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DEVELOPMENTAL LOGIC

by

Harvey Jack Schiller

A Dissertation Submitted to the Faculty of the Graduate School
of Loyola University of Chicago in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy

February

1977

ACKNOWLEDGEMENTS

The author wishes to thank the members of his dissertation committee: Dr. Anne McCreary Juhasz, Professor, Director, Dr. Rosemary V. Donatelli, Associate Professor, Dr. Jack Kavanagh, Associate Professor, Dr. Ronald R. Morgan, Assistant Professor, and Fr. Richard VandeVelde, S. J., Assistant Professor, without whose cooperation this dissertation could not have been produced.

FORWARD

Albert Einstein once said, "Imagination is more important than Knowledge." But Imagination cannot exist in chains. It is a child of the sun, the wind, the boundless horizons of Men's minds.

To Miriam, for killing me softly with her song, I dedicate this work on the mathematical theory of the evolution of Intelligence. For it has always lived upon the edge of the horizon of my mind, where the known and unknown appear to meet.

LIFE

Harvey Jack Schiller was born 29 October 1945 in Chicago, Illinois.

He earned a B.S. in Mathematics from the University of Illinois in 1966; an M.S. in Mathematics from Illinois in 1968; and had to leave school, interrupting his doctoral studies in Mathematics at Illinois, in 1970. In 1966, while at Illinois, Harvey was awarded Distinguished Honors in Mathematics. In addition to University Scholarships throughout his studies at Illinois, during 1974-1975, Harvey was awarded a University Fellowship at Loyola.

From 1970 through 1974, in order to support himself, Harvey worked in the applications of Operations Research to the treatment of Civil Engineering problems.

Harvey has always had a great interest in the Mathematical Biosciences, particularly in the mathematical treatment of the evolution of Intelligence. His numerous research papers span such disparate topics as Automata Theory, through Comparative and Developmental Ethology, to the Monte Carlo simulation of lava-lite flows, epidemiological processes, and congestion waves in networks.

Harvey is a member of the American Educational Research Association, the American Mathematical Society, the

American Statistical Association, the Association for Computing Machinery, and the Operations Research Society of America.

Harvey is currently an Assistant Professor at the University of Health Sciences/Chicago Medical School.

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CHAPTER I

PROBLEM

What is intelligence? And how may this elusive construct be measured? An answer may be provided for the first question by simply defining the intelligence of a system to be the structural complexity of the system. Admittedly, this definition leaves much to the imagination; and only becomes less tenuous with an answer to the second question - how does one operationalize the measurement of intelligence? It will be the objective of this dissertation to take the first steps toward a constructurally valid¹ realization of that goal.

The more sophisticated models of intelligence appear to differentiate between cognitive structure and the information integrated within that structure. The differentiation between structure and content is fundamental to the arguments presented in this dissertation. Unfortunately, there appear to exist few operational measures of structure; the vast preponderance of devices measures content. Indeed, one could ascribe the wide interest exhibited in the work

¹The term constructurally valid is stipulated to mean having construct validity.

of Jean Piaget, to the realization on the part of educators, that such work promises exciting new perspectives on the construct of intelligence, together with operational means of its structural measurement.

There is a need for the development of devices which measure cognitive structure, as opposed to cognitive content. There are numerous reasons for this need. The crux of the matter is this. It appears that cognitive structure is more reflective of problem-solving capabilities than is cognitive content. Since problem-solving is accepted as the highest form of learning behavior, it is reasonable to assume that measure of cognitive structure, rather than cognitive content, is of greater interest as one considers behaviors of increasingly greater intelligence.

Purpose of the Study

The need thus dictates the purpose. In order to map cognitive structure, it is the aim of this dissertation to first construct a model in which the behavior of the primitive logic operators or logic primitives may be embedded; and second, to design a problem-solving paradigm in the form of a game of strategy which will serve as a representation of the model.

This representation should exhibit the "track" of a subject's logic, in addition to the resulting conclusions of cognition. Finally, the representation of the model

will be tested to determine its validity with respect to the known ontogenesis¹ of cognitive structure; and to determine whether it also measures the same construct of intelligence as that of Raven's Progressive Matrices.

Thus, briefly, a model of cognitive structure will be constructed, and a representation of that model will be tested.

Background

Piaget² has theorized that, while the ontogenesis of cognitive structure may not proceed at identical rates in all humans, the sequence of the ontogenesis is invariant, independent of the human. Since language is a function of cognitive structure,³ the ontogenesis of language is interdependent upon the ontogenesis of cognitive structures.⁴

¹The term ontogenesis is stipulated here to mean the development or course of development of an individual organism.

²Jean Piaget, Genetic Epistemology (New York: Columbia University Press, 1970); Biology and Knowledge (Chicago: University of Chicago Press, 1971); and Psychology and Epistemology (New York: Grossman Press, 1971).

³Henry Edelheit, "The Relationship of Language Development to Problem-Solving Ability," Journal of the American Psychoanalytic Association 20 (January 1972): 145-155; David A. Freeman, "Relation of Language Development to Problem-Solving Ability," Bulletin of the Menninger Clinic 36 (November 1972): 583-595; and John E. Taplin, Herman Staudenmayer, and Judith L. Taddonio, "Developmental Changes in Conditional Reasoning: Linguistic or Logical?," Journal of Experimental Child Psychology 17 (April 1974): 360-373.

⁴Hans G. Furth and Janis Youniss, "Formal Operations

Recent theories of developmental psycholinguistics postulate the construct existence of hierachal organization in the structure of language.¹ Theoretically, the surface structure of language is comprised of lexicon; the deep structure is comprised of syntax; and the very deep structure is comprised of the primitive logic operators (logic primitives), both those genetically-programmed and those derived from internalized schema environmentally-induced. Thus, one might expect the convergence which exists between the ontogenesis of language and the ontogenesis of cognitive structure, especially as one proceeds from the interface to the compiler level of hierachal organization.²

and Language: A Comparison of Deaf and Hearing Adolescents," International Journal of Psychology 6 (No. 1, 1971): 49-64; H. J. A. Rimoldi, Analysis of the Interrelationship Between Logical Structure, Language and Thinking (Chicago, Illinois: Loyola Psychometric Laboratory, Publication No. 51, n. d.); and "Logical Structures and Languages in Thinking Processes," International Journal of Psychology 6 (No. 1, 1971): 65-77.

¹P. S. Dale, Language Development: Structure and Function (Hinsdale, Illinois: Dryden Press, 1972); Frederick Francois, "Syntactical Concepts in the Description of Children's Language," Bulletin de Psychologie 26 (Nos. 5-9, 1972-1973): 301-311; and Paula Menyuk, The Acquisition and Development of Language (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1971).

²Two definitions are appropriate here. The term interface is stipulated here to mean the communication mode between the organism and its environment. And the term compiler is stipulated here to mean the set of algorisms which translates between real information and its representation by logic primitives.

Since the evolution of language and cognitive structure is so integroplexed (interwoven, interdependent), the allusion is to the hierachal organization of the integroplex or synthesis of both language and cognitive structure.

Developmental analyses of psycholinguistics are integral to an understanding of the much broader scope of cognitive evolution in humans. In fact, one could easily view the work in infrahuman primate language,¹ developmental psycholinguistics,² and automata theory,³ as but various points on the same continuum.

Language itself is a medium by which cultural (as opposed to genetic) information is transmitted. For example, questions predicated upon highly developed syntactic and logic structures would therefore be capable of "learning" more information than questions predicated upon less highly developed syntactic and logic structures. Although language is an all-inclusive term signifying both structure (deep structure and very deep structure) and content (surface structure), its usual use in this dissertation should be interpreted as structure.

¹Roger Brown, A First Language: The Early Stages (Cambridge, Massachusetts: Harvard University Press, 1973); J. D. Fleming, "The State of the Apes," Psychology Today 7 (1974): 31-50; and R. S. Fouts, "Acquisition and Testing of Gestural Signs in Four Young Chimpanzees," Science 180 (1973): 978-980.

²John Anderson, "Computer Simulation of a Language Acquisition System: A First Report," paper presented at the Third Loyola Symposium on Cognitive Psychology, Chicago, Illinois, 30 April 1974; and W. J. Mayer and June Shane, "The Form and Function of Children's Questions," Journal of Genetic Psychology 123 (1973): 285-296.

³Zelig Harris, Mathematical Structures of Language (New York: Interscience, 1968).

The distinction between structure and content of a system is not a superficial one. It is fundamental to the thinking which has generated this study, and there will be occasion to make use of this distinction later. Now, since language is so reflective of the level of cognitive development in a child,¹ the information "learned" by a question generated by a child of higher cognitive development should be greater than the information "learned" by a question generated by a child of lower cognitive development.

It has been shown² that the correlation among sequential behaviors tends to be greater with increasing intelligence. Since greater information can be "learned" by a sequence of highly correlated questions,³ one may then infer that the information "learned" by a sequence of questions generated by a child of higher cognitive development is greater than the information "learned" by a sequence of

¹Johanna S. DeStefano, "Linguistics and Logical Reasoning," *Theory into Practice* 12 (December 1973): 272-277; and Georges Mounin, "Les Rapports Entre Le Langage et la Pensee: Point De Vue D'un Linguiste [Relations Between Language and Thought: A Linguist's Point of View]," *International Journal of Psychology* 6 (No. 1, 1971): 13-24.

²Harvey Jack Schiller, "On Behavioral Prediction Via Stochastic Integral Equations." [Notes in Mathematical Biology.], Chicago, Illinois, 1973.

³Harvey Jack Schiller, "On the Information of Correlated Behaviors." [Notes in Mathematical Biology.], Chicago, Illinois, 1973.

questions generated by a child of lower cognitive development. In this context, a strategy may be defined as the sequence of questions generated to obtain given information.

Now, given the same problem, one would thus expect the child of greater cognitive development to employ a more optimal question-asking strategy than would his less evolved counterpart. In view of Piagetian theory,¹ then, since cognitive development is dependent upon ontogenesis, one would therefore expect that question-asking strategies in problem-solving are dependent upon ontogenesis.

Constructs

A fundamental construct used in this dissertation is the stage of Piagetian ontogenesis. For the sake of ease, the stage of Piagetian ontogenesis was determined by school grade level. There were six such sampling strata used in the study, ranging from pre-school level to adult; each stratum containing between 18 and 24 subjects. It was felt that such stratification opted for the greatest natural differentiation of subjects according to the usual meaning of "Piagetian developmental stages," in the absence of any quick and valid differentiating device. The

¹M. A. Arbib and Roy M. Kahn, "A Developmental Model of Information Processing in the Child," Perspectives in Biology and Medicine 12 (Spring 1969): 397-415; and Stuart A. Offenbach, "A Developmental Study of Hypothesis Testing and Cue Selection Strategies," Developmental Psychology 10 (July, 1974): 484-490.

stages of Piagetian ontogenesis used in this study are exhibited in Table 1.

Such terminology, i.e., Piagetian ontogenesis, is no more an abuse of the term Piagetian than Newtonian mechanics is of the term Newtonian. Both nomenclatures simply refer to the individual to whom is ascribed the implementation of the conceptual foundation upon which subsequent, more advanced work was done.

Suppose one were to play the following game: Given an unknown integer between 1 and 8, determine this unknown integer by asking questions which are answerable only with yes or no. Clearly, if there are two unknown integers, the problem-to-be-solved would be more complex than if there were only one unknown integer. Similarly, in the case of the one unknown integer, a question-asking strategy depicted in Illustration 1 is certainly more optimal than the question-asking strategy depicted in Illustration 2. This easily follows since, on the average, one would require only 3.000 questions to solve the problem represented in Illustration 1, in comparison to 4.375 questions to solve the problem represented in Illustration 2.

To provide the necessary structure for these constructs, a mathematical model will be generated which incorporates newly derived measures of complexity and measures of question-asking strategy. Both of these kind of measures will provide absolute, criterion-referent quan-

TABLE 1
STAGES OF PIAGETIAN ONTOGENESIS
USED IN THIS STUDY

Stage	Number of Subjects
Adult (A)	18
High School (H)	23
Junior High (J)	20
Intermediate (I)	21
Primary (P)	23
Kindergarten (K)	24

IS UNKNOWN INTEGER i ...

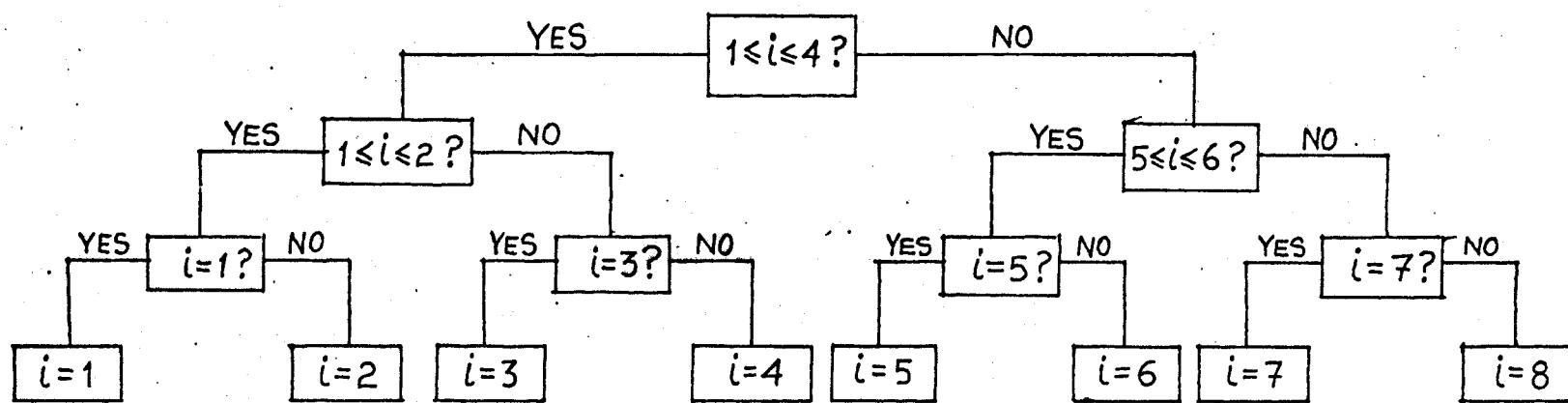


Illustration 1. An Optimal Question-Asking Strategy

IS UNKNOWN INTEGER $i \dots$

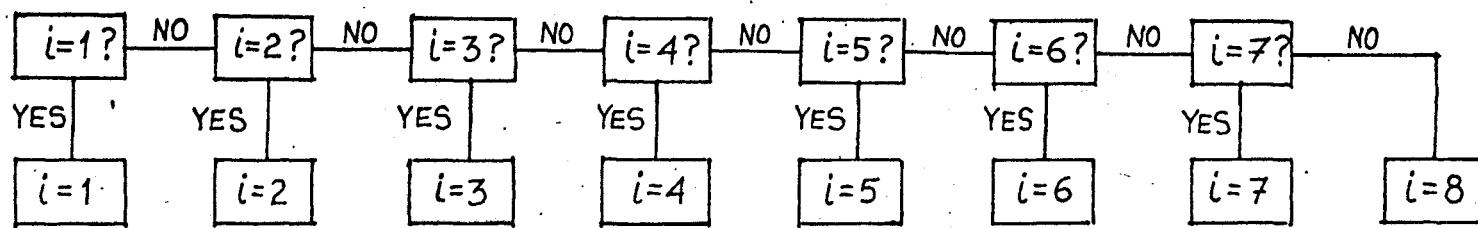


Illustration 2. A Not-So-Optimal Question-Asking Strategy

tification; thus admitting for the first time, a theoretic model of ontogenetic relativity.

Research Hypotheses

1. In the solution of problems of a given complexity, question-asking strategies become more optimal with greater Piagetian ontogenesis.
2. In the solution of problems of a given complexity, by subjects of equivalent Piagetian ontogenesis, question-asking strategies become more optimal with greater intelligence as measured by Raven's Progressive Matrices.

Implication for Education

Curricula may be considered as an organization of sequential problem-solving (learning). Should this research determine the existence of developmental-dependence in the ontogenesis of logical primitives, it would then follow that curricula should be dependent upon ontogenesis.

In addition, this work would then be fundamental to the development of new Piagetian measures of cognitive development which are predicated upon the operation of logic primitives in the context of games of strategy.

Summary

The purpose of this research project was to construct a model of cognitive structure in which the behavior of the logic primitives could be embedded, and to formulate a rep-

resentation of that model which could be empirically tested.

The representation of the model is in the form of logic problems-to-be-solved. The model admits for the absolute measure of the structural complexity of problems-to-be-solved, and the absolute measure of the optimality of the subject's problem-solving strategies. In this way, the "track of the subject's logic" may be analyzed.

There are two primary research hypotheses: First, that problem-solving strategies become more optimal with increasing Piagetian ontogenesis; and second, that for subjects of equivalent Piagetian ontogenesis, problem-solving strategies become more optimal with increasing intelligence as measured by Raven's Progressive Matrices.

Overview

In Chapter 2 -- Construction and Representation of the Model -- the evolution of the model, together with a description of the representation, and its Piagetian connections, will be discussed.

In Chapter 3 -- Method -- the testing materials, the subject sampling scheme, the testing procedure, and the statistical tests used will be discussed.

In Chapter 4 -- Results and Implications -- the results of the experiment, and their bearing upon the research hypotheses will be discussed. In addition, the implications of this research to Education will be advanced.

In Chapter 5 -- Summary -- a comprehensive summary
of the dissertation will be given.

CHAPTER II

CONSTRUCTION AND REPRESENTATION OF THE MODEL

In order to facilitate the construction of the model, the existence of a certain abstract Boolean structure was postulated. By defining an equivalence relation over this structure, the set of all subsets generated by combinatorially varying the contents of this abstract structure, was partitioned into isomorphs or supersets, each of which then contains structurally equivalent articepts or subsets.

The isomorphs are essentially Boolean representations of the logic primitives. And thus, by representing the isomorphs as "Whodunits?," the solution of the ensuing logic puzzle can be reduced to discovering which articept is the unknown solution to the logic puzzle. The puzzle is solved by strategically constructing clues, to which answers are then provided, and eliminating untenable hypotheses concerning the possible solution articepts of the puzzle.

Mathematical functions have been constructed to provide an absolute measure of the structural complexity of isomorphs. Such measures tell one how complex a particular logic puzzle is; and thus provide a template for logic test design and construction. Similarly, mathematical functions have been constructed to provide an absolute measure of the

optimality of the problem-solving strategy used to solve the puzzle. Such measures tell one how intelligent a particular solution strategy is; and thus provide a differentiating device which allows one to analyze the "track of a subject's logic" rather than only its resultant conclusions.

Construction of the Model

Traditional measures in contemporary education only consider the child's answer. Piaget has notably improved this situation by considering the child's reason for his answer. The Piagetian measure derived for this dissertation, will consider the child's questions as leading to the answer. The process of interest to this dissertation will thus be the track of human logic (the operation of the logic primitives) as it is exhibited in question-asking strategy during problem-solving.

Briefly, the subject must solve what is generically a "Whodunit?" by employing optimal question-asking strategies. In any problem-solving situation, the problem-solver must first generate a set of possible solutions, or suspects to the "Whodunit?" (tenable hypotheses); second, the problem-solver must then eliminate all those possible solutions or suspects (tenable hypotheses) which do not satisfy all the conditions required by the solution constraints or clues (information). The tenable hypoth-

eses, in the context of this dissertation, are articepts (artificial concepts). The subject's self-generated sequence of questions or clues (question-asking strategy) must be optimally contrived to then eliminate those articepts which are not consistent with known information.

Questions in their own right, of course, cannot exist. They must be in reference to a particular context. That context will now be described. Consider the articept generator space exhibited in illustration 3.

black	white
<hr/>	
COLOR	
<hr/>	
pyramid	cube
<hr/>	
FORM	
<hr/>	
large	small
<hr/>	
SIZE	

Illustration 3. The Generator Space $G_{2,2,2}$

By forming the Cartesian product space generated by the generating elements of generator space $G_{2,2,2}$, taken

three at a time, one has the structure depicted in illustration 4. The vertices of the Cartesian product space generated by the generating elements of generator space $G_{2,2,2}$, taken three at a time, comprise the elements of articept space \mathcal{I}_3 . The elements of articept space \mathcal{I}_3 are exhibited in illustration 5.

By discarding one of the three dimensions,¹ the depicted articept space \mathcal{I}_3 may be collapsed to a two dimensional (2-D) articept space \mathcal{I}_2 . There are six such two-dimensional (2-D) articept spaces. One may observe that the elements of a 2-D articept space \mathcal{I}_2 are the vertices of the various Cartesian product spaces depicted in illustration 6.

By forming the Cartesian product spaces generated by the generating elements of generator space $G_{2,2,2}$, taken one at a time, one has the twelve structures depicted in illustration 7.

Similarly, by discarding an additional dimension, the two-dimensional (2-D) articept space \mathcal{I}_2 may be collapsed to a one-dimensional (1-D) articept space \mathcal{I}_1 . There are twelve such one-dimensional (1-D) articept spaces. One may observe that the elements of a 1-D articept space \mathcal{I}_1 are the vertices of the various Cartesian product spaces depicted in illustration 7. The reader may

¹For example, this can be exhibited by considering only those elements of one color, say white.

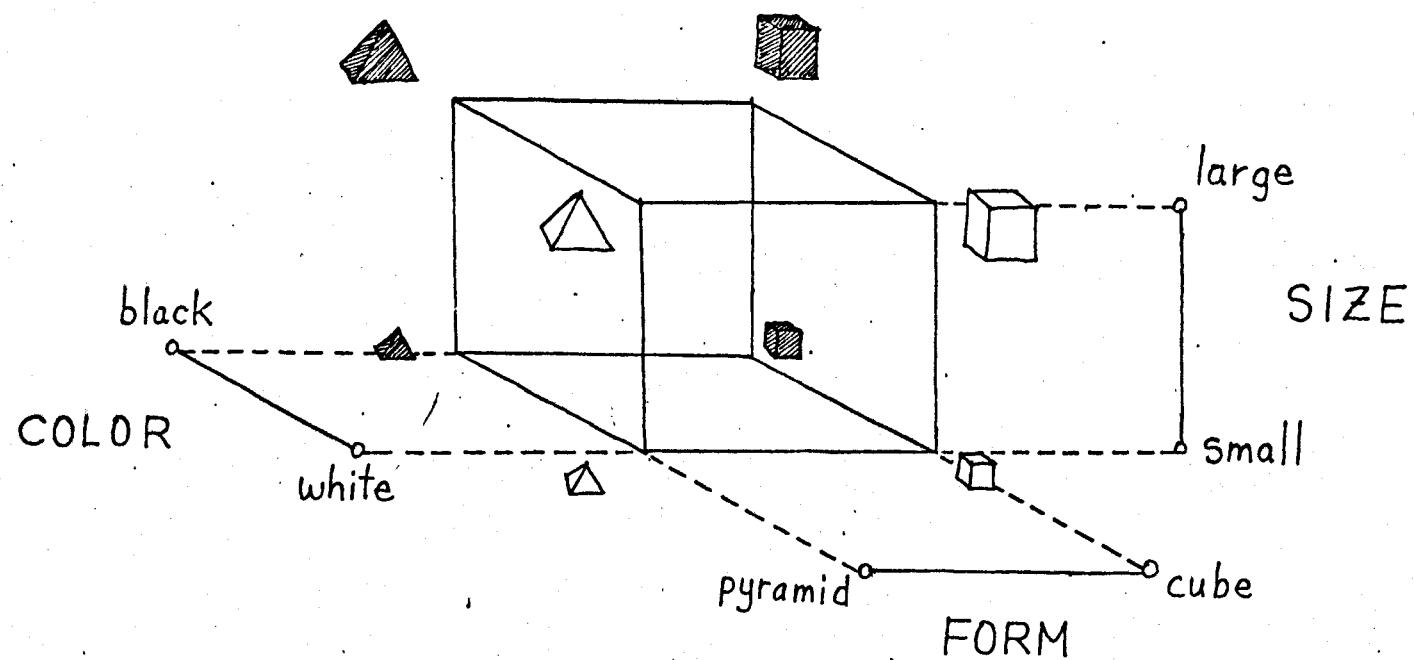
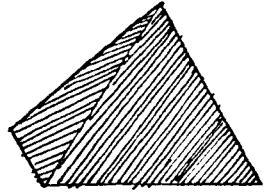


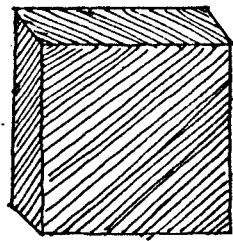
Illustration 4. Cartesian Product Space Generated by the Generating Elements of Generator Space $G_{2,2,2}$, Taken Three at a Time



LARGE
BLACK
PYRAMID



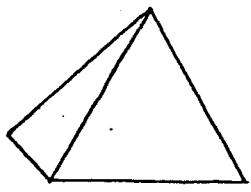
SMALL
BLACK
PYRAMID



LARGE
BLACK
CUBE



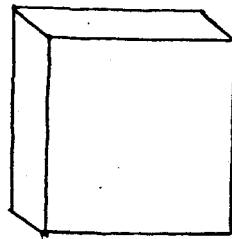
SMALL
BLACK
CUBE



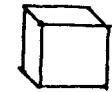
LARGE
WHITE
PYRAMID



SMALL
WHITE
PYRAMID



LARGE
WHITE
CUBE



SMALL
WHITE
CUBE

Illustration 5. The Elements of Articept Space 13

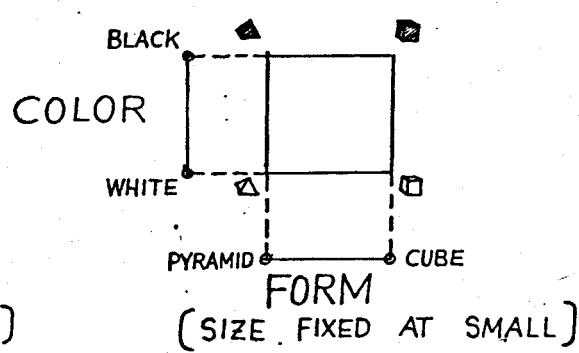
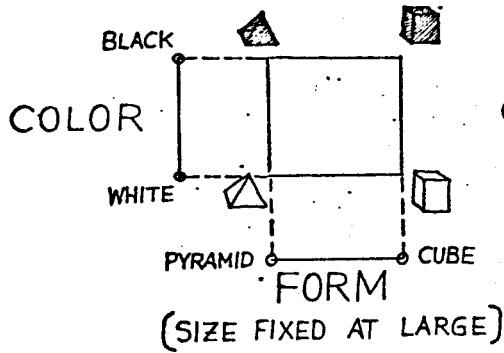
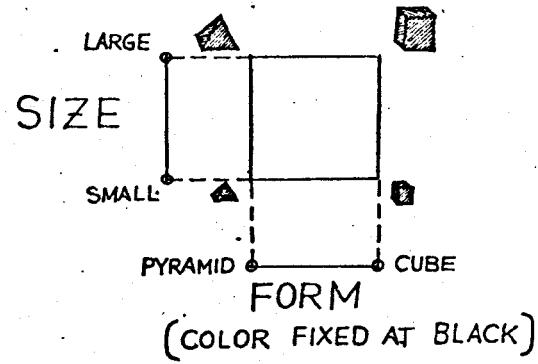
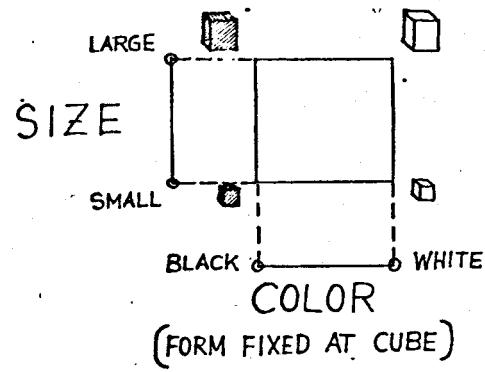
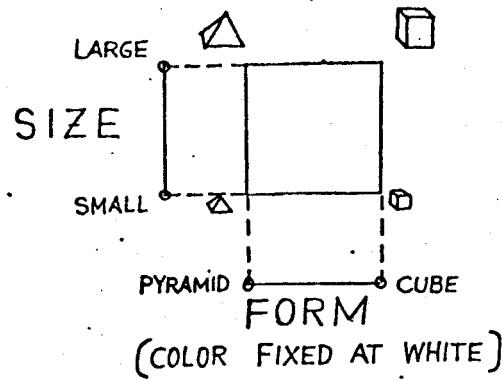
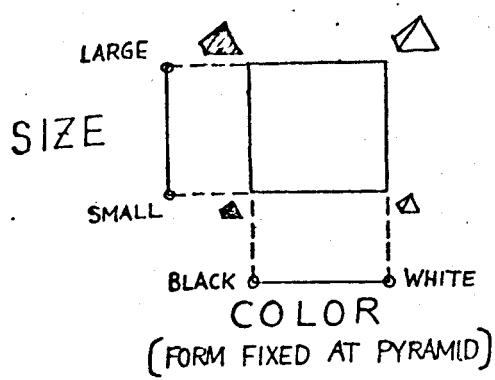


Illustration 6. Cartesian Product Spaces Generated by the Generating Elements of Generator Space $G_{2,2,2}$, Taken Two at a Time

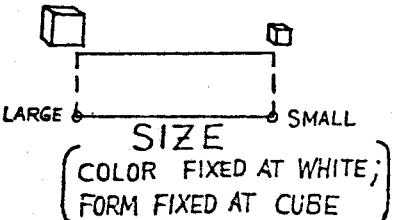
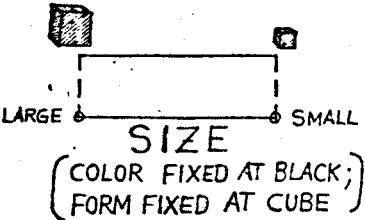
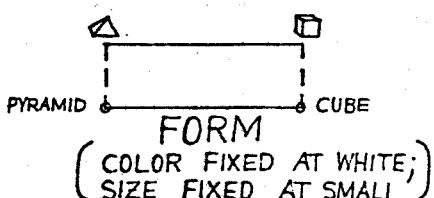
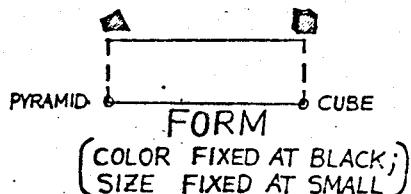
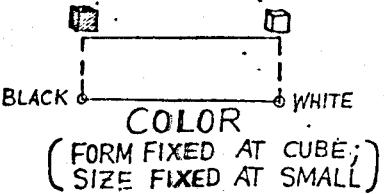
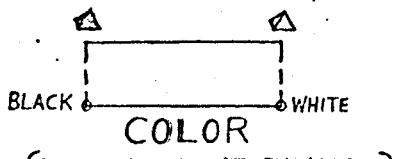
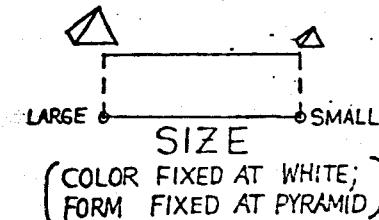
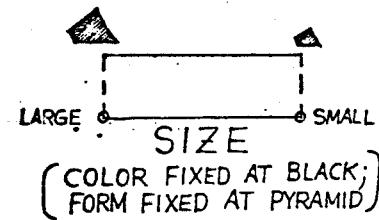
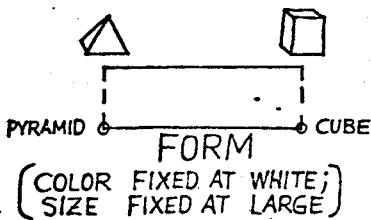
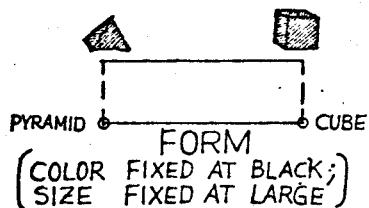
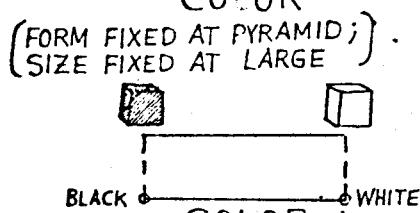
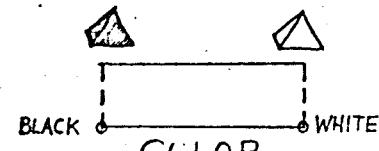


Illustration 7. Cartesian Product Spaces
Generated by the Generating Elements of Generator
Space $G_{2,2,2}$, Taken One at a Time

find a complete computer listing of all possible articept spaces 1_1 , 1_2 , 1_3 in appendix 5, pages 161-234.

By considering the set of all possible subsets (power set) of articept space 1_d , one may generate a wholly new articept space 2^{1_d} (articept power space), whose elements are essentially minimal Boolean forms. As the elements of 1_d were formed by conjunction of the possible dimensional levels of the articept generator space $G_{2,2,2}$; the elements of 2^{1_d} are formed by disjunction of the possible elements of the articept space 1_d . What do articeps belonging to this new articept power space look like? Consider the 3-D articept space 2^{1_3} , which consists of 256 articeps.

For example, one may form the disjunction of [large black pyramid] with [small black pyramid], which yields the articept, [black pyramids]. This is exhibited in illustration 8. The reader may find a complete computer listing of all possible articept spaces 2^{1_1} , 2^{1_2} , 2^{1_3} in appendix 5, pages 161-234.

Now, by construction, the articept space 2^{1_d} is a Boolean Algebra, thus making it tractable to mathematical analysis. The fundamental generating elements of the deep structure and very deep structure are the primitive logic operators (logic Primitives).

By hypothesis, these logic primitives may be repre-

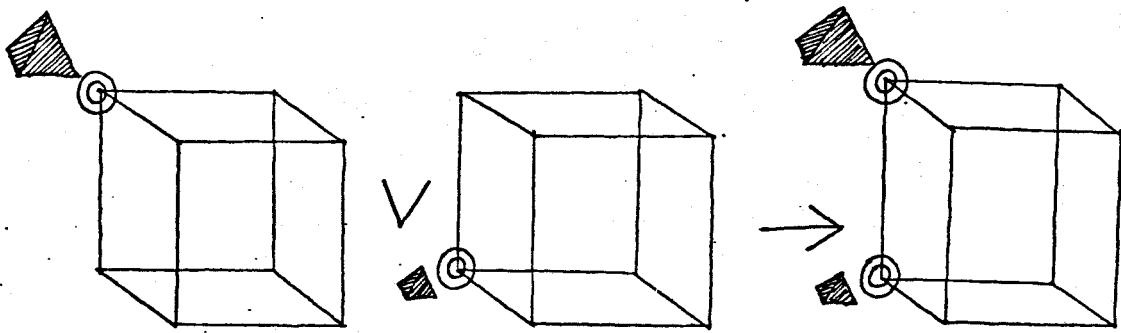


Illustration 8. The Disjunction of [Large Black Pyramid] with [Small Black Pyramid], which Yields the Antecept, [Black Pyramids]

sented by Boolean forms.¹ Thus, one may represent and simulate the operations of human logic primitives by the solution of a system of Boolean representations within the Boolean Algebra of articept space 2^{1_d} .

What is of equal import, the articept space 2^{1_d} may be exhaustively partitioned into disjunct equivalence classes (isomorphs) of structurally-identical articeps. That is, the articeps within a given isomorph are all structurally-identical to each other. The concept of articeps being structurally-identical to one another will be elucidated upon shortly; parenthetically however, as an illustration, consider the articept space 2^{1_2} , where $l_2 = \text{small}$. This is exhibited in illustration 9.

Observe that although the four articeps,



differ from each other in content; they are identical in structure, since they are each "twins." However, the two articeps



are not structurally-identical, since the articept on the

¹This is a plausible extension of the Stone Representation Theorem. The plausibility of the hypothesis is supported by the observation that the asymptotic ontogenesis of all exhibited human logics may be embedded within a Boolean Algebra.

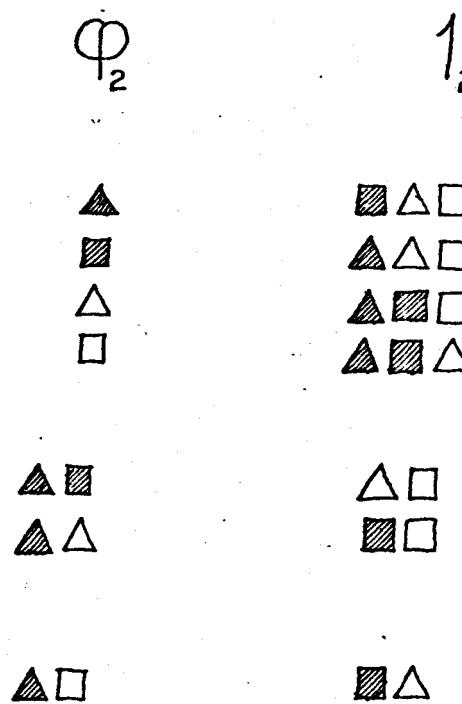


Illustration 9. The Elements of
Articept Space 2^2 , where 1_2 = Small

left-hand side is a twin (the form of pyramid is homogeneous), while the articept on the right-hand side is neither homogeneous with respect to color nor form. For reasons which will become apparent later, it is also stipulated that complements with respect to the articept space $\mathbb{1}_d$ are structurally-identical. Thus, the two complementary articepts



will also be considered structurally-identical. This is exhibited in illustration 10.

The existence of the structural-identity of articepts within a given isomorph immediately confers upon each isomorph a unique structural representation by Boolean form. That is, each isomorph may be represented by a uniquely different structure and Boolean form. Given an articept space $\mathbb{1}_d$ (articept universe), one may completely generate all of the articepts within a given isomorph by first finding all of the structure-invariant transformations T_d over the articept space $2^{\mathbb{1}_d}$. These transformations preserve structure but yet change content. Thus, one may completely separate structure from content. For example, consider the 2-D articept space $\mathbb{1}_2$, where the articept universe chosen is $\mathbb{1}_2 = \text{white}$, depicted in illustration 11.

The articept space $2^{\mathbb{1}_2}$ generated by $\mathbb{1}_2$, consists of

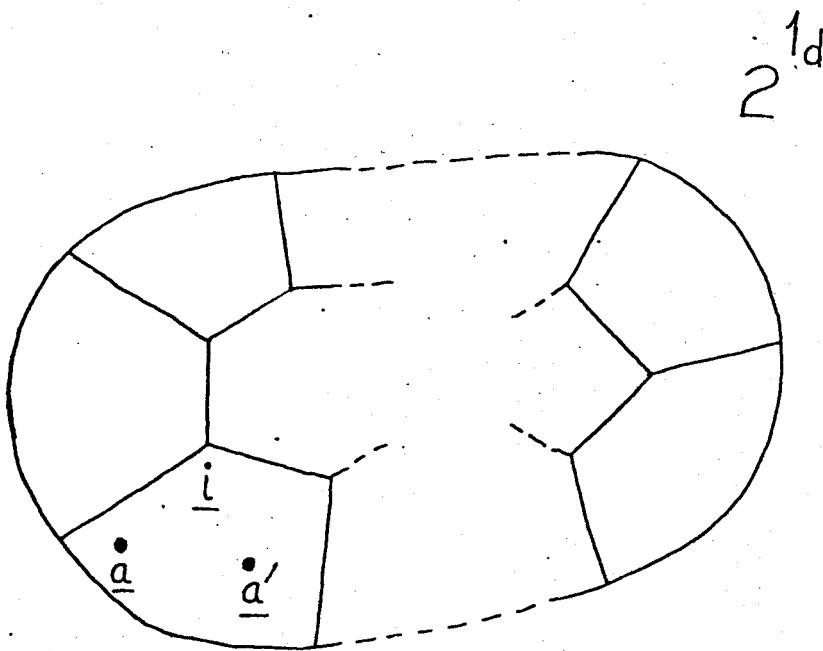
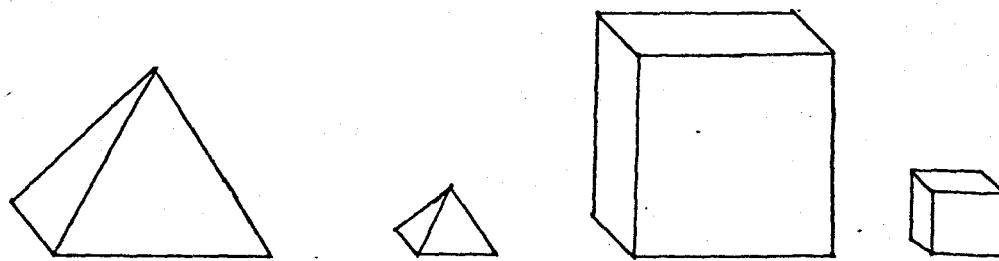


Illustration 10. Schematically depicted is Articept Space 2^4 exhaustively partitioned into distinct isomorphs, one of which is labelled Isomorph i . The two articepts, a and a' within Isomorph i , are structurally-identical to each other.



LARGE
(WHITE)
PYRAMID

SMALL
(WHITE)
PYRAMID

LARGE
(WHITE)
CUBE

SMALL
(WHITE)
CUBE

Illustration 11. A 2-D Articept Space, $l_2 = \text{White}$

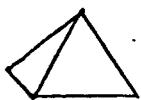
sixteen articepts which may be partitioned into four isomorphs, in the fashion shown in illustration 12. The reader may find a complete listing of the isomorphs for all possible Boolean universes $d=1, 2, 3$, in appendix 6, pages 235-250.

The essence of this discussion is that, given an articept universe \mathcal{I}_d , each isomorph may be represented by a unique Boolean structure; and by varying the invariant transformations T_d over $2^{\mathcal{I}_d}$, all the articepts within that isomorph may be generated--all articepts within a given isomorph being structurally-identical. Thus, the isomorph may be viewed as structure dissociated from content; and the articept may be viewed as content dissociated from structure.

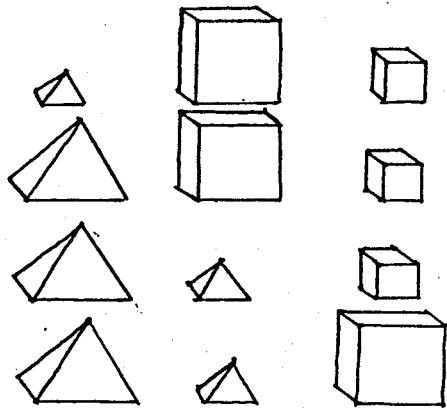
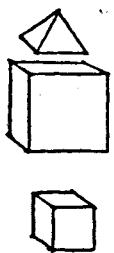
Schematic representations of the Boolean graphs of the isomorphic structures of the articept spaces 2^{1_1} , 2^{1_2} , 2^{1_3} are exhibited in illustrations 13, 14, 15, respectively.

One may observe that the articepts of any given isomorph are isomorphic to one another. That is, for example, given isomorph 13 (ISO 13) in articept space 2^{1_3} , the articepts depicted in illustration 16 are all isomorphic. They are the content, reflecting the structure of the embedding isomorph. The Boolean graphs representing each articept are transformed under T_3 into other articepts of that isomorph. The invariant-transformation T_3 may thus be seen to "turn" the Boolean graph "around."

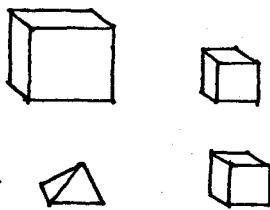
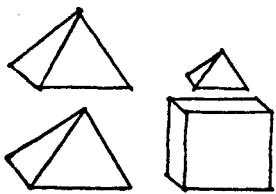
ISO 1

 φ_2 1_2 

ISO 2



ISO 3



ISO 4



Illustration 12. The Four Isomorphs of
Articent Space 2^{1_2} (Boolean Universe 1_2 = White)

ISO 1

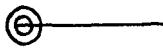
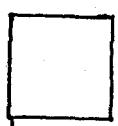
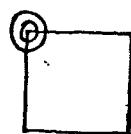
ISO 2

Illustration 13. Isomorphic Structure of Articept Space 2

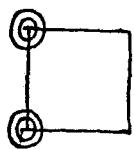
ISO 1



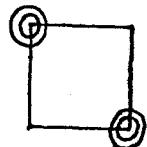
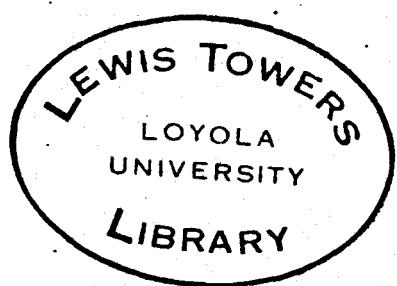
ISO 2



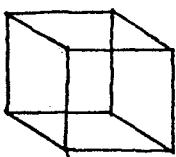
ISO 3



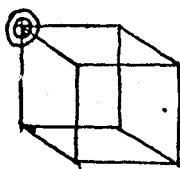
ISO 4

Illustration 14. Isomorphic Structure of Articept Space $2^{1/2}$ 

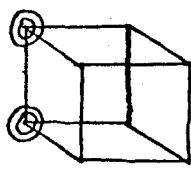
ISO 1



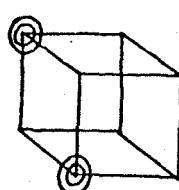
ISO 2



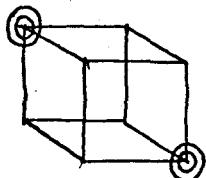
ISO 3



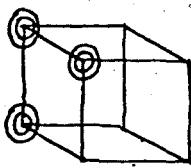
ISO 4



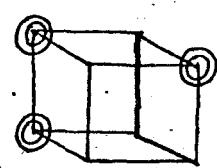
ISO 5



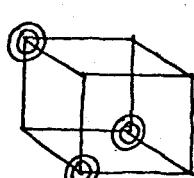
ISO 6



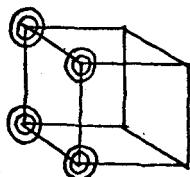
ISO 7



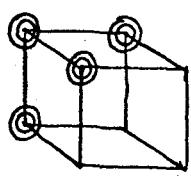
ISO 8



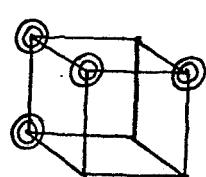
ISO 9



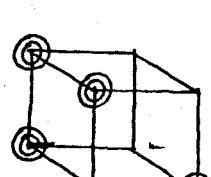
ISO 10



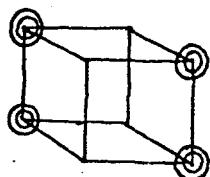
ISO 11



ISO 12



ISO 13



ISO 14

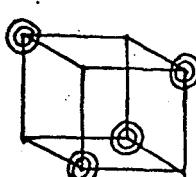
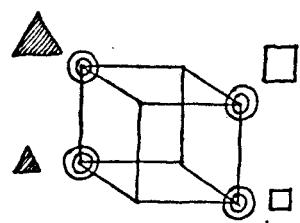
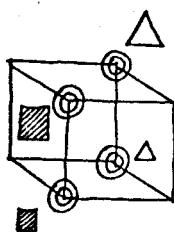


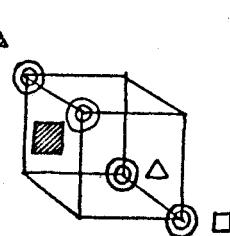
Illustration 15. Isomorphic Structure of Articept Space 2^3



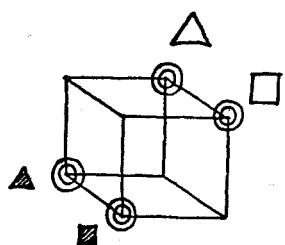
$$Z = \triangle \triangle \square \square$$



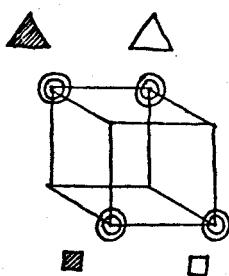
$$Z = \square \square \triangle \triangle$$



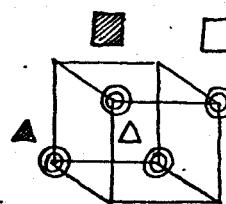
$$Z = \triangle \square \square \triangle \square$$



$$Z = \triangle \square \triangle \square$$



$$Z = \triangle \triangle \square \square$$



$$Z = \triangle \triangle \square \square$$

$$Z = x_i^{e_i} x_j^{e_j} \vee \bar{x}_i^{\bar{e}_i} \bar{x}_j^{\bar{e}_j}$$

Illustration 16. Schematic Representation of the Boolean Graphs of the Articepts of Isomorph 13 (ISO 13) in Articept Space 2^3

Some idea of the magnitude of the Boolean structures discussed here, is exhibited in table 2.

