# DISCRETE-TIME MEAN-VARIANCE PORTFOLIO SELECTION WITH TRANSACTION COSTS

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# Abstract

Transaction cost is a realistic feature in financial markets, which however is often ignored for the convenience of modeling and analysis. This thesis incorporates proportional transaction costs into the mean-variance formulation, and studies the optimal asset allocation policy in two kinds of single-period markets under the influence of transaction costs. The optimal asset allocation strategy is completely characterized in a market consisting of one riskless asset and one risky asset. Analytical expression for the optimal portfolio is derived, and the so-called "burn-money" phenomenon is observed by examining the stability of the optimal portfolio. In the market consisting of one riskless asset and two risky assets, we provide a detailed scheme for obtaining the optimal portfolio, whose analytical solution can be very complicated. We also study the no-transaction region and some special asset allocation strategies by the scheme.
**Key Words**: asset allocation, portfolio section, mean-variance formulation, transaction costs, no-transaction region, Sharpe Ratio.

# Notations and Assumptions

$b$: the coefficient of buying transaction cost

$s$: the coefficient of selling transaction cost

*For every \$1 worth of stock you buy, you pay \$*$(1+b)$*; for every \$1 stock you sell, you receive \$*$(1-s)$*.*

$e_0$: the single-period deterministic return of the bank account

$e_i$: the single-period random return of a stock

$\sigma_i$: volatility of a stock

$\rho$: the correlation between the return of stock 1 and stock 2

$x_0$: holdings in the bank account

$x_i$: holdings in stock $i$

Denote

$$\begin{cases} A_i = (1-s)e_i - (1+b)e_0 \\ A_i' = (1-s)(e_i - e_0) \\ A_i'' = (1+b)e_i - (1-s)e_0 \\ A_i''' = (1+b)(e_i - e_0) \end{cases} \qquad \begin{cases} B_1 = e_0[x_0 + (1+b)x_1]; B_2 = e_0[x_0 + (1-s)x_1] \\ \beta_1 = e_0[x_0 + (1+b)x_1 + (1+b)x_2] \\ \beta_2 = e_0[x_0 + (1-s)x_1 + (1+b)x_2] \\ \beta_3 = e_0[x_0 + (1+b)x_1 + (1-s)x_2] \\ \beta_4 = e_0[x_0 + (1-s)x_1 + (1-s)x_2]. \end{cases}$$

Assume $E[A_i] > 0$, for $i = 1, 2$.

*If you own \$(1+b) in bank account,* $E[A_i]$ *means the expected excess monetary profit if you were to invest the money in stock* $i$*.*

# Contents

# Chapter 1

# Introduction

One prominent problem in mathematical finance is portfolio selection. Portfolio selection is to seek the best allocation of wealth among a basket of securities. The mean-variance model by Markowitz (1959,1989) [7] [8] provided a fundamental framework for the study of portfolio selection in a single-period market. The most important contribution of this model is that it quantifies the risk by using the variance, which enables investors to seek the highest return after specifying their acceptable risk level (Zhou and Li (2000) [15]). As a tribute to the importance of his contribution, Markowitz was rewarded the Nobel Prize for Economics in 1990. An analytical solution of the mean-variance efficient frontier in the single period was obtained in Markowitz (1956) [6] and in Merton (1972) [10].

After Markowitz's pioneering work, single-period portfolio selection was soon extended to multi-period settings. See for example, Mossin (1968) [11], Samuelson (1969) [12] and Hakansson (1971) [2]. Researches on multi-period portfolio selections have been dominated by those of maximizing expected utility functions of the terminal wealth, namely maximizing $E[U(X(T))]$

where $U$ is a utility function of a power, log, exponential or quadratic form. The term $(E[x(T)])^2$ in Markowitz's original mean-variance formulation however, is of the form $U(E[x(T)])$ where $U$ is nonlinear. This posed a major difficulty to multi-period mean-variance formulations due to their non-separability in the sense of dynamic programming. This difficulty was solved by Li and Ng (2000) [3] by embedding the original problem into a tractable auxiliary problem. In a separate paper, similar embedding technique was used again to study the continuous-time mean-variance portfolio selection by Zhou and Li (2000) [15].

Another development in portfolio selection is the extension of a frictionless market to one with transaction costs. Historically, Merton (1971) [9] pioneered in applying continuous-time stochastic models to the study of portfolio selection. In the absence of transaction costs, he showed that the optimal investment policy of a $CRRA$ investor is to keep a constant fraction of total wealth in the risky asset. In 1976, Magil and Constantinides [5] incorporated proportional transaction costs into Merton's model and proposed that the shape of the no-transaction region is a wedge. Almost all the subsequent work along this direction has concentrated on the infinite horizon problem. See for example, Shreve and Soner (1994) [13]. Theoretical analysis on the finite horizon problem has been possible only very recently. See Liu and Loewenstein (2002) [4], Dai and Yi (2006) [1]. The continuous-time mean-variance model with transaction costs have recently been studied in Xu [14].

To the best of our knowledge, no results have been reported in the literature with regard to the discrete-time mean-variance model with transaction costs. The work presented in this thesis is an effort to extend Markowitz's mean-variance formulation to incorporate transaction costs in a discrete-time market setting. Li and Ng (2000) [3] solved the multi-period discrete-time mean-variance problem without transaction costs. In their paper, the original non-separable problem is embedded into a tractable auxiliary problem, and the method of dynamic programming is then applied to the auxiliary problem to obtain the solution. In this thesis, we consider proportional transaction costs, where transaction fees are charged as a fixed percentage of the amount transacted. We will follow the embedding technique in Li and Ng (2000) [3], and provide solution to the last investment stage of the multi-period problem with transaction costs. The solution we obtained will be needed when applying dynamic programming going backward in time-steps to solve the multi-period problem. We leave this to future research work.

We first look at the market consisting of one risky asset and one riskless asset, and then we move on to examine the market consisting of two risky assets and one riskless asset. In the market consisting one risky and one riskless asset, we present a complete analytical solution. We also derive the analytical expressions of the boundaries of the "no-transaction region". We show that if the initial holdings fall out of this no-transaction region, then the optimal asset allocation strategy is to bring the allocation to the nearest boundary of the no-transaction region.

It is to be noted that a feature results from transaction costs is that wealth can be disposed of by the investor of his own free will. This is achieved by continuingly buying and selling a stock and paying for the transaction fees. In the market consisting of one risky and one riskless asset, such phenomenon is indeed observed. It happens when the target investment return is too low.

In this case, the one-step solution is found to be unstable. As a result, a sequence of continuing buying and selling of the stock is required until the solution reaches stable state. As money is deliberately disposed of in this process, we call this phenomenon the "burn-money phenomenon".

To rule out the burn-money phenomenon, we assume the target investment return is of a sufficient high level in the market consisting of 2 risky assets and 1 riskless asset. In this market, we work out a complete scheme to find the optimal asset allocation strategy. We also derive a necessary and sufficient condition for a certain asset allocation strategy to be within the no-transaction region. One particular strategy is discussed in this market: when the Sharpe Ratio (with transaction costs) of the first stock is much higher then the Sharpe Ratio of the second stock, we find out that the optimal strategy implies we should not invest in the second stock at all. This confirms our intuition that stocks with higher Sharpe Ratio is preferable over stocks with lower Sharpe Ratio. Before we move on to examine the first market, we introduce the general problem settings in the rest of this introductory chapter.

## 1.1 Multi-period mean-variance formulation

Mathematically, a general mean-variance formulation for multi-period portfolio selection without transaction costs can be posed as one of the following two forms:

$$
\begin{aligned}
(P1(\sigma)) \qquad & \max_{u_t} E[x_T] \\
& s.t. \quad Var[x_T] \leq \sigma \\
& x_{t+1} = \sum_{i=1}^{n} e_t^i u_t^i, \\
& \sum_{i=1}^{n} u_t^i = x_t, \quad t = 0, 1, ..., T-1;
\end{aligned} \tag{1.1.1}
$$

and

$$
\begin{aligned}
(P2(\epsilon)) \qquad & \min_{u_t} Var[x_T] \\
& s.t. \quad E[x_T] \geq \epsilon \\
& x_{t+1} = \sum_{i=1}^{n} e_t^i u_t^i, \\
& \sum_{i=1}^{n} u_t^i = x_t, \quad t = 0, 1, ..., T-1.
\end{aligned} \tag{1.1.2}
$$

Here initial total wealth $x_0$ is given. $x_T$ represents final total wealth. $u_t^i$ is the amount invested in the $i$-th asset at the $t$-th period. The sequence of vectors $\mathbf{u_t}$ is our control. An equivalent formulation to either $(P1(\sigma))$ or $(P2(\epsilon))$ is

$$
\begin{aligned}
(E(\omega)) \qquad & \max_{u_t} E[x_T] - \omega Var[x_T] \\
& s.t. \quad x_{t+1} = \sum_{i=1}^{n} e_t^i u_t^i, \\
& \sum_{i=1}^{n} u_i = x_t, \quad t = 0, 1, ..., T-1.
\end{aligned} \tag{1.1.3}
$$

In Li and Ng (2000)'s paper, an auxiliary problem is constructed for

$(E(\omega))$. This auxiliary problem takes the following form.

$$\begin{aligned}
(A(\lambda)) \qquad & \max_{u_t} E\{-x_T^2 + \lambda x_T\} \\
& s.t. \quad x_T = \sum_{i=1}^{n} e_t^i u_t^i, \\
& \sum_{i=1}^{n} u_t^i = x_t, \quad t = 0, 1, ..., T-1.
\end{aligned} \tag{1.1.4}$$

Li and Ng (2000) established the necessary and sufficient conditions for a solution to $(A(\lambda))$ to be a solution of $(E(\omega))$. They also used the problem setting of $(A(\lambda))$ to obtain analytical solutions to $(E(\omega))$ in their paper. The problem setting of $(A(\lambda))$ was favored over $(E(\omega))$ because of the separable structure of $(A(\lambda))$ in the sense of dynamic programming. We will adopt the problem setting of $(A(\lambda))$ is our subsequent discussion.

## 1.2 The last stage with transaction costs

When transaction cost is considered, total wealth $x_t$ will not be enough to describe the state of the current investment. Instead, we have to specify the holdings $x_i$ in each individual asset at each time period. The terminal wealth will be calculated as the monetary value of the final portfolio, which is equal to the total cash amount when long stocks are sold and short stocks are bought back. In addition, the constraints in the optimization problem will become non-smooth. Despite these differences, it is still possible to apply the method of dynamic programming to the problem setting with transaction costs, if we adapt the objective function $\max_{u_t} E\{-x_T^2 + \lambda x_T\}$ from the separable auxiliary problem constructed above. In order to obtain solutions to the multi-period problem by the method of dynamic programming, we should start from the last investment stage of the problem. After we obtain the solution to the last stage, we can then go backwards stage by stage and obtain the sequence of optimal investment strategies. The solution to the last investment stage of the multi-period problem with transaction costs is what we deal with in this thesis.

In a market consisting of one riskless asset and $n$ risky assets, the problem setting for the last stage of the multi-period mean-variance formulation with transaction costs can be written as

$$
\begin{aligned}
&\max_{u_i} E\{-x_T^2 + \lambda x_T\} \\
&s.t. \quad x_T = e_0 u_0 + (1-s)e_1 u_1^+ - (1+b)e_1 u_1^- \\
&\qquad\qquad + (1-s)e_2 u_2^+ - (1+b)e_2 u_2^- \\
&\qquad\qquad + (1-s)e_3 u_3^+ - (1+b)e_3 u_3^- \\
&\qquad\qquad \cdots\cdots \\
&\qquad\qquad + (1-s)e_n u_n^+ - (1+b)e_n u_n^- \\
&\qquad u_0 = x_0 - (1+b)(u_1 - x_1)^+ + (1-s)(u_1 - x_1)^- \\
&\qquad\qquad - (1+b)(u_2 - x_2)^+ + (1-s)(u_2 - x_2)^- \\
&\qquad\qquad - (1+b)(u_3 - x_3)^+ + (1-s)(u_3 - x_3)^- \\
&\qquad\qquad \cdots\cdots \\
&\qquad\qquad - (1+b)(u_n - x_n)^+ + (1-s)(u_n - x_n)^-,
\end{aligned}
\tag{1.2.1}
$$

or in a more compact form

$$
\begin{aligned}
&\max_{u_i} E\{-x_T^2 + \lambda x_T\} \\
&s.t. \quad x_T = e_0 u_0 + (1-s)\sum_{i=1}^n e_i u_i^+ - (1+b)\sum_{i=1}^n e_i u_i^- \\
&u_0 = x_0 - (1+b)\sum_{i=1}^n (u_i - x_i)^+ + (1-s)\sum_{i=1}^n (u_i - x_i)^-.
\end{aligned}
\tag{1.2.2}
$$

Here $x_i$ denotes the initial amount invested in the $i$-th asset. $u_i$'s are our controls, namely, we would like to adjust each $x_i$ to the amount $u_i$. $x_T$ is the final total monetary wealth. $\lambda$ is the same as in the multi-period setting without transaction costs. It is to be noted that the value of $\lambda$ is chosen at the very beginning of the investment horizon and will remain constant throughout all investment stages. In particular, if we assume the investor's position is known at the beginning of the last investment stage, then we should have no information about how big $\lambda$ is, relative to the investor's position. As it turns out, in our subsequent discussions, this relation between $\lambda$ and the investor's current position is critical in determining the investor's strategies.

# Chapter 2

# One Risky Asset

## 2.1 Optimal strategies

Consider a market consisting of one riskless (bank account) and one risky asset (a stock). Assume at the initial time, the amount an investor holds in bank account is $x_0$, and the single-period return for the bank account is a deterministic number $e_0$; the amount he holds in stock is $x_1$, the return of the stock is a random variable $e_1$. Suppose our strategy is to adjust the amount in stock from $x_1$ to an optimal amount $u_1$. (In case $u_1 = x_1$, no adjustment is needed.) In the process of buying or selling stocks, transaction fees are charged. We treat transaction costs in the following manner: when we buy \$1 worth of stock, we pay \$$(1+b)$; when we sell \$1 worth of stock, we receive \$$(1-s)$. The optimization problem in this market can be written as

$$
\begin{aligned}
&\max_{u_1}\ E\{-x_T^2 + \lambda x_T\} \\
&\quad s.t.\ x_T = e_0[x_0 - (1+b)(u_1 - x_1)^+ + (1-s)(x_1 - u_1)^+] \\
&\qquad\qquad + (1-s)e_1(u_1)^+ - (1+b)e_1(-u_1)^+
\end{aligned}
\tag{2.1.1}
$$

Let $\lambda' = \frac{1}{2}\lambda$, and

$$\begin{cases} P_1 = B_1 + \dfrac{x_1 E[(A_1)^2]}{E[A_1]}, \\ P_2 = B_2 + \dfrac{x_1 E[(A_1')^2]}{E[A_1']}. \end{cases}$$

**Theorem 2.1.1** *Solution to (2.1.1), the Main Theorem of Chapter 2.*

*(1) When $x_1 \geq 0$, the optimal $u_1^*$ in (2.1.1) is given by*

$$\begin{cases} u_1^* = \dfrac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]} > x_1, & \text{when } \lambda' > P_1, \\ u_1^* = x_1, & \text{when } P_2 \leq \lambda' \leq P_1, \\ u_1^* = \dfrac{(\lambda' - B_2)E[A_1']}{E[(A_1')^2]} \in (0, x_1), & \text{when } B_2 \leq \lambda' < P_2, \\ u_1^* = 0 \quad \text{burn money}, & \text{when } \lambda' < B_2. \end{cases} \tag{2.1.2}$$

*Let $V$ be the objective value, the corresponding optimal objective value $V^*$ is given by*

$$V^* = \begin{cases} -\dfrac{(\lambda' - B_1)^2 VAR[A_1]}{E[(A_1)^2]} + \lambda'^2, & \lambda' > P_1, \\ -E[(e_0 x_0 + (1-s)e_1 x_1 - \lambda')^2] + \lambda'^2, & P_2 \leq \lambda' \leq P_1, \\ -\dfrac{(\lambda' - B_2)^2 VAR[A_1']}{E[(A_1')^2]} + \lambda'^2, & B_2 \leq \lambda' < P_2 \\ \lambda'^2, & \lambda' < B_2. \end{cases} \tag{2.1.3}$$

*(2) When $x_1 < 0$, the optimal $u_1^*$ in (2.1.1) is given by*

$$\begin{cases} u_1^* = \dfrac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]} > 0, & \text{when } \lambda' \geq B_1, \\ u_1^* = 0 \quad \text{burn money}, & \text{when } \lambda' < B_1. \end{cases} \tag{2.1.4}$$

*The corresponding optimal objective value $V^*$ is given by*

$$V^* = \begin{cases} -\dfrac{(\lambda' - B_1)^2 VAR[A_1]}{E[(A_1)^2]} + \lambda'^2, & \lambda' \geq B_1, \\ \lambda'^2, & \lambda' < B_1. \end{cases} \tag{2.1.5}$$

The rest of this chapter is mostly to establish this theorem. Part (1) in Theorem 2.1.1 corresponds to the case when the investor starts with a long position in the stock; part (2) corresponds to the case when the investor starts with a short position in the stock. In order to obtain the results in Theorem 2.1.1, we look at the following 6 cases.

