Optimal Allocation of Simple Step-Stress Model with Weibull Distributed Lifetimes under Type-I Censoring

LO, Kwok Yuen

A Thesis Submitted in Partial Fulfillment

of the Requirements for the Degree of

Master of Philosophy

in

Statistics

The Chinese University of Hong Kong

August 2010

Thesis/Assessment Committee

Professor LEUNG Pui Lam William (Chair)

Professor CHAN Ping Shing Ben (Thesis Supervisor)

Professor WU Ka Ho Eden (Committee Member)

Professor NG Hon Keung Tony (External Examiner)

Abstract of thesis entitled:

Optimal Experimental Design of Simple Step-Stress Model with Weibull Distributed Lifetimes under Type-I Censoring

Submitted by LO Kwok Yuen

for the degree of Master of Philosophy in Statistics

at The Chinese University of Hong Kong in August 2010

ABSTRACT

In accelerated life-testing (ALT) experiment, step-stress model is the most common model for exploring the relationship between lifetimes and stress levels of the products with large mean failure times under normal operating conditions. In this thesis, by assuming log-linear relation between the model parameters and stress levels, we consider a simple step-stress model under cumulative exposure assumption with Weibull distributed lifetimes in the present of Type-I censored data. Maximum likelihood estimators (MLEs) are used to estimate the model parameters. Expected Fisher information matrix is derived and used to find asymptotic variance-covariance matrix of MLEs. Some numerical techniques for step-stress model are introduced. The optimal allocation schemes for the simple step-stress model are determined based on different optimal criteria by using line graphs and nomographs. Sensitivity analysis on optimal allocation proportions against parameters is provided.

摘要

加速壽命測試實驗模型之目的，為尋求在正常情況下而又很長壽命的產品與應力水平之間關係，其中最常用的是逐步應力模型。在這篇論文中，首先在以下假設中介紹簡易逐步應力模型：模型參數與應力水平之間存在對數線性關係、產命壽命跟隨累積暴露假設以及韋伯分佈、以及數據集當中存在著第一類刪失數據。利用最大似然估計法可估計模型參數。然後計算其費雪資訊矩陣，可得出近似漸近協方差矩陣。一些應用在逐步應力模型的數值技巧進行了介紹。然後，我們透過不同最優化準則，以線圖和諾模圖表達簡易逐步應力模型的最佳分配方案。最後，我們利用錯誤估計引申的比例，隨著參數變化，對不同的準則下的最佳分配比例進行敏感度分析。

DECLARATION

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

ACKNOWLEDGEMENT

I would like to express gratitude to my supervisor, Prof. Chan Ping-Shing, Ben, for his valuable guidance and supervision during the course of this research program. I would also like to thank the entire staff of the Department of Statistics. This research is partially supported by HKRGC GRF project no. 2150567.

Lo Kwok Yuen

Department of Statistics

The Chinese University of Hong Kong

August, 2010

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background

Most industrial experiments are used to test the reliability of products. However, under normal conditions, the mean failure times of products are too large to wait for all the failures of products. Therefore, it is almost impossible to get enough information about the lifetime distribution and its unknown parameters in a reasonable experimental time. To overcome this problem, the products are tested under condition with higher stress than normal. Such testing technique is called Accelerated Life Testing (ALT). The accelerated test can be under higher constant stress or linearly increasing stress levels such as, for example, temperature, pressure, load, vibration, etc. Thus, the main purpose of ALT is to speed up the failure at more extreme conditions and the lifetimes of products in normal condition can be estimated by extrapolation with an appropriate model.

Nelson (1980) pointed out that step-stress testing is a special case of accelerated testing that allows testing stress level x_i changing to another level x_{i+1} at a given time τ_i, for $i = 1, \ldots, k$, where $\tau_1 < \tau_2 < \ldots < \tau_k$, or upon the occurrence of a specified number of failures n_i. They are called time-step-stress testing and failure-step-stress testing, respectively, where the former is considered in this thesis. Both cases can reduce test time obviously and ensure enough failures occuring within a short period of time. For further explanation and examples, see Miller and Nelson (1983) and Gouno and Balakrishnan (2001).

The simplest step-stress model is called simple step-stress models, involving two stress levels x_1, x_2 and one changing time τ_1 only during the whole test time. Since Nelson and Kielpinski (1976), using intuitive argument, showed that the optimal allocation about lognormal model uses only two stress levels, which are the lowest and highest, for their optimality criteria, simple step-stress models are applied with exponential distributions and their generalizations, Weibull distributions in many statistical literatures in recent decades. Bai, Kim and Lee (1989), Bai and Kim (1993) proposed the optimal changing time τ_1 with Type-I censoring data under exponential distributions and Weibull distributions respectively; Balakrishnan, Kundu, Ng and Kannan (2007) and Kateri and Balakrishnan (2008) discussed the interval estimation of parameters based on the maximum likelihood estimators (MLE) with Type-II censoring data under exponential distributions and Weibull distributions, respectively. Gouno, Sen and Balakrishnan (2004) determined D-optimal and V-optimal τ_1 with progressive Type-I censoring data under exponential distributions.

However, in Bai, Kim and Lee (1989) and Bai and Kim (1993), they did not consider the case when no failures are observed under some stress levels such that some parameters are inestimable. In this thesis, besides considering a simple step-stress model under assumptions with Weibull distributed lifetimes in the presence of Type-I censored data, we aim to analyze the likelihood under different situations and to study the characteristics of MLE, such as their asymptotic variance-covariance matrices.

1.2 Scope of the thesis

The scope of this thesis is the following. In Chapter 2, the model and its assumptions are introduced, and the simple step-stress model is described. In Chapter 3, the likelihood function is given under different cases, and parameters are estimated by MLE. The Fisher information matrix of the MLE is derived and used to determine the asymptotic variance-covariance matrix. After that, numerical improvement strategies for step-stress models are suggested for solving MLE. In Chapter 4, optimal criteria are given and the optimal design with different criteria and quantities are determined and shown by line graphs and nomographs. The sensitivity of proportions of optimal designs against parameters are discussed. Finally, in Chapter 5, we draw conclusions and raise some further research problems.

Chapter 2

Lifetime Model

2.1 Introduction

In this chapter, we introduce our lifetime model in detail. Starting from presenting the basic definitions of Weibull distributions in Section 2.2, we state the lifetime model with two important assumptions: the linear assumption of mean log-lifetime, and the cumulative exposure assumption in Section 2.3. Then the lifetime model is shown explicitly. Graphs are sketched for a better understanding the model of step-stress tests. Additionally, the Type-I censoring scheme is introduced at the end of this chapter.

2.2 Weibull Distribution

Weibull distribution is one of the most common distributions to model the lifetime data. Its cumulative density function (c.d.f.) is given by

$$F(t) \; = \; 1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}}, \qquad t > 0; \quad \theta, \beta > 0$$

its probability density function (p.d.f.) is given by

$$f(t) = \frac{\beta}{\theta^\beta} t^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^\beta}, \qquad t > 0; \quad \theta, \beta > 0$$

and its hazard function is given by

$$h(t) = \frac{\beta}{\theta^\beta} t^{\beta-1}, \qquad t > 0; \quad \theta, \beta > 0$$

where θ is the scale parameter and β is the shape parameter. It is easy to see that the hazard function is an increasing function of t when $\beta > 1$ and decreasing when $\beta < 1$. It is a constant when $\beta = 1$. Therefore it reduces to an exponential distribution when $\beta = 1$.

2.3 Step-Stress Experiment

In reliability experiments, the experimenters are always interested in the relationship between lifetimes and some operating conditions such as voltages, loads and temperatures. These are known as factors or covariates. It is called a constant stress experiment if the test is run under a specified level of factor x_0 and then the times to failure of items are observed. In the Weibull setup, the most common linkage between the lifetime and the factor x is through the following relationship: (see, for example, Lawless (2003))

$$\log \theta = \alpha_0 + \alpha_1 x \tag{2.1}$$

Therefore, the c.d.f. of the lifetime distribution under this model is given by

$$F(t;x) = 1 - e^{-\left(te^{-\alpha_0-\alpha_1 x}\right)^{\beta}}$$

In these days, the products are so reliable and it may take a long time to fail. In order to accelerate the failure, step-stress tests have been proposed (Nelson (1980)).

The step-stress experiment is set up as follows. At the beginning of the experiment, the stress is set to x_1 until a pre-specified time τ_1. After τ_1, the stress will change from x_1 to x_2. The experiment continues until another pre-fixed time τ_2. After τ_2, the stress will change from x_2 to x_3, and so on. It is known as a progressive step-stress test if there is more than one change of stress levels in the experiment. It is termed as simple step-stress test if only one change is made in it.

In this thesis, we are going to study the optimal design of the simple step-stress test when the lifetime is Weibull distributed.

A common assumption in the literature of step-stress test is called cumulative exposure assumption. Based on this assumption, the remaining lifetime of a test unit only depends on the current cumulative fraction failed and the current stress. That means if different test units have distinct exposure histories but the same age, then they share the same remaining life distribution. Therefore, lifetime T

can be described in terms of the c.d.f.:

$$G(t) = \begin{cases} G_1(t) = F_1(t) & 0 < t < \tau_1 \\ G_2(t) = F_2(s + t - \tau_1) & \tau_1 \le t < \infty \end{cases}$$

where $F_i(t)$ is the c.d.f. under stress level x_i, $i = 1, 2$, and s is the solution of the following equation:

$$F_2(s) = F_1(\tau_1)$$

Accordingly, if lifetime T is assumed to be Weibull distributed with c.d.f.:

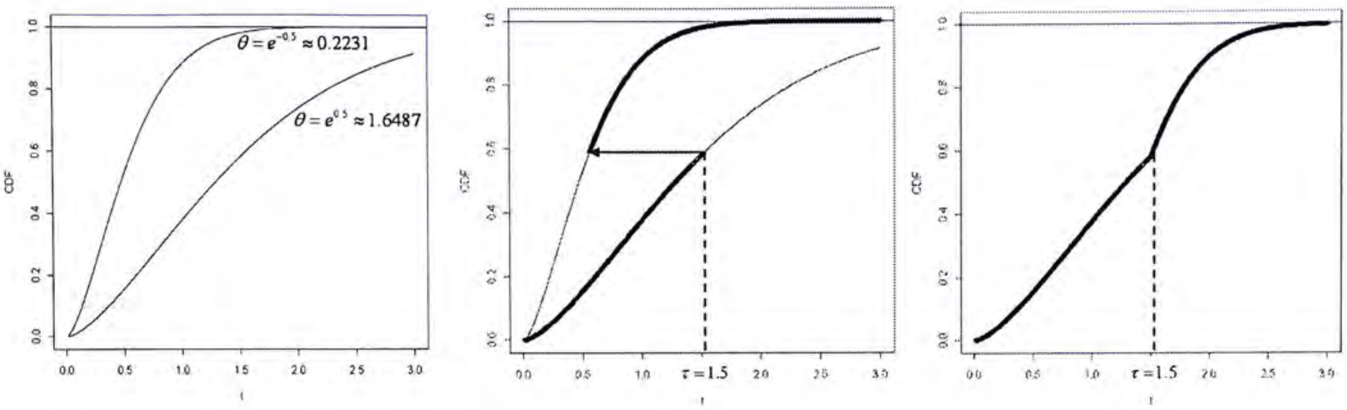
$$F_i(t; x_i) = 1 - e^{-\left(te^{-\alpha_0 - \alpha_1 x_i}\right)^\beta} \qquad 0 < t < \infty$$

under the constant stress level x_i, $i = 1, 2$, then the lifetime T under the simple step-stress set up with cumulative exposure assumption has c.d.f. and p.d.f., respectively, as follows:

$$G(t) = \begin{cases} G_1(t) = 1 - e^{-\left(te^{-\alpha_0 - \alpha_1 x_1}\right)^\beta} & 0 < t < \tau_1 \\ G_2(t) = 1 - e^{-\left[\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta + (t - \tau_1) e^{-\alpha_0 - \alpha_1 x_2}\right]^\beta} & \tau_1 \le t < \infty \end{cases} \tag{2.2}$$

$$g(t) = \begin{cases} g_1(t) = \beta e^{-\beta(\alpha_0 + \alpha_1 x_1)} t^{\beta - 1} e^{-\left(te^{-\alpha_0 - \alpha_1 x_1}\right)^\beta} & 0 < t < \tau_1 \\ g_2(t) = \beta e^{-\beta(\alpha_0 + \alpha_1 x_2)} \left(e^{\alpha_1 (x_2 - x_1)} + t - \tau_1\right)^{\beta - 1} & \\ \qquad \times e^{-\left[\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta + (t - \tau_1) e^{-\alpha_0 - \alpha_1 x_2}\right]^\beta} & \tau_1 \le t < \infty \end{cases} \tag{2.3}$$

Figure 2.1a shows the c.d.f.'s of Weibull constant stress models, where the upper curve has a smaller scale parameter θ_i. Therefore, when the stress increases, the parameter θ_1 of the Weibull distribution decreases to θ_2. Figure 2.1b shows



(a) (b) (c)

Figure 2.1: c.d.f. of Weibull simple step-stress model

the relationship between the constant stress models and the simple step-stress model. The arrow represents when the stress changes, the units following the distribution under stress x_1 jump to the distribution under stress x_2 and restart to follow it from the fraction failed at x_1. Therefore the darkline in Figure 2.1c, which is the lifetime c.d.f. of our model, consists of two segments.

The simulation can be done by finding the inverse of the c.d.f. $G(t)$ and the algorithm is addressed in the Appendix A.