In order to facilitate the construction of the model, the existence of a certain abstract Boolean structure was postulated. By defining an equivalence relation over this structure, the set of all subsets or power set generated by combinatorically varying the contents of this abstract structure, was partitioned into isomorphs or supersets, each of which then contained structurally equivalent articepts or subsets.

Representation of the Model: Logiktrak

What does all of this Boolean cerebration have to do with Piagetian measures of cognitive development?

The Logiktrak is generally a "Whodunit?" problem-to-be-solved where the subject (S) is the detective (problem-solver). The solution is the problem-to-be-solved; the "who" in the "Whodunit?," is an unknown articept in articept space 2^{1d} , selected by the experimenter (E). Thus, the subject (S) must generate his or her own clues in the form of a most optimal question-asking strategy in order to gain information and discover the unknown articept. Suppose an articept is selected by the experimenter (E) from articept space 2^{13} . Then, initially, there are 256 articepts which are tenable as unknowns. With the first question, the subject (S) may eliminate a number of arti-

TABLE 2
TABLE OF BOOLEAN MAGNITUDES

d	Number of Articepts in 1^d	Number of Isomorphs in 2^1d	Number of Articepts in 2^2d
1	2	2	4
2	4	4	16
3	8	14	256
4	16	50 (?)*	65,536
.	.	.	.
.	.	.	.
.	.	.	.
n	2^n	-	2^{2n}

*The number of isomorphs in 2^1d , $d > 3$ is uncertain.
Evidence beyond the scope of this paper tends to favor the model represented by the difference equation.

$$y_d = 2C_{d-1}^{2d-1} - C_{d-1}^{2d-2}$$

where y_d is the number of isomorphs in 2^1d .

The solution to the above equation is given by

$$y_d = \left(\frac{3d^2 - 2d}{4d^2 - 2d} \right) \left(\frac{(2d)!}{(d!)^2} \right)$$

Observe that

$$\text{ASYM } y_d \underset{d \sim \infty}{=} y_\infty = \frac{3}{4} \frac{(2d)!}{(d!)^2}$$

cepts as being untenable. With the second question, he may again eliminate further articepts as being untenable. This process continues until but one articept is considered tenable--hopefully, the unknown articept selected by the experimenter (E). Each question posed by the subject (S) is in essence a Boolean representation. The question-asking strategy itself is thus a sequentially-solved system of Boolean representations.

What kinds of questions are subjects allowed to ask?

Subjects are allowed to ask any question answerable with yes or no. By defining a measure function over the Boolean Algebra $2^{\mathcal{U}^d}$, subjects are also allowed to ask almost¹ any question answerable with a number.

Measure of Structural Complexity

Recall that each isomorph may be structurally represented by a unique Boolean form. Epagogically, one may realize that, in a certain sense, some isomorphs are structurally more complex than other isomorphs. For example, given the Boolean universe \mathbb{I}_3 , it is certainly obvious that the articept selected from isomorph one (ISO 1) is

¹From Piagetian theory, for example, Charles J. Brainerd, "Mathematical and Behavioral Foundation of Number," Journal of General Psychology 88 (April 1973): 221-281; and Jean Piaget, The Child's Conception of Number (New York: W. W. Norton & Co., 1965), the concept of number may be used to represent either enumeration (counting) or identification (coding). It is only in the context of enumeration that number is used in this dissertation.

less complex than the articept selected from isomorph fourteen (ISO 14), where the articeps in question are depicted in illustration 17.

Roughly speaking, the more difficult an isomorph is to solve for, the more complex it is. Perplexity reflects complexity.

By construction, if an articept Z lies within a given isomorph, the complement of articept Z with respect to Boolean universe $\mathbb{1}_d$, $\bar{Z} = \mathbb{1}_d - Z$, also lies within that same given isomorph; thus, complementation is arbitrarily added to the class of structure-invariant transformations over an articept space $2^{\mathbb{1}_d}$. The reader may find a complete listing of structure-invariant transformation $T_d: 2^{\mathbb{1}_d} \rightarrow 2^{\mathbb{1}_d}$, for $d=1,2,3$, in appendix 7, pages 251-260. Some idea of the magnitude of the structure-invariant transformations discussed here, is exhibited in table 3.

An articept Z and its complement \bar{Z} completely partition a Boolean universe $\mathbb{1}_d$. This is exhibited in illustration 18.

Since an articept is merely the content-reflection of the structure of the isomorph within which it is embedded, it follows that the structural identity of an isomorph may be uniquely represented by the configuration of the partition of articept space $\mathbb{1}_d$ into the two components--articept Z and its complement, articept \bar{Z} , thusly $\mathbb{1}_d = Z \oplus \bar{Z}$. For example, consider the Boolean universe $\mathbb{1}_2$.

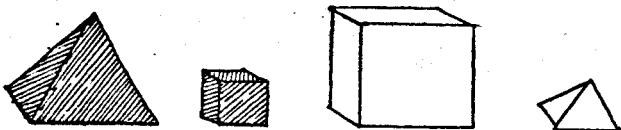
φ_3 

Illustration 17. The left-hand side articept is the Null Set selected from ISO 1, and the right-hand side articept is selected from ISO 14. Obviously, ISO 1 is less complex than ISO 14.

TABLE 3
TABLE OF TRANSFORMATION MAGNITUDES

d	Number of Structure-invariant Transformations $ T_d $, Over Articest Space $T_d: 2^{1_d} \rightarrow 2^{1_d}$	Number of Transformations, $ J_d $, Variant And Invariant Mapping $J_d: 1_d \rightarrow 1_d$	$\frac{ T_d }{ J_d }$
1	2	2	1
2	8	24	1/3
3	48	40,320	$\pm 1/840$
n	$2^n n!$	$(2^n)!$	---

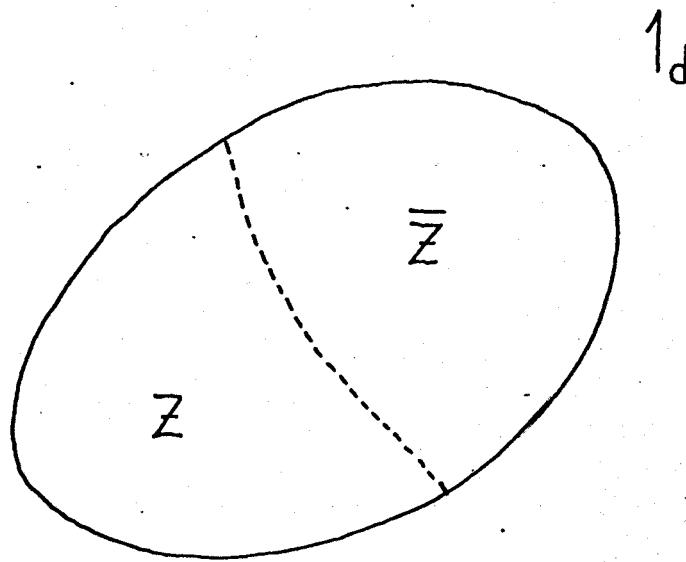


Illustration 18. The Partition of a Boolean Universe l_d into Isomorphic Articeps,
 $l_d = Z \oplus \bar{Z}$.

Then one may represent the isomorph structure by Boolean forms, as depicted in illustration 19.

There exists a one-to-one relation (isomorphism) between Boolean Algebra 2^d and Algebra of Switching Circuits \mathcal{U}_d (Switching Algebra).¹ This isomorphism may be taken advantage of, since there has been a great deal of work done on Switching Algebras in conjunction with optimizing computer design. One of the paramount problems in optimizing computer design is in the minimization of (1) arithmetic operation time, (2) cost of total switching components and (3) space occupied by total switching components. In view of the isomorphism, one may then consider the switching circuit in \mathcal{U}_d isomorphic to the artifact in 2^d , and make use of the concepts of Switching Algebras. Thus, it is entirely conceivable that, in a certain sense, time and cost may be feasible measures of the complexity of artifacts. Other possible avenues of attack to this problem may utilize derivative concepts from the Redfield-Polya Theorem of enumeration,² Winograd's Theorem establishing a lower bound on time required for

¹L. S. Bobrow and M. A. Arbib, Discrete Mathematics (Philadelphia: W. B. Saunders Co., 1974); and T. G. Room and J. M. Mack, The Sorting Process: A Study in Mathematical Structure (London: Methuen, 1966).

²N. G. DeBruijn, "Polya's Theory of Counting," in Applied Combinatorial Mathematics, ed. E. F. Beckenbach (New York: John Wiley & Sons, 1964).

IS01

$$\Phi_2 \oplus 1_2$$

$$IS02 \quad x_i^{e_i} x_j^{e_j} \oplus x_i^{e_i} \bar{x}_j^{\bar{e}_j} \vee \bar{x}_i^{\bar{e}_i} x_j^{e_j} \vee \bar{x}_i^{\bar{e}_i} \bar{x}_j^{\bar{e}_j}$$

$$IS03 \quad x_i^{e_i} x_j^{e_j} \vee x_i^{e_i} \bar{x}_j^{\bar{e}_j} \oplus \bar{x}_i^{\bar{e}_i} x_j^{e_j} \vee \bar{x}_i^{\bar{e}_i} \bar{x}_j^{\bar{e}_j}$$

$$IS04 \quad x_i^{e_i} x_j^{e_j} \vee \bar{x}_i^{\bar{e}_i} \bar{x}_j^{\bar{e}_j} \oplus \bar{x}_i^{\bar{e}_i} x_j^{e_j} \vee x_i^{e_i} \bar{x}_j^{\bar{e}_j}$$

$$(i, j = 1, 2; e_i, e_j = 0, 1; \bar{e}_i = e_i + 1 \text{ MOD } 2)$$

Illustration 19. Boolean Representation of Isomorphs in 2^2

NOTE: The " \oplus " operation denotes partition of the articept space 1_d into the two component articeps, Z and \bar{Z} . " \oplus " behaves as a disjunction which is impassable to the usual Boolean operations (i.e., distribution).

addition,¹ the Krohn-Rhodes theory of machine decomposition,² Ferdinand's work on the statistical complexity of systems³ and Mowshowitz's work on the information of graphs.⁴ Unfortunately, all of the preceding works are not easily applicable to non-ergodic, multi-valued logics. In an attempt to generate logically consistent measures of complexity, several different measures of complexity were derived, each with construct validity with respect to different conceptual foundations.⁵ A complete discussion of these recently derived measures of complexity is beyond the scope of this dissertation. Briefly, however, measures of complexity (partition functions) must satisfy the following axiomatic model:

¹S. Winograd, "On the Time Required to Perform Addition," Journal of the Association for Computing Machinery 12 (No. 2, 1965): 277-285.

²M. A. Arbib, ed., Algebraic Theory of Machines, Languages, and Semigroups, with a major contribution by K. Krohn and J. L. Rhodes (New York: Academic Press, 1968); and K. B. Krohn and J. L. Rhodes, "Algebraic Theory of Machines," in Proceedings of a Symposium in the Mathematical Theory of Automata, Brooklyn, N. Y., 1962 (New York: John Wiley & Sons, 1962), pp. 341-384.

³A. E. Ferdinand, "A Statistical Mechanical Approach to Systems Analysis," IBM Journal of Research and Development 14 (No. 5, 1970): 539-547.

⁴Abbe Mowshowitz, "Entropy and the Complexity of Graphs: I. An Index of the Relative Complexity of a Graph," Bulletin of Mathematical Biophysics 30 (1968): 175-204.

⁵Harvey Jack Schiller, "On Measures of Structural Complexity of Automata," [Notes in Automata Theory.] Chicago, Illinois, 1975.

Let 2^{1_d} be a d-dimensional articept space; and let \mathcal{A}_v be an isomorph in articept space 2^{1_d}

$$\mathcal{A}_v \subset 2^{1_d}$$

Then the isomorphs exhaustively partition the articept space

$$\bigcup_v \mathcal{A}_v = 2^{1_d}$$

and they are mutually disjunct

$$\mathcal{A}_\alpha \mathcal{A}_\beta = \emptyset_d, \quad (\alpha \neq \beta)$$

Consider an articept Z_{i_v} of the isomorph \mathcal{A}_v

$$\text{Then } Z_{i_v} \in \mathcal{A}_v$$

$$\text{And } \sum_v \sum_i Z_{i_v} = 2^{1_d}$$

Let $T_d : 2^{1_d} \rightarrow 2^{1_d}$ be the class of all structurally-invariant transformations over articept space 2^{1_d} .

$$\text{Then if } T_\tau \in T_d, \quad T_\tau Z_{i_v} \in \mathcal{A}_v$$

Let $H : 2^{1_d} \rightarrow R$, be a measure of complexity

(partition function)

Then H must satisfy the following conditions:

$$(1) \quad H(Z_i) = H(Z_j) \quad \text{iff} \quad \exists \alpha \ni T_\alpha Z_i = Z_j$$

$$(2) \quad \min_z H(z) = H(\emptyset_d)$$

$$(3) \quad \max_z H(z) = H(Z_{MAX})$$

Where in the Boolean representation

$$I_d = Z \oplus \bar{Z}$$

$$Z = \sum \beta_{ijk} x_1^i x_2^j x_3^k$$

$$\bar{Z} = \sum \bar{\beta}_{ijk} x_1^i x_2^j x_3^k, \quad \bar{\beta}_{ijk} = 1 + \beta_{ijk} \bmod 2$$

the coefficients of Z_{MAX} must satisfy the following condition:

$$i+j+k = \{0\} \otimes \{1\}, \quad \forall i,j,k$$

(4) Given a fixed analog¹ isomorph for any $d_i \in \mathbb{Z}^+$

$$\left| \frac{\partial H}{\partial d_i} \right| > 0$$

$$(5) \quad H(Z) = H(\bar{Z}) = H(1-Z)$$

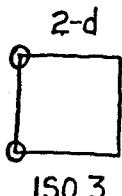
(6) the ordered sequence

$$H(Z_{\alpha_1}) \leq H(Z_{\alpha_2}) \leq \dots \leq H(Z_{\alpha_{|P^d/T_d|}})$$

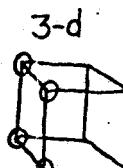
corresponds with trajectory of ontogenetic evolution of intelligence in humans

¹An analog isomorph is that isomorph which would be formed by the expansion or contraction of one variable of an existing isomorph. For example, the analogs of the 1-d isomorph \circ are:

ISO 2



ISO 3



ISO 4

Via use of these partition functions, it is possible to establish an ordering of complexity among the isomorphs from a given articept space 2^d . However, since all the partition functions do not perfectly agree upon the order of complexity among isomorphs, since each partition function has a different conceptual foundation, the order of complexity must necessarily be an approximate one.

Exhibited in table 4, the reader will find a table of quantitative comparison of several measures of complexity (all numerical quantities have been converted to bits for the reader's convenience).

In order to avoid introducing contamination by an experimental artifact, only those isomorphs whose partition functions exhibit zero variance, (i.e., invariant isomorphs, $\sigma_k^2 = 0$) have been used in this research.

In order to integrate this new theory with Piagetian Genetic Epistemology (evolution of logic), and recent work in concept-learning, consider Piaget's binary logic operations, or as they are better known in the study of mathematical logic, the connectives of the propositional calculus. This particular logic structure is the very kernel of all Piaget's work in the ontogenetic evolution of the scientific method of hypothesis-testing in human logic;¹

¹William M. Bart and Peter W. Airasian, "Determination of the Ordering Among Seven Piagetian Tasks by an Ordering-Theoretic Method," Journal of Educational Psychology

TABLE 4

TABLE OF QUANTITATIVE COMPARISON
OF SEVERAL MEASURES OF COMPLEXITY

Di- men- sion d	Iso- morph 1	Number of Articepts in Isomorph	Schematic Represen- tation of Isomorph	Conceptual Foundation of Structural Complexity		
				Orbits	Partitions	Gates
1	1	2	—	0.000	0.000	0
	2	2	—	1.000	1.000	4
2	1	2	□	0.000	0.000	0
	2	8	□	1.500	1.585	8
	3	4	□	1.500	1.500	4
	4	2	□	2.000	2.000	14
3	1	2	□	0.000	0.000	0
	2	16	□	1.750	2.000	11
	3	24	□	2.000	2.164	8
	4	24	□	2.333	2.497	17
	5	8	□	2.500	2.585	22
	6	48	□	2.250	2.376	12
	7	48	□	2.583	2.645	21
	8	16	□	2.750	2.807	27
	9	6	□	2.000	2.000	4
	10	8	□	2.500	2.585	20
	11	24	□	2.500	2.585	14
	12	24	□	2.667	2.723	22
	13	6	□	2.667	2.667	14
	14	2	□	3.000	3.000	34

so it is well worth singular treatment here. It is exhibited in illustration 21.

A little reflection will show that Piaget's binary logic operations are nothing other than the elements of articept space 2^1_2 ! A compact synthesis of Piaget's work interpreted in light of this new theory would posit that logic primitives of lesser complexity as determined by partition functions emerge first in the evolving cognitive structure of humans, followed sequentially by the emergence of those logic primitives of greater complexity as determined by partition functions. In fact, one may define a perfect partition function as one which perfectly predicts the order of ontogenetic evolution of the logic primitives in humans.

Via the partition functions, it is possible to order the basic syntactic operators: existence (\exists), negation (\neg), conjunction (\wedge), disjunction (\vee), condition (\rightarrow) and bicondition (\leftrightarrow), according to their predicted ontogenetic evolution. Consider articept space 2^1_2 ; from illustration 22. One sees that existence is subsumed under isomorph one (ISO 1); negation is subsumed under isomorph three (ISO 3); conjunction, disjunction and condition are all subsumed under isomorph two (ISO 2); and bicondition is subsumed under isomorph four (ISO 4).

66 (April 1974): 277-284; and Daniel N. Osherson, Logical Abilities in Children, 2 vols. (Potomac, Maryland: Lawrence Erlbaum Associates, 1974).

x_1	x_2	TAUT-OLOGY	$p \vee q$	$p < q$	p	$p \supset q$	q	$p \equiv q$	$p \wedge q$
P	q	(1 ₂)	$(p \vee q)$	$(p \vee \bar{q})$	(P)	$(\bar{p} \vee q)$	(q)	$(pq \vee \bar{p}\bar{q})$	(Pq)
T	T	T	T	T	T	T	T	T	T
T	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	F	F	F
F	F	T	F	T	F	T	F	T	F

x_1	x_2	p/q	$p \not\equiv q$	\bar{q}	$p \not\supset q$	\bar{p}	$p \not\equiv q$	$p \downarrow q$	CONTRA-DICTION (Φ ₂)
P	q	$(\bar{p} \vee \bar{q})$	$(\bar{p}q \vee p\bar{q})$	(\bar{q})	$(p\bar{q})$	(P)	$(\bar{p}q)$	$(\bar{p}\bar{q})$	
T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F

Illustration 20. Mappings of $\mathcal{T} \times \mathcal{T}$ ($\mathcal{T} = \{T, F\}$). The Connectives of the Propositional Calculus

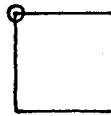
ISOMORPH

BOOLEAN
REPRESENTATION

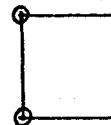
CORRESPONDING
CONNECTIVES OF THE
PROPOSITIONAL CALCULUS
OR
PIAGET'S BINARY LOGIC
OPERATORS



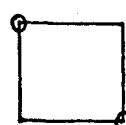
$$1_2 \oplus \Phi_2$$

TAUTOLOGY, 1_2 CONTRADICTION, Φ_2 

$$x_1^{i_1} x_2^{i_2} \oplus x_1^{\bar{i}_1} \vee x_2^{\bar{i}_2}$$

 $P \wedge q$  $P \neq q$  $P \downarrow q$  $P \not\rightarrow q$  P / q  $P \subset q$  $P \vee q$  $P \supset q$ 

$$x_3^{i_3} \oplus x_3^{\bar{i}_3}$$

 p  q  \bar{p}  \bar{q} 

$$x_1^{i_1} x_2^{i_2} \vee x_1^{\bar{i}_1} x_2^{\bar{i}_2} \oplus x_1^{\bar{i}_1} x_2^{i_2} \vee x_1^{\bar{i}_1} x_2^{\bar{i}_2}$$

 $P \equiv q$  $P \not\equiv q$

Illustration 21. Relationship of Isomorphs in 2^{1_2}
to Connectives of the Propositional Calculus (Piaget's Binary
Logic Operators)

The Logiktrak is generically quite similar to the popular Parker Brothers game, Clue--but without the dice. The solution to the "Whodunit?" is in the form of artifacts or artificial concepts comprised of canonical Boolean forms. The binary dimensions used in the Logiktrak are color, form, and size; as opposed to Who? (the murderer), Where? (the scene of the crime), and How? (the murder weapon) in Clue.

In view of the rich algebraic structure provided by the Logiktrak, certain solution problems-to-solve are more complex than others. The various classes of problems-to-solve are termed isomorphs. And the complexity of the isomorphs varies from isomorph to isomorph. In order to develop an absolute scale of complexity for the various solution problems, mathematical functions which provide absolute measures of the structural complexity of isomorphs, were constructed. Absolute, criterion-referent measurement of complexity is a significant advance upon the usual relative, norm-referent measurement of complexity.

Analysis exhibits that Piaget's work in the ontogenetic evolution of the scientific method in human logic may be quite succinctly represented by the algebraic structure of space 2^2 ; and that the sequence of ontogenetic emergence of the logic primitives in humans may be predicted by the ordering imposed by the measures of structural complexity derived for this work.

This is the order of ontogenetic evolution as determined by much of the work in concept-learning.¹ It also corresponds perfectly with the order of ontogenetic evolution as determined by the monotonically increasing values of the partition functions. The "lumping" of conjunction, disjunction and condition operators results from the boundedness (finiteness) of the Boolean universe \mathcal{B}_2 . In most concept-learning studies, the apparent differentiation between the three operators is an experimental artifact due to the subject's ignorance of the boundedness (finiteness) constraints imposed by articept generator space $G_{2,2,2}$.

Measure of Optimality of Problem-Solving Strategy

After constructing a representation of the model, and deriving absolute measures of the structural complexity of the problems-to-be-solved, absolute measures of the optimality of problem-solving strategies necessarily had to be developed.

Various conceptual foundations have been applied in the development of measures of diagnostic problem-solving. A long-time favorite measure has been the time taken to solve the problem. Since time to solution is not validly indicative of the development of cognitive structure, it

¹Scott G. Paris, "Comprehension of Language Connectives and Propositional Logical Relationships," Journal of Experimental Child Psychology 16 (October 1973): 278-291.

is a measure best left to more primitive designs.

The second stage in the evolution of diagnostic problem-solving measures is number of trials to solution or, generically put, number of questions to solution. This approach is more sophisticated than the former; however, number of questions to solution does not reflect strategy, and is thus too primitive a measure from which to draw inferences.

The third stage in the evolution of diagnostic problem-solving measures was derived by Rimoldi and his co-workers;¹ and has its conceptual foundation based upon use of the statistical artifact exhibited in illustration 23.

Where a_{ji} is the probability of the referent system having asked question j in order i ; $a_{j\Phi}$ denotes the probability that question j was never asked by the referent system, thus Φ is the null question.

$$\sum_{i=1}^{\Phi} a_{ji} = \frac{1}{N}, \quad \forall j = 1, 2, \dots, N$$

The referent system may be either group norm, "ideal solutions" or solutions representative of some fixed, known logic. Thus, given a subject's question-asking sequence,

¹H. J. A. Rimoldi, J. V. Haley, and M. M. Fogliatto, The Test of Diagnostic Skills (Chicago, Illinois: Loyola Psychometric Laboratory, Publication No. 25, 1962).

QUESTION.

	1	2	...	N
1	a_{11}	a_{21}	...	a_{N1}
2	a_{12}	a_{22}	...	a_{N2}
⋮	⋮	⋮	⋮	⋮
N	a_{1N}	...		a_{NN}
Φ	$a_{1Φ}$...		$a_{NΦ}$

$N+1 \times N$

Illustration 22. Matrix of System-Referent
Question-Asking Strategies

$Q_k = \{q_{k_1}, q_{k_2}, \dots, q_{k_N}\}$, the question-asking strategy would be computed as follows:

$$S_k = \sum_{l=1}^N a_{q_{k_l}} l \leq 1$$

Although this approach is far more reflective of question-asking strategy than the former two, it is still contaminated in the following respects: (1) norm-referent measurement, (2) the bounded and delimited universe of possible questions from which the subject must select his questions, (3) the most prevalent referent strategies are accorded the greatest merit and (4) stochastically speaking, the Rimoldian measure is probably Memoryless, and is certainly not even Markovian.

The fourth stage in the evolution of diagnostic problem-solving measures is a measure of question-asking strategy which was originally used in the comparative analysis of structurally-different problem-solving strategies.¹ This measure is a random variable of the number of questions to solution, ranging over all possible paths to the solution. The flaw of this measure is quite simply that it is pred-

¹The concept stochastic continuity has been developed to describe the path-dependence of stochastic processes. This concept is also of great import in the measure of the evolution of intelligence. The concept was introduced in Harvey Jack Schiller, "On the Evolution of Intelligence: Stochastic Continuity," [Notes in Automata Theory.] Chicago, Illinois, 1974.

icated upon the analysis of strategies which may be a priori determined algorithmically. This presupposes a discrete set of strategies from which the subject must select in order to solve the problem. In addition, it has an even lower order of stochastic continuity¹ than the Rimoldian measure. And it is insensitive to a continuous spectrum of question-asking strategies.

The fifth stage in the evolution of diagnostic problem-solving measures is a newly derived measure of question-asking strategy which has construct validity with respect to the following axiomatic model.

Let $X_\sigma = \{X_1, X_2, \dots, X_\eta, \dots X_{\eta_\sigma}\}$ be a strategy, where X_η is the η^{th} state of the strategy. If X_σ is a solution strategy for the solution S , then $S \in X_\sigma$. Let \mathcal{S} be the class of all solution strategies. And let $\Sigma: \mathcal{S} \rightarrow R$ be a measure of question-asking strategy. Then Σ must satisfy the following conditions:

(1) If $X, Y \in \mathcal{S}$ and X is more optimal than Y ,

$$\Sigma(X) < \Sigma(Y)$$

(2) $\Sigma \in C_o(\mathcal{S})$

¹P. Laughlin, "Selection Strategies in Concept Attainment," in Contemporary Issues in Cognitive Psychology, ed. R. L. Solso (Washington, D. C.: V. H. Winston & Sons, 1973).

(3) Σ is non-ergodic by X

(4) Σ is criterion-referent

$$(5) \Sigma(X_R) = \max_{X \in \mathcal{S}} \Sigma(X),$$

Where X_R is a memoryless strategy

$$(6) \Sigma(X_1) = \min_{X \in \mathcal{S}} \Sigma(X),$$

Where X_1 is an illegal identification¹

¹An identification is a question where the concept of number is used for coding (identification), rather than counting (enumeration).

An example of an identification in \mathbb{Z}_3^3 may be given by the following:

Given that the space \mathbb{Z}_3^3 has representation

$$\mathbb{Z}_3^3 = \sum_{i_1 i_2 i_3} \beta_{i_1 i_2 i_3}^{i_1 i_2 i_3} X_1^{i_1} X_2^{i_2} X_3^{i_3}$$

Where

$$\beta_{i_1 i_2 i_3} = \sum_{\mathbb{Z}} (X_1^{i_1} X_2^{i_2} X_3^{i_3}) = \begin{cases} 1, & X_1^{i_1} X_2^{i_2} X_3^{i_3} \in \mathbb{Z} \\ 0, & X_1^{i_1} X_2^{i_2} X_3^{i_3} \in \bar{\mathbb{Z}} \end{cases}$$

Then give the binary number

$$B_\beta = \sum_{B(i_1 i_2 i_3)} \beta_{i_1 i_2 i_3} 2^{B(i_1 i_2 i_3)}$$

Thus, any problem may be solved with one identification.

$$(7) \frac{\partial X}{\partial C} \Big|_{X,t} > 0$$

Where C is a measure of complexity

$$(8) \frac{\partial X}{\partial T} \Big|_t < 0$$

Where T is a measure of ontogenetic evolution
~~X~~, the measure of optimality of question-asking strategy has conceptual foundation based upon the minimization of the integral

$$J_\psi = \int_0^{\eta^*} \psi(\eta) d\eta$$

where $\psi(\eta)$ is the strategy function, and is given by the number of existing tenable hypotheses upon knowledge of the answer given to the η^{th} question; and η^* is that question where $\psi(\eta^*) = 1$.

Summary

The purpose of this experiment was to construct a model of cognitive structure in which the behavior of the logic primitives could be embedded, and to formulate a representation of that model which could be empirically tested.

In order to facilitate the construction of the model,

the existence of a certain algebraic structure was postulated. By defining an equivalence relation over this structure, the set of all subsets generated by combinatorically varying the contents of this algebraic structure, was partitioned into isomorphs or supersets, each of which then contains structurally equivalent articepts or subsets.

The representation of the model is in the form of logic problems-to-be-solved. The Logiktrak is generically quite similar to the popular Parker Brothers game, Clue--but without the dice. The solution to the "Whodunit?" is in the form of articepts or artificial concepts comprised of canonical Boolean forms. The binary dimensions used in the Logiktrak are color, form, and size; as opposed to Who? (the murderer), Where? (the scene of the crime), and How? (the murder weapon) in Clue.

In view of the rich algebraic structure provided by the Logiktrak, certain problems-to-be-solved are more complex than others. The various equivalence classes of problems-to-be-solved are termed isomorphs. And the complexity of isomorphs varies from isomorph to isomorph. In order to develop an absolute scale of complexity for the various isomorphs, mathematical functions which provide absolute measure of the structural complexity of isomorphs, were constructed. Absolute, criterion-referent measurement of complexity is a significant advance upon the usual relative norm-referent measurement of complexity.

Analysis reveals that Piaget's work in the ontogenetic evolution of the scientific method of hypothesis-testing in human logic may quite succinctly be represented by the algebraic structure of space 2^4_2 ; and that the sequence of ontogenetic emergence of the logic primitives in humans may be predicted by the measures of structural complexity derived for this work.

After constructing a representation of the model, and deriving absolute measures of the structural complexity of the problems-to-be-solved, absolute measures of the optimality of problem-solving strategies necessarily had to be developed. This was in response to the necessity of providing a differentiating device which allows one to analyze the "track of a subject's logic" rather than only its resultant conclusions.

There are two primary research hypotheses: first, that problem-solving strategies become more optimal with increasing Piagetian ontogenesis; and second, that for subjects of equivalent Piagetian ontogenesis, problem-solving strategies become more optimal with increasing intelligence.

CHAPTER III

METHOD

The purpose of this experiment was to construct a model of cognitive structure in which the behavior of the logic primitives could be embedded, and to formulate a representation of that model which could be empirically tested.

The representation of the model is in the form of logic problems-to-be-solved. The model admits for the absolute measure of the structural complexity of problems-to-be-solved, and absolute measure of the optimality of the subject's problem-solving strategies. In this way, the "track" of the subject's logic may be analyzed. The representation of the model which is then empirically tested, is appropriately known as the Logiktrak. There are two primary research hypotheses: first, that problem-solving strategies become more optimal with increasing Piagetian ontogenesis; and second, that for subjects of equivalent Piagetian ontogenesis, problem-solving strategies become more optimal with increasing intelligence as measured by Raven's Progressive Matrices.

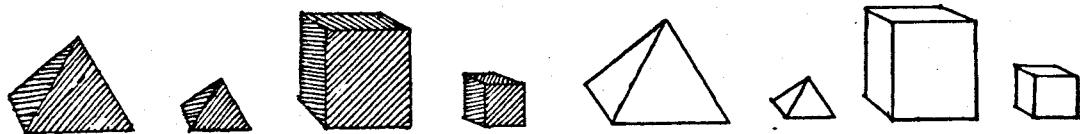
In this chapter, the following will be discussed: the design and pre-testing of the Logiktrak materials, the experimental procedures, and the method of analysis.

Logiktrak Materials

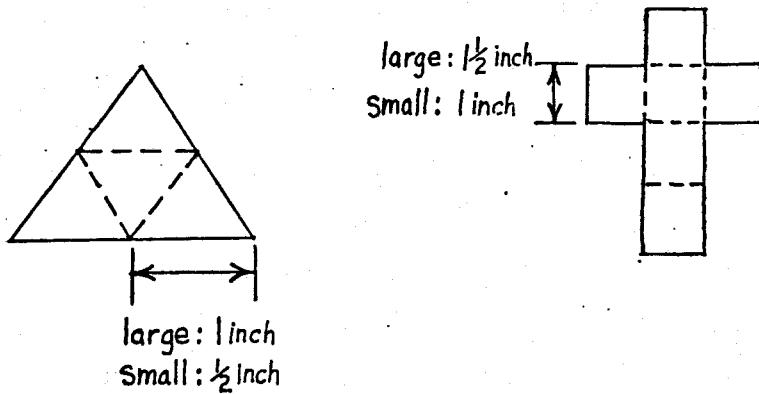
In order to administrate the Logiktrak, the following materials were used.

Kit of Toys

The kit of toys is really a set of miniature replicas of the elements of articept space $\frac{1}{3}$, depicted here:

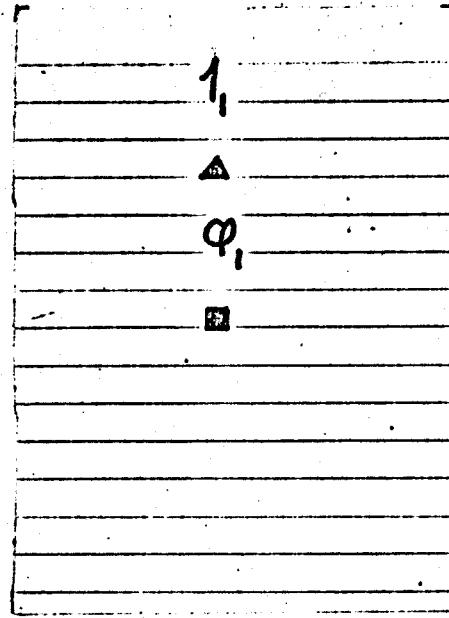


The replicas are made of sturdy cardboard with the dimensions used depicted here:

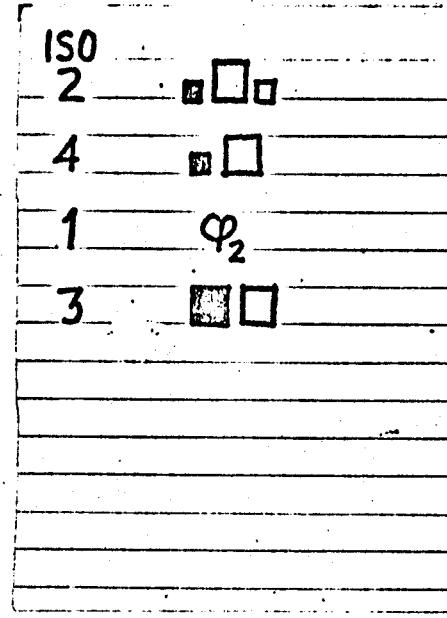


Kit of Problems

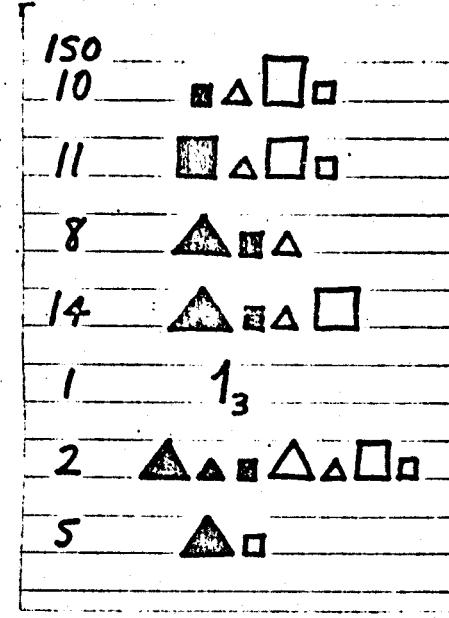
The kit of problems is a set of 3" x 4" lined index cards, each inscribed with a complete test comprised of randomly generated articepts representing the appropriate isomorphs of the articept space dimension required. Sample test cards are depicted in illustration 24.



$$1_1 = \triangle \blacksquare$$



$$1_2 = \blacksquare \blacksquare \square \square$$



$$1_3 \quad \sigma^2 = 0$$

Illustration 23. Sample Test Cards Randomly Generated for the articept
Universe 1_1 = Small Black, 1_2 = Cubes, 1_3

The articepts given in each test are generated in random sequence from the following isomorphs, exhibited in illustration 24.

Logiktrak Administration Manual

The Logiktrak Administration Manual contains a detailed description of both materials and procedure necessary for administration of the Logiktrak. The Logiktrak Administration Manual is included in appendix 4, pages 149-160, for the reader's convenience.

Pretest Results

A pilot study was conducted in advance of the final experimentation, in order to perfect the Standardized Test, and to determine the reliability of the Logiktrak.

One had to be certain that the necessarily primitive problem-solving behavior of very young children in particular, was due in fact to their primitive cognitive structure, and not to experimental artifact introduced by imperfections in the Standardized Text.

The reliability study implemented a single subject repetition design on perfectly parallel test items. Data taken from several very intelligent adults,¹ revealed that

¹According to the model of intelligence posited in this dissertation, the variation in intelligence is monotonically positive with greater intelligence. That is, the presence of homoscedasticity cannot be assumed. On subsequent tests, i.e., Raven's Advanced Progressive Matrices,

ARTICEPT SPACE
DIMENSION
 d

ISOMORPHS

1

ISO 1

ISO 2

2



ISO 1



ISO 2

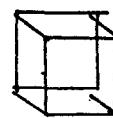


ISO 3



ISO 4

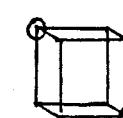
3



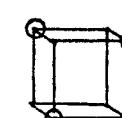
ISO 1



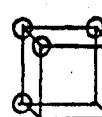
ISO 2



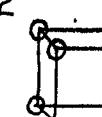
ISO 5



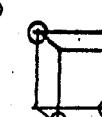
ISO 8



ISO 10



ISO 11



ISO 14

67

Illustration 24. Isomorphs Used to Generate Test Problems

the intra-subject variances were quite low; and hence that the Logiktrak possesses an inordinately great reliability. This data is exhibited in table 5.

Experiment

The reader is reminded that the experiment consisted of having subjects of a spectrum (ranging from pre-school to adult levels) of Piagetian ontogenetic levels solve logic problems in order to determine the dependence of problem-solving upon ontogenesis; and also, to determine, for a given ontogenesis level, the dependence of problem-solving upon intelligence as measured by Raven's Progressive Matrices.

Subjects

Since this study was primarily concerned with Developmental Education, a broad spectrum of subject ages was required. The subject sampling scheme used is exhibited in table 6.

All subjects were selected from the same socioeconomic stratum--all were culled from populations statistically (in the demographic sense) representative of the north suburban area of Chicago. This demographic population is among the highest ranking in terms of education

it turned out that each subject pretested was in the upper 5% of the normal distribution of intelligence.

TABLE 4
RELIABILITY STUDY DATA

		Isomorphs			
Subject	Statistic	ISO 1	ISO 2	ISO 3	ISO 4
S1	\bar{x}	2.00	5.13	6.31	7.58
	s^2	0.00	0.13	2.16	0.00
	k	3	8	9	5
S2	\bar{x}	2.00	5.00	6.58	7.58
	s^2	0.00	0.00	1.00	0.00
	k	2	4	2	2
S3	\bar{x}	2.00	5.15	6.58	7.58
	s^2	0.00	0.06	1.00	0.00
	k	2	4	2	1
S4	\bar{x}	2.00	5.00	5.58	7.08
	s^2	0.00	0.00	0.00	0.25
	k	2	4	1	2

 $\bar{x} \equiv$ Mean $s^2 \equiv$ Variancek \equiv Cell Size

NOTE: All isomorphs taken from articept universe l_2 ; all subjects very intelligent adults.

TABLE 6
SUBJECT SAMPLING SCHEME

Sampling Level	Approximate Grade Range	Number of Subjects
Kindergarten	K	24
Primary	1-2-3	23
Intermediate	4-5-6	21
Junior High	7-8	20
High	9-10-11-12	23
Adult	-----	18
Total		129

and wealth in the entire United States.¹

In the case of non-adults, permission was first obtained from the proper school district administrators, and then secondly from the parents of the students in the potential subject pool. The reader may find these parental permission forms in appendix 2, pages 138-145. The number of subjects selected from each stratum was primarily determined by logistic considerations, i.e., time.

Since the study was primarily a developmental one, no attempt was made to control for such variables as sex, ethnic background, age (within strata), school performance, personality or the like. If these variables had been controlled for, it would have required inordinately great sample sizes to provide for reliable resolution of the data. In the absence of any consideration of higher order effects due to these variables, it was tacitly assumed that neither the Raven's Progressive Matrices nor the Logiktrak scores would be dependent upon these variables.

Procedure

Recall that the object of the Logiktrak is for the subject to discover the unknown articept in space 2^{1d} by generating optimal question-asking strategy. In practice,

¹From private communication with Professor Pierre De Vise, Department of Urban Sciences, University of Illinois at Chicago Circle, Friday, 21 March 1975.

during the administration of the Logiktrak, the subject is allowed to manipulate, in any fashion, toys representing scaled replicas of the elements of articept space 1_d . Since any incapability to enumerate the elements of articept space 1_d , or perceptually differentiate between the binary levels of each dimension in generator space $G_{2,2,2}$ would contaminate the results of the testing, potential subjects were first administered a mini-qualifying quiz to make certain that each subject could:

1. Count to eight
2. Distinguish between black and white toys
3. Distinguish between pyramids and cubes
4. Distinguish between large and small toys

Throughout the experiment, surface structure (lexicon) of the subject's language was always used in order to concretize the concepts involved so that no experimental artifacts could contaminate the procedure. Upon successful completion of the mini-qualifying quiz, subjects were then administered the appropriate form of the Raven's Progressive Matrices (RPM), a highly reliable, nonverbal, acultural measure of cognitive development.¹ Apparently, it occupies the same niche in the British Empire which the Stanford-Binet and the Wechsler occupy here in the United States. The Raven's Progressive Matrices was administered

¹ Jerry S. Carlson, "A Note on the Relationship Between Raven's Coloured Progressive Matrices Test and Operational Thought," Psychology in the Schools 10 (April 1973): 211-214.

according to the following scheme, exhibited in table 7.

The Logiktrak was then administered within two weeks of the date of administration of the Raven's. The Logiktrak was administered according to the following scheme: The Standardized Text was recited to the subject (this may be found on pages 5-8 in the Logiktrak Administration Manual).

A randomly generated articept universe 1_1 was selected. A randomized sequence of all four possible articeps from 2^1 was then administered to the subject. If the subject experienced undue difficulty, they did not go on to any problem from a two-dimensional articept universe 2^{12} .

Then, a randomly generated articept universe 1_2 was selected. A randomized sequence of articeps representing all four possible isomorphs was then administered to the subject. If the subject experienced undue difficulty, they did not go on to any problems from the three-dimensional articept universe 2^{13} .

Then, a randomized sequence of articeps representing all seven possible invariant isomorphs was administered to the subject. This is exhibited in illustration 25.

The subject asked the administrator questions in order to solve each problem. These questions were transcribed by the administrator and answered. Care was exercised that the exact question asked by the subject was faithfully transcribed, and the correct answer to that ques-

TABLE 7
ADMINISTRATION SCHEME FOR RAVEN'S PROGRESSIVE MATRICES

Sampling Level	Approximate Grade Range	Form of Raven's to be Administered	Test Setting	Usual Time of Test	Remarks
Kindergarten	K	Colored Progressive Matrices	individually	untimed, 15-30 minutes	Raven's Colored Progressive Matrices is usually administered individually, except that most children from age 8 up can take the test in small groups; range: ages 5-11 and defective adults
Primary	1-2-3	Colored Progressive Matrices	individually	untimed, 15-30 minutes	
Intermediate	4-5-6	Colored Progressive Matrices	group	untimed, 15-30 minutes	
Junior High	7-8	Standard Progressive Matrices	group	untimed, about 45 minutes	Raven's Standard Progressive Matrices; range: ages 8-05
High	9-10-11-12	Standard Progressive Matrices	group	untimed, about 45 minutes	
Adult	----	Standard Progressive Matrices	group	untimed, about 45 minutes	

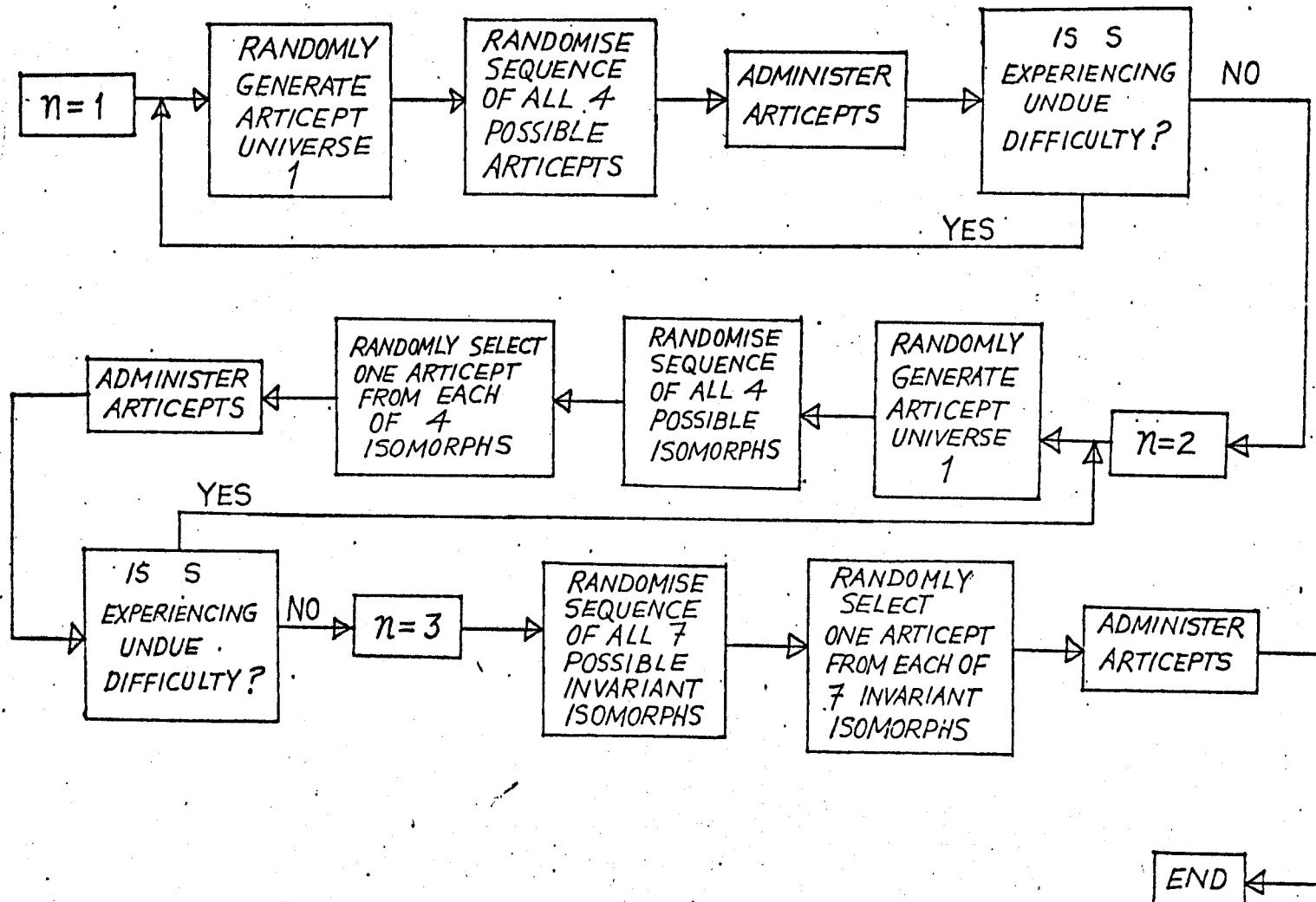


Illustration 25. Administration Scheme for Logiktrak

tion was given to the subject. The subject's questions were swiftly written down in the form of Boolean equations, as opposed to the laborious method of using traditional orthography.

During the course of the test, the subject was allowed free use of pencil and paper; and no time limit was imposed either for the asking of questions, or the solution of problems. Furthermore, the subject was allowed to ask the administrator to repeat any questions which the subject may have already asked, together with the answers to those questions. The reader may find a complete manual for the administration of the Logiktrak in appendix 4, pages 149-160.

Method of Analysis

For the reader's convenience, the two primary research hypotheses are spelled out completely here.

1. In the solution of problems of a given complexity, problem-solving strategies become more optimal with greater Piagetian ontogenesis, as measured by school grade level.
2. In the solution of problems of a given complexity, by subjects of equivalent Piagetian ontogenesis, as measured by school grade level, problem-solving strategies become more optimal with greater intelligence, as measured by Raven's Progressive Matrices.

