10-1982

Ideals and Centralizing Mappings in Prime Rings

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PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 2, October 1982

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ABSTRACT. Let R be a prime ring and U be a nonzero ideal of R. If T is a nontrivial automorphism or derivation of R such that $uu^T - u^T u$ is in the center of R and u^T is in U for every u in U, then R is commutative. If R does not have characteristic equal to two, then U need only be a nonzero Jordan ideal.

If R is a ring, a mapping T of R to itself is called *centralizing* on a subset S of R if $ss^T - s^T s$ is in the center of R for every s in S. There has been considerable interest in centralizing automorphisms and derivations defined on rings. Miers [**4**] has studied these mappings defined on C^*-algebras. In [**5**] Posner proved that if a prime ring has a nontrivial centralizing derivation, then the ring must be commutative. The same result was obtained for centralizing automorphisms in [**3**]. In this paper it is shown that the automorphism or derivation need only be centralizing and invariant on a nonzero ideal in the prime ring in order to ensure that the ring is commutative. Also, if R is of characteristic not two, then the mapping need only be centralizing and invariant on a nonzero Jordan ideal. For derivations this gives a short proof of a result related to that of Awtar [**1**, Theorem 3]. Awtar proved that if R is a prime ring of characteristic not two with a nontrivial derivation and a nonzero Jordan ideal U such that the derivation is centralizing on U, then U is contained in the center of R.

Jeffrey Bergen deserves many thanks for his suggestions concerning the results and proofs in this paper.

From now on assume that R is a prime ring and let Z be the center of R. Let $[x,y] = xy - yx$ and note the important identity $[x,yz] = y[x,z] + [x,y]z$. The following lemmas will be used in the proofs of the main results.

LEMMA 1. *If $b[a,r] = 0$ for all r in R, then $b = 0$ or a is in Z.*

PROOF. Assume that $b[a,r] = 0$ for all r in R. Replace r by xy to obtain $b[a,xy] = bx[a,y] + b[a,x]y = bx[a,y] = 0$ for all x and y in R. Since R is prime, $b = 0$ or $[a,y] = 0$ for all x in R.

LEMMA 2. *If D is a derivation of R such that $u^D = 0$ for all u in a nonzero right ideal U of R, then $r^D = 0$ for all r in R.*

PROOF. Let u be a nonzero element in U and x be an element in R. Then ux is in U and $0 = (ux)^D = u^D x + u(x^D) = u(x^D)$. Now replace x by sr to obtain $0 = u(sr)^D = [u(s^D)]r + us(r^D) = us(r^D)$ for all r and s in R. Since R is prime and u is nonzero, $r^D = 0$ for all r in R.

Received by the editors October 21, 1981 and, in revised form, January 12, 1982.
1980 *Mathematics Subject Classification*. Primary 16A70; Secondary 16A12, 16A68, 16A72.
Key words and phrases. Ideal, centralizing automorphism, centralizing derivation, prime ring, commutative.

0002-9939/82/0000-0069/$01.75

LEMMA 3. *If T is a homomorphism of R such that $u^T = u$ for all u in a nonzero right ideal U of R, then $r^T = r$ for every r in R.*

PROOF. Let u be a nonzero element in U and r, s be in R. Since U is a right ideal, us and usr are in U. Then $(usr)^T = usr = (us)^T r^T = usr^T$. Hence $us(r - r^T) = 0$ for all s and r in R. Thus $r = r^T$ for all r in R.

LEMMA 4. *If R contains a nonzero commutative right ideal U, then R must be commutative.*

PROOF. Let u be in U and assume that u^2 is not zero. Such an element exists for if not, then by a variation of Levitzki's theorem [**2**, Lemma 1.1], R has a nonzero nilpotent ideal and this is impossible in a prime ring. U is a right ideal and so ur and us are in U for every r and s in R. Since U is commutative, $u^2sr = u(us)r = us(ur) = ur(us) = u(ur)s = u^2rs$. Hence $u^2[r, s] = 0$ for all r and s in R. By Lemma 1, every r in R is in Z. Therefore R is commutative.

THEOREM. *Let R be a prime ring and U be a nonzero ideal of R. If R has a nontrivial automorphism or derivation T such that $uu^T - u^Tu$ is in the center of R and u^T is in U for every u in U, then R is commutative.*

PROOF. By Lemma 2 or Lemma 3, T is nontrivial on U. Since U is a nonzero ideal in a prime ring, U is itself a prime ring. U is then commutative by the author's result in [**3**] for automorphisms or by Posner's result [**5**] for derivations. By Lemma 4, R is commutative.

COROLLARY. *If U is a nonzero Jordan ideal in a prime ring R of characteristic not two and T is a nontrivial automorphism or derivation of R which is centralizing and invariant on U, then R is commutative.*

PROOF. Every nonzero Jordan ideal in a prime ring of characteristic not two contains a nonzero ideal [**2**, Theorem 1.1]. Apply the theorem to this ideal.

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