algebraic equations which take into account the singular nature of the solutions.

The finite element methods applied to generate finite difference schemes for singularly perturbed problems are generally based on Galerkin and Petrov-Galerkin finite element methods. The adjustment of these methods to singular perturbed problems relies on the use of a set of special trial functions satisfying some singularly perturbed equations with simple coefficients (constants or linear functions) or on the use of special elements which are refined in the zone of layers. Finite element methods were applied for the numerical solutions of some singularly perturbed problems by Szymczak and Babuska [82], Lube and Weiss [53] and O' Riordan, Hargty and Stynes [62].

For the fitted grids techniques, that was introduced by Bakhvalov [5], the requirement of the ε uniform convergence is achieved with a suitable mesh. The mesh is commonly chosen in such a way that the error of an approximation or of a numerical solution is ε uniformly bounded or the variation of the solution in the neighboring points is estimated by Mh, where h is a maximal stepsize. The application of such grid allows one to interpolate the numerical solution ε uniformly to the whole domain including layers.

A more detailed discussion about fitted operator method and fitted mesh method is given in the next section.

1.3.3 Finite Difference Methods

Early finite difference methods for problems involving singularly perturbed differential equations used standard finite difference operator on a uniform mesh and refined the mesh more and more to capture the boundary or interior layers as the singular perturbation parameter decreased in magnitude. The methods were inefficient to obtain accurate solutions, and hence, they are not ε-uniform.

Two approaches have generally be taken to construct ε-uniform finite difference methods, i.e., fitted operator methods and fitted mesh methods.

The fitted operator methods involve replacing the standard finite difference operator by a finite difference operator, called the fitted operator, that reflects the singularly perturbed nature of the differential operator. For example, for the linear problem, such methods can be constructed by choosing their coefficients so that some or all of the exponential functions in the null space of the difference operator, or part of it, are also in the null space of the finite difference operator. The corresponding numerical methods are obtained by applying the operator to obtain a system of finite differential equations on a standard mesh. Allen et al. [1] first suggested using such methods to solve the problem of the flow of a viscous fluid past a cylinder. The first successful mathematical analysis of ε-uniform finite difference methods was given in [32] for a linear two point boundary value problem. Further development of these kind of methods were performed by Lorenz [52], Berger, Han, Kellog [8] and others. An comprehensive discussion of ε-uniform fitted operator methods is given in Doolan et al. [19], [55], Farrell et al. [20] and Tobiska [66].

The fitted mesh methods use a mesh that is adapted to the singular perturbation. A standard finite difference operator is applied on the fitted mesh to obtain a system of finite difference equations, which is then solved in the usual way to obtain approximate solutions. It is often sufficient to construct a piecewise uniform mesh which is first introduced by Shishkin [71] to obtain approximate solutions. The piecewise uniform mesh is a union of a finite number of uniform meshes having different mesh parameters. This is the simplest adapting mesh method. Miller et al. [56] presented the first numerical results using the fitted mesh method. Further application and development of the fitted mesh methods can be found in [20, 37, 85, 38, 49, 21].

In practice, fitted mesh methods are frequently used whenever possible because of their simpler implementation. Moreover, the fitted mesh methods can be easily generalized to multidimensional and nonlinear problems. In this thesis, we use the fitted mesh methods to compute the solutions and the first derivatives of singular perturbed problems with appearing of the interior and boundary layers.

1.4 Norms and Notation

A maximum or minimum principle is a useful tool for deriving a priori bounds on the solutions of the differential equations and their derivatives. The one is referred to Protter and Weinberger [63] for a comprehensive discussion of these comparison principles. In this thesis, the ε-uniform error estimates are obtained using the maximum principle [68]. The key step in obtaining these estimates is the establishment of suitable bounds on the derivatives of the smooth and singular components of the solution. The error estimates obtained in this thesis are valid at each point of the mesh or domain.

The choice of the maximum norm as the measurement of error is due to the need to measure the error in the very small domains in which the boundary or interior layers occurs. Other norms, such as the root mean square, involve averages of the error which smooth out rapid changes in the solutions and therefore may fail to capture the local behavior of the error in these layers. Further discussion about the choice of an appropriate norm may be found in Farrell et al. [20], Hegarty et al. [27].

We define the parameter-uniform or ε-uniform methods as methods generated numerical solutions that converge uniformly for all values of the parameter ε, instead of for a given single value of the singularly perturbation parameter ε, in the range $(0, 1]$ and that require a parameter-uniform amount of computational work to compute each numerical solution. If the method is ε-uniform, the difference between the exact solution u_ε and the numerical solution z_ε^N satisfies an estimate of the following form: for some positive integer N_0, all integers $N \geq N_0$ and all $\varepsilon \in (0, 1]$, we have

$$\| \bar{z}_\varepsilon^N - u_\varepsilon \|_{\bar{\Omega}} \leq CN^{-p},$$

where C, N_0 and p are positive constants independent of ε and N. Here $\bar{z}_\varepsilon^N$ denotes the piecewise linear interpolant on the whole domain $\bar{\Omega}$ of the mesh function z_ε^N defined on the mesh $\bar{\Omega}^N$ and $\| \, . \, \|_{\bar{\Omega}}$ denotes the maximum norm on the whole domain $\bar{\Omega}$.

We call such numerical method the robust method that approximates ε-uniformly the solution of the problem and its first derivative; a strong definition of the robust method see in Chapter 4.

1.5 Mathematical Methods for Financial Derivatives

Finance plays an important role now in modern society, either in banking or in corporations. Modeling of instruments in financial market by mathematical methods has been a rapidly growing research area for both mathematicians and financiers. There are two divisions for financial markets: stocks and derivatives. Financial derivatives are a significant aspect of our economy. Options are one of the most common derivative securities in financial markets.

There have been many approaches developed for financial derivatives. It is well known that many important derivatives lack a closed-form analytical solution and their estimation has to be performed by approximation procedures. For this purpose, a number of analytical approximation methods have been suggested in the literature, especially for pricing American options, such as the quadratic approximation approach [54, 7], compound option approximation [24, 13], the method of interpolation between bounds [35, 12], and the analytical methods of lines [15]).

Analytical approximations usually cannot be made arbitrarily accurate. Alternatively, numerical methods are most widely used for valuation of a wide variety of derivative securities. In the financial literature, three major numerical approaches have been developed: binomial tree model [18, 64], Finite difference method [10, 11, 67], and Monte Carlo simulation [9]. Generally speaking, both the binomial tree method and the Monte Carlo simulation approximate the underlying stochastic process directly, while the Finite difference scheme and analytical approximation are used to solve the Black-Scholes equation with appropriate boundary conditions that characterize various options pricing problems.

Some detailed review and comparison of the alternative option valuation techniques are available in [25, 12]. The first numerical method for the Black-Scholes equation was the lattice technique introduced by J.C. Cox [18] and Hull and White [31]. This approach is equivalent to an explicit time-steping scheme. Several researchers have reported numerical methods for the Black-Scholes equation based on traditional finite difference methods and the constant coefficient heat equations [6, 17].

The best known analysis of convergence for standard finite difference methods involves the concepts of consistency and stability [34]. It is well known that most finite difference methods are stable and accurate, and hence their solutions converge to the exact solutions as the mesh number $N \to \infty$.

However, for the dimensionless formulation of the Black-Scholes equation for the value of a European call option for some values of parameters, most current finite difference and finite element methods can not fulfill the same stability and monotone properties with the exact solution of the original differential equation. Experimentally, the convergence behaviors do not behave uniformly well regardless of the value of the singular perturbation parameter [58]. The classical finite different and finite element methods are not parameter-uniform [57]. So methods with new attributes are required.

Recently, some asymptotic and numerical methods were designed for the singularly perturbed Black-Scholes equation with appearing of different layers in solutions. For example, Lin and Shishkin proposed a specific numerical technique to evaluate error bounds for the remainder term in the asymptotic expansion of the solution of the singularly perturbed Black-Scholes equation with a weak transient layer in solution [48]. This approach is based on using the computed numerical solutions of a robust difference scheme for the Black-Scholes equation in a bounded domain; error bounds for solutions of this robust scheme are independent of the singular perturbation parameters. Miller and Shishkin studied the Black-Scholes equation in dimensionless variables with both boundary and initial parabolic layers appear in the solution [58]. They proved that the errors in the maximum norm of an upwind finite difference method on uniform meshes are unsatisfactorily large, while the errors in the maximum norm of the same upwind finite difference method on piecewise-uniform meshes, appropriately fitted to the initial layer in some neighbourhood of the layer, don't depend on the value of the singular perturbation parameter ε. They considered the problem with smooth initial conditions instead of the piecewise-smooth initial conditions.

We will design a singularity splitting scheme based on the method of additive splitting of the singularity of the transient layer type for the singularly perturbed Black-Scholes equation of a European call option which contains singularity of interior layer type due to the piecewise-smooth initial conditions. Our key idea is: to represent the solution of the singularly perturbed problem with interior layer as sum of functions, which come from the singular part and regular part of the solution. We compute the solution of the singular function analytically and the solution of the regular function numerically. This allows us to approximate parameter-uniformly both the solution of the problem and its first order derivative in x.

We will also extend the singularity splitting method to singularly perturbed boundary value problems with piecewise smooth initial conditions with appearing of various intensity of singularities, e.g. the singularity of the boundary layer is stronger than that of the interior layer, which makes it difficult to construct and study special numerical methods suitable for the adequate description of the singularity of the interior layer type. Special technique is constructed that make it possible to approximate ε-uniformly the solution of the boundary value problem on the whole domain, its first order derivative in x on the whole domain except the discontinuity point, however, outside a neighbourhood of the boundary layer, and also the normalized derivative (the first order spatial derivative multiplied by the parameter ε) in a finite neighbourhood of the boundary layer. Numerical experiments illustrates the efficiency of the constructed scheme.

1.6 Scope of the Thesis

The scope of this thesis is as follow:

1, We transform the Black-Scholes equation of a European call option with appropriately specified final and boundary conditions to an initial boundary value problem in the dimensionless form. There are singularities in this problem: the unbounded domain, the no-smooth initial condition and the wide ranges of values of the free parameters. For certain ranges of values of these parameters, the solution of the problem may have an initial layer and may cause serious errors in current numerical approximations.

2, We prove that it is impossible to construct a parameter-uniform numerical method using a standard finite difference operator on a rectangular mesh for the the singularly perturbed Black-Scholes equation with interior layer type which coming from the discontinuity of the first derivative of the initial condition. We construct a parameter-uniform numerical method theoretically which we call the method of splitting of singularity *(or briefly, the singularity splitting method)* for the problem. Numerical experiments prove that the solution and its first order derivative obtained by using this method converged ε-uniformly.

3, Moreover, we extend the *the singularity splitting method* to a singularly perturbed boundary value problem whose solution has two types of layers, the bounary layer and the interior layer which coming from the piecewise smooth initial condition. The singularity of the boundary layer is stronger than that of the interior layer, which makes it difficult to construct and study special numerical methods suitable for the adequate description of the singularity of the interior layer type. Using the method of special meshes that condense in a neighbourhood of the boundary layer and the method of additive splitting of the singularity of the interior layer type, a special finite difference scheme is designed that make it possible to approximate ε-uniformly the solution of the boundary value problem on the whole domain, its first order derivative in x on the whole domain except the discontinuity point (outside a neighbourhood of the boundary layer), and also the normalized derivative (the first order spatial derivative multiplied by the parameter ε) in a finite neighbourhood of the boundary layer.

In all, what we are mainly concerned with here are the construction of the ε-uniform technique, *the singularity splitting method* theoretically for the singularly perturbed Black-Scholes equation of a European call option with nonsmooth initial condition (with appearing of interior layer in solution) and it is application in problems with boundary layers in solution. Experimental results are provided to support the constructed scheme. The results here could be useful in real financial market. Moreover, the methods discussed here may have theoretical value to other singular perturbation problems that arise in mathematics and its applications.

Chapter 2

Singularly Perturbed Black-Scholes Equation

In this chapter, we derive the dimensionless singularly perturbed parabolic Black-Scholes Equation for the value of a European call option and assess the singularities of the problem. We also investigate the dependence of the solution errors on the smoothness of the initial functions for various boundary conditions, e.g. Dirichlet problem and Cauchy problem.

2.1 Black-Scholes Equation for European Call Options

The value of a European option satisfies the Black-Scholes equation with appropriately special final and boundary conditions [88]. The Black-Scholes equation governing the call option $C(S,\, t)$ is

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0, \quad (S,t) \in I\!R^+ \times [0,T),$$

where S is the current value of the underlying asset and t is the time. S and t are the independent variables. The value of the option also depends on σ the volatility of the underlying asset; E, the exercise price; T, the expiry time and r, the interest rate. The domain of the independent variables S, t is $(0,\, \infty) \times (0,\, T]$.

To uniquely specify the problem, prescribed boundary conditions and initial conditions must be presented. In financial problems, the boundary conditions are usually specified as the solution at $S = 0$,

$$C(0,\, t) = 0$$

and the solution at $+\infty$ is,

$$C(S,\, T) \sim S \quad as \quad S \to +\infty.$$

The Black-Scholes equation is a backward equation, meaning that the signs for the t derivative and the second S derivative in the equation are the same when written on the same side of the equals sign. Therefore a final condition has to be imposed. This is usually the payoff function at expiry $t = T$,

$$C(S,\, T) = max(S - E,\, 0).$$

Typical ranges of values of T in years, r in percent per annum and σ in percent per annum arising in practice are

$$\frac{1}{12} \leq T \leq 1,$$

$$0.01 \leq r \leq 0.2,$$

$$0.01 \leq \sigma \leq 0.5.$$

2.2 Transformation of the Equation

Standard approaches (see, e.g. [5], [7]) to the reformulation of the problem lead to new problems in which the free parameters of the problem appear in the coefficients of the equation, the initial and boundary conditions or the definition of the solution domain. Here we reformulate in such a way that the two independent parameters appear only in the coefficients of the equation. This enables us to study the range of problems of financial relevance in a systematic way.

The independent variables S, t are changed to the new independent variables x, τ by the transformation

$$S = E\,e^x, \quad t = T - \tau r^{-1},$$

and the dependent variable $C(S,\,t)$ to the new dependent variable $v(x,\,\tau)$ by the transformation

$$C(S,t) = E\,v(x,\,\tau).$$

We arrive at the following problem for the dimensionless equation:

$$L_{(2.2.1a)}\,v(x,\tau) \equiv \left\{ \frac{\partial^2}{\partial x^2} + (k-1)\,\frac{\partial}{\partial x} - k - k\,\frac{\partial}{\partial \tau} \right\} v(x,\tau) = 0, \quad (x,\,\tau) \in I\!R \times (0,\,\tau^*] \tag{2.2.1a}$$

with the initial condition

$$v(x,0) = \varphi_v(x), \quad x \in I\!R, \tag{2.2.1b}$$

where

$$\varphi_v(x) = \max(e^x - 1,\,0), \quad x \in I\!R,$$

and with the condition at infinity

$$\left.\begin{array}{lll} v(x,\tau) \to 0 & \text{for} & x \to -\infty \\ v(x,\tau) \to e^x & \text{for} & x \to \ \ \infty \end{array}\right\}, \quad \tau \in (0,\tau^*]. \tag{2.2.1c}$$

Here $k = r2\sigma^{-2}$, $\tau^* = rT$.

The exact solution of the dimensionless Black-Scholes equation (2.2.1) satisfying the given initial and boundary conditions is

$$\begin{aligned} v(x,\tau) &= e^{-\frac{1}{2}(k-1)x-\frac{1}{4k}(k+1)^2\tau}u(x,\tau) \\ &= e^{x+\frac{(k-1)(k+1)^2}{4k}\tau}N(d^+) - e^{\frac{1}{4}\left((k-1)^2-\frac{(k+1)^2}{k}\right)}N(d^-), \end{aligned}$$

where

$$\begin{aligned} d^+ &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}, \\ d^- &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}, \end{aligned}$$

and N is the cumulative distribution function for the normal distribution with mean 0 and the standard deviation 1, which is given by

$$N(d^+) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d^+} e^{-\frac{1}{2}s^2}ds.$$

Under the condition $T, r = \mathcal{O}(1)$ and for σ taking any value from the half-open interval $(0, \sqrt{2r})$, we have an initial-value problem for the singularly perturbed parabolic equation

$$L_{(2.2.2)}\, v(x,t) \equiv \left\{\varepsilon\frac{\partial^2}{\partial x^2} + (1-\varepsilon)\frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial \tau}\right\} v(x,\tau) = 0, \qquad (2.2.2)$$

$$(x,\tau)\in I\!R\times (0,\tau^*]$$

with conditions (2.2.1b) and (2.2.1c); $\varepsilon = \sigma^2\, 2^{-1}\, r^{-1}$.

The range of values of ε corresponding to the above ranges of parameters T, r and σ is

$$0.00025 \le \varepsilon \le 12.5,$$

which lies approximately in the interval $[2^{-12},\, 2^4]$.

Equation (2.2.2) is defined on the axis $I\!R$. This is a singularly perturbed convection-diffusion equation with the perturbation parameter ε, $\varepsilon \in (0, 1]$. The problem (2.2.2), (2.2.1b), (2.2.1c) is a singularly perturbed problem which has different types of singularities.

2.3 Singularities in the Continuous Problem

In order to obtain accurate numerical approximations of the solution and its derivatives it is necessary to take account of the singularities of the problem. Each of the following singularities is a potential source of numerical difficulties:

1, The domain of the exact solution is infinite in the space variable, so artificial boundaries and boundary conditions may be required to define the numerical solutions on a finite domain, depending on whether the method is explicit or implicit in the time-like variable τ.

2,The initial function in condition (2.2.1b) has a discontinuity of the first kind at $x = 0$ which may cause numerical errors and may propagate into the solution domain.

3, The presence of large and (or) small parameters multiplying the coefficients of the differential equation may give rise to boundary and (or) interior layers in the solution and its derivatives, which, if not treated appropriately, will cause errors in the numerical solution.

To study the effect of these singularities on the errors in the numerical approximations, it is necessary to isolate them from each other in order to deal with them one at a time.

2.4 On Considering the Dirichlet Problem

Here, we are focused on approximations to the solution of the singularly perturbed problem with a nonsmooth initial condition, ignoring other types of singularities. We consider the Dirichlet problem for the singularly perturbed Black-Scholes equation on the bounded domain $\overline{G} = \{(x,t) : \ |x| < 1, \ t \in [0,1]\}$ with the initial condition of controlled restricted smoothness; the initial function $\varphi_0(x)$, $x \in [-1,1]$ belongs to a Hölder space H^α with $\alpha \in (0,2]$.

2.4.1 Problem Formulation

Let it be required to study the applicability of classical numerical methods (see, e.g., [68]) to the solution of problem (2.2.2), (2.2.1b), (2.2.1c). It is of interest to evaluate the error component of a numerical solution which is generated by nonsmooth initial data.

On a bounded domain $\overline{G}$, where

$$\overline{G} = G \cup S, \quad G = G(l) = \{(x,t) : \ \ |x| < l, \ \ t \in (0,T]\} \tag{2.4.1}$$

we consider the initial boundary value problem

$$\begin{aligned} &L_{(2.4.2)}\, u(x,t) \equiv \left\{\varepsilon \frac{\partial^2}{\partial x^2} + (1-\varepsilon)\frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial t}\right\} u(x,t) = 0, \quad (x,t) \in G, \\ &u(x,t) = \varphi(x,t), \quad (x,t) \in S. \end{aligned} \tag{2.4.2}$$

On the set S_0, which is the lower base of the boundary S ($S = S_0 \cup S^l$, S^l is the lateral boundary, $S_0 = \overline{S}_0$), the initial function $\varphi_0(x) = \varphi(x,t)$, $(x,t) \in S_0$, is of controlled bounded smoothness in a neighborhood of the point $x = 0$. Outside this neighborhood the function $\varphi(x,t)$ is sufficiently smooth on S_0 and $\overline{S}^{\,l}$. At the corner points $S^* = S_0 \cap \overline{S}^l$, the function $\varphi(x,t)$ is continuous. Other compatibility conditions on the set S^* are not assumed.

Let for simplicity

$$\varphi(x,t) = \varphi_0(x), \quad (x,t) \in S, \tag{2.4.3}$$

where

$$\varphi_0(x) = \varphi_0(x;\alpha) = \left\{ \begin{array}{ll} x^\alpha, & x \in [0,l], \\ 0, & x < 0 \end{array} \right\}, \quad |x| \le l, \quad \alpha \in (0,2].$$

The initial function $\varphi_0(x)$, $x \in [-l,l]$, belongs to a Hölder space H^α with $\alpha \in (0,2]$.

2.4.2 Finite Difference Schemes

To solve problem (2.4.2), (2.4.1), we use the classical finite difference scheme on uniform meshes [68]. On the set $\overline{G}$, we introduce a uniform mesh as

$$\overline{G}_h = \overline{\omega}_1 \times \overline{\omega}_0, \tag{2.4.4}$$

where $\overline{\omega}_1$ and $\overline{\omega}_0$ are uniform meshes on $[-l,l]$ and $[0,T]$ with N_1+1 and N_0+1 numbers of mesh points, respectively. Problem (2.4.2), (2.4.1) is approximated by the difference scheme [68]

$$\begin{aligned} \Lambda_{(2.4.5)}\, z(x,t) &\equiv \{\varepsilon\delta_{x\overline{x}} + (1-\varepsilon)\delta_x - 1 - \delta_{\overline{t}}\}\, z(x,t) = 0, \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{2.4.5}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$; $\delta_{x\overline{x}}\, z(x,t)$ and $\delta_x\, z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the second and first difference derivatives, $\delta_{x\overline{x}}\, z(x,t) = h^{-1}\big[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)\big]$, $\delta_x\, z(x,t) = h^{-1}\big[z(x+h,t) - z(x,t)\big]$, $\delta_{\overline{x}}\, z(x,t) = h^{-1}\big[z(x,t) - z(x-h,t)\big]$, $\delta_{\overline{t}}\, z(x,t) = h_t^{-1}\big[z(x,t) - z(x,t-h_t)\big]$, h and h_t are stepsizes in x and t, $h = 2lN_1^{-1}$, $h_t = TN_0^{-1}$.

The difference scheme (2.4.5), (2.4.4) is monotone [68] ε-uniformly.

Using the technique from [72, 75] one can prove that the solution of the finite difference scheme (2.4.5), (2.4.4) is convergent ε-uniformly to the solution of problem (2.4.2), (2.4.1) when $N_1, N_0 \to \infty$. However, the order of ε-uniform convergence, as a rule, is too small and much less than its real values. By this argument, the experimental technique from [20] allows one to derive realistic error bounds and orders of ε-uniform convergence.

When studying discrete solutions of problem (2.4.5), (2.4.4), we will use the experimental technique from [20].

2.4.3 Numerical Results and Discussion

This section displays the numerical results for problem (2.4.2), (2.4.1) with parameters as $l = 1.0$, $T = 1.0$. We use meshes with $N_1 = N_0 = N$, $N = 2^i$, $i = 2, 3, \ldots, 11$ to make computations for $\varepsilon = 2^{-j}$, $j = 0, 1, \ldots, 20$, and for different values of $\alpha \le 1$, namely, for $\alpha = 1$, $\alpha = 1/2$, $\alpha = 1/4$, $\alpha = 1/8$.

To compute errors of discrete solutions and orders of convergence, we use the formulae

$$
\begin{aligned}
E_\varepsilon^N &= ||z_\varepsilon^*(x,t) - z_\varepsilon^N(x,t)||_{\overline{G}_h^N}, \\
D_\varepsilon^N &= ||z_\varepsilon^N(x,t) - z_\varepsilon^{2N}(x,t)||_{\overline{G}_h^N}, \qquad (2.4.6) \\
p_\varepsilon^N &= \log_2 \tfrac{D_\varepsilon^N}{D_\varepsilon^{2N}}.
\end{aligned}
$$

Here $\overline{G}_h^N$ is the mesh $\overline{G}_{h(2.4.4)}$ with $N+1$ nodes in x and t, $z_\varepsilon^N(x,t)$ is the solution of scheme (2.4.5) on $\overline{G}_h^N$, $z_\varepsilon^*(x,t)$ is the solution on the finest mesh $\overline{G}_h^N$ with $N = N^* \equiv 2048$; D_ε^N is the double-mesh difference.

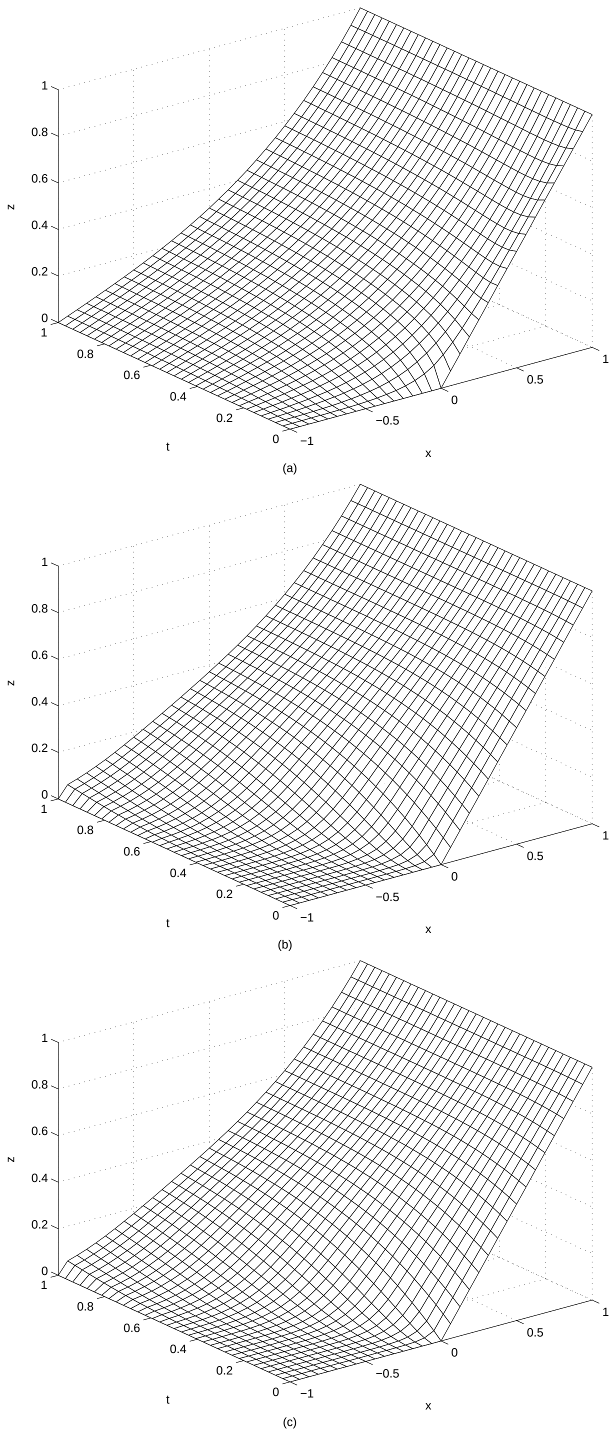


Figure 2.1: Plot of the numerical results z for $\alpha = 1,\ N = 32$. (a): $\varepsilon = 2^0$; (b): $\varepsilon = 2^{-10}$; (c): $\varepsilon = 2^{-20}$.

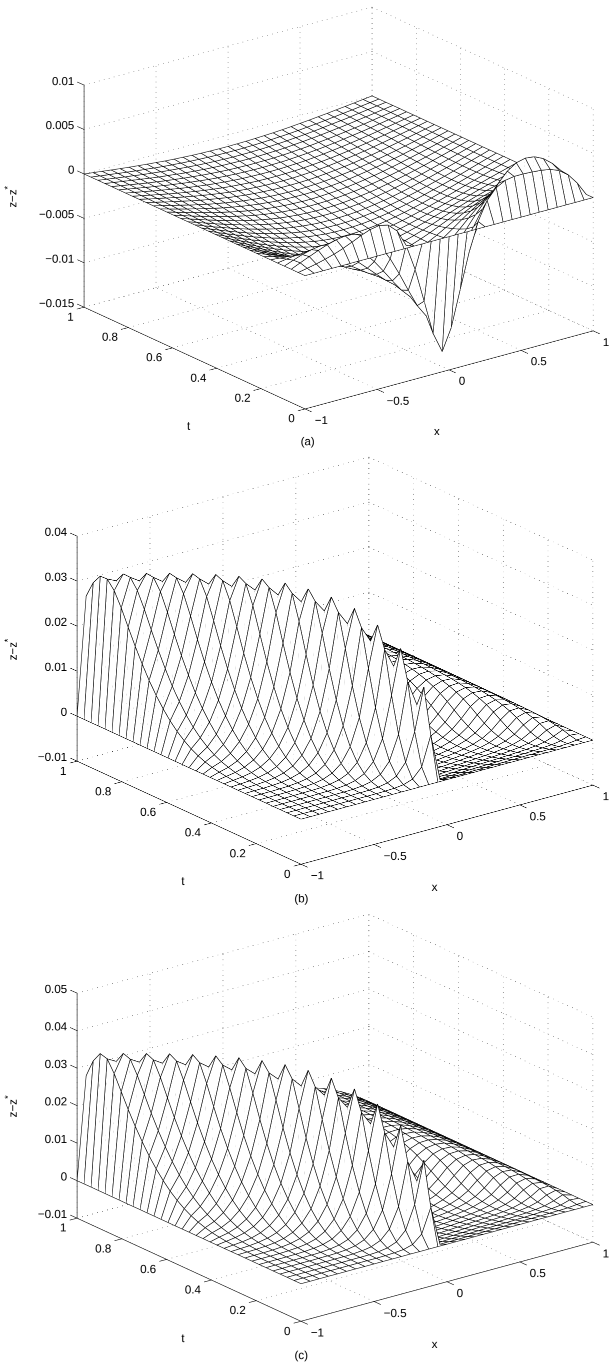


Figure 2.2: Plot of the maximum errors $z - z^*$ for $\alpha = 1$, $N^* = 2048$, $N = 32$. (a): $\varepsilon = 2^0$; (b): $\varepsilon = 2^{-10}$; (c): $\varepsilon = 2^{-20}$.

Table 2.1: Computed maximum pointwise errors E_ε^N for various values of ε and N; E^N is the maximum error for each N (for $\alpha = 1$).

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.1328-01	0.8553-02	0.5572-02	0.3564-02	0.2104-02	0.9567-03
2^{-1}	0.9576-02	0.6084-02	0.3951-02	0.2525-02	0.1491-02	0.6783-03
2^{-2}	0.6699-02	0.4274-02	0.2788-02	0.1787-02	0.1057-02	0.4818-03
2^{-3}	0.8357-02	0.4367-02	0.2182-02	0.1262-02	0.7508-03	0.3436-03
2^{-4}	0.1337-01	0.7214-02	0.3677-02	0.1765-02	0.7673-03	0.2577-03
2^{-5}	0.1880-01	0.1060-01	0.5569-02	0.2724-02	0.1197-02	0.4044-03
2^{-6}	0.2419-01	0.1426-01	0.7808-02	0.3933-02	0.1761-02	0.6011-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.4036-01	0.2785-01	0.1835-01	0.1134-01	0.6251-02	0.2598-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.4047-01	0.2796-01	0.1845-01	0.1143-01	0.6318-02	0.2637-02
E^N	**0.4047-01**	**0.2796-01**	**0.1845-01**	**0.1143-01**	**0.6318-02**	**0.2637-02**

Numerical solutions for $\alpha = 1$, $N = 32$, $\varepsilon = 2^0, 2^{-10}, 2^{-20}$ with the initial condition (2.4.3) are given in Fig. 2.1. For small ε, i.e. $\varepsilon = 2^{-10}$, 2^{-20}, boundary layer appears in the solution of the small region close to the left boundary; Interior layer arises from the discontinuity of the first derivative of the initial function move in the direction of the characteristic solution of problem (2.4.2), (2.4.1), (2.4.3). The layers become more apparent with the decrease of ε.

Fig. 2.2 is the plots of maximum pointwise errors $z - z^*$ corresponding to the solutions in Fig. 2.1 with same parameters. The errors appear obviously large in the direction of the characteristic solution of problem (2.4.2), (2.4.1), (2.4.3).

Note that there is a sharp change of errors between the solutions at left boundary and the solutions at the adjacent x-grid for all values of t. In considering the boundary layer appearance in Fig. 2.1 (b) and (c), we extend l to 2.0 to avoid the incompatibility.

Table 2.2: Computed order of convergence p_ε^N for various values of ε and N; p^N is the minimum order for each N (for $\alpha = 1$).

	Number of intervals N				
ε	32	64	128	256	512
2^0	0.6483	0.5916	0.5501	0.5260	0.5138
2^{-1}	0.7036	0.6204	0.5653	0.5333	0.5173
2^{-2}	0.7478	0.6545	0.5873	0.5457	0.5236
2^{-3}	0.8703	0.9311	0.6383	0.5659	0.5348
2^{-4}	0.8019	0.8875	0.9397	0.9687	0.9837
2^{-5}	0.7149	0.8225	0.8992	0.9447	0.9712
2^{-6}	0.6208	0.7388	0.8359	0.9049	0.9484
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
2^{-14}	0.4089	0.4415	0.4638	0.4792	0.4925
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
2^{-20}	0.4074	0.4402	0.4610	0.4739	0.4820
p^N	**0.4074**	**0.4402**	**0.4610**	**0.4739**	**0.4820**

The values of the computed maximum pointwise errors E_ε^N for problem (2.4.1) on condition (2.4.2) and (2.4.3) for $\alpha = 1$ and for various values of ε and N are presented in Table. 2.1. By examining each row of the table, it is obvious that E_ε^N decrease monotonically as the number of mesh elements N increase. Consider each column of the table in turn, the errors E_ε^N are nonmonotone and increase to a stabilized state as ε decrease. The value E^N in the last lines of the tables is the maximal value of errors E_ε^N with respect to ε, corresponding to the value N. Note that E^N is the stabilized state with respect to the minimal ε for fixed value N. These results validate that for $\alpha = 1$ the solution is ε-uniformly convergent.

