Duality Theory by Sum of Epigraphs of Conjugate Functions in Semi-Infinite Convex Optimization

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Abstract

In this thesis, we intend to give a survey on some recent results on the use of sum of epigraphs of conjugate functions in the duality theory of semi-infinite convex optimization. The thesis is divided into two parts. In the first part, we study the sum of epigraphs constraint qualification (the SECQ). We explain its relationship with other known constraint qualifications in convex optimization theory, for example, the strong conical hull intersection property (the strong CHIP) and the linear regularity. We give an overview of some sufficient conditions for the SECQ. In the second part, we establish the connection between the weakly* sum of epigraphs of conjugate functions and the Fenchel duality in the setting of semi-infinite optimization theory. We explain an extension of the Fenchel Duality Theorem which is applicable in this setting.

摘要

本文介紹一些涉及共軛函數的上圖和 (sum of epigraphs) 於半無窮凸優化對偶理論之結果。我們的討論分為兩部分。第一部分是關於上圖和約束規範 (the sum of epigraphs constraint qualification, abbrev. the SECQ)。我們將解釋它與凸優化理論中一些已知約束規範之間的關係，例如强 CHIP (the Strong CHIP) 和線性正則性 (the linear regularity)。我們也介紹一些 SECQ 的充分條件。第二部分是討論共軛函數的上圖弱星和 (weakly* sum of epigraphs) 與 Fenchel 對偶在半無窮優化理論中的聯繫。其中，我們解釋 Fenchel 對偶定理的一個推廣。

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Contents

Chapter 1

Introduction

In [9], under the setting of Banach space, Burachik and Jeyakumar utilized the epigraphs to give a sufficient condition for the strong conical hull intersection property (the strong CHIP) of two closed convex sets. They also gave some examples to show that this sufficient condition is weaker than some classical interior-point type conditions which are essential for the validity of the strong CHIP. Since the strong CHIP is crucial for some duality results (such as the dual formulation of best approximation problems, see [12, 13, 14]), the result of Jeyakumar et. al. established its own importance in the duality theory of convex optimization. In [17, 18], Li and Ng defined the notion of strong CHIP for an infinite system of closed convex sets in the setting of general normed linear space, and made extensive investigation on this property. In particular, they utilized this property in the study of general systems of infinite convex inequalities. This gave an application of the strong CHIP in the semi-infinite optimization theory. In view of the result of Jeyakumar et. al. in [9] that we have mentioned, it is both useful and interesting to see whether their work can be extended to cover the case when an infinite system of closed convex sets is considered, under some more general spaces. In [19], Li, Ng and Pong defined a new type of constraint qualification, known as the sum of epigraphs constraint qualification (the SECQ), for an arbitrary system of closed convex sets under the setting of normed linear space. It turns out that the SECQ is a generalization of the property that Jeyakumar et. al. considered, while it allows the discussion for an infinite system of closed convex sets. Moreover, they proved that this property still serve as a sufficient condition for the strong CHIP (for arbitrary system of closed convex sets). As a result, this links up the SECQ and semi-infinite optimization theory.

On the other hand, in [10], Jeyakumar et. al. utilized a more general version of the property of epigraphs than they studied in [9], to give a sufficient condition for the "Fenchel duality" for two functions on a Banach space. Recall that, the celebrated Fenchel Duality Theorem, states that, under some interior-point type conditions, one can transform a primal minimization problem into its dual maximization problem, and an optimal solution can be obtained in the dual problem. This special type of transformation between two optimization problems is called the Fenchel duality. Due to the fact that there are minimization problems in which its dual problem is easier to be handled (see [11, Example 25.2] for example), the Fenchel Duality Theorem is an important and useful result in the duality theory. Jeyakumar et. al. showed that the property of epigraphs that they considered in [10] is weaker than some classical interior-point type conditions for Fenchel duality. Since the property that they studied is not of interior-point type, their result gives further insight into the Fenchel duality. Using similar techniques that were used in [10], their result can be generalized to cover the case when finitely many functions are considered. From both theoretical and application points of view, it is meaningful to extend the above mentioned results to cover the case when infinitely many functions are in consideration. However, it turns out that such extension is non-trivial. Indeed, in [20], Li and Ng used the notion of weakly* sum of epigraphs to give a generalization for the mentioned result of Jeyakumar et. al. Their result showed that it is still possible to talk about Fenchel duality under the setting of semi-infinite optimization problems.

In this thesis, we intend to give a survey on the development of the uses of sum of epigraphs in the semi-infinite optimization theory, which are mainly from [19] and [20]. The outline of this thesis is shown as follows:

In Chapter 2, necessary notations and tools which are needed in our analysis will be given. Most of them are mainly concerned with convex analysis and set-valued analysis. Especially in the last section, the weakly* sum of sets in dual space will be discussed. In particular, a version of subdifferential sum rule of infinitely many proper convex continuous functions will be shown, which serves as a generalization of the well-known subdifferential sum rule of finitely many continuous functions. This is the result by Ng and Zheng in [32].

In Chapter 3, we will collect the work of Li, Ng and Pong in [19]. First, we give the definition of the SECQ and show some of its basic but useful properties. After that, as an extension of the result by Jeyakumar et. al. in [9], we will show that the SECQ is a sufficient condition for the strong CHIP. Also, since the converse implication holds under certain assumptions, some sufficient conditions for these assumptions will be given. Other than the strong CHIP, the relationship between the SECQ and the linear regularity will be studied. A new characterization of the linear regularity will be proved. In the last section, we discuss some interior-point type conditions for the SECQ.

The extension of the Fenchel Duality Theorem to the case for infinitely many proper convex lower-semicontinuous functions will be the main subject of Chapter 4, in which we show the results by Li and Ng in [20]. Using the weakly* sum of epigraphs of conjugate functions, we first prove a characterization of a property that is stronger than the Fenchel duality. This in turn gives a condition to ensure the Fenchel duality for an infinite system of functions. Next, we restrict our attention to two special classes of functions: continuous functions and non-negative functions. Sufficient conditions for the extended Fenchel duality of these two classes of functions will be provided.

Chapter 2

Notations and Preliminaries

2.1 Introduction

In this chapter, we introduce some notations and tools that will be used in this thesis. Most of them come from convex analysis and set-valued analysis.

2.2 Basic notations

Throughout this chapter, let X be a real normed linear space, unless otherwise stated. Let X^* denote the continuous dual of X, that is, the set of all continuous linear functional on X, equipped with the sup-norm. For any $x^* \in X^*$, we will use the notation $\langle x^*, x \rangle$ to denote the value $x^*(x)$.

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}_+$ be the set of all non-negative real numbers. The extended real line will be denoted by $\mathbb{R} \cup \{\pm\infty\}$. Let $\mathbf{B}(x, r)$ and $\mathbf{B_X}$ denote the closed ball with center x and radius r and the closed unit ball, respectively. Recall that the weakly* topology $\sigma(X^*, X)$ on X^* is the weakest topology on X^* such that for any $x \in X$, the linear functional $x^* \mapsto \langle x^*, x \rangle$ is continuous. Given a set A in X, the interior (resp. closure) of A, under the norm topology, is denoted by $\operatorname{int} A$ (resp. $\overline{A}$). In the case when A is a subset of X^*, let

$\overline{A}^{w^*}$ stand for the closure of A under the weakly* topology on X^*.

The following definitions are important.

Definition 2.2.1. *Let A be a subset of X. Then,*

(i) *we say that A is convex if*

$$\lambda x + (1-\lambda)y \in A \quad \text{for any } x, y \in A \text{ and } \lambda \in [0,1].$$

(ii) *we say that A is a cone if*

$$\alpha A = A \quad \text{for any } \alpha \in \mathbb{R}_+.$$

Remark 2.2.1. *In the literature, there are some authors who defined the term "cone" as the convex cone in our sense. We remark that the convexity is not included in our definition of cone.*

Let $A \subseteq X$. The linear hull (resp. affine hull, convex hull, convex conical hull) of A, denoted by $\operatorname{span} A$ (resp. $\operatorname{aff} A$, $\operatorname{co} A$, $\operatorname{cone} A$), is defined as the smallest linear subspace (resp. affine set, convex set, convex cone) in X which contains A as a subset. The relative interior of A is defined by

$$\operatorname{rint} A := \{x \in X : \exists \delta > 0 \text{ s.t. } \mathbf{B}(x,\delta) \cap \operatorname{aff} A \subseteq A\}.$$

Given another set $D \subseteq X$. The relative interior of A with respect to D is defined by

$$\operatorname{rint}_D A := \{x \in X : \exists \delta > 0 \text{ s.t. } \mathbf{B}(x,\delta) \cap \operatorname{aff} D \subseteq A\}.$$

When $\operatorname{aff} D = X$, $\operatorname{rint}_D A = \operatorname{int} A$. Also, if $\operatorname{aff} D = \operatorname{aff} A$, then $\operatorname{rint}_D A = \operatorname{rint} A$.

Given a convex set $C \subseteq X$ and $x \in C$. The dimension of C, denoted as $\dim C$, is defined by

$$\dim C := \dim(\operatorname{aff}(C - x)),$$

which is equal to the dimension of the subspace parallel to the affine hull of C. We remark that the choice of x does not affect the dimension of C (see [31, Page 3]).

Let f be an extended real-valued function on X. The epigraph of f is defined by

$$\text{epi}\, f := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \le \alpha\},$$

and the strict epigraph of f is defined by

$$\text{epi}_s\, f := \{(x, \alpha) \in X \times \mathbb{R} : f(x) < \alpha\},$$

while the graph of f is defined by

$$\text{gph}\, f := \{(x, \alpha) \in X \times \mathbb{R} : f(x) = \alpha\}.$$

The domain of f, denoted by $\text{dom}\, f$, is the set of all $x \in X$ such that $f(x) \neq +\infty$, that is

$$\text{dom}\, f := \{x \in X : f(x) < +\infty\}.$$

The function f is said to be proper if $\text{dom}\, f \neq \emptyset$ and $f > -\infty$ on X.

Another important definition is stated as follows.

Definition 2.2.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$. The function f is said to be convex if*

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda) f(y) \quad \textit{for any } x, y \in X \textit{ and } \lambda \in [0, 1].$$

Following [31], we define the following sets:

$$\Gamma(X) := \{\text{all proper convex lower semicontinuous functions on } X\};$$
$$\Gamma(X^*) := \{\text{all proper convex weakly}^* \text{ lower semicontinuous functions on } X^*\}.$$

Also, we define the following subsets of $\Gamma(X)$:

$$\Gamma_c(X) := \{f \in \Gamma(X) : \ f \text{ is real-valued and continuous on } X\},$$
$$\Gamma_+(X) := \{f \in \Gamma(X) : \ f \text{ is nonnegative-valued on } X\}.$$

Here, an extended real-valued function f is said to be nonnegative-valued if $f(X) \subseteq [0, +\infty]$ (so we allow that f takes the value $+\infty$).

Given $f : X \to \mathbb{R} \cup \{\pm\infty\}$. It is well known that f is convex (resp. lower semicontinuous) if and only epi f is a convex set (resp. lower semicontinuous). Moreover, if X is replaced by X^*, then f is weakly* lower semicontinuous if and only if epi f is weakly* closed. See [31] for the proof. Further, the lower semicontinuous regularization of the function f on X, denoted by $\overline{f}$, is defined as the function which satisfies the following:

$$\operatorname{epi}(\overline{f}) = \overline{\operatorname{epi} f}$$

(when X^* is considered in place to X, then we write $\overline{f}^{w^*}$ instead of $\overline{f}$ and consider the weakly* closure in the last set equality. The function $\overline{f}^{w^*}$ is called the weakly* lower semicontinuous regularization of f). Also, we use co f to denote the convex regularization of f, which is defined as the function such that

$$\operatorname{epi}(\operatorname{co} f) = \operatorname{co}(\operatorname{epi} f).$$

The lower semicontinuous convex regularization of f is then denoted by $\overline{\operatorname{co} f}$.

Next, we introduce the notion of conjugate function, which plays an important role in establishing duality results in convex optimization (as we will see in Chapter 3 and 4). Given $f : X \to \mathbb{R} \cup \{\pm\infty\}$. The conjugate function of f, denoted by f^*, is a function on X^* defined by

$$f^*(x^*) := \sup\{\langle x^*, z\rangle - f(z) : z \in X\} \quad \text{for any } x^* \in X^*.$$

Similarly, for a function $g : X^* \to \mathbb{R} \cup \{\pm\infty\}$, the conjugate function of g is a function on X defined by

$$g^*(x) := \sup\{\langle z^*, x\rangle - g(z^*) : z^* \in X^*\} \quad \text{for any } x \in X.$$

Below we give one simple but important example of conjugate function. Recall that given a set $A \subseteq X$, one defines the indicator function and support function of A, denoted by δ_A and σ_A, respectively, in the following way:

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\sigma_A(x^*) := \sup_{a \in A} \langle x^*, a \rangle \quad \text{for any } x^* \in X^*.$$

It is easy to verify by definitions that

$$\delta_A^*(x^*) = \sigma_A(x^*) \quad \text{for any } x^* \in X^*. \tag{2.2.1}$$

To end this section, we state the following equivalences. Let $f, g \in \Gamma(X)$. Then,

$$f \leq g \;\Leftrightarrow\; g^* \leq f^* \;\Leftrightarrow\; \operatorname{epi} f^* \subseteq \operatorname{epi} g^*, \tag{2.2.2}$$

where the backward implication of the first equivalence follows from the assumption that $f, g \in \Gamma(X)$ (see [31, Theorems 2.3.1(iii) and 2.3.3]), and the remaining implications follow easily from definitions.

2.3 On the properties of subdifferentials

In this section, we will state some properties of subdifferential of a convex function. Firstly, let us recall the following definition:

Definition 2.3.1 (cf. [25, 31]). *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function and $x \in X$. Then, the subdifferential of f at x, denoted by $\partial f(x)$, is defined by*

$$\partial f(x) := \begin{cases} \{x^* \in X^* : f(y) - f(x) - \langle x^*, y - x \rangle \geq 0, \ \forall y \in X\} & \text{if } x \in \operatorname{dom} f, \\ \emptyset & \text{otherwise.} \end{cases}$$

In fact, one has the following generalization of the concept of subdifferential, known as the ε-subdifferential:

Definition 2.3.2 (cf. [31]). *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function and $x \in X$. Fix $\varepsilon \geq 0$. Then, the ε-subdifferential of f at x, is defined by*

$$\partial f(x) := \begin{cases} \{x^* \in X^* : f(y) - f(x) + \varepsilon - \langle x^*, y - x\rangle \geq 0, \ \ \forall y \in X\} & \text{if } x \in \operatorname{dom} f, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is clear that $\partial f(x) = \partial_\varepsilon f(x)$ if $\varepsilon = 0$. So, one can regard the term "ε-subdifferential" as a generalization of subdifferential of a function. Among many useful and beautiful properties of subdifferential, we collect some of them which are helpful towards our analysis. Recall that given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x, h \in X$, the directional derivative of f at x in the direction h is defined by

$$d_+ f(x)(h) := \lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t}.$$

Proposition 2.3.1 (cf. [28, Theorem 4.1.3 and Proposition 4.1.6]). *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then, the following statements hold.*

(i) *If $x_0 \in \operatorname{dom} f$, then*

$$\partial f(x_0) = \{x^* \in X^* : \langle x^*, z\rangle \leq d_+ f(x_0)(z), \ \ \forall z \in X\}. \tag{2.3.1}$$

(ii) *Suppose that $x_0 \in \operatorname{int} \operatorname{dom} f$ and f is continuous at x_0. Then, $\partial f(x)$ is a nonempty weakly* compact convex set, and it holds that*

$$d_+ f(x_0)(z) = \max\{\langle x^*, z\rangle : x^* \in \partial f(x_0)\} \quad \textit{for any} \ \ z \in X. \tag{2.3.2}$$

Moreover, $d_+ f(x_0)(\cdot)$ is a continuous sublinear real-valued function on X.

2.4 On the properties of normal cones

First, we state the definition of normal cone, which is well-known in the literature (see [25, 28]).

Definition 2.4.1. *Let C be a subset of X and $x \in X$. Then, the normal cone of C at x, denoted by $N_C(x)$, is defined as*

$$N_C(x) := \begin{cases} \{x^* \in X^* : \langle x^*, c - x\rangle \leq 0, \quad \forall c \in C\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases}$$

The following proposition shows a relation between normal cone and indicator function of a set A in X.

Proposition 2.4.1. *Let A be a subset of X. Then, $\partial\delta_A(x) = N_A(x)$ for any $x \in X$.*

Proof. First, we observe that when $x \notin A$, $\delta_A(x) = +\infty$. Thus, it follows from the definitions that $\partial\delta_A(x) = \emptyset = N_A(x)$. On the other hand, fix $x \in A$. For any $x^* \in X^*$, one has the following equivalences:

$$\begin{aligned} x^* \in \partial\delta_A(x) &\Leftrightarrow \forall z \in Z, \ \ \delta_A(z) - \delta_A(x) - \langle x^*, z - x\rangle \geq 0 \\ &\Leftrightarrow \forall z \in Z, \ \ \delta_A(z) - \langle x^*, z - x\rangle \geq 0 \\ &\Leftrightarrow \forall z \in A, \ \ \delta_A(z) - \langle x^*, z - x\rangle \geq 0 \\ &\Leftrightarrow \forall z \in A, \ \ \langle x^*, z - x\rangle \leq 0 \\ &\Leftrightarrow x^* \in N_A(x). \end{aligned}$$

So, the result readily follows. □

Given a set $J \subseteq X$, let $|J|$ denote the cardinality of J. Recall that for any finite collection of non-empty sets $\{A_i : i \in I\}$ in X, the Minkowski sum of A_i's is defined by

$$\sum_{i\in I} A_i := \begin{cases} \{\sum_{i\in I} a_i : a_i \in A_i, \ \forall i \in I\}, & \text{if } I \neq \emptyset; \\ \{0\} & \text{if } I = \emptyset. \end{cases}$$

We next state a kind of definition for a sum of arbitrary (possibly infinite) collection of sets. Such definition was used in [18, 19] to make studies on problems in semi-infinite optimization, as we will see in Chapter 3.

Definition 2.4.2. *Let $\{A_i : i \in I\}$ be a collection of sets in X with $0 \in \bigcap_{i\in I} A_i$. Then,*

$$\sum_{i\in I} A_i := \begin{cases} \{\sum_{j\in J} a_j : a_j \in A_j, J \subseteq I, |J| < +\infty\}, & \text{if } I \neq \emptyset; \\ \{0\} & \text{if } I = \emptyset. \end{cases}$$

Remark 2.4.1. *Given a collection of sets $\{A_i : i \in I\}$ with $0 \in \bigcap_{i\in I} A_i$. It is easy to see that when I is a finite set, the Minkowski sum and the arbitrary sum defined in Definition 2.4.2 of $\{A_i : i \in I\}$ are the same. By abuse of notation, we use the same notation for these two kinds of sum. In case it is a must to distinguish them explicitly, we will clarify it.*

With the use of Definition 2.4.2, one can get the following set inclusion, which follows easily from definitions.

Proposition 2.4.2. *Let $\{A_i : i \in I\}$ be a collection of subsets of X with $A := \bigcap_{i\in I} A_i \neq \emptyset$. Then, $\sum_{i\in I} N_{A_i}(x) \subseteq N_A(x)$ for each $x \in A$.*

We will state a characterization of elements in ε-subdifferential. Before we do so, let us mention a well-known inequality. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function on X. By the definition of conjugate function, the following inequality is immediate:

$$f(x^*) + f(x) \geq \langle x^*, x\rangle \quad \text{for any } x \in X \text{ and } x^* \in X^*. \tag{2.4.1}$$

The above inequality is known as Young's inequality. The converse of this inequality is a special case of the following equivalences:

Proposition 2.4.3. *Let $f \in \Gamma(X)$. Then, for any $\varepsilon \geq 0$ and $x \in \operatorname{dom} f$,*

$$x^* \in \partial_\varepsilon f(x) \Leftrightarrow f^*(x^*) + f(x) \leq \langle x^*, x\rangle + \varepsilon \Leftrightarrow (x^*, \langle x^*, x\rangle - f(x) + \varepsilon) \in \operatorname{epi} f^*. \tag{2.4.2}$$

In particular, it holds that

$$x^* \in \partial f(x) \Leftrightarrow f^*(x^*) + f(x) = \langle x^*, x\rangle \Leftrightarrow (x^*, \langle x^*, x\rangle - f(x)) \in \operatorname{epi} f^*. \tag{2.4.3}$$

Moreover, one has that for any nonempty closed convex sets $A \subseteq X$ and $x \in A$,

$$x^* \in N_A(x) \;\Leftrightarrow\; \sigma_A(x^*) = \langle x^*, x\rangle \;\Leftrightarrow\; (x^*, \langle x^*, x\rangle) \in \operatorname{epi}\sigma_A. \qquad (2.4.4)$$

Proof. Note that (2.4.3) follows from (2.4.1) and (2.4.2) by taking $\varepsilon = 0$. To prove (2.4.2), let $\varepsilon \geq 0$ and $x \in \operatorname{dom} f$. Then, for any $x^* \in X^*$,

$$\begin{aligned} x^* \in \partial_\varepsilon f(x) \;&\Leftrightarrow\; \forall z \in X,\; f(z) - f(x) + \varepsilon - \langle x^*, z - x\rangle \geq 0 \\ &\Leftrightarrow\; \forall z \in X,\; \langle x^*, x\rangle - f(x) + \varepsilon \geq \langle x^*, z\rangle - f(z) \\ &\Leftrightarrow\; \langle x^*, x\rangle - f(x) + \varepsilon \geq f^*(x^*) \\ &\Leftrightarrow\; (x^*, \langle x^*, x\rangle - f(x) + \varepsilon) \in \operatorname{epi} f^*. \end{aligned}$$

So, (2.4.2) holds.

Next, let $A \subseteq X$ be a nonempty closed convex set and $x \in A$. Then, $\delta_A \in \Gamma(X)$. Noting that it always holds that $\sigma_A(x^*) \geq \langle x^*, x\rangle$ and $\partial\,\delta_A(x) = N_A(x)$ (by Proposition 2.4.1), (2.4.4) can be obtained by applying (2.4.3) to the case when $f = \delta_A$. □

We remark that the equality in (2.4.3) is known as the Young's equality.

Given a subspace Z of a normed linear space X and $x^* \in X^*$. The restriction of x^* on Z is denoted by $x^*|_Z$. Note that $x^*|_Z \in Z^*$.

Definition 2.4.3. *Let Z be a subspace of X and $A \subseteq X^*$. Then,*

$$A|_Z := \{x^*|_Z : x^* \in A\}.$$

The following result follows easily from definitions and the Hahn-Banach theorem.

Proposition 2.4.4. *Let Z be a subspace of X and $C \subseteq Z$ be a closed convex set. Then, for each $x \in C$,*

$$N_C(x)|_Z = N_C^Z(x),$$

where $N_C^Z(x) := \{z^* \in Z^* : \langle z^*, c - x\rangle \leq 0,\; \text{for any } c \in C\}$.

2.5 Some computation rules for conjugate functions

We will compute the conjugates of some special functions in this section. Before we do that, let us recall the notion of infimal convolution.

Definition 2.5.1 (cf. [30, 31]). *Let $f_1, \dots, f_m : X \to \mathbb{R} \cup \{+\infty\}$ be finitely many proper functions. Then, the infimal convolution of $f_1, \dots, f_m$, denoted by $f_1 \Box \cdots \Box f_m$, is a function on X defined by*

$$(f_1 \Box \cdots \Box f_m)(x) := \inf\{\sum_{i=1}^{m} f_i(x_i) : \sum_{i=1}^{m} x_i = x \quad \text{and} \quad x_i \in X \text{ for all } i\}, \quad \forall x \in X.$$

Moreover, the infimal convolution $f_1 \Box \cdots \Box f_m$ is said to be exact at $x \in X$ if $(f_1 \Box \cdots \Box f_m)(x) \in \mathbb{R}$ and the infimum is attained, that is, there exist some $x_1, \dots, x_m \in X$ with $\sum_{i=1}^{m} x_i = x$ such that

$$(f_1 \Box \cdots \Box f_m)(x) = \sum_{i=1}^{m} f_i(x_i).$$

The infimal convolution $f_1 \Box \cdots \Box f_m$ is said to be exact if it is exact at any $x \in X$ with $(f_1 \Box \cdots \Box f_m)(x) \in \mathbb{R}$.

The next two theorems are for computing the conjugate of infimal convolution of functions and that of the norm function.

Theorem 2.5.1. *Let $f_1, \dots, f_m : X \to \mathbb{R} \cup \{+\infty\}$ be finitely many proper functions. Then, $(f_1 \Box \cdots \Box f_m)^* = \sum_{i=1}^{m} f_i^*$ on X^*.*

Proof. Let $x^* \in X^*$. Then,

$$\begin{aligned}
(f_1\square\cdots\square f_m)^*(x^*) &= \sup_{z\in X}(\langle x^*, z\rangle - (f_1\square\cdots\square f_m)(z)) \\
&= \sup_{z\in X}(\langle x^*, z\rangle - \inf\{\sum_{i=1}^m f_i(z_i) : z_1,\dots,z_m \in X, \sum_{i=1}^m z_i = z\}) \\
&= \sup_{z\in X}\ \sup\{\sum_{i=1}^m(\langle x^*, z_i\rangle - f_i(z_i)) : z_1,\dots,z_m \in X, \sum_{i=1}^m z_i = z\} \\
&= \sup\{\sum_{i=1}^m(\langle x^*, z_i\rangle - f_i(z_i)) : z_1,\dots,z_m \in X\} \\
&= \sum_{i=1}^m(\sup\{\langle x^*, z_i\rangle - f_i(z_i) : z_i \in X\}) \\
&= \sum_{i=1}^m f_i^*(x^*).
\end{aligned}$$

□

Theorem 2.5.2. *Let $\|\cdot\| : X \to \mathbb{R}$ be the norm function on X. Then, $\|\cdot\|^* = \delta_{\mathbf{B_{X^*}}}$.*

Proof. Let $x^* \in X^*$. Then, $\|\cdot\|^*(x^*) = \sup_{z\in X}(\langle x^*, z\rangle - \|z\|)$. Suppose that $x^* \in \mathbf{B_{X^*}}$. It follows that $\langle x^*, z\rangle \le \|z\|$ for each $z \in X$. Hence,

$$0 = \langle x^*, 0\rangle - \|0\| \le \sup_{z\in X}(\langle x^*, z\rangle - \|z\|) \le 0,$$

which implies that $\|\cdot\|^*(x^*) = \sup_{z\in X}(\langle x^*, z\rangle - \|z\|) = 0$.

On the other hand, we turn to consider the case $x^* \notin \mathbf{B_{X^*}}$. Then, there exists some $z_0 \in X$ such that $\langle x^*, z_0\rangle > \|z_0\|$. Thus, given any $\lambda > 0$, one has

$$\sup_{z\in X}(\langle x^*, z\rangle - \|z\|) \ge \langle x^*, \lambda z_0\rangle - \|\lambda z_0\| = \lambda(\langle x^*, z_0\rangle - \|z_0\|).$$

By taking limit as $\lambda \to +\infty$, it follows that $\|\cdot\|^*(x^*) = \sup_{z\in X}(\langle x^*, z\rangle - \|z\|) = +\infty$. □

As an application of the two theorems that we have just proved, we consider the following example:

Example 2.5.1. Let $A \subseteq X$. Recall that the distance function d_A from the set A is defined by

$$d_A(x) := \inf_{a \in A} \|x - a\| \quad \text{for any } x \in X.$$

We will calculate the conjugate function of d_A. To do this, we first note that for any $x \in X$,

$$d_A(x) = \inf\{\|x - a\| : a \in A\} = \inf\{\|x - a\| + \delta_A(a) : a \in X\} = (\|\cdot\| \square \delta_A)(x).$$

Then, by (2.2.1), and Theorems 2.5.1 and 2.5.2, we get

$$d_A^* = (\delta_A \square \|\cdot\|)^* = \delta_A^* + \|\cdot\|^* = \sigma_A + \delta_{\mathbf{B_{X^*}}} \quad \text{on } X^*. \tag{2.5.1}$$

2.6 On the properties of epigraphs

The epigraph is crucial throughout our analysis. We will prove some of its important properties in this section. The first one is a result by Jeyakumar in [15]. It reveals a useful relationship between $\operatorname{epi} f^*$ and $\partial_\varepsilon f$.

Proposition 2.6.1 ([15, Lemma 2.1]). *Let $f \in \Gamma(X)$ and $x \in \operatorname{dom} f$. Then,*

$$\operatorname{epi} f^* = \bigcup_{\varepsilon \geq 0} \{(x^*, \langle x^*, x\rangle - f(x) + \varepsilon) : x^* \in \partial_\varepsilon f(x)\}. \tag{2.6.1}$$

Proof. Fix $x \in \operatorname{dom} f$. Let $(z^*, \alpha) \in \operatorname{epi} f^*$. Then, $f^*(z^*) \leq \alpha$. Take $\varepsilon := \alpha - \langle z^*, x\rangle + f(x)$. Thus, for any $z \in X$,

$$\langle z^*, x\rangle - f(x) + \varepsilon = \alpha \geq f^*(z^*) \geq \langle z^*, z\rangle - f(z),$$

which implies that

$$f(z) - f(x) + \varepsilon - \langle z^*, z - x\rangle \geq 0.$$

Hence, $\varepsilon \geq 0$ and $z^* \in \partial_\varepsilon f(x)$, and so (z^*, α) is in the RHS of (2.6.1).

