Effective Aspects of Positive Semi-Definite Real and Complex Polynomials

An academic exercise presented by

Mok Hoi Nam

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Supervisor: A/P To Wing Keung

Department of Mathematics

National University of Singapore

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Summary

The question of whether a positive semidefinite polynomial can be written as a sum of squares of rational functions was posed by Hilbert in the early 1900s, and this question, together with some related questions regarding positivity of polynomials have been of interest to many. Pólya gave a constructive proof with certain conditions: if the polynomial p is both positive definite and even, then for sufficiently large N, $p \cdot (\sum x_i^2)^N$ has positive coefficients. $(\sum x_i^2)^N$ is termed as a uniform denominator, and we are interested in several related questions to Pólya's theorem, such as: are there polynomials which can be written as a sum of rational functions but not with a uniform denominator? An effective bound for the exponent N has been given by Reznick, but can this result be extended for positive semi-definite polynomials? What about real-valued positive semi-definite bihomogeneous polynomials on $\mathbb{C}^n$?

We conduct a survey of results on the relations among certain subsets of real and complex positive semi-definite polynomials which are relevant to the above questions. In particular, we determine the minimum degree at which we have strict inclusion for number of variables up to 4, and collate them in tabular form. We also modify existing results of Reznick for effective aspects of real-valued bihomogeneous positive definite polynomials. Lastly, we obtain necessary as well as sufficient conditions for Pólya semi-stability of positive semi-definite polynomials with effective estimates.

Statement of Author's Contribution

Chapter 2 is a literature survey of the relations among certain subsets of positive semi-definite polynomials, and presented in tabular forms. As far as we know, this is the first time such tables have been compiled.

Chapter 3 contains modifications to known results by Bruce Reznick, and they are new. As for Chapter 4, Section 4.1 to Section 4.5 have been included in a joint paper by the author and A/P To Wing Keung, which has been accepted by Journal of Complexity. The author also illustrates an application of the results of Section 4.1-4.4 in Section 4.6 for certain positive semi-definite real-valued bihomogeneous polynomials.

Chapter 1

Introduction

1.1 Overview

In 1900, Hilbert asked whether a real positive semi-definite (psd) polynomial in n variables can be written as a sum of squares of rational functions, and this was known as Hilbert's 17th problem. It was well-known by the late 19th century that the set of real homogeneous polynomials (forms) that are positive semi-definite is equal to the set of real forms that can be represented as a sum of squares of polynomials (sos) when the number of variables is 2 or when the degree of the polynomials is 2. Hilbert also proved that every real positive semi-definite form of degree 4 in 3 variables can be written as a sum of three squares of quadratic forms, hence leading to his question above.

In 1920s, Artin solved Hilbert's 17th problem in the affirmative by a non-constructive method, and later Pólya [18] presented a proof in a special case: if p is both positive definite and even, then for sufficiently large N, $p \cdot (\sum x_i^2)^N$ has positive coefficients, i.e., it is a sum of squares of monomials. This implies that p is a sum of squares of rational functions with uniform denominator $(\sum x_i^2)^N$.

The above leads to some closely related questions: Are there real psd polynomials which can be written as a sum of rational functions but not with the uniform denominator

$(\sum x_i^2)^N$? What about a psd polynomial that is not a sum of squares of polynomials? If we were to extend the results for real positive semi-definite polynomials to their complex analogues, that is, real-valued bihomogeneous psd polynomials, what will they be? What is the minimum degree for which we have strict inclusion for the set of sos in the set of psd polynomials in n variables?

Chapter 2 gives a survey of the literature on current results of the above questions, and we present them in a tabular form (Table 2.1). We also construct examples for the cases where we have strict inclusions. We investigate the relationships between the following sets: the set of psds, the set of psds which can be written as a sum of rational functions, the set of psds which can be written as a sum of rational functions with uniform denominator $(\sum x_i^2)^N$, and the set of sos. Table 2.1 presents the above inclusions for the cases $n = 2, 3$ and for $n \geq 4$.

For Table 2.2, if set A is a strict subset of set B in Table 2.1, we show, with examples, the minimum degree for which there is a polynomial p that belongs to set B but not A. We show the minimum degree for the cases $n = 2, 3, 4$, for the above-mentioned sets of psd polynomials as in Table 2.1, in both real and complex n variables. The case (ii) of Table 2.2, where we investigate the minimum degree at which there exists an example in 4 real variables that is a psd which can be written as a sum of rational functions but not with uniform denominator, is an exception as we are unable to give a conclusion to the exact degree.

For a positive definite form p of degree d, Reznick [23] has proved effectively that $(\sum x_i^2)^N p$ is a positive linear combination over $\mathbb{R}$ of a set of $(2N + d)$-th powers of linear forms with rational coeffcients, and hence it is also a sos. The restriction to positive definite forms is necessary, as there exist psd forms p in $n \geq 4$ variables such that $(\sum x_i^2)^N p$ can never be a sum of squares of forms for any N, due to the existence of bad points, which was studied by Delzell [8]. In a paper by Reznick [25], he showed that there is no *single* form h so that if p is a psd form, then hp is sos. Furthermore, there is not even a finite set of forms so that if p is a psd form, then any h from this finite set of forms will ensure that hp is sos. The proofs for these two results require the existence of forms which are psds but not sos. Hence if we are able to show the existence of forms which are positive definite but not sos, then the above results will hold for positive definite forms. This is shown in Section 3.1, for the case of real variables as well as their complex analogues.

For a real positive definite form p of degree m in n variables, Reznick [23] has shown that if

$$N \geq \frac{nm(m-1)}{(4\log 2)\epsilon(p)} - \frac{n+m}{2}, \tag{1.1}$$

where $\epsilon(p)$ is a measure of how 'close 'p is to having a zero, then $(\sum x_i^2)^N p$ is a sum of $(m+2N)$-th powers of linear forms, and hence sos. Then using the method by Reznick [23], To and Yeung [30] have shown that for a real-valued bihomogeneous positive definite polynomial of degree m in n complex variables, if

$$N_c \geq \frac{nm(2m-1)}{\epsilon(p)\log 2} - n - m, \tag{1.2}$$

then $||z||^{2N_c}p$ is a sum of $2(m+N_c)$-th powers of norms of homogeneous linear polynomials. For a real-valued bihomogeneous positive definite polynomial in n complex variables, p can be written as a difference of squared norms, i.e., $p = ||g||^2 - ||h||^2$. Furthermore, there exists some real constant $c < 1$ such that $||h||^2 \leq c||g||^2$. Using this, we modify the proof of ([23], Theorem 3.11) in Section 3.2 of this thesis and see that the bound N_c can be slightly improved for some values of c.

Lastly, in an attempt to find an effective bound for the exponent N for a real-valued bihomogeneous psd polynomial p in n complex variables such that $||z||^{2N}p$ is a sos, we turn our attention to Pólya's theorem, which says that if f is real, homogeneous and positive definite on the standard simplex Δ_n, then for sufficiently large N, all the coefficients of $(x_1 + \cdots + x_n)^N f$ are positive. Such a polynomial f is said to be Pólya stable. In 2001, Powers and Reznick [20] have found an effective bound for N, and more recently also for the case when p has simple zeros (zeros only at the vertices of Δ_n), in [21] and [22].

Pólya's theorem and Powers and Reznick's effective bound can be extended to a result for certain psd bihomogeneous polynomials in $\mathbb{C}^n$, where their real analogues satisfy the conditions of Pólya's theorem.

In Chapter 4 of this thesis, we show the necessary conditions (Theorem 4.2.2) for a positive semi-definite real polynomial p to be Pólya semi-stable, as well as the sufficient conditions with effective estimates (Theorem 4.3.10). We also show that these necessary and sufficient conditions coincide for the case when the set of zeros of p is finite and the case when $n = 3$, hence obtaining a characterization of such Pólya semi-stable polynomials. Section 4.5 also shows an application of Theorem 4.3.10 for a general simplex. The contents of the sections 4.1-4.5 have been written in the paper 'Effective Pólya semi-positivity for non-negative polynomials on the simplex '. This paper is a joint effort between the author and Associate Professor To Wing Keung, and it has been accepted for publication in Journal of Complexity.

Similar to the extension of Pólya's theorem for certain bihomogeneous psd polynomials in $\mathbb{C}^n$, we can extend the necessary and sufficient conditions with effective estimates in Chapter 4 to a result for certain bihomogenous psd polynomials in $\mathbb{C}^n$. This will be the content of the last section, Section 4.6.

1.2 Some Notations and Definitions

Let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers. For positive integers n and d, we consider the index set $\mathcal{I}(n, d) := \{\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n_{\geq 0} \mid |\gamma| = d\}$, where $|\gamma| = \gamma_1 + \cdots + \gamma_n$. A homogeneous polynomial (form) f of degree d in $\mathbb{R}^n$ is given by

$$f(x_1, \cdots, x_n) = \sum_{\gamma \in \mathcal{I}(n,d)} a_\gamma x^\gamma, \tag{1.3}$$

where each $a_\gamma \in \mathbb{R}$, and $x^\gamma := x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}$. A homogeneous polynomial is known as a form, and the set of homogeneous polynomials on $\mathbb{R}^n$ of degree d is denoted by $H_d(\mathbb{R}^n)$. Also, we denote by $H_d(\mathbb{C}^n)$ the complex vector space of homogeneous holomorphic polynomials on $\mathbb{C}^n$ of degree d. A real-valued bihomogeneous polynomial on $\mathbb{C}^n$ of degree d

in z and $\bar{z}$ is of the form

$$p(z) = \sum_{I,J\in\mathcal{I}(n,d)} c_{IJ} z^I \bar{z}^J \tag{1.4}$$

where c_{IJ} are complex coefficients such that $c_{IJ} = \overline{c_{JI}}$ and the set of such polynomials is denoted by $BH_d(\mathbb{C}^n)$.

The cone of positive semidefinite forms in $H_d(\mathbb{R}^n)$ is denoted by

$$P_d(\mathbb{R}^n) = \{p \in H_d(\mathbb{R}^n) \mid p(x) \geq 0 \ \forall x \in \mathbb{R}^n\}, \tag{1.5}$$

and the cone of real-valued positive semidefinite bihomogeneous polynomials on $BH_d(\mathbb{C}^n)$ is similarly denoted by

$$P_d(\mathbb{C}^n) = \{p \in BH_d(\mathbb{C}^n) \mid p(z) \geq 0 \forall z \in \mathbb{C}^n\}. \tag{1.6}$$

For positive definite forms in $H_d(\mathbb{R}^n)$ (resp. $BH_d(\mathbb{C}^n)$), we have

$$\begin{aligned} PD_d(\mathbb{R}^n) &= \{p \in H_d(\mathbb{R}^n) \mid p(x) > 0 \forall x \in \mathbb{R}^n\}, & (1.7)\\ PD_d(\mathbb{C}^n) &= \{p \in BH_d(\mathbb{C}^n) \mid p(z) > 0 \forall z \in \mathbb{C}^n\}. & (1.8)\end{aligned}$$

The sets of sum of squares (sos) and sum of squares of rational functions in $H_d(\mathbb{R}^n)$ are denoted by

$$\begin{aligned} \Sigma_d(\mathbb{R}^n) &= \{p \in H_d(\mathbb{R}^n) \mid p = \sum_k h_k^2\}, & (1.9)\\ PQ_d(\mathbb{R}^n) &= \{p \in H_d(\mathbb{R}^n) \mid (\sum_j g_j^2)p = \sum_k f_k^2\}. & (1.10)\end{aligned}$$

for $h_k, f_k, g_j \in H_{d/2}(\mathbb{R}^n)$. Similarly, the set of sums of squared norms and the quotients of squared norms in $BH_d(\mathbb{C}^n)$ are denoted by

$$\begin{aligned} \Sigma_d(\mathbb{C}^n) &= \{p \in BH_d(\mathbb{C}^n) \mid p = \sum_k |h_k|^2\}, & (1.11)\\ PQ_d(\mathbb{C}^n) &= \{p \in BH_d(\mathbb{C}^n) \mid (\sum_j |g_j|^2)p = \sum_k |f_k|^2\}. & (1.12)\end{aligned}$$

for $h_k, f_k, g_k \in H_d(\mathbb{C}^n)$.

We also denote

$$PQD_d(\mathbb{R}^n) = \{p \in H_d(\mathbb{R}^n) \mid (\sum_i x_i^2)^N p = \sum_k h_k^2\} \tag{1.13}$$

to be the set of sums of squares of rational functions with uniform denominator, and similarly, the set of quotients of squared norms with uniform denominator is denoted by

$$PQD_d(\mathbb{C}^n) = \{p \in BH_d(\mathbb{C}^n) \mid (\sum_i |z_i|^2)^N p = \sum_k |f_k|^2\}. \tag{1.14}$$

We note that all the sums mentioned are finite sums.

Chapter 2

Subsets of the Set of Positive Semi-definite Polynomials

In the first section of this chapter, we consider the inclusion of the subsets of the set of positive semi-definite homogeneous real (resp. complex) polynomials with respect to the number of variables n. The four subsets (defined in Section 1.2) are

- the set of positive semi-definite forms $P_d(\mathbb{K}^n)$,
- the set of forms in $P_d(\mathbb{K}^n)$ that are sums of squares of rational functions (resp. quotients of squared norms) $PQ_d(\mathbb{K}^n)$,
- the set of forms in $P_d(\mathbb{K}^n)$ that are sums of squares of rational functions with uniform denominators (resp. quotients of squared norms with uniform denominators) $PQD_d(\mathbb{K}^n)$, and
- the set of forms in $P_d(\mathbb{K}^n)$ that are sums of squares of monomials (resp. sum of squared norms) $\Sigma_d(\mathbb{K}^n)$.

Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

In the second section, we again consider the inclusion of the above mentioned subsets and select the cases in which we have strict inclusions. We then determine the minimum degree m_d at which examples occur.

2.1 Table of subsets of PSD - w.r.t variables

The table below shows the inclusion of subsets of the set of positive semi-definite forms (in n real and complex variables), with 'E' signifying that the two sets in the leftmost column are the same set, while 'S' means that we have strict inclusion for the two sets in the leftmost column. The symbol $\mathbb{R}$ indicates that we are looking at real polynomials for a column, while $\mathbb{C}$ indicates that we are looking at real-valued bihomogeneous complex polynomials. The letters in parenthesis in the table indicates the part of the proof for each entry. For example, '$PQD_d \subset PQ_d$, $n = 3$, $\mathbb{C}$, S (g) ' means that for 3 complex variables, $PQD_d(\mathbb{C}^3)$ is a proper subset of $PQ_d(\mathbb{C}^3)$ and the proof is in part (g).

	$n = 2$		$n = 3$		$n \geq 4$	
	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{C}$
$PQ_d \subset P_d$	E (a)	S (b)	E (a)	S (b)	E (a)	S (b)
$PQD_d \subset PQ_d$	E (c)	E (d)	E (e)	S (g)	S (e)	S (h)
$\Sigma_d \subset PQD_d$	E (c)	S (f)	S (i)	S (j)	S (k)	S (l)

Table 2.1: Inclusion table for subsets of PSDs - w.r.t variables

Proof. (a) This is basically Hilbert's Seventeenth problem which was solved affirmatively by Artin in the 1920s. Hence for $n \geq 2$, all positive semidefinite forms must be a sum of squares of rational functions for real variables.

(b) For $n = 2$, the Hermitian function

$$p(z_1, z_2) = (|z_1|^2 - |z_2|^2)^2 \tag{2.1}$$

is a positive semi-definite form in $P_2(\mathbb{C}^2)$, but not a quotient of squared norms. Clearly, since p is a square it is greater than or equal to zero. There are two methods to see why p is not a quotient of squared norms. Firstly, based on the fact that if a Hermitian function P is a quotient of squared norms, then the zero set of the function must be a complex analytic set, and it is easy see that the zero set of p is a circle which is not a complex analytic set, implying that p is not a quotient of squared norms. Secondly, we can use the jet pullback method introduced by D'Angelo [12]. Choose the curve $z(t)$ to be $t \to (1, 1+t)$, then $z * p = 2|t|^2 + t^2 + \bar{t}^2 + \cdots$. The presence of terms in $z * p$ of lowest order 2 other than $2|t|^2$ causes the jet pullback property to fail, and hence p is not a quotient of squared norms.

For $n = 3$, the Hermitian function

$$q(z_1, z_2, z_3) = |z_1|^4|z_2|^2 + |z_1|^2|z_2|^4 + |z_3|^6 - 3|z_1z_2z_3|^2 \in P_3(\mathbb{C}^3), \tag{2.2}$$

is positive semi-definite, by using the arithmetic geometric mean inequality. However, it is not a quotient of squared norms as there exists a curve given by $z(t)$: $t \to (t, t+t^2, t)$ such that $z * q = 2|t|^8 + t^2|t|^6 + \bar{t}^2|t|^6 + \cdots$ violates the jet pullback property.

For $n = 4$, the Hermitian function

$$r(z_1, z_2, z_3, z_4) = |z_1|^4|z_2|^2|z_4|^2 + |z_1|^2|z_2|^4|z_4|^2 + |z_3|^6|z_4|^2 - 3|z_1z_2z_3z_4|^2 \in P_3(\mathbb{C}^4), \tag{2.3}$$

is psd as well, since $r = |z_4|^2 q$ where q is as in (2.2) and $|z_4|^2$ is nonnegative. Again, it is not a quotient of squared norms as there exists a curve given by $z(t) : t \to (t, t+t^2, t, t)$ such that $z * r = 2|t|^{10} + t^2|t|^8 + \bar{t}^2|t|^8 + \cdots$ violates the jet pullback property. Clearly, for $n \geq 4$, the Hermitian function

$$\begin{aligned} r_n(z_1, \cdots, z_n) &= |z_1|^4|z_2|^2|z_4|^2 \cdots |z_n|^2 + |z_1|^2|z_2|^4|z_4|^2 \cdots |z_n|^2 && (2.4) \\ &\quad + |z_3|^6|z_4|^2 \cdots |z_n|^2 - 3|z_1z_2z_3z_4 \cdots z_n|^2 && (2.5) \end{aligned}$$

is positive semi-definite, since $r_n = |z_4|^2 \cdots |z_n|^2 q$ where q is as in (2.2) and $|z_4|^2 \cdots |z_n|^2$ is nonnegative. It is not a quotient of squared norms by the jet pullback property, since there exists a curve given by $z(t) : t \to (t, t+t^2, t, t, \cdots, t)$, (where there are $(n-1)$ t terms) such that $z * r = 2|t|^{2n+2} + t^2|t|^{2n} + \bar{t}^2|t|^{2n} + \cdots$ violates the jet pullback property.

(c) If $p(x,y) \in P_d(\mathbb{R}^2)$, then let $f(t) = p(t,1) \geq 0$ for all real t, so that the roots of f can be seen to be either real with even multiplicity, or complex conjugate pairs. Hence $f(t) = A(t)^2(Q(t)+iR(t))(Q(t)-iR(t)) = (A(t)Q(t))^2 + (A(t)R(t))^2$. Upon homogenization of f, $p(x,y)$ is also a sum of two polynomial squares. This shows that $P_d(\mathbb{R}^2) = \Sigma_d(\mathbb{R}^2)$. Clearly, such forms in $P_d(\mathbb{R}^2)$ can be written as a sum of squares of rational functions with $(\sum x_i^2)^N = 1$ as the denominator, for $N \geq 1$.

(d) We refer to Theorem 2 of D'Angelo's paper [13]:

([13], Theorem 2). Let R be a positive semi-definite Hermitian symmetric polynomial in one complex variable. Then R is a quotient of squared norms if and only if one of the following three distinct conditions holds:

(1) R is identically zero.

(2) R is positive definite and a quotient of squared norms, $\cdots$

(3) The zero set of R is finite, and

$$R(z) = \prod_{j=1}^{N} |z - w_j|^{2k_j} r(z)$$

where r is postive definite and a quotient of squared norms.

