1972

The Myhill Functor, Input-Reduced Machines, and Generalised Krohn-Rhodes Theory

Robert H. Yacobellis
Loyola University Chicago, ryacobellis@luc.edu

PROCEEDINGS

OF THE

SIXTH ANNUAL

PRINCETON CONFERENCE

ON

INFORMATION SCIENCES AND SYSTEMS

Papers Presented

March 23-24, 1972

Program Co-Directors: M. E. Van Valkenburg
Murray Edelberg

Department of Electrical Engineering
Princeton University
Princeton, New Jersey 08540

TABLE OF CONTENTS

Session C-7: Societal Engineering
Chairman: Professor L. E. Franks

Session D-1: Automata and Cellular Arrays
Chairman: Professor R. Kieburtz

THE MYHILL FUNCTOR, INPUT-REDUCED MACHINES, AND GENERALIZED KROHN-RHODES THEORY

J. A. GOGUEN
IBM T. J. Watson Research Center
Yorktown Heights, New York

R. H. YACOBELLIS
Bell Telephone Laboratories, Incorporated
Naperville, Illinois

Summary

This paper reports recent progress in a program of extending Krohn-Rhodes theory, and its necessary preliminaries, to systems with structure other than the discrete. It is intended to provide a clear idea of the line of research and its motivation, with some idea of the methods. Further details may be found in future joint papers and in the forthcoming thesis of the second author.

1. Generalizing Machine Theory

Roughly the same results have been proved separately for several different types of discrete time system. For other types these same results are unknown. For example, transition systems, transducers and acceptors, with finite or arbitrary cardinality state and/or input sets, have engendered a large, and now almost classical, literature. The same models with the additional assumption of linearity have been less thoroughly studied, especially for such variations of linear as bilinear and affine, and when rings are used rather than fields. But many classical automaton results are known here, and research is actively proceeding. Topological machines, in which input, state, and output objects are topological spaces, and the transition and output functions are continuous, have been studied very little. They are interesting as models of nonlinear but smooth systems.

This paper discusses a general theory which gives special results for all the above cases. Many of these results are new. The method is that of Goguen,[4] to prove all results for machines having any "sufficiently nice" structure. This requires reformulating machine theory into the language of abstract structure, category theory, thus treating the universal properties of constructions as in modern algebra,[11] rather than their particular details. As usual, this method clarifies and extends existing results, while suggesting new ones.

Among the first nontrivial things done with a class of systems are to characterize the behaviors and seek minimal realizations. The work recently done by Goguen[2] for machines with "sufficiently nice" structure, including linear and topological, is summarized here in Section 3. The present paper treats the Myhill semigroup construction and aspects of Krohn-Rhodes theory in the same framework. As with the state minimization results, many of the applications are new. In Section 6 we discuss input minimization of machines.

The technical apparatus required for the general development is quite extensive. But fortunately, for expository purposes, the main ideas are adequately conveyed by the universal property formulations of results and constructions in just the discrete case. This we do, except in the final sections, which discuss machines, behaviors, and semigroups in diagonal closed categories.

2. The Language of Categories

Category theory provides a "language of structure" in which to do our theory of "machines with sufficiently nice structure." This section gives a dictionary for that language in more intuitive English. Of course, category theory is a totally rigorous branch of mathematics and all terms have precise technical definitions. The actual proofs of assertions in this paper are embedded in this framework. But the reader can usually appreciate the intuitive context of our results with these "basic doctrines"[5] of category theory: (1) any mathematical structure is represented by a <u>category</u>, (2) any mathematical construction is represented by a <u>functor</u>, (3) any canonical construction is represented by an <u>adjoint functor</u>, and (4) any natural translation from one construction to another is represented by a <u>natural transformation</u>.

