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Invariant and coinvariant spaces for the algebra of symmetric polynomials in non-commuting variables

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Abstract

We analyze the structure of the algebra $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ of symmetric polynomials in non-commuting variables in so far as it relates to $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$, its commutative counterpart. Using the “place-action” of the symmetric group, we are able to realize the latter as the invariant polynomials inside the former. We discover a tensor product decomposition of $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ analogous to the classical theorems of Chevalley, Shephard-Todd on finite reflection groups.

Résumé. Nous analysons la structure de l’algèbre $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ des polynômes symétriques en des variables non-commutatives pour obtenir des analogues des résultats classiques concernant la structure de l’anneau $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ des polynômes symétriques en des variables commutatives. Plus précisément, au moyen de “l’action par positions”, on réalise $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ comme sous-module de $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$. On découvre alors une nouvelle décomposition de $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ comme produit tensoriel, obtenant ainsi un analogues des théorèmes classiques de Chevalley et Shephard-Todd.

1 Introduction

One of the more striking results of invariant theory is certainly the following: if W is a finite group of $n \times n$ matrices (over some field \mathbb{K} containing \mathbb{Q}), then there is a W -module decomposition of the polynomial ring $S = \mathbb{K}[\mathbf{x}]$, in variables $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$, as a tensor product

$$S \simeq S_W \otimes S^W \tag{1}$$

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if and only if W is a group generated by (pseudo) reflections. As usual, S is afforded a natural W -module structure by considering it as the symmetric space on the defining vector space X^* for W , e.g., $w \cdot f(\mathbf{x}) = f(\mathbf{x} \cdot w)$. It is customary to denote by S^W the ring of W -invariant polynomials for this action. To finish parsing (1), recall that S_W stands for the **coinvariant space**, i.e., the W -module

$$S_W := S / \langle S_+^W \rangle \tag{2}$$

defined as the quotient of S by the ideal generated by constant-term free W -invariant polynomials. We give S an \mathbb{N} -grading by degree in the variables \mathbf{x} . Since the W -action on S preserves degrees, both S^W and S_W inherit a grading from the one on S , and (1) is an isomorphism of graded W -modules. One of the motivations behind the quotient in (2) is to eliminate trivially redundant copies of irreducible W -modules inside S . Indeed, if \mathcal{V} is such a module and f is any W -invariant polynomial with no constant term, then $\mathcal{V}f$ is an isomorphic copy of \mathcal{V} living within $\langle S_+^W \rangle$. Thus, the coinvariant space S_W is the more interesting part of the story.

The context for the present paper is the algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials, with W -module structure on T obtained by considering it as the tensor space on the defining space X^* for W . In the special case when W is the symmetric group \mathfrak{S}_n , we elucidate a relationship between the space S^W and the subalgebra T^W of W -invariants in T . The subalgebra T^W was first studied in [4, 20] with the aim of obtaining noncommutative analogs of classical results concerning symmetric function theory. Recent work in [2, 15] has extended a large part of the story surrounding (1) to this noncommutative context. In particular, there is an explicit \mathfrak{S}_n -module decomposition of the form $T \simeq T_{\mathfrak{S}_n} \otimes T^{\mathfrak{S}_n}$ [2, Theorem 8.7]. See [7] for a survey of other results in noncommutative invariant theory.

By contrast, our work proceeds in a somewhat complementary direction. We consider $\mathcal{N} = T^{\mathfrak{S}_n}$ as a tower of \mathfrak{S}_d -modules under the “place-action” and realize $S^{\mathfrak{S}_n}$ inside \mathcal{N} as a subspace Λ of invariants for this action. This leads to a decomposition of \mathcal{N} analogous to (1). More explicitly, our main result is as follows.

Theorem 1. *There is an explicitly constructed subspace \mathcal{C} of \mathcal{N} so that \mathcal{C} and the place-action invariants Λ exhibit a graded vector space isomorphism*

$$\mathcal{N} \simeq \mathcal{C} \otimes \Lambda. \tag{3}$$

An analogous result holds in the case $|\mathbf{x}| = \infty$. An immediate corollary in either case is the Hilbert series formula

$$\text{Hilb}_t(\mathcal{C}) = \text{Hilb}_t(\mathcal{N}) \prod_{i=1}^{|\mathbf{x}|} (1 - t^i). \tag{4}$$

Here, the **Hilbert series** of a graded space $\mathcal{V} = \bigoplus_{d \geq 0} \mathcal{V}_d$ is the formal power series defined as

$$\text{Hilb}_t(\mathcal{V}) = \sum_{d \geq 0} \dim \mathcal{V}_d t^d,$$

where \mathcal{V}_d is the **homogeneous degree d component** of \mathcal{V} . The fact that (4) expands as a series in $\mathbb{N}[[t]]$ is not at all obvious, as one may check that the Hilbert series of \mathcal{N} is

$$\text{Hilb}_t(\mathcal{N}) = 1 + \sum_{k=1}^{|\mathbf{x}|} \frac{t^k}{(1-t)(1-2t)\cdots(1-kt)}. \quad (5)$$

In Sections 2 and 3, we recall the relevant structural features of S and T . Section 4 describes the place-action structure of T and the original motivation for our work. Our main results are proven in Sections 5 and 6. We underline that the harder part of our work lies in working out the case $|\mathbf{x}| < \infty$. This is accomplished in Section 6. If we restrict ourselves to the case $|\mathbf{x}| = \infty$, both \mathcal{N} and Λ become Hopf algebras and our results are then consequences of a general theorem of Blattner, Cohen and Montgomery. As we will see in Section 5, stronger results hold in this simpler context. For example, (4) may be refined to a statement about “shape” enumeration.

2 The algebra $S^\mathfrak{S}$ of symmetric functions

2.1 Vector space structure of $S^\mathfrak{S}$

We specialize our introductory discussion to the group $W = \mathfrak{S}_n$ of permutation matrices (writing $|\mathbf{x}| = n$). The action on $S = \mathbb{K}[\mathbf{x}]$ is simply the **permutation action** $\sigma \cdot x_i = x_{\sigma(i)}$ and $S^{\mathfrak{S}_n}$ comprises the familiar symmetric polynomials. We suppress n in the notation and denote the subring of symmetric polynomials by $S^\mathfrak{S}$. (Note that upon sending n to ∞ , the elements of $S^\mathfrak{S}$ become formal series in $\mathbb{K}[[\mathbf{x}]]$ of bounded degree; we call both finite and infinite versions “functions” in what follows to affect a uniform discussion.) A monomial in S of degree d may be written as follows: given an r -subset $\mathbf{y} = \{y_1, y_2, \dots, y_r\}$ of \mathbf{x} and a **composition** of d into r parts, $\mathbf{a} = (a_1, a_2, \dots, a_r)$ ($a_i > 0$), we write $\mathbf{y}^\mathbf{a}$ for $y_1^{a_1} y_2^{a_2} \cdots y_r^{a_r}$. We assume that the variables y_i are naturally ordered, so that whenever $y_i = x_j$ and $y_{i+1} = x_k$ we have $j < k$. Reordering the entries of a composition \mathbf{a} in decreasing order results in a partition $\lambda(\mathbf{a})$ called the **shape** of \mathbf{a} . Summing over monomials $\mathbf{y}^\mathbf{a}$ with the same shape leads to the monomial symmetric function

$$m_\mu = m_\mu(\mathbf{x}) := \sum_{\lambda(\mathbf{a})=\mu, \mathbf{y} \subseteq \mathbf{x}} \mathbf{y}^\mathbf{a}.$$

Letting $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ run over all partitions of $d = |\mu| = \mu_1 + \mu_2 + \cdots + \mu_r$ gives a basis for $S_d^\mathfrak{S}$. As usual, we set $m_0 := 1$ and agree that $m_\mu = 0$ if μ has too many parts (i.e., $n < r$).