The above theorem considers the case when $n = 1$, but is equivalent to the result in the bihomogeneous case when $n = 2$. Given a positive semi-definite polynomial R, if it is a quotient of squared norms, then we can have the factorized representation in point (3) of the theorem. By point (3) of Theorem 2, $r(z)$ is positive definite and a quotient of squared norms. Hence by an earlier result of Catlin and D'Angelo (see Theorem 0 of [13]), for positive definite $r(z)$, there is an integer k and a holomorphic homogeneous polynomial vector-valued mapping A such that

$$r(z) = \frac{||A(z)||^2}{||z||^{2k}} \tag{2.6}$$

We multiply R with uniform denominator $||z||^{2d}$ for some integer $d \geq k$, and obtain:

$$||z||^{2d} R(z) = \prod_{j=1}^{N} |z - w_j|^{2k_j} r(z) ||z||^{2d}. \tag{2.7}$$

We combine (2.6) and (2.7), and we have:

$$\begin{aligned} ||z||^{2d}R(z) &= \prod_{j=1}^{N} |z-w_j|^{2k_j}\Big(\frac{||A(z)||^2}{||z||^{2k}}\Big)||z||^{2d} \\ &= \prod_{j=1}^{N} |z-w_j|^{2k_j}||A(z)||^2||z||^{2(d-k)}. \end{aligned}$$

Since the product of sos is sos, and $|z - w_j|$ has even powers, the right hand side of the above equation is sos. This gives the result that for $n = 2$, all positive semi-definite forms that are quotient of squared norms have uniform denominators.

(e) By Artin's result [1], any positive semi-definite form can be written as a sum of squares of rational functions for real variables. For $n = 3$, it is a consequence of [6] that there are no bad points for a form h, such that for any positive semi-definite form f, $h^2 f$ is a sos. Hence by Scheiderer ([7], Cor 3.12), such a form h^N can be the uniform denominator $(\sum x_i^2)^N$. This enables us to see that all positive semi-definite forms that can be written as a sum of squares of rational functions are forms with uniform denominators.

On the other hand, for $n = 4$, such bad points are known to exist. Take for example,

$$p(x, y, z, w) = w^2(x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2) \in P_8(\mathbb{R}^4).$$

It can be shown that $(1, 0, 0, 0)$ is a bad point for $p(x, y, z, w)$, i.e, there does not exist any form h such that $h^2 p$ is a sos. Specifically, it can be shown that $p(x, y, z, w) \cdot (x^2 + y^2 + z^2 + w^2)^r$ is not a sos for any r. In a similar manner, for a n variable real polynomial

$$P(x, y, z, x_1, \cdots, x_{n-3}) = (x_1 \cdots x_{n-3})^2(x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2),$$

$(1, 0, \cdots, 0)$ is a bad point, and there does not exist any form h such that $h^2 P$ is sos. Specifically, the uniform denominator $(\sum x_i^2)$ with any exponent r multiplied with P is also not sos. Hence for $n \geq 4$, $PQD_d(\mathbb{R}^n)$ is a proper subset of $PQ_d(\mathbb{R}^n)$.

(f) The following is an example of a polynomial in $PQD_4(\mathbb{C}^2)$ but not in $\Sigma_4(\mathbb{C}^2)$:

$$r_b(z_1, z_2) \;=\; (|z_1|^2 \;+\; |z_2|^2)^2 \;-\; b|z_1|^2\,|z_2|^2, \tag{2.8}$$

which is similar to an example given by D'Angelo [12]. We write $x = |z_1|^2$ and $y = |z_2|^2$, and obtain $r_b = (x+y)^2 - bxy$. Clearly, r_b is non-negative when $b \leq 4$ and positive away from the origin when $b < 4$. By point (d) of Table 4.1, if r_b is bihomogeneous and positive definite, then it is a quotient of squared norms with uniform denominator. We can also show that r_b is not a quotient of squared norms when $b = 4$. To do so, we show that the jet pullback property fails when $b = 4$. Let $z(t) = (1+t, t)$, then

$$z * r_b(t, \bar{t}) = (|1+t|^2 + |t|^2)^2 - 4|t|^2|1+t|^2 = 2|t|^2 + t^2 + \bar{t}^2 + \cdots .$$

Inspection of the coefficient of the term $|z_1 z_2|^2$ will show that r_b is sos when $b \leq 2$. Hence for $2 < b < 4$, r_b is a positive semidefinite polynomial that can be written as a quotient of squared norms with uniform denominator but not as a sos.

(g) We claim that $PQD_d(\mathbb{C}^3)$ is a proper subset of $PQ_d(\mathbb{C}^3)$, and consider the following example which is an element in $PQ_d(\mathbb{C}^3)$ but not an element in $PQD_d(\mathbb{C}^3)$.

$$p(z) = p(z_1, z_2, z_3) = \;|z_3|^2\Big((|z_1|^2 + |z_2|^2)^2 - b|z_1 z_2|^2\Big) = |z_3|^2(r_b(z))$$

where $2 < b < 4$, and $r_b(z)$ is as defined in (2.8). From point (f), r_b is a quotient of squared norms but not sos. Then $p(z)$ is also a quotient of squared norms but not sos. Suppose $p(z)$ is a quotient of squared norms with uniform denominator, which means

$$(|z_1|^2 + |z_2|^2 + |z_3|^2)^N \cdot p(z) = \sum_i |h_i|^2$$

for some $N \in \mathbb{N}$, $h_i \in H_{3+r}(\mathbb{C}^3)$. Let the monomial in each $|h_i|^2$ with $|z_3|^{2+2N}$ be $|\hat{h}_i|^2|z_3|^{2+2N}$. By comparing coefficients of $|z_3|$, we have

$$|z_3|^{2+2N} \cdot p(z) = \sum_i |\hat{h}_i|^2 |z_3|^{2+2N}$$

This implies $p(z)$ is sos which is a contradiction. Hence $p(z)$ is a quotient of squared norms but not a quotient of squared norms with uniform denominator.

(h) We consider the following polynomial

$$\begin{aligned} p(z_1, z_2, z_3, z_4) &= |z_4|^2 \Big(M_c(z_1, z_2, z_3) \Big) \\ &= |z_4|^2 \Big(|z_1|^4 |z_2|^2 + |z_1|^2 |z_2|^4 + |z_3|^6 - (3-\epsilon)|z_1 z_2 z_3|^2 \Big) \end{aligned} \tag{2.9}$$

where $0 < \epsilon < 3$, and M_c is a complex analogue of Motzkin's polynomial with a modification in the coefficient of $|z_1 z_2 z_3|^2$. By arithmetic-geometric inequality $a + b + c \geq 3(abc)^{\frac{1}{3}}$, we let $(a, b, c) = (|z_1|^4|z_2|^2, |z_1|^2|z_2|^4, |z_3|^6)$, and obtain

$$|z_1|^4|z_2|^2 + |z_1|^2|z_2|^4 + |z_3|^6 \geq 3|z_1 z_2 z_3|^2 > (3-\epsilon)|z_1 z_2 z_3|^2 \tag{2.10}$$

Hence M_c is positive semi-definite and p is positive semi-definite as well because $|z_4|^2$ is non-negative. Next, if we were to write p as a difference of squared norms, we have $p = ||g||^2 - ||h||^2$, where $||g||^2 = |z_1|^4|z_2|^2 + |z_1|^2|z_2|^4 + |z_3|^6$, and $||h||^2 = (3-\epsilon)|z_1 z_2 z_3|^2$. By (2.10), clearly there exists a constant $0 < c < 1$ such that $||h||^2 \leq c||g||^2$. This is equivalent to saying there exist a constant C (which can be written in terms of c) such that

$$\frac{||g||^2 + ||h||^2}{||g||^2 - ||h||^2} \leq C, \tag{2.11}$$

for all points of $\mathbb{C}^3$ which are not zeros of p. Hence by Varolin's result [9], M_c is a quotient of squared norms which implies p is a quotient of squared norms as well. Then we will show that although p is a quotient of squared norms, there is no integer r such that the uniform denominator is $||z||^{2r}$. We prove by contradiction. Suppose $p \cdot (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)^r = \sum |h_i|^2$ is sos for some $r \in \mathbb{N}$, $h_i \in H_{4+r}(\mathbb{C}^4)$. Then the component of $p \cdot (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)^r$ with the highest degree of $|z_4|$ is $|z_4|^{2r+2} M_c(z_1, z_2, z_3)$. Let the monomial in each $|h_i|^2$ with $|z_4|^{2r+2}$ be $|\hat{h}_i|^2 |z_4|^{2r+2}$. By comparing coefficients, we have

$$|z_4|^{2r+2} M_c(z_1, z_2, z_3) = \sum_i |\hat{h}_i|^2 |z_4|^{2r+2}$$

This implies that we have $M_c(z_1, z_2, z_3)$ as a sos. To see that M_c is not sos, simply set $|z_i|^2$ to x_i^2, and by the term inspection method in the real case which shows that the Motzkin polynomial is not sos, similarly, we have $0 > -(3-\epsilon) = \sum_k F_k^2$ ([24], page 7). Hence M_c is not sos, and by contradiction, p does not have a representation as a quotient of squared norms with the uniform denominator. By similar arguments, we can generalize the above counterexample $p(z)$ for $n \geq 4$:

$$\begin{aligned}
& p_n(z_1, z_2, z_3, z_4, \cdots, z_n) \\
= \quad & |z_4 \cdots z_n|^2 \Big(M_c(z_1, z_2, z_3) \Big) \\
= \quad & |z_4 \cdots z_n|^2 \Big(|z_1|^4 |z_2|^2 + |z_1|^2 |z_2|^4 + |z_3|^6 - (3-\epsilon)|z_1 z_2 z_3|^2 \Big)
\end{aligned}$$

By above, since M_c is a quotient of squared norms, then p_n is also a quotient of squared norms. However, $||z||^r$ is not the uniform denominator for p_n for all r. We prove by contradiction. Suppose

$$p \cdot (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + \cdots + |z_n|^2)^r = \sum |h_i|^2$$

is sos for some $r \in \mathbb{N}$, $h_i \in H_{n+r}(\mathbb{C}^n)$. Then the component of $p \cdot (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + \cdots + |z_n|^2)^r$ with the highest degree of $|z_k|$ is $|z_k|^{2r+2} M_c(z_1, z_2, z_3)$, for $4 \leq k \leq n$. Let the monomial in each $|h_i|^2$ with $|z_k|^{2r+2}$ be $|\hat{h}_i|^2 |z_k|^{2r+2}$. By comparing coefficients, we have

$$|z_k|^{2r+2} M_c(z_1, z_2, z_3) = \sum_i |\hat{h}_i|^2 |z_k|^{2r+2}$$

Hence we have $M_c(z_1, z_2, z_3)$ as a sos, which has been shown to be false. In conclusion, there are counterexamples in $n \geq 4$ variables where they are quotient of squared norms but not with uniform denominator.

(i) For a fixed degree d, as the sets $P_d(\mathbb{R}^3)$, $PQ_d(\mathbb{R}^3)$ and $PQD_d(\mathbb{R}^3)$ are equal by points (a) and (e) of Table 4.1, we only need to show an example which is positive semi-definite but not sos to justify the claim that $\Sigma_d(\mathbb{R}^3)$ is a proper subset of $PQD_d(\mathbb{R}^3)$.

The counter example is the celebrated Motzkin's polynomial:

$$M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2. \tag{2.12}$$

This polynomial has been shown in [24], by arithmetic-geometric inequality and term inspection to be positive semi-definite but not sos.

(j) Consider the polynomial

$$M_c(z_1, z_2, z_3) = |z_1|^4|z_2|^2 + |z_1|^2|z_2|^4 + |z_3|^6 - (3 - \epsilon)|z_1z_2z_3|^2$$

in (2.9), where $0 < \epsilon < 3$. In point (h) of Table 4.1, we have already shown that M_c is not sos, hence we only need to show that M_c is a quotient of squared norms with uniform denominator, that is

$$(|z_1|^2 + |z_2|^2 + |z_3|^2)^m M_c(z_1, z_2, z_3) \tag{2.13}$$

is sos for some integer m. We do the following substitution: $x = |z_1|^2$, $y = |z_2|^2$ and $z = |z_3|^2$, then we check that the resulting polynomial of (2.13) in x, y and z:

$$(x + y + z)^m(x^2y + xy^2 + z^3 - (3 - \epsilon)xyz) \tag{2.14}$$

has nonnegative coefficients for some integer m. Using Matlab, Figure 2.1 shows the graph of m against ϵ, while Figure 2.2 shows the graph of $\log(m)$ against $\log(\epsilon)$ with values of ϵ near 0, with best fitted linear line $y = -1.0163x + 2.6468$. Hence we have approximately,

$$m = \frac{e^{2.6468}}{\epsilon^{1.0163}} \tag{2.15}$$

For example, the form in (2.14) with $\epsilon = 0.1$ and $m = 146$ has nonnegative coefficients and hence is a sos, which implies that $M_{c,\epsilon=0.1}$ is a quotient of squared norms with uniform denominator but not a sos.

(k) From Parrilo's thesis [19], we see that Motzkin's example when multiplied with the uniform denominator $(x^2 + y^2 + z^2)$ has the following explicit decomposition into

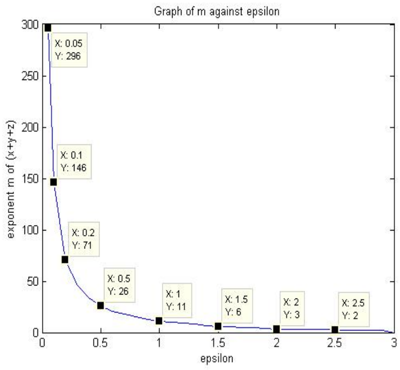

Figure 2.1: Graph of m against ϵ

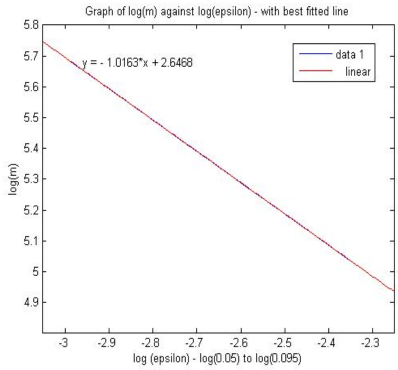

Figure 2.2: Graph of $\log(m)$ against $\log(\epsilon)$

sum of squares:

$$
\begin{aligned}
& (x^2+y^2+z^2)M(x,y,z) \\
= \;& (x^2+y^2+z^2)(x^4y^2+x^2y^4+z^6-3x^2y^2z^2) \\
= \;& y^2z^2(x^2-z^2)^2+x^2z^2(y^2-z^2)^2+(x^2y^2-z^4)^2+\frac{1}{4}x^2y^2(y^2-x^2)^2 \\
& +\frac{3}{4}x^2y^2(x^2+y^2-2z^2)^2
\end{aligned} \tag{2.16}
$$

To obtain a form in 4 real variables such that when multiplied with the uniform denominator, it is a sos, we simply substitute $z^2 = t^2 + s^2$ into (2.16), and obtain the following sum of squares:

$$
\begin{aligned}
& (x^2+y^2+t^2+s^2)M_4(x,y,t,s) \\
= \;& (x^2+y^2+t^2+s^2)(x^4y^2+x^2y^4+(t^2+s^2)^3-3x^2y^2(t^2+s^2)) \\
= \;& y^2(t^2+s^2)(x^2-t^2-s^2)^2+x^2(t^2+s^2)(y^2-t^2-s^2)^2 \\
& +(x^2y^2-(t^2+s^2)^2)^2+\frac{1}{4}x^2y^2(y^2-x^2)^2+\frac{3}{4}x^2y^2(x^2+y^2-2t^2-2s^2)^2
\end{aligned} \tag{2.17}
$$

Clearly, $M_4(x,y,t,s)$ as defined in (2.17) is positive semi-definite. This can be seen by applying arithmetic-geometric inequality $\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}}$ to $(a,b,c) = (x^4y^2, x^2y^4, (t^2+s^2)^3)$. Next, we claim that M_4 is not sos. Suppose not, then

$M_4 = \sum h_i^2(x,y,t,s)$ for some h_i. Let $s=0$ and assume that the rest of the variable are nonzero. Clearly, $M_4(x,y,t,0) = M(x,y,z) = \sum h_i^2(x,y,t,0)$, implying that Motzkin's example in three variables $M(x,y,z)$ is sos. Hence we get a contradiction. Then $M_4(x,y,t,s)$ is positive semi-definite which is not sos but a sum of squares of rational functions with uniform denominator.

For $n \geq 4$, the same argument for the case $n=4$ applies, and we have

$$\begin{aligned}
& (x^2+y^2+z_1^2+\cdots+z_{n-2}^2)M_n(x,y,z_1,\cdots,z_{n-2}) \\
= \; & (x^2+y^2+z_1^2+\cdots+z_{n-2}^2)(x^4y^2+x^2y^4+(z_1^2+\cdots+z_{n-2}^2)^3 \\
& -3x^2y^2(z_1^2+\cdots+z_{n-2}^2)) \\
= \; & y^2(z_1^2+\cdots+z_{n-2}^2)(x^2-z_1^2-\cdots-z_{n-2}^2)^2 \\
& +x^2(z_1^2+\cdots+z_{n-2}^2)(y^2-z_1^2-\cdots-z_{n-2}^2)^2+(x^2y^2-(z_1^2+\cdots+z_{n-2}^2)^2)^2 \\
& +\frac{1}{4}x^2y^2(y^2-x^2)^2+\frac{3}{4}x^2y^2(x^2+y^2-2z_1^2-\cdots-2z_{n-2}^2)^2.
\end{aligned}$$

Clearly, M_n is a positive semi-definite polynomial which is not a sos, but a sum of squares of rational functions with uniform denominator.

(l) Consider the polynomial

$$M_{4c}(z_1,z_2,z_3,z_4) = |z_1|^4|z_2|^2+|z_1|^2|z_2|^4+(|z_3|^2+|z_4|^2)^3-(3-\epsilon)|z_1z_2|^2(|z_3|^2+|z_4|^2)$$

which is a generalized form of M_c in point (j) of Table 4.1. Clearly, M_{4c} is also not sos, and we only need to show that M_c is a quotient of squared norms with uniform denominator, that is

$$(|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2)^m M_{4c}(z_1,z_2,z_3,z_4)$$

is sos for some integer m. Again, we do the following substitution: $x=|z_1|^2$, $y=|z_2|^2$, $z=|z_3|^2$ and $w=|z_4|^2$, and show that

$$(x+y+z+w)^m(x^2y+xy^2+(z+w)^3-(3-\epsilon)xy(z+w))$$

has nonnegative coefficients for some integer m. Using Matlab, we obtain (2.15) as in point (j), and see that exponent m of the uniform denominator has an exponential relationship with ϵ as ϵ approaches 0. In fact, the polynomial M_{4c} can be generalized to n variables:

$$\begin{aligned} & M_{nc}(z_1, z_2, z_3, \cdots, z_n) \\ = \; & |z_1|^4|z_2|^2 + |z_1|^2|z_2|^4 + (|z_3|^2 + \cdots + |z_n|^2)^3 - (3-\epsilon)|z_1z_2|^2(|z_3|^2 + \cdots + |z_n|^2) \end{aligned}$$

and the same argument follows. Hence for $n \geq 4$, there exist examples of positive semi-definite forms that are quotient of squared norms with uniform denominator but not sos.

□

2.2 Table of subsets of PSD - w.r.t degree

We consider further the cases in Table 4.1 where the inclusion is strict and obtain the minimum degree $d_{\min}$ where set A is a proper subset of set B, i.e., suppose A is a proper subset of B, then we determine the minimum degree at which a polynomial p is an element of B but not an element of A. The key to the following table is: for n real (resp. complex) variables, each entry in the table shows the minimum degree $d_{\min}$ for $A \subsetneqq B$ and the part of the proof in parenthesis, where A and B are sets in the leftmost column. The word 'Equal' indicates that the two sets are equal. We remark that part (ii) is the only case that is not determined precisely, i.e., we do not know whether $d_{\min}$ is 4 or 6.

