An Efficient Valuation of Participating Life Insurance Contracts under Lévy Process

WONG, Shiu Fung

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Thesis/Assessment Committee

Professor CHAN Ngai Hang (Chair)

Professor WONG Hoi Ying (Thesis Supervisor)

Professor WONG Po Shing(Committee Member)

Professor Siu Tak Kuen (External Examiner)

The Chinese University of Hong Kong

Graduate School

The undersigned certify that we have read a thesis, entitled "An Efficient Valuation of Participating Life Insurance Contracts under Lévy Process" submitted to the Graduate School by Wong, Shiu Fung ( ) in partial fulfillment of the requirements for the degree of Master of Philosophy in Risk Management Science. We recommend that it be accepted.

Prof. H. Y. Wong
Supervisor

Prof. N. H. Chan

Prof. P. S. Wong

Prof. T. K. Siu
External Examiner

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Abstract

Participating life insurance policies are investment/saving plans which specify a benchmark return, an annual minimum rate of return guarantee and a surplus distribution. Thus, it is essentially a contract embedded with strong path-dependent option features. These contracts make up a significant part of the life insurance market in Europe and North America. In this thesis, we propose a very efficient FFT network to value these contracts when the underlying asset follows Lévy process. The proposed method gets rid of the problems generated by standard Monte Carol (MC) simulation and allows for earlier termination of the contract. A basic network on the underlying asset resembling a combination of combining multinomial trees is first constructed through the characteristic function of the process using Fast-Fourier Transformation (FFT). The basic network is then transformed into the one that takes into account the strong path-dependency of a participating policy using a Markov Chain approximation. The valuation is then carried out efficiently within the resulting network. The main advantage is that it can easily accomplish the tasks involving surrender rights, different participation rates, guaranteed return rates and/or terminal bonus rates. The convergence of the FFT network is also proven. Numerical examples show that the proposed scheme is much more efficient than Monte Carlo simulation.

摘要

分紅壽險保單是一種預設基準回報、保證最底年息以及期滿分紅的投資/儲蓄計劃。因此，這類保單內含期權性質和很強的路徑依賴特質。此類保單在歐洲和北美的保險業市場佔有一定的地位。在這論文中，我們提出一種非常高效率的快速傅立葉轉換(Fast Fourier Transform, FFT) 網絡來為這種保單定價，而保單中的基準資產是跟隨Lévy過程的。這方法避免了由蒙地卡羅模擬法衍生出來的問題以及容許提早終結保單。我們首先用快速傅立葉轉換把基準資產過程的特徵函數建構出一個類似合斂多項樹的基本網絡。然後用馬可夫鏈近似法把基本網絡擴展成一個包括強路徑依賴特質變量的網絡。保單定價就在此擴展後的網絡上進行。此方法最大的好處是在有提早終結、不同參與比率、保證年息和/或期滿分紅的情況下仍舊可以使用。實驗計算結果總結此方法比蒙地卡羅模擬法更為有效率。

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Chapter 1

Introduction

Participating life insurance policies are popular in many developed countries such as the US, Canada, Australia, Japan and many European countries in the recent decade. These policies are relatively less risky but provide competitive return compared to other equity-linked products. Participating life insurance policies provide guaranteed premiums, death benefits, cash values and a dividend based on the profits earned by the insurance company issuing the policy. These policies are particularly interested by those who concern about their pension or retirement plans against inflation. While the payoffs are linked to equity indices which represent the economy, the policies can track inflation to a certain extent.

In this thesis we concentrate on classical contingent claim valuation of the most common policy design. Grosen and Jørgensen (2000,2002) [1, 2], Bacinello (2003i, 2003ii, 2005) [3, 4, 5], Ballotta (2005) [6] and Ballotta et al. (2006) [7] mention several possible contract designs and custom features can be included in participating policies.

Unlike non-participating life insurance policies with predictable cash flows, participating life insurance policies represent variational liabilities to the issuers. The potential risk of participating policies should be carefully addressed. Adequate and consistent models for pricing and liability valuation have to be built to meet the requirements of market-oriented accounting principles for insurance liabilities, such as International Finance Reporting Standards(IFRS), from the International Accounting Standards Board (IASB). The pricing and risk analysis of participating policies are focused on modelling underlying asset.

Apart from classic Black and Schole geometric Brownian motion model [8], assorted continuous path models have been considered in modelling underlying assets in participating policies. Since life insurance products usually have long maturities, models capture long term movement are under concerns. Ballotta (2005) [6] developed closed-form valuation on part of participating policies which underlying assets follow Lévy processes. Siu (2005, 2007) [9, 10] investigated regime switching models in underlying assets with surrender rights. Siu et al. (2007) [11] considered Asymmetric Power GARCH.

Derivative pricing with Lévy processes become practical and feasible after Carr and Madan (1999) [12] derive a simple expression for European options by using Fast Fourier Transform(FFT). Lewis (2001) [13] applied Fourier Transform to European simple or exotic options (without path-dependence) under Lévy processes with known characteristic functions. In addition to the FFT proposed by Cooley and Tukey (1965) [14], FFT implemented with Fourier-cosine series was recently adopted to financial product pricing, by Fang and Oosterlee (2008) [15].

This research develops an efficient numerical approach for valuating participating policies which's underlying assets follows Lévy processes. Because of the strong path dependent feature in participating policies, no closed-form pricing formula is found. We create a network of underlying asset's dynamics with transition probabilities directly calculated from taking inverse Fourier Transform of its characteristic function using FFT. The network discretizes in asset price domain and time domain. Therefore, we can price participating policies according to this network which mimics the dynamics of underlying asset. The network is expanded to cope with two path dependent variables. This approach addressed several problems in traditional pricing methods like tree and simulation. The network pricing approach allows jumps in asset dynamics, which cannot be included in tree models, by mean of transition probabilities. It also allows Bermudan surrender rights, which can be troublesome in simulation, as time in network is discretized and can be adjusted to fit Bermudan surrender time.

Duan and and Simonato (2001) [16] used a Markov Chain approximation of GARCH(1,1) model in pricing American option. This method innovates our development of network pricing scheme. The major different between our proposed method and theirs is we use FFT in finding transition probabilities and they used statistical properties of GARCH models. Our scheme is much practical when facing different kinds of asset dynamics while most dynamics used in financial markets are not analytically tractable.

We refer Kwok (2009) [17] and Wong and Guan (2010) [18] for similar lattice approach in pricing exotic options. The convergence is trivial in pricing options with barrier like American barrier options, American lookback options and American Asian options. The product we encountered in this thesis has no barrier embedded. Therefore, the convergence of the FFT network is questionable. We have proven the convergence for all payoffs which are dominated by a finite first moment function.

In the next chapter, we introduce the payoff structure of participating policy. Chapter 3 reviews use of Lévy processes and Fast Fourier Transform(FFT) technique in financial product pricing. Chapter 4 proposes network pricing scheme and proves the scheme converges. Chapter 5 puts the proposed scheme into practice and compare the results with simulation. Last chapter concludes and discuss extension of custom features on policy.

Chapter 2

Participating policy

When people consider their pension or retirement plans, they look for the one with a stable payoff and a fair return which can beat inflation, meaning protection against adverse movement of the investment. Stock market is a good indicator of the economy and inflation for one to invest. The protection can be constructed by using derivatives like put options. Gerber and Shiu (1998,1999) [19, 20] introduced dynamic fund protection(DFP) which extends this option protection concept to provide protection at multiple time points contingent on a underlying asset. The most popular contracts are participating life insurance policies. This kind of policies is an insurance product which guarantees the contract holder a minimum annual return. The cash value of the policy is specified by a benchmark return from an underlying asset or index. The policyholder will receive the cash value of the policy at maturity or his beneficiary will receive a death benefit at the policyholder's premature death. These contacts make up a significant proportion of life insurance market in many countries like the US, Canada, Australia, Japan and other countries in continental Europe.