2.2 Dimension enumeration

A fundamental result in the invariant theory of \mathfrak{S}_n is that $S^\mathfrak{S}$ is generated by a family $\{f_k\}_{1 \leq k \leq n}$ of algebraically independent symmetric functions, having respective degrees

$\deg f_k = k$. (One may choose $\{m_k\}_{1 \leq k \leq n}$ for such a family.) It follows that the Hilbert series of $S^\mathfrak{S}$ is

$$\text{Hilb}_t(S^\mathfrak{S}) = \prod_{i=1}^n \frac{1}{1-t^i}. \quad (6)$$

Recalling that the Hilbert series of S is $(1-t)^{-n}$, we see from (1) and (6) that the Hilbert series for the coinvariant space $S_\mathfrak{S}$ is the well-known t -analog of $n!$:

$$\prod_{i=1}^n \frac{1-t^i}{1-t} = \prod_{i=1}^n (1+t+\cdots+t^{i-1}). \quad (7)$$

In particular, contrary to the situation in (4), the series $\text{Hilb}_t(S)/\text{Hilb}_t(S^\mathfrak{S})$ in $\mathbb{Q}[[t]]$ *obviously* belongs to $\mathbb{N}[[t]]$.

2.3 Algebra and coalgebra structures of $S^\mathfrak{S}$

Given partitions μ and ν , there is an explicit multiplication rule for computing the product $m_\mu \cdot m_\nu$. In lieu of giving the formula, see [2, §4.1], we simply give an example

$$m_{21} \cdot m_{11} = 3m_{2111} + 2m_{221} + 2m_{311} + m_{32} \quad (8)$$

and highlight two features relevant to the coming discussion.

First, we note that if $n < 4$, then the first term is equal to zero. However, if n is sufficiently large then analogs of this term always appear with positive integer coefficients. If $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ and $\nu = (\nu_1, \nu_2, \dots, \nu_s)$ with $r \leq s$, then the partition indexing the left-most term in $m_\mu m_\nu$ is denoted by $\mu \cup \nu$ and is given by sorting the list $(\mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s)$ in increasing order; the right-most term is indexed by $\mu + \nu := (\mu_1 + \nu_1, \dots, \mu_r + \nu_r, \nu_{r+1}, \dots, \nu_s)$. Taking $\mu = 31$ and $\nu = 221$, we would have $\mu \cup \nu = 32211$ and $\mu + \nu = 531$.

Second, we point out that the leftmost term (indexed by $\mu \cup \nu$) is indeed a *leading term* in the following sense. An important partial order on partitions takes

$$\lambda \leq \mu \quad \text{iff} \quad \sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i \quad \text{for all } k.$$

With this ordering, $\mu \cup \nu$ is the least partition occurring with nonzero coefficient in the product of $m_\mu m_\nu$. That is, $S^\mathfrak{S}$ is **shape-filtered**: $(S^\mathfrak{S})_\lambda \cdot (S^\mathfrak{S})_\mu \subseteq \bigoplus_{\nu \geq \lambda \cup \mu} (S^\mathfrak{S})_\nu$. Here $(S^\mathfrak{S})_\lambda$ denotes the subspace of $S^\mathfrak{S}$ indexed by partitions of shape λ (the linear span of m_λ), which we point out in preparation for the noncommutative analog.

The ring $S^\mathfrak{S}$ is afforded a coalgebra structure with counit $\varepsilon : S^\mathfrak{S} \rightarrow \mathbb{K}$ and coproduct $\Delta : S_d^\mathfrak{S} \rightarrow \bigoplus_{k=0}^d S_k^\mathfrak{S} \otimes S_{d-k}^\mathfrak{S}$ given, respectively, by

$$\varepsilon(m_\mu) = \delta_{\mu,0} \quad \text{and} \quad \Delta(m_\nu) = \sum_{\lambda \cup \mu = \nu} m_\lambda \otimes m_\mu.$$

If $|\mathbf{x}| = \infty$, Δ and ε are algebra maps, making $S^\mathfrak{S}$ a graded connected Hopf algebra.

3 The algebra \mathcal{N} of noncommutative symmetric functions

3.1 Vector space structure of \mathcal{N}

Suppose now that \mathbf{x} denotes a set of non-commuting variables. The algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials is graded by degree. A degree d **noncommutative monomial** $\mathbf{z} \in T_d$ is simply a length d “word”:

$$\mathbf{z} = z_1 z_2 \cdots z_d, \quad \text{with each } z_i \in \mathbf{x}.$$

In other terms, \mathbf{z} is a function $\mathbf{z} : [d] \rightarrow \mathbf{x}$, with $[d]$ denoting the set $\{1, 2, \dots, d\}$. The permutation-action on \mathbf{x} clearly extends to T , giving rise to the subspace $\mathcal{N} = T^{\mathfrak{S}}$ of noncommutative \mathfrak{S} -invariants. With the aim of describing a linear basis for the homogeneous component \mathcal{N}_d , we next introduce set partitions of $[d]$ and the type of a monomial $\mathbf{z} : [d] \rightarrow \mathbf{x}$. Let $\mathbf{A} = \{A_1, A_2, \dots, A_r\}$ be a set of subsets of $[d]$. Say \mathbf{A} is a **set partition** of $[d]$, written $\mathbf{A} \vdash [d]$, iff $A_1 \cup A_2 \cup \dots \cup A_r = [d]$, $A_i \neq \emptyset$ ($\forall i$), and $A_i \cap A_j = \emptyset$ ($\forall i \neq j$). The **type** $\tau(\mathbf{z})$ of a degree d monomial $\mathbf{z} : [d] \rightarrow \mathbf{x}$ is the set partition

$$\tau(\mathbf{z}) := \{\mathbf{z}^{-1}(x) : x \in \mathbf{x}\} \setminus \{\emptyset\} \quad \text{of } [d],$$

whose parts are the non-empty fibers of the function \mathbf{z} . For instance,

$$\tau(x_1 x_8 x_1 x_5 x_8) = \{\{1, 3\}, \{2, 5\}, \{4\}\}.$$

Note that the type of a monomial is a set partition with at most n parts. In what follows, we lighten the heavy notation for set partitions, writing, e.g., the set partition $\{\{1, 3\}, \{2, 5\}, \{4\}\}$ as 13.25.4. We also always order the parts in increasing order of their minimum elements. The **shape** $\lambda(\mathbf{A})$ of a set partition $\mathbf{A} = \{A_1, A_2, \dots, A_r\}$ is the (integer) partition $\lambda(|A_1|, |A_2|, \dots, |A_r|)$ obtained by sorting the part sizes of \mathbf{A} in increasing order, and its **length** $\ell(\mathbf{A})$ is its number of parts (r). Observing that the permutation-action is *type preserving*, we are led to index the **monomial** linear basis for the space \mathcal{N}_d by set partitions:

$$m_{\mathbf{A}} = m_{\mathbf{A}}(\mathbf{x}) := \sum_{\tau(\mathbf{z})=\mathbf{A}, \mathbf{z} \in \mathbf{x}^{[d]}} \mathbf{z}$$

For example, with $n = 2$, we have $m_1 = x_1 + x_2$, $m_{12} = x_1^2 + x_2^2$, $m_{1.2} = x_1 x_2 + x_2 x_1$, $m_{123} = x_1^3 + x_2^3$, $m_{12.3} = x_1^2 x_2 + x_2^2 x_1$, $m_{13.2} = x_1 x_2 x_1 + x_2 x_1 x_2$, $m_{1.2.3} = 0$, and so on. (We set $m_{\emptyset} := 1$, taking \emptyset as the unique set partition of the empty set, and we agree that $m_{\mathbf{A}} = 0$ if \mathbf{A} is a set partition with more than n parts.)