Conversely, let (z^*, α) be in the RHS of (2.6.1). Then, there exists some $\varepsilon \geq 0$ such that $z^* \in \partial_\varepsilon f(x)$ and $\alpha = \langle z^*, x\rangle - f(x) + \varepsilon$. So, by (2.4.2), it follows that $(z^*, \alpha) \in \operatorname{epi} f^*$. Therefore, (2.6.1) holds. □

The following proposition is about the epigraph of infimal convolution of functions.

Proposition 2.6.2 (cf. [30, Theorem 2.2]). *Let $f_1, \dots, f_m : X \to \mathbb{R} \cup \{+\infty\}$ be finitely many proper functions. Then,*

(i) $\mathit{epi}_s\,(f_1 \square \cdots \square f_m) = \mathit{epi}_s\, f_1 + \cdots + \mathit{epi}_s\, f_m$.

(ii) $\mathit{epi}\,(f_1 \square \cdots \square f_m) \supseteq \mathit{epi}\, f_1 + \cdots + \mathit{epi}\, f_m$. *Moreover, the set equality holds if and only if the infimal convolution $f_1 \square \cdots \square f_m$ is exact.*

Proof. By definitions it is straight forward to check **(i)** and the first assertion in **(ii)**. Thus, we only need to prove the second assertion in **(ii)**. Suppose

$$\operatorname{epi}\,(f_1 \square \cdots \square f_m) \subseteq \operatorname{epi} f_1 + \cdots \operatorname{epi} f_m\,. \tag{2.6.2}$$

Let $x \in X$ be such that $(f_1 \square \cdots \square f_m)(x) \in \mathbb{R}$. Then, $(x, (f_1 \square \cdots \square f_m)(x)) \in \operatorname{epi}\,(f_1 \square \cdots \square f_m)$ and it follows from (2.6.2) that

$$(x, (f_1 \square \cdots \square f_m)(x)) = \sum_{i=1}^{m} (x_i, \alpha_i)$$

for some $(x_i, \alpha_i) \in \operatorname{epi} f_i$ (where $i \in \{1, \dots, m\}$). Hence,

$$\sum_{i=1}^{m} f_i(x_i) \leq \sum_{i=1}^{m} \alpha_i = (f_1 \square \cdots \square f_m)(x) \leq \sum_{i=1}^{m} f_i(x),$$

which implies $(f_1 \square \cdots \square f_m)(x) = \sum_{i=1}^{m} f_i(x)$, that is, $f_1 \square \cdots \square f_m$ is exact at x. So, we have proved that $f_1 \square \cdots \square f_m$ is exact.

Conversely, suppose that $f_1 \square \cdots \square f_m$ is exact. We have to show (2.6.2). Let $(x, \alpha) \in \operatorname{epi}\,(f_1 \square \cdots \square f_m)$. Consider the case when $(x, \alpha) \in \operatorname{epi}_s\,(f_1 \square \cdots \square f_m)$, it follows from **(i)** that

$$\operatorname{epi}_s\,(f_1 \square \cdots \square f_m) = \operatorname{epi}_s f_1 + \cdots + \operatorname{epi}_s f_m \subseteq \operatorname{epi} f_1 + \cdots + \operatorname{epi} f_m,$$

and so $(x, \alpha) \in \operatorname{epi} f_1 + \cdots + \operatorname{epi} f_m$. For the case when $(x, \alpha) \notin \operatorname{epi}_s\,(f_1 \square \cdots \square f_m)$, one has $(f_1 \square \cdots \square f_m)(x) = \alpha$. Thus, by the exactness, $(f_1 \square \cdots \square f_m)(x) =$

$\sum_{i=1}^m f_i(x)$ for some $x_1, \ldots, x_m \in X$ with $\sum_{i=1}^m x_i = x$. Hence,

$$(x, \alpha) = (x, (f_1 \square \cdots \square f_m)(x)) = \sum_{i=1}^m (x_i, f_i(x_i)) \in \operatorname{epi} f_1 + \cdots + \operatorname{epi} f_m.$$

Therefore, combining the two cases, one can see that (2.6.2) holds. This completes the proof. □

The next two theorems concern about the epigraph of infimal convolution of conjugate functions and that of the conjugate of sum function.

Theorem 2.6.3. *Let $f_1, \ldots, f_m$ be finitely many proper functions on X. Then,*

$$\overline{\operatorname{epi}(f_1^* \square \cdots \square f_m^*)}^{w^*} = \overline{\sum_{i=1}^m \operatorname{epi} f_i^*}^{w^*}. \tag{2.6.3}$$

Proof. By Proposition 2.6.2**(ii)**, we have $\operatorname{epi}(f_1^* \square \cdots \square f_m^*) \supseteq \sum_{i=1}^m \operatorname{epi} f_i^*$, which implies that

$$\overline{\operatorname{epi}(f_1^* \square \cdots \square f_m^*)}^{w^*} \supseteq \overline{\sum_{i=1}^m \operatorname{epi} f_i^*}^{w^*}.$$

In order to prove the converse set inclusion, it suffices to show that

$$\operatorname{epi}(f_1^* \square \cdots \square f_m^*) \subseteq \overline{\sum_{i=1}^m \operatorname{epi} f_i^*}^{w^*}. \tag{2.6.4}$$

Noting that, it is easy to verify (from definitions) that

$$(f_1^* \square \cdots \square f_m^*)(x^*) = \inf\{\beta \in \mathbb{R} : (x^*, \beta) \in \sum_{i=1}^m \operatorname{epi} f_i^*\} \quad \text{for any } x^* \in X^*. \tag{2.6.5}$$

Also, it is straight forward to check that

$$\{(z^*, \xi) \in X^* \times \mathbb{R} : \xi \geq \inf\{\beta \in \mathbb{R} : (z^*, \beta) \in \sum_{i=1}^m \operatorname{epi} f_i^*\}\} \subseteq \overline{\sum_{i=1}^m \operatorname{epi} f_i^*}^{w^*}. \tag{2.6.6}$$

This together with (2.6.5) show that (2.6.4) holds. □

Theorem 2.6.4. *Let $f_1, \ldots, f_m \in \Gamma(X)$. Then,*

$$(\sum_{i=1}^m f_i)^* = \overline{f_1^* \square \cdots \square f_m^*}^{w^*} \tag{2.6.7}$$

and

$$\operatorname{epi}(\sum_{i=1}^{m} f_i)^* = \overline{\operatorname{epi}(f_1^*\square\cdots\square f_m^*)}^{w^*}. \tag{2.6.8}$$

Proof. By [31, Theorem 2.3.3] and the assumption that each $f_i \in \Gamma(X)$, we have $f_i^{**} = f_i$ for all $i \in I$. Then, using Theorem 2.5.1, one has

$$(f_1^*\square\cdots\square f_m^*)^* = (f_1^{**} + \cdots + f_m^{**}) = f_1 + \cdots + f_m.$$

Taking conjugation on both sides, we get

$$\begin{aligned}(f_1 + \cdots + f_m)^* &= (f_1^*\square\cdots\square f_m^*)^{**} \\ &= (\overline{\operatorname{co}(f_1^*\square\cdots\square f_m^*)}^{w^*})^{**} \\ &= \overline{\operatorname{co}(f_1^*\square\cdots\square f_m^*)}^{w^*} \\ &= \overline{f_1^*\square\cdots\square f_m^*}^{w^*},\end{aligned}$$

where the second equality follows from [31, Theorem 2.3.1(iv)], the third equality comes from [31, Theorem 2.3.3], and the last equality is by the convexity of $f_1^*\square\cdots\square f_m^*$ (see [31, Theorem 2.3.1(ix)]). Therefore, This shows (2.6.7). In particular, (2.6.7) and the definition of $\overline{f_1^*\square\cdots\square f_m^*}^{w^*}$ imply that

$$\operatorname{epi}(f_1 + \cdots + f_m)^* = \overline{\operatorname{epi}(f_1^*\square\cdots\square f_m^*)}^{w^*}.$$

So, (2.6.8) holds. This completes the proof. □

Combining Theorems 2.6.3 and 2.6.4, we get the following theorem:

Theorem 2.6.5. *Let $f_1, \ldots, f_m \in \Gamma(X)$. Then,*

$$\operatorname{epi}(\sum_{i=1}^{m} f_i)^* = \overline{\sum_{i=1}^{m} \operatorname{epi} f_i^*}^{w^*}. \tag{2.6.9}$$

Proof. It immediately follows from (2.6.3) and (2.6.8). □

2.7 Set-valued analysis

We collect some results from set-valued analysis in this section. First, recall the following definition of lower semicontinuity of a set-valued mapping.

Definition 2.7.1 (cf. [2, 26]). *Let Q and Y be metric spaces. Let $F : Q \to 2^Y\backslash\{\emptyset\}$ be a set-valued mapping and $t_0 \in Q$. Then,*

(i) *The set-valued mapping F is said to be lower semicontinuous at t_0 if for any $y_0 \in F(t_0)$ and $\varepsilon > 0$, there exists some open neighborhood $U(t_0)$ of t_0 in Q such that for any $t \in U(t_0)$, $F(t) \cap B(y_0, \varepsilon) \neq \emptyset$.*

(ii) *The set-valued mapping F is said to be lower semicontinuous on Q if it is lower semicontinuous at any $t \in Q$.*

(iii) *The limit inferior of F at t_0, denoted by $\liminf_{t\to t_0} F(t)$, is defined by*

$$\liminf_{t\to t_0} F(t) := \{z \in Y : \forall w \in Q, \exists z_w \in F(w) \text{ s.t. } \lim_{w\to t_0} z_w = z\}.$$

(iv) *Let $\{C_n : n \in \mathbb{N}\}$ be a sequence of nonempty subsets of X. Then,*

$$\liminf_{n\to\infty} C_n := \{z \in Y : \forall n \in \mathbb{N}, \exists c_n \in C_n \text{ s.t. } \lim_{n\to\infty} c_n = z\}.$$

Remark 2.7.1. *Given a set-valued mapping $F : Q \to 2^Y\backslash\{\emptyset\}$ and $t_0 \in Q$. It is direct from definitions that for any sequence $\{t_n\}_{n\in\mathbb{N}}$ in Q with $t_n \longrightarrow t_0$ as $n \longrightarrow \infty$, one has*

$$\liminf_{t\to t_0} F(t) \subseteq \liminf_{n\to\infty} F(t_n).$$

The next theorem states an equivalent formulation of the lower semicontinuity of a set-valued mapping at a point.

Theorem 2.7.1 (cf. [18, Proposition 3.1]). *Let Q and Y be metric spaces. Let $F : Q \to 2^Y\backslash\{\emptyset\}$ be a set-valued mapping and $t_0 \in Q$. Then, the following statements are equivalent.*

(i) *F is lower semicontinuous at t_0,*

(ii) $F(t_0) \subseteq \liminf_{t\to t_0} F(t)$.

(iii) *For any $y_0 \in F(t_0)$, $\lim_{t\to t_0} d_{F(t)}(y_0) = 0$.*

Proof. We first prove **(i)** $\Rightarrow$ **(ii)**. Suppose that F is lower semicontinuous at t_0. Let $y_0 \in F(t_0)$. Given any $n \in \mathbb{N}$, it follows from the lower semicontinuity of F at t_0 that there exists some open neighborhood $U_n(t_0)$ of t_0 such that

$$B(y_0, \frac{1}{n}) \cap F(t) \neq \emptyset \quad \text{for any } t \in U_n(t_0).$$

Since Q is a metric space, one can assume that $U_n(t_0) \supseteq U_{n+1}(t_0)$ for any $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} U_n(t_0) = \{t_0\}$. Now, we construct $y_t \in F(t)$ according to the following selection:

$$\begin{aligned} & y_t \in F(t), && \text{if } t \notin U_1(t_0), \\ & y_t \in B(y_0, \frac{1}{n}) \cap F(t), && \text{if } t \in U_n(t_0) \backslash U_{n+1}(t_0), \\ & y_t = y_0, && \text{if } t = t_0. \end{aligned}$$

Then, $\lim_{t\to t_0} y_t = y_0$. So, $y_0 \in \liminf_{t\to t_0} F(t)$.

Now we turn to prove **(ii)** $\Rightarrow$ **(iii)**. Fix $y_0 \in F(t_0)$. By **(ii)**, one can pick some $\{y_t\}_{t\in Q}$ such that $y_t \in F(t)$ and $\lim_{t\to t_0} y_t = y_0$. Note that

$$0 \le d_{F(t)}(y_0) \le d(y_0, y_t).$$

As $\lim_{t\to t_0} y_t = y_0$, one has $\lim_{t\to t_0} d_{F(t)}(y_0) = 0$.

Finally, we prove the implication **(iii)** $\Rightarrow$ **(i)**. Let $y_0 \in F(t_0)$ and $\varepsilon > 0$. By **(iii)**, there exists some open neighborhood $U(t_0)$ of t_0 such that for any $t \in U(t_0)$, $d_{F(t)}(y_0) < \varepsilon$. It is clear that $d_{F(t)}(y_0) < \varepsilon$ implies $B(y_0, \varepsilon) \cap F(t) \neq \emptyset$. Thus, F is lower semicontinuous at t_0 and so **(i)** holds. This completes the proof. $\square$

The following proposition is about set-valued mapping that is convex-valued.

Proposition 2.7.2. *Let Q be a metric space and X be a finite dimensional normed linear space. Let $F : Q \to 2^Y \setminus \{\emptyset\}$ be a set-valued mapping such that $F(t)$ is convex for every $t \in Q$. Fix $t_0 \in Q$ and let $D \subseteq Y$ be a nonempty compact subset of Y such that $D \subseteq \operatorname{int}(\liminf_{t\to t_0} F(t))$. Then, there exists some neighborhood of t_0, denoted by $U(t_0)$, such that $D \subseteq \operatorname{int} F(t)$ for any $t \in U(t_0)$.*

Proof. Suppose to the contrary that for any neighborhood $U(t_0)$ of t_0, there exists some $t_u \in U(t_0)$ such that $D \not\subseteq \operatorname{int} F(t_u)$. Then, one can pick a sequence $\{t_n\}_{n\in\mathbb{N}}$ in X such that

$$t_n \in B(t_0, 1/n) \quad \text{and} \quad D \not\subseteq \operatorname{int} F(t_n) \quad \text{for each } n \in \mathbb{N}. \tag{2.7.1}$$

Then, $\lim_{n\to\infty} t_n = t_0$, which implies that $\liminf_{t\to t_0} F(t) \subseteq \liminf_{n\to\infty} F(t_n)$, thanks to Remark 2.7.1. Hence,

$$D \subseteq \operatorname{int}(\liminf_{t\to t_0} F(t)) \subseteq \operatorname{int}(\liminf_{n\to\infty} F(t_n)). \tag{2.7.2}$$

Now, by [26, Proposition 4.15], (2.7.2) implies that there exists some $N_0 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq N_0$, $D \subseteq \operatorname{int} F(t_n)$, which contradicts to (2.7.1). $\square$

2.8 Weakly* sum of sets in dual spaces

In this section, we summarize some properties of the weakly* sum of a collection of sets in X^* (see definitions below), which is in general different from the sum that was defined in Definition 2.4.2.

Let I be an arbitrary index set. Let $\mathcal{F}(I)$ be the collection of all finite subsets of I. Then, under the set inclusion relation, $\mathcal{F}(I)$ becomes a directed set. Given a collection of extended real-valued numbers $\{a_i : i \in I\}$ in $\mathbb{R} \cup \{+\infty\}$, one defines the sum of a_i's, denoted by $\sum_{i\in I} a_i$, as the limit of the net $\{\sum_{j\in J} a_j\}_{J\in\mathcal{F}(I)}$, that is

$$\sum_{i\in I} a_i = \lim_{J\in\mathcal{F}(I)} \sum_{j\in J} a_j,$$

provided that the limit on the right-hand side exists as a (extended real-valued) number in $\mathbb{R} \cup \{+\infty\}$. For convenience, we define $\sum_{j\in J} a_j = 0$ for $J = \emptyset$. Note that $\sum_{i\in I} a_i$ coincides with the ordinary sum of a_i's if I is a finite set. Also, it is easy to see that for any collection $\{a_i : i \in I\} \subseteq [0, \infty]$, one has that $\sum_{i\in I} a_i$ exists in $\mathbb{R} \cup \{+\infty\}$ and

$$\sum_{i\in I} a_i = \sup_{J\in\mathcal{F}(I)} \sum_{j\in J} a_j. \tag{2.8.1}$$

Below we state some remarks concerning about the properties of $\sum_{i\in I} a_i$ that we have just defined, which follows from direct checking of definitions:

Remark 2.8.1. *Let $\{a_i : i \in I\} \subseteq \mathbb{R} \cup \{+\infty\}$. Suppose that $\sum_{i\in I} a_i$ exists in $\mathbb{R}$. Then $a_i \in \mathbb{R}$ for all $i \in I$. Also, given any $J \in \mathcal{F}(I)$, $\sum_{i\in I\setminus J} a_i$ exists in $\mathbb{R}$ and*

$$\sum_{i\in I\setminus J} a_i = \sum_{i\in I} a_i - \sum_{j\in J} a_j. \tag{2.8.2}$$

Remark 2.8.2. *Let $\{a_i, b_i, c_i : i \in I\} \subseteq \mathbb{R}$ be such that $a_i \le b_i \le c_i$ for all $i \in I$. If both of $\sum_{i\in I} a_i$ and $\sum_{i\in I} c_i$ exist in $\mathbb{R}$, then it follows that $\sum_{i\in I} b_i$ exists in $\mathbb{R}$. (In fact, by the assumption on $\{a_i, b_i, c_i : i \in I\}$, we have*

$$0 \le b_i - a_i \le c_i - a_i \quad \text{for all } i \in I.$$

Then, it follows from (2.8.1) and the existence of $\sum_{i\in I} a_i$ and $\sum_{i\in I} c_i$ in $\mathbb{R}$ that $\sum_{i\in I}(b_i - a_i)$ exists in $\mathbb{R}$. Therefore, $\sum_{i\in I} b_i$ $(= \sum_{i\in I} a_i + \sum_{i\in I}(b_i - a_i))$ exists in $\mathbb{R}$.)

Let $\{f_i : i \in I\}$ be a collection of functions on X with values in $\mathbb{R} \cup \{+\infty\}$. Let $D_f := \{x \in X : \sum_{i\in I} f_i(x) \in \mathbb{R} \cup \{+\infty\}\}$. Then, we can define the sum function f of f_i's on D_f by

$$f(x) := \sum_{i\in I} f_i(x) \quad \text{for any } x \in D_f.$$

In particular, when $\{f_i : i \in I\} \subseteq \Gamma_+(X)$, one sees that $D_f = X$ and

$$f(x) = \sup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in I} f_j(x) \quad \text{for any } x \in X, \tag{2.8.3}$$

thanks to (2.8.1).

We are ready to give the definition of weakly* sum of elements in X^*. Let $x^* \in X^*$ and $\{x_i^* : i \in I\}$ be a collection of elements in X^*. Then, we say that x^* is the weakly* sum of x_i^*'s if

$$\langle x^*, x\rangle = \sum_{i\in I}\langle x_i^*, x\rangle \quad \text{for any } x \in X.$$

In this case, we write $x^* = \sum_{i\in I}^* x_i^*$. Next, let $\{A_i : i \in I\}$ be a collection of subsets in X^*. We define the weakly* sum of A_i's by

$$\sum_{i\in I}^* A_i := \{x^* \in X^* : \exists\{x_i^*\}_{i\in I} \subseteq X^* \text{ s.t. } x_i^* \in A_i \ \forall i \in I, \text{ and } x^* = \sum_{i\in I}^* x_i^*\}.$$

Moreover, the collection $\{A_i : i \in I\}$ is said to be weakly* summable if for any $a_i^* \in A_i$ $(i \in I)$, there exists some $a^* \in X^*$ such that $a^* = \sum_{i\in I}^* a_i^*$.

Remark 2.8.3. *It is straight forward to check that $\sum_{i\in I}^* A_i = \sum_{i\in I} A_i$ if I is finite. Also, the set $\sum_{i\in I}^* A_i$ is convex if each A_i is convex.*

The next proposition gives a sufficient condition for the weakly* closedness of the weakly* sum of sets in X^*. It is originated from the proof of [32, Proposition 2.3]. We give the statement in a more general form and provide a proof.

Proposition 2.8.1. *Let X be a real normed linear space. Let $\{A_i : i \in \mathbb{N}\}$ be a collection of weakly* compact set in X^*. Suppose that there exist some collection of real-valued functions $\{g_i : i \in \mathbb{N}\}$ and $\{h_i : i \in \mathbb{N}\}$ on X such that for any $x \in X$ and $x_i^* \in A_i$ $(i \in \mathbb{N})$,*

$$\sum_{i\in\mathbb{N}} g_i(x) \quad \textit{and} \quad \sum_{i\in\mathbb{N}} h_i(x) \quad \textit{exist in } \mathbb{R} \tag{2.8.4}$$

and

$$g_i(x) \le \langle x_i^*, x\rangle \le h_i(x). \tag{2.8.5}$$

Then, $\sum_{i\in\mathbb{N}}^* A_i$ is weakly closed.*

Proof. To prove the weakly* closedness of $\sum_{i\in\mathbb{N}}^* A_i$, it suffices to show that

$$\overline{\sum_{i\in\mathbb{N}}^* A_i}^{w^*} \subseteq \sum_{i\in\mathbb{N}}^* A_i. \tag{2.8.6}$$

Let $z^* \in \overline{\sum_{i\in\mathbb{N}}^* A_i}^{w^*}$. Then, there exist some directed set $\mathbb{D}$ and $\{a_\alpha^*\}_{\alpha\in\mathbb{D}} \subseteq \sum_{i\in\mathbb{N}}^* A_i$ such that $a_\alpha^* \longrightarrow^{w^*} z^*$. For each $\alpha \in \mathbb{D}$, write $a_\alpha^* = \sum_{i\in\mathbb{N}}^* x_{i,\alpha}^*$, where $x_{i,\alpha}^* \in A_i$ $(i \in \mathbb{N})$. Since $\{x_{1,\alpha}^*\}_{\alpha\in\mathbb{D}} \subseteq A_1$, it follows from the weakly* compactness of A_1 that there exist some directed set $\mathbb{D}_1 \subseteq \mathbb{D}$ and subnet $\{x_{1,\alpha}^*\}_{\alpha\in\mathbb{D}_1}$ of $\{x_{1,\alpha}^*\}_{\alpha\in\mathbb{D}}$ such that

$$x_{1,\alpha}^* \longrightarrow^{w^*} z_1^*$$

for some $z_1^* \in A_1$. Noting that $\{x_{2,\alpha}^*\}_{\alpha\in\mathbb{D}_1} \subseteq A_2$, the weakly* compactness of A_2 implies that we can find some directed set $\mathbb{D}_2 \subseteq \mathbb{D}_1$ and subnet $\{x_{2,\alpha}^*\}_{\alpha\in\mathbb{D}_2}$ of $\{x_{2,\alpha}^*\}_{\alpha\in\mathbb{D}_1}$ such that

$$x_{2,\alpha}^* \longrightarrow^{w^*} z_2^*$$

for some $z_2^* \in A_2$. Inductively, one can find a sequence of nets $\{x_{i,\alpha}^*\}_{\alpha\in\mathbb{D}_i}$ (where $i \in \mathbb{N}$) such that the following statements hold:

(a) The net $\{x_{1,\alpha}^*\}_{\alpha\in\mathbb{D}_1}$ is a subnet of $\{x_{1,\alpha}^*\}_{\alpha\in\mathbb{D}}$.

(b) For each $i \in \mathbb{N}$, $\{x_{i,\alpha}^*\}_{\alpha\in\mathbb{D}_{i+1}}$ is a subnet of $\{x_{i,\alpha}^*\}_{\alpha\in\mathbb{D}_i}$.

(c) For each $i \in \mathbb{N}$, $\{x_{i,\alpha}^*\}_{\alpha\in\mathbb{D}_i}$ is weakly* convergent to some $z_i^* \in A_i$.

We next show that

$$z^* = \sum_{i\in\mathbb{N}}^* z_i^*. \tag{2.8.7}$$

Let $x \in X$. By **(c)**, (2.8.4), (2.8.5) and Remark 2.8.2, one sees that $\sum_{i\in\mathbb{N}}\langle z_i^*, x\rangle$ exists in $\mathbb{R}$. Now, fix $\varepsilon > 0$. Again, by (2.8.4), there exist some $J_0 \subseteq \mathbb{N}$ with $|J_0| < \infty$ such that for any $J \subseteq \mathbb{N}$ with $|J| < \infty$ and $J_0 \subseteq J$ such that

$$\left|\sum_{i\in\mathbb{N}\setminus J} g_i(x)\right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left|\sum_{i\in\mathbb{N}\setminus J} h_i(x)\right| < \frac{\varepsilon}{3}.$$

Combining the two inequalities above with Remark 2.8.2 and (2.8.5) (as $x^*_{i,\alpha} \in A_i$ for any $i \in \mathbb{N}$ and $\alpha \in \mathbb{D}$), it follows that for any $\alpha \in \mathbb{D}$, $\sum_{i\in\mathbb{N}\setminus J}\langle x^*_{i,\alpha}, x\rangle$ exists in $\mathbb{R}$ and

$$-\frac{\varepsilon}{3} < \sum_{i\in\mathbb{N}\setminus J} g_i(x) \leq \sum_{i\in\mathbb{N}\setminus J} \langle x^*_{i,\alpha}, x\rangle \leq \sum_{i\in\mathbb{N}\setminus J} h_i(x) < \frac{\varepsilon}{3},$$

Thus,

$$\left|\sum_{i\in\mathbb{N}\setminus J} \langle x^*_{i,\alpha}, x\rangle\right| < \frac{\varepsilon}{3} \quad \text{for any } \alpha \in \mathbb{D}. \tag{2.8.8}$$

Fix $J \subseteq \mathbb{N}$ with $|J| < \infty$ and $J_0 \subseteq J$. Let $\overline{j} := \max\{j : j \in J\}$. By **(a)**, **(b)** and the fact that the net $\{\sum^*_{i\in\mathbb{N}} x^*_{i,\alpha}\}_{\alpha\in\mathbb{D}}$ is weakly* convergent to z^*, one has that $\{\sum^*_{i\in\mathbb{N}} x^*_{i,\alpha}\}_{\alpha\in\mathbb{D}_{\overline{j}}}$ is weakly* convergent to z^*. Also, for any $j \in J$, the net $\{x_{j,\alpha}\}_{\alpha\in\mathbb{D}_{\overline{j}}}$ is weakly* convergent to z^*_j. Hence, one can pick some $\overline{\alpha} \in \mathbb{D}_{\overline{j}}$ such that

$$\left|\sum_{i\in\mathbb{N}}\langle x^*_{i,\overline{\alpha}}, x\rangle - \langle z^*, x\rangle\right| < \frac{\varepsilon}{3}, \tag{2.8.9}$$

and

$$|\langle x^*_{j,\overline{\alpha}} - z^*_j, x\rangle| < \frac{\varepsilon}{3(|J|+1)} \quad \text{for any } j \in J. \tag{2.8.10}$$

Combining (2.8.8), (2.8.9) and (2.8.10), one has

$$\begin{aligned}
&\left|\sum_{j\in J}\langle z^*_j, x\rangle - \langle z^*, x\rangle\right| \\
&\leq \left|\sum_{j\in J}\langle z^*_j, x\rangle - \sum_{i\in\mathbb{N}}\langle x^*_{i,\overline{\alpha}}, x\rangle\right| + \left|\sum_{i\in\mathbb{N}}\langle x^*_{i,\overline{\alpha}}, x\rangle - \langle z^*, x\rangle\right| \\
&\leq \left|\sum_{i\in\mathbb{N}\setminus J}\langle x^*_{i,\overline{\alpha}}, x\rangle\right| + \sum_{j\in J}|\langle x^*_{j,\overline{\alpha}} - z^*_j, x\rangle| + \left|\sum_{i\in\mathbb{N}}\langle x^*_{i,\overline{\alpha}}, x\rangle - \langle z^*, x\rangle\right| \\
&< \frac{\varepsilon}{3} + |J|\cdot\frac{\varepsilon}{3(|J|+1)} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily chosen, we have proved that $\sum_{i\in\mathbb{N}}\langle z^*_i, x\rangle = \langle z^*, x\rangle$. So, (2.8.7) holds, which in turn implies that $z^* \in \sum^*_{i\in\mathbb{N}} A_i$, thanks to **(c)**. Therefore, (2.8.6) is seen to hold. □

The next theorem will be useful in Chapter 4. Moreover, the theorem itself is an important result: It generalizes the well-known subdifferential sum rule for finitely many continuous convex functions to the case when infinitely many of them are considered. For the finite case, one may consult [31] or Theorem 3.4.3 in Chapter 3 for more details.