Chapter 3

Maximum Likelihood Estimation of Model Parameters

3.1 Introduction

To investigate the relationship between the lifetime T and the stress x_i, unknown parameters $\alpha_0, \alpha_1, \beta$ of the model are necessary to be estimated. Maximum likelihood estimation is used to estimate the parameters. In Section 3.2, the likelihood is given for different situations and the score functions for $\alpha_0, \alpha_1, \beta$ are derived in order to obtain the MLEs. Then, the asymptotic variance-covariance matrix is obtained by inverting the expected Fisher information matrix in Section 3.3. Finally, some numerical techniques for solving MLEs in step-stress models are studied in Section 3.4.

3.2 Maximum Likelihood Estimation

Let n_1, n_2 be the numbers of failures occured before τ_1 and between τ_1 and τ_2 respectively, and $n_c = N - n_1 - n_2$ be the number of remaining units after experiment, where all n's are random. Then, the likelihood of the observed failure times $0 < T_{1:N} < \ldots < T_{n_1:N} < \tau_1 < T_{n_1+1:N} < \ldots < T_{n_1+n_2:N} < \tau_2$ is given by

$$L(\alpha_0, \alpha_1, \beta) = \begin{cases} (1 - G(\tau_2))^N & (n_1 = n_2 = 0) \\ \prod_{j=1}^{n_1} g_1(T_{j:N})(1 - G(\tau_2))^{N-n_1} & (n_1 > 0, n_2 = 0) \\ \prod_{j=1}^{n_2} g_2(T_{n_1+j:N})(1 - G(\tau_2))^{N-n_2} & (n_1 = 0, n_2 > 0) \\ \prod_{j=1}^{n_1} g_1(T_{j:N}) \prod_{j=1}^{n_2} g_2(T_{n_1+j:N})(1 - G(\tau_2))^{N-n_1-n_2} & (n_1 > 0, n_2 > 0) \end{cases}$$

where $G(\cdot)$ and $g_i(\cdot)$, $i = 1, 2$ are given in (2.2) and (2.3), respectively. Since the likelihood has no information about g_1 for $n_1 = 0$ and g_2 for $n_2 = 0$, it is evident that MLEs of α_0, α_1 do not exist under these two conditions. Moreover, MLEs also do not exist if the number of observed failure is less than the number of parameters. Therefore, MLEs of all parameters are estimable when $n_1 > 0$, $n_2 > 0$ and $n_1 + n_2 \geq 3$ only. Besides, since $n_c = N - n_1 - n_2$ and $n_c \geq 0$, the MLEs are only defined on

$$\begin{cases} A_1 = \{1 \leq n_1 \leq N - 1\} \\ A_2 = \{\max(1, 3 - n_1) \leq n_2 \leq N - n_1\} \end{cases}$$

Therefore, we can only obtain MLEs and their variance-covariance matrix under the constraints A_1, A_2. Under these constraints the log-likelihood is given by:

$$\begin{aligned}
\log L(\alpha_0, \alpha_1, \beta) &= (n_1+n_2)\log\beta - \beta n_1(\alpha_0+\alpha_1 x_1) - \beta n_2(\alpha_0+\alpha_1 x_2) \\
&\quad +(\beta-1)\sum_{j=1}^{n_1}\log T_{j:N} + (\beta-1)\sum_{j=1}^{n_2}\log\left(\tau_1 e^{\alpha_1(x_2-x_1)} + T_{n_1+j:N} - \tau_1\right) \\
&\quad -e^{-\beta(\alpha_0+\alpha_1 x_1)}\sum_{j=1}^{n_1} T_{j:N}^{\beta} - e^{-\beta(\alpha_0+\alpha_1 x_2)}\sum_{j=1}^{n_2}\left(\tau_1 e^{\alpha_1(x_2-x_1)} + T_{n_1+j:N} - \tau_1\right)^{\beta} \\
&\quad -e^{-\beta(\alpha_0+\alpha_1 x_2)}(N-n_1-n_2)\left(\tau_1 e^{\alpha_1(x_2-x_1)} + \tau_2 - \tau_1\right)^{\beta}
\end{aligned}$$

Then the score functions are derived by partial differentiating $\log L$ with respect to each parameter :

$$\begin{aligned}
\frac{\partial \log L}{\partial \alpha_0} &= -\beta(n_1+n_2) + \beta e^{-\beta(\alpha_0+\alpha_1 x_1)} H_1(\beta) + \beta e^{-\beta(\alpha_0+\alpha_1 x_2)}(H_2(\alpha_1,\beta) + H_3(\alpha_1,\beta)) \\
\frac{\partial \log L}{\partial \alpha_1} &= -\beta(n_1 x_1 + n_2 x_2) + \beta x_1 e^{-\beta(\alpha_0+\alpha_1 x_1)} H_1(\beta) \\
&\quad +\beta x_2 e^{-\beta(\alpha_0+\alpha_1 x_2)}(H_2(\alpha_1,\beta) + H_3(\alpha_1,\beta)) + (\beta-1)e^{\alpha_1(x_2-x_1)}\tau_1(x_2-x_1)H_2(\alpha_1,-1) \\
&\quad -\beta\tau_1 e^{-\{\alpha_0\beta+\alpha_1[x_1+x_2(\beta-1)]\}}(x_2-x_1)(H_2(\alpha_1,\beta-1) + H_3(\alpha_1,\beta-1)) \\
\frac{\partial \log L}{\partial \beta} &= \frac{n_1+n_2}{\beta} - \alpha_0(n_1+n_2) - \alpha_1(n_1 x_1 + n_2 x_2) \\
&\quad +\sum_{j=1}^{n_1}\log T_{j:N} + \sum_{j=1}^{n_2}\log\left(\tau_1 e^{\alpha_1(x_2-x_1)} + T_{n_1+j:N} - \tau_1\right) \\
&\quad +e^{-\beta(\alpha_0+\alpha_1 x_1)}[(\alpha_0+\alpha_1 x_1)H_1(\beta) - H_1'(\beta)] \\
&\quad +e^{-\beta(\alpha_0+\alpha_1 x_2)}[(\alpha_0+\alpha_1 x_2)(H_2(\alpha_1,\beta) + H_3(\alpha_1,\beta)) - (H_2'(\alpha_1,\beta) + H_3'(\alpha_1,\beta))]
\end{aligned}$$

where for any z

$$\begin{aligned}
H_1(\beta) &= \sum_{j=1}^{n_1} T_{j:N}^{\beta} && (3.1)\\
H_1'(\beta) &= \frac{\partial H_1(\beta)}{\partial \beta} = \sum_{j=1}^{n_1} T_{j:N}^{\beta} \log T_{j:N} && (3.2)\\
H_2(\alpha_1, z) &= \sum_{j=1}^{n_2} \left(\tau_1 e^{\alpha_1(x_2-x_1)} + T_{n_1+j:N} - \tau_1\right)^z && (3.3)\\
H_2'(\alpha_1, z) &= \frac{\partial H_2(\alpha_1, z)}{\partial z}\\
&= \sum_{j=1}^{n_2} \left(\tau_1 e^{\alpha_1(x_2-x_1)} + T_{n_1+j:N} - \tau_1\right)^z \log\left(\tau_1 e^{\alpha_1(x_2-x_1)} + T_{n_1+j:N} - \tau_1\right)\\
& && (3.4)\\
H_3(\alpha_1, z) &= (N - n_1 - n_2)\left(\tau_1 e^{\alpha_1(x_2-x_1)} + \tau_2 - \tau_1\right)^z && (3.5)\\
H_3'(\alpha_1, z) &= \frac{\partial H_3(\alpha_1, z)}{\partial z}\\
&= (N - n_1 - n_2)\left(\tau_1 e^{\alpha_1(x_2-x_1)} + \tau_2 - \tau_1\right)^z \log\left(\tau_1 e^{\alpha_1(x_2-x_1)} + \tau_2 - \tau_1\right)\\
& && (3.6)
\end{aligned}$$

Then the MLEs of $\alpha_0, \alpha_1, \beta$ can be found by solving the likelihood equations

$$\begin{pmatrix} \frac{\partial \log L}{\partial \alpha_0} \\ \frac{\partial \log L}{\partial \alpha_1} \\ \frac{\partial \log L}{\partial \beta} \end{pmatrix} = \mathbf{0} \qquad (3.7)$$

simultaneously. Since the solution cannot be found as analytical form, numerical methods such as Newton's method and Fisher scoring method will be used for solving the MLEs. We discuss these numerical methods in detail in Section 3.4.

3.3 Fisher Information Matrix

To assess the precision of the MLEs, we obtain the asymptotic variance-covariance matrix of the MLEs of $\alpha_0, \alpha_1, \beta$, which is the inverse of the expected Fisher information matrix $\mathscr{I}(\alpha_0, \alpha_1, \beta)$.

As mentioned in the previous section, we can only find variance-covariance matrix under constraint A_1, A_2. However, since inestimable cases are rare when N is large enough,

$$
\begin{aligned}
P(A_1, A_2) &= 1 - P(n_1 = 0) - P(n_2 = 1|n_1 = 1)P(n_1 = 1) \\
&\quad - \sum_{k_1=1}^{N-1} P(n_2 = 0|n_1 = k_1)P(n_1 = k_1) - P(n_1 = N) \\
&= 1 - e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^{\beta} N} \\
&\quad -N(N-1)(1 - e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^{\beta}})(1 - e^{\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^{\beta} - \omega^{\beta}})e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^{\beta} - \omega^{\beta}(N-2)} \\
&\quad -(1 - e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^{\beta}} + e^{-\omega^{\beta}})^N + e^{-\omega^{\beta} N} \qquad (3.8) \\
&\to 1 \qquad \text{as } N \to \infty
\end{aligned}
$$

where $\omega = \tau_1 e^{-\alpha_0 - \alpha_1 x_1} + (\tau_2 - \tau_1)e^{-\alpha_0 - \alpha_1 x_2}$ and

$$
n_1 \sim Bin\left(N, 1 - e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^{\beta}}\right) \qquad n_2|n_1 \sim Bin\left(N - n_1, 1 - e^{\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^{\beta} - \omega^{\beta}}\right) \qquad (3.9)
$$

and the detail derivations are presented in Appendix C, the inestimable cases are neglected and the Fisher information matrix, can be used to obtain the asymptotic variance-covariance matrix. This technique is very common in analysis of censored data. Interested readers can refer to the case of exponential distributions in Lawless (2003).

The Fisher information matrix is defined as the negative expectation of the Hessian matrix of the log-likelihood (Lawless (2003)), i.e.:

$$\mathscr{I}(\alpha_0, \alpha_1, \beta) = -\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \alpha_0^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha_0 \partial \alpha_1}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha_0 \partial \beta}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha_0 \partial \alpha_1}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha_1^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha_1 \partial \beta}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha_0 \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha_1 \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) \end{pmatrix} \tag{3.10}$$

where the second derivatives of log-likelihood functions are:

$$\begin{aligned}
\frac{\partial^2 \log L}{\partial \alpha_0^2} &= -\beta \frac{\partial \log L}{\partial \alpha_0} - \beta^2 (n_1 + n_2) \\
\frac{\partial^2 \log L}{\partial \alpha_0 \partial \alpha_1} &= -\beta \frac{\partial \log L}{\partial \alpha_1} - \beta^2 (n_1 x_1 + n_2 x_2) + \beta(\beta - 1) e^{\alpha_1 (x_2 - x_1)} \tau_1 (x_2 - x_1) H_2(\alpha_1, -1) \\
\frac{\partial^2 \log L}{\partial \alpha_0 \partial \beta} &= \frac{1}{\beta} \cdot \frac{\partial \log L}{\partial \alpha_0} - \beta e^{-\beta(\alpha_0 + \alpha_1 x_1)} [(\alpha_0 + \alpha_1 x_1) H_1(\beta) - H_1'(\beta)] \\
&\quad -\beta e^{-\beta(\alpha_0 + \alpha_1 x_2)} [(\alpha_0 + \alpha_1 x_2)(H_2(\alpha_1, \beta) + H_3(\alpha_1, \beta)) \\
&\quad -(H_2'(\alpha_1, \beta) + H_3'(\alpha_1, \beta))]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \log L}{\partial \alpha_1^2} &= -\beta^2 x_1^2 e^{-\beta(\alpha_0+\alpha_1 x_1)} H_1(\beta) - \beta^2 x_2^2 e^{-\beta(\alpha_0+\alpha_1 x_2)} (H_2(\alpha_1,\beta) + H_3(\alpha_1,\beta)) \\
&+(\beta-1) e^{\alpha_1(x_2-x_1)} \tau_1 (x_2-x_1)^2 H_2(\alpha_1,-1) \\
&-(\beta-1) e^{2\alpha_1(x_2-x_1)} \tau^2 (x_2-x_1)^2 H_2(\alpha_1,-2) \\
&+\beta[x_1+(2\beta-1)x_2] e^{-\{\alpha_0\beta+\alpha_1[x_1+x_2(\beta-1)]\}} \tau_1 (x_2-x_1) \\
&\quad (H_2(\alpha_1,\beta-1)+H_3(\alpha_1,\beta-1)) \\
&-\beta(\beta-1) e^{-\{\alpha_0\beta+\alpha_1[2x_1+x_2(\beta-2)]\}} \tau_1^2 (x_2-x_1)^2 (H_2(\alpha_1,\beta-2)+H_3(\alpha_1,\beta-2))) \\
\frac{\partial^2 \log L}{\partial \alpha_1 \partial \beta} &= \frac{1}{\beta}\left[\frac{\partial \log L}{\partial \alpha_1} + e^{\alpha_1(x_2-x_1)} \tau_1 (x_2-x_1) H_2(\alpha_1,-1)\right] \\
&-\beta x_1 e^{-\beta(\alpha_0+\alpha_1 x_1)} [(\alpha_0+\alpha_1 x_1) H_1(\beta) - H_1'(\beta)] \\
&-\beta x_2 e^{-\beta(\alpha_0+\alpha_1 x_2)} \left[(\alpha_0+\alpha_1 x_2)(H_2(\alpha_1,\beta)+H_3(\alpha_1,\beta)) - (H_2'(\alpha_1,\beta)+H_3'(\alpha_1,\beta))\right] \\
&+\beta e^{-\{\alpha_0\beta+\alpha_1[x_1+x_2(\beta-1)]\}} \tau_1 (x_2-x_1) \left[(\alpha_0+\alpha_1 x_2)(H_2(\alpha_1,\beta-1)+H_3(\alpha_1,\beta-1))\right. \\
&\quad \left. -(H_2'(\alpha_1,\beta-1)+H_3'(\alpha_1,\beta-1))\right] \\
\frac{\partial^2 \log L}{\partial \beta^2} &= -\frac{n_1+n_2}{\beta^2} - e^{-\beta(\alpha_0+\alpha_1 x_1)} \left[(\alpha_0+\alpha_1 x_1)^2 H_1(\beta) - 2(\alpha_0+\alpha_1 x_1) H_1'(\beta) + H_1''(\beta)\right] \\
&-e^{-\beta(\alpha_0+\alpha_1 x_2)} \left[(\alpha_0+\alpha_1 x_2)^2 H_2(\alpha_1,\beta) - 2(\alpha_0+\alpha_1 x_2) H_2'(\alpha_1,\beta) + H_2''(\alpha_1,\beta)\right] \\
&-e^{-\beta(\alpha_0+\alpha_1 x_2)} \left[(\alpha_0+\alpha_1 x_2)^2 H_3(\alpha_1,\beta) - 2(\alpha_0+\alpha_1 x_1) H_3'(\alpha_1,\beta) + H_3''(\alpha_1,\beta)\right]
\end{aligned}$$