Table 2.2 gives the computed orders of convergence p_ε^N and p^N for $\alpha = 1$ and for various values of ε and N. It shows that the order of local (i.e. for fixed ε) convergence rate depends on the value of ε nonmonotonically and $p^N > 0.45$ for $N > 128$. However the order of convergence is always less than 1.0.

2.4.4 Conclusion

Summarizing the results of the numerical experiments, we come to the following conclusions:

1. The solution converges ε-uniformly for $\alpha = 1$. Moreover, the order of local (i.e., for fixed ε) convergence and solution errors depend on the value of ε nonmonotonically.

2. For other values of α, the qualitative behavior of the solution error and the convergence order are similar to the case for $\alpha = 1$.

3. The order of ε-uniform convergence for decreasing α becomes worse and solution errors grow.

4. It is not necessary to use condensing meshes for ε-uniform convergence of the classical scheme in a neighborhood of the set where initial conditions are not sufficiently smooth. However, the order of such convergence essentially depends on the value α, which defines the class H^{α}.

2.5 On Considering the Cauchy Problem

In this section, we develop an approach to construct a discrete approximation of a solution to problem (2.2.2), (2.2.1b), (2.2.1c) in the case of the Cauchy problem with nonsmooth initial data, for simplicity, assuming that the initial function is bounded.

2.5.1 Problem Formulation

Let it be required to study the applicability of classical numerical methods (see, e.g., [68]) to the solution of problem (2.2.2), (2.2.1b), (2.2.1c). It is of interest to evaluate the error component of a numerical solution which is generated by nonsmooth initial data.

In an unbounded domain $\overline{G}$, where

$$\overline{G} = \overline{G}^{\infty}, \quad \overline{G} = G \cup S, \quad G = I\!R \times (0, T], \tag{2.5.7}$$

We consider the initial value problem[1]

$$L_{(2.5.8)} u(x,t) \equiv \Big\{\varepsilon \frac{\partial^2}{\partial x^2} + (1-\varepsilon)\frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial t}\Big\} u(x,t) = 0, \; (x,t) \in G, \tag{2.5.8a}$$

$$u(x,t) = \varphi_{(2.5.8)}(x), \quad (x,t) \in S. \tag{2.5.8b}$$

The initial function $\varphi(x)$ is bounded on $I\!R$:

$$|\varphi(x)| \le M, \quad x \in I\!R, \tag{2.5.8c}$$

is smooth for $|x| > 0$ and belongs to a Hölder space H^{α} with $\alpha \in (0,\, 3]$.

Our primary interest is to find a solution of problem (2.5.8), (2.5.7) but in a bounded subdomain $\overline{G}^{\,l}$, where

$$\overline{G}^{\,l}_{(2.5.9)} = G^{\,l} \cup S^{\,l}, \quad G^{\,l} = (-l,\, l) \times (0, T]. \tag{2.5.9}$$

Also, on set (2.5.9) we consider the auxiliary initial boundary value problem

$$L_{(2.5.8)}\, u(x,t) = 0, \quad (x,t) \in G^{\,l}, \tag{2.5.10}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S^{\,l}.$$

[1]Here and below M (m) denote sufficiently large (small) positive constants independent of the parameter ε and parameters of difference schemes. The notation $L_{(j.k)}$ $(m_{(j.k)},\, G_{h(j.k)})$ means that this operator (constant, grid) is introduced in formula $(j.k)$.

On the set S_0^l, which is the lower base of the boundary S^l ($S^l = S_0^l \cup S_1^l$, S_1^l is the lateral boundary, $S_0^l = \overline{S}_0^l$), the initial function $\varphi_0(x) = \varphi(x,t)$, $(x,t) \in S_0^l$, satisfies the condition

$$\varphi_0(x) = \varphi_{(2.5.8)}(x), \quad x \in S_0^l,$$

$\varphi(x,t)$ is sufficiently smooth on $\overline{S}_1^l$. At the corner points $S^* = S_0^l \cap \overline{S}_1^l$, the function $\varphi(x,t)$ is continuous and satisfies compatibility conditions.

Let for simplicity

$$\varphi(x,t) = \varphi_0(x), \quad (x,t) \in S,$$

$$\varphi_0(x) = \varphi_0(x;\ \alpha,\ \beta) = \left\{ \begin{array}{ll} \beta^{-2\alpha}\,(x+\beta)^\alpha\,(\beta-x)^\alpha, & |x| \le \beta \\ 0, \quad \beta < |x| \le l & \end{array} \right\}, \quad |x| \le l,$$

where $0 < \beta < l$, $\alpha \in (0,\ 3]$.

Our aim is, using classical approximations of differential equation (2.5.8a), to construct a numerical method which allows us to find a solution of problem (2.5.8), (2.5.7) on the set $\overline{G}^l_{(2.5.9)}$. For the constructed method it is required to study a behaviour of the solution error depending on the perturbation parameter ε and the parameter α which defines the smoothness of the initial function.

2.5.2 Finite Difference Schemes

To solve problem (2.5.8), (2.5.7), we use the classical finite difference scheme on uniform meshes on $\overline{G}$; we refer to such a scheme as formal. On the set $\overline{G}$, we introduce a uniform mesh

$$\overline{G}_{h(2.5.11)} = \overline{G}_h^\infty = \omega_1 \times \overline{\omega}_{0(2.5.11)}, \tag{2.5.11}$$

where ω_1 and $\overline{\omega}_{0(2.5.11)}$ are uniform meshes on $I\!R$ and $[0,T]$ with step-sizes h and h_t, respectively; $h = (N_1^*)^{-1}$, $h_t = T N_0^{-1}$, i.e., $N_1^* + 1$ and $N_0 + 1$ are the number of mesh points on a unit interval of the x-axis and the number of mesh points on the segment $[0, T]$.

Problem (2.5.8), (2.5.7) is approximated by the difference scheme [68]

$$\begin{aligned} \Lambda_{(2.5.12)}\, z(x,t) &\equiv \left\{\varepsilon\delta_{x\overline{x}} + (1-\varepsilon)\delta_x - 1 - \delta_{\overline{t}}\right\} z(x,t) = 0, \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x), \quad (x,t) \in S_h. \end{aligned} \tag{2.5.12}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$; $\delta_{x\overline{x}}\, z(x,t)$ and $\delta_x\, z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the second and first (forward and backward) difference derivatives,

$$\delta_{x\overline{x}}\, z(x,t) = h^{-1}\big[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)\big], \quad \delta_x\, z(x,t) = h^{-1}\left[z(x+h,t) - z(x,t)\right],$$
$$\delta_{\overline{x}}\, z(x,t) = h^{-1}\left[z(x,t) - z(x-h,t)\right], \quad \delta_{\overline{t}}\, z(x,t) = h_t^{-1}\left[z(x,t) - z(x,t-h_t)\right].$$

We denote the solution of scheme (2.5.12), (2.5.11) by $u_h(x,t)$, $(x,t) \in \overline{G}_{h(2.5.11)}$.

The formal finite difference scheme (2.5.12), (2.5.11) is monotone ε-uniformly [68].

For problem (2.5.8), (2.5.7), we shall construct a grid approximation on finite meshes (i.e. meshes with a finite number of mesh points) so that its solution converges to the solution of the problem (2.5.8), (2.5.7) ε-uniformly on the set $\overline{G}^{\,l}$ as the number of nodes of the finite mesh grows. We refer to such a scheme with a finite number of nodes as constructive.

In the case of problem (2.5.10), (2.5.9), for its solving we use the classical scheme. On the set $\overline{G}^{\,l}$ we introduce a uniform mesh as

$$\overline{G}^{\,l}_h = \overline{\omega}_1 \times \overline{\omega}_{0(2.5.13)}, \tag{2.5.13}$$

where $\overline{\omega}_1$ and $\overline{\omega}_{0(2.5.13)}$ are uniform meshes on $[-l, l]$ and $[0, T]$ with $N_1{+}1$ and $N_0{+}1$ numbers of mesh points, respectively. Problem (2.5.10), (2.5.9) is approximated by the difference scheme [68]

$$\begin{aligned} \Lambda_{(2.5.12)}\, z(x,t) &= 0, \quad (x,t) \in G_h^l, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h^l. \end{aligned} \tag{2.5.14}$$

2.5.3 Constructive Scheme

To derive a constructive scheme for problem (2.5.8), (2.5.7) we use an approach proposed in [76].

To have an approximate solution on $\overline{G}^{\,l}$, we consider the equation in the larger domain $\overline{G}^{\,L}$, where

$$\overline{G}^{\,L}_{(2.5.15)} = G^L \cup S^L; \quad G^L = (-L, L) \times (0, T], \quad L > l, \tag{2.5.15}$$

with the exact initial condition on S_0^L ($S^L = S_0^L \cup S_1^L$, $S_0^L = \overline{S}_0^{\,L}$, S_1^L is the lateral boundary) and with some boundary condition that is an "extension" of the initial condition onto S_1^L.

On $\overline{G}^{\,L}$, we construct a uniform mesh

$$\overline{G}^{\,L}_{h(2.5.16)} = \overline{\omega}_1 \times \overline{\omega}_{0(2.5.16)}, \tag{2.5.16}$$

where $\overline{\omega}_{0(2.5.16)} = \overline{\omega}_{0(2.5.11)}$, $\overline{\omega}_1$ is a mesh on $[-L, L]$ with $N_1 + 1$ mesh points. Let $u_h^L(x,t)$, $(x,t) \in \overline{G}_h^L$, be a solution of the difference scheme

$$\begin{aligned} \Lambda_{(2.5.12)} \quad & z(x,t) = 0, && (x,t) \in G_h^L, \\ & z(x,t) = \varphi(x), && (x,t) \in S_{0h}^L, \\ & z(x,t) = \varphi^*_{(2.5.17)}(x,t), && (x,t) \in S_{1h}^L. \end{aligned} \tag{2.5.17}$$

Here $\varphi^*_{(2.5.17)}(x,t)$, $(x,t) \in S_1^L$, is an "extension" of the function $\varphi_{(2.5.8)}(x)$, $x = \pm L$, on the set S_1^L. We assume the following condition to be fulfilled: $|\varphi^*_{(2.5.17)}(x,t)| \le M, \quad (x,t) \in S^L$ (see condition (2.5.8c)).

We suppose that

$$\overline{G}^L_{h(2.5.16)} = \overline{G}^L_{(2.5.15)} \bigcap \ \overline{G}_{h(2.5.11)}, \tag{2.5.18}$$

that is, the step-sizes of the meshes $\overline{G}^L_h$ and $\overline{G}_h$ are the same, and $\overline{G}^L_h \subset \overline{G}_h$. The value L is chosen to satisfy the condition

$$L = l + \sigma, \tag{2.5.19a}$$

where $\sigma = \sigma(N_1)$; we set

$$\sigma = m \ln N_1\,, \tag{2.5.19b}$$

m is an arbitrary constant, $N_1 = N_{1(2.5.16)}$.

Thus, the constructive scheme (2.5.17), (2.5.16), (2.5.19) is constructed.

By using the technique of majorant functions, it is possible to show that, under condition (2.5.18), the following estimate is valid:

$$|u(x,t) - u^L(x,t)| \le M\, N_1^{-1} \ln N_1, \quad (x,t) \in \overline{G}^{\,l},$$

and hence

$$|u(x,t) - u^L_h(x,t)| \le \max_{\overline{G}^L_h \bigcap \overline{G}^{\,l}} |u^L(x,t) - u^L_h(x,t)| + M\, N_1^{-1} \ln N_1, \quad (x,t) \in \overline{G}^L_h \cap \overline{G}^{\,l}. \tag{2.5.20}$$

Here $u^L_h(x,t)$, $(x,t) \in \overline{G}^L_h$ is the solution of difference scheme (2.5.17), (2.5.16), (2.5.19), $u^L(x,t)$, $(x,t) \in \overline{G}^{\,L}$ is the solution of problem (2.5.10), (2.5.9) for $l = L$:

$$\begin{aligned} L_{(2.5.8)}\;\; u(x,t) &= 0, && (x,t) \in G^L, \quad \textit{where} \;\; \overline{G}^{\,L} = \overline{G}^L_{(2.5.15)}, \\ u(x,t) &= \varphi^*_{(2.5.17)}(x,t), && (x,t) \in S^L. \end{aligned} \tag{2.5.21}$$

Using the technique from [72, 75] one can prove that the solution $u^L_h(x,t)$, $(x,t) \in \overline{G}^L_h$ of the finite difference scheme (2.5.17), (2.5.16), (2.5.19) is convergent on $\overline{G}^L_h$ ε-uniformly to the solution $u^L(x,t)$ of problem (2.5.21), (2.5.15) as N_1, $N_0 \to \infty$. However, the order of ε-uniform convergence, as a rule, is too small and much less than its real values. By this argument, the experimental technique from [20] allows one to derive realistic error bounds and orders of ε-uniform convergence.

When studying convergence of discrete solutions of problem (2.5.17), (2.5.16), (2.5.19) to the solution of problem (2.5.8), (2.5.7) on $\overline{G}^l$, we will use the experimental technique from [20].

2.5.4 Numerical Results and Discussion

We choose the main parameters in auxiliary problem (2.5.10), (2.5.9) as $l = 2$, $T = 1$. Solving discrete problem (2.5.14), (2.5.13), we use meshes with $N_1 = N_0 = N$, $N = 2^i$, $i = 2, 3, \ldots, 11$. We make computations for $\varepsilon = 2^{-j}$, $j = 0, 1, \ldots, 20$, and for different values of $\alpha \leq 3$, namely, for $\alpha = 1/2$, 1, 2, 3. To compute errors of discrete solutions and orders of convergence, we use the formulae (2.4.6) with $\overline{G}_h^N$ is the mesh $\overline{G}^l_{h(2.5.13)}$ with $N+1$ nodes in x and t, $z_\varepsilon^N(x,t)$ is the solution of scheme (2.5.14) on $\overline{G}_h^N$, $z_\varepsilon^*(x,t)$ is the solution on the finest mesh $\overline{G}_h^N$ with $N = N^* \equiv 2048$; D_ε^N is the double-mesh difference. As an example, we show experimental results for $\alpha = 1$.

The plots of the numerical solutions for problem (2.5.14), (2.5.13) with $\alpha = 1$ are presented on Fig. 2.3 for $\varepsilon = 2^0$, $\varepsilon = 2^{-4}$, $\varepsilon = 2^{-8}$, $N = N_0 = 16$ respectively.

The values of the computed errors E_ε^N and the maximum error E^N for each N for problem (2.5.8) for $\alpha = 1$ and for various values of ε and N are presented in Table. 2.3. Examining each row of the table we see that the computed maximum pointwise error E_ε^N decreases monotonically as the number of mesh elements N increases. Consider each column of the table in turn, the computed maximum pointwise error E_ε^N is nonmonotone and increases to a stabilized state as ε decreases. These results validate that for $\alpha = 1$ the solution is ε-uniformly convergent.

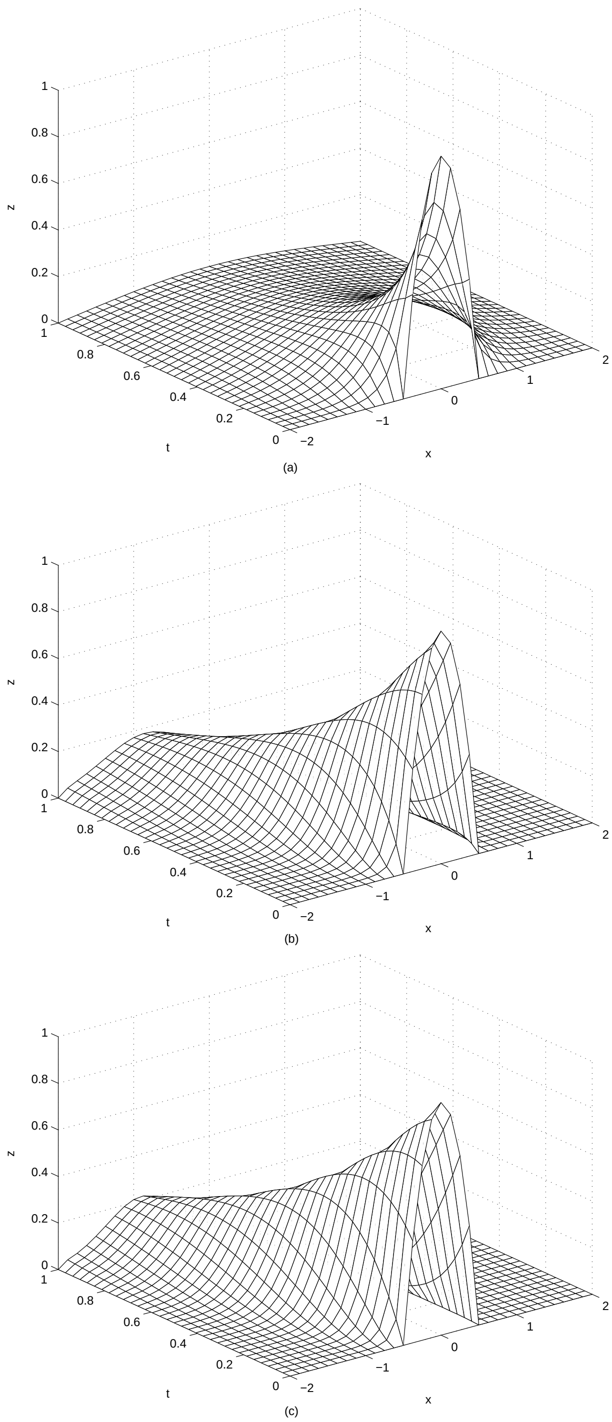


Figure 2.3: Plot of the numerical results z for $\alpha = 1$, $N = 32$. (a): $\varepsilon = 2^0$; (b): $\varepsilon = 2^{-4}$; (c): $\varepsilon = 2^{-8}$.

Table 2.3: Computed maximum pointwise errors E_ε^N and E^N for various values of ε and N; E^N is the maximum error for each N (for $\alpha = 1, \beta = 2^{-1}$).

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.6296-01	0.3871-01	0.2406-01	0.1492-01	0.8651-02	0.3900-02
2^{-1}	0.5798-01	0.3189-01	0.1857-01	0.1109-01	0.6305-02	0.2817-02
2^{-2}	0.5057-01	0.2625-01	0.1458-01	0.8398-02	0.4669-02	0.2064-02
2^{-3}	0.5955-01	0.3096-01	0.1557-01	0.7434-02	0.3560-02	0.1554-02
2^{-4}	0.9319-01	0.5148-01	0.2668-01	0.1293-01	0.5652-02	0.1903-02
2^{-5}	0.1242+00	0.7060-01	0.3740-01	0.1834-01	0.8063-02	0.2723-02
2^{-6}	0.1452+00	0.8424-01	0.4510-01	0.2218-01	0.9744-02	0.3339-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.1715+00	0.1239+00	0.8486-01	0.5384-01	0.3025-01	0.1276-01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.1718+00	0.1242+00	0.8517-01	0.5411-01	0.3046-01	0.1288-01
E^N	**0.1718+00**	**0.1242+00**	**0.8517-01**	**0.5411-01**	**0.3046-01**	**0.1288-01**

Table 2.4: Computed order of convergence p_ε^N and p^N for various values of ε and N; p^N is the minimum order for each N (for $\alpha = 1, \beta = 2^{-1}$).

	Number of intervals N				
ε	32	64	128	256	512
2^0	0.7690	0.7063	0.6289	0.5725	0.5389
2^{-1}	0.9232	0.8266	0.7128	0.6243	0.5680
2^{-2}	0.9781	0.9069	0.7997	0.6924	0.6113
2^{-3}	0.9045	0.9244	0.9448	0.7710	0.6744
2^{-4}	0.7775	0.8592	0.9195	0.9574	0.9782
2^{-5}	0.7267	0.8131	0.8960	0.9447	0.9719
2^{-6}	0.6806	0.7916	0.8893	0.9477	0.9521
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
2^{-14}	0.6347	0.6237	0.4052	0.4337	0.4566
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
2^{-20}	0.6345	0.6225	0.4028	0.4305	0.4504
p^N	**0.6345**	**0.6225**	**0.4028**	**0.4305**	**0.4504**

Table 2.4 gives the computed orders of convergence p_ε^N and p^N for $\alpha = 1$ and for various values of ε and N. We can see from Table 2.4 that the order of local (i.e. for fixed ε) convergence rate depends on the value of ε nonmonotonically and $p^N \geq 0.40$ for $N \geq 128$.

2.5.5 Conclusion

Summarizing the results of the numerical experiments, we come to the following conclusions:

1. The solution of difference scheme (2.5.14), (2.5.13) converges to the solution of problem (2.5.10), (2.5.9) ε-uniformly for $\alpha = 1$. Moreover, the order of local (i.e., for fixed ε) convergence and solution errors depend on the value of ε nonmonotonically. For other values of α, the qualitative behaviour of the solution error and the convergence order are similar to the case for $\alpha = 1$. The order of ε-uniform convergence for decreasing α becomes worse and solution errors grow.

2. In the case of constructive difference scheme (2.5.17), (2.5.16), (2.5.19), which approximates the solution of problem (2.5.8), (2.5.7) on $\overline{G}^l$, the behaviour of errors of the problem solutions is similar to one for difference scheme (2.5.14), (2.5.13).

3. To have the ε-uniform convergence on $\overline{G}^l$ for the constructed schemes, it is not necessary to use meshes condensing in a neighbourhood of the set where initial conditions are not sufficiently smooth. However, the order of such convergence essentially depends on the value α, which defines the class H^α.

Chapter 3

Approximation of the Solution and Its Derivative for the Singularly Perturbed Black-Scholes Equation

A problem for the Black-Scholes equation that arises in financial mathematics, by a transformation of variables, is leaded to the Cauchy problem for a singularly perturbed parabolic equation with variables x, t and a perturbation parameter ε, $\varepsilon \in (0, 1]$. This problem has several singularities such as: the unbounded domain; the piecewise smooth initial function (its first order derivative in x has a discontinuity of the first kind at the point $x = 0$); an interior (moving in time) layer generated by the piecewise smooth initial function for small values of the parameter ε; etc.

In this chapter, a grid approximation of the solution and its first order derivative is studied in a finite domain including the interior layer. On a uniform mesh, using the method of additive splitting of a singularity of the interior layer type, a special difference scheme is constructed that allows us to approximate ε-uniformly both the solution of the boundary value problem and its first order derivative in

x with convergence orders close to 1 and 0.5, respectively. The efficiency of the constructed scheme is illustrated by numerical experiments.

3.1 Introduction

The Black-Scholes equation with the value of European call option $C = C(S, t')$ is [88],

$$\frac{\partial C}{\partial t'} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r\,S\,\frac{\partial C}{\partial S} - r\,C = 0, \quad (S, t') \in I\!R^+ \times [0, T), \tag{3.1.1a}$$

with the final condition

$$C(S, T) = \max(S - E, 0), \quad S \in I\!R^+, \tag{3.1.1b}$$

and the boundary conditions at $S = 0$ and at infinity $S = +\infty$

$$C(0, t') = 0; \quad C(S, t') \to S \quad \text{for} \quad S \to \infty, \quad t' \in [0, T). \tag{3.1.1c}$$

Here S and t' are the current values of the underlying asset and time, σ, E, T and r are the volatility, exercise price, expiry time and the interest rate, respectively.

For the problem (3.1.1), in addition to the solution itself, some of the partial derivatives of the solution are of interest [88].

When studying this problem, a standard approach is a transformation of the equation by the changes of variables.

By the transformations

$$S = E\,e^x, \quad t' = T - \tau\,r^{-1}, \quad C = E\,v(x, \tau) \tag{3.1.1d}$$

and introducing the notation $k = 2\,\sigma^{-2}\,r$, $\tau^* = r\,T$, we come to the following problem for the dimensionless parabolic equation in the new variables x, τ:

$$L_{(3.1.2)}\,v(x,\tau) \equiv \left\{ \frac{\partial^2}{\partial x^2} + (k-1)\,\frac{\partial}{\partial x} - k - k\,\frac{\partial}{\partial \tau} \right\} v(x,\tau) = 0, \quad (3.1.2)$$
$$(x,\,\tau) \in I\!R \times (0,\,\tau^*]$$

with the initial condition

$$v(x,0) = \varphi_v(x), \quad x \in I\!R, \qquad (3.1.3a)$$

where

$$\varphi_v(x) = \max(e^x - 1,\,0), \quad x \in I\!R,$$

and with the condition at infinity

$$\left.\begin{array}{lll} v(x,\tau) \to 0 & \text{for} & x \to -\infty \\ v(x,\tau) \to e^x & \text{for} & x \to \ \ \infty \end{array}\right\}, \quad \tau \in (0,\tau^*]. \qquad (3.1.3b)$$

Under the condition $T, r = \mathcal{O}(1)$ and for σ taking an arbitrary value from the half-open interval $(0, \sqrt{2r})$, we come to the Cauchy problem for the singularly perturbed parabolic equation

$$L_{(3.1.4)}\,v(x,t) \equiv \left\{\varepsilon\frac{\partial^2}{\partial x^2} + (1-\varepsilon)\frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial \tau}\right\} v(x,\tau) = 0, \qquad (3.1.4)$$
$$(x,\tau) \in I\!R \times (0,\tau^*]$$

with conditions (3.1.3). Here $\varepsilon = 2^{-1}\,\sigma^2\,r^{-1}$ is a dimensionless "perturbation" parameter, $\varepsilon \in (0,1]$.

The initial function in condition (3.1.3a) is continuous; its first derivative in x has a discontinuity of the first kind at the point $x = 0$

$$\left[\frac{d}{dx}\varphi_v(0)\right] = 1,$$

where the jump of the derivative is defined by the relation

$$\left[\frac{d}{dx}\varphi_v(0)\right] = \lim_{x\searrow 0}\frac{d}{dx}\varphi_v(x) - \lim_{x\nearrow 0}\frac{d}{dx}\varphi_v(x).$$

The initial function and the solution itself for this problem grow (exponentially) without bound as $x \to \infty$. If the parameter $\varepsilon = 1$ then the problem (3.1.4), (3.1.3) becomes the one of reaction-diffusion type, and for $\varepsilon < 1$, it is of convection-diffusion type. For small values of the parameter ε, an interior (moving in time) layer with the typical width of $\varepsilon^{1/2}$ appears in a neighbourhood of the characteristic (of the operator $L_1 \equiv (1-\varepsilon)\dfrac{\partial}{\partial x} - 1 - \dfrac{\partial}{\partial\tau}$) passing through the point $(0,0)$.

Thus, the Cauchy problem (3.1.4), (3.1.3) is a singularly perturbed problem with different types of singularities. In the present paper we are interested in approximations to both the solution and its first order derivative in a finite subdomain that contains the singularity of the interior layer type.

Boundary value problems in bounded domains for singularly perturbed parabolic reaction-diffusion equations with a discontinuous initial condition have been considered in [28, 37, 70, 71, 73, 74]. To construct schemes that converge ε-uniformly, the method of condensing meshes (in a neighbourhood of boundary layers), and also either the fitted operator method [28, 70, 71, 73] or the method of additive splitting of a singularity [37, 74] (in a neighbourhood of the points at which the initial function is discontinuous) were applied.

In [70, 71, 73, 74], approximations to the normalized derivatives $\varepsilon\,(\partial/\partial x)\,u(x,t)$, i.e., the first order spatial derivative multiplied by the parameter ε, were considered. For this purpose, the method of additive splitting of the singularity generated by the discontinuity of the initial function was used; however, the approximation of the derivative $(\partial/\partial x)\,u(x,t)$ itself was not considered.

A boundary value problem on a segment for singularly perturbed parabolic convection-diffusion equations with a piecewise smooth initial condition has been considered in [77, 78]. In [78], by using the method of special meshes that condense in a neighbourhood of the boundary layer and the method of the additive splitting of a singularity of the interior layer type, special difference schemes are constructed that make it possible to approximate ε-uniformly the solution of the problem on the entire set under consideration, the normalized derivative on the entire set except for the discontinuity point $(0, 0)$, and the first spatial derivative on the same set but outside a small neighbourhood of the boundary layer.

In the present chapter, instead of the Cauchy problem (3.1.4), (3.1.3), we consider a singularly perturbed boundary value problem for equation (3.1.4) with a non-smooth initial condition similar to (3.1.3), namely, the problem (3.2.2), (3.2.1) (see the formulation of this problem in Section 3.2). The technique from [78] is used for studying the problem (3.2.2), (3.2.1). Note that in a problem of the type (3.2.2), (3.2.1) considered in a finite domain, except for the interior layer, an additional singularity appears, namely, a boundary layer with the typical width of ε. The singularity of the boundary layer is more strong than that of the interior layer, which makes it difficult to construct special numerical methods suitable for the adequate description of the singularity of the interior-layer type. In contrast to [78], here conditions are defined that allow us to investigate each singularity of the problem separately. For the boundary value problem (3.2.2), (3.2.1), we construct a finite difference scheme that approximates the solution and its first order derivative in x. To construct ε-uniform approximations for the solution and its first derivative in a finite subdomain including only the interior layer singularity, it suffices to use a uniform mesh and the method of the additive splitting of the singularity of the interior layer type. The efficiency of the scheme constructed in this paper is verified with numerical experiments.

The numerical method constructed for problem (3.2.2), (3.2.1), after the transformation to the original variables S, t' and the function C (see the change (3.1.1d)), allows us to approximate the solution of problem (3.1.1) and its first derivative $(\partial/\partial\, S)\, C(S,t')$ in a finite neighbourhood of the point (E,T) (the point of discontinuity of the derivative in condition (3.1.1b)), including the interior layer (appearing for small values of the dimensionless quantity $\sigma^2\, r^{-1}$). Errors in the approximation of the solution and derivative (for $(S,t') \neq (E,T)$) are independent of the value $\sigma^2\, r^{-1}$; these errors (in the maximum norm) are defined only by the number of nodes in the mesh used for the numerical solution of the discrete problem.

3.2 Problem Formulation

On the set $\overline{G}$ with the boundary S,

$$\overline{G} = G \cup S, \quad G = D \times (0,\, T], \quad D = \{x:\ x \in (-d,\, d)\}, \tag{3.2.1}$$

we consider the Dirichlet problem for the singularly perturbed parabolic convection-diffusion equation [1]

$$L_{(3.2.2a)}\, u(x,t) \;=\; f(x,t), \quad (x,t) \in G, \tag{3.2.2a}$$

$$u(x,t) \;=\; \varphi(x,t), \quad (x,t) \in S. \tag{3.2.2b}$$

Here $$L_{(3.2.2a)} \equiv \varepsilon\, a\, \frac{\partial^2}{\partial x^2} + b\, \frac{\partial}{\partial x} - c - q\, \frac{\partial}{\partial t},$$
a, b, $q > 0$, $c \geq 0$, the right-hand side $f(x,t)$ is a sufficiently smooth function on $\overline{G}$; the parameter ε takes arbitrary values in the half-open interval $(0,\, 1]$. The boundary function $\varphi(x,t)$ is sufficiently smooth on the sets $\overline{S}_0^{\,-}$, $\overline{S}_0^{\,+}$, $\overline{S}^{\,L}$ and continuous

[1] The notation $L_{(j.k)}$ ($m_{(j.k)}$, $M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, meshes) are introduced in formula $(j.k)$.

on S; the first order derivative in x of the function $\varphi(x,t)$ has a discontinuity of the first kind on the set $S^{(*)} = \{(0,\,0)\}$, i.e.,

$$\left[\frac{\partial}{\partial x}\varphi(x,t)\right] \neq 0, \quad (x,t) \in S^{(*)}. \tag{3.2.2c}$$

Here

$$S_0^- = \{(x,t):\ x \in [-d,\,0),\ t = 0\},$$

$$S_0^+ = \{(x,t):\ x \in (0,\,d],\ t = 0\}, \quad S_0 = \overline{S}_0^- \bigcup \overline{S}_0^+,$$

S_0 and S^L are the lower and lateral parts of the boundary S, $S^L = \Gamma \times (0,\,T]$, $\Gamma = \overline{D} \setminus D$.

The solution of problem (3.2.2) is a function $u \in C(\overline{G}) \bigcap C^{2,1}(G)$ satisfying the differential equation on G and the boundary conditions on S.