Theorem 2.8.2 ([32, Proposition 2.3]). *Let $\{f, f_i : i \in I\} \subseteq \Gamma_c(X)$ be such that*

$$f(x) = \sum_{i \in I} f_i(x) \quad \text{for any } x \in X. \tag{2.8.11}$$

Then, for any $x \in X$, the collection $\{\partial f_i(x) : i \in I\}$ is weakly summable and*

$$\partial f(x) = \overline{\sum_{i\in I}^{*} \partial f_i(x)}^{w^*}. \tag{2.8.12}$$

Moreover, if assume in addition that I is countable, then, for each $x \in X$, $\sum_{i\in I}^{*} \partial f_i(x)$ is weakly closed and so*

$$\partial f(x) = \sum_{i\in I}^{*} \partial f_i(x) \tag{2.8.13}$$

Proof. Let $x, h \in X$. Then, by [28, Theorem 4.1.3(a)], it follows that, for each $i \in I$, the function $t \longmapsto \frac{f_i(x+th)-f_i(x)}{t}$ is increasing on $\mathbb{R}\backslash\{0\}$ and hence the directional derivative $d_+ f_i(x)(h)$ exists, which satisfies the following inequalities:

$$f_i(x) - f_i(x-h) \leq d_+ f_i(x)(h) = \inf_{t>0} \left(\frac{f_i(x+th) - f_i(x)}{t} \right) \leq f_i(x+h) - f_i(x). \tag{2.8.14}$$

Thus, by (2.8.11) and Remark 2.8.2, $\sum_{i\in I} d_+ f_i(x)(h)$ exists in $\mathbb{R}$. Also, we remark that

$$t \longmapsto \frac{f_i(x+th) - f_i(x)}{t} - d_+ f_i(x)(h) \text{ is non-negative and increasing on } (0, +\infty). \tag{2.8.15}$$

We now show that

$$d_+ f(x)(h) = \sum_{i \in I} d_+ f_i(x)(h). \tag{2.8.16}$$

Note that for any $t > 0$, we have

$$\begin{aligned}\sum_{i\in I}\left(\frac{f_i(x+th)-f_i(x)}{t}-d_+f_i(x)(h)\right) &= \left|\sum_{i\in I}\left(\frac{f_i(x+th)-f_i(x)}{t}-d_+f_i(x)(h)\right)\right| \\ &= \left|\frac{f(x+th)-f(x)}{t}-\sum_{i\in I}d_+f_i(x)(h)\right|,\end{aligned}$$

where the first equality follows (2.8.15), while (2.8.11) gives the second equality. So, for any $t > 0$, the sum $\sum_{i\in I}(\frac{f_i(x+th)-f_i(x)}{t} - d_+f_i(x)(h))$ exists in $\mathbb{R}$. Let $\varepsilon > 0$ and fix some $t_0 > 0$. Since $\sum_{i\in I}(\frac{f_i(x+t_0h)-f_i(x)}{t_0} - d_+f_i(x)(h))$ exists in $\mathbb{R}$, one can find some $I_0 \subseteq I$ with $|I_0| < +\infty$ such that

$$\sum_{i\in I\setminus I_0}\left(\frac{f_i(x+t_0h)-f_i(x)}{t_0}-d_+f_i(x)(h)\right) < \frac{\varepsilon}{2}.$$

This and (2.8.15) implies that for any $t \in (0, t_0]$,

$$\sum_{i\in I\setminus I_0}\left(\frac{f_i(x+th)-f_i(x)}{t}-d_+f_i(x)(h)\right) < \frac{\varepsilon}{2}. \tag{2.8.17}$$

On the other hand, since $|I_0| < +\infty$ and $d_+f_i(x)(h) = \inf_{t>0}(\frac{f_i(x+th)-f_i(x)}{t})$, one can pick some $\delta \in (0, t_0)$ such that

$$\sum_{i\in I_0}\left(\frac{f_i(x+th)-f_i(x)}{t}-d_+f_i(x)(h)\right) < \frac{\varepsilon}{2} \quad \text{for any } t\in(0,\delta). \tag{2.8.18}$$

Using (2.8.17) and (2.8.18), we have, for any $t \in (0, \delta)$,

$$\begin{aligned}&\left|\frac{f(x+th)-f(x)}{t}-\sum_{i\in I}d_+f_i(x)(h)\right| \\ &= \sum_{i\in I}\left(\frac{f_i(x+th)-f_i(x)}{t}-d_+f_i(x)(h)\right) \\ &= \sum_{i\in I_0}\left(\frac{f_i(x+th)-f_i(x)}{t}-d_+f_i(x)(h)\right) + \sum_{i\in I\setminus I_0}\left(\frac{f_i(x+th)-f_i(x)}{t}-d_+f_i(x)(h)\right) \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily chosen, we have proved that $\lim_{t\to 0+} \frac{f(x+th)-f(x)}{t} = \sum_{i\in I} d_+f_i(x)(h)$, that is, (2.8.16) holds for any $x, h \in X$.

We next show that the collection $\{\partial f_i(x) : i \in I\}$ is weakly* summable. By the continuity of f_i and Proposition 2.3.1(**i**), we see that $\partial f_i(x) \neq \emptyset$ for each $i \in I$. Now, for each $i \in I$, let $x_i^* \in \partial f_i(x)$. It follows from (2.3.1) that

$$-d_+f_i(x)(-z) \le \langle x_i^*, z\rangle \le d_+f_i(x)(z) \quad \text{for any } z \in X. \tag{2.8.19}$$

By (2.8.16) (applied to the case when $h = z$ or $h = -z$) and Remark 2.8.2, it follows that $\sum_{i\in I}\langle x_i^*, z\rangle$ exists in $\mathbb{R}$ and

$$-d_+f(x)(-z) \le -\sum_{i\in I} d_+f_i(x)(-z) \le \sum_{i\in I}\langle x_i^*, z\rangle \le \sum_{i\in I} d_+f_i(x)(z) = d_+f(x)(z). \tag{2.8.20}$$

Define the function $x^* : X \to \mathbb{R}$ by

$$x^*(z) = \sum_{i\in I}\langle x_i^*, z\rangle \quad \text{for each } z \in X.$$

It is clear that x^* is linear. Also, by (2.8.20), one has

$$|x^*(z)| = |\sum_{i\in I}\langle x_i^*, z\rangle| \le \max\{|d_+f(x)(z)|, |d_+f(x)(-z)|\} \quad \text{for any } z \in X. \tag{2.8.21}$$

By the assumption that $f \in \Gamma_c(X)$ and Propositon 2.3.1(**ii**), one can see that $d_+f(x)(\cdot)$ is a continuous sublinear functional on X. Thus, x^* is continuous on X, thanks to (2.8.21) and the linearity of x^*. Hence, $x^* \in X^*$, and so $x^* = \sum_{i\in I}^* x_i^*$. Therefore, we have shown that $\{\partial f_i(x) : i \in I\}$ is weakly* summable.

We turn to prove that (2.8.12) holds. Let $z^* \in \sum_{i\in I}^* \partial f_i(x)$. Then, by (2.8.20), we see that $\langle z^*, z\rangle \le d_+f(x)(z)$ for any $z \in X$. Hence, it follows from (2.3.1) that $z^* \in \partial f(x)$. Thus, the set inclusion $\sum_{i\in I}^* \partial f_i(x) \subseteq \partial f(x)$ holds. Since the set $\partial f(x)$ is weakly* closed (see Proposition 2.3.1(**ii**)), one has

$$\overline{\sum_{i\in I}^* \partial f_i(x)}^{w^*} \subseteq \partial f(x).$$

It remains to show the set inclusion

$$\partial f(x) \subseteq \overline{\sum_{i\in I}^{*} \partial f_i(x)}^{w^*}. \tag{2.8.22}$$

Suppose to the contrary that there exists some $x_0^* \in X^*$ such that $x_0^* \in \partial f(x)$ but $x_0^* \notin \overline{\sum_{i\in I}^{*} \partial f_i(x)}^{w^*}$. Since $\overline{\sum_{i\in I}^{*} \partial f_i(x)}^{w^*}$ is weakly* closed and convex (by Remark 2.8.3), one can use the the separation theorem to find some $y_0 \in X$ such that

$$\sup\{\langle y^*, y_0\rangle : y^* \in \overline{\sum_{i\in I}^{*} \partial f_i(x)}^{w^*}\} < \langle x_0^*, y_0\rangle. \tag{2.8.23}$$

Noting that, since each f_i is continuous, it follows from (2.3.2) that there exists some $z_i^* \in \partial f_i(x)$ such that $\langle z_i^*, y_0\rangle = d_+ f_i(x)(y_0)$. Let $a^* := \sum_{i\in I}^{*} z_i^*$ (such a^* is well-defined, thanks to the weakly* summability of the collection $\{\partial f_i(x) : i \in I\}$, which we have just proved). Thus, $a^* \in \sum_{i\in I}^{*} \partial f_i(x)$. Using (2.8.16), we get

$$\langle a^*, y_0\rangle = \sum_{i\in I} \langle z_i^*, y_0\rangle = \sum_{i\in I} d_+ f_i(x)(y_0) = d_+ f(x)(y_0).$$

Consequently,

$$\begin{aligned} \langle x_0^*, y_0\rangle &\leq d_+ f(x)(y_0) \\ &= \langle a^*, y_0\rangle \\ &\leq \sup\{\langle y^*, y_0\rangle : y^* \in \overline{\sum_{i\in I}^{*} \partial f_i(x)}^{w^*}\}, \end{aligned}$$

where the first inequality follows from (2.3.1) and the fact that $x_0^* \in \partial f(x)$. This contradicts (2.8.23). Now, this contradiction shows us that (2.8.22) holds. Therefore, (2.8.12) is established.

Finally, assume in addition that I is countable. We will show that (2.8.13) holds. Without loss of generality, assume that $I = \mathbb{N}$. Fix $x \in X$. By the result that we have just proved, we have

$$\partial f(x) = \overline{\sum_{i\in \mathbb{N}}^{*} \partial f_i(x)}^{w^*}.$$

Thus, it suffices for us to show that $\sum_{i\in\mathbb{N}}^{*}\partial f_i(x)$ is weakly* closed. Note that, for each $i \in I$, $\partial f_i(x)$ is weakly* compact, thanks to Proposition 2.3.1**(ii)** and the continuity of f_i. Then, it follows from Proposition 2.8.1 (applied to the case when $A_i := \partial f_i(x)$, $g_i(\cdot) := -d_+ f_i(x)(-\cdot)$ and $h_i(\cdot) := d_+ f_i(x)(\cdot)$ for all $i \in I$) that $\sum_{i\in I}^{*}\partial f_i(x)$ is weakly* closed. Since $x \in X$ is arbitrary, we have proved that (2.8.13) is valid. This completes the proof. $\square$

Chapter 3

Sum of Epigraph Constraint Qualification (SECQ)

3.1 Introduction

In [9], Burachik and Jeyakumar utilized the epigraphs of support functions to provide a new sufficient condition for the strong conical hull intersection propery (the strong CHIP) of two closed convex sets in a Banach space. Their result is stated as follows:

Theorem 3.1.1 ([9]). *Let X be a Banach space. Let C, D be two closed convex sets in X with $C \cap D \neq \emptyset$. Suppose that*

$$\operatorname{epi} \sigma_{C \cap D} = \operatorname{epi} \sigma_C + \operatorname{epi} \sigma_D. \tag{3.1.1}$$

Then C and D satisfies the strong CHIP.

In their paper, they also provided some examples to show that the condition (3.1.1) is weaker than some classical interior-point type conditions which are essential for the validity of the strong CHIP. Since the strong CHIP is crucial for some duality results such as the dual formulation of best approximation problems (see [12, 13, 14] for more details), the importance of the condition (3.1.1) in the duality theory of convex optimization is shown by Theorem 3.1.1. In [17, 18], under the setting of general normed linear space, Li and Ng extended the concept of strong CHIP to cover the case when an arbitrary system of closed convex sets is considered. The properties and consequences of this generalized notion of strong CHIP were investigated. In particular, they utilized the strong CHIP to study some general systems of infinite convex inequalities. This shows that the strong CHIP is useful in semi-infinite convex optimization theory. Concerning the result of Theorem 3.1.1 by Jeyakumar et. al. and the extended notion of strong CHIP by Li and Ng, it is natural to ask whether we can extend the condition (3.1.1) to be defined for arbitrary system of closed convex sets, under the setting of normed linear spaces.

In [19], Li, Ng and Pong defined a new type of constraint qualification, known as the sum of epigraphs constraint qualification (the SECQ), for an arbitrary system of closed convex sets under the setting of normed linear space. According to the definition of the SECQ (which we will see in the next section), it reduces to (3.1.1) when the system consists of two closed convex sets. They proved that the SECQ still serve as sufficient condition for the strong CHIP, which generalized Theorem 3.1.1. In view of the application in semi-infinite optimization theory, it is useful to study the properties of the SECQ and its relationship with other types of constraint qualification.

In the chapter, we give a survey on the result by Li, Ng and Pong in [19], which mainly concerns the SECQ. In the next section, we will first give the definition of the SECQ and prove some of its simple but useful properties. After that, the relationship between the SECQ and the strong CHIP will be investigated. We will show that the SECQ serves as a sufficient condition for the strong CHIP. While the converse implication holds under some additional assumptions, we also study conditions which can ensure these assumptions. Next, we study the relationship between the SECQ and the linear regularity. In particular, by considering the epigraph of conjugate of distance function, a new characterization of the linear regularity will be given. In the last section, some interior-point type sufficient conditions for the SECQ will be shown.

3.2 Definition of the SECQ and its basic properties

Throughout this chapter, unless otherwise stated, let X be a real normed linear space. We will study the basic properties of a constraint qualification, known as the sum of epigraph constraint qualification (the SECQ). To begin with, let us state the definition of the SECQ. Here, the definition of infinite sum of sets is given in Definition 2.4.2.

Definition 3.2.1. *Let X be a normed linear space. Let $\{C_i : i \in I\}$ be a collection of sets in X with $\bigcap_{i\in I} C_i \neq \emptyset$. Then, we say that $\{C_i : i \in I\}$ satisfies the sum of epigraph constraint qualification (SECQ) if*

$$\operatorname{epi} \sigma_{\bigcap_{i\in I} C_i} = \sum_{i\in I} \operatorname{epi} \sigma_{C_i}. \tag{3.2.1}$$

As we can see, given a collection of convex sets $\{C_i : i \in I\}$, the SECQ concerns about the decompostion of epigraph of $\sigma_{\bigcap_{i\in I} C_i}$ into a sum of each epigraph of σ_{C_i}. Historically, such property was first studied by Burachik and Jeyakumar in [9] for the case when $|I| = 2$. After that, in [19], Li, Ng and Pong stated the definition above, where $|I|$ is allowed to be greater than two, or infinite.

Below we give an example of a system of closed convex sets which satisfies the SECQ.

Example 3.2.1. Let $X = \mathbb{R}$. Let $C := [0, 1]$ and $D := [-1, 0]$. Then, $C \cap D =$

$\{0\}$. By direct computation, one gets the following set equalities:

$$\begin{aligned}\operatorname{epi}\sigma_C &= \{(x,\alpha)\in X\times\mathbb{R} : \max\{x,0\}\le\alpha\},\\ \operatorname{epi}\sigma_D &= \{(x,\alpha)\in X\times\mathbb{R} : \max\{-x,0\}\le\alpha\},\\ \operatorname{epi}\sigma_{C\cap D} &= \mathbb{R}\times[0,+\infty).\end{aligned}$$

Hence,

$$\operatorname{epi}\sigma_{C\cap D} = \operatorname{epi}\sigma_C + \operatorname{epi}\sigma_D.$$

So, $\{C,D\}$ satisfies the SECQ.

We aim at giving some useful equivalent formulation of the SECQ. In order to do that, we have to do some preparatory works. First, we prove the following lemma (see [19, Lemma 2.2]), which is about taking conjugation of the pointwise supremum of a collection of functions in $\Gamma(X)$.

Lemma 3.2.1. *Let $\{f_i : i\in I\}$ be a family of functions in $\Gamma(X)$. Assume that there exists some $x_0\in X$ such that $\sup_{i\in I} f_i(x_0) < +\infty$. Then, for each $x^*\in X^*$, one has $(\sup_{i\in I} f_i)^*(x^*) = \overline{\operatorname{co}(\inf_{i\in I} f_i^*)}^{w^*}(x^*)$.*

Proof. Since $f_i\in\Gamma(X)$, one has $f_i^{**} = f_i$ for each $i\in I$ (see [31, Theorem 2.3.3]). Next, we note that, for each $x\in X$,

$$\begin{aligned}(\inf_{i\in I} f_i^*)^*(x) &= \sup_{x^*\in X^*}\{\langle x^*,x\rangle - \inf_{i\in I} f_i^*(x^*)\}\\ &= \sup_{x^*\in X^*}\sup_{i\in I}\{\langle x^*,x\rangle - f_i^*(x^*)\}\\ &= \sup_{i\in I}\sup_{x^*\in X^*}\{\langle x^*,x\rangle - f_i^*(x^*)\}\\ &= \sup_{i\in I}(f_i^*)^*(x)\\ &= \sup_{i\in I} f_i^{**}(x)\\ &= \sup_{i\in I} f_i(x).\end{aligned}$$

Hence, by [31, Theorem 2.3.1] one has

$$\left(\overline{\mathrm{co}(\inf_{i\in I} f_i^*)}^{w^*}\right)^* = \left(\inf_{i\in I} f_i^*\right)^* = \sup_{i\in I} f_i\,, \tag{3.2.2}$$

and so

$$\left(\overline{\mathrm{co}(\inf_{i\in I} f_i^*)}^{w^*}\right)^{**} = \left(\sup_{i\in I} f_i\right)^*.$$

Thus it remains to show that $\phi = \phi^{**}$, where $\phi := \overline{\mathrm{co}(\inf_{i\in I} f_i^*)}^{w^*}$. By [31, Theorem 2.3.3], it suffices to show that $\phi \in \Gamma(X^*)$. Since it is obvious that ϕ is of weakly* closed convex graph, it remains to show the properness of ϕ. By (3.2.2) and the assumption, there exists some $x_0 \in X$ such that $\phi^*(x_0) < +\infty$. Also, by the properness of f_i, we have that $\phi^* = \sup_{i\in I} f_i > -\infty$ on X. Hence, ϕ^* is proper. This in turn implies that ϕ is proper (which is direct from definition of conjugate function). The proof is completed. □

Using the previous lemma, we can get an explicit form of epigraph of $(\sup_{i\in I} f_i)^*$ in terms of the epigraphs of f_i's, which was proved in [19, Lemma 2.3] by Ng et. al. The following lemma is an intermediate step in their proof. We isolate it and give a proof for the sake of completeness.

Lemma 3.2.2. *Let $\{f_i : i \in I\}$ be a collection of extended real-valued functions on X. Then,*

$$\bigcup_{i\in I} \mathrm{epi}\, f_i^* \subseteq \mathrm{epi}\,(\inf_{i\in I} f_i^*) \subseteq \overline{\mathrm{co}\bigcup_{i\in I} \mathrm{epi}\, f_i^*}^{w^*}. \tag{3.2.3}$$

In particular, it holds that $\overline{\mathrm{co}(\mathrm{epi}\,(\inf_{i\in I} f_i^*))}^{w^*} = \overline{\mathrm{co}\bigcup_{i\in I} \mathrm{epi}\, f_i^*}^{w^*}$.

Proof. The last assertion follows from (3.2.3) by considering weakly* closed convex regularizations. Also, The first inclusion in (3.2.3) holds because $f_j^* \geq \inf_{i\in I} f_i^*$ and so $\mathrm{epi}\, f_j^* \subseteq \mathrm{epi}\,(\inf_{i\in I} f_i^*)$ for each $j \in I$.

We now turn to prove the second inclusion in (3.2.3). Suppose on the contrary that there exists some $(x_0^*, \alpha_0) \in X^* \times \mathbb{R}$ such that

$$(x_0^*, \alpha_0) \in \mathrm{epi}\,(\inf_{i\in I} f_i^*) \setminus \overline{\mathrm{co}\bigcup_{i\in I} \mathrm{epi}\, f_i^*}^{w^*}. \tag{3.2.4}$$

This implies in particular that $f_i^*(x_0^*) > \alpha_0$ for each i. It follows that

$$(\inf_{i\in I} f_i^*)(x_0^*) \geq \alpha_0$$

and so the equality holds thanks to the fact that $(x_0^*, \alpha_0) \in \operatorname{epi}(\inf_{i\in I} f_i^*)$. Consequently, one can find a sequence $\{j_n\}_{n\in\mathbb{N}} \subseteq I$ such that $\lim_{n\to\infty} f_{j_n}^*(x_0^*) = \alpha_0$ and $f_{j_n}^*(x_0^*) \in \mathbb{R}$ for all $n \in \mathbb{N}$. Hence, $(x_0^*, f_{j_n}^*(x_0^*)) \longrightarrow^{w^*} (x_0^*, \alpha_0)$ as $n \to \infty$. Noting for each $n \in \mathbb{N}$,

$$(x_0^*, f_{j_n}^*(x_0^*)) \in \operatorname{epi} f_{j_n}^* \subseteq \operatorname{co} \bigcup_{i\in I} \operatorname{epi} f_i^*,$$

it follows by passing to limits that $(x_0^*, \alpha_0) \in \overline{\operatorname{co} \bigcup_{i\in I} \operatorname{epi} f_i^*}^{w^*}$, which clearly contradicts (3.2.4). Therefore the second inclusion in (3.2.3) holds. □

Here is the result that we have just mentioned.

Lemma 3.2.3 ([19, Lemma 2.3]). *Let $\{f_i : i \in I\}$ be a family of functions in $\Gamma(X)$. Assume that there exists some $x_0 \in X$ such that $(\sup_{i\in I} f_i)(x_0) < +\infty$. Then,*

$$\operatorname{epi}(\sup_{i\in I} f_i)^* = \overline{\operatorname{co} \bigcup_{i\in I} \operatorname{epi} f_i^*}^{w^*}. \tag{3.2.5}$$

Proof. Since $\sup_{i\in I} f_i(x_0) < +\infty$ for some $x_0 \in X$, we can use Lemma 3.2.1 to conclude that $(\sup_{i\in I} f_i)^* = \overline{\operatorname{co}(\inf_{i\in I} f_i^*)}^{w^*}$. Therefore, one has

$$\operatorname{epi}(\sup_{i\in I} f_i)^* = \operatorname{epi}(\overline{\operatorname{co}(\inf_{i\in I} f_i^*)}^{w^*}) = \overline{\operatorname{co}(\operatorname{epi}(\inf_{i\in I} f_i^*))}^{w^*} = \overline{\operatorname{co} \bigcup_{i\in I} \operatorname{epi} f_i^*}^{w^*},$$

where the second equality comes from the definition of weakly* lower semicontinuous convex regularization of a function and the last equality follows from Lemma 3.2.2. This completes the proof. □

The proof of the following lemma is elementary.

Lemma 3.2.4. *Let $A_i \subseteq X$ be a convex cone for each $i \in I$. Then, $\operatorname{co} \bigcup_{i\in I} A_i = \sum_{i\in I} A_i$.*

With the use of Lemma 3.2.3 and 3.2.4, we get the following important proposition. For the case when I is finite, this proposition is a special case of Theorem 2.6.5 (by letting $f_i := \delta_{C_i}$ for all $i \in I$).

Proposition 3.2.5 ([19, Proposition 2.4]). *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i\in I} C_i \neq \emptyset$. Then,*

$$\operatorname{epi}\sigma_C = \overline{\sum_{i\in I} \operatorname{epi}\sigma_{C_i}}^{w^*}. \tag{3.2.6}$$

Proof. Since C_i is a non-empty closed convex set, $\delta_{C_i} \in \Gamma(X)$ for each $i \in I$. Also, one can check by the definition that $\delta_C = \sup_{i\in I} \delta_{C_i}$ on X. By the assumption, take $x_0 \in C$ and it follows that $(\sup_{i\in I} \delta_{C_i})(x_0) = 0 < +\infty$. Then, by applying Lemma 3.2.3 to the family of functions $\{\delta_{C_i} : i \in I\}$, we see that

$$\operatorname{epi}\sigma_C = \operatorname{epi}(\sup_{i\in I} \delta_{C_i})^* = \overline{\operatorname{co}\bigcup_{i\in I} \operatorname{epi}\delta^*_{C_i}}^{w^*} = \overline{\operatorname{co}\bigcup_{i\in I} \operatorname{epi}\sigma_{C_i}}^{w^*} = \overline{\sum_{i\in I} \operatorname{epi}\sigma_{C_i}}^{w^*}$$

where the last equality follows from Lemma 3.2.4 and the fact that each $\operatorname{epi}\sigma_{C_i}$ is a convex cone. □

The next corollary is what we want, which gives some equivalent conditions for the SECQ.

Corollary 3.2.6 ([19, Corollary 2.5]). *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X. Assume that $C := \bigcap_{i\in I} C_i \neq \emptyset$. Then, the following statements are equivalent:*

(i) *$\{C_i : i \in I\}$ satisfies the SECQ.*

(ii) *$\sum_{i\in I} \operatorname{epi}\sigma_{C_i}$ is weakly* closed.*

(iii) $\operatorname{epi}\sigma_C \subseteq \sum_{i\in I} \operatorname{epi}\sigma_{C_i}$.

Proof. The equivalence **(i)** ⇔ **(ii)** follows readily from (3.2.6) and the definition of SECQ. Moreover, since it is trivial that $\operatorname{epi}\sigma_C \supseteq \sum_{i\in I} \operatorname{epi}\sigma_{C_i}$, we have **(i)** ⇔ **(iii)**. □

In view of Corollary 3.2.6, we see that the SECQ (for a system of closed convex sets with non-empty intersection) is closely related to the weakly* closedness of the sum of epigraphs. Moreover, in order to check the SECQ, it suffices to check one-sided set inclusion for the set equality as was required in the definition. We will use this corollary several times throughout our analysis.

Before we end this section, we show the following proposition, which was proved in [19, Proposition 2.6]. It states the important fact that SECQ is translational invariant.

Proposition 3.2.7. *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i\in I} C_i \neq \emptyset$. Then, the following statements are equivalent:*

(i) *$\{C_i : i \in I\}$ satisfies the SECQ.*

(ii) *For each $x \in X$, the collection $\{C_i - x : i \in I\}$ satisfies the SECQ.*

Proof. The implication **(ii)** $\Rightarrow$ **(i)** can be seen by taking $x = 0$ in **(ii)**. We now turn to prove the implication **(i)** $\Rightarrow$ **(ii)**. Suppose that $\{C_i : i \in I\}$ satisfies the SECQ and fix $x \in X$. We have to show that $\operatorname{epi} \sigma_{C-x} = \sum_{i\in I} \operatorname{epi} \sigma_{C_i - x}$. Let $(x^*, \alpha) \in X^* \times \mathbb{R}$. Note that we have the following equivalences:

$$\begin{aligned}
(x^*, \alpha) \in \operatorname{epi} \sigma_{C-x} &\iff \sigma_{C-x}(x^*) \leq \alpha \\
&\iff \sup_{z\in C} \langle x^*, z - x\rangle \leq \alpha \\
&\iff \sup_{z\in C} \langle x^*, z\rangle \leq \alpha + \langle x^*, x\rangle \\
&\iff \sigma_C(x^*) \leq \alpha + \langle x^*, x\rangle \\
&\iff (x^*, \alpha + \langle x^*, x\rangle) \in \operatorname{epi} \sigma_C
\end{aligned}$$

Using the same argument as shown above, one can prove that

$$(x^*, \alpha + \langle x^*, x\rangle) \in \sum_{i\in I} \operatorname{epi} \sigma_{C_i} \iff (x^*, \alpha) \in \sum_{i\in I} \operatorname{epi} \sigma_{C_i - x}.$$

By the equivalences above and (3.2.1), it follows that if **(i)** is assumed, then the following equivalences hold:

$$\begin{aligned}(x^*, \alpha) \in \operatorname{epi} \sigma_{C-x} &\iff (x^*, \alpha + \langle x^*, x\rangle) \in \operatorname{epi} \sigma_C \\ &\iff (x^*, \alpha + \langle x^*, x\rangle) \in \sum_{i\in I} \operatorname{epi} \sigma_{C_i} \\ &\iff (x^*, \alpha) \in \sum_{i\in I} \operatorname{epi} \sigma_{C_i - x}.\end{aligned}$$

Therefore, $\operatorname{epi} \sigma_{C-x} = \sum_{i\in I} \operatorname{epi} \sigma_{C_i-x}$ and the proof is completed. □

3.3 Relationship between the SECQ and other constraint qualifications

This section is devoted to study the relationship between the SECQ and other types of constraint qualifications.

3.3.1 The SECQ and the strong CHIP

We first recall the following definition of the strong conical hull intersection property (the strong CHIP). It was first given by Deustch et. al. in [13] under the Hilbert space setting, for the case when I is finite. Later, in [17, 18], Li and Ng extended the definition to the case when I is allowed to be infinite and under the setting of a normed linear space.

Definition 3.3.1. *Let X be a normed linear space. Let $\{C_i : i \in I\}$ be a collection of convex sets in X with $\bigcap_{i\in I} C_i \neq \emptyset$.*

(i) *The collection $\{C_i : i \in I\}$ is said to have strong conical hull intersection property (strong CHIP) at $x \in \bigcap_{i\in I} C_i$ if $N_{\bigcap_{i\in I} C_i}(x) = \sum_{i\in I} N_{C_i}(x)$.*

(ii) *The collection $\{C_i : i \in I\}$ is said to have strong conical hull intersection property if $\{C_i : i \in I\}$ has strong CHIP at x, for each $x \in \bigcap_{i\in I} C_i$.*

The strong CHIP was originally proposed for establishing dual formulation of constrained best approximation problems in the setting of Hilbert space, see [12, 13, 14, 16, 17]. After that, some authors also studied its relation with other concepts in optimization like the basic constraint qualification, bounded linear regularity and Jameson's (G) property, etc., see [7, 16, 17] for examples. Due to its importance in convex optimization theory, sufficient conditions for the strong CHIP were extensively studied in the literature. In particular, in [19], Li and Ng gave some interior-point type conditions to ensure the strong CHIP for an infinite system of closed convex sets in normed linear spaces.

