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Conventions, Definitions, Identities, and Other Useful Formulae

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1. Curvature tensors

Consider a $d + 1$ dimensional manifold \mathcal{M} with metric $g_{\mu\nu}$. The covariant derivative on \mathcal{M} that is metric-compatible with $g_{\mu\nu}$ is ∇_μ .

Christoffel Symbols

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \quad (1)$$

Riemann Tensor

$$R^\lambda{}_{\mu\sigma\nu} = \partial_\sigma \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\sigma}^\lambda + \Gamma_{\mu\nu}^\kappa \Gamma_{\kappa\sigma}^\lambda - \Gamma_{\mu\sigma}^\kappa \Gamma_{\kappa\nu}^\lambda \quad (2)$$

Ricci Tensor

$$R_{\mu\nu} = \delta^\sigma{}_\lambda R^\lambda{}_{\mu\sigma\nu} \quad (3)$$

Schouten Tensor

$$S_{\mu\nu} = \frac{1}{d-1} \left(R_{\mu\nu} - \frac{1}{2d} g_{\mu\nu} R \right) \quad (4)$$

$$\nabla^\nu S_{\mu\nu} = \nabla_\mu S^\nu{}_\nu \quad (5)$$

Weyl Tensor

$$C^\lambda{}_{\mu\sigma\nu} = R^\lambda{}_{\mu\sigma\nu} + g^\lambda{}_\nu S_{\mu\sigma} - g^\lambda{}_\sigma S_{\mu\nu} + g_{\mu\sigma} S^\lambda{}_\nu - g_{\mu\nu} S^\lambda{}_\sigma \quad (6)$$

Commutators of Covariant Derivatives

$$[\nabla_\mu, \nabla_\nu] A_\lambda = R_{\lambda\sigma\mu\nu} A^\sigma \quad (7)$$

$$[\nabla_\mu, \nabla_\nu] A^\lambda = R^\lambda{}_{\sigma\mu\nu} A^\sigma \quad (8)$$

Bianchi Identity

$$\nabla_\kappa R_{\lambda\mu\sigma\nu} - \nabla_\lambda R_{\kappa\mu\sigma\nu} + \nabla_\mu R_{\kappa\lambda\sigma\nu} = 0 \quad (9)$$

$$\nabla^\nu R_{\lambda\mu\sigma\nu} = \nabla_\mu R_{\lambda\sigma} - \nabla_\lambda R_{\mu\sigma} \quad (10)$$

$$\nabla^\nu R_{\mu\nu} = \frac{1}{2} \nabla_\mu R \quad (11)$$

Bianchi Identity for Weyl

$$\nabla^\nu C_{\lambda\mu\sigma\nu} = (d-2) \left(\nabla_\mu S_{\lambda\sigma} - \nabla_\lambda S_{\mu\sigma} \right) \quad (12)$$

$$\begin{aligned} \nabla^\lambda \nabla^\sigma C_{\lambda\mu\sigma\nu} = \frac{d-2}{d-1} & \left[\nabla^2 R_{\mu\nu} - \frac{1}{2d} g_{\mu\nu} \nabla^2 R - \frac{d-1}{2d} \nabla_\mu \nabla_\nu R - \left(\frac{d+1}{d-1} \right) R_\mu{}^\lambda R_{\nu\lambda} \right. \\ & \left. + C_{\lambda\mu\sigma\nu} R^{\lambda\sigma} + \frac{(d+1)}{d(d-1)} R R_{\mu\nu} + \frac{1}{d-1} g_{\mu\nu} \left(R^{\lambda\sigma} R_{\lambda\sigma} - \frac{1}{d} R^2 \right) \right] \end{aligned} \quad (13)$$

2. Conventions for Differential Forms

p-Form Components

$$\mathbf{A}_{(p)} = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \quad (14)$$

Exterior Derivative

$$\left(d\mathbf{A}_{(p)} \right)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]} \quad (15)$$

$$B_{[\mu_1 \dots \mu_n]} := \frac{1}{n!} \left(B_{\mu_1 \dots \mu_n} + \text{permutations} \right) \quad (16)$$

Hodge-Star

$$\left(\star \mathbf{A}_{(p)} \right)_{\mu_1 \dots \mu_{d+1-p}} = \frac{1}{p!} \epsilon_{\mu_1 \dots \mu_{d+1-p}}{}^{\nu_1 \dots \nu_p} A_{\nu_1 \dots \nu_p} \quad (17)$$

$$\star \star = (-1)^{p(d+1-p)+1} \quad (18)$$

Wedge Product

$$\left(\mathbf{A}_{(p)} \wedge \mathbf{B}_{(q)} \right)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p! q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]} \quad (19)$$

3. Euler Densities

Let \mathcal{M} be a manifold with dimension $d + 1 = 2n$ an even number. Normalized so that $\chi(S^{2n}) = 2$.