For simplicity, we assume that compatibility conditions that ensure the smoothness of the solution for fixed values of ε [40] are fulfilled on the set $S_* = S_0 \bigcap \overline{S}^L$. Let $\overline{G}^\delta$ be the δ-neighbourhood of the set S_*, i.e.,

$$\overline{G}^\delta = \{(x,t):\ r\big((x,t),\,S_*\big) \leq \delta\},$$

where $r\big((x,t),\,S_*\big)$ is the distance from the point (x,t) to the set S_*. We suppose that the following inclusion holds on the set $\overline{G}^\delta$:

$$u \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}^\delta), \quad l \geq 2, \quad \alpha \in (0,\,1). \tag{3.2.3}$$

It follows from [40] that, under the condition (3.2.3), the solution of the problem (for sufficiently smooth functions $f(x,t)$ on $\overline{G}$ and $\varphi(x,t)$ on $\overline{S}_0^-$, $\overline{S}_0^+$, $\overline{S}^L$) is smooth on the set

$$\overline{G}^* = \overline{G} \setminus S^{(*)}, \tag{3.2.4}$$

i.e., $u \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}^*)$. The derivative $(\partial/\partial x)u(x,t)$ is continuous on $\overline{G}^*$, bounded on $\overline{G}^*$ for fixed values of ε and has a discontinuity on the set $S^{(*)}_{(3.2.2c)}$.

Under the condition

$$a = c = p = 1, \quad b = 1 - \varepsilon, \quad f(x,t) = 0, \quad (x,t) \in \overline{G}$$

the equation (3.2.2a) becomes the equation (3.1.4).

We are interested in an approximation of the solution $u(x,t)$, $(x,t) \in \overline{G}$, and of the derivative $(\partial/\partial x)u(x,t)$, $(x,t) \in \overline{G}^{\,*}$. Let us describe the behaviour of the solution and derivatives more precisely.

Let $S^L = S^l \bigcup S^r$, S^l and S^r be the left and right parts of the boundary S^L, and let

$$S^{\gamma} = \{(x,t): \ x = \gamma(t), \ (x,t) \in \overline{G}\}, \quad \gamma(t) = -b\,q^{-1}\,t, \quad t \geq 0$$

be the characteristic of the reduced equation passing through the point (0, 0). When the parameter ε tends to zero, boundary and interior layers with the typical length scales ε and $\varepsilon^{1/2}$, respectively, appear in a neighbourhood of the sets S^l and S^{γ}; as opposed to the boundary layer, the interior layer is weak (the first order derivative in x of the interior-layer function is bounded ε-uniformly).

For simplicity, we assume that the characteristic S^{γ} does not meet the boundary S^l. The derivative $(\partial/\partial x)u(x,t)$ (denoted by $p(x,t)$) in a neighbourhood of the set S^l grows without bound as $\varepsilon \to 0$. It is convenient to consider the quantity $P(x,t) = \varepsilon\,(\partial/\partial x)u(x,t)$, i.e., the normalized first derivative in x, in the m-neighbourhood of the set S^l, instead of the derivative $(\partial/\partial x)u(x,t)$, because this quantity is bounded ε-uniformly. Outside a neighbourhood of the set S^l, the derivative $(\partial/\partial x)u(x,t)$ is bounded ε-uniformly. The quantity $P(x,t)$ will be called the diffusion flux (or briefly, the flux). Outside of a neighbourhood of the set S^l, the derivative $p(x,t)$ is bounded ε-uniformly on $\overline{G}^{\,*}$. For small values of the parameter ε, the derivative $p(x,t)$ is more "informative" (on the set where it is bounded) than the flux $P(x,t)$.

It is well known (see, e.g., [20]) that even in the case of singularly perturbed problems with sufficiently smooth data, solutions of classical finite difference schemes do not converge ε-uniformly; for small values of the parameter ε, errors in the discrete solutions are commensurable with the actual solutions of the differential problem. The diffusion fluxes obtained on the basis of such schemes also do not converge ε-uniformly. It will be shown in Section 3.3 that for a boundary value problem whose solution is regular, classical difference schemes do not allow one to obtain ε-uniformly convergent approximations of the derivative in x.

Due to this it would be interesting to construct a difference scheme that allows us to approximate ε-uniformly both the solution on the whole domain $\overline{G}$ and diffusion fluxes in this domain excluding the discontinuity point $S^{(*)}$. Also, it will be interesting to determine conditions under which the boundary layer does not appear, and for such a problem, to find the ε-uniform approximation of the derivative in x on the set $\overline{G}^*$.

Definition. Let

$$\overline{G}_0^* = \overline{G}_0^*(m) = \overline{G}^* \cap \{x \geq -d + m\} \tag{3.2.5}$$

be the set $\overline{G}^*$ excluding an m-neighbourhood [2] of the set $\overline{S}^l$ (the m-neighbourhood of the boundary layer). If the interpolants constructed using the solution of some finite difference scheme converge on $\overline{G}$ ε-uniformly, we say that the discrete solution $\big($the difference scheme) converges on $\overline{G}$ uniformly with respect to the parameter ε (or, briefly, ε-uniformly) in $C(\overline{G})\big)$. If, moreover, the interpolants of the diffusion fluxes (the first order derivatives in x) converge ε-uniformly on $\overline{G}$ $\big(\varepsilon$-uniformly on $\overline{G}_0^*\big)$, we say that the difference scheme converges ε-uniformly in $C^{1(n)}(\overline{G}^*)$ $\big(\varepsilon$-uniformly in $C^1(\overline{G}_0^*)\big)$.

[2] Throughout this chapter, M, M_i (or m) denote sufficiently large (small) positive constants that do not depend on ε and on the discretization parameters.

Thus, it is attractive to find numerical methods that converge ε-uniformly in $C^{1(n)}(\overline{G}^*)\cap C^1(\overline{G}_0^*)$, where $\overline{G}_0^* = \overline{G}_0^*(m)$, moreover, it is required that the value m could be chosen sufficiently small.

Our aim is to construct a difference scheme for problem (3.2.2), (3.2.1) that converges ε-uniformly in $C^{1(n)}(\overline{G}^*)\cap C^1(\overline{G}_0^*)$, and also to determine conditions under which the boundary layer does not appear, and in this case to construct a difference scheme that converges ε-uniformly in $C^1(\overline{G}^*)$.

In that case when the method converges ε-uniformly in $C^{1(n)}(\overline{G}^*)\cap C^1(\overline{G}_0^*)$, we say that this method is robust.

Some preliminary results related to this problem are given in [42, 43]. To investigate the problem, a technique similar to that developed in [78] is used. In the present chapter, in contrast to [78], the main attention is given to the study of a singularity of the interior-layer type, because the boundary layer does not arise in the original problem (3.1.1).

3.3 Difficulties on Approximation of the Derivative in x

Let us discuss difficulties arising in the approximation of derivatives for the regular components of the problem solution, i.e., when the solution of a singularly perturbed problem is regular (not containing the singular component) and sufficiently smooth. In this case the solution of a classical finite difference scheme on a uniform mesh converges to the exact solution ε-uniformly. However, its difference derivatives are no longer convergent ε-uniformly; the error in the derivative of the solution of the grid problem can have the order of the derivative of the solution itself for the differential problem.

Consider the stationary problem

$$\begin{aligned} L_{(3.3.1)}\, u(x) &\equiv \left\{\varepsilon \frac{d^2}{dx^2} + \frac{d}{dx}\right\} u(x) = f(x), \quad x \in D, \qquad (3.3.1a)\\ u(x) &= \varphi(x), \quad x \in \Gamma. \end{aligned}$$

Here

$$\overline{D} = [0,\, 1]. \qquad (3.3.1b)$$

Let $u(x) = x^2$, $x \in \overline{D}$, be a solution of problem (3.3.1); this solution has no singular component.

To solve problem (3.3.1), we apply the classical difference scheme [68]. On the segment $\overline{D}$, we introduce the uniform mesh

$$\overline{D}_h = \overline{D}_{h(3.3.2)} \qquad (3.3.2)$$

with the step-size $h = N^{-1}$. On the mesh $\overline{D}_h$, we consider the difference scheme

$$\begin{aligned} \Lambda_{(3.3.3)}\, z(x) &\equiv \{\varepsilon\, \delta_{x\,\overline{x}} + \delta_x\}\; z(x) = f(x), \quad x \in D_h, \qquad (3.3.3)\\ z(x) &= \varphi(x), \quad x \in \Gamma_h. \end{aligned}$$

The solution of problem (3.3.3), (3.3.2) has the explicit form

$$\begin{aligned} z(x_i) = x_i^2 + N^{-1}\big\{1 - x_i - [(1+\varepsilon^{-1}\,N^{-1})^{-i} - (1+\varepsilon^{-1}\,N^{-1})^{-N}] \,\times \\ \times\, [1 - (1+\varepsilon^{-1}\,N^{-1})^{-N}]^{-1}\big\}, \quad x_i = i\,N^{-1} \in \overline{D}_h, \quad i \le N. \end{aligned}$$

Thus, the function $z(x)$, $x \in \overline{D}_h$, converges to $u(x)$, $x \in \overline{D}$, ε-uniformly with the estimate

$$|\; u(x) - z(x) \;| \le N^{-1}, \quad x \in \overline{D}_h.$$

For the first discrete derivative

$$\delta_x\, z(x_i) = 2\,x_i + N^{-1}\,(\varepsilon + N^{-1})^{-1}\,(1+\varepsilon^{-1}\,N^{-1})^{-i}\ \times [1-(1+\varepsilon^{-1}\,N^{-1})^{-N}]^{-1},$$

$$x_i = i\,N^{-1} \in \overline{D}_h, \quad i \le N-1$$

we have the error

$$\left|\frac{d}{dx}\,u(x_i) - \delta_x\,z(x_i)\right| = N^{-1}\,(\varepsilon + N^{-1})^{-1}\,(1+\varepsilon^{-1}\,N^{-1})^{-i}\ \times$$

$$\times\,[1-(1+\varepsilon^{-1}\,N^{-1})^{-N}]^{-1}, \quad x_i \in \overline{D}_h, \quad x_i < 1;$$

$$\max_{\overline{D}_h,\ x<1}\left|\frac{d}{dx}\,u(x) - \delta_x\,z(x)\right| \ge m\,N^{-1}\,(\varepsilon + N^{-1})^{-1}.$$

Thus, the discrete derivative does not converge ε-uniformly: when $\varepsilon \le N^{-1}$, the error in this discrete derivative is of the order of the derivative itself.

If the solution of the boundary value problem (3.2.2), (3.2.1) is regular, moreover, the regular component is of the order of unity in a neighbourhood of the boundary layer, then the error in the approximation of the derivative $(\partial/\partial x)u(x,t)$, in general, grows unboundedly under the condition $(\varepsilon{+}N^{-1})^{-1}N_0^{-1} \to \infty$ as $N,\ N_0 \to \infty$, where $N+1$ and N_0+1 are the number of nodes with respect to x and t in the uniform mesh on $\overline{G}$.

3.4 A Priori Estimates of the Solution and Derivatives

3.4.1 Preliminaries

In this section, we obtain some bounds on the solution of the boundary value problem (3.2.2), (3.2.1) and its derivatives. To drive these bounds, we use the technique developed in [78] for singularly perturbed parabolic equations with a piecewise smooth initial function, and also the technique developed for singularly perturbed convection-diffusion problems with sufficiently smooth data (see, e.g., [57, 66, 72, 78], and the bibliography therein). We assume that the functions $f(x,t)$ and $\varphi(x,t)$ are sufficiently smooth on the sets $\overline{G}$ and $\overline{S}^L$, $\overline{S}_0^+$ and $\overline{S}_0^-$, respectively, moreover, compatibility conditions are fulfilled on the set S_* that ensure the sufficient smoothness of the solution of the problem in a neighbourhood of the lateral boundary of the set $\overline{G}$. When derive the estimates, it is convenient to consider the problem solution in neighbourhoods of the boundary and interior layers and also outside these neighbourhoods.

Deriving the estimates, we assume that the following condition holds:

The data of the boundary value problem (3.2.2), (3.2.1) *satisfy the condition*

$$f \in C^{l_1,l_1}(\overline{G}), \quad \varphi \in \left\{ C^{l_1}(\overline{S}_0^-) \bigcup C^{l_1}(\overline{S}_0^+) \bigcup C^{l_1}(\overline{S}^L) \right\} \bigcap C(S), \tag{3.4.1a}$$

the condition (3.2.3) *holds for the solution of this problem, where*

$$l_1 = l + \alpha, \quad l \geq 2\,K - 1, \quad K \geq 2, \quad \alpha \in (0,1). \tag{3.4.1b}$$

Also, we assume that the following condition is fulfilled

$$\frac{\partial^k}{\partial x^k}\,\varphi(x,t),\ \frac{\partial^{k_0}}{\partial t^{k_0}}\,\varphi(x,t) = 0, \quad (x,t) \in S_*, \quad k + k_0 \leq l, \tag{3.4.2}$$

$$\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}\,f(x,t) = 0, \quad (x,t) \in S_*, \quad k,\, k_0 \leq l-1,$$

where $l = l_{(3.4.1)}$. This condition is sufficient for the inclusion (3.2.3).

We represent the set $\overline{G}$ as the sum of overlapping sets

$$\overline{G} = \bigcup_j \overline{G}^{\,j}, \quad j = 1,\, 2,\, 3, \tag{3.4.3a}$$

where

$$G^{\,1} = G^{\,1}(m^1) = \{(x,t):\ |x-\gamma(t)| < m^1,\ \ t \in (0,T]\}, \tag{3.4.3b}$$

$$G^{\,2} = G^{\,2}(m^2) = \{(x,t):\ x \in (-d, -d + m^2),\ \ t \in (0,T]\},$$

$$G^{\,3} = G^{\,3}(m^3) = G \setminus \left\{G^1(m^3) \bigcup G^2(m^3)\right\}.$$

Here m^1, m^2, m^3 are sufficiently arbitrary constants satisfying the condition $m^3 < m^1,\ m^2$, the sets $G^{\,1}$ and $G^{\,2}$ are neighbourhoods of the interior and boundary layers, respectively. Let $\overline{G}^{\,1} \bigcap \overline{G}^{\,2} = \emptyset$.

The solution of problem (3.2.2), (3.2.1) considered on the set $\overline{G}^{\,j}$ will be also denoted by the $u^j(x,t)$, $j = 1,\, 2,\, 3$.

3.4.2 The Estimate of the Problem Solution on the Set $\overline{G}^{\,3}$

Lemma 3.4.1 *Let conditions* (3.4.1), (3.4.2) *be satisfied. Then the solution of the boundary value problem satisfies the estimate*

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u^3(x,t) \right| \le M, \quad (x,t) \in \overline{G}^{\,3}, \quad k + 2k_0 \le K, \tag{3.4.4}$$

where $K = K_{(3.4.1)}$.

Proof. By virtue of the maximum principle (see, e.g., [40, 72, 57, 66, 63], and also the bibliography therein), the solution of problem (3.2.2), (3.2.1) satisfies the estimate

$$|\, u(x,t)\, | \le M, \quad (x,t) \in \overline{G}.$$

Write the set $\overline{G}^{\,3}$ as the sum of sets

$$\overline{G}^{\,3} = \overline{G}^{\,3l} \bigcup \overline{G}^{\,3r},$$

where $\overline{G}^{\,3l}$ and $\overline{G}^{\,3r}$ are sets that are located from the left and right of the characteristic S^{γ}. Let us find the estimate of derivatives of the function $u^3(x,t)$ on the set $\overline{G}^{\,3l}$. We have $S^{3l} \bigcap S \subset S_0$, moreover, $S^{3l} \bigcap S^L = \emptyset$.

On the set $\overline{G}^{\,3l}$, we pass to new variables $\xi = \varepsilon^{-1} x$, $\tau = \varepsilon^{-1} t$. In the new variables, we have the function $\widetilde{u}^3(\xi,\tau)$, $(\xi,\tau) \in \overline{\widetilde{G}}^{\,3l}$, where $\widetilde{u}^3(\xi(x),\tau(t)) = u(x,t)$ and $\overline{\widetilde{G}}^{\,3l}$ is the image of the set $\overline{G}^{\,3l}$. This function $\widetilde{u}^3(\xi,\tau)$ is the solution of the regular parabolic equation

$$\left\{ a\,\frac{\partial^2}{\partial \xi^2} + b\,\frac{\partial}{\partial \xi} - c - q\,\frac{\partial}{\partial \tau} \right\} \widetilde{u}^3(\xi,\tau) = \widetilde{F}(\xi,\tau), \quad (\xi,\tau) \in \widetilde{G}^{\,3l},$$

that satisfies the condition

$$\widetilde{u}^{\,3}(\xi,\,\tau) = \widetilde{\varphi}(\xi,\,\tau), \quad (\xi,\,\tau) \in \widetilde{S}^{\,3\,l}, \quad \tau = 0.$$

Here $\widetilde{S}^{\,3\,l}$ is the boundary of the set $\widetilde{G}^{\,3\,l}$, $\widetilde{F}(\xi,\,\tau) = \varepsilon\,\widetilde{f}(\xi,\,\tau)$, $\widetilde{f}(\xi,\,\tau)$ and $\widetilde{\varphi}(\xi,\,\tau)$ are functions $f(x,\,t)$ and $\varphi(x,\,t)$ which are written in the variables ξ, τ.

Using interior a priori estimates and estimates up to smooth parts of the boundary (see, e.g., [4, 40]), we find the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial \xi^k \partial \tau^{k_0}}\, \widetilde{u}^{\,3}(\xi,\,\tau) \right| \le M, \quad (\xi,\,\tau) \in \overline{\widetilde{G}}^{\,3\,l}, \quad k + 2k_0 \le K, \tag{3.4.5a}$$

where $K \le l$, $l = l_{(3.4.1)}$. Note that the constants m^1, m^2, m^3, in (3.4.3) can be chosen sufficiently arbitrary. Returning to the variables x, t, we obtain

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u^{\,3}(x,\,t) \right| \le M\varepsilon^{\,-k-k_0}, \quad (x,\,t) \in \overline{G}^{\,3\,l}, \quad k + 2k_0 \le K. \tag{3.4.5b}$$

Let us make more precise estimate (3.4.5b). Note that by virtue of condition (3.4.1), it is possible apply the differentiation operator $\partial^{k+k_0}/\partial x^k \partial t^{k_0}$, $k + 2k_0 \le K$

to the boundary value problem (3.2.2), (3.2.1) considered on the set $\overline{G}^{3l}$. Derivatives with respect to t of the problem solution on the set S_0^{3l} are computed according to the differential equation (3.2.2a). For example, the mixed derivative $(\partial^{k+1}/\partial x^k\,\partial t)\,u^3(x,\,t)$ is defined by the relation

$$\frac{\partial^{k+1}}{\partial x^k \partial t}\,u^3(x,\,t) = q^{-1}\left\{\varepsilon\,a\,\frac{\partial^2}{\partial x^2} + b\,\frac{\partial}{\partial x} - c\right\}\frac{\partial^k}{\partial x^k}\,\varphi(x,\,t) - q^{-1}\,\frac{\partial^k}{\partial x^k}\,f(x,\,t),$$
$$(x,\,t) \in S_0^{3\,l},$$

where $S^{3\,l} = S_0^{3\,l}\bigcup S_L^{3\,l}$, $S_0^{3\,l}$ and $S_L^{3\,l}$ are the lower and lateral parts of the boundary $S^{3\,l}$; $\overline{G}^{3\,l} = G^{3\,l}\bigcup S^{3\,l}$.

Note that the sets S_{L1}^{3l} and S_{L2}^{3l} which are the left and right parts of the boundary S_L^{3l} where $S_L^{3l} = S_{L1}^{3l}\bigcup S_{L2}^{3l}$, are output and characteristic boundaries respectively. Characteristics of the reduced differential equation leave the set G^{3l} through the output boundary S_{L1}^{3l}.

The derivatives $\dfrac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\,u^3(x,\,t)$, $k + 2k_0 \le K$ are bounded ε-uniformly on S_0^{3l}, however by virtue of the estimate (3.4.5b), these derivatives are not bounded ε-uniformly on S_L^{3l}. The derivatives $\dfrac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\,u^3(x,\,t)$ on the set G^{3l} satisfy the parabolic equation whose right hand side is also bounded ε-uniformly. Thus, the function

$$u_{K_1}^3(x,\,t) = \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\,u^3(x,\,t), \quad (x,\,t) \in \overline{G}^{3\,l}, \quad k + 2k_0 = K_1, \quad K_1 \le K,$$

can be written as the sum of the functions

$$u_{K_1}^3(x,\,t) = U_{K_1}^3(x,\,t) + V_{K_1}^3(x,\,t), \quad (x,\,t) \in \overline{G}^{3\,l},$$

where $U_{K_1}^3(x,\,t)$ and $V_{K_1}^3(x,\,t)$ are the regular and singular components of the function $u_{K_1}^3(x,\,t)$. The function $V_{K_1}^3(x,\,t)$ is the solution of a homogeneous equation with the homogeneous initial condition. The function $U_{K_1}^3(x,\,t)$ on the set $S_0^{3\,l}$ coincides with $u_{K_1}^3(x,\,t)$ which is sufficiently smooth on $S_0^{3\,l}$ and ε-uniformly bounded that implies ε-uniform boundedness of the function $U_{K_1}^3(x,\,t)$ on $\overline{G}^{3\,l}$.

The function $V^3_{K_1}(x,\,t)$ on the set $S^{3\,l}_L$ satisfies an estimate that is similar to (3.4.5b). Applying the majorant functions technique (see [4, 40, 57]), we verify that the function $V^3_{K_1}(x,\,t)$ decreases exponentially as the value $\varepsilon^{-1/2}\, r\big((x,\,t),\, S^{3\,l}_L\big)$ increases, where $r\big((x,\,t),\, S^{3\,l}_L\big)$ is the distance from the point $(x,\,t)$ to the set $S^{3\,l}_L$. Thus, the function $u^3_{K_1}(x,\,t)$ is bounded ε-uniformly on the set $\overline{G}^{\,3\,l}$ outside sufficiently small neighbourhood of the set $S^{3\,l}_L$.

By virtue of arbitrary choice of the values m^1, m^2, m^3 in (3.4.3), choosing the values m_1, m_2, m_3, it is possible to change sizes of the set $\overline{G}^{\,3l}$, $\overline{G}^{\,3l} = \overline{G}^{\,3l}(m_1, m_2, m_3)$. We estimate the function $u^3(x,t)$ on some set $\overline{G}^{\,3l}$ outside a sufficiently small neighbourhood of the set S^{3l}_L, i.e., we obtain an estimate on the whole set $\overline{G}^{\,3l}(m'_1, m'_2, m'_3)$, where, in general, $m_i \neq m'_i$; $\overline{G}^{\,3l}(m'_1, m'_2, m'_3) \subset \overline{G}^{\,3l}(m_1, m_2, m_3)$. Since the choice of the values m_i is arbitrary, we thus obtain an estimate on the set of our interest $\overline{G}^{\,3l}$. Thus, we obtain estimate (1.4.4) for the function $u^3(x,t)$ on the set $\overline{G}^{\,3l}$.

Let us find an estimate for derivatives of the function $u^3(x,\,t)$ on the set

$$\overline{G}^{\,3\,r}.$$

Here $\overline{G}^{\,3\,r} = G^{3\,r} \bigcup S^{3\,r}$; $S^{3\,r} = S^{3\,r}_0 \bigcup S^{3\,r}_L$, $S^{3\,r}_0$ and $S^{3\,r}_L$ are the lower and lateral parts of the boundary $S^{3\,r}$. We have $S^{3\,r} \bigcap S = S^{3\,r}_0 \bigcup S^{3\,r}_{L2}$ where $S^{3\,r}_L = S^{3\,r}_{L1} \bigcup S^{3\,r}_{L2}$, $S^{3\,r}_{L1}$ and $S^{3\,r}_{L2}$ are the left and right parts of the boundary $S^{3\,r}_L$. The sets $S^{3\,r}_{L1}$ and $S^{3\,r}_{L2}$ are characteristic and input parts of the boundary $S^{3\,r}_L$; characteristics of the reduced differential equation enter in the set G^{3r} through the input boundary S^{3r}_{L2}.

Decompose the function $u^3(x,\,t)$, $(x,\,t) \in \overline{G}^{\,3\,r}$ as the sum

$$u^3(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}^{\,3\,r}, \tag{3.4.6a}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular parts of the solution. The function $U(x,t)$, $(x,t) \in \overline{G}^{\,3\,r}$, is the restriction to $\overline{G}^{\,3\,r}$ of the function $U^0(x,t)$,

$(x,t) \in \overline{G}^0$, $U(x,t) = U^0(x,t)$, $(x,t) \in \overline{G}^{3\,r}$. The function $U^0(x,t)$, $(x,t) \in \overline{G}^0$, is the solution of the boundary value problem

$$\begin{aligned} L^0 U^0(x,t) &= f^0(x,t), \quad (x,t) \in G^0, \\ U^0(x,t) &= \varphi^0(x,t), \quad (x,t) \in S^0. \end{aligned} \tag{3.4.7}$$

Here $\overline{G}^0$ is the half-strip ($x \le d$) that is a continuation of $\overline{G}^{3\,r}$ beyond the side $S^{3\,r}_{L1}$; the data of problem (3.4.7) are smooth continuations of the data of problem (3.2.2), (3.2.1) considered on the set $\overline{G}^{3\,r}$; such continuations preserve properties (3.4.1) on the set $\overline{G}^0$. The functions $f^0(x,t)$, $(x,t) \in \overline{G}$ and $\varphi^0(x,t)$, $(x,t) \in S^0$ are considered to be equal to zero outside the m_1-neighbourhood of the set $\overline{G}$; $L^0 = L_{(3.2.2a)}$. The function $V(x,t)$, $(x,t) \in \overline{G}^{3\,r}$ is the solution of the problem

$$\begin{aligned} L_{(3.2.2a)}\, V(x,t) &= 0, \quad (x,t) \in G^{3\,r}, \\ V(x,t) &= \varphi(x,t) - U(x,t), \quad (x,t) \in S^{3\,r}, \end{aligned} \tag{3.4.8}$$

the function $V(x,t)$ is equal to zero on the set $S^{3\,r}_0 \bigcup S^{3\,r}_{L2}$.

Write the function $U(x,\, t)$ as the sum of the functions

$$U(x,\, t) = U_0(x,\, t) + \varepsilon\, U_1(x,\, t) + \cdots + \varepsilon^n\, U_n(x,\, t) + v_U(x,\, t), \tag{3.4.6b}$$

$$(x,\, t) \in \overline{G}^{3\,r},$$

corresponding to the representation of the function $U^0(x,\, t)$ in the form

$$U^0(x,t) = U^0_0(x,t) + \varepsilon U^0_1(x,t) + \cdots + \varepsilon^n U^0_n(x,t) + v^0_U(x,t), \quad (x,t) \in \overline{G}^0, \tag{3.4.9a}$$

which is the solution of the boundary value problem (3.4.7). Here $v^0_U(x,t)$ is the remainder term and

$$U(x,t) = U^0(x,t), \ldots, v_U(x,t) = v^0_U(x,t), \quad (x,t) \in \overline{G}^{3\,r}.$$

In (3.4.9a) the functions $U_0^0(x,t)$, $U_i^0(x,t)$, $i=1,\dots,n$ are solutions of the problems

$$\begin{aligned} L^0_{(3.4.9)} U_0^0(x,t) &= f^0(x,t), \quad (x,t)\in G^0, \\ U_0^0(x,t) &= \varphi^0(x,t), \quad (x,t)\in S^0; \\ L^0_{(3.4.9)} U_i^0(x,t) &= -a\,\frac{\partial^2}{\partial x^2}\,U_{i-1}^0(x,t), \quad (x,t)\in G^0, \\ U_i^0(x,t) &= 0, \quad (x,t)\in S^0, \quad i=1,\dots,n. \end{aligned} \tag{3.4.9b}$$

Here $L^0_{(3.4.9)}$ is the operator $L^0_{(3.4.7)}$ for $\varepsilon=0$

$$L^0_{(3.4.9)} \equiv b\,\frac{\partial}{\partial x} - c - q\,\frac{\partial}{\partial t}, \quad (x,t)\in G^0.$$

By virtue of condition (3.4.2), apart from the compatibility conditions on the set S_*, ensuring the smoothness of the solution $u(x,t)$ of problem (3.2.2), (3.2.1) on $\overline{G}$ outside a neighbourhood of the set $S^{(*)}$, the problem data satisfy additional conditions on the set $S_*^0 = S_0^0 \bigcap S^{0L}$ that ensure the sufficient smoothness of the functions $U_0^0(x,t)$, $U_i^0(x,t)$, $i=1,\dots,n$ and $v_U^0(x,\,t)$ on $\overline{G}^{\,0}$.

For $n=K_{(3.4.1)}-1$ the inclusion $U\in C^{l^2,\,l^2}(\overline{G}^{\,0})$ holds, where $l^2=K$, $l=2\,K-1$. Condition (3.4.1b) then gives $V\in C^{l^2,l^2}(\overline{G}^{\,3r})$.

The functions $U_i^0(x,\,t)$ in (3.4.9a) and their derivatives $\dfrac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\,U_i^0(x,t)$, $k+2k_0\le K$ are bounded ε-uniformly. For the remainder term $v_U^0(x,\,t)$ we obtain the estimate

$$\left|v_U^0(x,\,t)\right| \le M\,\varepsilon^K, \quad (x,t)\in\overline{G}^{\,0}.$$

Taking into account derivatives of the function $v_U^0(x,\,t)$, we find the estimate for the function $U(x,\,t), (x,t)\in\overline{G}^{\,3r}$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\,U(x,t)\right| \le M, \quad (x,t)\in\overline{G}^{\,3\,r}, \quad k+k_0\le K, \tag{3.4.10}$$

where $K=K_{(3.4.1)}$. For the function $V(x,\,t)$ on the boundary S^{3r}_{L1}, an estimate is fulfilled which is similar to (3.4.5b), moreover, the function $V(x,\,t)$ and its derivatives decrease exponentially as the value $\varepsilon^{-1/2}\,r\big((x,\,t),\,S^{3\,r}_{L1}\big)$ increases. Thus, the function $u^3(x,\, t)$ and its derivatives are bounded ε-uniformly on the set $\overline{G}^{\,3\,r}$ outside sufficiently small neighbourhood of the boundary S^{3r}_{L1}. By virtue of appropriate choice of the values m^1, m^2, m^3, we obtain the estimate (3.4.4) for the function $u^3(x,\, t)$ on the set $\overline{G}^{\,3\,r}$.

The proof of the lemma is complete.

3.4.3 The Estimate of the Problem Solution on the Set $\overline{G}^{\,2}$

We represent the solution on the set $\overline{G}^{\,2}$ as the decomposition into two functions

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}^{\,2}, \tag{3.4.11}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular components of the solution.

The function $U(x,t)$ is the restriction of the function $U^{\,0}(x,t)$, $(x,t) \in \overline{G}^{\,0}$ to the set $\overline{G}^{\,2}$. Here, $U^{\,0}(x,t)$ is a solution of problem

$$\begin{aligned} &L_{(3.2.2)}\, U^{\,0}(x,t) = f^{\,0}(x,t), \quad (x,t) \in\, G^{\,0}, \\ &U^{\,0}(x,t) = \varphi^{\,0}(x,t), \quad (x,t) \in\, S^{\,0}. \end{aligned} \tag{3.4.12}$$

The domain $G^{\,0}$ is an extension of G beyond the boundary $S^{\,l}$. The right hand side of equation (3.4.12) is a smooth continuation of the function $f(x,t)$. The function $\varphi^{\,0}(x,t)$ is smooth on each piecewise smooth part of the set $S^{\,0}$, and it coincides with $\varphi(x,t)$ on the set $S_0 \bigcup S^r$. The functions $f^{\,0}(x,\, t)$ and $\varphi^{\,0}(x,\, t)$ outside a m-neighbourhood of the set $\overline{G}$ are assumed to be equal to zero. The function $V(x,t)$, $(x,\, t) \in \overline{G}$ is a solution of problem

$$L_{(3.2.2)}\, V(x,t) = 0, \quad (x,t) \in\, G,$$

$$V(x,t) = \varphi(x,t) - U^{\,0}(x,t), \quad (x,t) \in S \bigcap S^{\,L}, \quad V(x,t) = 0, \quad (x,t) \in S \setminus S^{\,L}.$$

Lemma 3.4.2 *Let conditions* (3.4.1), (3.4.2) *be satisfied. Then for the functions* $U(x,t)$ *and* $V(x,t)$ *on the set* $\overline{G}^{\,2}$, *the following estimates are valid:*

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U(x,\, t)\right| \le M, \tag{3.4.13a}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, V(x,\, t)\right| \le M\,\varepsilon^{-k} \exp\Big(-m\,\varepsilon^{-1}\, r\big((x,t),\, \overline{S}^{l}\big)\Big), \tag{3.4.13b}$$

$$(x,t) \in \overline{G}^{\,2}, \quad k + 2\,k_0 \le K,$$

where $r\big((x,t),\, \overline{S}^{\,l}\big)$ *is the distance from the point* (x,t) *to the set* $\overline{S}^{\,l}$, m *is an arbitrary constant from the interval* $(0,\, m_0)$, $m_0 = a^{-1}\, b$, $K = K_{(3.4.1)}$.

Proof. Let us estimate the function $U(x,\, t)$, $(x,\, t) \in \overline{G}^{\,2}$.

Consider the function $U^{\,0}(x,\, t)$ on the set $\overline{G}^{\,(3)} \subset \overline{G}^{\,0}$ where

$\overline{G}^{\,(3)} = G^{\,(3)} \bigcup S^{\,(3)}, \quad G^{\,(3)} = G^{\,0} \bigcap \big\{(x,\, t):\, -d - m^4 < x < \gamma(t) - m^1,\ t \in (0,\, T]\big\};$

$m^1 = m^1_{(3.4.3)}$, m^4 is a sufficiently arbitrary constant. For the data of problem (3.4.12), conditions similar to those given in Lemma 3.4.1 for problem (3.2.2), (3.2.1) are fulfilled.