Proof. (i) We are interested in complex polynomials in $z = (z_1, \cdots, z_n)$ (n variables) that are real-valued. This is equivalent to saying that the matrix of coefficients of such a polynomial is Hermitian. For $d = 1$ and for all n, we show that the set of psds is equal to the set of Σ_1 for complex polynomials that are real-valued. For every Hermitian matrix H, it is a known fact that it can be orthogonally diagonalized – $H = UCU^*$, with U as a unitary matrix where the column vectors are orthogonal

	$n = 2$		$n = 3$		$n = 4$	
	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{C}$
$PQ_d \subset P_d$	Equal	(i) 2	Equal	(i) 2	Equal	(i) 2
$PQD_d \subset PQ_d$	Equal	Equal	Equal	(i) 2	(ii) 4 or 6	(i) 2
$\Sigma_d \subset PQD_d$	Equal	(i) 2	(iii) 6	(i) 2	(iii) 4	(i) 2

Table 2.2: Inclusion table for subsets of PSDs - w.r.t degree

eigenvectors of H, and C is a real diagonal matrix. If p is a real-valued positive semidefinite bihomogeneous polynomial in n complex variables, then we can write $p = zHz^*$ where z is a row vector. Then it is clear that a change of basis (under the unitary transformation U) will enable us to write p as a difference of squared norms – $p = \sum_i |f_i|^2 - \sum_j |g_j|^2$, where f_i, g_i are orthogonal linear polynomials. This implies that f_i and g_j have common zeros, otherwise p is not positive semi-definite. However, since f_i and g_i are orthogonal they would not have common zeros. Hence p is a sum of squared norms.

We note that since $P_1(\mathbb{C}^n) = \Sigma_1(\mathbb{C}^n)$ for all n, and we have the following examples for $n = 2, 3, 4$ with degree 2 respectively, the containment $PQ_d \subset P_d$ is strict in the first row of Table 2.

$$\begin{aligned} p_2(z_1, z_2) &= (|z_1|^2 - |z_2|^2)^2, \\ p_3(z_1, z_2, z_3) &= |z_3|^2(|z_1|^2 - |z_2|^2)^2, \\ p_4(z_1, z_2, z_3, z_4) &= |z_3 z_4|^2(|z_1|^2 - |z_2|^2)^2, \end{aligned}$$

with z_3 and z_4 as constants in p_3 and p_4, the argument is the same as Table 4.1 (b), showing that $PQ_2(\mathbb{C}^n)$ is a proper subset of $P_2(\mathbb{C}^n)$ for $n = 2, 3, 4$ respectively. Also, there are forms for $n = 3$ and $n = 4$, $d = 2$, which show that the containment

$PQD_2(\mathbb{C}^n) \subset PQ_2(\mathbb{C}^n)$ is strict in the second row of Table 4.2. They are

$$\begin{aligned} r_{3b}(z_1, z_2, z_3) &= |z_3|^2(|z_1|^2 + |z_2|^2)^2 - b|z_3|^2|z_1 z_2|^2, \quad \text{and} \\ r_{4b}(z_1, z_2, z_3, z_4) &= |z_3 z_4|^2(|z_1|^2 + |z_2|^2)^2 - b|z_3|^2|z_1 z_2|^2, \end{aligned}$$

where $2 < b < 4$, and the argument follows from point (f) of Table 4.1. For the last row in Table 4.2, the following forms for $n = 2, 3, 4$ with degree $d = 2$ respectively show that the containment $\Sigma_2(\mathbb{C}^n) \subset PQD_2(\mathbb{C}^n)$ is strict. For $n = 2$, let $r_b = (|z_1|^2 + |z_2|^2)^2 - b|z_1 z_2|^2$, $2 < b < 4$. The argument in Table 4.1 (h) tells us that it is a positive semi-definite form with a representation as a quotient of squared norms but not a sos. Next, for $n = 3$, let

$$R_b = (|z_1|^2 + |z_2|^2 + |z_3|^2)^2 - b|z_1 z_2|^2 = r_b + 2|z_3|^2(|z_1|^2 + |z_2|^2) + |z_3|^4,$$

where $2 < b < 4$. Similar to the argument in Table 4.1 (h), R_b is non-negative when $b \leq 4$ and positive definite when $b < 4$. Hence it is a quotient of squared norms with uniform denominator when $b < 4$. It is also not a quotient of squared norms when $b = 4$ by jet pullback property with $z(t) = (1 + t, t, 1)$. Inspection of the coefficient of the term $|z_1 z_2|^2$ will also show that R_b is not sos when $b > 2$. Lastly, for $n = 4$, let $R_{4b} = (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)^2 - b|z_1 z_2|^2$. The argument is the same as the $n = 3$ case.

(ii) For $d = 2$, any positive semi-definite n-ary quadratic form p can be diagonalized as a sum of rank$(p) \leq n$ squares of linear forms. We also have a counterexample for the strict containment $PQD_6 \subset PQ_6$, which is Table 4.1 part (e), with w as a constant. As there are no known examples for $n = 4$, we are unsure at the moment whether the containment $PQD_4 \subset PQ_4$ is strict. Hence the minimum degree is either 4 or 6.

(iii) It is clear that for a positive semi-definite polynomial to be a sos, the degree has to be even. It is also easy to prove that $\Sigma_{n,2} = P_{n,2}$. By 1888, Hilbert gave the result that

$\Sigma_{3,4} = P_{3,4}$ is the only case in which they are equal, and there are counterexamples in $P_{3,6}$ and $P_{4,4}$ in which they are not sos. A famous one for $n = 3$ is Motzkin's polynomial $M(x, y, z)$, and we can observe that it is a quotient of sos with uniform denominator by the explicit decomposition of $(x^2+y^2+z^2)(x^4y^2+x^2y^4+z^6-3x^2y^2z^2)$ into a sos in Table 4.1 part (k). For $n = 4$, we have the following example by Choi and Lam [3]:

$$x^2y^2 + x^2z^2 + y^2z^2 + w^4 - 4wxyz$$

which is positive semi-definite by arithmetic-geometric inequality but not sos. A survey of such examples and history can be found in the survey paper [24]. The softwares Yalmip (http://control.ee.ethz.ch/∼joloef/wiki/pmwiki.php) or SOSTOOLS (http://www.cds.caltech.edu/sostools/) will also show the existence of a sos decomposition of $(x^2 + y^2 + z^2 + w^2)^2(x^2y^2 + x^2z^2 + y^2z^2 + w^4 - 4wxyz)$.

□

Chapter 3

Uniform denominators and their effective estimates

In this chapter we consider some modifications to theorems in two papers by Reznick and write the complex analogues for these results.

3.1 On the absence of a uniform denominator

3.1.1 For real variables

Reznick [25] gave the following theorem and corollary for positive semidefinite forms p:

Theorem 1. Suppose the set $P_d(\mathbb{R}^n) \setminus \Sigma_d(\mathbb{R}^n)$ is not empty. Then there does not exist a non-zero form h so that if $p \in PD_d(\mathbb{R}^n)$ then hp is sos.

Corollary 2. Suppose the set $P_d(\mathbb{R}^n) \setminus \Sigma_d(\mathbb{R}^n)$ is not empty. Then there does not exist a finite set of non-zero forms $\mathcal{H} = \{h_1, \cdots, h_N\}$ so that $p \in P_d(\mathbb{R}^n)$ then $h_k p$ is sos for some $h_k \in \mathcal{H}$.

We can see that the above theorem and corollary can be adapted so that it applies to $p \in PD_d(\mathbb{R}^n)$, and the proof is the same as the one given in [25]. It is clear that we only need to verify that $PD_d(\mathbb{R}^n) \setminus \Sigma_d(\mathbb{R}^n)$ is non-empty, which is shown by the example below.

Example 3.1.1. Motzkin's example $x^4y^2+x^2y^4+z^6-3x^2y^2z^2$ is well-known as a positive semi-definite form but not a sos. To modify this example so that it is positive definite, we add $\epsilon(x^6+y^6)$ to Motzkin's example, where $\epsilon>0$. Firstly, $p=x^4y^2+x^2y^4+z^6-3x^2y^2z^2+\epsilon(x^6+y^6)$ is positive definite since $\epsilon(x^6+y^6)\geq 0$, and the zero set of p is trivial as it contains only the origin. Next, we want to show that p is not a sos. We prove by contradiction. Assume that p is a sos, that is, $p=\sum_k h_k^2(x,y,z)$ would hold for suitable $h_k\in H_3(\mathbb{R}^3)$. Similar to the proof in ([24], p. 257), we write p as a ternary sextic, and h_k as follows:

$$\begin{aligned} h_k(x,y,z) &= A_kx^3+B_kx^2y+C_kxy^3+D_ky^3+E_kx^2z \\ &\quad +F_kxyz+G_ky^2z+H_kxz^2+I_kyz^2+J_kz^3 \end{aligned}$$

The coefficient of x^6 is ϵ, hence the corresponding coefficient in $\sum_k h_k^2$, $\sum_k A_k^2$ is bounded by ϵ. Hence, for all k, $A_k\leq\epsilon$. Next, the coefficient of x^4z^2 in p is zero, hence for $\sum_k h_k^2$, $\sum_k(E_k^2+2A_kH_k)$ is zero. Since each A_k is bounded by ϵ, for arbitrary value of H_k, E_k must also be small for all k. Continuing, we compare the coefficients of x^2z^4 in $\sum_k h_k^2$ and p, where we obtain $\sum_k(2E_kJ_k+H_k^2)=0$. Here we observe that E_k is small, and J_k is bounded by 1, hence H_k is small.

Using similar arguments, when taking a small value for ϵ such that H_k, I_k, E_k and G_k are small as well, we compare the coefficient of $x^2y^2z^2$ in $\sum_k h_k^2$ and p. We have:

$$\sum_k 2C_kH_k+2B_kI_k+2E_kG_k+F_k^2=-3.$$

By the above, if H_k, I_k, E_k and G_k are small, then we see that $\sum_k F_k^2<0$. This is a contradiction, hence p is not a sos.

In conclusion, Example 4.1.1 shows that $PD_d(\mathbb{R}^n)\setminus\Sigma_d(\mathbb{R}^n)$ is non-empty, and we have Theorem 1 and Corollary 2 of [25] for $p\in PD_d(\mathbb{R}^n)$.

3.1.2 For complex variables

We also show the complex analogue of Theorem 1 and Corollary 2 as in Section 4.1.1, along the lines of the proof in [25].

Theorem 3.1.2. Suppose the set $P_d(\mathbb{C}^n) \setminus \Sigma_d(\mathbb{C}^n)$ is not empty. Then there does not exist a non-zero form h so that if $p \in PD_d(\mathbb{C}^n)$ then hp is sos.

Proof. We prove by contradiction. Suppose such a non-zero form h exists, and hence there exists a point $w \in \mathbb{C}^n$ such that $h(w) \neq 0$. By making an invertible linear change of variables, take $w = (1, 0, \cdots, 0)$. Without loss of generality, it can be assumed that $h(z_1, 0, \cdots, 0) = \alpha|z_1|^{2m}$, where $|\alpha| > 0$ and m is even. Let $p \in PD_d(\mathbb{C}^n) \setminus \Sigma_d(\mathbb{C}^n)$. Then by assumption,

$$h(z_1, z_2, \cdots, z_n)p(z_1, rz_2, \cdots, rz_n)$$

is an sos for $r \in \mathbb{N}$. The change of variables $z_i \to z_i/r$ for $i \geq 2$ gives

$$h(z_1, r^{-1}z_2, \cdots, r^{-1}z_n)p(z_1, z_2, \cdots, z_n)$$

as an sos as well. Clearly,

$$\lim_{r\to\infty} h(z_1, r^{-1}z_2, \cdots, r^{-1}z_n) = h(z_1, 0, \cdots, 0) = \alpha|z_1|^{2m}$$

and since $\Sigma_{m+d}(\mathbb{C}^n)$ is a closed cone, then

$$\lim_{r\to\infty} h(z_1, r^{-1}z_2, \cdots, r^{-1}z_n)p(z_1, z_2, \cdots, z_n) = \alpha|z_1|^{2m}p(z_1, z_2, \cdots, z_n)$$

is an sos. Hence $p \in \Sigma_d(\mathbb{C}^n)$, which is a contradiction. □

Corollary 3.1.3. Suppose the set $PD_d(\mathbb{C}^n) \setminus \Sigma_d(\mathbb{C}^n)$ is not empty. Then there does not exist a finite set of non-zero forms $\mathcal{H} = \{h_1, \cdots, h_N\}$ so that if $p \in PD_d(\mathbb{C}^n)$, then $h_k p$ is sos for some $h_k \in \mathcal{H}$.

Proof. Suppose there exists a finite set of non-zero forms $\mathcal{H}$, and by assumption, for each $k \leq N$, there exists a non-zero $p \in PD_d(\mathbb{C}^n) \setminus \Sigma_d(\mathbb{C}^n)$ where $h_k p$ is sos, that is,

$h_k p = \sum_j |f_j|^2$ for some homogenous holomorphic polynomial $f_j(z)$. Since p is positive definite, by Catlin and D'Angelo's result ([5], Theorem 2), there exists an integer m such that $||z||^{2m}p$ is a squared norm. Hence p can be represented as a quotient of squared norms of holomorphic homogeneous polynomials, that is,

$$p = \frac{\sum_i |g_i|^2}{||z||^{2m}}$$

for some $g_i(z) \in H_{m+d}(\mathbb{C}^n)$. Then clearly, h_k can be represented as follows:

$$h_k = \frac{\sum_j |f_j|^2}{p} = \frac{\sum_j |f_j|^2}{\frac{\sum_i |g_i|^2}{||z||^{2m}}} = \frac{||z||^{2m} \sum_j |f_j|^2}{\sum_i |g_i|^2}$$

By above, each h_k is a quotient of squared norms (also positive semi-definite), and there exist a squared norm $||G_k||^2$ so that $||G_k||^2 h_k$ is sos. We define $h = \prod_k ||G_k||^2 h_k$, and see that

$$hp = \Big(\prod_{l \neq k} ||G_l||^2 h_l\Big) \cdot ||G_k||^2 \cdot h_k p$$

This shows that hp is a product of sos factors and hence is sos for every $p \in PD_d(\mathbb{C}^n)$. This contradicts Theorem 4.1.2 and hence proving the non-existence of the finite set of non-zero forms $\mathcal{H}$ in this corollary. □

Remark 3.1.4. Since the above set $\mathcal{H}$ does not exist for $PD_d(\mathbb{C}^n)$, then consequently such a set does not exist for $P_d(\mathbb{C}^n)$ since $PD_d(\mathbb{C}^n) \subset P_d(\mathbb{C}^n)$.

3.2 Effective estimates for complex variables

Let $p \in PD_m(\mathbb{C}^n)$. Then To and Yeung [30] have given an effective bound s_o such that

$$||z||^{2s} p(z) \in \Sigma_{m+s}(\mathbb{C}^n), \tag{3.1}$$

for any integer $s \geq s_o$, adapted from the methods of Reznick [23]. Explicitly, the bound is

$$s_o := \frac{nm(2m-1)}{(\log 2)\epsilon(p)} - n + m, \tag{3.2}$$

where

$$\epsilon(p) := \frac{\inf\{p(u) \mid u \in S^{2n-1}\}}{\sup\{p(u) \mid u \in S^{2n-1}\}} \in \mathbb{R}^{+}. \tag{3.3}$$

For $z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$, by writing $z_i = x_i + \sqrt{-1}y_i$, $1 \leq i \leq n$, we obtain an identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ given by $(z_1, \cdots, z_n) \longleftrightarrow (x_1, \cdots, x_n, y_1, \cdots, y_n)$. In this section, by the above identification, we modify Theorem 3.11 of [23] to obtain a slightly improved bound compared to (3.2) for complex variables. Firstly, we need the following remark and lemma:

Remark 3.2.1. A complex polynomial $p \in BH_d(\mathbb{C}^n)$ can be written as a difference of squares of norms, i.e,

$$p = \sum |g_i|^2 - \sum |h_i|^2 = ||g||^2 - ||h||^2,$$

where $g = (g_1, \cdots, g_j)$ and $h = (h_1, \cdots, h_k)$ are tuples of holomorphic homogeneous polynomials. If p is positive definite, it can be seen that $||h||^2 \leq c||g||^2$, for some real constant $c < 1$. For $u \in S^{2n-1}$, it is easy to see that

1. $(1-c)\max ||g(u)||^2 \leq \max p(u) \leq \max ||g(u)||^2$
2. $(1-c)\min ||g(u)||^2 \leq \min p(u) \leq \min ||g(u)||^2$

Combining the two points above will give the following:

$$(1-c)\epsilon(||g||^2) \leq \epsilon(p) \leq \frac{1}{1-c}\epsilon(||g||^2) \tag{3.4}$$

We recall that $\Delta(p(x)) = \sum_i \frac{\partial^2}{\partial x_i^2}$, the Laplacian that is the sum of all unmixed second partial derivatives.

Lemma 3.2.2. Let $p = ||g||^2 - ||h||^2$. Then for all l, $\Delta^l(||g||^2) \geq 0$ and $\Delta^l(||h||^2) \geq 0$.

Proof. We have $||g||^2 = \sum_i |g_i|^2$, and we let $g_i = u_i + iv_i$ where u_i and v_i are the real and imaginary parts of g_i respectively. Then

$$\begin{aligned}
\Delta(||g||^2) &= \sum_i \Delta(|g_i|^2) = \sum_i \Delta(u_i^2 + v_i^2) \\
&= \sum_i \sum_j^{2n} \frac{\partial^2}{\partial x_j^2}(u_i^2 + v_i^2) \\
&= \sum_i \sum_j^{2n} \frac{\partial}{\partial x_j}\Big(2u_i \frac{\partial u_i}{\partial x_j} + 2v_i \frac{\partial v_i}{\partial x_j}\Big) \\
&= \sum_i \sum_j^{2n} \Big(2(\frac{\partial u_i}{\partial x_j})^2 + 2u_i \frac{\partial^2 u_i}{\partial x_j^2} + 2(\frac{\partial v_i}{\partial x_j})^2 + 2v_i \frac{\partial^2 v_i}{\partial x_j^2}\Big)
\end{aligned}$$

Since g_i is holomorphic, u_i and v_i are harmonic with respect to each pair of variable x_{2k-1}, x_{2k}, and hence

$$\frac{\partial^2 u_i}{\partial x_{2k-1}^2} + \frac{\partial^2 u_i}{\partial x_{2k}^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 v_i}{\partial x_{2k-1}^2} + \frac{\partial^2 v_i}{\partial x_{2k}^2} = 0, \qquad \forall k.$$

Hence,

$$\Delta(||g||^2) = \sum_i \sum_j^{2n} 2\Big((\frac{\partial u_i}{\partial x_j})^2 + (\frac{\partial v_i}{\partial x_j})^2\Big) \geq 0$$

Similarly, $\Delta(||h||^2) \geq 0$. Since $\Delta(||g||^2)$ is a squared norm, then by induction, $\Delta^l(||g||^2)$ is also a squared norm. Hence $\Delta^l(||g||^2) \geq 0$ and $\Delta^l(||h||^2) \geq 0$. □

Before we give the modification of Theorem 3.11 from [23], as aforementioned, by the identification of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, we replace $p(z, \bar{z}) \in BH_d(\mathbb{C}^n)$ by $p(x) \in P_{2d}(\mathbb{R}^{2n})$. Also, we need the following theorem, which is Theorem 3.9 from [23].

Theorem 3.2.3. If $p \in H_{2d}(\mathbb{R}^{2n})$ and $s \geq 2d$, then

$$\Phi_s^{-1}(p) = \frac{1}{(s)_{2d} 2^{2d}} \sum_{l \geq 0} \frac{(-1)^l}{2^{2l} l! (n+s-1)_l} \Delta^l(p) G_{2n}^l \tag{3.5}$$

where $G_{2n}^l(x_1, \cdots, x_{2n}) = (x_1^2 + \cdots + x_{2n}^2)^l$.

Now, we give the modification to Theorem 3.11 of [23].

Theorem 3.2.4. Suppose $p \in PD_{2m}(\mathbb{R}^{2n})$. If

$$s \geq \frac{nm(2m-1)}{\sinh^{-1}(\frac{1}{\sqrt{1-c^2}}(K - c\sqrt{1+K^2}))} - n + m, \tag{3.6}$$

where $K = (1-c)\epsilon(||g||^2) + c$, c and $\epsilon(||g||^2)$ are as in Remark 4.2.1, then $\Phi_s^{-1}(p) \in P_{2m}(\mathbb{R}^{2n})$.