3.2 Dimension enumeration and shape grading

Above, we determined that $\dim \mathcal{N}_d$ is the number of set partitions of d into at most n parts. These are counted by the (length restricted) **Bell numbers** $B_d^{(n)}$. Consequently,

(5) follows from the fact that its right-hand side is the ordinary generating function for length restricted Bell numbers. See [10, §2]. We next highlight a finer enumeration, where we grade \mathcal{N} by shape rather than degree.

For each partition μ , we may consider the subspace \mathcal{N}_μ spanned by those $m_{\mathbf{A}}$ for which $\lambda(\mathbf{A}) = \mu$. This results in a direct sum decomposition $\mathcal{N}_d = \bigoplus_{\mu \vdash d} \mathcal{N}_\mu$. A simple dimension description for \mathcal{N}_d takes the form of a **shape Hilbert series** in the following manner. View commuting variables q_i as marking parts of size i and set $\mathbf{q}_\mu := q_{\mu_1} q_{\mu_2} \cdots q_{\mu_r}$. Then

$$\text{Hilb}_{\mathbf{q}}(\mathcal{N}_d) = \sum_{\mu \vdash d} \dim \mathcal{N}_\mu \mathbf{q}_\mu, = \sum_{\mathbf{A} \vdash [d]} q_{\lambda(\mathbf{A})}. \quad (9)$$

Here, \mathbf{q}_μ is a marker for set partitions of shape $\lambda(\mathbf{A}) = \mu$ and the sum is over all partitions into at most n parts. Such a shape grading also makes sense for $S_d^{\mathfrak{S}}$. Summing over all $d \geq 0$ and all μ , we get

$$\text{Hilb}_{\mathbf{q}}(S^{\mathfrak{S}}) = \sum_{\mu} \mathbf{q}_\mu = \prod_{i \geq 1}^n \frac{1}{1 - q_i}. \quad (10)$$

Using classical combinatorial arguments, one finds the enumerator polynomials $\text{Hilb}_{\mathbf{q}}(\mathcal{N}_d)$ are naturally collected in the **exponential generating function**

$$\sum_{d=0}^{\infty} \text{Hilb}_{\mathbf{q}}(\mathcal{N}_d) \frac{t^d}{d!} = \sum_{m=0}^n \frac{1}{m!} \left(\sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right)^m. \quad (11)$$

See [1, Chap. 2.3], Example 13(a). For instance, with $n = 3$, we have

$$\text{Hilb}_{\mathbf{q}}(\mathcal{N}_6) = q_6 + 6 q_5 q_1 + 15 q_4 q_2 + 15 q_4 q_1^2 + 10 q_3^2 + 60 q_3 q_2 q_1 + 15 q_2^3,$$

thus $\dim \mathcal{N}_{222} = 15$ when $n \geq 3$. Evidently, the \mathbf{q} -polynomials $\text{Hilb}_{\mathbf{q}}(\mathcal{N}_d)$ specialize to the length restricted Bell numbers $B_d^{(n)}$ when we set all q_k equal to 1.

In view of (10), (11), and Theorem 1, we claim the following refinement of (4).

Corollary 2. *Sending n to ∞ , the shape Hilbert series of the space \mathcal{C} is given by*

$$\text{Hilb}_{\mathbf{q}}(\mathcal{C}) = \sum_{d \geq 0} d! \exp \left(\sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right) \Big|_{t^d} \prod_{i \geq 1} (1 - q_i), \quad (12)$$

with $(-)|_{t^d}$ standing for the operation of taking the coefficient of t^d .

This refinement of (4) will follow immediately from the isomorphism $\mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ in Section 5, which is shape-preserving in an appropriate sense. Thus we have the expansion

$$\begin{aligned} \text{Hilb}_{\mathbf{q}}(\mathcal{C}) &= 1 + 2 q_2 q_1 + (3 q_3 q_1 + 2 q_2^2 + 3 q_2 q_1^2) \\ &\quad + (4 q_4 q_1 + 9 q_3 q_2 + 6 q_3 q_1^2 + 10 q_2^2 q_1 + 4 q_2 q_1^3) + \cdots \end{aligned}$$

3.3 Algebra and coalgebra structures of \mathcal{N}

Since the action of \mathfrak{S} on T is multiplicative, it is straightforward to see that \mathcal{N} is a subalgebra of T . The *multiplication rule* in \mathcal{N} , expressing a product $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ as a sum of basis vectors $\sum_{\mathbf{C}} m_{\mathbf{C}}$, is easy to describe. Since we make heavy use of the rule later, we develop it carefully here. We begin with an example (digits corresponding to $\mathbf{B} = \mathbf{1.2}$ appear in bold):

$$\begin{aligned} m_{\mathbf{13.2}} \cdot m_{\mathbf{1.2}} &= m_{\mathbf{13.2.4.5}} + m_{\mathbf{134.2.5}} + m_{\mathbf{135.2.4}} \\ &\quad + m_{\mathbf{13.24.5}} + m_{\mathbf{13.25.4}} + m_{\mathbf{135.24}} + m_{\mathbf{134.25}} \end{aligned} \tag{13}$$

Notice that the shapes indexing the first and last terms in (13) are the partitions $\lambda(13.2) \cup \lambda(1.2)$ and $\lambda(13.2) + \lambda(1.2)$. As was the case in $S^{\mathfrak{S}}$, one of these shapes, namely $\lambda(\mathbf{A}) + \lambda(\mathbf{B})$, will always appear in the product, while appearance of the shape $\lambda(\mathbf{A}) \cup \lambda(\mathbf{B})$ depends on the cardinality of \mathbf{x} .

Let us now describe the multiplication rule. Given any $D \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we write D^{+k} for the set

$$D^{+k} := \{a + k : a \in D\}.$$

By extension, for any set partition $\mathbf{A} = \{A_1, A_2, \dots, A_r\}$ we set $\mathbf{A}^{+k} := \{A_1^{+k}, A_2^{+k}, \dots, A_r^{+k}\}$. Also, we set $\mathbf{A}_{\hat{i}} := \mathbf{A} \setminus \{A_i\}$. Next, if \mathcal{X} is a collection of set partitions of D , and A is a set disjoint from D , we extend \mathcal{X} to partitions of $A \cup D$ by the rule

$$A \diamond \mathcal{X} := \bigcup_{\mathbf{B} \in \mathcal{X}} \{A\} \cup \mathbf{B}.$$

Finally, given partitions $\mathbf{A} = \{A_1, A_2, \dots, A_r\}$ of C and $\mathbf{B} = \{B_1, B_2, \dots, B_s\}$ of D (disjoint from C), their **quasi-shuffles** $\mathbf{A} \omega \mathbf{B}$ are the set partitions of $C \cup D$ recursively defined by the rules:

- $\mathbf{A} \omega \emptyset = \emptyset \omega \mathbf{A} := \mathbf{A}$, where \emptyset is the unique set partition of the empty set;
- $\mathbf{A} \omega \mathbf{B} := \bigcup_{i=0}^s (A_1 \cup B_i) \diamond (\mathbf{A}_{\hat{1}} \omega (\mathbf{B}_{\hat{i}}))$, taking B_0 to be the empty set.