Euler Number

$$\chi(\mathcal{M}) = \int_{\mathcal{M}} d^{2n}x \sqrt{g} \mathcal{E}_{2n} \quad (20)$$

$$= \int_{\mathcal{M}} \mathbf{e}_{2n} \quad (21)$$

Euler Density

$$\mathcal{E}_{2n} = \frac{1}{(8\pi)^n \Gamma(n+1)} \epsilon_{\mu_1 \dots \mu_{2n}} \epsilon_{\nu_1 \dots \nu_{2n}} R^{\mu_1 \mu_2 \nu_1 \nu_2} \dots R^{\mu_{2n-1} \mu_{2n} \nu_{2n-1} \nu_{2n}} \quad (22)$$

$$\mathbf{e}_{2n} = \frac{1}{(4\pi)^n \Gamma(n+1)} \epsilon_{a_1 \dots a_{2n}} \mathbf{R}^{a_1 a_2} \wedge \dots \wedge \mathbf{R}^{a_{2n-1} a_{2n}} \quad (23)$$

Curvature Two-Form

$$\mathbf{R}^a{}_b = \frac{1}{2} R^a{}_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \quad (24)$$

Examples

$$\begin{aligned} \mathcal{E}_2 &= \frac{1}{8\pi} \epsilon_{\mu\nu} \epsilon_{\lambda\rho} R^{\mu\nu\lambda\rho} \\ &= \frac{1}{4\pi} R \end{aligned} \quad (25)$$

$$\begin{aligned} \mathcal{E}_4 &= \frac{1}{128\pi^2} \epsilon_{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} R^{\mu\nu\alpha\beta} R^{\lambda\rho\gamma\delta} \\ &= \frac{1}{32\pi^2} \left(R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} - 4 R^{\mu\nu} R_{\mu\nu} + R^2 \right) \\ &= \frac{1}{32\pi^2} C^{\mu\nu\lambda\rho} C_{\mu\nu\lambda\rho} - \frac{1}{8\pi^2} \left(\frac{d-2}{d-1} \right) \left(R^{\mu\nu} R_{\mu\nu} - \frac{d+1}{4d} R^2 \right) \end{aligned} \quad (26)$$

4. Hypersurfaces

Let $\Sigma \subset \mathcal{M}$ be a d dimensional hypersurface whose embedding is described locally by an outward-pointing, unit normal vector n^μ . Rather than keeping track of the signs associated with n^μ being either spacelike or timelike, we will just assume that n^μ is spacelike. Indices are lowered and raised using $g_{\mu\nu}$ and $g^{\mu\nu}$, and symmetrization of indices is implied when appropriate.

First Fundamental Form / Induced Metric on Σ

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \quad (27)$$

Projection onto Σ

$$\perp T^{\mu\dots}{}_{\nu\dots} = h^\mu{}_\lambda \dots h^\sigma{}_\nu \dots T^{\lambda\dots}{}_{\sigma\dots} \quad (28)$$

Second Fundamental Form / Extrinsic Curvature of Σ

$$K_{\mu\nu} = \perp(\nabla_\mu n_\nu) = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\lambda n_\sigma = \frac{1}{2} \mathcal{L}_n h_{\mu\nu} \quad (29)$$

Trace of Extrinsic Curvature

$$K = \nabla_\mu n^\mu \quad (30)$$

‘Acceleration’ Vector

$$a^\mu = n^\nu \nabla_\nu n^\mu \quad (31)$$

Surface-Forming Normal Vectors

$$n_\mu = \frac{1}{\sqrt{g^{\nu\lambda} \partial_\nu \alpha \partial_\lambda \alpha}} \partial_\mu \alpha \quad \Rightarrow \quad \perp \nabla_{[\mu} n_{\nu]} = 0 \quad (32)$$