Similar to the estimation of derivatives for the function $u^{\,3}(x,\, t)$ on the set $\overline{G}^{\,3\,l}$, we find the estimate of the derivatives $U^0(x,\, t)$ on $\overline{G}^{\,(3)}$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U^{\,0}(x,\, t)\right| \le M, \quad (x,\, t) \in \overline{G}^{\,(3)}, \quad k + 2\,k_0 \le K.$$

From this estimate, it follows that estimate (3.4.13a) holds.

Applying the majorant functions technique, we find

$$|V(x,\, t)| \le M \exp\Big(-m\,\varepsilon^{-1}\, r\big((x,\, t),\, \overline{S}^{\,l}\big)\Big), \quad (x,\, t) \in \overline{G},$$

where $m = m_{(3.4.13b)}$. The function $V(x,\, t)$ is sufficiently smooth on $\overline{G}$, moreover, its derivatives with respect to t are bounded ε-uniformly, and they decrease exponentially as the value $\varepsilon^{-1}\, r\big((x,\, t),\, \overline{S}^{\,l}\big)$ increases. Taking into account estimates for the function $V(x,\,t)$ that are derived in the variables $\xi = \varepsilon^{-1}\,x$, $\tau = \varepsilon^{-1}\,t$, we find the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, V(x,\,t)\right| \le M\varepsilon^{-k} \exp\Big(-m\,\varepsilon^{-1}\, r\big((x,\,t),\,\overline{S}^{\,l}\big)\Big), \quad (x,\,t)\in\overline{G}, \quad k+2\,k_0 \le K.$$

From this estimate, it follows that estimate (3.4.13b) holds, and the proof of the lemma is complete.

3.4.4 The Estimate of the Problem Solution on the Set $\overline{G}^{\,1}$

Before to formulate a lemma about estimates for the solution of the problem (3.2.2) on the set $\overline{G}^{\,1}$, we make some auxiliary constructs. We shall consider the problem in new variables in which characteristics of the reduced differential equation are parallel to the t-axis.

On the set $\overline{G}^{\,1}$, we introduce the new variables

$$\widetilde{u}(\xi,\,t) = u\big(x(\xi,t),\,t\big)\,\exp(\alpha\,t), \quad (\xi,\,t)\in\overline{\widetilde{G}}^{\,1}, \quad \xi = x-\gamma(t), \quad (x,\,t)\in\overline{G}^{\,1}. \quad (3.4.14)$$

Here $\gamma(t) = -b\,q^{-1}\,t$, $\alpha = c\,q^{-1}$, and $\widetilde{G}^{\,1}$ is the image of the set G^1. In the new variables, problem (3.2.2) considered on the set $\overline{G}^{\,1}$ is transformed into a problem for the singularly perturbed heat equation

$$\begin{aligned} L_{(3.4.15a)}\widetilde{u}(\xi,\,t) \;&\equiv\; \left\{\varepsilon\,a\,\frac{\partial^2}{\partial\xi^2} - q\,\frac{\partial}{\partial t}\right\}\widetilde{u}(\xi,\,t) = \widetilde{f}(\xi,\,t), \quad (\xi,\,t)\in\widetilde{G}^{\,1}, \quad (3.4.15a)\\ \widetilde{u}(\xi,\,t) \;&=\; \begin{cases} \widetilde{u}^{\,3}(\xi,\,t), & (\xi,\,t)\in\widetilde{S}^{\,1}\setminus\widetilde{S}, \\ \widetilde{\varphi}(\xi,\,t), & (\xi,\,t)\in\widetilde{S}^{\,1}\bigcap\widetilde{S}. \end{cases} \end{aligned}$$

Here, the function $\widetilde{u}(\xi,\,t)$, $(\xi,\,t)\in\overline{\widetilde{G}}^{\,1}$, is the solution of the problem; $\widetilde{S}^{\,1} = \overline{\widetilde{G}}^{\,1}\setminus\widetilde{G}^{\,1}$, and $\widetilde{v}(\xi,\,t)$ is the image of the function $v(x,t)$,

$$\widetilde{v}(\xi,\,t) = v\big(x(\xi,\,t),\;t\big)\,\exp(\alpha\,t), \qquad (3.4.15b)$$

where $v(x,t)$ is one of the functions $u(x,t)$, $f(x,t)$, $(x,\, t) \in \overline{G}^{\,1}$, $\varphi(x,\, t)$, $(x,\, t) \in S^1 \bigcap \{t=0\}$, and $u^3(x,t)$, $(x,\, t) \in \overline{G}^{\,1} \bigcap \overline{G}^{\,3}$, where $u^3(x,t) = u(x,t)$, $(x,\, t) \in \overline{G}^{\,3}$.

We represent the solution of boundary value problem (3.4.15) $\widetilde{u}(\xi,\, t)$ as the sum of functions

$$\widetilde{u}(\xi,\, t) = \widetilde{U}^{\,1}(\xi,\, t) + \widetilde{W}^{\,1}(\xi,\, t), \quad (\xi,\, t) \in \overline{\widetilde{G}}^{\,1}, \tag{3.4.16a}$$

where $\widetilde{U}^{\,1}(\xi,t)$ and $\widetilde{W}^{\,1}(\xi,t)$ are the regular (sufficiently smooth) and singular components of the solution, respectively.

The function $\widetilde{W}^{\,1}(\xi,t)$ is the solution of the Cauchy problem

$$\begin{aligned} L_{(3.4.15)}\, \widetilde{W}^{\,1}(\xi,\, t) &= 0, \qquad (\xi,t) \in I\!R \times (0,\, T], \\ \widetilde{W}^{\,1}(\xi,\, t) &= \widetilde{\Phi}^1_W(\xi), \quad \xi \in I\!R, \quad t = 0. \end{aligned} \tag{3.4.17}$$

Here

$$\widetilde{\Phi}^1_W(\xi) = 2^{-1}\left[\frac{\partial}{\partial \xi}\, \widetilde{\varphi}(0,\, 0)\right] |\xi|, \quad \xi \in I\!R,$$

and $\left[\frac{\partial}{\partial \xi}\, \widetilde{\varphi}(0,\, 0)\right]$ is the jump of the derivative $\frac{\partial}{\partial \xi}\, \widetilde{\varphi}(\xi,\, t)$,

$$\left[\frac{\partial}{\partial \xi}\, \widetilde{\varphi}(0,\, 0)\right] = \frac{\partial}{\partial \xi}\, \widetilde{\varphi}(+0,\, 0) - \frac{\partial}{\partial \xi}\, \widetilde{\varphi}(-0,\, 0).$$

The function $\widetilde{U}^{\,1}(\xi,\, t)$ is the solution of the problem

$$\begin{aligned} L_{(3.4.15)}\, \widetilde{U}^{\,1}(\xi,\, t) &= \widetilde{f}(\xi,\, t), \quad (\xi,\, t) \in \widetilde{G}^1, \\ \widetilde{U}^{\,1}(\xi,\, t) &= \begin{cases} \widetilde{u}^3(\xi,\, t) - \widetilde{W}^{\,1}(\xi,\, t) & (\xi,\, t) \in \widetilde{S}^1, \quad t > 0, \\ \widetilde{\varphi}(\xi,\, t) - \widetilde{\Phi}^1_W(\xi,\, t) & (\xi,\, t) \in \widetilde{S}^1, \quad t = 0. \end{cases} \end{aligned}$$

Thus, the function $u(x,t)$, $(x,t) \in \overline{G}^{\,1}$ can be represented as the sum of the functions corresponding to representation (3.4.16a)

$$u(x,t) = U^1(x,t) + W^1(x,t), \quad (x,\, t) \in \overline{G}^{\,1}, \tag{3.4.18}$$

where, $U^1(x,t)$ and $W^1(x,t)$ are the regular and singular components of the solution $u(x,t)$.

Lemma 3.4.3 *Let conditions* (3.4.1) *be satisfied. Then the components in representation* (3.4.18) *satisfy the estimates*

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U^1(x,t) \right| \le M \left[1 + \varepsilon^{(2-k-k_0)/2}\, t^{(2-k-k_0)/2} + \varepsilon^{(2-k)/2}\, t^{(2-k-2k_0)/2} \right],$$

$$(x,t) \in \overline{G}^{\,1}, \tag{3.4.19}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} W^1(x,t) \right| \le M \left[1 + \varepsilon^{(1-k-k_0)/2}\, t^{(1-k-k_0)/2} + \varepsilon^{(1-k)/2}\, t^{(1-k-2k_0)/2} \right] \times$$

$$\times \exp(-m\,\varepsilon^{-1/2}\, |x - \gamma(t)|\,), \quad (x,t) \in \overline{G}^{*}, \quad k + 2k_0 \le K,$$

where $K = K_{(3.4.1)}$*,* m *is an arbitrary constant.*

Proof. Write the solution of boundary value problem (3.4.15) as the sum of the functions

$$\widetilde{u}(\xi,\, t) = \widetilde{U}(\xi,\, t) + \widetilde{W}(\xi,\, t), \quad (\xi,\, t) \in \overline{\widetilde{G}}^{\,1}, \tag{3.4.20a}$$

where $\widetilde{U}(\xi,t)$ and $\widetilde{W}(\xi,t)$ are the regular (”sufficiently“ smooth) and singular parts of the solution. The function $\widetilde{W}(\xi,t)$ is the solution of the Cauchy problem

$$L_{(3.4.15)}\widetilde{W}(\xi,\, t) = 0, \quad (\xi,t) \in \widetilde{G}^{\infty}, \quad \widetilde{W}(\xi,\, t) = \widetilde{\Phi}_W(\xi), \quad \xi \in I\!R, \quad t = 0. \tag{3.4.21}$$

Here, $\quad \widetilde{G}^{\infty} = I\!R \times (0,\, T]$,

$$\widetilde{\Phi}_W(\xi) = 2^{-1} \sum_{k=1}^{K-1} (k!)^{-1} \left[\frac{\partial^k}{\partial \xi^k}\, \widetilde{\varphi}(0,\, 0) \right] \left\{ \begin{array}{rl} \xi^k, & \xi \ge 0 \\ -\xi^k, & \xi < 0 \end{array} \right\}, \quad \xi \in I\!R;$$

and $\left[\dfrac{\partial^k}{\partial \xi^k}\, \widetilde{\varphi}(0,0) \right]$ is the jump of the derivative $\dfrac{\partial^k}{\partial \xi^k}\, \widetilde{\varphi}(\xi,t)$,

$$\left[\frac{\partial^k}{\partial \xi^k}\, \widetilde{\varphi}(0,0) \right] = \frac{\partial^k}{\partial \xi^k}\, \widetilde{\varphi}(+0,0) - \frac{\partial^k}{\partial \xi^k}\, \widetilde{\varphi}(-0,0).$$

The function $\widetilde{U}(\xi,\,t)$ is the solution of the problem

$$\begin{aligned} L_{(3.4.15)}\widetilde{U}(\xi,\,t) &= \widetilde{f}(\xi,\,t), \quad (\xi,\,t) \in \widetilde{G}^{1}, \\ \widetilde{U}(\xi,\,t) &= \begin{cases} \widetilde{u}^{3}(\xi,\,t) - \widetilde{W}(\xi,\,t), & (\xi,\,t) \in \widetilde{S}^{1}, \quad t > 0, \\ \widetilde{\varphi}(\xi,\,t) - \widetilde{\Phi}_{W}(\xi), & (\xi,\,t) \in \widetilde{S}^{1}, \quad t = 0. \end{cases} \end{aligned}$$

According to the construction, the function $\widetilde{U}(\xi,\,t)$ is sufficiently smooth on the boundary $\widetilde{S}^{1}$, and it satisfies compatibility conditions on the set of corner points. The function $\widetilde{U}(\xi,\,t)$ has ε-uniformly bounded derivatives with respect to ξ up to the Kth order and ε-uniformly bounded derivatives with respect to t up to the $K/2$th order. For $\widetilde{U}(\xi,\,t)$, we have the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial \xi^{k} \partial t^{k_0}}\, \widetilde{U}(\xi,\,t) \right| \le M, \quad (\xi,\,t) \in \overline{\widetilde{G}}^{1}, \quad k + 2k_0 \le K, \tag{3.4.22}$$

which is established with account for the smoothness of the problem data from [4, 40] by the way similar to the estimation of the function $u^{3}(x,t)$ on $\overline{G}^{3l}$.

Represent the function $\widetilde{W}(\xi,\,t)$ in the form

$$\widetilde{W}(\xi,\,t) = \sum_{k=1}^{K-1} \widetilde{W}_{k}(\xi,\,t), \quad (\xi,\,t) \in \overline{\widetilde{G}}^{\infty}, \tag{3.4.20b}$$

here $\widetilde{W}_{k}(\xi,\,t)$ is a solution of problem (3.4.21) with $\widetilde{\Phi}_{W}(\xi)$ defined by

$$\widetilde{\Phi}_{k}(\xi) = 2^{-1}\,(k!)^{-1} \left[\frac{\partial^{k}}{\partial \xi^{k}}\, \widetilde{\varphi}(0,\,0) \right] \left\{ \begin{array}{cc} \xi^{k}, & \xi \ge 0 \\ -\xi^{k}, & \xi < 0 \end{array} \right\}, \quad \xi \in I\!R, \quad k = 1, 2, ..., K-1.$$

The functions $\widetilde{W}_{k}(\xi,\,t)$ can be found explicitly. For example, for $\widetilde{W}_{1}(\xi,\,t)$ we have the representation

$$\widetilde{W}_{1}(\xi,\,t) = \widetilde{W}^{1}_{(3.4.16a)}(\xi,\,t) \equiv 2^{-1} \left[\frac{\partial}{\partial \xi}\, \widetilde{\varphi}(0,0) \right] \widetilde{w}_{1}(\xi,\,t), \tag{3.4.20c}$$

$$\widetilde{w}_1(\xi,\, t) = \xi\, v(2^{-1}\,\varepsilon^{-1/2} a^{-1/2}\, q^{1/2}\, \xi\, t^{-1/2}) +$$

$$+2\,\pi^{-1/2}\,\varepsilon^{1/2}\,a^{1/2}\,q^{-1/2}\,t^{1/2}\,\exp(-4^{-1}\,\varepsilon^{-1}\,a^{-1}\,q\,\xi^2\,t^{-1}), \quad (\xi, t) \in \overline{\widetilde{G}}^{\,\infty},$$

$$v(\xi) = erf(\xi) = 2\,\pi^{-1/2} \int_0^{\xi} \exp(-\alpha^2)\, d\alpha, \quad \xi \in I\!R.$$

Note that the first-order derivative with respect to ξ of the function $\widetilde{W}_1(\xi,\, t)$ is bounded on $\overline{\widetilde{G}}^{\,\infty}$ and has a discontinuity at the point (0, 0).

Represent the function $u(x,t)$, $(x,t) \in \overline{G}^{\,1}$ as the sum of the form (3.4.18)

$$u(x,t) = U^{\,1}(x,t) + W^{\,1}(x,t), \quad (x,t) \in \overline{G}^{\,1},$$

where $U^{\,1}(x,t)$ and $W^{\,1}(x,t)$ are defined by the relations

$$U^{\,1}(x,t) = U(x,t) + \sum_{k=2}^{K-1} W_k(x,t), \quad (x,t) \in \overline{G}^{\,1}, \tag{3.4.23a}$$

$$W^{\,1}(x,t) = W_1(x,t), \quad (x,t) \in \overline{G}.$$

Here the functions $U(x,t)$, $W_k(x,t)$ correspond to the components in representations (3.4.20a), (3.4.20b).

The function $W^{\,1}(x,t)$ in (3.4.18) is defined by the relation

$$W_1(x,t) = W^1_{(3.4.18)}(x,t) = \widetilde{W}^1_{(3.4.16a)}\big(\xi(x,t),\, t\big)\, \exp(-\alpha\, t) = \tag{3.4.23b}$$

$$= W^1_{(3.4.23\mathrm{b})}(x,t) \equiv$$

$$\equiv 2^{-1}\left[\frac{\partial}{\partial x}\,\varphi(0,\,0)\right]\,\Big\{\big(x-\gamma(t)\big)\,v\Big(2^{-1}\,\varepsilon^{-1/2}\,a^{-1/2}\,q^{1/2}\,\big(x-\gamma(t)\big)t^{-1/2}\Big) +$$

$$+2\,\pi^{-1/2}\,\varepsilon^{1/2}\,a^{1/2}\,q^{-1/2}\,t^{1/2}\,\exp\Big(-4^{-1}\,\varepsilon^{-1}\,a^{-1}\,q\,\big(x-\gamma(t)\big)^2\,t^{-1}\Big)\Big\}\,\exp(-\alpha\, t),$$

$$(x,t) \in I\!R \times [0,\, T], \quad \alpha = \alpha_{(3.4.14)}.$$

Taking into account estimate (3.4.22) and the explicit form of the functions $\widetilde{W}_k(\xi,t)$,

$k = 1, \ldots, K-1$, we find the following estimates on the components in representation (3.4.18) written in the variables ξ, t:

$$\left| \frac{\partial^{k+k_0}}{\partial \xi^k \partial t^{k_0}} \widetilde{U}^1(\xi, t) \right| \le M \left[1 + \varepsilon^{(2-k)/2} t^{(2-k-2k_0)/2} \right], \quad (\xi, t) \in \overline{\widetilde{G}}^1,$$

$$\left| \frac{\partial^{k+k_0}}{\partial \xi^k \partial t^{k_0}} \widetilde{W}^1(\xi, t) \right| \le M \left[1 + \varepsilon^{(1-k)/2} t^{(1-k-2k_0)/2} \right] \exp(-m\, \varepsilon^{-1/2} |\xi|),$$

$$(\xi, t) \in \overline{\widetilde{G}}^*, \quad k + 2k_0 \le K.$$

By virtue of these estimates, we come to the estimates (3.4.19).

The lemma is proved.

3.4.5 Theorem of Estimates on the Solution of the Boundary Value Problem

The following theorem is a corollary of Lemmas 3.4.1–3.4.3.

Theorem 3.4.1 *Let conditions* (3.4.1), (3.4.2) *be satisfied. Then the solution of the boundary value problem and its components in representations* (3.4.11), (3.4.18) *satisfy the estimates* (3.4.4), (3.4.13), *and* (3.4.19).

Remark 3.4.1 Let us give the condition under which the boundary layer does not arise.

Let us define the set

$$G^4 = G^4(m) = \big\{ (x, t);\ x > \gamma(t) - \gamma(T) - d + m \big\}, \quad \overline{G}^4 = G^4 \bigcup S^4, \quad (3.4.24a)$$

where $m < d + \gamma(T)$. Introduce the set

$$\overline{G}^5 = G^5 \bigcup S^5, \quad G^5 = G^5(m) = G \setminus \overline{G}^4(m). \quad (3.4.24b)$$

Let the functions $f(x,t)$ and $\varphi(x,t)$ satisfy the conditions

$$\begin{aligned} f(x,t) &= 0, \quad (x,\,t) \in \overline{G}^{\,5}, \\ \varphi(x,t) &= 0, \quad (x,\,t) \in\, S \bigcap \overline{G}^{\,5}. \end{aligned} \tag{3.4.25}$$

Then the boundary layer is absent, i.e., the singular component is absent in the representation (3.4.11):

$$V(x,t) = 0, \quad (x,\,t) \in \overline{G}^{\,2}, \tag{3.4.26}$$

and $u(x,t) = U(x,t)$ on the set $\overline{G}^{\,2}$. For the solution $u(x,t)$, the estimate (3.4.13a) takes place, moreover,

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u(x,t) \right| \le\, M\,\varepsilon^{K_1}, \quad (x,t) \in \overline{G}^{\,2}, \quad k + 2k_0 \le K, \tag{3.4.27}$$

where $\overline{G}^{\,2} = \overline{G}^{\,2}_{(3.4.3)}(m^2)$, the constant m^2 can be chosen sufficiently small, $K = K_{(3.4.1)}$, and the constant K_1 in (3.4.27) can be chosen sufficiently large.

Emphasize that condition (3.4.25) is only sufficient for the relation (3.4.26). The boundary layer in representation (3.4.11) can be absent also in that case when the condition (3.4.25) is violated.

Remark 3.4.2 Consider the approximation of the solution of problem (3.2.2), (3.2.1) on the set $\overline{G}^{\,1}$ for small values of the parameter ε. The function $\varphi(x,t)$ at $t = 0$ can be approximated by a sufficiently smooth function $\varphi^{\lambda}(x,t)$ satisfying the condition

$$\begin{aligned} &\left|\varphi(x,t) - \varphi^{\lambda}(x,t)\right| \le M\,\lambda, \\ &\left| \frac{\partial^k}{\partial x^k}\, \varphi^{\lambda}(x,t) \right| \le M\, \left[1 + \lambda^{1-k}\right], \quad (x,t) \in S_0, \quad k \le K, \end{aligned}$$

moreover, $\varphi^{\lambda}(x,t) = \varphi(x,t)$ for $\left|x\right| \ge m\,\lambda$. Here λ is a sufficiently small parameter that determines the proximity of the functions $\varphi(x,t)$ and $\varphi^{\lambda}(x,t)$. Let $u^{\lambda}(x,t)$, $(x,t) \in \overline{G}$ is a solution of problem (3.2.2), (3.2.1) with $\varphi(x,t)$ equal to $\varphi^{\lambda}(x,t)$.

Denote $u_0^\lambda(x,t)$ a solution of the problem for the reduced equation from (3.2.2):

$$\begin{aligned} L_{(3.4.28)}\, u_0^\lambda(x,t) &\equiv \left\{ b\,\frac{\partial}{\partial x} - c - p\,\frac{\partial}{\partial t} \right\} u_0^\lambda(x,t) = f(x,t), \qquad (3.4.28)\\ &\qquad (x,t) \in \overline{G} \setminus \{S_0 \bigcup S^r\},\\ u_0^\lambda(x,t) &= \varphi^\lambda(x,t), \quad (x,t) \in S_0 \bigcup S^r. \end{aligned}$$

Under the assumptions of Theorem 3.4.1, we have the following estimates for the functions $u^\lambda(x,t)$ and $u_0^\lambda(x,t)$:

$$\begin{aligned} &\left| u(x,t) - u^\lambda(x,t) \right| \le M\,\lambda, \qquad (x,t) \in \overline{G}, \qquad (3.4.29)\\ &\left| u^\lambda(x,t) - u_0^\lambda(x,t) \right| \le M\,\lambda^{-1}\,\varepsilon, \quad (x,t) \in \overline{G}^{\,1},\\ &\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u_0^\lambda(x,t) \right| \le M\left[1 + \lambda^{1-k-k_0}\right], \quad (x,t) \in \overline{G}, \quad k + 2k_0 \le K, \end{aligned}$$

where $K = K_{(3.4.1)}$.

3.5 Classical Grid Approximations of the Problem on Uniform and Piecewise Uniform Meshes

In this section, we construct a scheme that makes it possible to approximate the solution of boundary value problem (3.2.2), (3.2.1) ε-uniformly. When the convergence of the schemes is investigated, we apply the technique developed for monotone schemes in the case of regular and singularly perturbed boundary value problems (see, e.g., [57, 66, 68] and the bibliography therein).

3.5.1 Difference Scheme Based on Classical Approximation

On the set $\overline{G}_{(3.2.1)}$, define the rectangular mesh

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega} \times \overline{\omega}_0, \tag{3.5.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are meshes on the segments $[-d,\, d]$ and $[0,\, T]$, respectively; the mesh $\overline{\omega}$ has an arbitrary distribution of nodes satisfying only the condition $h \le MN^{-1}$, where $h = \max_i h^i$, $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$; the mesh $\overline{\omega}_0$ is uniform with the step-size $h_0 = TN_0^{-1}$. Here $N+1$ and N_0+1 are the numbers of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively.

We approximate the boundary value problem (3.2.2) by the difference scheme

$$\begin{aligned} \Lambda_{(3.5.2)}\, z(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{3.5.2}$$

Here

$$\Lambda_{(3.5.2)} \equiv \varepsilon\, a\, \delta_{\overline{x}\widehat{x}} + b\, \delta_x - c - q\, \delta_{\overline{t}},$$

$$\delta_{\overline{x}\widehat{x}} z(x,t) = z_{\overline{x}\widehat{x}}(x,t) = 2(h^i + h^{i-1})^{-1}[\delta_x z(x,t) - \delta_{\overline{x}} z(x,t)], \quad (x,t) = (x^i,t) \in G_h,$$

is the second difference derivative on a nonuniform mesh, $\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the first (forward and backward) difference derivatives,

$$\delta_x z(x,t) = \left(h^i\right)^{-1} \left(z(x^{i+1},t) - z(x^i,t)\right),$$

$$\delta_{\overline{x}} z(x,t) = \left(h^{i-1}\right)^{-1} \left(z(x^i,t) - z(x^{i-1},t)\right),$$

$$\delta_{\overline{t}}\, z(x,t) = h_0^{-1} \left(z(x^i,t) - z(x^i,t-h_0)\right).$$

Difference scheme (3.5.2), (3.5.1) is ε–uniformly monotone, and it satisfies the discrete maximum principle (see [57, 68]). Due to this, to justify the convergency of discrete solutions, we can apply the majorant functions technique [68].

The following version of the comparison theorem [68] holds.

Theorem 3.5.1 *Let the functions $z^1(x,t)$, $z^2(x,t)$, $(x,t) \in \overline{G}_h$ satisfy the conditions*

$$\Lambda\, z^1(x,t) < \Lambda\, z^2(x,t), \quad (x,t) \in G_h, \quad z^1(x,t) > z^2(x,t), \quad (x,t) \in S_h.$$

Then $z^1(x,t) > z^2(x,t)$, $(x,t) \in \overline{G}_h$.

Now we show results on the convergence of difference scheme (3.5.2), (3.5.1). Under the investigation of the scheme, we assume that the following condition is fulfilled:

$$\begin{gathered}\textit{The solution of problem (3.2.2) and its components}\\ \textit{in representations (3.4.11), (3.4.18)}\\ \textit{satisfy the estimates (3.4.4), (3.4.13), (3.4.19) for } K \geq 2.\end{gathered} \tag{3.5.3}$$

Sufficient conditions for (3.5.3) are established in Theorem 3.4.1.

Lemma 3.5.1 *Let condition (3.5.3) be satisfied for $K = 4$. Then the solution of scheme (3.5.2) on the uniform grid*

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0 \tag{3.5.4}$$

converges for fixed values of the parameter ε

$$|u(x,t) - z(x,t)| \leq M\left[(\varepsilon + N^{-1})^{-1}\, N^{-1} + N^{-1/2} + N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h. \tag{3.5.5}$$

Under the additional condition (3.4.25) (the boundary layer is absent) the scheme (3.5.2), (3.5.4) converges ε-uniformly

$$|u(x,t) - z(x,t)| \leq M\left[N^{-1/2} + N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h. \tag{3.5.6}$$

Proof. Estimation of errors for solutions of the difference scheme is realized by the standard way [68]. With using a priori estimates of the boundary value problem, we estimate $\psi(x,t)$ where

$$\Lambda\, w_h(x,t) = (\Lambda - L)\, u(x,t) \equiv \psi(x,t), \quad (x,t) \in G_h;$$

the $\psi(x,t)$ is an error of the approximation for the operator L by the difference operator Λ on the solution of the boundary value problem,

$$w_h(x,t) = u(x,t) - z(x,t), \quad (x,t) \in \overline{G}_h,$$

is an error of the discrete solution.

Here we use a function $w(x,t)$ as the majorant function for the error $w_h(x,t)$; the function $w(x,t)$ on the set $\overline{G}_h$ is chosen satisfying the condition:

$$\Lambda w(x,t) \le -M\Bigg\{\varepsilon \min\left[\max_{x\in\overline{D}}\left|\frac{\partial^2}{\partial x^2}\,u(x,t)\right|,\ N^{-2}\max_{x\in\overline{D}}\left|\frac{\partial^4}{\partial x^4}\,u(x,t)\right|\right] +$$
$$+ \min\left[\max_{x\in\overline{D}}\left|\frac{\partial}{\partial x}\,u(x,t)\right|,\ N^{-1}\max_{x\in\overline{D}}\left|\frac{\partial^2}{\partial x^2}\,u(x,t)\right|\right] +$$
$$\min\left[\max_{x\in\overline{D}}\left|\frac{\partial}{\partial t}\,u(x,t)\right|,\ N_0^{-1}\max_{x\in\overline{D}}\left|\frac{\partial^2}{\partial t^2}\,u(x,t)\right|\right]\Bigg\}, \quad (x,t) \in G_h.$$

The function $w(x,t)$ is constructed with regard estimates for derivatives of the solution of the boundary value problem (3.2.2), (3.2.1)

$$w(x,t) = \sum_{j=1}^{4} w_j(x,t), \quad (x,t) \in \overline{G}_h.$$

Here

$$w_1(x,t) = M\left[N^{-1} + N_0^{-1}\right] t,$$

$$w_2(x,t) = M\,\varepsilon^{-1/2}\,N^{-1}\,t^{1/2},$$

$$w_3(x,t) = M\,(\varepsilon + N^{-1})^{-1}\,N^{-1}\,\exp\big(-m\,\varepsilon^{-1}(x+d)\big),$$

$$w_4(x,t) = \begin{cases} M_0\,\varepsilon^{1/2}\,t^{1/2}, & t \le t_0,\\ M_0\,\varepsilon^{1/2}\,t_0^{1/2} + M_1\,\varepsilon^{1/2}\,t_0\,(t_0^{-1/2} - t^{-1/2}), & t > t_0, \end{cases}$$

where $t_0 = (\varepsilon^{1/2} + N^{-1})^{-2} N^{-2} + N_0^{-1}$, $m = m_{(3.4.13)}$, the constants M, M_0, M_1 are chosen sufficiently large so that to satisfy assumptions of Theorem 3.5.1. Note that the functions $w_1(x,t)$ and $w_3(x,t)$ majorize errors related to the approximation of the components $U^1_{(3.4.18)}(x,t)$, $U_{(3.4.11)}(x,t)$, $u^3_{(3.4.4)}(x,t)$ and $V_{(3.4.11)}(x,t)$ respectively, and the functions $w_2(x,t)$ and $w_4(x,t)$ majorize errors related to the approximation of the derivatives, respectively, $\partial/\partial x$ and $\partial^2/\partial^2 x$, $\partial/\partial t$ of the component $W^1_{(3.4.18)}(x,t)$.

Using a priori estimates (3.4.4), (3.4.13), (3.4.19) for the solution of problem (3.2.2) and using the comparison theorem 3.5.1, we find the estimate

$$|u(x,t) - z(x,t)| \le \tag{3.5.7}$$

$$\le M\left[(\varepsilon + N^{-1})^{-1} N^{-1} + (\varepsilon^{1/2} + N^{-1})^{-1} N^{-1} + N_0^{-1} + \varepsilon^{1/2} N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h.$$

Thus, the difference scheme (3.5.2), (3.5.4) converges for fixed values of the parameter ε.

Taking into account the estimates (3.4.4), (3.4.19), (3.4.29) where $\lambda = (\varepsilon + N^{-1} + N_0^{-1})^{1/2}$, in the case of scheme (3.5.2), (3.5.4) we obtain the estimate

$$|u(x,t) - z(x,t)| \le M\left[(\varepsilon + N^{-1})^{-1} N^{-1} + \varepsilon^{1/2} + N^{-1/2} + N_0^{-1/2}\right], \tag{3.5.8}$$

$$(x,t) \in \overline{G}_h.$$

From this estimate and estimate(3.5.7) it follows that estimate (3.5.5) holds.

Under condition (3.4.25), we have (3.4.26). In this case, using the estimates (3.4.4), (3.4.13), (3.4.19), we obtain the estimates that are similar to those (3.5.7), (3.5.8):

$$|u(x,t) - z(x,t)| \le M\left[(\varepsilon^{1/2} + N^{-1})^{-1} N^{-1} + N_0^{-1} + \varepsilon^{1/2} N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h,$$

$$|u(x,t) - z(x,t)| \le M\left[\varepsilon^{1/2} + N^{-1/2} + N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h.$$

From these estimates it follows that estimate (3.5.6) holds.

The lemma is proved.

Lemma 3.5.2 *Let condition* (3.5.3) *be satisfied for* $K = 4$. *Then under the condition*

$$\left[\frac{\partial}{\partial x}\,\varphi(x,t)\right] = 0, \qquad (x,t) \in S^{(*)}, \tag{3.5.9}$$

for the solution of difference scheme (3.5.2), (3.5.4)*, the following estimate holds:*

$$|u(x,t) - z(x,t)| \le M\left[(\varepsilon + N^{-1})^{-1}\, N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h. \tag{3.5.10}$$

Under the additional condition (3.4.25)*, we have the estimate*

$$|u(x,t) - z(x,t)| \le M\left[N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h. \tag{3.5.11}$$

Proof. In the case of condition (3.5.9), we have

$$W^1(x,t) = 0, \quad (x,t) \in \overline{G}^{\,1}, \tag{3.5.12}$$

and under the additional condition (3.4.25), we have (3.4.26).

Taking into account a priori estimates of the problem solution and its components in representations (3.4.11), (3.4.18) (see estimates (3.4.19) and (3.4.13)), and using the comparison theorem 3.5.1, we obtain the estimates (3.5.10) and (3.5.11) in the case of conditions (3.5.17) and (3.5.12) respectively, that completes the proof.

The following theorem results from Lemmas 3.5.1 and 3.5.2.