The above research hypotheses are then transformed into tractable statistical hypotheses, which make the collected data amenable to analysis by the appropriate, com-

patible statistical tests. The collected data were representable in the following form exhibited in table 8.

The reader may observe, that for purposes of computation, only four manipulatable variables were of interest; stratum, (i.e., "grade" level--K, P, I, J, H, A), subject, articept space dimension, (I.E., $d=1,2,3$), and isomorph, (i.e., for $d=1$, $1 \leq i \leq 2$; for $d=2$, $1 \leq i \leq 4$; for $d=3$, $i=1,2,5,8,10,11,14$). The sole dependent variable, then, was the strategy parameter, λ .

For a given articept space dimension and isomorph, lower values of λ denote more optimal strategies. Since the younger children often discontinued the Logiktrak between dimensions, between articeps, or even within articeps, as when they could not attain the solution and gave up, there is a lot of missing data. This could not be helped, since it was felt that any kind of forced participation would not necessarily yield the subject's optimal strategy, and hence provide invalid data. This was in addition, of course, to the primary concern for the subject. In recognition of the salient problems which missing data would present for the computation of statistical means and the like, it was decided that an entirely new approach would have to be made to the analysis of the collected data.

δ -filters

In order to treat missing data, the following assumption

TABLE 8
SCHEMATIC REPRESENTATION OF THE COLLECTED DATA

		d = 1				d = 2				d = 3						
S	SPM	ISO	ISO	ISO	ISO	ISO	ISO	ISO	ISO	ISO	ISO	ISO	ISO	ISO	ISO	
		1	1	2	2	1	2	3	4	1	2	5	8	10	11	
															14	
TJ	58	1	1	2	2	2	6.585	7.585	5.585	4	10	18.807	23.977	20.129	22.884	28.639

Strata - A

tion was arbitrarily made:

Missing data will be treated as a failure to solve the problem represented by the unknown articept.

Now, consider the binary search strategy when the subject successfully reduces the number of tenable hypotheses by one half on succeeding questions. This particular strategy is quite an efficient yet facile one. For obvious reasons, it was dubbed Fool's Strategy and is shown in illustration 26.

Based upon the functional constructed to generate strategy parameters, the following values were found for Fool's Strategy, and are exhibited in table 9.

By considering Fool's Strategy to be an absolute referent strategy, one may then establish relativity¹ of the optimality of comparative strategies. Arbitrarily, the optimality of Fool's Strategy has been set to $\delta=1$. Thus, one has the following table, exhibited in table 10.

Thus, suppose the strategy parameter λ , in the solution of a given problem in a d-dimensional articept space

¹Certainly, the complexity of problems-to-be-solved varies with both articept space dimension (d) and isomorph (i). Thus, when speaking of the concept of relativity, the stipulation must be made for a given articept space dimension and isomorph. Via correction for the measure of articept complexity, it is theoretically possible to establish an inter-dimensional and inter-isomorph concept of relativity of the optimality of comparative solution strategies. In view of the various possible foundations which one may choose for constructing the measure of complexity, it would be premature to give "weights" to the different Logiktrak problems at this stage in the evolution of the theory.

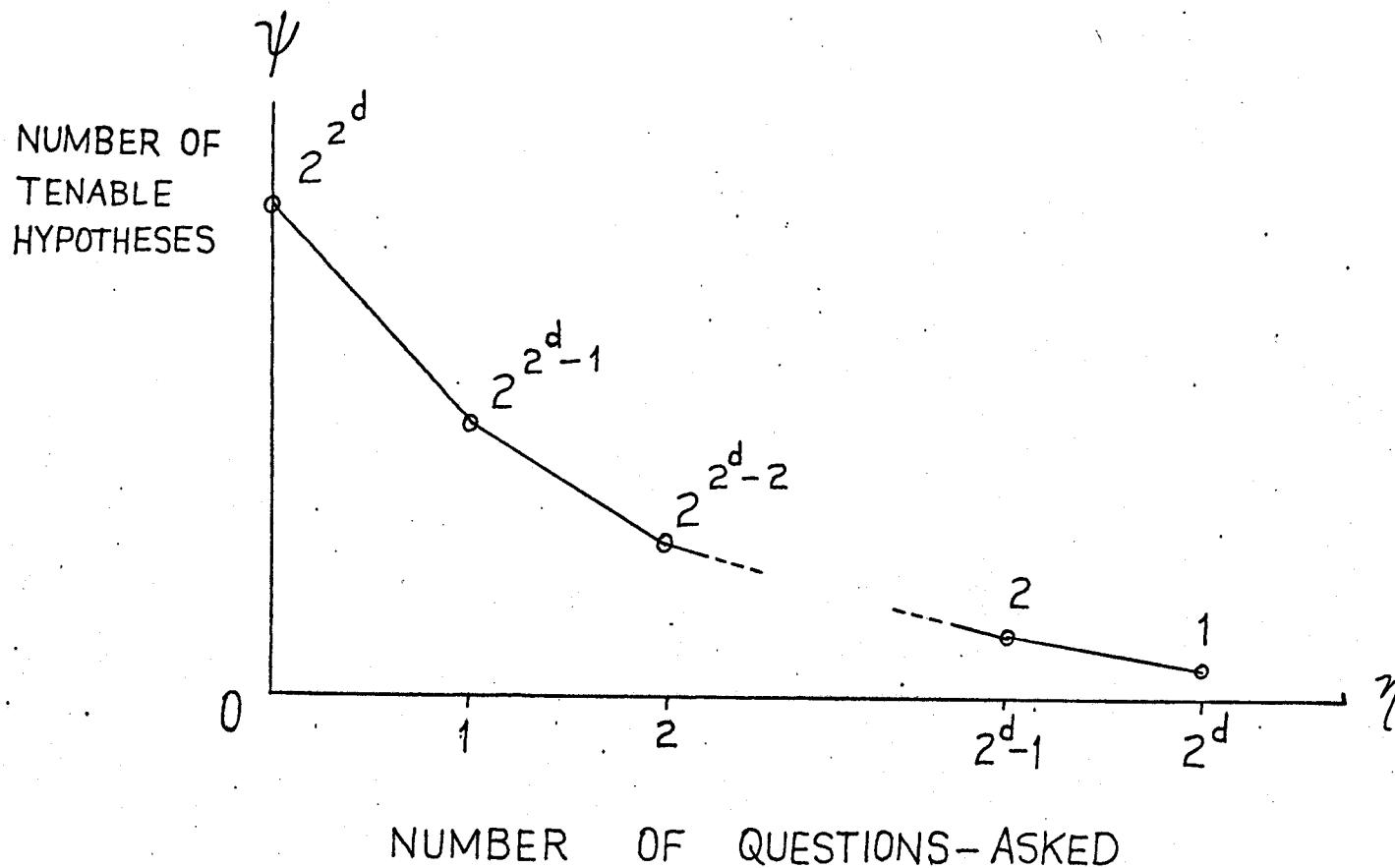


Illustration 26. Fool's Strategy

TABLE 9
STRATEGY PARAMETERS
FOR FOOL'S STRATEGY

d	1	2	3
λ	2	8	32

TABLE 10
 δ -FILTERS FOR FOOL'S STRATEGY

δ	d		
	1	2	3
0.50	1	4	16
1.00	2	8	32
1.50	3	12	48
2.00	4	16	64

2^1 were $\Sigma = \Sigma_{\psi}$.¹ Let $\Sigma_{d, \delta=1}$ represent the strategy parameter generated by the solution of that same problem via Fool's Strategy Ψ_F . Then, δ_{ψ} , the δ -value of strategy ψ is given by:

$$\delta_{\psi} = \frac{\Sigma_{\psi}}{\Sigma_{d, \delta=1}}$$

And thus strategy ψ would "pass" through the δ -filter for any $\delta \geq \delta_{\psi}$, and "fail" for any $\delta < \delta_{\psi}$. Thus, for example, one may see from table 10 that a strategy parameter of $\Sigma_{\psi} = 10$, in a two-dimensional articept space 2^1 , would pass through the δ -filters of both $\delta = 1.50$ and $\delta = 2.00$; however, it would fail the δ -filters of both $\delta = 1.00$ and $\delta = 0.50$. Succinctly, then, the more optimal the strategy ψ , the smaller the δ -filter through which it will pass. Lower values of δ_{ψ} thus reflect more optimal strategies. And δ_{ψ}^{-1} is essentially the ratio of how-many-times-more-optimal strategy ψ is than Fool's Strategy Ψ_F , for a given articept space dimension and isomorph.

Statistical Tests

Since the assumption of normality is at best a frail one in questions concerning the evolution of human intel-

¹The abstract representation of a given strategy will be the symbol ψ , together with the appropriate identifying sub-or-superscripts.

lignce, the decision was made to attack this problem via some form of appropriate non-parametric statistical test of hierachal order. This tact seemed appropriate, since it had been hypothesized essentially, that

$$\tilde{\psi}_A > \tilde{\psi}_H > \tilde{\psi}_J > \tilde{\psi}_I > \tilde{\psi}_P > \tilde{\psi}_K , \text{ where } \tilde{\psi}_A$$

represents the Adult (A) stratum optimal solution strategies, and so on. The " $>$ " operation in this case implies the relation "is more optimal than."

The first primary research hypothesis may be restated as follows: The hierachal order of solution strategy optimality will be ordered, from most optimal to least optimal: Adult (A), High School (H), Junior High School (J), Intermediate (I), Primary (P) and Kindergarten (K). Let

$P_A^*(\delta)$ represent the proportion of Adult (A) solution strategies passing a δ -filter, and so on.

Then the first primary research hypothesis is equivalent to the following:

$$H_1: P_A^*(\delta) > P_H^*(\delta) > P_J^*(\delta) > P_I^*(\delta) > P_P^*(\delta) > P_K^*(\delta)$$

Since the collected data afford only a sample observation proportion, $P(\delta)$, the observed order is necessarily but an inference concerning the real order. The statistical test chosen must reflect the possible error of sample observation and other statistical noise. In view of the complication introduced by an ordered hierarchy of

random variables, a specially-constructed Parity Test for Hierarchical Order was used.¹ For a given δ -filter, there are $6! = 720$ possible permutations of the set

$\tilde{\Psi} = \{ \tilde{\Psi}_A, \tilde{\Psi}_H, \tilde{\Psi}_J, \tilde{\Psi}_I, \tilde{\Psi}_P, \tilde{\Psi}_K \}$. The greater the statistical distance from H_1 of the observed permutation of set $\tilde{\Psi}$ as depicted by the corresponding $P_g(\delta)$, the greater the probability that H_1 is not true. And conversely, the smaller the statistical distance from H_1 of the observed permutation of set $\tilde{\Psi}$ as depicted by the corresponding $P_g(\delta)$, the greater the probability that H_1 is true. This statistical distance may be measured by the parity, N , or number of transpositions necessary to return the observed order to the order dictated by H_1 .

$\tilde{\Psi}_A > \tilde{\Psi}_H > \tilde{\Psi}_J > \tilde{\Psi}_I > \tilde{\Psi}_P > \tilde{\Psi}_K$. A partial parity table for a six-tiered hierarchical order under the Parity Test, is given in table 11.

The α error may thus be easily calculated from the formula

$$\alpha = \sum_0^{N_c} v_N$$

where N_c is the observed parity and $n=6$.

¹Harvey Jack Schiller, "On a Parity Test of Hierarchical Order," [Notes in Mathematics.] Chicago, Illinois, 1975.

TABLE 11

A PARTIAL PARITY TABLE FOR A
6-TIERED HIERARCHAL ORDER

Parity, N	Number of Possible States, V_N
0	1
1	5
2	14
3	26
.	.
.	.
.	.
15	1
Σ	720

The second primary research hypothesis presented a much easier problem. It may be restated as follows: For a given stratum, a statistically significant positive correlation will exist between $P_S(\delta)$, the observed proportion of solution strategies passing through a δ -filter for subject S, and U_S , the score of subject S on the appropriate form of Raven's Progressive Matrices.

Spearman's ρ for rank-order correlation was applied to the collected data; measuring the rank-order correlation between $P_\delta(1)$, the observed proportion of solution strategies passing through a 1-filter ($\delta = 1.00$) for subjects $S \in \delta$, where δ is a given strata and U_δ , the score of subjects $S \in \delta$, on the appropriate form of Raven's Progressive Matrices.

Summary

There are two primary research hypotheses: first, that problem-solving strategies become more optimal with increasing Piagetian ontogenesis, as measured by school grade level; and second, that for subjects of equivalent Piagetian ontogenesis, as measured by school grade level, problem-solving strategies become more optimal with increasing intelligence as measured by Raven's Progressive Matrices.

The design of the Logiktrak, the experimental paradigm used, consisted of a set of miniature replicas of the

elements of articept space 1_3 , together with an accompanying set of logic problems-to-be-solved, conveniently inscribed upon cards. These materials were pre-tested in order to ensure the reliability of the instrument.

A narrow stratum subject sampling scheme was used. All subjects were selected from one socioeconomic stratum. Since the study was developmental, there were six strata used, ranging from pre-school through adult, each stratum containing between 18 and 24 subjects.

All subjects were first pre-tested to ensure they could count to eight, distinguish black from white, pyramid from cube, and large from small. This mini-qualifying test ensured the absence of experimental noise. Successful subjects were then administered the appropriate form of the Raven's Progressive Matrices, and within two weeks time, the Logiktrak. The administration of the Logiktrak entailed a series of "Whodunit?" logic problems-to-be-solved, which the subject solved by generating "clues" to which the administrator provided the correct answers.

Logic problems-to-be-solved in the form of articeps were randomly generated from isomorphs of given complexity. The order of presentation of isomorphs within a given dimensional space was likewise randomly determined. The articept spaces used to generate the isomorphs, were sequentially, 2^{1_1} , 2^{1_2} , 2^{1_3} . Subjects were allowed free use of pencil and paper to minimize memory-induced noise; and

no temporal constraints were imposed to minimize temporal-induced noise. A Logiktrak Administration Manual was designed to facilitate the administration of the Logiktrak.

In order to accomodate the problem generated by missing data and distribution uncertainty, a new statistical procedure was developed. The first research hypothesis was examined via a new parity test; and the second research hypothesis was examined via Spearman's ρ .

CHAPTER IV

RESULTS AND IMPLICATIONS

The purpose of this experiment was to construct a model of cognitive structure in which the behavior of the logic primitives could be embedded, and to formulate a representation of that model which could be empirically tested.

The representation of the model, Logiktrak, is in the form of logic problems-to-be-solved. The model admits for the absolute measure of the structural complexity of problems-to-be-solved, and the absolute measure of the optimality of the subject's problem-solving strategies. In this way, the subject's "track of logic" may be analyzed. In this chapter, the results of the empirical testing will be presented and discussed.

For the reader's convenience, both primary research hypotheses and the respective statistical tests used to affirm or disaffirm those hypotheses are repeated here.

First, in the solution of problems of a given complexity, problem-solving strategies become more optimal with greater Piagetian ontogenesis, as measured by school grade level; second, in the solution of problems of a given complexity, by subjects of equivalent Piagetian ontogenesis,

as measured by school grade level, problem-solving strategies become more optimal with greater intelligence, as measured by Raven's Progressive Matrices.

The first primary research hypothesis was examined by a new parity test developed by this author to test the validity of hierachal structures. The second primary research hypothesis was examined by Spearman's ρ .

To analyze the data pertaining to the first primary research hypothesis, the proportions of solution strategies passing through various δ -filters was computed for each of the six strata (Adult, High School, Junior High, Intermediate, Primary, Kindergarten). This data is summarized in table 12.

A quick glance at the collected data in table 11 reveals that the observed parity $N_c = 0$, for all calculated δ -filters. Thus, $\alpha = (6!)^{-1} = \frac{1}{720}$, or $\alpha = 0.0013888\dots$. This is an α error of about 0.10%. One may easily see that the first primary research hypothesis may be accepted for all practical α .

To analyze the data pertaining to the second primary research hypothesis, the proportion of solution strategies passing through a 1-filter ($\delta = 1.00$) was computed for each subject. This data was then rank-correlated against each subject's respective Raven's score, using Spearman's ρ , for each stratum separately. The results of statistical testing concerning the second primary research hypothesis

TABLE 12

PROPORTIONS OF SOLUTION STRATEGIES
PASSING THROUGH VARIOUS δ -FILTERS

Strata	0.50	1.00	1.50
A	0.256	0.771	0.890
H	0.197	0.667	0.879
J	0.154	0.450	0.567
I	0.093	0.333	0.504
P	0.042	0.300	0.467
K	0.033	0.169	0.256

are summarized in table 13. Again, one may easily see that the second primary research thesis cannot be accepted for any practical α .

Implications

The implications of the work within this dissertation may be divided into four basic areas: pure mathematics, comparative and developmental ethology, Piagetian measures of cognitive developmental ethology, and curricular design.

In addition to providing experimental proof of the acceptance of the first primary research hypothesis, and the rejection of the second; evidence has now been exhibited for the construction and development of absolute measures of Boolean structure, and absolute measures of problem-solving-strategy, together with their attendant implications of Piagetian theory of the ontogenetic evolution of cognitive structure. The existence of such measures provides conceptual foundation of a Theory of Relativistic Intelligence.

It would appear that human knowledge has received the greatest impetus in its evolution, from the proof of existence of relativistic constructs. For example, Newtonian physics had reached a conceptual cul-de-sac in the waning years of the 19th century. Many paradoxes abounded, which could not be resolved solely upon the basis of then-

TABLE 13
STATISTICAL SUMMARY OF SECOND
PRIMARY RESEARCH HYPOTHESIS

Strata	n	$D^2 + T$	Level of Statistical Significance
A	18	1,053	None
H	23	1,366	None
J	20	1,254	None
I	21	1,768	None
P	23	1,678	None
K	24	1,964	None

existing models.¹ It remained for Albert Einstein, in 1905, to revolute the theoretical construct of relativity, by deriving the equation of energy and matter ($E=MC^2$), and thus penetrate the conceptual barrier imposed by then-existing homocentric thought.

Einstein's monumental feat marked a great discontinuity in the phylogenetic evolution of human intelligence. Ultimately, Einstein's Theory of Relativity has generated a culture-shock of such magnitude, that Man's perspective of the relationship between himself and his world, has been radically transformed--prosthetic evolution.

The kernel of this concept is simply this: That measurement is yet artificial so long as it is relative (norm-referent); it can only transcend these confines when it has become absolute (criterion-referent), thus expanding the scope of Man's world, then his thought, and ultimately, Man himself. Such a construct of relativity then necessarily provides one with ratio scales of measurement, generating a conceptual foundation from which yet unborn models may emerge in conceptoclastic fashion. It is the contention of this dissertation, that the Theory of Relativistic Intelligence promulgated here, provides such a conceptual foundation. The capability to provide absolute, i.e., cri-

¹ The term model is stipulated here to mean a system of relations satisfying a set of axioms, so that the axioms may be interpreted as true statements about the system.

terion-referent, measurement of the various theoretical constructs relating to Piagetian theory of the ontogenetic evolution of cognitive structure, similarly allows one to penetrate the conceptual barrier imposed by existing homocentric thought.

It is the powerful ramifications of such a Theory of Relativistic Intelligence, which shall now be discussed, together with their immediate implications to Developmental Education.

Pure Math

The construction of absolute measures of complexity of Boolean structures may introduce new dimension to the areas of automata theory (Switching Algebras), Boolean Algebra, combinatorial topology and graph theory. A specific discussion of these new dimensions is beyond the scope of this dissertation; however, one may briefly sketch the following concepts.

The absolute measure of complexity of Boolean structures allows one to define a measure function over spaces whose elements may be represented by Boolean structures. In particular, the logic primitives of cognitive structures may now theoretically be analyzed and synthesized. This is especially important in the case of cognitive structures which are non-human. Thus, one may mathematically represent the evolution of comparative and developmental

cognitive structures.

Comparative and Developmental Ethology

The Logiktrak and other like-Piagetian measures of the development of cognitive structure, may be adapted for nonverbal, acultural populations. Such experimental paradigms would be capable of differentiation on the intra-taxonomic level (the usual measurement of individual differences), as well as differentiation on the inter-taxonomic level. This would provide a pan-taxonomic scale, yielding a valid hierarchy of the phylogenetic evolution of cognitive structures with respect to the existence of logic primitives.

For example, one could require a perfect Piagetian measure to be capable of differentiating between the ontogenetic evolution of the cognitive structure of two distinct individuals from a given specie (individual difference); and also be capable of differentiating between the phylogenetic evolution of the cognitive structure of individuals representative of two distinct species. The former application would certainly have great impact on developmental ethology; while the latter application would certainly have great impact on comparative ethology. Such a device may ultimately be capable of answering the questions: "Which of these humans is the more intelligent?" and "Which is the more intelligent, a chimpanzee or a dolphin?"

The matter of phylogenetic evolution in processes of learning is now only beginning to be systematically explored.¹ However, it is not unlikely that as one proceeds up the scale of phylogenetic evolution, the development of the logic primitives similarly becomes more advanced. And thus the evolution of logic primitives offers a natural measure of the phylogenetic evolution of life on Earth.

Via consideration of the rate of phylogenetic evolution of cognitive structures in both Pongidae (orangutan, gorilla, and chimpanzee) and Hominidae (man), one may be able to then calculate the elapsed time since the bifurcation of proto-Pongidae and proto-Hominidae.² This would provide data ancillary to that which is found by the exciting, recent methods of comparing macromolecular structures in the two species.³

While reviewing the work of Piaget and other developmentalists, one cannot help but be singularly impressed by the lack of connection and inference concerning human be-

¹M. E. Bitterman, "The Comparative Analysis of Learning," Science 188 (1975): 699-709.

²Pongidae is the zoologic family containing the great apes. Hominidae is the zoologic family containing Man. About twenty million years ago, during the geological epoch known as the Miocene, the suborder Catarrhini of the order Primates further fragmented into the two families, Pongidae and Hominidae.

³M. C. King and A. C. Wilson, "Evolution at Two Levels in Humans and Chimpanzees," Science 188 (1975): 107-116.

havior (the development of cognitive structure in particular), which was drawn from the corresponding generative behaviors of the higher infrahuman primates. There appears to be little reference in the literature, to any of the work in the comparative and developmental ethology of lower life-forms.¹ Such a situation bodes poorly for graduate education in the human behavioral sciences.

It is hardly prudent to assume that one has the broad scope necessary to properly integrate and understand the problems of developmental psychology, together with its implications to developmental education, without having studied the comparative and developmental ethology of the lower life-forms. Such a tenet has far-reaching implications to graduate education in the human behavioral sciences. The farthest extent many graduate programs in the human behavioral sciences currently go toward this end, is perhaps an elective survey course in comparative psychology. The integration of animal ethology and child behavior, for instance, has to a great extent been the effort of workers in zoology "reaching" across to psychology--certainly not the reverse situation. And yet, the ethological study of child behavior

¹A survey of 30 bibliographies of work representative of Piaget and other developmentalists revealed that, out of a total of 901 bibliographic references, 2 were direct citations of work in the comparative and developmental ethology of lower life-forms.

is integroplexed¹ to the work of Piaget and other developmentalists.

Piagetian Measures of Cognitive Development

The ontogenetic evolution of the logic primitives in man may now be viewed from a Piagetian perspective, to better fit in with the existing matrix of human knowledge.

Cursory examination of both the theoretical model derived for this dissertation, and also the experimental data, lends itself to the following empirical conclusion: The logic connectives of the predicate calculus emerge in the following order in humans: (i) existence (\exists), (ii) negation (\neg), (iii) conjunction (\wedge), disjunction (\vee), condition (\rightarrow) and (iv) bicondition (\leftrightarrow).

An analysis of Piaget's logical operations or schema², reveals that Piaget based the conceptual foundation of his work upon the assumption of Abelian structure; that is, Piaget's schema commute.³ The ignorance of the existence of non-Abelian logic structures should not be continued. Indeed, the assumption of Abelian structure for logic primi-

¹The term integroplexed is stipulated here to mean completely interwoven and interdependent.

²William M. Bart, "A Comparison of Premise Types in Hypothetico-Deductive Thinking at the Stage of Formal Operations," Journal of Psychology 81 (May 1972): 45-51.

³Two operators A and B may be said to commute or lie embedded within an Abelian structure if their sequence of application is irrelevant; that is, if $AB=BA$.

tives is an unrealistic one.¹ It presupposes simple, ergodic behaviors of logic primitives, inferring that learning does not occur.

The benchmark of evolution has always been the adaptive modification of behavior, or learning. Since the existence of commutativity essentially reflects a more simple

¹This author has developed numerous Piagetian tasks where Abelian behaviors do not appear to exist. For example, very young children often persist in trying to pass a knot through a hole from one direction, when it obviously failed to pass through from the obverse direction. Similarly, it is not obvious to very young children that the distance from A to B is identical to the distance from B to A, along the same path.

These particular instances are exemplars of a more generalized genus of ontogenetically-derived behaviors which this author has termed anisotropic. Essentially, the child has not yet "learnt" that various properties of space are not direction dependent. Strangely enough, the integration of the great body of human knowledge in the mathematical sciences requires that one "revert" to the cognitive axiom (or logic primitive derivative) that certain properties of space may not necessarily be isotropic. Further exemplars of other genuses of ontogenetically-derived behaviors may be provided by the operations represented by the group of rotations of a rigid body (rotors), which are not Abelian, and surprisingly, few adults (a survey revealed $p < 10^{-2}$) can correctly solve rotor problems.

In addition to the existence of non-Abelian logic primitives, one may also consider the existence of logic primitives which lack other algebraic properties, i.e., associativity, distributivity, et al.

The classic Piagetian chemistry experiments where the subject had to solve for the correct reactants in order to precipitate a given color, have been modified by this author to provide for the existence of logic primitives which are embedded in sub-algebraic structures.

This is not to say that Piaget's foundations are disputed entirely; merely that if Piagetian theoretic models are to gain greater predictive and construct validity, greater effort must be made toward the logical axiomization of the models. Presently, they are not mathematically rich enough to allow the evolution of developmental theories based upon other-than-normal human ontogenetic logics.

intelligence than its non-existence, the degree of commutativity in the logic primitives of a cognitive structure (intelligence) should be a constructurally valid measure of evolution.

One may then consider the dependence of commutativity upon evolution. It may be that the emergence of non-Abelian schema in certain humans may be the primitive differentiation of *Homo sapiens* into the next phase of speciation. Certainly, one must consider the possibility that speciation may first occur in mutation of the logic primitives of human cognitive structure.

Via the development of absolute measures of complexity of Boolean structures, one may determine a criterion-referent measure of the complexity of a problem-to-be-solved, provided that problem may be embedded (representation) within a logic structure. Via the development of absolute measures of problem-solving strategy, one may determine a criterion-referent measure of the optimality of problem-solving strategy completely independent of group norms. The power afforded by absolute measurement allows one to construct relativistic models of intelligence.

One may then construct logic games of strategy, of known structural complexity, where information is controlled, capable of play by one or more players. Such games would provide operational measure of the development of cognitive structure.

One could construct semantic logic puzzles of known structural complexity, whose semantic content may be varied to minimize cultural noise (culture-dependent artifact and contamination).

One could construct artificial theorems embedded within equally artificial logics, of known structural complexity, amenable to ontogenetic analysis of the process of logical "proof."

And one could also construct absolute measures to provide ontogenetic analysis of Piaget's classic experimental criterion for scientific hypothesis-testing, together with a more detailed ontogenetic analysis of the logic primitives of the simple predicate calculus than has been attempted in this dissertation.

In brief summary, Piagetian measures of cognitive development would provide absolute, i.e., ratio scale, criterion-referent, measures of problem-solving strategy. The paradigm itself must be representable by Boolean structure, so that an isomorphism may be established between the logic primitives of human cognitive structure and the Boolean structure representation of the Piagetian device. Via this assumed isomorphism, the complexity of the Boolean structure may be determined, and the problem-solving strategy of the human cognitive structure may thence be "tracked" and measured for optimality. Admittedly, this is but a rough sketch of an ambitious endeavor; however, it may prove of some

value to note the directions in which this research must inevitably follow.

The construction of a Piagetian criterion-referent test, appropriately normed for ontogenetic development in children, would prove of immeasurable aid in differentiating certain learning-disabled children. This reason alone should provide sufficient incentive for further work in this area. The development of criterion-referent, diagnostic problem-solving tests, appropriately normed for ontogenetic development of cognitive structure, would differentiate between retarded ontogenesis of the logic primitives, as opposed to compiler-interface dysfunction¹, which appears common to most learning disabilities. Usually the child is of at least normal ontogenetic cognitive development; however, there is either perceptual (input) dysfunction or sensory-motor (output) dysfunction, impairing the child's capability to communicate with the environment. It is common knowledge in Illinois' educational circles, that funds would be easily available for valid educational research promising such a differentiating device.

Certainly, as rejection of the second primary research

¹The term compiler-interface dysfunction is stipulated here to mean dysfunction at the hierachal level of the cognitive structure which is the boundary between the internal representation of the logic primitives and their schema derivatives, which is essentially cognition; and the sensory-motor inputs and outputs which provide channels for the flow of information input and output.

hypothesis¹ exhibited, one cannot assume any statistically significant common loading of Raven's Progressive Matrices and the Logiktrak. Thus, each instrument appears to measure different perspectives of that elusive construct, intelligence.

Curricular Design

The affirmation of the first primary research hypothesis² has essentially two basic and immediate implications to curricular design: First, that curricula are ontogenetic-dependent; and second, that human cognitive structure, as evinced through the ontogenetic evolution of logic primitives in humans, does not approximate an asymptote³ at the onset of puberty, as postulated by Piaget⁴, but continues to develop, well into the third and fourth decade of life.

If these results may be provisionally accepted pending further life-span research, then the following considerations must be seriously entertained. From the ontogenet-

¹The second primary research hypothesis is that for subjects of equivalent Piagetian ontogenesis, problem-solving strategies become more optimal with increasing intelligence, as measured by Raven's Progressive Matrices.

²The first primary research hypothesis is that problem-solving strategies become more optimal with increasing Piagetian ontogenesis.

³The term asymptote is stipulated here to mean a constant level, exhibiting no further ontogenetic evolution.

⁴V. M. Bondarovskaya and M. L. Smul'son, "Some Features of the Development of Problem-Solving Strategies," *Voprosy Psichologii* 19 (September 1973): 58-65.

ic-dependence of curricula and the positive monotonic ontogenesis of human cognitive structure, it follows that the educational timetable imposed by Society is severely awry; students should be encouraged to attend universities and seats of higher learning, i.e., graduate and professional schools, when they are older, and their cognitive structures have more greatly evolved, since they can then better take advantage of the precious opportunity, than in the post-pubescent period when curricula are less efficiently learned.

Consider the example of medical education. Medical schools do not usually prefer to admit students who are beyond the age of about 26. Should the results of this dissertation be considered valid, it would appear that medical school admissions policies are logically specious. It also follows that curricula should be sequenced, organized, and presented with respect to logic structure rather than content.

Since information of a logic structure may be integrated most optimally by logic primitives of a cognitive structure of slightly greater complexity, it immediately follows that adults can integrate structure more optimally than content; while younger children can integrate content more optimally than structure. With greater cognitive structure, the integration of logic structures becomes more optimal with increasingly greater logic structure of the in-

formation-to-be-learned.

Thus, more evolved cognitive structures integrate conceptual information more optimally than elemental information. Curricula designed for adults should emphasize conceptual foundations, together with patterns (unifying principles) for embedding the more elemental information into an integrated matrix of concepts. Curricula designed for adults must necessarily then emphasize structure; for the more highly evolved logic primitives of the adolescent's or adult's cognitive structure cannot optimally integrate content which is devoid of structure. Younger children, not having such greatly evolved cognitive structures, cannot integrate the logic structures of curricula which are more complex (or of greater intelligence) than their own. Thus curricula designed for younger children must necessarily then emphasize content.

Thus, for example, one would venture to say that contrary to most educational research, adults can learn a second language even more efficiently than a child can, provided the language curriculum is constructed with respect to its logic structure, i.e., syntax. Children learn by acquiring content, i.e., lexicon, since they cannot integrate the logic aspects of the new language.

Similarly, to provide further example, curriculum in the life-sciences can most optimally be integrated by providing advance organizers at various hierachal levels of

complexity to structure the information into "hypertexts of progressively deeper" material. The recently revised Britannica 3 is vaguely structured upon such a design.¹ Britannica 3 is partitioned into three levels of complexity: propaedia--outline of knowledge, micropaedia--basic knowledge and macropaedia--knowledge in depth. Unfortunately, the vast preponderance of curricula in the life-sciences emphasizes content, instead of structure. The absence of a continuous, underlying connecting logic is often readily apparent. Although no self-respecting life-science curriculum would dare omit discussion of the scientific method of hypothesis-testing (the very kernel of Piaget's work), few curricula are structured to present their contents in such a fashion that all information-to-be-integrated is analyzed via the scientific method of hypothesis-testing.

A particularly observable example of this hypocrisy may be found in the study of anatomy. Anatomy may be viewed from either a comparative or developmental perspective, or a combination of the two.

Now, since the morphology of an anatomical structure has, over the course of ages, evolved to its present state because that particular morphology was best suited to the behavior of that particular taxon, it follows that compara-

¹F. S. Pierce, "Launching the New Britannica," The Chicago Guide, March 1974, pp. 100-105.

tive and developmental anatomy might comply with the scientific method of hypothesis-testing by analyzing why the particular anatomical structures possess the morphology which they do, and what morphological changes might be expected to occur if the effect of certain evolutionary variables were to be considered.

Such an educational attack would probably require the development of Monte Carlo simulators of biological processes, so that different evolutionary trajectories could be accelerated, say about 10^{11} -fold, or at the rate of one million years per minute (elapsed real-time). Thus, the process of evolution of a morphologic structure and its attendant behavior could be simulated on a computer's CRT (Cathode Ray Tube or "TV Screen") so that one could experiment with the evolutionary process by manipulating the underlying variables and observing the resulting changes. Such a "simu-lab" (simulation laboratory) would provide for experimentation and discovery learning by allowing the student to actually see "Why. . . ?" and "What if . . . ?"

The emphasis of a curriculum structured upon the tenets of the scientific method of hypothesis-testing must necessarily be upon the twin preludes to Discovery--"Why . . . ?" and "What if . . . ?" It is no mere accident that as the proportion of a teacher's questions increase from "What (predicate) . . . ?" up through "Why . . . ?" to "What if . . . ?," the quality of teaching, and hence, learning, correspondingly

increases.¹

Recently, in medical science education in particular, an attempt has been made to construct criterion-referent diagnostic problem-solving tests of student educational progress.² Since diagnostic problem-solving requires a cognitive synthesis of integrated information, such tests may ultimately prove, in the long run, to be the most valid indicators of learning. Unfortunately, presently they appear to be quite primitive.

In addition, the focus of these tests has primarily been upon patient-management. That is, the student may or may not be given the previous medical history of a patient, then the student is required to properly diagnose an existing symptomology of that patient, and decide the appropriate strategy of patient-management for the patient.

Anthropomorphic computerized simulators have recently been under development to provide both a plethora of variables for diagnosis, and realistic medical environment for diagnosis.³

Admittedly, embedding a problem in medical diagnosis

¹Harvey Jack Schiller, "On the Kinds of Questions Kinds of Teachers Ask," [Notes in Education.] Chicago, Illinois, 1974.

²C. H. McGuire and F. H. Wezeman, Use of Simulation in Instruction and Evaluation in Medicine (World Health Organization Report 74.171).

³U. S., Department of Health, Education, and Welfare, Research and Development: Advances in Education, 1968.

within a logic structure so that the techniques developed in this dissertation may be applied, may not be possible at this stage in the evolution of the Theory. The embedding process is clearly the crux of the whole matter. It does appear though, that all problems in medical diagnosis may ultimately be reduced, mathematically, to a multistage decision algorism--or super-flo-graph. The graph itself, however, does possess an inherent logic structure; and is therefore mathematically tractable to analysis. The structural complexity of the graph, together with the optimality of any solution strategy of the diagnostic problem represented by the graph, may then be determined.

What has been sketched out for the reader is an existence proof of a particular curricular design (a solution strategy) for medical science education (a problem-to-be-solved). It is the most urgent suggestion of this author, that medical science education seriously consider the optimality of this approach. The development of such a medical science curriculum could not be immediate; but it can never be accomplished if it is not soon begun.

Some final thoughts concerning the direction of future Piagetian impact upon curricular development are in order. Although the Piagetian model has been the object of an exacting mathematical critique in this work, its viability should not be derided. It is a profound synthesis of singular thought concerning the evolution of intelligence. It is a

fecund theory generating much debate, as evidenced by the recent contrasting articles of Steiner¹, who argues that cognitive structure really does exist; and Phillips and Kelly², who attempt to debunk the Piagetian concept of ontogenetic evolution of cognitive structure. Certainly, as this dissertation has endeavored to show, the ontogenesis of intelligence in man is directly manifested in the increasing structural complexity of the evolving logic primitives. Ideally, one would expect that if curricula could be structured upon the concept of hypothesis-testing, the knowledge of the structural complexity of the student's logic primitives would enable curricula to be modified so that Education could really provide individualized instruction in the purest sense of the word.

Summary

The first research hypothesis was affirmed, namely, that problem-solving strategies become more optimal with increasing Piagetian ontogenesis.

The second research hypothesis was not affirmed, namely, no statistical proof was exhibited that for subjects of equivalent Piagetian ontogenesis, problem-solving strat-

¹Gerhard Steiner, "On the Psychological Reality of Cognitive Structures," Child Development 45 (December 1974): 891-900.

²D. C. Phillips and Mavis E. Kelly, "Hierarchical Theories of Development in Education and Psychology," Harvard Educational Review 45 (August 1975): 351-375.

egies become more optimal with increasing intelligence, as measured by Raven's Progressive Matrices.

The affirmation of the first primary research hypothesis has essentially two basic and immediate implications to curricular design: first, that curricula are ontogenetic-dependent; and second, that human cognitive structure, as evinced through the ontogenetic evolution of logic primitives in humans, does not approximate an asymptote at puberty, as predicted by Piaget. Among the many and varied applications of this research are: the construction of Piagetian measures of intelligence.

The applications of this research may be subsumed under a Theory of Relativistic Intelligence. The theory provides for absolute measures of the complexity of problems-to-be-solved which can be embedded within a Boolean structure. The theory also provides for absolute measures of the optimality of problem-solving strategies. Such functions are readily extendable to providing absolute measures of the complexity of games, both probabilistic and deterministic; and absolute measures of the optimality of the solution strategies to those games. Piagetian measures of hypothesis-testing behavior may thus be embedded within algebraic structures, turned into games of logic, and used as differentiating devices to measure intelligence. These Piagetian measures would necessarily then, provide absolute measurement of the complexity of the problem representation

together with absolute measurement of the optimality of the solution strategy used.

The application of the theory may be extended to pure Mathematics in a straight-forward fashion; in particular, to the areas of the complexity of Boolean structures, the optimality of solution strategies in game theory, and probabilistic analyses of multi-valued logics.

The proof of existence of a relativistic model of intelligence implies that one may construct a pan-taxonomic scale of intelligence based upon Piagetian measures. The behavior of the logic primitives with respect to the phylogenetic spectrum of evolution may then provide a measure of evolutionary distance.

In order to gain a more "inclusive" perspective of the phylogenetic evolution of intelligence, and more particularly, the logic primitives, it is suggested that graduate programs concerned with the human behavioral sciences also include extensive study of comparative and developmental ethology, since human behavior can best be understood in the context of the primitive proto-behaviors in lower life-forms from which human behaviors were derived.

Secondary research hypotheses concomitantly validated during the experiment, indicated that the sequence of ontogenetic evolution of the logic primitives in humans may be predicted by the monotonic increase in the measure of complexity of the isomorph representing each logic primitive.

The order of emergence of the logic primitives is seen to be: existence (\exists), negation (\neg), conjunction (\wedge), disjunction (\vee), condition (\rightarrow), and bicondition (\leftrightarrow).

The observation is also made that Piaget mistakenly assumed an Abelian or commutative structure to describe the logic used by young children. In fact, non-Abelian logics appear to predominate in young children. The assumption of Abelian structure also precludes the existence of learning behaviors in automata.

The development of the theory also admits the construction of semantic logic puzzles, Logiktrak II, which appear to entirely eliminate the experimental noise contaminating the resolution in measuring individual differences; the construction of measurable artificial logics via which one may study the process of proof; and the construction of Piagetian measures represented by other algebraic structures to study the process of hypothesis-testing in the ontogenetic evolution of the scientific method of enquiry in humans. It is suggested that such measures may be a valuable differentiating device in the diagnosis of certain learning disabilities.

In view of the evidence exhibited that the ontogenetic evolution of the logic primitives does not appear to asymptote out at puberty, it would appear that the educational timetable imposed by society is suboptimal. Information can be more optimally integrated by human cognitive structures

at later age than the post-pubescent period now voguishly misappropriated.

Also in corollary to the finding that curricula are ontogenetic-dependent, it would appear that curricula should be designed to emphasize conceptual as opposed to elemental information in order to be optimally integrated into the more developed cognitive structure of adults. The example of biological sciences curricula is cited; where minimal appeal is made to a hypothesis-testing hypertextual approach. The bulk of the information is structured elementally rather than conceptually, and rarely is the positive evolutionary advantage of the evolving morphological or behavioral structure discussed. It is suggested that simu-labs be constructed to provide simulation of the evolutionary process in a laboratory environment.

Finally, it is proposed that the technology developed in this research, be applied to the further development of measures of diagnostic problem-solving in medical education.

CHAPTER V

SUMMARY

Problem

The purpose of this experiment was to construct a model of cognitive structure in which the behavior of the logic primitives could be embedded, and to formulate a representation of that model which could be empirically tested.

Construction and Representation of the Model

In order to facilitate the construction of the model, the existence of a certain algebraic structure was postulated. By defining an equivalence relation over this structure, the set of all subsets generated by combinatorically varying the contents of this algebraic structure, was partitioned into isomorphs or supersets, each of which then contains structurally equivalent articepts or subsets.

The representation of the model is in the form of logic problems-to-be-solved. The model admits for the absolute measure of the structural complexity of problems-to-be-solved, and the absolute measure of the optimality of the subject's problem-solving strategies. In this way, the "track of the subject's logic" may be analyzed.

The Logiktrak is generically quite similar to the

popular Parker Brothers game, Clue--but without the dice.

The solution to the "Whodunit?" is in the form of artefacts or artificial concepts comprised of canonical Boolean forms. The binary dimensions used in the Logiktrak are color, form, and size; as opposed to Who? (the murderer), Where? (the scene of the crime), and How? (the murder weapon) in Clue.

In view of the rich algebraic structure provided by the Logiktrak, certain problems-to-solve are more complex than others. The various equivalence classes of problems-to-solve are termed isomorphs. And the complexity of isomorphs varies from isomorph to isomorph. In order to develop an absolute scale of complexity for the various isomorphs, mathematical functions which provide absolute measure of the structural complexity of isomorphs, were constructed. Absolute, criterion-referent measurement of complexity is a significant advance upon the usual relative, norm-referent measurement of complexity.

Analysis reveals that Piaget's work in the ontogenetic evolution of the scientific method of hypothesis-testing in human logic may quite succinctly be represented by the algebraic structure of space 2^1_2 ; and that the sequence of ontogenetic emergence of the logic primitives in humans may be predicted by the measures of structural complexity derived for this work.

After constructing a representation of the model, and deriving absolute measures of the structural complexity of

the problems-to-be-solved, absolute measures of the optimality of problem-solving strategies necessarily had to be developed. This was in response to the necessity of providing a differentiating device which allows one to analyze the "track of a subject's logic" rather than only its resultant conclusions.

Method

There are two primary research hypotheses: first, that problem-solving strategies become more optimal with increasing Piagetian ontogenesis, as measured by school grade level; and second, that for subjects of equivalent Piagetian ontogenesis, as measured by school grade level, problem-solving strategies become more optimal with increasing intelligence as measured by Raven's Progressive Matrices.

The design of the Logiktrak, the experimental paradigm used, consisted of a set of miniature replicas of the elements of articept space Γ_3 , together with an accompanying set of logic problems-to-be-solved, conveniently inscribed upon cards.

The Logiktrak had first been pre-tested in order to ensure the reliability of the instrument.

A narrow strata subject sampling scheme was used. All subjects were selected from one socioeconomic strata. Since the study was developmental, there were six strata used, ranging from pre-school through adult, each strata contain-

ing between 18 and 24 subjects.

All subjects were first pre-tested to ensure they could count to eight, distinguish black from white, pyramid from cube, and large from small. This mini-qualifying test ensured the absence of experimental noise. Successful subjects were then administered the appropriate form of the Raven's Progressive Matrices, and within two weeks time, the Logiktrak. The administration of the Logiktrak entailed a series of "Whodunit?" logic problems-to-be-solved, which the subject solved by generating "clues" to which the administrator provided the correct answers.

Logic problems-to-be-solved in the form of articepts were randomly generated from isomorphs of given complexity. The order of presentation of isomorphs within a given dimensional space was likewise randomly determined. The articept spaces used to generate the isomorphs, were sequentially, $2^1, 2^{1_2}, 2^{1_3}$. Subjects were allowed free use of pencil and paper to minimize memory-induced noise; and no temporal constraints were imposed to minimize temporal-induced noise. A Logiktrak Administration Manual was designed to facilitate the administration of the Logiktrak.

In order to accomodate the problem generated by missing data, and distribution uncertainty, a new statistical procedure was developed. The first research hypothesis was examined via a new parity test; and the second research hypothesis was examined via Spearman's ρ .

Results and Implications

The first research hypothesis was affirmed, namely, that problem-solving strategies become more optimal with increasing Piagetian ontogenesis.

The second research hypothesis was not affirmed, namely, no statistical proof was exhibited that for subjects of equivalent Piagetian ontogenesis, problem-solving strategies become more optimal with increasing intelligence, as measured by Raven's progressive Matrices.

The affirmation of the first primary null hypothesis has essentially two basic and immediate implications to curricular design: first, that curricula are ontogenetic-dependent; and second, that human cognitive structure, as evinced through the ontogenetic evolution of logic primitives in humans, does not approximate an asymptote at puberty, as predicted by Piaget. Among the many and varied applications of this research are: the construction of Piagetian measures of intelligence.

The applications of this research may be subsumed under a Theory of Relativistic Intelligence. The theory provides for absolute measures of the complexity of problems-to-be-solved which can be embedded within a Boolean structure. The theory also provides for absolute measures of the optimality of problem-solving strategies. Such functions are readily extendable to providing absolute measures of the

complexity of games, both probabilistic and deterministic; and absolute measures of the optimality of the solution strategies to those games. Piagetian measures of hypothesis-testing behavior may thus be embedded within algebraic structures, turned into games of logic, and used as differentiating devices to measure intelligence. These Piagetian measures would necessarily then, provide absolute measurement of the complexity of the problem representation together with absolute measurement of the optimality of the solution strategy used.

The application of the theory may be extended to pure Mathematics in a straight-forward fashion; in particular, to the areas of the complexity of Boolean structures, the optimality of solution strategies in game theory, and probabilistic analyses of multi-valued logics.

The proof of existence of a relativistic model of intelligence implies that one may construct a pan-taxonomic scale of intelligence based upon Piagetian measures. The behavior of the logic primitives with respect to the phylogenetic spectrum of evolution may then provide a measure of evolutionary distance.

In order to gain a more "inclusive" perspective of the phylogenetic evolution of intelligence, and more particularly, the logic primitives, it is suggested that graduate programs concerned with the human behavioral sciences also include extensive study of comparative and developmental ethology,

since human behavior can best be understood in the context of the primitive proto-behaviors in lower life-forms from which human behaviors were derived.

Secondary research hypotheses concomitantly validated during the experiment, indicated that the sequence of ontogenetic evolution of the logic primitives in humans may be predicted by the monotonic increase in the measure of complexity of the isomorph representing each logic primitive. The order of emergence of the logic primitives is seen to be: existence (\exists), negation (\neg), conjunction (\wedge), disjunction (\vee), condition (\rightarrow), and bicondition (\leftrightarrow).

The observation is also made that Piaget mistakenly assumed an Abelian or commutative structure to describe the logic used by young children. In fact, non-Abelian logics appear to pre-dominate in young children. The assumption of Abelian structure also precludes the existence of learning behaviors in automata.

The development of the theory also admits the construction of semantic logic puzzles, Logiktrak II, which appear to entirely eliminate the experimental noise contaminating the resolution in measuring individual differences; the construction of measurable artificial logics via which one may study the process of proof; and the construction of Piagetian measures represented by other algebraic structures to study the process of hypothesis-testing in the ontogenetic evolution of the scientific method of enquiry in humans. It is

suggested that such measures may be a valuable differentiating device in the diagnosis of certain learning disabilities.