(1) $x_1 \geq 0$, (2) $x_1 < 0$,

$$\begin{cases} u_1 > x_1, & \text{case 1;} \\ 0 \leq u_1 \leq x_1, & \text{case 2;} \\ u_1 < 0, & \text{case 3;} \end{cases} \qquad \begin{cases} u_1 > 0, & \text{case 4;} \\ x_1 \leq u_1 \leq 0, & \text{case 5;} \\ u_1 < x_1, & \text{case 6.} \end{cases} \tag{2.1.6}$$

We examine the two parts separately in subsequent discussion.

### 2.1.1 Optimal strategy with a long position in stock

In this part, we assume

$$x_1 \geq 0.$$

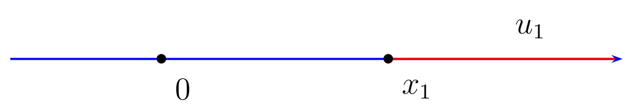
We distinguish the following 3 kinds of strategies, each of which corresponds to a different form of objective function.

$$\begin{cases} u_1 > x_1, & \text{case 1;} \\ 0 \leq u_1 \leq x_1 & \text{case 2;} \\ u_1 < 0, & \text{case 3;} \end{cases}$$

Case 1 represents the strategy to purchase more stocks; Case 2 represents the strategy to sell off some stocks but avoid a short position in stock; Case 3 represents the strategy to sell more stocks than we currently own (short sell), and therefore assumes a short position in stock.

Under different parameter settings (parameters include $b, s, e_0, e_1$ and $\lambda$), we wish to identify the strategy that dominates all other strategies, namely gives a better objective value than the rest to $E\{-x_T^2 + \lambda x_T\}$. For a given parameter setting, the best strategy among the 3 is the optimal strategy.

**Case 1.** $x_1 \geq 0, u_1 > x_1$. The strategy of buying more stocks.




In this case,

$$\begin{aligned} x_T &= e_0[x_0 - (1+b)(u_1 - x_1)] + (1-s)e_1 u_1 \\ &= [(1-s)e_1 - (1+b)e_0]u_1 + e_0[x_0 + (1+b)x_1]. \end{aligned} \tag{2.1.7}$$

Let

$$\begin{cases} A_1 = (1-s)e_1 - (1+b)e_0 \\ B_1 = e_0[x_0 + (1+b)x_1]. \end{cases} \tag{2.1.8}$$

So now $x_T$ can be written as

$$x_T = A_1 u_1 + B_1. \tag{2.1.9}$$

Here $A_1$ has the following financial meaning. Suppose an investor has \$$(1+b)$ cash amount in his hands. He has two investment options. If he puts the money in the bank, he will get a sure return of \$$(1+b)e_0$ at the end of the single-period investment horizon; If he invests the money in the stock, with the money he can purchase \$1-worth of stock due to buying transaction costs. At the end of the investment horizon, the \$1-worth of stock will become $\$e_1$. After he cashes in the holdings in stock, $\$(1-s)e_1$ is what he will get in monetary terms due to selling transaction costs. So $A_1$ means the excess return of investment in the risky asset over the riskless asset. It is thus reasonable to assume

$$E[A_1] > 0,$$

for otherwise, investing in stock will yield a lower expected return yet the investor has to bear a higher level of risk, making investment in stocks much like a lottery game or a unfair gambling game.

To solve the maximization problem, we have

$$\begin{aligned}
& \frac{dE[-x_T^2 + \lambda x_T]}{du_1} = 0 \\
\Rightarrow \quad & E[-2x_T \frac{dx_T}{du_1} + \lambda \frac{dx_T}{du_1}] = 0 \\
\Rightarrow \quad & E[-2(A_1 u_1 + B_1)A_1 + \lambda A_1] = 0 \\
\Rightarrow \quad & -E[(A_1)^2]u_1 + (\lambda' - B_1)E[A_1] = 0 \qquad (\lambda' = \frac{\lambda}{2}) \\
\Rightarrow \quad & u_1 = \frac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]}.
\end{aligned}$$

From above calculations, we know that if we adopt the strategy to buy more stocks, the best values of $u_1$ are given by

$$\begin{cases} u_1 = \dfrac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]}, & \text{when } \dfrac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]} > x_1, \\ u_1 = x_1, & \text{when } \dfrac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]} \leq x_1. \end{cases} \tag{2.1.10}$$

As $E[A_1] > 0$, the above results are equivalent to

$$\begin{cases} u_1 = \dfrac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]}, & \text{when } \lambda' > B_1 + \dfrac{x_1 E[(A_1)^2]}{E[A_1]}, \\ u_1 = x_1, & \text{when } \lambda' \leq B_1 + \dfrac{x_1 E[(A_1)^2]}{E[A_1]}. \end{cases} \tag{2.1.11}$$

In the following, for simplicity reason let us denote

$$P_1 \doteq B_1 + \frac{x_1 E[(A_1)^2]}{E[A_1]}.$$

With the values of $u_1$ obtained in (2.1.1), we can now calculate the optimal objective value of $V_1 = E\{-x_T^2 + \lambda x_T\}$ under the strategy of buying more stocks. The optimal objective values are summarized below followed by a detailed calculation.

$$\begin{cases} V_{1\{\lambda' > P_1\}} = -\dfrac{(\lambda' - B_1)^2 VAR[A_1]}{E[(A_1)^2]} + \lambda'^2, \\ V_{1\{\lambda' \leq P_1\}} = -E[(e_0 x_0 + (1-s)e_1 x_1 - \lambda')^2] + \lambda'^2. \end{cases} \tag{2.1.12}$$

Note that

$$\begin{cases} V_{1\{\lambda' > P_1\}} & \text{corresponds to the case when } u_1 = \dfrac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]}; \\ V_{1\{\lambda' \leq P_1\}} & \text{corresponds to the case when } u_1 = x_1. \end{cases}$$

Calculation for (2.1.12)

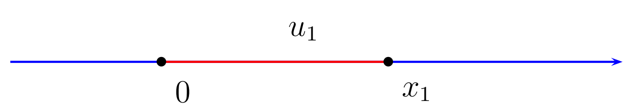
$V_{1\{\lambda' \leq P_1\}}$. In this case, $u_1 = x_1$.

$$\begin{aligned} V_{1\{\lambda' \leq P_1\}} &= E[-x_T^2 + \lambda x_T] \\ &= E[-(e_0 x_0 + (1-s)e_1 x_1)^2 + 2\lambda'(e_0 x_0 + (1-s)e_1 x_1)] \\ &= -E[(e_0 x_0 + (1-s)e_1 x_1 - \lambda')^2] + \lambda'^2 \end{aligned}$$

$V_{1\{\lambda'>P_1\}}$. In this case, $u_1 = \frac{(\lambda'-B_1)E[A_1]}{E[(A_1)^2]}$.

$$\begin{aligned}
V_{1\{\lambda'>P_1\}} \quad &= E[-x_T^2 + \lambda x_T] \\
&= E[-(A_1u_1 + B_1)^2 + \lambda(A_1u_1 + B_1)] \\
&= E[-A_1^2u_1^2 - 2A_1B_1u_1 - B_1^2 + \lambda(A_1u_1 + B_1)] \\
&\qquad\qquad \text{(both } u_1 \text{ and } B_1 \text{ are deterministic numbers)} \\
&= -E[A_1^2]u_1^2 - 2E[A_1]B_1u_1 - B_1^2 + \lambda(E[A_1]u_1 + B_1) \\
&= -\frac{(\lambda'-B_1)^2E^2[A_1]}{E[A_1^2]} - \frac{2B_1(\lambda'-B_1)E^2[A_1]}{E[A_1^2]} + \frac{2\lambda'(\lambda'-B_1)E^2[A_1]}{E[A_1^2]} - B_1^2 + 2\lambda' B_1 \\
&= \frac{[-(\lambda'-B_1)^2 - 2B_1(\lambda'-B_1) + 2\lambda'(\lambda'-B_1)]E^2[A_1]}{E[A_1^2]} - B_1^2 + 2\lambda' B_1 \\
&= \frac{[-(\lambda'-B_1)^2 + 2(\lambda'-B_1)^2]E^2[A_1]}{E[A_1^2]} - B_1^2 + 2\lambda' B_1 \\
&= \frac{(\lambda'-B_1)^2E^2[A_1]}{E[A_1^2]} - B_1^2 + 2\lambda' B_1 \\
&= -\frac{(\lambda'-B_1)^2VAR[A_1]}{E[(A_1)^2]} + \lambda'^2.
\end{aligned}$$

**Case 2.** $x_1 \geq 0, 0 \leq u_1 \leq x_1$. The strategy of selling some stocks.




In this case,

$$\begin{aligned}
x_T &= e_0[x_0 + (1-s)(x_1 - u_1)] + (1-s)e_1u_1 \\
&= (e_1 - e_0)(1-s)u_1 + e_0[x_0 + (1-s)x_1].
\end{aligned} \tag{2.1.13}$$

Let

$$\begin{cases} A_1' = (e_1 - e_0)(1-s) \\ B_2 = e_0[x_0 + (1-s)x_1]. \end{cases} \tag{2.1.14}$$

So now $x_T$ can be written as

$$x_T = A_1' u_1 + B_2. \tag{2.1.15}$$

With similar calculations as in case 1, we can derive that if we adopt this strategy, the best values of $u_1$ are given by

$$\begin{cases} u_1 = 0, & \text{when } \dfrac{(\lambda' - B_2)E[A_1']}{E[(A_1')^2]} < 0, \\ u_1 = \dfrac{(\lambda' - B_2)E[A_1']}{E[(A_1')^2]}, & \text{when } 0 \leq \dfrac{(\lambda' - B_2)E[A_1']}{E[(A_1')^2]} \leq x_1, \\ u_1 = x_1, & \text{when } \dfrac{(\lambda' - B_2)E[A_1']}{E[(A_1')^2]} > x_1. \end{cases} \tag{2.1.16}$$

Since $E[A_1] > 0 \Rightarrow E[A_1'] > 0$, the above results are equivalent to

$$\begin{cases} u_1 = 0, & \text{when } \lambda' < B_2, \\ u_1 = \dfrac{(\lambda' - B_2)E[A_1']}{E[(A_1')^2]}, & \text{when } B_2 \leq \lambda' \leq B_2 + \dfrac{x_1 E[(A_1')^2]}{E[A_1']}, \\ u_1 = x_1, & \text{when } \lambda' > B_2 + \dfrac{x_1 E[(A_1')^2]}{E[A_1']}. \end{cases} \tag{2.1.17}$$

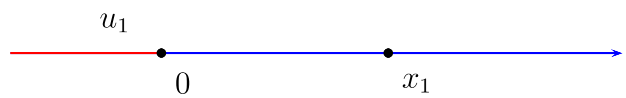
In the following, denote

$$P_2 \doteq B_2 + \frac{x_1 E[(A_1')^2]}{E[A_1']}.$$

The optimal objective value of $V_2 = E\{-x_T^2 + \lambda x_T\}$ under the strategy of selling some stocks can be calculated in the same way as in case 1. These optimal objective values are summarized below.

$$\begin{cases} V_{2\{\lambda' < B_2\}} = -(\lambda' - B_2)^2 + \lambda'^2, \\ V_{2\{B_2 \leq \lambda' \leq P_2\}} = -\dfrac{(\lambda' - B_2)^2 VAR[A_1']}{E[(A_1')^2]} + \lambda'^2, \\ V_{2\{\lambda' > P_2\}} = -E[(e_0 x_0 + (1-s)e_1 x_1 - \lambda')^2] + \lambda'^2. \end{cases} \tag{2.1.18}$$

**Case 3.** $x_1 \geq 0, u_1 < 0$. The strategy of short selling.




In this case,

$$\begin{aligned} x_T &= e_0[x_0 + (1-s)(x_1 - u_1)] + (1+b)e_1u_1 \\ &= [(1+b)e_1 - (1-s)e_0]u_1 + e_0[x_0 + (1-s)x_1]. \end{aligned} \tag{2.1.19}$$

Let

$$A_1'' = (1+b)e_1 - (1-s)e_0. \tag{2.1.20}$$

So now $x_T$ can be written as

$$x_T = A_1''u_1 + B_2. \tag{2.1.21}$$

With similar calculations as before, we can derive that if we adopt this strategy, the best values of $u_1$ are given by

$$\begin{cases} u_1 = \dfrac{(\lambda' - B_2)E[A_1'']}{E[(A_1'')^2]}, & \text{when } \dfrac{(\lambda' - B_2)E[A_1'']}{E[(A_1'')^2]} < 0, \\ u_1 = 0, & \text{when } \dfrac{(\lambda' - B_2)E[A_1'']}{E[(A_1'')^2]} \geq 0. \end{cases} \tag{2.1.22}$$

Again $E[A_1] > 0 \Rightarrow E[A_1''] > 0$, the above results are equivalent to

$$\begin{cases} u_1 = \dfrac{(\lambda' - B_2)E[A_1'']}{E[(A_1'')^2]}, & \text{when } \lambda' < B_2, \\ u_1 = 0, & \text{when } \lambda' \geq B_2. \end{cases} \tag{2.1.23}$$

The optimal objective value of $V_3 = E\{-x_T^2 + \lambda x_T\}$ under the strategy of short selling stocks can be calculated in the same way as before. These optimal objective values are summarized below.

$$\begin{cases} V_{3\{\lambda' < B_2\}} = -\dfrac{(\lambda' - B_2)^2 VAR[A_1'']}{E[(A_1'')^2]} + \lambda'^2, \\ V_{3\{\lambda' \geq B_2\}} = -(\lambda' - B_2)^2 + \lambda'^2. \end{cases} \tag{2.1.24}$$

**The division of regions**

**Lemma 2.1.2** *$P_1 \geq P_2$, $P_2 \geq B_2$.*

Proof. $P_1$ and $P_2$ are given by

$$\begin{cases} P_1 = B_1 + \dfrac{x_1 E[(A_1)^2]}{E[A_1]} \\ P_2 = B_2 + \dfrac{x_1 E[(A_1')^2]}{E[A_1']}. \end{cases} \tag{2.1.25}$$

To see the above result let us look at the difference of the two.

$$\begin{aligned}
& P_1 - P_2 \\
= \quad & B_1 - B_2 + \frac{x_1 E[(A_1)^2]}{E[A_1]} - \frac{x_1 E[(A_1')^2]}{E[A_1']} \\
= \quad & (b+s)x_1 e_0 + \frac{x_1 E[(A_1)^2]}{E[A_1]} - \frac{x_1 E[(A_1')^2]}{E[A_1']} \\
= \quad & x_1(E[A_1'] - E[A_1]) + \frac{x_1 E[(A_1)^2]}{E[A_1]} - \frac{x_1 E[(A_1')^2]}{E[A_1']} \\
= \quad & x_1\Big( (\frac{E[(A_1)^2]}{E[A_1]} - E[A_1]) - (\frac{E[(A_1')^2]}{E[A_1']} - E[A_1']) \Big) \\
= \quad & x_1\Big( \frac{Var[A_1]}{E[A_1]} - \frac{Var[A_1']}{E[A_1']} \Big) \\
= \quad & x_1\Big( \frac{(1-s)^2 Var[e_1]}{E[A_1]} - \frac{(1-s)^2 Var[e_1]}{E[A_1']} \Big) \\
= \quad & x_1(1-s)^2 Var[e_1]\Big( \frac{1}{E[A_1]} - \frac{1}{E[A_1']} \Big) \\
= \quad & \frac{x_1(1-s)^2 Var[e_1]}{E[A_1]E[A_1']}(E[A_1'] - E[A_1]) \geq 0.
\end{aligned}$$

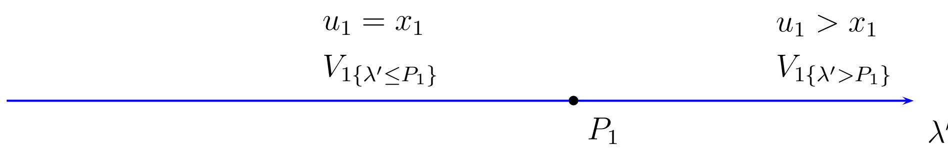
Because

$$E[A_1] > 0, E[A_1'] > 0 \text{ and } E[A_1'] > E[A_1].$$

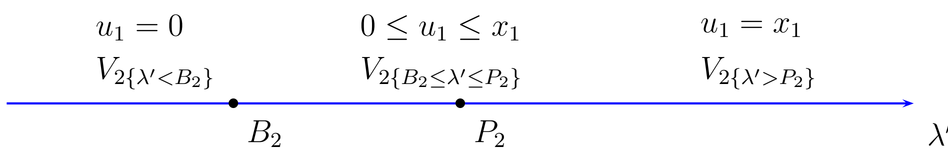
In the first 3 cases, we have assumed that $x_1 \geq 0$, so it is clear that $P_2 > B_2$. $\square$

With Lemma 2.1.2, the first 3 cases are summarized graphically here.