If $\mathbf{A} \vdash [c]$ and $\mathbf{B} \vdash [d]$, we abuse notation and write $\mathbf{A} \omega \mathbf{B}$ for $\mathbf{A} \omega \mathbf{B}^{+c}$. As shown in [2, Prop. 3.2], the multiplication rule for $m_{\mathbf{A}}$ and $m_{\mathbf{B}}$ in \mathcal{N} is

$$m_{\mathbf{A}} \cdot m_{\mathbf{B}} = \sum_{\mathbf{C} \in \mathbf{A} \omega \mathbf{B}} m_{\mathbf{C}}. \tag{14}$$

The subalgebra \mathcal{N} , like its commutative analog, is freely generated by certain monomial symmetric functions $\{m_{\mathbf{A}}\}_{\mathbf{A} \in \mathcal{A}}$, where \mathcal{A} is some carefully chosen collection of set partitions. This is the main theorem of Wolf [20]. We use two such collections later, our choice depending on whether or not $|\mathbf{x}| < \infty$.

The operation $(-)^{+k}$ has a left inverse called the **standardization** operator and denoted by “ $(-)^{\downarrow}$ ”. It maps set partitions \mathbf{A} of any cardinality d subset $D \subseteq \mathbb{N}$ to set

partitions of $[d]$, by defining \mathbf{A}^\downarrow as the pullback of \mathbf{A} along the unique increasing bijection from $[d]$ to D . For example, $(18.4)^\downarrow = 13.2$ and $(18.4.67)^\downarrow = 15.2.34$. The coproduct Δ and counit ε on \mathcal{N} are given, respectively, by

$$\Delta(m_{\mathbf{A}}) = \sum_{\mathbf{B} \cup \mathbf{C} = \mathbf{A}} m_{\mathbf{B}^\downarrow} \otimes m_{\mathbf{C}^\downarrow} \quad \text{and} \quad \varepsilon(m_{\mathbf{A}}) = \delta_{\mathbf{A}, \emptyset},$$

where $\mathbf{B} \cup \mathbf{C} = \mathbf{A}$ means that \mathbf{B} and \mathbf{C} form complementary subsets of \mathbf{A} . In the case $|\mathbf{x}| = \infty$, the maps Δ and ε are algebra maps, making \mathcal{N} a graded connected Hopf algebra.

4 The place-action of \mathfrak{S} on \mathcal{N}

4.1 Swapping places in T_d and \mathcal{N}_d

On top of the permutation-action of the symmetric group $\mathfrak{S}_{\mathbf{x}}$ on T , we also consider the “place-action” of \mathfrak{S}_d on the degree d homogeneous component T_d . Observe that the permutation-action of $\sigma \in \mathfrak{S}_{\mathbf{x}}$ on a monomial \mathbf{z} corresponds to the functional composition

$$\sigma \circ \mathbf{z} : [d] \xrightarrow{\mathbf{z}} \mathbf{x} \xrightarrow{\sigma} \mathbf{x}$$

(notation as in Section 3.1). By contrast, the **place-action** of $\rho \in \mathfrak{S}_d$ on \mathbf{z} gives the monomial

$$\mathbf{z} \circ \rho : [d] \xrightarrow{\rho} [d] \xrightarrow{\mathbf{z}} \mathbf{x},$$

composing ρ on the right with \mathbf{z} . In the linear extension of this action to all of T_d , it is easily seen that \mathcal{N}_d (even each \mathcal{N}_μ) is an invariant subspace of T_d . Indeed, for any set partition $\mathbf{A} = \{A_1, A_2, \dots, A_r\} \vdash [d]$ and any $\rho \in \mathfrak{S}_d$, one has

$$m_{\mathbf{A}} \cdot \rho = m_{\rho^{-1} \cdot \mathbf{A}} \tag{15}$$

(see [15, §2]), where as usual $\rho^{-1} \cdot \mathbf{A} := \{\rho^{-1}(A_1), \rho^{-1}(A_2), \dots, \rho^{-1}(A_r)\}$.

4.2 The place-action structure of \mathcal{N}

Notice that the action in (15) is shape-preserving and transitive on set partitions of a given shape (i.e., \mathcal{N}_μ is an \mathfrak{S}_d -submodule of \mathcal{N}_d for each $\mu \vdash d$). It follows that there is exactly one copy of the trivial \mathfrak{S}_d -module inside \mathcal{N}_μ for each $\mu \vdash d$, that is, a basis for the place-action invariants in \mathcal{N}_d is indexed by partitions. We choose as basis the functions

$$\mathbf{m}_\mu := \frac{1}{(\dim \mathcal{N}_\mu) \mu^\dagger} \sum_{\lambda(\mathbf{A}) = \mu} m_{\mathbf{A}}, \tag{16}$$

with $\mu^\dagger = a_1! a_2! \dots$ whenever $\mu = 1^{a_1} 2^{a_2} \dots$. The rationale for choosing this normalizing coefficient will be revealed in (20).

To simplify our discussion of the structure of \mathcal{N} in this context, we will say that \mathfrak{S} acts on \mathcal{N} rather than being fastidious about underlying in each situation that individual

\mathcal{N}_d 's are being acted upon on the right by the corresponding group \mathfrak{S}_d . We denote the set $\mathcal{N}^{\mathfrak{S}}$ of **place-invariants** by Λ in what follows. To summarize,

$$\Lambda = \text{span}\{\mathbf{m}_\mu : \mu \text{ a partition of } d, d \in \mathbb{N}\}. \quad (17)$$

The pair (\mathcal{N}, Λ) begins to look like the pair $(S, S^{\mathfrak{S}})$ from the introduction. This was the observation that originally motivated our search for Theorem 1.

We next decompose \mathcal{N} into irreducible place-action representations. Although this can be worked out for any value of n , the results are more elegant when we send n to infinity. Recall that the **Frobenius characteristic** of a \mathfrak{S}_d -module \mathcal{V} is a symmetric function

$$\text{Frob}(\mathcal{V}) = \sum_{\mu \vdash d} v_\mu s_\mu,$$

where s_μ is a Schur function (the character of “the” irreducible \mathfrak{S}_d representation \mathcal{V}_μ indexed by μ) and v_μ is the multiplicity of \mathcal{V}_μ in \mathcal{V} . To reveal the \mathfrak{S}_d -module structure of \mathcal{N}_μ , we use (15) and techniques from the theory of combinatorial species.

Proposition 3. *For a partition $\mu = 1^{a_1} 2^{a_2} \cdots k^{a_k}$, having a_i parts of size i , we have*

$$\text{Frob}(\mathcal{N}_\mu) = h_{a_1}[h_1] h_{a_2}[h_2] \cdots h_{a_k}[h_k], \quad (18)$$

with $f[g]$ denoting plethysm of f and g , and h_i denoting the i^{th} homogeneous symmetric function.

Recall that the **plethysm** $f[g]$ of two symmetric functions is obtained by linear and multiplicative extension of the rule $p_k[p_\ell] := p_{k\ell}$, where the p_k 's denote the usual power sum symmetric functions (see [12, I.8] for notation and details).