Proof. Since $\epsilon(||g||^2) = \epsilon(\lambda||g||^2)$, we scale $||g||^2$ such that $1 \geq ||g(u)||^2 \geq \epsilon(||g||^2)$ for $u \in S^{2n-1}$. We want to show that $\Phi_s^{-1}(p) \geq 0$. By Theorem 4.2.3, we have

$$\begin{aligned}
I \quad &:= \Big((s)_{2m}2^{2m}\Phi_s^{-1}(p)\Big)(u) \\
&= p(u) - \sum_{l\geq 1} \frac{(-1)^l \Delta^l(p)(u) G_{2n}^l(u)}{2^{2l} l!(n+s-1)_l} \\
&\geq \epsilon(p) - \frac{\Delta(||g||^2 - ||h||^2)(u)}{2^2(n+s-1)} + \frac{\Delta^2(||g||^2 - ||h||^2)(u)}{2^4 2!(n+s-1)_2} + \cdots \qquad (\text{since } G_{2n}(u) = 1) \\
&\geq (1-c)\epsilon(||g||^2) - \frac{\Delta(||g||^2)(u)}{2^2(n+s-1)} - \frac{\Delta^2(||h||^2)(u)}{2^4 2!(n+s-1)_2} + \cdots \\
&\qquad \text{(by Remark 4.2.1 and Lemma 4.2.2)} \\
&\geq (1-c)\epsilon(||g||^2) - \sum_{l\geq 1}^{m} \frac{(2n)^{2l-1}(2m)_{4l-2} M_g}{2^{4l-2}(2l-1)!(n+s-1)_{2l-1}} - \sum_{l\geq 1}^{m} \frac{(2n)^{2l}(2m)_{4l} M_h}{2^{4l}(2l)!(n+s-1)_{2l}}
\end{aligned} \tag{3.7}$$

where $\Big|||g||^2(u)\Big| \leq M_g = 1$ and $\Big|||h||^2(u)\Big| \leq M_h$, and the bounds for $\Delta^l(||g||^2)(u)$ and $\Delta^l(||h||^2)(u)$ are from Theorem 4.14 of [23]. Next, we apply the following inequalities to (3.7):

$$\begin{aligned}
(2m)_{4l} &\leq (2m)^{2l}(2m-1)^{2l}, \\
(2m)_{4l-2} &\leq (2m)^{2l-1}(2m-1)^{2l-1}, \text{ and} \\
(n+s-1)_l &\geq (n+s-l)^l \geq (n-m+s)^l.
\end{aligned} \tag{3.8}$$

Thus we have

$$I \geq (1-c)\epsilon(||g||^2) - M_g \sum_{l\geq 1}^{m} \frac{1}{(2l-1)!}\Big(\frac{nm(2m-1)}{(n-m+s)}\Big)^{2l-1} - M_h \sum_{l\geq 1}^{m} \frac{1}{(2l)!}\Big(\frac{nm(2m-1)}{(n-m+s)}\Big)^{2l} \tag{3.9}$$

Let $A = \frac{nm(2m-1)}{(n-m+s)}$. By letting the sum go to infinity, we have strict inequality:

$$\begin{aligned}
I &> (1-c)\epsilon(||g||^2) - M_g \sum_{l\geq 1}^{\infty} \frac{A^{2l-1}}{(2l-1)!} - M_h \sum_{l\geq 1}^{\infty} \frac{A^{2l}}{(2l)!} \\
&\geq (1-c)\epsilon(||g||^2) - M_g \sinh(A) - M_h \Big[\cosh(A) - 1\Big] \\
&\qquad \text{(By Taylor series of sinh and cosh)} \\
&\geq (1-c)\epsilon(||g||^2) - \sinh(A) - c\Big[\cosh(A) - 1\Big] \qquad (3.10) \\
&\qquad \Big(\text{since } M_g \text{ is scaled to 1, } c||g||^2 \geq ||h||^2, \text{ we have } cM_g \geq M_h, \text{ hence } -M_h \geq -c.\Big)
\end{aligned}$$

From the lower bound of s as given in (3.6), we can rearrange, using the trigonometric identity $\cosh^2(x) - \sinh^2(x) = 1$, to get:

$$A = \frac{nm(2m-1)}{(n-m+s)} \leq \sinh^{-1}\Big((1-c)\epsilon(||g||^2) + c\Big) - \tanh^{-1}(c) \qquad (3.11)$$

By using trigonometric identity $\sinh(\theta + A) = \sinh(\theta)\cosh(A) + \cosh(\theta)\sinh(A)$ where $\theta = \tanh^{-1}(c)$, we have

$$\sinh(\theta + A) = \sinh(A) + c\cosh(A) \qquad (3.12)$$

Substitute (3.12) into (3.10), we have

$$I \geq (1-c)\epsilon(||g||^2) - \sinh(\theta + A) + c \geq 0$$

where the last inequality can be seen by substitution of (3.11). Hence $\Phi_s^{-1}(p) \in P_{2m}(\mathbb{R}^{2n})$.

□

With this theorem replacing Proposition 2.2.3 of [30] ([23], Proprosition 3.11), the proof of (3.1) will follow accordingly.

Remark 3.2.5. We would like to compare the bound s_1 in (3.6) with the following two bounds, for n complex variables and degree m in z and $\bar{z}$:

$$s_2 \geq \frac{nm(2m-1)}{\epsilon(p)\ln(2)} - n + m, \qquad (3.13)$$

$$s_3 \geq \frac{nm(2m-1)}{\ln(1+\epsilon(p))} - n + m, \tag{3.14}$$

where s_2 is the bound from [30], and s_3 is simply a variation of the denominator in s_2. As the three bounds s_1, s_2, and s_3 are different only in the denominator, we shall plot only the denominator to see which bound is larger. Here we write $\epsilon(p) = (1-c)\epsilon(||g||^2)$ for all three bounds for purpose of comparison. The plots in Figure 3.2.5 show that the denominator of s_1 is larger than the denominators of s_2 and s_3 for small c. Graphically,

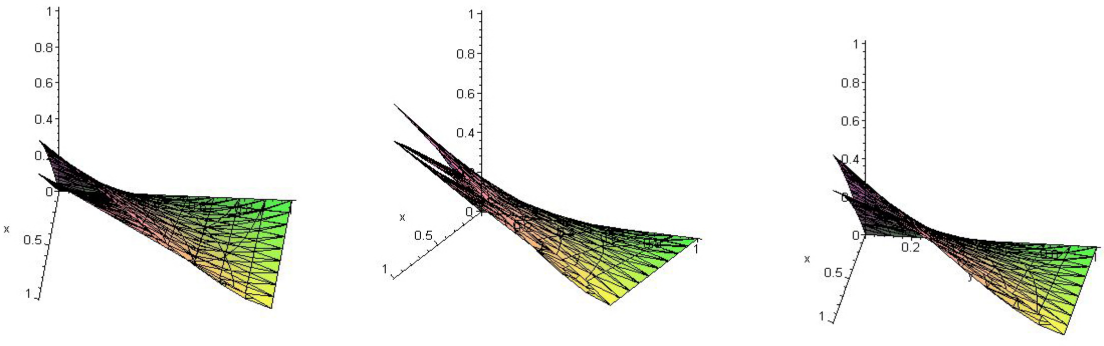

Figure 3.1: s_1 and s_2, all three bounds, s_1 and s_3

we can observe that as $\epsilon(||g||^2)$ (x-axis) increases, the denominator of s_1 is larger than denominators of s_2 and s_3 for increasing values of c (y-axis). This implies that s_1 has a lower bound, for these values of c. For example, when $\epsilon(||g||^2) = 0.6$, s_1 is greater than s_2 and s_3 for $0 < c < 0.32$, whereas, when $\epsilon(||g||^2) = 0.8$, s_1 is greater than s_2 and s_3 for $0 < c < 0.41$.

Chapter 4

Effective Pólya Semi-stability for Non-negative Polynomials on the Simplex

Let $f \in \mathbb{R}[x_1, \cdots, x_n]$ be a homogeneous polynomial which is positive on the standard simplex

$$\Delta_n := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \cdots, n; \sum_{i=1}^{n} x_i = 1\},$$

i.e., $f(x) > 0$ for all $x \in \Delta_n$. Pólya [18] showed that there exists a positive integer N_o such that all the coefficients of

$$(x_1 + \cdots + x_n)^N f(x_1, \cdots, x_n) \tag{4.1}$$

are positive for all positive integers $N \geq N_o$. As such, we simply say a polynomial f is *Pólya stable* if it satisfies the above property. Powers and Reznick [20] gave an explicit lower bound for N_o (see related works in [2], [10], [11], [15], [16], [17], [28]), and Catlin and D'Angelo ([4], [5]) have generalized it to a result for several complex variables. Pólya's theorem and the effective estimates of N_o in [20] has a wide range of applications in the works ([14], [26], [27], [29]), among others (see [21] for a description of these applications and the aforementioned related works). Powers and Reznick have further investigated

([21] and [22]) analogous properties of f when f is non-negative on Δ with corner zeros. Hence we would like to investigate analogous properties of f when f is not necessarily positive on Δ_n.

In this chapter, we consider homogeneous polynomials $f \in \mathbb{R}[x_1, \cdots, x_n]$ which are non-negative on Δ_n, and obtain necessary and/or sufficient conditions for such an f to be *Pólya semi-stable*, that is, for some positive integer N_o, all the coefficients of $(x_1 + \cdots + x_n)^N f$ are non-negative for all integers $N \geq N_o$. We are also interested in obtaining effective estimates on N_o. We explain our approach as follows: First we see that one only needs to consider those polynomials f such that $Z(f) \cap \Delta_n$ consists of faces of Δ_n. Note also that any polynomial f admits a unique decomposition into "positive"and "negative"parts according to the signs of the coefficients of its monomial terms. Roughly speaking, the necessary (resp. sufficient) conditions for Pólya semi-stability amount to the following: for each face in $Z(f) \cap \Delta_n$ and each negative monomial term of f, there exists a corresponding positive monomial term of f with lower (resp. strictly lower) vanishing orders along the face. The main difficulty in deriving the effective estimate for N_o lies in the coefficients of those monomial terms of (4.1) whose exponents, upon suitable normalizations, are close to $Z(f) \cap \Delta_n$. The sufficient conditions allow us to handle these coefficients by using an iterative process involving induction on the dimensions of the faces in $Z(f) \cap \Delta_n$.

The first section in this chapter consist of some preliminaries such as notations and definitions. Our second section in this chapter gives some necessary conditions for such an f to be Pólya semi-stable. These necessary conditions are expressed in terms of vanishing orders of the monomial terms of f along the faces of Δ_n (see Theorem 4.2.2 for the precise statement).

Next, Section 4.3 gives sufficient conditions for such an f to be Pólya semi-stable, and we also obtain explicit lower bound on N_o under such conditions. These sufficient conditions are given in Theorem 4.3.10. Using these results, in Section 4.4 we obtain a simple characterization of the Pólya semi-stable polynomials in the low dimensional case when $n \leq 3$ as well as the case (in any dimension) when the zero set $Z(f)$ of f in Δ_n consists of a finite number of points (cf. Corollary 4.4.1 and Corollary 4.4.3). In Section 4.5, we give an application of our results to the representations of non-homogeneous polynomials which are non-negative on a general simplex (cf. Corollary 4.5.1). Lastly, Section 4.6 shows that the necessary as well as sufficient conditions obtained in Theorem 4.2.2 and Theorem 4.3.10 can be extended to certain complex analogues of the positive semidefinite forms on $\mathbb{R}^n$.

The contents of sections 4.1-4.5 have been written in the paper 'Effective Pólya semi-positivity for non-negative polynomials on the simplex '. This paper is a joint efffort between the author and Associate Professor To Wing Keung, and it has been accepted for publication in the Journal of Complexity.

4.1 Preliminaries

Let $P_d(\Delta_n)$ be the set of homogeneous polynomials in $H_d(\mathbb{R}^n)$ which are non-negative on Δ_n, i.e.,

$$P_d(\Delta_n) := \{f \in H_d(\mathbb{R}^n) \mid f(x) \geq 0 \ \forall x \in \Delta_n\}.$$

The set of polynomials in $H_d(\mathbb{R}^n)$ that have only non-negative coefficients is denoted by

$$\Sigma_d^+(\mathbb{R}^n) := \{f \in H_d(\mathbb{R}^n) \mid f(x) = \sum_{\gamma \in \mathcal{I}(n,d)} a_\gamma x^\gamma \text{ with each } a_\gamma \geq 0\}.$$

Note that we always have $\Sigma_d^+(\mathbb{R}^n) \subset P_d(\Delta_n)$. For each $f = \sum_{\gamma \in \mathcal{I}(n,d)} a_\gamma x^\gamma \in H_d(\mathbb{R}^n)$, we let

$$\begin{aligned} \Lambda^+ &:= \{\alpha \in \mathcal{I}(n,d) \mid a_\alpha > 0\}, \\ \Lambda^- &:= \{\beta \in \mathcal{I}(n,d) \mid a_\beta < 0\}, \end{aligned}$$

and we write $b_\beta = -a_\beta > 0$ for each $\beta \in \Lambda^-$. Then it is easy to see that f admits the following unique decomposition into 'positive' and 'negative' parts given by

$$\begin{aligned} f &= f^+ - f^-, \quad \text{where} \\ f^+ &:= \sum_{\alpha\in\Lambda^+} a_\alpha x^\alpha \quad \text{and} \quad f^- := \sum_{\beta\in\Lambda^-} b_\beta x^\beta. \end{aligned} \tag{4.2}$$

Note that both $f^+, f^- \in \Sigma_d^+(\mathbb{R}^n)$, and we have $f \in P_d(\Delta_n)$ if and only if $f^+(x) \geq f^-(x)$ for all $x \in \Delta_n$.

For each index set $I \subsetneqq \{1, 2, \cdots, n\}$, one has an associated face F_I of Δ_n given by

$$F_I := \{x = (x_1, \cdots, x_n) \in \Delta_n \mid x_i = 0 \text{ for all } i \in I\}.$$

We also call F_I a k-face of Δ_n, where $k = n - |I| - 1$. Here $|I|$ denotes the cardinality of the set I. In particular, a 0-face is simply a vertex of Δ_n. We can identify F_I as the standard simplex Δ_{k+1} of $\mathbb{R}^{k+1}$ by setting the coordinates $x_i = 0$ for $i \in I$. Note that the boundary of the simplex Δ_n in the hyperplane $x_1 + \cdots + x_n = 1$ in $\mathbb{R}^n$ consists of n $(n-2)$-faces. Clearly, the boundary of each i-face (identified as Δ_{i+1}) consists of $i+1$ $(i-1)$-faces.

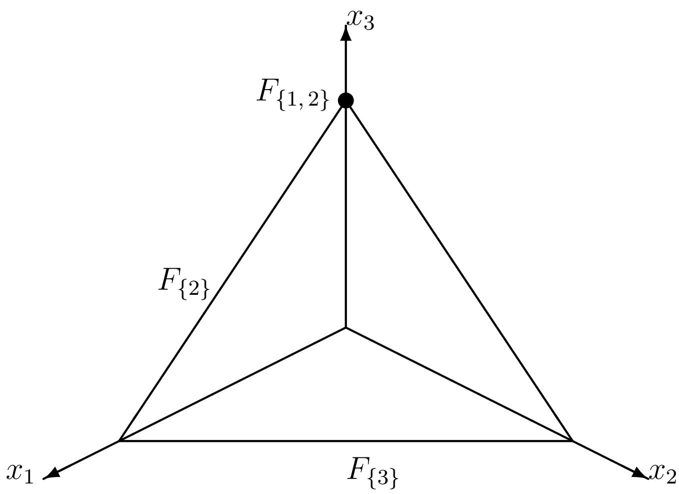

Figure 4.1: Δ_3, with three faces shown

Example 4.1.1. Figure 4.1 shows Δ_3 with axes x_1, x_2 and x_3. The boundary of Δ_3 clearly consists of the lines

$$x_1 + x_2 = 0; \quad x_2 + x_3 = 0; \quad x_1 + x_3 = 0$$

and each of them is Δ_2. The three vertices $(0,0,1)$, $(1,0,0)$ and $(0,1,0)$ are also simplexes. For $x_2 = 0$, we see that $F_{\{2\}}$ is the line $x_1 + x_3 = 0$ by definition. Also, another face is $F_{\{1,2\}}$ which is the the vertex $(0,0,1)$, with $x_1 = x_2 = 0$, as shown in Figure 2.1.

It is also easy to see that faces of Δ_n satisfy the following properties:

(i) If $I \subset J$, then $F_I \supset F_J$.

(ii) $F_I \cap F_J = F_{I \cup J}$.

Proof. (i) Given $I \subset J$ for indexes I and J with $|I| = k$ and $|J| = l$. We have $\{1, \cdots, n\} = I \cup \{i_1, \cdots, i_{n-k}\}$ and $\{1, \cdots, n\} = J \cup \{j_1, \cdots, j_{n-l}\}$. Clearly,

$$I' = \{i_1, \cdots, i_{n-k}\} \supset \{j_1, \cdots, j_{n-l}\} = J'$$

by the inequality $k < l$. Since $\Delta_{n-l} \subset \Delta_{n-k}$, we have $F_I \supset F_J$.
(ii) $F_I \cap F_J$ can be described by the equation $\sum x_m = 1$, for all $m \notin J$, $m \notin I$. Hence the index for $F_I \cap F_J$ is $I \cup J$. $\square$

For each fixed $f = \sum_{\gamma \in \mathcal{I}(n,d)} a_\gamma x^\gamma \in H_d(\mathbb{R}^n)$, we denote its zero set by

$$\begin{aligned} Z(f) \quad &:= \quad \{x \in \mathbb{R}^n \mid f(x) = 0\}, \quad \text{so that} \\ Z(f) \cap \Delta_n \quad &= \quad \{x \in \Delta^n \mid f(x) = 0\}. \end{aligned}$$

To facilitate the comparison of vanishing orders of monomial terms of f along faces of Δ_n, we introduce the following definition.

Definition 4.1.2. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ be n-tuples in $\mathbb{Z}^n_{\geq 0}$, and let $I \subset \{1, 2, \cdots, n\}$. Then we say that

$$\beta \succeq_I \alpha \text{ if and only if } \beta_i \geq \alpha_i \text{ for all } i \in I. \tag{4.3}$$

Moreover, we say that

$$\beta \succ_I \alpha \text{ if and only if } \beta \succeq_I \alpha \text{ and there exists } i_0 \in I \text{ such that } \beta_{i_0} > \alpha_{i_0}. \tag{4.4}$$

Definition 4.1.3. Let $f \in H_d(\mathbb{R}^n)$. Then f is said to be Pólya semi-stable if $(x_1 + \cdots + x_n)^N f$ has only non-negative coefficients for all sufficiently large positive integers N, i.e., there exists a natural number N_o such that

$$(x_1 + \cdots + x_n)^N f \in \Sigma^+_{N+d}(\mathbb{R}^n),$$

for all $N \geq N_o$.

Remark 4.1.4. Since $x_1 + \cdots + x_n = 1$ on Δ_n, it is easy to see that if $f \in H_d(\mathbb{R}^n)$ is Pólya semi-stable, then $f \in P_d(\Delta_n)$. Thus, in discussing necessary and/or sufficient conditions for $f \in H_d(\mathbb{R}^n)$ to be Pólya semi-stable, we only need to consider the case when $f \in P_d(\Delta_n)$.

4.2 Necessary conditions for Pólya semi-stability

Lemma 4.2.1. Let $g \in \Sigma^+_d(\mathbb{R}^n)$. Then $Z(g) \cap \Delta_n$ consists of a finite union of faces of Δ_n. More precisely, we have

$$Z(g) \cap \Delta_n = \bigcup_{I \in \Phi_g} F_I,$$

where $\Phi_g := \{I \subsetneq \{1, \cdots, n\} \mid \alpha \succ_I (0, \cdots, 0)\ \forall \alpha \in \Lambda^+\}$.