Theorem 3.5.2 *Let condition* (3.5.3) *be satisfied for* $K = 4$. *Then the solution of the scheme* (3.5.2), (3.5.4) *satisfies the estimate* (3.5.5)*, and in the case of conditions* (3.5.9), (3.4.25) *it satisfies the estimates* (3.5.10), (3.5.6) *and* (3.5.11) *respectively.*

3.5.2 Solution of the Problem with Boundary Layer

We now consider the case when the solution of the problem has a boundary layer. On the set $\overline{G}$, we construct the mesh condensing in a neighbourhood of the boundary layer (similar to that constructed in [20, 57, 72, 78]):

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega}^* \times \overline{\omega}_0, \tag{3.5.13a}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(3.5.1)}$, $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a piecewise uniform mesh on $[-d,\, d]$, and σ is a parameter depending on ε and N. We choose the value σ satisfying the condition

$$\sigma = \sigma(\varepsilon,\, N) = \min\,[\,\beta,\, 2\,m^{-1}\,\varepsilon\,\ln N\,], \tag{3.5.13b}$$

where β is an arbitrary number in the half-open interval $(0,\, d]$ and $m = m_{(3.4.13)}$. The segment $[-d,\, d]$ is divided into two parts: $[-d,\, -d+\sigma]$ and $[-d+\sigma,\, d]$; on each part, the step-size is constant and is equal to $h^{(1)} = 2\,d\,\sigma\,\beta^{-1}\,N^{-1}$ on the segment $[-d, -d+\sigma]$ and to $h^{(2)} = 2\,d\,(2\,d-\sigma)\,(2\,d-\beta)^{-1}\,N^{-1}$ on the segment $[-d+\sigma, d]$, $\sigma \le d$. The piecewise uniform mesh is constructed.

The following theorem holds.

Theorem 3.5.3 *Let condition* (3.5.3) *be satisfied for* $K = 4$*. Then the solution of the scheme* (3.5.2), (3.5.13) *converges* ε*-uniformly; the discrete solution satisfies the estimate*

$$|u(x,t) - z(x,t)| \le M\,\left[\,N^{-1/2} + N_0^{-1/2}\,\right], \quad (x,t) \in \overline{G}_h. \tag{3.5.14}$$

Under the additional condition (3.5.9), the following estimate holds:

$$|u(x,t) - z(x,t)| \le M\,\left[\,N^{-1}\,\ln N + N_0^{-1}\,\right], \quad (x,t) \in \overline{G}_h. \tag{3.5.15}$$

The proof of this theorem is similar to the proof of Theorem 3.5.2. The estimate (3.5.14) is found similar to the derivation of estimate (3.5.6). In the case of condition (3.5.9), we have $W^1_{(3.4.18)}(x,t) = 0$. The estimate of the error generated by the components $U^1_{(3.4.18)}(x,t)$, $U_{(3.4.11)}(x,t)$, $u^3_{(3.4.4)}(x,t)$ is performed similar to estimate (3.5.11). To estimate the error generated by the component $V_{(3.4.11)}(x,t)$, the technique from [20, 57] is used.

3.5.3 Solution of the Problem without Interior and Boundary Layers

Consider the boundary value problem in that case when the conditions (3.4.26), (3.5.9) are satisfied (i.e., the interior and boundary layers are absent).

Theorem 3.5.4 *Let condition* (3.5.3) *be satisfied for $K = 6$, and also the conditions* (3.4.26), (3.5.9)*. Then in the case of the difference scheme* (3.5.2) *on the uniform mesh* (3.5.4)*, the following estimates are satisfied for the flux and for the discrete derivative:*

$$| P(x,t) - P^h(x,t) | \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}_h, \quad x \ne d; \tag{3.5.16a}$$

$$| p(x,t) - p^h(x,t) | \le \tag{3.5.16b}$$

$$\le \; M\,[\,N^{-1/2} + N_0^{-1/2}\,]\,[(\varepsilon + N^{-1})^{-1}(1 + 2\,a^{-1}\,b\,d\,\varepsilon^{-1}\,N^{-1})^{-i} + 1],$$

$$(x,t) \in \overline{G}_h, \quad x = x_i \ne d;$$

$$| p(x,t) - p^h(x,t) | \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \tag{3.5.17}$$

$$(x,t) \in \overline{G}_h, \quad x \in [-d + \beta_0,\, d).$$

Here $\beta_0 > 0$ is an arbitrary sufficiently small constant, and $M_{(3.5.17)} = M(\beta_0)$, $x_i = -d + i\,h$, $i \ge 0$, $h = 2\,d\,N^{-1}$.

Proof. On the uniform mesh (3.5.4), the discrete derivative $p^h(x,t)$ is the solution of the equation

$$\Lambda_{(3.5.2)}\, p^h(x,t) = \delta_x\, f(x,t), \quad (x,t) \in G_h, \quad x \le d - 2\,h.$$

On the set S_{0h}, it holds that

$$p^h(x,t) = \delta_x\, \varphi(x,t), \quad (x,t) \in S_{0h}, \quad x \ne d;$$

moreover, by virtue of (3.5.9), we have

$$\left| \frac{\partial}{\partial x}\, \varphi(x,t) - \delta_x\, \varphi(x,t) \right| \le M\, N^{-1}, \quad (x,t) \in S_{0h}, \quad x \ne d.$$

In the case of the additional condition (3.4.26), and taking into account estimate (3.5.11), we find estimate for $p^h(x,t)$ on the lateral boundary of the grid domain:

$$|p(x,t) - p^h(x,t)| \le \begin{cases} M\, [N^{-1} + N_0^{-1}], & (x,t) \in \overline{G}_h, \quad x = d - h, \\ M\, (\varepsilon + N^{-1})^{-1}\, [N^{-1} + N_0^{-1}], & (x,t) \in S_h^l. \end{cases} \tag{3.5.18}$$

Thus, the difference scheme

$$\Lambda_{(3.5.2)}\, p^h(x,t) = \delta_x\, f(x,t), \quad (x,t) \in G_h, \quad x \le d - 2\,h, \tag{3.5.19}$$

$$p^h(x,t) = \begin{cases} \delta_x\, z(x,t), & (x,t) \in \overline{G}_h \bigcap S^l, \\ \delta_x\, z(x,t), & (x,t) \in G_h, \quad x = d - h, \\ \delta_x\, \varphi(x,t), & (x,t) \in S_{0h}, \quad x \ne d, \end{cases}$$

where $z(x,t)$, $(x,t) \in \overline{G}_h$ is the solution of difference scheme (3.5.2), (3.5.4), in the case of conditions (3.4.26), (3.5.9), approximates the boundary value problem

$$L_{(3.2.2)}\, p(x,t) = \frac{\partial}{\partial x}\, f(x,t), \quad (x,t) \in G, \tag{3.5.20}$$

$$p(x,t) = \begin{cases} \dfrac{\partial}{\partial x}\, u(x,t), & (x,t) \in S^l, \\ \dfrac{\partial}{\partial x}\, u(x,t), & (x,t) \in G, \quad x = d - h, \\ \dfrac{\partial}{\partial x}\, \varphi(x,t), & (x,t) \in S_0, \quad x \le d - h. \end{cases}$$

The estimate (3.5.16) is found similar to the derivation of estimate (3.5.6).

Let

$$w_{ph}(x,t) = p(x,t) - p^h(x,t), \quad (x,t) \in \overline{G}_h, \quad x \neq d$$

be the error of the grid derivative. Represent the function $w_{ph}(x,t)$ as the sum of the functions

$$w_{ph}(x,t) = w_{ph}^R(x,t) + w_{ph}^S(x,t),$$

where $w_{ph}^R(x,t)$ and $w_{ph}^S(x,t)$ are the regular and singular parts of the error of the derivative. The function $w_{ph}^R(x,t)$ corresponds to the boundary value problem (3.5.20) and such difference scheme (3.5.19) for which the following condition is fulfilled on the boundary S^l:

$$p^h(x,t) = p(x,t), \quad (x,t) \in S_h^l,$$

that corresponds to the condition

$$w_{ph}^R(x,t) = 0, \quad (x,t) \in S_h^l.$$

The function $w_{ph}^S(x,t)$ corresponds to the boundary value problem and the difference scheme of the form (3.5.19) and (3.5.20) with homogeneous equations and homogeneous boundary conditions on the lower and right parts of the boundary.

For the function $w_{ph}^R(x,t)$, we obtain the estimate which is similar to estimate (3.5.6):

$$|w_{ph}^R(x,t)| \leq M \left[N^{-1/2} + N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h, \quad x \neq d. \tag{3.5.21}$$

For the function $w_{ph}^S(x,t)$, taking into account estimate (3.5.18) on S_h^l, and using the grid majorant function

$$w(x,t) = M \left[N^{-1/2} + N_0^{-1/2}\right] (\varepsilon + N^{-1})^{-1} (1 + 2\,a^{-1}\,b\,d\,\varepsilon^{-1}\,N^{-1})^{-i}, \tag{3.5.22a}$$

$$(x,t) \in \overline{G}_h, \quad x = i\,h,$$

we obtain the estimate

$$|w^S_{ph}(x,t)| \le w(x,t), \quad (x,t) \in \overline{G}_h, \quad x \ne d. \tag{3.5.22b}$$

The estimate (3.5.16b) follows from estimates (3.5.21) and (3.5.22).

Note that

$$w_{(3.5.22a)}(x,t) \le M, \quad (x,t) \in \overline{G}_h, \quad x \ge -d + \beta_0.$$

From this inequality and estimate (3.5.16b), it follows that the estimate (3.5.17) holds.

The theorem is proved.

Remark 3.5.1 Let the assumptions of Theorem 3.5.4 be fulfilled, where the condition (3.4.26) is changed to (3.4.25). In this case, the solution of the boundary value problem satisfies estimate (3.4.27). The solution of the difference scheme (3.5.2), (3.5.4) satisfies the estimate

$$\mid z(x,t) \mid \le M\,(\varepsilon + N^{-1})^{K_1}, (x,t) \in \overline{G}^{\,2}_h,$$

where $\overline{G}^{\,2}_h = \overline{G}^{\,2} \bigcap \overline{G}_h$, K_1 is an arbitrary constant, $M = M(K_1)$. Under the condition (3.4.25), for the singular component $w^S_{ph}(x,t)$ we obtain the estimate

$$|w^S_{ph}(x,t)| \le M\left[N^{-1/2} + N_0^{-1/2}\right](\varepsilon + N^{-1})^{K_2}, \quad (x,t) \in \overline{G}_h, \quad x \ne d,$$

where the constant K_2 can be chosen sufficiently large. Similar to the derivation of estimate (3.5.16b), we obtain the following estimate for the discrete derivative $p(x,\, t)$:

$$\Big| p(x,\, t) - p^h(x,\, t) \Big| \le\; M\left[N^{-1/2} + N_0^{-1/2}\right], \quad (x,\, t) \in \overline{G}_h, \quad x \ne\; d, \tag{3.5.23}$$

which is stronger than (3.5.17). Thus, in the case of scheme (3.5.2), (3.5.4), the conditions (3.4.25) and (3.5.9)are sufficient for the ε-uniform convergence of the derivative $p^h(x,\, t)$ on the whole set $\overline{G}_h$, $x \ne\; d$.

3.5.4 Approximation of the Solution and Derivatives

We consider the approximation of the functions $u(x,t)$, $p(x,t)$, $P(x,t)$, $(x,t) \in \overline{G}$, using the interpolants constructed on the basis of the functions $z(x,t)$, $p^h(x,t)$, $P^h(x,t)$.

Let $z(x,t)$, $(x,t) \in \overline{G}_h$, be a solution of some scheme. For the function $z(x,t)$, we construct its extension $\overline{z}(x,t)$ to $\overline{G}$; $\overline{z}(x,t)$ is a bilinear interpolant on the elementary rectangles generated by the lines that pass through the nodes of the mesh $\overline{G}_h$ in parallel to the coordinate axes. Further, we construct the interpolant $\overline{p}^h(x,t)$, $(x,t) \in \overline{G}$, for the discrete derivative $p^h(x,t)$, $(x,t) \in \overline{G}_h$, $x \neq d$. At the interior points of the elementary rectangles, we assume $\overline{p}^h(x,t) = \overline{p}^h_z(x,t) = (\partial/\partial x)\,\overline{z}(x,t)$; the function $\overline{p}^h(x,t)$ is continuous on the upper and on the lower sides of the rectangles, and it is defined according to continuity on the left sides of the elementary rectangles. But if the rectangles are adjacent, by their right sides, to the set $\overline{S}^{\,r}$ $\big($where $S^{\,r}$ is the right side of the boundary $S^{\,L}$, $S^{\,L} = S^{\,l} \bigcup S^{\,r}\big)$, then we also define according to continuity the function $\overline{p}^h(x,t)$ on these sides. Hence, we have constructed the function $\overline{p}^h(x,t)$, $(x,t) \in \overline{G}$. The interpolant $\overline{p}^h(x,t)$, in general, has discontinuities on the lines that are parallel to the t-axis and pass through the nodes of the mesh G_h. We define the interpolant of the diffusion flux by the relation

$$\overline{P}^h(x,t) = \overline{P}^h_z(x,t) = \varepsilon\,\overline{p}^h(x,t), \quad (x,t) \in \overline{G}.$$

Definition. In that case when the interpolants constructed on the basis of the solution of the difference scheme approximate ε-uniformly the solution of the differential problem, its diffusion flux and the derivative with respect to x on the set $\overline{G}$, and also on the set $\overline{G}$ outside a β_0-neighbourhood of the set S^l, we say that the difference scheme approximates the solution of the differential problem, its diffusion flux and the derivative with respect to x respectively on the set $\overline{G}$, and also on the set $\overline{G}$ outside the β_0-neighbourhood of the set S^l ε-uniformly. We also briefly say that the difference scheme approximates the solution, the derivative and the diffusion flux ε-uniformly.

Theorem 3.5.5 *Let the assumptions of Theorem* 3.5.4 *be fulfilled. Then, the difference scheme* (3.5.2), (3.5.4) *approximates the solution of problem* (3.2.2), (3.2.1), *its derivative and the diffusion flux ε-uniformly with the estimates*

$$| u(x,t) - \overline{z}(x,t) | \le M\,[\,N^{-1} + N_0^{-1}\,], \quad (x,t) \in \overline{G}, \tag{3.5.24a}$$

$$| p(x,t) - \overline{p}^{\,h}(x,t) | \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \tag{3.5.24b}$$

$$(x,t) \in \overline{G}, \quad x \ge -d + \beta_0;$$

$$| P(x,t) - \overline{P}^{\,h}(x,t) | \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}, \tag{3.5.24c}$$

where $\beta_0 = \beta_{0(3.5.17)}$, $M_{(3.5.24b)} = M(\beta_0)$.

Under the assumptions of Theorem 3.5.5, the grid solution, its derivative and the diffusion flux satisfy the estimates (3.5.11), (3.5.16), (3.5.17). The statement of this theorem follows from these estimates and a priori estimates of the boundary value problem (see estimates of Theorem 3.4.1 that correspond to the assumptions of Theorem 3.5.5). Here we used the triangle inequality; for example,

$$\max_{\overline{G}} |u(x,t) - \overline{z}(x,t)| \le \max_{\overline{G}} |u(x,t) - \overline{u}_h(x,t)| + \max_{\overline{G}_h} |\overline{u}_h(x,t) - z(x,t)|,$$

where $u_h(x,t) = u(x,t)$, $(x,t) \in \overline{G}_h$, $\overline{u}_h(x,t)$, $(x,t) \in \overline{G}$ is an interpolant that is constructed using $u_h(x,t) = u(x,t)$.

Remark 3.5.2 Let the assumptions of Theorem 3.5.4 be fulfilled, where the condition (3.4.26) is changed to (3.4.25). Then, in the case of scheme (3.5.2), (3.5.4), the interpolant $\overline{p}^{\,h}$ satisfies the estimate

$$| p(x,t) - \overline{p}^{\,h}(x,t) | \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}.$$

3.6 Decomposition Scheme for the Solution and Derivatives

To construct a difference scheme that approximates the first order derivative $p(x,t) = (\partial/\partial x)u(x,t)$ on the set $\overline{G}^*$, we use the method of the additive splitting of a singularity such as the interior layer function [78] (or briefly, the singularity splitting method).

3.6.1 Construction of the Singularity Splitting Method

We represent the solution of problem (3.2.2), (3.2.1) as the sum of functions

$$u(x,t) = u_1(x,t) + u_2(x,t), \quad (x,\, t) \in \overline{G}. \tag{3.6.1a}$$

Here $u_1(x,t)$ and $u_2(x,t)$ are components of the solution of boundary value problem (3.2.2), (3.2.1), including singularities of the boundary and interior layers types, respectively. We call the functions $u_1(x,t)$ and $u_2(x,t)$ the components containing the boundary and interior layers, respectively. We represent the function $u_2(x,t)$ as the sum of functions

$$u_2(x,t) = u_2^1(x,t) + u_2^2(x,t), \quad (x,\, t) \in \overline{G}, \tag{3.6.1b}$$

where $u_2^1(x,t)$ and $u_2^2(x,t)$ are the regular and singular parts of the component $u_2(x,t)$, containing the interior layer;

$$u_2^2(x,t) = W^1_{(3.4.23b)}(x,t), \quad (x,\, t) \in \overline{G}.$$

The functions $u_1(x,t)$, $u_2^1(x,t)$ are solutions of the following problems

$$\begin{aligned} L_{(3.2.2a)}\, u_2^1(x,t) &= f_2(x,t), \quad (x,\, t) \in G, \\ u_2^1(x,t) &= \varphi_2(x,t), \quad (x,\, t) \in S; \end{aligned} \tag{3.6.1c}$$

$$\begin{aligned} L_{(3.2.2a)}\, u_1(x,t) &= f_1(x,t), \quad (x,\, t) \in G, \\ u_1(x,t) &= \varphi_1(x,t), \quad (x,\, t) \in S. \end{aligned} \tag{3.6.1d}$$

The functions $f_i(x,t)$, $\varphi_i(x,t)$, $i = 1,\, 2$ are defined by the relations

$$\begin{aligned} & f_2(x,t) = f(x,t)\, \eta\,(x,t), \\ & f_1(x,t) = f(x,t) - f_2(x,t), \quad (x,t) \in \overline{G}; \\ & \varphi_2(x,t) = (\varphi(x,t) - u_2^2(x,t))\, \eta\,(x,t), \\ & \varphi_1(x,t) = \varphi(x,t) - \varphi_2(x,t) - u_2^2(x,t), \quad (x,t) \in S. \end{aligned} \tag{3.6.1e}$$

Here $\eta\,(x,t)$, $(x,t) \in \overline{G}$ is a sufficiently smooth function, that vanishes in a neighbourhood of the boundary layer

$$\left. \begin{aligned} & \eta\,(x,t) = 0, \quad (x,t) \in \overline{G}^{\,5}_{(3.4.24)}(2^{-1}\, m_1) \\ & \eta\,(x,t) = 1, \quad (x,t) \in \overline{G}^{\,4}_{(3.4.24)}(m_1) \end{aligned} \right\}, \quad 0 \le \eta(x,t) \le 1, \quad (x,t) \in \overline{G},$$

where m_1 is an arbitrary number in the interval $\Big(0,\, 2^{-1}\left(d + \gamma(T)\right)\Big)$.

Remark 3.6.1 The data of problem (3.6.1c) on the set $\overline{G}^{\,5}$ satisfy the condition similar to (3.4.25) $\left(f_2(x,t) = 0,\ \varphi_2(x,t) = 0,\ (x,t) \in \overline{G}^{\,5}\right)$, moreover, for $t = 0$, the first order derivative in x of the function $\varphi_2(x,t)$ is continuous. For the singular components of the solution to problem (3.6.1c) in representations similar to (3.4.11), (3.4.18), conditions of the type (3.4.26), (3.5.12) are satisfied. For this problem, a difference scheme, similar to (3.5.2), on the uniform mesh (3.5.4) ensures the ε-uniform convergence in $C^1(\overline{G})$.

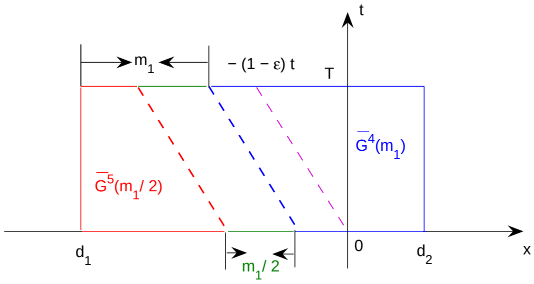


Figure 3.1: Illustration of sets $\overline{G}^{5}_{(3.4.24)}(2^{-1}\, m_1)$ and $\overline{G}^{4}_{(3.4.24)}(m_1)$.

An illustration of sets $\overline{G}^{5}_{(3.4.24)}(2^{-1}\, m_1)$ and $\overline{G}^{4}_{(3.4.24)}(m_1)$ is shown on Fig 3.1.

The data of problem (3.6.1d) are sufficiently smooth, moreover, the functions $f_1(x,t)$ and $\varphi_1(x,t)$ vanish on the set $\overline{G}^{4}(m_1)$.

For this problem, a difference scheme, similar to (3.5.2), on the piecewise uniform mesh (3.5.13) gives the ε-uniform convergence in $C^{1(n)}(\overline{G})$.

To solve problem (3.6.1d), we use the difference scheme

$$\begin{aligned} \Lambda_{(3.5.2)} z_1(x,\, t) &= f_1(x,t), \quad (x,t) \in G_h, \\ z_1(x,t) &= \varphi_1(x,t), \quad (x,t) \in S_h, \end{aligned} \tag{3.6.2a}$$

where $\overline{G}_h$ is the piecewise uniform mesh (3.5.13).

To solve problem (3.6.1c), we use the difference scheme

$$\begin{aligned} \Lambda_{(3.5.2)} z_2^1(x,\, t) &= f_2(x,t), \quad (x,t) \in G_h, \\ z_2^1(x,t) &= \varphi_2(x,t), \quad (x,t) \in S_h, \end{aligned} \tag{3.6.2b}$$

where $\overline{G}_h$ is the uniform mesh (3.5.4).

Further, we construct the special interpolants into which the singular component, i.e., the function of the interior layer type, enters in the explicit form

$$u_0^h(x,t) = \overline{z}_1(x,t) + u_2^h(x,t), \tag{3.6.2c}$$

$$u_2^h(x,t) = \overline{z}_2^1(x,t) + u_2^2(x,t), \quad (x,t) \in \overline{G};$$

$$p_0^h(x,t) = \overline{p}_{z_1}^h(x,t) + p_2^h(x,t), \tag{3.6.2d}$$

$$p_2^h(x,t) = \overline{p}_{z_2^1}^h(x,t) + \frac{\partial}{\partial x}\, u_2^2(x,t), \quad (x,t) \in \overline{G}^*;$$

$$P_0^h(x,t) = \varepsilon\, p_0^h(x,t), \quad (x,t) \in \overline{G}^*, \tag{3.6.2e}$$

where $\overline{z}_1(x,t)$, $\overline{p}_{z_1}^h(x,t)$ and $\overline{z}_2(x,t)$, $\overline{p}_{z_2}^h(x,t)$ are bilinear interpolants that are constructed using the functions $z_1(x,t)$, $(x,t) \in \overline{G}_{h(3.5.13)}$ and $z_2(x,t)$, $(x,t) \in \overline{G}_{h(3.5.4)}$ $\big($similarly to the construction of interpolants in Subsection 5.4$\big)$. The use of the interpolants allows us to find the solution on the set $\overline{G}$, its first derivative in x and the diffusion flux on the set $\overline{G}^*$.

The function $u_0^h(x,t)$, $(x,t) \in \overline{G}$, is called the solution of the difference scheme $\{(3.6.2), (3.5.4), (3.5.13)\}$, and the functions $p_0^h(x,t)$ and $P_0^h(x,t)$, $(x,t) \in \overline{G}^*$, are called the derivative and the diffusion flux, respectively, corresponding to this scheme. The scheme $\{(3.6.2), (3.5.4), (3.5.13)\}$ is the scheme of the decomposition method for the solution in the case of the additive splitting of a singularity of the interior-layer type $\big($briefly, we call this scheme by the scheme of the singularity splitting method$\big)$.

If the condition (3.4.25) or the following (stronger) condition are fulfilled:

$$f(x,t) = 0, \quad (x,t) \in \overline{G}, \tag{3.6.3}$$

$$\varphi(x,t) = 0, \quad (x,t) \in S, \quad x < 0,$$

then the scheme is simplified if we take

$$u_2^2(x,t) = W^1_{(3.4.23\text{b})}(x,t) + 2^{-1}\left[\frac{\partial}{\partial x}\,\varphi(0,\,0)\right]\big(x-\gamma(t)\big)\,\exp\big(-\alpha\,t\big), \quad (3.6.4)$$

$$(x,t)\in\overline{G}, \quad \alpha=\alpha_{(3.4.14)}, \quad \gamma(t)=\gamma_{(3.4.14)}(t);$$

$u_2^2(x,t) = 0$ for $x < 0$, $t = 0$. In this case, the component $u_1(x,t)$ that contains the boundary layer is absent in the representation (3.6.1a). Then the solution of problem (3.2.2), (3.2.1) takes the form

$$u(x,t) = u_2(x,t) = u_2^1(x,t) + u^2_{2(3.6.4)}(x,t), \quad (x,t)\in\overline{G}, \qquad (3.6.5\text{a})$$

where $u_2^1(x,t)$ is the solution of the problem

$$\begin{aligned} L_{(3.2.2\text{a})}\,u_2^1(x,t) &= 0, && (x,t)\in G \qquad (3.6.5\text{b})\\ u_2^1(x,t) &= \varphi_2(x,t), && (x,t)\in S; \end{aligned}$$

Here $\varphi_2(x,t) = \varphi(x,t) - u^2_{2(3.6.4)}(x,t)$, $(x,t)\in S$.

To solve problem (3.6.5), we use the difference scheme

$$\begin{aligned} \Lambda_{(3.5.2)}\,z_2^1(x,t) &= 0, && (x,t)\in G_h, \qquad (3.6.6\text{a})\\ z_2^1(x,t) &= \varphi_2(x,t), && (x,t)\in S_h, \end{aligned}$$

where $\overline{G}_h$ is the uniform mesh (3.5.4).

Further, we construct the following special interpolants (similar to (3.6.2c))

$$\begin{aligned} u_0^h(x,t) &= \overline{z}{}_2^1(x,t) + u^2_{2(3.6.4)}(x,t), && (x,t)\in\overline{G},\\ p_0^h(x,t) &= \overline{p}{}^h_{z_2^1}(x,t) + \frac{\partial}{\partial x}u^2_{2(3.6.4)}(x,t), && (x,t)\in\overline{G}^{\,*}, \qquad (3.6.6\text{b})\\ P_0^h(x,t) &= \varepsilon\,p_0^h(x,t), && (x,t)\in\overline{G}^{\,*}, \end{aligned}$$

where, $\overline{z}{}_2^1(x,t)$ is a bilinear interpolant constructed using the function

$$z_2^1(x,t) = z^1_{2(3.6.6\text{a})}(x,t).$$

The scheme (3.6.6), (3.5.4) is the scheme of the singularity splitting method under condition (3.4.25) or (3.6.3).

Note that condition (3.6.3) is satisfied in the case of problem (3.1.2), (3.1.3).

3.6.2 Error Estimates for the Constructed Scheme

In the case of schemes {(3.6.2), (3.5.4), (3.5.13)} and (3.6.6), (3.5.4), we give estimates of errors for solutions and derivatives that follow from results of Theorem 3.5.5 and Remark 3.5.2.

In the case of scheme {(3.6.2), (3.5.4), (3.5.13)}, we have the estimates [20]

$$\left|u(x,t) - u_0^h(x,t)\right| \le M\left[N^{-1}\ln N + N_0^{-1}\right], \quad (x,\,t) \in \overline{G}, \tag{3.6.7a}$$

$$\left|P(x,t) - P_0^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \qquad (x,\,t) \in \overline{G}^{\,*}, \tag{3.6.7b}$$

$$\left|p(x,t) - p_0^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \qquad (x,\,t) \in \overline{G}_0^{\,*}, \tag{3.6.7c}$$

where $\overline{G}_0^{\,*} = \overline{G}^{\,*}_{0(3.2.5)}(m)$, m is an arbitrary sufficiently small constant, $M_{(3.6.7c)} = M(m)$.

In the case of scheme (3.6.6), (3.5.4) under condition (3.4.25) or (3.6.3), the following estimates are valid:

$$\left|u(x,t) - u_0^h(x,t)\right| \le M\left[N^{-1} + N_0^{-1}\right], \quad (x,\,t) \in \overline{G}, \tag{3.6.8}$$

$$\left|p(x,t) - p_0^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \quad (x,\,t) \in \overline{G}^{\,*}.$$

Thus, scheme {(3.6.2), (3.5.4), (3.5.13)} converges ε-uniformly in $C^1(\overline{G}_0^{\,*})$, and scheme (3.6.6), (3.5.4) under condition (3.4.25) or (3.6.3) converges ε-uniformly in $C^1(\overline{G}^{\,*})$.

In the case of difference scheme $\{(3.6.2), (3.5.4), (3.5.13)\}$, the component $u_{2(3.6.1a)}(x,t)$ that involves the interior layer, and its derivative in x satisfy the estimates

$$\left|u_2(x,t) - u_2^h(x,t)\right| \le M\left[N^{-1} + N_0^{-1}\right], \quad (x,\, t) \in \overline{G}, \tag{3.6.9}$$

$$\left|p_2(x,t) - p_2^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \quad (x,\, t) \in \overline{G}^*,$$

where $p_2(x,t) = \dfrac{\partial}{\partial x}u_2(x,t)$, $p_2^h(x,t) = p^h_{2(3.6.2\mathrm{d})}(x,t)$. Thus, the component involving the interior layer converges ε-uniformly in $C^1(\overline{G}^*)$.

We summarize above results in the following theorem.

Theorem 3.6.1 *Let the assumptions of Theorem* 3.5.4 *be fulfilled. Then the difference scheme* $\{(3.6.2), (3.5.4), (3.5.13)\}$ $\big($*the difference scheme* (3.6.6), (3.5.4) *under condition* (3.4.25) *or* (3.6.3)$\big)$ *approximates the solution of the problem* (3.2.2), (3.2.1), *the derivative* $p(x,t)$ *and the diffusion flux* $P(x,t)$ $\big($*the solution of the problem* (3.2.2), (3.2.1) *and the derivative* $p(x,t)\big)$ ε*-uniformly with the estimates* (3.6.7) *and* (3.6.9), *respectively* $\big($*with the estimates* (3.6.8)$\big)$.

Note that the order of the ε-uniform convergence of schemes $\{(3.6.2), (3.5.4), (3.5.13)\}$, and (3.6.6), (3.5.4) under condition (3.4.25) or (3.6.3) is essentially better than it is for the scheme (3.5.2), (3.5.13) $\big($see the estimates (3.5.14), (3.6.7), (3.6.8)$\big)$.

3.6.3 Conclusion

In the case of problem (3.1.1), i.e., the Cauchy problem for the Black-Scholes equation, the scheme of the singularity splitting method makes it possible to obtain the approximation of the solution $C(S,t')$ in a finite neighbourhood of the point (E,T) containing the interior layer, and also of its derivative $(\partial/\partial\, S)\, C(S,t')$ in this neighbourhood excluding the point (E,T), with errors independent of the dimensionless value $\sigma^2 r^{-1}$ for $\sigma^2 r^{-1} \leq M$. The interpolants approximating the solution $C(S,t')$ and its derivative $(\partial/\partial S)\,C(S,t')$ converge in the maximum norm uniformly with respect to the value $\sigma^2 r^{-1}$ at a rate of convergence with the order close to 1 and 0.5, respectively.

3.7 Numerical Experiments

3.7.1 Problem in Presence of Interior Layer

In this section, we present experimental results for the problem

$$\begin{aligned} L_{(3.7.1)}\, u(x,t) \;&\equiv\; \left\{ \varepsilon \frac{\partial^2}{\partial x^2} + (1-\varepsilon)\frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial t} \right\} u(x,t) = 0, \quad (x,t) \in G, \quad (3.7.1\text{a}) \\ u(x,t) \;&=\; \varphi(x,t), \quad (x,t) \in S, \end{aligned}$$

that has the same singularity of the solution as problem (3.1.4), (3.1.3) in a neighbourhood of the interior layer. Note that the problem (3.1.4), (3.1.3) is equivalent to problem (3.1.1). Here

$$G = D \times (0,\, T], \quad D = \{x: \; x \in (-d,\, d)\}, \qquad (3.7.1\text{b})$$

$$T = 1, \quad d = 2; \quad \varphi(x,0) = \begin{cases} 0, & -d < x \leq 0, \\ x + m\,x^2, & 0 < x < d; \end{cases}$$

$$\varphi(-d,t) = 0,$$

$$\varphi(d,t) = e^{-t}\Big(md^2 + \big(2mt(1-\varepsilon)+1\big)d + \big(2m\varepsilon + (1-\varepsilon)\big)t + m(1-\varepsilon)^2 t^2 \Big),$$

where $m = 4^{-1}$.

The jump of the first order derivative of the function $\varphi_{(3.7.1)}(x,t)$ at the point $(0,t)$ is the same as for the function $\varphi_{(3.1.3a)}(x)$ for $x = 0$ and is equal to 1 for both functions.