Let Par denote the combinatorial species of set partitions. So $\text{Par}[n]$ denotes the set partitions of $[n]$ and permutations $\sigma: [n] \rightarrow [n]$ are transferred in a natural way to permutations $\text{Par}[\sigma]: \text{Par}[n] \rightarrow \text{Par}[n]$. The number $\text{fix Par}[\sigma]$ of fixed points of this permutation is the same as the character $\chi_{\text{Par}[n]}(\sigma)$ of the \mathfrak{S}_n -representation given by $\text{Par}[n]$. Given a partition $\mu = 1^{a_1} 2^{a_2} \cdots k^{a_k}$, put $z_\mu := 1^{a_1} a_1! 2^{a_2} a_2! \cdots k^{a_k} a_k!$. (There are $n!/z_\mu$ permutations in \mathfrak{S}_n of cycle type μ .) The **cycle index series** for Par is defined by

$$Z_{\text{Par}} = \sum_{n \geq 0} \sum_{\mu \vdash n} \text{fix Par}[\sigma_\mu] \frac{p_\mu}{z_\mu},$$

where σ_μ is any permutation with cycle type μ and $p_\mu := p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ (taking p_i as the i -th power sum symmetric function).

Proof. Recall that the Schur and power sum symmetric functions are related by

$$s_\lambda = \sum_{\mu \vdash |\lambda|} \chi_\lambda(\sigma_\mu) \frac{p_\mu}{z_\mu},$$

so $Z_{\text{Par}} = \text{Frob}(\text{Par})$. Because Par is the composition $\mathbf{E} \circ \mathbf{E}_+$ of the species of sets and nonempty sets, we also know that its cycle index series is given by plethystic substitution: $Z_{\mathbf{E} \circ \mathbf{E}_+} = Z_{\mathbf{E}}[Z_{\mathbf{E}_+}]$. See Theorem 2 and (12) in [1, I.4]. Combining these two results will give the proof.

First, we are only interested in that piece of $\text{Frob}(\text{Par})$ coming from set partitions of shape μ . For this we need weighted combinatorial species. If a set partition has shape μ , give it the weight $q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$ in the cycle index series enumeration. The relevant identity is

$$Z_{\mathbf{P}}(\mathbf{q}) = \exp \sum_{k \geq 1} \frac{1}{k} \left(\exp \left(\sum_{j \geq 1} q_j^k \frac{p_{jk}}{j} \right) - 1 \right)$$

(cf. Example 13(c) of Chapter 2.3 in [1]). Collecting the terms of weight \mathbf{q}_μ gives $\text{Frob}(\mathcal{N}_\mu)$. We get

$$\text{coeff}_{\mathbf{q}_\mu} [Z_{\text{Par}}(\mathbf{q})] = \prod_{i=1}^k \left(\sum_{\lambda \vdash a_i} \frac{p_\lambda}{z_\lambda} \right) \left[\sum_{\nu \vdash i} \frac{p_\nu}{z_\nu} \right].$$

Standard identities [12, (2.14')] in I.2] between the h_i 's and p_j 's finish the proof. \square

As an example, we consider $\mu = 222 = 2^3$. Since

$$h_2 = \frac{p_1^2}{2} + \frac{p_2}{2} \quad \text{and} \quad h_3 = \frac{p_1^3}{6} + \frac{p_1 p_2}{2} + \frac{p_3}{3},$$

a plethysm computation (and a change of basis) gives

$$\begin{aligned} h_3[h_2] &= \frac{p_1^3}{6} \left[\frac{p_1^2}{2} + \frac{p_2}{2} \right] + \frac{p_1 p_2}{2} \left[\frac{p_1^2}{2} + \frac{p_2}{2} \right] + \frac{p_3}{3} \left[\frac{p_1^2}{2} + \frac{p_2}{2} \right] \\ &= \frac{1}{6} \left(\frac{p_1^2}{2} + \frac{p_2}{2} \right)^3 + \frac{1}{2} \left(\frac{p_1^2}{2} + \frac{p_2}{2} \right) \left(\frac{p_2^2}{2} + \frac{p_4}{2} \right) + \frac{1}{3} \left(\frac{p_3^2}{2} + \frac{p_6}{2} \right) \\ &= s_6 + s_{42} + s_{222}. \end{aligned}$$

That is, \mathcal{N}_{222} decomposes into three irreducible components, with the trivial representation s_6 being the span of \mathbf{m}_{222} inside Λ .

4.3 Λ meets $S^{\mathfrak{S}}$

We begin by explaining the choice of normalizing coefficient in (16). Analyzing the **abelianization** map $\mathbf{ab} : T \rightarrow S$ (the map making the variables \mathbf{x} commute), Rosas and Sagan [15, Thm. 2.1] show that $\mathbf{ab}|_{\mathcal{N}}$ satisfies:

$$\mathbf{ab}(m_{\mathbf{A}}) = \lambda(\mathbf{A})! m_{\lambda(\mathbf{A})}. \tag{19}$$

In particular, \mathbf{ab} maps onto $S^{\mathfrak{S}}$ and

$$\mathbf{ab}(\mathbf{m}_\mu) = m_\mu. \tag{20}$$

Note that \mathbf{ab} is also an algebra map. The reader may wish to use (19) to compare (8) and (13). Formula (20) suggests that a natural right-inverse to $\mathbf{ab}|_{\mathcal{N}}$ is given by

$$\iota : S^{\mathfrak{S}} \hookrightarrow \mathcal{N}, \quad \text{with} \quad \iota(m_\mu) := \mathbf{m}_\mu \quad \text{and} \quad \iota(1) = 1. \quad (21)$$

This fact, combined with the observation that $\iota(S^{\mathfrak{S}}) = \Lambda$, affords a quick proof of Theorem 1 when $|\mathbf{x}| = \infty$. We explain this now.

5 The coinvariant space of \mathcal{N} (Case: $|\mathbf{x}| = \infty$)

5.1 Quick proof of main result

When $|\mathbf{x}| = \infty$, the pair of maps (\mathbf{ab}, ι) have further properties: the former is a Hopf algebra map and the latter is a coalgebra map [2, Props. 4.3 & 4.5]. Together with (20) and (21), these properties make ι a **coalgebra splitting** of $\mathbf{ab} : \mathcal{N} \rightarrow S^{\mathfrak{S}} \rightarrow 0$. A theorem of Blattner, Cohen, and Montgomery immediately gives our main result in this case.

Theorem 4 ([5], Thm. 4.14). *If $H \xrightarrow{\pi} \overline{H} \rightarrow 0$ is an exact sequence of Hopf algebras that is split as a coalgebra sequence, and the splitting map ι satisfies $\iota(\overline{1}) = 1$, then H is isomorphic to a crossed product $A \# \overline{H}$, where A is the left Hopf kernel of π . In particular, $H \simeq A \otimes \overline{H}$ as vector spaces.*

For the technical definition of crossed products, we refer the reader to [5, §4]. We mention only that: (i) the crossed product $A \# \overline{H}$ is a certain algebra structure placed on the tensor product $A \otimes \overline{H}$; and (ii) the **left Hopf kernel** is the subalgebra

$$A := \{h \in H : (\text{id} \otimes \pi) \circ \Delta(h) = h \otimes \overline{1}\}.$$

We take $H = \mathcal{N}$, $\overline{H} = S^{\mathfrak{S}}$, and $\pi = \mathbf{ab}$. Since our ι is a coalgebra splitting, the coinvariant space \mathcal{C} we seek seems to be the left Hopf kernel of \mathbf{ab} . Before setting off to describe \mathcal{C} more explicitly, we point out that the left Hopf kernel is graded: the maps Δ , id , and \mathbf{ab} are graded, as is the map $\mathcal{C} \# \Lambda \xrightarrow{\simeq} \mathcal{N}$ used in the proof of Theorem 4 (which is simply $a \otimes \overline{h} \mapsto a \cdot \iota(\overline{h})$). Theorem 1 follows immediately from this result.