Covariant Derivative on Σ compatible with $h_{\mu\nu}$

$$\mathcal{D}_\mu T^{\alpha\dots}_{\beta\dots} = \perp \nabla_\mu T^{\alpha\dots}_{\beta\dots} \quad \forall \quad T = \perp T \quad (33)$$

Intrinsic Curvature of (Σ, h)

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] A^\lambda = \mathcal{R}^\lambda_{\sigma\mu\nu} A^\sigma \quad \forall \quad A^\lambda = \perp A^\lambda \quad (34)$$

Gauss-Codazzi

$$\perp R_{\lambda\mu\sigma\nu} = \mathcal{R}_{\lambda\mu\sigma\nu} - K_{\lambda\sigma} K_{\mu\nu} + K_{\mu\sigma} K_{\nu\lambda} \quad (35)$$

$$\perp \left(R_{\lambda\mu\sigma\nu} n^\lambda \right) = \mathcal{D}_\nu K_{\mu\sigma} - \mathcal{D}_\sigma K_{\mu\nu} \quad (36)$$

$$\perp \left(R_{\lambda\mu\sigma\nu} n^\lambda n^\sigma \right) = -\mathcal{L}_n K_{\mu\nu} + K_\mu^\lambda K_{\lambda\nu} + \mathcal{D}_\mu a_\nu - a_\mu a_\nu \quad (37)$$

Projections of the Ricci tensor

$$\perp (R_{\mu\nu}) = \mathcal{R}_{\mu\nu} + \mathcal{D}_\mu a_\nu - a_\mu a_\nu - \mathcal{L}_n K_{\mu\nu} - K K_{\mu\nu} + 2K_\mu^\lambda K_{\nu\lambda} \quad (38)$$

$$\perp (R_{\mu\nu} n^\mu) = \mathcal{D}^\mu K_{\mu\nu} - \mathcal{D}_\nu K \quad (39)$$

$$R_{\mu\nu} n^\mu n^\nu = -\mathcal{L}_n K - K^{\mu\nu} K_{\mu\nu} + \mathcal{D}_\mu a^\mu - a_\mu a^\mu \quad (40)$$

Decomposition of the Ricci scalar

$$R = \mathcal{R} - K^2 - K^{\mu\nu} K_{\mu\nu} - 2\mathcal{L}_n K + 2\mathcal{D}_\mu a^\mu - 2a_\mu a^\mu \quad (41)$$

Lie Derivatives along n^μ

$$\mathcal{L}_n K_{\mu\nu} = n^\lambda \nabla_\lambda K_{\mu\nu} + K_{\lambda\nu} \nabla_\mu n^\lambda + K_{\mu\lambda} \nabla_\nu n^\lambda \quad (42)$$

$$\perp (\mathcal{L}_n \mathcal{F}^{\mu\dots}_{\nu\dots}) = \mathcal{L}_n \mathcal{F}^{\mu\dots}_{\nu\dots} \quad \forall \quad \perp \mathcal{F} = \mathcal{F} \quad (43)$$

5. Sign Conventions for the Action

These conventions follow Weinberg, keeping in mind that he defines the Riemann tensor with a minus sign relative to our definition. They are appropriate when using signature $(-, +, \dots, +)$. The $d + 1$ -dimensional Newton’s constant is $2\kappa^2 = 16\pi G_{d+1}$. The sign on the boundary term follows from our definition of the extrinsic curvature.

Gravitational Action

$$I_G = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} (R - 2\Lambda) + \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma} K \quad (44)$$

$$= \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} (\mathcal{R} + K^2 - K^{\mu\nu} K_{\mu\nu} - 2\Lambda) \quad (45)$$

Gauge Field Coupled to Particles

$$I_M = -\frac{1}{4} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} F^{\mu\nu} F_{\mu\nu} \quad (46)$$

$$- \sum_n m_n \int dp \left(-g_{\mu\nu}(x_n(p)) \frac{dx_n^\mu(p)}{dp} \frac{dx_n^\nu(p)}{dp} \right)^{1/2} \quad (47)$$

$$+ \sum_n e_n \int dp \frac{dx_n^\mu(p)}{dp} A_\mu(x_n(p)) \quad (48)$$

Gravity Minimally Coupled to a Gauge Field

$$I = \int_{\mathcal{M}} d^{d+1}x \sqrt{g} \left[\frac{1}{2\kappa^2} (R - 2\Lambda) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] + \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma} K \quad (49)$$

6. Hamiltonian Formulation

The canonical variables are the metric $h_{\mu\nu}$ on Σ and its conjugate momenta $\pi^{\mu\nu}$. The momenta are defined with respect to evolution in the *spacelike* direction n^μ , so this is not the usual notion of the Hamiltonian as the generator of time translations.