As we have mentioned in the introduction, Jeyakumar et. al. showed that the SECQ is a sufficient condition for the strong CHIP of two closed convex sets (see [9, Theorem 3.1]). A natural question that comes to mind is whether the same implication is preserved under the setting of arbitrary system of closed convex sets. The answer is affirmative. Before we show this result, let us prove the following theorem, which gives an equivalent condition for the strong CHIP. One may take a note for its similarity with Corollary 3.2.6.

Theorem 3.3.1. *Suppose that $\{C_i : i \in I\}$ is a collection of closed convex sets in X with $C := \bigcap_{i\in I} C_i \neq \emptyset$. Let $x \in C$. Then, the following statements are equivalent.*

(i) *$\{C_i : i \in I\}$ has strong CHIP at x.*

(ii) *$N_C(x) \subseteq \sum_{i\in I} N_{C_i}(x)$.*

Proof. The result immediately follows from Proposition 2.4.2 and the definition of strong CHIP. □

The next theorem shows that the SECQ implies the strong CHIP. Also, under some additional assumption, these two properties become equivalent.

Theorem 3.3.2 ([19, Theorem 3.1]). *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i\in I} C_i \neq \emptyset$. If $\{C_i : i \in I\}$ satisfies SECQ, then it has the strong CHIP; if, in addition, assume that $\operatorname{dom}\sigma_C \subseteq \operatorname{Im}\partial\delta_C$, that is*

$$\operatorname{dom}\sigma_C \subseteq \bigcup_{x\in C} N_C(x), \tag{3.3.1}$$

then the converse implication holds.

Proof. Suppose that $\{C_i : i \in I\}$ satisfies the SECQ. In view of Theorem 3.3.1, it suffices to show that

$$N_C(x) \subseteq \sum_{i\in I} N_{C_i}(x) \quad \text{for any } x \in C.$$

Fix $x \in C$ and let $x^* \in N_C(x)$. Then by (2.4.4), it follows that $(x^*, \langle x^*, x\rangle) \in \operatorname{epi}\sigma_C$. Thus, by (3.2.1), there exist some finite subset $J \subseteq I$ and $(x_j^*, \alpha_j) \in \operatorname{epi}\sigma_{C_j}$ for each $j \in J$ such that

$$(x^*, \langle x^*, x\rangle) = \textstyle\sum_{j\in J}(x_j^*, \alpha_j).$$

In particular, one has $\langle x^*, x\rangle = \sum_{j\in J}\langle x_j^*, x\rangle = \sum_{j\in J}\alpha_j$. Noting $\langle x_j^*, x\rangle \leq \sigma_{C_j}(x_j^*) \leq \alpha_j$ (as $x \in C$) for each $j \in J$, it follows that each $\langle x_j^*, x\rangle = \alpha_j$, that is $x_j^* \in N_{C_j}(x)$, thanks to (2.4.4). Therefore, $x^* = \sum_{j\in J} x_j^* \in \sum_{i\in I} N_{C_i}(x)$ and so $N_C(x) \subseteq \sum_{i\in I} N_{C_i}(x)$ is shown for each $x \in C$. Thus, $\{C_i : i \in I\}$ has the strong CHIP.

Next, we assume that (3.3.1) holds and $\{C_i : i \in I\}$ has the strong CHIP. To show that $\{C_i : i \in I\}$ satisfies the SECQ, by Corollary 3.2.6, it suffices to show that $\operatorname{epi}\sigma_C \subseteq \sum_{i\in I}\operatorname{epi}\sigma_{C_i}$. Let $(x^*, \alpha) \in \operatorname{epi}\sigma_C$. Then, $\sigma_C(x^*) \leq \alpha$, which implies that $x^* \in \operatorname{dom}\sigma_C$. So, by (3.3.1), there exists some $x \in C$ such that $x^* \in N_C(x)$. Since $\{C_i : i \in I\}$ has the strong CHIP, there exist some finite subset $J \subseteq I$ and $x_j^* \in N_{C_j}(x)$ for each $j \in J$ such that $x^* = \sum_{j\in J} x_j^*$. Hence, $\sum_{j\in J}\langle x_j^*, x\rangle = \langle x^*, x\rangle \leq \alpha$. Also, by (2.4.4), $\langle x_j^*, x\rangle = \sigma_{C_j}(x_j^*)$ for each $j \in J$. Now, define $c := \frac{1}{|J|}(\alpha - \sum_{j\in J}\langle x_j^*, x\rangle)$ (≥ 0) and $\alpha_j := \langle x_j^*, x\rangle + c$, where

$|J|$ denotes the cardinality of J. Then, for each $j \in J$, $(x_j^*, \alpha_j) \in \operatorname{epi} \sigma_{C_j}$ and $\sum_{j \in J} \alpha_j = \alpha$. So, $(x^*, \alpha) = \sum_{j \in J} (x_j^*, \alpha_j) \in \sum_{i \in I} \operatorname{epi} \sigma_{C_i}$. Therefore, the set inclusion $\operatorname{epi} \sigma_C \subseteq \sum_{i \in I} \operatorname{epi} \sigma_{C_i}$ holds as was required to show. □

The result of Theorem 3.3.2 provides a way to check that a collection of closed convex sets does not satisfy the SECQ. We give an example for illustration.

Example 3.3.1 ([29, Example 11.1]). Let $X := \mathbb{R}^2$ and $x_0 := (1, 0) \in \mathbb{R}^2$. Define $C := x_0 - \mathbf{B_X}$ and $D := \mathbf{B_X} - x_0$. Then, $C \cap D = \{0\}$. Note that

$$N_C(0) = (-\infty, 0] \times \{0\} \quad \text{and} \quad N_D(0) = [0, +\infty) \times \{0\},$$

and

$$N_{C \cap D}(0) = N_{\{0\}}(0) = \mathbb{R}^2.$$

Hence,

$$N_{C \cap D} = \mathbb{R}^2 \neq \mathbb{R} \times \{0\} = N_C(0) + N_D(0).$$

So, $\{C, D\}$ does not satisfy the strong CHIP. This and Theorem 3.3.2 imply that $\{C, D\}$ does not satisfy the SECQ.

Remark 3.3.1. *In [29, Example 11.2], Burachik and Simons show in details that* $\operatorname{epi} \sigma_C + \operatorname{epi} \sigma_D$ *is not closed in* $\mathbb{R}^2$ *by finding the explicit form of* $\operatorname{epi} \sigma_C + \operatorname{epi} \sigma_D$, *where C and D are the sets as defined in Example 3.3.1. Since* $\operatorname{epi} \sigma_{C \cap D}$ *is closed, it follows that*

$$\operatorname{epi} \sigma_{C \cap D} \neq \operatorname{epi} \sigma_C + \operatorname{epi} \sigma_D.$$

So, $\{C, D\}$ does not satisfy the SECQ. This provides a more direct way to show that $\{C, D\}$ does not satisfy the SECQ, in the sense that we do not have to consult Theorem 3.3.2. However, as shown in [29, Example 11.2], it may not be easy to find the explicit form of $\operatorname{epi} \sigma_C + \operatorname{epi} \sigma_D$. *Therefore, the method that we have used in Example 3.3.1 may be easier to work with, if it is not difficult to check that the strong CHIP does not hold.*

In view of Theorem 3.3.2, it is meaningful to give some sufficient conditions which can ensure that (3.3.1) holds. The following theorem is of this type, which is from [19, Proposition 3.2]. Recall that for a proper extended real-valued function f on X^* and $x_0^* \in X^*$, the continuity of f at x_0^* means that for any $\alpha > 0$, there exists some neighborhood V_0 (in the norm topology) of x_0^* such that $f(x^*) > \alpha$ for any $x^* \in V_0$.

Theorem 3.3.3 ([19, Proposition 3.2]). *Let C be a non-empty, closed and convex subset of X. Then,* (3.3.1) *holds if at least one of the following conditions holds:*

- **(i)** *The set C can be expressed in the form $C = D+K$ for some weakly compact convex set D and closed convex cone K in X.*
- **(ii)** $\dim C < +\infty$, $\operatorname{Im} \partial\delta_C$ *is convex and* $\sigma_C|_{\operatorname{span} C}$ *is continuous on* $(\operatorname{span} C)^*\backslash\{0\}$.

Proof. We first prove that the condition **(i)** implies (3.3.1). Thus, suppose $C = D + K$ as in **(i)**. To prove (3.3.1), let $x^* \in \operatorname{dom} \sigma_C$. Then,

$$\sup_{d\in D}\langle x^*, d\rangle + \sup_{k\in K}\langle x^*, k\rangle = \sup_{d\in D, k\in K}\langle x^*, d+k\rangle = \sup_{a\in C}\langle x^*, a\rangle = \sigma_C(x^*) < +\infty.$$

Thus, we must have $\sigma_K(x^*) = \sup_{k\in K}\langle x^*, k\rangle < +\infty$. Since K is a cone and $0 \in K$, it follows that $\sigma_K(x^*) = 0$. Hence, $\sigma_D(x^*) = \sigma_C(x^*) \in \mathbb{R}$. Pick a sequence $\{d_n\}_{n\in\mathbb{N}}$ from D such that $\lim_{n\to\infty}\langle x^*, d_n\rangle = \sigma_D(x^*)$. Since D is weakly compact, one can, without loss of generality, assume that $d_n \to^w d_0$ for some $d_0 \in D$. Hence, $d_0 \in D + K = C$ and $\sigma_C(x^*) = \sigma_D(x^*) = \lim_{n\to\infty}\langle x^*, d_n\rangle = \langle x^*, d_0\rangle$. By (2.4.4), we see that $x^* \in N_C(d_0)$ and so $x^* \in \bigcup_{x\in C} N_C(x)$. Since $x^* \in \operatorname{dom} \sigma_C$ is arbitrarily chosen, we have proved that (3.3.1) holds.

We now turn to prove that the condition **(ii)** implies (3.3.1). Consider the following three cases:

Case 1: C is bounded in X.

Note that C is closed and $\operatorname{span} C$ is of finite dimension by **(ii)**. Hence, C is (norm-) compact and so C is weakly compact in X. Since C is convex, the condition **(i)** holds and so does (3.3.1) with $D = C$ and $K = \{0\}$. Thus, the result follows.

Case 2: C is a subspace of X.

Let $x^* \in \operatorname{dom} \sigma_C$. Then, $\sup_{c \in C} \langle x^*, c \rangle = \sigma_C(x^*) < +\infty$. By the assumption in this case, this implies that $\sup_{c \in C} \langle x^*, c \rangle = 0$ and so $x^* \in N_C(0) \subseteq \bigcup_{x \in C} N_C(x)$. Since x^* is arbitrarily chosen, we have proved that (3.3.1) holds for this case.

Case 3: C is unbounded and is not a subspace of X.

In this case, C is a proper subset of $\operatorname{span} C$. Let $\hat{\delta}_C$ denote the indicator function of C in $\operatorname{span} C$, and $\hat{\sigma}_C$ the support function of C in $(\operatorname{span} C)^*$. Then, $(\hat{\delta}_C)^* = \hat{\sigma}_C = \sigma_C|_{\operatorname{span} C}$ on $(\operatorname{span} C)^*$. Since $C \subsetneq \operatorname{span} C$, one can easily show by virtue of the Hahn-Banach theorem that

$$\operatorname{dom} \sigma_C = \{z^* \in X^* : z^*|_{\operatorname{span} C} \in \operatorname{dom} \hat{\sigma}_C\}. \tag{3.3.2}$$

We show next that

$$\operatorname{Im} \partial \delta_C = \{z^* \in X^* : z^*|_{\operatorname{span} C} \in \operatorname{Im} \partial \hat{\delta}_C\}. \tag{3.3.3}$$

Let $y^* \in \operatorname{Im} \partial \delta_C$. Then, there exists some $c_0 \in C$ such that $y^* \in \partial \delta_C(c_0)$, that is $y^* \in N_C(c_0)$ by Proposition 2.4.1. Noting that

$$N_C(c_0)|_{\operatorname{span} C} = N_C^{\operatorname{span} C}(c_0) = \partial \hat{\delta}_C(c_0) \subseteq \operatorname{Im} \partial \hat{\delta}_C,$$

where the first equality follows from Proposition 2.4.4 (by applying to $\operatorname{span} C$ in place of Z), we have $y^*|_{\operatorname{span} C} \in \operatorname{Im} \partial \hat{\delta}_C$. So, $\operatorname{Im} \partial \delta_C \subseteq \{z^* \in X^* : z^*|_{\operatorname{span} C} \in \operatorname{Im} \partial \hat{\delta}_C\}$. The converse set inclusion follows from direct checking of definition and the fact that $C \subsetneq \operatorname{span} C$. Thus, (3.3.3) holds. This together with (3.3.2) and the Hahn-Banach extension theorem imply that

$$\operatorname{dom} \sigma_C|_{\operatorname{span} C} = \operatorname{dom} \hat{\sigma}_C \quad \text{and} \quad \operatorname{Im} \partial \delta_C|_{\operatorname{span} C} = \operatorname{Im} \partial \hat{\delta}_C. \tag{3.3.4}$$

We claim further that

$$\operatorname{dom} \hat{\sigma}_C \subseteq \operatorname{Im} \partial \hat{\delta}_C. \tag{3.3.5}$$

By applying [3, Proposition 2.4.3] to the non-empty, proper, closed, convex and unbounded subset C in the finite dimensional normed linear space $\operatorname{span} C$, we get

$$\operatorname{dom}\hat{\sigma}_C\backslash\{0\} = \operatorname{int}(\operatorname{dom}\hat{\sigma}_C) \neq \emptyset. \tag{3.3.6}$$

Also, $\operatorname{Im}\partial\hat{\delta}_C$ is convex, thanks to (3.3.4) and the convexity assumption of $\operatorname{Im}\partial\delta_C$ in **(ii)**. (Indeed, let $a^*, b^* \in \operatorname{Im}\partial\hat{\delta}_C$ and $\lambda \in [0,1]$. Then, by (3.3.4), take $\overline{a}^*$, $\overline{b}^* \in \operatorname{Im}\partial\delta_C$ such that $\overline{a}^*|_{\operatorname{span} C} = a^*$ and $\overline{b}^*|_{\operatorname{span} C} = b^*$. By the convexity of $\operatorname{Im}\partial\delta_C$, we have $\lambda\overline{a}^* + (1-\lambda)\overline{b}^* \in \operatorname{Im}\partial\delta_C$. So, $\lambda a^* + (1-\lambda)b^* \in \operatorname{Im}\partial\delta_C|_{\operatorname{span} C} = \operatorname{Im}\partial\hat{\delta}_C$ by (3.3.4).) Thus, we can conclude by [31, Proposition 1.2.1(ii) and Corollary 1.3.4] that

$$\operatorname{int}(\operatorname{Im}\partial\hat{\delta}_C) = \operatorname{int}(\overline{\operatorname{Im}\partial\hat{\delta}_C}). \tag{3.3.7}$$

Furthermore, since $\hat{\sigma}_C = (\hat{\delta}_C)^*$, one can apply [31, Theorem 3.1.2] to get

$$\operatorname{dom}(\hat{\sigma}_C) = \operatorname{dom}(\hat{\delta}_C)^* \subseteq \overline{\operatorname{Im}\partial\hat{\delta}_C}. \tag{3.3.8}$$

Combining (3.3.6), (3.3.7) and (3.3.8), it follows that

$$\operatorname{dom}\hat{\sigma}_C\backslash\{0\} = \operatorname{int}(\operatorname{dom}\hat{\sigma}_C) \subseteq \operatorname{int}(\overline{\operatorname{Im}\partial\hat{\delta}_C}) = \operatorname{int}(\operatorname{Im}\partial\hat{\delta}_C) \subseteq \operatorname{Im}\partial\hat{\delta}_C.$$

Finally, it is obvious that $0 \in \operatorname{Im}\partial\hat{\delta}_C$, so our claimed (3.3.5) follows. This proves our claim.

Let $x^* \in \operatorname{dom}\sigma_C$. By (3.3.4) and (3.3.5), it follows that $x^*|_{\operatorname{span} C} \in \operatorname{Im}\partial\hat{\delta}_C$. So, $x^* \in \operatorname{Im}\partial\delta_C$, thanks to (3.3.3). Thus, there exists $x \in C$ such that $x^* \in \partial\delta_C(x)$, and so $x^* \in N_C(x)$ by Proposition 2.4.1. Therefore, (3.3.1) holds for the present case.

From these three cases, we see that (3.3.1) holds in any case provided that **(ii)** holds. The proof is completed. □

Combining Theorems 3.3.2 and 3.3.3, we arrive at:

Corollary 3.3.4. *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i\in I} C_i \neq \emptyset$ such that at least one of the following conditions is satisfied.*

(i) *There exist some weakly compact convex set $D \subseteq X$ and closed convex cone K such that $C = D + K$.*

(ii) $\dim C < +\infty$, $\operatorname{Im} \partial\delta_C$ *is convex and* $\sigma_C|_{\operatorname{span} C}$ *is continuous on* $(\operatorname{span} C)^* \backslash \{0\}$.

Then, $\{C_i : i \in I\}$ *satisfies SECQ if and only if it has strong CHIP.*

3.3.2 The SECQ and the linear regularity

We now turn to study the relationship between the SECQ and another concept, known as the linear regularity. The definition is shown as follows, see [4, 5, 6] for the case when I is finite, and [19] for I being allowed to be infinite.

Definition 3.3.2. *Let X be a normed linear space. Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i\in I} C_i \neq \emptyset$. Then, the collection $\{C_i : i \in I\}$ is said to be linearly regular if there exists some $\tau > 0$ such that*

$$d_C(x) \leq \tau \sup_{i\in I} d_{C_i}(x) \quad \textit{for all } x \in X; \tag{3.3.9}$$

The notion of linear regularity was used to give some norm convergence results for projection algorithms, which is useful in solving the convex feasibility problems. Please refer to [4, 5, 6] and the reference therein for details.

Following [19], we first give a characterization of linear regularity via the use of epigraphs. In order to get that, we need the following lemma, which concerns about the conjugate function of a (positive) scalar multiple of distance function. In the remainder of this subsection, let $\Lambda^* := \mathbf{B}_{\mathbf{X}^*} \times \mathbb{R}$ and $\Lambda^*_+ := \mathbf{B}_{\mathbf{X}^*} \times \mathbb{R}_+$.

Lemma 3.3.5 ([19, Lemma 4.3]). *Let S be a non-empty, closed and convex subset of X and let $\gamma > 0$. Then, $(\gamma d_S)^* = \delta_{\gamma \mathbf{B}_{\mathbf{X}^*}} + \sigma_S$ on X^* and*

$$\operatorname{epi}(\gamma d_S)^* = \operatorname{epi}\sigma_S \cap \gamma\Lambda^*. \tag{3.3.10}$$

If $0 \in S$ in addition, then

$$\operatorname{epi}(\gamma d_S)^* = \operatorname{epi}\sigma_S \cap \gamma\Lambda^*_+. \tag{3.3.11}$$

Proof. Let $x^* \in X^*$. By Example 2.5.1, we have

$$d_S^*(x^*) = \sigma_S(x^*) + \delta_{\mathbf{B}_{\mathbf{X}^*}}(x^*).$$

Then, by [31, Theorem 2.3.1(v)] (applied to get the first and third equality), it follows that

$$(\gamma d_S)^*(x^*) = \gamma d_S^*(\frac{1}{\gamma}x^*) = \gamma\sigma_S(\frac{1}{\gamma}x^*) + \gamma\delta_{\mathbf{B}_{\mathbf{X}^*}}(\frac{1}{\gamma}x^*) = \sigma_S(x^*) + \gamma\delta_{\mathbf{B}_{\mathbf{X}^*}}(\frac{1}{\gamma}x^*).$$

Since $\gamma\delta_{\mathbf{B}_{\mathbf{X}^*}}(\frac{1}{\gamma}x^*) = \delta_{\gamma\mathbf{B}_{\mathbf{X}^*}}(x^*)$ (by direct checking of definitions), one gets

$$(\gamma d_S)^*(x^*) = \sigma_S(x^*) + \delta_{\gamma\mathbf{B}_{\mathbf{X}^*}}(x^*).$$

Next, we prove that (3.3.10) holds. To do this, we first let $(x^*, \alpha) \in \operatorname{epi}\sigma_S \cap \gamma\Lambda^*$. Then, $\sigma_S(x^*) \leq \alpha$ and $\delta_{\gamma\mathbf{B}_{\mathbf{X}^*}}(x^*) = 0$. Hence, $(\gamma d_S)^*(x^*) = \sigma_S(x^*) + \delta_{\gamma\mathbf{B}_{\mathbf{X}^*}}(x^*) \leq \alpha$. So, $(x^*, \alpha) \in \operatorname{epi}(\gamma d_S)^*$. Conversely, let $(x^*, \alpha) \in \operatorname{epi}(\gamma d_S)^*$. Then, $\sigma_S(x^*) + \delta_{\gamma\mathbf{B}_{\mathbf{X}^*}}(x^*) = (\gamma d_S)^*(x^*) \leq \alpha$. In particular, it follows that $\delta_{\gamma\mathbf{B}_{\mathbf{X}^*}} = 0$. Hence, $x^* \in \gamma\mathbf{B}_{\mathbf{X}^*}$ and $\sigma_S(x^*) \leq \alpha$. Thus, $(x^*, \alpha) \in \operatorname{epi}\sigma_S \cap \gamma\Lambda^*$. Therefore, we conclude that (3.3.10) holds.

It remains to show (3.3.11). In view of the (3.3.10), it is sufficient to show that

$$\operatorname{epi}(\gamma d_S)^* \subseteq \operatorname{epi}\sigma_S \cap \gamma\Lambda_+^*.$$

Let $(x^*, \alpha) \in \operatorname{epi}(\gamma d_S)^*$. Since $0 \in S$, $\sigma_S(x^*) \geq 0$. Further, by (3.3.10), $(x^*, \alpha) \in \operatorname{epi}\sigma_S$ and so $\sigma_S(x^*) \leq \alpha$. Therefore, one can see that $0 \leq \sigma_S(x^*) \leq \alpha$ and $x^* \in \gamma\mathbf{B}_{\mathbf{X}^*}$ (by (3.3.10)). So, $(x^*, \alpha) \in \operatorname{epi}\sigma_S \cap \gamma\Lambda_+^*$. This completes the proof. □

Theorem 3.3.6 ([19, Theorem 4.4]). *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i\in I} C_i \neq \emptyset$. Let $\gamma > 0$. Consider the following statements:*

(i) *For any $x \in X$, $d_C(x) \leq \gamma \sup_{i\in I} d_{C_i}(x)$.*

(ii) $\operatorname{epi}\sigma_C \cap \Lambda^* \subseteq \overline{\operatorname{co}\bigcup_{i\in I}(\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*)}^{w^*}$.

(iii) $\operatorname{gph}\sigma_C \cap \Lambda^* \subseteq \overline{\operatorname{co} \bigcup_{i\in I}(\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*)}^{w^*}$.

(ii*) $\operatorname{epi}\sigma_C \cap \Lambda^*_+ \subseteq \overline{\operatorname{co} \bigcup_{i\in I}(\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*_+)}^{w^*}$.

(iii*) $\operatorname{gph}\sigma_C \cap \Lambda^*_+ \subseteq \overline{\operatorname{co} \bigcup_{i\in I}(\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*_+)}^{w^*}$.

Then, the following assertions hold.

(a) *(i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(iii).*

(b) *If* $0 \in S$, *then (i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(iii)* $\Leftrightarrow$ *(ii*)* $\Leftrightarrow$ *(iii*).*

Proof. We first prove **(a)**. Note that, by (2.2.2), one sees that

$$\textbf{(i)} \quad \Leftrightarrow \quad \operatorname{epi} d_C^* \subseteq \operatorname{epi}(\gamma \sup_{i\in I} d_{C_i})^*.$$

Now, by using (3.2.5) and (3.3.10), one has

$$\operatorname{epi}(\gamma \sup_{i\in I} d_{C_i})^* = \operatorname{epi}(\sup_{i\in I} \gamma d_{C_i})^* = \overline{\operatorname{co} \bigcup_{i\in I} \operatorname{epi}(\gamma d_{C_i})^*}^{w^*} = \overline{\operatorname{co} \bigcup_{i\in I} (\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*)}^{w^*}.$$

Similarly, (3.3.10) also entails $\operatorname{epi} d_C^* = \operatorname{epi}\sigma_C \cap \Lambda^*$. So, the equivalence of **(i)** and **(ii)** follows.

Since $\operatorname{gph}\sigma_C \subseteq \operatorname{epi}\sigma_C$, it is obvious that **(ii)** $\Rightarrow$ **(iii)**. Conversely, suppose that **(iii)** holds. Then,

$$\begin{aligned}
\operatorname{epi}\sigma_C \cap \Lambda^* &\subseteq \operatorname{gph}\sigma_C \cap \Lambda^* + \{0\} \times \mathbb{R}_+ \\
&\subseteq \overline{\operatorname{co} \bigcup_{i\in I} (\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*)}^{w^*} + \{0\} \times \mathbb{R}_+ \\
&= \overline{\operatorname{co} \bigcup_{i\in I} (\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*)}^{w^*},
\end{aligned}$$

where the last equality follows from the fact that each $\operatorname{epi}\sigma_{C_i} + \{0\} \times \mathbb{R}_+ = \operatorname{epi}\sigma_{C_i}$ and direct checking of definitions. So, **(ii)** holds.

Finally, we assume that $0 \in S$. Then, the equivalences in **(b)** can be seen by following the same argument that was used to prove **(a)** with the use of (3.3.11) in place of (3.3.10). The proof is completed. □

The next theorem is an application of the previous theorem. It gives another important characterization of the linear regularity in Banach space. For the case when finitely many sets are considered, it reduces to [23, Theorem 4.2] by Ng and Yang.

Theorem 3.3.7 ([19, Theorem 4.5]). *Let X be a Banach space. Let $\{C_i : i \in I\}$ be a collection of closed convex sets with $C := \bigcap_{i\in I} C_i \neq \emptyset$ and $\gamma > 0$. Consider the following statements.*

- **(i)** *For each $x \in X$, $d_C(x) \leq \gamma \sup_{i\in I} d_{C_i}(x)$.*
- **(ii)** *For each $x \in C$, $N_C(x) \cap \mathbf{B}_{\mathbf{X}^*} \subseteq \overline{\operatorname{co} \bigcup_{i\in I}(N_{C_i}(x) \cap \gamma \mathbf{B}_{\mathbf{X}^*})}^{w^*}$.*
- **(iii)** *For each $x \in C$, $N_C(x) \cap \mathbf{B}_{\mathbf{X}^*} \subseteq \operatorname{co} \bigcup_{i\in I}(N_{C_i}(x) \cap \gamma \mathbf{B}_{\mathbf{X}^*})$.*
- **(iv)** *For each $x \in C$, $N_C(x) = \sum_{i\in I} N_{C_i}(x)$ and*

$$\left(\sum_{i\in I} N_{C_i}(x)\right) \cap \mathbf{B}_{\mathbf{X}^*} \subseteq \operatorname{co} \bigcup_{i\in I}(N_{C_i}(x) \cap \gamma \mathbf{B}_{\mathbf{X}^*}).$$

*Then, **(ii)** $\Rightarrow$ **(i)**. If assume in addition that I is a compact metric space and the set-valued map $i \mapsto C_i$ is lower semicontinuous, then **(i)** $\Leftrightarrow$ **(ii)**. Furthermore, if I is finite, then **(i)** $\Leftrightarrow$ **(ii)** $\Leftrightarrow$ **(iii)** $\Leftrightarrow$ **(iv)**.*

Proof. We first prove the implication **(ii)** $\Rightarrow$ **(i)**. Suppose that **(ii)** holds. In order to prove **(i)**, it suffices to show by Theorem 3.3.6 that

$$\operatorname{gph} \sigma_C \cap \Lambda^* \subseteq \overline{\operatorname{co} \bigcup_{i\in I}(\operatorname{epi} \sigma_{C_i} \cap \gamma \Lambda^*)}^{w^*}. \tag{3.3.12}$$

To do this, let $(z^*, \sigma_C(z^*)) \in \operatorname{gph} \sigma_C \cap \Lambda^*$. Then, $z^* \in \mathbf{B}_{\mathbf{X}^*}$. Also, $(\delta_C)^*(z^*) = \sigma_C(z^*) \in \mathbb{R}$ and so $z^* \in \operatorname{dom} \delta_C^*$.