5.2 Atomic set partitions.

Recall the main result of Wolf [20] that \mathcal{N} is freely generated by some collection of functions. We announce our first choice for this collection now, following the terminology of [3]. Let Π denote the set of all set partitions (of $[d]$, $\forall d \geq 0$). The **atomic set partitions** Π are defined as follows. A set partition $\mathbf{A} = \{A_1, A_2, \dots, A_r\}$ of $[d]$ is *atomic* if there does not exist a pair (s, c) ($1 \leq s < r$, $1 \leq c < d$) such that $\{A_1, A_2, \dots, A_s\}$ is a set partition of $[c]$. Conversely, \mathbf{A} is not atomic if there are set partitions \mathbf{B} of $[d']$ and \mathbf{C} of $[d'']$ splitting \mathbf{A} in two: $\mathbf{A} = \mathbf{B} \cup \mathbf{C}^{+d'}$. We write $\mathbf{A} = \mathbf{B}|\mathbf{C}$ in this situation. A **maximal splitting** $\mathbf{A} = \mathbf{A}'|\mathbf{A}'' \cdots |\mathbf{A}^{(t)}$ of \mathbf{A} is one where each $\mathbf{A}^{(i)}$ is atomic. For example, the partition 17.235.4.68 is atomic, while 12.346.57.8 is not. The maximal splitting of

the latter would be $12|124.35|1$, but we abuse notation and write $12|346.57|8$ to improve legibility.

It follows from [3, Corollary 9] that \mathcal{N} is freely generated by the atomic monomial functions $\{m_{\mathbf{A}} : \mathbf{A} \in \dot{\Pi}\}$. We now introduce an order on Π that will make this explicit. First we introduce the *restricted growth function* associated to a set partition (see Section 6.1): if $\mathbf{A} = \{A_1, A_2, \dots, A_r\} \vdash [d]$, define $w(\mathbf{A}) \in \mathbb{N}^d$ by

$$w(\mathbf{A}) = w_1 w_2 \cdots w_d, \quad \text{with} \quad w_i := k \iff i \in A_k. \quad (22)$$

For example, $w(\mathbf{13.24}) = \mathbf{1212}$ and $w(\mathbf{17.235.4.68}) = \mathbf{12232414}$. Now, given two atomic set partitions $\mathbf{A} \vdash [c]$ and $\mathbf{B} \vdash [d]$, we put:

- $\mathbf{A} \succ \mathbf{B}$ when $c > d$; or
- $\mathbf{A} \succ \mathbf{B}$ when $c = d$ and $w(\mathbf{A}) >_{\text{lex}} w(\mathbf{B})$.

Finally, given two set partitions \mathbf{A} and \mathbf{B} , put $\mathbf{A} > \mathbf{B}$ if $\lambda(\mathbf{A}) <_{\text{lex}} \lambda(\mathbf{B})$ in the usual lexicographic order on integer partitions. If $\lambda(\mathbf{A}) = \lambda(\mathbf{B})$, then determine maximal splittings of \mathbf{A} and \mathbf{B} , view them as words in the atomic set partitions and use the lexicographic order induced by \succ . The following chain of set partitions of shape 3221 illustrates our total ordering on Π :

$$1|23|45|678 < 13.2|456|78 < 13.24|568.7 < 13.24|578.6 < 17.235.4.68 < 17.236.4.58.$$

In fact, $1|23|45|678$ is the unique minimal element of Π of shape 3221.

Define the **leading term** of a sum $\sum_{\mathbf{C}} \alpha_{\mathbf{C}} m_{\mathbf{C}}$ to be the monomial $m_{\mathbf{C}_0}$ such that \mathbf{C}_0 is greatest (according to $>$ above) among all \mathbf{C} with $\alpha_{\mathbf{C}} \neq 0$. Combined with (14), our definition of $>$ makes it clear that the leading term of $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ is $m_{\mathbf{A|B}}$ and that \mathcal{N} is freely generated by the atomic monomial functions. Moreover, it is clear that multiplication in \mathcal{N} is shape-filtered. Since the left Hopf kernel \mathcal{C} is a subalgebra, \mathcal{C} is shape-filtered as well. Finally, the isomorphism $\mathcal{C} \# \Lambda \xrightarrow{\simeq} \mathcal{N}$ constructed in the proof of Theorem 4 is also shape-filtered. These facts give Corollary 2 immediately.

5.3 Explicit description of the Hopf algebra structure of \mathcal{C}

We begin by partitioning $\dot{\Pi}$ into two sets according to length,

$$\dot{\Pi}_{(1)} := \{ \mathbf{A} \in \dot{\Pi} : \ell(\mathbf{A}) = 1 \} \quad \text{and} \quad \dot{\Pi}_{(>1)} := \{ \mathbf{A} \in \dot{\Pi} : \ell(\mathbf{A}) > 1 \}.$$

It is easy to find elements of the left Hopf kernel \mathcal{C} . For instance, if \mathbf{A} and \mathbf{B} belong to $\dot{\Pi}_{(1)}$, then the Lie bracket $[m_{\mathbf{A}}, m_{\mathbf{B}}]$ belongs to \mathcal{C} . Indeed,

$$\begin{aligned} \Delta([m_{\mathbf{A}}, m_{\mathbf{B}}]) &= \Delta(m_{\mathbf{A|B}} - m_{\mathbf{B|A}}) \\ &= m_{\mathbf{A|B}} \otimes 1 + m_{\mathbf{A}} \otimes m_{\mathbf{B}} + m_{\mathbf{B}} \otimes m_{\mathbf{A}} + 1 \otimes m_{\mathbf{A|B}} \\ &\quad - m_{\mathbf{B|A}} \otimes 1 - m_{\mathbf{B}} \otimes m_{\mathbf{A}} - m_{\mathbf{A}} \otimes m_{\mathbf{B}} - 1 \otimes m_{\mathbf{B|A}} \\ &= (m_{\mathbf{A|B}} - m_{\mathbf{B|A}}) \otimes 1 + 1 \otimes (m_{\mathbf{A|B}} - m_{\mathbf{B|A}}). \end{aligned}$$

Since $\mathbf{ab}(m_{\mathbf{A}|\mathbf{B}}) = \mathbf{ab}(m_{\mathbf{B}|\mathbf{A}})$, we have

$$(\mathrm{id} \otimes \mathbf{ab}) \circ \Delta([m_{\mathbf{A}}, m_{\mathbf{B}}]) = [m_{\mathbf{A}}, m_{\mathbf{B}}] \otimes 1$$

as desired. Similarly, the difference of monomial functions $m_{13.2} - m_{12.3}$ belongs to \mathcal{C} . The leading term here is indexed by $13.2 \in \dot{\Pi}_{(>1)}$. These two simple examples essentially exhaust the different ways in which an element can belong to \mathcal{C} . The following discussion makes this precise.

