A Study of Matching Mechanisms

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Abstract

One of the main features of many markets and social processes is their bilateral structure and the need to match agents from one side of the market with the other side of the market, e.g. students and colleges, employees and firms, marriageable men and women, residents and hospitals. Our work mainly focuses on two types of applications of matching market: one is the school matching problem for matching students to schools (could be either high schools or colleges) while the other issue arises in the area of Internet advertising.

The first study is motivated by our investigations of college enrollment in mainland China and Hong Kong. Two college admissions (CA) mechanisms, namely the Boston mechanism and the Gale-Shapley (GS) mechanism, were intensively studied and compared in recent literature on school matching. A widely accepted conclusion in previous works is that the Boston mechanism should be replaced by the more efficient GS mechanism without hesitation. However, we found that JUPAS, the practical college admissions system used in Hong Kong, did not adopt the "pure" GS mechanism as suggested. Inspired by this, we propose a generalized CA model which aims to strike a balance between students' eligibility and interests by adjusting an additional parameter we named as *reciprocating factor*. Two extreme CA mechanisms, namely the Boston mechanism and the GS mechanism, can be easily incorporated into our generalized model. Furthermore, we find that the classic marriage problem can also be extended in the same way by assuming *interdependent* participating agents' preferences.

The second application focuses on the recent booming Internet advertising market. In sponsored search, a number of advertising slots is available on a search results page, and have to be allocated among a set of advertisers competing to display an advertisement on the page. This gives rise to a bipartite matching market that is typically cleared by the way of an auction. The recent trend in sponsored search research is to design more expressive and efficient mechanism by considering the heterogeneity of advertisers' valuations. However, a potential issue is largely ignored in existing literature when considering the fierce competition between *multiple* search engines in the market. This motivates our work to model the comprehensive interaction of *heterogenous* search engines, advertisers and end users in a *competitive* environment. By applying game theoretical approach, we prove the existence of Nash equilibrium prices pair in duopoly and compare them with the optimal price when one search engine monopolizes the market. We further carry out extensive simulation to illustrate the comparative results of expected revenues and social welfare under competition and monopoly. Both our analytical and simulation results could provide some insight in regulating the search engine market and protecting the interests of advertisers and end users.

摘要

雙向結構是許多市場和社會進程所具有的一個顯著特性，在此架構下我們通常需要基于某種原則將處于市場中某一方的參與者與另一方的參與者進行一對一，多對一或是多對多的匹配，例如考生同學校，雇員同公司，單身男生和女生，實習醫生同醫院之間的匹配。本文主要分析了雙向機制的兩類具體應用：一爲學校招生機制的研究，另一爲互聯網搜索廣告的匹配分析。

第一部分的研究主要基于我們對當前香港及大陸地區高考招生機制的調研。相關的文獻在分析學校招生機制時，主要總結和比較了兩類廣泛使用的機制，即波士頓機制（BM）和 Gale-Shapley（GS）機制，并且一般都認爲 BM 應該被效率更高的 GS 機制所取代。然而，我們發現當前香港地區所采用的招生機制即 JUPAS 并未完全采取“單純的”GS 機制，而是一類介乎 BM 與 GS 之間的“混合”機制。有鑒于此，我們提出了一個通用的學校招生模型，在此模型中通過調節一個額外參數即*交互因子*（reciprocating factor），可以使得學校在評定學生時綜合考慮學生的能力與興趣。而 BM 與 GS 機制亦可看作此通用模型的兩種極端情況。我們更進一步發現，通過假定參與各方喜好具有相關性，此模型還可擴展至經典的婚姻匹配問題。