Bulk Lagrangian Density

$$\mathcal{L}_{\mathcal{M}} = \frac{1}{2\kappa^2} (K^2 - K^{\mu\nu} K_{\mu\nu} + \mathcal{R} - 2\Lambda) \quad (50)$$

Momentum Conjugate to $h_{\mu\nu}$

$$\pi^{\mu\nu} = \frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial (\mathcal{L}_n h_{\mu\nu})} = \frac{1}{2\kappa^2} (h^{\mu\nu} K - K^{\mu\nu}) \quad (51)$$

Momentum Constraint

$$\mathcal{H}_\mu = \frac{1}{\kappa^2} \perp (n^\nu G_{\mu\nu}) = 2\mathcal{D}^\nu \pi_{\mu\nu} = 0 \quad (52)$$

Hamiltonian Constraint

$$\mathcal{H} = -\frac{1}{\kappa^2} n^\mu n^\nu G_{\mu\nu} = 2\kappa^2 \left(\pi^{\mu\nu} \pi_{\mu\nu} - \frac{1}{d-1} \pi^2 \right) + \frac{1}{2\kappa^2} (\mathcal{R} - 2\Lambda) = 0 \quad (53)$$

7. Conformal Transformations

The dimension of spacetime is $d+1$. Indices are raised and lowered using the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

Metric

$$\hat{g}_{\mu\nu} = e^{2\sigma} g_{\mu\nu} \quad (54)$$

Christoffel

$$\widehat{\Gamma}_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} + \Theta_{\mu\nu}^{\lambda} \quad (55)$$

$$\Theta_{\mu\nu}^{\lambda} = \delta^{\lambda}_{\mu} \nabla_{\nu} \sigma + \delta^{\lambda}_{\nu} \nabla_{\mu} \sigma - g_{\mu\nu} \nabla^{\lambda} \sigma \quad (56)$$

Riemann Tensor

$$\widehat{R}^{\lambda}_{\mu\rho\nu} = R^{\lambda}_{\mu\rho\nu} + \delta^{\lambda}_{\nu} \nabla_{\mu} \nabla_{\rho} \sigma - \delta^{\lambda}_{\rho} \nabla_{\mu} \nabla_{\nu} \sigma + g_{\mu\rho} \nabla_{\nu} \nabla^{\lambda} \sigma - g_{\mu\nu} \nabla_{\rho} \nabla^{\lambda} \sigma \quad (57)$$

$$+ \delta^{\lambda}_{\rho} \nabla_{\mu} \sigma \nabla_{\nu} \sigma - \delta^{\lambda}_{\nu} \nabla_{\mu} \sigma \nabla_{\rho} \sigma + g_{\mu\nu} \nabla_{\rho} \sigma \nabla^{\lambda} \sigma - g_{\mu\rho} \nabla_{\nu} \sigma \nabla^{\lambda} \sigma \quad (58)$$

$$+ \left(g_{\mu\rho} \delta^{\lambda}_{\nu} - g_{\mu\nu} \delta^{\lambda}_{\rho} \right) \nabla^{\alpha} \sigma \nabla_{\alpha} \sigma \quad (59)$$

Ricci Tensor

$$\widehat{R}_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} \nabla^2 \sigma - (d-1) \nabla_{\mu} \nabla_{\nu} \sigma + (d-1) \nabla_{\mu} \sigma \nabla_{\nu} \sigma \quad (60)$$

$$- (d-1) g_{\mu\nu} \nabla^{\lambda} \sigma \nabla_{\lambda} \sigma \quad (61)$$

Ricci Scalar

$$\widehat{R} = e^{-2\sigma} \left(R - 2d \nabla^2 \sigma - d(d-1) \nabla^{\mu} \sigma \nabla_{\mu} \sigma \right) \quad (62)$$

Schouten Tensor

$$\widehat{S}_{\mu\nu} = S_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} \sigma + \nabla_{\mu} \sigma \nabla_{\nu} \sigma - \frac{1}{2} g_{\mu\nu} \nabla^{\lambda} \sigma \nabla_{\lambda} \sigma \quad (63)$$