A general participating life insurance policy guarantees the policy holder a death benefit and a minimum rate of return, which is determined by the underlying asset or index. In addition to the guaranteed minimum amounts, the participating policy accumulates dividends each year. The dividends are not guaranteed but are paid once the underlying asset outperforms the guaranteed return. In our discussion, we ignore lapses and mortality. However,surrender right of contracts and mortality within our pricing scheme will be discussed in a later section.

Consider a fund which puts all the premium from the policyholders to an underlying asset, $A(t)$, with the guaranteed return by the insurance company. This fund provides 100% of the profit to policyholders. This is called the unsmoothed asset share, i.e. an 100% participation rate participating policy. Let $P^*(t)$ be the value of this unsmoothed asset share for $0 \leq t \leq T$, where T is the maturity time. At time zero, the value of this unsmoothed asset share is equal to the initial premium, P_0.

$$\begin{aligned} P^*(0) &= P_0, \\ P^*(t) &= P^*(t-1)(1+r_P(t)), \\ r_P(t) &= \max\{r_G, \beta r_A(t)\}, \end{aligned} \tag{2.1}$$

r_G and $\beta \in (0,1)$ are the guaranteed rate of return and the return ratio, respectively. $r_A(t)$ is the annual rate of return of the reference asset which can be calculated from its value, $A(t)$ for $0 \leq t \leq T$,

$$r_A(t) = \frac{A(t) - A(t-1)}{A(t-1)}.$$

Participating policy will invest a fixed proportion, $\alpha \in (0,1)$, of the policy account in the unsmoothed asset share and $(1-\alpha)$ of it as reserve. This participating account is defined as

$$\begin{aligned} P(t) &= \alpha P^*(t) + (1-\alpha)P(t-1), \alpha \in (0,1), \qquad (2.2) \\ P(0) &= P_0. \end{aligned}$$

At the contract's maturity, a terminal bonus might be paid by the issuer. The terminal bonus is a fraction of the surplus earned from the reference asset over the policy reserve, which can be expressed as $\gamma(A(T) - P(T))$ where γ is the bonus rate usually taking value between 0 and 1. However, it is possible that the reference asset underperforms the participating account and the issuer cannot fully repaid the participating account value. The issuer would then pay the value of the reference asset instead in this case. These two terminal features can be summarized as follows:

$$C(T) = \begin{cases} A(T) & \text{if} A(T) < P(T), \\ P(T) + \gamma(A(T) - P(T)) & \text{otherwise,} \end{cases}$$

where $C(T)$ is the terminal payoff of the contract. The payoff can be rewritten in an option-like payoff:

$$\begin{aligned} C(T) &= P(T) + \gamma \max(A(T) - P(T), 0) - \max(P(T) - A(T), 0) \\ &= A(T) + (\gamma - 1) \max(A(T) - P(T), 0). \end{aligned} \tag{2.3}$$

From the previous section, the terminal payoff depends on the value of the underlying reference asset, $A(T)$, and participating account,$P(T)$, only. The value of the participating account at maturity can be calculated recursively as

$$\begin{aligned} P(T) &= \alpha P^*(T) + (1 - \alpha P(T-1)) \\ &= \alpha \sum_{k=0}^{T-k} (1-\alpha)^k P^*(T-k) + (1-\alpha)^T P_0 \\ &= P_0 \left[\alpha \sum_{k=0}^{T-1} (1-\alpha)^k \prod_{t=1}^{T-k} (1 + r_P(t)) + (1-\alpha)^T \right]. \end{aligned}$$

This means that the terminal payoff is actually depending on the path of $A(t)$ instead of just the single terminal value of it. The variable relationship is shown in figure 2.1. This highly path-dependent feature causes the contact to lack of closed-form pricing formula and the early exercise boundary is uneasy to be visualized once early unwind is allowed.

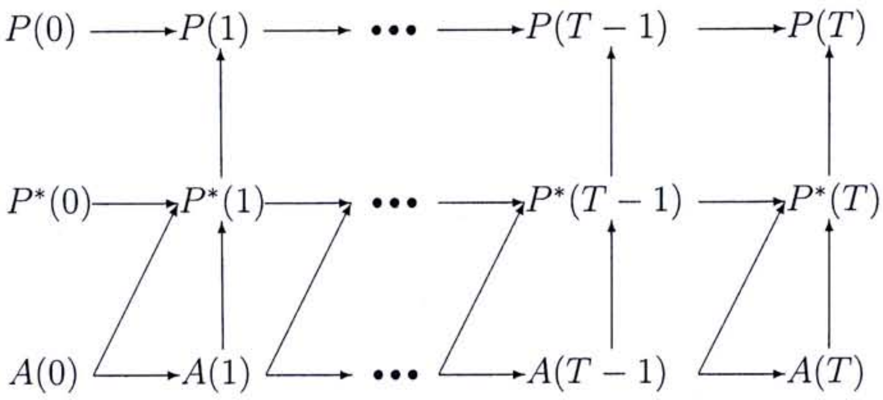

Figure 2.1: Variable relationship in participating policy

Chapter 3

Lévy Process and its use in financial modelling

Although Bachelier[21] first suggested Brownian motions to model stock prices, financial modelling using stochastic processes was not widely adopted to the market until 1970's. Black and Scholes (1973) [8] proposed geometric brownian motion(GBM) model for assets and risk neutral pricing. Merton (1976) [22] introduced a jump in asset value process which further generalizes diffusion model to account for unusual dramatic movements observed in the market. The jump diffusion model is proved empirically better than GBM for describing stock movements. Plenty of models are then introduced thereafter to capture different movement behaviors in different markets. Whereas, GBM becomes a benchmark model for financial product pricing.

3.1 Lévy process in asset modelling

In the late 1980s, Lévy process was first proposed for modelling financial data. Let's first look at the definition of Lévy processes.

Definition 3.1. *(Properties of Lévy processes) An adapted real-valued stochastic process X_t, with $X_0 = 0$, is called a Lévy process if it has the following properties:*

(i) *Independent increment. For any choice of* $n \geq 1$ *and* $0 \leq t_0 < t_1 < \cdots < t_n$, *the random variables,* $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent;*

(ii) *Time-homogeneous. The distribution of the random variable,* $X_{t+s} - X_s$, *does not depend upon* s.

(iii) *Stochastically continuous. For any* $\epsilon > 0$, $\Pr\{|X_{s+t} - X_s| > \epsilon\} \to 0$ *as* $t \to 0$.

(iv) *Càdlàg. It is everywhere right-continuous and has left limits everywhere.*

One can observe that Black and Scholes's diffusion model and Merton's jump model are both under the scope of Lévy processes.

Let's consider some common Lévy processes which consist of Brownian motions with drift and a jump process. In the jump process part, we define possible jump sizes to be non-zero real numbers. Let N_t^A be the cumulative number of jumps in time interval $[0, t]$ with jump size in a closed interval $A \in \mathbb{R}\backslash\{0\}$. N_t^A is a random variable as well as a measure. If we fix A, N_t^A is a Poisson random variable with mean $t \int_A \mu(x)dx$. The measure $\mu(x)dx$ is a Lévy measure measuring the relative occurrence of different jump sizes.

We can distinguish two types of Lévy processes by looking at the Poisson arrival rate of jumps. For type I, the unit time arrival rate is finite, $\int_{\mathbb{R}} \mu(x)dx < \infty$. We can write the mean jump arrival rate to be $\lambda = \int_{\mathbb{R}} \mu(x)dx$. Take Merton's jump-diffusion model(1976) as an example, the jump size is normally distributed, $\mu(x) = \lambda f(x) = \lambda \exp[-(x-m)^2/2s^2]/\sqrt{2\pi s^2}$

In type II, the unit time arrival rate is infinite, $\int_{\mathbb{R}} \mu(x)dx = \infty$, no Poisson intensity can be defined. The jumps in this case is infinitely frequent and small which causes $\mu(x)$ cannot be integrated at the origin. We can use a general integral representation to describe the process at some large $|x|$.

$$X_t = \omega t + \sigma W_t + \int_0^t \int_{\mathbb{R}\backslash\{0\}} (x\nu(ds, dx) - x\mathbb{I}_{\{|x|<1\}}\mu(s)ds\ dx)$$

$\nu(ds, dx)$ is a differential form of the integer-valued random measure of $N_t^A = \nu([0,t], A)$. The Lévy process X_t can then be decomposed in the above form.