where

$$H_1''(\beta) = \frac{\partial^2 H_1(\beta)}{\partial \beta^2} = \sum_{i=1}^{n_1} T_{j:N}^{\beta} (\log T_{j:N})^2 \tag{3.11}$$

$$\begin{aligned}
H_2''(\alpha_1,\beta) &= \left.\frac{\partial^2 H_2(\alpha_1,\beta)}{\partial z^2}\right|_{z=\beta} \\
&= \sum_{i=1}^{n_2} \left(\tau_1 e^{\alpha_1(x_2-x_1)} + T_{n_1+j:N} - \tau_1\right)^{\beta} \left[\log\left(\tau_1 e^{\alpha_1(x_2-x_1)} + T_{n_1+j:N} - \tau_1\right)\right]^2
\end{aligned} \tag{3.12}$$

$$
\begin{aligned}
H_3''(\alpha_1,\beta) &= \left.\frac{\partial^2 H_3(\alpha_1,\beta)}{\partial z^2}\right|_{z=\beta} \\
&= (N-n_1-n_2)\left(\tau_1 e^{\alpha_1(x_2-x_1)}+\tau_2-\tau_1\right)^\beta \left[\log\left(\tau_1 e^{\alpha_1(x_2-x_1)}+\tau_2-\tau_1\right)\right]^2
\end{aligned}
\tag{3.13}
$$

It is noted that all second derivatives only contain random variables $\frac{\partial \log L}{\partial \theta}, n_1, n_2$, H's in (3.1) - (3.6), (3.11) - (3.13). To find the expectation of the second derivatives of log-likelihood, the following facts can be used:

- The asymptotic properties about the score functions:

$$E\left(\frac{\partial \log L}{\partial \alpha_0}\right) = E\left(\frac{\partial \log L}{\partial \alpha_1}\right) = E\left(\frac{\partial \log L}{\partial \beta}\right) = 0$$

- By (3.9) and the law of total expectation, the expectations of n_1, n_2 are:

$$E(n_1) = N\left[1 - e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\right] \qquad E(n_2) = N\left[e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} - e^{-\omega^\beta}\right] \tag{3.14}$$

- The formulae of expectations of F's are located in Appendix B. They all are multiples of N, i.e., in the form of N multiplied by a function independent to N.

- Since C's are multiples of $(N - n_1 - n_2)$, by (3.14), their expectations:

$$E(C) = Ne^{-\omega^\beta} \times \text{ a constant}$$

  also are multiples of N.

The expected Fisher information matrix can be computed by obtaining the values of $E(n_1), E(n_2)$, $E(F)$'s and $E(C)$'s and hence the asymptotic variance-covariance matrix. After that, the Newton's method, and the Fisher scoring method can be used to find MLEs accurately. Besides this, the optimal experimental schemes can be determined under different stresses x_1, x_2, different censoring times τ_2 and different values of parameters in Chapter 4. Moreover, since the expected Fisher information matrix is a multiple of N, the optimal experimental schemes will be the same for any N given other quantities are the same.

3.4 Numerical Methods improving Newton's method

As we cannot get the analytical form for solving the system of likelihood equations (3.7), some numerical techniques are needed to obtain the solution. The Newton-Raphson method, or Newton's method, is one of the most common methods for solving nonlinear equations. It uses Taylor series expansion to approximate the equations to first order so that its updating formulae:

$$\boldsymbol{\theta}_{(\text{new})} = \boldsymbol{\theta}_{(\text{old})} - \left(\frac{\partial^2 \log L}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} \right)^{-1}_{\boldsymbol{\theta}_{(\text{old})}} \cdot \left. \frac{\partial \log L}{\partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta}_{(\text{old})}} \tag{3.15}$$

where $\boldsymbol{\theta} = (\alpha_0, \alpha_1, \beta)^T$. (See, for example, Burden and Faires (1993) and Demidenko (2004)) One of the advantages of Newton's method is the quadratic convergence of the solution, which means the number of accurate digits of the roots doubles in each step. However, algorithm usually fails to obtain the solution, especially when datasets from step-stress model are considered. To overcome the problem of convergence, the following improvements are suggested.

3.4.1 Initial values

The choice of initial values of the algorithm is an important issue for the algorithm to converge. Kateri and Balakrishnan (2008) introduces some methods for the step-stress model with Type-II censoring, which is also useful in our situation with Type-I censoring:

- Starting from the exponential step-stress model, i.e., set the initial values to be (Bai, Kim and Lee (1989)):

$$\alpha_0^{(0)} = \frac{-x_1 \log\left(\frac{\sum_{j=1}^{n_2}(T_{n_1+j:N}-\tau_1)+(\tau_2-\tau_1)(N-n_1-n_2)}{n_2}\right) + x_2 \log\left(\frac{\sum_{j=1}^{n_1} T_{j:N}+(N-n_1)\tau_1}{n_1}\right)}{x_2 - x_1}$$

$$\alpha_1^{(0)} = \frac{\log\left(\frac{\sum_{j=1}^{n_2}(T_{n_1+j:N}-\tau_1)+(\tau_2-\tau_1)(N-n_1-n_2)}{n_2}\right) - \log\left(\frac{\sum_{j=1}^{n_1} T_{j:N}+(N-n_1)\tau_1}{n_1}\right)}{x_2 - x_1}$$

$$\beta^{(0)} = 1$$

  This initial guess is very useful for the convergence of the Newton's method if the true value of β is less than 2. However, if we have no idea about β and the dataset seems to be far away from exponential distribution, usually the algorithm fails to converge. Then we have the following:

- Starting from the Simple Weibull Type-I censoring model, we treat the data after τ_1 as the censored data and find the MLE of β and θ_1 first (Lawless (2003)):

$$\beta^{(0)} : \quad \text{Solve } \frac{\sum_{j=1}^{n_1} T_{j:N}^{\beta^{(0)}} \log T_{j:N} + (N-n_1)\tau_1^{\beta^{(0)}} \log \tau_1}{\sum_{j=1}^{n_1} T_{j:N}^{\beta^{(0)}} + (N-n_1)\tau_1^{\beta^{(0)}}} - \frac{1}{\beta^{(0)}} - \frac{\sum_{j=1}^{n_1} \log T_{j:N}}{n_1} = 0$$

$$\theta_1^{(0)} = \left(\frac{\sum_{j=1}^{n_1} T_{j:N}^{\beta^{(0)}} + (N-n_1)\tau_1^{\beta^{(0)}}}{n_1}\right)^{\frac{1}{\beta}}$$

then $\theta_2^{(0)}$ can be estimated by solving for the $\left(\frac{n_1+n_2}{N}\right)^{\text{th}}$ quantile, i.e., solving $G(T_{n_1+n_2:N}; \theta_1^{(0)}, \theta_2^{(0)}, \beta^{(0)}) = \frac{n_1+n_2}{N}$, and the solution is:

$$\theta_2^{(0)} = \frac{T_{n_1+n_2:N} - \tau_1}{\left[-\log\left(1 - \frac{n_1+n_2}{N}\right)\right]^{\frac{1}{\beta}} - \frac{\tau_1}{\theta_1^{(0)}}}$$

finally $\alpha_0^{(0)}$ and $\alpha_1^{(0)}$ can be obtained by solving

$$\log \theta_i = \alpha_0 + \alpha_1 x_i$$

for $i = 1, 2$ simultaneously.

3.4.2 Fisher-Scoring method

If Newton's algorithm fails to converge, Demidenko (2004) recommends using Fisher scoring method instead of Newton's method to solve general MLE problems. The Fisher information matrix $\mathscr{I}$ shown in (3.10) is the expected negative Hessian, therefore it is reasonable to modify the Newton's method by replacing the negative Fisher information matrix $-\mathscr{I}(\boldsymbol{\theta}_{(\text{old})})$ instead of the Hessian matrix in (3.15) of the solving function. Fisher scoring method is the suitable numerical method for solving the MLE problem based on the following three reasons:

1. The Fisher information matrix is the inverse of the asymptotic covariance matrix of the MLE, which is always positive definite. Therefore, for any dataset, we can obtain the solution by Fisher scoring algorithm.

2. The Fisher information matrix at the final iteration leads to a better estimate of asymptotic covariance matrix of the MLE than the sample covariance matrix.

3. Use of the Fisher information matrix simplifies the formation of different versions of likelihood maximization algorithm, such as EM algorithm.

However, the number of iterations of Fisher scoring method is usually larger than the Newton's method since Fisher scoring method is linear convergent, which the convergence rate is lower (Demidenko (2004)). Thus, for the cases which estimates cannot be found by Newton's algorithm, Fisher-scoring algorithm is used.

Chapter 4

Optimal Experimental Design

4.1 Introduction

The main purpose of step-stress test is to investigate the relationship between lifetimes and stress levels. Therefore, we need to estimate the parameters $\alpha_0, \alpha_1, \beta$ in the model introduced in Chapter 2. Moreover, the accuracy of estimates is an important issue so we must discuss how to get the best estimates we want. Besides the sample size, the design of experiment is one of the most important factors affecting the accuracy of the MLEs. Therefore, after introducing different optimal criteria in Section 4.2, we find the optimal experimental schemes, or the optimal value $\eta = \frac{\tau_1}{\tau_2}$, of the simple step-stress model under different criteria and different values of parameters and initial settings in Section 4.3 with aid of line graphs and nomographs. In Section 4.4, sensitivity analysis is provided to study the effect due to incorrect guesses of parameters.

4.2 Optimal Criteria

To achieve different goals in estimating parameters, we need different optimality criteria to have the best performance of estimation. In this section, we introduce three different optimality criteria: *(i)* determinant-optimality, or D-optimality, *(ii)* minimum-variance of slope parameter α_1, or V-optimality, and *(iii)* trace-optimality, or A-optimality . Besides, predicting quantile of log-lifetime distributions is also important in reliability studies so the criterion of minimizing variance of MLE of quantile of the lifetime distribution is used in some literatures. See Bai and Kim (1993), Gouno and Balakrishnan (2001) and Ng, Balakrishnan and Chan (2007) for detail.

Criterion 1. determinant-optimality (D-optimality)

> Under this criterion, we choose the allocation scheme which maximizes the determinant of expected Fisher information matrix $\mathscr{I}(\alpha_0, \alpha_1, \beta)$ given in (3.10). Note that the determinant of $\mathscr{I}$ is the reciprocal of the asymptotic variance-covariance matrix of MLEs. Maximizing it is equivalent to minimizing the determinant of variance-covariance matrix of MLEs, and hence the volume of the Wald-type joint confidence region of $(\alpha_0, \alpha_1, \beta)$. Therefore, D-optimality is a natural way to optimize the accuracy of estimates.

Criterion 2. minimum-variance of MLE of slope parameter α_1 (V-optimality)

> Under this criterion, we choose the allocation scheme which minimizes the variance of MLE of slope parameter α_1. Many literatures about optimal

allocation such as Ng, Balakrishnan and Chan (2006) state that the variance of $\hat{\alpha}_1$ is related with the determinant of variance-covariance matrix. Moreover, the slope parameter is especially important in regression analysis so concerning the estimation of α_1 and minimizing its variance is one of the criteria considered.

Criterion 3. trace-optimality (A-optimality)

Under this criterion, we choose the allocation scheme which minimizes the trace of $\mathscr{I}^{-1}(\alpha_0, \alpha_1, \beta)$, which is the sum of variances of parameter estimates. A-optimality uses variances, which is a kind of marginal variation measures, to conclude the overall variability of estimates. Therefore it can be compared with different allocation schemes without using the whole variance-covariance matrix.