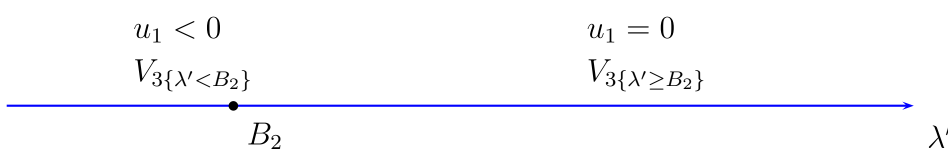
Case 1.




Case 2.




Case 3.




Case 1.

$$\begin{cases} V_{1\{\lambda' > P_1\}} = -\dfrac{(\lambda' - B_1)^2 VAR[A_1]}{E[(A_1)^2]} + \lambda'^2, \\ V_{1\{\lambda' \le P_1\}} = -E[(e_0 x_0 + (1-s)e_1 x_1 - \lambda')^2] + \lambda'^2. \end{cases}$$

Case 2.

$$\begin{cases} V_{2\{\lambda' < B_2\}} = -(\lambda' - B_2)^2 + \lambda'^2, \\ V_{2\{B_2 \le \lambda' \le P_2\}} = -\dfrac{(\lambda' - B_2)^2 VAR[A_1']}{E[(A_1')^2]} + \lambda'^2, \\ V_{2\{\lambda' > P_2\}} = -E[(e_0 x_0 + (1-s)e_1 x_1 - \lambda')^2] + \lambda'^2. \end{cases}$$

Case 3.

$$\begin{cases} V_{3\{\lambda' < B_2\}} = -\dfrac{(\lambda' - B_2)^2 VAR[A_1'']}{E[(A_1'')^2]} + \lambda'^2, \\ V_{3\{\lambda' \ge B_2\}} = -(\lambda' - B_2)^2 + \lambda'^2. \end{cases}$$

**The dominate strategy**

**Lemma 2.1.3** *When* $\lambda' < \dfrac{B_2 + P_2}{2}$*, the strategy of* $u_1 = 0$ *dominates the strategy of* $u_1 = x_1$*; When* $\lambda' > \dfrac{B_2 + P_2}{2}$*, the strategy of* $u_1 = x_1$ *dominates the strategy of* $u_1 = 0$*; When* $\lambda' = \dfrac{B_2 + P_2}{2}$*, the two strategies* $u_1 = 0$ *and* $u_1 = x_1$ *will yield the same objective value.*

Proof. The objective value can be written as

$$\begin{cases} V_{\{u_1=0\}} = -E[(\lambda' - B_2)^2] + \lambda'^2, & u_1 = 0, \\ V_{\{u_1=x_1\}} = -E[(A_1' x_1 + B_2 - \lambda')^2] + \lambda'^2, & u_1 = x_1. \end{cases} \tag{2.1.26}$$

So we have

$$\begin{aligned} & V_{\{u_1=x_1\}} - V_{\{u_1=0\}} \\ = \quad & E[2\lambda' A_1' x_1 - (A_1')^2 (x_1)^2 - 2A_1' B_2 x_1] \\ = \quad & x_1 \Big( 2\lambda' E[A_1'] - 2B_2 E[A_1'] - E[(A_1')^2] x_{T-1} \Big) \\ = \quad & 2E[A_1'] x_1 \Big( \lambda' - (B_2 + \frac{E[(A_1')^2] x_{T-1}}{2E[A_1']}) \Big) \\ = \quad & 2E[A_1'] x_1 \Big( \lambda' - (\frac{B_2 + P_2}{2}) \Big). \end{aligned}$$

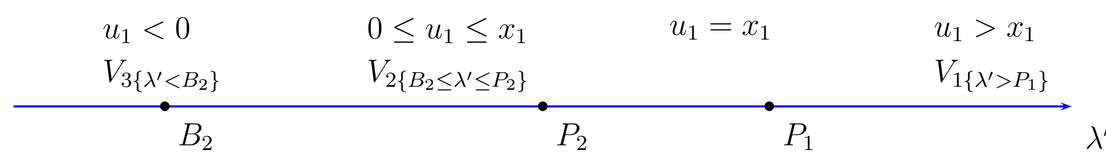
Since both $E[A_1']$ and $x_1$ are greater than 0, the results follow immediately. $\square$

**Lemma 2.1.4** *Among the 3 strategies, (i) When* $\lambda' > P_1$*,* $u_1 > x_1$ *dominates; (ii) When* $P_1 \le \lambda' \le P_2$*,* $u_1 = x_1$ *dominates; (iii) When* $P_2 \ge \lambda' \ge B_2$*,* $0 \le u_1 \le x_1$ *dominates; (iv) When* $\lambda' < B_2$*,* $u_1 < 0$ *dominates;*

Proof. The result for the case when $\lambda' \ge B_2$ is self-evident. The case when $\lambda' < B_2$ can be deduced from Lemma (2.1.3). $\square$

Making use of lemma (2.1.2) (2.1.3) and (2.1.4), case 1, 2 and 3 can now be combined.

Case 1, 2 and 3 combined.




Now we have complete information of the optimal control and the value function in the different parameter regions when $x_1 \ge 0$. The results can be summarized by the following.

$$\begin{cases} u_1 = \dfrac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]} > x_1, & \text{when } \lambda' > P_1, \\ u_1 = x_1, & \text{when } P_2 \le \lambda' \le P_1, \\ u_1 = \dfrac{(\lambda' - B_2)E[A_1']}{E[(A_1')^2]} \in [0, x_1), & \text{when } B_2 \le \lambda' < P_2, \\ u_1 = \dfrac{(\lambda' - B_2)E[A_1'']}{E[(A_1'')^2]} < 0, \quad \text{unstable}, & \text{when } \lambda' < B_2. \end{cases}$$

$$V = \begin{cases} -\dfrac{(\lambda' - B_1)^2 VAR[A_1]}{E[(A_1)^2]} + \lambda'^2, & \lambda' > P_1, \\ -E[(e_0 x_0 + (1-s)e_1 x_1 - \lambda')^2] + \lambda'^2, & P_2 \le \lambda' \le P_1, \\ -\dfrac{(\lambda' - B_2)^2 VAR[A_1']}{E[(A_1')^2]} + \lambda'^2, & B_2 \le \lambda' < P_2 \\ -\dfrac{(\lambda' - B_2)^2 VAR[A_1'']}{E[(A_1'')^2]} + \lambda'^2 \quad \text{(unstable)}, & \lambda' < B_2 \end{cases} \tag{2.1.27}$$

**The no-transaction region**

**Remark 2.1.5** *$P_1$ and $P_2$ can be seen as a sort of buying and selling boundaries respectively. The interval $\lambda' > P_1$ is the buying region; the interval $P_2 \leq \lambda' \leq P_1$ corresponds to no transaction region; the interval $\lambda' < P_2$ is the selling region. When transactions costs are zero, $b = s = 0$, we have $P_1 = P_2$, hence the no transaction region vanishes without transaction costs.*

Proof. We have seen in Lemma(2.1.2) that

$$P_1 - P_2 = \frac{x_1(1-s)^2 Var[e_1]}{E[A_1]E[A'_1]}(E[A'_1] - E[A_1]).$$

When $b = s = 0$, we have $E[A_1] = E[A'_1]$, the result follows. $\square$.

**Remark 2.1.6** *Both $B_1$ and $B_2$ are combinations of our positions in bank and in stock.*

$$\begin{cases} B_1 = e_0[x_0 + (1+b)x_1], \\ B_2 = e_0[x_0 + (1-s)x_1]. \end{cases} \tag{2.1.28}$$

*The value of $B_1$ remains unchanged when we buy stocks; the value of $B_2$ remains unchanged when we sell stock.*

**Theorem 2.1.7** *The optimal strategy in $\lambda' > P_1$ and $B_2 \leq \lambda' < P_2$ brings the current position in bank and stock to the buying and selling boundaries $\lambda' = P_1$ and $\lambda' = P_2$ respectively.*

Proof. In $\lambda' > P_1$, our original position $x_0$ and $x_1$ gives

$$\begin{aligned} \lambda' > P_1 &= B_1 + \frac{x_1 E[(A_1)^2]}{E[A_1]} \\ &= e_0[x_0 + (1+b)x_1] + \frac{x_1 E[(A_1)^2]}{E[A_1]}. \end{aligned}$$

The optimal strategy in this (buying) region is to increase $x_1$ to

$$u_1 = \frac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]}.$$

Let us denote the new position in bank by $u_0$, we have

$$u_0 = x_0 - (1+b)(u_1 - x_1).$$

So the corresponding new $P_1$, which we denote by $P_1'$, becomes

$$\begin{aligned} P_1' &= e_0[u_0 + (1+b)u_1] + \frac{u_1 E[(A_1)^2]}{E[A_1]} \\ &= e_0[x_0 - (1+b)(u_1 - x_1) + (1+b)u_1] + \frac{u_1 E[(A_1)^2]}{E[A_1]} \\ &= e_0[x_0 + (1+b)x_1] + \frac{u_1 E[(A_1)^2]}{E[A_1]} \\ &= B_1 + \frac{u_1 E[(A_1)^2]}{E[A_1]} \\ &= B_1 + (\lambda' - B_1) \\ &= \lambda'. \quad \square \end{aligned}$$

The above calculation shows that when $\lambda' > P_1$, our optimal strategy brings our positions in bank and in stock to the buying boundary $\lambda' = P_1$. In the case when $B_2 \leq \lambda' < P_2$, the optimal strategy brings current position to the selling boundary $\lambda' = P_2$. The calculation is similar to above. In the case when $\lambda' < B_2$, the optimal strategy is to short stocks. This falls into the case when our new position in stock is negative. It will be seen in the following discussion that this strategy is unstable. It will result in a sequence of continuing buying and selling of the stock until the holdings in the stock become 0, in which case, we again have $\lambda' = P_2$.

### 2.1.2 Optimal strategy with a short position in stock

In this case, we assume

$$x_1 < 0.$$

We distinguish the following 3 kinds of strategies, each of which corresponds to a different form of objective function.

$$\begin{cases} u_1 \geq 0, & \text{case 4;} \\ x_1 \leq u_1 \leq 0, & \text{case 5;} \\ u_1 \leq x_1, & \text{case 6.} \end{cases}$$

Case 4 represents the strategy to purchase more stocks and eventually avoid a short position in stock; Case 5 represents the strategy to buy some more stocks but still maintain a short position in stock; Case 6 represents the strategy to sell even more stocks. All the calculations in this part are similar to the previous part, and hence are omitted. We provide the summary of the results of these 3 cases here.

In the following, for simplicity reason let us denote

$$\begin{cases} P_3 \doteq B_1 + \frac{x_1 E[(A_1''')^2]}{E[A_1''']}, \\ P_4 \doteq B_2 + \frac{x_1 E[(A_1'')^2]}{E[A_1'']}. \end{cases}$$

**Case 4.**

$$\begin{cases} u_1 = \dfrac{(\lambda' - B_1)E[A_1]}{E[(A_1)^2]}, & \text{when} \quad \lambda' \geq B_1, \\ u_1 = 0, & \text{when} \quad \lambda' < B_1. \end{cases} \tag{2.1.29}$$

**Case 5.**

$$\begin{cases} u_1 = 0, & \text{when} \quad \lambda' > B_1, \\ u_1 = \dfrac{(\lambda' - B_1)E[A_1''']}{E[(A_1''')^2]}, & \text{when} \quad P_3 \leq \lambda' \leq B_1, \\ u_1 = x_1, & \text{when} \quad \lambda' < P_3. \end{cases} \tag{2.1.30}$$

**Case 6.**

$$\begin{cases} u_1 = \dfrac{(\lambda' - B_2)E[A_1'']}{E[(A_1'')^2]}, & \text{when} \quad \lambda' \le P_4, \\ u_1 = 0, & \text{when} \quad \lambda' > P_4. \end{cases} \tag{2.1.31}$$

**The division of regions**

**Lemma 2.1.8** $P_3 < P_4$; $P_4 < B_1$.

Proof. To see the above result let us look at the difference of the pairs.

$$\begin{aligned} & P_3 - P_4 \\ = \quad & B_1 - B_2 + \frac{x_1 E[(A_1''')^2]}{E[A_1''']} - \frac{x_1 E[(A_1'')^2]}{E[A_1'']} \\ = \quad & (b+s)x_1 e^0_{T-1} + \frac{x_1 E[(A_1''')^2]}{E[A_1''']} - \frac{x_1 E[(A_1'')^2]}{E[A_1'']} \\ = \quad & x_1(E[A_1''] - E[A_1''']) + \frac{x_1 E[(A_1''')^2]}{E[A_1''']} - \frac{x_1 E[(A_1'')^2]}{E[A_1'']} \\ = \quad & x_1\Big( (\frac{E[(A_1''')^2]}{E[A_1''']} - E[A_1''']) - (\frac{E[(A_1'')^2]}{E[A_1'']} - E[A_1''])\Big) \\ = \quad & x_1\Big(\frac{Var[A_1''']}{E[A_1''']} - \frac{Var[A_1'']}{E[A_1'']}\Big) \\ = \quad & x_1\Big(\frac{(1+b)^2 Var[e^1_{T-1}]}{E[A_1''']} - \frac{(1+b)^2 Var[e^1_{T-1}]}{E[A_1'']}\Big) \\ = \quad & x_1(1+b)^2 Var[e^1_{T-1}]\Big(\frac{1}{E[A_1''']} - \frac{1}{E[A_1'']}\Big) \\ = \quad & \frac{x_1(1+b)^2 Var[e^1_{T-1}]}{E[A_1'']E[A_1''']}(E[A_1''] - E[A_1''']) < 0. \end{aligned}$$

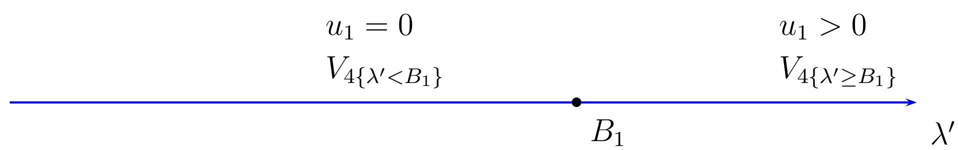
Because $x_1 < 0, E[A_1''] > 0, E[A_1'''] > 0$ and $E[A_1''] > E[A_1''']$. In the same way we have

$$
\begin{aligned}
& B_1 - P_4 \\
= \quad & B_1 - B_2 - \frac{x_1 E[(A_1'')^2]}{E[A_1'']} \\
= \quad & (b+s)x_1 e^0_{T-1} - \frac{x_1 E[(A_1'')^2]}{E[A_1'']} \\
= \quad & x_1(E[A_1''] - E[A_1''']) - \frac{x_1 E[(A_1'')^2]}{E[A_1'']} \\
= \quad & x_1\Big( - E[A_1'''] - (\frac{E[(A_1'')^2]}{E[A_1'']} - E[A_1''])\Big) = x_1\Big( - E[A_1'''] - \frac{Var[A_1'']}{E[A_1'']}\Big) > 0.
\end{aligned}
$$

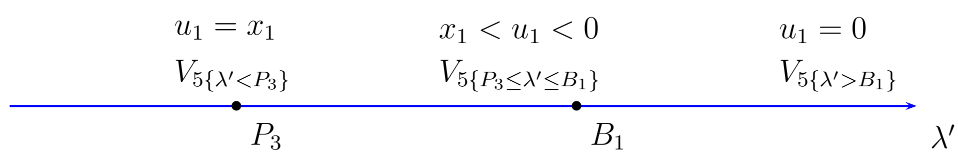
Because $x_1 < 0$, $E[A_1''] > 0$, and $E[A_1'''] > 0$. $\square$

We summarize the above results here graphically.