It can be seen that the Wiener process and the jump process are independent. ω and σ are constant. For a given $\{X_t : t \geq 0\}$, $\{\omega, \sigma, \mu(x)\}$ forms a unique Lévy-Khintchine triplet which can be fully representing a Lévy process. Before stating the Lévy-Khintchine representation, we define the characteristic function of a stochastic process.

Definition 3.2. *(Characteristic function) The characteristic function of process X_t is defined as*

$$\phi_t(z) = \mathbb{E}[\exp(izX_t)] \qquad z \in \mathbb{C}, i = \sqrt{-1}$$

Let $f_t(x)$ be the transition probability density for X_t to be x. The characteristic function is the generalized Fourier transform of the transition density inside regular strip $a < Im(z) < b$, where $a < b$.

$$\phi_t(z) = \mathcal{F}[f_t(x)] \triangleq \int_{\mathbb{R}} \exp(izx) f_t(x)\, dx$$

Definition 3.3. *(Infinite divisibility) A probability distribution F is infinitely divisible if X is a random variable with distribution F, for any positive integer n there exist n independent identically distributed random variables, $X_1, \ldots, X_n$ such that $X_1 + \cdots + X_n \xrightarrow{d} X$.*

Definition 3.4. *(Infinite divisible characteristic functions) An infinite divisible characteristic function is the characteristic function of any infinitely divisible distribution.*

Theorem 3.5. *(Lévy-Khintchine Representation) If the process X_t is stable and $\phi_t(z)$ is an infinitely divisible characteristic function, the characteristic function has the representation*

$$\phi_t(z) = \exp\left\{ iz\omega t - \frac{1}{2}z^2\sigma^2 t + t\int_{\mathbb{R}\setminus\{0\}} [e^{izx} - 1 - izx\mathbb{I}_{\{|x|<1\}}]\mu(x)dx \right\}$$

When we take the stock price evolution to be $S_T = S_0 \exp[(r - q)T + X_T$, where r is the interest rate, q is the dividend yield and X_T is a Lévy process. The evolution is full characterized by the characteristic function of X_T. In this thesis, we will focus on three Lévy processes which includes pure diffusion model, jump-diffusion model with finite activities and jump-diffusion model with infinite activities. To illustrate the aforementioned three types, we use Black-Scholes (1973)[8] GBM model, Merton's(1976)[22] jump-diffusion model and Carr, Madan and Chang's(1998)[23] variance gamma model. Their corresponding characteristic functions are listed in table 3.1.

Lévy Process	Characteristic Function $\phi_T(z)$
Pure Diffusion	
Geometric Brownian Motion	$\exp[iz\omega T - \frac{1}{2}z^2\sigma^2 T]$
Jump-Diffusion with finite activities	
Merton's Jump-Diffusion	$\exp[iz\omega T - \frac{1}{2}z^2\sigma^2 T + \lambda T(e^{iz\omega_J - z^2\sigma_J^2/2} - 1)]$
Jump-Diffusion with infinite activities	
Geometric Brownian Motion	$\exp[iz\omega T]\left(1 - iz\nu\theta + \frac{1}{2}z^2\nu\sigma^2\right)^{-T/\nu}$

Table 3.1: Characteristic Functions for Lévy Process

3.2 Lévy process in derivative pricing

The most common problem encountered in derivative pricing is option pricing. Option prices are evaluated by taking the expectations on terminal payoffs. Taking the simplest European Vanilla option as example, the expectation can be solved if the density function of the underlying is analytically known, e.g. Black-Scholes(1973) model. However, the density functions of most Lévy processes cannot be expressed analytically. Carr and Madan (1999) [12] suggested a Fast Fourier Transformation(FFT) approach to value options when the characteristic function of the underlying asset is known. This approach can virtually deal with all kinds of distribution because the characteristic functions are always well defined and this approach only requires the numerical values of the characteristic functions.

Suppose that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and filtration $\{\mathcal{F}_t, 0 \le t \le T\}$ are given. We consider the price process $S_t = S_0 e^{X_t}$ defined on this probability space, where X_t is a Lévy process. We assume $\mathcal{F}_t = \sigma(S_s, 0 \le s \le t) = \sigma(X_s, 0 \le s \le t)$ and $\mathcal{F} = \mathcal{F}_T$. Our pricing will be on a risk free probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which is an equivalent martingale measure of S_t satisfying $\mathbb{Q} \sim \mathbb{P}$ and $e^{-rt}S_t$ is $(\mathcal{F}_t, \mathbb{Q})$-martingale, where r is the risk free interest rate. We refer the work by Gerber and Shiu(1994) [24] who adopted Esscher transform, which introduced by Esscher(1932) [25], in choosing an equivalent martingale measure. Pricing of derivatives throughout this thesis will be under this risk neutral measure $\mathbb{Q}$.

3.2.1 Review of FFT methods in option pricing

Consider an European call option with maturity T and strike at K written on an underlying asset S_t. The risk-neutral price of this call option is

$$C_t = e^{-r(T-t)}\mathbb{E}_t[\max(S_T - K, 0)].$$

Denote $s_T = \ln(S_T)$ and $k = \ln(K)$. We can write the above expectation in integral form as

$$C_t = e^{-r(T-t)} \int_k^{\infty} (e^{s_T} - e^k, 0) f_t(s_T)\, ds_T,$$

where $f_t(s_T)$ is the risk-neutral density function of the log asset price.

By taking Fourier Transformation on the strike, the transformed option price can be expressed in terms of the characteristic function of the log asset price. To make sure C_t is square integrable in the transformation, Carr and Madan(1999) suggested to add a damping factor, $\exp(\alpha k)$ to the integral part and undamp it afterwards. The Fourier Transform of the damped call option price is

$$
\begin{aligned}
\mathcal{F}[\exp(\alpha k)C_t]] &= \int_{-\infty}^{\infty} e^{izk} e^{\alpha k} C_T \, dk \\
&= \int_{-\infty}^{\infty} e^{izk} e^{\alpha k} e^{-r(T-t)} \int_{k}^{\infty} (e^{s_T} - e^k, 0) f_t(s_T) \, ds_T \, dk \\
&= \int_{-\infty}^{\infty} e^{-r(T-t)} f_t(s_T) \int_{-\infty}^{s_T} (e^{s_T+\alpha k} - e^{(1+\alpha)k} e^{izk} \, dk \, ds_T \\
&= \int_{-\infty}^{\infty} e^{-r(T-t)} f_t(s_T) \left(\frac{e^{(\alpha+1+iz)s_T}}{(\alpha+iz)(\alpha+1+iz)} \right) ds_T \\
&= e^{-r(T-t)} \frac{\phi_t(z-(\alpha+1)i)}{(\alpha+iz)(\alpha+1+iz)},
\end{aligned}
$$

where $\phi_t(z)$ is the characteristic function of the log asset price.

The option price can be found by performing inverse Fourier Transformation with undamping factor $\exp(-\alpha k)$,

$$
\begin{aligned}
C_t &= \mathcal{F}^{-1}[\exp(-\alpha k)\mathcal{F}[\exp(\alpha k)C_t]] \\
&= \frac{1}{\pi} \int_0^{\infty} \exp(-\alpha k) e^{-r(T-t)} \frac{\phi_t(z-(\alpha+1)i)}{(\alpha+iz)(\alpha+1+iz)} \, dz.
\end{aligned}
$$

In order to make use of FFT to evaluate the integral efficiently, we first discretize the integral by quadrature rules. Different from expectation approach, a spectrum of option prices with different strike prices will be yield by mean of FFT.

3.2.2 Expectation using FFT

In the previous part, the payoff function after Fourier Transform is analytically trackable. However, in some cases, Fourier Transform cannot be performed analytically. This will yield two numerical integrations in Fourier Transforming the payoff function and inverse Fourier Transforming to the expectation. It is undesirable to do numerical integration twice even though it can be speeded up by using FFT. We now demonstrate a numerical expectation using characteristic functions.