Weyl Tensor

$$\widehat{C}^{\lambda}_{\mu\rho\nu} = C^{\lambda}_{\mu\rho\nu} \quad (64)$$

Normal Vector

$$\widehat{n}^{\mu} = e^{-\sigma} n^{\mu} \quad \widehat{n}_{\mu} = e^{\sigma} n_{\mu} \quad (65)$$

Extrinsic Curvature

$$\widehat{K}_{\mu\nu} = e^{\sigma} \left(K_{\mu\nu} + h_{\mu\nu} n^{\lambda} \nabla_{\lambda} \sigma \right) \quad (66)$$

$$\widehat{K} = e^{-\sigma} \left(K + d n^{\lambda} \nabla_{\lambda} \sigma \right) \quad (67)$$

8. Small Variations of the Metric

Consider a small perturbation to the metric of the form $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$. All indices are raised and lowered using the unperturbed metric $g_{\mu\nu}$ and its inverse. All quantities are expressed in terms of the perturbation to the metric with lower indices, and never in terms of the perturbation to the inverse metric. As in the previous sections, ∇_{μ} is the covariant derivative on \mathcal{M} compatible with $g_{\mu\nu}$ and \mathcal{D}_{μ} is the covariant derivative on a hypersurface Σ compatible with $h_{\mu\nu}$.

Inverse Metric

$$g^{\mu\nu} \rightarrow g^{\mu\nu} - g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} g^{\lambda\rho} \delta g_{\alpha\lambda} \delta g_{\beta\rho} + \dots \quad (68)$$

Square Root of Determinant

$$\sqrt{g} \rightarrow \sqrt{g} \left(1 + \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} + \dots \right) \quad (69)$$

Variational Operator

$$\delta(g_{\mu\nu}) = \delta g_{\mu\nu} \quad \delta^2(g_{\mu\nu}) = \delta(\delta g_{\mu\nu}) = 0 \quad (70)$$

$$\delta(g^{\mu\nu}) = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \quad \delta^2(g^{\mu\nu}) = \delta \left(-g^{\mu\lambda} g^{\nu\rho} \delta g_{\lambda\rho} \right) = 2 g^{\mu\alpha} g^{\nu\beta} g^{\lambda\rho} \delta g_{\alpha\lambda} \delta g_{\beta\rho} \quad (71)$$

$$\mathcal{F}(g + \delta g) = \mathcal{F}(g) + \delta \mathcal{F}(g) + \frac{1}{2} \delta^2 \mathcal{F}(g) + \dots + \frac{1}{n!} \delta^n \mathcal{F}(g) + \dots \quad (72)$$

Christoffel (All Orders)

$$\delta \Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\rho} \left(\nabla_{\mu} \delta g_{\rho\nu} + \nabla_{\nu} \delta g_{\mu\rho} - \nabla_{\rho} \delta g_{\mu\nu} \right) \quad (73)$$

$$\delta^2 \Gamma_{\mu\nu}^{\lambda} = -g^{\lambda\alpha} g^{\rho\beta} \delta g_{\alpha\beta} \left(\nabla_{\mu} \delta g_{\rho\nu} + \nabla_{\nu} \delta g_{\mu\rho} - \nabla_{\rho} \delta g_{\mu\nu} \right) \quad (74)$$

$$\delta^n \Gamma_{\mu\nu}^{\lambda} = \frac{n}{2} \delta^{n-1} \left(g^{\lambda\rho} \right) \left(\nabla_{\mu} \delta g_{\rho\nu} + \nabla_{\nu} \delta g_{\mu\rho} - \nabla_{\rho} \delta g_{\mu\nu} \right) \quad (75)$$

Riemann Tensor

$$\delta R^{\lambda}_{\mu\sigma\nu} = \nabla_{\sigma} \delta \Gamma_{\mu\nu}^{\lambda} - \nabla_{\nu} \delta \Gamma_{\mu\sigma}^{\lambda} \quad (76)$$

Ricci Tensor

$$\delta R_{\mu\nu} = \nabla_{\lambda} \delta \Gamma_{\mu\nu}^{\lambda} - \nabla_{\nu} \delta \Gamma_{\mu\lambda}^{\lambda} \quad (77)$$

$$= \frac{1}{2} \left(\nabla^{\lambda} \nabla_{\mu} \delta g_{\lambda\nu} + \nabla^{\lambda} \nabla_{\nu} \delta g_{\mu\lambda} - g^{\lambda\rho} \nabla_{\mu} \nabla_{\nu} \delta g_{\lambda\rho} - \nabla^2 \delta g_{\mu\nu} \right) \quad (78)$$