Proof. Since $g \in \Sigma^+_d(\mathbb{R}^n)$, we may write $g = \sum_{\alpha \in \Lambda^+} a_\alpha x^\alpha$ (i.e., we have $\Lambda^- = \emptyset$). First we show that $Z(g) \cap \Delta_n \supset \bigcup_{I \in \Phi_g} F_I$. Let $x \in \bigcup_{I \in \Phi_g} F_I$. Then $x \in F_I$ for some $I \in \Phi_g$. For any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \Lambda^+$, it follows from the definition of Φ_g that $\alpha \succ_I (0, \cdots, 0)$, which implies that $\alpha_i > 0$ for some $i = i(\alpha) \in I$. It follows readily that $a_\alpha x^\alpha = a_\alpha x_1^{\alpha_1} \cdots x_i^{\alpha_i} \cdots x_n^{\alpha_n} = 0$. By varying $\alpha \in \Lambda^+$, we conclude that $g(x) = 0$. Thus we have $Z(g) \cap \Delta_n \supset \bigcup_{I \in \Phi_g} F_I$. Next we proceed to show that $Z(g) \cap \Delta_n \subset \bigcup_{I \in \Phi_g} F_I$. Let $x = (x_1, \cdots, x_n) \in Z(g) \cap \Delta_n$, so that $g(x) = 0$. Since $a_\alpha x^\alpha \geq 0$ with $a_\alpha > 0$ for each

$\alpha \in \Lambda^+$, it follows that $x^\alpha = 0$ for each $\alpha \in \Lambda^+$. Thus, for each $\alpha = (\alpha_1, \cdots, \alpha_n) \in \Lambda^+$, there exists $i = i(\alpha)$ with $1 \le i \le n$ such that $x_i = 0$ and $\alpha_i > 0$. Let $I = \{i \mid x_i = 0\}$. Then it follows readily that $I \neq \emptyset$, $x \in F_I$, and $\alpha \succ_I (0, \cdots, 0)$ for all $\alpha \in \Lambda^+$ (which implies that $I \in \Phi_g$). By varying x, we have $Z(g) \cap \Delta_n \subset \bigcup_{I \in \Phi_g} F_I$. □

Let $f = f^+ - f^- = \sum_{\alpha \in \Lambda^+} a_\alpha x^\alpha - \sum_{\beta \in \Lambda^-} b_\beta x^\beta \in P_d(\Delta_n)$ be as in (4.2). For each $N \ge 0$ and $\gamma \in \mathcal{I}(n, N+d)$, we denote by A_γ^N the coefficient of x^γ in $(x_1 + \cdots + x_n)^N f$, so that we have

$$(x_1 + \cdots + x_n)^N f = \sum_{\gamma \in \mathcal{I}(n,N+d)} A_\gamma^N x^\gamma. \tag{4.5}$$

Furthermore, we denote the coefficient of x^γ in $(x_1+\cdots+x_n)^N f^+$ (resp. $(x_1+\cdots+x_n)^N f^-$) by $A_\gamma^{N,+}$ (resp. $B_\gamma^{N,-}$), so that we have

$$\begin{aligned} (x_1 + \cdots + x_n)^N f^+ &= \sum_{\gamma \in \mathcal{I}(n,N+d)} A_\gamma^{N,+} x^\gamma, \quad \text{and} \\ (x_1 + \cdots + x_n)^N f^- &= \sum_{\gamma \in \mathcal{I}(n,N+d)} B_\gamma^{N,-} x^\gamma. \end{aligned}$$

Clearly, for each N and γ, we have

$$A_\gamma^N = A_\gamma^{N,+} - B_\gamma^{N,-}. \tag{4.6}$$

Similarly, for each $\alpha \in \Lambda^+$ and $\beta \in \Lambda^-$, we also write

$$\begin{aligned} (x_1 + \cdots + x_n)^N \cdot a_\alpha x^\alpha &= \sum_{\gamma \in \mathcal{I}(n,N+d)} A_\gamma^{N,\alpha} x^\gamma, \\ (x_1 + \cdots + x_n)^N \cdot b_\beta x^\beta &= \sum_{\gamma \in \mathcal{I}(n,N+d)} B_\gamma^{N,\beta} x^\gamma. \end{aligned}$$

One easily sees that each $A_\gamma^{N,\alpha} \ge 0$ and $B_\gamma^{N,\beta} \ge 0$. Moreover, one has

$$A_\gamma^{N,+} = \sum_{\alpha \in \Lambda^+} A_\gamma^{N,\alpha} \quad \text{and} \quad B_\gamma^{N,-} = \sum_{\beta \in \Lambda^-} B_\gamma^{N,\beta}. \tag{4.7}$$

From the calculations by Pólya and given in ([20], p. 223), it follows that for each $\alpha = (\alpha_1, \cdots, \alpha_n) \in \Lambda^+$, $\beta = (\beta_1, \cdots, \beta_n) \in \Lambda^-$ and $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathcal{I}(n, N+d)$, one has

$$A_\gamma^{N,\alpha} = \frac{N!(N+d)^d}{\gamma_1!\cdots\gamma_n!} \cdot a_\alpha \cdot \prod_{i=1}^n \Big(\frac{\gamma_i}{N+d}\Big)\Big(\frac{\gamma_i - 1}{N+d}\Big)\cdots\Big(\frac{\gamma_i - (\alpha_i - 1)}{N+d}\Big), \tag{4.8}$$

$$B_\gamma^{N,\beta} = \frac{N!(N+d)^d}{\gamma_1!\cdots\gamma_n!} \cdot b_\beta \cdot \prod_{i=1}^n \Big(\frac{\gamma_i}{N+d}\Big)\Big(\frac{\gamma_i - 1}{N+d}\Big)\cdots\Big(\frac{\gamma_i - (\beta_i - 1)}{N+d}\Big). \tag{4.9}$$

The following theorem gives the necessary conditions for a polynomial to be Pólya semi-stable.

Theorem 4.2.2. Let $f \in P_d(\Delta_n)$ be Pólya semi-stable. Then f satisfies the following two properties:

(Z1) $Z(f) \cap \Delta_n$ consists of a finite union of faces of Δ_n, and

(Z2) For each face $F_I \subset Z(f) \cap \Delta_n$ and each $\beta \in \Lambda^-$, there exists $\alpha = \alpha(\beta, I) \in \Lambda^+$ depending on β and I such that $\beta \succeq_I \alpha$.

Proof. Let $f \in P_d(\Delta_n)$ be Pólya semi-stable. Then there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$(x_1 + \cdots + x_n)^N f \in \Sigma^+_{N+d}(\mathbb{R}^n).$$

Since $\sum x_i = 1$ on Δ_n, it follows that

$$Z((x_1 + \cdots + x_n)^N f) \cap \Delta_n = Z(f) \cap \Delta_n.$$

Together with Lemma 4.2.1 (applied to $(x_1 + \cdots + x_n)^N f$), it follows readily that $Z(f) \cap \Delta_n$ is a finite union of faces of Δ_n. Hence f satisfies (Z1). Next we prove (Z2) by contradiction. Suppose (Z2) does not hold. Then there exist a face $F_I \subset Z(f) \cap \Delta_n$ and $\beta \in \Lambda^-$ such that

$$\beta \not\succeq_I \alpha \quad \text{for all } \alpha \in \Lambda^+, \tag{4.10}$$

i.e., there exist $i_0 = i_0(\alpha, \beta, I)$ such that $\alpha_{i_0} > \beta_{i_0}$. Now we fix an integer $i_1 \in \{1, 2, \cdots, n\} \setminus I$. For each positive integer $N \geq 1$, we let $\gamma = \gamma(N) = (\gamma_1, \cdots, \gamma_n)$ be defined by

$$\gamma_i := \begin{cases} \beta_i & \text{if } i \neq i_1, \\ N + \beta_{i_1} & \text{if } i = i_1. \end{cases} \tag{4.11}$$

It is easy to see that $\gamma \in \mathcal{I}(n, N+d)$. For each $\alpha \in \Lambda^+$, since there exists $i_0 \in I$ such that $\alpha_{i_0} > \beta_{i_0} = \gamma_{i_0}$, it follows that one of the factors in (4.8) is zero, i.e., we have $A_\gamma^{N,\alpha} = 0$. Together with (4.7), it follows that we have

$$A_\gamma^{N,+} = 0. \tag{4.12}$$

On the other hand, it follows from (4.11) that $\gamma_j \geq \beta_j$ for all $1 \leq j \leq n$. Together with (4.9), it follows that $B_\gamma^{N,\beta} > 0$. Since we also have $B_\gamma^{N,\beta'} \geq 0$ for any other $\beta' \in \Lambda^-$, it follows from (4.7) that we have $B_\gamma^{N,-} > 0$. Together with (4.12) and (4.6), it follows that $A_\gamma^N < 0$. Thus for each $N \geq 1$, we have constructed a $\gamma = \gamma(N) \in \mathcal{I}(n, N+d)$ such that $A_\gamma^N < 0$, which contradicts the Pólya semi-stability of f. Hence f satisfies (Z2). $\square$

We construct a polynomial in $P_3(\Delta_4)$ which satisfies (Z1) and (Z2) but is not Pólya semi-stable. This illustrates that the necessary conditions (Z1) and (Z2) in Theorem 4.2.2 for Pólya semi-stability are not sufficient conditions for Pólya semi-stability.

Example 4.2.3. Let $f \in \mathbb{R}[x, y, z, w]$ be given by

$$\begin{aligned} f(x,y,z,w) &:= x^3 + xy^2 + xz^2 + xw^2 + x^2y + y^3 + yz^2 + yw^2 - 2xzw - 2yzw \\ &= (x+y)(x^2 + y^2 + (z-w)^2). \end{aligned}$$

Clearly, $f \in P_3(\Delta_4)$. It can be easily seen that

$$Z(f) \cap \Delta_4 = \{(x,y,z,w) \in \Delta_4 \mid x = y = 0\} = F_{\{1,2\}}.$$

Hence f satisfies (Z1). Moreover, the three faces of Δ_4 in $Z(f) \cap \Delta_4$ are $F_I = F_{\{1,2\}}$, $F_J = F_{\{1,2,3\}}$ and $F_K = F_{\{1,2,4\}}$. We list the 4-tuples in Λ^+ and Λ^- as follows:

$$\begin{aligned} \Lambda^+ &= \{(3,0,0,0), (1,2,0,0), (2,1,0,0), (0,3,0,0), \\ &\qquad (1,0,2,0), (1,0,0,2), (0,1,2,0), (0,1,0,2)\}. \\ \Lambda^- &= \{(1,0,1,1), (0,1,1,1)\}. \end{aligned}$$

Clearly, for each $\beta \in \Lambda^-$ and each I, J or K, there exist $\alpha(I), \alpha(J), \alpha(K) \in \Lambda^+$ (depending also on β) such that $\beta \succeq_I \alpha(I)$, $\beta \succeq_J \alpha(J)$ and $\beta \succeq_K \alpha(K)$. As an example, when $\beta = (1,0,1,1)$, it is easy to check that one may let

$$\alpha(I) = (1,0,2,0), \qquad \alpha(J) = (1,0,0,2) \quad \text{and} \quad \alpha(K) = (1,0,2,0).$$

The case when $\beta = (0,1,1,1)$ is similar and will thus be left to the reader. Hence we see that the polynomial $f \in P_3(\Delta_4)$ satisfies (Z2). On the other hand, for each even positive integer $N = 2m$, we let $\gamma = (1,0,m+1,m+1) \in \mathcal{I}(4, N+3)$ and consider the associated monomial $A_\gamma^N xz^{m+1}w^{m+1}$ in $(x+y+z+w)^N f$. Then the terms in f contributing to this monomial are xz^2, xw^2 and $-2xzw$, and we have

$$\begin{aligned} A_\gamma^N &= \frac{(2m)!}{0!0!(m-1)!(m+1)!} + \frac{(2m)!}{0!0!(m+1)!(m-1)!} - 2 \cdot \frac{(2m)!}{0!0!m!m!} \\ &= -\frac{2 \cdot (2m)!}{m!(m+1)!} < 0. \end{aligned}$$

Hence f is not Pólya semi-stable.

4.3 Sufficient conditions for Pólya semi-stability with effective estimates

In this section, we establish the sufficient conditions for Pólya semi-stability with effective estimates. Let f be in $P_d(\Delta_n)$ satisfying (Z1) and the following condition:

(Z2′) For each face $F_I \subset Z(f) \cap \Delta_n$ and each $\beta \in \Lambda^-$, there exists $\alpha = \alpha(\beta, I) \in \Lambda^+$ depending on β and I such that $\beta \succ_I \alpha$.

In subsection 4.3.1, we will show that $A_\gamma^N \geq 0$ for all sufficiently large N and all $\gamma \in \mathcal{I}(n, N+d)$ such that $\frac{\gamma}{N+d}$ is sufficiently close to $Z(f) \cap \Delta_n$, where A_γ^N is as in (4.5). This will be achieved by an iterative process which involves induction on the dimensions of the faces in $Z(f) \cap \Delta_n$. In subsection 4.3.2, we will handle those γ's such that $\frac{\gamma}{N+d}$ stays away from $Z(f) \cap \Delta_n$. Lastly, we establish the sufficient conditions for Pólya semi-stability with effective estimates.

4.3.1 $\frac{\gamma}{N+d}$ being sufficiently close to $Z(f) \cap \Delta$

Lemma 4.3.1. Let $f \in P_d(\Delta_n)$ be such that f satisfies (Z1). Then we have

$$Z(f) \cap \Delta_n = Z(f^+) \cap \Delta_n. \tag{4.13}$$

Proof. For any $x \in Z(f^+) \cap \Delta_n$, since $f \in P_d(\Delta_n)$, we have $0 = f^+(x) \geq f^-(x) \geq 0$, and thus $f^+(x) = f^-(x) = 0$. Hence $f(x) = 0$ and $x \in Z(f) \cap \Delta_n$. Thus we have $Z(f^+) \cap \Delta_n \subset Z(f) \cap \Delta_n$. Conversely, since f satisfies (Z1), we may write $Z(f) \cap \Delta_n = \bigcup_{I \in \Phi} F_I$ for an index set Φ. Recall from section 4.1 that for each face $F_I \subset Z(f) \cap \Delta_n$ with the associated index $I \subset \{1, 2, \cdots, n\}$, F_I can be identified with the standard simplex Δ_k of $\mathbb{R}^k$ with $k = n - |I|$ by setting the coordinates $x_i = 0$ for all $i \in I$. Then one easily sees that the restriction $f|_{\mathbb{R}^k} \in H_d(\mathbb{R}^k)$, and $f|_{\mathbb{R}^k}$ vanishes on $\Delta_k \cong F_I$. Together with the homogeneity of $f|_{\mathbb{R}^k}$, it follows that $f|_{\mathbb{R}^k}$ vanishes on the non-empty open cone in $\mathbb{R}^k$ defined by Δ_k. Hence $f|_{\mathbb{R}^k}$ is the zero polynomial, and it follows that $f^+|_{\mathbb{R}^k} = f^-|_{\mathbb{R}^k} = 0$. Thus, $F_I \subset Z(f^+) \cap \Delta_n$. By varying $I \in \Phi$, we see that $Z(f) \cap \Delta_n \subset Z(f^+) \cap \Delta_n$. $\square$

Let $f = f^+ - f^- = \sum_{\alpha \in \Lambda^+} a_\alpha x^\alpha - \sum_{\beta \in \Lambda^-} b_\beta x^\beta \in P_d(\Delta_n)$ be as in (4.2). Note that if $f^- = 0$, then $f \in \Sigma_d^+(\mathbb{R}^n)$, and thus f is necessarily Pólya semi-stable. Therefore, when considering sufficient conditions for Pólya semi-stability, we will always assume that $f^- \neq 0$ (and thus also $f^+ \neq 0$). Then we have

$$a_{\max} := \max_{\alpha \in \Lambda^+} a_\alpha > 0, \quad a_{\min} := \min_{\alpha \in \Lambda^+} a_\alpha > 0, \quad \text{and} \quad b_{\max} := \max_{\beta \in \Lambda^-} b_\beta > 0. \tag{4.14}$$

Also, we define

$$c = c(f) := \sup_{x \in \Delta_n \setminus Z(f)} \frac{f^-(x)}{f^+(x)} \leq 1. \tag{4.15}$$

For any $f \in P_d(\Delta_n$ satisfying (Z1), we let $k = k(f)$ be the maximum dimension of the faces of Δ_n that lie in $Z(f) \cap \Delta_n$, i.e.,

$$k := n - 1 - \min\{|I| \mid F_I \subset Z(f) \cap \Delta_n\} \leq n - 2. \tag{4.16}$$

Let F_I be the face of Δ_n associated to an index set $I \subset \{1, 2, \cdots, n\}$. For $r > 0$ we consider the following tubular neighbourhood of F_I in Δ_n given by

$$F_I(r) := \{x = (x_1, \cdots, x_n) \in \Delta_n \mid x_i \leq r \ \forall i \in I\}. \tag{4.17}$$

From now on, we fix an $f \in P_d(\Delta_n)$ such that f satisfies (Z1) and (Z2$'$). By (Z1), we may write

$$Z(f) \cap \Delta_n = \widetilde{F}_0 \cup \widetilde{F}_1 \cup \cdots \cup \widetilde{F}_k, \tag{4.18}$$

where k is as defined in (4.16), and for each $0 \leq \ell \leq k$, $\widetilde{F}_\ell$ is the finite union of the ℓ-faces in $Z(f) \cap \Delta_n$. For each $0 \leq \ell \leq k$, we let Φ_ℓ be the set of indexes corresponding to the ℓ-faces in $Z(f) \cap \Delta_n$, so that we have $\widetilde{F}_\ell = \bigcup\limits_{I \in \Phi_\ell} F_I$. For $r > 0$, we also denote the following tubular neighborhoods of the $\widetilde{F}_\ell$'s as well as that of $Z(f) \cap \Delta_n$ in Δ_n by

$$\begin{aligned} \widetilde{F}_\ell(r) : &= \bigcup_{I \in \Phi_\ell} F_I(r), \quad \text{and} \\ \widetilde{Z(f)}(r) : &= \bigcup_{\ell=0}^{k} \widetilde{F}_\ell(r). \end{aligned} \tag{4.19}$$

To carry out the iterative process, we define two finite sequences of numbers $\{\epsilon_i\}_{0 \leq i \leq k}$ and $\{N_\ell\}_{0 \leq \ell \leq k}$ recursively (see (4.27) below), so that for all $N \geq N_\ell$ and all $\gamma \in \mathcal{I}(n, N+d)$ such that $\frac{\gamma}{N+d} \in \widetilde{F}_\ell(\epsilon_\ell)$, one has $A^N_\gamma \geq 0$, where A^N_γ is as in (4.5). First we consider the case when $\ell = 0$ in the following lemma.

Lemma 4.3.2. Let

$$\epsilon_0 := \frac{a_{\min}}{|\Lambda^-| b_{\max} + nd a_{\min}}, \quad \text{and} \quad N_0 := \frac{2d}{\epsilon_0} - d. \tag{4.20}$$

Then for any $N \geq N_0$ and any $\gamma \in \mathcal{I}(n, N+d)$ satisfying $\frac{\gamma}{N+d} \in \widetilde{F}_0(\epsilon_0)$, we have $A^N_\gamma \geq 0$. Here $|\Lambda^-|$, $a_{\min}$ and $b_{\max}$ are as in (4.14).