From [3, Theorem 15], we learn that \mathcal{N} is cofree cocommutative with minimal cogenerated set indexed by the Lyndon words in $\dot{\Pi}$. (This result and the previously mentioned freeness result may also be deduced from the techniques developed in [9].) Since single letters are Lyndon words, we know there are primitive elements associated to each atomic set partition. Recall that an element h in a Hopf algebra is **primitive** if $\Delta(h) = h \otimes 1 + 1 \otimes h$. Let $\mathrm{Prim}(\mathcal{N})$ denote the set of primitive elements in \mathcal{N} —a Lie algebra under the commutator bracket.

Bearing the free and cofree cocommutative results in mind, a classical theorem of Milnor and Moore [13] guarantees that \mathcal{N} is isomorphic to the universal enveloping algebra $\mathfrak{U}(\mathfrak{L}(\dot{\Pi}))$ of the free Lie algebra $\mathfrak{L}(\dot{\Pi})$ on the set $\dot{\Pi}$. In the isomorphism $\mathfrak{L}(\dot{\Pi}) \xrightarrow{\cong} \mathrm{Prim}(\mathcal{N})$, one may map $\mathbf{A} \in \dot{\Pi}_{(1)}$ to $m_{\mathbf{A}}$ since these monomial functions are already primitive. The choice of where to send $\mathbf{A} \in \dot{\Pi}_{(>1)}$ is the subject of the next proposition.

Proposition 5. *For each $\mathbf{A} \in \dot{\Pi}_{(>1)}$, there is a primitive element $\tilde{m}_{\mathbf{A}}$ of \mathcal{N} ,*

$$\tilde{m}_{\mathbf{A}} = m_{\mathbf{A}} - \sum_{\mathbf{B} \in \dot{\Pi}} \alpha_{\mathbf{B}} m_{\mathbf{B}},$$

satisfying: (i) if $\mathbf{B} \in \dot{\Pi}$ or $\lambda(\mathbf{B}) \neq \lambda(\mathbf{A})$, then $\alpha_{\mathbf{B}} = 0$; and (ii) $\sum_{\mathbf{B}} \alpha_{\mathbf{B}} = 1$.

Proof. Suppose $\mathbf{A} \in \dot{\Pi}_{(>1)}$. A primitive $\tilde{m}_{\mathbf{A}}$ exists by the Milnor–Moore theorem, as explained above.

(i). Since $\mathcal{N} = \bigoplus_{\mu} \mathcal{N}_{\mu}$ is a coalgebra grading by shape, we may assume $\lambda(\mathbf{B}) = \lambda(\mathbf{A})$ for any nonzero coefficients $\alpha_{\mathbf{B}}$. Now, since there are linearly independent primitive elements in \mathcal{N} associated to every atomic set partition, we may use Gaussian elimination and our ordering on $\dot{\Pi}$ to ensure that $\alpha_{\mathbf{B}} = 0$ for any $\mathbf{B} \in \dot{\Pi}$.

(ii). Define linear maps $\Delta_{+}^j : \mathcal{N}_{+} \rightarrow \mathcal{N} \otimes \mathcal{N}$ recursively by

$$\begin{aligned} \Delta_{+}(h)^1 &:= \Delta(h) - h \otimes 1 - 1 \otimes h, \\ \Delta_{+}^{j+1}(h) &:= (\Delta_{+} \otimes \mathrm{id}^{\otimes j}) \circ \Delta_{+}^j(h) \quad \text{for } j > 0. \end{aligned}$$

Assume that (i) is satisfied for $\tilde{m}_{\mathbf{A}}$ and that $\mathbf{A} = \{A_1, A_2, \dots, A_r\}$. Since $\Delta_{+}(\tilde{m}_{\mathbf{A}}) = 0$, we have $\Delta_{+}^j(m_{\mathbf{A}}) = \Delta_{+}^j(\sum_{\mathbf{B}} \alpha_{\mathbf{B}} m_{\mathbf{B}})$ for all $j > 1$. Now,

$$\Delta_{+}^r(m_{\mathbf{A}}) = \sum_{\sigma \in \mathfrak{S}_r} m_{A_{\sigma 1} \downarrow} \otimes m_{A_{\sigma 2} \downarrow} \otimes \cdots \otimes m_{A_{\sigma r} \downarrow}.$$

Indeed, the same holds for any \mathbf{B} with $\lambda(\mathbf{B}) = \lambda(\mathbf{A})$:

$$\Delta_+^r \left(\sum_{\mathbf{B}} \alpha_{\mathbf{B}} m_{\mathbf{B}} \right) = \left(\sum_{\mathbf{B}} \alpha_{\mathbf{B}} \right) \sum_{\sigma \in \mathfrak{S}_r} m_{A_{\sigma_1} \downarrow} \otimes m_{A_{\sigma_2} \downarrow} \otimes \cdots \otimes m_{A_{\sigma_r} \downarrow}.$$

Conclude that $\sum_{\mathbf{B}} \alpha_{\mathbf{B}} = 1$. □

We say an element $h \in \mathcal{N}_\mu$ has the “zero-sum” property if it satisfies (ii) from the proposition. Put $\tilde{m}_{\mathbf{A}} := m_{\mathbf{A}}$ for $\mathbf{A} \in \dot{\Pi}_{(1)}$. We next describe the coinvariant space \mathcal{C} .

Corollary 6. *Let \mathcal{C} be the Lie ideal in $\mathfrak{L}(\dot{\Pi})$ given by $\mathcal{C} = [\mathfrak{L}(\dot{\Pi}), \mathfrak{L}(\dot{\Pi})] \oplus \dot{\Pi}_{(>1)}$. If $\varphi : \mathfrak{U}(\mathfrak{L}(\dot{\Pi})) \rightarrow \mathcal{N}$ is the Milnor–Moore isomorphism given by putting $\varphi(\mathbf{A}) := \tilde{m}_{\mathbf{A}}$ for all $\mathbf{A} \in \dot{\Pi}$ and extending multiplicatively, then the left Hopf kernel \mathcal{C} is the Hopf subalgebra $\varphi(\mathfrak{U}(\mathcal{C}))$.*

Proof. We first show that $\varphi(\mathfrak{U}(\mathcal{C})) \subseteq \mathcal{C}$. We certainly have $\tilde{m}_{\mathbf{A}} \in \mathcal{C}$ for all $\mathbf{A} \in \dot{\Pi}_{(>1)}$, since the zero-sum property means $\mathbf{ab}(\tilde{m}_{\mathbf{A}}) = 0$. Next suppose $f \in [\mathfrak{L}(\dot{\Pi}), \mathfrak{L}(\dot{\Pi})]$ is a sum of Lie brackets $[\mathbf{A}] = [[\dots [\mathbf{A}', \mathbf{A}''], \dots], \mathbf{A}^{(t)}]$. In this case, $\varphi(f) \in \mathcal{C}$ because each $\varphi([\mathbf{A}])$ is primitive and \mathbf{ab} is an algebra map. Indeed, $\mathbf{ab}([\tilde{m}_{\mathbf{A}'}, \tilde{m}_{\mathbf{A}''}]) = 0$. The inclusion follows, since $\mathfrak{U}(\mathcal{C})$ is generated by elements of these two types.