Ricci Scalar

$$\delta R = -R^{\mu\nu} \delta g_{\mu\nu} + \nabla^{\mu} \left(\nabla_{\nu} \delta g_{\mu\nu} - g^{\lambda\rho} \nabla_{\mu} \delta g_{\lambda\rho} \right) \quad (79)$$

Surface Forming Normal Vector

$$\delta n_{\mu} = \frac{1}{2} n_{\mu} n^{\nu} n^{\lambda} \delta g_{\nu\lambda} = \frac{1}{2} \delta g_{\mu\nu} n^{\nu} + c_{\mu} \quad (80)$$

$$c_{\mu} = \frac{1}{2} n_{\mu} n^{\nu} n^{\lambda} \delta g_{\nu\lambda} - \frac{1}{2} \delta g_{\mu\nu} n^{\nu} = -\frac{1}{2} h_{\mu}^{\lambda} \delta g_{\lambda\nu} n^{\nu} \quad (81)$$

Extrinsic Curvatures

$$\begin{aligned} \delta K_{\mu\nu} &= \frac{1}{2} n^{\alpha} n^{\beta} \delta g_{\alpha\beta} K_{\mu\nu} + \delta g_{\lambda\rho} n^{\rho} \left(n_{\mu} K^{\lambda}_{\nu} + n_{\nu} K_{\mu}^{\lambda} \right) \\ &\quad - \frac{1}{2} h_{\mu}^{\lambda} h_{\nu}^{\rho} n^{\alpha} \left(\nabla_{\lambda} \delta g_{\alpha\rho} + \nabla_{\rho} \delta g_{\lambda\alpha} - \nabla_{\alpha} \delta g_{\lambda\rho} \right) \end{aligned} \quad (82)$$

$$\delta K = -\frac{1}{2} K^{\mu\nu} \delta g_{\mu\nu} - \frac{1}{2} n^{\mu} \left(\nabla_{\nu} \delta g_{\mu\nu} - g^{\nu\lambda} \nabla_{\mu} \delta g_{\nu\lambda} \right) + \mathcal{D}_{\mu} c^{\mu} \quad (83)$$

9. The ADM Decomposition

The conventions and notation in this section (and the next) are different than what was used in the preceding sections. We consider a d -dimensional spacetime with metric h_{ab} .

We start by identifying a scalar field t whose isosurfaces Σ_t are normal to the timelike unit vector given by

$$u_a = -\alpha \partial_a t , \quad (84)$$

where the lapse function α is

$$\alpha := \frac{1}{\sqrt{-h^{ab} \partial_a t \partial_b t}} . \quad (85)$$

An observer whose worldline is tangent to u_a experiences an acceleration given by the vector

$$a_b = u^c \cdot {}^d \nabla_c u_b , \quad (86)$$

which is orthogonal to u_a . The (spatial) metric on the $d - 1$ dimensional surface Σ_t is given by

$$\sigma_{ab} = h_{ab} + u_a u_b . \quad (87)$$

The intrinsic Ricci tensor built from this metric is denoted by \mathcal{R}_{ab} , and its Ricci scalar is \mathcal{R} . The covariant derivative on Σ_t is defined in terms of the d dimensional covariant derivative as

$$D_a V_b := \sigma_a^c \sigma_b^e ({}^d \nabla_c V_e) \quad \text{for any} \quad V_b = \sigma_b^c V_c . \quad (88)$$

The extrinsic curvature of Σ_t embedded in the ambient d dimensional spacetime (the constant r surfaces from the previous section) is

$$\theta_{ab} := -\sigma_a^c \sigma_b^d ({}^d \nabla_c u_d) = -{}^d \nabla_a u_b - u_a a_b = -\frac{1}{2} \mathcal{L}_u \sigma_{ab} . \quad (89)$$

This definition has an additional minus sign, compared to the extrinsic curvature $K_{\mu\nu}$ for the constant r surfaces of the previous section. This is merely for compatibility with the standard conventions in the literature.