Proof. We fix a positive integer $N \geq N_0$ and a $\gamma \in \mathcal{I}(n, N+d)$ satisfying $\frac{\gamma}{N+d} \in \widetilde{F}_0(\epsilon_0)$. Recall from (4.19) that $\widetilde{F}_0(\epsilon_0) = \bigcup_{I \in \Phi_0} F_I(\epsilon_0)$. Thus we have $\frac{\gamma}{N+d} \in F_I(\epsilon_0)$ for some $I \in \Phi_0$. Note that $|I| = n-1$, since F_I is a 0-face (vertex) of Δ_n. Upon permuting the x_i's if necessary, we will assume without loss of generality that $I = \{1, 2, \cdots, n-1\}$. Then for each $\beta = (\beta_1, \cdots, \beta_n) \in \Lambda^-$, it follows from (Z2′) that there exists $\alpha = (\alpha_1, \cdots, \alpha_n) \in \Lambda^+$ and i_0 satisfying $1 \le i_0 \le n-1$, depending on β, and such that

$$\beta_{i_0} > \alpha_{i_0}, \quad \text{and} \quad \beta_i \ge \alpha_i \ \text{ for } 1 \le i \ne i_0 \le n-1. \tag{4.21}$$

Note that it follows from (4.21) that we necessarily have $\beta_n < \alpha_n$. We estimate $B_\gamma^{N,-}$ by bounding each $B_\gamma^{N,\beta}$. For this purpose, we will only consider those $B_\gamma^{N,\beta}$'s which are positive. Note that for such $B_\gamma^{N,\beta} \ne 0$, it follows from (4.9) that we must have $\gamma_i \ge \beta_i$ for all $1 \le i \le n$. Formally it follows from (4.8) and (4.9) that

$$\frac{B_\gamma^{N,\beta}}{A_\gamma^{N,\alpha}} = \frac{b_\beta}{a_\alpha}\Big[\prod_{\substack{i=1\\ i\ne i_0}}^{n-1}\prod_{j=\alpha_i}^{\beta_i-1}\frac{\gamma_i - j}{N+d}\Big]\cdot\Big[\prod_{j=\alpha_{i_0}}^{\beta_{i_0}-1}\frac{\gamma_{i_0}-j}{N+d}\Big]\cdot\frac{1}{\displaystyle\prod_{j=\beta_n}^{\alpha_n-1}\frac{\gamma_n - j}{N+d}}, \tag{4.22}$$

where for each $1 \le i \ne i_0 \le n-1$, the factor $\prod_{j=\alpha_i}^{\beta_i-1}\frac{\gamma_i-j}{N+d}$ is understood to be 1 if $\alpha_i = \beta_i$. Since $\beta_i \le \gamma_i \le N+d$ for each i and $\frac{\gamma_i}{N+d} \le \epsilon_0$ for $1 \le i \le n-1$ (in particular, one has $\frac{\gamma_{i_0}-(\beta_{i_0}-1)}{N+d} \le \epsilon_0$), it follows that one has

$$0 \le \Big[\prod_{\substack{i=1\\ i\ne i_0}}^{n-1}\prod_{j=\alpha_i}^{\beta_i-1}\frac{\gamma_i - j}{N+d}\Big]\cdot\Big[\prod_{j=\alpha_{i_0}}^{\beta_{i_0}-1}\frac{\gamma_{i_0}-j}{N+d}\Big] \le \epsilon_0. \tag{4.23}$$

Note that since $F_I \subset Z(f) \cap \Delta_n$, it follows from Lemma 4.3.1 that $a_\alpha x^\alpha$ vanishes on F_I. This implies that one has $\alpha_n \le d-1$. Since $\gamma \in \mathcal{I}(n, N+d)$ and $\frac{\gamma}{N+d} \in F_I(\epsilon_0)$, it follows that for each $\beta_n \le j \le \alpha_n - 1 < d-1$, one has

$$\begin{aligned}
\frac{\gamma_n - j}{N+d} &\ge \frac{N+d-\gamma_1-\cdots-\gamma_{n-1}}{N+d} - \frac{d}{N+d} \\
&\ge \frac{N}{N+d} - (n-1)\epsilon_0 \quad (\text{since } \frac{\gamma_i}{N+d} \le \epsilon_0 \text{ for } 1 \le i \le n-1) \\
&\ge \frac{\frac{2d}{\epsilon_0} - d}{\frac{2d}{\epsilon_0}} - (n-1)\epsilon_0 \quad (\text{since } N \ge N_0 = \frac{2d}{\epsilon_0} - d) \\
&= 1 - (n - \frac{1}{2})\epsilon_0 > 0,
\end{aligned} \tag{4.24}$$

where the last inequality follows readily from (4.20). Recall that the Bernoulli inequality implies that $(1-x)^m \ge 1 - mx \ge 0$ for any non-negative integer m and any x such that

$x < \frac{1}{m}$. It is also easily seen from (4.20) that $(n - \frac{1}{2})\epsilon_0 < \frac{1}{d-1}$. Together with (4.24), we have

$$\begin{aligned} \prod_{j=\beta_n}^{\alpha_n - 1} \frac{\gamma_n - j}{N+d} &\geq \big(1 - (n - \frac{1}{2})\epsilon_0\big)^{d-1} \\ &\geq 1 - (d-1)(n - \frac{1}{2})\epsilon_0 \\ &\geq 1 - nd\epsilon_0 > 0, \end{aligned} \tag{4.25}$$

where the inequality $1 - nd\epsilon_0 > 0$ follows readily from (4.20). Together with (4.14), (4.22), (4.23), (4.25), and noting that $A_\gamma^{N,\alpha} \leq A_\gamma^{N,+}$, we have

$$\frac{B_\gamma^{N,\beta}}{A_\gamma^{N,+}} \leq \frac{B_\gamma^{N,\beta}}{A_\gamma^{N,\alpha}} \leq \frac{b_{\max} \cdot \epsilon_0}{a_{\min}(1 - nd\epsilon_0)}. \tag{4.26}$$

Upon summing (4.26) over each $\beta \in \Lambda^-$, we have

$$\frac{B_\gamma^{N,-}}{A_\gamma^{N,+}} \leq |\Lambda^-| \cdot \frac{b_{\max} \cdot \epsilon_0}{a_{\min} \cdot (1 - nd\epsilon_0)} = 1,$$

where the last equality follows from (4.20). Hence $B_\gamma^{N,-} \leq A_\gamma^{N,+}$, and we have $A_\gamma^N \geq 0$. □

Next we define two sequences of numbers $\{\epsilon_\ell\}_{0 \leq \ell \leq k}$ and $\{N_\ell\}_{0 \leq \ell \leq k}$ recursively as follows: Let ϵ_0 and N_0 be as in (4.20). For $1 \leq \ell \leq k$, let

$$\epsilon_\ell := \min\big\{\epsilon_{\ell-1}, \frac{a_{\min} \cdot \epsilon_{\ell-1}^{d-1}}{2^{d-1}|\Lambda^-| b_{\max}}\big\} \quad \text{and} \quad N_\ell := \frac{2d}{\epsilon_{\ell-1}} - d. \tag{4.27}$$

It is easy to see that $\epsilon_0 \geq \epsilon_1 \geq \cdots \geq \epsilon_k$, while $N_0 \leq N_1 \leq \cdots \leq N_k$.

Proposition 4.3.3. For a given fixed integer ℓ satisfying $0 \leq \ell \leq k$, let N_ℓ and ϵ_ℓ be as in (4.27). Then for any positive integer $N \geq N_\ell$ and any $\gamma \in \mathcal{I}(n, N+d)$ satisfying $\frac{\gamma}{N+d} \in \widetilde{F}_\ell(\epsilon_\ell)$, we have $A_\gamma^N \geq 0$.

Proof. We prove this proposition by induction on ℓ. This proposition in the case when $\ell = 0$ was proved in Lemma 4.3.2. Next we make the induction hypothesis that Proposition 4.3.3 holds for the cases when the running indexes take the values $0, 1, \cdots, \ell - 1$. Now we let N be a positive integer such that $N \geq N_\ell$ and we let $\gamma \in \mathcal{I}(n, N+d)$ be such that $\frac{\gamma}{N+d} \in \widetilde{F}_\ell(\epsilon_\ell)$. Write $\widetilde{F}_\ell(\epsilon_\ell) = \bigcup_{I\in\Phi_\ell} F_I(\epsilon_\ell)$ as in (4.19). Then $\frac{\gamma}{N+d} \in F_I(\epsilon_\ell)$ for some $I \in \Phi_\ell$. Since $F_I \subset Z(f) \cap \Delta_n$, it follows readily that $F_J \subset Z(f) \cap \Delta_n$ for any $J \supset I$. Thus we have $\bigcup_{J \supsetneqq I} F_J(\epsilon_{\ell-1}) \subset \bigcup_{j=0}^{\ell-1} \widetilde{F}_j(\epsilon_j)$. In particular, since $N \geq N_\ell$ (and thus $N \geq N_j$ for all $j < \ell$), it follows from the induction hypothesis that we must have $A^N_\gamma \geq 0$ if $\frac{\gamma}{N+d} \in \bigcup_{J \supsetneqq I} F_J(\epsilon_{\ell-1})$. It remains to consider the case when $\frac{\gamma}{N+d} \in F_I(\epsilon_\ell) \setminus \bigcup_{J \supsetneqq I} F_J(\epsilon_{\ell-1})$. It is easy to check that

$$F_I(\epsilon_\ell) \setminus \bigcup_{J \supsetneqq I} F_J(\epsilon_{\ell-1}) = \{x \in \Delta_n \mid x_i \leq \epsilon_\ell \text{ for } i \in I, \text{ and } x_i > \epsilon_{\ell-1} \text{ for } i \notin I\}. \tag{4.28}$$

As in the proof of Lemma 4.3.2, we estimate $B^{N,-}_\gamma$ by bounding each non-zero $B^{N,\beta}_\gamma$, which from (4.9), must satisfy the inequality $\gamma_i \geq \beta_i$ for each $1 \leq i \leq n$. Recall also that for each $\beta \in \Lambda^-$, it follows from (Z2′) that there exists $\alpha = \alpha(\beta) \in \Lambda^+$ and $i_0 \in I$ such that $\beta_{i_0} > \alpha_{i_0}$ and $\beta_i \geq \alpha_i$ for all $i \in I$. Formally and as in (4.22), it follows from (4.8) and (4.9) that

$$\frac{B^{N,\beta}_\gamma}{A^{N,\alpha}_\gamma} = \frac{b_\beta}{a_\alpha}\Big[\prod_{\substack{i\in I\\ i\neq i_0}} \prod_{j=\alpha_j}^{\beta_i-1} \frac{\gamma_i - j}{N+d}\Big] \cdot \Big[\prod_{j=\alpha_{i_0}}^{\beta_{i_0}-1} \frac{\gamma_{i_0} - j}{N+d}\Big] \cdot \frac{\displaystyle\prod_{i\notin I}\prod_{j=0}^{\beta_i-1} \frac{\gamma_i - j}{N+d}}{\displaystyle\prod_{i\notin I}\prod_{j=0}^{\alpha_i-1} \frac{\gamma_i - j}{N+d}} \tag{4.29}$$

As in (4.23), it follows from the inequalities $\beta_i \leq \gamma_i \leq N+d$, $1 \leq i \leq n$, and $\frac{\gamma_{i_0}}{N+d} \leq \epsilon_\ell$ that one has

$$0 \leq \Big[\prod_{\substack{i\in I\\ i\neq i_0}} \prod_{j=\alpha_j}^{\beta_i-1} \frac{\gamma_i - j}{N+d}\Big] \cdot \Big[\prod_{j=\alpha_{i_0}}^{\beta_{i_0}-1} \frac{\gamma_{i_0} - j}{N+d}\Big] \cdot \prod_{i\notin I}\prod_{j=0}^{\beta_i-1} \frac{\gamma_i - j}{N+d} \leq \epsilon_\ell. \tag{4.30}$$

For each $i \notin I$ and each $0 \leq j \leq \alpha_i - 1 < d$, it follows from (4.28) that $\gamma_i > \epsilon_{\ell-1}$, and thus as in (4.24), we have

$$\begin{aligned} \frac{\gamma_i - j}{N+d} &\geq \epsilon_{\ell-1} - \frac{d}{N+d} \\ &\geq \epsilon_{\ell-1} - \frac{d}{\frac{2d}{\epsilon_{\ell-1}}} \quad (\text{since } N \geq N_\ell = \frac{2d}{\epsilon_{\ell-1}} - d) \\ &= \frac{\epsilon_{\ell-1}}{2}. \end{aligned} \tag{4.31}$$

As in Lemma 4.3.2, since $F_I \subset Z(f) \cap \Delta_n$, it follows that there are at most $d-1$ factors in the product $\prod_{i \notin I} \prod_{j=0}^{\alpha_i - 1} \frac{\gamma_i - j}{N+d}$. Together with (4.29), (4.30), (4.31) and as in (4.26), we have

$$\frac{B_\gamma^{N,\beta}}{A_\gamma^{N,+}} \leq \frac{B_\gamma^{N,\beta}}{A_\gamma^{N,\alpha}} \leq \frac{b_{\max}}{a_{\min}} \cdot \epsilon_\ell \cdot \frac{1}{\left(\frac{\epsilon_{\ell-1}}{2}\right)^{d-1}} = \frac{2^{d-1} b_{\max} \epsilon_\ell}{a_{\min} \epsilon_{\ell-1}^{d-1}}. \tag{4.32}$$

Then by summing (4.32) over $\beta \in \Lambda^-$, we have

$$\frac{B_\gamma^{N,-}}{A_\gamma^{N,+}} \leq |\Lambda^-| \cdot \frac{2^{d-1} b_{\max} \epsilon_\ell}{a_{\min} \epsilon_{\ell-1}^{d-1}} \leq 1, \tag{4.33}$$

where the last equality follows from (4.27), and it follows that we have $A_\gamma^N \geq 0$. $\square$

Lemma 4.3.4. For each $0 \leq \ell \leq k$, we have

$$\epsilon_\ell \geq \min\left\{\left(\frac{a_{\min}}{2^{d-1}|\Lambda^-| b_{\max}}\right)^{\frac{(d-1)^\ell - 1}{d-2}}, 1\right\} \cdot \epsilon_0^{(d-1)^\ell} \tag{4.34}$$

(The exponent $\frac{(d-1)^\ell - 1}{d-2}$ is understood to be equal to ℓ when $d = 2$).

Proof. First we remark that the inequality in (4.34) in the case when $\ell = 0$ is obvious. It is easy to see from (4.20) and (4.27) that $\epsilon_\ell < 1$ for all $0 \leq \ell \leq k$. Let $\kappa := \frac{a_{\min}}{2^{d-1}|\Lambda^-| b_{\max}}$. Then for $1 \leq \ell \leq k$, we have, from (4.27),

$$\begin{aligned}
\epsilon_\ell &= \min\{\kappa \cdot \epsilon_{\ell-1}^{d-1}, \epsilon_{\ell-1}\} \\
&\geq \min\{\kappa, 1\} \cdot \epsilon_{\ell-1}^{d-1} \quad \text{(since } \epsilon_{\ell-1} \leq 1\text{)} \\
&\geq \min\{\kappa, 1\} \cdot \left(\min\{\kappa, 1\} \epsilon_{\ell-2}^{d-1}\right)^{d-1} \quad \text{(by iterating the above inequality)} \\
&\geq \cdots \\
&\geq \min\{\kappa, 1\}^{1+(d-1)+\cdots+(d-1)^{(\ell-1)}} \cdot \epsilon_0^{(d-1)^\ell} \\
&= \min\left\{\kappa^{\frac{(d-1)^\ell - 1}{d-2}}, 1\right\} \cdot \epsilon_0^{(d-1)^\ell}.
\end{aligned}$$

$\square$

In summary, we have

Proposition 4.3.5. Let f be in $P_d(\Delta_n)$ satisfying (Z1) and (Z2'). Let

$$\epsilon_Z := \min\Big\{\Big(\frac{a_{\min}}{2^{d-1}|\Lambda^-|b_{\max}}\Big)^{\frac{(d-1)^k-1}{d-2}}, 1\Big\}\cdot \epsilon_0^{(d-1)^k} \quad \text{and} \quad N_Z := \frac{2d}{\epsilon_Z} - d. \tag{4.35}$$

Then for any positive integer $N \geq N_Z$ and any $\gamma \in \mathcal{I}(n, N+d)$ satisfying $\frac{\gamma}{N+d} \in \widetilde{Z(f)}(\epsilon_Z)$, we have $A_\gamma^N \geq 0$.

Proof. From Lemma 3.4 and (4.27), we easily see that $\epsilon_Z \leq \epsilon_k \leq \cdots \leq \epsilon_{k-1} \leq \epsilon_0$, and thus $N_Z \geq N_\ell$ for each $0 \leq \ell \leq k$. Then the proposition follows readily from Proposition 3.3 and the inclusion $\widetilde{Z(f)}(\epsilon_Z) \subset \bigcup_{0\leq \ell \leq k} \widetilde{F_\ell}(\epsilon_\ell)$. □

4.3.2 $\frac{\gamma}{N+d}$ being away from $Z(f) \cap \Delta$

Next we consider those $\gamma \in \mathcal{I}(n, N+d)$ for sufficiently large N and such that $\frac{\gamma}{N+d}$ stays away from $Z(f) \cap \Delta_n$.

Definition 4.3.6. We define a metric on Δ_n by the following:

$$dist(y, z) = ||y - z||, \quad \forall y, z \in \Delta_n \tag{4.36}$$

where $||\cdot||$ is the Euclidean norm. Then the distance between a point x and a set of points say S is

$$dist(x, S) = \inf_{z\in S} dist(x, z) = \inf_{z\in S} ||x - z||. \tag{4.37}$$

Proposition 4.3.7. Suppose $f \in P_d(\Delta_n)$ satisfies (Z1) and (Z2$'$). For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $dist(x, Z(f) \cap \Delta_n) < \delta$, then

$$\frac{f^-(x)}{f^+(x)} < \varepsilon. \tag{4.38}$$

In particular, we have $c < 1$, where $c = c(f)$ is as defined in (4.15).

Proof. Since f satisfies (Z1), we may write $Z(f) \cap \Delta_n = \widetilde{F_0} \cup \widetilde{F_1} \cup \cdots \cup \widetilde{F_k}$ with each $\widetilde{F_\ell} = \bigcup_{I\in\Phi_\ell} F_I$ as in (4.18). Suppose we have a sequence of numbers $\delta_0, \cdots, \delta_k$, and we take $\delta = \min\{\delta_0, \cdots, \delta_k\}$. Then for a given point x,

$$dist(x, F_I(\delta)) \leq dist(x, F_I(\delta_\ell)) \tag{4.39}$$

where F_I is an ℓ-face of $Z(f) \cap \Delta_n$. From the decomposition of the tubular neighborhoods of $Z(f) \cap \Delta_n$ in (4.19), it is easy to see that to prove (4.38), it suffices to show that for any given $\varepsilon > 0$, there exist positive numbers δ_ℓ, $0 \leq \ell \leq k$, such that

$$\frac{f^-(x)}{f^+(x)} < \varepsilon \quad \forall x \in \widetilde{F}_\ell(\delta_\ell) \setminus Z(f). \tag{4.40}$$

Let $\varepsilon > 0$ be a given number. To prove (4.40) by induction on ℓ, we define the δ_ℓ's recursively as follows: Set

$$\begin{aligned} \delta_0 : &= \frac{a_{\min}\varepsilon}{|\Lambda^-|b_{\max} + nda_{\min}\varepsilon}, \quad \text{and} \\ \delta_\ell : &= \min\left\{\delta_{\ell-1}, \frac{a_{\min}\varepsilon\delta_{\ell-1}^{d-1}}{|\Lambda^-|b_{\max}}\right\} \quad \text{for } 1 \leq \ell \leq k. \end{aligned} \tag{4.41}$$

First we consider the case when $\ell = 0$. Take $x = (x_1, \cdots, x_n) \in \widetilde{F}_0(\delta_0) \setminus Z(f)$. Then $x \in F_I(\delta_0)$ for some $I \in \Phi_0$. Upon permuting the x_i's if necessary, we will assume without loss of generality that $I = \{1, \cdots, n-1\}$, and thus we have $0 \leq x_i \leq \delta_0$ for $1 \leq i \leq n-1$, which implies that $x_n \geq 1 - (n-1)\delta_0$. For any $\beta = (\beta_1, \cdots, \beta_n) \in \Lambda^-$, it follows from (Z2$'$) that there exists $\alpha = (\alpha_1, \cdots, \alpha_n) \in \Lambda^+$ and i_0 with $1 \leq i_0 \leq n-1$ and satisfying (4.21), and in particular, one has $\beta_n < \alpha_n$. Then similar to (4.22), (4.23) and (4.25), we have

$$\begin{aligned} \frac{b_\beta x^\beta}{f^+(x)} \leq \frac{b_\beta x^\beta}{a_\alpha x^\alpha} &= \frac{b_\beta}{a_\alpha} \cdot \Big(\prod_{\substack{i=1 \\ i \neq i_0}}^{n-1} x_i^{\beta_i - \alpha_i}\Big) \cdot x_{i_0}^{\beta_{i_0} - \alpha_{i_0}} \cdot \frac{1}{x_n^{\alpha_n - \beta_n}} \\ &\leq \frac{b_{\max}}{a_{\min}} \cdot 1 \cdot \delta_0 \cdot \frac{1}{(1-(n-1)\delta_0)^{d-1}} \\ &\leq \frac{b_{\max}\delta_0}{a_{\min}(1-(d-1)(n-1)\delta_0)} \quad \text{(by Bernoulli inequality).} \end{aligned} \tag{4.42}$$