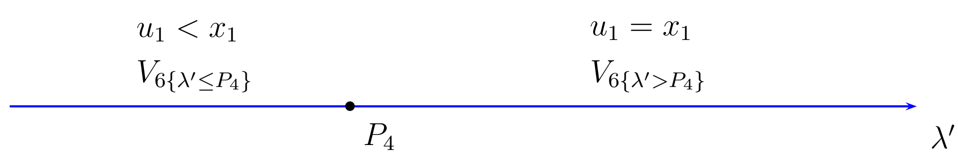
Case 4.




Case 5.




Case 6.




Case 4.

$$
\begin{cases}
V_{4\{\lambda' \geq B_1\}} = -\dfrac{(\lambda' - B_1)^2 VAR[A_1]}{E[(A_1)^2]} + \lambda'^2, \\
V_{4\{\lambda' < B_1\}} = -(\lambda' - B_1)^2 + \lambda'^2.
\end{cases}
$$

Case 5.

$$\begin{cases} V_{5\{\lambda'>B_1\}} = -(\lambda'-B_1)^2+\lambda'^2, \\ V_{5\{P_3\le\lambda'\le B_1\}} = -\dfrac{(\lambda'-B_1)^2 VAR[A''']}{E[(A''')^2]}+\lambda'^2, \\ V_{5\{\lambda'<P_3\}} = -E[(e_0x_0+(1-s)e_1x_1-\lambda')^2]+\lambda'^2. \end{cases}$$

Case 6.

$$\begin{cases} V_{6\{\lambda'\le P_4\}} = -\dfrac{(\lambda'-B_2)^2 VAR[A'']}{E[(A'')^2]}+\lambda'^2, \\ V_{6\{\lambda'>P_4\}} = -E[(e_0x_0+(1-s)e_1x_1-\lambda')^2]+\lambda'^2. \end{cases}$$

**The dominate strategy**

**Lemma 2.1.9** *When* $\lambda' > \dfrac{B_1+P_3}{2}$*, the strategy of* $u_1=0$ *dominates the strategy of* $u_1=x_1$*; When* $\lambda', \dfrac{B_1+P_3}{2}$*, the strategy of* $u_1=x_1$ *dominates the strategy of* $u_1=0$*; When* $\lambda' = \dfrac{B_1+P_3}{2}$*, the two strategies* $u_1=0$ *and* $u_1=x_1$ *will yield the same objective value.*

Proof. Same as lemma 2.1.3. □

**Lemma 2.1.10** *Let*

$$P_5 = \frac{B_1\sqrt{E[(A_1'')^2]}-B_2\sqrt{E[(A_1''')^2]}}{\sqrt{E[(A_1'')^2]}-\sqrt{E[(A_1''')^2]}},$$

*then we have* $P_3 < P_5 < P_4$.

Proof. Since we have $B_2 = B_1-(b+s)e_0x_1$, and $E[A_1'']-E[A_1''']=(b+s)e_0$,

$P_5$ can be rewritten as

$$
\begin{aligned}
P_5 &= B_1 + \frac{(b+s)e_0\sqrt{E[(A_1''')^2]}}{\sqrt{E[(A_1'')^2]}-\sqrt{E[(A_1''')^2]}}x_1 \\
&= B_1 + \frac{(E[A_1'']-E[A_1'''])\sqrt{E[(A_1''')^2]}}{\sqrt{E[(A_1'')^2]}-\sqrt{E[(A_1''')^2]}}x_1 \\
&= B_1 + \frac{(E[A_1'']-E[A_1'''])\sqrt{E[(A_1''')^2]}(\sqrt{E[(A_1'')^2]}+\sqrt{E[(A_1''')^2]})}{(\sqrt{E[(A_1'')^2]}-\sqrt{E[(A_1''')^2]})(\sqrt{E[(A_1'')^2]}+\sqrt{E[(A_1''')^2]})}x_1 \\
&= B_1 + \frac{(E[A_1'']-E[A_1'''])\sqrt{E[(A_1''')^2]}(\sqrt{E[(A_1'')^2]}+\sqrt{E[(A_1''')^2]})}{E[(A_1'')^2]-E[(A_1''')^2]}x_1 \\
&= B_1 + \frac{(E[A_1'']-E[A_1'''])\sqrt{E[(A_1''')^2]}(\sqrt{E[(A_1'')^2]}+\sqrt{E[(A_1''')^2]})}{E^2[A_1'']-E^2[A_1''']}x_1 \\
&\qquad \text{as } VAR[A_1''] = VAR[A_1'''] \\
&= B_1 + \frac{\sqrt{E[(A_1''')^2]}(\sqrt{E[(A_1'')^2]}+\sqrt{E[(A_1''')^2]})}{E[A_1'']+E[A_1''']}x_1 \\
&= B_1 + \frac{\sqrt{\frac{E[(A_1'')^2]}{E[(A_1''')^2]}}+1}{\frac{E[A_1'']}{E[A_1''']}+1}\frac{E[(A_1''')^2]}{E[A_1''']}x_1 \\
&= B_1 + K\frac{E[(A_1''')^2]}{E[A_1''']}x_1,
\end{aligned}
$$

where

$$
K = \frac{\sqrt{\frac{E[(A_1'')^2]}{E[(A_1''')^2]}}+1}{\frac{E[A_1'']}{E[A_1''']}+1}.
$$

Because $E[A_1''] > E[A_1'''] > 0$ and $VAR[A_1''] = VAR[A_1'''] = (1+b)^2E[e_1]$, we have

$$
\frac{E[(A_1'')^2]}{E[(A_1''')^2]} = \frac{VAR[A_1'']+E^2[A_1'']}{VAR[A_1''']+E^2[A_1''']} < \frac{E^2[A_1'']}{E^2[A_1''']}.
$$

So we can see $K < 1$. As $x_1 < 0$, and

$$
P_3 = B_1 + \frac{E[(A_1''')^2]}{E[A_1''']}x_1,
$$

we conclude $P_5 > P_3$. The proof for $P_5 < P_4$ is the same. □

**Lemma 2.1.11** *When $x_1 < 0$, (i) if $\lambda' > B_1$, the strategy $u_1 > 0$ dominates; (ii) if $P_5 \leq \lambda' \leq B_1$, the strategy $x_1 < u_1 < 0$ dominates; (iii) if $\lambda' < P_5$, the strategy $u_1 < x_1$ dominates. In particular, when $\lambda' = P_5$, the investor is indifferent between the strategy of buying more stocks and the strategy of selling some stocks.*

Proof. When $x_1 < 0$, we look at case 4, 5 and 6. In the region $\lambda' > B_1$, by Lemma 2.1.9, case 5 dominates case 6. The strategy of case 5 in this region is $u_1 = 0$. Case 4 clearly shows that the strategy of $u_1 > 0$ is better than the strategy of $u_1 = 0$ in this region. Hence $u_1 > 0$ dominates all if $\lambda' > B_1$. By comparing the objective value of $V_5$ and $V_6$, it can be seen that on the right of $P_5$ case 5 dominates case 6; on the left of $P_5$ case 6 dominates case 5. The argument for the rest of the result is thus similar. □

## 2.2 The burn-money phenomenon

In case 4, 5 and 6 in the previous section, it is observed that the strategy of $u_1 = x_1$ never dominates. This means no-transaction region does not exist when the initial holding in stock is negative. In other words, we should continue trading for as long as the holding in stock is negative, until it eventually becomes 0. In fact, we have the following theorem.

**Theorem 2.2.1** *When $x_1 < 0$, if $\lambda' = P_3$, case 6 dominates and the best strategy is to sell some stocks so that $\lambda' = P_4$. On the other hand, when $\lambda' = P_4$, case 5 dominates the best strategy is to buy some stocks so that $\lambda' = P_3$.*

Proof. Same procedure as in Theorem 2.1.6. □.

**Remark 2.2.2** *Equation 2.1.27 revisited. In equation 2.1.27 we summarized the optimal strategy when the investor starts off with a long position* *in the stock. When $\lambda' < B_2$, the optimal strategy was found to be short selling the stock. This strategy actually results in a new position such that $x_1$(new) $< 0$ and $\lambda' = P_4$. This new position is not in the no-transaction region. According to Theorem 2.2.1, at this new position, the investor would find himself better off if he is to sell some stocks so that $\lambda' = P_3$. However, at the yet new position, the investor would find again that he needs to buy some stocks to change his position to $\lambda' = P_4$. As a result, a series of buying and selling follows. Same is also true when $x_1 < 0$ and $\lambda' < B_1$.*

**Theorem 2.2.3** *A strategy is stable if it brings the investor's position to the no-transaction region. If (1) $x_1 > 0$ and $\lambda' < B_2$ or (2) $x_1 < 0$ and $\lambda' < B_1$, the one-step optimal strategies are unstable. A series of buying and selling will take place and eventually the holding in stock will become zero; the objective function will approach the value $\lambda'^2$.*

It is to be noted that $B_1$ ($B_2$) is the risk-free return the investor would get if he closes his position in the stock immediately and put all the money in bank when $x_1 > 0$ ($x_1 < 0$). $\lambda'$ is actually the target return of the investor. If the investor's target return is less than the risk-free return he can get, then he can simply "burn" some money and close his position in stock ($u_1 = 0$) to enjoy a sure return of $\lambda'$. Such phenomena would never happen in reality. It happens here because in mean variance formulation, the objective function is penalized when the actual return deviates from the target return, both from above and from below! A way around this is to define risk as semi-variance instead of variance. Upside deviation from the target return should not be penalized.

**Remark 2.2.4** *The unstable strategies will eventually approach a stable state where $P_3 = P_4 = B_1$.*

Consider a buy and a sell combination as one round of trading. If we start with $\lambda' = P_3$, after one round of trading we still have $\lambda' = P_3$, however the value of $B_1$ will keep decreasing. So the eventual stable state happens when $P_3 = P_4 = B_1$. This holds only when $x_1 = 0$, which means at the stable state, $u_1^* = 0$, ie. all wealth must be invested in the bank account. The discussion so far has explained the strategy of $u_1^* = 0$ (burn money) in Theorem 2.2.1.

# Chapter 3

# Two Risky Assets

In this chapter, we study the market consisting of 2 risky assets and 1 risk free asset. Suppose an investor starts off with a position of $x_0$, $x_1 \geq 0$ and $x_2 \geq 0$ in the risk free asset and the two risky assets respectively. For simplicity we assume the two stocks are non-negatively correlated ($\rho \geq 0$), and no short-selling of stocks is allowed. Our goal is to solve the following optimization problem.

$$
\begin{aligned}
&\max_{u_1,u_2} E\{-x_T^2 + \lambda x_T\} \\
&s.t. x_T = e_0[x_0 - (1+b)(u_1 - x_1)^+ + (1-s)(x_1 - u_1)^+] \\
&\qquad\qquad - (1+b)(u_2 - x_2)^+ + (1-s)(x_2 - u_2)^+] \\
&\qquad\qquad + (1-s)e_1 u_1 + (1-s)e_2 u_2 \\
&\qquad\qquad u_1 \geq 0,\ u_2 \geq 0
\end{aligned}
\tag{3.0.1}
$$

We utilize a two-step optimization technique to solve the problem. We first treat $u_2$ as given and find the optimal $u_1$ as a function of $u_2$. We then substitute this function into the problem so that it becomes an optimization problem of one variable ($u_2$).

## 3.1 Characterization of optimal strategies

**The first step: $u_1$ as a function of $x_0', x_1, u_2$**

In our first step of solving the above question, we assume the optimal $u_2$ is achieved, and in the adjusting process, $x_0$ becomes $x_0'$. We then look at what is the optimal choice of $u_1$ when we treat $x_0'$ and $u_2$ as given. The optimization problem can be written as

$$\begin{aligned}
&\max_{u_1} E\{-x_T^2 + \lambda x_T\} \\
&s.t.\ \ x_T = e_0[x_0' - (1+b)(u_1 - x_1)^+ + (1-s)(x_1 - u_1)^+] \\
&\qquad\qquad + (1-s)e_1 u_1 + (1-s)e_2 u_2 \\
&\qquad u_1 \geq 0
\end{aligned}$$

In order to get rid of the nonlinearity in the constrains, we consider different cases of $u_1$, namely (1). $u_1 \geq x_1$ (buying more of stock 1); (2). $u_1 < x_1$ (selling some of stock 1).