In view of the evidence exhibited that the ontogenetic evolution of the logic primitives does not appear to asymptote out at puberty, it would appear that the educational timetable imposed by society is suboptimal. Information can be more optimally integrated by human cognitive structures at later age than the post-pubescent period now voguishly misappropriated.

Also in corollary to the finding that curricula are ontogenetic-dependent, it would appear that curricula should be designed to emphasize conceptual as opposed to elemental information in order to be optimally integrated into the more developed cognitive structure of adults. The example of biological sciences curricula is cited, where minimal appeal is made to a hypothesis-testing hypertextual approach. The bulk of the information is structured elementally rather than conceptually, and rarely is the positive evolutionary advantage of the evolving morphological or behavioral structure discussed. It is suggested that simu-labs be constructed to provide simulation of the evolutionary process in a laboratory environment.

Finally, it is proposed that the technology developed in this research, be applied to the further development of measures of diagnostic problem-solving in medical education.

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APPENDIX 1

131
Overview of Dissertation/Harv. Schiller

Developmental analysis of question-asking strategies in concept-learning: Implications to Piagetian genetic epistemology

It is my hypothesis that the question-asking strategies of children, in the learning of concepts, are peculiar to their respective Piagetian stages of cognitive development.

Hopefully, this study will show that there exists a natural hierarchy of question-asking strategies in concept-learning, and that this hierarchy is developmental in nature, being derivable from the concepts of Piagetian genetic epistemology; and is intimately dependent upon the structure of the concepts-to-be-learned.

This is accomplished via validation of the following two primary hypotheses: first, that for concepts of a given complexity, question-asking strategies become more optimal with greater Piagetian developmental stages; and second, for subjects of a given Piagetian developmental stage, question-asking strategies become more optimal with greater intelligence as measured by Raven's Progressive Matrices. The question-asking paradigm itself, is generically similar to that of the well-known game of "Twenty Questions".

132
Synopsis for Educators

since this is primarily a developmental study, a broad spectrum of subject ages will be required. The subject sampling scheme is as follows:

sampling level	approximate grade range	number of subjects required
Kindergarten	K	20
Primary	1-2-3	20
Intermediate	4-5-6	20
Junior High	7-8	20
High	9-10-11-12	20
Adult	-----	20
Total		120

All subjects should be selected from the same socioeconomic stratum, preferably, all subjects would be residents of a given north suburban area.

The study would require about 1 hour of the subject's time on each of 2 occasions, or about 2 hours altogether.

The first portion of the study will consist of developing rapport with the students, checking potential subjects out on a quick

qualifying quiz (which takes about 30 seconds), and then administering the appropriate form of the Raven's Progressive Matrices--a highly reliable, nonverbal, acultural measure of cognitive development. Apparently, it occupies the same niche in the British Empire which the Stanford-Binet and the Wechsler occupy here in the States.

The Raven's would be administered according to the following scheme:

sampling level	approximate grade range	form of Raven's to be administered	test setting	usual time of test	Remarks
Kindergarten	K	Coloured Progressive Matrices	individually	untimed, 15-30 min.	Raven's Coloured Progressive Matrices is usually administered individually, except that most children from age 8 up can take the test in small groups; range: ages 5-11 and defective adults.
Primary	1-2-3	Coloured Progressive Matrices	individually	untimed, 15-30 min.	H W
Intermediate	4-5-6	Coloured Progressive Matrices	group	untimed, 15-30 min.	
Junior High	7-8	Standard Progressive Matrices	group	untimed, about 45 minutes	Raven's Standard Progressive Matrices range: ages 8-65
High	9-10-11-12	Standard Progressive Matrices	group	untimed, about 45 minutes	
Adult	-----	Standard Progressive Matrices	group	untimed, about 45 minutes	

The second portion of the study consists of administering the Cognitive Development Paradigm (CDP), an instrument which was designed to measure cognitive development as it is reflected in the evolution of logic operators. The subject is administered a series of 'logic puzzles' which he or she must then solve by posing questions, which are then truthfully answered by the test administrator. Some very sophisticated mathematics have gone into the development of processes for constructing criterion-referenced 'logic puzzles', and for measuring the optimality of question-asking strategies.

The 'logic puzzles' themselves are really artificial concepts (articeps) which the subject must discover by constructing his or her own 'clews' to solve the 'mystery'. In this perspective, in pretesting exercises, the CDP has been viewed as a game rather than as a test. The CDP has also exhibited great stability (reliability) so interproblem learning does not seem to be a contaminant.

The time required to administer a CDP varies with the sampling level and the individual subject. A rough schedule is as follows:

sampling level	approximate grade range	time
Kindergarten	K	20-30 minutes
Primary	1-2-3	30-40 minutes
Intermediate	4-5-6	40-50 minutes
Junior High	7-8	50-60 minutes
High	9-10-11-12	60-70 minutes
Adult	-----	70-80 minutes

The CDP should be administered within 2 weeks of the date of administration of the Raven's; and small, quiet rooms would be necessary for the testing procedure.

All data collected during this study would of course be considered confidential; and the results of this study, together with whatever educational consulting problems in the area of cognitive development you may wish to pursue, will gladly be provided.

It should be impressed that this is not an affective study, but rather a cognitive study. The greatest care and effort will be taken to ensure that a compassionate and thoroughly competent presence is at all times maintained. I will comply with whatever constraints and rules you and your staff direct, the sole proviso being that the integrity of the design of the CDP shall remain my province.

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Should any question or reservation arise concerning my competence or integrity, the following professors at Loyola University will gladly provide particulars:

Name	Title and/or Department
Dr. Anne Juhasz	Educational Foundations
Dr. Jack Kavanagh	Educational Foundations
Fr. Richard Vande Velde	Mathematics (Chairman)
Dr. Rosemary V. Donnatelli	Educational Foundations (Chairwoman)
Dr. Ronald Morgan	Educational Foundations
Dr. Jeanne Foley	Clinical Psychology (Associate Dean of the Graduate School)
Dr. Joy Rogers	Educational Foundations
Prof. Leon Chestang	School of Social Service Adminis- tration, University of Chicago

APPENDIX 2

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LOYOLA UNIVERSITY OF CHICAGO

SCHOOL OF EDUCATION



Lewis Towers • 820 North Michigan Avenue, Chicago, Illinois 60611 • (312) 944-0900

To Whom It May Concern:

Harvey Jack Schiller is a graduate student in the School of Education at Loyola University. I am his graduate advisor, and the supervisor on his doctoral dissertation. In order to proceed with his study, he will require feedback from children, which will be used to clarify the instructions for a block-sorting task.

Since the permission of both the cooperating institution and the child's parent will probably be required, any questions or reservations which you may have concerning Harvey's competence and integrity should be directed to me.

This dissertation is being conducted under the usual auspices of Loyola University.

Very truly yours,

Anne M. Juhasz

Dr. Anne Juhasz

Department of Foundations
School of Education
Loyola University
820 North Michigan Ave.
Chicago, Ill 60611

Phone: 670-3047

Dear Parent,

We are affording Loyola University an opportunity to conduct certain educational research activities at Deborah.

Your child's participation is requested in order to provide information which will be used to clarify the instructions for a toy-sorting game. About 1/2 hour-1 hour of the child's time would be required.

If you would allow your child to participate in this project, would you please fill out the remainder of this sheet and have it returned to Deborah. If you should have any questions, please call 539-5907.

Thank You,

Laura Hewett
Supervisor of Childrens' Programs
Deborah Boys' Club

Child's name _____

Parents' signature _____

Date _____

GLENVIEW PUBLIC SCHOOLS

JOHN H. SPRINGMAN SCHOOL

• 2701 CENTRAL ROAD

GLENVIEW, ILLINOIS 60025

Wayne W. Buchholz, Principal

(312) 724-7000

Dear

Mr. Harvey Schiller of Loyola University will be conducting research at our school into some of the educational implications of the developmental theories of Jean Piaget.

We are requesting your permission to allow your child's participation in a research study that will provide information which will be used in a developmental analysis of question-asking strategies in the solution of 'logic puzzles'. Participation would include a series of two one-hour sessions. The first session entails discovering 'patterns', and the second entails solving 'logic puzzles'.

Because modern educators are focusing increasingly upon the child's reasoning processes during problem-solving activities, rather than upon the answer itself; studies such as this one may help us in our efforts to provide the best possible education for your child. The direct educational implications of this study may provide both new perspectives on curricular design for early childhood education and the diagnosis of certain learning disabilities; while the indirect educational implications of this study may provide new perspectives on the evolution of intelligence.

Altogether, about two hours of your child's time would be required. The study would be conducted during school hours; and each child would meet with Mr. Schiller on an individual basis, sometime between the middle of October and the middle of December. It is understood that all data will be held confidential; and upon conclusion of the study, the educational implications will be made available to the school, and to interested parties upon request.

If you would allow your child to participate in this study, please fill out the attached form and return it in the enclosed self-addressed envelope. If you should have any questions, please call school at 724-7000 (ask for Springman School), or Mr. Harvey Schiller at 539-6974.

Thank you!

W. W. Buchholz, Principal
Springman School

Dear Dr. Buchholz,

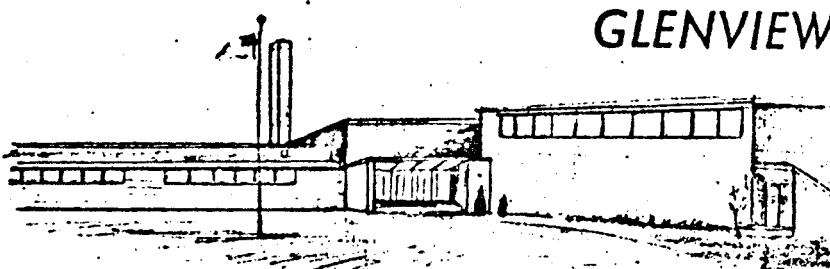
My son/daughter, _____, has my permission to
(child's name)
participate in this research study.

(date)

(parent's signature)

GLENVIEW PUBLIC SCHOOLS

RUGEN ELEMENTARY SCHOOL



Richard W. Clifford, Principal

901 SHERMER ROAD
GLENVIEW, ILLINOIS 60025
(312) 724-7000

Dear

Mr. Harvey Schiller of Loyola University will be conducting research at our school into some of the educational implications of the developmental theories of Jean Piaget.

We are requesting your permission to allow your child's participation in a research study that will provide information which will be used in a developmental analysis of question-asking strategies in the solution of 'logic puzzles'. Participation would include a series of two one-hour sessions. The first session entails discovering 'patterns', and the second entails solving 'logic puzzles'.

Because modern educators are focusing increasingly upon the child's reasoning processes during problem-solving activities, rather than upon the answer itself; studies such as this one may help us in our efforts to provide the best possible education for your child. The direct educational implications of this study may provide both new perspectives on curricular design for early childhood education and the diagnosis of certain learning disabilities; while the indirect educational implications of this study may provide new perspectives on the evolution of intelligence.

Altogether, about two hours of your child's time would be required. The study would be conducted during school hours; and each child would meet with Mr. Schiller on an individual basis, sometime between the middle of October and the middle of December. It is understood that all data will be held confidential; and upon conclusion of the study, the educational implications will be made available to the school, and to interested parties upon request.

If you would allow your child to participate in this study, please fill out the attached form and return it to your child's teacher. If you should have any questions, please call Rugen School at 724-7000 (ask for Rugen School), or Mr. Harvey Schiller at 539-6974.

Thank you!

W.H. Andre

W.H. Andre, Principal

Rugen School

Dear Mr. Andre,

My son/daughter, _____
(child's name)
to participate in this research study.

(parent's signature)

(date)

West Northfield
School District No. 31

WNS

3131 TECHNY ROAD
NORTHBROOK, ILLINOIS
60062
PHONE 272-6880

October 18, 1974

Dear Parents:

Please be informed that District 31 has agreed to cooperate in a study conducted by Loyola University related to the educational implications of the developmental theories of Jean Piaget. The research project will be directed by Mr. Harvey Schiller, a doctoral candidate.

The study will be conducted during school hours and involve a total of 20 children from grades 4, 5, and 6, and 20 children from grades 7 and 8. The students will be randomly selected and meet with Mr. Schiller for a maximum of two hours over a period of two and one-half months. The children will be involved in discovering patterns and solving logic puzzles.

It should be understood that children participating in the study will not be identified in any way and that individual responses will be held in strict confidence. The completed study will be available to the district.

Questions related to the study should be directed to Mr. Harvey Schiller at 539-6974.

If you would prefer that your child NOT participate in the study, please call the Administration Office at 272-6880 by Thursday, October 23.

Sincerely,

Allen P. Zak

Dr. Allen P. Zak,
Superintendent

APZ/mar

CROW ISLAND SCHOOL
WINNETKA, ILLINOIS

November 20, 1974

Dear Parents:

I want you to know that Crow Island School has agreed to cooperate in a research study conducted by Loyola University related to the educational implications of the developmental theories of Jean Piaget. The research project will be directed by Mr. Harvey Schiller, a doctoral candidate at Loyola University. Dr. Lola May, our math consultant, will be working with Mr. Schiller.

The study will be conducted during school hours and involve a total of twenty children from Mrs. Stephenson's 2nd Grade and Miss Hedges Senior Kindergarten class. The students will be randomly selected and will meet with Mr. Schiller for a maximum of two hours over a period of two and one-half months. The children will be involved in discovering patterns and solving logic puzzles.

It should be understood that children participating in the study will not be identified in any way and that individual responses will be held in strict confidence. The completed study will be available to the district.

Questions about the study should be directed to me.

If you would prefer that your child NOT participate in the study, please call me at Crow Island (446-0353) no later than Monday, November 25th.

Thank you for your help.

Dr. Donald Crowe, Principal

CROW ISLAND SCHOOL

APPENDIX 3

INFOPAK

SAMPLE LEVEL					
(NAME)	K	P	I	JrH	H
(BIRTHDAY)					

(HOW CAN YOU BE CONTACTED)

(SCHOOL & GRADE)

CHECK OFF THE GAMES AND ACTIVITIES YOU'RE REALLY INTERESTED IN OR LIKE:

- | | | |
|-----------------------------------|--|--------------------------------------|
| <input type="checkbox"/> CHESS | <input type="checkbox"/> LOGIC PUZZLES | <input type="checkbox"/> 'CLUE' |
| <input type="checkbox"/> CHECKERS | <input type="checkbox"/> GAMES OF STRATEGY | <input type="checkbox"/> WFF'N PROOF |

ANY OTHERS? _____

HAVE YOU EVER STUDIED BOOLEAN ALGEBRA OR MATHEMATICAL LOGIC?

RAVEN'S

CPM	SPM

(DATE GIVEN)

Y M

(AGE)

A	Ab	B	C	D	E	Σ	%-ile	GRADE

EXPECTED

DEVIATIONS

COGNITIVE DEVELOPMENT PARADIGM

(DATE GIVEN)

Y M

(AGE)

TEXT		
A	B	C

REMARKS _____

SHEET OF **SHEETS.**

(NAME)

n 1 *iso* *z* *y_c*

APPENDIX 4

LOGIKTRAK

ADMINISTRATION MANUAL

**HARVEY JACK SCHILLER
LOYOLA UNIVERSITY**

ADMINISTRATION MANUAL

PREFACE

The Logiktrak is simple elegance incarnate. It is a simple test to give, and yet it is constructed upon an elegant conceptual foundation.

It should be born in mind by the prospective experimenter (E) that any subject (S) to whom the Logiktrak is given, should be treated according to the highest standards of professional ethics and human compassion.

No part of the Logiktrak may be administered or made use of without prior consent of Harvey Jack Schiller.

The Logiktrak is administered individually, and since it is not a test of fixed length, but rather a paradigm, the time of administration depends wholly upon the age of the subject and the number and complexity of the problems which the subject is required to solve.

MATERIAL NEEDED

Kit of Toys



Pencil

Paper

(Optional: ZOT/NOTZOT partition)

CONDITIONS

The room where the Logiktrak is given should be

quiet, well-lit and well-ventilated. Care should also be taken that E and S are not disturbed during the testing period.

E should maintain a friendly disposition to S at all times. An aura of warmth will encourage S to treat the Logiktrak as a game and hence promote far better performance than staid neutrality on E's part.

QUALIFYING MINI-QUIZ FOR LOGIKTRAK

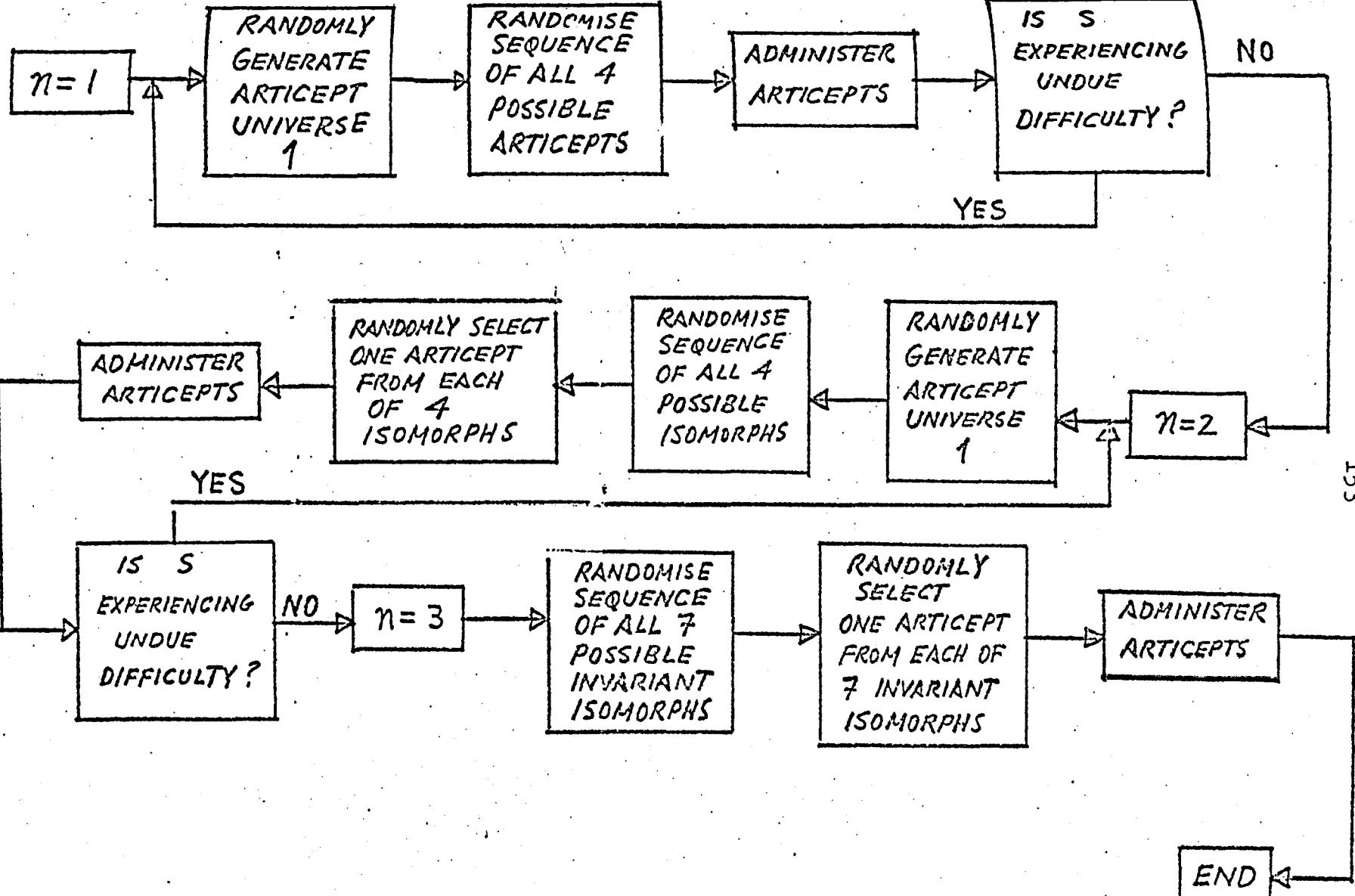
Before S is allowed to take the Logiktrak, S must first be able to:

1. Count to eight,
2. Distinguish between black and white toys,
3. Distinguish between tetrahedrons and cubes and
4. Distinguish between large and small toys.

Any S who cannot pass this qualifying mini-quiz, should not be given the Logiktrak.

PROCEDURE

The design used in the dissertation is flowcharted on the next page.



三

As one may see by referring to the computer print-out in the Appendix, those articepts (artificial concepts) of equivalent complexity are grouped together into the same isomorph. The universe 2^1 comprised of all possible articepts is thus partitioned into isomorphs or equivalence classes of articepts. Since any articept may be described by its colour (black or white), form (tetrahedron or cube) and size (large or small), by choosing a universe 1 where all of the toys have either two, one or none constant dimension $n=1$, $n=2$ or $n=3$. For example, if $n=1$, a possible universe 1 would be

$$1 = \{\triangle, \square\}$$

(note that colour (white) and size (large) are the two constant dimensions in 1).

The isomorphs would then be:

$$\text{IS01} = \{\varnothing, 1\}, \quad \text{IS02} = \{\triangle, \square\}$$

Thus, the four articepts generated by 1 would then be:

$$2^1 = \begin{Bmatrix} \varnothing & 1 \\ \triangle & \square \end{Bmatrix}$$

STANDARDISED TEXT

The first thing one must know in order to play any game is the rules. In the case of giving the Logiktrak, the ascertainment by E that S understands the rules and objectives of the game is of the utmost importance; be-

cause one must be certain, especially in the case of young children, that S's performance is not predetermined by a poor conceptualization of the game, but rather is reflective of S's true cognitive development. Even so, the possibility of experimental artifact must loom as a source of contamination.

The following standardized text should be adhered to, except in cases of vocabulary more appropriate to S's level of abstraction.

See these eight toys? There are lots of ways we could separate all these toys into two piles. Like this ... (illustrate). So that these toys are in this pile, and those toys are in that pile, or like this ... (illustrate by consecutively moving toys from one pile to the other).

Now if we wanted to tell all the toys in this pile apart from all the toys in that pile, we could give all the toys in this pile one name, and all the toys in the other pile a different name (illustrate). Let's call all the toys in this pile ZOTS, and let's call all the toys in the other pile NOTZOTS--because they're not ZOTS (illustrate)! Then maybe these toys are ZOTS, and those toys are NOTZOTS (illustrate), or maybe these toys are ZOTS and those toys are NOTZOTS (illustrate by consecutively moving toys from one pile to the other), or ...

Now every toy that is not in the ZOT pile has got to be in the NOTZOT pile (illustrate); and every toy that is not in the NOTZOT pile has got to be in the ZOT pile (illustrate). Every toy that is not a ZOT is a NOTZOT (illustrate); and every toy that is not a NOTZOT is a ZOT (illustrate). Every toy must either be in one pile or the other. Every toy is either a ZOT or a NOTZOT.

Now I know what is in the ZOT pile and in the NOTZOT pile but you don't! I know what ZOTS are and what NOTZOTS are but you don't. What you have to do is figure out which toys are in the ZOT pile and which toys are in the NOTZOT pile. You have to figure out which toys are ZOTS and which toys are NOTZOTS.

Now you can figure out the answer to this puzzle by asking me questions just like a detective. You can ask me any question you want, but I can only answer questions with yes or no or with a number. Remember --the kinds of questions you can ask me are questions I can answer with yes or no, or questions I can answer with a number.

I'll be writing down all your questions, so any time you want me to repeat what you've asked, just say so, and I'll be glad to give them to you. Here's a pencil and paper just in case you want to write your clues down. Any time you feel like telling me what you're thinking, go right ahead, because I'm interested in that. Also, there's no time limit, so take your time--don't rush. The only thing I do ask, is that you do the best you can and don't waste your questions.

Do you understand the game? 'No' (repeat the instructions). 'Yes' Good! Then let's see if you know them. Why don't you tell me what the rules are!

Once E is sure that S knows the rules and objectives of the game,

E should begin the Logiktrak by randomly selecting a set of problems from a universe $\{ \}$ in $n=1$.

Let's start off with some real easy ones. Everyone starts off with easy ones so they can learn the rules and build up their confidence.

Upon selection of the universe $\{ \}$, E should then show S all four possible ZOTS and NOTZOTS.

For example, maybe both of these are ZOTS and there aren't any NOTZOTS (illustrate), or maybe none of these are ZOTS and both of these are NOTZOTS (illustrate), or maybe this is a ZOT and that is a NOTZOT (illustrate), or maybe this is a ZOT and that is a NOTZOT (illustrate).

With very young children, there still may be some inertia at generating their first questions. If the cartoons don't give the child the necessary stimulus to ask their first questions, then E is advised not to go any further.

Upon solution of the first problem, E should then tell S that

Now there is something new in the ZOT pile and something new in the NOTZOT pile. Can you figure out what they are, just like before?

This procedure should be repeated after each problem, as long as necessary.

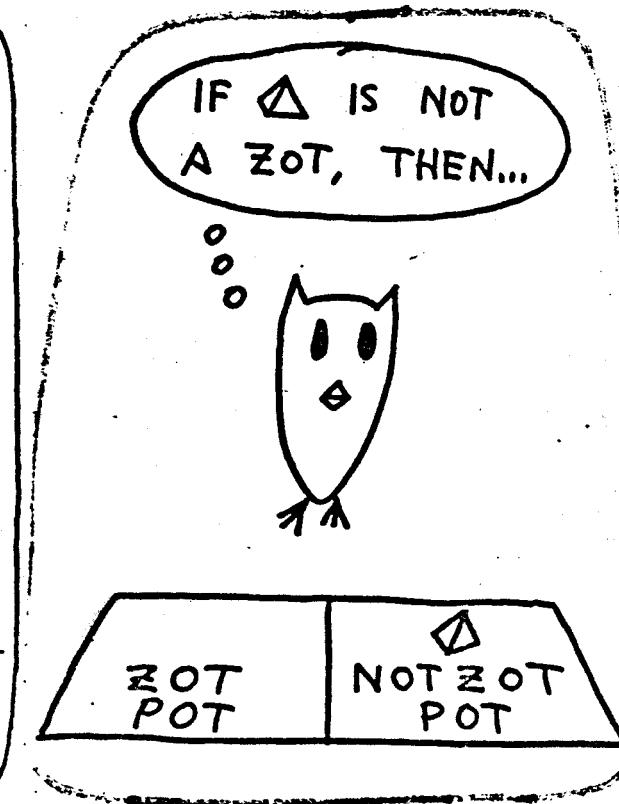
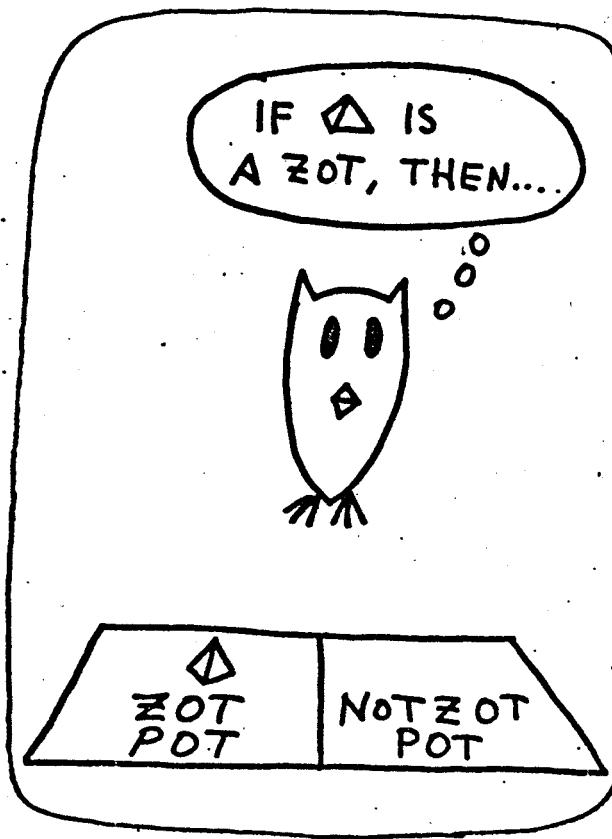
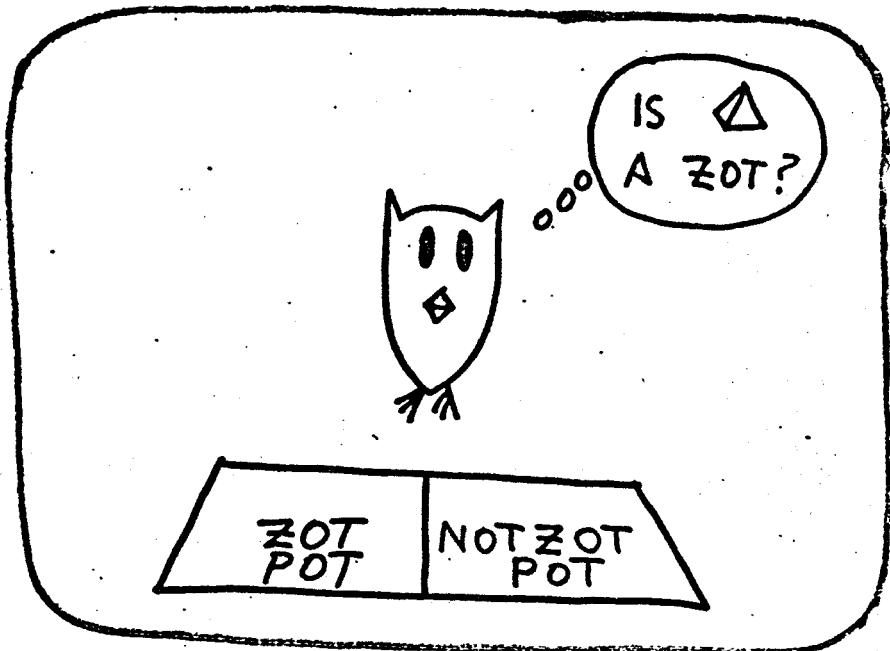
Data

E should accurately record S's questions, together with the resulting answers to these questions. The questions involved in the solution of a problem should all be properly identified and kept together, separate from those questions involved in the solution of any other problems.

Kinds of Questions

The only questions E is allowed to answer are called legal questions. Legal questions are questions answerable by yes or no, or by a number. Not all questions answerable by a number are really legal questions. There are basically two different ways in which the concept of number may be used. First, the concept of number may be used in counting or enumeration. In this sense, a question is legally answerable by a number. Second, the concept of number may be used in coding or identification. In this sense, a question is not legally answerable by a number. For example, S could list all the possible articepts in ¹ 2 and ask E to give the number (identify) of the solution articept. This would not be a legal question.

Whenever there is any doubt as to the legality of a question, it should not be allowed; however, it should



still be recorded. After all, the Logiktrak primarily follows the track of S's logic, and is only secondarily concerned with S's individual questions.

E should repeat all questions to S before answering, for purposes of verification. E should also remove any ambiguity between what S asks and what E thinks S is asking, by carefully rephrasing S's questions so that S will be asking exactly what he or she means to ask. And E should exercise caution that he or she doesn't invent questions for S to ask.

Reliability

From data collected in pilot testing, the Logiktrak appears to have an extremely high reliability; that is, intra-subject learning during the Logiktrak does not occur. The variances of error are provided in the table on the next page.

Apparently, the question-asking strategies of very intelligent people, which should theoretically prove to be the most unstable, were in fact quite stable.

Reliability Study Data

(All isomorphs taken from articept universe $\frac{1}{2}$; all subjects very intelligent adults.)

Subject	Statistic	Isomorphs		
		Singlet/ Triplet	Easy Doublet	Hard Doublet
S1	\bar{X}	2.00	5.13	6.31
	S^2	0.00	0.13	2.16
	K	3	8	9
S2	\bar{X}	2.00	5.00	6.58
	S^2	0.00	0.00	1.00
	K	2	4	2
S3	\bar{X}	2.00	5.15	6.58
	S^2	0.00	0.06	1.00
	K	2	4	2
S4	\bar{X}	2.00	5.00	5.58
	S^2	0.00	0.00	0.00
	K	2	4	1

\bar{X} = Mean

S^2 = Variance

K = Cell Size

APPENDIX 5

ARTICEPTS OF 1 DIMENSION(S):

XXXXXX
XXXXXX X XXX
XXXXXXX XXXXX
XXXXXXXXXXXX XXXXXXXX
UNIT SET # 1 FOR DIMENSION 1.

THERE ARE 4 ARTICEPTS FOR THE ABOVE UNIT SET!

162

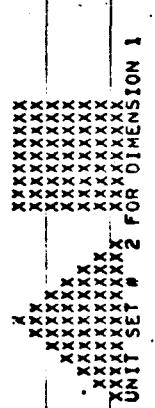
X
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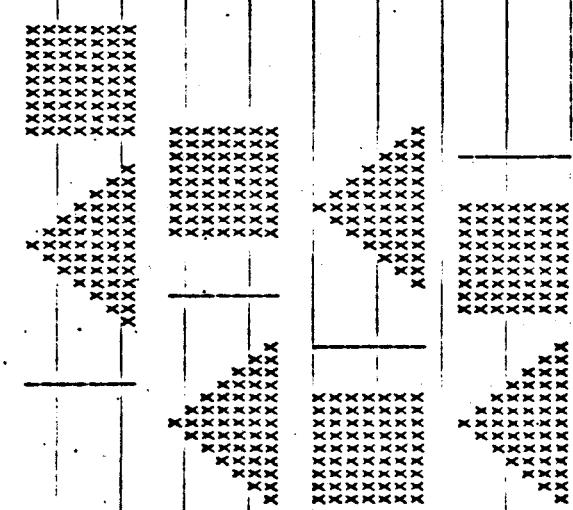
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ARTICEPTS OF 1 DIMENSION(S)!



THERE ARE 4 ARTICEPTS FOR THE ABOVE UNIT SET!

163



ARTICENTS OF 1 DIMENSION(S)!

X X X X X
X X X X X
X X X X X
X X X X X
X X X X X
UNIT SET # 3 FOR DIMENSION 1

THERE ARE 4 ARTICENTS FOR THE ABOVE UNIT SET!

164

ARTICLES OF : 1 DIMENSION(S)!

X X X X X
X X X X X
X X X X X
X X X X X
UNIT SET # 4 FOR DIMENSION 1.

THERE ARE 4 ARTICLES FOR THE ABOVE UNIT SET!

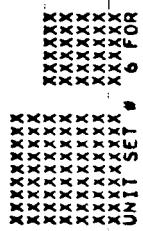
165

ARTICLES OF 1 DIMENSION(S)!

XXXX
XXXXXX
XXXXXXX
XXXXXXX UNIT SET # 5 FOR DIMENSION 1

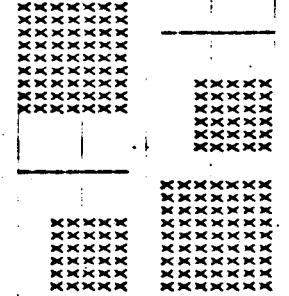
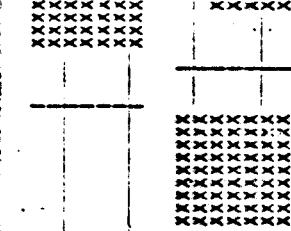
THERE ARE 4 ARTICLES FOR THE ABOVE UNIT SET!

ARTICENTS OF 1 DIMENSION(S)!



UNIT SET # 6 FOR DIMENSION 1

THERE ARE 4 ARTICENTS FOR THE ABOVE UNIT SET!



167

ARTICCEPTS OF 1 DIMENSION(S):



XXXXXX
XXXXXX
XXXXXX
XXXXXX UNIT SET # 7 FOR DIMENSION 1

THERE ARE 4 ARTICCEPTS FOR THE ABOVE UNIT SET!

168

ARTICENTS OF 1 DIMENSION(S):

XXXXX
XXXXX
XXXXX
XXXXX
XXXXX
UNIT SET # 8 FOR DIMENSION 1

THERE ARE 4 ARTICEPTS FOR THE ABOVE UNIT SET!

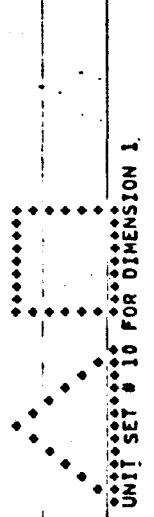
ARTICEPTS OF 1 DIMENSION(S)!

UNIT SET # 9 FOR DIMENSION 1

THERE ARE 4 ARTICEPTS FOR THE ABOVE UNIT SET!

170

ARTICEPTS OF 1 DIMENSION(S)!



UNIT SET # 10 FOR DIMENSION 1.

THERE ARE 4 ARTICEPTS FOR THE ABOVE UNIT SET!

171

ARTICCEPTS OF 1 DIMENSION(S)!

UNIT SET # 11 FOR DIMENSION 1

THERE ARE 4 ARTICCEPTS FOR THE ABOVE UNIT SET!

ARTICEPTS OF 1 DIMENSION(S)!

UNIT SET # 12 FOR DIMENSION 1

THERE ARE 4 ARTICEPTS FOR THE ABOVE UNIT SET!

173

ARTICLES OF 2 DIMENSION(S)!

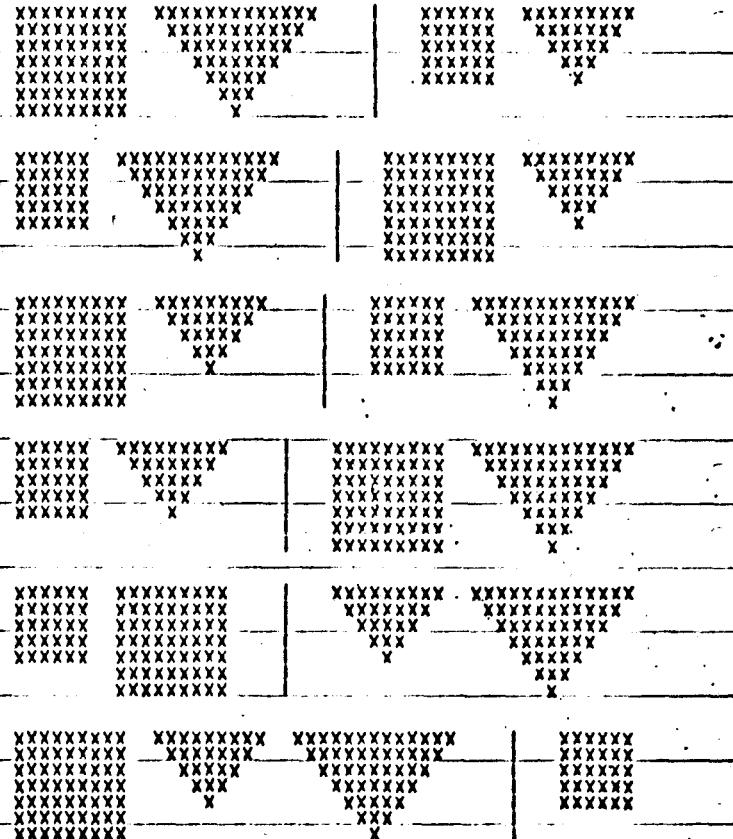
X X X X X X X X X X X X X X X X
X X X X X X X X X X X X X X X X
X X X X X X X X X X X X X X X X
X X X X X X X X X X X X X X X X
X X X X X X X X X X X X X X X X
UNIT SET # 1 FOR DIMENSION 2

THERE ARE 16 ARTICLES FOR THE ABOVE UNIT SET!

174

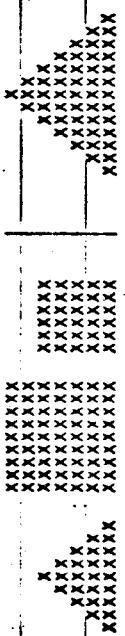
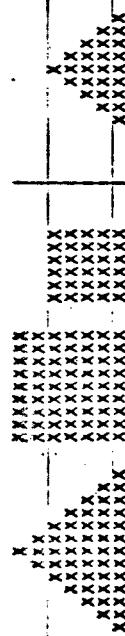
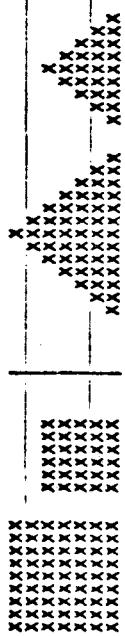
175

ARTICLES OF 2 DIMENSION(S)

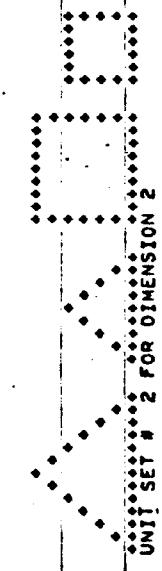


176

ARTICLES OF 2 DIMENSION(S)!



ARTICEPTS OF 2 DIMENSION(S)!

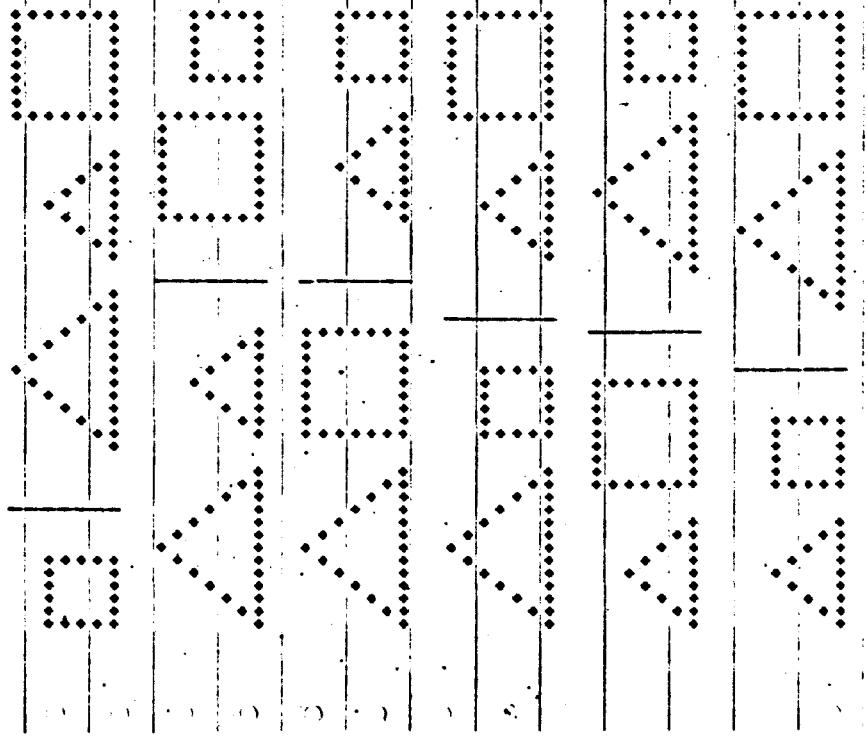


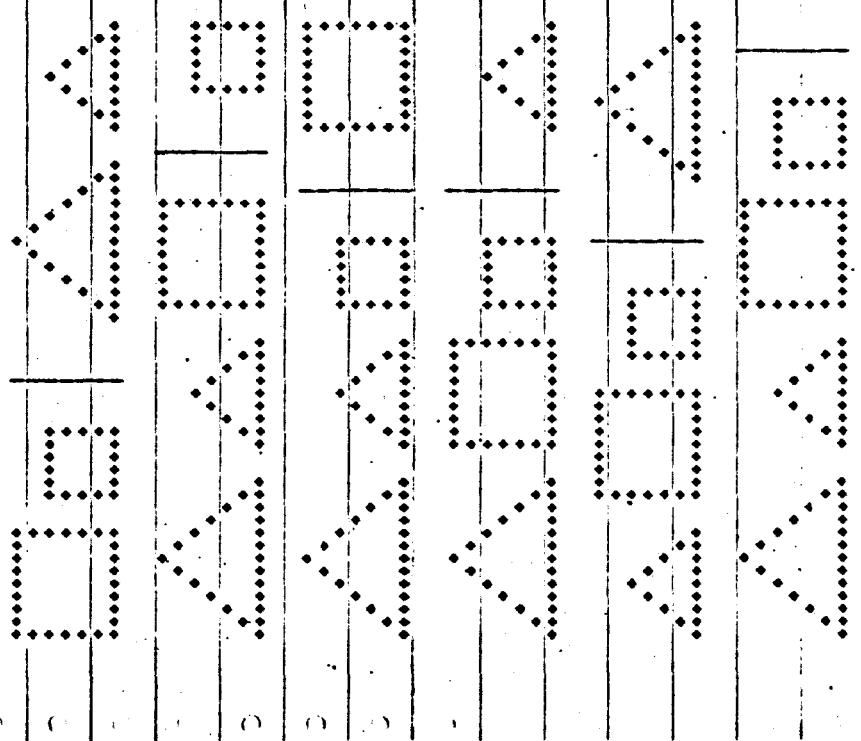
UNIT SET # 2 FOR DIMENSION 2

THERE ARE 16 ARTICEPTS FOR THE ABOVE UNIT SET!

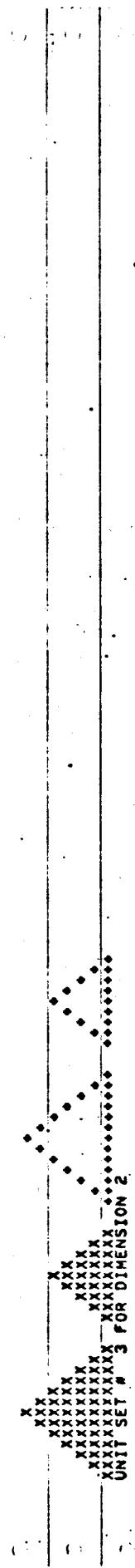
ARTICLES OF 2 DIMENSION(S):

178





ARTICEPTS OF 2 DIMENSION(S)!

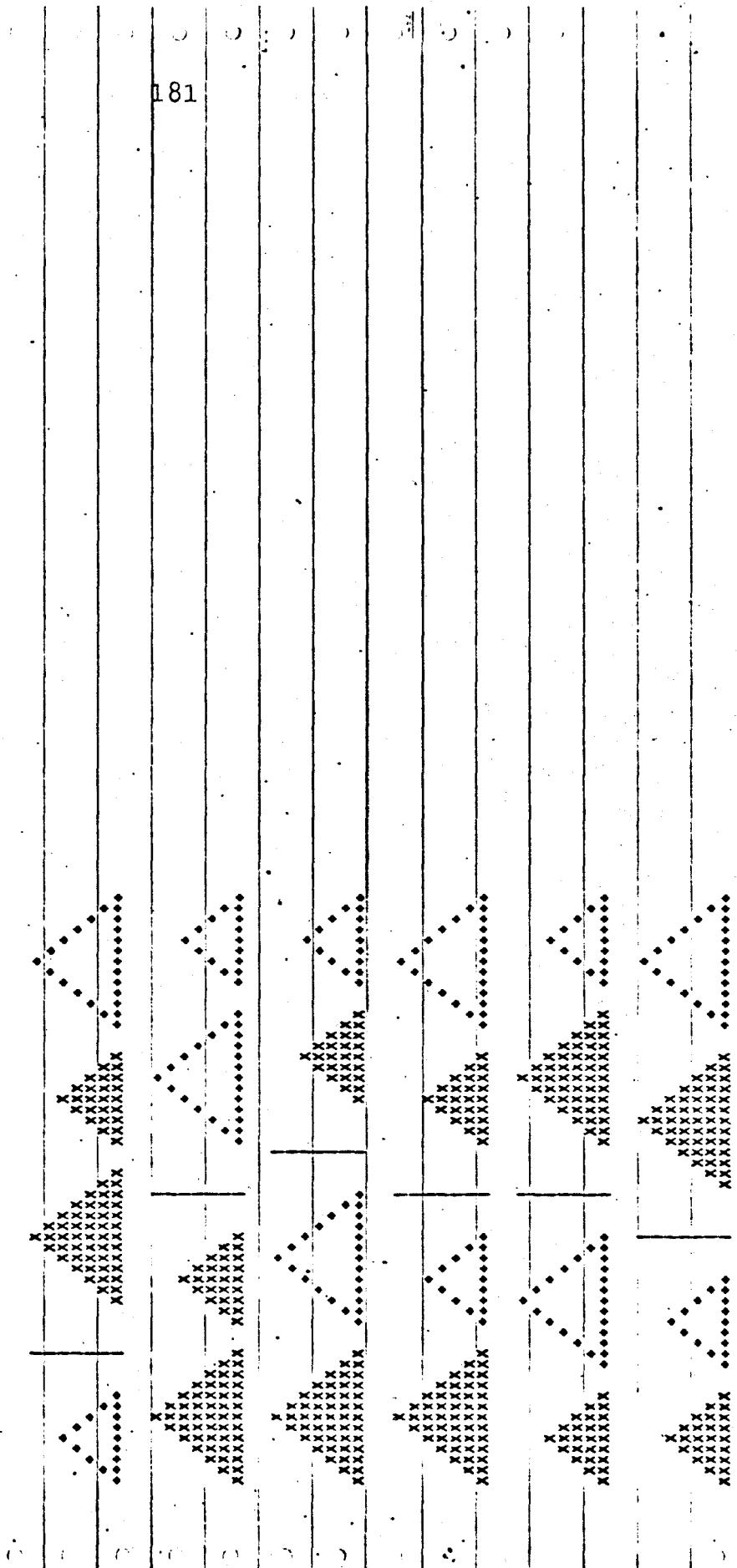


THERE ARE 16 ARTICEPTS FOR THE ABOVE UNIT SET!