It remains to show that $\mathcal{C} \subseteq \varphi(\mathfrak{U}(\mathcal{C}))$. To begin, note that $\mathfrak{L}(\dot{\Pi})/\mathcal{C}$ is isomorphic to the abelian Lie algebra generated by $\dot{\Pi}_{(1)}$. The universal enveloping algebra of this latter object is evidently isomorphic to $S^{\mathfrak{S}}$. (Send $\mathbf{A} = \{[d]\}$ to m_d .) The Poincaré–Birkhoff–Witt theorem guarantees that the map $\varphi(\mathfrak{U}(\mathcal{C})) \otimes S^{\mathfrak{S}} \rightarrow \mathcal{N}$ given by $a \otimes b \mapsto a \cdot \iota(b)$ is onto \mathcal{N} . Conclude that $\mathcal{C} \subseteq \varphi(\mathfrak{U}(\mathcal{C}))$, as needed. □

Before turning to the case $|\mathbf{x}| < \infty$, we remark that we have left unanswered the question of finding a systematic procedure (e.g., a closed formula in the spirit of Möbius inversion) that constructs a primitive element $\tilde{m}_{\mathbf{A}}$ for each $\mathbf{A} \in \dot{\Pi}_{(>1)}$. This is accomplished in [11].

6 The coinvariant space of \mathcal{N} (Case: $|\mathbf{x}| \leq \infty$)

6.1 Restricted growth functions

We repeat our example of Section 3.3 in the case $n = 3$. The leading term with respect to our previous order would be $m_{13.2.4.5}$, except that this term does not appear because 13.2.4.5 has more than $n = 3$ parts:

$$m_{13.2} \cdot m_{1.2} = 0 + m_{134.2.5} + m_{135.2.4} + m_{13.24.5} + m_{13.25.4} + m_{135.24} + m_{134.25}.$$

Fortunately, the map w from set partitions to words on the alphabet $\mathbb{N}_{>0}$ reveals a more useful leading term, underlined below:

$$m_{121} \cdot m_{12} = 0 + m_{12113} + m_{12131} + m_{12123} + m_{12132} + m_{12121} + \underline{m_{12112}}. \quad (23)$$

Notice that the words appearing on the right in (23) all begin by 121 and that the concatenation $\underline{121}\underline{12}$ is the lexicographically smallest word appearing there. This is generally true and easy to see: if $w(\mathbf{A}) = u$ and $w(\mathbf{B}) = v$, then uv is the lexicographically smallest element of $w(\mathbf{A} \cup \mathbf{B})$.

The map w maps set partitions to **restricted growth functions**, i.e., the words $w = w_1w_2 \cdots w_d$ satisfying $w_1 = 1$ and $w_i \leq 1 + \max\{w_1, w_2, \dots, w_{i-1}\}$ for all $2 \leq i \leq d$. We call them restricted growth words here. See [16, 17, 19] and [6, 8] for some of their combinatorial properties and applications. These words are also known as “*rhyme scheme words*” in the literature; see [14] and [18, A000110]. Before looking for a coinvariant space \mathcal{C} within \mathcal{N} , we first fix the representatives of Λ . Consider the partition $\mu = 3221$. Of course, \mathbf{m}_μ is the sum of all set partitions of shape μ , but it will be nice to have a single one in mind when we speak of \mathbf{m}_μ . A convenient choice turns out to be 123.45.67.8: if we use the length plus lexicographic order on $w(\Pi)$, then it is easy to see that $w(123.45.67.8) = 11122334$ is the minimal element of Π of shape 3221. We are led to introduce the words

$$w(\mu) := 1^{\mu_1} 2^{\mu_2} \cdots k^{\mu_k}$$

associated to partitions $\mu = (\mu_1, \mu_2, \dots, \mu_k)$; we call such restricted growth words **convex words** since $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$.

6.2 Proof of main theorem

We say that a restricted growth word is **non-splittable** if $w_i \cdots w_{n-1}w_n$ is not a restricted growth word for any $i > 1$. The **maximal splitting** of a restricted growth word w is the maximal deconcatenation $w = w' | w'' | \cdots | w^{(r)}$ of w into non-splittable words $w^{(i)}$. For example, 12314 is non-splittable while 11232411 is a string of four non-splittable words $1 | 12324 | 1 | 1$.

It is easy to see that if a, b, c , and d are non-splittable, then $ac = bd$ if and only if $a = b$ and $c = d$. Together with the remarks on $\mathbf{A} \cup \mathbf{B}$ following (23), this implies that if $\{u_1, u_2, \dots, u_r\}$ and $\{v_1, v_2, \dots, v_s\}$ are two sets of non-splittable words, then

$$m_{u_1} m_{u_2} \cdots m_{u_r} \quad \text{and} \quad m_{v_1} m_{v_2} \cdots m_{v_s}$$

share the same leading term (namely, $m_{u_1 | u_2 | \cdots | u_r}$) if and only if $r = s$ and $u_i = v_i$ for all i . In other words, our algebra \mathcal{N} is *non-splittable word-filtered* and freely generated by the monomial functions $\{m_{w(\mathbf{A})} : w(\mathbf{A}) \text{ is non-splittable}\}$. This is one of the collections of monomial functions originally chosen by Wolf [20].

We aim to index \mathcal{C} by the restricted growth words that don't end in a convex word. Toward that end, we introduce the notion of **bimodal words**. These are words with a maximal (but possibly empty) convex prefix, followed by one non-splittable word. The **bimodal decomposition** of a restricted growth word w is the expression of w as a product $w = w' | w'' | \cdots | w^{(r)} | w^{(r+1)}$, where $w', w'', \dots, w^{(r)}$ are bimodal and $w^{(r+1)}$ is a possibly empty convex word (which we call a **tail**). For a given word w , this decomposition is accomplished by first splitting w into non-splittable words, then recombining, from

left to right, consecutive non-splittable words to form bimodal words. For instance, the maximal splitting of 1122212 into non-splittable words is $1|1222|12$. The first two factors combine to make one bimodal word; the last factor is a convex tail: $1122212 \mapsto \widehat{1|1222} \widehat{1}2$. Similarly,

$$1231231411122311 \mapsto 123|12314|1|1|1223|1|1 \mapsto \widehat{123|12314} \widehat{1|1|1223} \widehat{1} \widehat{1}.$$

Suppose now that u and v are restricted growth words and that the bimodal decomposition of u is tail-free. Then by construction, the bimodal decomposition of uv is the concatenation of the respective bimodal decompositions of u and v . We are ready to identify \mathcal{C} as a subalgebra of \mathcal{N} .

Theorem 7. *Let \mathcal{C} be the subalgebra of \mathcal{N} generated by $\{m_v : v \text{ is bimodal}\}$. Then \mathcal{C} has a basis indexed by restricted growth words w whose bimodal decompositions are tail-free. Moreover, the map $\varphi : \mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ given by $m_w m_{w''} \cdots m_{w^{(r)}} \otimes \mathbf{m}_\mu \mapsto m_w |w''| \cdots |w^{(r)}| w(\mu)$ is a vector space isomorphism.*

Proof. The advertised map is certainly onto, since $\{m_w : w \in w(\Pi)\}$ is a basis for \mathcal{N} and every restricted growth word has a bimodal decomposition $w' |w''| \cdots |w^{(r)}| w(\mu)$. It remains to show that the map is one-to-one.

Note that the monomial functions $\{m_v : v \text{ is bimodal}\}$ are algebraically independent: certainly, the leading term in a product $m_{v_1} m_{v_2} \cdots m_{v_s}$ (with v_i bimodal) is $m_{v_1 |v_2| \cdots |v_s}$; now, since every word has a unique bimodal decomposition, no (nontrivial) linear combination of products of this form can be zero. Finally, the leading term in the simple tensor $m_w m_{w''} \cdots m_{w^{(r)}} \otimes \mathbf{m}_\mu$ is the basis vector $m_w |w''| \cdots |w^{(r)}| w(\mu)$, so no (nontrivial) linear combination of these will vanish under the map φ . \square

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