Now we consider a ‘time flow’ vector field t^a , which satisfies the condition

$$t^a \partial_a t = 1 . \quad (90)$$

The vector t^a can be decomposed into parts normal and along Σ_t as

$$t^a = \alpha u^a + \beta^a , \quad (91)$$

where α is the lapse function (85) and $\beta^a := \sigma^a_b t^b$ is the shift vector. An important result in the derivations that follow relates the Lie derivative of a scalar or spatial tensor (one that is orthogonal to u^a in all of its indices) along the time flow vector field, to Lie derivatives along u^a and β^a . Let S be a scalar. Then

$$\mathcal{L}_t S = \mathcal{L}_{\alpha u} S + \mathcal{L}_{\beta} S = \alpha \mathcal{L}_u S + \mathcal{L}_{\beta} S . \quad (92)$$

Rearranging this expression then gives

$$\mathcal{L}_u S = \frac{1}{\alpha} (\mathcal{L}_t S - \mathcal{L}_{\beta} S) . \quad (93)$$

Similarly, for a spatial tensor with all lower indices we have

$$\mathcal{L}_t W_{a\dots} = \alpha \mathcal{L}_u W_{a\dots} + \mathcal{L}_{\beta} W_{a\dots} . \quad (94)$$

This is not the case when the tensor has any of its indices raised. In a moment, these identities will allow us to express certain Lie derivatives along u^a in terms of regular time derivatives and Lie derivatives along the shift vector β^a .

Next, we construct the coordinate system that we will use for the decomposition of the equations of motion. The adapted coordinates (t, x^i) are defined by

$$\partial_t x^a := t^a . \quad (95)$$

The x^i are d dimensional coordinates along the surface Σ_t . If we define

$$P_i^a := \frac{\partial x^a}{\partial x^i} , \quad (96)$$

then it follows from the definition of the coordinates that $P_i^a \partial_{at} = 0$ and we can use P_i^a to project tensors onto Σ_t . For example, in the adapted coordinates the spatial metric, extrinsic curvature, and acceleration and shift vectors are

$$\sigma_{ij} = P_i^a P_j^b \sigma_{ab} \quad (97)$$

$$\theta_{ij} = P_i^a P_j^b \theta_{ab} \quad (98)$$

$$a_j = P_j^b a_b \quad (99)$$

$$\beta_i = P_i^a \beta_a = P_i^a t_a . \quad (100)$$

The line element in the adapted coordinates takes a familiar form:

$$h_{ab} dx^a dx^b = h_{ab} \left(\frac{\partial x^a}{\partial t} dt + \frac{\partial x^a}{\partial x^i} dx^i \right) \left(\frac{\partial x^b}{\partial t} dt + \frac{\partial x^b}{\partial x^j} dx^j \right) \quad (101)$$

$$= h_{ab} (t^a dt + P_i^a dx^i) (t^b dt + P_j^b dx^j) \quad (102)$$

$$= t^a t_a dt^2 + 2t_a dt P_i^a dx^i + h_{ab} P_i^a P_j^b dx^i dx^j \quad (103)$$

$$= (-\alpha^2 + \beta^i \beta_i) dt^2 + 2\beta_i dt dx^i + \sigma_{ij} dx^i dx^j \quad (104)$$

$$\Rightarrow h_{ab} dx^a dx^b = -\alpha^2 dt^2 + \sigma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt) . \quad (105)$$

Thus, in the adapted coordinate system we can express the components of the (d dimensional) metric h_{ab} and its inverse h^{ab} as

$$h_{ab} = \left(\begin{array}{c|c} -\alpha^2 + \beta^i \beta_i & \sigma_{ij} \beta^j \\ \hline \sigma_{ij} \beta^j & \sigma_{ij} \end{array} \right) \quad (106)$$

$$h^{ab} = \left(\begin{array}{c|c} -\frac{1}{\alpha^2} & \frac{1}{\alpha^2} \beta^i \\ \hline \frac{1}{\alpha^2} \beta^i & \sigma^{ij} - \frac{1}{\alpha^2} \beta^i \beta^j \end{array} \right) \quad (107)$$

$$\det(h_{ab}) = -\alpha^2 \det(\sigma_{ij}) \quad (108)$$

Obtaining the components of the inverse is a short algebraic calculation. Note that the spatial indices ' i, j, \dots ' in the adapted coordinates are lowered and raised using the spatial metric σ_{ij} and its inverse σ^{ij} .