Categories are denoted $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc., and the class of objects of $\mathcal{A}$ is denoted $|\mathcal{A}|$. <u>Morphisms</u> (or <u>maps</u>), in a category, thought of as "preserving the structure of objects," are indicated as arrows, $A \xrightarrow{f} B$ from <u>source</u> to <u>target</u> object, and are <u>composed</u> in the order natural to diagrams, $f:A \to B$ and $g:B \to C$ composing to give $fg:A \to C$. Composition is assumed associative, with an identity A for each object <u>A</u>. Application is indicated as usual, i.e., for $a \in A$, $f(a) \in B$, and $g(f(a)) = (fg)(a) \in C$, but also "categorically" as $af \in B$ and $afg \in C$. The set of maps from A to B in $\mathcal{C}$ is denoted $\mathcal{C}(A,B)$. A functor F from $\mathcal{A}$ to $\mathcal{B}$ is indicated $F:\mathcal{A} \to \mathcal{B}$. Speaking more technically now, $F:\mathcal{A} \to \mathcal{B}$ is <u>left adjoint</u> to $G:\mathcal{B} \to \mathcal{A}$ iff there is a natural isomorphism $\phi:\mathcal{B}(F(A),B) \simeq \mathcal{A}(A,G(B))$ of set-valued functors of A,B. One writes $F \dashv G$. For $f:F(A) \to B$, $\phi(f):A \to G(B)$ is called the <u>adjoint transform</u> of f. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is <u>reflective</u> iff the inclusion functor $\mathcal{B} \subseteq \mathcal{A}$ has a left adjoint. Technical references for category theory include References 3, 10, 12, and 13.

3. Machines and Behaviors

A <u>machine</u> is $M = \langle X,S,Y,\delta,\lambda,\sigma \rangle$, where X, S, Y are sets, and $\delta:S \times X \to S$, $\lambda:S \to Y$, $\sigma:1 \to S$ are functions, with 1 a one point set $\{\cdot\}$. A <u>machine morphism</u> $M \to M'$ is $\langle a,b,c \rangle$ where $a:X \to X'$, $b:S \to S'$, $c:Y \to Y'$ are functions such that the equations $(b \times a)\delta' = \delta b$, $b\lambda' = \lambda c$, $\sigma b = \sigma'$ hold. Given a machine M, let $\delta^+:X^* \to S$ be the usual recursive extension of δ to strings of inputs, using σ as starting state, i.e., $\delta^+(\Lambda) = \sigma$ and $\delta^+(wx) =$

$\delta(\delta^+(w),x)$ for $x \in X$, $w \in X^*$, where X^* is the monoid of all strings over X, Λ is the empty string, and $\sigma:1 \to S$ is identified with its image $\sigma(\cdot) \in S$. Call M reachable iff δ^+ is surjective. Let $\mathcal{M}$ be the category of all reachable machines with morphisms having their first (or input) component surjective.

The external behavior of M, denoted E(M), is the composite $\delta^+\lambda:X^* \to Y$. In general, a behavior is a function $f:X^* \to Y$, and a morphism of behaviors $f \to f'$ is a pair $\langle a,c \rangle$ where $a:X \to X'$ and $c:Y \to Y'$ such that $a^*f' = fc$, with $a^*(x_1 \ldots x_n) = a(x_1) \ldots a(x_n)$, concatenation in X'^*. Let $\mathcal{B}$ be the category of behaviors with morphisms having first component surjective. For $\langle a,b,c \rangle$ in $\mathcal{M}$, let $E(\langle a,b,c \rangle) = \langle a,c \rangle$ in $\mathcal{B}$. Then $E:\mathcal{M} \to \mathcal{B}$ is a functor, called the external behavior functor.

Theorem: There is a functor $N:\mathcal{B} \to \mathcal{M}$ right adjoint and left inverse to E.

These conditions determine N uniquely up to state set isomorphism as the Nerode minimal state realization construction. N being a left inverse means $(fN)E = f$, i.e., fN realizes f, for all behaviors f. Let $\mathcal{FSM}$ be the full subcategory of $\mathcal{M}$ with objects having S finite, and let $\mathcal{FSB}$ be the full subcategory of $\mathcal{B}$ with objects E(M) for M in $\mathcal{FSM}$. Then E and N restricted to these categories are still adjoint, and this gives the classical situation. More generally, any right-adjoint-left-inverse is a sort of minimal realization functor, and exhibits a number of useful properties: see Reference 2.

Most of the above results are proved in Reference 4, though without having possibly infinite X and Y. The extension to affine and topological cases is discussed later. Note that to get acceptors we let $Y = \{0,1\}$, and to get transition systems, we let Y = S and λ the identity. The methods of Reference 4 show adjointness here too.