In the first case $u_1 \geq x_1$, the original optimization problem can be rewritten as

$$\begin{aligned}
&\max_{u_1 \geq 0} E\{-x_T^2 + \lambda x_T\} \\
&s.t.\ \ x_T = e_0[x_0' - (1+b)(u_1 - x_1)^+] + (1-s)e_1 u_1 + (1-s)e_2 u_2 \\
&\qquad\quad = [(1-s)e_1 - (1+b)e_0]u_1 + e_0[x_0' + (1+b)x_1] + (1-s)e_2 u_2 \\
&\qquad\quad = A_1 u_1 + (1-s)e_2 u_2 - B_1,
\end{aligned}$$

where

$$\begin{cases} A_1 = (1-s)e_1 - (1+b)e_0 \\ B_1 = e_0[x_0' + (1+b)x_1] \end{cases}$$

To solve this problem, we equate the derivative of the value function with respect to $u_1$ with zero.

$$\begin{aligned}
& \frac{dE[-x_T^2+\lambda x_T]}{du_1}=0 \\
\Rightarrow \quad & E[-2x_T\frac{dx_T}{du_1}+\lambda\frac{dx_T}{du_1}]=0 \\
\Rightarrow \quad & E[-2(A_1u_1+(1-s)e_2u_2-B_1)A_1+\lambda A_1]=0 \\
\Rightarrow \quad & -E[(A_1)^2]u_1-(1-s)E[e_2A_1]u_2+(\lambda'-B_1)E[A_1]=0 \qquad (\lambda'=\frac{\lambda}{2}) \\
\Rightarrow \quad & u_1=\frac{(\lambda'-B_1)E[A_1]-(1-s)E[e_2A_1]u_2}{E[(A_1)^2]}
\end{aligned}$$

In order for the above optimal solution to be attainable, we require

$$\frac{(\lambda'-B_1)E[A_1]-(1-s)E(e_2A_1)u_2}{E[(A_1)^2]}\geq x_1,$$

This is equivalent to

$$\lambda'\geq B_1+\frac{E[A_1^2]x_1+(1-s)E[e_2A_1]u_2}{E[A_1]}\triangleq P_1.$$

So we have

$$\begin{cases} u_1=\dfrac{(\lambda'-B_1)E[A_1]-(1-s)E[e_2A_1]u_2}{E[(A_1)^2]}, & \text{when } \lambda'\geq P_1, \\ u_1=x_1, & \text{when } \lambda'<P_1. \end{cases} \tag{3.1.1}$$

Under the second case $0\leq u_1\leq x_1$, the original optimization problem can be rewritten as

$$\begin{aligned}
& \max_{u_1\geq 0} E\{-x_T^2+\lambda x_T\} \\
& s.t. \;\; x_T=e_0[x_0'+(1-s)(x_1-u_1)^+]+(1-s)e_1u_1+(1-s)e_2u_2 \\
& \qquad\quad =(1-s)(e_1-e_0]u_1+e_0[x_0'+(1-s)x_1]+(1-s)e_2u_2 \\
& \qquad\quad =A_1'u_1+(1-s)e_2u_2-B_2,
\end{aligned}$$

where

$$\begin{cases} A'_1 = (1-s)(e_1 - e_0) \\ B_2 = e_0[x'_0 + (1-s)x_1] \end{cases}$$

With similar procedure as above, we can get the optimal solution $u_1$,

$$u_1 = \frac{(\lambda' - B_2)E[A'_1] - (1-s)E[e_2 A'_1]u_2}{E[(A'_1)^2]}$$

In order for this optimal solution to be attainable, we require

$$0 \leq \frac{(\lambda' - B_2)E[A'_1] - (1-s)E[e_2 A'_1]u_2}{E[(A'_1)^2]} \leq x_1$$

This is equivalent to

$$P_3 \triangleq B_2 + \frac{(1-s)E[e_2 A'_1]u_2}{E[A'_1]} \leq \lambda' \leq B_2 + \frac{E[A'^2_1]x_1 + (1-s)E[e_2 A'_1]u_2}{E[A'_1]} \triangleq P_2$$

So we have

$$\begin{cases} u_1 = x_1, & \text{when } \lambda' > P_2, \\ u_1 = \dfrac{(\lambda' - B_2)E[A'_1] - (1-s)E[e_2 A'_1]u_2}{E[(A'_1)^2]}, & \text{when } P_3 \leq \lambda' \leq P_2, \\ u_1 = 0, & \text{when } \lambda' < P_3. \end{cases} \quad (3.1.2)$$

**Lemma 3.1.1** $P_1 \geq P_2 \geq P_3$.

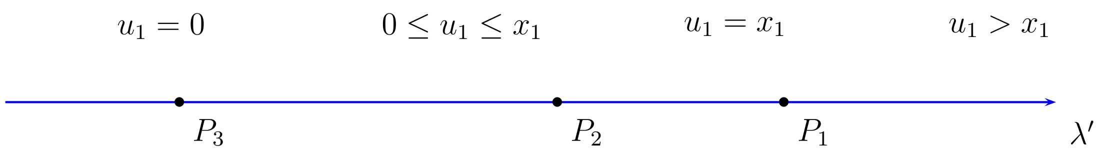
Proof. The result that $P_2 \geq P_3$ is straightforward, as

$$P_2 - P_3 = \frac{E[A'^2_1]x_1}{E[A'_1]} \geq 0.$$

The following calculation establishes the fact that $P_1 \geq P_2$.

$$\begin{aligned} P_1 - P_2 &= B_1 - B_2 + \frac{E[A_1^2]x_1}{E[A_1]} - \frac{E[A'^2_1]x_1}{E[A'_1]} + (1-s)u_2[\frac{E[e_2 A_1]}{E[A_1]} - \frac{E[e_2 A'_1]}{E[A'_1]}] \\ &= x_1(E[A'_1] - E[A_1] + \frac{E[A_1^2]}{E[A_1]} - \frac{E[A'^2_1]}{E[A'_1]}) + (1-s)u_2[\frac{E[e_2 A_1]}{E[A_1]} - \frac{E[e_2 A'_1]}{E[A'_1]}] \\ &= x_1(\frac{VAR[A_1]}{E[A_1]} - \frac{VAR[A'_1]}{E[A'_1]}) + (1-s)u_2 \frac{COV[e_2, A_1]E[A'_1] - COV[e_2, A'_1]E[A_1]}{E[A_1]E[A'_1]} \\ &= x_1(1-s)^2 VAR(e_1)\frac{E[A'_1] - E[A_1]}{E[A_1]E[A'_1]} + (1-s)^2 COV[e_1, e_2]u_2 \frac{E[A'_1] - E[A_1]}{E[A_1]E[A'_1]} \\ &\geq 0. \quad \square \end{aligned}$$

With the above lemma, we can now plot the following graph for our first step in solving the problem.




**The second step: $u_1$ as a function of $x_0, x_1, x_2, u_2$**

In the second step, we shall: 1. express optimal $u_1$ as a function of $x_0$, $x_1$, $x_2$ and $u_2$; 2. divide the regions according to the value of $u_2$. In this way, we can find out the optimal $u_2$ in each interval. By comparing the optimal objective value in every interval, we can then identify the global optimal solution of $u_2$ and hence the global optimal $u_1$. There are four cases.

**Case 1.** $u_2 \ge x_2$, $u_1 \ge x_1$.

In this case, we have from equation (3.1.1)

$$u_1 = \frac{(\lambda' - \beta_1)E[A_1] - E[A_1A_2]u_2}{E[A_1^2]}.$$

We require

$$\begin{aligned}
&\lambda' \ge P_1 \\
\Rightarrow\ &\lambda' \ge e_0[x_0 - (1+b)(u_2 - x_2) + (1+b)x_1] + \frac{E[A_1^2]x_1 + (1-s)E[e_2A_1]u_2}{E[A_1]} \\
\Rightarrow\ &u_2 \le \frac{(\lambda' - \beta_1)E[A_1] - E[A_1^2]x_1}{(1-s)E[e_2A_1] - (1+b)e_0E[A_1]} = \frac{(\lambda' - \beta_1)E[A_1] - E[A_1^2]x_1}{E[A_1A_2]} \triangleq Q_1,
\end{aligned}$$

where

$$\beta_1 = e_0[x_0 + (1+b)x_1 + (1+b)x_2].$$

**Case 2.** $u_2 \geq x_2$, $u_1 \leq x_1$.

In this case we have from equation (3.1.2)

$$u_1 = \frac{(\lambda' - \beta_2)E[A_1'] - E[A_1' A_2]u_2}{E[A_1'^2]}.$$

We require

$$\begin{aligned}
&\lambda' \leq P_2 \\
\Rightarrow\ &\lambda' \geq e_0[x_0 - (1+b)(u_2 - x_2) + (1-s)x_1] + \frac{E[A_1'^2]x_1 + (1-s)E[e_2 A_1']u_2}{E[A_1']} \\
\Rightarrow\ &u_2 \geq \frac{(\lambda' - \beta_2)E[A_1'] - E[A_1'^2]x_1}{(1-s)E[e_2 A_1'] - (1+b)e_0 E[A_1']} = \frac{(\lambda' - \beta_2)E[A_1'] - E[A_1'^2]x_1}{E[A_1' A_2]} \triangleq Q_2,
\end{aligned}$$

where

$$\beta_2 = e_0[x_0 + (1-s)x_1 + (1+b)x_2].$$

**Case 3.** $u_2 \leq x_2$, $u_1 \geq x_1$.

In this case we have from equation (3.1.1)

$$u_1 = \frac{(\lambda' - \beta_3)E[A_1] - E[A_1 A_2']u_2}{E[A_1^2]}.$$

We require

$$\begin{aligned}
&\lambda' \geq P_1 \\
\Rightarrow\ &\lambda' \geq e_0[x_0 + (1-s)(x_2 - u_2) + (1+b)x_1] + \frac{E[A_1^2]x_1 + (1-s)E[e_2 A_1]u_2}{E[A_1]} \\
\Rightarrow\ &u_2 \leq \frac{(\lambda' - \beta_3)E[A_1] - E[A_1^2]x_1}{(1-s)E[e_2 A_1] - (1-s)e_0 E[A_1]} = \frac{(\lambda' - \beta_3)E[A_1] - E[A_1^2]x_1}{E[A_1 A_2']} \triangleq Q_3,
\end{aligned}$$

where

$$\beta_3 = e_0[x_0 + (1+b)x_1 + (1-s)x_2].$$

**Case 4.** $u_2 \leq x_2$, $u_1 \leq x_1$.

In this case we have from equation (3.1.2)

$$u_1 = \frac{(\lambda' - \beta_4)E[A_1'] - E[A_1'A_2']u_2}{E[A_1'^2]}.$$

We require

$$\begin{aligned}
&\lambda' \leq P_2 \\
\Rightarrow\ &\lambda' \geq e_0[x_0 + (1-s)(x_2 - u_2) + (1-s)x_1] + \frac{E[A_1'^2]x_1 + (1-s)E[e_2A_1']u_2}{E[A_1']} \\
\Rightarrow\ &u_2 \geq \frac{(\lambda' - \beta_2)E[A_1'] - E[A_1'^2]x_1}{(1-s)E[e_2A_1'] - (1-s)e_0E[A_1']} = \frac{(\lambda' - \beta_2)E[A_1'] - E[A_1'^2]x_1}{E[A_1'A_2']} \triangleq Q_4,
\end{aligned}$$

where

$$\beta_4 = e_0[x_0 + (1-s)x_1 + (1-s)x_2].$$

### Division of regions and the dominate strategy

**Lemma 3.1.2** *$Q_1 < Q_2$ and $Q_3 < Q_4$.*

Proof. We present the proof for $Q_1 < Q_2$ here, the proof for $Q_3 < Q_4$ is almost identical.

$$\frac{E[A_1A_2]}{E[A_1]}Q_1 = \lambda' - \beta_1 - \frac{E[A_1^2]}{E[A_1]}x_1 \tag{1}$$

$$\frac{E[A_1'A_2]}{E[A_1']}Q_2 = \lambda' - \beta_2 - \frac{E[A_1'^2]}{E[A_1']}x_1 \tag{2}$$

$$\begin{aligned}
(1) - (2) &= \beta_2 - \beta_1 + \left(\frac{E[A_1'^2]}{E[A_1']} - \frac{E[A_1^2]}{E[A_1]}\right)x_1 \\
&= -e_0(b+s)x_1 + \left(\frac{E[A_1'^2]}{E[A_1']} - \frac{E[A_1^2]}{E[A_1]}\right)x_1 \\
&= (E[A_1] - E[A_1'])x_1 + \left(\frac{E[A_1'^2]}{E[A_1']} - \frac{E[A_1^2]}{E[A_1]}\right)x_1
\end{aligned}$$

$$
\begin{aligned}
&= (\frac{E[A_1'^2] - E^2[A_1']}{E[A_1']} - \frac{E[A_1^2] - E^2[A_1]}{E[A_1]})x_1 \\
&= (\frac{VAR[A_1']}{E[A_1']} - \frac{VAR[A_1]}{E[A_1]})x_1 \\
&= (1-s)^2 VAR[e_1]\frac{E[A_1] - E[A_1']}{E[A_1]E[A_1']}x_1 < 0.
\end{aligned}
$$

Let $M$ and $N$ denote the coefficients in front of $Q_1$ $Q_2$ in (1) (2),

$$
\begin{cases}
M = \dfrac{E[A_1 A_2]}{E[A_1]} = E[A_2] + \dfrac{COV[A_1 A_2]}{E[A_1]}, \\
N = \dfrac{E[A_1' A_2]}{E[A_1']} = E[A_2] + \dfrac{COV[A_1' A_2]}{E[A_1']}, .
\end{cases}
$$

The above result says

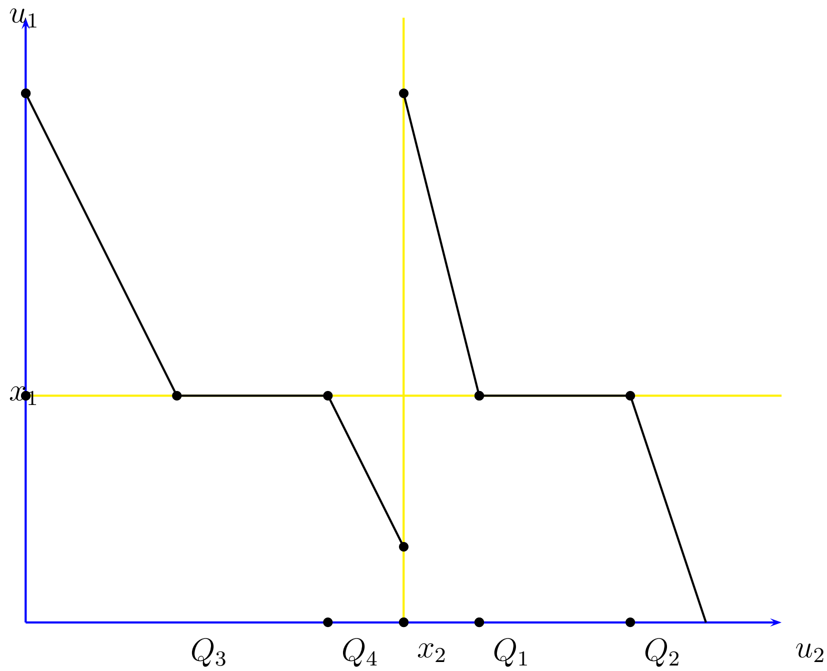
$$MQ_1 < NQ_2.$$

The following calculation shows $M > N$.

$$
\begin{cases}
M = E[A_2] + (1-s)^2 \dfrac{COV[e_1 e_2]}{E[A_1]}, \\
N = E[A_2] + (1-s)^2 \dfrac{COV[e_1 e_2]}{E[A_1']}.
\end{cases}
$$

By our assumption, $E[A_i] > 0$ and $COV[e_1 e_2] > 0$, hence $M > 0$ $N > 0$, so we can conclude $Q_1 < Q_2$. $\square$

With the above results, we can see on the two sides of $x_2$, the relative positions of $Q_1$ and $Q_2$, and also that of $Q_3$ and $Q_4$. We plot the graph of $u_1$ as a (linear) function of $u_2$ separately on the two sides of $x_2$.