Denote a general payoff function $V(T)$ depends only on the terminal value of the underlying asset S_T which's return follows Lévy process, i.e. $V(T) = g^*(S_T) = g(s_T)$ where $s_T = \ln S_T$. The risk-neutral price at time t is the expectation of the terminal payoff under risk-neutral measure discounted by risk-free rate,

$$\begin{aligned} V(t) &= \mathbb{E}[e^{-r(T-t)} g^*(S_T)|\mathcal{F}_t] \\ &= \int_0^\infty e^{-r(T-t)} g(s(T)) f(s_T)\, ds_T\,, \end{aligned}$$

where $f(s_T)$ is the density function of s_T under risk-neutral measure given filtration $\mathcal{F}_t$. We discretize the log asset value into n levels and hence write the integral into a sum,

$$\begin{aligned} \overline{V(t)} &= \mathbb{E}^n[e^{-r(T-t)} g(s_T)|\mathcal{F}_t] \\ &= \sum_{i=0}^{n} e^{-r(T-t)} g(\bar{s}_i) p(\bar{s}_i), \end{aligned}$$

where $\bar{s}_i$ is the discretize value of s_T for $i = 0, \ldots, n$, Δs is the size of discretized levels of s_T and $p(\bar{s}_i)$ is the probability of terminal value s_T falls into the bin $[\bar{s}_i - \Delta s/2, \bar{s}_i + \Delta s/2)$. The probabilities, $p(\bar{s}_i)$, are evaluated by the inversion formula of characteristic function numerically.

Proposition 3.6. *Given a non-negative real-valued continuous payoff function $g(s_T)$ is dominated by a real-valued function $h(s_T)$ such that $\mathbb{E}[h(s_T)] \leq c < \infty$,*

$$\lim_{s^* \to \infty} \lim_{n \to \infty} \overline{V(T)} = V(T)$$

where s^* is the computational domain radius center at $s_t = \ln S_t$, i.e. the computational domain is $[s_t - s^*, s_t + s^*]$.

Proof. Since the discrete density function generated by FFT converges pointwisely to the continuous one, when $n \to \infty$, the discretized random variable $\bar{s}_T$ converges in distribution to the continuous one s_T. Therefore, under the computational domain, the discretized expected payoff is

$$\lim_{n\to\infty}\sum_{i=0}^{n} e^{-r(T-t)}g(\bar{s}_i)p(\bar{s}_i) = \mathbb{E}[g(s_T)1_{S_T\in[\exp(s_t-s^*),\exp(s_t+s^*)]}|\mathcal{F}_t]\,.$$

The error between the cropped domain expectation and original continuous expectation is

$$\begin{aligned} V(T) - \lim_{n\to\infty}\overline{V(T)} &= \mathbb{E}[V(T)1_{S_T\notin[\exp(s_t-s^*),\exp(s_t+s^*)]}|\mathcal{F}_t] \\ &\leq \mathbb{E}[h(T)1_{S_T\notin[\exp(s_t-s^*),\exp(s_t+s^*)]}|\mathcal{F}_t] \\ &\leq c\Pr(S_T\notin[\exp(s_t-s^*),\exp(s_t+s^*)|\mathcal{F}_t)\,. \end{aligned}$$

When the computational domain radius s^* goes to infinity, the tail probability $\Pr(S_T\notin[\exp(s_t-s^*),\exp(s_t+s^*))$ goes to 0. Therefore, the error diminish to 0 when we expand the computational domain. □

There is practical interest to compute the price of an derivative with a different initial value, for example in Greeks calculation. The error will increase if we keep using the domain centered at s_t. However, it is possible for us to expand the domain in order to keep the error tolerance satisfied.

Corollary 3.7. *When the pricing error of numerical expectation is satisfied by using the computational domain $[s_t-s^*, s_t+s^*]$, the domain can be expanded to keep the pricing error from a range of initial stock value under the tolerance level.*

Proof. Consider the computational domain $[s_t-s^*, s_t+s^*]$, the error of pricing original derivative is bounded by

$$\epsilon = c\Pr(S_T\notin[\exp(s_t-s^*),\exp(s_t+s^*)|\mathcal{F}_t)\,.$$

Assume we are interested in pricing the derivative with initial stock price $\hat{S}_t = e^{s_t+\Delta\hat{s}} > S_t$. The equivalent error bound is

$$\epsilon' = c\Pr(S_T\notin[\exp(\hat{s}_t-s^*),\exp(\hat{s}_t+s^*)|S_t=\hat{S}_t) = \epsilon\,.$$

By extending the computational domain from $[s_t - s^*, s_t + s^*]$ to $[s_t - s^*, \hat{s}_t + s^*]$, the error bound of derivative pricing with initial stock value $\hat{S}_t$ is

$$c \Pr(S_T \notin [\exp(s_t - s^*), \exp(\hat{s}_t + s^*)|S_t = \hat{S}_t) < \epsilon' .$$

Therefore, pricing with initial stock value ranged in $[S_t, \hat{S}_t]$ using the expanded computational domain $[s_t - s^*, \hat{s}_t + s^*]$ have error lower than the original error bound ϵ. $\square$

Chapter 4

Network methodology

In this chapter we propose a network approach to price participating contracts. This network approach makes use of the Markovian property of Lévy processes to construct a progressive network. The network approach is a two-step method. Firstly, we discretize the underlying reference asset values and the time domain. This is used to define the network which mimics the dynamics of the asset. It is required that the asset dynamics has Markovian property which can be assumed from the Lévy properties. We then expand the network to a multi-dimension network which records the participating account and the unsmoothed asset share account which are required for the pricing of participating contracts. The pricing will be done under the expanded network as an numerical expectation.

4.1 Asset dynamic: Network Approach

Under the assumption of underlying reference asset follows Lévy process, we use a Markov chain process to approximate the asset price in the first network setup. Let $A(t)$ be the price at time t of an asset, which can be a stock or an index, follows an exponential Lévy process defined as

$$A(t) = A(0)\exp[(r-q)t + X(t)]$$

where r is the risk free rate, q is the dividend rate and $X(t)$ is a general Lévy process defined in section 3. The asset prices can be approximated by n different vales

$$A(t) \approx \bar{A}(t) \in \{A_1, A_2, \ldots, A_n\}.$$

Denote the Markov transition probability matrix from time τ_i to time τ_j by Π_{ij}. The elements $\pi_{ij}(k,l)$ is the transition probability of the asset from A_k at time τ_i to A_l at time τ_j. The elements are ordered as follows:

$$\Pi_{ij} = \begin{pmatrix} \pi_{ij}(1,1) & \pi_{ij}(1,2) & \ldots & \pi_{ij}(1,n) \\ \pi_{ij}(2,1) & \pi_{ij}(2,2) & \ldots & \pi_{ij}(2,n) \\ \vdots & & \ddots & \vdots \\ \pi_{ij}(n,1) & \pi_{ij}(n,2) & \ldots & \pi_{ij}(n,n) \end{pmatrix}.$$

Under constant parameter Lévy process, the Markov chain is time-homogenous. If we assume time intervals are equal, the Markov transition probability matrices are identical. Hence, denote the transition probability matrix by Π with elements $\pi(k,l)$ where $k, l = 1, 2, \ldots, n$. The dynamic of the asset value can be represented by a network as shown in figure 4.1.