4. Traits and The Myhill Functor

It is convenient to use a structure conveying somewhat less information than the external behavior. For then the canonical reconstruction of the original data, while not in general faithful, exhibits a certain minimality. A trait is $T = \langle X,M,Y,i,o \rangle$ where X, Y are sets, M is a monoid, $i:X \to M$ is injective and $o:M \to Y$ is a function, such that X generates M, in the sense that every function $\underline{h}:X \to M'$ extends to at most one monoid morphism $h:M \to M'$ (if we had said "exactly one," M would be freely generated by X, and thus isomorphic to X^*). A trait morphism $T \to T'$ is $\langle a,g,c \rangle$ where $a:X \to X'$ and $c:Y \to Y'$ are functions and $g:M \to M'$ is a monoid morphism such that $ai' = ig$ and $go' = oc$. (Note that g determines a because i' is injective.) Call a trait T firm iff $o(m_1 m m_2) = o(m_1 m' m_2)$ for all m_1, $m_2 \in M$ implies $m = m'$. Let $\mathcal{TR}$ denote the category of firm traits with morphisms having surjective first component.

Given $f:X^* \to Y$, define the Myhill congruence on X^* as usual by $w \sim_f w'$ iff $f(uwv) = f(uw'v)$ for all $u, v \in X^*$, and call the quotient $X^*/\sim_f = M_f$ the Myhill monoid of f. Let $q_f:X^* \to M_f$ be the quotient, and also write $[w]_f$ or even $[w]$ for $q_f(w)$. Let $X_f = q_f(X) \subseteq M_f$, and let $i_f:X_f \to M_f$ be the inclusion. Define $o_f:M_f \to Y$ by $o_f([w]) = f(w)$, which is easily seen to be well-defined. Call $T_f = \langle X_f,M_f,Y,i_f,o_f \rangle$ the Myhill trait of f (one must check that X_f generates). Note that T_f is firm.

Given $\langle a,c \rangle:f \to f'$ in $\mathcal{B}$, define $g:M_f \to M_{f'}$ by $g([w]_f) = [a^*(w)]_{f'}$, which again is easily seen well-defined and a monoid morphism. Now for $[x]_f \in X_f$ define $a_o([x]_f) = g([x]_f) = [a(x)]_{f'}$, and check that $\langle a_o,g,c \rangle$ is a trait morphism $T_f \to T_{f'}$. This gives rise to the Myhill functor $My:\mathcal{B} \to \mathcal{TR}$. It can be composed with the forgetful functor from traits to monoids to obtain what should be regarded as the classical Myhill functor. Letting $\mathcal{FSTR}$ be the full subcategory of $\mathcal{TR}$ with finite monoids in the objects, it is easily checked that the restriction and composites $My:\mathcal{FSB} \to \mathcal{FSTR}$ and $EMy:\mathcal{FSM} \to \mathcal{FSTR}$ exist.

We assume the reader sufficiently familiar with the importance of the Myhill monoid as a summary of a machine's activity not to require further exhortation here. (See Reference 1 for further details.)

5. The Behavior-Trait Adjunction

The map $i:X \to M$ of a trait $\underline{T} = \langle X,M,Y,i,o \rangle$ determines a monoid morphism $\bar{i}:X^* \to M$ uniquely from the condition $i = j\bar{i}$, where $j:\underline{X} \to X^*$ is the canonical inclusion (in fact, $\bar{i}(x_1 \ldots x_n) = x_1 \cdots x_n$, multiplication in M) because X^* is free. Now define $B(T) = \bar{i}o$: $X^* \to Y$, the behavior of T. Any $\langle a,g,c \rangle:T \to T'$ in $\mathcal{TR}$ gives $\langle a,c \rangle:B(T) \to B(T')$ in $\mathcal{B}$, and thus $B:\mathcal{TR} \to \mathcal{B}$ is a functor.

Theorem: My is left adjoint to B.

But B is not a left inverse, so this is not a minimal realization situation in the sense that the machine-behavior adjunction was. However, merely being an adjoint entitles a functor to a number of benefits; for example, B preserves products and My preserves colimits. We see in the next section that this adjunction is "almost" a minimal realization situation, and actually induces one.