4.3 Optimal Stress-changing-time Proportion

To find the optimal testing plan, the definition of "optimal" should be clearly stated. Bai, Kim and Lee (1989) and Gouno, Sen and Balakrishnan (2004) aim to search the optimal stress-changing-time proportion

$$\eta = \frac{\tau_1}{\tau_2}$$

which is a common technique to determine optimal experimental schemes in step-stress testing. In this section, we search the optimal η by determining Fisher information matrices and hence the asymptotic variance-covariance matrices for different pre-fixed values of τ_2, x_1, x_2 and the parameters $\alpha_0, \alpha_1, \beta$.

It is well known that the Fisher information matrix is a multiple of N so the results of optimal experimental scheme among different N are the same. Thus, in this section, we find the optimal η by considering:

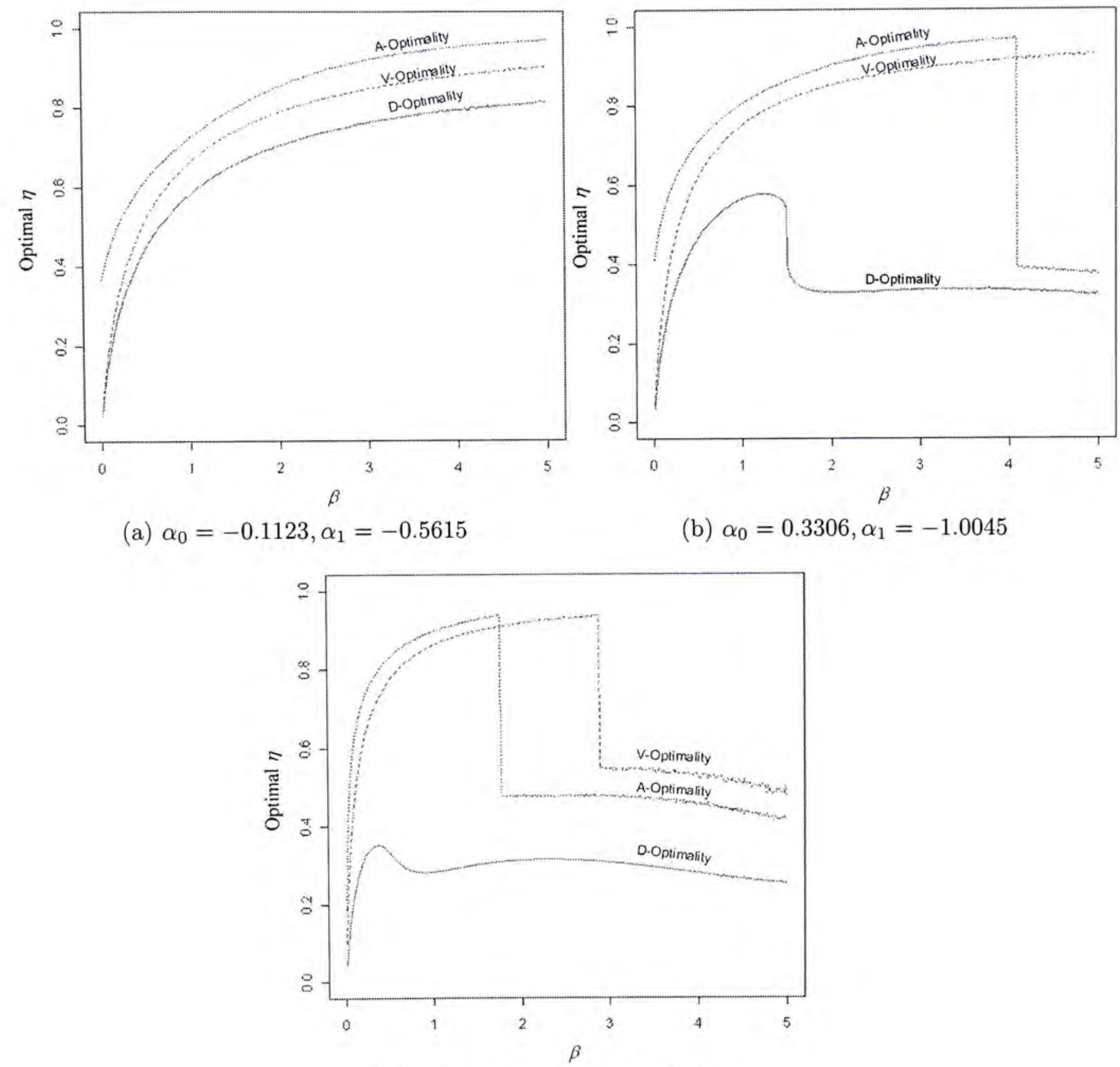
- different β given $\alpha_0, \alpha_1, \tau_2, x_1, x_2$.
- different α_0, α_1 given β, τ_2, x_1, x_2.
- different x_1 given $\alpha_0, \alpha_1, \beta, \tau_2, x_2$.
- different τ_2 given $\alpha_0, \alpha_1, \beta, x_1, x_2$.

4.3.1 Optimal η versus the shape parameter β

Figure 4.1 shows the curves of optimal η against β under different optimal criteria for $(\alpha_0, \alpha_1) = (-0.1123, -0.5615), (0.3306, -1.0045), (1.1436, -1.8175)$, with fixed $x_1 = 0, x_2 = 1, \tau_2 = 0.7$. In this figure, the following are observed:

- Optimal η increases with β when β is small but it drops down when β is larger than certain values. After that it decreases with increasing β. If α_1 is more negative, the drop appears at a smaller β. The drop takes place because the optimums of η-curve against β are swapped for different β.

  For example, we consider the curve for V-optimality in Figure 4.1c. The variances of $\hat{\alpha}_1$ against η under three particular values of $\beta = 2, 2.9, 4$, respectively, representing the cases of higher optimal η, when optimal η is

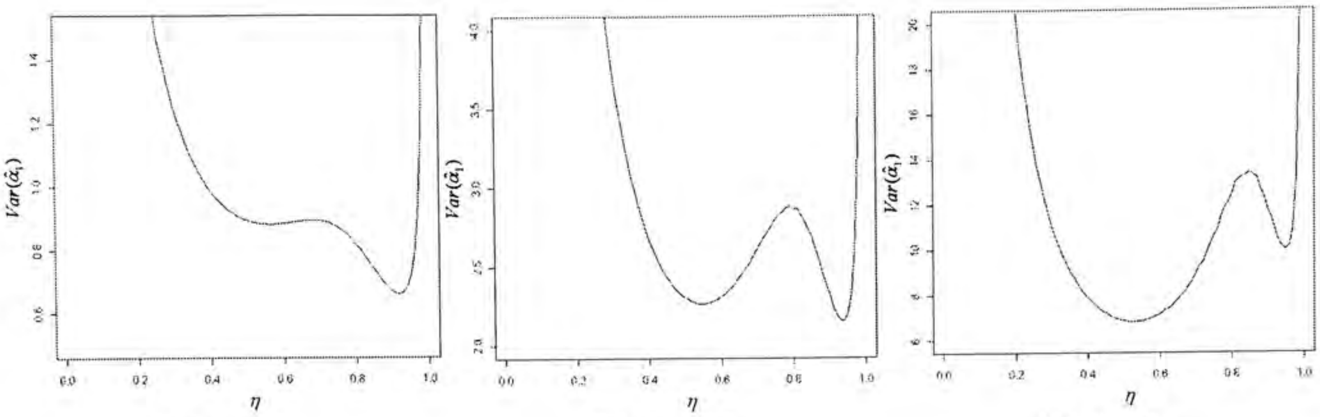


(a) $\alpha_0 = -0.1123, \alpha_1 = -0.5615$

(b) $\alpha_0 = 0.3306, \alpha_1 = -1.0045$

(c) $\alpha_0 = 1.1436, \alpha_1 = -1.8175$

Figure 4.1: Line graph for optimal η against β



(a) $\beta = 2$ (b) $\beta = 2.9$ (c) $\beta = 4$

Figure 4.2: Line graph for variance of $\hat{\alpha}_1$ against η ($\alpha_0 = 1.1436, \alpha_1 = -1.8175$)

jumping, and lower optimal η, are plotted in Figure 4.2. When the distributions under two different stresses are not too far away from each other, i.e., α_1 is not too negative or β is small, say $\beta = 2$, according to Figure 4.2a, the minimum of the variance attains at η close to 1, i.e., scheduling large proportion of time under x_1 is preferred. When the distributions are significantly different, say for example by Figure 4.2b, when β passes through a certain value 2.9, another minimum appears at small η and gradually replaces the previous minimum. In Figure 4.2c, when the distributions differ extremely, optimal η close to 0 suggests that large proportion of time should put under stress x_2.

- Moreover, when β is small, the optimal η under V-optimality always smaller than the one under A-optimality and the one under D-optimality is the smallest among three. However, when β is as large as beyond the jumps, the order of V-optimality and A-optimality is swapped and D-optimality remains at the smallest position.

4.3.2 Optimal η versus the parameters α_0, α_1

To show optimal η versus two parameters, nomographs, which are two-dimensional graphical calculating devices to allow the approximate graphical computation of optimal η under specific values of parameters, are suggested in many literatures of optimal design problems such as Bai, Kim and Lee (1989), and Bai and Kim (1993).

Moreover, since the ranges of possible α_0, α_1 are too large, Bai, Kim and Lee (1989) suggested showing the optimal η by nomographs with p_i, for $i = 1, 2$, which are the probabilities that a test unit fails before τ_2 while testing only at stress x_i, i.e.:

$$p_i = 1 - \exp\left[-\left(\tau_2 e^{-\alpha_0 - \alpha_1 x_i}\right)^{\beta}\right] \tag{4.1}$$

Then, α_0, α_1 can be written in terms of p_1 and p_2 as:

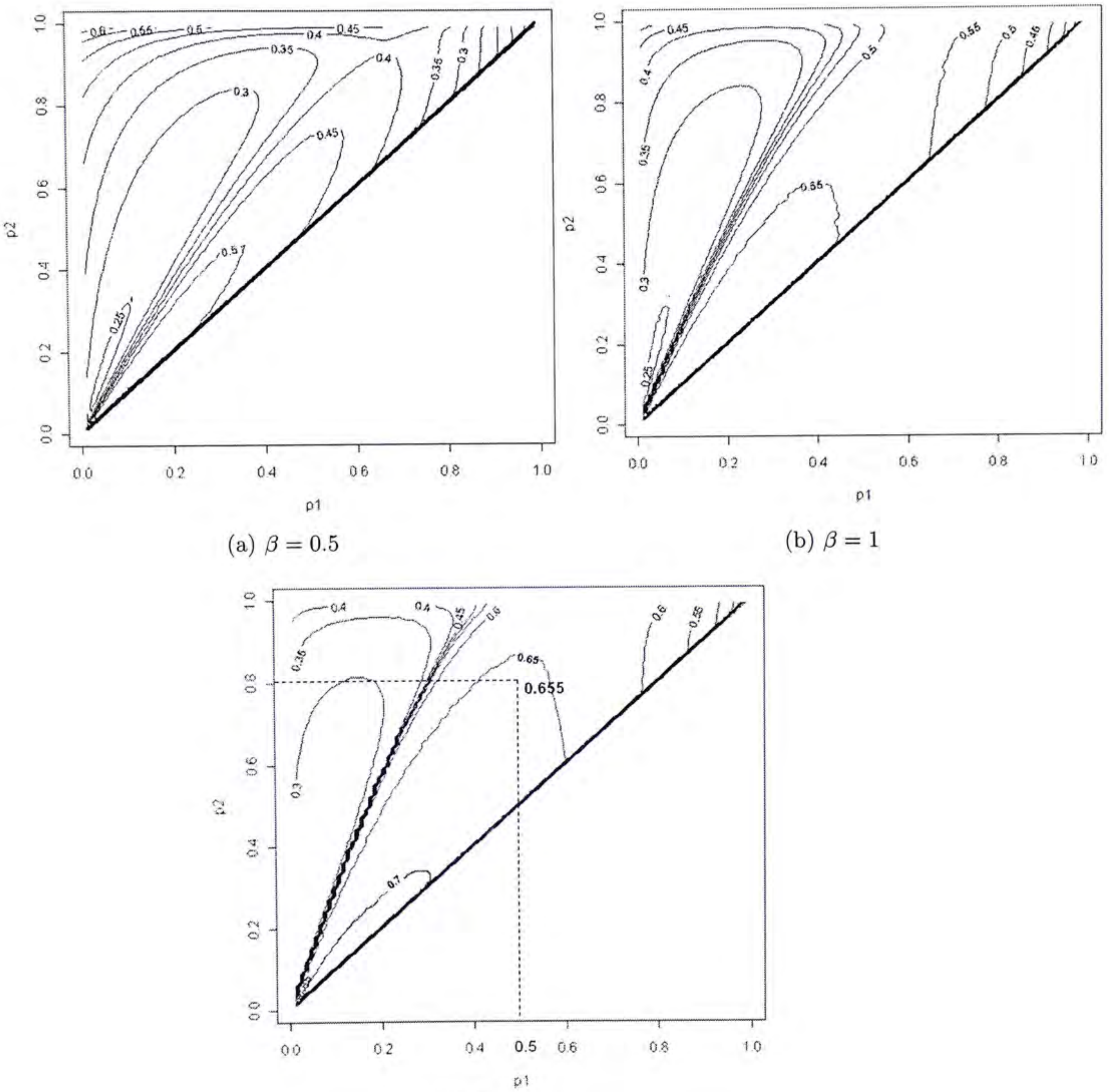
$$\begin{aligned}
\alpha_0 &= \log \tau_2 - \frac{x_1 \log(-\log(1-p_2)) - x_2 \log(-\log(1-p_1))}{\beta(x_1 - x_2)} \\
\alpha_1 &= -\frac{\log(-\log(1-p_2)) - \log(-\log(1-p_1))}{\beta(x_1 - x_2)}
\end{aligned}$$

Thus, when we know τ_2, x_1, x_2 and parameters p_1, p_2, β, the optimal plan can be found.