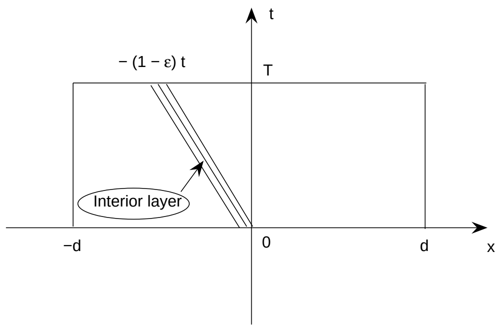


Figure 3.2: Problem (3.7.1) with appearance of interior layer.

Note that $\varphi(x,t) = w(x,t)$ for $(x,t) \in S,\ \ x \geq 0,$ where

$$w(x,t) = e^{-t}\Big(mx^2 + \big(2mt(1-\varepsilon)+1\big)x + \big(2m\varepsilon + (1-\varepsilon)\,\big)t + m(1-\varepsilon)^2t^2\Big),$$

$$(x,t) \in I\!R \times [0,\,T].$$

Here the function $w(x,t)$ is the solution of the Cauchy problem

$$\begin{aligned} L_{(3.7.1)}\,w(x,t) &= 0, \qquad (x,t) \in I\!R \times (0,\,T], \\ w(x,0) &= \varphi_w(x), \quad x \in I\!R, \end{aligned} \tag{3.7.2}$$

where $\varphi_w(x) = x + m\,x^2,\ \ x \in I\!R.$

The choice of boundary conditions for $x = -d,\, d$ ensures that compatibility conditions are fulfilled for the data of problem (3.7.1) and prevents the appearance of the boundary layer and of the interior layer in a neighborhood of the characteristic passing through the point $(d,0)$.

Under the chosen data of the problem, the singularity of the solution generated by the jump of the derivative of the initial function is not "polluted" by other

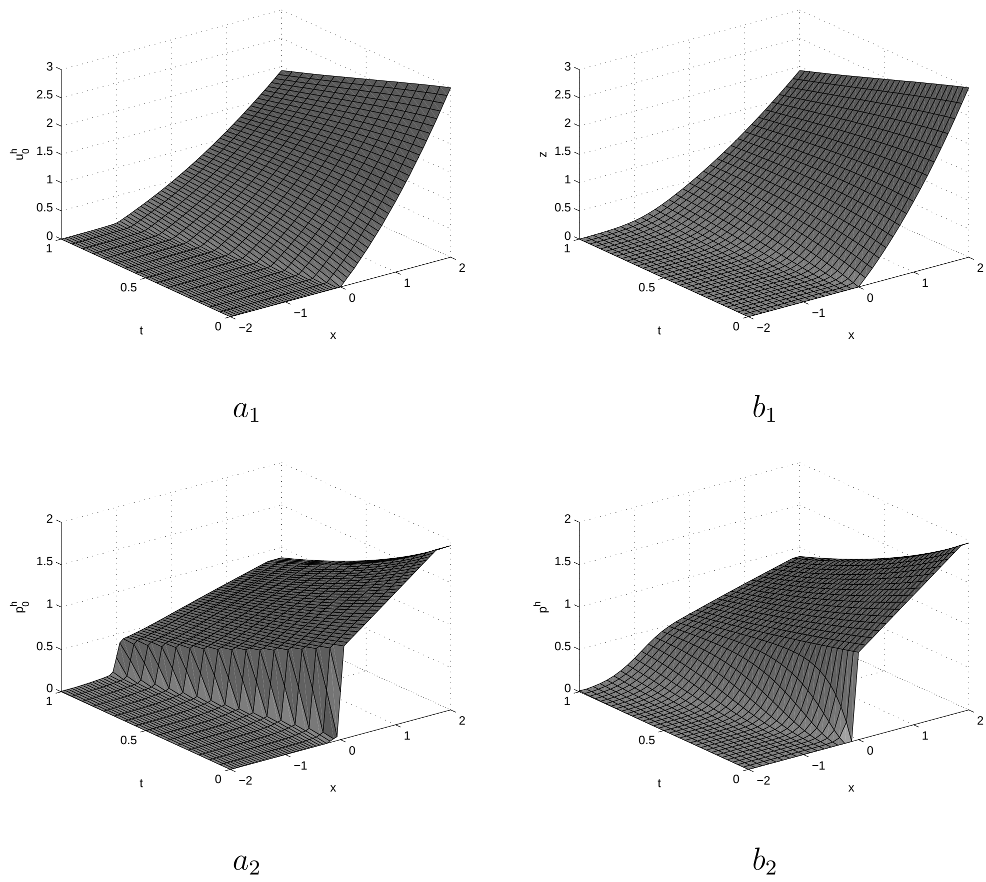


Figure 3.3: Plots of the solutions a_1, b_1 and the derivatives a_2, b_2; plots of a_1, a_2 and b_1, b_2 are generated by Schemes A and B, respectively, for $\varepsilon = 2^{-10}$, $N = 16$ and $N_0 = 16$.

singularities, that allows us to study numerically the efficiency of the constructed difference scheme in a domain containing the interior layer and to compare this scheme with the classical finite difference scheme.

The data of problem (3.7.1) satisfy condition (3.6.3) $\big($and condition (3.4.25)$\big)$. Thus, for the numerical solution of this problem, it is possible to apply the simplified scheme (3.6.6), (3.5.4), i.e., the scheme of the singularity splitting method under condition (3.6.3) or (3.4.25) $\big($we denote it briefly by Scheme A$\big)$.

In order to estimate the efficiency of the developed method, we compare solutions generated using Scheme A in accuracy with discrete solutions of problem (3.7.1) generated using the classical finite difference scheme (3.5.2), (3.5.4) $\big($we denote it briefly by Scheme B$\big)$.

The plots of the solutions a_1, b_1 and the derivatives a_2, b_2 computed using Scheme A $\big($see a_1, $a_2\big)$ and Scheme B $\big($see b_1, $b_2\big)$ are presented on Fig. 3.3 for $\varepsilon = 2^{-10}$, $N = 16$ and $N_0 = 16$.

3.7.2 Error Estimates of the Discrete Solutions

To analyze errors in the discrete solutions, a technique similar to that given in [20] is used, however, it is modified with regard to the singularity splitting method. Computations are made for values of $\varepsilon = 2^{-j}$, $j = 0, 1, \ldots, 20$ on grids with the number of nodes $N = N_0$ for $N = 2^i$, $i = 5, 6, \ldots, 10$. The numerical solution $u_{0,\varepsilon}^{h,\,N^F}(x,t)$, generated by Scheme A on the finest mesh $\overline{G}_h^{N^F}$ with $N = N_0 = N^F = 2048$ for each value of ε is used as the exact solution of problem (3.7.1).

Errors in the numerical solutions in the maximum norm for each value of ε and N are computed by the formula

$$E_\varepsilon^N = E_\varepsilon^N\big(u_{0,\varepsilon}^{h,\,N}(\cdot)\big) = \big\|u_{0,\varepsilon}^{h,\,N^F}(x,t) - u_{0,\varepsilon}^{h,\,N}(x,t)\big\|_{\overline{G}_h^N} \tag{3.7.3}$$

for Scheme A and by the formula

$$E_\varepsilon^N = E_\varepsilon^N\big(z_\varepsilon^N(\cdot)\big) = \big\|u_{0,\varepsilon}^{h,\,N^F}(x,t) - z_\varepsilon^N(x,t)\big\|_{\overline{G}_h^N} \tag{3.7.4}$$

for Scheme B. Here the function $u_{0,\varepsilon}^{h,\,N}(x,t) = u_{0(3.6.6),\,\varepsilon}^{h,\,N}(x,t)$ in (3.7.3) and the function $z_\varepsilon^N(x,t)$ in (3.7.4) are the numerical solutions obtained, respectively, by Schemes A and B.

Table 3.1: Errors $E_\varepsilon^N = E_\varepsilon^N(u_{0,\varepsilon}^{h,N})$ and $E^N = E^N(u_{0,\varepsilon}^{h,N})$ for the solutions generated by Scheme A.

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.1320-02	0.6633-03	0.3328-03	0.1666-03	0.8338-04	0.4170-04
2^{-1}	0.1140-02	0.5863-03	0.2972-03	0.1496-03	0.7507-04	0.3760-04
2^{-2}	0.2782-02	0.1426-02	0.7219-03	0.3632-03	0.1822-03	0.9123-04
2^{-3}	0.3904-02	0.2006-02	0.1017-02	0.5120-03	0.2569-03	0.1287-03
2^{-4}	0.4614-02	0.2384-02	0.1214-02	0.6131-03	0.3082-03	0.1545-03
2^{-5}	0.5034-02	0.2623-02	0.1345-02	0.6823-03	0.3440-03	0.1727-03
2^{-6}	0.5288-02	0.2769-02	0.1429-02	0.7289-03	0.3688-03	0.1856-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.5557-02	0.2948-02	0.1544-02	0.8004-03	0.4116-03	0.2103-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.5558-02	0.2948-02	0.1544-02	0.8006-03	0.4117-03	0.2104-03
E^N	**0.5558-02**	**0.2948-02**	**0.1544-02**	**0.8006-03**	**0.4117-03**	**0.2104-03**

Table 3.2: Errors $E_\varepsilon^N = E_\varepsilon^N(z_\varepsilon^N)$ and $E^N = E^N(z_\varepsilon^N)$ for the solutions generated by Scheme B.

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.9868-02	0.5482-02	0.3296-02	0.2116-02	0.1419-02	0.9756-03
2^{-1}	0.8081-02	0.4425-02	0.2560-02	0.1584-02	0.1036-02	0.7016-03
2^{-2}	0.6600-02	0.3718-02	0.2093-02	0.1237-02	0.7772-03	0.5126-03
2^{-3}	0.9033-02	0.4972-02	0.2627-02	0.1354-02	0.6878-03	0.3850-03
2^{-4}	0.1242-01	0.7186-02	0.3933-02	0.2072-02	0.1065-02	0.5406-03
2^{-5}	0.1515-01	0.9293-02	0.5348-02	0.2919-02	0.1535-02	0.7887-03
2^{-6}	0.1706-01	0.1105-01	0.6736-02	0.3866-02	0.2107-02	0.1107-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.1963-01	0.1403-01	0.9948-02	0.7029-02	0.4956-02	0.3486-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.1964-01	0.1404-01	0.9960-02	0.7045-02	0.4978-02	0.3516-02
E^N	**0.1964-01**	**0.1404-01**	**0.9960-02**	**0.7045-02**	**0.4978-02**	**0.3516-02**

Tables 3.1 and 3.2 contain the values E_ε^N of errors in the solutions generated by Schemes A and B for various values of ε and N. The value E^N in the last rows of the tables is the maximal value of the errors E_ε^N with respect to ε, corresponding to the given value of N.

Tables 3.3 and 3.4, which are similar to tables 3.1 and 3.2, demonstrate errors in the first derivatives computed by the formula

$$E_\varepsilon^N = E_\varepsilon^N\left(p_{0,\,\varepsilon}^{h,\,N}(\cdot)\right) = \left\| p_{0,\,\varepsilon}^{h,\,N^F}(x,t) - p_{0,\,\varepsilon}^{h,\,N}(x,t)\right\|_{\overline{G}_h^{N\,*}}, \tag{3.7.5}$$
$$\overline{G}_h^{N\,*} = \overline{G}_h^{N} \setminus S^{(*)}, \quad S^{(*)} = S^{(*)}_{(3.2.2c)}$$

for Scheme A and by the formula

$$E_\varepsilon^N = E_\varepsilon^N\left(p_{z,\,\varepsilon}^{N}(\cdot)\right) = \left\| p_{0,\,\varepsilon}^{h,\,N^F}(x,t) - p_{z,\,\varepsilon}^{N}(x,t)\right\|_{\overline{G}_h^{N\,\{*\}}}, \tag{3.7.6}$$
$$\overline{G}_h^{N\,\{*\}} = \overline{G}_h^{N} \setminus S^{\{*\}}, \quad S^{\{*\}} = \{(x,0):\ x = x_{i-1},\, x_i,\, x_{i+1};\ x_i = 0\}$$

for Scheme B. Here, $p_{z,\,\varepsilon}^{N}(x,t)$ in (3.7.6) is the first difference derivative

$$p_{z,\,\varepsilon}^{N}(x_i,\,t_j) = \frac{z_\varepsilon^N(x_{i+1},\,t_j) - z_\varepsilon^N(x_i,\,t_j)}{x_{i+1}-x_i}, \quad i = 0,\ldots,N, \quad j = 0,\ldots,N_0. \tag{3.7.7}$$

The function $p_{0,\,\varepsilon}^{h,\,N^F}(x,t)$ in formulae (3.7.5) and (3.7.6) and the function $p_{0,\,\varepsilon}^{h,\,N}(x,t)$ in formula (3.7.5) are the special interpolants of the first order derivative of the solution computed by formula (3.6.6), respectively, on the finest mesh $\overline{G}_h^{N^F}$ and on the mesh $\overline{G}_h^{N}$ for fixed value of ε.

Analyzing the values of errors for the solutions in Tables 3.1 and 3.2, and for the first derivatives in Table 3.3, we observe the ε-uniform convergence, since, with decreasing ε, the errors are stabilized for each value of N approximately for one and the same values of ε, i.e., the errors are independent of the value of the parameter ε, moreover, the values of E^N (the last row) decrease as N increases. However,

Table 3.3: Errors $E_\varepsilon^N = E_\varepsilon^N(p_{0,\varepsilon}^{h,N})$ and $E^N = E^N(p_{0,\varepsilon}^{h,N})$ for the first discrete derivatives generated by Scheme A.

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.1562-01	0.7931-02	0.4073-02	0.2106-02	0.1100-02	0.5835-03
2^{-1}	0.1562-01	0.7910-02	0.4016-02	0.2034-02	0.1048-02	0.5470-03
2^{-2}	0.1620-01	0.8583-02	0.4431-02	0.2253-02	0.1136-02	0.5705-03
2^{-3}	0.1701-01	0.9262-02	0.4858-02	0.2492-02	0.1263-02	0.6355-03
2^{-4}	0.1754-01	0.9768-02	0.5198-02	0.2690-02	0.1370-02	0.6915-03
2^{-5}	0.1788-01	0.1011-01	0.5446-02	0.2840-02	0.1454-02	0.7359-03
2^{-6}	0.1806-01	0.1034-01	0.5616-02	0.2947-02	0.1515-02	0.7688-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.1831-01	0.1066-01	0.5915-02	0.3966-02	0.2697-02	0.1850-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.1831-01	0.1066-01	0.5920-02	0.3974-02	0.2708-02	0.1865-02
E^N	**0.1831-01**	**0.1066-01**	**0.5920-02**	**0.3974-02**	**0.2708-02**	**0.1865-02**

in Table 3.4 the first derivative of the solution generated by Scheme B does not converge at all; the values of E^N practically do not change as N increases.

In Tables 3.5 and 3.6, the values of q_ε^N are shown that are the convergence orders for the solutions computed by Schemes A and B, respectively.

In analogous Table 3.7, one can see the convergence orders for the first discrete derivatives generated by Scheme A for various values of ε and N. The value q^N in last rows of the tables is the minimal value of q_ε^N with respect to ε, corresponding to the given value of N. The convergence order for the discrete solutions is defined by the formula

$$q_\varepsilon^N = \log_2 \frac{E_\varepsilon^N}{E_\varepsilon^{2N}}. \tag{3.7.8}$$

The quantities E_ε^N, E_ε^{2N} are defined by formula (3.7.3) for Scheme A and by formula (3.7.4) for Scheme B.

Table 3.4: Errors $E_\varepsilon^N = E_\varepsilon^N(p_{z,\varepsilon}^N)$ and $E^N = E^N(p_{z,\varepsilon}^N)$ for the first discrete derivatives generated by Scheme B.

ε	Number of intervals N					
	32	64	128	256	512	1024
2^{0}	0.1080+00	0.7864-01	0.6165-01	0.5118-01	0.4465-01	0.4000-01
2^{-1}	0.1296+00	0.1013+00	0.7555-01	0.5941-01	0.5017-01	0.4405-01
2^{-2}	0.1394+00	0.1219+00	0.9764-01	0.7388-01	0.5826-01	0.4964-01
2^{-3}	0.1302+00	0.1315+00	0.1178+00	0.9572-01	0.7301-01	0.5768-01
2^{-4}	0.1045+00	0.1228+00	0.1275+00	0.1157+00	0.9473-01	0.7256-01
2^{-5}	0.7901-01	0.9761-01	0.1189+00	0.1254+00	0.1147+00	0.9424-01
2^{-6}	0.8714-01	0.7835-01	0.9407-01	0.1170+00	0.1244+00	0.1141+00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.9914-01	0.9833-01	0.9985-01	0.1005+00	0.1006+00	0.1002+00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.9918-01	0.9839-01	0.1000+00	0.1008+00	0.1012+00	0.1014+00
E^N	**0.1394+00**	**0.1315+00**	**0.1275+00**	**0.1254+00**	**0.1244+00**	**0.1238+00**

Table 3.5: Convergence orders $q_\varepsilon^N = q_\varepsilon^N(u_{0,\varepsilon}^{h,N})$ and $q^N = q^N(u_{0,\varepsilon}^{h,N})$ for the solutions of Scheme A.

ε	Number of intervals N				
	32	64	128	256	512
2^{0}	0.9928	0.9950	0.9983	0.9986	0.9997
2^{-1}	0.9593	0.9802	0.9903	0.9948	0.9975
2^{-2}	0.9641	0.9821	0.9910	0.9952	0.9979
2^{-3}	0.9606	0.9800	0.9901	0.9949	0.9972
2^{-4}	0.9526	0.9736	0.9856	0.9923	0.9963
2^{-5}	0.9405	0.9636	0.9791	0.9880	0.9941
2^{-6}	0.9334	0.9544	0.9712	0.9829	0.9906
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.9146	0.9331	0.9479	0.9595	0.9688
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.9148	0.9331	0.9475	0.9595	0.9685
q^N	**0.9148**	**0.9331**	**0.9475**	**0.9595**	**0.9685**

Table 3.6: Convergence orders $q_\varepsilon^N = q_\varepsilon^N(z_\varepsilon^N)$ and $q^N = q^N(z_\varepsilon^N)$ for the solutions of Scheme B.

	Number of intervals N				
ε	32	64	128	256	512
2^{0}	0.8481	0.7340	0.6394	0.5765	0.5405
2^{-1}	0.8689	0.7895	0.6926	0.6125	0.5623
2^{-2}	0.8279	0.8290	0.7587	0.6705	0.6005
2^{-3}	0.8614	0.9204	0.9562	0.9772	0.8371
2^{-4}	0.7894	0.8696	0.9246	0.9602	0.9782
2^{-5}	0.7051	0.7971	0.8735	0.9272	0.9607
2^{-6}	0.6266	0.7141	0.8011	0.8757	0.9285
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.4845	0.4960	0.5011	0.5041	0.5076
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.4843	0.4953	0.4995	0.5010	0.5016
q^N	**0.4843**	**0.4953**	**0.4995**	**0.5010**	**0.5016**

Table 3.7: Convergence orders $q_\varepsilon^N = q_\varepsilon^N(p_{0,\varepsilon}^{h,N})$ and $q^N = q^N(p_{0,\varepsilon}^{h,N})$ for the first discrete derivatives generated by Scheme A.

	Number of intervals N				
ε	32	64	128	256	512
2^{0}	0.9778	0.9614	0.9516	0.9370	0.9147
2^{-1}	0.9816	0.9779	0.9814	0.9567	0.9380
2^{-2}	0.9164	0.9538	0.9758	0.9879	0.9937
2^{-3}	0.8770	0.9310	0.9631	0.9804	0.9909
2^{-4}	0.8445	0.9101	0.9504	0.9734	0.9864
2^{-5}	0.8226	0.8925	0.9393	0.9659	0.9824
2^{-6}	0.8046	0.8806	0.9303	0.9599	0.9786
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.7804	0.8498	0.5767	0.5563	0.5438
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.7804	0.8485	0.5750	0.5534	0.5381
q^N	**0.7804**	**0.8485**	**0.5750**	**0.5534**	**0.5381**

The convergence order for the discrete derivatives is defined by formula (3.7.8) where the quantities E_ε^N, E_ε^{2N} are defined by formula (3.7.5) for Scheme A and by formula (3.7.6) for Scheme B.

Orders of the rate of ε-uniform convergence for the solutions generated by Schemes A and B (see Tables 3.5 and 3.6) are close, respectively, to 1 and 0.5; for the first derivative generated by Scheme A (see Table 3.7), the order of the rate of ε-uniform convergence is close to 0.5.

3.7.3 Conclusion

Thus, it follows from our numerical experiments that the solution and its first order derivative obtained by Scheme A, i.e., the scheme based on the method of the additive splitting of a singularity, converge ε-uniformly at the rate of convergence of the order close to 1 and 0.5, respectively. Whereas the convergence rate for the solutions of the classical finite difference Scheme B yields to that for scheme A, and the derivatives computed by Scheme B do not converge even for fixed values of the parameter ε. The numerical experiments, consistent with the theoretical results, illustrate the efficiency of the singularity splitting method for the approximation of the interior layer generated by the discontinuity of the first order derivative $\frac{\partial}{\partial x}\varphi(x,0)$ in problem (3.7.1).

The numerical experiments showed that the special difference scheme constructed in this paper, i.e., the scheme of the method of the additive splitting of a singularity on uniform meshes, is effective both for small values of N and for its sufficiently large values, for which results of the theoretical study become apparent.

In the case of the Cauchy problem for the Black-Scholes equation, the interpolants constructed using solutions of the special difference scheme, which approximates the solution $C(S,t')$ of problem (3.1.1) and its derivative $(\partial/\partial\ S)\,C(S,t')$ (for $(S,t') \neq (E,T)$) in a neighbourhood of the interior layer, converge at the rate of $\sigma^2\,r^{-1}-$uniform convergence with orders close to 1 and 0.5, respectively.

Chapter 4

Parameter-Uniform Method for the Singularly Perturbed Black–Scholes Equation in Presence of Interior and Boundary Layers

4.1 Introduction

We have introduced in former chapters that mathematical modeling in financial mathematics leads to the Cauchy problem for the parabolic Black-Scholes equation [88] with respect to the value $C = C(S, t')$, i.e., a European call option, where S and t' are the current values of the underlying asset and time. Along with the solution $C = C(S, t')$ itself, the first partial derivative $(\partial/\partial S)\, C(S, t')$ of the solution is of interest. The change of variables leads to the Cauchy problem for the dimensionless parabolic equation, i.e., the singularly perturbed parabolic equation with the perturbation parameter $\varepsilon = 2^{-1}\,\sigma^2\, r^{-1}$, $\varepsilon \in (0, 1]$; σ and r are the volatility and the interest rate, respectively. For finite values of the parameter ε, the solution of the Cauchy problem has singularities of different types that are generated by the unboundedness of the domain where the problem is defined, the discontinuity of the first derivative of the initial function and its unbounded growth at infinity. For small values of the parameter ε, an additional singularity arises, such as an interior layer which moving in time. In this problem, primarily, we are interested in approximations to both the solution and its first order derivative in a neighbourhood of the interior layer generated by the piecewise smooth initial function [46, 44].

In the present chapter, in order to construct adequate grid approximations for the singularity of the interior layer type, we consider, instead of the Cauchy problem for the dimensionless Black-Scholes equation, a "simpler" singularly perturbed boundary value problem with a piecewise smooth initial condition, i.e., the problem (4.2.2), (4.2.1) (see its formulation in Section 2). In this boundary value problem in a bounded domain, except the interior layer, an additional singularity appears, namely, a boundary layer with the typical width of ε; the typical width of the interior layer is $\varepsilon^{1/2}$. Moreover, the singularity of the boundary layer is stronger than that of the interior layer, which makes it difficult to construct and study special numerical methods suitable for the adequate description of the singularity of the interior layer type. We are interested in approximation to both the solutions and the first order discrete derivatives of the boundary value problem in the boundary layer and outside the boundary layer.

Boundary value problems in bounded domains for parabolic equations with a discontinuous initial condition have been studied in [28, 70, 74]; however, an approximation of the derivative itself was not considered. A boundary value problem on an interval for singularly perturbed parabolic convection-diffusion equations with a piecewise smooth initial condition has been considered in [78]; approximations of the solution and the derivative were investigated. Here, in contrast to those papers, a finite difference scheme based on the solution decomposition method is constructed that allows us to resolve each singularity of the problem separately [47, 45, 79, 80].

4.2 Grid Approximation of the Boundary Value Problem

4.2.1 Problem Formulation

On the set $\overline{G}$ with the boundary S

$$\overline{G} = G \bigcup S, \quad G = D \times (0,\, T], \quad D = \{x:\ x \in (d_1,\, d_2)\}, \tag{4.2.1}$$

we consider the Dirichlet problem for a singularly perturbed parabolic convection-diffusion equation

$$L_{(4.2.2a)}\, u(x,t) \;=\; f(x,t), \quad (x,t) \in G, \tag{4.2.2a}$$

$$u(x,t) \;=\; \varphi(x,t), \quad (x,t) \in S. \tag{4.2.2b}$$

Here $L_{(4.2.2a)} \equiv \varepsilon\, a\, \dfrac{\partial^2}{\partial x^2} + b\, \dfrac{\partial}{\partial x} - c - q\, \dfrac{\partial}{\partial t}, \quad \varepsilon \in (0,\, 1], \quad a,\ b,\ q > 0,\ c \geq 0.$ The right hand side function $f(x,t)$ is sufficiently smooth on $\overline{G}$. The boundary function $\varphi(x,t)$ is sufficiently smooth function on the sets $\overline{S}_0^{\,-}$.

Here

$$S_0^- = \{(x,t):\ x \in [d_1,\, 0),\ t = 0\},$$

$$S_0^+ = \{(x,t):\ x \in (0,\, d_2],\ t = 0\}, \quad S_0 = \overline{S}_0^{\,-} \bigcup \overline{S}_0^{\,+},$$

S_0 and S^L are the lower and lateral parts of the boundary S, $S^L = \Gamma \times (0,\, T]$, $\Gamma = \overline{D} \backslash D$. The first derivative in x of the function $\varphi(x,t)$ has a jump discontinuity at the point $S^{(*)} = \{(0,\, 0)\}$

$$\left[\frac{\partial}{\partial x}\, \varphi(x,t)\right] \equiv \lim_{x_1 \searrow x} \frac{\partial}{\partial x} \varphi(x_1,t) - \lim_{x_1 \nearrow x} \frac{\partial}{\partial x} \varphi(x_1,t) \neq 0, \quad (x,t) \in S^{(*)}. \tag{4.2.2c}$$

Our aim is for problem (4.2.2), (4.2.1) to construct a finite difference scheme that approximates ε-uniformly both the solution of the problem and its first derivative.

4.2.2 Approximations of the Problem on Uniform Mesh

On the set $\overline{G}_{(4.2.1)}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega} \times \overline{\omega}_0, \tag{4.2.3}$$

where $\overline{\omega}$ is a mesh on $[d_1,\, d_2]$ with an arbitrary distribution of nodes and $\overline{\omega}_0$ is the uniform mesh on $[0,\, T]$.

We approximate the boundary value problem (4.2.2) by the difference scheme

$$\begin{aligned} \Lambda_{(4.2.4)}\, z(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{4.2.4}$$

Here

$$\Lambda_{(4.2.4)} \equiv \varepsilon\, a\, \delta_{\overline{x}\widehat{x}} + b\, \delta_x - c - q\, \delta_{\overline{t}}.$$

On the uniform mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{4.2.5}$$

we obtain the estimate

$$|u(x,t) - z(x,t)| \le M\left[\varepsilon^{-1} N^{-1} + N^{-1/2} + N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h, \tag{4.2.6}$$

i.e. the scheme (4.2.4), (4.2.5) converges for fixed values of the parameter ε.

4.2.3 Approximations of the Problem on Piecewise Uniform Mesh

On the set $\overline{G}$ we construct the piecewise uniform mesh condensing in a neighborhood of the boundary layer

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega}^* \times \overline{\omega}_0, \tag{4.2.7a}$$

where $\overline{\omega}_0$ is uniform mesh on the segment $[0,\, T]$ with the step-size $h_0 = T\, N_0^{-1}$. Here $N_0 + 1$ is the number of nodes in the mesh $\overline{\omega}_0$. The mesh $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a piecewise-uniform mesh on $[d_1,\, d_2]$, σ is a parameter depending on ε and N. We choose the value σ satisfying the condition

$$\sigma = \sigma(\varepsilon,\, N) = \min\,[\,\beta,\, 2\, m^{-1}\, \varepsilon\, \ln N\,], \tag{4.2.7b}$$

where β is an arbitrary number in the half-interval $(0,\, (d_2 - d_1)/2]$, m is an arbitrary constant from the interval $(0,\, m_0)$, $m_0 = a^{-1}\, b$. The segment $[d_1,\, d_2]$ is divided in two parts: $[d_1,\, d_1 + \sigma]$, $[d_1 + \sigma,\, d_2]$; in each part the step-size is constant and equal to $h^{(1)} = (d_2 - d_1)\, \sigma\, \beta^{-1}\, N^{-1}$ on the segment $[d_1, d_1 + \sigma]$ and $h^{(2)} = (d_2 - d_1)\,(d_2 - d_1 - \sigma)\,(d_2 - d_1 - \beta)^{-1}\, N^{-1}$ on the segment $[d_1 + \sigma, d_2]$, $\sigma \le (d_2 - d_1)/2$.

On the piecewise uniform mesh (4.2.7) we have the estimate

$$|\; u(x,t) - z(x,t)\; | \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}_h^{\,s}, \tag{4.2.8}$$

i.e. the scheme (4.2.4), (4.2.7) converges ε-uniformly.

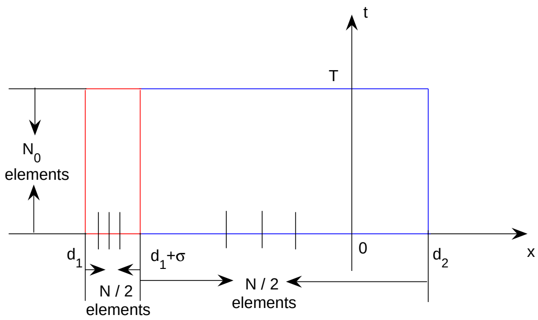


Figure 4.1: Constructed piecewise uniform meshes for problem (4.2.2), (4.2.1) with appearance of boundary and interior layers.

4.2.4 Decomposition Scheme Approximating the Derivative

We represent the solution of the boundary value problem (4.2.2), (4.2.1) as the sum of functions

$$u(x,t) = u_1(x,t) + u_2(x,t), \quad (x,\, t) \in \overline{G}. \tag{4.2.9}$$

Here $u_1(x,t)$ and $u_2(x,t)$ are components of the solution of the boundary value problem (4.2.2), (4.2.1), including singularities of the boundary and interior layers types respectively.

We represent the interior layer component $u_2(x,t)$ as the sum of functions

$$u_2(x,t) = u_2^1(x,t) + u_2^2(x,t), \quad (x,\, t) \in \overline{G}, \tag{4.2.10}$$

where $u_2^1(x,t)$ and $u_2^2(x,t)$ are the regular and singular parts of the function $u_2(x,t)$;

$$u_2^2(x,t) = W(x,t), \quad (x,\, t) \in \overline{G},$$

$$W(x,t) = 2^{-1}\left[\frac{\partial}{\partial x}\varphi(0,0)\right]\Big\{\big(x-\gamma(t)\big)v\Big(2^{-1}\varepsilon^{-1/2}a^{-1/2}\,q^{1/2}\,\big(x-\gamma(t)\big)t^{-1/2}\Big)+$$

$$+2\,\pi^{-1/2}\,\varepsilon^{1/2}\,a^{1/2}\,q^{-1/2}\,t^{1/2}\,\exp\Big(-4^{-1}\,\varepsilon^{-1}\,a^{-1}\,q\,\big(x-\gamma(t)\big)^2\,t^{-1}\Big)\Big\}\;\exp(-\alpha\,t),$$

$$(x,t)\in I\!R\times[0,\,T].\quad v(\xi)=erf(\xi)=2\,\pi^{-1/2}\int_0^\xi \exp(-\alpha^2)\,d\alpha,\quad \xi\in I\!R.$$

The functions $u_1(x,t)$ and $u_2^1(x,t)$ are solutions of the following problems

$$L_{(4.2.2a)}\,u_2^1(x,t)=f_2(x,t),\;(x,\,t)\in G,\quad u_2^1(x,t)=\varphi_2(x,t),\;(x,\,t)\in S;\quad (4.2.11)$$

$$L_{(4.2.2a)}\,u_1(x,t)=f_1(x,t),\;(x,\,t)\in G,\quad u_1(x,t)=\varphi_1(x,t),\;(x,\,t)\in S.\quad (4.2.12)$$

The functions $f_i(x,t)$, $\varphi_i(x,t)$, $i=1,2$, are defined by the relations

$$f_2(x,t)=f(x,t)\,\eta\,(x,t),\qquad (4.2.13)$$

$$f_1(x,t)=f(x,t)-f_2(x,t),\quad (x,t)\in\overline{G};$$

$$\varphi_2(x,t)=(\varphi(x,t)-u_2^2(x,t))\,\eta\,(x,t),\qquad (4.2.14)$$

$$\varphi_1(x,t)=\varphi(x,t)-\varphi_2(x,t)-u_2^2(x,t),\quad (x,t)\in S.$$

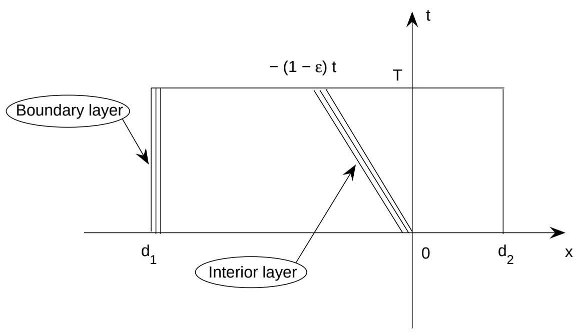
Here $\eta\,(x,t)$, $(x,t)\in\overline{G}$, is a sufficiently smooth function that vanishes in a neighbourhood of the boundary layer

$$\left.\begin{array}{l}\eta\,(x,t)=0,\quad (x,t)\in\overline{G}^{\,5}_{(4.2.16)}(2^{-1}\,m_1)\\ \eta\,(x,t)=1,\quad (x,t)\in\overline{G}^{\,4}_{(4.2.15)}(m_1)\end{array}\right\},\quad 0\le\eta(x,t)\le 1,\quad (x,t)\in\overline{G},$$

where $\quad m_1\in\big(0,\,2^{-1}\,(-d_1+\gamma(T))\big)$;

$$G^4=G^4(m)=\big\{(x,t);\,x\,>\,\gamma(t)-\gamma\,(T)+d_1+m\big\},\quad \overline{G}^{\,4}=G^{\,4}\bigcup S^{\,4},\;(4.2.15)$$

$$\overline{G}^{\,5}=G^{\,5}\bigcup S^{\,5},\quad G^{\,5}=G^{\,5}(m)=G\setminus\overline{G}^{\,4}(m);\quad m\,<\,-d_1+\gamma\,(T).\qquad (4.2.16)$$



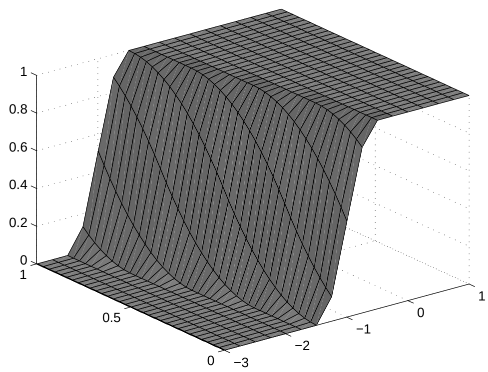


Figure 4.2: η function with $N = 32$, $\varepsilon = 2^{-10}$, $m_{(4.2.15)} = 0.9$.