Consider the special case when $z^* \in \operatorname{Im} \partial\delta_C$. So, there exists $x \in C$ such that (see (2.4.4) and Proposition 2.4.1)

$$z^* \in \partial\delta_C(x) = N_C(x) \quad \text{and} \quad \langle z^*, x\rangle = \sigma_C(z^*). \tag{3.3.13}$$

Thus, $z^* \in N_C(x) \cap \mathbf{B}_{\mathbf{X}^*}$. By **(ii)**, there exists some directed set $\mathbb{D}$ and some net $\{\overline{z}^*_\nu\}_{\nu\in\mathbb{D}}$ in $\operatorname{co}\bigcup_{i\in I}(N_{C_i}(x) \cap \gamma\mathbf{B}_{\mathbf{X}^*})$ such that $\overline{z}^*_\nu \longrightarrow^{w^*} z^*$, and for each $\nu \in \mathbb{D}$,

$$\overline{z}^*_\nu = \sum_{j\in J_\nu} \lambda_j x^*_j$$

for some finite $J_\nu \subseteq I$, $\lambda_j \in [0,1]$, $\sum_{j\in J_\nu} \lambda_j = 1$, and each $x^*_j \in N_{C_j}(x) \cap \gamma\mathbf{B}_{\mathbf{X}^*}$. Note, as in (3.3.13), $(x^*_j, \langle x^*_j, x\rangle) \in \operatorname{epi}\sigma_{C_j}$ and so $(x^*_j, \langle x^*_j, x\rangle) \in \operatorname{epi}\sigma_{C_j} \cap \gamma\Lambda^*$. On the other hand, since $\overline{z}^*_\nu \longrightarrow^{w^*} z^*$, one can see that $\langle \overline{z}^*_\nu, x\rangle \longrightarrow \langle z^*, x\rangle$. Note, by (3.3.13), that

$$(z^*, \sigma_C(z^*)) = (z^*, \langle z^*, x\rangle) = w^*\text{-}\lim_\nu (\overline{z}^*_\nu, \langle \overline{z}^*_\nu, x\rangle) = w^*\text{-}\lim_\nu \sum_{j\in J_\nu} \lambda_j (x^*_j, \langle x^*_j, x\rangle),$$

Therefore,

$$(z^*, \sigma_C(z^*)) \in \overline{\operatorname{co}\bigcup_{i\in I}(\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*)}^{w^*}. \tag{3.3.14}$$

We remark that (3.3.14) holds for $z^* \in \operatorname{Im}\partial\delta_C$ and $\|z^*\| \leq 1$.

Next, we turn to consider the general case. Since X is a Banach space and $z^* \in \operatorname{dom}\delta^*_C$, one can apply [31, Theorem 3.1.4] to obtain a sequence $\{(z_n, z^*_n)\}_{n\in\mathbb{N}}$ from $\operatorname{gph}\partial\delta_C$ (which implies that $z^*_n \in \operatorname{Im}\partial\delta_C$) such that z^*_n converges to z^* in the norm (and so also in the weakly* topology) and $\sigma_C(z^*_n)$ converges to $\sigma_C(z^*)$ in $\mathbb{R}$. Noting that for each $n \in \mathbb{N}$, $z^*_n \in N_C(z_n)$, we have $\sigma_C(z^*_n) = \langle z^*_n, z_n\rangle$ as in (3.3.13). We first restrict our attention to the case when $\|z^*_n\| \leq 1$ for infinitely many $n \in \mathbb{N}$. Then, by considering subsequence if necessary, one can assume that $\|z^*_n\| \leq 1$ for all $n \in \mathbb{N}$. So, for each $n \in \mathbb{N}$, (3.3.14) holds for z_n in place of z^*, that is

$$(z^*_n, \sigma_C(z^*_n)) \in \overline{\operatorname{co}\bigcup_{i\in I}(\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*)}^{w^*}. \tag{3.3.15}$$

By taking limit on the left hand side of (3.3.15), one sees that (3.3.14) holds. Finally, we consider the case when there are only finitely many $n \in \mathbb{N}$ such that $\|z^*_n\| \leq 1$. Without loss of generality, one can assume that $\|z^*_n\| > 1$ for all $n \in \mathbb{N}$. So, by the condition that $\|z^*_n\| \longrightarrow \|z^*\|$ and $z^* \in \mathbf{B}_{\mathbf{X}^*}$, we must have $\|z^*\| = 1$.

Since $\operatorname{Im}\partial\delta_C$ is a cone, we see that $\frac{z_n^*}{\|z_n^*\|} \in \operatorname{Im}\partial\delta_C$. Hence, for each $n \in \mathbb{N}$, one can apply (3.3.14) to $\frac{z_n^*}{\|z_n^*\|}$ in place of z^*. So,

$$\left(\frac{z_n^*}{\|z_n^*\|}, \sigma_C\left(\frac{z_n^*}{\|z_n^*\|}\right)\right) \in \overline{\operatorname{co}\bigcup_{i\in I}(\operatorname{epi}\sigma_{C_i} \cap \gamma\Lambda^*)}^{w^*}, \tag{3.3.16}$$

Again, (3.3.14) holds by taking limit on the left hand side of (3.3.16). Therefore, we conclude that (3.3.12) holds in any case, which implies that **(i)** holds.

Now, assume in addition that I is a compact metric space and the set-valued map $i \mapsto C_i$ is lower semicontinuous on I. We will prove the implication **(i)** $\Rightarrow$ **(ii)**. Fix $a \in C$. By [2, Corollary 1.4.17] and the lower semicontinuity of $i \mapsto C_i$, one sees that for any $z \in X$, the function $i \mapsto d_{C_i}(z)$ is upper semicontinuous on I. So, by [31, Theorem 2.4.18], we have $\partial(\sup_{i\in I} d_{C_i})(a) = \overline{\operatorname{co}\bigcup_{i\in I}\partial d_{C_i}(a)}^{w^*}$. Since **(i)** holds and $d_C(a) = 0 = \gamma\sup_{i\in I} d_{C_i}(a)$, it follows from the definition of subdifferential that $\partial d_C(a) \subseteq \partial(\gamma\sup_{i\in I} d_{C_i})(a)$. Furthermore, we recall that $\partial d_C(a) = N_C(a) \cap \mathbf{B}_{\mathbf{X}^*}$ and each $\partial d_{C_i}(a) = N_{C_i}(a) \cap \mathbf{B}_{\mathbf{X}^*}$ (see [31, Theorem 3.8.3]). Hence,

$$\begin{aligned}
N_C(a) \cap \mathbf{B}_{\mathbf{X}^*} &= \partial d_C(a) \\
&\subseteq \partial(\gamma \sup_{i\in I} d_{C_i})(a) \\
&= \gamma\partial(\sup_{i\in I} d_{C_i})(a) \\
&= \gamma\overline{\operatorname{co}\bigcup_{i\in I}\partial d_{C_i}(a)}^{w^*} \\
&= \gamma\overline{\operatorname{co}\bigcup_{i\in I}(N_{C_i}(a) \cap \mathbf{B}_{\mathbf{X}^*})}^{w^*} \\
&= \overline{\operatorname{co}\bigcup_{i\in I}(N_{C_i}(a) \cap \gamma\mathbf{B}_{\mathbf{X}^*})}^{w^*},
\end{aligned}$$

Since $a \in C$ is arbitrary, we see that **(ii)** holds.

We next consider the special case when I is finite and prove that **(i)** $\Leftrightarrow$ **(ii)** $\Leftrightarrow$ **(iii)** $\Leftrightarrow$ **(iv)**. Write $I = \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$. The equivalence **(i)** $\Leftrightarrow$ **(ii)** is true by what we have just shown together with the fact that I is compact with respect to the discrete metric. Also, noting that the set $\mathrm{co}\bigcup_{i=1}^{m}(N_{C_i}(c)\cap\gamma\mathbf{B}_{\mathbf{X}^*})$ is weakly* closed, it is clear that the equivalence **(ii)** $\Leftrightarrow$ **(iii)** holds. It remains to show that **(iii)** $\Rightarrow$ **(iv)** (as the converse implication is obvious). Suppose that **(iii)** holds. By Theorem 3.3.1, it suffices to show that

$$N_C(x)\subseteq\sum_{i=1}^{m}N_{C_i}(x)\quad\text{for each } x\in C. \tag{3.3.17}$$

Fix $x\in C$ and let $x^*\in N_C(x)$. Note that $0\in\sum_{i=1}^{m}N_{C_i}(x)$ and so we assume that $x^*\neq 0$. Since $N_C(x)$ is a cone, one sees that $\frac{x^*}{\|x^*\|}\in N_C(x)\cap\mathbf{B}_{\mathbf{X}^*}$. By **(iii)**, it follows that

$$\tfrac{x^*}{\|x^*\|}=\gamma\textstyle\sum_{i=1}^{m}\lambda_i x_i^*$$

for some $\{\lambda_i\}_{i=1}^{m}\subseteq[0,1]$ with $\sum_{i=1}^{m}\lambda_i=1$ and $x_i^*\in N_{C_i}(x)\cap\mathbf{B}_{\mathbf{X}^*}$ for all i. Thus, $x^*=\sum_{i=1}^{m}\gamma\lambda_i\|x^*\|x_i^*$. By the fact that $N_{C_i}(x)$ is a cone for all i, one concludes that $x^*\in\sum_{i=1}^{m}N_{C_i}(x)$. So, (3.3.17) is established and the proof is completed. □

The following theorem is from [19, Theorem 4.6], which reveals the fact that together with an additional assumption, linear regularity becomes a sufficient condition for the SECQ.

Theorem 3.3.8 ([19, Theorem 4.6]). *Let $\{C_i : i\in I\}$ be a collection of closed convex sets in X with $C:=\bigcap_{i\in I}C_i\neq\emptyset$. Suppose that $\{C_i : i\in I\}$ is linearly regular and that*

$$\overline{\mathrm{co}\bigcup_{i\in I}(\mathrm{epi}\,\sigma_{C_i}\cap\Lambda^*)}^{w^*}\subseteq\sum_{i\in I}\mathrm{epi}\,\sigma_{C_i}. \tag{3.3.18}$$

Then, $\{C_i : i\in I\}$ satisfies the SECQ.

Proof. Since each $\mathrm{epi}\,\sigma_{C_i}$ is a convex cone, (3.3.18) implies that

$$\overline{\mathrm{co}\bigcup_{i\in I}(\mathrm{epi}\,\sigma_{C_i}\cap\gamma\Lambda^*)}^{w^*}\subseteq\sum_{i\in I}\mathrm{epi}\,\sigma_{C_i}\quad\text{for all }\gamma>0.$$

Hence, thanks to the regularity assumption, the implication **(i)** $\Rightarrow$ **(ii)** of Theorem 3.3.6 entails that $\mathrm{epi}\,\sigma_C\cap\Lambda^*\subseteq\sum_{i\in I}\mathrm{epi}\,\sigma_{C_i}$. Therefore, by Corollary 3.2.6, we see that $\{C_i : i\in I\}$ satisfies the SECQ. □

We have to give some sufficient conditions for (3.3.18). The following proposition will be helpful for its proof. It states that (3.3.18) is translational invariant, in the sense that the set equality will be preserved even if we translate the whole collection $\{C_i : i \in I\}$.

Proposition 3.3.9. *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X. Then, the following statements are equivalent.*

(i) $\overline{\operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i} \cap \Lambda^*)}^{w^*} \subseteq \sum_{i \in I} \operatorname{epi} \sigma_{C_i}$.

(ii) *For each $x \in X$,* $\overline{\operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i - x} \cap \Lambda^*)}^{w^*} \subseteq \sum_{i \in I} \operatorname{epi} \sigma_{C_i - x}$.

Proof. Following the same argument as that was shown in the proof of Proposition 3.2.7, one can show that, for each $z \in X$ and $(z^*, \beta) \in X^* \times \mathbb{R}$, we have

$$(z^*, \beta) \in \sum_{i \in I} \operatorname{epi} \sigma_{C_i - z} \Leftrightarrow (z^*, \beta + \langle z^*, z \rangle) \in \sum_{i \in I} \operatorname{epi} \sigma_{C_i},$$
$$(z^*, \beta) \in \operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i - z} \cap \Lambda^*) \Leftrightarrow (z^*, \beta + \langle z^*, z \rangle) \in \operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i} \cap \Lambda^*).$$

To finish the proof, it suffices to prove that **(i)** $\Rightarrow$ **(ii)** because the implication **(ii)** $\Rightarrow$ **(i)** is trivial. Suppose that **(i)** holds and fix $x \in X$. Let $(x^*, \alpha) \in \overline{\operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i - x} \cap \Lambda^*)}^{w^*}$. Then, one can find a directed set $\mathbb{D}$ and a net $\{(x^*_\nu, \alpha_\nu)\}_{\nu \in \mathbb{D}} \subseteq \operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i - x} \cap \Lambda^*)$ such that $(x^*_\nu, \alpha_\nu) \longrightarrow^{w^*} (x^*, \alpha)$. Note that for each $\nu \in D$, one has $(x^*_\nu, \alpha_\nu + \langle x^*_\nu, x \rangle) \in \operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i} \cap \Lambda^*)$. Since $(x^*_\nu, \alpha_\nu + \langle x^*_\nu, x \rangle) \longrightarrow^{w^*} (x^*, \alpha + \langle x^*, x \rangle)$, it follows from **(i)** that $(x^*, \alpha + \langle x^*, x \rangle) \in \sum_{i \in I} \operatorname{epi} \sigma_{C_i}$, which in turn implies that $(x^*, \alpha) \in \sum_{i \in I} \operatorname{epi} \sigma_{C_i - x}$. So, **(ii)** holds. □

The following technical lemma will be used in the next theorem and Section 3.4.

Lemma 3.3.10 ([19, Lemma 4.7]). *Let X be a normed linear space and I be a metric space. Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X. Assume* *that the set-valued map $i \mapsto Z \cap C_i$ is lower semicontinuous on I. Suppose that there exist a directed set $\mathbb{D}$, a net $\{i_\nu\}_{\nu\in\mathbb{D}} \subseteq I$, $(x^*_{i_\nu}, \alpha_{i_\nu}) \in \operatorname{epi}\sigma_{C_{i_\nu}}$, $i_0 \in I$, $(x^*_0, \alpha_0) \in X^* \times \mathbb{R}$ and a subspace Z of X such that $i_\nu \longrightarrow i_0$, $\alpha_{i_\nu} \longrightarrow \alpha_0$, $x^*_{i_\nu}|_Z \longrightarrow x^*_0|_Z$ and $\{x^*_{i_\nu}|_Z\}_{\nu\in\mathbb{D}}$ is bounded. Then, $(x^*_0, \alpha_0) \in \operatorname{epi}\sigma_{Z\cap C_{i_0}}$.*

Proof. We have to show that $\sigma_{Z\cap C_{i_0}}(x^*_0) \le \alpha_{i_0}$. To do this, it suffices to show that for each $z \in Z\cap C_{i_0}$, $\langle x^*_0, z\rangle \le \alpha_0$. Fix $z \in Z \cap C_{i_0}$. Since $i \mapsto Z\cap C_i$ is lower semicontinuous at i_0, it follows from Theorem 2.7.1 that for each $j \in I$, there exists some $z_j \in Z \cap C_j$ such that $z_j \longrightarrow z$ as $j \to i_0$. In particular, $z_{i_\nu} \in Z \cap C_{i_\nu}$ for any $\nu \in \mathbb{D}$, and $z_{i_\nu} \longrightarrow z$ (thanks to $i_\nu \longrightarrow i_0$). Note that

$$\langle x^*_0, z\rangle = \langle x^*_0 - x^*_{i_\nu}, z\rangle + \langle x^*_{i_\nu}, z - z_{i_\nu}\rangle + \langle x^*_{i_\nu}, z_{i_\nu}\rangle \le \langle x^*_0 - x^*_{i_\nu}, z\rangle + \|x^*_{i_\nu}|_Z\|_Z \|z - z_{i_\nu}\| + \alpha_{i_\nu}. \tag{3.3.19}$$

By passing to limits in (3.3.19), we obtain $\langle x^*_0, z\rangle \le \alpha_0$. □

We are ready to give the promised sufficient condition for (3.3.18). This is from [19, Theorem 4.8].

Theorem 3.3.11 ([19, Theorem 4.8]). *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i\in I} C_i \neq \emptyset$. Suppose that I is a compact metric space and the set-valued map $i \mapsto C_i$ is lower semicontinuous on I. Then,* (3.3.18) *holds if at least one of the following conditions holds.*

(i) *I is finite.*

(ii) *There exists some $i_0 \in I$ such that C_{i_0} is of finite dimension.*

Proof. In view of the result of Proposition 3.3.9, we can assume (by translating the sets if necessary) that $0 \in \bigcap_{i\in I} C_i$. Then, for any $i \in I$, σ_{C_i} is a non-negative real-valued function on X^*. Then, it readily follows that

$$\operatorname{epi}\sigma_{C_i} \cap \Lambda^* = \operatorname{epi}\sigma_{C_i} \cap \Lambda^*_+ \quad \text{for each } i \in I.$$

So, (3.3.18) is equivalent to

$$\overline{\operatorname{co}\bigcup_{i\in I}(\operatorname{epi}\sigma_{C_i}\cap\Lambda_+^*)}^{w^*}\subseteq\sum_{i\in I}\operatorname{epi}\sigma_{C_i} \tag{3.3.20}$$

Thus, the proof will be completed if we can show (3.3.20) for each of the cases **(i)** and **(ii)**.

For **(i)**: Let I be finite, say $I=\{1,2,\dots,m\}$, where $m\in\mathbb{N}$. Let $(\overline{z}^*,\overline{\alpha})\in\overline{\operatorname{co}\bigcup_{i=1}^m(\operatorname{epi}\sigma_{C_i}\cap\Lambda_+^*)}^{w^*}$. Then, there exist a directed set $\mathbb{D}$ and a net $\{(\overline{z}_\nu^*,\overline{\alpha}_\nu)\}_{\nu\in\mathbb{D}}$ in $\operatorname{co}\bigcup_{i=1}^m(\operatorname{epi}\sigma_{C_i}\cap\Lambda_+^*)$ such that $(\overline{z}_\nu^*,\overline{\alpha}_\nu)\longrightarrow^{w^*}(\overline{z}^*,\overline{\alpha})$. For each $\nu\in\mathbb{D}$, there exist some $(z_{\nu,i}^*,\alpha_{\nu,i})\in\operatorname{epi}\sigma_{C_i}\cap\Lambda_+^*$, $\lambda_{\nu,i}\in[0,1]$ with $\sum_{i=1}^m\lambda_{\nu,i}=1$ such that

$$(\overline{z}_\nu^*,\overline{\alpha}_\nu)=\sum_{i=1}^m\lambda_{\nu,i}(z_{\nu,i}^*,\alpha_{\nu,i}). \tag{3.3.21}$$

By passing to subnet if necessary, one can assume that $0\le\overline{\alpha}_\nu\le\mu$ for any $\nu\in\mathbb{D}$, where $\mu:=\overline{\alpha}+1$. Note that for any $\nu\in\mathbb{D}$ and $i\in\{1,\dots,m\}$,

$$0\le\lambda_{\nu,i}\alpha_{\nu,i}\le\overline{\alpha}_\nu\le\mu. \tag{3.3.22}$$

Also, since $\lambda_{\nu,i}\in[0,1]$ and $\operatorname{epi}\sigma_{C_i}$ is a cone, we have

$$(\lambda_{\nu,i}z_{\nu,i}^*,\lambda_{\nu,i}\alpha_{\nu,i})\in\operatorname{epi}\sigma_{C_i}\cap\Lambda_+^*. \tag{3.3.23}$$

By the weakly* compactness of $\mathbf{B}_{\mathbf{X}^*}$ (which follows from Banach-Alaoglu Theorem, see [21, Theorem 2.6.18]), (3.3.22) and (3.3.23), one can assume, without loss of generality, that for each $i\in\{1,\dots,m\}$, there exist some $y_i^*\in\mathbf{B}_{\mathbf{X}^*}$ and $\beta_i\in[0,\mu]$ such that

$$\lambda_{\nu,i}z_{\nu,i}^*\longrightarrow^{w^*}y_i^*\quad\text{and}\quad\lambda_{\nu,i}\alpha_{\nu,i}\longrightarrow\beta_i.$$

In particular, it follows that $(\lambda_{\nu,i}z_{\nu,i}^*,\lambda_{\nu,i}\alpha_{\nu,i})\longrightarrow^{w^*}(y_i^*,\beta_i)$. By the weakly* closedness of $\operatorname{epi}\sigma_{C_i}$ and (3.3.23), one sees that $(y_i^*,\beta_i)\in\operatorname{epi}\sigma_{C_i}$. So, by taking limits in (3.3.21), we obtain

$$(\overline{z}^*,\overline{\alpha})=w^*\text{-}\lim_\nu\sum_{i=1}^m\lambda_{\nu,i}(z_{\nu,i}^*,\alpha_{\nu,i})=\sum_{i=1}^m(y_i^*,\beta_i)\in\sum_{i=1}^m\operatorname{epi}\sigma_{C_i}.$$

Since $(\overline{z}^*, \overline{\alpha}) \in \overline{\mathrm{co} \bigcup_{i=1}^{m} (\mathrm{epi}\, \sigma_{C_i} \cap \Lambda_+^*)}^{w^*}$ is arbitrary, we have proved that the set inclusion (3.3.20) holds for the case **(i)**.

Next, we turn to consider the case **(ii)**: there exists some $i_0 \in I$ such that C_{i_0} is of finite dimension. Define $Y := \mathrm{span}\, C_{i_0} (= \mathrm{aff}\, C_{i_0})$ and let m be the dimension of Y. We will show that (3.3.20) holds. Let $(\overline{z}^*, \overline{\alpha}) \in \overline{\mathrm{co} \bigcup_{i\in I} (\mathrm{epi}\, \sigma_{C_i} \cap \Lambda_+^*)}^{w^*}$. Then there exist a directed set $\mathbb{D}$ and a net $\{(\overline{z}_\nu^*, \overline{\alpha}_\nu)\}_{\nu\in\mathbb{D}} \subseteq \mathrm{co} \bigcup_{i\in I} (\mathrm{epi}\, \sigma_{C_i} \cap \Lambda_+^*)$ such that $(\overline{z}_\nu^*, \overline{\alpha}_\nu) \longrightarrow^{w^*} (\overline{z}^*, \overline{\alpha})$. Note that $\mathrm{co} \bigcup_{i\in I} (\mathrm{epi}\, \sigma_{C_i} \cap \Lambda_+^*)|_{Y\times\mathbb{R}} \subseteq Y^* \times \mathbb{R}$ and $Y^* \times \mathbb{R}$ is of dimension $m+1$. Hence, by the virtue of Carathéodory Theorem [25, Corollary 17.1.1], one can express each $(\overline{z}_\nu^*|_Y, \overline{\alpha}_\nu)$ as a convex combination of $m+2$ elements from $\bigcup_{i\in I} (\mathrm{epi}\, \sigma_{C_i} \cap \Lambda_+^*)|_{Y\times\mathbb{R}}$. So, for each $\nu \in \mathbb{D}$, there exist $\{i_j^\nu\}_{j=1}^{m+2} \subseteq I$, $\{\lambda_{\nu,j}\}_{j=1}^{m+2} \subseteq [0,1]$ with $\sum_{j=1}^{m+2} \lambda_{\nu,j} = 1$, and

$$(z_{\nu,j}^*, \alpha_{\nu,j}) \in \mathrm{epi}\, \sigma_{C_{i_j^\nu}} \cap \Lambda_+^* \quad \text{for each}\, j = 1, \ldots, m+2 \tag{3.3.24}$$

such that

$$(\overline{z}_\nu^*|_Y, \overline{\alpha}_\nu) = \sum_{j=1}^{m+2} \lambda_{\nu,j} (z_{\nu,j}^*|_Y, \alpha_{\nu,j}). \tag{3.3.25}$$

In particular, since $\mathrm{epi}\, \sigma_i$ is a cone for all $i \in I$, (3.3.24) implies that

$$(\lambda_{\nu,j} z_{\nu,j}^*, \lambda_{\nu,j} \alpha_{\nu,j}) \in \mathrm{epi}\, \sigma_{C_{i_j^\nu}} \quad \text{for each}\, \nu \in \mathbb{D} \text{ and } j = 1, \ldots, m+2. \tag{3.3.26}$$

Since $\overline{\alpha}_\nu \longrightarrow \overline{\alpha}$, one can assume without loss of generality that $0 \le \overline{\alpha}_\nu \le \mu$ for all $\nu \in \mathbb{D}$, where $\mu := \alpha + 1$. Hence, for each $j = 1, \ldots, m+2$, one has

$$\{\lambda_{\nu,j} z_{\nu,j}^*\}_{\nu\in\mathbb{D}} \subseteq \mathbf{B}_{\mathbf{X}^*} \text{ and } \{\lambda_{\nu,j} \alpha_{\nu,j}\}_{\nu\in\mathbb{D}} \subseteq [0, \mu]. \tag{3.3.27}$$

By the weakly* compactness of $\mathbf{B}_{\mathbf{X}^*}$ and passing to subsets if necessary, one can assume that there exist some $\{y_j^*\}_{j=1}^{m+2} \subseteq \mathbf{B}_{\mathbf{X}^*}$ and $\{\beta_j\}_{j=1}^{m+2} \subseteq [0, \mu]$ such that

$$\lambda_{\nu,j} z_{\nu,j}^* \longrightarrow^{w^*} y_j^* \text{ and } \lambda_{\nu,j} \alpha_{\nu,j} \longrightarrow \beta_j \quad \text{for each}\, j = 1, \ldots, m+2. \tag{3.3.28}$$

Furthermore, by the compactness of I, one can assume that there exist $\{\overline{i}_j\}_{j=1}^{m+2} \subseteq I$ such that

$$i_j^\nu \longrightarrow \overline{i}_j \quad \text{for each}\, j = 1, \ldots, m+2. \tag{3.3.29}$$

In view of (3.3.26), (3.3.27), (3.3.28) and (3.3.29), we can apply Lemma 3.3.10 (to X in place of Z) to see that

$$(y_j^*, \beta_j) \in \operatorname{epi} \sigma_{C_{\bar{i}_j}} \quad \text{for each } j = 1, \ldots, m+2. \tag{3.3.30}$$

Now, by taking limits in (3.3.25), we obtain

$$(\overline{z}^*|_Y, \overline{\alpha}) = \sum_{j=1}^{m+2} (y_j^*|_Y, \beta_j). \tag{3.3.31}$$

Take $d^* := \overline{z}^* - \sum_{j=1}^{m+2} y_j^*$. Then, $d^*|_Y = 0$ and hence $(d^*, 0) \in Y^\perp \times \mathbb{R}_+$. Note that $Y^\perp \times \mathbb{R}_+ \subseteq \operatorname{epi} \sigma_{C_{i_0}}$, thanks to the fact that $0 \in C_{i_0} \subseteq Y$. So, $(d^*, 0) \in \operatorname{epi} \sigma_{C_{i_0}}$. Since $(\overline{z}^*, \overline{\alpha}) = (d^*, 0) + \sum_{j=1}^{m+2} (y_j^*, \beta_j)$ (by (3.3.31)), it follows from (3.3.30) that

$$(\overline{z}^*, \overline{\alpha}) \in \operatorname{epi} \sigma_{C_{i_0}} + \sum_{j=1}^{m+2} \operatorname{epi} \sigma_{C_{\bar{i}_j}} \subseteq \sum_{i \in I} \operatorname{epi} \sigma_{C_i}.$$

Therefore, (3.3.20) follows. This completes the proof. □

The following corollary is an immediate consequence of Theorems 3.3.8 and 3.3.11.

Corollary 3.3.12. *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i \in I} C_i \neq \emptyset$. Suppose that I is a compact metric space and the set-valued map $i \mapsto C_i$ is lower semicontinuous on I. Assume that $\{C_i : i \in I\}$ is linearly regular. Then, $\{C_i : i \in I\}$ satisfies the SECQ if at least one of the following conditions holds.*

(i) *I is finite.*

(ii) *There exists some $i_0 \in I$ such that C_{i_0} is of finite dimension.*

Below we give an example to show that the linear regularity of $\{C_i : i \in I\}$ may not imply the SECQ, when I is infinite.

Example 3.3.2 ([19, Example 4.1]). Let $X := \mathbb{R}^2$ and $I := \mathbb{N}$. For each $i \in \mathbb{N}$, define

$$C_i := \{x \in \mathbb{R}^2 : \|x\| \leq \frac{1}{i}\}.$$

Then, $C := \bigcap_{i \in \mathbb{N}} C_i = \{0\}$. Note that

$$d(x, C_i) = \max\{0, \|x\| - \frac{1}{i}\} \quad \text{for any } x \in \mathbb{R}^2 \text{ and } i \in \mathbb{N}.$$

Then,

$$d(x, C) = d(x, 0) = \|x\| = \sup_{i \in \mathbb{N}} d(x, C_i) \quad \text{for any } x \in \mathbb{R}^2.$$

So, $\{C_i : i \in I\}$ is linearly regular. On the other hand, one has that

$$N_C(0) = N_{\{0\}}(0) = \mathbb{R}^2 \quad \text{and} \quad N_{C_i}(0) = \{0\} \quad \text{for any } i \in \mathbb{N}.$$

Hence,

$$N_C(0) = \mathbb{R}^2 \neq \{0\} = \sum_{i \in I} N_{C_i}(0).$$

So, $\{C_i : i \in \mathbb{N}\}$ does not satisfy the strong CHIP. This implies that $\{C_i : i \in I\}$ does not satisfy the SECQ, thanks to Theorem 3.3.2.