As a summary,

when $Q_1 > x_2$,

$$\begin{cases} u_1 = \dfrac{(\lambda' - \beta_1)E[A_1] - E[A_1A_2]u_2}{E[A_1^2]}, & x_2 \leq u_2 \leq Q_1, \\ u_1 = x_1, & Q_1 \leq u_2 \leq Q_2, \\ u_1 = \dfrac{(\lambda' - \beta_2)E[A_1'] - E[A_1'A_2]u_2}{E[A_1'^2]}, & Q_2 \leq u_2; \end{cases}$$

when $Q_4 < x_2$,

$$\begin{cases} u_1 = \dfrac{(\lambda' - \beta_3)E[A_1] - E[A_1A_2']u_2}{E[A_1^2]}, & u_2 \leq Q_3, \\ u_1 = x_1, & Q_3 \leq u_2 \leq Q_4, \\ u_1 = \dfrac{(\lambda' - \beta_4)E[A_1'] - E[A_1'A_2']u_2}{E[A_1'^2]}, & Q_4 \leq u_2, \end{cases}$$

where

$$\begin{cases} A_1 = (1-s)e_1 - (1+b)e_0; \\ A_1' = (1-s)(e_1 - e_0), \end{cases} \qquad \begin{cases} A_2 = (1-s)e_2 - (1+b)e_0; \\ A_2' = (1-s)(e_2 - e_0), \end{cases}$$

and

$$\begin{cases} Q_1 = \dfrac{(\lambda' - \beta_1)E[A_1] - E[A_1^2]x_1}{E[A_1A_2]}, & \beta_1 = e_0[x_0 + (1+b)x_1 + (1+b)x_2], \\ Q_2 = \dfrac{(\lambda' - \beta_2)E[A_1'] - E[A_1'^2]x_1}{E[A_1'A_2]}, & \beta_2 = e_0[x_0 + (1-s)x_1 + (1+b)x_2], \\ Q_3 = \dfrac{(\lambda' - \beta_3)E[A_1] - E[A_1^2]x_1}{E[A_1A_2']}, & \beta_3 = e_0[x_0 + (1+b)x_1 + (1-s)x_2], \\ Q_4 = \dfrac{(\lambda' - \beta_4)E[A_1'] - E[A_1'^2]x_1}{E[A_1'A_2']}, & \beta_4 = e_0[x_0 + (1-s)x_1 + (1-s)x_2]. \end{cases}$$

**Lemma 3.1.3** *$Q_1$ and $Q_3$, $Q_2$ and $Q_4$ have the following relationships.*

$$\begin{cases} Q_1 = x_2 \Leftrightarrow Q_3 = x_2; \\ Q_1 > x_2 \Leftrightarrow Q_1 > Q_3 > x_2; \\ Q_1 < x_2 \Leftrightarrow Q_1 < Q_3 < x_2. \end{cases} \qquad \begin{cases} Q_2 = x_2 \Leftrightarrow Q_4 = x_2; \\ Q_2 > x_2 \Leftrightarrow Q_2 > Q_4 > x_2; \\ Q_2 < x_2 \Leftrightarrow Q_2 < Q_4 < x_2. \end{cases}$$

Proof. From the expression of $Q_1$ and $Q_2$ we have

$$\frac{(1-s)E[e_2A_1] - (1+b)e_0E[A_1]}{E[A_1]}Q_1 = \lambda' - \beta_1 - \frac{E[A_1^2]}{E[A_1]}x_1 \tag{3}$$

$$\frac{(1-s)E[e_2A_1] - (1-s)e_0E[A_1]}{E[A_1]}Q_3 = \lambda' - \beta_3 - \frac{E[A_1^2]}{E[A_1]}x_1 \tag{4}$$

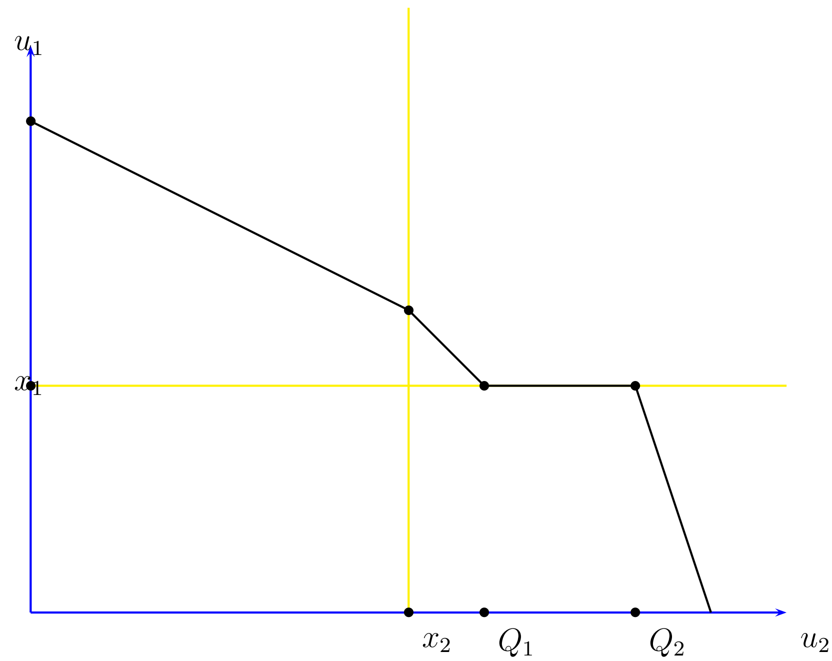
Take (4) $-$ (3), we get

$$\begin{aligned} &\Big(\frac{(1-s)E[e_2A_1]}{E[A_1]} - (1+b)e_0\Big)Q_1 - \Big(\frac{(1-s)E[e_2A_1]}{E[A_1]} - (1-s)e_0\Big)Q_3 \\ &= \beta_1 - \beta_3 \\ &= e_0(b+s)x_2. \end{aligned}$$

It is clear from above calculation that $Q_2 = x_2 \Leftrightarrow Q_4 = x_2$. As the coefficient in front of $Q_1$ is less than $Q_3$, the rest of the results follows. The proof for the rest of the results is the same and is omitted. $\square$

With the above 2 lemmas, we can plot the graph of optimal $u_1$ as a function of $u_2$. It can be summarized into 3 scenarios:

**Scenario 1,** $Q_1 \geq x_2$.




In this scenario, $u_2$ is divided into 4 regions. In the first region $0 \leq u_2 \leq x_2$ we have $u_1 \geq x_1$ and $u_2 \leq x_2$, and so

$$u_1 = \frac{(\lambda' - \beta_3)E[A_1] - E[A_1 A_2']u_2}{E[A_1^2]}.$$

$$x_T = A_1 u_1 + A_2' u_2 + \beta_3.$$

In order to find the optimal $u_2$ in this region, we let

$$\begin{aligned}
&\frac{dE[-x_T^2 + \lambda x_T]}{du_2} = 0 \\
\Rightarrow &E[(\lambda' - x_T)\frac{dx_T}{du_2}] = 0 \\
\Rightarrow &E[(\lambda' - x_T)(A_1\frac{du_1}{du_2} + A_2')] = 0 \\
\Rightarrow &E[(\lambda' - x_T)A_2'] = 0, \qquad \text{(from step 1, } E[(\lambda' - x_T)A_1] = 0)
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow E[(\lambda' - A_1u_1 - A_2'u_2 - \beta_3)A_2'] = 0 \\
&\Rightarrow (\lambda' - \beta_3)E[A_2'] - E[A_1A_2']u_1 - E[A_2'^2]u_2 = 0 \\
&\Rightarrow (\lambda' - \beta_3)E[A_2'] - \frac{(\lambda' - \beta_3)E[A_1]E[A_1A_2'] - E^2[A_1A_2']u_2}{E[A_1^2]} - E[A_2'^2]u_2 = 0 \\
&\Rightarrow u_2 = \frac{(\lambda' - \beta_3)(E[A_1^2]E[A_2'] - E[A_1]E[A_1A_2'])}{E[A_1^2]E[A_2'^2] - E^2[A_1A_2']}.
\end{aligned}
$$

Let

$$
R_{11} = \frac{(\lambda' - \beta_3)(E[A_1^2]E[A_2'] - E[A_1]E[A_1A_2'])}{E[A_1^2]E[A_2'^2] - E^2[A_1A_2']},
$$

the optimal $u_2$ in this region is given by

$$
\begin{cases}
u_2 = 0, & \text{if } R_{11} < 0, \\
u_2 = R_{11}, & \text{if } 0 \le R_{11} \le x_2, \\
u_2 = x_2, & \text{if } R_{11} > x_2.
\end{cases}
\tag{3.1.3}
$$

In the second region $x_2 \le u_2 \le Q_1$ we have $u_1 \ge x_1$ and $u_2 \ge x_2$, and so

$$
u_1 = \frac{(\lambda' - \beta_1)E[A_1] - E[A_1A_2]u_2}{E[A_1^2]}.
$$

$$
x_T = A_1u_1 + A_2u_2 + \beta_1.
$$

Let

$$
R_{12} = \frac{(\lambda' - \beta_1)(E[A_1^2]E[A_2] - E[A_1]E[A_1A_2])}{E[A_1^2]E[A_2^2] - E^2[A_1A_2]},
$$

with similar calculations, the optimal $u_2$ in this region is given by

$$
\begin{cases}
u_2 = x_2, & \text{if } R_{12} < x_2, \\
u_2 = R_{12}, & \text{if } x_2 \le R_{12} \le Q_1, \\
u_2 = Q_1, & \text{if } R_{12} > Q_1.
\end{cases}
\tag{3.1.4}
$$

In the third region $Q_1 \le u_2 \le Q_2$ we have $u_1 = x_1$ and $u_2 \ge x_2$, and so

$$\begin{aligned} u_1 &= x_1, \\ x_T &= A_1 x_1 + A_2 u_2 + \beta_1 \qquad (\text{or} = A_1' x_1 + A_2 u_2 + \beta_2). \end{aligned}$$

Let

$$R_{13} = \frac{(\lambda' - \beta_1) E[A_2] - E[A_1 A_2] x_1}{E[A_2^2]} = \frac{(\lambda' - \beta_2) E[A_2] - E[A_1' A_2] x_1}{E[A_2^2]},$$

with similar calculations, the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = Q_1, & \text{if } R_{13} < Q_1, \\ u_2 = R_{13}, & \text{if } Q_1 \le R_{13} \le Q_3, \\ u_2 = Q_3, & \text{if } R_{13} > Q_3. \end{cases} \tag{3.1.5}$$

In the fourth region $Q_2 \le u_2 \le \dfrac{(\lambda' - \beta_2) E[A_1']}{E[A_1' A_2]} (\triangleq Q_5)$. we have $u_1 \le x_1$ and $u_2 \ge x_2$, and so

$$u_1 = \frac{(\lambda' - \beta_2) E[A_1'] - E[A_1' A_2] u_2}{E[A_1'^2]}.$$

$$x_T = A_1' u_1 + A_2 u_2 + \beta_2.$$

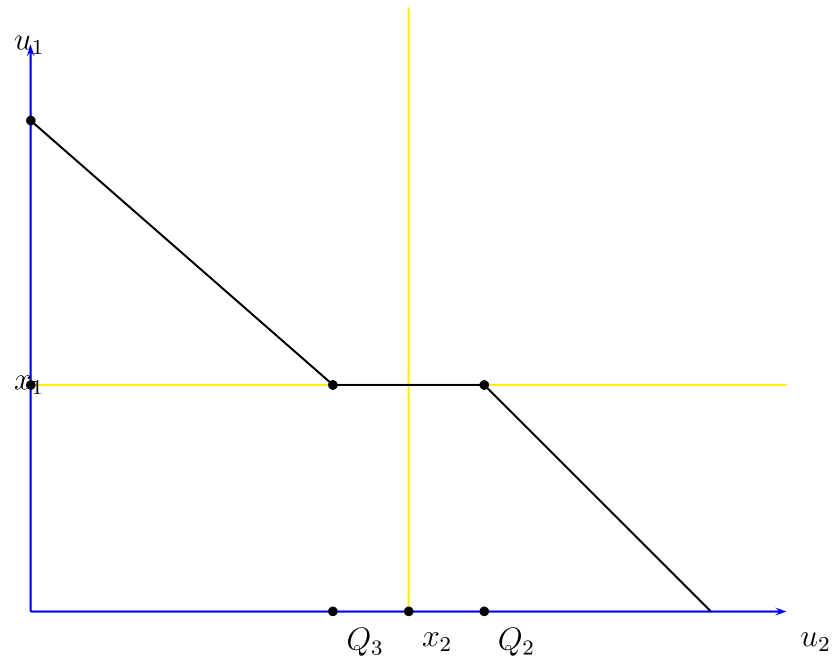
Let

$$R_{14} = \frac{(\lambda' - \beta_2)(E[A_1'^2] E[A_2] - E[A_1'] E[A_1' A_2])}{E[A_1'^2] E[A_2^2] - E^2[A_1' A_2]},$$

with similar calculations, the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = Q_2, & \text{if } R_{14} < Q_2, \\ u_2 = R_{14}, & \text{if } Q_2 \le R_{14} \le Q_5, \\ u_2 = Q_5, & \text{if } R_{14} > Q_5. \end{cases} \tag{3.1.6}$$

**Scenario 2,** $Q_1 \le x_2 \le Q_2$.




In this scenario, $u_2$ is also divided into 4 regions. The calculations are the same as before, hence only the results are summarized here. In the first region $0 \le u_2 \le Q_3$ we have $u_1 \ge x_1$ and $u_2 \le x_2$, and so

$$u_1 = \frac{(\lambda' - \beta_3)E[A_1] - E[A_1 A_2']u_2}{E[A_1^2]}.$$

$$x_T = A_1 u_1 + A_2' u_2 + \beta_3.$$

Let

$$R_{21} = \frac{(\lambda' - \beta_3)(E[A_1^2]E[A_2'] - E[A_1]E[A_1 A_2'])}{E[A_1^2]E[A_2'^2] - E^2[A_1 A_2']},$$

the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = 0, & \text{if } R_{21} < 0, \\ u_2 = R_{21}, & \text{if } 0 \le R_{21} \le Q_3, \\ u_2 = Q_3, & \text{if } R_{21} > Q_3. \end{cases} \tag{3.1.7}$$

In the second region $Q_3 \leq u_2 \leq x_2$ we have $u_1 = x_1$ and $u_2 \leq x_2$, and so

$$\begin{aligned} u_1 &= x_1, \\ x_T &= A_1 x_1 + A_2' u_2 + \beta_3, \qquad (\text{or} = A_1' x_1 + A_2' u_2 + \beta_4). \end{aligned}$$

Let

$$R_{22} = \frac{(\lambda' - \beta_3)E[A_2'] - E[A_1 A_2'] x_1}{E[A_2'^2]} = \frac{(\lambda' - \beta_4)E[A_2'] - E[A_1' A_2'] x_1}{E[A_2'^2]},$$

the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = Q_3, & \text{if } R_{22} < Q_3, \\ u_2 = R_{22}, & \text{if } Q_3 \leq R_{22} \leq x_2, \\ u_2 = x_2, & \text{if } R_{22} > x_2. \end{cases} \tag{3.1.8}$$

In the third region $x_2 \leq u_2 \leq Q_2$ we have $u_1 = x_1$ and $u_2 \geq x_2$, and so

$$\begin{aligned} u_1 &= x_1 \\ x_T &= A_1 x_1 + A_2 u_2 + \beta_1 = A_1' x_1 + A_2 u_2 + \beta_2. \end{aligned}$$

Let

$$R_{23} = \frac{(\lambda' - \beta_1)E[A_2] - E[A_1 A_2] x_1}{E[A_2^2]} = \frac{(\lambda' - \beta_2)E[A_2] - E[A_1' A_2] x_1}{E[A_2^2]},$$

the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = x_2, & \text{if } R_{23} < x_2, \\ u_2 = R_{23}, & \text{if } x_2 \leq R_{23} \leq Q_2, \\ u_2 = Q_3, & \text{if } R_{23} > Q_2. \end{cases} \tag{3.1.9}$$

In the fourth region $Q_2 \leq u_2 \leq Q_5$ we have $u_1 \leq x_1$ and $u_2 \geq x_2$, and so

$$u_1 = \frac{(\lambda' - \beta_2)E[A_1'] - E[A_1' A_2] u_2}{E[A_1'^2]}.$$

$$x_T = A_1' u_1 + A_2 u_2 + \beta_2.$$

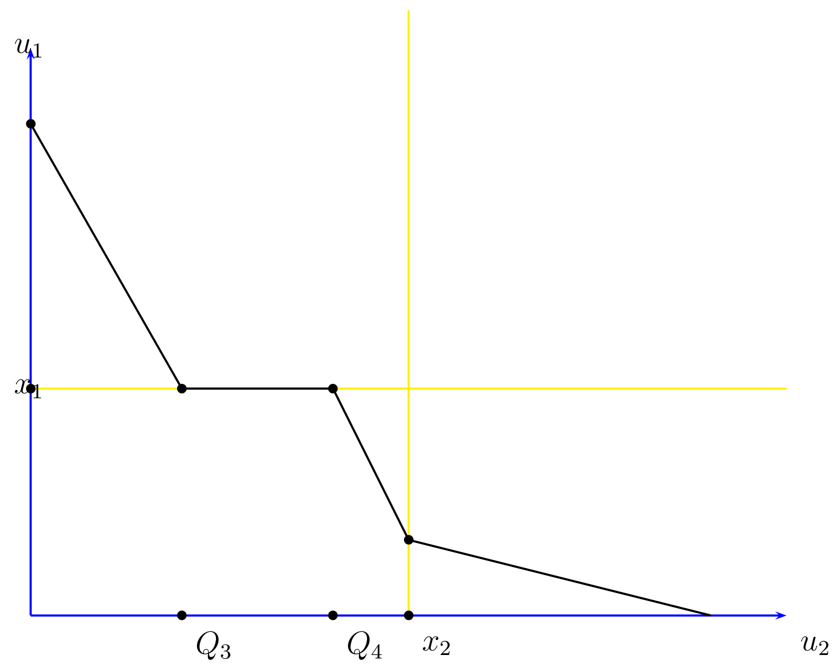
Let

$$R_{24} = \frac{(\lambda' - \beta_2)(E[A_1'^2]E[A_2] - E[A_1']E[A_1'A_2])}{E[A_1'^2]E[A_2^2] - E^2[A_1'A_2]},$$

the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = Q_2, & \text{if } R_{24} < Q_2, \\ u_2 = R_{24}, & \text{if } Q_2 \leq R_{24} \leq Q_5, \\ u_2 = Q_5, & \text{if } R_{23} > Q_5. \end{cases} \tag{3.1.10}$$