Clearly, we can again restrict to the finite state case, obtaining $B:\mathcal{FSTR} \to \mathcal{FSB}$ right adjoint to $My:\mathcal{FSB} \to \mathcal{FSTR}$. The notion of trait is close to Krohn-Rhodes notion of the "normal form" of a behavior,[8] but of course our results on universality are new.

6. Input Reduced Machines

There are situations in which one wants a minimal set of controls for a sequential process. For example, a minimal control set will optimize the reliability and cost of a link for remote controlling an industrial process or an artificial satellite. This section shows how to find such sets for discrete systems. The extension to linear and continuous systems is discussed later.

A behavior $f:X^* \to Y$ is input reduced iff for all $u, v \in X^*$ and $x, x' \in X$, $f(uxv) = f(ux'v)$ implies $x = x'$. A machine is input reduced iff its behavior is, and a trait is input reduced iff its behavior is.

Proposition: For $f \in |\mathcal{B}|$ and $T \in |\mathcal{TR}|$, fMyB and TBMy are input reduced. In fact, $T \simeq$ TBMy iff T is input reduced and $f \simeq$ fMyB iff f is input reduced.

Proposition: The input reduced behaviors are a reflective subcategory $\mathcal{IRB}$ of $\mathcal{B}$, that is, the inclusion has a left adjoint; and $\mathcal{IRB}$ is equivalent to the (full sub) category of input-reduced traits. In fact, the left adjoint to the inclusion $\mathcal{IRB} \subseteq \mathcal{B}$ is MyB.

The adjunction MyB $\dashv \subseteq$ is a minimal realization situation, in the technical sense[5] that $\mathcal{IRB} \subseteq \mathcal{B}$ is a right-adjoint-left-inverse. We can compose with E $\dashv$ N (Section 3) to obtain others.

Theorem: The functor $N:\mathcal{IRB} \to \mathcal{M}$ is right-adjoint-left-inverse to EMyB:$\mathcal{M} \to \mathcal{IRB}$. The input reduced state reduced machines are a (full) reflective subcategory of $\mathcal{M}$; in fact, the left adjoint to the inclusion is EMyB.

This gives the input reduced state reduced realizations promised earlier. It might be noted that the minimal input set X_f may very well contain symbols having no effect on certain, or even on all, states. Such inputs may be necessary for states to persist through several "clock pulses." Also note that if encoding letters in X by strings from X_f were allowed, the problem of input minimality would be trivial and unrelated to the Myhill monoid.

7. Categorical Krohn-Rhodes Theory

Original interest in Krohn-Rhodes theory sprang from the novel decompositions it gave for behaviors and monoids.[8,9] More recent work has concerned complexity.[1] Generalizations to linear and topological systems should have the same applications. This section presents the major theorem of Krohn and Rhodes in categorical language, suggesting the form of the generalized theory.

Say $f:X^* \to Y$ divides $g:W^* \to Z$, written $f|g$, or g simulates f iff there is a monoid homomorphism $h:X^* \to W^*$ and a set map $b:Z \to Y$ such that $f = hgb$. f divides g length preserving, written $f|g(\ell p)$, iff $f|g$ with $h = a^*$ for some $a:X \to W$. The series connection of $f:X^* \to Y$ and $g:Y^* \to Z$ is the composite function $f^e g$, where $f^e(x_1 \ldots x_n) = f(x_1)f(x_1x_2) \ldots f(x_1 \ldots x_n)$. The parallel connection of $f:X^* \to Y$ and $g:W^* \to Z$ is $f\times g:(X\times W)^* \to Y\times Z$, where $f\times g(\langle x_1,y_1\rangle \ldots \langle x_n,y_n\rangle) = \langle f(x_1 \ldots x_n), g(y_1 \ldots y_n)\rangle$. The series parallel closure of a family of behaviors $\mathcal{F}$, SP($\mathcal{F}$), is the smallest family of behaviors containing $\mathcal{F}$ which is closed under series and parallel connection and length preserving division. A behavior f is irreducible iff whenever f divides a series or parallel connection of two behaviors g and h, it divides a finite parallel connection of g with itself (or h with itself).[14] Let IRR(f) be the collection of all irreducible behaviors which divide a behavior f. Then one form of the Krohn-Rhodes Theorem is that $f \in$ SP(IRR(f) $\cup\, \mathcal{D} \cup$ {U}), where $\mathcal{D}$ is a collection of delay behaviors and U is an identity-reset behavior.