The optimal experimental schemes for the ordered pair $(p_1, p_2) \in \{(0,1) \times (0,1) : p_1 < p_2\}$ are considered with choices of $\tau_2 = 0.7, x_1 = 0, x_2 = 1, \beta = 0.5, 1, 1.5$. The optimal η for each (p_1, p_2) are calculated and presented as nomographs presented in Figure 4.3-4.5 by different optimal criteria. For example, if the test-situation is given as above and the values of $\alpha_0, \alpha_1, \beta$ are -0.1123, -0.5616, 1.5 respectively, then by (4.1), $p_1 \approx 0.5, p_2 \approx 0.8$. Hence according to Figure 4.3c, under D-optimality, the optimal η is approximately 0.655 so that the optimal stress-changing time $\tau_1 = \eta\tau_2 = 0.7 \times 0.655 = 0.4585$.

The summary of findings of these nomographs is as follows:

- Each graph has a critical line from the left bottom corner to the ceiling. By the explanation of the first phenomenon stated in the previous subsection, left side of the line represents significant differences between p_1 and p_2, which have optimal η close to 0. On the other side, p_1 differs p_2 not too much, so we have optimal η close to 1. Therefore in each nomograph, the critical line separates two different areas which have different levels of optimal η.
- The critical line moves left when β increases. Moreover, optimal η raises and drops, respectively, in the right and left side with increasing β. This matches the second phenomenon stated in the previous subsection.

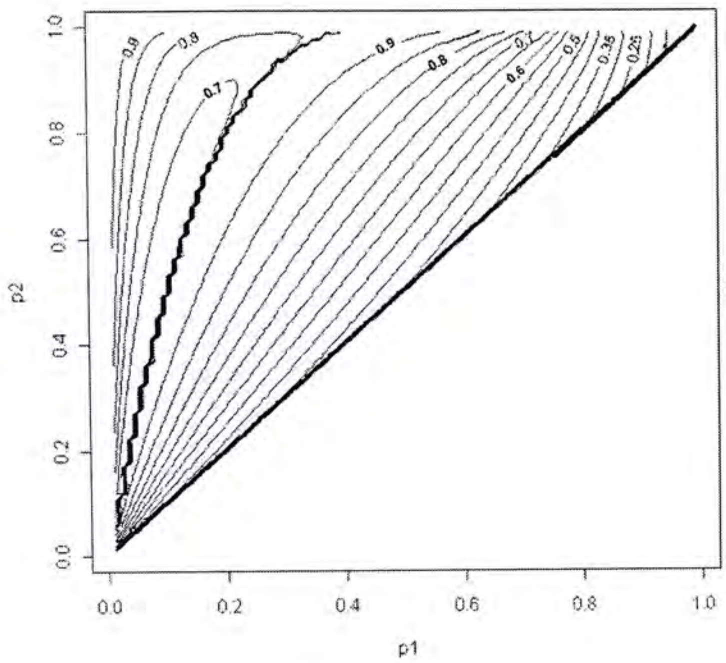


(a) $\beta = 0.5$

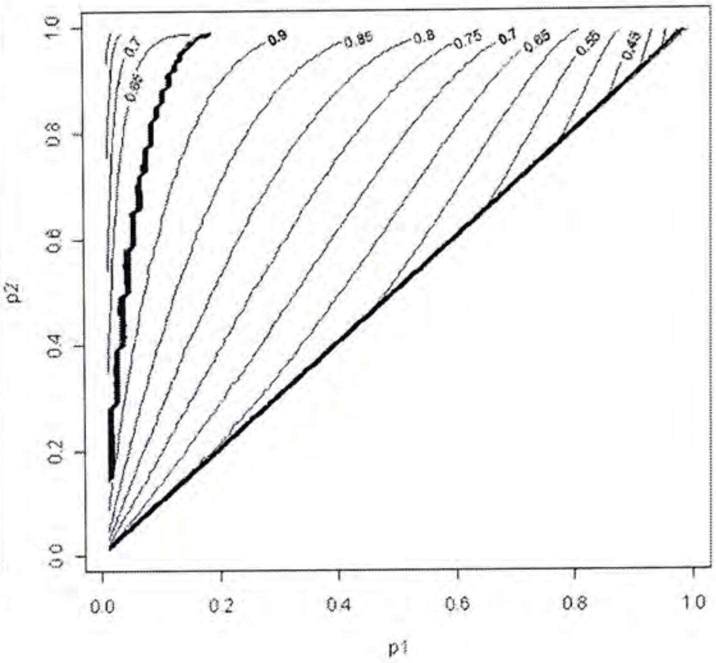
(b) $\beta = 1$

(c) $\beta = 1.5$

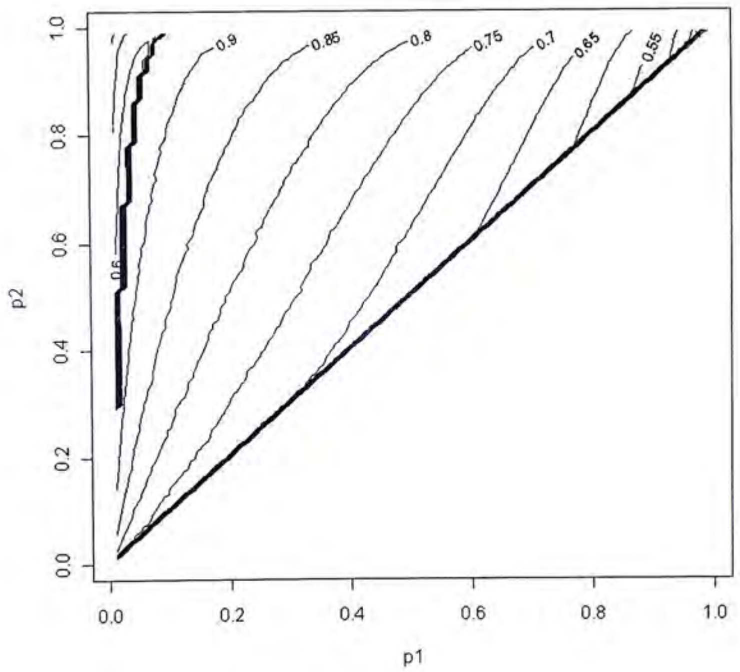
Figure 4.3: Nomograph for optimal η by D-optimality



(a) $\beta = 0.5$

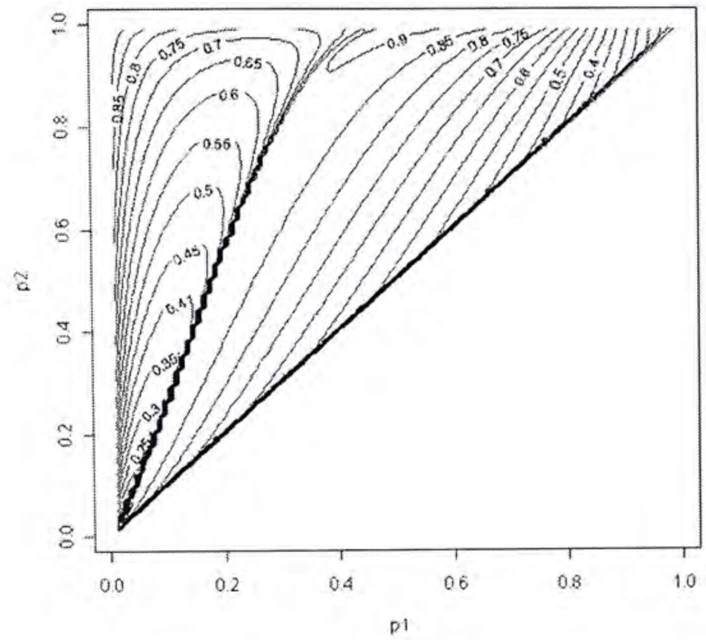


(b) $\beta = 1$

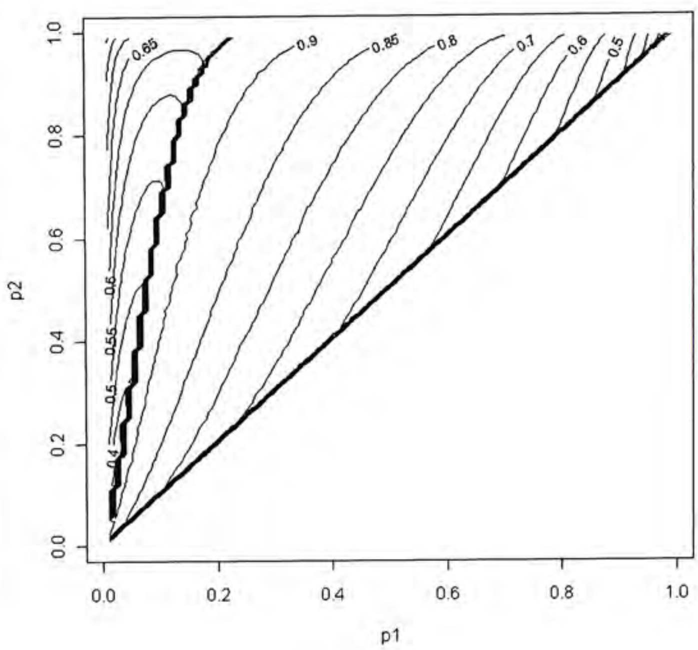


(c) $\beta = 1.5$

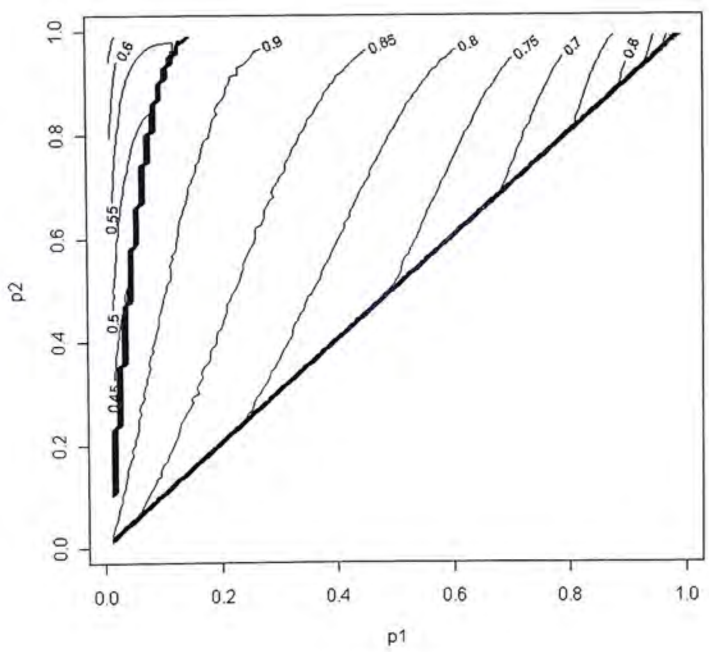
Figure 4.4: Nomograph for optimal η by V-optimality



(a) $\beta = 0.5$



(b) $\beta = 1$



(c) $\beta = 1.5$

Figure 4.5: Nomograph for optimal η by A-optimality

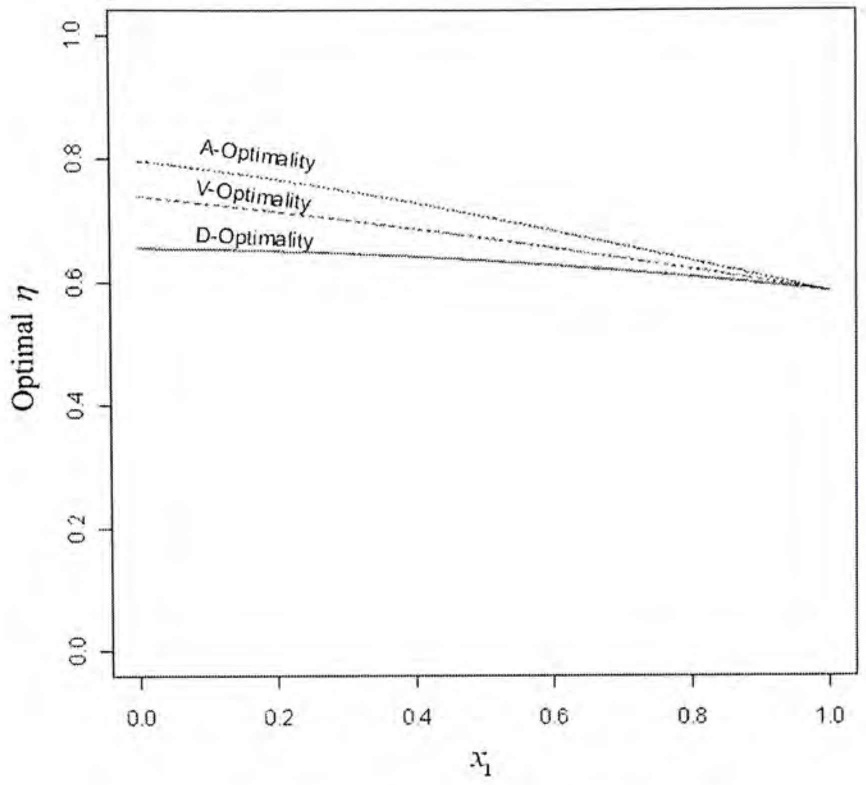


Figure 4.6: Line graph for optimal η vs. x_1

4.3.3 Optimal η versus the initial stress level x_1

Note that when x_2 is set to be 1, x_1 can be seen as extrapolation amount of stress from normal conditions. Formally, x_1 is called the standardized stress. For normal conditions, $x_1 = 0$. This approach is frequently used in literatures such as Bai, Kim and Lee (1989) and Bai and Kim (1993). Moreover, engineers usually set the initial stress level higher than or same as normal stress, i.e., $0 \leq x_1 < x_2$. Therefore, $0 \leq x_1 < 1$ with $x_2 = 1$ is considered in the sensitivity analysis.