3.4 Interior-point conditions for the SECQ

As we have seen in Theorem 3.3.2, the SECQ serves as a sufficient condition for the strong CHIP. Since sufficient conditions for strong CHIP were extensively studied in the literature, it is natural to compare the SECQ with some known sufficient conditions for the strong CHIP, while most of them are interior-point type conditions. In this section, we will study the relationship between the SECQ and some interior-point type conditions. It turns out that the SECQ is weaker than that. Our analysis will be divided into two parts, the first part will be devoted to study the case when I is finite, and the case when I is allowed to be infinite will be discussed in the second part.

3.4.1 I is finite

Given a collection of closed convex sets $\{C_i : i \in I\}$ in X, we assume that I is finite throughout this subsection. First, we prove the following theorem, which is more general than what we need at this stage. It gives a formula for computing the conjugate of the sum of two functions under some additional assumptions. Indeed, this theorem will be further elaborated in Chapter 4. Recall that given a set A in X, $\operatorname{core} A := \{a \in A : \forall x \in X, \exists \delta > 0 \text{ s.t. } \forall \lambda \in [0, \delta), x + \lambda a \in A\}$.

Theorem 3.4.1. *Let $f, g \in \Gamma(X)$ be such that with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Suppose that at least one of the following conditions hold:*

(i) *There exists some $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ such that g is continuous at x_0*

(ii) *X is a Banach space and $\operatorname{dom} f \cap \operatorname{int}(\operatorname{dom} g) \neq \emptyset$.*

(iii) *X is a Banach space and $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$.*

(iv) *X is a Banach space and $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$.*

(v) *X is a Banach space and $\bigcup_{\lambda \geq 0} \lambda(\operatorname{dom} f - \operatorname{dom} g)$ is a closed subspace of X.*

Then, $(f+g)^* = f^* \square g^*$ with exact infimal convolution. Moreover, one has

$$\operatorname{epi}(f+g)^* = \operatorname{epi} f^* + \operatorname{epi} g^*. \tag{3.4.1}$$

Proof. It is easy to check that **(ii)** implies **(iii)**. By [31, Theorem 2.8.7], one sees that any of **(i)** and **(iii)**-**(v)** implies that $(f+g)^* = f^* \square g^*$ with exact infimal convolution, and so does **(ii)**. Further, (3.4.1) follows from the first assertion and Proposition 2.6.2**(ii)**. □

Using the previous theorem, the following result follows, which discusses the SECQ for two sets. It is the result by Jeyakumar et. al. (see [9, Proposition 3.1]).

Theorem 3.4.2 ([9, Proposition 3.1]). *Let $C_1, C_2 \subseteq X$ be two closed convex sets with $C_1 \cap C_2 \neq \emptyset$. Suppose that at least one of the following conditions hold:*

(i) $C_1 \cap \operatorname{int} C_2 \neq \emptyset$.

(ii) *X is a Banach space and $0 \in \operatorname{core}(C_1 - C_2)$.*

(iii) *X is a Banach space and $\bigcup_{\lambda \geq 0} \lambda(C_1 - C_2)$ is a closed subspace of X.*

Then, $\{C_1, C_2\}$ satisfies the SECQ.

Proof. This follows from Theorem 3.4.1 (applied to the case when $f = \delta_{C_1}$ and $g = \delta_{C_2}$) and the fact that $(\delta_{C_1} + \delta_{C_2})^* = \delta^*_{C_1 \cap C_2} = \sigma_{C_1 \cap C_2}$ on X^*. □

Below we give an example of a system of two sets which satisfies the SECQ (and so as the strong CHIP, by Theorem 3.3.2), while it does not satisfy the conditions **(i)**-**(iii)** in Theorem 3.4.2. In view of the result of Theorem 3.3.2 and 3.4.2, we see that for a system of two sets, the SECQ is the weakest sufficient condition for the strong CHIP when compare with conditions **(i)**-**(iii)**.

Example 3.4.1. Recall the setting of Example 3.2.1: Let $X := \mathbb{R}$. Let $C := [0, 1]$ and $D := [-1, 0]$. Then, $C \cap D = \{0\}$. We have seen that $\{C, D\}$ satisfies the SECQ. Now, note that

$$N_C(0) = (-\infty, 0], \quad N_D(0) = [0, +\infty), \quad N_{C\cap D}(0) = \mathbb{R}.$$

Hence,

$$N_{C\cap D}(0) = \mathbb{R} = N_C(0) + N_D(0).$$

So, $\{C, D\}$ has the strong CHIP. On the other hand, note that

$$C \cap \operatorname{int} D = \emptyset = D \cap \operatorname{int} C.$$

Also, $\operatorname{core}(C - D) = (0, 2)$, and thus $0 \notin \operatorname{core}(C - D)$. Moreover, $\bigcup_{\lambda \geq 0} \lambda(C - D) = [0, +\infty)$, which is not a subspace. Therefore, we see that $\{C, D\}$ does not satisfy the conditions **(i)**-**(iii)**.

For the case when $|I| > 2$, we give the following theorem, which follows easily from part of Theorem 3.4.1 and the fact that for any $m \in \mathbb{N}$ and $f_1, \cdots, f_m \in \Gamma(X)$ with $\bigcap_{i=1}^m (\operatorname{dom} f_i) \neq \emptyset$,

$$(f_1 \Box \cdots \Box f_{m-1}) \Box f_m = (f_1 \Box \cdots \Box f_m),$$

where the exactness of either one side will imply that of the other side.

Theorem 3.4.3. *Let $I := \{1, \ldots, m\}$ for some $m \in \mathbb{N}$. Let $\{f_i : i \in I\} \subseteq \Gamma(X)$ be such that $\bigcap_{i \in I} (\operatorname{dom} f_i) \neq \emptyset$. Suppose that at least one of the following conditions hold.*

(i) *There exists some $i_0 \in I$ and $x_0 \in X$ such that all functions in $\{f_i : i \in I \backslash \{i_0\}\}$ are continuous at x_0.*

(ii) *X is a Banach space and there exists some $i_0 \in I$ such that $\operatorname{dom} f_{i_0} \cap \operatorname{int} (\bigcap_{i \in I \backslash \{i_0\}} (\operatorname{dom} f_i)) \neq \emptyset$.*

Then,

$$(\sum_{i=1}^m f_i)^* = f_1^* \Box \cdots \Box f_m^* \quad \text{with exact infimal convolution.}$$

Moreover, one has

$$\operatorname{epi} (\sum_{i=1}^m f_i)^* = \sum_{i=1}^m \operatorname{epi} f_i^*. \tag{3.4.2}$$

Theorem 3.4.4. *Let $\{C_i : i \in I\}$ be a finite collection of closed convex sets in X with $\bigcap_{i \in I} C_i \neq \emptyset$. Suppose that there exists some $i_0 \in I$ such that $C_{i_0} \cap \operatorname{int} (\bigcap_{i \in I \backslash \{i_0\}} C_i) \neq \emptyset$. Then, $\{C_i : i \in I\}$ satisfies the SECQ.*

3.4.2 I is infinite

In this subsection, we suppose that the index set I is a compact metric space, while I is allowed to be infinite. In [18], Li and Ng gave some interior-point type conditions for the strong CHIP of infinite system of closed convex sets in a general normed linear space. By considering similar type of conditions, in [19], they gave sufficient conditions for the SECQ when I is allowed to be infinite. Here, we will show their results and we begin with the following two definitions.

Definition 3.4.1 ([18, 19]). *Let $D \subseteq X$ be a closed convex set and $\{C_i : i \in I\}$ be a collection of closed convex sets in X. Then, the collection $\{D, C_i : i \in I\}$ is said to be a closed convex set system with base set D (abbreviated as CCS-system with base set D) if $D \cap \bigcap_{i\in I} C_i \neq \emptyset$.*

Definition 3.4.2 (see [19]). *Let $\{D, C_i : i \in I\}$ be a CCS-system with base set D in X and $m \in \mathbb{N}$. The CCS-system is said to satisfy m-D-interior point condition if*

$$D \cap \left(\bigcap_{j\in J} \operatorname{rint}_D C_j \right) \neq \emptyset \qquad \text{for any } J \subseteq I \text{ with } |J| \leq m.$$

Suppose that we are now given a CCS-system $\{D, C_i : i \in I\}$ with base set D. We will show some sufficient conditions for the SECQ of this CCS-system. In order to simplify the proofs that we are going to show, we assume that $0 \in D \cap \bigcap_{i\in I} C_i$ (we are allowed to do so as the SECQ and the sufficient conditions that we consider are all translational invariant). Now, we first prove the following technical lemma. Recall that given a set $A \subseteq X$, the orthogonal complement of A is defined by

$$A^{\perp} := \{x^* \in X^* : \langle x^*, a\rangle = 0, \ \ \forall a \in A\}.$$

Lemma 3.4.5 ([19, Lemma 5.2]). *Let $\{D, C_i : i \in I\}$ be a CCS-system with base set D such that $0 \in D \cap \bigcap_{i\in I} C_i$. Let $m \in \mathbb{N}$ and $Z := \operatorname{span} D$. Suppose that the following conditions are satisfied.*

(i) *Z is finite dimensional.*

(ii) *The set-valued map $i \mapsto Z \cap C_i$ is lower semicontinuous on I.*

(iii) *The CCS-system $\{D, C_i : i \in I\}$ satisfies the m-D-interior point condition.*

Furthermore, let $(x^*, \xi) \in X^* \times \mathbb{R}$ *and a sequence* $\{(x_k^*, \xi_k)\}_{k\in\mathbb{N}} \subseteq \operatorname{epi}\sigma_D + \sum_{i\in I}\operatorname{epi}\sigma_{C_i}$ *be such that*

$$(x_k^*|_Z, \xi_k) \longrightarrow^{\|\cdot\|_{Z\times\mathbb{R}}} (x^*|_Z, \xi), \tag{3.4.3}$$

where each $(x_k^*|_Z, \xi_k)$ *can be expressed in the form*

$$(x_k^*|_Z, \xi_k) = (v_k^*|_Z, \beta_k) + \sum_{j=1}^{m}(w_{i_j^k}^*|_Z, \gamma_{i_j^k}), \tag{3.4.4}$$

with

$$(v_k^*, \beta_k) \in \operatorname{epi}\sigma_D \qquad \text{and} \qquad (w_{i_j^k}^*, \gamma_{i_j^k}) \in \operatorname{epi}\sigma_{C_{i_j^k}} \tag{3.4.5}$$

for some $i_1^k, \ldots, i_m^k \in I$. *Then,*

$$(x^*, \xi) \in \operatorname{epi}\sigma_D + \sum_{i\in I}\operatorname{epi}\sigma_{Z\cap C_i}.$$

Proof. By the compactness of I, one can (by passing to subsequences if necessary) assume that for each $j \in \{1, \ldots, m\}$, there exists some $i_j \in I$ such that $i_j^k \longrightarrow i_j$ as $k \longrightarrow \infty$. By the assumption **(iii)**, there exists some $z_0 \in D \cap \bigcap_{j=1}^m \operatorname{rint}_D C_{i_j}$. Hence, one can find some $\delta' > 0$ such that $B(z_0, \delta') \cap Z \subseteq C_{i_j}$ for any $j \in \{1, \ldots, m\}$. In particular, this implies that

$$B(z_0,\delta)\cap Z \subseteq \operatorname{rint}_Z(B(z_0,\delta')\cap Z) \subseteq \operatorname{rint}_Z(C_{i_j}\cap Z) \subseteq C_{i_j}\cap Z \quad \text{for any } j \in \{1,\ldots,m\}, \tag{3.4.6}$$

where $\delta := \delta'/2$. Since $B(z_0, \delta) \cap Z$ is a closed and bounded set in the finite dimensional space Z, we see that $B(z_0, \delta) \cap Z$ is compact. Thus, by (3.4.6), **(ii)** and Proposition 2.7.2 (applied to the space Z in place of X), there exists some $K_0 \in \mathbb{N}$ such that

$$B(z_0, \delta) \cap Z \subseteq C_{i_j^k} \cap Z \qquad \text{for any } k \in \mathbb{N} \text{ with } k \geq K_0. \tag{3.4.7}$$

Now, we show that for each $j \in \{1, \ldots, m\}$, the sequence $\{w_{i_j^k}^*|_Z\}_{k\in\mathbb{N}}$ is bounded in Z^*. To show this, we first note that, for each $j \in \{1, \ldots, m\}$ and

$k \geq K_0$, one can apply (3.4.5) and (3.4.7) to get

$$\begin{aligned}\gamma_{i_j^k} \geq \sigma_{C_{i_j^k}}(w_{i_j^k}^*) \geq \sigma_{Z\cap C_{i_j^k}}(w_{i_j^k}^*) &= \sup_{z\in Z\cap C_{i_j^k}} \langle w_{i_j^k}^*, z\rangle \\ &\geq \sup_{z\in Z\cap B(z_0,\delta)} \langle w_{i_j^k}^*, z\rangle \\ &= \langle w_{i_j^k}^*, z_0\rangle + \delta \sup_{z\in Z\cap \mathbf{B}_{\mathbf{X}^*}} \langle w_{i_j^k}^*, z\rangle \\ &= \langle w_{i_j^k}^*, z_0\rangle + \delta\|w_{i_j^k}^*|_Z\|_Z,\end{aligned}$$

and so

$$\gamma_{i_j^k} - \langle w_{i_j^k}^*, z_0\rangle \geq \delta\|w_{i_j^k}^*|_Z\|_Z.$$

Since $(v_k^*, \beta_k) \in \operatorname{epi}\sigma_D$ and so $\beta_k \geq \sigma_D(v_k^*) \geq \langle v_k^*, z_0\rangle$, it follows from (3.4.4) that for any $k \in \mathbb{N}$ with $k \geq K_0$,

$$\begin{aligned}0 \leq \sum_{j=1}^m \delta\|w_{i_j^k}^*|_Z\|_Z \leq \sum_{j=1}^m (\gamma_{i_j^k} - \langle w_{i_j^k}^*, z_0\rangle) &\\ = (\xi_k - \beta_k) - \langle (x_k^* - v_k^*)|_Z, z_0\rangle &\\ \leq \xi_k - \langle x_k^*|_Z, z_0\rangle. &\end{aligned}$$

Combining with the fact that $\{\xi_k - \langle x_k^*|_Z, z_0\rangle\}_{k\in\mathbb{N}}$ is bounded (thanks to (3.4.3)), we can conclude that $\{w_{i_j^k}^*|_Z\}_{k\in N}$ is a bounded sequence in Z^* for each $j \in \{1, \dots, m\}$.

Note that $\{x_k^*|_Z\}_{k\in\mathbb{N}}$ is bounded in Z^* by (3.4.3). So, it follows from (3.4.4) and the boundedness of the sequence $\{w_{i_j^k}^*|_Z\}_{k\in\mathbb{N}}$ that the sequence $\{v_k^*|_Z\}_{k\in\mathbb{N}}$ is bounded in Z^*. By considering subsequences if necessary, one assume henceforth that there exist some $\tilde{v}^*, \tilde{w}_1^*, \dots, \tilde{w}_m^* \in Z^*$ such that $v_k^*|_Z \longrightarrow^{\|\cdot\|_Z} \tilde{v}^*$, and $w_{i_j^k}^*|_Z \longrightarrow^{\|\cdot\|_Z} \tilde{w}_j^*$ for all $j \in \{1, \dots, m\}$. Moreover, observe that $\beta_k \geq \sigma_D(v_k^*) \geq 0$ and $\gamma_{i_j^k} \geq \sigma_{C_{i_j^k}}(w_{i_j^k}^*) \geq 0$. Thus, by the boundedness of the sequence $\{\xi_k\}_{k\in\mathbb{N}}$, it follows from (3.4.4) that the sequences $\{\beta_k\}_{k\in\mathbb{N}}$ and $\{\gamma_{i_j^k}\}_{k\in\mathbb{N}}$ (for $j \in \{1, \dots, m\}$) are bounded. Without loss of generality, we can assume further that there exist $\beta \geq 0$ and $\gamma_1, \dots, \gamma_m \geq 0$ such that $\beta_k \longrightarrow \beta$ and $\gamma_{i_j^k} \longrightarrow \gamma_j$ for all $j \in \{1, \dots, m\}$.

By Hahn-Banach extension theorem, let $v^* \in X^*$ be an extension of $\tilde{v}^*$ from Z to X, and for each $j \in \{1, \ldots, m\}$, $w_j^* \in X^*$ be an extension of $\tilde{w}_j^*$ from Z to X. So it follows from (3.4.4) that

$$x^*|_Z = v^*|_Z + \sum_{j=1}^{m} w_j^*|_Z \qquad \textit{and} \qquad \xi = \beta + \sum_{j=1}^{m} \gamma_j \,. \tag{3.4.8}$$

It is an easy consequence from the Hahn-Banach extension theorem that

$$\operatorname{epi} \sigma_D|_{Z\times\mathbb{R}} = \operatorname{epi} \hat{\sigma}_D,$$

where $\hat{\sigma}_D$ is the support function of D in the space Z. Since $\{(v_k^*, \beta_k)\}_{k\in\mathbb{N}} \subseteq \operatorname{epi} \sigma_D$ and $(v_k^*|_Z, \beta_k) \longrightarrow (v^*|_Z, \beta)$, one has

$$(v^*|_Z, \beta) \in \overline{(\operatorname{epi} \sigma_D)|_{Z\times\mathbb{R}}}^{\|\cdot\|_{Z\times\mathbb{R}}} = \overline{\operatorname{epi} \hat{\sigma}_D}^{\|\cdot\|_{Z\times\mathbb{R}}} = \operatorname{epi} \hat{\sigma}_D.$$

So, $(v^*, \beta) \in \operatorname{epi} \sigma_D$, thanks to the fact that $D \subseteq Z$. Also, one can make use of (3.4.5) and apply Lemma 3.3.10 to see that $(w_j^*, \gamma_j) \in \operatorname{epi} \sigma_{Z\cap C_{i_j}}$ for each $j \in \{1, \ldots, m\}$. Define $d^* := x^* - v^* - \sum_{j=1}^{m} w_j^*$. Then, $(d^*, 0) \in Z^\perp \times \{0\}$ by (3.4.8). Thus,

$$(x^*, \xi) = (d^*, 0) + (v^*, \beta) + \sum_{j=1}^{m} (w_j^*, \gamma_j) \in Z^\perp \times \{0\} + \operatorname{epi} \sigma_D + \sum_{i\in I} \operatorname{epi} \sigma_{Z\cap C_i}.$$

It is direct from definition that $Z^\perp \times \{0\} \subseteq \operatorname{epi} \sigma_D$. Therefore, with the set inclusion $Z^\perp \times \{0\} + \operatorname{epi} \sigma_D \subseteq \operatorname{epi} \sigma_D$, we see that $(x^*, \xi) \in \operatorname{epi} \sigma_D + \sum_{i\in I} \operatorname{epi} \sigma_{Z\cap C_i}$. This completes the proof. □

The following theorem is the main result in this subsection, this is from [19, Theorem 5.3].

Theorem 3.4.6 ([19, Theorem 5.3]). *Let $\{D, C_i : i \in I\}$ be a CCS-system with base set D such that $0 \in D \cap \bigcap_{i\in I} C_i$. Let $m \in \mathbb{N}$ and $Z := \operatorname{span} D$. Consider the following conditions:*

(i) *Z is of finite dimension m.*

(ii) *The set-valued map $i \mapsto Z \cap C_i$ is lower semicontinuous on I.*

(iii) *$\{D, C_i : i \in I\}$ satisfies the $(m+1)$-D-interior point condition.*

(iv) *For each $i \in I$, $\{D, C_i\}$ satisfies the SECQ.*

(iii*) *$\{D, C_i : i \in I\}$ satisfies the m-D-interior point condition.*

(iv*) *For any $J \subseteq I$ with $|J| = \min\{m+1, |I|\}$, $\{D, C_j : j \in J\}$ satisfies the SECQ.*

Then the following statements hold.

(a) *If **(i)**, **(ii)**, **(iii)** and **(iv)** hold, then $\{D, C_i : i \in I\}$ satisfies the SECQ.*

(b) *Suppose that **(i)**, **(ii)**, **(iii*)** and **(iv*)** hold. Assume in addition that D is bounded. Then, $\{D, C_i : i \in I\}$ satisfies the SECQ.*

Proof. We first prove **(a)**. To begin with, we will show the set inclusion

$$\overline{\operatorname{epi}\sigma_D + \sum_{i\in I}\operatorname{epi}\sigma_{C_i}}^{w^*} \subseteq \operatorname{epi}\sigma_D + \sum_{i\in I}\operatorname{epi}\sigma_{Z\cap C_i}. \tag{3.4.9}$$

Let $(x^*, \xi) \in \overline{\operatorname{epi}\sigma_D + \sum_{i\in I}\operatorname{epi}\sigma_{C_i}}^{w^*}$. Note that

$$\begin{aligned}(\overline{\operatorname{epi}\sigma_D + \sum_{i\in I}\operatorname{epi}\sigma_{C_i}}^{w^*})|_{Z\times\mathbb{R}} &\subseteq \overline{(\operatorname{epi}\sigma_D + \sum_{i\in I}\operatorname{epi}\sigma_{C_i})|_{Z\times\mathbb{R}}}^{w^*} \\ &= \overline{(\operatorname{epi}\sigma_D + \sum_{i\in I}\operatorname{epi}\sigma_{C_i})|_{Z\times\mathbb{R}}}^{\|\cdot\|_{Z\times\mathbb{R}}},\end{aligned}$$

where the set inclusion follows from direct checking of definition, and the set equality holds by the finite dimensionality of Z. Thus, one can pick a sequence $\{(x_k^*, \xi_k)\}_{k\in\mathbb{N}} \subseteq \operatorname{epi}\sigma_D + \sum_{i\in I}\operatorname{epi}\sigma_{C_i}$ such that

$$(x_k^*|_Z, \xi_k) \longrightarrow^{\|\cdot\|_{Z\times\mathbb{R}}} (x^*|_Z, \xi). \tag{3.4.10}$$

On the other hand, it is easy to see that $\sum_{i\in I}\operatorname{epi}\sigma_{C_i} = \operatorname{cone}(\bigcup_{i\in I}\operatorname{epi}\sigma_{C_i})$. Observe that the set $\operatorname{cone}(\bigcup_{i\in I}\operatorname{epi}\sigma_{C_i})|_{Z\times\mathbb{R}}$ is a convex cone generated by the collection $\{(\operatorname{epi}\sigma_{C_i})|_{Z\times\mathbb{R}} : i \in I\}$ of subcones in the $m+1$-dimensional space $Z^* \times \mathbb{R}$. By the Carathéodory Theorem [25, Corollary 17.1.2], it follows that for each $k \in \mathbb{N}$,

$$(x_k^*|_Z, \xi_k) = (v_k^*|_Z, \beta_k) + \sum_{j=1}^{m+1} (w_{i_j^k}^*|_Z, \gamma_{i_j^k}) \tag{3.4.11}$$

with

$$(v_k^*, \beta_k) \in \operatorname{epi} \sigma_D \quad \text{and} \quad (w_{i_j^k}^*, \gamma_{i_j^k}) \in \operatorname{epi} \sigma_{C_{i_j^k}} \quad \text{for some } i_1^k, \ldots, i_{m+1}^k \in I. \tag{3.4.12}$$

Hence, thanks to the assumption **(i)**, **(ii)**, **(iii)** and Lemma 3.4.5 (applied to $m+1$ in place of m), one has $(x^*, \xi) \in \operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{Z \cap C_i}$. So, (3.4.9) holds. In particular, it follows from Proposition 3.2.5 that

$$\operatorname{epi} \sigma_{D \cap \bigcap_{i \in I} C_i} = \overline{\operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{C_i}}^{w^*} \subseteq \operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{Z \cap C_i}. \tag{3.4.13}$$

Since $D \subseteq Z$ and thanks to the assumption **(iv)**, it follows that

$$\begin{aligned} \operatorname{epi} \sigma_{D \cap \bigcap_{i \in I} C_i} \subseteq \operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{Z \cap C_i} &\subseteq \operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{D \cap C_i} \\ &= \operatorname{epi} \sigma_D + \sum_{i \in I} (\operatorname{epi} \sigma_D + \operatorname{epi} \sigma_{C_i}) \\ &= \operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{C_i}. \end{aligned}$$

Therefore, by Corollary 3.2.6, we conclude that $\{D, C_i : i \in I\}$ satisfies the SECQ.

We now turn to prove **(b)**. Suppose that the conditions stated in **(b)** hold. In view of **(iv*)**, one assumes without loss of generality that $|I| > m+1$. We claim that (3.4.9) holds in this case. Let $(x^*, \xi) \in \overline{\operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{C_i}}^{w^*}$. By following the same procedure as shown in the proof of **(a)**, pick $\{(x_k^*, \xi_k)\}_{k \in \mathbb{N}}$, $\{(v_k^*, \beta_k)\}_{k \in \mathbb{N}}$ and $\{(w_{i_j^k}^*, \gamma_{i_j^k})\}_{k \in \mathbb{N}}$ for $j \in \{1, \ldots, m+1\}$ such that (3.4.10), (3.4.11) and (3.4.12) holds.

Fix $k \in \mathbb{N}$. Note that for any $z \in D \cap \bigcap_{j=1}^{m+1} C_{i_j^k}$,

$$\xi_k = \beta_k + \sum_{j=1}^{m+1} \gamma_{i_j^k} \geq \sigma_D(v_k^*) + \sum_{j=1}^{m+1} \sigma_{C_{i_j^k}}(w_{i_j^k}^*) \geq \langle v_k^*, z \rangle + \sum_{j=1}^{m+1} \langle w_{i_j^k}^*, z \rangle = \langle x_k^*, z \rangle,$$

which implies that

$$\xi_k \geq \sup_{z \in D \cap \bigcap_{j=1}^{m+1} C_{i_j^k}} \langle x_k^*, z \rangle = \sigma_{D \cap \bigcap_{j=1}^{m+1} C_{i_j^k}}(x_k^*).$$

Since $D \cap \bigcap_{j=1}^{m+1} C_{i_j^k}$ is compact (as $D \cap \bigcap_{j=1}^{m+1} C_{i_j^k}$ is a closed and bounded set in the finite dimensional space Z), there exists $z_0 \in D \cap \bigcap_{j=1}^{m+1} C_{i_j^k}$ such that

$$\xi_k \geq \sigma_{D \cap \bigcap_{j=1}^{m+1} C_{i_j^k}}(x_k^*) = \langle x_k^*, z_0 \rangle. \tag{3.4.14}$$

Thus, $x_k^* \in N_{D \cap \bigcap_{j=1}^{m+1} C_{i_j^k}}(z_0)$ by (2.4.4). Thanks to assumption **(iv*)**, one knows that $\{D, C_{i_j^k} : j = 1, \ldots, m+1\}$ satsifies the SECQ and so the strong CHIP by Theorem 3.3.2:

$$N_{D \cap \bigcap_{j=1}^{m+1} C_{i_j^k}}(z_0) = N_D(z_0) + \sum_{j=1}^{m+1} N_{C_{i_j^k}}(z_0) = N_D(z_0) + \text{cone}\left(\bigcup_{j=1}^{m+1} N_{C_{i_j^k}}(z_0) \right),$$

and it follows that

$$x_k^*|_Z \in N_D(z_0)|_Z + \text{cone}\left(\bigcup_{j=1}^{m+1} N_{C_{i_j^k}}(z_0)|_Z \right).$$

By the virtue of Carathéodory Theorem [25, Corollary 17.1.2] (applied to the m-dimensional space Z), there exists some $L_k \subseteq \{i_j^k : j = 1, \ldots, m+1\}$ with $|L_k| = m$, $s_{k,l}^* \in N_{C_l}(z_0)$ (so $\langle s_{k,l}^*, z_0 \rangle = \sigma_{C_l}(s_{k,l}^*)$ by (2.4.4)) for each $l \in L_k$, and $y_k^* \in N_D(z_0)$ (so $\langle y_k^*, z_0 \rangle = \sigma_D(y_k^*)$ by (2.4.4)) such that $x_k^*|_Z = y_k^*|_Z + \sum_{l \in L_k} s_{k,l}^*|_Z$. Then, as $z_0 \in Z$,

$$\xi_k \geq \langle x_k^*, z_0 \rangle = \langle y_k^*, z_0 \rangle + \sum_{l \in L_k} \langle s_{k,l}^*, z_0 \rangle = \sigma_D(y_k^*) + \sum_{l \in L_k} \sigma_{C_l}(s_{k,l}^*).$$

Hence, $(y_k^*, \xi_k - \sum_{l \in L_k} \sigma_{C_l}(s_{k,l}^*)) \in \text{epi}\, \sigma_D$. Noting that,

$$(x_k^*|_Z, \xi_k) = (y_k^*|_Z, \xi_k - \sum_{l \in L_k} \sigma_{C_l}(s_{k,l}^*)) + \sum_{l \in L_k} (s_{k,l}^*|_Z, \sigma_{C_l}(s_{k,l}^*)),$$

we can then use Lemma 3.4.5 to conclude that $(x^*, \xi) \in \text{epi}\, \sigma_D + \sum_{i \in I} \text{epi}\, \sigma_{Z \cap C_i}$. So, (3.4.9) holds.