4.1.1 Transition probability by FFT

The Markov transition probability matrix is evaluated by using inversion formula of characteristic function. In basic asset network setup, the discretized values are set to be evenly distributed with grid size ΔA. The continuous value of A is approximated by A_i if A is fallen into the bin $[A_i - \Delta A/2, A_i + \Delta A/2)$. Therefore, the transition probability to A_j given current value A_i is

$$\pi(i,j) \quad = \quad \Pr\{A(t) \in [A_j - \Delta A/2, A_j + \Delta A/2) | A(0) = A_i\}.$$

When the characteristic function of $A(t)$, $\Phi_A(z)$, is given, the density function $f_A(x)$ can be evaluated using inversion formula,

$$f_A(x) \quad = \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} \Phi_A(z)\, dz.$$

Since $A(t)$ is a real valued function having support $[0, \infty)$, the characteristic function $\Phi_A(z)$ is symmetric,

$$f_A(x) = \frac{1}{\pi}\int_0^\infty e^{-izx}\Phi_A(z)\,dz\,.$$

FFT is an efficient algorithm for computing the sum

$$X(k) = \sum_{n=1}^{N} e^{-i2\pi(k-1)\frac{n-1}{N}}x(n) \quad \text{for } k = 1, \dots, N\,,$$

where N is an integer of power 2. Applying Simpson's rule, the integral in inversion formula is discretized into a sum which can be evaluated by FFT,

$$f_A(x_k) = \frac{1}{\pi}\sum_{n=1}^{N} e^{-i2\pi(k-1)\frac{n-1}{N}}e^{ibz_n}\Phi(z_n)\frac{\eta}{3}[3 + (-1)^n - \delta_{n-1}],$$

where η is partition size of numerical integration, b indicates the truncation point of the integral meaning the range of integral is from $-b$ to b, δ_n is the Kronecker delta function which is 1 for $n = 0$ and zero otherwise.

After obtaining the numerical density function, Simpson's rule is applied again to solve for the probabilities,

$$\pi(i, j) = \frac{\Delta A}{8}\left(f_A(A_{j-1}) + 6f_A(A_j) + f_A(A_{j+1})\right)\,.$$

4.1.2 Example in American option pricing

Let $\overrightarrow{A}$ be an $n \times 1$ vector collecting the discretized values of the asset price such that

$$\overrightarrow{A} = [A_1, A_2, \dots, A_{n-1}, A_n]' \in \mathbb{R}^{n\times 1}.$$

The vector of prices of American put option, $\overrightarrow{V}_{\text{put}}(t)$, can be approximated by

$$\overrightarrow{V}_{\text{put}}(t) = \max\{\max[K\mathbb{I} - \overrightarrow{A}, \bar{0}], e^{-r}\Pi\overrightarrow{V}_{\text{put}}(t+1)]\}$$

where the terminal payoff is

$$\overrightarrow{V}_{\text{put}}(T) = \max\{K\mathbb{I} - \overrightarrow{A}, \bar{0}\}\,.$$

The "max(.)" operator here is a vector operator comparing the elements inside the vector one by one. $\bar{0}$ is an $\mathbb{R}^{n\times 1}$ vector with all elements 0. $\mathbb{I}$ is $\mathbb{R}^{n\times 1}$ vector with all elements 1.

Early exercise is not optimal at anytime in American call options for non-dividend paying stocks. The checking of earlier exercise can be omitted. Hence the pricing formula of American can get rid of the recursive operation and can be simplified in to

$$\overrightarrow{V}_{\text{call}}(0) = e^{-rT}\Pi^T \max\{\overrightarrow{A} - K, 0\}$$

which is the same as its European counterpart.

4.2 Extended Network for Participating Contract

Making use of the Markov property of Lévy processes, we can construct the underlying dynamics as mentioned in the previous section. A similar approach has been used for pricing path-dependent options by extending the network into higher dimensions, see Wong and Guan (2010) [18]. Numerical methods like lattice exotic derivative pricing was addressed by Kwok (2009) [17]. This section is to demonstrate an extension of the network approach to price participating contracts which have stronger path-dependence feature. Let's look at a simpler form of participating contracts by ignoring lapse and mortality.

As discussed in Chapter 2, there are two extra path depending variables $P(t)$ and $P^*(t)$, namely participating account and unsmoothed asset share, in valuating participating policies. Although the unsmoothed asset share is not directly accounted for the terminal payoff, the participating account depends on unsmoothed asset share for every time steps. Note that the only randomness is from the underlying asset.

To accommodate participating contracts, we extend the network states from only the underlying reference asset to a triplet of all depending variables $(A(t), P^*(t), P(t))$. Here we use n_1, n_2 and n_3 different values to approximate the underlying asset, unsmoothed asset share and the participating account respectively,

$$\begin{aligned} A(t) &\approx \bar{A}(t) \in \{A_1, A_2, \ldots, A_{n_1}\} \\ P^*(t) &\approx \bar{P}^*(t) \in \{P_1^*, P_2^*, \ldots, P_{n2}^*\} \\ P(t) &\approx \bar{P}(t) \in \{P_1, P_2, \ldots, P_{n_3}\}. \end{aligned}$$

The discretized states of this extended network is a collection of all combinations of discretized individual variables.

$$(\bar{A}(t), \bar{P}^*(t), \bar{P}(t)) \in \{(A_1, P_1^*, P_1), \ldots, (A_{n_1}, P_{n_2}^*, P_{n_3})\}$$

where n_1, n_2 and n_3 are some positive integers. We choose n_1 to be a power of 2 integer in order to use Fast Fourier Transform.

Without lost of generality, we assume the interval of evaluation dates of participation contracts are equal through out the period. Denote the Markov transition probability matrix by Π^*. The elements $\pi_{i_1 i_2 i_3, j_1 j_2 j_3}$ is the transition probability from state with value $(A_{i_1}, P_{i_2}^*, P_{i_3})$ to $(A_{j_1}, P_{j_2}^*, P_{j_3})$. The elements are ordered as follows:

$$\Pi^* = \begin{pmatrix} \pi_{111,111} & \pi_{111,112} & \cdots & \pi_{111,n_1n_2n_3} \\ \pi_{112,111} & \pi_{112,112} & \cdots & \pi_{112,n_1n_2n_3} \\ \vdots & \ddots & & \vdots \\ \pi_{n_1n_2n_3,111} & \pi_{n_1n_2n_3,112} & \cdots & \pi_{n_1n_2n_3,n_1n_2n_3} \end{pmatrix}.$$

One can note that there are maximum n_1 non-zero elements in a row because given a state of $(A_{i_1}, P_{i_2}^*, P_{i_3})$, there is only one possible resultant state if A_{i_1} goes to A_{j_1}. With the terminal payoff function given in (2.3), denote the payoff as a function of underlying asset value, $A(T)$, and participating policy account,

$P(T)$ as follows:

$$C_\gamma(A(T), P(T)) = A(T) + (\gamma - 1)\max(A(T) - P(T), 0)\,.$$

Hence, an $n_1 \times n_2 \times n_3$ vector, $\overrightarrow{C}(T)$, representing the terminal payoff at different states is

$$\overrightarrow{C}(T) = \begin{pmatrix} \overrightarrow{c}(A_1) \\ \overrightarrow{c}(A_2) \\ \vdots \\ \overrightarrow{c}(A_n) \end{pmatrix}, \quad \text{where} \quad \overrightarrow{c}(A_i) = \begin{pmatrix} C_\gamma(A_i, P_1) \\ \vdots \\ C_\gamma(A_i, P_{n_3}) \\ \vdots \\ \vdots \\ C_\gamma(A_i, P_1) \\ \vdots \\ C_\gamma(A_i, P_{n_3}) \end{pmatrix} \in \mathbb{R}^{n_2 n_3 \times 1}.$$

Note that there are n_2 repeating elements for each state inside the vector. It is because we expanded the network according to unsmoothed asset share, P^*, which is not taken into account for the terminal payoff.

The price of participating policy under this network approach is

$$\overrightarrow{C}(t) = e^{-r(T-t)}\Pi^{T-t}\overrightarrow{C}(T).$$

In the network construction, we can take the current value of underlying asset, unsmoothed asset share and participating account be A_i, P_j^* and P_k respectively, the current price of the participating policy is the (i, j, k)-th element of the price vector $\overrightarrow{C}(t)$.

4.3 Practical network construction

The Markov approximation is described in the previous sections. We will look at how the transition probabilities are calculated and some difficulties encountered in practical implementation of the network in this section.