Figure 4.6 shows the curves of optimal η versus different x_1 under different optimal criteria for $\alpha_0 = -0.1123, \alpha_1 = -0.5615, \beta = 1.5, x_2 = 1, \tau_2 = 0.7$. All curves decrease gently for $0 \leq x_1 < 1$. When x_1 approaches 1, optimal η under different criteria converges to the same value.

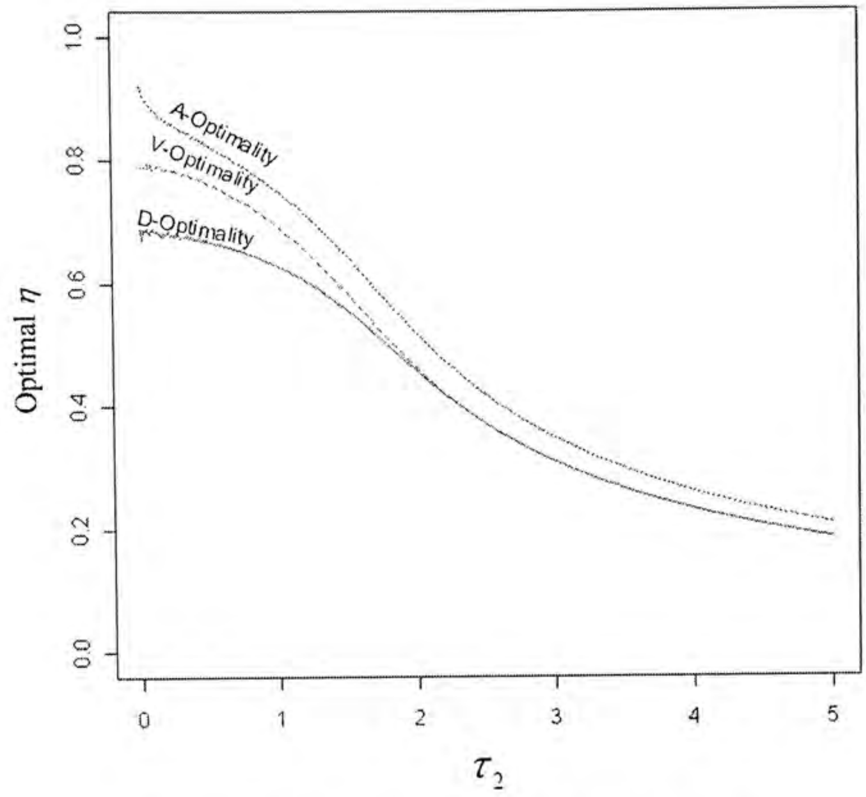


Figure 4.7: Line graph for optimal η vs. τ_2

4.3.4 Optimal η versus the censoring time τ_2

Figure 4.7 shows the curves of optimal η against different τ_2 under different optimal criteria for $\alpha_0 = -0.1123, \alpha_1 = -0.5615, \beta = 1.5, x_1 = 0, x_2 = 1$. When τ_2 is large, the optimal η under D-optimality is as same as the one under V-optimality. In general the result of D-optimality is different from V-optimality under Type-I censoring (see, for example, see Gunno, Sen and Balakrishnan (2004)). However, when the censoring time continues to increase, the optimal scheme becomes the case of complete data and the results.

It is interesting to note that the decreasing trend of optimal η . That means if we have extra time to conduct experiment, we are willing to put more time to observe failures in higher stress.

4.4 Sensitivity Analysis

Since incorrect guesses of parameters may give a non-optimal experimental plan which can worsen the precision of parameter estimations. Therefore it is important in validating the optimal experimental schemes. Sensitivity analysis is a useful technique for systematically changing parameters in a model to determine the effects of such changes. In this section we investigate the effects by determining the ratios of the quantities under different criteria such as determinant, variance and trace of the optimal model, or the optimal η, due to incorrect guesses of the parameters.

4.4.1 Effects of the shape parameter β

The determinant ratios under D-optimality, the variance ratios under V-optimality and the trace ratios under A-optimality due to the incorrect guess of β are, respectively, computed and tabulated in Table 4.1, 4.2 and 4.3, where the true values of $\beta = 0.5, 1, 1.5$, and the given values $x_1 = 0, x_2 = 1, \tau_2 = 0.7$, parameters $\alpha_0 = -0.1123, \alpha_1 = -0.5615$.

From these tables, the errors due to incorrect guesses are small if the guesses are not too far away from the true β. For example, if true $\beta = 1.5$ but it is wrongly set as 1, by Table 4.1, the model determinant is 96.83% of the optimal determinant. Moreover, by Table 4.2 and 4.3, the variance of MLE of α_1 and the model trace only inflate 4.88% and 4.74% respectively.

However, the optimal experimental scheme is sensitive when β is severely under-guessed. To make clear how the ratios change with respect to β, the line

Table 4.1: Determinant Ratios due to incorrect guess of β

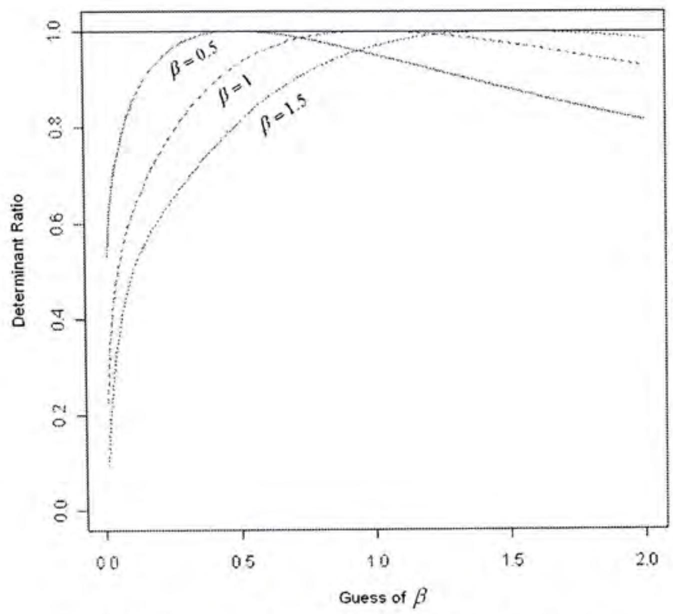
Guess of β	True $\beta = 0.5$	True $\beta = 1$	True $\beta = 1.5$	Guess of β	True $\beta = 0.5$	True $\beta = 1$	True $\beta = 1.5$
0.1	0.8368	0.6071	0.4878	1.1	0.9368	0.9985	0.9812
0.2	0.9339	0.7343	0.6045	1.2	0.9222	0.9946	0.9900
0.3	0.9771	0.8223	0.6830	1.3	0.9079	0.9889	0.9959
0.4	0.9954	0.8858	0.7510	1.4	0.8935	0.9818	0.9991
0.5	**1.0000**	0.9297	0.8087	1.5	0.8796	0.9739	**1.0000**
0.6	0.9968	0.9600	0.8571	1.6	0.8659	0.9652	0.9992
0.7	0.9887	0.9800	0.8959	1.7	0.8526	0.9559	0.9970
0.8	0.9777	0.9920	0.9266	1.8	0.8388	0.9457	0.9937
0.9	0.9651	0.9982	0.9504	1.9	0.8255	0.9354	0.9892
1	0.9511	**1.0000**	0.9683	2	0.8144	0.9264	0.9841

Table 4.2: Variance Ratios due to incorrect guess of β

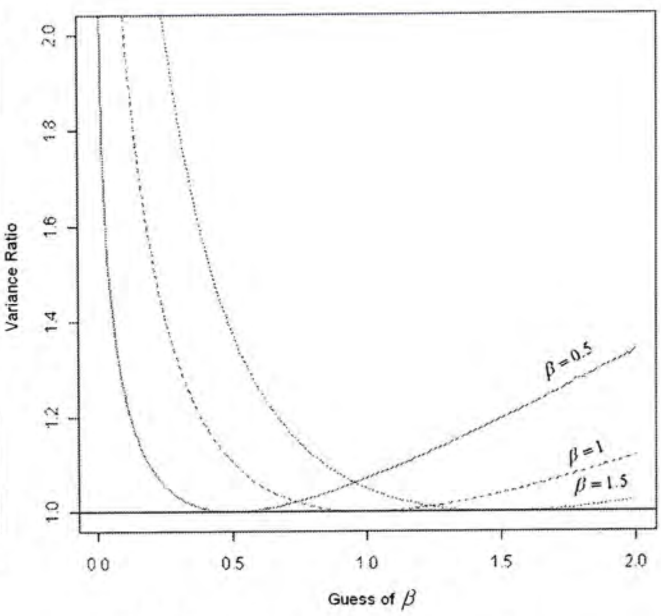
Guess of β	True $\beta = 0.5$	True $\beta = 1$	True $\beta = 1.5$	Guess of β	True $\beta = 0.5$	True $\beta = 1$	True $\beta = 1.5$
0.1	1.2525	2.0327	3.1009	1.1	1.0924	1.0021	1.0281
0.2	1.0914	1.5490	2.2676	1.2	1.1161	1.0075	1.0148
0.3	1.0304	1.3146	1.8295	1.3	1.1410	1.0156	1.0060
0.4	1.0061	1.1833	1.5552	1.4	1.1683	1.0264	1.0014
0.5	**1.0000**	1.1047	1.3788	1.5	1.1945	1.0383	**1.0000**
0.6	1.0043	1.0571	1.2594	1.6	1.2219	1.0517	1.0013
0.7	1.0152	1.0279	1.1769	1.7	1.2493	1.0661	1.0046
0.8	1.0306	1.0110	1.1188	1.8	1.2806	1.0834	1.0095
0.9	1.0489	1.0025	1.0778	1.9	1.3090	1.0998	1.0167
1	1.0697	**1.0000**	1.0488	2	1.3405	1.1184	1.0238

Table 4.3: Trace Ratios due to incorrect guess of β

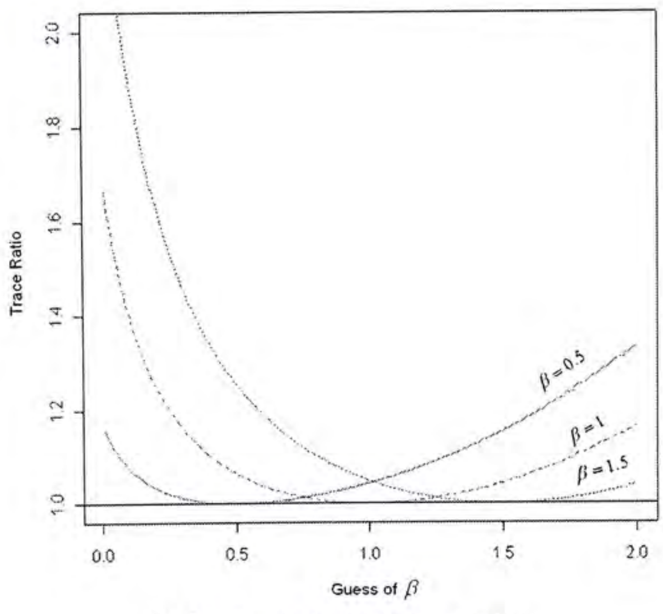
Guess of β	True $\beta = 0.5$	True $\beta = 1$	True $\beta = 1.5$	Guess of β	True $\beta = 0.5$	True $\beta = 1$	True $\beta = 1.5$
0.1	1.0756	1.4043	1.9127	1.1	1.0600	1.0019	1.0295
0.2	1.0333	1.2519	1.6377	1.2	1.0784	1.0070	1.0161
0.3	1.0124	1.1618	1.4625	1.3	1.1011	1.0160	1.0068
0.4	1.0027	1.1037	1.3427	1.4	1.1257	1.0278	1.0017
0.5	**1.0000**	1.0646	1.2554	1.5	1.1533	1.0429	**1.0000**
0.6	1.0022	1.0379	1.1904	1.6	1.1844	1.0615	1.0016
0.7	1.0080	1.0198	1.1401	1.7	1.2191	1.0837	1.0065
0.8	1.0170	1.0082	1.1021	1.8	1.2549	1.1076	1.0154
0.9	1.0285	1.0020	1.0713	1.9	1.2931	1.1341	1.0259
1	1.0431	**1.0000**	1.0474	2	1.3343	1.1635	1.0367



(a) Line graph for determinant ratios



(b) Line graph for variance ratios



(c) Line graph for trace ratios

Figure 4.8: Line graph for ratios due to incorrect guess of β

graphs of ratios against $0 < \beta < 2$ are plotted in Figure 4.8. We can easily observe that the error of under-guessed β is much larger than the one of over-guessed β. Therefore, when we have several possible choices of β, choose the highest one to prevent high estimation variance.

4.4.2 Effects of the parameters α_0, α_1

We consider the parameterization in terms of p_1 and p_2 instead of α_0 and α_1. The determinant ratios under D-optimality, the variance ratios under V-optimality and the trace ratios under A-optimality due to the incorrect guesses of p_1, p_2 are computed and tabulated in Table 4.4, 4.5, 4.6 respectively, where the true values of $p_1 = 0.5, p_2 = 0.8$, and the given values $x_1 = 0, x_2 = 1, \tau_2 = 0.7$, and parameter $\beta = 1.5$.