Finally, noting that for each $i \in I$, one can find some $J \subseteq I$ with $i \in J$ such that $|J| = m + 1$ (such J must exist because $|I| > m + 1$). Thus, by **(iv*)**,

$$\operatorname{epi} \sigma_{Z \cap C_i} \subseteq \operatorname{epi} \sigma_{D \cap \bigcap_{j \in J} C_j} = \operatorname{epi} \sigma_D + \sum_{j \in J} \operatorname{epi} \sigma_{C_j}. \tag{3.4.15}$$

Therefore, by Proposition 3.2.6, (3.4.9) and (3.4.15), one gets

$$\begin{aligned} \operatorname{epi} \sigma_{D \cap \bigcap_{i \in I} C_i} = \overline{\operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{C_i}}^{w^*} &\subseteq \operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{Z \cap C_i} \\ &\subseteq \operatorname{epi} \sigma_D + \sum_{i \in I} \operatorname{epi} \sigma_{C_i}. \end{aligned}$$

So, $\{D, C_i : i \in I\}$ satisfies the SECQ, thanks to Corollary 3.2.6. This completes the proof of **(b)**. □

Chapter 4

Duality theory of semi-infinite optimization via weakly* sum of epigraph of conjugate functions

4.1 Introduction

In the last chapter, we have studied the sum of epigraphs constraint qualification (SECQ) of a collection of closed convex sets, in which the epigraphs of support functions are considered. Note that, as seen in Proposition 2.4.1, one has $\delta_A^* = \sigma_A$ for any non-empty set A in a real normed linear space X. So, given a collection of closed convex sets $\{C_i : i \in I\}$ with $C := \bigcap_{i \in I} C_i \neq \emptyset$, we can reformulate the definition of SECQ for $\{C_i : i \in I\}$ (see (3.2.1)) as follows:

$$\operatorname{epi} \delta_C^* = \sum_{i \in I} \operatorname{epi} \delta_{C_i}^*, \tag{4.1.1}$$

where $\delta_C = \sum_{i \in I} \delta_{C_i}$. Recall that δ_A is a proper lower semicontinuous convex function on X if A is a nonempty closed convex set. Motivated by (4.1.1) as was shown above, it is natural to ask the following questions: Given a collection of functions $\{f_i : i \in I\}$ in $\Gamma(X)$ and let $f := \sum_{i \in I} f_i$ on X. When does it hold that $\operatorname{epi} f^* = \sum_{i\in I} \operatorname{epi} f_i^*$? Moreover, what are the consequences if such set equality holds? Based on these two questions, we will study the "sum" of epigraphs of conjugate functions and see how is it related to optimization theory in this chapter. (Here, we remark that the term "sum" that is used in this chapter is different from the one that was studied in Chapter 3, as will be seen later.)

To begin with, let us state the following form of the famous Fenchel Duality Theorem (cf. [25, Theorem 31.1]):

Theorem 4.1.1. *Let X be a finite dimensional space. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two proper convex functions. Suppose that* $\operatorname{rint}(\operatorname{dom} f) \cap \operatorname{rint}(\operatorname{dom} g) \neq \emptyset$. *Then,*

$$\inf\{f(x) + g(x) : x \in X\} = \max\{-f^*(x^*) - g^*(-x^*) : x^* \in X^*\}. \tag{4.1.2}$$

The Fenchel Duality Theorem enables ones to transform a primal minimization problem to its dual maximization problem, and an optimal solution can be found for the dual problem. The critiria stated in the theorem for such transformation is an interior-point type condition. Since there are minimization problems in which their dual maximization problems are easier to be handled (see [11, Example 25.2] for example), the Fenchel Duality Theorem is proved to be useful in the optimization theory. Note that, in the language of conjugate function and infimal convolution, the equality (4.1.2) can be written as follows:

$$(f + g)^*(0) = -(f^* \Box g^*)(0) \quad \text{with exact infimal convolution.}$$

So, it is immediate for us to see that (4.1.2) is closely related to the following statement:

$$\text{For any } x^* \in X^*, \ \ (f + g)^*(x^*) = -(f^* \Box g^*)(x^*) \ \text{ with exact infimal convolution.} \tag{4.1.3}$$

In the literature, many researchers were interested in generalizing the Fenchel Duality Theorem. More precisely, they seek weaker sufficient conditions for

(4.1.2), or more generally, for (4.1.3), under the setting of some more general spaces. See [1, 8, 22, 24, 27] for some of those successful generalizations. In fact, in Theorem 3.4.1, we have stated some sufficient conditions for (4.1.3). Moreover, generalization of (4.1.3) to the setting of finitely many functions was discussed in Theorem 3.4.3. However, among many of those generalized sufficient conditions for the Fenchel duality, most of them are still interior-point type conditions. In [10], Burachik and Jeyakumar provided a characterization of (4.1.3) in terms of the sum of the epigraphs of the conjugate functions of f and g, and that theorem is stated as follows:

Theorem 4.1.2. *[10, Theorem 1] Let X be a Banach space. Let $f, g \in \Gamma(X)$ be such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Then, the following statements are equivalent:*

- **(i)** $(f+g)^* = f^* \Box g^*$ *with exact infimal convolution.*
- **(ii)** $\operatorname{epi} f^* + \operatorname{epi} g^*$ *is weakly* closed.*
- **(iii)** *For each $\varepsilon \geq 0$ and $x \in \operatorname{dom} f \cap \operatorname{dom} g$,*

$$\partial_\varepsilon (f+g)(x) = \bigcup \{\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x) : \varepsilon_1, \varepsilon_2 \geq 0, \varepsilon_1 + \varepsilon_2 = \varepsilon\}.$$

The merit of this theorem is that it avoids the classical interior-point type conditions and gives a complete characterization of (4.1.3). Moreover, Jeyakumar et. al. gave examples in [10] to show that the condition **(ii)** in Theorem 4.1.2 is weaker than some classical interior-point type conditions. So, their result offers further insight to the Fenchel Duality Theorem. Inspired by consideration of semi-infinite optimization problems, it is interesting to extend Theorem 4.1.2, make it applicable in the setting of a (possibly infinite) collection of $\{f_i : i \in I\}$ in $\Gamma(X)$, where X is a normed linear space. It turns out that such generalization is non-trivial. In [20], Li and Ng used the notion of weakly* sum of sets to study the Fenchel duality for an infinite collection of proper convex lower semicontinuous functions in normed linear space, and they gave a complete generalization of

Theorem 4.1.2. Their result shows that the Fenchel duality can still be discussed in the setting of semi-infinite convex optimization theory.

In this chapter, we give an overview of the work that was done by Li and Ng in [20]. First, we define the Fenchel duality for an infinite system of proper extended real-valued functions. Then, we study a generalization of Theorem 4.1.2, which utilized the notion of weakly* sum of epigraphs of conjugate functions and ε-subdifferentials. After that, we consider two special classes of functions: continuous functions and non-negative functions. Sufficient conditions for the generalized Fenchel duality of these two classes of functions will be studied.

4.2 Fenchel duality in semi-infinite convex optimization

Unless otherwise stated, let X be a real normed linear space in the rest of this chapter. In this section, the main theorem is the generalization of Theorem 4.1.2 for a system of functions $\{f_i : i \in I\}$ in $\Gamma(X)$, where $|I|$ is allowed to be infinite. Before we show this, we first state the following definition of Fenchel duality for an arbitrary system of proper extended real-valued functions.

Definition 4.2.1. *Let $\{f, f_i : i \in I\}$ be a collection of proper extended real-valued functions on X such that*

$$f(x) = \sum_{i \in I} f_i(x), \quad \forall x \in X.$$

Then, this collection of functions is said to satisfy the Fenchel duality if the following equality holds:

$$\inf \{f(x) : x \in X\} = \max\{-\sum_{i \in I} f_i^*(x_i^*) : \sum_{i \in I}^{*} x_i^* = 0\}. \tag{4.2.1}$$

Remark 4.2.1. *It is easy to see that* (4.2.1) *is equivalent to*

$$f^*(0) = \min\{\sum_{i \in I} f_i^*(x_i^*) : \sum_{i \in I}^{*} x_i^* = 0\}.$$

Consider the following equality:

$$f^*(x^*) = \min\{\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}{}^* x_i^* = x^*\}, \quad \forall x^* \in X^*, \tag{4.2.2}$$

which is obviously equivalent to

$$\inf\{f(x) - \langle x^*, x\rangle : x \in X\} = \max\{-\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}{}^* x_i^* = x^*\}, \quad \forall x^* \in X^*.$$

Using Young's inequality, one can show that the following inequality always hold:

$$f^*(x^*) \le \inf\{\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}{}^* x_i^* = x^*\}, \quad \forall x^* \in X^*. \tag{4.2.3}$$

So, it follows that (4.2.2) holds if and only if

$$f^*(x^*) = \sum_{i\in I} f_i^*(x_i^*) \quad \text{for some } \{x_i^*\}_{i\in I} \subseteq X^* \text{ with } \sum_{i\in I}{}^* x_i^* = x^*. \tag{4.2.4}$$

In order to generalize Theorem 4.1.2, we will study some equivalent formulations of (4.2.2) in terms of weakly* sum of epigraphs and ε-subdifferentials, which in turns gives new sufficient conditions for Fenchel duality for infinite system of functions in $\Gamma(X)$. We need the following proposition, which is from [20, Lemma 3.1].

Proposition 4.2.1. *Let $\{f, f_i : i \in I\} \subseteq \Gamma(X)$ be such that $f = \sum_{i\in I} f_i$ on X, that is,*

$$f(x) = \sum_{i\in I} f_i(x) \quad \textit{for any } x \in X. \tag{4.2.5}$$

Then,

$$\overline{\sum_{i\in I}{}^* \operatorname{epi} f_i^*}^{w^*} \subseteq \operatorname{epi} f^*.$$

Proof. By the weakly* closedness of epi f^*, it suffices to show that

$$\sum_{i\in I}{}^* \operatorname{epi} f_i^* \subseteq \operatorname{epi} f^*. \tag{4.2.6}$$

Let $(x^*, \alpha) \in X^* \times \mathbb{R}$ be such that $(x^*, \alpha) \in \sum^*_{i\in I} \operatorname{epi} f_i^*$. Then, for each $i \in I$, there exists some $(x_i^*, \alpha_i) \in \operatorname{epi} f_i^*$ such that

$$\alpha = \sum_{i\in I} \alpha_i \qquad \text{and} \qquad \langle x^*, z\rangle = \sum_{i\in I}{}^{*} \langle x_i^*, z\rangle \quad \text{for any } z \in X. \tag{4.2.7}$$

Fix $z \in \operatorname{dom} f$. By the Young's inequality (see (2.4.1)) and the fact that $(x_i^*, \alpha_i) \in \operatorname{epi} f_i^*$, one gets

$$\langle x_i^*, z\rangle - f_i(z) \leq f_i^*(x_i^*) \leq \alpha_i \qquad \text{for any } i \in I. \tag{4.2.8}$$

With the use of (4.2.5), (4.2.7) and (4.2.8), it follows from Remark 2.8.2 that $\sum_{i\in I} f_i^*(x_i^*)$ exists in $\mathbb{R}$ and

$$\langle x^*, z\rangle - f(z) = \sum_{i\in I} \langle x_i^*, z\rangle - \sum_{i\in I} f_i(z) \leq \sum_{i\in I} f_i^*(x_i^*) \leq \sum_{i\in I} \alpha_i = \alpha.$$

By taking supremum over all $z \in \operatorname{dom} f$, we have

$$f^*(x^*) = \sup_{z\in \operatorname{dom} f} (\langle x^*, z\rangle - f(z)) \leq \alpha,$$

Thus, $(x^*, \alpha) \in \operatorname{epi} f^*$ and so (4.2.6) holds. □

Here is the main theorem that we mentioned. Noting that Theorem 4.1.2 was proved in [10] under the setting of Banach space, we remark that such assumption is not necessary, as it can be observed in the following theorem.

Theorem 4.2.2 ([20, Theorem 3.2]). *Let $\{f, f_i : i \in I\}$ be defined as that in Proposition 4.2.1. Then, the following statements are equivalent.*

(i) *For any $x \in X$ and $\varepsilon \geq 0$,*

$$\partial_\varepsilon f(x) \subseteq \bigcup \{\sum_{i\in I}{}^{*} \partial_{\varepsilon_i} f_i(x) : \varepsilon_i \geq 0, \sum_{i\in I} \varepsilon_i = \varepsilon\}. \tag{4.2.9}$$

(ii) *For any $x \in \operatorname{dom} f$ and $\varepsilon \geq 0$,*

$$\partial_\varepsilon f(x) = \bigcup \{\sum_{i\in I}{}^{*} \partial_{\varepsilon_i} f_i(x) : \varepsilon_i \geq 0, \sum_{i\in I} \varepsilon_i = \varepsilon\}. \tag{4.2.10}$$

(iii) $\operatorname{epi} f^* = \sum_{i\in I}^* \operatorname{epi} f_i^*$.

(iv) *For any* $x^* \in X^*$,

$$f^*(x^*) = \min\{\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}^* x_i^* = x^*\}. \tag{4.2.11}$$

That is,

$$\inf_{x\in X}(f(x) - \langle x^*, x\rangle) = \max\{-\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}^* x_i^* = x^*\}. \tag{4.2.12}$$

*Moreover, any of the statements **(i)** to **(iv)** imply that*

(v) *The collection of functions* $\{f, f_i : i \in I\}$ *satisfies the Fenchel duality:*

$$\inf\{f(x) : x \in X\} = \max\{-\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}^* x_i^* = 0\}.$$

Proof. It is clear that **(iv)** implies **(v)**, by taking $x^* = 0$ in (4.2.12). We now prove the equivalences. First, we will prove that **(i)** $\Rightarrow$ **(ii)**. Suppose that **(i)** holds. Let $x \in \operatorname{dom} f$ and $\varepsilon \geq 0$. In order to show (4.2.10), it suffices for us to show that

$$\partial_\varepsilon f(x) \supseteq \bigcup\{\sum_{i\in I}^* \partial_{\varepsilon_i} f_i(x) : \varepsilon_i \geq 0, \sum_{i\in I} \varepsilon_i = \varepsilon\}. \tag{4.2.13}$$

Let x^* be in the RHS of (4.2.13). Then, for each $i \in I$, there exist some $\varepsilon_i \geq 0$ and $x_i^* \in \partial_{\varepsilon_i} f_i(x)$ such that

$$x^* = \sum_{i\in I}^* x_i^* \quad \text{and} \quad \sum_{i\in I} \varepsilon_i = \varepsilon. \tag{4.2.14}$$

Noting that, for each $i \in I$, one has

$$\langle x_i^*, x\rangle - \varepsilon_i \leq f_i^*(x_i^*) + f_i(x) - \varepsilon_i \leq \langle x_i^*, x\rangle, \tag{4.2.15}$$

where the first inequality is by the Young's inequality, and the second inequality follows from (2.4.2) and the fact that $x_i^* \in \partial_{\varepsilon_i} f_i(x)$. Since $\sum_{i\in I}\langle x_i^*, x\rangle = \langle x^*, x\rangle \in \mathbb{R}$ and $\sum_{i\in I}(\langle x_i^*, x\rangle - \varepsilon_i) = \langle x^*, x\rangle - \varepsilon \in \mathbb{R}$ (thanks to (4.2.14)), it follows from

(4.2.15) and Remark 2.8.2 that $\sum_{i\in I}(f_i^*(x_i^*) + f_i(x) - \varepsilon_i)$ exists in $\mathbb{R}$ (and so $\sum_{i\in I} f_i^*(x_i^*)$ exists in $\mathbb{R}$) and

$$\sum_{i\in I} f_i(x_i^*) + f(x) - \varepsilon = \sum_{i\in I}(f_i^*(x_i^*) + f_i(x) - \varepsilon_i) \leq \sum_{i\in I}\langle x_i^*, x\rangle = \langle x^*, x\rangle. \quad (4.2.16)$$

Now, for any $z \in \operatorname{dom} f$, the Young's inequality implies that

$$\langle x_i^*, z\rangle - f_i(z) \leq f_i^*(x_i^*) \quad \text{for any } i \in I. \quad (4.2.17)$$

Summing over all $i \in I$ in (4.2.17) gives

$$\langle x^*, z\rangle - f(z) = \sum_{i\in I}(\langle x_i^*, z\rangle - f_i(z)) \leq \sum_{i\in I} f_i^*(x_i^*). \quad (4.2.18)$$

By taking supremum in (4.2.18) over all $z \in \operatorname{dom} f$, we get

$$f^*(x^*) = \sup_{z\in\operatorname{dom} f}(\langle x^*, z\rangle - f(z)) \leq \sum_{i\in I} f_i^*(x_i^*). \quad (4.2.19)$$

Thus, (4.2.16) and (4.2.19) imply that

$$f^*(x^*) + f(x) - \varepsilon \leq \langle x^*, x\rangle.$$

Hence, by (2.4.2), we see that $x^* \in \partial_\varepsilon f(x)$. So, (4.2.13) holds. This completes the proof of **(i)** $\Rightarrow$ **(ii)**.

Next, we prove the implication **(ii)** $\Rightarrow$ **(iii)**. Assume that **(ii)** holds. We have to show that $\operatorname{epi} f^* = \sum_{i\in I}^* \operatorname{epi} f_i^*$. In view of Proposition 4.2.1, it suffices to show that

$$\operatorname{epi} f^* \subseteq \sum_{i\in I}^* \operatorname{epi} f_i^*. \quad (4.2.20)$$

Let $(x^*, \alpha) \in \operatorname{epi} f^*$. Take $x \in \operatorname{dom} f$. Then, define $\varepsilon := \alpha + f(x) - \langle x^*, x\rangle$. It follows that

$$(x^*, \langle x^*, x\rangle - f(x) + \varepsilon) = (x^*, \alpha) \in \operatorname{epi} f^*.$$

So, by (2.4.2), $x^* \in \partial_\varepsilon f(x)$. Using **(ii)**, one can find some $\varepsilon_i \geq 0$ and $x_i^* \in \partial_{\varepsilon_i} f_i(x)$ ($i \in I$) such that

$$x^* = \sum_{i\in I}^* x_i^* \quad \text{and} \quad \varepsilon = \sum_{i\in I} \varepsilon_i. \quad (4.2.21)$$

Thus, (4.2.5), (4.2.21) and the definition of ε give

$$(x^*, \alpha) = \sum_{i\in I}^{*} (x_i^*, \langle x_i^*, x\rangle - f_i(x) + \varepsilon_i).$$

Hence, $(x^*, \alpha) \in \sum_{i\in I}^{*} \operatorname{epi} f_i^*$, thanks to (2.4.2) and the fact that $x_i^* \in \partial_{\varepsilon_i} f_i(x)$ for each $i \in I$. So, (4.2.20) holds as was required to show.

We now turn to prove **(iii)** $\Rightarrow$ **(iv)**. Let $x^* \in X^*$. We have to show that (4.2.11) holds. When $f^*(x^*) = +\infty$, (4.2.3) gives the desired result. We now consider the case when $f^*(x^*) < +\infty$. It suffices for us to show that (4.2.4) holds. Since $f \in \Gamma(X)$, it follows from [31, Corollary 2.3.2] that $f^* \in \Gamma(X^*)$. Thus, $f^*(x^*) \in \mathbb{R}$. Hence, $(x^*, f^*(x^*)) \in \operatorname{epi} f^*$. It then follows from **(iii)** that for each $i \in I$, there exists some $(x_i^*, \alpha_i) \in \operatorname{epi} f_i^*$ such that

$$(x^*, f^*(x^*)) = \sum_{i\in I}^{*} (x_i^*, \alpha_i). \tag{4.2.22}$$

Observe that for each $i \in I$, one can use the Young's inequality and $(x_i^*, \alpha_i) \in \operatorname{epi} f_i^*$ to see that for any $z \in \operatorname{dom} f$,

$$\langle x_i^*, z\rangle - f_i(z) \le f_i^*(x_i^*) \le \alpha_i. \tag{4.2.23}$$

Since $\sum_{i\in I} \alpha_i = f^*(x^*) \in \mathbb{R}$ and $\sum_{i\in I}(\langle x_i^*, z\rangle - f_i(z)) \in \mathbb{R}$ (thanks to (4.2.1) and (4.2.22)), it follows from (4.2.23) and Remark 2.8.2 that $\sum_{i\in I} f_i^*(x_i^*) \in \mathbb{R}$ and

$$\langle x^*, z\rangle - f(z) = \sum_{i\in I}(\langle x_i^*, z\rangle - f_i(z)) \le \sum_{i\in I} f_i^*(x_i^*) \le \sum_{i\in I} \alpha_i = f^*(x^*).$$

Taking supremum over all $z \in \operatorname{dom} f$ gives

$$f^*(x^*) \le \sum_{i\in I} f_i^*(x_i^*) \le f^*(x^*),$$

that is, $f^*(x^*) = \sum_{i\in I} f_i^*(x_i^*)$. So, (4.2.4) holds. The proof of **(iii)** $\Rightarrow$ **(iv)** is complete.

It remains to show the implication **(iv)** $\Rightarrow$ **(i)**. Suppose that **(iv)** holds. Let $x \in X$ and $\varepsilon \ge 0$. We have to show (4.2.9). Since $\partial_\varepsilon f(x) = \emptyset$ when $x \notin \operatorname{dom} f$, it suffices to consider the case when $x \in \operatorname{dom} f$. Let $x^* \in \partial_\varepsilon f(x)$. By (4.2.11), there exists some collection $\{x_i^* \in X^* : i \in I\}$ such that

$$\sum_{i \in I}^{*} x_i^* = x^* \quad \text{and} \quad f^*(x^*) = \sum_{i \in I} f_i^*(x_i^*). \tag{4.2.24}$$

This together with (4.2.5) and the Young's inequality that

$$\varepsilon \geq f^*(x^*) + f(x) - \langle x^*, x \rangle = \sum_{i \in I} (f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x \rangle).$$

Now, pick some $\varepsilon_i \geq 0$ $(i \in I)$ such that

$$\varepsilon = \sum_{i \in I} \varepsilon_i \quad \text{and} \quad \varepsilon_i \geq f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x \rangle.$$

Then, it follows from (2.4.2) that $x_i^* \in \partial_{\varepsilon_i} f_i(x)$ for each $i \in I$, and so $x^* = \sum_{i \in I}^{*} x_i^* \in \sum_{i \in I}^{*} \partial_{\varepsilon_i} f_i(x)$. Therefore, x^* is in the RHS of (4.2.9) as was required to show. This finishes the proof. □

4.3 Sufficient conditions for Fenchel duality in semi-infinite convex optimization

Throughout this section, we let $\{f, f_i : i \in I\} \subseteq \Gamma(X)$ be such that

$$f(x) = \sum_{i \in I} f_i(x) \quad \text{for any } x \in X. \tag{4.3.1}$$

As we have just shown in Theorem 4.2.2, one of the sufficient conditions for the Fenchel duality of $\{f, f_i : i \in I\}$ is the following set equality:

$$\operatorname{epi} f = \sum_{i \in I}^{*} \operatorname{epi} f_i.$$

In this section, we will study the sufficient conditions of this set equality for two special classes of functions: continuous real-valued functions and non-negative functions.

4.3.1 Continuous real-valued functions

Recall that $\Gamma_c(X) = \{f \in \Gamma(X) : f \text{ is a real-valued continuous function on } X\}$. In this subsection, we assume that

$$\{f, f_i : i \in I\} \subseteq \Gamma_c(X). \tag{4.3.2}$$

The following theorem is an analog of Theorem 2.6.5. Note that the completeness of X is used in the following proof.

Theorem 4.3.1 ([20, Theorem 4.1]). *Let X be a Banach space. Assume that* (4.3.1) *and* (4.3.2) *hold. Then,*

$$\text{epi}\, f^* = \overline{\sum_{i\in I}^{*} \text{epi}\, f_i^*}^{w^*}.$$

Proof. In view of Proposition 4.2.1, it suffices for us to show that

$$\text{epi}\, f^* \subseteq \overline{\sum_{i\in I}^{*} \text{epi}\, f_i^*}^{w^*}. \tag{4.3.3}$$

It is clear that the set $\overline{\sum_{i\in I}^{*} \text{epi}\, f_i^*}^{w^*}$ is weakly* closed. Also, by Remark 2.8.3, $\overline{\sum_{i\in I}^{*} \text{epi}\, f_i^*}^{w^*}$ is convex (as the weakly* closure of a convex set is still convex). We now show that $\overline{\sum_{i\in I}^{*} \text{epi}\, f_i^*}^{w^*} \neq \emptyset$. Let $x \in X$. Since each f_i is continuous at $x \in X$, by Proposition 2.3.1**(ii)**, we see that $\partial f_i(x) \neq \emptyset$. Take $x_i^* \in \partial f_i(x)$ for each $i \in I$. By Theorem 2.8.2, we have that

$$x^* = \sum_{i\in I}^{*} x_i^* \quad \text{for some } x^* \in \partial f(x). \tag{4.3.4}$$

Note, by (2.4.3), that

$$f^*(x^*) = \langle x^*, x\rangle - f(x) \in \mathbb{R} \quad \text{and} \quad f_i^*(x^*) = \langle x_i^*, x\rangle - f_i(x) \in \mathbb{R} \quad \text{for each } i \in I.$$

Thus, thanks to (4.3.1) and (4.3.4),

$$f^*(x^*) = \langle x^*, x\rangle - f(x) = \sum_{i\in I}(\langle x_i^*, x\rangle - f_i(x)) = \sum_{i\in I} f_i^*(x^*).$$

So,

$$(x^*, f^*(x^*)) = \sum_{i\in I}^{*}(x_i^*, f_i^*(x_i^*)) \in \sum_{i\in I}^{*} \operatorname{epi} f_i^*,$$

which implies that $\overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*} \neq \emptyset$.

Suppose that (4.3.3) does not hold. Then, there exists some $(x_0^*, \alpha_0) \in \operatorname{epi} f^* \backslash \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}$. Since $\overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}$ is non-empty, convex and weakly* closed, one can use the separation theorem to find some $(z_0, \gamma_0) \in X \times \mathbb{R}$ such that

$$\sup\{\langle y^*, z_0\rangle + \beta\gamma_0 : (y^*, \beta) \in \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}\} < \langle x_0^*, z_0\rangle + \alpha_0\gamma_0. \tag{4.3.5}$$

We consider the following cases:

Case 1: $\gamma_0 > 0$.

In this case, one can take some $(y^*, \beta) \in \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}$ and consider large positive value of β, then (4.3.5) would lead to contradiction.

Case 2: $\gamma_0 = 0$.

By (4.3.5), one has

$$\sup\{\langle y^*, z_0\rangle : (y^*, \beta) \in \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}\} < \langle x_0^*, z_0\rangle. \tag{4.3.6}$$

Since $(x_0^*, \alpha_0) \in \operatorname{epi} f^*$, we have

$$x_0^* \in \operatorname{dom} f^*. \tag{4.3.7}$$

By the virtue of Brøndsted-Rockafellar Theorem (cf. [31, Theorem 3.1.2]) and the assumption that X is a Banach space, we have

$$\operatorname{dom} f^* \subseteq \overline{\operatorname{Im} \partial f}.$$

This together with (4.3.6) and (4.3.7) imply that there exist $a \in X$ and $a^* \in \partial f(a)$ such that

$$\sup\{\langle y^*, z_0\rangle : (y^*, \beta) \in \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}\} < \langle a^*, z_0\rangle.$$

Moreover, by Theorem 2.8.2, we have that $\partial f(a) = \overline{\sum_{i\in I}^{*} \partial f_i(a)}^{w^*}$, and so we can find some $\overline{a}^* \in \sum_{i\in I}^{*} \partial f_i(a)$ such that

$$\sup\{\langle y^*, z_0\rangle : (y^*, \beta) \in \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}\} < \langle \overline{a}^*, z_0\rangle. \tag{4.3.8}$$

Write $\overline{a}^* = \sum_{i\in I}^{*} a_i^*$ for some $a_i^* \in \partial f_i(a)$. For each $i \in I$, define $r_i := \langle a_i^*, a\rangle - f_i(a)\ (\in \mathbb{R})$. Then, by (2.4.3), one has $r_i = f_i^*(a_i^*)$ (and so $(a_i^*, r_i) \in \operatorname{epi} f_i^*$) for all $i \in I$. Also, (4.3.1) implies that $r := \sum_{i\in I} r_i = \langle \overline{a}^*, a\rangle - f_i(a) \in \mathbb{R}$. Hence,

$$(\overline{a}^*, r) = \sum_{i\in I}^{*} (a_i^*, r_i) \in \sum_{i\in I}^{*} \operatorname{epi} f_i^*.$$

So, we get

$$\langle \overline{a}^*, z_0\rangle \le \sup\{\langle y^*, z_0\rangle : (y^*, \beta) \in \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}\},$$

which contradicts (4.3.8).

Case 3: $\gamma_0 < 0$.