An appropriate selection of η function can prevent interaction between the boundary and interior layers so that there is no overlapping between the two layers. The plot of a constructed η function is shown in Fig. 4.2.

To solve problem (4.2.12), we use finite difference scheme on the piecewise uniform mesh(4.2.7):

$$\begin{aligned} \Lambda_{(4.2.4)} z_1(x,t) &= f_1(x,t), \quad (x,t) \in G_{h(4.2.7)}, \qquad (4.2.17a) \\ z_1(x,t) &= \varphi_1(x,t), \quad (x,t) \in S_h. \end{aligned}$$

To solve problem (4.2.11), we use the difference scheme on the uniform mesh (4.2.5):

$$\begin{aligned} \Lambda_{(4.2.4)} z_2^1(x,t) &= f_2(x,t), \quad (x,t) \in G_{h(4.2.5)}, \qquad (4.2.17b) \\ z_2^1(x,t) &= \varphi_2(x,t), \quad (x,t) \in S_h. \end{aligned}$$

Further, we construct the special interpolants into which the singular part $u_2^2(x,t)$, i.e., the function of the interior layer type, enters in the explicit form as follows:

$$\begin{aligned} u_0^h(x,t) &= \overline{z}_1(x,t) + u_2^h(x,t), \qquad (4.2.17c) \\ u_2^h(x,t) &= \overline{z}_2^1(x,t) + u_2^2(x,t),\ (x,t) \in \overline{G}; \\ p_0^h(x,t) &= \overline{p}_{z_1}^h(x,t) + p_2^h(x,t), \qquad (4.2.17d) \\ p_2^h(x,t) &= \overline{p}_{z_2^1}^h(x,t) + \frac{\partial}{\partial x} u_2^2(x,t),\ (x,t) \in \overline{G}^*; \\ P_0^h(x,t) &= \varepsilon\, p_0^h(x,t), \ \ (x,t) \in \overline{G}^*; \ \ \overline{G}^* = \overline{G} \setminus S^{(*)}. \qquad (4.2.17e) \end{aligned}$$

here $\overline{z}_1(x,t)$, $\overline{p}_{z_1}^h(x,t)$ and $\overline{z}_2^1(x,t)$, $\overline{p}_{z_2^1}^h(x,t)$ are bilinear interpolants that are constructed using the functions $z_1(x,t)$, $(x,t) \in \overline{G}_{h(4.2.7)}$ and $z_2^1(x,t)$, $(x,t) \in \overline{G}_{h(4.2.5)}$. The use of the interpolants allows us to find the solution on the set $\overline{G}$, its first derivative in x and the diffusion flux on the set $\overline{G}^*$.

The function $u_0^h(x,t)$, $(x,t) \in \overline{G}$, is called the solution of the difference scheme (4.2.17), (4.2.5), (4.2.7), and the functions $p_0^h(x,t)$ and $P_0^h(x,t) = \varepsilon\, p_0^h(x,t)$,

$(x,t) \in \overline{G}^*$, are called the derivative and the diffusion flux (the normalized derivative), respectively, corresponding to this scheme.

The scheme (4.2.17), (4.2.5), (4.2.7) is the solution decomposition scheme with the additive splitting of a singularity of the interior-layer type. Briefly, we call this scheme the singularity splitting scheme.

In the case of scheme (4.2.17), (4.2.5), (4.2.7), we have the estimates

$$\left|u(x,t) - u_0^h(x,t)\right| \le M\left[N^{-1}\ln N + N_0^{-1+\nu_0}\right], \quad (x,\, t) \in \overline{G}; \quad (4.2.18a)$$

$$\left|P(x,t) - P_0^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \qquad (x,\, t) \in \overline{G}^*; (4.2.18b)$$

$$\left|p(x,t) - p_0^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \qquad (x,\, t) \in \overline{G}_0^*, (4.2.18c)$$

where $\overline{G}_0^* = \overline{G}_0^*(m) = \overline{G}^* \cap \{x \ge d_1 + m\}$, m is an arbitrary sufficiently small constant, and $M_{(4.2.18c)} = M(m)$. The interior layer component $u_{2(4.2.9)}(x,t)$ and its derivative in x satisfy the estimates

$$\left|u_2(x,t) - u_2^h(x,t)\right| \le M\left[N^{-1} + N_0^{-1+\nu_0}\right], \quad (x,\, t) \in \overline{G}, \qquad (4.2.19)$$

$$\left|p_2(x,t) - p_2^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \quad (x,\, t) \in \overline{G}^*,$$

where $p_2(x,t) = \dfrac{\partial}{\partial x}u_2(x,t)$, $p_2^h(x,t) = p^h_{2(4.2.17d)}(x,t)$. In (4.2.18) and (4.2.19), $\nu_0 \in (0,\, 1)$.

Thus, the interior layer component converges ε-uniformly in $C^1(\overline{G}^*)$.

4.3 Numerical Experiments

In this section, we present experimental results for the boundary value problem (4.2.2) with singularities of interior layer and boundary layer types. In order to study the effect of these singularities on the errors in the numerical approximations, we isolate them from each other and deal with them one at a time.

4.3.1 Problem in Presence of Boundary Layer

In this section, we consider a problem with singularity of boundary layer type. We present the experimental results for the boundary value problem

$$L_{(4.3.1)}\,u(x,t) \;\equiv\; \left\{\varepsilon\,\frac{\partial^2}{\partial x^2} + (1-\varepsilon)\,\frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial t}\right\} u(x,t) = 0,\ (x,t)\in G, \quad (4.3.1\text{a})$$

$$u(x,t) \;=\; \varphi(x,t), \quad (x,t)\in S,$$

Here

$$G = D\times(0,\,T], \quad D=\{x:\ x\in(d_1,\,d_2)\}, \qquad (4.3.1\text{b})$$

$$T=1, \quad d_1=-3, \quad d_2=1; \quad \varphi(x,0)=0, \quad d_1\ < x\ < d_2;$$

$$\varphi(d_1,t)=t, \quad \varphi(d_2,t)=0.$$

Table 4.1: Errors E_ε^N of $z_{1(4.2.17b)}(x,t)$ for the solution generated by Scheme B′.

	Number of intervals N					
ε	32	64	128	256	512	1024
2^{0}	0.1706-02	0.8546-03	0.4339-03	0.2209-03	0.1123-03	0.5689-04
2^{-1}	0.7430-02	0.3723-02	0.1864-02	0.9333-03	0.4668-03	0.2334-03
2^{-2}	0.1753-01	0.1126-01	0.5967-02	0.3061-02	0.1552-02	0.7811-03
2^{-3}	0.1841-01	0.1276-01	0.8256-02	0.5254-02	0.3061-02	0.1868-02
2^{-4}	0.2167-01	0.1499-01	0.9629-02	0.6082-02	0.3560-02	0.2047-02
2^{-5}	0.2385-01	0.1643-01	0.1056-01	0.6592-02	0.3889-02	0.2230-02
2^{-6}	0.2516-01	0.1726-01	0.1109-01	0.6877-02	0.4072-02	0.2333-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.2664-01	0.1818-01	0.1167-01	0.7185-02	0.4270-02	0.2442-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.2664-01	0.1818-01	0.1168-01	0.7186-02	0.4270-02	0.2443-02
E^N	**0.2664-01**	**0.1818-01**	**0.1168-01**	**0.7186-02**	**0.4270-02**	**0.2443-02**

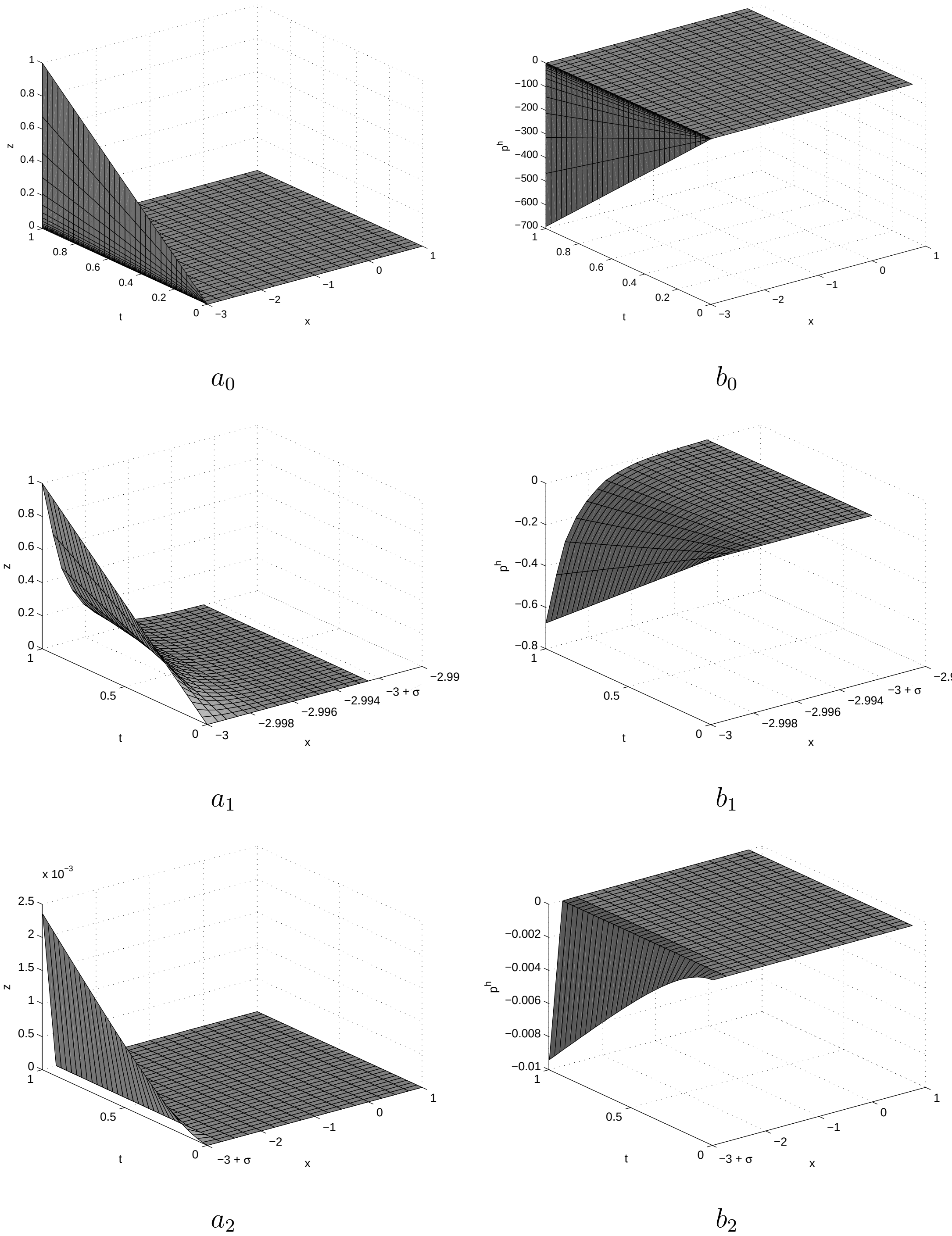


Figure 4.3: Plots of the solutions and the derivatives for $N = N_0 = 32$, $\varepsilon = 2^{-10}$, $\sigma = 0.0075$ generated by Scheme B′ applied to Problem (4.3.1), (a_0): Solution in $[-3, 1]$; (a_1): Zoom of the solution in $[-3, -3+\sigma]$; (a_2): Solution in $[-3+\sigma, 1]$, (b_i), $i = 0, 1, 2$ are the corresponding plots for derivatives.

Table 4.2: Errors E_ε^N of $z_{1(4.2.17b)}(x,t)$ for the first discrete derivatives generated by Scheme B′.

ε	Number of intervals N					
	32	64	128	256	512	1024
2^0	0.5963-01	0.3073-01	0.1560-01	0.7856-02	0.4491-02	0.2821-02
2^{-1}	0.1045+00	0.5805-01	0.3064-01	0.1575-01	0.7985-02	0.4020-02
2^{-2}	0.1440+00	0.9767-01	0.5565-01	0.2981-01	0.1544-01	0.7858-02
2^{-3}	0.1089+00	0.8439-01	0.5855-01	0.3744-01	0.2261-01	0.1403-01
2^{-4}	0.1047+00	0.8201-01	0.5736-01	0.3688-01	0.2234-01	0.1301-01
2^{-5}	0.1027+00	0.8092-01	0.5683-01	0.3664-01	0.2224-01	0.1296-01
2^{-6}	0.1018+00	0.8043-01	0.5659-01	0.3653-01	0.2220-01	0.1294-01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.1011+00	0.8001-01	0.5639-01	0.3645-01	0.2217-01	0.1293-01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.1011+00	0.8001-01	0.5639-01	0.3645-01	0.2217-01	0.1293-01
E^N	**0.1011+00**	**0.8001-01**	**0.5639-01**	**0.3645-01**	**0.2217-01**	**0.1293-01**

We solve problem (4.3.1) using the classical finite difference scheme (4.2.4) on piecewise uniform meshes (4.2.7) $\big($we denote it briefly by Scheme B′$\big)$. The parameters which we take for the following results are $\beta_{4.2.7b} = 2.0$, $m_{4.2.7b} = 0.9$.

An appropriate choice of β in the rectangle $\overline{G}$ is important to define the transition parameter $\sigma_{4.2.7b}$ fitted to the boundary layer, which determines the point of transition from a fine to a coarse mesh. More detailed discussion about appropriate selection of β is introduced in [20].

The plots of the solutions a_0 and the derivatives b_0 computed using Scheme B′ are presented on Fig. 4.3 for $\varepsilon = 2^{-10}$, $N = 32$ and $N_0 = 32$. The derivative $p^h(x,t) = p^N_{z,\varepsilon(4.3.4)}(x,t)$ in the boundary layer requires scaling, as for small values of ε the derivative is excessively large within the layer region. The appropriate scaling is $\varepsilon p^N_{z,\varepsilon(4.3.4)}(x,t)$. The plot of $\varepsilon p^N_{z,\varepsilon(4.3.4)}(x,t)$ is shown in b_1.

Error analysis of the discrete solutions and derivatives are made for values of $\varepsilon =$

Table 4.3: Convergence orders q_ε^N of $z_{1(4.2.17b)}(x,t)$ for the solutions generated by Scheme B′.

	Number of intervals N				
ε	32	64	128	256	512
2^{0}	0.9973	0.9779	0.9740	0.9760	0.9811
2^{-1}	0.9969	0.9981	0.9980	0.9995	1.0000
2^{-2}	0.6386	0.9161	0.9630	0.9799	0.9905
2^{-3}	0.5289	0.6281	0.6520	0.7794	0.7125
2^{-4}	0.5317	0.6385	0.6628	0.7727	0.7984
2^{-5}	0.5377	0.6377	0.6798	0.7613	0.8024
2^{-6}	0.5437	0.6382	0.6894	0.7560	0.8036
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.5512	0.6395	0.6997	0.7508	0.8062
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.5512	0.6383	0.7008	0.7510	0.8056
q^N	**0.5512**	**0.6383**	**0.7008**	**0.7510**	**0.8056**

Table 4.4: Convergence orders q_ε^N of $z_{1(4.2.17b)}(x,t)$ for the first discrete derivatives generated by Scheme B′.

	Number of intervals N				
ε	32	64	128	256	512
2^{0}	0.9564	0.9781	0.9897	0.8068	0.6708
2^{-1}	0.8481	0.9219	0.9601	0.9800	0.9901
2^{-2}	0.5601	0.8115	0.9006	0.9491	0.9744
2^{-3}	0.3679	0.5274	0.6451	0.7276	0.6884
2^{-4}	0.3524	0.5158	0.6372	0.7232	0.7800
2^{-5}	0.3439	0.5098	0.6332	0.7203	0.7791
2^{-6}	0.3399	0.5072	0.6315	0.7185	0.7787
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.3375	0.5047	0.6295	0.7173	0.7779
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.3375	0.5047	0.6295	0.7173	0.7779
q^N	**0.3375**	**0.5047**	**0.6295**	**0.7173**	**0.7779**

2^{-j}, $j = 0, 1, \ldots, 20$ on grids with the number of nodes $N = N_0$ for $N = 2^i$, $i = 5, 6, \ldots, 10$. The numerical solution $u_{0,\varepsilon}^{h,N^F}(x,t)$, generated by Scheme B′ on the finest mesh $\overline{G}_h^{N^F}$ with $N = N_0 = N^F = 2048$ for each value of ε is used as the exact solution of problem (4.3.1).

Errors in the numerical solutions in the maximum norm for each value of ε and N for Scheme B′ are computed by the formula

$$E_\varepsilon^N = E_\varepsilon^N\left(z_\varepsilon^N(\cdot)\right) = \left\|\overline{z}_\varepsilon^{N^F}(x,t) - z_\varepsilon^N(x,t)\right\|_{\,\overline{G}_h^N} \tag{4.3.2}$$

Here the function $z_\varepsilon^N(x,t) = z_{\varepsilon\,(4.2.4)}^N(x,t)$ and the function $\overline{z}_\varepsilon^N(x,t)$ in (4.3.2) are the numerical solution obtained by Schemes B′ and its bilinear interpolation respectively.

Table 4.1 shows the values of E_ε^N of errors in solution generated by Schemes B′ for different values of ε and N. The value E^N in the last row of the table is the maximal value of the errors E_ε^N for some fixed value of ε to the given value of N.

Table 4.2 displays errors for the first discrete derivatives computed by the formula

$$E_\varepsilon^N = E_\varepsilon^N\left(p_{z,\varepsilon}^N(\cdot)\right) = \left\|\overline{p}_{z,\varepsilon}^{N^F}(x,t) - p_{z,\varepsilon}^N(x,t)\right\|_{\,\overline{G}_h^N} \tag{4.3.3}$$

Here, $p_{z,\varepsilon}^N(x,t)$ in (4.3.3) is the first difference derivative

$$p_{z,\varepsilon}^N(x_i,\, t_j) = \frac{z_\varepsilon^N(x_{i+1},\, t_j) - z_\varepsilon^N(x_i,\, t_j)}{x_{i+1} - x_i}, \quad i = 0, \ldots, N, \quad j = 0, \ldots, N_0. \tag{4.3.4}$$

The function $z_\varepsilon^N(x,t)$ is the solution obtained from formula (4.2.17b).

The results in Tables 4.1 and 4.2 suggest that the method is ε-uniform convergent for the solution $z_{\varepsilon\,(4.2.4)}^N(x,t)$ and the first discrete derivative $p_{z,\varepsilon(4.3.3)}^N(x,t)$.

The convergence order q_ε^N for the discrete solutions and derivatives are defined by the formula

$$q_\varepsilon^N = \log_2 \frac{E_\varepsilon^N}{E_\varepsilon^{2N}}. \tag{4.3.5}$$

The quantities E_ε^N, E_ε^{2N} are defined by formula (4.3.2) for solutions and by formula (4.3.3) for derivatives for Scheme B′.

The corresponding computed order of convergence for solutions and derivatives in Tables 4.3 and 4.4 indicate that the ε-uniform order of convergence is at least 0.5 for all $N \le 32$.

4.3.2 Problem in Presence of Interior Layer

We have discussed in Chapter 3 the problem (3.7.1) in presence of interior layer in the domain $[-2,\, 2]$ on condition (3.4.25) or condition (3.6.3). Here, we consider the boundary layer problem (4.2.2), (4.2.1) with the same singularity of interior layer type as problem (3.7.1). We present the experimental results for the boundary value problem

$$\begin{aligned} L_{(4.3.6)}\, u(x,t) \;&\equiv\; \left\{\varepsilon\,\frac{\partial^2}{\partial x^2} + (1-\varepsilon)\,\frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial t}\right\} u(x,t) = 0, \quad (x,t) \in G, \quad (4.3.6\text{a}) \\ u(x,t) \;&=\; \varphi(x,t), \quad (x,t) \in S, \end{aligned}$$

that has the same singularity of the solution as problem (3.1.4), (3.1.3) in a neighbourhood of the interior layer. Note that the problem (4.3.6) is equivalent to problem (3.7.1).

Here

$$G = D \times (0,\, T], \quad D = \{x :\; x \in (d_1,\, d_2)\}, \qquad (4.3.6\text{b})$$

$$T = 1, \quad d_1 = -3, \quad d_2 = 1; \quad \varphi(x,0) = \begin{cases} 0, & d_1 < x \le 0, \\ x + m\,x^2, & 0 < x < d_2; \end{cases}$$

$$\varphi(d_1,t) = 0,$$

$$\varphi(d_2,t) = e^{-t}\Big(md_2^2 + \big(2mt(1-\varepsilon)+1\big)d_2 + \big(2m\varepsilon + (1-\varepsilon)\,\big)t + m(1-\varepsilon)^2t^2\Big),$$

where $m = 4^{-1}$.

Jump of the derivative to the function $\varphi_{(4.3.6)}(x,t)$ in the point $(0,t)$ is the same as for the function $\varphi_{(3.1.3a)}(x)$ in $x = 0$ and equals to 1 for both functions.

Note that

$$\varphi(x,t) = w(x,t), \quad (x,t) \in S, \quad x \geq 0.$$

Here

$$w(x,t) = e^{-t}\Big(mx^2 + \big(2mt(1-\varepsilon)+1\big)x + \big(2m\varepsilon + (1-\varepsilon)\,\big)t + m(1-\varepsilon)^2 t^2\Big),$$
$$(x,t) \in I\!\!R \times [0,\, T]$$

is the solution of the Cauchy problem

$$\begin{aligned} L_{(3.7.1)}\, w(x,t) &= 0, \qquad (x,t) \in I\!\!R \times (0,\, T], \\ w(x,0) &= \varphi_w(x), \quad x \in I\!\!R, \end{aligned} \tag{4.3.7}$$

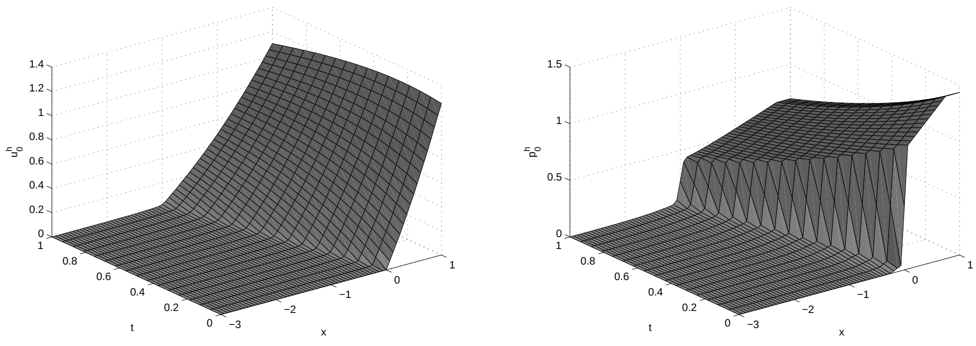
where $\varphi_w(x) = x + m\,x^2, \;\; x \in I\!\!R$.

Choice of boundary conditions for $x = d_1$, d_2 ensures compatibility conditions for data of problem (4.3.6) and prevents appearance of the boundary layer and the interior layer that is going from the point $(d_2, 0)$.

We solve problem (4.3.6) with the scheme of the splitting singularity method (3.6.3), (3.4.25) (Method A) as stated in chapter 4.

Fig. 4.4 shows plots of the solutions a_1 and the derivatives a_2 computed using Scheme A for $\varepsilon = 2^{-10}$, $N = 16$ and $N_0 = 16$.

The maximum errors in the solutions and the first discrete derivatives are shown in Tables 4.5 and 4.6 which are computed by formula (3.7.3) and (3.7.5) respectively. The corresponding orders of convergence computed by formula (3.7.8) for the solutions and the first discrete derivatives are shown in Tables 4.7 and 4.8.



a_1 a_2

Figure 4.4: Plots of the solution and the derivative for $\varepsilon = 2^{-10}$ generated by Scheme A applied to Problem (4.3.6), (a_1): Solution for $N = N_0 = 16$ with 3-node advanced interpolation in x-coordinate; (a_2): First discrete derivative for $N = N_0 = 16$ with 3-node advanced interpolation in x-coordinate .

Table 4.5: Errors $E^N_{\varepsilon(4.3.11)} = E^N_\varepsilon(u^{h,N}_{0,\varepsilon})$ and $E^N = E^N(u^{h,N}_{0,\varepsilon})$ for the solutions generated by Scheme A

	Number of intervals N					
ε	32	64	128	256	512	1024
2^{0}	0.1081-02	0.5445-03	0.2735-03	0.1370-03	0.6854-04	0.3428-04
2^{-1}	0.8732-03	0.4312-03	0.2144-03	0.1069-03	0.5339-04	0.2668-04
2^{-2}	0.2353-02	0.1210-02	0.6142-03	0.3094-03	0.1553-03	0.7781-04
2^{-3}	0.3529-02	0.1850-02	0.9487-03	0.4806-03	0.2419-03	0.1214-03
2^{-4}	0.4325-02	0.2295-02	0.1185-02	0.6025-03	0.3038-03	0.1525-03
2^{-5}	0.4820-02	0.2577-02	0.1336-02	0.6804-03	0.3434-03	0.1725-03
2^{-6}	0.5105-02	0.2743-02	0.1427-02	0.7288-03	0.3688-03	0.1856-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.5415-02	0.2939-02	0.1544-02	0.8002-03	0.4114-03	0.2102-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.5416-02	0.2940-02	0.1544-02	0.8006-03	0.4117-03	0.2104-03
E^N	**0.5416-02**	**0.2940-02**	**0.1544-02**	**0.8006-03**	**0.4117-03**	**0.2104-03**

Table 4.6: Errors $E^N_{\varepsilon(4.3.12)} = E^N_\varepsilon(p^{h,N}_{0,\varepsilon})$ and $E^N = E^N(p^{h,N}_{0,\varepsilon})$ for the first discrete derivatives generated by Scheme A.

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.1641-01	0.8314-02	0.4187-02	0.2106-02	0.1100-02	0.5835-03
2^{-1}	0.1563-01	0.7817-02	0.3974-02	0.2034-02	0.1048-02	0.5470-03
2^{-2}	0.1646-01	0.8685-02	0.4474-02	0.2272-02	0.1145-02	0.5749-03
2^{-3}	0.1768-01	0.9590-02	0.5022-02	0.2574-02	0.1304-02	0.6562-03
2^{-4}	0.1845-01	0.1024-01	0.5443-02	0.2816-02	0.1434-02	0.7237-03
2^{-5}	0.1892-01	0.1067-01	0.5743-02	0.2995-02	0.1533-02	0.7764-03
2^{-6}	0.1920-01	0.1096-01	0.5948-02	0.3122-02	0.1605-02	0.8152-03
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-14}	0.1954-01	0.1135E-01	0.6278-02	0.3958-02	0.2686-02	0.1835-02
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-20}	0.1954-01	0.1135-01	0.6280-02	0.3974-02	0.2708-02	0.1865-02
E^N	**0.1954-01**	**0.1135-01**	**0.6280-02**	**0.3974-02**	**0.2708-02**	**0.1865-02**

Table 4.7: Convergence orders $q^N_{\varepsilon(4.3.15)} = q^N_\varepsilon(u^{h,N}_{0,\varepsilon})$ and $q^N = q^N(u^{h,N}_{0,\varepsilon})$ for the solutions of Scheme A

	Number of intervals N				
ε	32	64	128	256	512
2^0	0.9894	0.9934	0.9974	0.9992	0.9996
2^{-1}	1.0180	1.0081	1.0040	1.0016	1.0008
2^{-2}	0.9595	0.9782	0.9892	0.9944	0.9970
2^{-3}	0.9317	0.9635	0.9811	0.9904	0.9946
2^{-4}	0.9142	0.9536	0.9759	0.9878	0.9943
2^{-5}	0.9033	0.9478	0.9735	0.9865	0.9933
2^{-6}	0.8962	0.9428	0.9694	0.9827	0.9906
⋮	⋮	⋮	⋮	⋮	⋮
2^{-14}	0.8816	0.9287	0.9482	0.9598	0.9688
⋮	⋮	⋮	⋮	⋮	⋮
2^{-20}	0.8814	0.9291	0.9475	0.9595	0.9685
q^N	**0.8814**	**0.9291**	**0.9475**	**0.9595**	**0.9685**

Table 4.8: Convergence orders $q^N_{\varepsilon(4.3.15)} = q^N_\varepsilon(p^{h,N}_{0,\varepsilon})$ and $q^N = q^N(p^{h,N}_{0,\varepsilon})$ for the first discrete derivatives generated by Scheme A.

	Number of intervals N				
ε	32	64	128	256	512
2^0	0.9810	0.9896	0.9914	0.9370	0.9147
2^{-1}	0.9996	0.9760	0.9663	0.9567	0.9380
2^{-2}	0.9224	0.9570	0.9776	0.9886	0.9940
2^{-3}	0.8825	0.9333	0.9642	0.9811	0.9907
2^{-4}	0.8494	0.9117	0.9508	0.9736	0.9866
2^{-5}	0.8264	0.8937	0.9392	0.9662	0.9815
2^{-6}	0.8089	0.8818	0.9299	0.9599	0.9773
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.7837	0.8543	0.6655	0.5593	0.5497
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.7837	0.8539	0.6602	0.5534	0.5381
q^N	**0.7837**	**0.8539**	**0.6602**	**0.5534**	**0.5381**

It follows the conclusion that for problem (4.3.6), the Scheme A is ε-uniform convergent for solutions and first discrete derivatives. The errors estimates data has a perfect match with the data for problem (3.7.1) in Chapter 3.

4.3.3 Problem in Presence of Interior and Boundary Layers

We have shown theoretical and numerical results in former sections for problem (4.2.2), (4.2.1) in presence of boundary and interior layer separately. In this section, we present experimental results for this boundary value problem with appearance of both interior and boundary layers.

We present experimental results for the boundary value problem (4.2.2), (4.2.1), where

$$\begin{aligned} L_{(4.3.8)}\, u(x,t) &= 0, \qquad (x,t) \in G, \qquad\qquad (4.3.8a)\\ u(x,t) &= \varphi(x,t), \quad (x,t) \in S. \end{aligned}$$

Here $L_{(4.3.8)} = L_{(4.2.2)}$ under the condition

$$a = c = p = 1, \quad b = 1 - \varepsilon; \qquad (4.3.8b)$$

$$G = D \times (0,\, T], \quad D = \{x:\ x \in (d_1,\, d_2)\} \qquad (4.3.8c)$$

with

$$T = 1.0, \quad d_1 = -3.0, \quad d_2 = 1.0; \quad \varphi(x,0) = \begin{cases} 0, & d_1 < x \le 0, \\ x + m\,x^2, & 0 < x < d_2; \end{cases}$$

$$\varphi(d_1, t) = t,$$

$$\varphi(d_2, t) = e^{-t}\Big(md_2^2 + \big(2mt(1-\varepsilon)+1\big)d_2 + \big(2m\varepsilon + (1-\varepsilon)\,\big)t + m(1-\varepsilon)^2t^2\Big)$$

with $m = 4^{-1}$.