180

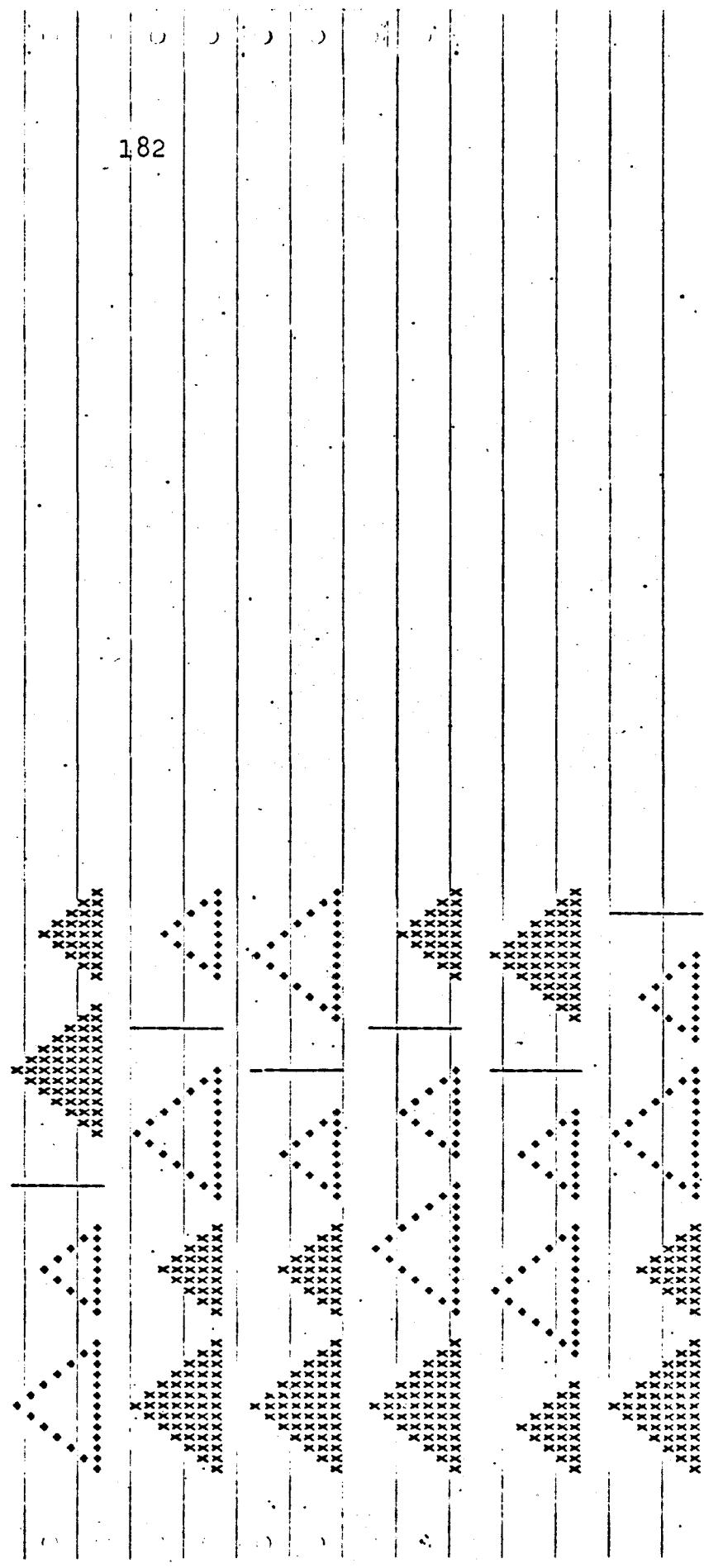
ARTIFACTS OF 2 DIMENSION(S)!

181



ARTICLES OF 2 DIMENSION(S)!

182



ARTICLES OF 2 DIMENSION(S)!

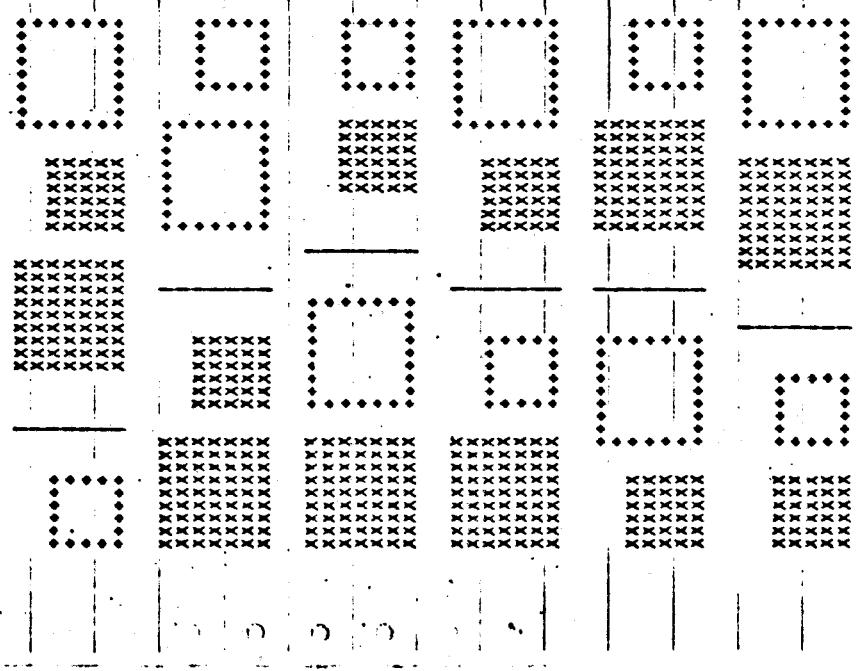


UNIT SET # 4 FOR DIMENSION 2

THERE ARE 16 ARTICLES FOR THE ABOVE UNIT SET!

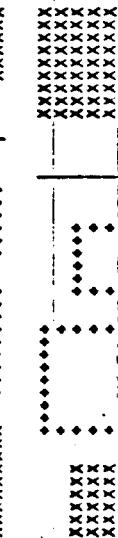
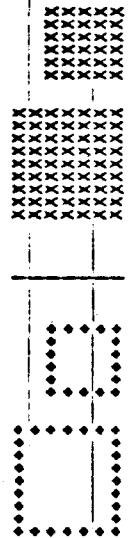
183

ARTICLES OF 2 DIMENSION(S)!

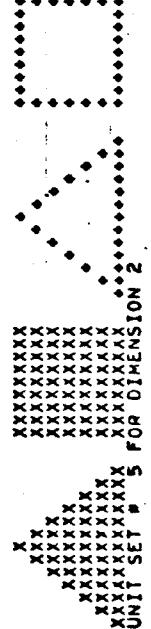


185

ARTICLES OF 2 DIMENSION(S)!

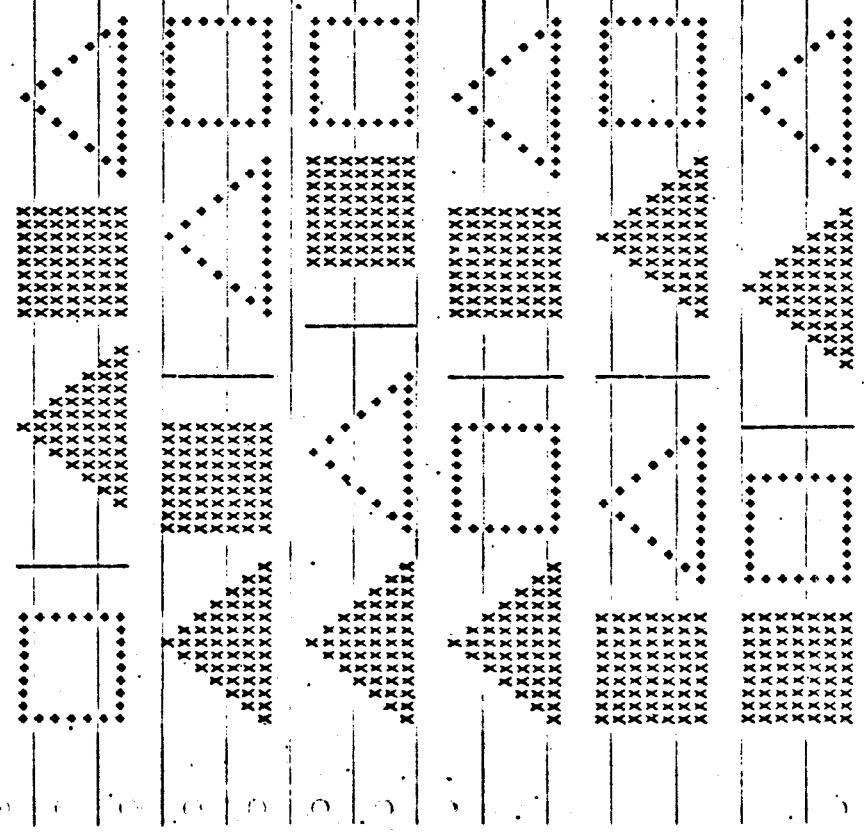


ARTICLES OF .2 DIMENSION(S)!

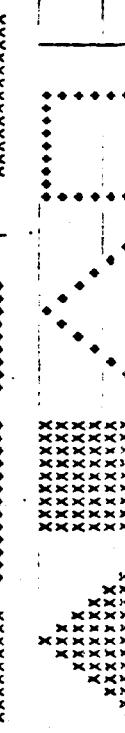
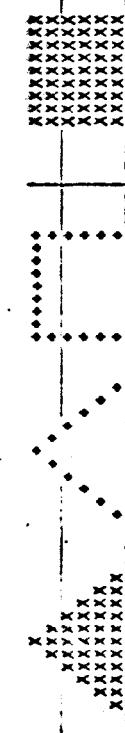


THERE ARE 16 ARTICLES FOR THE ABOVE UNIT SET!

ARTICLEPTS OF 2 DIMENSION(S)!



ARTICLES OF 2 DIMENSION(S)!

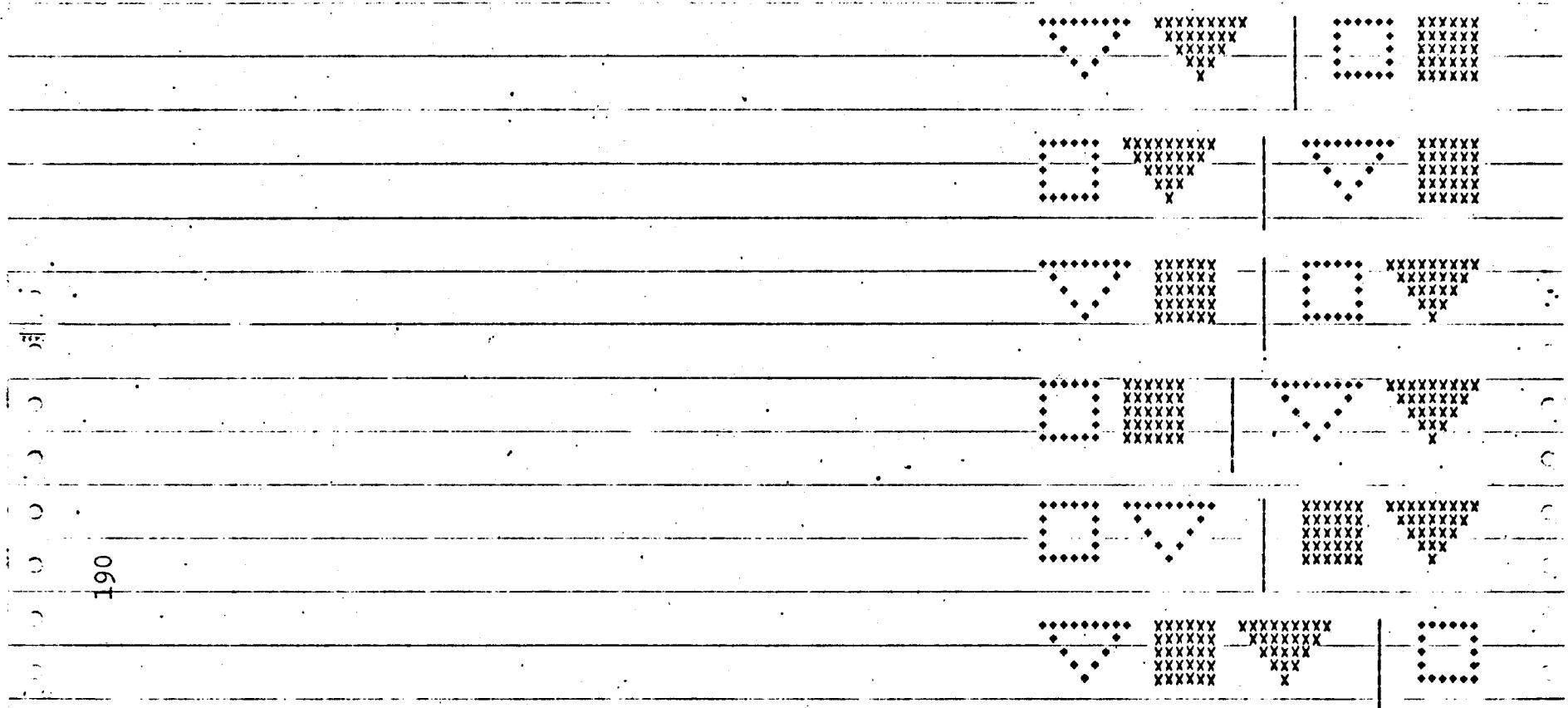


ARTICLES OF 2 DIMENSION(S):

X X X X X
X X X X X
X X X X X
X X X X X
X X X X X
UNIT SET # 6 FOR DIMENSION 2

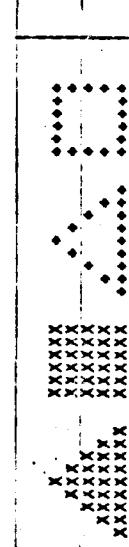
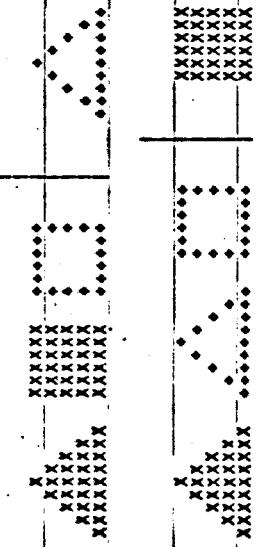
THERE ARE 16 ARTICLES FOR THE ABOVE UNIT SET!

189



ARTICLES OF 2 DIMENSIONAL

191



192

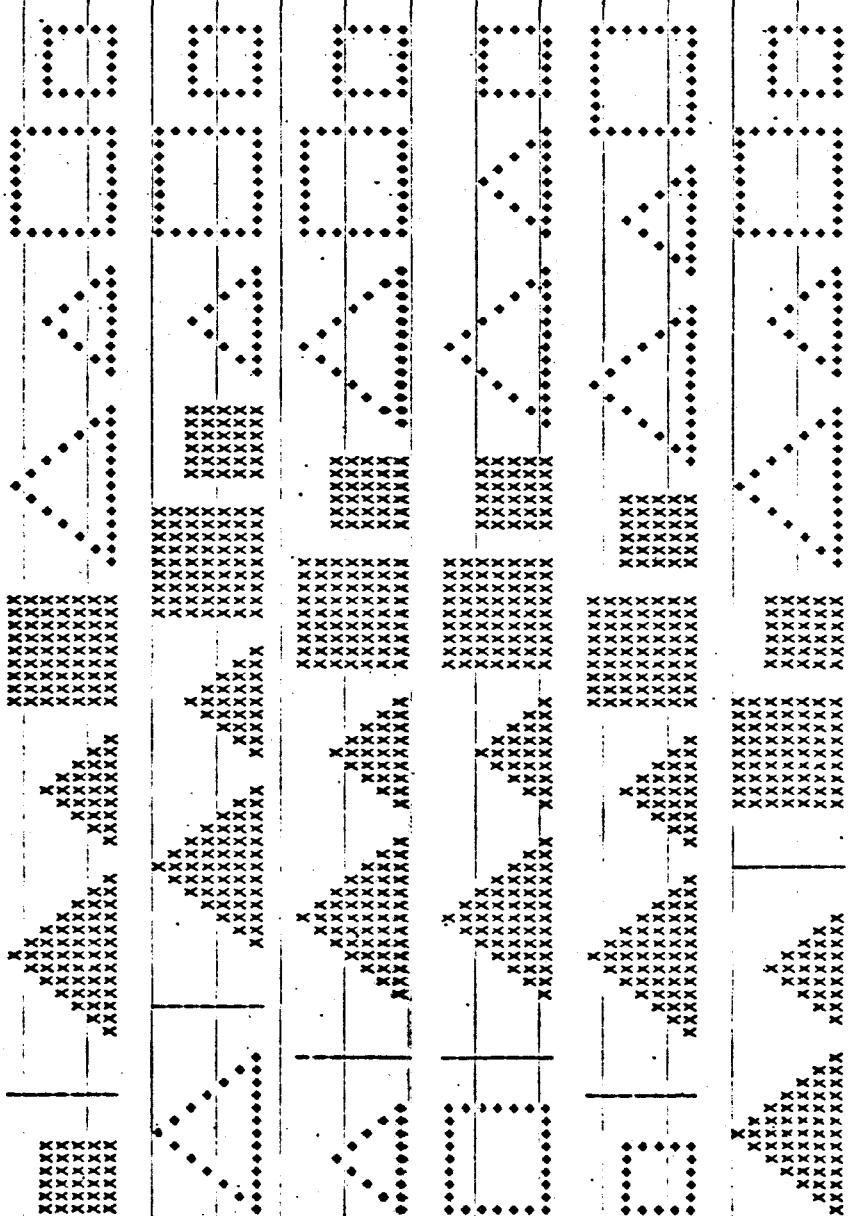
PARTICULARS OF . . . 3 DIMENSION(S)!

FOR DIMENSION 3

THERE ARE 256 ARTICLES FOR THE ABOVE UNIT SET!

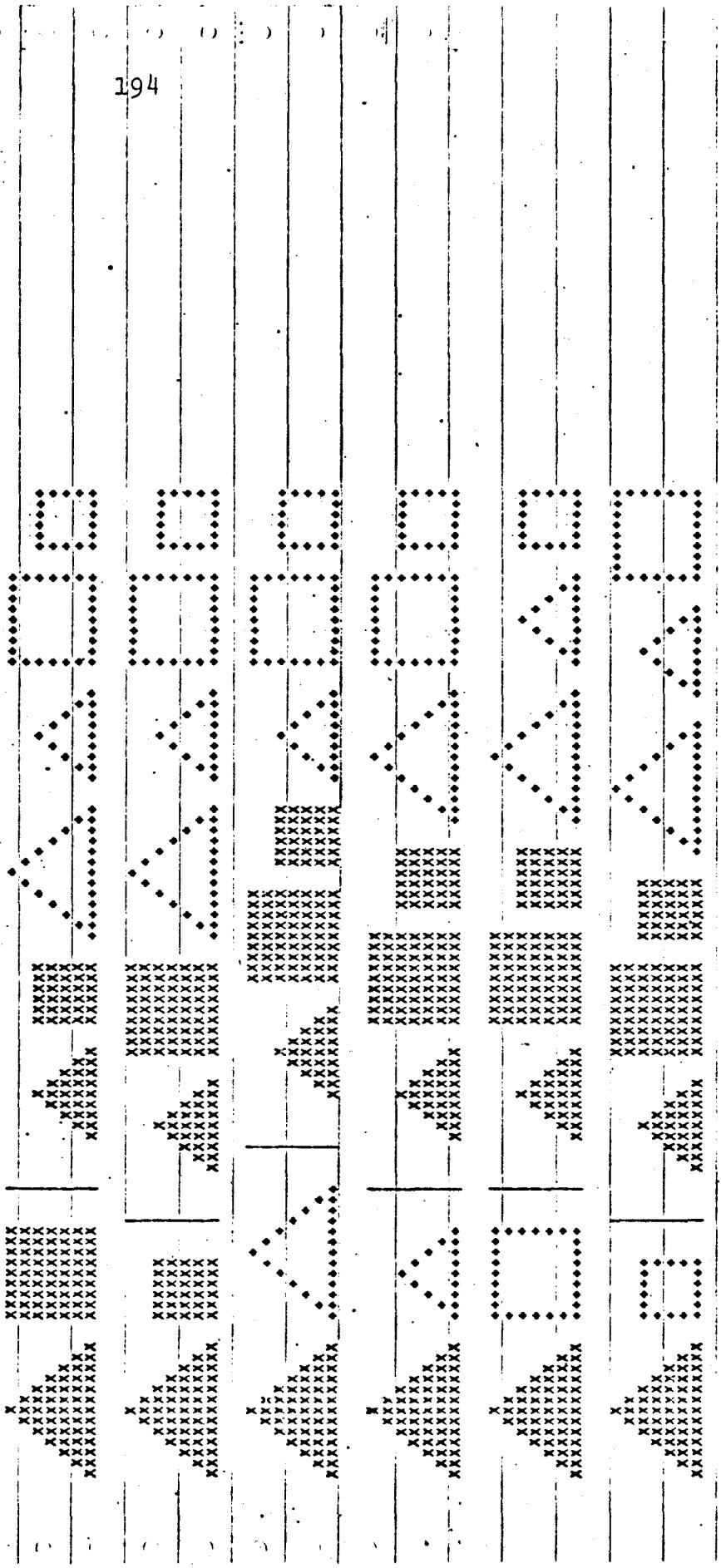
ARTICENTS OF 3 DIMENSION(S)!

193

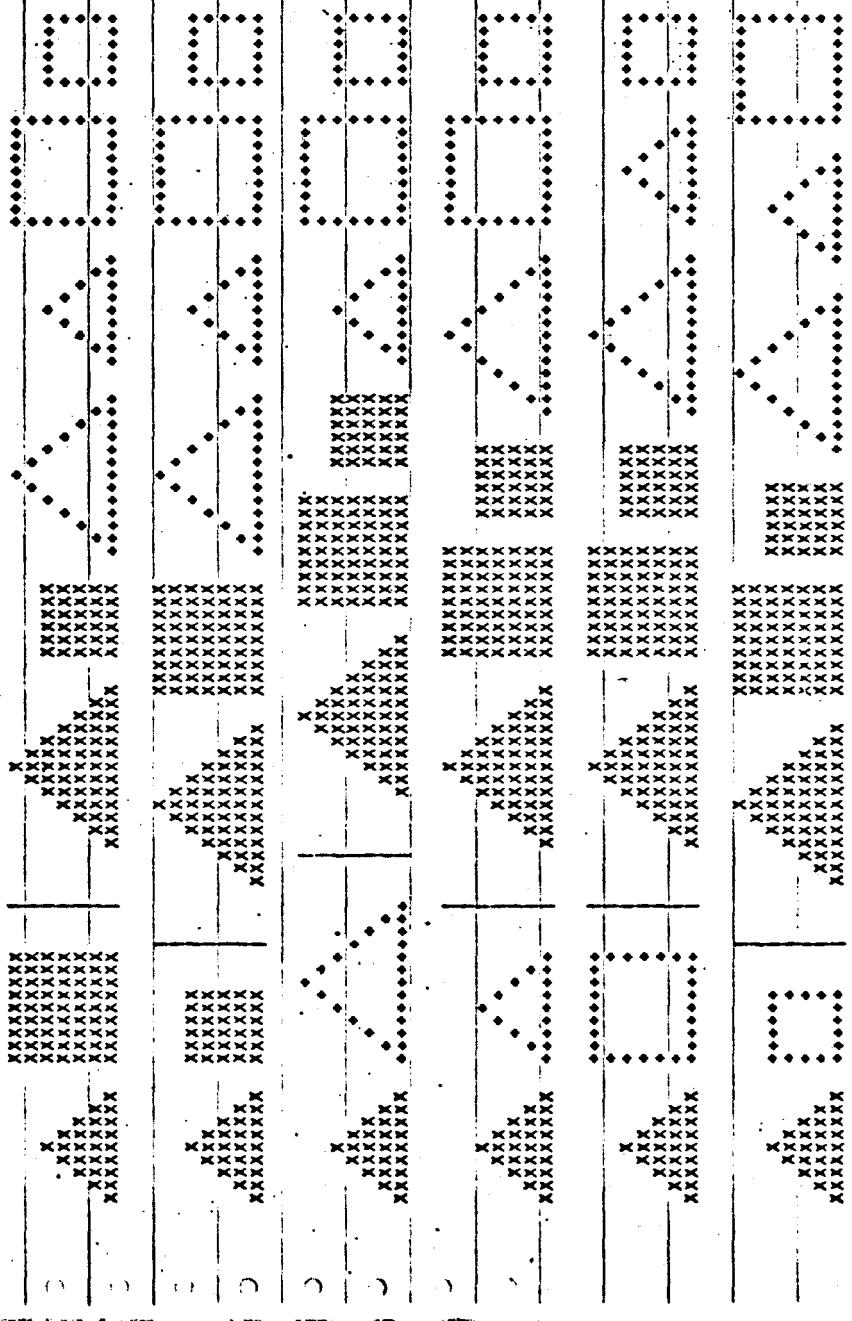


ARTICENTS OF 3 DIMENSION(S)!

194

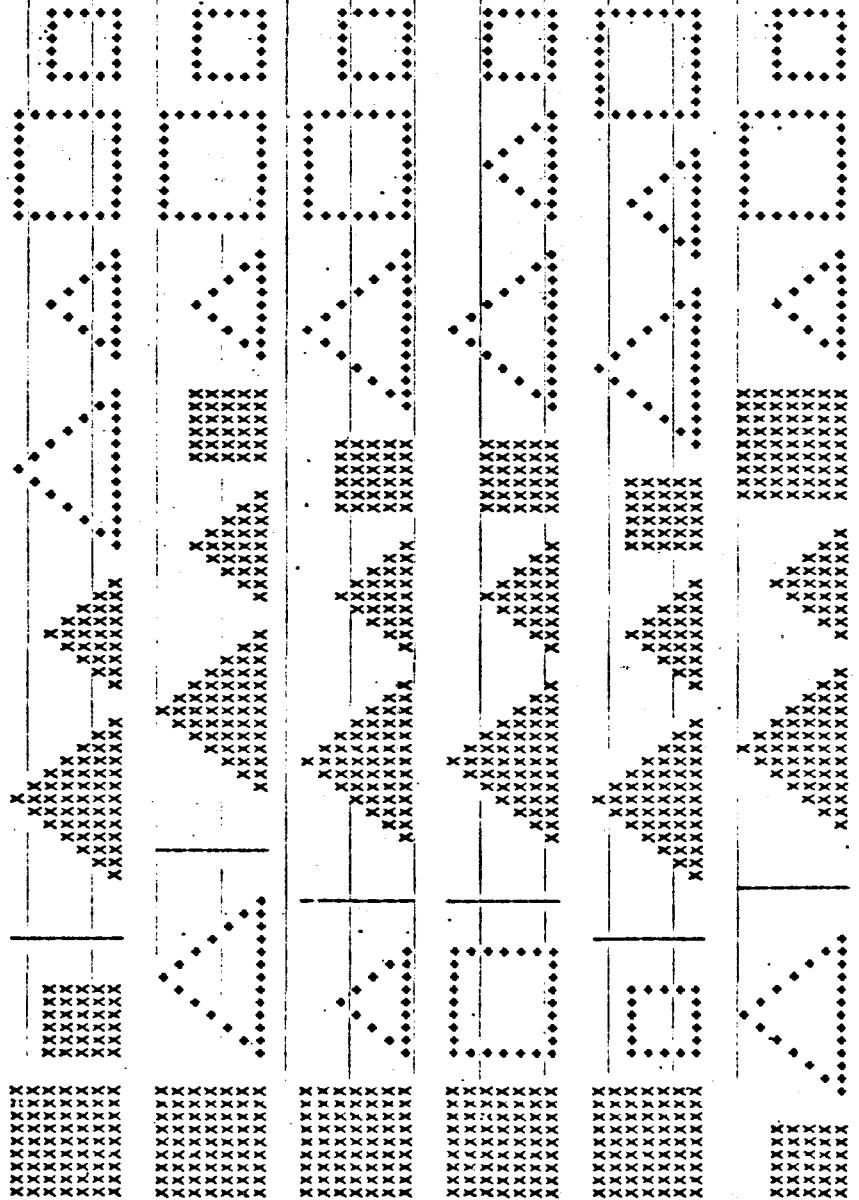


ARTICLES OF 3 DIMENSION(S)!!



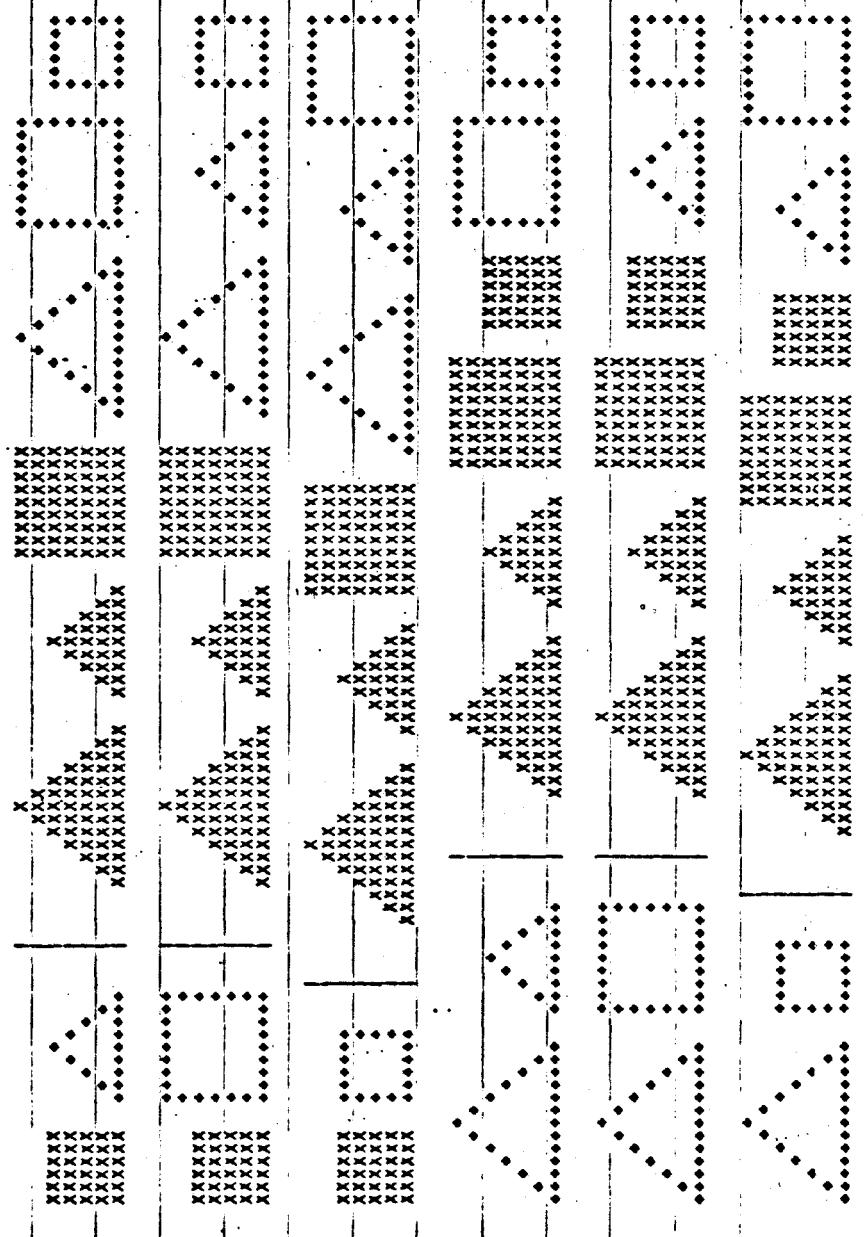
ARTICLETS OF 3 DIMENSION(S) I

196



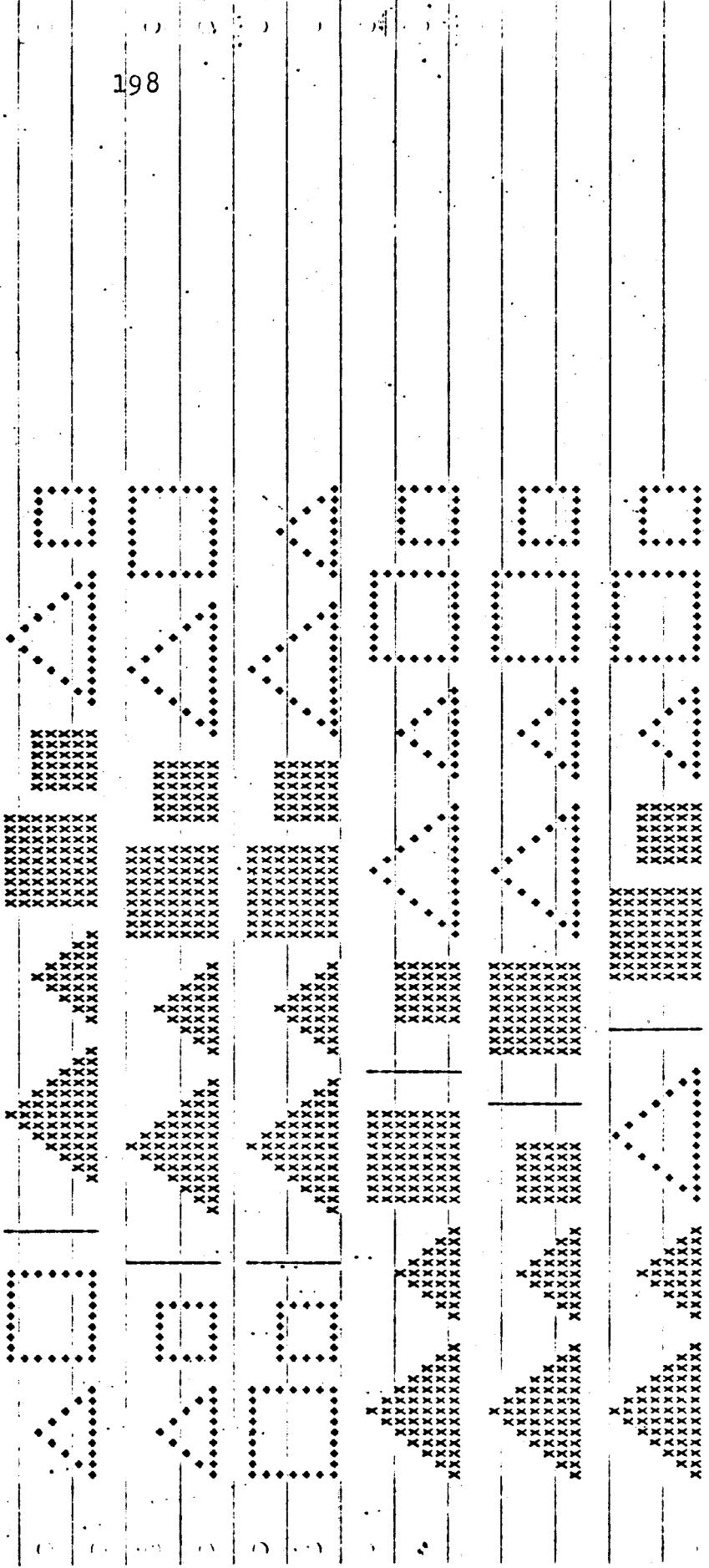
ARTICLES OF 3 DIMENSION(S) I

197

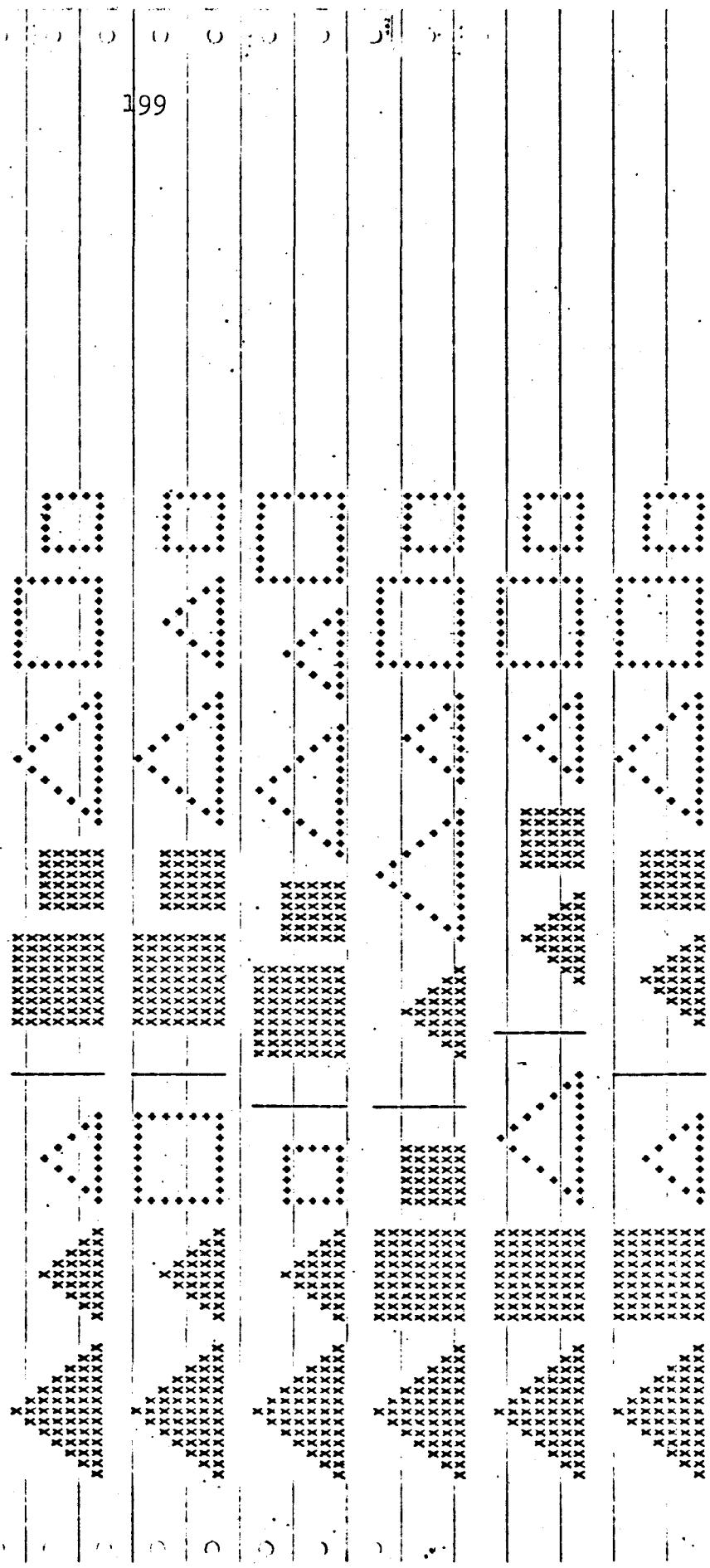


198

ARTIFACTS OF 3 DIMENSION(S)!

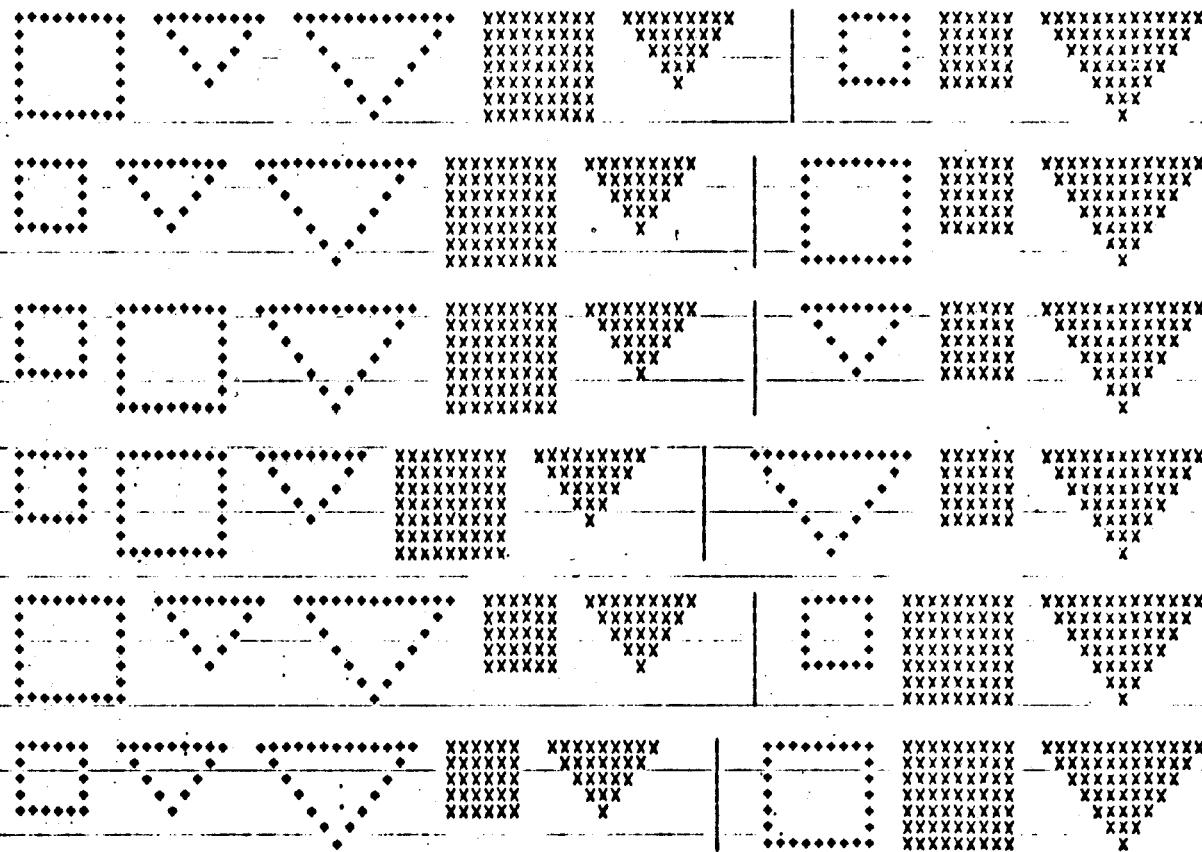


ARTICLES OF 3 DIMENSION(S)!

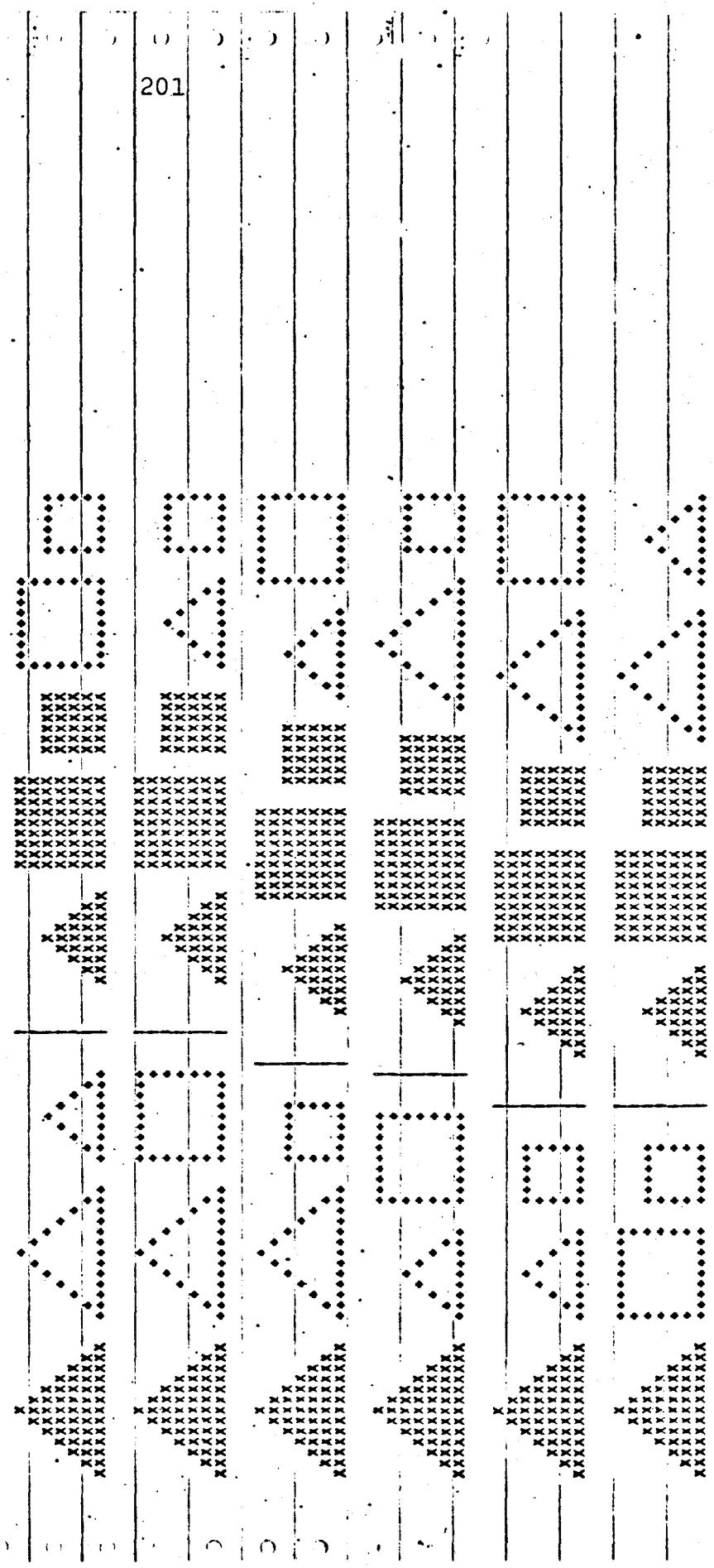


ARTICLES OF . 3 DIMENSION(S)

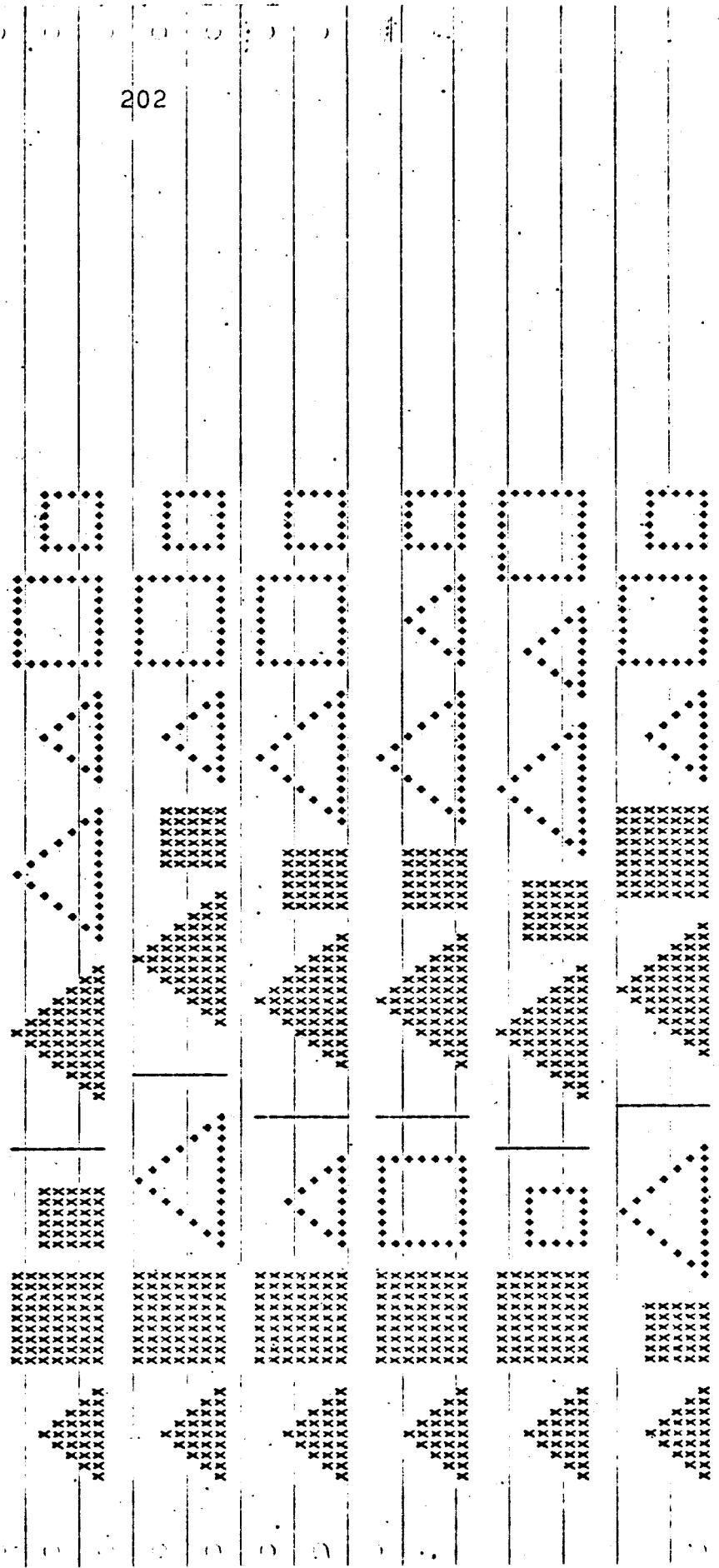
200



ARTIFACTS OF 3 DIMENSION(S)!