Upon summing (4.42) over $\beta \in \Lambda^-$, we have

$$\frac{f^-(x)}{f^+(x)} \leq |\Lambda^-| \cdot \frac{b_{\max}\delta_0}{a_{\min}(1-(d-1)(n-1)\delta_0)} < \frac{|\Lambda^-|b_{\max}\delta_0}{a_{\min}(1-dn\delta_0)} = \varepsilon, \tag{4.43}$$

where the last equality follows from a simple calculation using (4.41), and thus (4.40) holds for the case when $\ell = 0$. Now we make the induction hypothesis that (4.40) holds for the cases when the running index takes the values $0, 1, \cdots, \ell-1$. Then to prove (4.40) for the case when the running index is ℓ, it follows from the induction hypothesis and the arguments in the beginning of the proof of Proposition 4.3.3 that we only need to consider those points $x = (x_1, \cdots, x_n) \in \Big(F_I(\delta_\ell) \setminus \bigcup_{J \supsetneq I} F_J(\delta_{\ell-1})\Big) \setminus Z(f)$ for some $I \in \Phi_\ell$, so that as in (4.28), one has $x_i \le \delta_\ell$ for $i \in I$ and $x_i > \delta_{\ell-1}$ for $i \notin I$. Then for each $\beta \in \Lambda^-$ (and a corresponding $\alpha = \alpha(\beta) \in \Lambda^+$ arising from (Z2$'$) as mentioned above), a consideration similar to (4.42) (cf. also (4.29), (4.30), (4.31) (4.32)) leads readily to the following:

$$\frac{b_\beta x^\beta}{f^+(x)} \le \frac{b_\beta x^\beta}{a_\alpha x^\alpha} < \frac{b_{\max}}{a_{\min}} \cdot \delta_\ell \cdot \frac{1}{\delta_{\ell-1}^{d-1}}. \tag{4.44}$$

Upon summing (4.44) over $\beta \in \Lambda^-$, we have

$$\frac{f^-(x)}{f^+(x)} < |\Lambda^-| \cdot \frac{b_{\max}\delta_\ell}{a_{\min}\delta_{\ell-1}^{d-1}} \le \varepsilon, \tag{4.45}$$

where the last inequality follows from (4.41). This finishes the proof of (4.38). Finally, it follows from (4.18) that the function g defined by

$$g(x) := \frac{f^-(x)}{f^+(x)}, \quad x \in \Delta_n \setminus Z(f) \cap \Delta_n,$$

extends to a continuous function on the compact set Δ_n, which we denote by the same symbol, such that $g(x) = 0$ on $Z(f) \cap \Delta_n$. By the extreme value theorem, we may take $c = g(x_0)$ for some $x_0 \in \Delta_n \setminus Z(f) \cap \Delta_n$. Then $f(x_0) > 0$ and thus $f^+(x_0) > f^-(x_0)$, which implies $c = g(x_0) < 1$. $\square$

Lemma 4.3.8. Let $0 \le r \le 1$, and suppose $f \in P_d(\Delta_n)$ satisfies (Z1). Then for any $x \in \Delta_n \setminus \widetilde{Z(f)}(r)$, we have

$$f^+(x) \ge a_{\min} r^d.$$

Proof. For any fixed $x = (x_1, \cdots, x_n) \in \Delta_n \setminus \widetilde{Z(f)}(r)$, we let $J = \{j \mid x_j < r\}$. If $J \ne \emptyset$ and the associated face F_J of Δ_n is a subset of $Z(f) \cap \Delta_n$, then we have $x \in \widetilde{F}_{|J|}(r)$ which contradicts $x \notin \widetilde{Z(f)}(r)$. Thus, $J = \emptyset$, or $F_J \not\subset Z(f) \cap \Delta_n = Z(f^+) \cap \Delta_n$, where the last equality follows from Lemma 4.3.1. In either case, it follows readily that there exists

$\alpha = (\alpha_1, \cdots, \alpha_n) \in \Lambda^+$ such that $\alpha_j = 0$ for each $j \in J$. In other words, one has $x_i \geq r$ whenever $\alpha_i > 0$. Hence, we have

$$f^+(x) \geq a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \geq a_{\min} r^d.$$

$\square$

Similar to ([20], p. 223), for any given real number t, we introduce the following polynomials associated to f^+ and f^- respectively given by

$$f_t^+(x) := \sum_{\alpha \in \Lambda^+} a_\alpha \prod_{i=1}^n x_i(x_i - t) \cdots (x_i - (\alpha_i - 1)t) \quad \text{and} \tag{4.46}$$

$$f_t^-(x) := \sum_{\beta \in \Lambda^-} b_\beta \prod_{i=1}^n x_i(x_i - t) \cdots (x_i - (\beta_i - 1)t), \tag{4.47}$$

where $x = (x_1, \cdots, x_n)$. From now on, we will always let $t = \frac{1}{N+d}$. Then from (4.8), one easily sees that

$$A_\gamma^{N,+} = \frac{N!(N+d)^d}{\gamma_1! \cdots \gamma_n!} f_t^+(\frac{\gamma}{N+d}) \quad \text{and} \quad A_\gamma^{N,-} = \frac{N!(N+d)^d}{\gamma_1! \cdots \gamma_n!} f_t^-(\frac{\gamma}{N+d}). \tag{4.48}$$

Proposition 4.3.9. Suppose $f \in P_d(\Delta_n)$ satisfies (Z1) and (Z2$'$). Let R be any given real number satisfying $0 < R < 1$, and let

$$N_R := \frac{d(d-1)a_{\max}}{2(1-c)a_{\min} R^d} - d. \tag{4.49}$$

Then for any positive integer $N \geq N_R$ and any $\gamma \in \mathcal{I}(n, N+d)$ satisfying $\frac{\gamma}{N+d} \in \Delta_n \setminus \widetilde{Z(f)}(R)$, we have $A_\gamma^N \geq 0$.

Proof. For any given $0 < R < 1$ and any $N \geq N_R$, we let $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathcal{I}(n, N+d)$ be such that $\frac{\gamma}{N+d} \in \Delta_n \setminus \widetilde{Z(f)}(R)$. Then by (4.6) and (4.48), we have

$$\begin{aligned}
\frac{\gamma_1! \cdots \gamma_n!}{N!(N+d)^d} A_\gamma^N &= f_t^+(\frac{\gamma}{N+d}) - f_t^-(\frac{\gamma}{N+d}) \\
&= \Big(f^+(\frac{\gamma}{N+d}) - f^-(\frac{\gamma}{N+d})\Big) - \Big(f^+(\frac{\gamma}{N+d}) - f_t^+(\frac{\gamma}{N+d})\Big) \\
&\quad + \Big(f^-(\frac{\gamma}{N+d}) - f_t^-(\frac{\gamma}{N+d})\Big).
\end{aligned} \tag{4.50}$$

By Lemma 4.3.7 and (4.15), we have

$$f^+(\frac{\gamma}{N+d}) - f^-(\frac{\gamma}{N+d}) \geq (1-c)f^+(\frac{\gamma}{N+d}) \geq (1-c)a_{\min}R^d. \tag{4.51}$$

From (4.46), it is easy to see that

$$f^-(\frac{\gamma}{N+d}) - f_t^-(\frac{\gamma}{N+d}) = \sum_{\beta\in\Lambda^-} b_\beta\Big[\prod_{j=1}^{n}\Big(\frac{\gamma_j}{N+d}\Big)^{\beta_j} - \prod_{j=1}^{n}\prod_{k=0}^{\beta_j-1}\Big(\frac{\gamma_j-k}{N+d}\Big)\Big] \tag{4.52}$$

Note that if $\beta_j > \gamma_j$ for some j, then we have $\prod_{k=0}^{\beta_j-1}\left(\frac{\gamma_j-k}{N+d}\right) = 0$, since one of the factors in the product is zero. It follows readily that we have $\left(\frac{\gamma_j}{N+d}\right)^{\beta_j} \geq \prod_{k=0}^{\beta_j-1}\left(\frac{\gamma_j-k}{N+d}\right)$ for each $\beta = (\beta_1, \cdots, \beta_n) \in \Lambda^-$ and each $1 \leq j \leq n$. Hence we have

$$f^-(\frac{\gamma}{N+d}) - f_t^-(\frac{\gamma}{N+d}) \geq 0. \tag{4.53}$$

Then following the argument in ([20], p. 223-224), we have

$$\begin{aligned}
& f^+(\frac{\gamma}{N+d}) - f_t^+(\frac{\gamma}{N+d}) \\
= \;& \sum_{\alpha\in\Lambda^+} a_\alpha\Big[\prod_{j=1}^{n}\Big(\frac{\gamma_j}{N+d}\Big)^{\alpha_j} - \prod_{j=1}^{n}\prod_{k=0}^{\alpha_j-1}\Big(\frac{\gamma_j-k}{N+d}\Big)\Big] \quad \text{(as in (4.52))} \\
\leq \;& a_{\max}\sum_{\alpha\in\mathcal{I}(n,N+d)} \frac{d!}{\alpha_1!\cdots\alpha_n!}\Big[\prod_{j=1}^{n}\Big(\frac{\gamma_j}{N+d}\Big)^{\alpha_j} - \prod_{j=1}^{n}\prod_{k=0}^{\alpha_j-1}\Big(\frac{\gamma_j-k}{N+d}\Big)\Big] \\
= \;& a_{\max}\cdot\Big[1 - \prod_{m=0}^{d-1}(1-\frac{m}{N+d})\Big] \\
\leq \;& a_{\max}\cdot\frac{d(d-1)}{2(N+d)},
\end{aligned} \tag{4.54}$$

where the second last line follows from the multinomial theorem and the iterated Vandermonde-Chu identity as given in ([20], p. 224), and the last line follows from the well-known inequality that $\prod(1-w_\ell) \geq 1 - \sum w_\ell$ if $0 \leq w_\ell \leq 1$. Finally, upon combining (4.50), (4.51), (4.53) and (4.54), we have

$$\begin{aligned}
\frac{\gamma_1!\cdots\gamma_n!}{N!(N+d)^d}A_\gamma^N \;&\geq\; (1-c)a_{\min}R^d - \frac{d(d-1)a_{\max}}{2(N+d)} + 0 \\
&\geq\; (1-c)a_{\min}R^d - \frac{d(d-1)a_{\max}}{2(N_R+d)} \quad (\text{since } N \geq N_R) \\
&=\; 0 \quad (\text{by (4.49)}).
\end{aligned}$$

□

The next theorem gives the sufficient conditions for $f \in P_d(\Delta_n)$ to be Pólya semi-stable with effective estimates as follows:

Theorem 4.3.10. Let $f \in P_d(\Delta_n)$. Suppose f satisfies (Z1) and (Z2'). Then there exists an effective constant $N_o = N_o(n, d, \frac{b_{\max}}{a_{\min}}, \frac{a_{\max}}{a_{\min}}, c, k, |\Lambda^-|)$ such that $(x_1 + \cdots + x_n)^N f \in \Sigma^+_{N+d}(\mathbb{R}^n)$ for all positive integers $N \geq N_o$ (cf. (4.14), 4.15), 4.16)). In particular, f is Pólya semi-stable. Explicitly, let

$$\mu := \max\{\Big(\frac{2^{d-1}|\Lambda^-|b_{\max}}{a_{\min}}\Big)^{\frac{(d-1)^k-1}{d-2}}, 1\}\Big(nd + \frac{|\Lambda^-|b_{\max}}{a_{\min}}\Big)^{(d-1)^k}. \tag{4.55}$$

Then N_o can be given by

$$N_o := \max\{2d\mu, \frac{d(d-1)a_{\max}\mu^d}{2(1-c)a_{\min}}\} - d. \tag{4.56}$$

Proof. Let f be in $P_d(\Delta_n)$ satisfying (Z1) and (Z2'), and let ϵ_Z, N_Z be as in Proposition 4.3.5. Let N_R be as in Proposition 4.3.8, and set $R := \epsilon_Z$. Then it is easy to see from (4.55), (4.20) and (4.35) that $\mu = \frac{1}{\epsilon_Z}$. Together with (4.56), (4.35) and (4.49), it follows readily that one has

$$N_o = \max\{N_Z, N_R\}.$$

For any positive integer $N \geq N_o$ and any $\gamma \in \mathcal{I}(n, N+d)$, let A^N_γ be as in (4.6). If $\frac{\gamma}{N+d} \in \widetilde{Z(f)}(\epsilon_Z)$, then by Proposition 4.3.5, we have $A^N_\gamma \geq 0$. On the other hand, if $\frac{\gamma}{N+d} \in \Delta_n \setminus \widetilde{Z(f)}(\epsilon_Z)$, then we also have $A^N_\gamma \geq 0$ by Proposition 4.3.8. Hence $(x_1 + \cdots + x_n)^N f \in \sum^+_{N+d}(\mathbb{R}^n)$. This finishes the proof of Theorem 4.3.9. □

Remark 4.3.11. (i) The bound N_o in Theorem 4.3.10 is obtained by taking maximum of two values. The first value can be considered as arising from the zero set $Z(f)$ of f, while the second value can be considered as arising from the strict positivity of f in the complement of some tubular neighbourhood of $Z(f) \cap \Delta_n$ in Δ_n, reminiscent of the strictly positive case in [18] and [20].

(ii) One can drop the dependence of N_o on the parameteres k and $|\Lambda^-|$ by replacing them by $\max\{n-3,0\}$ and $\binom{n+d-1}{d-1}-1$ respectively in (4.55) and (4.56). To see this, we first note that the value of the expression for N_o increases with the values of k, $|\Lambda^-|$ and d. Since $\Lambda^+ \neq \emptyset$, it follows that one always has $|\Lambda^-| \leq \binom{n+d-1}{d-1} - 1$. Also, we always have $k \leq n-2$ as in (4.16). On the other hand, an $(n-2)$-face of Δ_n corresponding to the equation $x_i = 0$ lies in $Z(f) \cap \Delta_n$ if and only if x_i is a factor of each monomial term of f (cf. Lemma 4.3.1). Thus, when $n \geq 3$ and $k = n-2$, by factoring out all the common factors of the monomial terms of f, one may write $f = x_1^{\sigma_1} \cdots x_n^{\sigma_n} \hat{f}$ with $\sigma_1, \cdots, \sigma_n \in \mathbb{Z}_{\geq 0}$ and such that $Z(\hat{f}) \cap \Delta_n$ consists of faces of dimensions $\leq n-3$, i.e., $k(\hat{f}) \leq n-3$. Note that $(x_1 + \cdots + x_n)f \in \Sigma^+_{N+d}(\mathbb{R}^n)$ if and only if $(x_1 + \cdots + x_n)\hat{f} \in \Sigma^+_{N+d-\sigma_1-\cdots-\sigma_n}(\mathbb{R}^n)$. Thus the value of $N_o = N_o(f)$ in Theorem 4.3.10 can be replaced by that of $N_o(\hat{f})$, which means that in (4.55) and (4.56), d is replaced by $d - \sigma_1 - \cdots - \sigma_n$, k is replaced by $k(\hat{f}) \leq n-3$, while the values of the other parameteres remain unchanged.

The following example illustrates a polynomial which is Pólya semi-stable, and satisfies (Z1) but does not satisfy (Z2'). This implies that the sufficient conditions (Z1) and (Z2') are not necessary conditions for Pólya semi-stability.

Example 4.3.12. Let

$$p(x_1, x_2, x_3, x_4) = x_1^2 x_3 x_4 + x_2^2 x_3 x_4 + x_1 x_2 x_3^2 + x_1 x_2 x_4^2 - x_1 x_2 x_3 x_4 \qquad (4.57)$$

be a polynomial in $P_4(\mathbb{R}^4)$. Clearly, by arithmetic-geometric inequality, p is positive semi-definite. Also, by the expansion of $(x_1+x_2+x_3+x_4)p(x_1,x_2,x_3,x_4)$, we can see that all the coefficients of $(x_1+x_2+x_3+x_4)p(x_1,x_2,x_3,x_4)$ are non-negative. Hence $p(x)$ is Pólya semi-stable. The zero set of $p(x)$ is

$$\begin{aligned} \{(x_1, x_2, x_3, x_4) \in \Delta_4 \quad | \quad & x_1 = x_2 = 0; x_3 = x_4 = 0; x_1 = x_3 = 0; \\ & x_1 = x_4 = 0; x_2 = x_3 = 0; x_2 = x_4 = 0; \} \end{aligned} \qquad (4.58)$$

and we have

$$\Lambda^+ = \{(2,0,1,1),(0,2,1,1),(1,1,2,0),(1,1,0,2)\}, \tag{4.59}$$

$$\Lambda^- = \{(1,1,1,1)\}. \tag{4.60}$$

Clearly, there exists a face F_I and $\beta \in \Lambda^-$ such that $\beta \not\succ_I \alpha$ for all $\alpha \in \Lambda^+$. Take $F_{\{3,4\}}$, and we can see that $(1,1,1,1) \not\succ_{\{3,4\}} \alpha$ for all $\alpha \in \Lambda^+$. Hence $p(x)$ does not satisfy (Z2'), and (Z2') cannot be a necessary condition for Pólya semi-stability.

Next we construct a family of polynomials $\{h_\epsilon\} \subset P_4(\Delta_4)$ to illustrate that the growth order of N_o with respect to c in Theorem 4.3.10, namely, $N_o \sim \frac{1}{1-c}$ as $c \to 1$, is sharp. Each h_ϵ will be such that $Z(h_\epsilon) \cap \Delta_4$ consists of a union of 0-faces and 1-faces of Δ_4, i.e., $k = 1$.

Example 4.3.13. For $0 < \epsilon < 4$, consider the polynomial in $\mathbb{R}[x,y,z,w]$ given by

$$h_\epsilon(x,y,z,w) := x^2yz + xy^2z + xyz^2 + w^4 - (4-\epsilon)xyzw.$$

By the arithmetic-geometric mean inequality, one easily sees that $h_\epsilon \in P_4(\Delta_4)$, and one has $Z(h_\epsilon) \cap \Delta_4 = \{(x,y,z,w) \in \Delta_4 \mid x = w = 0\} \cup \{(x,y,z,w) \in \Delta_4 \mid y = w = 0\} \cup \{(x,y,z,w) \in \Delta_4 \mid z = w = 0\}$. Hence h_ϵ satisfies (Z1). Clearly there are three 1-faces F_{I_1}, F_{I_2}, F_{I_3} and three 0-faces F_{I_4}, F_{I_5}, F_{I_6} in $Z(h_\epsilon) \cap \Delta_4$ with associated indexes given by

$$I_1 = \{1,4\}, \quad I_2 = \{2,4\}, \quad I_3 = \{3,4\},$$
$$I_4 = \{1,2,4\}, \quad I_5 = \{1,3,4\}, \quad I_6 = \{2,3,4\}.$$

We list the 4-tuples in Λ^+ and Λ^- as follows:

$$\Lambda^+ = \{(2,1,1,0),(1,2,1,0),(1,1,2,0),(0,0,0,4)\}.$$
$$\Lambda^- = \{(1,1,1,1)\}.$$

Let $\beta = (1,1,1,1)$. It is easy to see that for each face F_{I_i}, there exists $\alpha(I_i) \in \Lambda^+$ such that $\beta \succ_{I_i} \alpha(I_i)$, $i = 1, \cdots, 6$ (as an example, one may take $\alpha(I_4) = (1,1,2,0)$).

Hence h_ϵ satisfies (Z2′). One can also easily check that $n = 4$, $d = 4$, $|\Lambda^-| = 1$, $k = 1$, $a_{\min} = a_{\max} = 1$, $b_{\max} = 4 - \epsilon$, $c = 1 - \frac{\epsilon}{4}$. For $0 < \epsilon < 3$, the constant N_o in Theorem 4.3.10 is then given by

$$N_o = \max\Big\{64(4-\epsilon)(20-\epsilon)^3, \frac{6\cdot 8^4\cdot(4-\epsilon)^4(20-\epsilon)^{12}}{\epsilon}\Big\} - 4.$$

As $\epsilon \to 0$ (or equivalently $c \to 1$), N_o is asymptotically $\sim \frac{6\cdot 8^4\cdot 4^4\cdot 20^{12}}{\epsilon}$ (or equivalently $\frac{6\cdot 8^4\cdot 4^3\cdot 20^{12}}{1-c}$). Thus N_o has growth order $N_o \sim \frac{1}{1-c}$ as $c \to 1$. Let $N = 4m$ for some positive integer m. Then the coefficient of $x^{m+1}y^{m+1}z^{m+1}w^{m+1}$ in $(x+y+z+w)^N h_\epsilon$ is

$$\begin{aligned} A^N_{(m+1,m+1,m+1,m+1)} &= \frac{(4m)!}{(m!)^4}\Big(\frac{3m}{m+1} + \frac{m(m-1)(m-2)}{(m+1)^3} - (4-\epsilon)\Big) \\ &\le \frac{(4m)!}{(m!)^4}\Big(\frac{3m}{m+1} + \frac{m}{m+1} - 4 + \epsilon\Big) \\ &= \frac{(4m)!}{(m!)^4}\Big(\epsilon - \frac{4}{m+1}\Big). \end{aligned}$$

Thus if $(x+y+z+w)^N h_\epsilon \in \Sigma^+_{N+4}(\mathbb{R}^4)$, we must have $\epsilon - \frac{4}{m+1} \ge 0$, which implies that $N = 4m \ge \frac{16}{\epsilon} - 4$ (or equivalently $\frac{4}{1-c} - 4$), and hence the minimum growth order of N is at least $\frac{1}{1-c}$, as $c \to 1$. Therefore the growth order of N_o in Theorem 4.3.10, namely $N_o \sim \frac{1}{1-c}$, is sharp.