4.3.1 Modified network-drift offsetting

Recall the asset value is discretized into n_1 different values in creating the network. It is necessary to truncate the asset value at a large value to capture as much information as possible. There exist a trend in the underlying asset process called the drift term. The problem is amplified in pricing long-term products which are very common in insurance market. The upward drift of the underlying asset requires setting a large value for the maximum value in the approximation which will lower the accuracy and efficiency. To prevent this happen, we build a Markov chain based on a shifted network in which the state of the next time step is shifted according to the drift of the asset dynamics. Under risk-neutral measure, the drift follows risk-free rate, the levels in time $t \geq 0$ are chosen to be

$$A(t) \approx \bar{A}(t) \in \{e^{rt}A_1, e^{rt}A_2, \ldots, e^{rt}A_{n_1}\}.$$

An illustrative modified network is shown in figure 4.2.

The same drifting problem appears in the other two pricing variables. Part of the drifts can be solved by expanding the account values. From equation (2.1), the unsmoothed asset share can be expressed as

$$\begin{aligned} P^*(t) &= (1 + r_P(t)t)P^*(0) \\ &= (1 + \max(r_G, \beta r_A(t))t)P^*(0) \\ &= (1 + \max(\beta r_A(t) - r_G)t + r_G t)P^*(0) \\ &\geq (1 + r_G t)P^*(0)\,. \end{aligned}$$

The variation of unsmoothed asset share is mainly given by the return of the underlying asset, $r_A(t)$ and captured in the grid. The minimum drift of $P^*(t)$ is $(1 + r_G t)$, i.e.

$$P^*(t) \approx \bar{P^*}(t) \in \{(1 + r_G t)P_1^*, (1 + r_G t)P_2^*, \ldots, (1 + r_G t)P_{n2}^*\}\,.$$

The drift cannot be evaluated explicitly by expressing $P(t)$. Since there are only upward movement in unsmoothed asset share and participating account, we shift up the grid in participating account by the minimum upward movement. The participating account defined in equation (2.2) can be expressed as

$$\begin{aligned} P(t) &= \alpha P^*(t) + (1-\alpha)P(0) \\ &= \alpha(1+r_p(t)t)P^*(0) + (1-\alpha)P(0) \\ &= P(0)(\alpha(1+r_p(t)t) + (1-\alpha)) \\ &= P(0)(1+\alpha((1+r_p(t)t)-1)) \\ &\geq P(0)(1+\alpha r_G t)\,. \end{aligned}$$

The minimum upward movement of participating account is $(1+\alpha r_G t)$. The new approximation is

$$P(t) \approx \bar{P}(t) \in \{(1+\alpha r_G t)P_1, (1+\alpha r_G t)P_2, \ldots, (1+\alpha r_G t)P_{n_3}\}\,.$$

The whole network is then modified to cope with the drift of the tracking variables. This method is equivalent to considering an adjusted dynamics of asset which drift is removed. Doing the same on unsmoothed asset share and participating account is valid because the grid values are shifted according to the support of the continuous counterparts. These shifts are summarized in table 4.1.

Variable	Shifting factor
$A(t)$	$\exp\{rt\}$
$P^*(t)$	$(1+r_G t)$
$P(t)$	$(1+\alpha r_G t)$

Table 4.1: Shifting size of modified network

4.3.2 Logarithmic scale network

Under exponential Lévy process, the underlying asset moves in an exponential manner. It is sensible to define the grids of the network in logarithmic scale. Define the logarithmic asset $a(t)$ as

$$\begin{aligned} a(t) &= \ln A(t) \\ &= \ln(A(0)e^{rt+X(t)}) \\ &= a(0) + rt + X(t)\,. \end{aligned}$$

The new approximation using n_1 different values with equal grid size, Δa, is

$$a(t) \approx \bar{a}(t) \in \{a_0, a_0 + \Delta a, \ldots, a_0 + (n_1 - 1)\Delta a\}\,.$$

The corresponding approximation of unsmoothed asset share and participating account are

$$\begin{aligned} p^*(t) &= \ln P^*(t) \approx \bar{p^*}(t) \in \{p_0^*, p_0^* + \Delta p^*, \ldots, p_0^* + (n_2 - 1)\Delta p^*\}\,, \\ p(t) &= \ln P(t) \approx \bar{p}(t) \in \{p_0, p_0 + \Delta p, \ldots, p_0 + (n_3 - 1)\Delta p\}\,. \end{aligned}$$

The setup of this logarithmic scale network is the same as the previous one except the shifting factors are taken natural log which become rt, $r_G t$ and $\alpha r_G t$ respectively.

The risk-neutral price of participating contract with maturity time T at time 0 under this scheme is

$$\begin{aligned} C(0) &= \mathbb{E}^{(n_1,n_2,n_3)}[e^{-rT}C_\gamma(A(T), P(T))|A(0), P^*(0), P(0)] \\ &= e^{-rT}\Pi^T C_\gamma(e^{a_{j1}}, e^{p_{j3}}) \end{aligned}$$

where the transition probability matrix Π is defined under new grids.

4.4 Incorporating surrender rights and mortality

We discussed pricing of simple contracts under different asset model assumptions in the previous sections. There may be some special features specified by the insurance company or the policy holder. We will show how early redemption, which is the most common feature can be incorporated in network pricing. As an insurance product, mortality has to be taken into account for the pricing scheme. In this section, we will also talk about how mortality can be adopted in our network pricing scheme.

4.4.1 Surrender right

In some participating policies, policy holders are allowed to stop the policy and redeem the cash value of the contract. We call this feature as surrender rights. This rights can occur in different time points. If one can exercise this right anytime before maturity, this right is said to be American. If one can exercise the right in some specified time before maturity, we call it Bermudan right. One can notice if exercise time points allowed in Bermudan type policies goes to infinity, the policies become American.

The criteria for one to surrender is rather subjective. We adopt a usual criteria saying surrender occurs if the cash value of the policy is higher than the expected value of holding the contract, meaning that surrender occurs when

$$C(A(\tau), P^*(\tau), P(\tau)) \geq e^{-r(T-\tau)}\mathbb{E}[C(A(T), P^*(T), P(T))|\mathcal{F}_\tau]$$

where $\tau \in \mathcal{T}$ and $\mathcal{T}$ is the set of time when surrender is allowed.

Assume surrender rights are Bermudan and can be exercised in time $\mathcal{T}$, where

$$\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_n\}.$$

In the network construction, we include $\mathcal{T}$ in time domain T, i.e.

$$T = \{t_1, t_2, \ldots, t_m\},$$

such that $t_i = \tau_j$ for some $i \in \{1, 2, \ldots, m\}$ and all $j \in \{1, 2, \ldots, m\}$ and t_m is the policy maturity.

Under the proposed network approach, participating policy with surrender rights is valuated backwardly by

$$\begin{aligned} &C(\bar{A}(t_i), \bar{P^*}(t_i), \bar{P}(t_i)) \\ &= \begin{cases} \max\left[f(\bar{A}(t_i), \bar{P^*}(t_i), \bar{P}(t_i)), \right. & \\ \qquad \left. e^{-r(t_{i+1}-t_i)}\Pi_{t_i,t_{i+1}} C(\bar{A}(t_{i+1}), \bar{P^*}(t_{i+1}), \bar{P}(t_{i+1}))\right] & \text{if } t_i \in \mathcal{T} \\ e^{-r(t_{i+1}-t_i)}\Pi_{t_i,t_{i+1}} C(\bar{A}(t_{i+1}), \bar{P^*}(t_{i+1}), \bar{P}(t_{i+1})) & \text{otherwise} \end{cases}, \end{aligned}$$

where

$$C(\bar{A}(t_m), \bar{P^*}(t_m), \bar{P}(t_m)) = f(\bar{A}(t_m), \bar{P^*}(t_m), \bar{P}(t_m))$$

and $f(\,.\,,\,.\,,\,.\,)$ is the terminal payoff function. This is to check whether surrender is worthy by comparing the expected value of holding and the value of unwind instantly, i.e. receiving terminal payoff, at every time points which allows surrender.

4.4.2 Mortality

Participating life insurance policies are not simply financial products. The policies will terminate on the mortality of policy holders. Here we demonstrate a method to incorporate mortality in the network pricing approach.