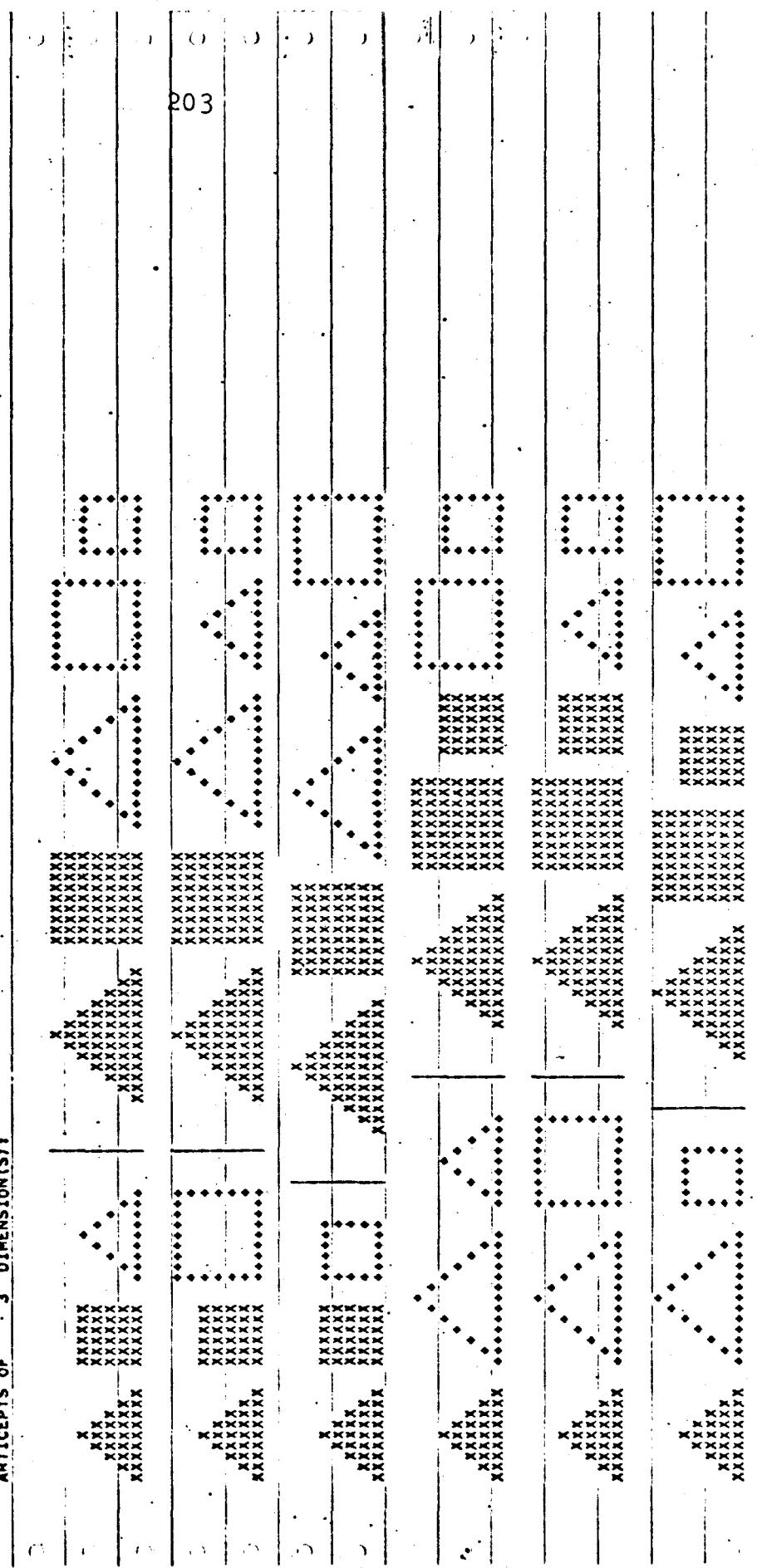


ARTICLES OF 3 DIMENSION(S) I



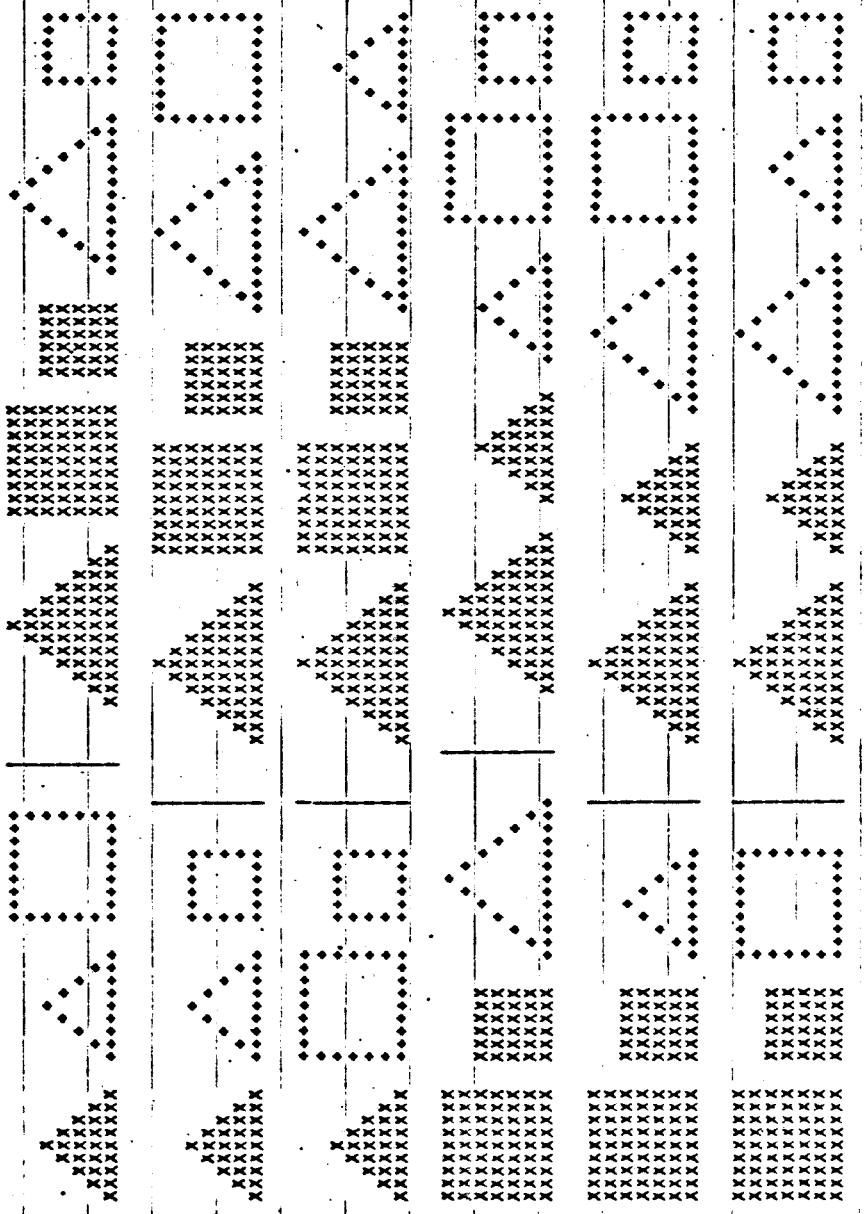
ARTIFICEPTS OF 3 DIMENSION(S)!

203

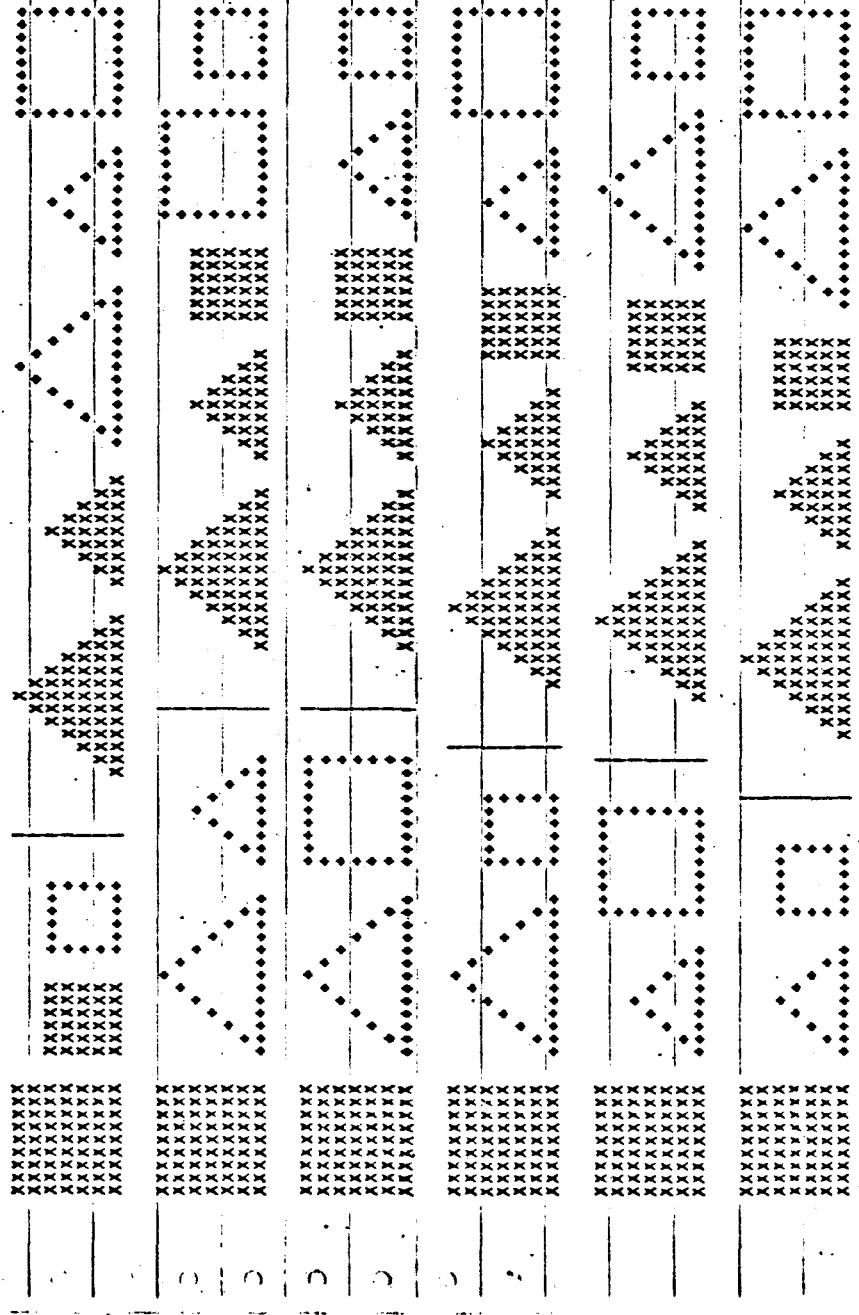


ARTIFACTS OF 3 DIMENSION(S)!

204

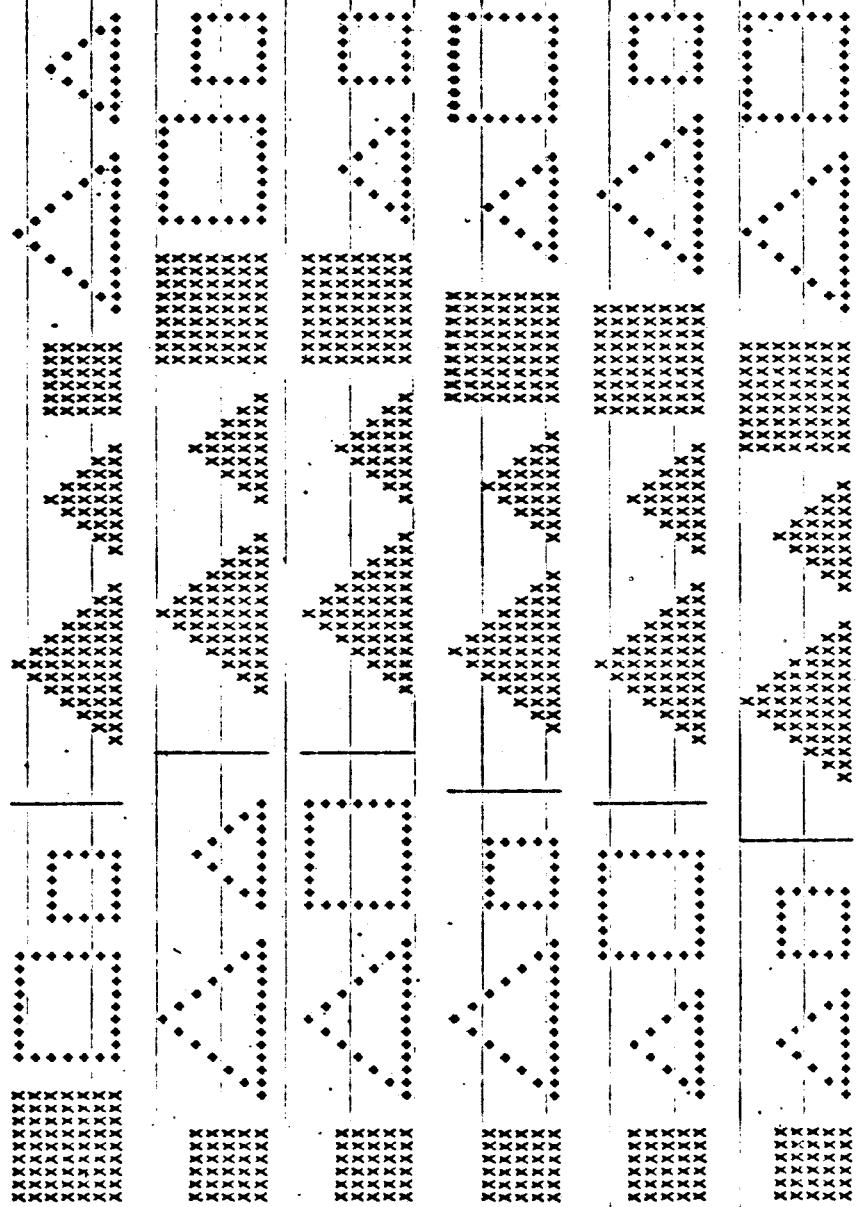


ARTIFACTS OF 3 DIMENSION(S)!



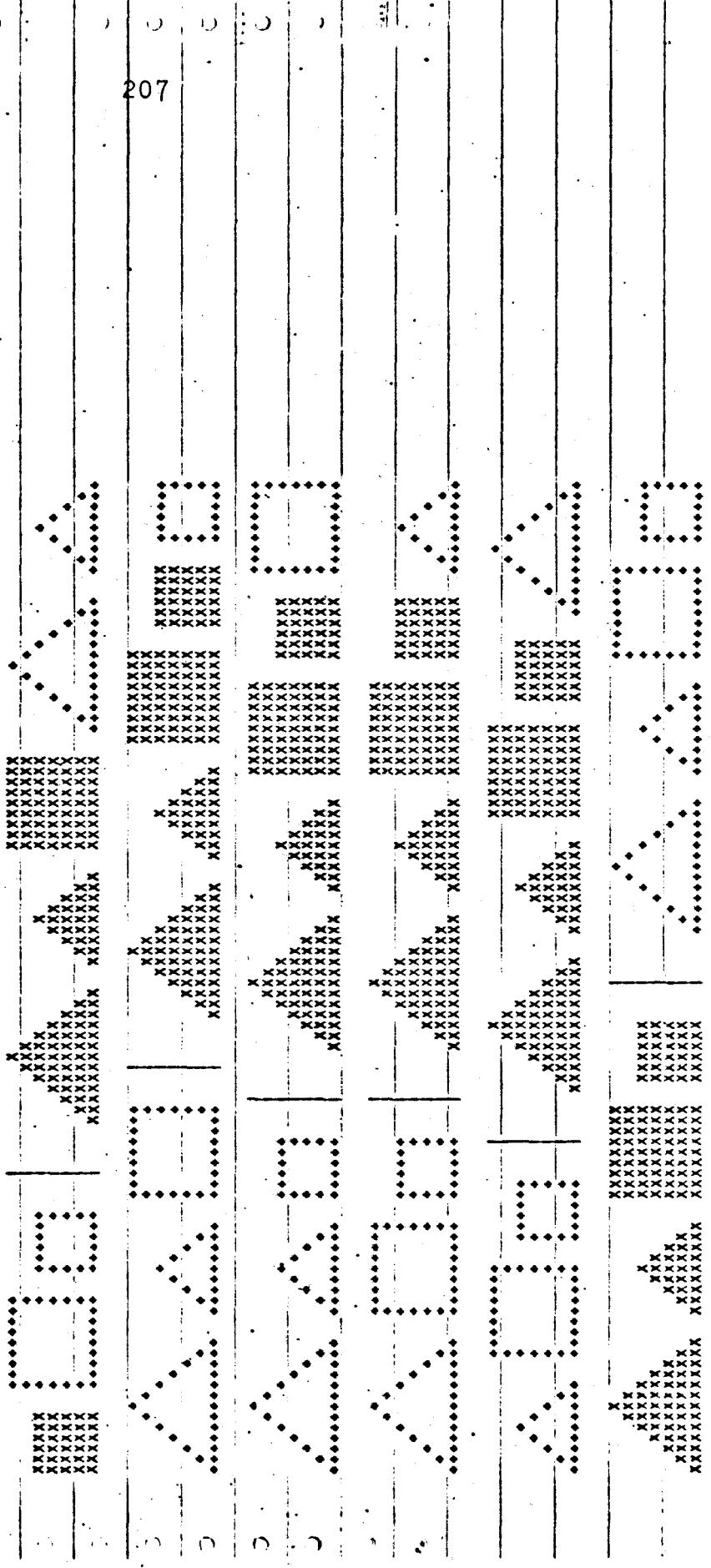
ARTICLES OF 3 DIMENSION(S)

206

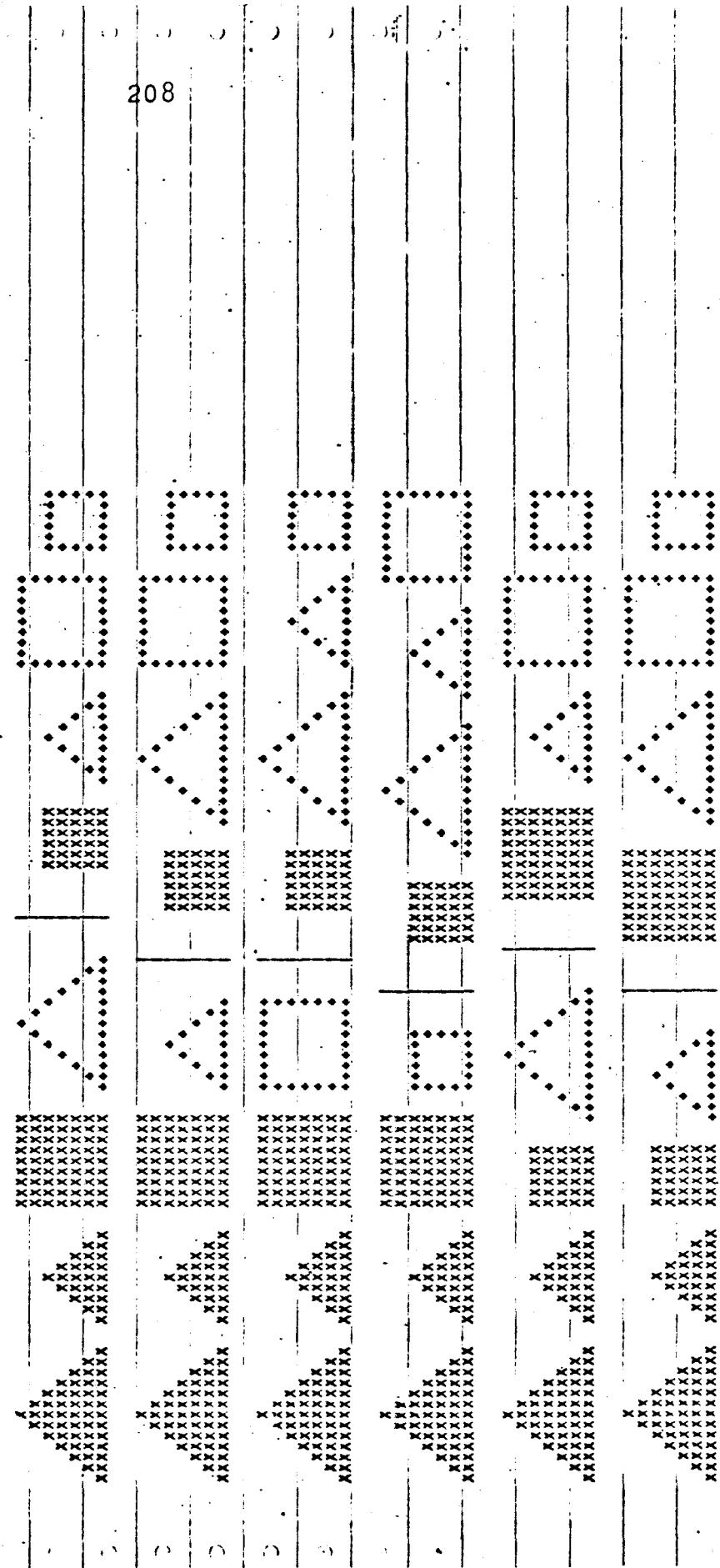


ARTIFACTS OF 3 DIMENSION(S)

207

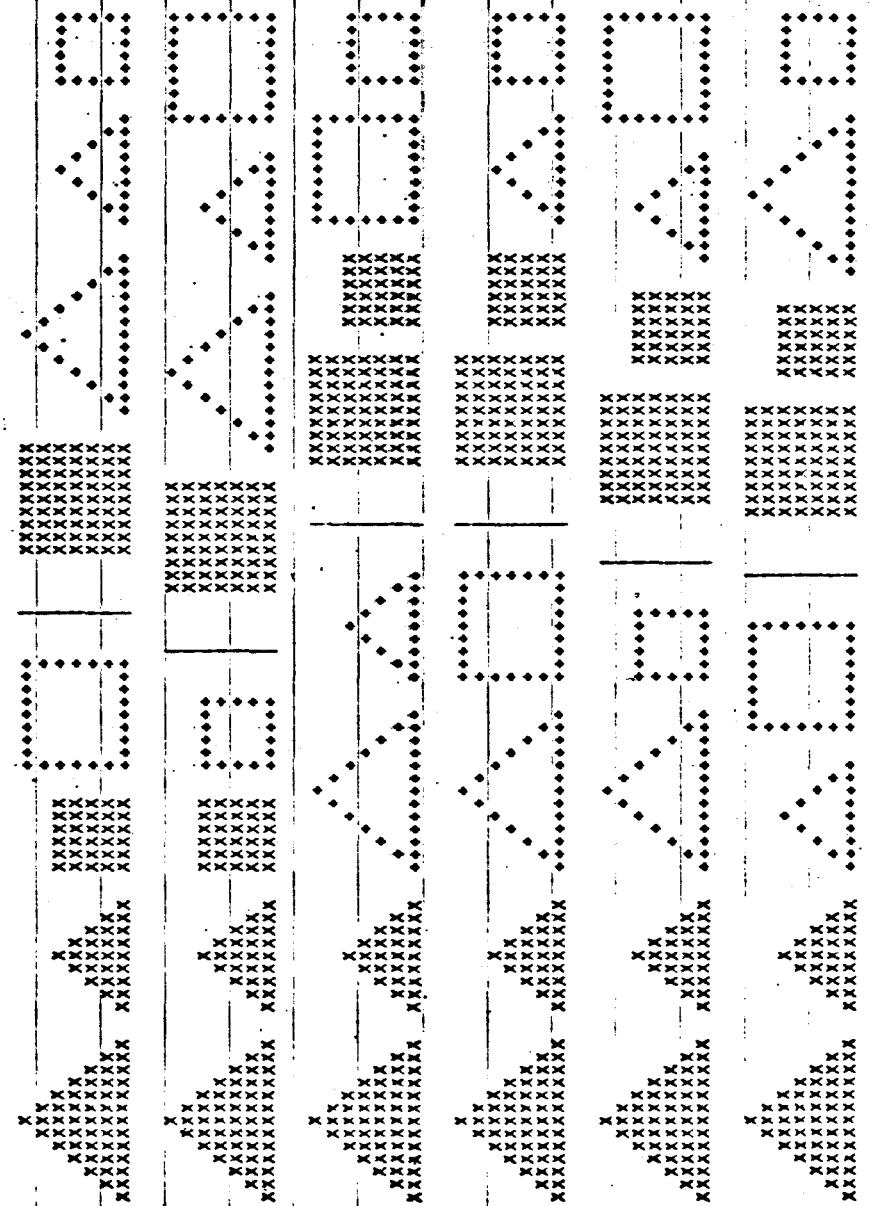


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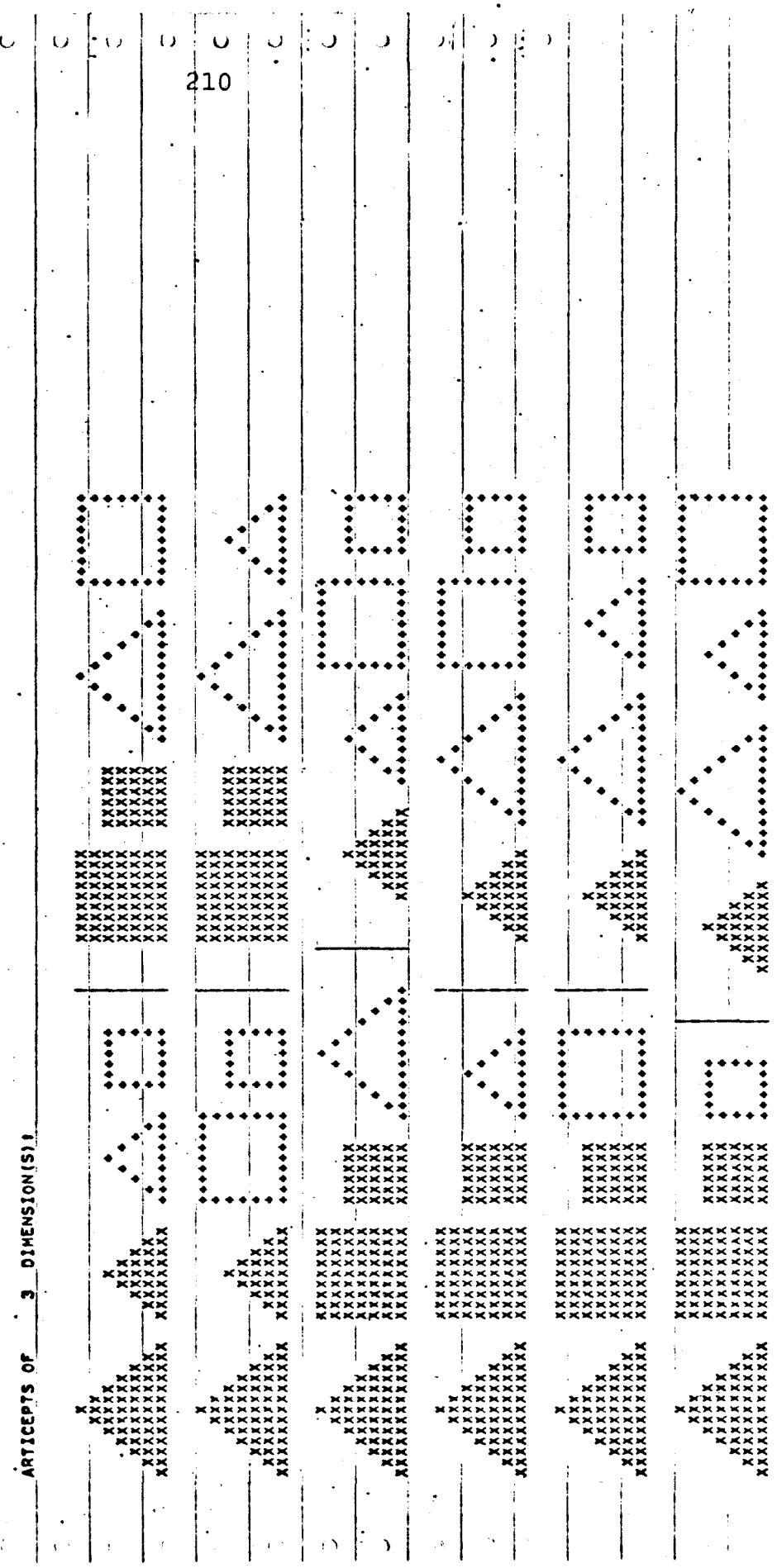


ARTIFACTS OF 3 DIMENSION(S)!

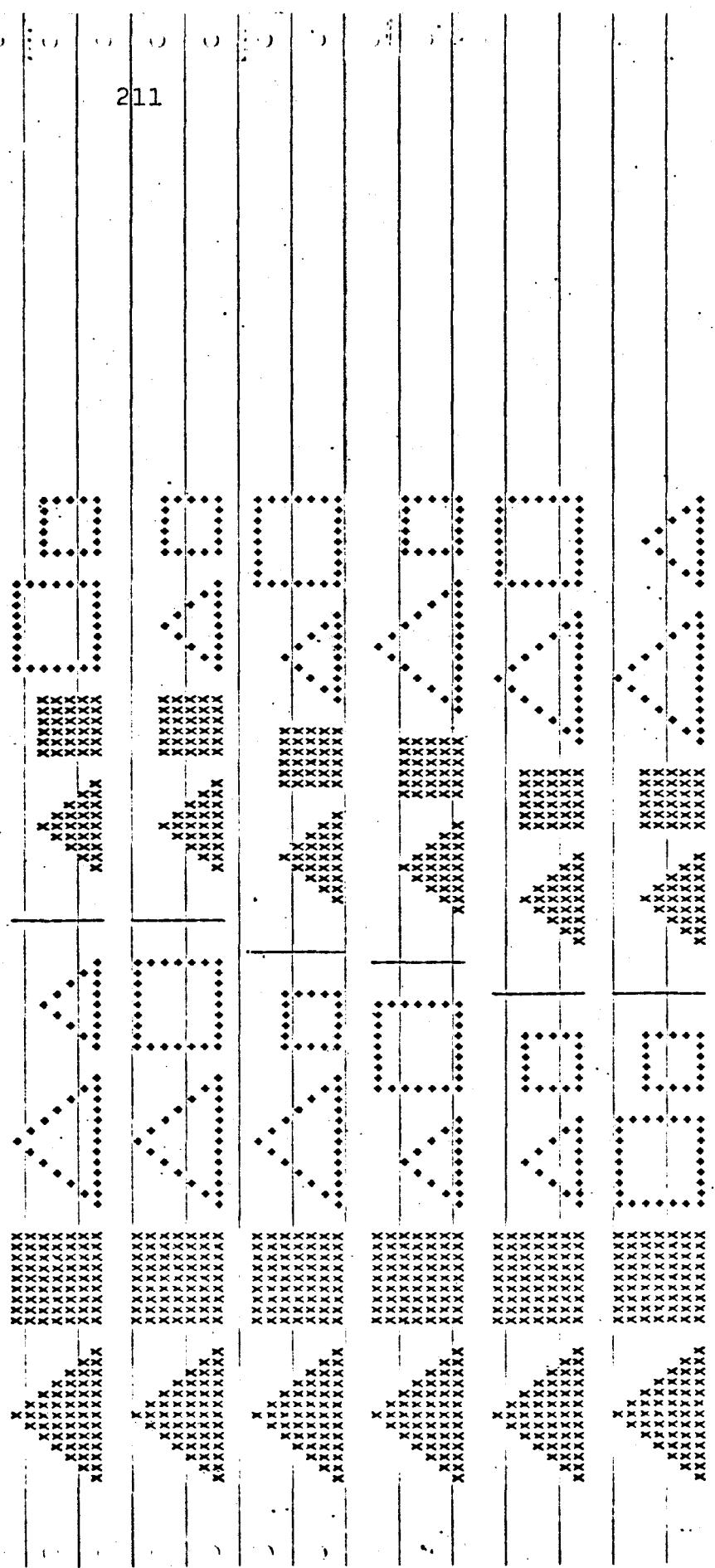
209



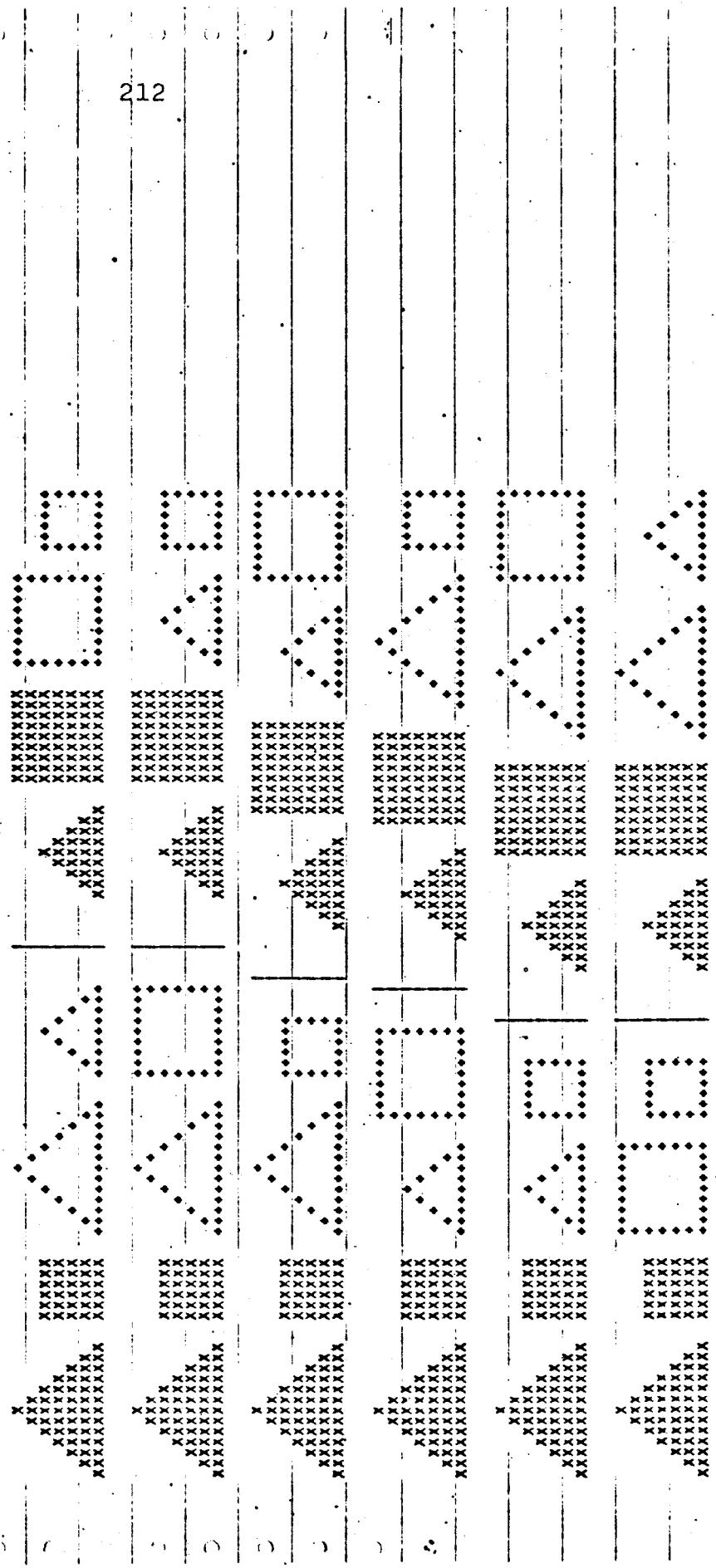
ARTICLETS OF 3 DIMENSION(S)



ARTICLETS OF 3 DIMENSION(S)

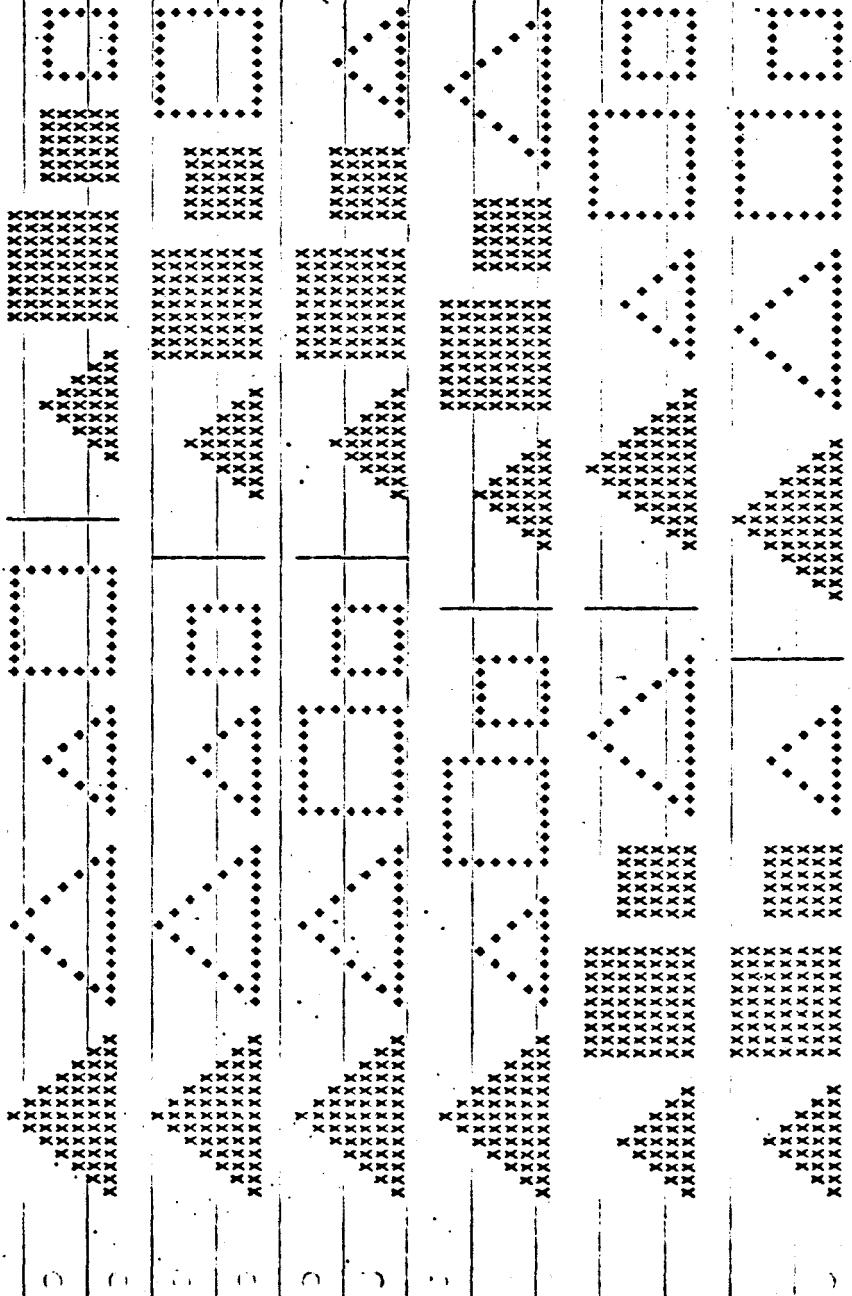


ARTICLES OF 3 DIMENSION(S)!

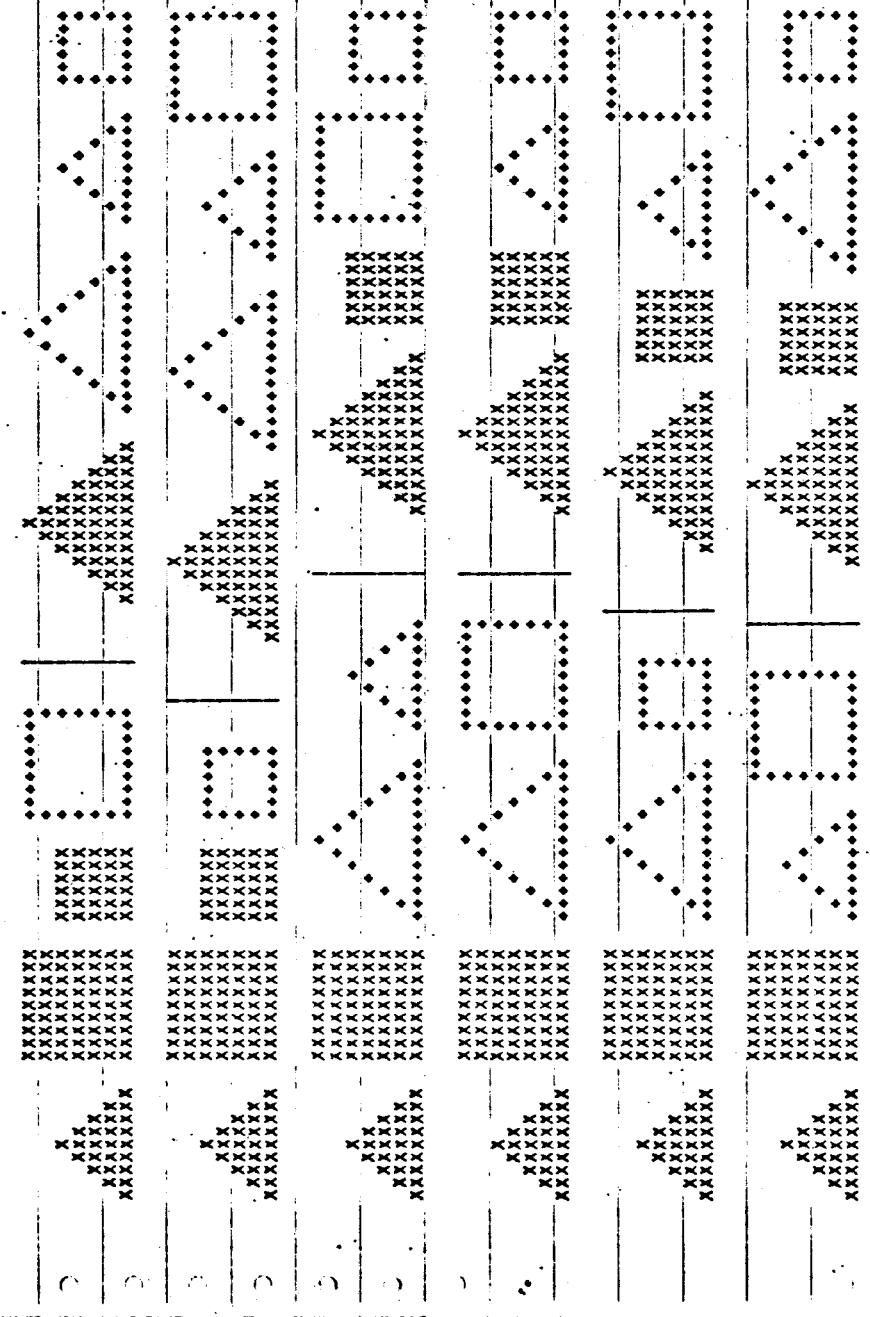


ARTICLES OF 3 DIMENSION(S)!

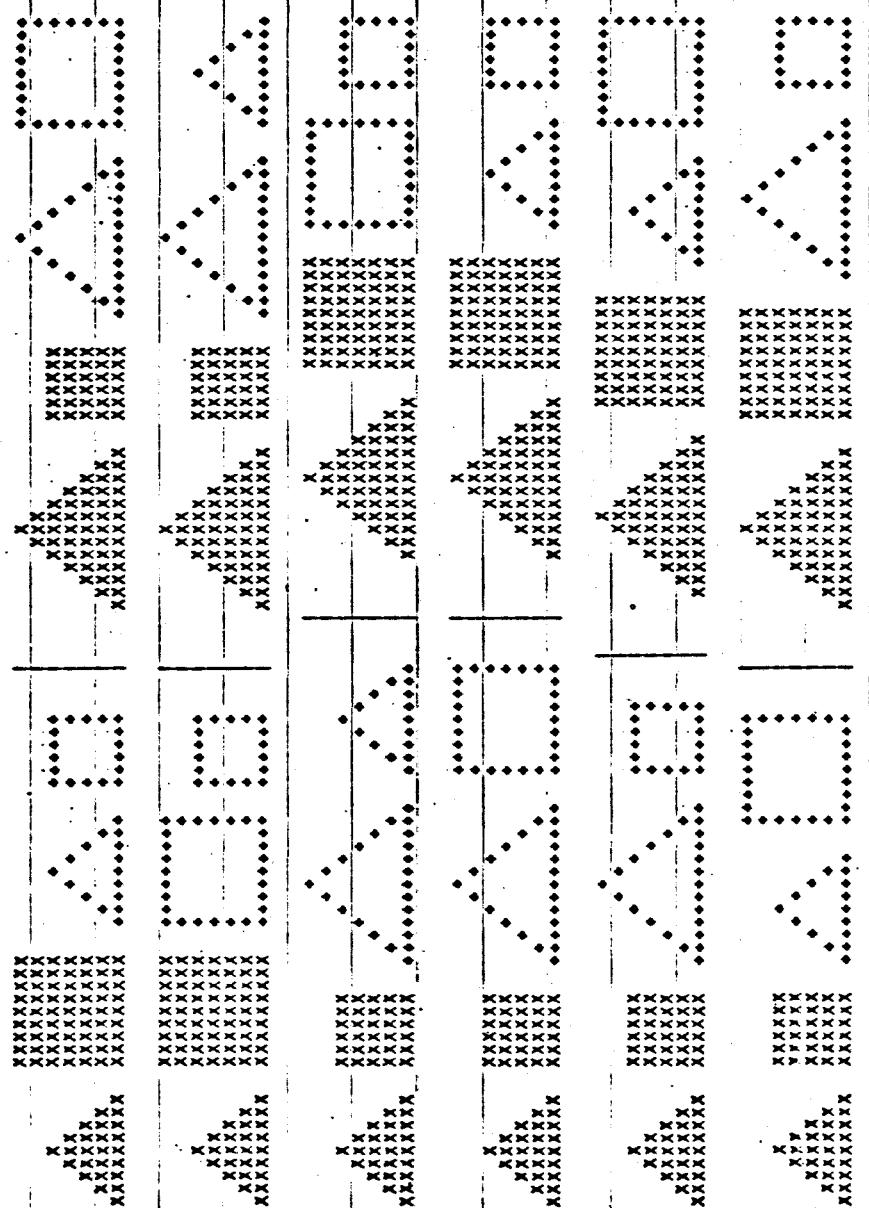
213



ARTICLES OF 3 DIMENSION(S)!!