Let $\mathcal{E}$ be the category with objects sets and morphisms extended behaviors $f^e:X^* \to Y^*$. Here $(X\times Y)^*$ is the Cartesian product of X^* and Y^*, and $(f\times g)^e$ is the unique morphism obtained from $f^e:X^* \to Y^*$ and $g^e:W^* \to Z^*$. Let $\mathcal{SP}$ be the least subcategory of $\mathcal{E}$ closed under products and containing all the "free" behaviors $a^*:X^* \to Y^*$. If $\mathcal{F}$ is a collection of extended behaviors, let $\mathcal{SP}(\mathcal{F})$ be the least subcategory of $\mathcal{E}$ containing $\mathcal{SP}$ and $\mathcal{F}$ and closed under product. Note that $\mathcal{SP}$(f) = $\mathcal{SP}$({f}) contains every behavior which f simulates (ℓp). The Krohn-Rhodes theorem then says $f \in \mathcal{SP}$(IRR(f) $\cup\, \mathcal{D} \cup$ {U}). If we give $\mathcal{SP}$ vertical morphisms as in the category of behaviors, or if we let a vertical morphism be a division relation, $\mathcal{SP}$ takes on a 2-category structure.[10] If we look at a trait $\langle X,M,Y,i,o\rangle$ as a morphism from X to Y, $\mathcal{TR}$ also has a 2-category structure. It seems likely that the Myhill adjunction preserves division, irreducibility, etc., and is actually some kind of 2-adjunction. We would then get similar Krohn-Rhodes results for traits (and for monoids via the forgetful functor). This is an area which we are currently exploring.

8. Mathematical Methods of Generalized Machine Theory

The first "basic doctrine" of Section 2 says any mathematical structure is represented by a category. The preceding theory concerned discrete structure, represented by the category $\mathcal{Set}$ of sets. We now generalize from $\mathcal{Set}$ to categories $\mathcal{C}$ representing other "suitable" structures.

$\mathcal{C}$ must be "closed," or have an "internal hom functor." This means for each pair A,B of objects in $\mathcal{C}$, the set of $\mathcal{C}$-morphisms from A to B should become an object [A,B] of $\mathcal{C}$. The functor [,] arises most easily as a right adjoint to a "monoidal" functor $\boxtimes:\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, so called because assumed to have an "identity" $I \in |\mathcal{C}|$, isomorphisms expressing the monoid laws, $a_{ABC}:A\boxtimes(B\boxtimes C) \to (A\boxtimes B)\boxtimes C$, $r_A:A\boxtimes I \to A$, and in the "symmetric" case we use also $c_{AB}: A\boxtimes B \to B\boxtimes A$. These isomorphisms are "coherent," i.e., any diagram of them commutes. This discussion motivates the closed symmetric monoidal category concept defined in Reference 3 or 7. It can be shown that such categories have natural "evaluation" transforms $\nu_{AB}: A\boxtimes[A,B] \to B$; in the case of *Set*, ν_{AB} takes $\langle a,f\rangle$ to f(a).

<u>Suitable</u> categories must also have countable coproducts, the universal construction corresponding to countable disjoint unions in *Set*; and an appropriate generalization of the usual surjective-injective set map factorizations. Examples include *Set* of course, with $\boxtimes$ Cartesian product; but also the category Mod_R of R-modules with $\boxtimes$ tensor product, for R a commutative ring with unit; the category *Kell* of Kelley (i.e., compactly generated Hausdorff) spaces with $\boxtimes$ Cartesian, and most interestingly, the affine category Aff_R with objects R-modules (again for R commutative with unit) and with R-affine morphisms (i.e., R-linear plus a constant), and the affine tensor product $A \textcircled{A}_R B = A\boxtimes_R B + A + B$. Affine machines are a natural and physically significant generalization of linear machines; see Reference 3.