**Scenario 3,** $Q_2 \leq x_2$.




In this scenario, $u_2$ is again divided into 4 regions. In the first region $0 \leq u_2 \leq Q_3$ we have $u_1 \geq x_1$ and $u_2 \leq x_2$, and so

$$u_1 = \frac{(\lambda' - \beta_3)E[A_1] - E[A_1A_2']u_2}{E[A_1^2]}.$$

$$x_T = A_1u_1 + A_2'u_2 + \beta_3.$$

Let

$$R_{31} = \frac{(\lambda' - \beta_3)(E[A_1^2]E[A_2'] - E[A_1]E[A_1 A_2'])}{E[A_1^2]E[A_2'^2] - E^2[A_1 A_2']},$$

the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = 0, & \text{if } R_{31} < 0, \\ u_2 = R_{31}, & \text{if } 0 \le R_{31} \le Q_3, \\ u_2 = Q_3, & \text{if } R_{31} > Q_3. \end{cases} \tag{3.1.11}$$

In the second region $Q_3 \le u_2 \le Q_4$ we have $u_1 = x_1$ and $u_2 \le x_2$, and so

$$\begin{aligned} u_1 &= x_1 \\ x_T &= A_1 x_1 + A_2' u_2 + \beta_3, \qquad \text{or} \\ &= A_1' x_1 + A_2' u_2 + \beta_4. \end{aligned}$$

Let

$$R_{32} = \frac{(\lambda' - \beta_3)E[A_2'] - E[A_1 A_2']x_1}{E[A_2'^2]} = \frac{(\lambda' - \beta_4)E[A_2'] - E[A_1' A_2']x_1}{E[A_2'^2]},$$

the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = Q_3, & \text{if } R_{32} < Q_3, \\ u_2 = R_{32}, & \text{if } Q_3 \le R_{32} \le Q_4, \\ u_2 = Q_4, & \text{if } R_{32} > Q_3. \end{cases} \tag{3.1.12}$$

In the third region $Q_4 \le u_2 \le x_2$ we have $u_1 \le x_1$ and $u_2 \le x_2$, and so

$$u_1 = \frac{(\lambda' - \beta_4)E[A_1'] - E[A_1' A_2']u_2}{E[A_1'^2]}.$$

$$x_T = A_1' u_1 + A_2' u_2 + \beta_4.$$

Let

$$R_{33} = \frac{(\lambda' - \beta_4)(E[A_1'^2]E[A_2'] - E[A_1']E[A_1' A_2'])}{E[A_1'^2]E[A_2'^2] - E^2[A_1' A_2']},$$

the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = Q_4, & \text{if } R_{33} < Q_4, \\ u_2 = R_{33}, & \text{if } Q_4 \leq R_{33} \leq x_2, \\ u_2 = x_2, & \text{if } R_{33} > x_2. \end{cases} \tag{3.1.13}$$

In the fourth region $x_2 \leq u_2 \leq Q_5$, we have $u_1 \leq x_1$ and $u_2 \geq x_2$, and so

$$u_1 = \frac{(\lambda' - \beta_2)E[A_1'] - E[A_1'A_2]u_2}{E[A_1'^2]}.$$

$$x_T = A_1'u_1 + A_2u_2 + \beta_2.$$

Let

$$R_{34} = \frac{(\lambda' - \beta_2)(E[A_1'^2]E[A_2] - E[A_1']E[A_1'A_2])}{E[A_1'^2]E[A_2^2] - E^2[A_1'A_2]},$$

the optimal $u_2$ in this region is given by

$$\begin{cases} u_2 = x_2, & \text{if } R_{34} < x_2, \\ u_2 = R_{34}, & \text{if } x_2 \leq R_{34} \leq Q_5, \\ u_2 = Q_5, & \text{if } R_{34} > Q_5. \end{cases} \tag{3.1.14}$$

**Remark 3.1.4** ***The complete scheme to obtain optimal solution.*** *The discussion so far have provided a complete scheme to obtain the optimal solution to (3.0.1). For whatever position an investor holds in this market, his position must fall into exactly one of the 3 scenarios. In each scenario, there are four regions which represents the 4 possible trading strategies the investor can choose (buy one sell another; hold one buy another, etc). Using the analytical expressions of $u_2$ and $u_1(u_2)$ in each region obtained in the above 3 scenarios, the investor can compute the best objective value from each of the 4 trading strategies. The best objective value among the 4, corresponds to the optimal strategy $u_1^*$ and $u_2^*$ to (3.0.1).*

## 3.2 Sharpe Ratio with transaction costs

Let $r$ be the risk-free interest rate, $\mu$ be the expected return rate of a stock and $\sigma$ the standard deviation of the return. The standard Sharpe Ratio (reward-to-variability ratio) is defined as

$$\frac{\mu - r}{\sigma}.$$

It is a measure of the excess return (or Risk Premium) per unit of risk in an investment asset or a trading strategy. In general, stocks with higher Sharpe Ratio are preferable over those with lower Sharpe Ratio, as the former offers more excess return than the latter to investors to compensate the same amount of risk. Inspired by a result we obtained from this thesis, here we define the Sharpe Ratio in a market with proportional transaction costs. We think the excess return in a market with transaction costs is no longer $\mu - r$, instead it should be $\frac{(1-s)\mu-(1+b)r}{1+b}$ in monetary terms.

**Definition 3.2.1** ***Sharpe Ratio with Transaction Costs.*** *Suppose the buying and selling proportional transaction cost coefficients are $b$ and $s$, then the Sharpe Ratio with Transaction Cost of a stock is defined as*

$$\frac{(1-s)\mu - (1+b)r}{\sigma}.$$

The result that inspired the above definition is the following theorem.

**Theorem 3.2.1** *Given $\lambda' \geq \beta_2$, if $\rho \dfrac{E[A_1]}{\sigma_{A_1}} \geq \dfrac{E[A_2']}{\sigma_{A_2'}}$, where $\rho$ is the correlation between the 2 risky assets, then the optimal strategy is to sell all the holdings in the second stock, in other words, $u_2^* = 0$.*

The condition that $\lambda' \geq \beta_2$ is to ensure the target return is of a reasonably high level, whereas $\dfrac{E[A_1]}{\sigma_{A_1}}$ is simply the Sharpe Ratio with transaction costs of the first stock. This theorem states that if the Sharpe Ratio of the first stock times $\rho$ ($0 \leq \rho \leq 1$) is still bigger than the Sharpe Ratio of the second stock, then the first stock is so preferable than the second one that no matter what position an investor holds currently, he should not invest in the second stock at all. To prove the theorem, we need a few lemmas.

**Lemma 3.2.2** *If* $\rho\dfrac{E[A_1]}{\sigma_{A_1}} \geq \dfrac{E[A_2']}{\sigma_{A_2'}}$, *where* $\rho$ *is the correlation between the 2 risky assets, then*

$$\begin{cases} E[A_1^2]E[A_2'] - E[A_1]E[A_1A_2'] \leq 0, \\ E[A_1^2]E[A_2] - E[A_1]E[A_1A_2] < 0, \\ E[A_1'^2]E[A_2] - E[A_1']E[A_1'A_2] < 0, \\ E[A_1'^2]E[A_2'] - E[A_1']E[A_1'A_2'] < 0, \\ E[A_1]E[A_2^2] - E[A_2]E[A_1A_2] > 0, \\ E[A_1]E[A_2'^2] - E[A_2']E[A_1A_2'] > 0. \end{cases}$$

Proof. We present the calculation for the first inequality.

$$\begin{aligned} & E[A_1^2]E[A_2'] - E[A_1]E[A_1A_2'] \\ &= (E^2[A_1] + VAR[A_1])E[A_2'] - E[A_1](E[A_1]E[A_2'] + COV[A_1A_2']) \\ &= VAR[A_1]E[A_2'] - E[A_1]COV[A_1A_2'] \\ &= \sigma_{A_1}^2 E[A_2'] - E[A_1]\rho\sigma_{A_1}\sigma_{A_2'} \\ &= \sigma_{A_1}^2\sigma_{A_2'}(\frac{E[A_2']}{\sigma_{A_2'}} - \rho\frac{E[A_1]}{\sigma_{A_1}}) \\ &\leq 0. \end{aligned}$$

The other 5 inequalities follow from the fact that $\dfrac{E[A_1']}{\sigma_{A_1'}} > \dfrac{E[A_1]}{\sigma_{A_1}}$, and $\dfrac{E[A_2']}{\sigma_{A_2'}} > \dfrac{E[A_2]}{\sigma_{A_2}}$. $\square$

**Lemma 3.2.3** *When $Q_1 \geq x_2$, if $\rho\frac{E[A_1]}{\sigma_{A_1}} \geq \frac{E[A_2']}{\sigma_{A_2'}}$, where $\rho$ is the correlation between the 2 risky assets, then the optimal strategy is to sell all the holdings in $x_2$, in other words, $u_2^* = 0$.*

Proof. When $Q_1 \geq x_2$, this is the first scenario we have discussed in previous section. We have 4 regions. From the above lemma, we see in region 1, 2 and 4, $R_{11}, R_{12} and R_{14} \leq 0$. So $u_2$ takes the left boundary of each region as the optimal solution within each region. In the following, we shall see same is also true for the 3rd region where $Q_1 \leq u_2 \leq Q_2$. The optimal $u_2$ in this region is given by $u_2 = Q_1$ if

$$R_{13} = \frac{(\lambda' - \beta_1)E[A_2] - E[A_1A_2]x_1}{E[A_2^2]} < Q_1.$$

We shall show the above $u_2$ satisfies $u_2 \leq Q_1$.

$$\begin{aligned}
&R_{13} \leq Q_1 \\
\Leftrightarrow &\frac{(\lambda' - \beta_1)E[A_2] - E[A_1A_2]x_1}{E[A_2^2]} \leq \frac{(\lambda' - \beta_1)E[A_1] - E[A_1^2]x_1}{E[A_1A_2]} \\
\Leftrightarrow &(\lambda' - \beta_1)E[A_2]E[A_1A_2] - E^2[A_1A_2]x_1 \leq (\lambda' - \beta_1)E[A_1]E[A_2^2] - E[A_1^2]E[A_2^2]x_1 \\
\Leftrightarrow &(\lambda' - \beta_1)(E[A_1]E[A_2^2] - E[A_2]E[A_1A_2]) \geq (E[A_1^2]E[A_2^2] - E^2[A_1A_2])x_1.
\end{aligned}$$

According to our assumption, $Q_1 \geq x_2$, we have

$$\begin{aligned}
&\frac{(\lambda' - \beta_1)E[A_1] - E[A_1^2]x_1}{E[A_1A_2]} \geq x_2 \geq 0 \\
\Rightarrow &(\lambda' - \beta_1) \geq \frac{E[A_1^2]}{E[A_1]}x_1 \\
\Rightarrow &(\lambda' - \beta_1)(E[A_1]E[A_2^2] - E[A_2]E[A_1A_2]) \geq \frac{E[A_1^2]}{E[A_1]}x_1(E[A_1]E[A_2^2] - E[A_2]E[A_1A_2]) \\
\Rightarrow &(\lambda' - \beta_1)(E[A_1]E[A_2^2] - E[A_2]E[A_1A_2]) \geq (E[A_1^2]E[A_2^2] - \frac{E[A_1^2]E[A_2]E[A_1A_2]}{E[A_1]})x_1 \\
\Rightarrow &(\lambda' - \beta_1)(E[A_1]E[A_2^2] - E[A_2]E[A_1A_2]) \geq (E[A_1^2]E[A_2^2] - \frac{E[A_1]E[A_1A_2]E[A_1A_2]}{E[A_1]})x_1 \\
\Rightarrow &(\lambda' - \beta_1)(E[A_1]E[A_2^2] - E[A_2]E[A_1A_2]) \geq (E[A_1^2]E[A_2^2] - E^2[A_1A_2])x_1.
\end{aligned}$$

So we have proved that within all four regions, optimal $u_2$ always takes the value of the left boundary in each region. Because all the regions are inclusive of the two endpoints, we can conclude the global optimal $u_2^* = 0$.

**Lemma 3.2.4** *When $Q_1 \leq x_2 \leq Q_2$, if $\rho \frac{E[A_1]}{\sigma_{A_1}} \geq \frac{E[A_2']}{\sigma_{A_2'}}$, where $\rho$ is the correlation between the 2 risky assets, then the optimal strategy is to sell all the holdings in $x_2$, in other words, $u_2^* = 0$.*

Proof. There are 2 cases. 1. $Q_3 > 0$; 2. $Q_3 \leq 0$.

Case 1, $Q_3 > 0$.

In this case, we have 4 regions. In the first region $0 \leq u_2 \leq Q_3$, as

$$Q_3 > 0 \Rightarrow \lambda' > \beta_3,$$

we have $\lambda' - \beta_3 > 0$, and by lemma 3.2.2,

$$E[A_1^2]E[A_2'] - E[A_1]E[A_1A_2'] \leq 0,$$

so $R_{21} < 0$ and the optimal $u_2 = 0$ in this region. Optimal $u_2$ is taken at the left boundary.

In the second region $Q_3 \leq u_2 \leq x_2$, optimal $u_2$ is given by $u_2 = Q_3$ if

$$R_{22} = \frac{(\lambda' - \beta_3)E[A_2'] - E[A_1A_2']x_1}{E[A_2'^2]} \leq Q_3.$$

We shall show that $R_{22} \leq Q_3$.

$$\begin{aligned}
&R_{22} \leq Q_3 \\
\Leftrightarrow &\frac{(\lambda' - \beta_3)E[A_2'] - E[A_1A_2']x_1}{E[A_2'^2]} \leq \frac{(\lambda' - \beta_3)E[A_1] - E[A_1^2]x_1}{E[A_1A_2']} \\
\Leftrightarrow &(\lambda' - \beta_3)E[A_2']E[A_1A_2'] - E^2[A_1A_2']x_1 \leq (\lambda' - \beta_3)E[A_1]E[A_2'^2] - E[A_1^2]E[A_2'^2]x_1 \\
\Leftrightarrow &(\lambda' - \beta_3)(E[A_1]E[A_2'^2] - E[A_2']E[A_1A_2']) \geq (E[A_1^2]E[A_2'^2] - E^2[A_1A_2'])x_1
\end{aligned}$$

According to our assumption, $Q_3 \geq 0$, we have

$$\begin{aligned}
&\frac{(\lambda' - \beta_3)E[A_1] - E[A_1^2]x_1}{E[A_1 A_2']} \geq 0 \Rightarrow (\lambda' - \beta_3) \geq \frac{E[A_1^2]}{E[A_1]}x_1 \\
\Rightarrow &(\lambda' - \beta_3)(E[A_1]E[A_2'^2] - E[A_2']E[A_1 A_2']) \geq \frac{E[A_1^2]}{E[A_1]}x_1(E[A_1]E[A_2'^2] - E[A_2']E[A_1 A_2']) \\
\Rightarrow &(\lambda' - \beta_3)(E[A_1]E[A_2'^2] - E[A_2']E[A_1 A_2']) \geq (E[A_1^2]E[A_2'^2] - \frac{E[A_1^2]E[A_2']E[A_1 A_2']}{E[A_1]})x_1 \\
\Rightarrow &(\lambda' - \beta_1)(E[A_1]E[A_2'^2] - E[A_2']E[A_1 A_2']) \geq (E[A_1^2]E[A_2^2] - \frac{E[A_1]E[A_1 A_2']E[A_1 A_2']}{E[A_1]})x_1 \\
\Rightarrow &(\lambda' - \beta_1)(E[A_1]E[A_2'^2] - E[A_2']E[A_1 A_2']) \geq (E[A_1^2]E[A_2'^2] - E^2[A_1 A_2'])x_1
\end{aligned}$$

So indeed, $R_{22} \leq Q_3$ in this region and $u_2 = Q_3$. Optimal $u_2$ is taken at the left boundary.