Assume the payoff method at mortality is the same as at maturity,

$$C(\bar{A}(\tau), \bar{P^*}(\tau), \bar{P}(\tau)) = f(\bar{A}(\tau), \bar{P^*}(\tau), \bar{P}(\tau)),$$

where τ is the time mortality occurs and $f(\,.\,,\,.\,,\,.\,)$ is the terminal payoff function. Define $p(\tau)$ be the probability of death occurs at time τ. This probability can be found empirically from life table or mortality table. Assume mortality event is independent of underlying asset prices and contract values. Participating policy with mortality can be valuated backwardly by

$$\begin{aligned} & C(\bar{A}(t_i), \bar{P}^*(t_i), \bar{P}(t_i)) \\ = \; & p(t_i) f(\bar{A}(t_i), \bar{P}^*(t_i), \bar{P}(t_i)) \\ & \quad + (1 - p(t_i)) e^{-r(t_{i+1} - t_i)} \Pi_{t_i, t_{i+1}} C(\bar{A}(t_{i+1}), \bar{P}^*(t_{i+1}), \bar{P}(t_{i+1})) \,, \end{aligned}$$

where

$$C(\bar{A}(t_m), \bar{P}^*(t_m), \bar{P}(t_m)) = f(\bar{A}(t_m), \bar{P}^*(t_m), \bar{P}(t_m)) \,.$$

4.5 Proof of convergence

Proposition 4.1. *Suppose a non-negative real-valued continuous payoff function $g(s_T)$ is dominated by a real-valued function $h(s_T)$ such that $\mathbb{E}[h(s_T)] \le c < \infty$. The expectation of $g(s_T)$ under FFT network converges to the true value*

$$\lim_{s^* \to \infty} \lim_{n \to \infty} \mathbb{E}^n[g(s_T) | \mathcal{F}_t] = \mathbb{E}[g(s_T) | \mathcal{F}_t] \,,$$

where the network time grid is $\mathcal{T} = \{t_0 = t, t_1, \ldots, t_m = T\}$, the computational domain $[s_t - s^*, s_t + s^*]$ and number of discretized level n.

Proof. From Proposition 3.6, a one time step progression in the network converges to the true value, i.e.

$$\lim_{s^* \to \infty} \lim_{n \to \infty} \mathbb{E}^n[g(s_{t_{i+1}}) | \mathcal{F}_{t_i}] = \mathbb{E}[g(s_{t_{i+1}}) | \mathcal{F}_{t_i}] \,.$$

By Markovian property of Lévy processes, the convergence can be proved recursively in time domain and hence proves the proposition. □

Theorem 4.2. *Consider a participating policy payoff $g(A_T, P_T)$ dominated by a real-valued function $h(A_T)$ such that $\mathbb{E}[h(A_T)] \leq c < \infty$, under FFT network*

$$\lim_{\bar{a},\bar{p}^*,\bar{p}\to\infty} \lim_{n_1,n_2,n_3\to\infty} \mathbb{E}^{n_1,n_2,n_3}[g(A_T, P_T)|\mathcal{F}_t] = \mathbb{E}[g(A_T, P_T)|\mathcal{F}_t] ,$$

where the network time grid is $\mathcal{T} = \{t_0 = t, t_1, \ldots, t_m = T\}$, the computational domains and numbers of discretized level are $([a_t - \bar{a}, a_t + \bar{a}], [p_t^* - \bar{p}^*, p_t^* + \bar{p}^*], [p_t - \bar{p}, p_t + \bar{p}])$ and (n_1, n_2, n_3) for asset, unsmoothed asset shares and participating account respectively.

Proof. The transition probabilities used in FFT network are generated by FFT. This means the discretized random variables in the network converges in distribution to the continuous one as discretization level goes to infinity. Recall the payoff of participating policy in (2.3),

$$\begin{aligned} C_T &= A_T + (\gamma - 1)\max(A_T - P_T, 0) \\ &\leq A_T . \end{aligned}$$

The one step error under FFT network is

$$\begin{aligned} &\mathbb{E}[g(A_{t_m}, P_{t_m})|\mathcal{F}_{t_{m-1}}] - \lim_{n_1,n_2,n_3\to\infty} \mathbb{E}^{n_1,n_2,n_3}[g(A_{t_m}, P_{t_m})|\mathcal{F}_{t_{m-1}}] \\ \leq\ & \mathbb{E}[A_{t_m}|\mathcal{F}_{t_{m-1}}](\Pr(A_{t_m} \notin [a_t - \bar{a}, a_t + \bar{a}]) \\ & + \Pr(P_{t_m}^* \notin [p_t^* - \bar{p}^*, p_t^* + \bar{p}^*]) + \Pr(P_{t_m}) \notin [p_t - \bar{p}, p_t + \bar{p}]) \\ \to\ & 0 \qquad \text{as } \bar{a}, \bar{p}^*, \bar{p} \to \infty \end{aligned}$$

The same argument holds for every time steps in the network. By backward induction in time domain,

$$\lim_{\bar{a},\bar{p}^*,\bar{p}\to\infty} \lim_{n_1,n_2,n_3\to\infty} \mathbb{E}^{n_1,n_2,n_3}[g(A_T, P_T)|\mathcal{F}_t] = \mathbb{E}[g(A_T, P_T)|\mathcal{F}_t] .$$