In adapted coordinates there are several results concerning the projections of Lie derivatives of scalars and tensors which will be important in what follows. The first, which is trivial, is that the Lie derivative of a scalar S along the time-flow vector t^a is just the regular time-derivative

$$\mathcal{L}_t S = t^a \partial_a S = \frac{\partial x^a}{\partial t} \frac{\partial S}{\partial x^a} = \partial_t S . \quad (109)$$

Next, we consider the projector P_i^a applied to the Lie derivative along t^a of a general vector W_a , which gives

$$P_i^a \mathcal{L}_t W_a = \partial_t W_a \quad \forall \quad W_a . \quad (110)$$

The important point is that this applies not just to spatial vectors but to *any* vector W_a , as a consequence of the result

$$P_i^a \mathcal{L}_t u_a = 0 . \quad (111)$$

Finally, we can show that the Lie derivative along t^a of any contravariant spatial vector satisfies

$$P^i_a \mathcal{L}_t V^a = \partial_t V^i \quad \forall \quad V^i = P^i_a V^a . \quad (112)$$

This follows from a lengthier calculation than what is required for the first two results.

Given these results, we can express various geometric quantities and their projections normal to and along Σ_t in terms of quantities intrinsic to Σ_t and simple time derivatives. First, the extrinsic curvature is

$$\theta_{ij} = -\frac{1}{2} P_i^a P_j^b \mathcal{L}_u \sigma_{ab} \quad (113)$$

$$= -\frac{1}{2} P_i^a P_j^b \left(\frac{1}{\alpha} (\mathcal{L}_t \sigma_{ab} - \mathcal{L}_\beta \sigma_{ab}) \right) \quad (114)$$

$$\Rightarrow \theta_{ij} = -\frac{1}{2\alpha} \left(\partial_t \sigma_{ab} - (D_a \beta_b + D_b \beta_a) \right) . \quad (115)$$

Since θ_{ab} is a spatial tensor, projections of its Lie derivative along u^a can be expressed in a similar manner

$$P_i^a P_j^b \mathcal{L}_u \theta_{ab} = \frac{1}{\alpha} (\partial_t \theta_{ab} - \mathcal{L}_\beta \theta_{ab}) . \quad (116)$$

Now we present the Gauss-Codazzi and related equations in adapted coordinates:

$$P_i^a P_j^b ({}^d R_{ab}) = \mathcal{R}_{ij} + \theta \theta_{ij} - 2\theta_i^k \theta_{jk} - \frac{1}{\alpha} (\partial_t \theta_{ij} - \mathcal{L}_\beta \theta_{ij}) - \frac{1}{\alpha} D_i D_j \alpha \quad (117)$$

$$P_i^a ({}^d R_{ab} u^b) = D_i \theta - D^j \theta_{ij} \quad (118)$$

$${}^d R_{ab} u^a u^b = \frac{1}{\alpha} (\partial_t \theta - \beta^i \partial_i \theta) - \theta^{ij} \theta_{ij} + \frac{1}{\alpha} D_i D^i \alpha \quad (119)$$

$${}^d R = \mathcal{R} + \theta^2 + \theta^{ij} \theta_{ij} - \frac{2}{\alpha} (\partial_t \theta - \beta^i \partial_i \theta) - \frac{2}{\alpha} D_i D^i \alpha . \quad (120)$$

10. Converting to ADM Variables

The metric is often presented in the form

$$h_{ab} dx^a dx^b = h_{tt} dt^2 + 2h_{ti} dt dx^i + h_{ij} dx^i dx^j . \quad (121)$$

We would like to relate these components to the ADM variables: the lapse function α , the shift vector β_i , and the spatial metric σ_{ij} . This is a fairly straightforward exercise in linear algebra. Comparing with (105), we first note that

$$\sigma_{ij} = h_{ij} . \quad (122)$$

The inverse spatial metric, σ^{ij} , is literally the inverse of h_{ij} , which is *not* the same thing as h^{ij}

$$\sigma^{ij} = (\sigma_{ij})^{-1} = (h_{ij})^{-1} \neq h^{ij} . \quad (123)$$

For the shift vector we have

$$h_{ti} = \sigma_{ij} \beta^j \quad \rightarrow \quad \sigma^{ik} h_{tk} = \sigma^{ik} \sigma_{kl} \beta^l = \beta^i \quad (124)$$

$$\Rightarrow \beta^i = \sigma^{ij} h_{tj} . \quad (125)$$

Finally, for the lapse we obtain

$$\alpha^2 = \sigma^{ij} h_{ti} h_{tj} - h_{tt} . \quad (126)$$