The effect of incorrect guesses is small if the guesses are not too far away from the true parameters. For example, if parameters p_1, p_2 are wrongly estimated as 0.35 and 0.75 respectively, the model determinant is 99.9484% of the optimal determinant, and the variance of $\hat{\alpha}_1$ and the model trace only inflate 2.1855% and 2.0865% from the optimum, respectively.

In conclusion of the tables, the result of sensitivity analysis for α_0, α_1 is different from the one for β. Under D-optimality, if the error of guess of p_1 is within 0.2, i.e., $0.3 \leq$ guess of $p_1 \leq 0.7$, its model determinant is at least 68% of the optimal one. On the other hand, under V-optimality and A-optimality, the variance of $\hat{\alpha}_1$ and the model trace inflates not more than 17% and 12% , respectively.

Table 4.4: Determinant ratios due to incorrect guesses of p_1, p_2 (True $p_1 = 0.5, p_2 = 0.8$)

	guess of p_1								
p_2	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7
0.35	0.989987	-	-	-	-	-	-	-	-
0.4	0.991899	0.992693	-	-	-	-	-	-	-
0.45	0.994465	0.993773	0.994382	-	-	-	-	-	-
0.5	0.996575	0.995596	0.996199	0.996966	-	-	-	-	-
0.55	0.998304	0.996522	0.997051	0.997926	0.998899	-	-	-	-
0.6	0.999229	0.998492	0.997682	0.998178	0.999268	0.999871	-	-	-
0.65	0.999998	0.999386	0.998846	0.999094	0.999510	0.999945	0.999747	-	-
0.7	0.998207	0.999996	0.999544	0.999464	0.999678	0.999993	0.999636	0.997870	-
0.75	0.989920	0.999484	0.999995	0.999822	0.999936	0.999982	0.999509	0.997718	0.993757
0.8	0.755379	0.996356	0.999728	0.999997	**1.000000**	0.999920	0.999394	0.997667	0.993881
0.85	0.688955	0.981958	0.998029	0.999677	0.999894	0.999830	0.999316	0.997619	0.993900
0.9	0.685723	0.790746	0.990216	0.998150	0.999553	0.999584	0.999178	0.997935	0.994328
0.95	0.700726	0.739515	0.946426	0.992789	0.998215	0.999298	0.999102	0.998225	0.995652

Table 4.5: Variance ratios due to incorrect guesses of p_1, p_2 (True $p_1 = 0.5, p_2 = 0.8$)

	guess of p_1								
p_2	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7
0.35	1.004494	-	-	-	-	-	-	-	-
0.4	1.000865	1.007928	-	-	-	-	-	-	-
0.45	1.000123	1.003245	1.013544	-	-	-	-	-	-
0.5	1.001976	1.000480	1.005922	1.018794	-	-	-	-	-
0.55	1.005462	1.000184	1.001922	1.010649	1.025892	-	-	-	-
0.6	1.011326	1.002346	1.000160	1.005843	1.018272	1.035995	-	-	-
0.65	1.020520	1.006686	1.000387	1.001694	1.009859	1.024925	1.048095	-	-
0.7	1.031575	1.013165	1.002636	1.000107	1.004722	1.016210	1.035456	1.064181	-
0.75	1.044766	1.021855	1.006707	1.000564	1.001134	1.009130	1.024736	1.049661	1.083700
0.8	1.062687	1.036001	1.014795	1.003662	**1.000000**	1.003713	1.015364	1.034902	1.067240
0.85	1.085340	1.050395	1.026071	1.009602	1.001510	1.000474	1.007669	1.022775	1.049661
0.9	1.118250	1.075762	1.043725	1.021726	1.007230	1.000475	1.001638	1.011677	1.032465
0.95	1.179258	1.117535	1.075659	1.044608	1.022685	1.007287	1.000394	1.002566	1.015639

Table 4.6: Trace ratios due to incorrect guesses of p_1, p_2 (True $p_1 = 0.5, p_2 = 0.8$)

	guess of p_1								
p_2	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7
0.35	1.000027	-	-	-	-	-	-	-	-
0.4	1.000778	1.001075	-	-	-	-	-	-	-
0.45	1.003075	1.000016	1.004172	-	-	-	-	-	-
0.5	1.006730	1.000539	1.001127	1.008843	-	-	-	-	-
0.55	1.010160	1.001548	1.000100	1.004445	1.015876	-	-	-	-
0.6	1.015462	1.004503	1.000194	1.001894	1.010073	1.024479	-	-	-
0.65	1.025301	1.008833	1.001553	1.000429	1.006002	1.018381	1.037171	-	-
0.7	1.031720	1.015279	1.003981	1.000029	1.002776	1.011832	1.028206	1.052668	-
0.75	1.041965	1.020865	1.008145	1.000710	1.000753	1.007229	1.020392	1.042412	1.073359
0.8	1.057203	1.030595	1.013227	1.003617	**1.000000**	1.003300	1.013608	1.031738	1.058782
0.85	1.070903	1.044403	1.021436	1.007508	1.000877	1.000771	1.007430	1.022186	1.046642
0.9	1.092645	1.058055	1.033338	1.015667	1.004503	1.000068	1.002757	1.013687	1.032638
0.95	1.128796	1.085417	1.052580	1.030623	1.013258	1.003211	1.000008	1.004353	1.018252

Chapter 5

Conclusion Remarks and Further Research

In recent years, many literatures, not only about step-stress test but also the other fields in reliability, usually ignore inestimable cases and large sample assumptions on MLE to build theories of reliability models. Ignorance of these cases may incur serious problems. This thesis acts as a remainder.

After that, the accuracy of MLEs, asymptotic variance-covariance matrix can be obtained by inverting their Fisher information matrix, which is derived in Section 3.3 and Appendix B. Finally, numerical techniques for step-stress models by previous literatures are presented in Section 3.4.

Optimal experimental schemes under different optimal criteria, parameters and settings of models are found in Section 4.3 by searching the optimal stress-changing-time proportion η. The results are presented in line graphs and nomographs which are very useful for determining optimal η. Finally, by observing the ratios of determinants, variances, or traces, the sensitivity analysis tells the effect due to incorrect guesses of α_0, α_1 is small but the large error occurs when β is under-guessed.

For further research, other lifetime distributions such as Pareto distributions and Birnbaum-Saunders distributions, and censoring schemes such as progressive and hybrid censoring can be considered and compared.

However, occurrence of inestimable dataset is possible while time-step-stress testing plan is used. To avoid the appearance of inestimable dataset, a change of testing plan is necessary. A good suggestion is the failure-step-stress testing plan, which changes testing stresses according to the number of failure happened. This can force the minimum number of failures happened under each stress to make sure that every datasets are estimable.

Appendix A

Simulation Algorithm for a Weibull Type-I Censored Simple Step-Stress Model

1. Simulate and sort by ascending order for a sample size N from the uniform distribution $U(0,1)$, labelled as $U_{1:N}, U_{2:N}, \ldots, U_{N:N}$

2. For $j = 1, \ldots, N$, set $T_{j:N} = e^{\alpha_0+\alpha_1 x_1}\left[-\log(1-U_{j:N})\right]^{\frac{1}{\beta}}$.

3. If $T_{j:N} > \tau_1$, set $T_{j:N} = e^{\alpha_0+\alpha_1 x_2}\left[-\log(1-U_{j:N})\right]^{\frac{1}{\beta}} - \tau_1 e^{-(\alpha_0+\alpha_1 x_1)} + \tau_1 e^{-(\alpha_0+\alpha_1 x_2)}$.

4. If $T_{j:N} > \tau_2$, set $T_{j:N} = \tau_2$

Finally, $T_{j:N}$ is the j^{th} ordered lifetime experiencing in the experiment.

Appendix B

Expected values of Fisher Information Matrix

In this section, $E(H_1(\beta))$'s and $E(H_2(\alpha_1, z))$'s are derived for determining Fisher information matrix in (3.10), where $H_1(\beta)$'s and $H_2(\alpha_1, z)$'s are located at (3.1) - (3.4), (3.11), (3.12). Firstly, $E(H_1(\beta))$'s, which involving the failure failed under the first stress, are considered. Since $H_1(\beta)$'s only depends on n_1 only, the expected $H_1(\beta)$'s can be obtained by the law of total expectation:

$$E(H_1(\beta)) = E_{n_1}(E_T(H_1(\beta)|n_1)) \tag{B.1}$$

By Balakrishnan, Kundu, Ng, Kannan (2007), considering the order statistics $T_{1:N}, \ldots, T_{N:N}$ of any random sample with p.d.f. $G(t)$, the conditional joint p.d.f. of $T_{1:N}, \ldots, T_{k_1:N}$ given $n_1 = k_1$ is identical to the joint p.d.f. of all order statistics from the random sample of size k_1 from the right-truncated density function:

$$\frac{g(t)}{G(\tau_1)} \qquad \text{for } 0 < t < \tau_1 \tag{B.2}$$

A proof of this result may refer to Arnold, Balakrishnan and Nagaraja (1992, P.23-24). Therefore,

$$
\begin{aligned}
E(H_1(\beta)|n_1) &= E\left(\sum_{j=1}^{n_1} T_{j:N}^{\beta} \middle| n_1\right) \\
&= \sum_{j=1}^{n_1} \int_0^{\tau_1} t^{\beta} \frac{\beta e^{-\beta(\alpha_0+\alpha_1 x_1)} t^{\beta-1} e^{-\left(te^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}}{1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}} dt \\
&= n_1 e^{-\beta(\alpha_0+\alpha_1 x_1)} \left(1 - \frac{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta} e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}}{1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}}\right)
\end{aligned}
$$

Similarly $E(H_1'(\beta))$ and $E(H_1''(\beta))$ can be evaluated by (B.1) and their conditional expectations given n_1 shown below:

$$
\begin{aligned}
E(H_1'(\beta)|n_1) &= n_1 e^{-\beta(\alpha_0+\alpha_1 x_1)} \left[(\alpha_0+\alpha_1 x_1) \left(1 - \frac{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta} e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}}{1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}}\right) \right. \\
&\quad \left. + \frac{1}{1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}} \left(\frac{1}{\beta} \int_0^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}} w \log w e^{-w} dw \right) \right] \\
E(H_1''(\beta)|n_1) &= n_1 e^{-\beta(\alpha_0+\alpha_1 x_1)} \left[(\alpha_0+\alpha_1 x_1)^2 \left(1 - \frac{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta} e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}}{1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}}\right) \right. \\
&\quad + \frac{1}{1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}} \left(\frac{2(\alpha_0+\alpha_1 x_1)}{\beta} \int_0^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}} w \log w e^{-w} dw \right. \\
&\quad \left. \left. + \frac{1}{\beta^2} \int_0^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}} w (\log w)^2 e^{-w} dw \right) \right]
\end{aligned}
$$

where the above improper integrals can be simplified by integration by parts and taking limits:

$$
\begin{aligned}
&\int_0^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} w(\log w)e^{-w}dw = \lim_{x\to 0}\int_x^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} w(\log w)e^{-w}dw \\
= \;& -e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\left[1+\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta \log\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right] \\
& +\lim_{x\to 0} e^{-x}(1+x\log x) + \lim_{x\to 0}\int_x^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} e^{-w}(\log w)dw \\
= \;& 1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\left[1+\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta \log\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right] + M_1\left(\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} w(\log w)^2e^{-w}dw = \lim_{x\to 0}\int_x^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} w(\log w)^2e^{-w}dw \\
= \;& -e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\left\{1+\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta \left[\log\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right]^2\right\} \\
& +\lim_{x\to 0} e^{-x}(1+x(\log x)^2) + \lim_{x\to 0}\int_x^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} e^{-w}\left((\log w)^2+2\log w\right)dw \\
= \;& 1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\left[1+\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta \log\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right] \\
& +M_2\left(\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right) + 2M_1\left(\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right)
\end{aligned}
$$

where the M's can be determined by formulae in the handbooks by Abromowitz and Stegun (1965) and Mathai (1993):

$$
\begin{aligned}
M_1(x) &= \int_0^x e^{-w}\log w dw \\
&= -\gamma - e^{-x}\log x - E_1(x) \\
M_2(x) &= \int_0^x e^{-w}(\log w)^2 dw \\
&= \log x(-2E_1(x) - \log x - e^{-x}\log x - 2\gamma) + 2x \cdot {}_3F_3(1,1,1;2,2,2;-x)
\end{aligned}
$$

where $\gamma = 0.577215665...$ is the Euler constant, $E_1(x)$ is the exponential integral, i.e.:

$$E_1(x) = \int_x^{\infty} \frac{e^{-w}}{w} dw$$

and ${}_3F_3(1,1,1;2,2,2;-x)$ is a generalized hypergeometric function, which can be expanded as the following series:

$$\begin{aligned} {}_3F_3(1,1,1;2,2,2;-x) &= \sum_{r=0}^{\infty} \frac{(1 \cdot 2 \cdot \ldots \cdot r)^3}{(2 \cdot 3 \cdot \ldots \cdot (r+1))^3} \frac{(-x)^r}{r!} \\ &= \sum_{r=0}^{\infty} \frac{(-x)^r}{(r+1)^3 \cdot r!} \end{aligned}$$

It is noted that although double precision is used in computer programs, calculating $M_2(x)$ for large x is inaccurate . Instead, $M_2(x) = 1.978111991$ for $x > 25$ is suggested since it converges to 1.978111991 when x is large.