□

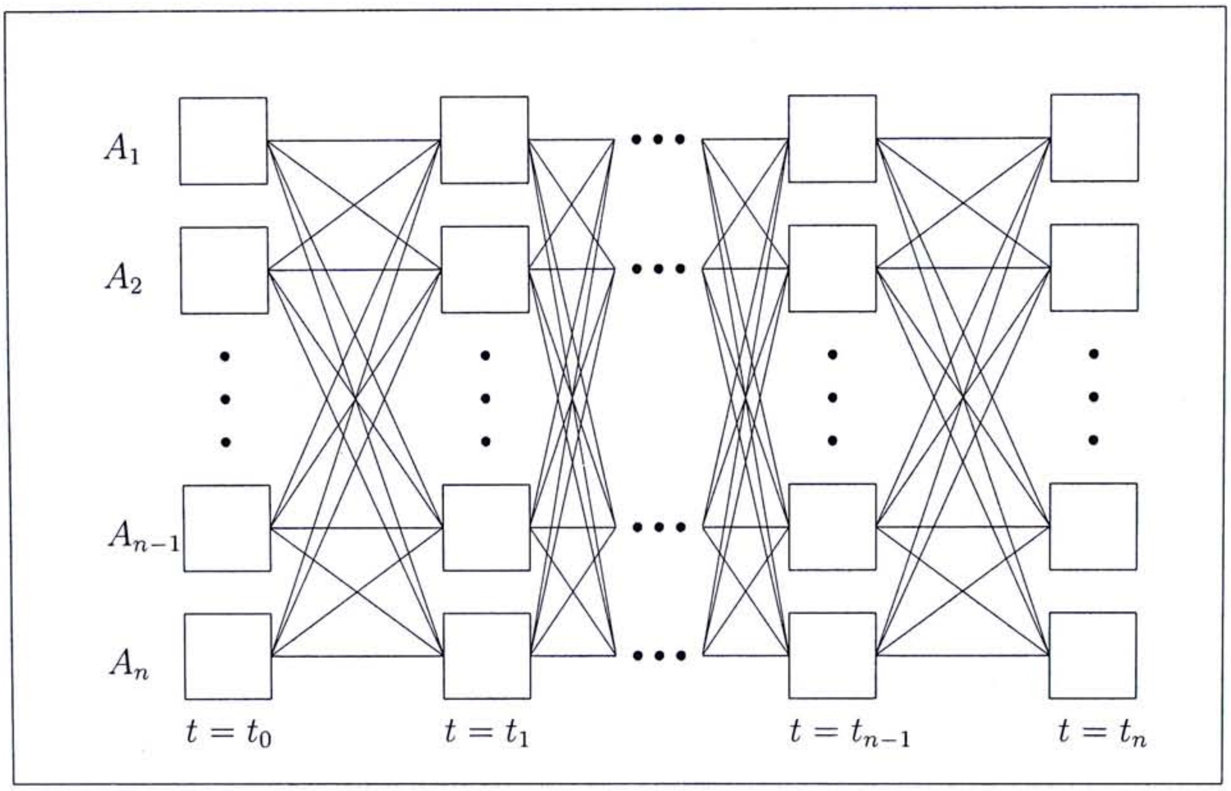

Figure 4.1: Network representation of asset dynamic

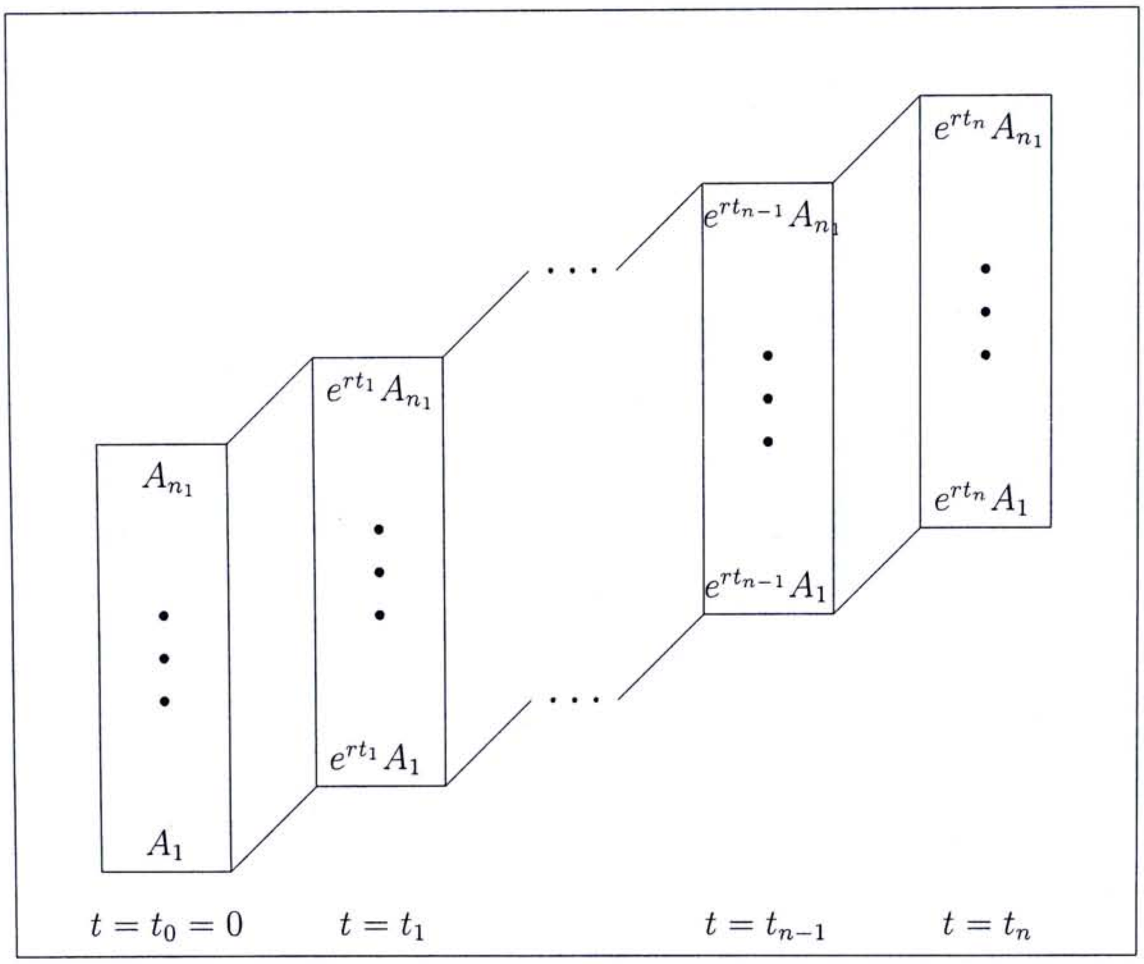

Figure 4.2: Illustration of modified network

Chapter 5

Numerical Results

We tried three different asset dynamics models, representing different types of Lévy processes discussed in Chapter 3, in our numerical analysis. A comparison will be made on pricing of participating policy using our proposed network approximation and Monte Carlo simulation. We assume there are no taxes, no transaction cost, no restrictions on borrowing or short selling and all securities are infinitely divisible. The participating policies priced in the following section have the same contract specifications as mentioned in Chapter 2 and the parameters are

$$A(0) = 1; \quad P^*(0) = 1; \quad P(0) = 1; \quad r = 5\%;$$
$$r_G = 4\%; \quad \alpha = 0.6; \quad \beta = 0.5; \quad \gamma = 0.7; \quad T = 10\text{years}\,.$$

Average of one million paths in simulation were used as a reference. Logarithmic scale network is used in this numerical experiment. We used same number of discretization levels in partitioning $A(t)$, $P^*(t)$ and $P(t)$. The number of discretization levels N is chosen to be a power of 2 integer, i.e. $N = 2^n$, in order to use FFT. The pricing programs are written in C++ using C++ Standard Template Library(STL) and uses software library "Fastest Fourier Transform in the West(FFTW)" for fast Fourier transformation. All computation is made on a 3.0GHz machine.

5.1 The Black and Scholes model

In Black and Scholes model(BS), we used 30% as the asset volatility σ. Table 5.1 summarized the results in BS model. The converging speed is not very fast but the computational time is about one-forth of using Monte Carlo simulation.

No. of nodes	Price	Time(sec.)	Diff.
2^5	1.2509	0.0508	32.73%
2^6	1.0255	0.9628	8.82%
2^7	0.9648	11.9970	2.38%
2^8	0.9438	165.639	0.15%
MC	0.9424	617.2960	

Table 5.1: Results: The Black and Scholes model

5.2 The Merton's Jump diffusion model

Parameters used in Merton's jump diffusion model are

$$\sigma = 30\%, \quad r_j = 0, \quad \sigma_j = 50\%, \quad \lambda = 1.75\,,$$

where r_j and σ_j are the mean and standard derivation of the jump process and λ is the poisson frequency of jumps. From table 5.2, we can see the converging speed is similar to the Black and Scholes model.

No. of nodes	Price	Time(sec.)	Diff.
2^5	1.2318	0.0872	33.31%
2^6	1.0052	0.7744	8.79%
2^7	0.9457	10.3345	2.35%
2^8	0.9258	149.5670	0.19%
MC	0.9240	661.7290	

Table 5.2: Results: The Merton's Jump diffusion model

5.3 Variance gamma model

Gamma distributed random variables are required in Monte Carlo simulation of variance gamma process. Therefore, the simulation time is much longer than that in the previous two models. However, the network approach used less time than the previous two models. It is mainly due to the leptokurtic behavior of variance gamma model. The non-zero elements in transition probability matrix is fewer than the other two models. Here are the parameters we used,

$$\sigma = 0.1213, \quad \nu = 0.1686, \quad , \theta = -0.1436 .$$

The results are shown in table 5.3.

No. of nodes	Price	Time(sec.)	Diff.
2^5	1.1149	0.0136	15.78%
2^6	0.9075	0.0759	5.76%
2^7	0.9920	0.4622	3.01%
2^8	0.9644	24.4756	0.14%
MC	0.9630	1271.32	

Table 5.3: Results: Variance gamma model

Chapter 6

Conclusion

This thesis has proposed a numerical method for pricing participating life insurance policies. The network approach makes use of FFT to calculate transition probabilities in Markov approximation if characteristic function of the underlying asset dynamics is given. We proved the convergence of the network price. This network has been applied to the contracts with complex guarantees and option-like features embedded. Numerical examples showed the application on different types of Lévy processes including pure diffusion, jump-diffusion with finite activities and jump-diffusion with infinite activities.

This network approach inherits the advantages from traditional lattice approach and solves the problems in tree approach. Since characteristic functions always exist in real-valued stochastic processes, the use of this network can be very extensive. We also provide some extensions on incorporating surrender rights and mortality.

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