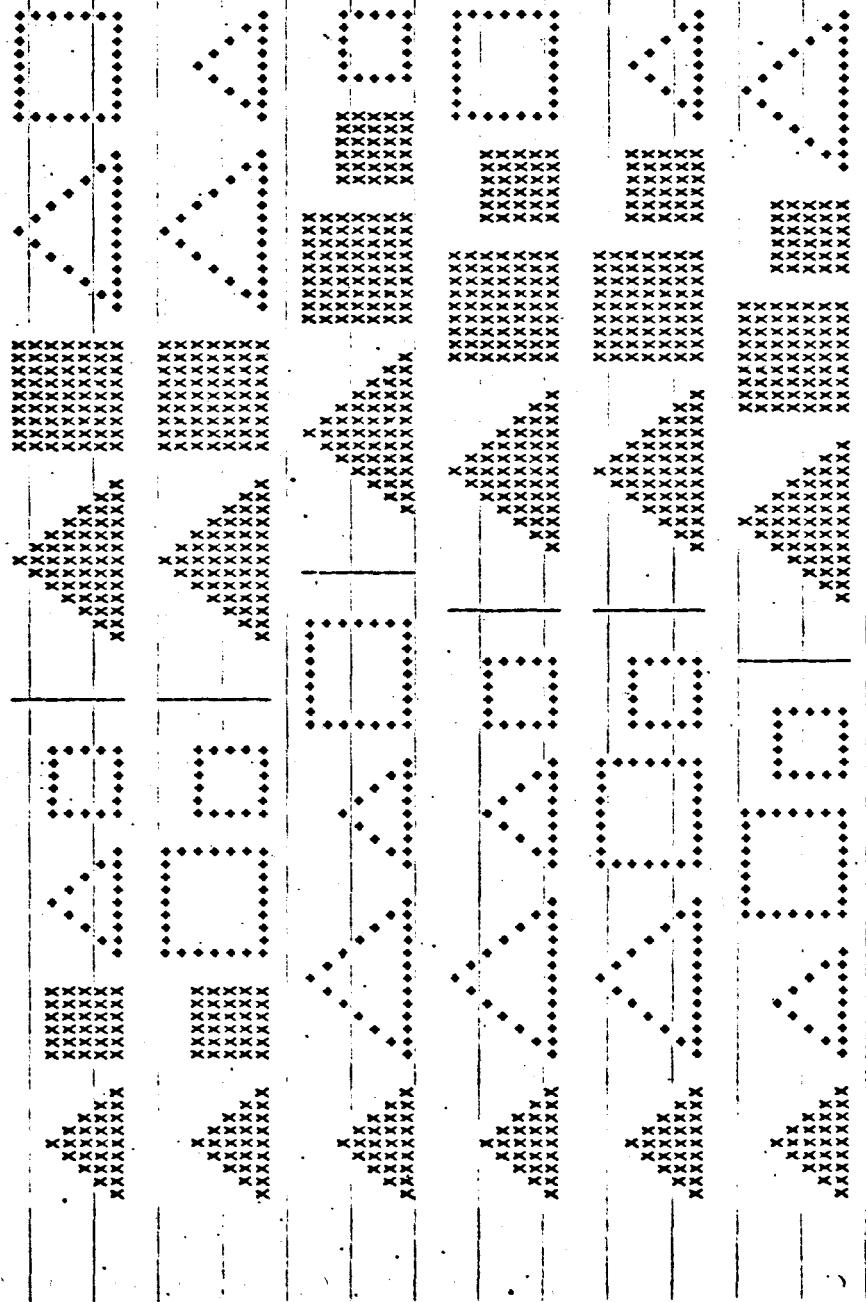


ARTIFACTS OF 3 DIMENSION(S)!



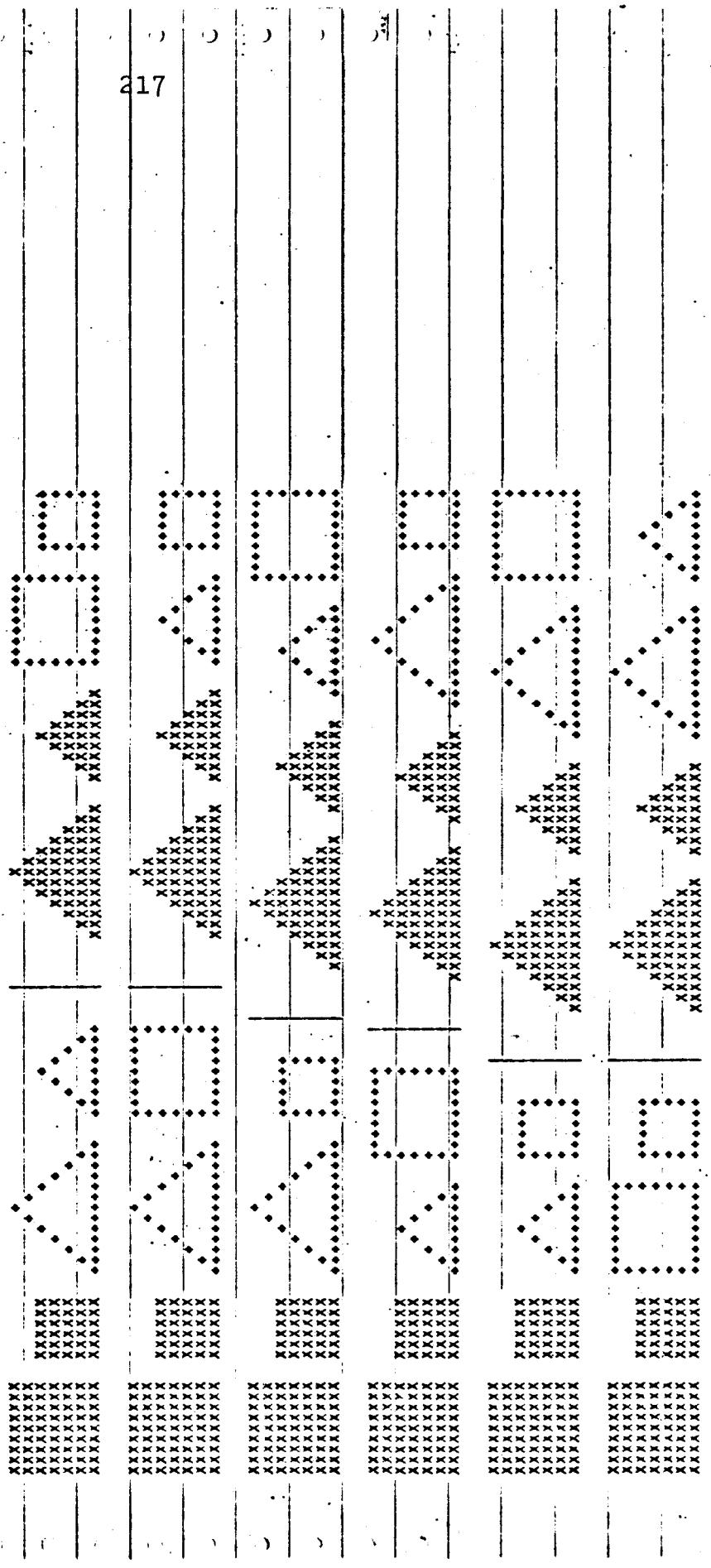
ARTICLES OF 3 DIMENSION(S)!

216



ARTICLES OF 3 DIMENSION(S)!

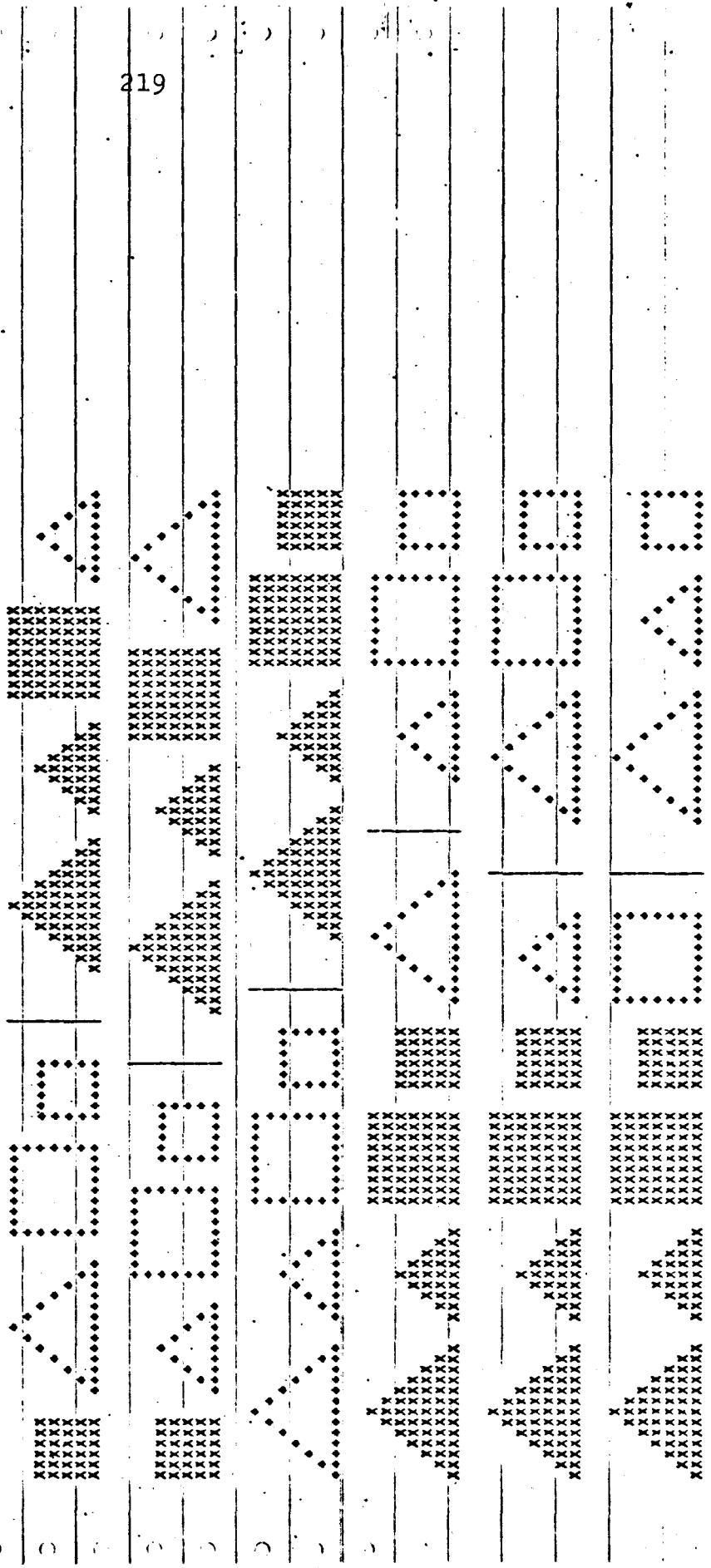
217



ARTICENTS OF 3 DIMENSION(S)!

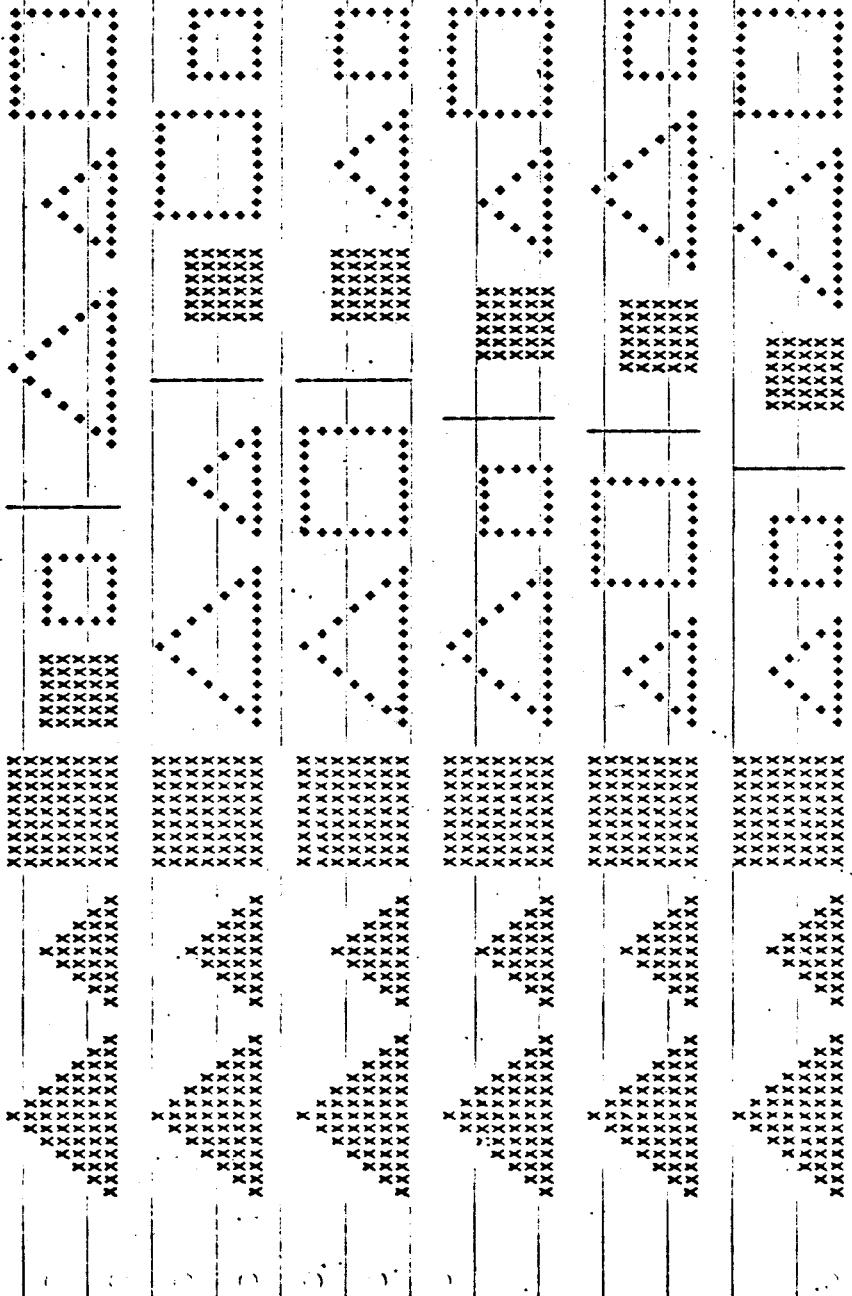
218

ARTICENTS OF 3 DIMENSION(S)!



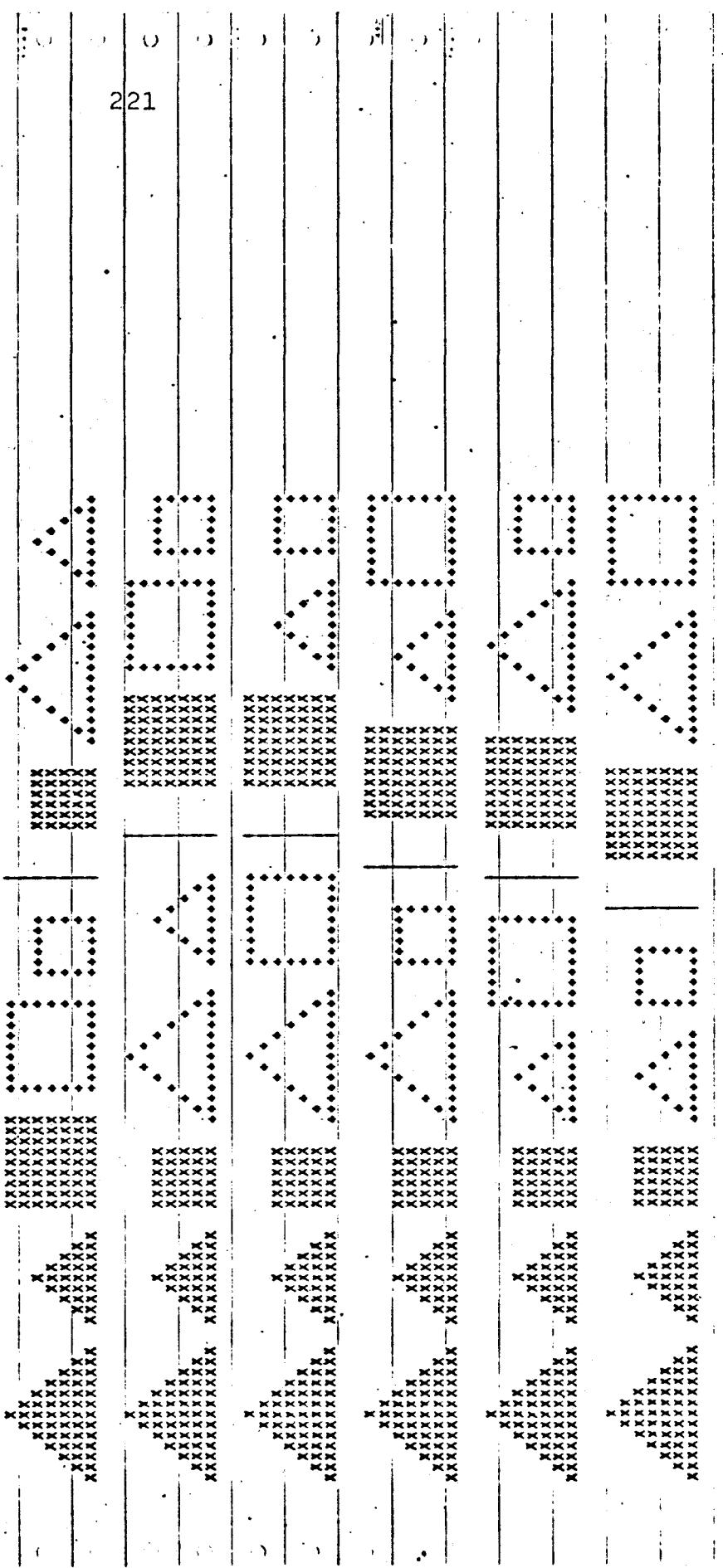
ARTIFACTS OF 3 DIMENSION(S)!

220



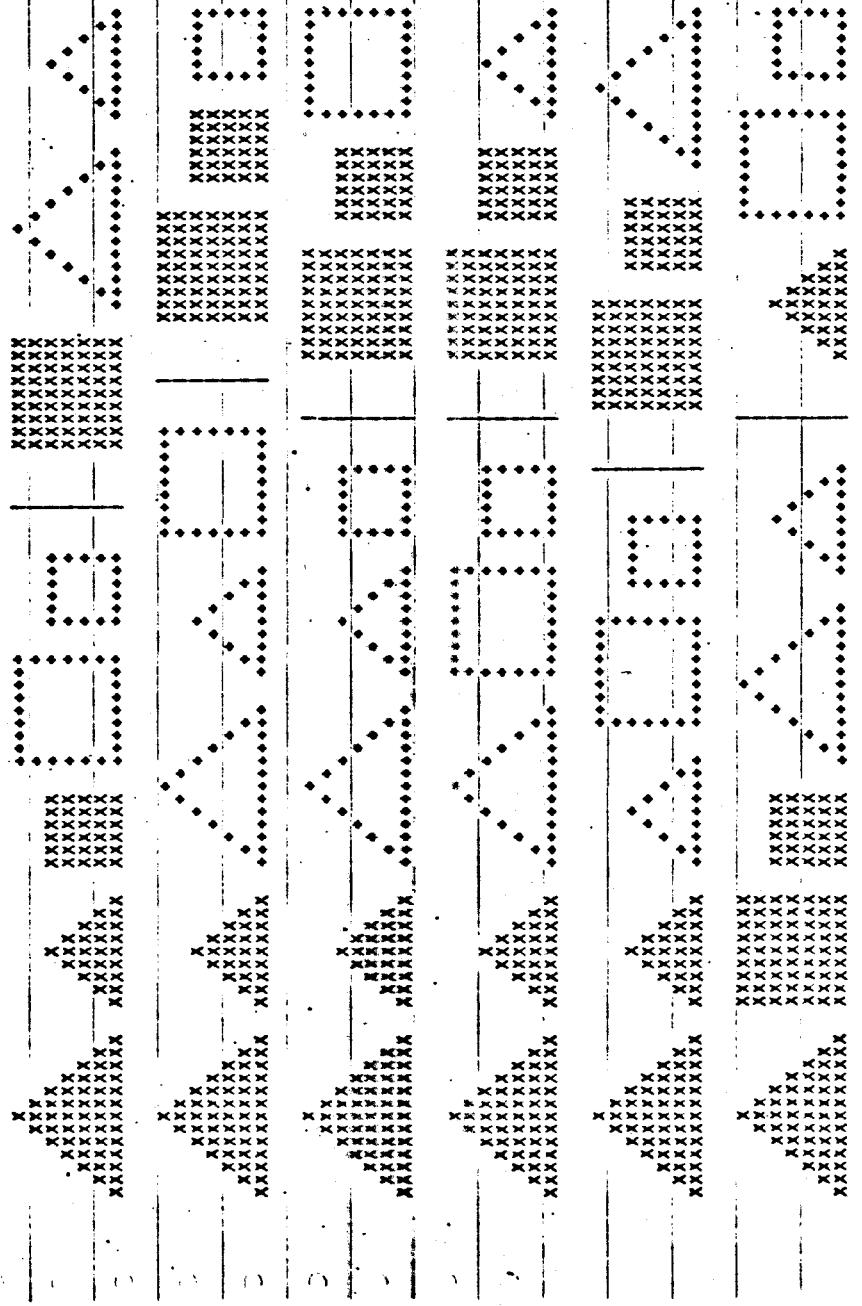
ARTICLES OF . . . 3 DIMENSION(S)!

221

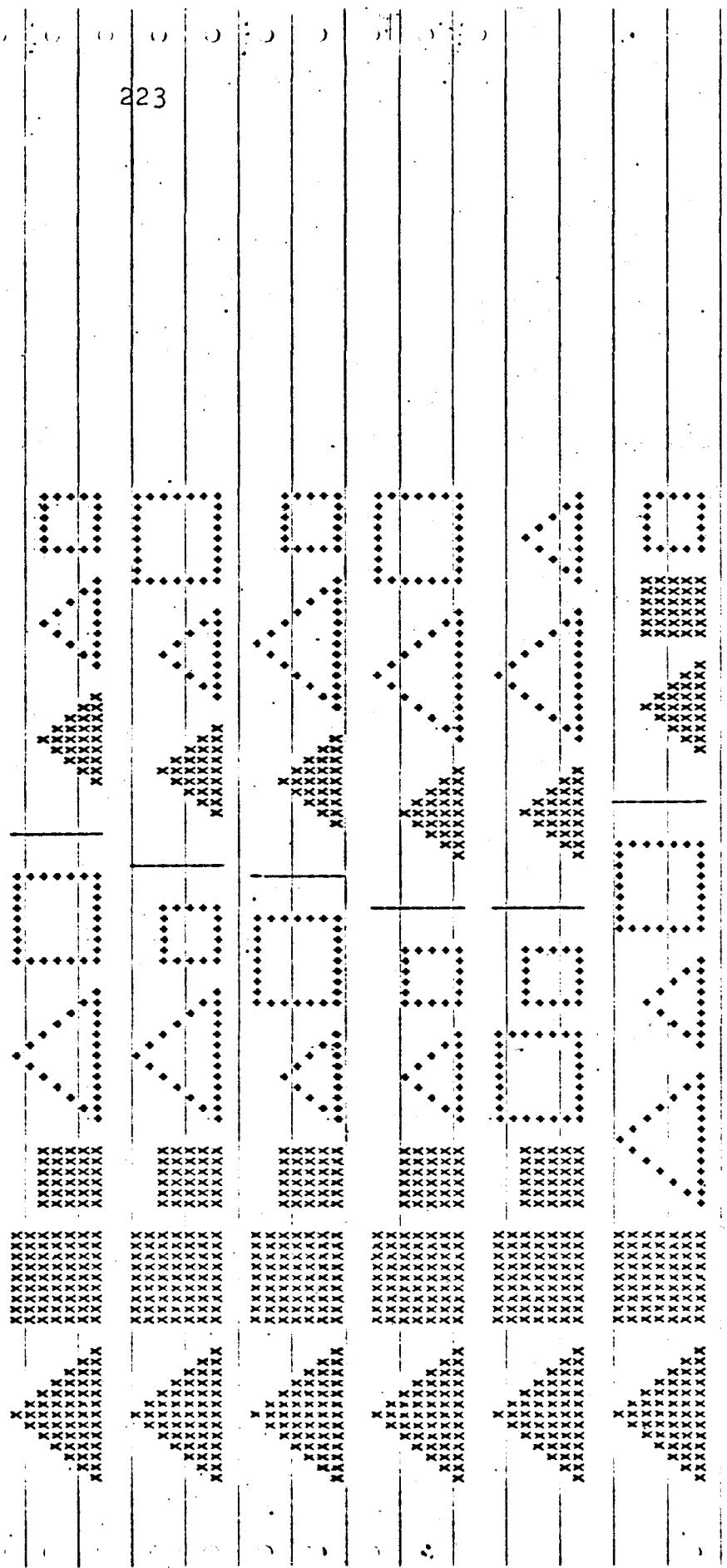


ARTICLES OF 3 DIMENSIONAL

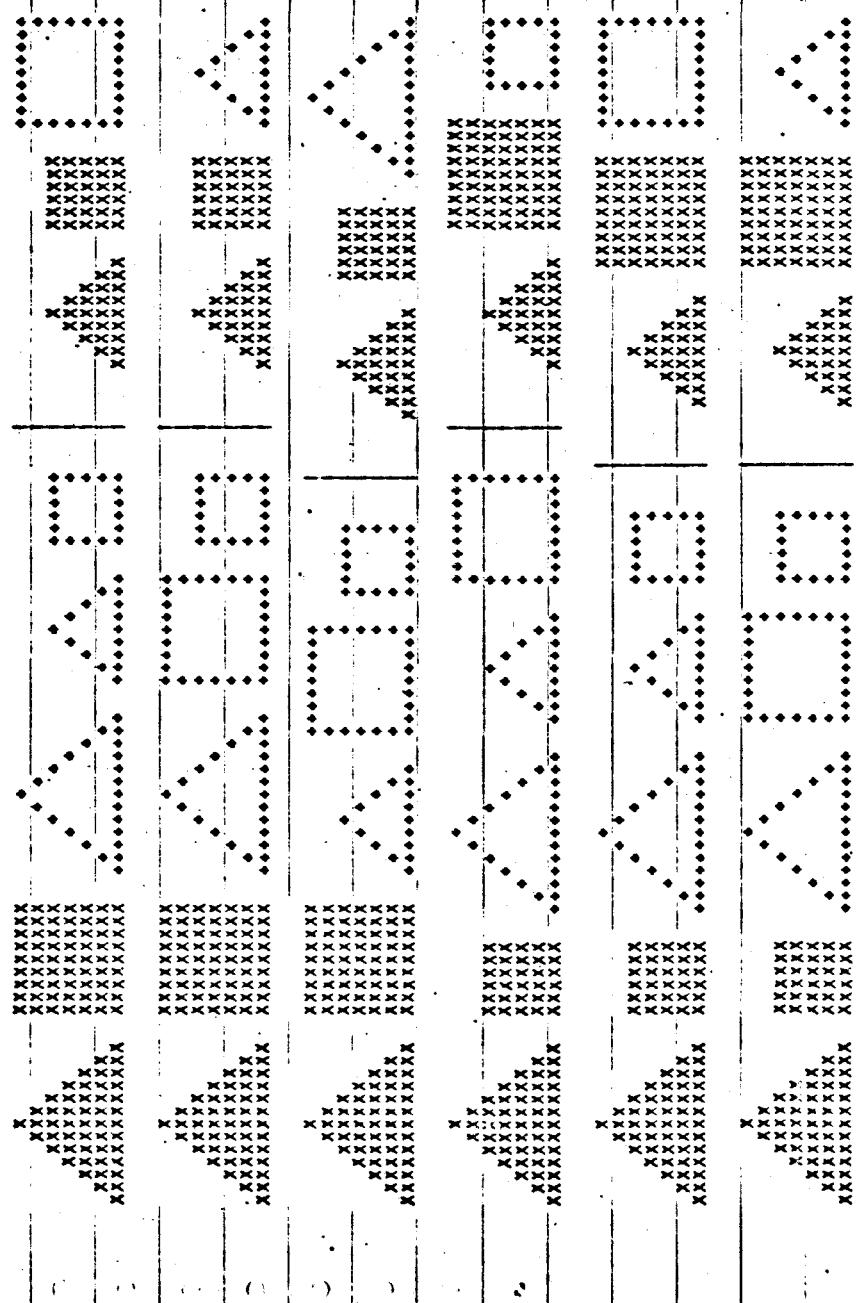
222



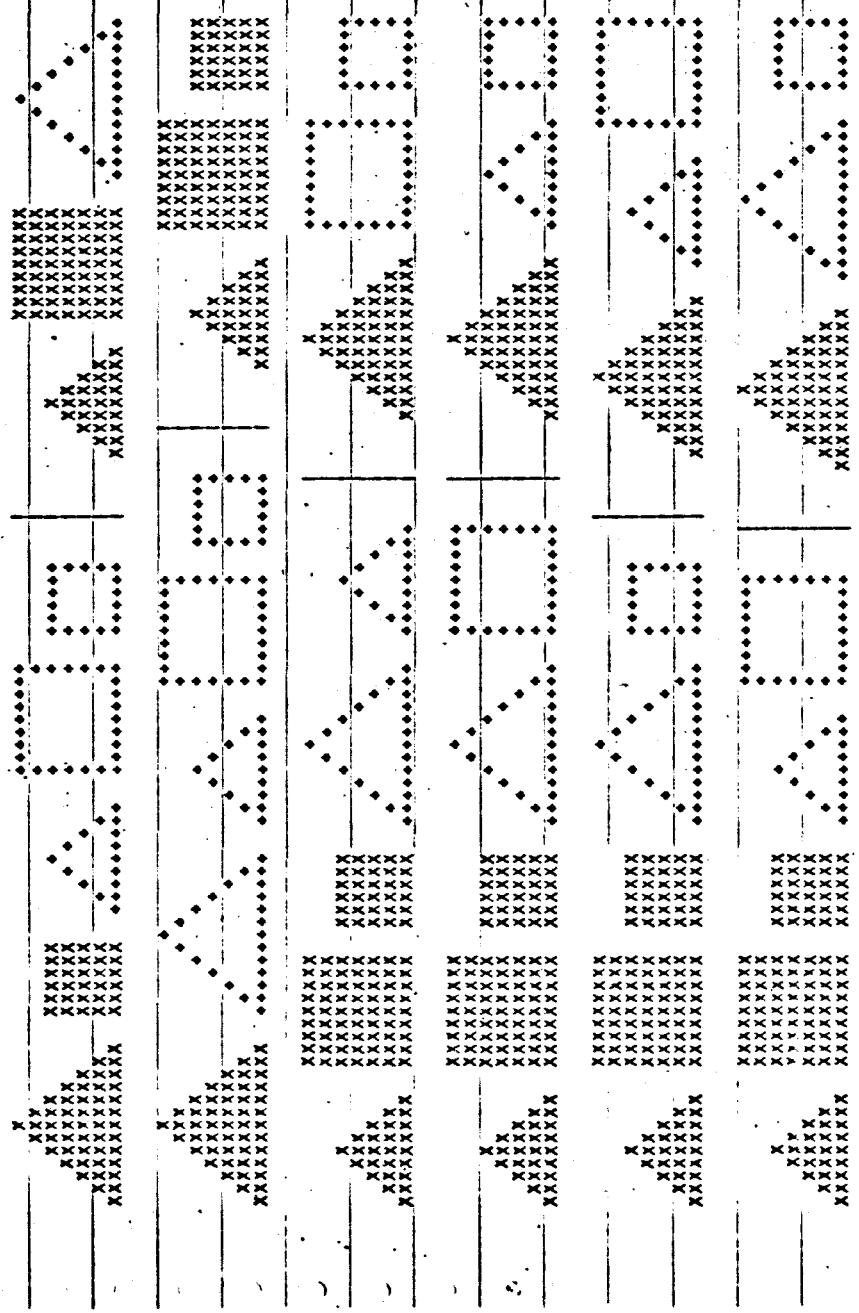
ARTICULENTS OF 3 DIMENSION(S):



ARTICLES OF 3 DIMENSION(S):



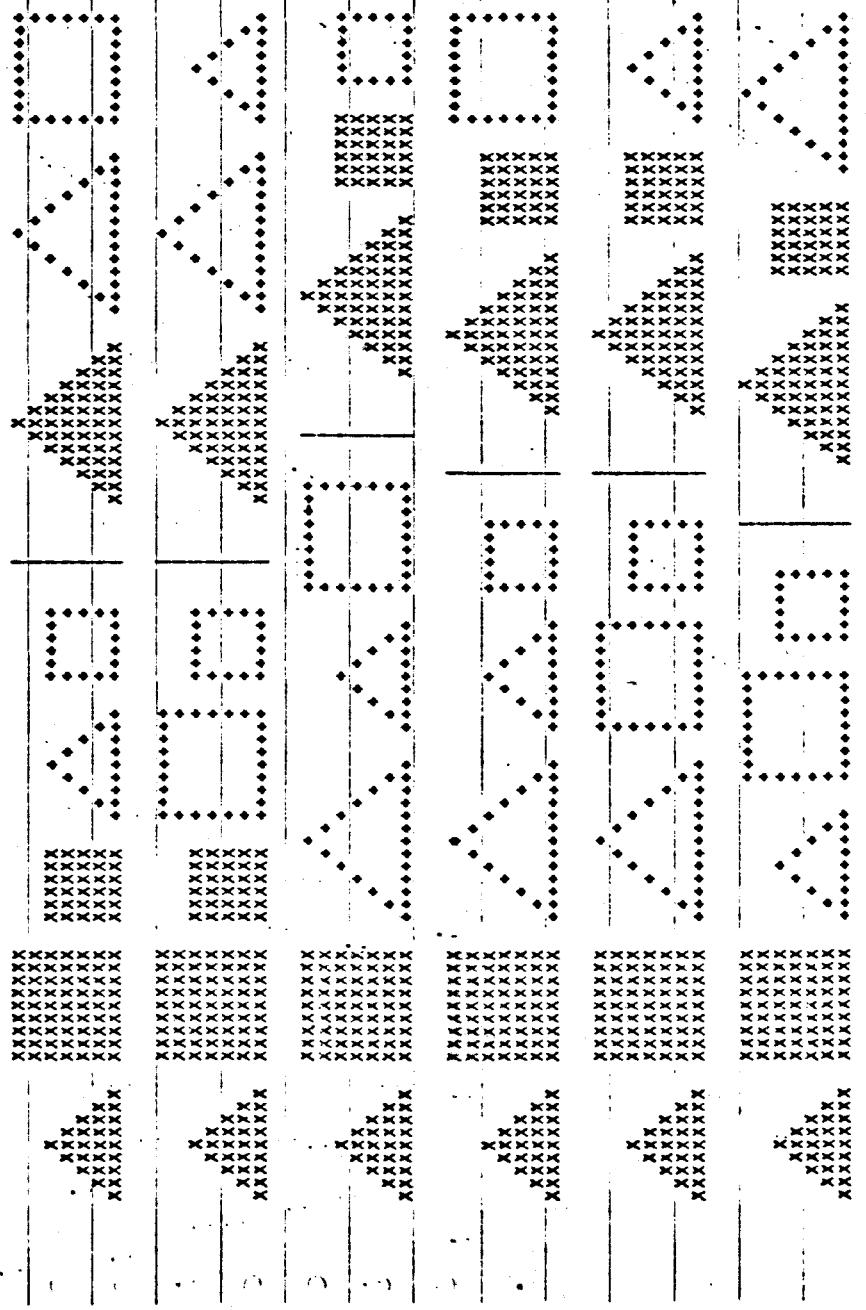
ARTICLES OF 3 DIMENSION(S) I



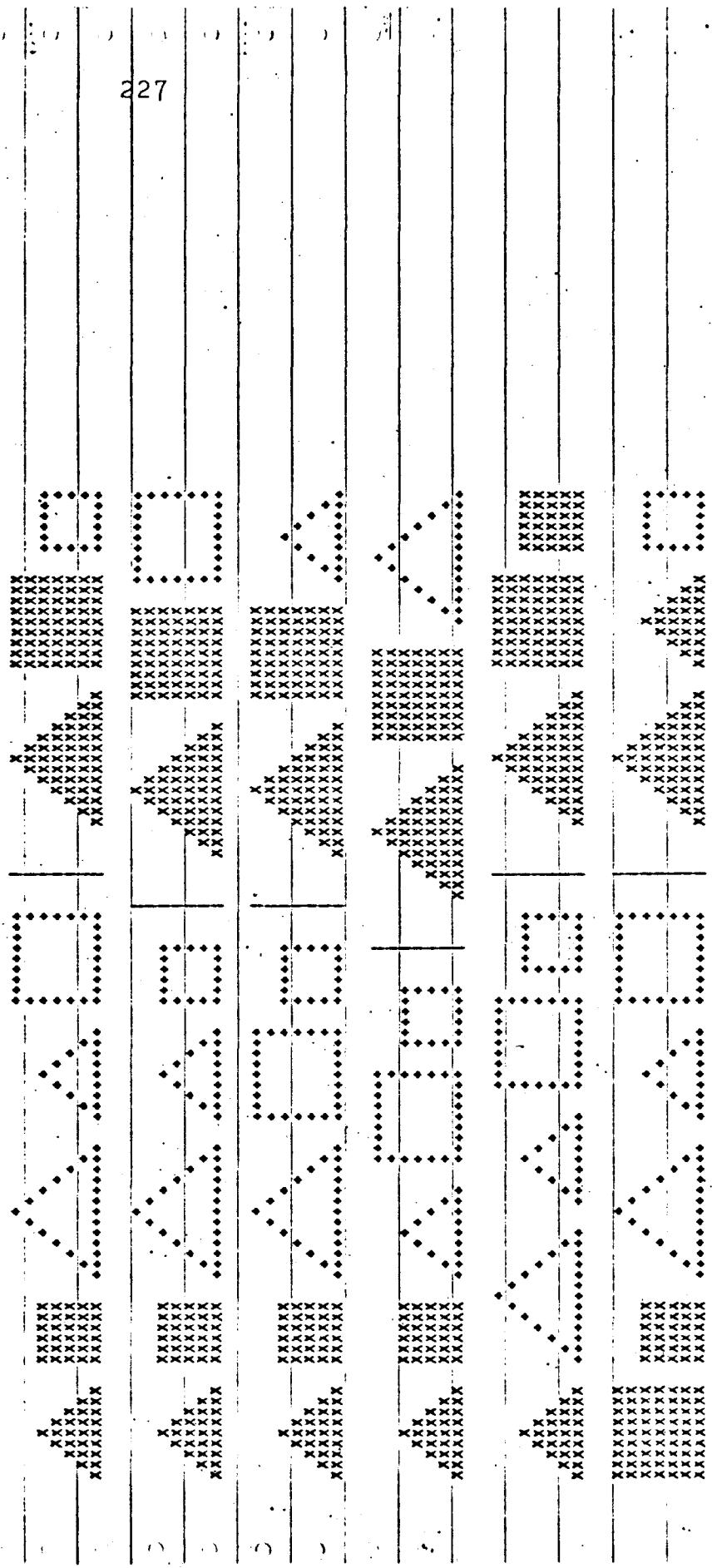
225

ARTIFICEPTS OF 3 DIMENSION(S)!!

226

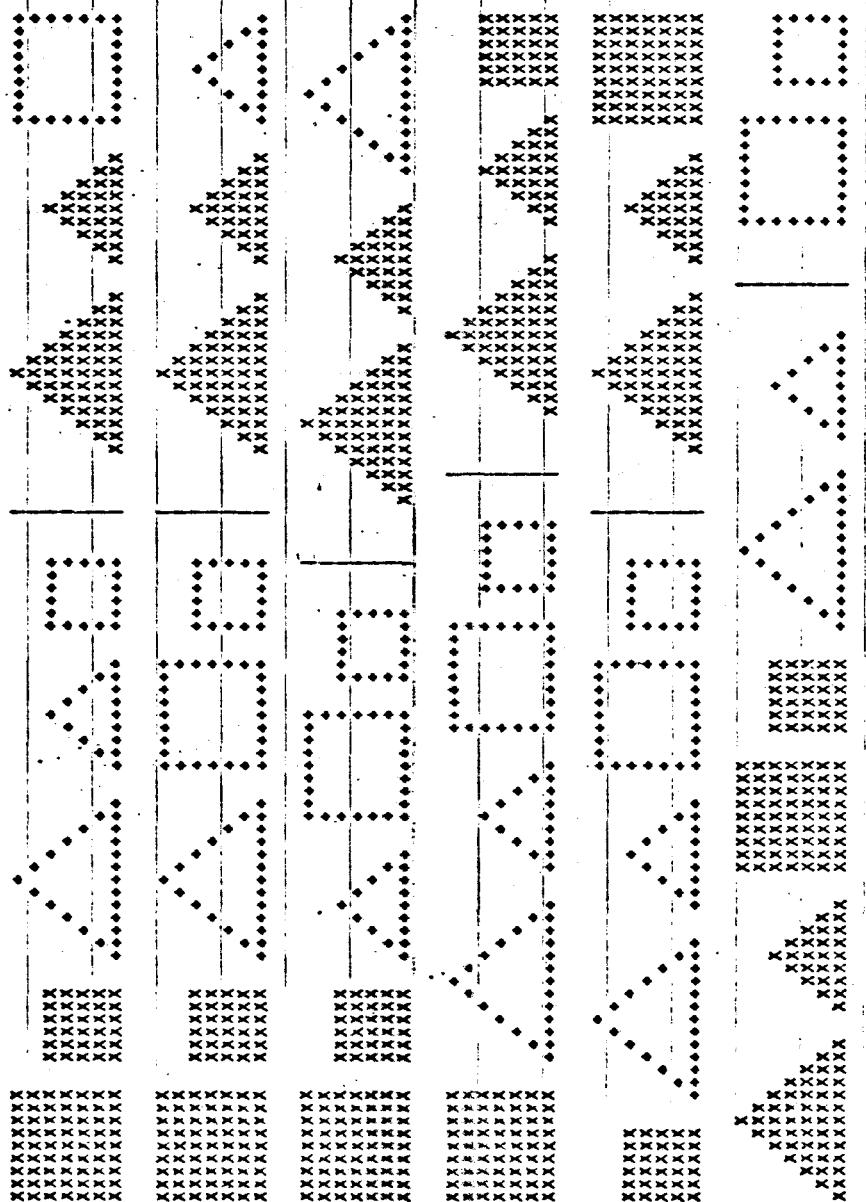


ARTICLES OF 3 DIMENSION(S)!

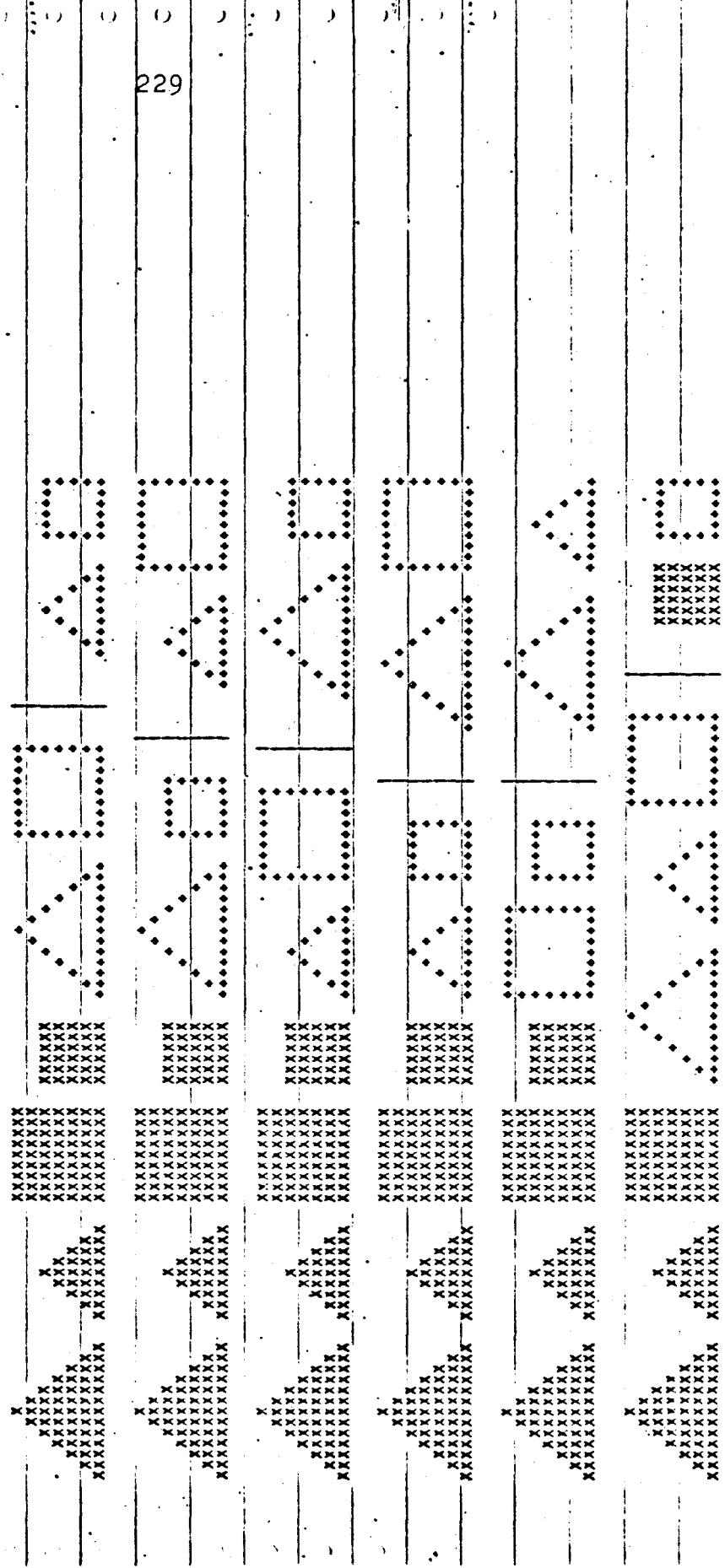


228

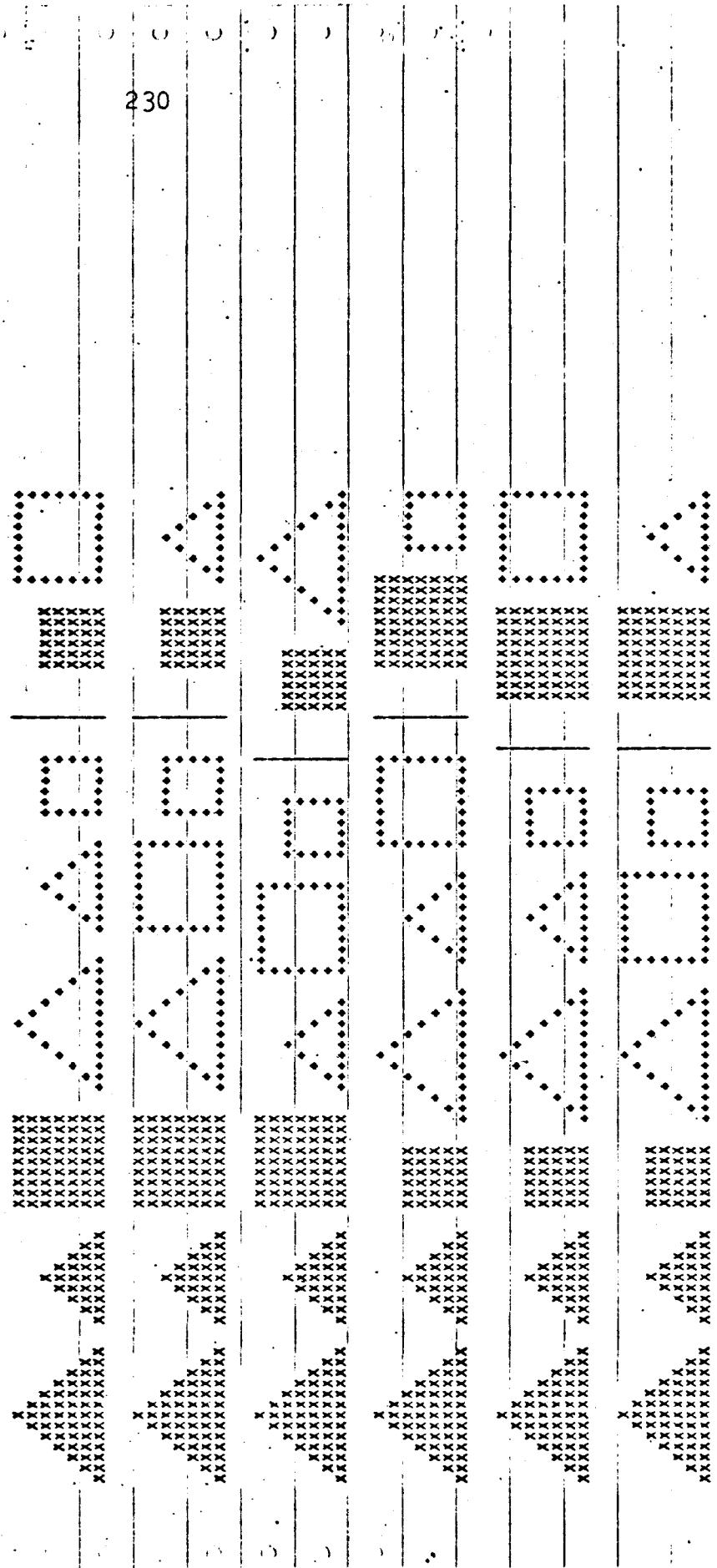
ARTIFACTS OF 3 DIMENSION(S):



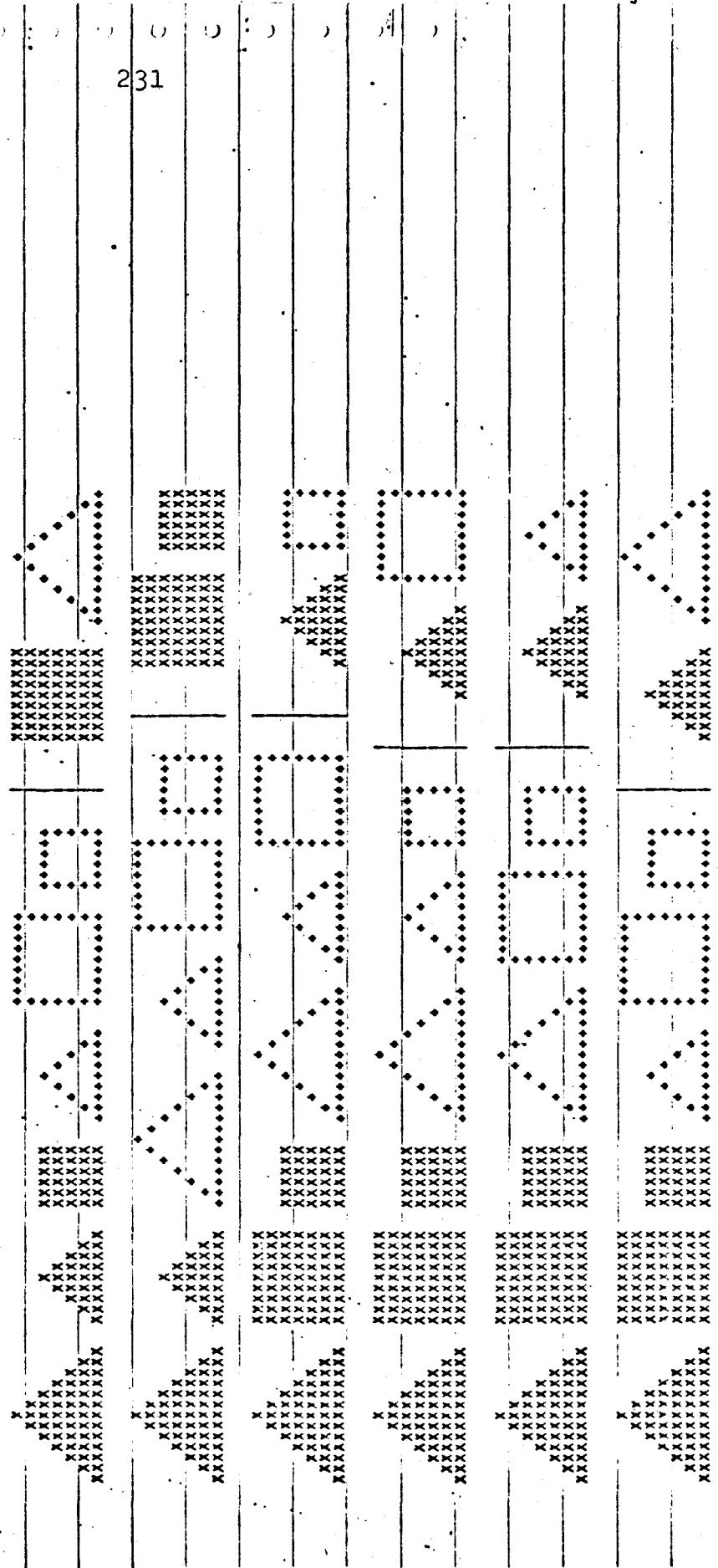
ARTIFICEPTS OF 3 DIMENSION(S)!