Note that

$$\varphi(x,t) = w(x,t), \quad (x,t) \in S, \quad x \ge 0.$$

Here

$$w(x,t) = e^{-t}\Big(mx^2 + \big(2mt(1-\varepsilon)+1\big)x + \big(2m\varepsilon + (1-\varepsilon)\,\big)t + m(1-\varepsilon)^2t^2\Big),$$

$$(x,t) \in I\!R \times [0,\, T]$$

is the solution of the Cauchy problem

$$\begin{aligned} L_{(3.7.1)}\, w(x,t) &= 0, \qquad (x,t) \in I\!R \times (0,\, T], \qquad\qquad (4.3.9)\\ w(x,0) &= \varphi_w(x), \quad x \in I\!R, \end{aligned}$$

where $\varphi_w(x) = x + m\,x^2$, $x \in I\!R$.

the choice of boundary conditions for $x = d_2$ ensures compatibility conditions for data of problem (3.7.1) and prevents appearance of the boundary layer and the interior layer that is going from the point $(d_2, 0)$.

We apply the scheme (4.2.9) on the piecewise uniform mesh (4.2.7) $\big($we denote it briefly by Scheme A$'\big)$ to problem (4.3.8) which has solution in presence of both interior and boundary layers.

Table 4.9: Errors $E_\varepsilon^N(u^h_{0(4.2.17c)})$for the solutions generated by Scheme A$'$ for $x \in [-3, -3+\sigma]$ (in the boundary layer)

	Number of intervals N					
ε	32	64	128	256	512	1024
2^{0}	0.1183-02	0.6584-03	0.3639-03	0.1948-03	0.1029-03	0.5349-04
2^{-1}	0.5367-02	0.2680-02	0.1337-02	0.6677-03	0.3337-03	0.1668-03
2^{-2}	0.1491-01	0.1005-01	0.5296-02	0.2724-02	0.1382-02	0.6961-03
2^{-3}	0.1491-01	0.1137-01	0.7649-02	0.4884-02	0.2881-02	0.1775-02
2^{-4}	0.1806-01	0.1334-01	0.8808-02	0.5699-02	0.3355-02	0.1940-02
2^{-5}	0.2018-01	0.1470-01	0.9683-02	0.6203-02	0.3662-02	0.2121-02
2^{-6}	0.2146-01	0.1551-01	0.1018-01	0.6484-02	0.3843-02	0.2222-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.2291-01	0.1642-01	0.1074-01	0.6787-02	0.4037-02	0.2330-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.2291-01	0.1642-01	0.1074-01	0.6788-02	0.4038-02	0.2331-02
E^N	**0.2291-01**	**0.1642-01**	**0.1074-01**	**0.6788-02**	**0.4038-02**	**0.2331-02**

The plots of the solutions and derivatives computed by Scheme A$'$ for different scaled layers , i.e. the interior and boundary layers are presented in Fig. 4.5.

For compare, the plots of the solutions and derivatives which are resulted from the classical finite difference scheme (4.2.4) on piecewise uniform mesh (4.2.7) for problem (3.7.1a) and (4.3.8b) are shown in Fig. 4.6.

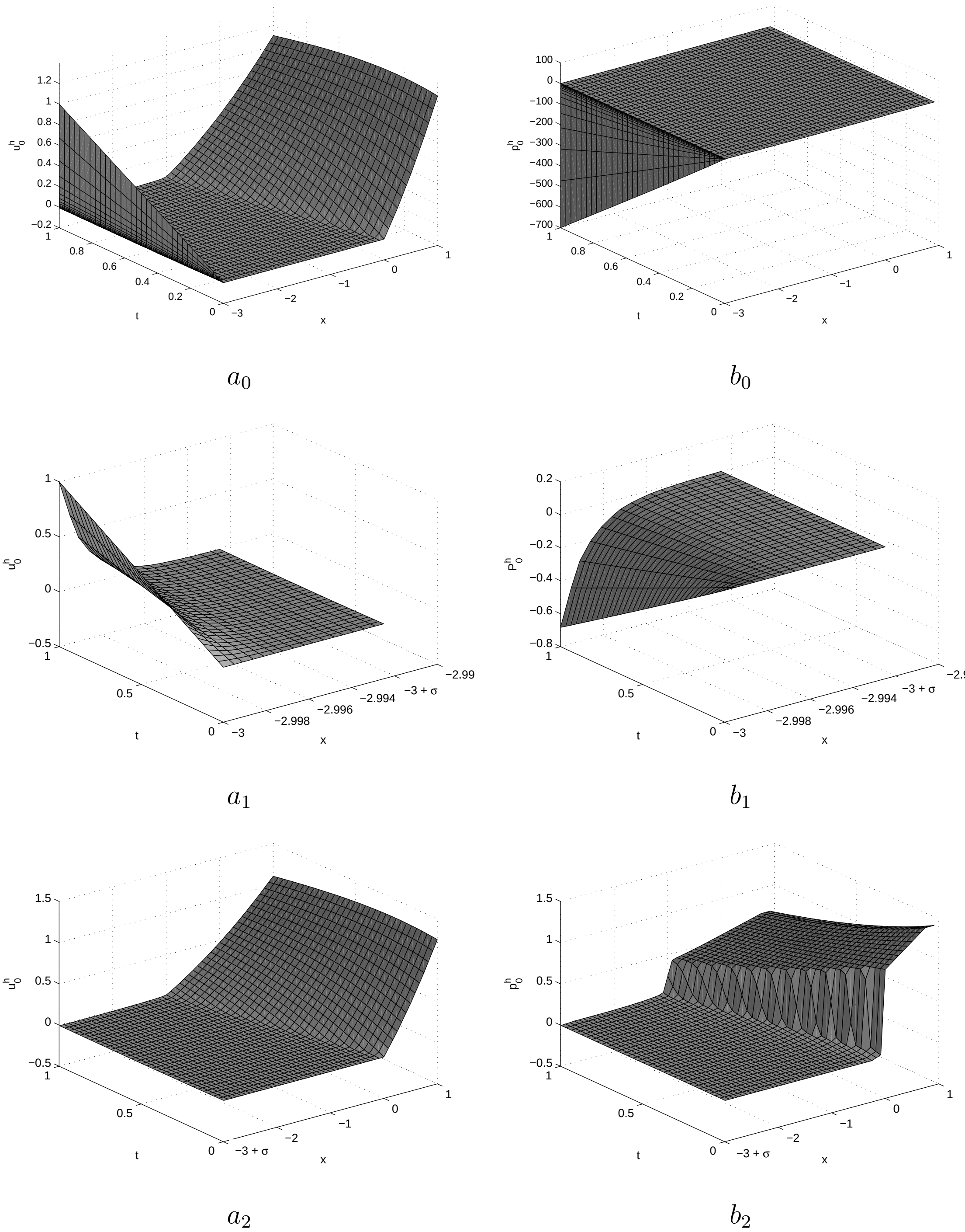


Figure 4.5: Plots of the solutions and the derivatives for $N = N_0 = 32$, $\varepsilon = 2^{-10}$, $\sigma = 0.0075$ generated by Scheme A′ applied to Problem (4.3.8), (a_0): Solution in $[-3, 1]$; (a_1): Zoom of the solution in $[-3, -3+\sigma]$; (a_2): Solution in $[-3+\sigma, 1]$ with 2-node advanced interpolation; (b_i), i = 0, 1, 2 are the corresponding plots for derivatives.

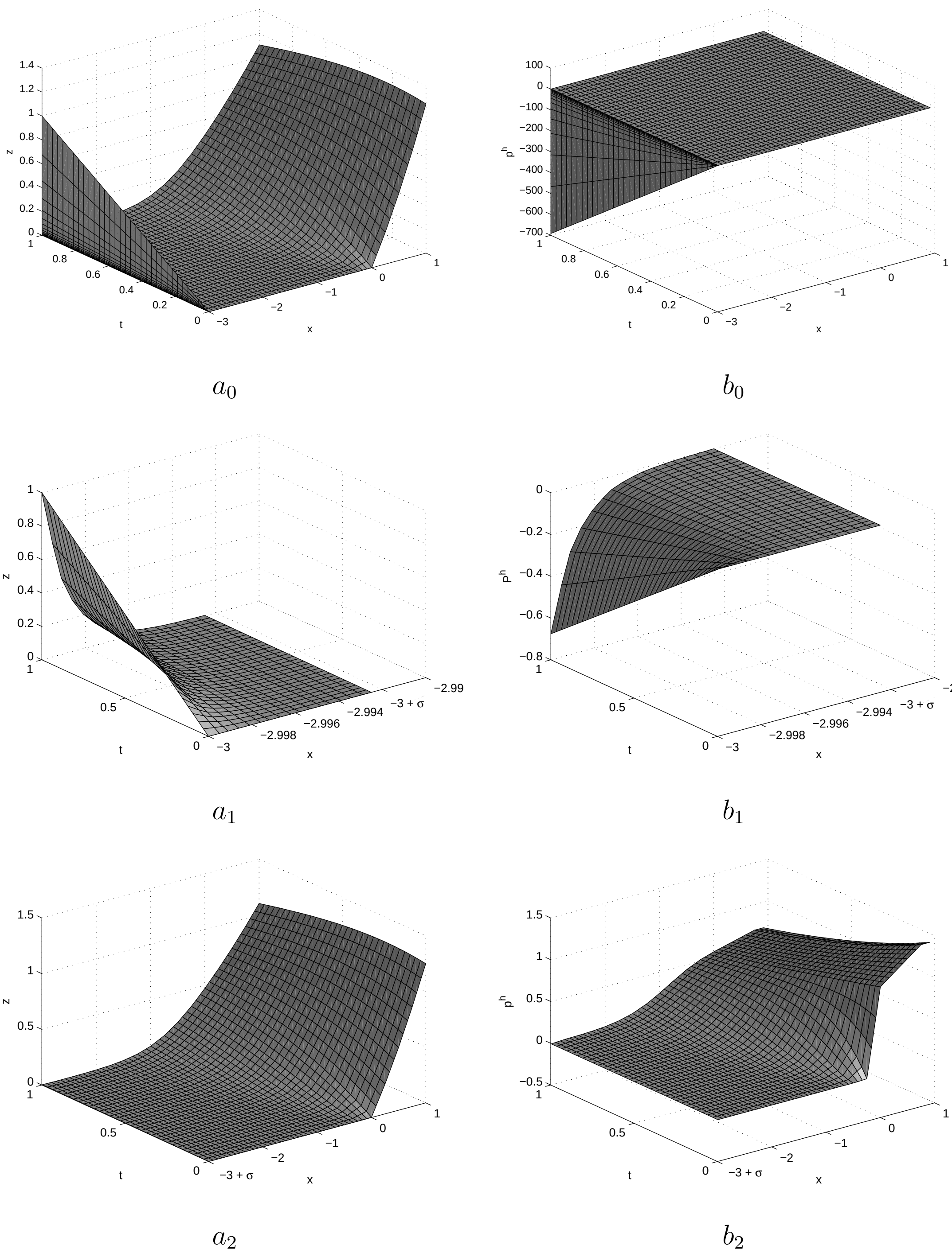


Figure 4.6: Plots of the solutions and the derivatives for $N = N_0 = 32$, $\varepsilon = 2^{-10}$, $\sigma = 0.0075$ generated by scheme (4.2.9) on the piecewise uniform mesh (4.2.7) applied to Problem (4.3.8), (a_0): Solution in $[-3, 1]$; (a_1): Zoom of the solution in $[-3, -3+\sigma]$; (a_2): Solution in $[-3+\sigma, 1]$ with 2-node bilinear interpolation; (b_i), $i = 0$, 1, 2 are the corresponding plots for derivatives.

Table 4.10: Errors $E_\varepsilon^N(u^h_{0(4.2.17c)})$ for the solutions generated by Scheme A$'$ for $x \in [-3+\sigma, 1]$ (outside the boundary layer)

ε	Number of intervals N					
	32	64	128	256	512	1024
2^0	0.9425-03	0.4698-03	0.2348-03	0.1173-03	0.5864-04	0.2946-04
2^{-1}	0.1166-02	0.6706-03	0.3580-03	0.1848-03	0.9383-04	0.4728-04
2^{-2}	0.1561-02	0.8450-03	0.4579-03	0.2379-03	0.1212-03	0.6115-04
2^{-3}	0.4049-02	0.2060-02	0.9938-03	0.4598-03	0.2072-03	0.8383-04
2^{-4}	0.5541-02	0.3035-02	0.1565-02	0.7818-03	0.3820-03	0.1848-03
2^{-5}	0.6548-02	0.3631-02	0.1913-02	0.9754-03	0.4870-03	0.2413-03
2^{-6}	0.7134-02	0.3987-02	0.2107-02	0.1078-02	0.5425-03	0.2706-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.7786-02	0.4394-02	0.2330-02	0.1191-02	0.6025-03	0.3042-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.7789-02	0.4396-02	0.2331-02	0.1191-02	0.6028-03	0.3044-03
E^N	**0.7789-02**	**0.4396-02**	**0.2331-02**	**0.1191-02**	**0.6028-03**	**0.3044-03**

To analyze errors in the discrete solutions, a technique similar to that given in [20] is used. However, it is modified with regard to the singularity splitting method. Computations are made for values of $\varepsilon = 2^{-j}$, $j = 0, 1, \ldots, 20$ on grids with the number of nodes $N = N_0$ for $N = 2^i$, $i = 5, 6, \ldots, 10$. The numerical solution $u^{h,N^F}_{0,\varepsilon}(x,t)$, generated by scheme scheme (4.2.9) with piecewise uniform mesh (4.2.7) on the finest mesh $\overline{G}_h^{N^F}$ with $N = N_0 = N^F = 2048$ for each value of ε is used as the exact solution of problem (4.3.8).

Errors for the numerical solutions in the boundary layer in the maximum norm for each value of ε and N are computed by the formula

$$E_\varepsilon^N = E_\varepsilon^N\big(z_\varepsilon^N(\cdot)\big) = \big\| u^{h,N^F}_{0,\varepsilon}(x,t) - z_\varepsilon^N(x,t) \big\|_{\overline{G}_h^N} \qquad (4.3.10)$$

Errors for the numerical solutions outside the boundary layer in the maximum norm for each value of ε and N are computed by the formula

$$E_\varepsilon^N = E_\varepsilon^N\big(u^{h,N}_{0,\varepsilon}(\cdot)\big) = \big\| u^{h,N^F}_{0,\varepsilon}(x,t) - u^{h,N}_{0,\varepsilon}(x,t) \big\|_{\overline{G}_h^N} \qquad (4.3.11)$$

Table 4.11: Errors $E_\varepsilon^N(P^h_{0(4.2.17e)})$ for the scaled first discrete derivatives generated by Scheme A′ for $x \in [-3, -3+\sigma]$ (in the boundary layer)

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.5629-01	0.2895-01	0.1468-01	0.7388-02	0.4465-02	0.2812-02
2^{-1}	0.1010+00	0.5603-01	0.2956-01	0.1519-01	0.7699-02	0.3876-02
2^{-2}	0.1411+00	0.9579-01	0.5456-01	0.2922-01	0.1513-01	0.7703-02
2^{-3}	0.1057+00	0.8250-01	0.5747-01	0.3685-01	0.2229-01	0.1387-01
2^{-4}	0.1015+00	0.8008-01	0.5625-01	0.3627-01	0.2202-01	0.1284-01
2^{-5}	0.9955-01	0.7897-01	0.5570-01	0.3602-01	0.2191-01	0.1279-01
2^{-6}	0.9865-01	0.7847-01	0.5546-01	0.3591-01	0.2187-01	0.1277-01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.9786-01	0.7804-01	0.5525-01	0.3582-01	0.2183-01	0.1276-01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.9786-01	0.7804-01	0.5525-01	0.3582-01	0.2183-01	0.1276-01
E^N	**0.9786-01**	**0.7804-01**	**0.5525-01**	**0.3582-01**	**0.2183-01**	**0.1276-01**

Here the function $u^{h,N}_{0,\varepsilon}(x,t) = u^{h,N}_{0(4.2.17c),\varepsilon}(x,t)$ in (4.3.11) and the function $z^N_\varepsilon(x,t)$ in (4.3.10) are the numerical solutions obtained by scheme (4.2.17d), (4.2.7) in the boundary layer and outside the boundary layer.

Tables 4.9 and 4.10 contain the value E^N_ε of errors of the solutions in the boundary layer and outside the boundary layer respectively, generated by schemes (4.2.17), (4.2.7) for various values of ε and N. The value E^N in the last rows of the tables is the maximal value of the errors E^N_ε with respect to ε, corresponding to the given value of N.

Similar to tables 4.9 and 4.10, tables 4.11 and 4.12 demonstrate errors of the first derivatives in the boundary layer and outside the boundary layer respectively, generated by schemes (4.2.17), (4.2.7) for various values of ε and N. The first

Table 4.12: Errors $E_\varepsilon^N(p^h_{0(4.2.17d)})$ for the first discrete derivatives generated by Scheme A′ for $x \in [-3+\sigma, 1]$ (outside the boundary layer)

	Number of intervals N					
ε	32	64	128	256	512	1024
2^0	0.1581-01	0.7964-02	0.4081-02	0.2107-02	0.1101-02	0.5837-03
2^{-1}	0.1563-01	0.7813-02	0.3981-02	0.2036-02	0.1049-02	0.5471-03
2^{-2}	0.1696-01	0.8216-02	0.4221-02	0.2141-02	0.1079-02	0.5413-03
2^{-3}	0.2619-01	0.1370-01	0.6788-02	0.3239-02	0.1510-02	0.6501-03
2^{-4}	0.2904-01	0.1636-01	0.8709-02	0.4445-02	0.2212-02	0.1084-02
2^{-5}	0.3030-01	0.1772-01	0.9795-02	0.5164-02	0.2642-02	0.1329-02
2^{-6}	0.3088-01	0.1843-01	0.1041-01	0.5585-02	0.2901-02	0.1477-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.3144-01	0.1919-01	0.1113-01	0.6134-02	0.3930-02	0.2633-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.3144-01	0.1919-01	0.1113-01	0.6137-02	0.3946-02	0.2656-02
E^N	**0.3144-01**	**0.1919-01**	**0.1113-01**	**0.6137-02**	**0.3946-02**	**0.2656-02**

derivatives in the boundary layer are computed by formula

$$E_\varepsilon^N = E_\varepsilon^N\left(p^{h,\,N}_{0,\,\varepsilon}(\cdot)\right) = \left\|p^{h,\,NF}_{0,\,\varepsilon}(x,t) - p^{h,\,N}_{0,\,\varepsilon}(x,t)\right\|_{\overline{G}_h^{N\,*}}, \qquad (4.3.12)$$
$$\overline{G}_h^{N\,*} = \overline{G}_h^{N} \setminus S^{(*)}, \quad S^{(*)} = S^{(*)}_{(4.2.2c)}$$

and the first derivatives outside the boundary layer by the formula

$$E_\varepsilon^N = E_\varepsilon^N\left(p^{N}_{z,\,\varepsilon}(\cdot)\right) = \left\|p^{h,\,NF}_{0,\,\varepsilon}(x,t) - p^{N}_{z,\,\varepsilon}(x,t)\right\|_{\overline{G}_h^{N\,\{*\}}}, \qquad (4.3.13)$$
$$\overline{G}_h^{N\,\{*\}} = \overline{G}_h^{N} \setminus S^{\{*\}}, \quad S^{\{*\}} = \{(x,0):\ x = x_{i-1},\, x_i,\, x_{i+1};\ x_i = 0\}$$

Here, $p^N_{z,\,\varepsilon}(x,t)$ in (4.3.13) is the first difference derivative

$$p^N_{z,\,\varepsilon}(x_i,\,t_j) = \frac{z^N_\varepsilon(x_{i+1},\,t_j) - z^N_\varepsilon(x_i,\,t_j)}{x_{i+1}-x_i}, \quad i = 0,\ldots,N, \quad j = 0,\ldots,N_0. \qquad (4.3.14)$$

The function $p^{h,\,NF}_{0,\,\varepsilon}(x,t)$ in formulae (4.3.12) and (4.3.13) and the function $p^{h,\,N}_{0,\,\varepsilon}(x,t)$ in formula (4.3.12) are the special interpolants of the first order derivative of the

Table 4.13: Convergence orders $q_\varepsilon^N(u^h_{0(4.2.17c)})$ for the solutions generated by Scheme A$'$ for $x \in [-3, -3+\sigma]$ (in the boundary layer)

	Number of intervals N				
ε	32	64	128	256	512
2^0	0.8454	0.8554	0.9015	0.9208	0.9439
2^{-1}	1.0019	1.0032	1.0017	1.0006	1.0004
2^{-2}	0.5691	0.9242	0.9592	0.9790	0.9894
2^{-3}	0.3910	0.5719	0.6472	0.7615	0.6988
2^{-4}	0.4370	0.5989	0.6281	0.7644	0.7903
2^{-5}	0.4571	0.6023	0.6425	0.7603	0.7879
2^{-6}	0.4685	0.6075	0.6508	0.7547	0.7904
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.4805	0.6125	0.6621	0.7495	0.7930
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.4805	0.6125	0.6619	0.7493	0.7927
q^N	**0.4805**	**0.6125**	**0.6619**	**0.7493**	**0.7927**

Table 4.14: Convergence orders $q_\varepsilon^N(u^h_{0(4.2.17c)})$ for the solutions generated by Scheme A$'$ for $x \in [-3+\sigma, 1]$ (outside the boundary layer)

	Number of intervals N				
ε	32	64	128	256	512
2^0	1.0044	1.0006	1.0012	1.0002	0.9931
2^{-1}	0.7980	0.9055	0.9540	0.9778	0.9888
2^{-2}	0.8854	0.8839	0.9447	0.9730	0.9870
2^{-3}	0.9749	1.0516	1.1119	1.1500	1.3055
2^{-4}	0.8684	0.9555	1.0013	1.0332	1.0476
2^{-5}	0.8507	0.9245	0.9718	1.0021	1.0131
2^{-6}	0.8394	0.9201	0.9668	0.9907	1.0035
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.8253	0.9152	0.9682	0.9831	0.9859
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.8252	0.9152	0.9688	0.9824	0.9857
q^N	**0.8252**	**0.9152**	**0.9688**	**0.9824**	**0.9857**

Table 4.15: Convergence orders $q_\varepsilon^N(P^h_{0(4.2.17e)})$ for the scaled first discrete derivatives generated by Scheme A$'$ for $x \in [-3, -3+\sigma]$ (in the boundary layer)

	Number of intervals N				
ε	32	64	128	256	512
2^{0}	0.9593	0.9797	0.9906	0.7265	0.6671
2^{-1}	0.8501	0.9226	0.9605	0.9804	0.9901
2^{-2}	0.5588	0.8120	0.9009	0.9495	0.9739
2^{-3}	0.3575	0.5216	0.6411	0.7253	0.6844
2^{-4}	0.3420	0.5096	0.6331	0.7200	0.7782
2^{-5}	0.3341	0.5036	0.6289	0.7172	0.7766
2^{-6}	0.3302	0.5007	0.6271	0.7154	0.7762
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.3265	0.4982	0.6252	0.7145	0.7747
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.3265	0.4982	0.6252	0.7145	0.7747
q^N	**0.3265**	**0.4982**	**0.6252**	**0.7145**	**0.7747**

Table 4.16: Convergence orders $q_\varepsilon^N(p^h_{0(4.2.17d)})$ for the first discrete derivatives generated by Scheme A$'$ for $x \in [-3+\sigma, 1]$ (outside the boundary layer)

	Number of intervals N				
ε	32	64	128	256	512
2^{0}	0.9893	0.9646	0.9537	0.9364	0.9155
2^{-1}	1.0004	0.9727	0.9674	0.9567	0.9391
2^{-2}	1.0456	0.9609	0.9793	0.9886	0.9952
2^{-3}	0.9348	1.0131	1.0674	1.1010	1.2158
2^{-4}	0.8279	0.9096	0.9703	1.0068	1.0290
2^{-5}	0.7739	0.8553	0.9236	0.9669	0.9913
2^{-6}	0.7446	0.8241	0.8983	0.9450	0.9739
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-14}	0.7122	0.7859	0.8596	0.6423	0.5778
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.7122	0.7859	0.8588	0.6371	0.5711
q^N	**0.7122**	**0.7859**	**0.8588**	**0.6371**	**0.5711**

solution computed by formula (4.2.17d), respectively, on the finest mesh $\overline{G}_h^{N^F}$ and on the mesh $\overline{G}_h^N$ for fixed value of ε.

The convergence order for the discrete solutions is defined by the formula

$$q_\varepsilon^N = \log_2 \frac{E_\varepsilon^N}{E_\varepsilon^{2N}}. \tag{4.3.15}$$

The quantities E_ε^N, E_ε^{2N} are defined by formula (4.3.10) for solution in the boundary layer and by formula (4.3.11) for solution outside the boundary layer.

Tables 4.13 and 4.14 are the corresponding orders of ε-uniform convergence for the solutions in the boundary layer and outside boundary respectively. The orders of the convergence rate to solutions goes to 0.8 for solution in the boundary layer and 1.0 for solution outside the boundary layer.

Similarly, Tables 4.15 and 4.16 display the orders of convergence for the first discrete derivatives of the solutions in the boundary layer and outside the boundary layer. The order of convergence goes to 0.8 and 0.5 respectively.

The error tables and orders of convergence for the solution and derivative in the boundary layer and outside boundary layer have a prefect match with considering the interior layer and boundary layer separately.

4.4 Conclusion

In this chapter, we consider the boundary value problem in bounded domain with appearing of interior and boundary layers with typical layer widths $\varepsilon^{1/2}$ and ε respectively. Using the method of piecewise uniform meshes that condense in a neighbourhood of the boundary layer and the method of additive splitting of the singularity of the interior layer type, a special finite difference scheme, i.e. Scheme A$'$, is constructed that make it possible to approximate ε-uniformly the solution of the boundary value problem on the whole domain, its first order derivative in x on the whole domain except the discontinuity point, however, outside a neighbourhood of the boundary layer, and also the normalized derivative (the first order spatial derivative multiplied by the parameter ε) in a finite neighbourhood of the boundary layer. Numerical experiments illustrates the efficiency of the constructed scheme.

Chapter 5

Conclusions and Future Work

A problem for the Black-Scholes equation with the value of a European call option that arises in financial mathematics, by transformation of variables, is reformulated to the Cauchy problem for a singularly perturbed parabolic equation with variables x, t and a perturbation parameter ε, $\varepsilon \in (0, 1]$. This problem has several singularities such as: the unbounded domain; the piecewise smooth initial function (its first order derivative in x has a discontinuity of the first kind at the point $x = 0$); an interior (moving in time) layer generated by the piecewise smooth initial function for small values of the parameter ε; etc.

In order to study the effect of these singularities on the errors in the numerical approximations, it is necessary to isolate them from each other in order to deal with them one at a time. The specific objective of the study was to construct difference schemes to approximate ε-uniformly the solution and its first order discrete derivative of the singularly perturbed Black-Scholes equation with the value of a European call option with nonsmooth initial conditions on various problems with appearing of different layers, i.e. interior layer and boundary layer.

5.1 Conclusion and Remarks

We prove that it is impossible to construct a parameter-uniform numerical method using a standard finite difference operator on a rectangular mesh for the the singularly perturbed Black-Scholes equation with interior layer type which coming from the discontinuity of the first derivative of the initial condition.

In Chapter 3, We construct a parameter-uniform numerical method theoretically which we call the method of splitting of singularity *(or briefly, the singularity splitting method)* for the problem. Numerical experiments prove that the solution and its first order derivative obtained by using this method converged ε-uniformly with a rate of convergence order close to 1.0 and 0.5 respectively. In comparison, the convergence rate of the solution obtained by the classical finite difference scheme is only 0.5. Moreover, The derivative computed by the classical finite difference scheme does not converge even for fixed values of the singular perturbation parameter ε.

We then finished the significant part of research related to financial mathematics, i.e. an accurate approximation of the first derivative of solution in a neighbourhood of a singularity appearing due to the discontinuity of the first derivative of the initial condition. This part is the most difficult for mathematicians and financial analysts. It is also significant step in applied mathematics, even for regular problems, when no singular perturbations are involved. Numerical technique and results have no analogy in existing experiments.

In Chapter 4, we considered the boundary value problem in bounded domains for parabolic equations coming from the Black-Scholes equation with a discontinuous initial condition. Thus, we have a boundary value problem with two different types of singularities, the discontinuity of the initial condition and the presence of small parameter multiplying the coefficients of the differential equation. Moreover, The singularity of the boundary layer is stronger than that of the interior layer, which makes it difficult to construct and study special numerical methods suitable for the adequate description of the singularity of the interior layer type. Using the method of special meshes that condense in a neighbourhood of the boundary layer and the method of additive splitting of the singularity of the interior layer type, a special finite difference scheme is designed that make it possible to approximate ε-uniformly the solution of the boundary value problem on the whole domain, its first order derivative in x on the whole domain except the discontinuity point (outside a neighbourhood of the boundary layer), and also the normalized derivative (the first order spatial derivative multiplied by the parameter ε) in a finite neighbourhood of the boundary layer.

About boundary layers only, this subject is more known (at least, for specialists in singular perturbed problems). But having two types of singularities, our problem became unpredictable and therefore interesting. This is a nontrivial extension that can also be applied to heat conduction and other problems.

This is the first time that the Black-Scholes equation is considered in a singular perturbation perspective. Construction and application of the singularity splitting method which ε-uniformly approximates the solution and derivative of the singular perturbed Black-Scholes equation are also novel.

This study is of considerable importance since it suggests that there are errors in existing finite difference methods for the Black-Scholes equation. These errors may results in predication of option prices that deviates significantly from actual prices.

Hopefully, our research will be of interest to other research areas beyond finance, e.g. fluid dynamics, when similar singularities occur.

5.2 Future Work

In this thesis, we focused on studying the one dimensional singularly perturbed Black-Scholes equation with European call options. The reason for starting with European options is because they are the simplest and their exact solution and derivatives can be expressed in simple closed analytical forms. On the other hand, in almost all other cases, the errors themselves must be approximated, which adds a further layer of difficulty to the study of the behaviour of the error for the relevant ranges of the free parameters.

It is known that an American option is determined by a linear complementarity problem involving the Black-Scholes differential operator and a constraint on the value of the option. Mathematically, it is a free boundary problem. So it is almost impossible to obtain an analytical solution for such a problem, numerical solutions are always sought in practice. So analyzing the parameter-uniform properties of the Black-Scholes equation with American options are even more on urging.

Future research work can be focused on:

1, To employ the singularity splitting method to other options, e.g. American options and Asian options;

2, To apply the method to high dimensional Black-Scholes equations with various options;

3, To improve the order of ε-uniform convergence rate. One feasible technique is to use defect-correction technique, which has proved to be useful for singularly perturbed parabolic convection-diffusion equations [29];

4, To possibly exploit the method to other research fields, e.g. fluid flow and electrodynamics.

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List of Publications

1). *Li S., Shishkina L.P. and Shishkin G.I.*, Numerical method for a singularly perturbed parabolic equation with a piecewise–smooth initial condition, the 6th Annual Workshop on Numerical Methods for Problems with Layer Phenomena, Department of Matematics & Statisitics in conjunction with Mathematics Applications Consortium for Science and Industry (MACSI), University of Limerick, Ireland, 8–9 February 2007.

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Degree: Doctor of Philosophy
Department: Mathematics
Thesis Title: Parameter-Uniform Numerical Methods for Problems with Layer Phenomena: Application in Mathematical Finance

Abstract

Mathematical modeling in financial mathematics leads to the Cauchy problem for the parabolic Black-Scholes equation with respect to the value of a European call option. By changing of variables, the problem is a singularly perturbed equation with the perturbation parameter ε, $\varepsilon \in (0, 1]$; For finite values of the parameter ε, the solution of the Cauchy problem has different types of singularities: the unbounded domain; the piecewise smooth initial function and its unbounded growth at infinity; an interior layer generated by the piecewise smooth initial function for small values of the parameter ε; etc.

Primarily, we are interested in approximations to both the solution and its first order derivative in a neighbourhood of the interior layer generated by the piecewise smooth initial function. For this purpose, a new method which we call *the method of additive splitting of a singularity(or briefly, the singularity splitting method)* of the interior layer type is constructed. The numerical results verifies that using singularity splitting method, we can approximate ε-uniformly both the solution of the boundary value problem and its first order derivative in x with convergence orders close to 1 and 0.5, respectively, whereas the classical finite difference method does not.

Moreover, in order to construct adequate grid approximations for the singularity of the interior layer type, we consider the boundary value problem in bounded domain with appearing of interior and boundary layers with typical layer widths $\varepsilon^{1/2}$ and ε respectively. The singularity of the boundary layer is stronger than that of the interior layer, which makes it difficult to construct and study special numerical methods suitable for the adequate description of the singularity of the interior layer type. Using the method of piecewise uniform meshes that condense in a neighbourhood of the boundary layer and the method of additive splitting of the singularity of the interior layer type, a special finite difference scheme is constructed that make it possible to approximate ε-uniformly the solution of the boundary value problem on the whole domain, its first order derivative in x on the whole domain except the discontinuity point, however, outside a neighbourhood of the boundary layer, and also the normalized derivative (the first order spatial derivative multiplied by the parameter ε) in a finite neighbourhood of the boundary layer.

Numerical experiments illustrates the efficiency of the constructed schemes.

Keywords: Black-Scholes Equation, Singular Perturbation, Boundary Layer, Interior Layer, Singularity Splitting Method, Piecewise Uniform Mesh.