All of Section 3 generalizes to suitable C. A machine in C has X, S, Y ε $|C|$ and δ,λ,σ morphisms in C, but replace 1 by I. Next define $X^* = \coprod_t \boxtimes^t X$, the countable coproduct of the iterated powers of X, and show it is the free monoid in C generated by X. A <u>monoid</u> in monoidal C is $\langle M,\mu,e\rangle$ with M ε $|C|$ and with $\mu: M\boxtimes M \to M$, $e: I \to M$ in C, satisfying associativity and identity laws. A <u>semigroup</u> in C has only μ and associativity.

By considering automata in C ("machines" without λ) we can define $\delta^+: X^* \to S$, and then the <u>behavior</u> $E(M) = \delta^+\lambda: X^* \to Y$. Morphisms of machines and behaviors are just as in Section 3 (use epic for surjective) giving categories $\mathcal{M}$, $\mathcal{B}$, and the behavior functor $E: \mathcal{M} \to \mathcal{B}$. Again there is a functor $N: \mathcal{B} \to \mathcal{M}$ right-adjoint-left-inverse to E, giving Nerode minimal state realizations. The construction crucially uses all the suitability assumptions; see Reference 3 or 4.

9. Monoids in Diagonal Closed Categories

Generalized Krohn-Rhodes theory uses monoid theory in closed categories sketched here. Parts of this theory need more structured closed categories than those of Section 8. We call them <u>diagonal</u>, because their main new feature is a "diagonal" natural transform $\nabla_X: X \to X\boxtimes X$, coherent with the closed symmetric monoidal structure. We again assume countable coproducts and reasonable factorizations. For some purposes we assume I is a terminal object, meaning each object A has a unique morphism $u_A: A \to I$, and also assume some coherence for these morphisms. These assumptions about I are not needed for just semigroup theory. Among the examples of Section 8, *Set*, *Kell*, and Aff_R are diagonal, but Lin_R is not.

The definitions of monoid and semigroup in a diagonal category are exactly the same as in monoidal category. Monoids in *Set* are ordinary monoids, in *Kell* are "continuous" monoids, and in Aff_R are generalized linear. Among the latter are the tensor, Grassman, and other "algebras" of modern mathematics (see Reference 11). We now describe some basic constructions for monoids in categories.

If M and M′ are monoids in a symmetric monoidal C, so is their product $M\boxtimes M'$, with identity

$$I \xrightarrow{r_I^{-1}} I\boxtimes I \xrightarrow{e\boxtimes e'} M\boxtimes M'$$

and multiplication

$$(M\boxtimes M')\boxtimes(M\boxtimes M') \xrightarrow{b} (M\boxtimes M)\boxtimes(M'\boxtimes M') \xrightarrow{\mu\boxtimes\mu'} M\boxtimes M',$$

where b is a combination of the associative and commutative laws a and c.

For any object A in closed symmetric monoidal C, [A,A] can be made a monoid, with multiplication the adjoint transform of $([A,A]\boxtimes[A,A])\boxtimes A \xrightarrow{b} (A\boxtimes[A,A])\boxtimes[A,A] \xrightarrow{\nu\boxtimes 1} A\boxtimes[A,A] \xrightarrow{\nu} A$ and identity the transform of $I\boxtimes A \xrightarrow{cr_A} A$, where b is made from a and c, and ν is evaluation.

If M is a monoid and A an object in diagonal C, then [A,M] is also a monoid, with multiplication and identity the adjoint transforms of the composites $([A,M]\boxtimes[A,M])\boxtimes A \xrightarrow{d} (A\boxtimes[A,M])\boxtimes(A\boxtimes[A,M]) \xrightarrow{\nu\boxtimes\nu} M\boxtimes M \xrightarrow{\mu} M$, and $I\boxtimes A \xrightarrow{u} I \xrightarrow{e} M$, where d is composed from ∇, a, c; $\nu: A\boxtimes[A,M] \to M$ is evaluation; u is the unique map; and μ, e are from M. The proof is quite long, but straightforward.