In the third region $x_2 \leq u_2 \leq Q_2$, the optimal $u_2$ is given by $u_2 = x_2$ if

$$R_{23} = \frac{(\lambda' - \beta_1)E[A_2] - E[A_1 A_2]x_1}{E[A_2^2]} < x_2.$$

We shall show that $R_{23} \leq x_2$.

$$\begin{aligned}
R_{23} \leq x_2 \Leftrightarrow &\frac{(\lambda' - \beta_1)E[A_2] - E[A_1 A_2]x_1}{E[A_2^2]} \leq x_2 \\
\Leftrightarrow &(\lambda' - \beta_1)E[A_2] \leq E[A_1 A_2]x_1 + E[A_2^2]x_2.
\end{aligned}$$

By our assumption $Q_1 \leq x_2$, we have

$$\begin{aligned}
&(\lambda' - \beta_1)E[A_1] \leq E[A_1^2]x_1 + E[A_1 A_2]x_2 \\
\Rightarrow &(\lambda' - \beta_1)E[A_2] \leq \frac{E[A_2]E[A_1^2]x_1 + E[A_2]E[A_1 A_2]x_2}{E[A_1]} \leq E[A_1 A_2]x_1 + E[A_2^2]x_2.
\end{aligned}$$

Because from lemma 3.1, we have

$$\begin{cases}
\dfrac{E[A_2]E[A_1^2]}{E[A_1]} \leq E[A_1 A_2], \\
\dfrac{E[A_2]E[A_1 A_2]}{E[A_1]} \leq E[A_2^2].
\end{cases}$$

So again, in this region $u_2 = x_2$. Optimal $u_2$ is taken at the left boundary.

In the fourth region $Q_2 \leq u_2 \leq Q_5$,

$$R_{24} = \frac{(\lambda' - \beta_2)(E[A_1'^2]E[A_2] - E[A_1']E[A_1'A_2])}{E[A_1'^2]E[A_2^2] - E^2[A_1'A_2]}.$$

As $Q_2 \geq x_2 \geq 0 \Rightarrow \lambda' - \beta_2 > 0$, and from Lemma 3.1,

$$E[A_1'^2]E[A_2] - E[A_1']E[A_1'A_2] < 0,$$

So $R_{24} < 0$, hence the optimal $u_2$ is again forced to take the left boundary $Q_2$.

Case 2. $Q_3 \leq 0$.

In this case, there are only 3 regions. In the second region $x_2 \leq u_2 \leq Q_2$ and the third region $Q_2 \leq u_2 \leq Q_5$, the proof is same as above. In the first region $0 \leq u_2 \leq x_2$, we shall show

$$R_{22} = \frac{(\lambda' - \beta_3)E[A_2'] - E[A_1A_2']x_1}{E[A_2'^2]} < 0.$$

Rewrite the desired result, we get

$$R_{22} < 0 \Leftrightarrow (\lambda' - \beta_3) < \frac{E[A_1A_2']}{E[A_2']}x_1.$$

From our assumption, $Q_3 \leq 0$, we have

$$Q_3 \leq 0 \Rightarrow (\lambda' - \beta_3) < \frac{E[A_1^2]}{E[A_1]}x_1.$$

From Lemma 3.1, we have

$$E[A_1^2]E[A_2'] - E[A_1]E[A_1A_2'] \leq 0 \Rightarrow \frac{E[A_1^2]}{E[A_1]} \leq \frac{E[A_1A_2']}{E[A_2']}.$$

Hence indeed $R_{22} < 0$ and we conclude that in this region $u_2 = 0$.

We have seen that under all cases when $Q_1 \leq x_2 \leq Q_2$, the optimal $u_2$ is to be taken at the left boundary of every region, thus we have proved the lemma. □

**Lemma 3.2.5** *When $Q_2 \leq x_2$, if $\lambda' > \beta_2$ and $\rho\frac{E[A_1]}{\sigma_{A_1}} \geq \frac{E[A_2']}{\sigma_{A_2'}}$, where $\rho$ is the correlation between the 2 risky assets, then the optimal strategy is to sell all the holdings in $x_2$, in other words, $u_2^* = 0$.*

Proof. This is the third scenario as discussed in previous section. The idea of proof is the same as before. We show that $u_2$ in each region is taken at the left boundary of that region, hence we can conclude that $u_2^* = 0$. The calculation is almost the same as in the previous lemmas, and is not repeated here. □

The preceding 3 lemmas, lemma 3.2.3, lemma 3.2.4 and lemma 3.2.5 lead to theorem 3.2.1.

## 3.3 No-transaction region

The no-transaction region is the region in which $u_1^* = x_1$ and $u_2^* = x_2$. In other words, the optimal strategy in the no-transaction region is to remain at the current position. In this section, we give a necessary and sufficient condition for a position to be inside the no-transaction region in this market.

**Theorem 3.3.1** *Suppose an investor starts off with a position of $x_0$, $x_1$ and $x_2$. This position is in the no-transaction region if and only if*

$$\begin{cases} \max\left(Q_3, R_{23}\right) \leq x_2 \leq \min\left(Q_2, R_{22}\right) & \textit{and} \\ \max\left(Q_3', R_{23}'\right) \leq x_1 \leq \min\left(Q_2', R_{22}'\right). \end{cases} \tag{3.3.1}$$

Here $Q_2$, $Q_3$, $R_{22}$ and $R_{23}$ are defined in 3.1 with regard to $x_2$. $Q'_2$ $Q'_3$ $R'_{22}$ and $R'_{23}$ are the counterparts with regard to $x_1$. They can be obtained from $Q_2$, $Q_3$, $R_{22}$ and $R_{23}$ respectively by changing $x_2$ to $x_1$ and $e_2$ to $e_1$. For clarity, they are listed here.

$$\begin{cases} Q_2 = \dfrac{(\lambda' - \beta_2)E[A'_1] - E[A'^2_1]x_1}{E[A'_1 A_2]}, \\ Q_3 = \dfrac{(\lambda' - \beta_3)E[A_1] - E[A^2_1]x_1}{E[A_1 A'_2]}, \\ R_{22} = \dfrac{(\lambda' - \beta_3)E[A'_2] - E[A_1 A'_2]x_1}{E[A'^2_2]}, \\ R_{23} = \dfrac{(\lambda' - \beta_2)E[A_2] - E[A'_1 A_2]x_1}{E[A^2_2]}. \end{cases} \qquad \begin{cases} Q'_2 = \dfrac{(\lambda' - \beta'_2)E[A'_2] - E[A'^2_2]x_2}{E[A'_2 A_1]}, \\ Q'_3 = \dfrac{(\lambda' - \beta'_3)E[A_2] - E[A^2_2]x_2}{E[A_2 A'_1]}, \\ R'_{22} = \dfrac{(\lambda' - \beta'_3)E[A'_1] - E[A_2 A'_1]x_2}{E[A'^2_1]}, \\ R'_{23} = \dfrac{(\lambda' - \beta'_2)E[A_1] - E[A'_2 A_1]x_2}{E[A^2_1]}. \end{cases}$$

$$\begin{cases} \beta_2 = e_0[x_0 + (1-s)x_1 + (1+b)x_2], \\ \beta_3 = e_0[x_0 + (1+b)x_1 + (1-s)x_2]. \end{cases} \qquad \begin{cases} \beta'_2 = e_0[x_0 + (1-s)x_2 + (1+b)x_1] = \beta_3, \\ \beta'_3 = e_0[x_0 + (1+b)x_2 + (1-s)x_1] = \beta_2. \end{cases}$$

Proof. (1) Inside no-transaction region $\Rightarrow$ (3.3.1).
If the position of $x_0$, $x_1$ and $x_2$ is inside the no-transaction region, this means $u^*_1 = x_1$ and $u^*_2 = x_2$. Such an optimal solution is only possible in scenario 2 in the previous section in which $Q_1 \leq x_2 \leq Q_2$. By lemma 3.1.3, this is equivalent to

$$Q_3 \leq x_2 \leq Q_2.$$

At the same time, the strategy of $u^*_2 = x_2$ must dominate all other strategies in all regions. In particular, in the second and third region of scenario 2, we must have $R_{22} \geq x_2$ and $R_{23} \leq x_2$ by 3.1.8 and 3.1.9. So we must have

$$\max(Q_3, R_{23}) \leq x_2 \leq \min(Q_2, R_{22}).$$

If we have exchanged the position of $x_2$ with $x_1$ in all our proceeding discussion, we must require the same condition on $x_1$, thus by symmetry, when $u_1^* = x_1$ and $u_2^* = x_2$, we must also have

$$\max(Q_3', R_{23}') \leq x_1 \leq \min(Q_2', R_{22}').$$

(2) (3.3.1)$\Rightarrow$ Inside no-transaction region.

By condition (3.3.1), $Q_3 \leq x_2 \leq Q_2$. By lemma 3.1.3, this is equivalent to $Q_1 \leq x_2 \leq Q_2$. So the position of such $x_0$, $x_1$ and $x_2$ falls into scenario 2 described in section 3.1. Condition 3.3.1 implies $R_{22} \geq x_2$ and $R_{23} \leq x_2$. By 3.1.8, the strategy $u_1 = x_1$ and $u_2 = x_2$ thus dominates all other strategies in region 2 and 3 in scenario 2. In what follows, we shall prove that this strategy also dominates any strategy in region 1 and 4. Condition 3.3.1 implies $Q_3' \leq R_{22}'$, and

$$\begin{aligned}
&Q_3' \leq R_{22}' \\
\Rightarrow &\frac{(\lambda' - \beta_3')E[A_2] - E[A_2^2]x_2}{E[A_2A_1']} \leq \frac{(\lambda' - \beta_3')E[A_1'] - E[A_2A_1']x_2}{E[A_1'^2]} \\
\Rightarrow &(\lambda' - \beta_3')E[A_2]E[A_1'^2] - E[A_2^2]E[A_1'^2]x_2 \leq (\lambda' - \beta_3')E[A_1']E[A_2A_1'] - E^2[A_2A_1']x_2 \\
\Rightarrow &(\lambda' - \beta_3')E[A_2]E[A_1'^2] - (\lambda' - \beta_3')E[A_1']E[A_2A_1'] \leq E[A_2^2]E[A_1'^2]x_2 - E^2[A_2A_1']x_2 \\
\Rightarrow &\frac{(\lambda' - \beta_3')(E[A_2]E[A_1'^2] - E[A_1']E[A_2A_1'])}{E[A_2^2]E[A_1'^2] - E^2[A_2A_1']} \leq x_2 \\
\Rightarrow &\frac{(\lambda' - \beta_2)(E[A_2]E[A_1'^2] - E[A_1']E[A_2A_1'])}{E[A_2^2]E[A_1'^2] - E^2[A_2A_1']} \leq x_2 \\
\Rightarrow &R_{24} \leq x_2.
\end{aligned}$$

As $x_2 \leq Q_2$, we get $R_{24} \leq Q_2$. By 3.1.10, the best strategy in region 4 is taken at the left boundary $u_2 = Q_2$. But in region 3 ($x_2 \leq u_2 \leq Q_2$), we have showed that the strategy $u_2 = x_2$ dominates all other strategies including $u_2 = Q_2$, hence we can conclude that $u_2 = x_2$ dominates all strategies in region 4.

Similarly, Condition 3.3.1 implies $Q'_2 \geq R'_{23}$, and

$$
\begin{aligned}
&Q'_2 \geq R'_{23}\\
\Rightarrow&\frac{(\lambda' - \beta'_2)E[A'_2] - E[A'^2_2]x_2}{E[A'_2A_1]} \geq \frac{(\lambda' - \beta'_2)E[A_1] - E[A'_2A_1]x_2}{E[A_1^2]}\\
\Rightarrow&(\lambda' - \beta'_2)E[A'_2]E[A_1^2] - E[A'^2_2]E[A_1^2]x_2 \geq (\lambda' - \beta'_2)E[A_1]E[A'_2A_1] - E^2[A'_2A_1]x_2\\
\Rightarrow&(\lambda' - \beta'_2)E[A'_2]E[A_1^2] - (\lambda' - \beta'_2)E[A_1]E[A'_2A_1] \geq E[A'^2_2]E[A_1^2]x_2 - E^2[A'_2A_1]x_2\\
\Rightarrow&\frac{(\lambda' - \beta'_2)(E[A'_2]E[A_1^2] - E[A_1]E[A'_2A_1])}{E[A'^2_2]E[A_1^2] - E^2[A'_2A_1]} \geq x_2\\
\Rightarrow&\frac{(\lambda' - \beta_3)(E[A'_2]E[A_1^2] - E[A_1]E[A'_2A_1])}{E[A'^2_2]E[A_1^2] - E^2[A'_2A_1]} \geq x_2\\
\Rightarrow&R_{21} \geq x_2.
\end{aligned}
$$

By the same argument as above, we see the best strategy in region 1 is taken at the right boundary $u_2 = Q_3$ and hence the strategy $u_2 = x_2$ dominates any strategy from region 1.

As we can see condition 3.3.1 implies that the strategy $u_1 = x_1$ and $u_2 = x_2$ dominates all strategies in the 4 regions, $u_1^* = x_1$ and $u_2^* = x_2$ is the optimal solution. Combining (1) and (2) above, Theorem 3.3.1 is proved. □

As a corollary to theorem, we state the following optimal trading strategy to end this chapter.

**Corollary 3.3.2** *If $x_0$, $x_1$ and $x_2$ satisfy*

$$
\begin{cases}
\max(Q_3, R_{23}) \leq x_2 \leq \min(Q_2, R_{22}) \qquad \textit{and}\\
\max(Q'_3, R'_{23}) \leq x_1 \leq \min(Q'_2, R'_{22}),
\end{cases}
\tag{3.3.2}
$$

*Then the optimal strategy is $u_1^* = x_1$ and $u_2^* = x_2$.*

# Chapter 4

# Conclusion

In this thesis, we have made use of the discrete-time mean-variance formulation to study the problem of optimal portfolio selection with transaction costs. We derived the optimal solution for the single-period market consisting of one riskless and one risky asset. In this market, we also discussed the burn-money phenomenon which occurs when the target investment return in the mean-variance formulation is too low. Such phenomenon will not be observed in a model without transaction costs.

In the single-period market consisting of one riskless asset and two risky assets, we defined the Sharpe Ratio with transaction costs. Our definition is inspired by a particular result we obtained with regard to an optimal trading strategy in this market. We also established a necessary and sufficient condition for a current position to be in the no-transaction region in this market.

There are a few areas in which future research work can be carried upon. Firstly, one can apply the method of dynamic programming to the results we obtained and search for a solution to the multi-period problem with transaction costs. Due to the different regions existed in our solution, one might need to assume a certain distribution of the stock returns. This can pose a major difficulty in applying the method of dynamic programming. Secondly, in order for the result to be of practical interest, the number of risky assets in the market could be extended to $n$. The no short-selling constraint can be removed. The correlation of assets can be both positive or negative. Thirdly, with regard to the burn-money phenomenon, the objective function can be modified such that upside deviation of return will not be penalized. In particular, one could use semi-variance to quantify risk instead of using variance. With this, we end this thesis.

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