Without loss of generality, we assume that $\gamma_0 = -1$. Thus, (4.3.5) becomes

$$\sup\{\langle y^*, z_0\rangle - \beta : (y^*, \beta) \in \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}\} < \langle x_0^*, z_0\rangle - \alpha_0. \tag{4.3.9}$$

By the Young's inequality and the fact that $(x_0^*, \alpha_0) \in \operatorname{epi} f^*$, one has

$$\langle x_0^*, z_0\rangle - f(z_0) \le f^*(x_0^*) \le \alpha_0,$$

and so

$$\langle x_0^*, z_0\rangle - \alpha_0 \le f(z_0). \tag{4.3.10}$$

Now, pick $y_i^* \in \partial f_i(z_0)$ for each $i \in I$. Then, by (2.4.3), we have

$$f_i^*(y_i^*) = \langle y_i^*, z_0\rangle - f_i(z_0) \quad \text{for all } i \in I. \tag{4.3.11}$$

In particular, $f_i^*(y_i^*) \in \mathbb{R}$ (and so $(y_i^*, f_i^*(y_i^*)) \in \operatorname{epi} f_i^*$) for all $i \in I$. Further, using Theorem 2.8.2, take some $c^* \in \partial f(z_0)$ such that $c^* = \sum_{i\in I}^{*} y_i^*$. Hence, (4.3.1) and (4.3.11) imply that

$$\langle c^*, z_0\rangle - f(z_0) = \sum_{i\in I} (\langle y_i^*, z_0\rangle - f_i(z_0)) = \sum_{i\in I} f_i^*(y_i^*).$$

Thus,

$$(c^*, \langle c^*, z_0\rangle - f(z_0)) = \sum_{i\in I}^{*} (y_i^*, f_i^*(y_i^*)) \in \sum_{i\in I}^{*} \operatorname{epi} f_i^*.$$

So, it follows from (4.3.9) that

$$\begin{aligned} f(z_0) &= \langle c^*, z_0\rangle - (\langle c^*, z_0\rangle - f(z_0)) \\ &\le \sup\{\langle y^*, z_0\rangle - \beta : (y^*, \beta) \in \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}\} \\ &< \langle x_0^*, z_0\rangle - \alpha_0, \end{aligned}$$

and this contradicts (4.3.10).

Combining the consequences of Case 1, 2 and 3, we can conclude that (4.3.5) leads to contradiction. Therefore, (4.3.3) is valid. This finishes the proof. □

The next theorem is from [20, Theorem 4.2], which gives a sufficient condition for the weakly* closedness of $\sum_{i\in I}^{*} \operatorname{epi} f_i^*$ under the setting of (4.3.2).

Theorem 4.3.2. *Let X be a Banach space. Assume that* (4.3.1) *and* (4.3.2) *hold. Suppose in addition that I is countable and*

$$\operatorname{dom} f^* \subseteq \operatorname{Im} \partial f. \tag{4.3.12}$$

Then,

$$\operatorname{epi} f^* = \sum_{i\in I}^{*} \operatorname{epi} f_i^*.$$

Proof. By Theorem 4.3.1, it suffices for us to show that

$$\operatorname{epi} f^* \subseteq \sum_{i\in I}^{*} \operatorname{epi} f_i^*. \tag{4.3.13}$$

We first prove that

$$\operatorname{gph} f^* \subseteq \sum_{i\in I}^{*} \operatorname{epi} f_i^*. \tag{4.3.14}$$

Let $(x^*, f^*(x^*)) \in \operatorname{gph} f^*$. Then, $x^* \in \operatorname{dom} f^*$. By (4.3.12), one has $x^* \in \partial f(a)$ for some $a \in X$. Since I is countable, it then follows from Theorem 2.8.2 that

$\partial f(a) = \sum_{i\in I} \partial f_i(a)$. Thus, $x^* = \sum_{i\in I}^* x_i^*$ for some $x_i^* \in \partial f_i(a)$, where $i \in I$. With the use of (2.4.3), we have $f^*(x^*) = \langle x^*, a\rangle - f(a)$ and $f_i^*(x_i^*) = \langle x_i^*, a\rangle - f_i(a)$ for all $i \in I$. Hence, $f^*(x^*) = \sum_{i\in I} f_i^*(x_i^*)$, and so,

$$(x^*, f^*(x^*)) = \sum_{i\in I}^* (x_i^*, f_i^*(x_i^*)) \in \sum_{i\in I}^* \operatorname{epi} f_i^*$$

Therefore, (4.3.14) holds.

With (4.3.14), we get

$$\operatorname{epi} f^* = \operatorname{gph} f^* + \{0\} \times [0, \infty) \subseteq \sum_{i\in I}^* \operatorname{epi} f_i^* + \{0\} \times [0, \infty) = \sum_{i\in I}^* \operatorname{epi} f_i^*,$$

where the last equality follows from the fact that $\operatorname{epi} f_i^* + \{0\} \times [0, \infty) = \operatorname{epi} f_i^*$ for each $i \in I$. So, (4.3.13) is established. □

Below is an immediate consequence of Theorems 4.2.2 and 4.3.2.

Theorem 4.3.3. *Let X be a Banach space. Assume that* (4.3.1) *and* (4.3.2) *hold. Suppose in addition that I is countable and* $\operatorname{dom} f^* \subseteq \operatorname{Im} \partial f$. *Then, for any* $x^* \in X^*$,

$$\inf_{x\in X} (f(x) - \langle x^*, x\rangle) = \max\{-\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}^* x_i^* = x^*\}.$$

In particular, the collection $\{f, f_i : i \in I\}$ satisfies the Fenchel duality:

$$\inf\{f(x) : x \in X\} = \max\{-\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}^* x_i^* = 0\}.$$

4.3.2 Nonnegative-valued functions

Throughout this subsection, we assume that

$$\{f, f_i : i \in I\} \subseteq \Gamma_+(X). \tag{4.3.15}$$

Note that, in this case, it follows from (2.8.3) that

$$f(x) = \sup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} f_j(x) \quad \text{for any } x \in X. \tag{4.3.16}$$

Similar to the case for continuous real-valued functions, one can prove the following set equality in the present case. In fact, we get a bit more.

Theorem 4.3.4 ([20, Theorem 4.3]). *Suppose that* (4.3.1) *and* (4.3.15) *hold. Then,*

$$\operatorname{epi} f^* = \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} \operatorname{epi} f_j^*}^{w^*} = \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*}. \tag{4.3.17}$$

Proof. Since each $\operatorname{epi} f_i^*$ contains the origin (thanks to (4.3.15)), one has that

$$\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} \operatorname{epi} f_j^* \subseteq \sum_{i\in I}^{*} \operatorname{epi} f_i^*.$$

Then, it follows from Proposition 4.2.1 that

$$\overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} \operatorname{epi} f_j^*}^{w^*} \subseteq \overline{\sum_{i\in I}^{*} \operatorname{epi} f_i^*}^{w^*} \subseteq \operatorname{epi} f^*. \tag{4.3.18}$$

For each $J \subseteq I$ with $|J| < \infty$, let $g_J := \sum_{j\in J} f_j$ on X. Then, (4.3.16) becomes

$$f(x) = \sup_{\substack{J\subseteq I,\\ |J|<\infty}} g_J(x) \quad \text{for any } x \in X.$$

By applying Lemma 3.2.3 to the collection of functions $\{g_J : J \subseteq I \text{ and } |J| < \infty\}$, we get

$$\operatorname{epi} f^* = \operatorname{epi} (\sup_{\substack{J\subseteq I,\\ |J|<\infty}} g_J)^* = \overline{\operatorname{co} \bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \operatorname{epi} g_J^*}^{w^*}. \tag{4.3.19}$$

Note that $\operatorname{epi} g_{J_1}^* \subseteq \operatorname{epi} g_{J_2}^*$ if $J_1 \subseteq J_2$ (In fact, if $J_1 \subseteq J_2$, then $g_{J_1} \le g_{J_2}$ on X, and so the set inclusion follows from (2.2.2)). In particular, this and the convexity of each $\operatorname{epi} g_J^*$ imply that the set $\bigcup_{J\subseteq I, |J|<\infty} \operatorname{epi} g_J^*$ is convex. Thus, (4.3.19) gives us that

$$\operatorname{epi} f^* = \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \operatorname{epi} g_J^*}^{w^*} = \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \overline{\sum_{j\in J} \operatorname{epi} f_j^*}^{w^*}}^{w^*}, \tag{4.3.20}$$

where the last set equality follows from Theorem 2.6.5. Now, observe that for any $J_0 \subseteq I$ with $|J_0| < \infty$,

$$\sum_{j\in J_0} \operatorname{epi} f_j^* \subseteq \bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} \operatorname{epi} f_j^*,$$

and so

$$\overline{\sum_{j\in J_0} \operatorname{epi} f_j^*}^{w^*} \subseteq \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} \operatorname{epi} f_j^*}^{w^*}.$$

Hence, one has

$$\overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \overline{\sum_{j\in J} \operatorname{epi} f_j^*}^{w^*}}^{w^*} \subseteq \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} \operatorname{epi} f_j^*}^{w^*}. \tag{4.3.21}$$

Combining (4.3.18), (4.3.20) and (4.3.21), one sees that (4.3.17) holds. The proof is completed. □

The next theorem is from [20, Theorem 4.4], which shows a sufficient condition for the set $\sum_{i\in I}^* \operatorname{epi} f_i$ being weakly* closed. Recall that for a set $A \subseteq X$, the diameter of A, denoted by $\operatorname{diam} A$, is defined as

$$\operatorname{diam} A := \sup\{\|x-y\| : x-y \in A\}.$$

Also, the negative polar of A is defined by

$$A^\circ := \{x^* \in X^* : \langle x^*, a\rangle \le 0 \ \ \forall a \in A\}.$$

(For a set $A \subseteq X^*$, its negative polar is defined as

$$A^\circ := \{x \in X : \langle a^*, x\rangle \le 0 \ \ \forall a^* \in A\}.)$$

Furthermore, let J be a finite set and $\{K_j\}_{j\in J}$ be a collection of closed convex cones in $X^*\times\mathbb{R}$. Let H be a subspace of X and $Z := H\times\mathbb{R}$. We define $\gamma(K_j|_Z; J)$ by

$$\gamma(K_j|_Z; J) := \inf\{\|\sum_{j\in J}(x_j^*|_H, \alpha_j)\| : \sum_{j\in J}\|(x_j^*|_H, \alpha_j)\| = 1, \ \text{each } (x_j^*, \alpha_j) \in K_j\}.$$

Remark 4.3.1. *Let $C, D \subseteq X$. It is direct from definition of negative polar that if $C \subseteq D$, then $D^\circ \subseteq C^\circ$. The same result holds for $C, D \subseteq X^*$.*

We first show the following lemma.

Lemma 4.3.5. *Let Z be a subspace of X. Let $\{D_i : i \in \mathbb{N}\}$ be a collection of subsets in X^* with $0 \in \bigcap_{i\in\mathbb{N}} D_i$. Suppose that $\sum_{i\in\mathbb{N}} \operatorname{diam} D_i < +\infty$. Then, $\sum_{i\in\mathbb{N}}^*(D_i|_Z) = (\sum_{i\in\mathbb{N}}^* D_i)|_Z$.*

Proof. It is straight forward to check that $(\sum_{i\in\mathbb{N}}^* D_i)|_Z \subseteq \sum_{i\in\mathbb{N}}^*(D_i|_Z)$. Conversely, let $d^* \in \sum_{i\in\mathbb{N}}^*(D_i|_Z)$. Write

$$d^* = \sum_{i\in\mathbb{N}}{}^* (d_i^*|_Z) \quad \text{for some } d_i^* \in D_i \ (i \in \mathbb{N}). \tag{4.3.22}$$

We claim that $\sum_{i\in\mathbb{N}}^* d_i^*$ exists in X^*. To do this, let $h \in X$. By the assumption that $0 \in \bigcap_{i\in\mathbb{N}} D_i$, we have that $\|d_i^*\| \le \operatorname{diam} D_i$ for all $i \in \mathbb{N}$. Thus, for any $i \in I$,

$$-(\operatorname{diam} D_i)\|h\| \le -\|d_i^*\|\|h\| \le \langle d_i^*, h\rangle \le \|d_i^*\|\|h\| \le (\operatorname{diam} D_i)\|h\|. \tag{4.3.23}$$

Since $\sum_{i\in\mathbb{N}}(\operatorname{diam} D_i)$ exists in $\mathbb{R}$, it follows from (4.3.23) and Remark 2.8.2 that $\sum_{i\in\mathbb{N}}\langle d_i^*, h\rangle$ exists in $\mathbb{R}$ and

$$|\sum_{i\in\mathbb{N}}\langle d_i^*, h\rangle| \le (\sum_{i\in\mathbb{N}}(\operatorname{diam} D_i))\|h\|. \tag{4.3.24}$$

Define $\overline{d}^* : X \to \mathbb{R}$ by

$$\overline{d}^*(x) := \sum_{i\in\mathbb{N}}\langle d_i^*, x\rangle \quad \text{for any } x \in X.$$

It is clear that $\overline{d}^*$ is linear. This and (4.3.24) imply that $\overline{d}^*$ is continuous on X. Hence, $\overline{d}^* \in X^*$ and so $\overline{d}^* = \sum_{i\in\mathbb{N}}^* d_i^*$, which proves our claim. Therefore, by (4.3.22) and the definition of $\overline{d}^*$, one has

$$d^* = \overline{d}^*|_Z \in (\sum_{i\in\mathbb{N}}{}^* D_i)|_Z$$

as was required to show. □

Theorem 4.3.6 ([20, Theorem 4.4]). *Suppose that* (4.3.1) *and* (4.3.15) *hold and* I *is a compact metric space. Assume that the following conditions hold.*

(i) *For each* $i \in I$*, there exist some weakly* compact convex set* D_i *with* $0 \in D_i$ *and some weakly* closed convex cone* K_i *in* $X^* \times \mathbb{R}$ *such that*

$$\operatorname{epi} f_i^* = D_i + K_i. \tag{4.3.25}$$

(ii) $\sum_{i\in I} \operatorname{diam} D_i < \infty$.

(iii) *There exist some* $i_0 \in I$ *and finite dimensional space* $H \subseteq X$ *such that* $K_i^\circ \subseteq Z := H \times \mathbb{R}$. *(Denote the dimension of* Z *by* m*)*

(iv) *For any* $J \subseteq I$ *with* $|J| = m$, $\gamma(K_j|_Z; J) > 0$.

(v) *The set-valued map* $i \mapsto K_i|_Z$ *is upper semicontinuous on* I*. That is, for each* $\bar{i} \in I$,

$$\limsup_{i\to\bar{i}}(K_i|_Z) \subseteq K_{\bar{i}}|_Z,$$

where

$$\begin{aligned}\limsup_{i\to\bar{i}}(K_i|_Z) := \{z^* \in Z^* :& \exists\{i_n\}_{n\in\mathbb{N}} \subseteq I \text{ and } \{k_{i_n}^*\}_{n\in\mathbb{N}} \subseteq X^* \\ & \text{with } i_n \longrightarrow \bar{i} \text{ and } k_{i_n}^* \in K_{i_n}\ \forall n \in \mathbb{N} \\ & \text{such that } k_{i_n}^*|_Z \longrightarrow^{\|\cdot\|_Z} z^* \text{ as } n \longrightarrow \infty\}.\end{aligned}$$

Then, the set $\sum_{i\in I}^* \operatorname{epi} f_i^*$ *is weakly* closed. Moreover, one has*

$$\operatorname{epi} f^* = \sum_{i\in I}^* \operatorname{epi} f_i^*. \tag{4.3.26}$$

Proof. In view of Theorem 4.3.4, one sees that (4.3.26) follows from the weakly* closedness of $\sum_{i\in I}^* \operatorname{epi} f_i^*$. In order to show that $\sum_{i\in I}^* \operatorname{epi} f_i^*$ is weakly* closed, it suffices to show that

$$\overline{\sum_{i\in I}^* \operatorname{epi} f_i^*}^{w^*} \subseteq \sum_{i\in I}^* \operatorname{epi} f_i^*. \tag{4.3.27}$$

For convenience, let $A_i := \operatorname{epi} f_i^*$ for each $i \in I$. Let $a^* \in \overline{\sum_{i\in I}^* A_i}^{w^*}$. Note that

$$\begin{aligned}
(\overline{\sum_{i\in I}^* A_i}^{w^*})|_Z = (\overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} A_j}^{w^*})|_Z &\subseteq \overline{(\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} A_j)|_Z}^{w^*} \\
&\subseteq \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} (A_j|_Z)}^{w^*} \\
&= \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{j\in J} (A_j|_Z)}^{\|\cdot\|_Z},
\end{aligned}$$

where the first set equality follows from Theorem 4.3.4, the first and second set inclusions are direct from definitions, and the finite dimensionality of Z gives us the last set equality. Hence, for each $k \in \mathbb{N}$, there exist some $I_k \subseteq I$ with $|I_k| < \infty$ and $a_{i,k}^* \in A_i$ for each $i \in I_k$ such that

$$\sum_{i\in I_k} a_{i,k}^*|_Z \longrightarrow^{\|\cdot\|_Z} a^*|_Z \quad \text{as } k \longrightarrow \infty. \tag{4.3.28}$$

Write $u_k^* := \sum_{i\in I_k} a_{i,k}^*$ for each $k \in \mathbb{N}$. Then, by **(i)**, one has that $u_k^* \in \sum_{i\in I_k} D_i + \sum_{i\in I_k} K_i$. Observe that

$$(\sum_{i\in I_k} K_i)|_Z = \sum_{i\in I_k} (K_i|_Z) = \operatorname{cone} \bigcup_{i\in I_k} (K_i|_Z) \subseteq Z,$$

where Z is of finite dimension m (by **(iii)**). Thus, by the virtue of Carathéodory Theorem [25, Corollary 17.1.2], we see that for each $k \in \mathbb{N}$,

$$u_k^*|_Z = \sum_{i\in I_k} d_{i,k}^*|_Z + \sum_{j=1}^{m} z_{i_{j,k}}^*|_Z \tag{4.3.29}$$

for some $d_{i,k}^* \in D_i$ $(i \in I_k)$, $\{i_{1,k}, \ldots, i_{m,k}\} \subseteq I$ and $z_{i_{j,k}}^* \in K_{i_{j,k}}$ for each $j \in \{1, \ldots, m\}$. Let $I' := \bigcup_{k\in\mathbb{N}} I_k$. Then, I' is countable. Also, for each $k \in \mathbb{N}$ and $i \in I' \backslash I_k$, let $d_{i,k}^* = 0$. Without loss of generality, one can assume that $I' = \mathbb{N}$. For each $k \in \mathbb{N}$, since $|I_k| < \infty$ and $d_{i,k}^* = 0$ for any $i \in I' \backslash I_k$, it follows that $\sum_{i\in\mathbb{N}}^* d_{i,k}^*$ exists in X^* and

$$\sum_{i\in\mathbb{N}}^* d_{i,k}^* = \sum_{i\in I_k} d_{i,k}^* \quad \text{on } X. \tag{4.3.30}$$

So, for each $k \in \mathbb{N}$, (4.3.29) becomes

$$u_k^*|_Z = \sum_{i\in\mathbb{N}}^{*} d_{i,k}^*|_Z + \sum_{j=1}^{m} z_{i_{j,k}}^*|_Z. \tag{4.3.31}$$

Now, we will show that:

$$\{z_{i_{j,k}}^*|_Z\}_{k\in\mathbb{N}} \text{ is a bounded sequence on } Z \text{ for each } j \in \{1,\dots,m\}. \tag{4.3.32}$$

To do this, suppose on the contrary that there exists some $j_0 \in \{1,\dots,m\}$ such that $\{z_{i_{j_0,k}}^*|_Z\}_{k\in\mathbb{N}}$ is unbounded. Then, by passing to subsequence if necessary, one can assume that

$$\sum_{l=1}^{m} \|z_{i_{l,k}}^*|_Z\|_Z \longrightarrow \infty \quad \text{as } k \longrightarrow \infty,$$

and

$$\sum_{l=1}^{m} \|z_{i_{l,k}}^*|_Z\|_Z > 0 \quad \text{for any } k \in \mathbb{N}.$$

Dividing both sides of (4.3.31) by $\sum_{l=1}^{m} \|z_{i_{l,k}}^*|_Z\|_Z$, we get that, for each $k \in \mathbb{N}$,

$$\frac{u_k^*|_Z}{\sum_{l=1}^{m} \|z_{i_{l,k}}^*|_Z\|_Z} = \frac{(\sum_{i\in\mathbb{N}}^{*} d_{i,k}^*)|_Z}{\sum_{l=1}^{m} \|z_{i_{l,k}}^*|_Z\|_Z} + \frac{\sum_{j=1}^{m} z_{i_{j,k}}^*|_Z}{\sum_{l=1}^{m} \|z_{i_{l,k}}^*|_Z\|_Z}. \tag{4.3.33}$$

Since $u_k^*|_Z \longrightarrow^{\|\cdot\|_Z} a^*|_Z$ as $k \longrightarrow \infty$ (see (4.3.28)), one knows that $\{u_k^*|_Z\}_{k\in\mathbb{N}}$ is bounded on Z. Thus,

$$\left\|\frac{u_k^*|_Z}{\sum_{l=1}^{m} \|z_{i_{l,k}}^*|_Z\|_Z}\right\|_Z \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,$$

which implies that

$$\frac{u_k^*|_Z}{\sum_{l=1}^{m} \|z_{i_{l,k}}^*|_Z\|_Z} \longrightarrow^{\|\cdot\|_Z} 0 \quad \text{as } k \longrightarrow \infty. \tag{4.3.34}$$

Noting that $\|d^*\| \leq \operatorname{diam} D_i$ for any $i \in I$ and $d^* \in D_i$, thanks to the assumption that each D_i contains the origin. This together with **(ii)** and (4.3.30) show that

$$\left\|\sum_{i\in\mathbb{N}}^{*} d_{i,k}^*\right\| = \left\|\sum_{i\in I_k} d_{i,k}^*\right\| \leq \sum_{i\in I_k} \|d_{i,k}^*\| \leq \sum_{i\in I} \operatorname{diam} D_i < \infty,$$

and so it follows that

$$\frac{(\sum^*_{i\in\mathbb{N}} d^*_{i,k})|_Z}{\sum_{l=1}^m \|z^*_{i_{l,k}}|_Z\|_Z} \longrightarrow^{\|\cdot\|_Z} 0 \quad \text{as } k \longrightarrow \infty. \tag{4.3.35}$$

Combining (4.3.33), (4.3.34) and (4.3.35), we get that

$$\frac{\sum_{j=1}^m z^*_{i_{j,k}}|_Z}{\sum_{l=1}^m \|z^*_{i_{l,k}}|_Z\|_Z} \longrightarrow^{\|\cdot\|_Z} 0 \quad \text{as } k \longrightarrow \infty.$$

On the other hand, it is clear that for each $k \in \mathbb{N}$, $\sum_{j=1}^m \|\frac{z^*_{i_{j,k}}|_Z}{\sum_{l=1}^m \|z^*_{i_{l,k}}|_Z\|_Z}\|_Z = 1$ and $\frac{z^*_{i_{j,k}}|_Z}{\sum_{l=1}^m \|z^*_{i_{l,k}}|_Z\|_Z} \in K_{i_{j,k}}$ for all $j \in \{1, \ldots, m\}$ (as K_i is a cone for all $i \in I$). Hence, $\gamma(K_j|_Z; \{1, \ldots, m\}) = 0$, which contradicts **(iv)**. So, (4.3.32) is seen to hold.

By the compactness of I, we can assume that for each $j \in \{1, \ldots, m\}$, $i_{j,k} \longrightarrow \bar{i}_j$ for some $\bar{i}_j \in I$. Also, by (4.3.32) and the fact that Z is of finite dimension, one assumes (by passing to subsequence if necessary) that for each $j \in \{1, \ldots, m\}$, $z^*_{i_{j,k}}|_Z \longrightarrow^{\|\cdot\|_Z} \overline{z}^*_j|_Z$ as $k \longrightarrow \infty$ for some $\overline{z}^*_j \in X^*$. Then, it follows from the upper semicontinuity of the set-valued map $i \mapsto K_i|_Z$ at $\bar{i}_j$ (see **(v)**) that

$$\overline{z}^*_j|_Z \in K_{\bar{i}_j}|_Z.$$

Hence, there exists some $\overline{w}^*_{\bar{i}_j} \in K_{\bar{i}_j}$ such that $\overline{w}^*_{\bar{i}_j}|_Z = \overline{z}^*_j$. Thus, by replacing $\overline{z}^*_j$ with $\overline{w}^*_{\bar{i}_j}$ if necessary, we assume that $\overline{z}^*_j \in K_{\bar{i}_j}$ for each $j \in \{1, \ldots, m\}$.

Next, we prove that

$$(a^* - \sum_{j=1}^m \overline{z}^*_j)|_Z \in (\sum_{i\in\mathbb{N}}{}^* D_i)|_Z. \tag{4.3.36}$$

Indeed, by **(ii)**, one sees that

$$\sum_{i\in\mathbb{N}} \text{diam}\,(D_i|_Z) \le \sum_{i\in\mathbb{N}} \text{diam}\, D_i \le \sum_{i\in I} \text{diam} D_i < \infty.$$

Also, for any $i \in \mathbb{N}$, $v^* \in D_i|_Z$ and $x \in Z$, we have

$$-(\text{diam}\,(D_i|_Z))\|x\|_Z \le \langle v^*, x\rangle \le (\text{diam}\,(D_i|_Z))\|x\|_Z.$$

Thus, it follows from **(i)** (which implies that $D_i|_Z$ is weakly* compact for all $i \in \mathbb{N}$) and Proposition 2.8.1 (applied to the space Z in place of X with $g_i(\cdot) := -\mathrm{diam}\,(D_i|_Z)\|\cdot\|$ and $h_i(\cdot) = \mathrm{diam}\,(D_i|_Z)\|\cdot\|$ for each $i \in \mathbb{N}$) that

$$\sum_{i\in\mathbb{N}}^{*}(D_i|_Z) = \overline{\sum_{i\in\mathbb{N}}^{*}(D_i|_Z)}^{w^*}.$$

Hence,

$$\sum_{i\in\mathbb{N}}^{*}(D_i|_Z) = \overline{\sum_{i\in\mathbb{N}}^{*}(D_i|_Z)}^{\|\cdot\|_Z}, \tag{4.3.37}$$

thanks to the finite dimensionality of Z. Moreover, by Lemma 4.3.5, one has

$$\sum_{i\in\mathbb{N}}^{*}(D_i|_Z) = (\sum_{i\in\mathbb{N}}^{*}D_i)|_Z. \tag{4.3.38}$$

Furthermore, since $(u_k^* - \sum_{j=1}^{m} z_{i_j,k}^*)|_Z \longrightarrow^{\|\cdot\|_Z} (a^* - \sum_{j=1}^{m} \overline{z}_j^*)|_Z$ as $k \longrightarrow \infty$, and $(\sum_{i\in\mathbb{N}}^{*} d_{i,k}^*)|_Z \in (\sum_{i\in\mathbb{N}}^{*} D_i)|_Z$ for all $k \in \mathbb{N}$, one has from (4.3.31) that

$$(a^* - \sum_{j=1}^{m}\overline{z}_j^*)|_Z \in \overline{(\sum_{i\in\mathbb{N}}^{*}D_i)|_Z}^{\|\cdot\|_Z}. \tag{4.3.39}$$

So, by (4.3.37), (4.3.38) and (4.3.39), we see that

$$(a^* - \sum_{j=1}^{m}\overline{z}_j^*)|_Z \in (\sum_{i\in\mathbb{N}}^{*}D_i)|_Z,$$

that is, (4.3.36) holds.

Noting that (4.3.36) implies

$$a^* \in \sum_{i\in\mathbb{N}}^{*}D_i + \sum_{j=1}^{m}K_{\overline{i}_j} + Z^{\perp}.$$

Since Z is a subspace, we have that $Z^\circ = Z^\perp$. Also, by **(iii)** and Remark 4.3.1, $Z^\circ \subseteq K_{i_0}^{\circ\circ}$. Thus, by applying the bipolar theorem [31, Theorem 1.1.9] to the closed convex cone K_{i_0}, it follows that

$$Z^\perp = Z^\circ \subseteq K_{i_0}^{\circ\circ} = K_{i_0}.$$

Then,

$$\begin{aligned}
\sum_{i\in\mathbb{N}}^{*} D_i + \sum_{j=1}^{m} K_{\bar{i}_j} + Z^{\perp} &\subseteq \sum_{i\in\mathbb{N}}^{*} D_i + \sum_{j=1}^{m} K_{\bar{i}_j} + K_{i_0} \\
&\subseteq \sum_{i\in I}^{*} D_i + \sum_{i\in I}^{*} K_i \\
&= \sum_{i\in I}^{*} (D_i + K_i) \\
&= \sum_{i\in I}^{*} A_i,
\end{aligned}$$

where the second set inclusion follows from the fact that $0 \in D_i \cap K_i$ for each $i \in I$, and the last set equality comes from **(i)**. Therefore, $a^* \in \sum_{i\in I}^{*} A_i$. So, (4.3.27) is established. The proof is complete. □

Similar to the case for continuous functions, we have the following theorem for nonnegative functions, which follows from Theorems 4.2.2 and 4.3.6.

Theorem 4.3.7. *Suppose that* (4.3.1) *and* (4.3.15) *hold and I is a compact metric space. Assume in addition that the conditions **(i)**-**(v)** stated in Theorem 4.3.6 hold. Then, for any $x^* \in X^*$,*

$$\inf_{x\in X}(f(x) - \langle x^*, x\rangle) = \max\{-\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}^{*} x_i^* = x^*\}.$$

In particular, the collection $\{f, f_i : i \in I\}$ satisfies the Fenchel duality:

$$\inf\{f(x) : x \in X\} = \max\{-\sum_{i\in I} f_i^*(x_i^*) : \sum_{i\in I}^{*} x_i^* = 0\}.$$

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