If M is a monoid in monoidal C, a right M-<u>action</u> is an object A and an $\alpha: A\boxtimes M \to A$ satisfying associativity and identity laws. Defining morphisms of (right) M-actions in the obvious way, we get a category Act^M of them. Examples: an X-automaton with state object S is an X^*-action on S with a "point" $\sigma: I \to S$; M itself is an M-action with $\mu: M\boxtimes M \to M$; a <u>right ideal</u> of M is a monic $U \to M$ in Act^M; I is an M-act with $I\boxtimes M \xrightarrow{u} I$; a <u>left zero</u> or <u>reset</u> of M is a right ideal of the form $g: I \to M$ in Act^M. <u>Left action</u> is defined dually, and a <u>bi-action</u> involves one of each. If C is closed and A ε $|C|$, [M,A] can be given a left M-action structure α using evaluation; it is in fact the cofree M-action generated by A. It is possible to go on and develop ideal theory. For example, U is a <u>right simple</u> ideal of M iff it contains no proper right subideal, and is <u>right cyclic</u> iff the image of some $I^* \to M$ in Act^M.

If M and N are monoids in diagonal C, $\phi: M\boxtimes N \to N$ is a <u>connecting morphism</u> iff it is a left M-action satisfying also identity and distributive conditions for N, the latter using ∇_M. The <u>semidirect product</u> $N\boxtimes_\phi M$ is $N\boxtimes M$ with multiplication the composite $(N\boxtimes M)\boxtimes(N\boxtimes M) \xrightarrow{d} (N\boxtimes(M\boxtimes N))\boxtimes(M\boxtimes M) \xrightarrow{N\boxtimes\phi\boxtimes 1} (N\boxtimes N)\boxtimes(M\boxtimes M) \xrightarrow{\mu_N\boxtimes\mu_M} N\boxtimes M$, where d uses ∇_M, a, and with identity as in the monoid product. It is interesting, though tedious, to verify $N \boxtimes_\phi M$ really is a monoid. Similarly, it can be verified that the left M-action $M \boxtimes [M,N] \xrightarrow{\alpha} [M,N]$ is a connecting homomorphism; the resulting semidirect product $[M,N] \boxtimes_\alpha M$ is called the <u>wreath product</u> of M and N, denoted N wr M.

<u>Division</u> of monoids is defined as usual, $M|N$ iff M is a quotient of a submonoid of N. When the category of monoids has pullbacks, division is a transitive relation, since the pullback of $N' \rightarrowtail N \twoheadleftarrow P'$, coming from $M|N$ and $N|P$ gives $M|P$. Therefore the monoids in C are a category with divisions as morphisms.

10. The Myhill Functor and Krohn-Rhodes Theory in Diagonal Closed Categories

Several aspects of the Myhill monoid construction, traits, and Krohn-Rhodes theory carry over to diagonal closed categories. For the Myhill construction, we take a behavior $f:X^* \to Y$ in diagonal C, and form $f^B:X^* \to [X^* \boxtimes X^*, Y]$, the adjoint transform of $a(c\mu \boxtimes X^*)\mu$, where μ is the natural multiplication on X^* corresponding to concatenation. The object $[X^* \boxtimes X^*, Y]$ is an X^*-biaction, and f^B is an X^*-biaction homomorphism. If we factor f^B as $f^B = q_f m$, where m is monic and q_f is a coequalizer, then the domain M_f of m is also an X^*-biaction. Furthermore, $(X^* \boxtimes q_f)(q_f \boxtimes M_f) = (q_f \boxtimes X^*)(M_f \boxtimes q_f)$ is a pushout, giving a monoid multiplication $\mu_f: M_f \boxtimes M_f \to M_f$ with q_f a monoid epimorphism. f factors as $q_f o_f$, where o_f is constructed from m, r, e_M, and evaluation, and if $f = hk$ is any other factorization of f through a monoid epimorphism $h:X^* \to N$, there is a unique monoid epimorphism $h':N \to M_f$ such that $hh' = q_f$. This minimality condition for M_f gives a Myhill adjunction with $T_f = \langle X_f, M_f, Y, i_f, o_f \rangle$ the trait of f, where X_f is the domain of i_f in the factorization $j_1 q_f = q i_f$. Then X_f generates M_f since q_f is epic, and T_f is firm in the sense that it has an isomorphic Myhill quotient.