ARTICLES OF . . . 3 DIMENSION(S)!

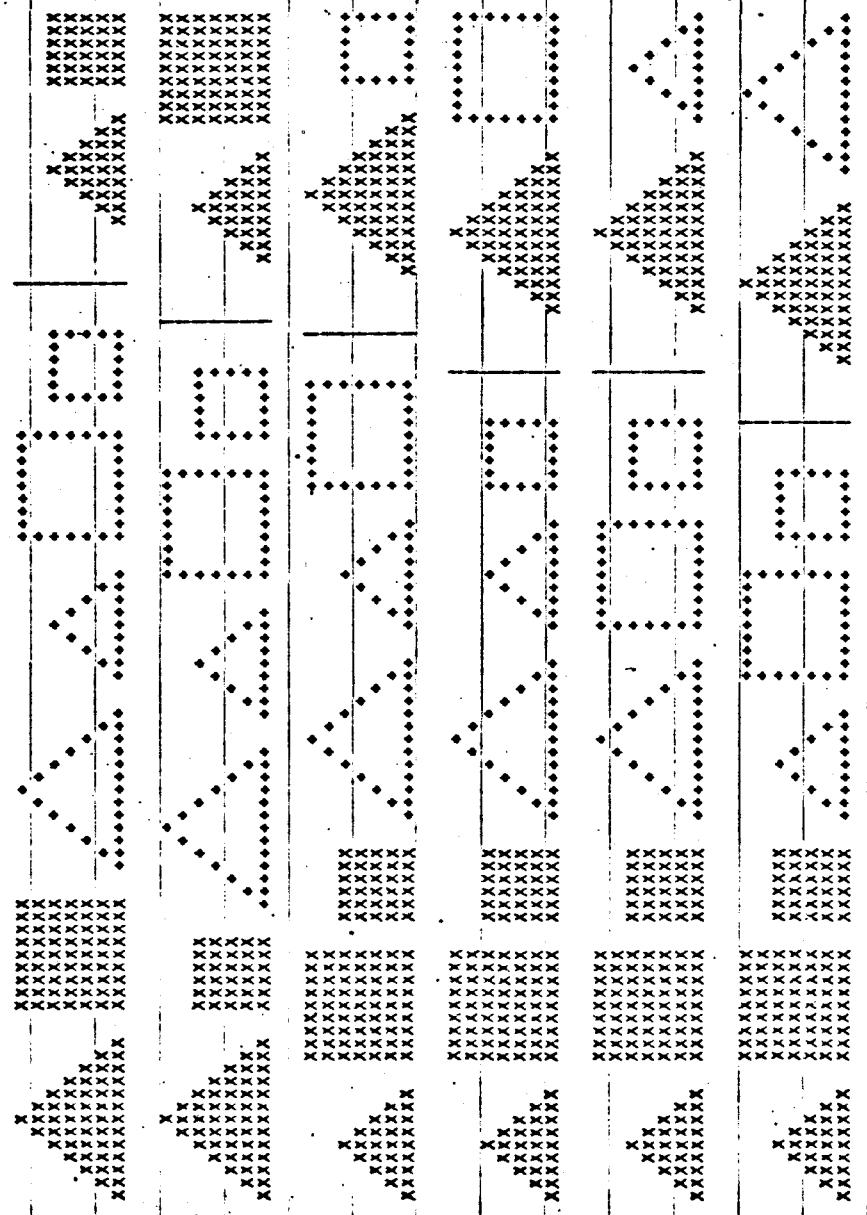


ARTICLES OF 3 DIMENSION(S).

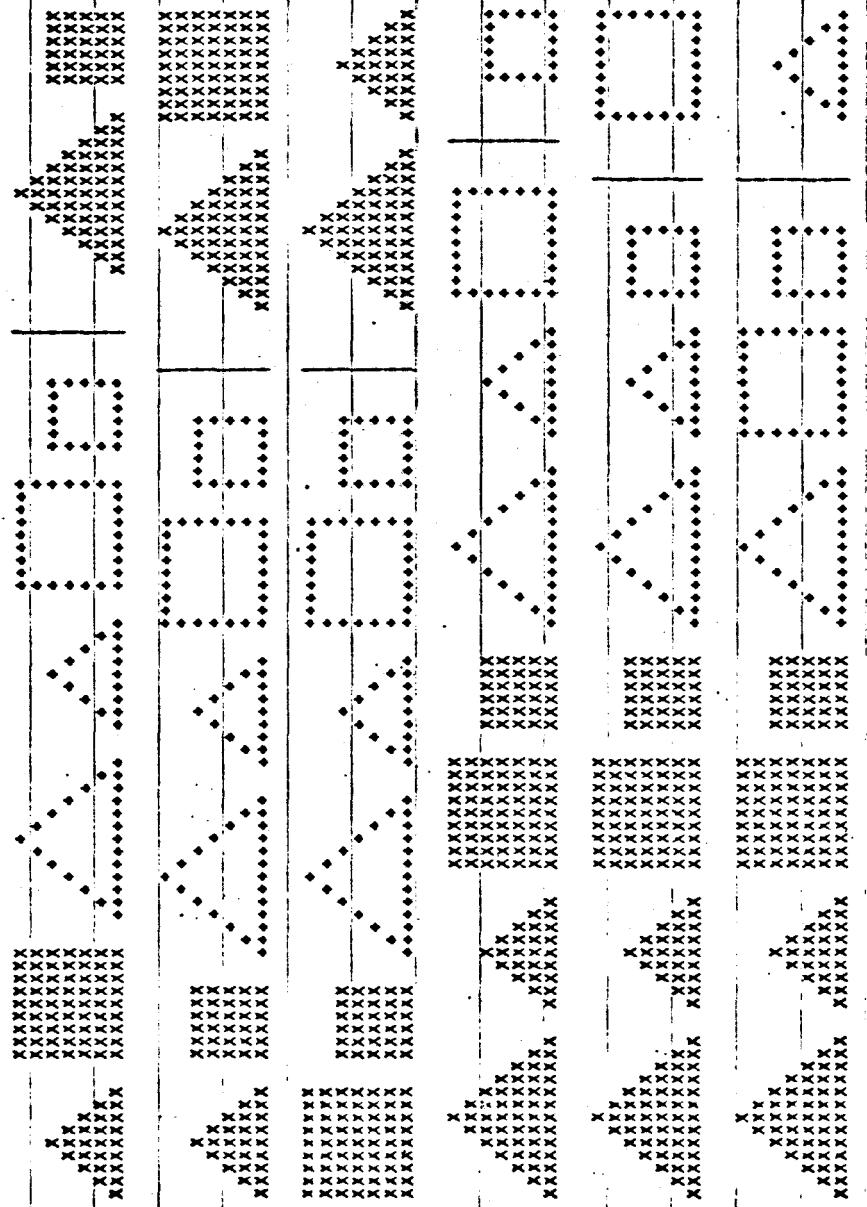


ARTICLES OF 3 DIMENSION(S):

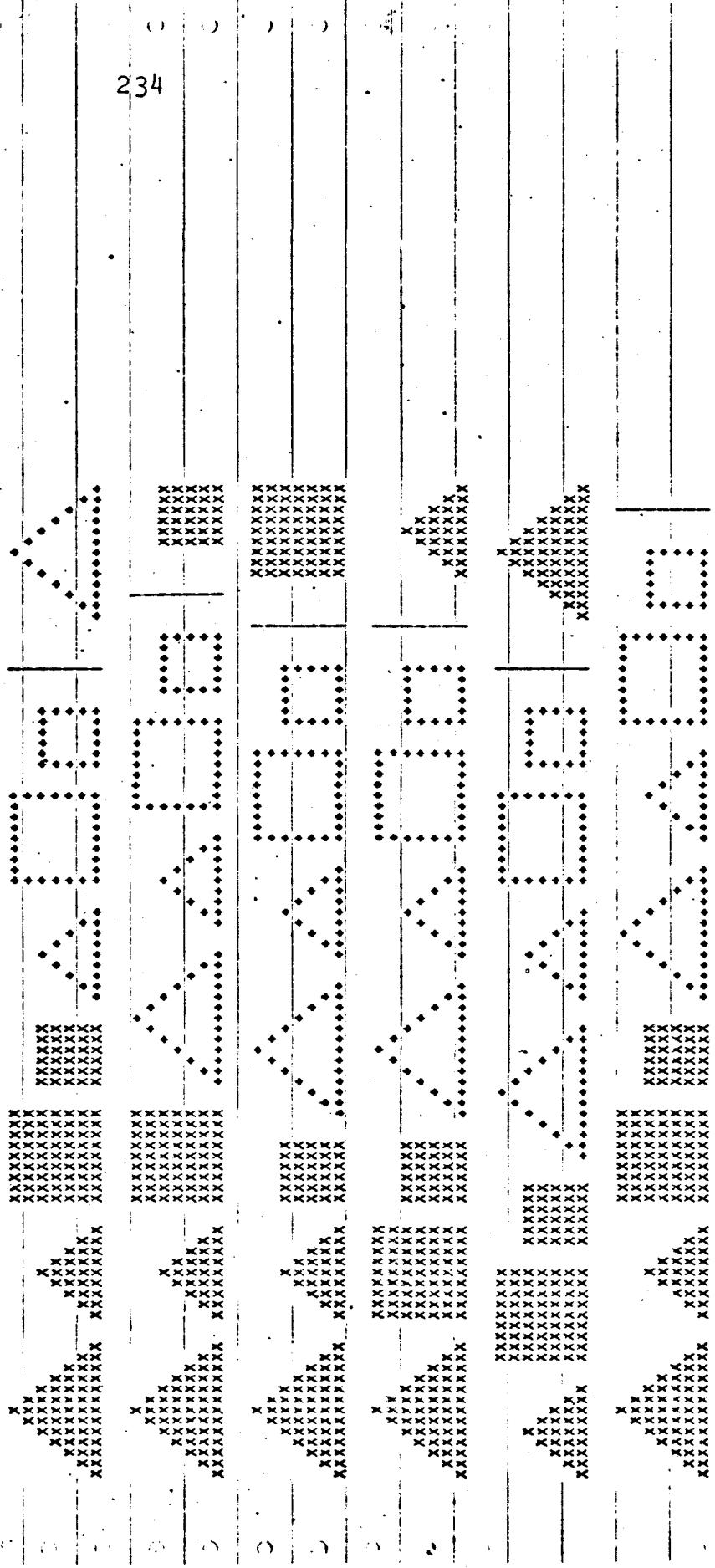
232



ARTICLES OF 3 DIMENSION(S)!



ARTIFACTS OF 3 DIMENSION(S) 1



APPENDIX 6

$$n=1 \quad 1 = \Delta \Delta$$

$$\text{IS01}_{(2)} \quad \varphi \quad 1$$

$$\text{IS02}_{(2)} \quad \Delta \quad \Delta$$

$$n=1 \quad 1 = \Delta \square$$

$$\text{IS01}_{(2)} \quad \varphi \quad 1$$

$$\text{IS02}_{(2)} \quad \Delta \quad \square$$

$$n=1 \quad 1 = \Delta \Delta$$

$$\text{IS01}_{(2)} \quad \varphi \quad 1$$

$$\text{IS02}_{(2)} \quad \Delta \quad \Delta$$

$$n=1 \quad 1 = \Delta \square$$

$$\text{IS01}_{(2)} \quad \varphi \quad 1$$

$$\text{IS02}_{(2)} \quad \Delta \quad \square$$

$$n=1 \quad 1 = \Delta \Delta$$

$$\text{IS01}_{(2)} \quad \varphi \quad 1$$

$$\text{IS02}_{(2)} \quad \Delta \quad \Delta$$

$$n=1 \quad 1 = \square \square$$

$$\text{IS01}_{(2)} \quad \varphi \quad 1$$

$$\text{IS02}_{(2)} \quad \square \quad \square$$

$n=1 \quad 1 = \blacksquare \square$

IS01₍₂₎ φ 1

IS02₍₂₎ ■ □

$n=1 \quad 1 = \blacksquare \square$

IS01₍₂₎ φ 1

IS02₍₂₎ ■ □

$n=1 \quad 1 = \triangle \Delta$

IS01₍₂₎ φ 1

IS02₍₂₎ △ ▲

$n=1 \quad 1 = \triangle \square$

IS01₍₂₎ φ 1

IS02₍₂₎ △ □

$n=1 \quad 1 = \triangle \square$

IS01₍₂₎ φ 1

IS02₍₂₎ △ □

$n=1 \quad 1 = \square \square$

IS01₍₂₎ φ 1

IS02₍₂₎ □ □

$n=2$

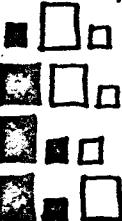
$$1 = \blacksquare \cdot \square \square$$

IS01 (2)

 Φ

1

IS02 (8)



IS03 (4)

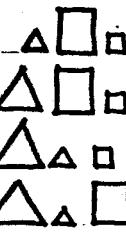


IS04 (2)



$n=2$ $1 = \Delta \Delta \square \square$ IS01₍₂₎ \varnothing

1

IS02₍₈₎ Δ Δ \square \square IS03₍₄₎ $\Delta \square$ IS04₍₂₎ $\Delta \square$ $\Delta \square$

$n=2$ $1 = \triangle \blacksquare \triangle \square$ IS01₍₂₎ φ

1

IS02₍₈₎ \triangle $\blacksquare \triangle \square$ \blacksquare $\triangle \triangle \square$ \triangle $\blacksquare \blacksquare \square$ \square $\triangle \blacksquare \triangle$ IS03₍₄₎ $\triangle \blacksquare$ $\triangle \square$ $\triangle \triangle$ $\blacksquare \square$ IS04₍₂₎ $\triangle \square$ $\blacksquare \triangle$

$n=2$ $1 = \Delta \Delta \Delta \Delta$

IS01 (2)

 \varnothing

1

IS02 (8)



IS03 (4)



IS04 (2)



$n=2$

1 ▲▲■■

IS01 (2)

Φ

1

IS02 (8)



IS03 (4)

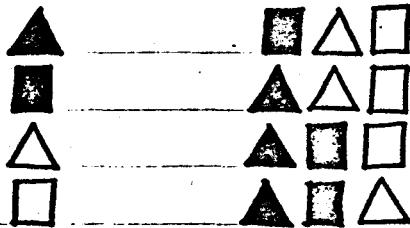
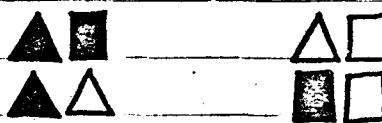


IS04 (2)



$n=2$ $1 = \triangle \blacksquare \triangle \blacksquare$ ISO1₍₂₎ φ

1

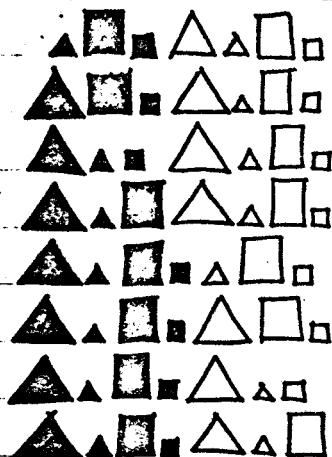
ISO2₍₈₎ISO3₍₄₎ISO4₍₂₎

IS01 (2)

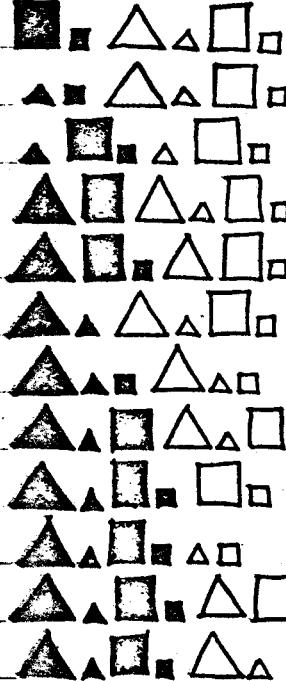
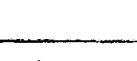
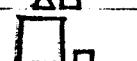
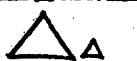
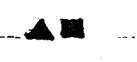
 φ

1

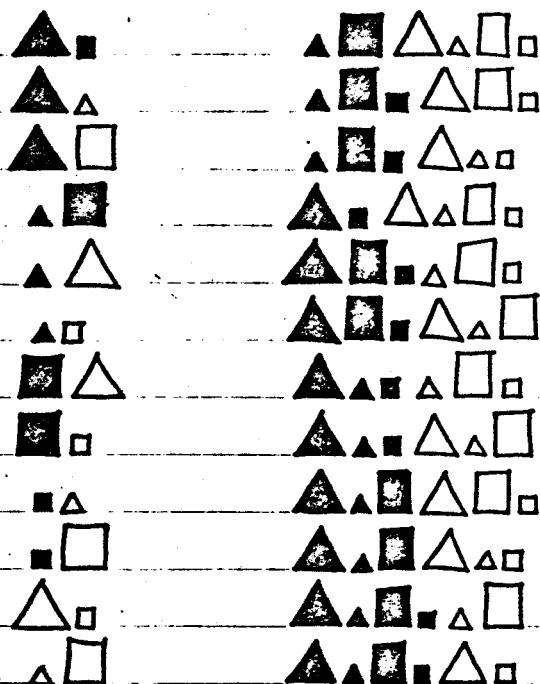
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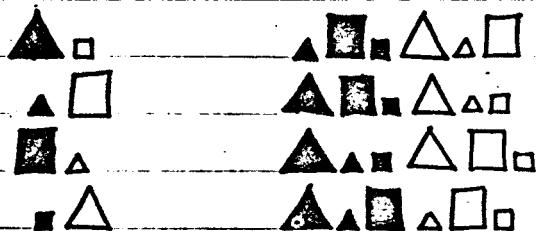
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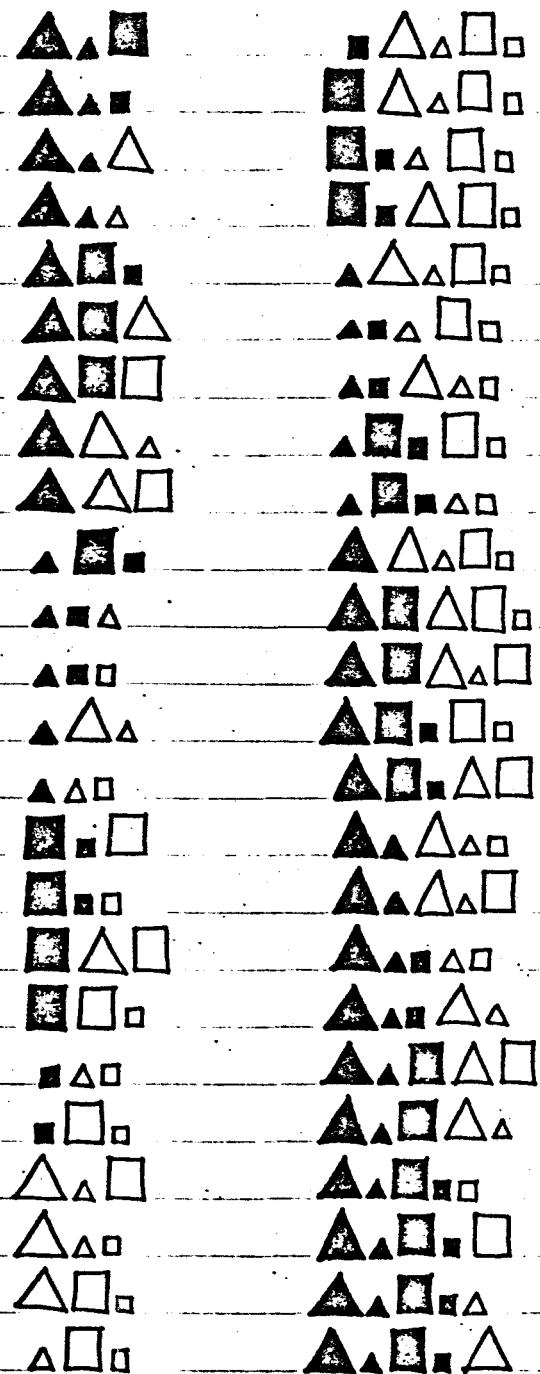
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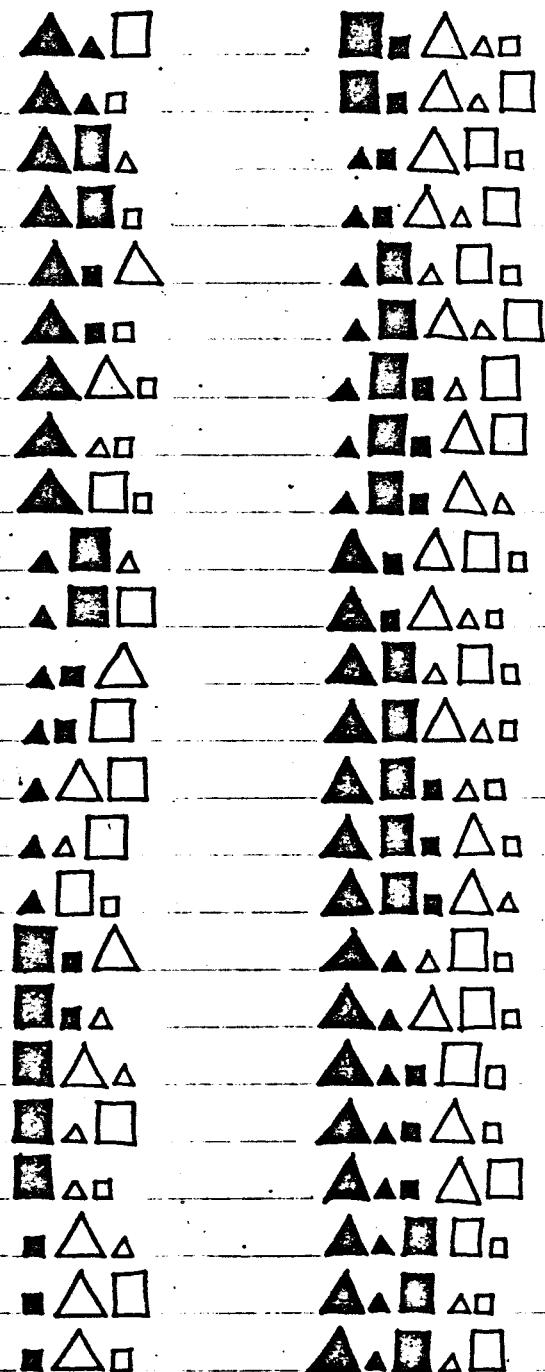
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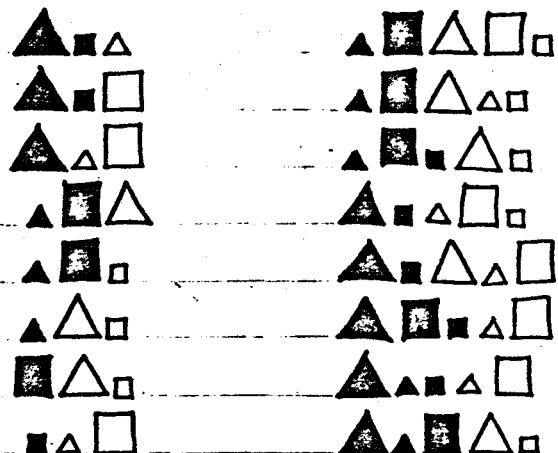
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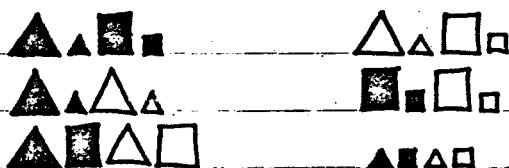
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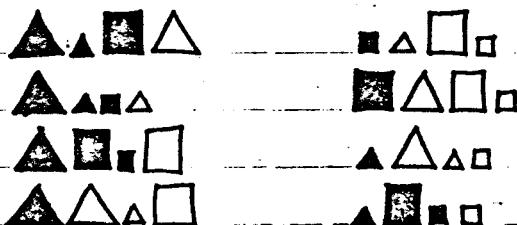
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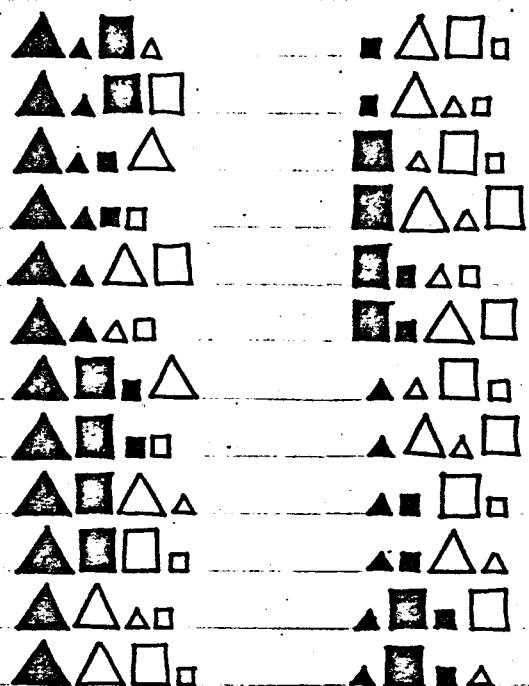
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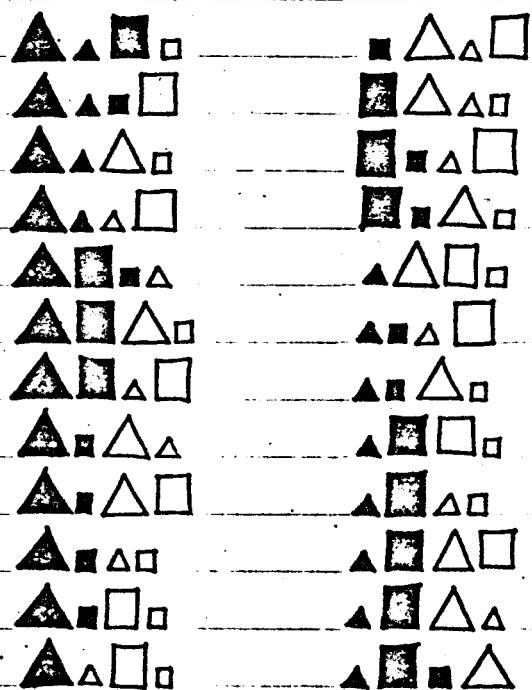
IS010 (8)



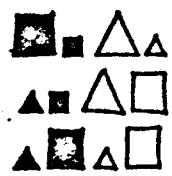
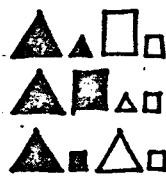
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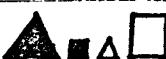
IS012 (24)



IS013 (6)



IS014 (2)



APPENDIX 7

ISOMORPH-INVARIANT

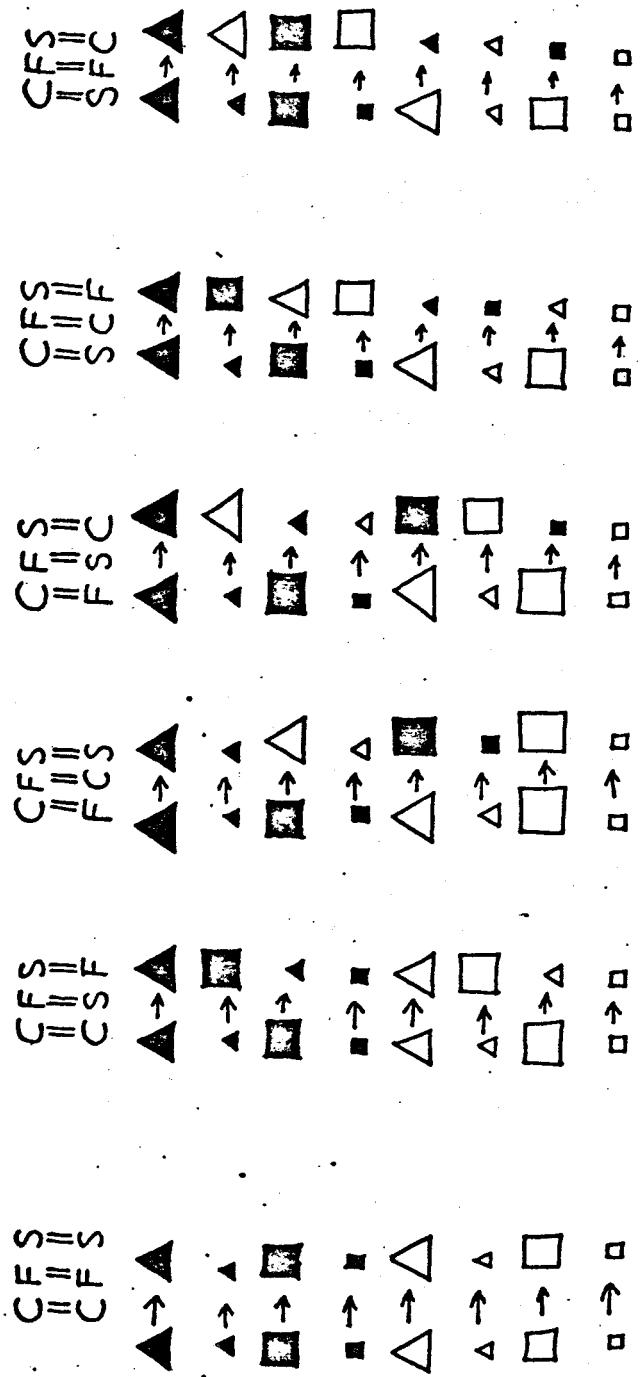
OPERATORS

(EXCLUDING THE INVERSION OPERATOR)

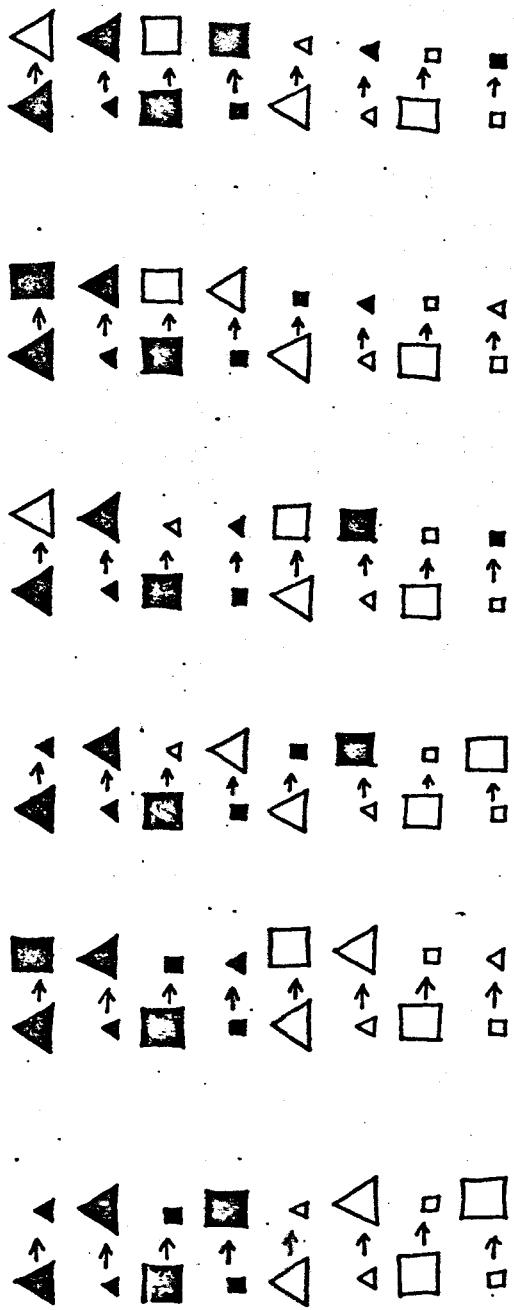
$$\bar{z} = 1 - z$$

ISOMORPH-INVARIANT OPERATORS
(C = COLOUR , F = FORM , S = SIZE)

(C=COLOUR, F=FORM, S=SIZE)



$\begin{array}{c} \text{CFS} \\ \equiv \\ \text{SFC} \end{array}$



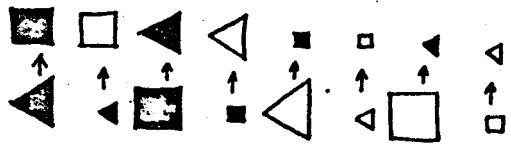
$\begin{array}{c} \text{CFS} \\ \equiv \\ \text{SCF} \end{array}$

$\begin{array}{c} \text{CFS} \\ \equiv \\ \text{FCS} \end{array}$

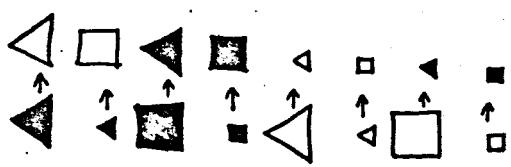
$\begin{array}{c} \text{CFS} \\ \equiv \\ \text{CSF} \end{array}$

$\begin{array}{c} \text{CFS} \\ \equiv \\ \text{CFS} \end{array}$

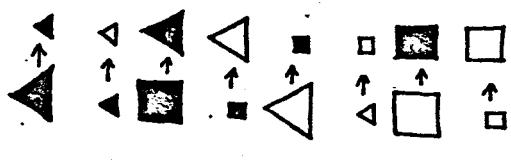
$\begin{smallmatrix} C & F & S \\ \times & = & C \end{smallmatrix}$



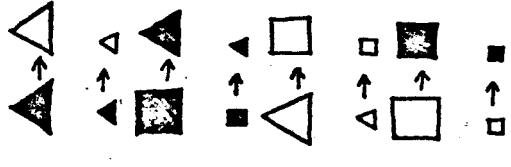
$\begin{smallmatrix} C & F & S \\ \times & = & F & C \end{smallmatrix}$



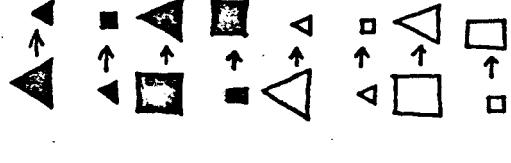
$\begin{smallmatrix} C & F & S \\ \times & = & F & C \end{smallmatrix}$



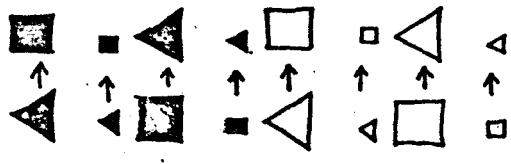
$\begin{smallmatrix} C & F & S \\ \times & = & F & C \end{smallmatrix}$



$\begin{smallmatrix} C & F & S \\ \times & = & C & S & F \end{smallmatrix}$



$\begin{smallmatrix} C & F & S \\ \times & = & C & F & S \end{smallmatrix}$



$\begin{matrix} C & F & S \\ \equiv & X & \times \\ C & F & C \end{matrix}$

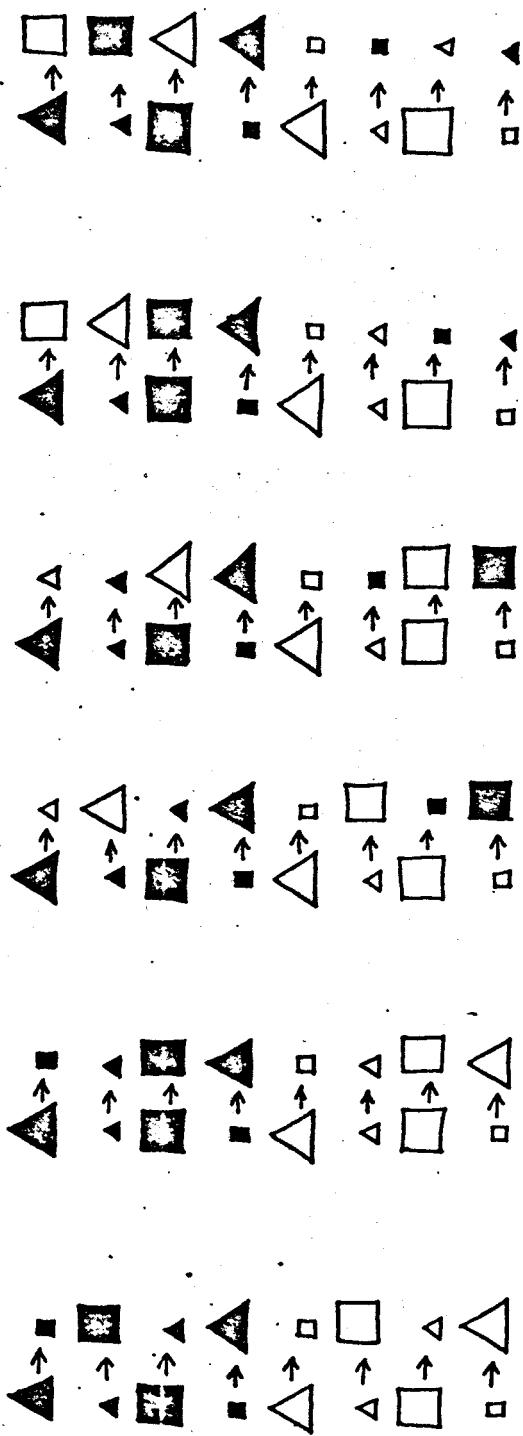
$\begin{matrix} C & F & S \\ \equiv & X & \times \\ S & C & F \end{matrix}$

$\begin{matrix} C & F & S \\ \equiv & X & \times \\ F & S & C \end{matrix}$

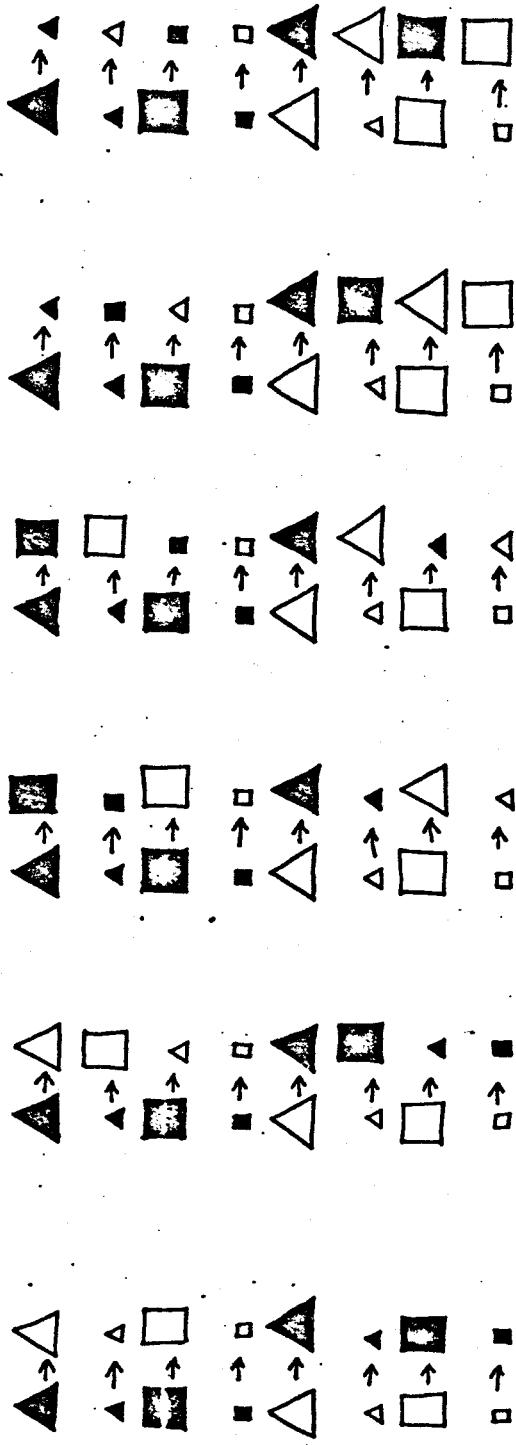
$\begin{matrix} C & F & S \\ \equiv & X & \times \\ F & C & S \end{matrix}$

$\begin{matrix} C & F & S \\ \equiv & X & \times \\ C & S & F \end{matrix}$

$\begin{matrix} C & F & S \\ \equiv & X & \times \\ C & F & S \end{matrix}$



$\begin{array}{c} \text{CFS} \\ \text{X} \\ \text{SFC} \end{array}$



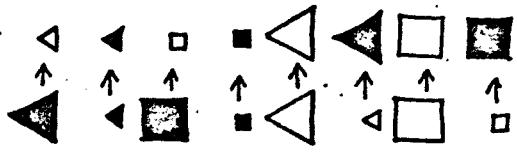
$\begin{array}{c} \text{CFS} \\ \text{X} \\ \text{SCF} \end{array}$

$\begin{array}{c} \text{CFS} \\ \text{X} \\ \text{FCS} \end{array}$

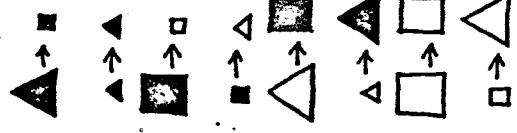
$\begin{array}{c} \text{CFS} \\ \text{X} \\ \text{CSF} \end{array}$

$\begin{array}{c} \text{CFS} \\ \text{X} \\ \text{FCS} \end{array}$

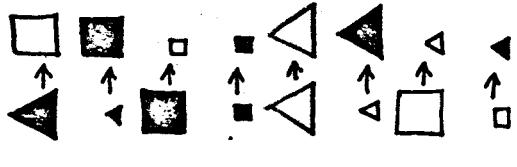
CFS
XSF



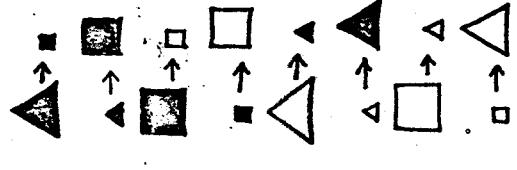
CFS
XSF



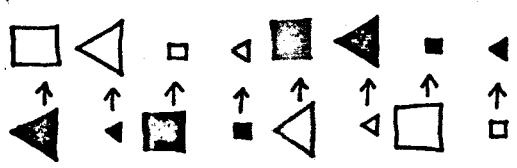
CFS
XSC



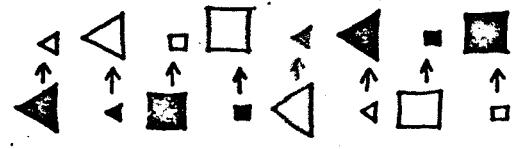
CFS
XCS



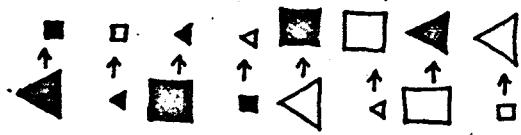
CFS
CSF



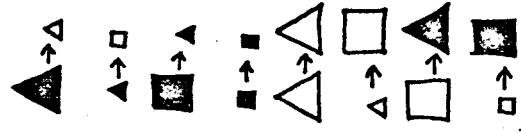
CFS
CFS



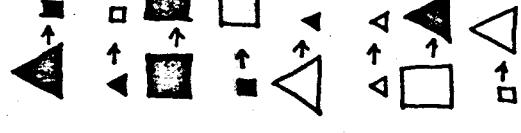
५०



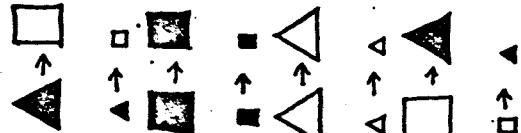
CFS = F
X SF



$\text{GFS} \equiv \text{GXFSC}$



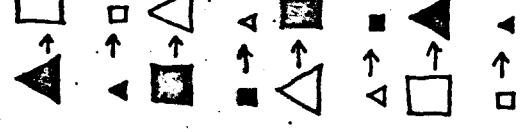
5=5
LxU
UxL



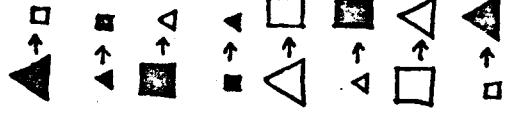
CELESTE



$$\begin{matrix} \mathbb{U} = \mathbb{U} \\ \mathbb{U} \times \mathbb{U} \\ \mathbb{U} \times \mathbb{U} \end{matrix}$$



CFS
XXS
SCF



CFS
XXX
SCF



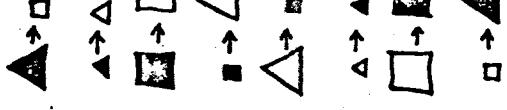
CFS
XXS
FSC



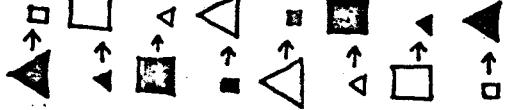
CFS
XXS
CSF



CFS
XXS
CSF



CFS
XXS
CSF



APPROVAL SHEET

The dissertation submitted by Harvey Jack Schiller has been read and approved by the following committee:

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Associate Professor, Foundations, Loyola

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The final copies have been examined by the director of the dissertation and the signature which appears below verifies the fact that any necessary changes have been incorporated and that the dissertation is now given final approval by the Committee with reference to content and form.

The dissertation is therefore accepted in partial full-filment of the requirements for the degree of Doctor of Philosophy.

Date

June 24, 1976

Anne McCreary Juhasz
Director's Signature