Remark 4.3.14. Powers and Reznick [21] have earlier constructed a similar family of polynomials such that the zero set of each polynomial in Δ_n consists of only 0-faces, and for which one can easily check that the minimum growth order of N is also at least $\frac{1}{1-c}$.

4.4 Characterization of Pólya semi-stable polynomials in some cases

In this section, we use Theorem 4.2.2 and Theorem 4.3.10 to deduce our characterization of Pólya semi-stable polynomials in the case when $Z(f) \cap \Delta_n$ consists of a finite number of points and as well as the case when $n = 3$.

Corollary 4.4.1. Let $f \in P_d(\Delta_n)$ be such that $|Z(f) \cap \Delta_n|$ is finite. Then f is Pólya semi-stable if and only if f satisfies (Z1) and (Z2). (Note that in this case, $Z(f) \cap \Delta_n$ necessarily consists of a union of vertices of Δ_n.)

Proof. Let $f \in P_d(\Delta_n)$ be such that $|Z(f) \cap \Delta_n|$ is finite and f satisfies (Z1), so that $Z(f) \cap \Delta_n$ consists of 0-faces of Δ_n. It is obvious that Corollary 4.4.1 will readily follow from Theorem 4.2.2 and Theorem 4.3.10 if one can show that such an f satisfies (Z2) if and only if it satisfies (Z2′). Clearly if f satisfies (Z2′), then it satisfies (Z2). Conversely, suppose f satisfies (Z2) but not (Z2′). Then it follows that there exists $\alpha = (\alpha_1, \cdots, \alpha_n) \in \Lambda^+$, $\beta = (\beta_1, \cdots, \beta_n) \in \Lambda^-$ and a 0-face $F_I \subset Z(f) \cap \Delta_n$ such that $\beta \succeq_I \alpha$, but $\beta \not\succ_I \alpha$. Upon permuting the coordinates of Δ_n if necessary, we will assume without loss of generality that $I = \{1, \cdots, n-1\}$. Then we have $\beta_i \geq \alpha_i$ for all $1 \leq i \leq n-1$, but there does not exist i_0 with $1 \leq i_0 \leq n-1$ such that $\beta_{i_0} > \alpha_{i_0}$. Hence we must have $\beta_i = \alpha_i$ for all $1 \leq i \leq n-1$. Since $|\alpha| = |\beta| = d$, it follows that $\beta_n = \alpha_n$. Thus we have $\alpha = \beta$, which is a contradiction, since the sets Λ^+ and Λ^- are disjoint by construction. □

Remark 4.4.2. In the case when $|Z(f) \cap \Delta_n|$ is finite, Powers and Reznick ([21] and [22]) showed the characterization of Pólya semi-stable polynomials with 'simple zeros' at vertices of Δ_n (with effective estimates). It is easy to see that such a polynomial with simple zeros at vertices of Δ_n necessarily satisfies (Z1) and (Z2′) but not vice versa (see [21] for the definition of 'simple zeros').

When $n = 2$, it is easy to see that $f \in P_d(\Delta_2)$ is Pólya semi-stable if and only if f can be expressed in the form

$$f(x_1, x_2) = x_1^{\sigma_1} x_2^{\sigma_2} \hat{f},$$

with $\sigma_1, \sigma_2 \in \mathbb{Z}_{\geq 0}$ and such that $\hat{f}(x) > 0$ on Δ_2. When $n = 3$, Theorem 4.2.2 and Theorem 4.3.10 lead to a simple characterization of Pólya semi-stable polynomials as follows:

Corollary 4.4.3. Let $f \in P_d(\Delta_3)$. Then f is Pólya semi-stable if and only if f can be expressed in the form

$$f(x_1, x_2, x_3) = x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \hat{f}, \tag{4.61}$$

for some $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{Z}_{\geq 0}$ and an $\hat{f} \in P_{d-\sigma_1-\sigma_2-\sigma_3}(\Delta_3)$ such that $|Z(\hat{f}) \cap \Delta_3|$ is finite and $\hat{f}$ satisfies (Z1) and (Z2).

Proof. First we remark that if $f \in P_d(\Delta_3)$ is factored into the form $f = x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \hat{f}$ as given in (4.61), then it is easy to see that f is Pólya semi-stable if and only if $\hat{f}$ is Pólya semi-stable. The 'if' part of Corollary 4.4.3 then follows as a direct consequence of the above remark and Corollary 4.4.1 (applied to $\hat{f}$). Conversely, suppose $f \in P_d(\Delta_3)$ is Pólya semi-stable. Then by factoring out all the common factors of the monomial terms of f, one can write $f = x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \hat{f}$ with $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{Z}_{\geq 0}$ and such that the monomial terms of $\hat{f}$ have no common factors. By the aforementioned remark, $\hat{f}$ is necessarily Pólya semi-stable, and thus by Theorem 4.2.2, $\hat{f}$ satisfies (Z1) and (Z2). Moreover, it follows from a simple dimension consideration that $Z(\hat{f}) \cap \Delta_3$ necessarily consists of a union of 0-faces and 1-faces of Δ_3. Since the monomial terms of $\hat{f}$ have no common factors, it follows that $Z(\hat{f}) \cap \Delta_3$ cannot contain any 1-faces. Hence $Z(\hat{f}) \cap \Delta_3$ consists of a union of 0-faces, and thus it is a finite set. This finishes the proof of the 'only if' part of Corollary 4.4.3. $\square$

4.5 Application to polynomials on a general simplex

Following the methods of Powers and Reznick ([20], Theorem 3), we proceed to show the upper bound for the degree N of a representation of a non-negative polynomial f on a general simplex S as a positive linear combination of powers of the barycentric coordinates of S.

Let S be a general n-simplex in $\mathbb{R}^n$ and let $\{v_0, \cdots, v_n\}$ be the set of vertices of S. If for some point $x \in S$, we have

$$(\lambda_0(x) + \cdots + \lambda_n(x))x = \lambda_0(x)v_1 + \cdots + \lambda_n(x)v_n, \tag{4.62}$$

and $\sum_{i=0}^{n} \lambda_i(x) = 1$, then $\{\lambda_0, \cdots, \lambda_n\}$ is the set of barycentric coordinates of S. We can also see that $\{\lambda_0, \cdots, \lambda_n\}$ is a set of linear polynomials in x and we also have $\lambda_i(v_j) = \delta_{ij}$.

Let $f \in \mathbb{R}[x_1, \cdots, x_n]$ be a (non-homogeneous) polynomial of degree d in $\mathbb{R}^n$. Then we can find a homogenization $\tilde{f} \in \mathbb{R}[y_0, \cdots, y_n]$ of degree d such that $\tilde{f}(\lambda_0, \cdots, \lambda_n) = f(x)$. $\tilde{f}$ can be constructed as follows: Given $f(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$, then we rewrite as

$$f(x) = \sum_{|\alpha| \leq d} a_\alpha \Big(\sum_{i=0}^{n} v_i \lambda_i(x) \Big)^{\alpha} \Big(\sum_{i=0}^{n} \lambda_i(x) \Big)^{d-|\alpha|} \tag{4.63}$$

and we have

$$\tilde{f}(y_0, \cdots, y_n) = \sum_{|\alpha| \leq d} a_\alpha \Big(\sum_{i=0}^{n} v_i y_i \Big)^{\alpha} \Big(\sum_{i=0}^{n} y_i \Big)^{d-|\alpha|}. \tag{4.64}$$

If f is non-negative on S, then it is easy to see that $\tilde{f} \in P_d(\Delta_{n+1})$. An immediate consequence of Theorem 4.3.10 is the following:

Corollary 4.5.1. Let S be a general n-simplex in $\mathbb{R}^n$, $\{v_0, \cdots, v_n\}$ be the set of vertices of S, and $\{\lambda_0, \cdots, \lambda_n\}$ be the set of barycentric coordinates of S. Let $f \in \mathbb{R}[x_1, \cdots, x_n]$ be of degree d and non-negative on the simplex S, and let $\tilde{f} \in \mathbb{R}[y_0, \cdots, y_n]$ be the homogenization of f. Suppose $\tilde{f}$ satisifes (Z1) and (Z2'). Then f admits a representation of the form

$$f = \sum_{|\alpha| \leq N} a_\alpha \lambda_0^{\alpha_0} \cdots \alpha_n^{\alpha_n} \quad \text{(with each } a_\alpha \geq 0) \tag{4.65}$$

for some $N \leq N_o$, where $N_o = N_o(\tilde{f})$ is as given in Theorem 4.3.10.

Proof. We can apply Theorem 4.3.10 to $\tilde{f}$ which satisfies (Z1) and (Z2') by assumption, and see that there exists N such that $(\sum y_i)^N \tilde{f}(y)$ has non-negative coefficients,

$$(\sum y_i)^N \tilde{f}(y) = \sum_{|\beta| = N} b_\beta y^\beta, \tag{4.66}$$

where $b_\beta \geq 0$ for all β. Substituting λ_i for y_i gives $f(x)$ on the left hand side of (4.66), and a representation of degree N on the right hand side. □

Remark 4.5.2. Using the approach in ([20], p. 226) which treated the case of positive polynomials on a convex compact polyhedron, we might ask whether the above Corollary

4.5.1 can be generalized to the case of polynomials which are non-negative on a convex compact polyhedron. However, no such generalization is possible, as the example by Handelman ([11], pg 57) shows.

4.6 Generalization for certain bihomogeneous polynomials

Let $p(z)$ be a real-valued bihomogeneous complex polynomial which has the following representation:

$$p(z) = \sum_{\alpha} a_\alpha \prod_{i=1}^{n} |z_i|^{2\alpha_i}, \tag{4.67}$$

where $a_\alpha \in \mathbb{R}$ for each $\alpha = (\alpha_1, \cdots, \alpha_n)$ and the associated real polynomial $\tilde{p}$ of p is defined as

$$\tilde{p}(x) := \sum_{\alpha} a_\alpha \prod_{i=1}^{n} x_i^{\alpha_i}. \tag{4.68}$$

In other words, $\tilde{p}$ is obtained from p by replacing each $|z_i|^2$ is replaced by x_i in (4.67).

Example 4.6.1. Let $p(z_1, z_2, z_3) = |z_1|^4|z_2|^2 + |z_1|^2|z_2|^4 + |z_3|^6 - 3|z_1z_2z_3|^2$. Clearly, it is a real-valued bihomogeneous complex polynomial of the representation in (4.67). Moreover, the associated real polynomial is $p(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + x_3^3 - 3x_1x_2x_3$.

Proposition 4.6.2. Let $p(z)$ be a positive semi-definite bihomogeneous real-valued complex polynomial on $\mathbb{C}^n$ which satisfies (4.67). Then $p \in PQD_d(\mathbb{C}^n)$ if and only if $\tilde{p}$ is Pólya semi-stable.

Proof. If $\tilde{p}$ is Pólya semi-stable, then it is clear to see that $(\sum x_i)^N\tilde{p} \in \sum_{N+d}^{+}(\mathbb{R}^n)$ for sufficiently large N, and the substitution of x_i by $|z_i|^2$ enables us to see that the associated real-valued bihomogeneous complex polynomial p is in $PQ_d(\mathbb{C}^n)$, by definition. On the other hand, suppose $p \in PQD_d(\mathbb{C}^n)$, and this means

$$(\sum_{i=1}^{n} |z_i|^2)^N p(z) = \sum_j b_j |g_j|^2 \tag{4.69}$$

where g_j are of the form $g_j = \sum_\beta c_{j,\beta} z^\beta$, b_j are all non-negative real numbers and N is some sufficiently large integer. By expanding each $|g_j|^2$ in (4.69), we have

$$(\sum_{i=1}^{n} |z_i|^2)^N p(z) = \sum_j b_j (\sum_\beta c_{j,\beta} z^\beta)(\sum_\beta \overline{c_{j,\beta} z^\beta}). \tag{4.70}$$

From (4.67), it is clear that we may write

$$(\sum_{i=1}^{n} |z_i|^2)^N p(z) = \sum_\gamma A_\gamma \prod_{i=1}^{n} |z_i|^{2\gamma_i}, \tag{4.71}$$

where each $A_\gamma \in \mathbb{R}$. Then on comparing the coefficients of $\prod_{i=1}^n |z_i|^{2\gamma_i}$ in (4.70) and (4.71), we have $A_\gamma = \sum_j b_j (\sum_\gamma |c_{j,\gamma}|^2)$, which is clearly non-negative. On replacing each $|z_i|^2$ with x_i in (4.71), we obtain

$$(x_1 + \cdots + x_n)^N \tilde{p}(x) = \sum_\gamma A_\gamma x^\gamma \tag{4.72}$$

hence showing that $\tilde{p}(x)$ is Pólya semi-stable. $\square$

Corollary 4.6.3. Let $p(z)$ be a positive semi-definite bihomogeneous real-valued complex polynomial on $\mathbb{C}^n$ which can be expressed as in (4.67). Suppose the associated real polynomial $\tilde{p}$ of p satisfies (Z1) and (Z2'), then $p \in PQD_d(\mathbb{C}^n)$.

Proof. Since $\tilde{p}(x)$ satisfies (Z1) and (Z2'), then by Theorem 4.3.10, it is Pólya semi-stable. By Proposition 4.6.2, we have $p \in PQD_d(\mathbb{C}^n)$. Furthermore, $(\sum_{i=1}^n |z_i|^2)^N p(z)$ is a sum of squared norms for all $N \geq N_o$, N_o as defined in Theorem 4.3.10. $\square$

Example 4.6.4. Consider the polynomial $M_c(z_1, z_2, z_3) = |z_1|^4|z_2|^2 + |z_1|^2|z_2|^4 + |z_3|^6 - (3-\epsilon)|z_1 z_2 z_3|^2$, $0 < \epsilon < 3$, which was introduced in Section 2.1 (j). From the proof in Chapter 2, we have shown M_c to be a quotient of squared norms with uniform denominator. Now, we consider the associated real polynomial $\tilde{M}_c$ of M_c, and we have

$$\tilde{M}_c = x_1^2 x_2 + x_1 x_2^2 + x_3^3 - (3-\epsilon) x_1 x_2 x_3. \tag{4.73}$$

It is clear that $\tilde{M}_c$ satisfies (Z1) and (Z2'), hence by Corollary 4.6.3, we can conclude that M_c is a quotient of squared norms with uniform denominator, which coincides with the results in Chapter 2.

Bibliography

[1] E. Artin, Über die Zerlegung definiter Funktionen in Quadrate., Hamb. Abh. **5** (1927), 100-115; see Collected Papers (S. Lang, J. Tate, eds.), Addison-Wesley 1965, reprinted by Springer-Verlag, New York, et. al., pp. 273-288.

[2] V. de Angelis and S. Tuncel, Handelman's theorem on polynomials with positive multiples, in Codes, Systems, and Graphical Models (Minneapolis, MN, 1999), **123** (2001), 439-445; Springer, New York.

[3] Choi, M.D., Lain, T.Y., Extremal positive semidefinite forms, Math. Ann., **231** (1977), 1-18.

[4] D. W. Catlin and J. P. D'Angelo, A stabilization theorem for Hermitian forms and applications to holomorphic mappings, Math. Res. Lett., **3** (1996), 149-166.

[5] D. W. Catlin and J. P. D'Angelo, Positivity conditions for bihomogeneous polynomials, Math. Res. Lett., **4** (1997), 555-567.

[6] Scheiderer, Claus., *On sums of squares in local rings*, J. reine angew. Math. 540 (2001), 205-227

[7] Scheiderer, Claus., *Sums of squares on real algebraic surfaces*, Manuscripta Math. 119, 395410 (2006)

[8] C. N. Delzell, *A constructive, continuous solution to Hilbert's 17th problem, and other results in semi-algebraic geometry*, Ph.D. thesis, Stanford University, 1980.

[9] Varolin, Dror., *Geometry of Hermitian Algebraic functions. Quotient of squared norms*, preprint.

[10] D. Handelman, Deciding eventual positivity of polynomials, Ergod. Th. & Dynam. Sys., **6** (1986), 57-79.

[11] D. Handelman, Representing polynomials by positive linear functions on compact convex polyhedra, Pac. J. Math., **132** (1988), 35-62.

[12] D'Angelo, John P., *Inequalities from Complex Analysis*, Carus Mathematical Monograph, No. 28, Mathematical Association of America, Washington, 2002.

[13] D'Angelo, John P., *Complex variables analogues of Hilbert's Seventeenth Problem*, International Journal of Mathematics Vol.16, No. 6 (2005) 609-627.

[14] E. de Klerk and D. Pasechnik, Approximation of the stability number of a graph via copositive programming, SIAM J. Optimization, **12** (2002), 875-892.

[15] J.A. de Loera, F. Santos, An effective version of Pólya's theorem on positive definite forms, J. Pure Appl. Algebra **108** (1996), 231-240

[16] J.A. de Loera, F. Santos, Erratum to "an effective version of Pólya's theorem on positive definite forms", J. Pure Appl. Algebra **155** (2000), 309-310.

[17] T. S. Motzkin and E. G. Strauss, Divisors of polynomials and power series with positive coefficients, Pacific J. Math, **29** (1969), 641-652.

[18] G. Pólya, Über positive Darstellung von Polynomen, in: Vierterhartschrigt d. Naturforschenden Gessellschaft in Zürich **73** (1928), 141-145; see: Collected Papers, Vol. 2, pp. 309-313, MIT Press, Cambridge, 1974.

[19] Parrilo, P.A., *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, PhD Thesis, California Institute of Technology, Pasadena, CA, May 2000.

[20] V. Powers and B. Reznick, A new bound for Pólya's theorem with applications to polynomials positive on polyhedra, J. Pure Appl. Algebra **164** (2001), 221-229.

[21] V. Powers and B. Reznick, A quantitative Pólya's theorem with corner zeros. ISSAC 2006, 285-289, ACM, New York, 2006.

[22] M. Castle, V. Powers and B. Reznick, A quantitative Pólya's theorem with zeros. Published as an extended abstract in MEGA 2007 conference proceedings.

[23] Reznick, B., *Uniform denominators in Hilbert's Seventeenth Problem*, Math. Z., 220 (1995), 75-98.

[24] Reznick, B., *Some concrete aspects of Hilbert's 17th Problem*, Contemp. Math. 253 (2000), 251-272.

[25] Reznick, B., *On the absence of uniform denominators in Hilbert's 17th Problem*, Proc. Amer. Math. Soc. 133 (2005), 2829-2834.

[26] M. Schweighofer, An algorithmic approach to Schmüdgen's Positivstellensatz, J. Pure and Appl. Alg., **166** (2002), 307-319.

[27] M. Schweighofer, On the complexity of Schmüdgen's Positivstellensatz, J. Complexity, **20** (2004), 529-543.

[28] M. Schweighofer, Certificates for nonnegativity of polynomials with zeros on compact semialgebraic sets, Manuscripta Math., **117** (2005), 407-428.

[29] M. Schweighofer, Optimization of polynomials on compact semialgebraic sets, SIAM J. Optimization, **15** (2005), 805-825.

[30] W.-K. To and S.-K. Yeung, *Effective isometric embeddings for certain Hermitian holomorphic line bundles*, Journal of the London Mathematical Society, (2) 73 (2006), 607-624.