If $f|g$, i.e., $f = hgb$, then $M_f | M_g$. For factoring g as $q_g o_g$, we have $f = hq_g \cdot o_g b$. Then factor $hq_g: X^* \to M_g$ through a monoid M as $hq_g = qo$. Since h and q_g are homomorphisms, $M \approx [X^* \boxtimes X^*, M_g]$, so o is a monoid monomorphism. Since $f = q(o\bar{o}_g b)$, minimality of M_f gives an epic $q':M \to M_f$ such that $q_f = qq'$. Thus M_f is a subquotient of M_g, via o and q'.

As in *Set*, the Myhill monoid of $f \times g$ divides $M_f \boxtimes M_g$, where $f \times g$ is $(\widehat{j_X \boxtimes j_W})(f \boxtimes g)$ for $f:X^* \to Y$ and $g:W^* \to Z$. We get $\widehat{j_X \boxtimes j_W}$ because $X \boxtimes W$ generates $(X \boxtimes W)^*$. For the series connection of $f:X^* \to Y$ and $g:Y^* \to Z$, we first define $f^e:X^* \to Y^*$ by components $f^e_i: \boxtimes^i X \to \boxtimes^i Y$ as $f^e_0 = 1_I : I \to I$, $f^e_1 = j_1 f : X \to Y$, and $f^e_i = (\nabla_{\boxtimes^{i-1}X} \boxtimes X)(f^e_{i-1} \boxtimes j_{i-1} f)$, where j_{i-1} is the inclusion of $\boxtimes^{i-1} X$ into X^*. Then f factors as $q_f o_f$, $h = f^e g$ as qo, and $f^e q : X^* \to M_g$ as $q'o'$, where $q':X^* \to M$. Since q' is a monoid epic, there is a unique $q'':M \to M_h$ such that $q'q'' = q_h$. M is a submonoid of M_f wr M_g, so M_h divides M_f wr M_g.

Further research on Krohn-Rhodes Theory is in progress. Some preliminary results use split exact sequences to examine irreducible monoids (those for which $M | S \times_\phi T$ implies either $M|S$ or $M|T$) in diagonal closed categories.

<u>References</u>

1. Arbib, M. A., ed., Algebraic Theory of Machines, Languages, and Semigroups, Academic Press, 1968.

2. Give'on, Y., and Zalcstein, Y., "Algebraic Structures in Linear Systems Theory," Jnl. of Cptr. and Sys. Sci, 4, 346-370, 1970.

3. Goguen, J. A., "Discrete Time Machines in Closed Monoidal Categories, I," Institute for Cptr. Res. Quarterly Report, University of Chicago, 1971.

4. Goguen, J. A., "Realization is Universal," to appear in Math. Sys. Th. also ICR Quarterly Report, Univ. of Chicago, 1970.

5. Goguen, J. A., "Systems and Minimal Realization," Proc. of 1971 IEEE Confr. on Decision and Control, Miami, 1971.

6. Kalman, R. E., Falb, P., and Arbib, M., Topics in Mathematical Systems Theory, McGraw Hill, 1969.

7. Kelley, G. M., and MacLane, S., "Coherence in Closed Categories," Journal of Pure and Applied Algebra, Vol. 1, No. 1:97-140 (1971).

8. Krohn, K., and Rhodes, J., "Algebraic Theory of Machines I Prime Decomposition for Finite Semigroups and Machines," <u>Transactions of the American Mathematical Society</u>, 116:450-464 (1965).

9. Lallement, Gerard, "On the Prime Decomposition Theorem for Finite Monoids," <u>Mathematical Systems Theory</u>, Vol. 5, No. 1: 8-12 (1969).

10. MacLane, S., Category Theory for the Working Mathematician, Springer, 1972.

11. MacLane, S., and Birkhoff, G., Algebra, Macmillan, 1967.

12. Mitchell, B., <u>Theory of Categories</u>, Academic Press (1965).

13. Pareigis, <u>Categories and Functors</u>, Academic Press (1970).

14. Zeigler, B., "A Note on Series Parallel Irreducibility," <u>Mathematical Systems Theory</u>, Vol. 5, No. 1: 1-3 (1971).