Therefore, $E(H_1(\beta))$'s can be obtained by expecting n_1 with formulae (3.14):

$$\begin{aligned} E(H_1(\beta)) &= Ne^{-\beta(\alpha_0+\alpha_1 x_1)} \left[1 - e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} - \left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\right] \\ E(H_1'(\beta)) &= Ne^{-\beta(\alpha_0+\alpha_1 x_1)} \{(\alpha_0 + \alpha_1 x_1) \\ &\quad \times \left[1 - e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} - \left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\right] \\ &\quad \left. + \frac{1}{\beta} \int_0^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} w \log w e^{-w} dw \right\} \end{aligned}$$

$$
\begin{aligned}
E(H_1''(\beta)) &= Ne^{-\beta(\alpha_0+\alpha_1 x_1)}\Big\{(\alpha_0+\alpha_1 x_1)^2 \\
&\quad \left[1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\right] \\
&\quad +\frac{2(\alpha_0+\alpha_1 x_1)}{\beta}\int_0^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} w\log w e^{-w}dw \\
&\quad +\frac{1}{\beta^2}\int_0^{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} w(\log w)^2 e^{-w}dw\Big\}
\end{aligned}
$$

Now the conditional expectation of $H_2(\alpha_1, z)$'s, which involving failures under the stress x_2, are considered. Using the law of total expectation again, the conditional expectations of $H_2(\alpha_1, z)$'s can be found with the following formulae:

$$
E(H_2(\alpha_1, z)) = E_{n_1}(E_{n_2}(E_T(F_{II,z}|n_2)|n_1)) \tag{B.3}
$$

Due to (B.2), it is clear that the conditional joint p.d.f. of $T_{k_1+1:N}, \ldots, T_{k_1+k_2:N}$ given $n_1 = k_1, n_2 = k_2$ is identical to the joint p.d.f. of all order statistics from the random sample of size k_2 from the truncated density function:

$$
\frac{g(t)}{G(\tau_2)-G(\tau_1)} \qquad \text{for } \tau_1 < t < \tau_2
$$

Therefore for any z,

$$\begin{aligned}
E(H_2(\alpha_1, z)|n_2) &= E\left(\sum_{j=1}^{n_2}\left(\tau_1 e^{\alpha_1(x_2-x_1)} + T_{n_1+j:N} - \tau_1\right)^z \middle| n_2\right) \\
&= \sum_{i=1}^{n_2}\int_{\tau_1}^{\tau_2}\left(e^{\alpha_1(x_2-x_1)}\tau_1 + t - \tau_1\right)^z \frac{\beta e^{-\beta(\alpha_0+\alpha_1 x_2)}\left(e^{\alpha_1(x_2-x_1)}\tau_1 + t - \tau_1\right)^{\beta-1}}{e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} - e^{-\omega^\beta}} \\
&\times \quad e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1} + (t-\tau_1)e^{-\alpha_0-\alpha_1 x_2}\right)^\beta} dt \\
&= \frac{n_2 e^{-z(\alpha_0+\alpha_1 x_2)}}{e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} - e^{-\omega^\beta}} \int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}^{\omega^\beta} w^{\frac{z}{\beta}} e^{-w} dw \\
&= \frac{n_2 e^{-z(\alpha_0+\alpha_1 x_2)}}{e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} - e^{-\omega^\beta}}\left[\Gamma\left(1+\frac{z}{\beta}, \left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right) - \Gamma\left(1+\frac{z}{\beta}, \omega^\beta\right)\right]
\end{aligned}$$

where $\omega = \tau_1 e^{-\alpha_0-\alpha_1 x_1} + (\tau_2 - \tau_1)e^{-\alpha_0-\alpha_1 x_2}$ and $\Gamma(s, x)$ is the incomplete gamma function, i.e.:

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$$

Similarly $E(H_2'(\alpha_1, \beta - 1))$, $E(H_2'(\alpha_1, \beta))$ and $E(H_2''(\alpha_1, \beta))$ can be evaluated by (B.3) and their conditional expectations given n_2 are the following:

$$\begin{aligned}
E(H_2'(\alpha_1, \beta-1)|n_2) &= \frac{n_2 e^{-z(\alpha_0+\alpha_1 x_2)}}{e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} - e^{-\omega^\beta}}\Bigg\{(\alpha_0 + \alpha_1 x_2) \\
&\times \left[\Gamma\left(2-\frac{1}{\beta}, \left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right) - \Gamma\left(2-\frac{1}{\beta}, \omega^\beta\right)\right] \\
&+\frac{1}{\beta}\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}^{\omega^\beta} w^{1-\frac{1}{\beta}} \log w e^{-w} dw\Bigg\} \qquad \text{(B.4)} \\
E(H_2'(\alpha_1, \beta)|n_2) &= \frac{n_2 e^{-z(\alpha_0+\alpha_1 x_2)}}{e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} - e^{-\omega^\beta}}\Bigg\{(\alpha_0 + \alpha_1 x_2) \\
&\times \left[\Gamma\left(2, \left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right) - \Gamma\left(2, \omega^\beta\right)\right] \\
&+\frac{1}{\beta}\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}^{\omega^\beta} w \log w e^{-w} dw\Bigg\} \qquad \text{(B.5)}
\end{aligned}$$

$$
\begin{aligned}
E(H_2''(\alpha_1,\beta)|n_2) &= \frac{n_2 e^{-z(\alpha_0+\alpha_1 x_2)}}{e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta} - e^{-\omega^\beta}} \left\{(\alpha_0+\alpha_1 x_2)^2\right. \\
&\left[\Gamma\left(2,\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right) - \Gamma\left(2,\omega^\beta\right)\right] \\
&+\frac{2(\alpha_0+\alpha_1 x_2)}{\beta}\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}^{\omega^\beta} w\log w e^{-w}dw \\
&\left.+\frac{1}{\beta^2}\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}^{\omega^\beta} w(\log w)^2 e^{-w}dw\right\} \qquad \text{(B.6)}
\end{aligned}
$$

where the proper integrals (B.5) and (B.6) can be simplified by integration by parts and taking limits:

$$
\begin{aligned}
\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}^{\omega^\beta} w\log w e^{-w}dw &= -e^{-\omega^\beta}\left(1+\omega^\beta\log\omega^\beta\right) \\
&+e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\left[1+\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta \log\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right] \\
&+M_1(\omega^\beta) - M_1\left(\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right) \\
\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}^{\omega^\beta} w(\log w)^2 e^{-w}dw &= -e^{-\omega^\beta}\omega^\beta(\log\omega^\beta)^2 \\
&+e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\left(\log\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right)^2 \\
&+M_2(\omega^\beta) - M_2\left(\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right) \\
&+2M_1(\omega^\beta) - 2M_1\left(\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta\right)
\end{aligned}
$$

However, the remaining integral (B.4) cannot be solved in analytical form so we should use numerical integration to approximate them by computers. The most frequently used called Simpson's composite rule is suggested (see Burden and Faires (1993)).

Therefore, $E(H_2(\alpha_1, z))$'s can be obtained by expecting n_2 with formulae (3.14):

$$\begin{aligned}
E(H_2(\alpha_1,z)) &= Ne^{-z(\alpha_0+\alpha_1 x_2)}\left[\Gamma\left(1+\frac{z}{\beta},\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}\right)-\Gamma\left(1+\frac{z}{\beta},\omega^{\beta}\right)\right] \\
E(H_2'(\alpha_1,\beta-1)) &= Ne^{-(\beta-1)(\alpha_0+\alpha_1 x_2)}\left[(\alpha_0+\alpha_1 x_2)\right. \\
&\quad\times\left(\Gamma\left(2-\frac{1}{\beta},\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}\right)-\Gamma\left(2-\frac{1}{\beta},\omega^{\beta}\right)\right) \\
&\quad\left.+\frac{1}{\beta}\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}^{\omega^{\beta}} w^{1-\frac{1}{\beta}}\log w e^{-w}dw\right] \\
E(H_2'(\alpha_1,\beta)) &= Ne^{-\beta(\alpha_0+\alpha_1 x_2)}\left[(\alpha_0+\alpha_1 x_2)\left(\Gamma\left(2,\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}\right)-\Gamma\left(2,\omega^{\beta}\right)\right)\right. \\
&\quad\left.+\frac{1}{\beta}\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}^{\omega^{\beta}} w\log w e^{-w}dw\right] \\
E(H_2''(\alpha_1,\beta)) &= Ne^{-\beta(\alpha_0+\alpha_1 x_2)}\left[(\alpha_0+\alpha_1 x_2)^2\left(\Gamma\left(2,\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}\right)-\Gamma\left(2,\omega^{\beta}\right)\right)\right. \\
&\quad+\frac{2(\alpha_0+\alpha_1 x_2)}{\beta}\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}^{\omega^{\beta}} w\log w e^{-w}dw \\
&\quad\left.+\frac{1}{\beta^2}\int_{\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^{\beta}}^{\omega^{\beta}} w(\log w)^2 e^{-w}dw\right]
\end{aligned}$$

Appendix C

Derivation of $P(A_1, A_2)$

Since $n_1 \sim Bin\left(N, 1 - e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta}\right)$ and $n_2|n_1 \sim Bin\left(N - n_1, 1 - e^{\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta - \omega^\beta}\right)$,

$$\begin{aligned}
P(n_1 = 0) &= e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta N} && \text{(C.1)} \\
P(n_1 = N) &= \left[1 - e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta}\right]^N && \text{(C.2)} \\
P(n_2 = 1|n_1 = 1)P(n_1 = 1) &= N(N-1)\left[1 - e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta}\right]\left[1 - e^{\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta - \omega^\beta}\right] \\
&\quad \times e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta - \omega^\beta (N-2)} && \text{(C.3)}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k_1=1}^{N-1} P(n_2 = 0|n_1 = k_1)P(n_1 = k_1) &= \sum_{k_1=0}^{N} e^{\left[\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta - \omega^\beta\right](N-k_1)} P(n_1 = k_1) \\
&\quad - e^{\left[\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta - \omega^\beta\right]N} P(n_1 = 0) - P(n_1 = N) \\
&= e^{\left[\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta - \omega^\beta\right]N} E\left(e^{\left[\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta - \omega^\beta\right](-n_1)}\right) \\
&\quad - e^{\left[\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta - \omega^\beta\right]N} e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta N} \\
&\quad - \left[1 - e^{-\left(\tau_1 e^{-\alpha_0 - \alpha_1 x_1}\right)^\beta}\right]^N
\end{aligned}$$

By the moment generating function of n_1,

$$E\left(e^{\left[\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta-\omega^\beta\right](-n_1)}\right) = \left\{e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}+\left[1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\right]e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta+\omega^\beta}\right\}^N$$

Therefore,

$$\sum_{k_1=1}^{N-1} P(n_2=0|n_1=k_1)P(n_1=k_1) = \left[1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}+e^{-\omega^\beta}\right]^N - e^{-\omega^\beta N} - \left[1-e^{-\left(\tau_1 e^{-\alpha_0-\alpha_1 x_1}\right)^\beta}\right]^N \tag{C.4}$$

Finally by subtracting (C.1) - (C.4) from 1, $P(A_1, A_2)$ can be obtained and the result follows.

Bibliography

[1] Abramowitz, M. and Stegun, I.A. (1965). *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables,* Dover, New York.

[2] Arnold, B.C.; Balakrishnan, N. and Nagaraja, H.N. (1992). *A First Course in Order Statistics,* John Wiley & Sons, New York.

[3] Bai, D.S., Kim, M.S. and Lee, S.H. (1989). Optimal Simple Step-Stress Accelerated Life Tests with Censoring, *IEEE Transactions on Reliability,* **38**, 528-532.

[4] Bai, D.S. and Kim, M.S. (1993). Optimal Simple Step-Stress Accelerated Life Tests for Weibull Distribution and Type I Censoring, *Naval Research Logistics,* **40**, 193-210.

[5] Balakrishnan, N., Kundu, D., Ng, H.K.T. and Kannan, N. (2007). Point and Interval Estimation for a Simple Step-Stress Model with Type-II Censoring, *Journal of Quality Technology,* **39**, 35-47.

[6] Burden, R.L. and Faires, J.D. (1993). *Numerical Analysis,* PWS, Boston.

[7] Demidenko, E. (2004). *Mixed Models: Theory and Applications,* John Wiley & Sons, New Jersey.

[8] Gouno, E. and Balakrishnan, N. (2001) Step-stress accelerated life test, *Handbook of statistics 20, Advances in Reliability,* Balakrishnan, N. and Rao, C.R., Eds., Elsevier, Amsterdam.

[9] Gouno, E., Sen, A. and Balakrishnan, N. (2004) Optimal Step-Stress Test Under Progressive Type-I Censoring, *IEEE Transactions on Reliability,* **53**, 388-393.

[10] Kateri, M. and Balakrishnan, N. (2008). Inference for a Simple Step-Stress Model With Type-II Censoring, and Weibull Distributed Lifetimes, *IEEE Transactions on Reliability,* **57**, 616-626.

[11] Lawless, J.F. (2003). *Statistical Models & Methods For Lifetime Data, Second Edition,* John Wiley & Sons, New York.

[12] Mathai, A.M. (1993). *A Handbook of Generalized Special Functions for Statistical and Physical Sciences,* Oxford Science Publications, New York.

[13] Miller, R. and Nelson, W. (1983). Optimal Simple Step-Stress Plans for Accelerated Life Testing, *IEEE Transactions on Reliability,* **32**, 59-65.

[14] Nelson, W. (1980). Accelerated Life Testing - Step-Stress Model and Data Analyses, *IEEE Transactions on Reliability,* **29**, 103-108.

[15] Ng, H.K.T., Balakrishnan, N. and Chan, P.S. (2007). Optimal Sample Size Allocation for Tests with Multiple Levels of Stress with Extreme Value Regression, *Naval Research Logistics,* **54**, 237-249.