第二部分的應用集中在日益繁榮的互聯網廣告市場。對于典型的贊助搜索，一般會有多個廣告商同時競爭搜索引擎返回頁面上的若干個廣告位。這一過程可看作典型的雙向市場并一般通過拍賣機制來完成廣告商與廣告位之間的匹配。最近這一領域的研究趨勢是通過對廣告商*异質性*的研究來設計分析表達性更强、效率更高的匹配機制。然而，大多數相關文獻都忽略了一個潜在問題，即事實上類似的拍賣機制同時在*多個*搜索引擎進行，廣告商可以自由選擇更符合自己利益的搜索引擎進行廣告投入。基于這一考慮，我們對于异質的搜索引擎、廣告商以及終端用戶在*競爭*環境下的交互進行了建模分析。通過博弈論方法，我們證明了在競爭環境下納什均衡價格的存在性，并將其同壟斷市場下的價格進行了分析和比較。更進一步，通過計算機仿真計算，我們分別闡釋了在競爭和壟斷情形下期望的搜索引擎收益與社會效益的對比結果。我們的分析及仿真結果可爲今後規範搜索引擎市場以保障廣告商及用戶權益提供可借鑒的結論。

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Chapter 1

Introduction of Matching Mechanisms

One of the main features of many markets and social processes is their *bilateral structure* and the need to match agents from one side of the market with the other side of the market, e.g. students and colleges, employees and firms, marriageable men and women, residents and hospitals. *Matching theory* for these two-sided market has therefore received a lot of attentions in recent years due to its wide scope of applications. Our work mainly focuses on two of these applications: one is the school matching problem for matching students to schools (could be either high schools or colleges) while the other issue arises in the area of Internet advertising where the advertising slots in search engine results page need to be matched with the "right" advertisers.

1.1 Background for College Admissions Problem

A class of two-sided matching models for studying bilateral market was first introduced by Gale and Shapley in 1962 in their seminal paper [1], in the context of the college admissions and the marriage problem.

In the college admission problem, there are a set of colleges $C = \{c_1, \ldots, c_m\}$ and a set of students $S = \{s_1, \ldots, s_n\}$. Each college c_i ($i \in \{1, \ldots, m\}$) has a limited quota q_i, which denotes the maximum number of students it can admit each year. Each student s_j ($j \in \{1, \ldots, n\}$) can apply to any of these colleges freely but can accept at most one offer as his/her final choice. We assume that each student has a *strict* preference ordering over these m colleges, and each college also has a strict ranking list over all students according to their test scores, interview performances or other particular criteria. By *strict* preference we mean that a student is NOT indifferent between any two colleges, and vice versa.[1] A *matching* is a function $\mu : S \to C \cup \{c_0\}$ such that no college is assigned to more students than its quota and each student is admitted by at most one college. We create a dummy college c_0 to capture all unmatched students. The college admission problem is then converted to designing a *direct mechanism* to implement a matching outcome for arbitrary input of preference lists from both sides of the market. Here the *direct mechanism* in college admissions means it requires students to reveal their preferences over schools *all at once* and selects a matching based on these submitted preferences and student priorities.

The marriage problem can be interpreted as a special case of the college admission problem when all the colleges have a unity quota. There are also two sets of agents: men side and women side in certain community and each person has a strict preference list over the members of the opposite sex. The marriage problem is trying to find a "proper" matching between both sides of agents based on their preference lists. One important criterion for "proper" matching proposed by Gale and Shapley in [1] is the *stability* condition: no pair of agents who are not matched to one another would both prefer to be. An unstable outcome would cause some man-woman pair to make a private

[1] In practice when there are ties in the preference lists, we can simply break them by some randomly generated lotteries. To make our exposition concise, we just make the strictness assumption here.

date and leave their current mates.

In terms of college admissions, three basic mechanisms are discussed extensively in relevant literature, which are the Boston mechanism, the Gale-Shapley (GS) mechanism and the top trading cycles (TTC) mechanism respectively. The Boston mechanism is common in practice but suffers a lot of criticism since it is not strategy-proof for students: students have no incentive to reveal their true preference. TTC mechanism is exempt from such problems but is not stable in general. The transition from Boston mechanism to GS mechanism is therefore suggested in literature such as [4, 6, 19], which arguably would lead to "unambiguous" efficiency gains. However, our investigation of some practical college admissions systems shows that certain hybrid mechanism may be more acceptable in society from the perspective of students' personal interest.

In our study, we propose a generalized model for college admissions, which considers the tradeoff between students' eligibility and interest by adjusting an additional parameter called *reciprocating factor*. The larger the reciprocating factor is, the more would the interest factor counts when inspecting the applicants. GS mechanism and Boston mechanism are just two particular cases of the generalized model when setting different reciprocating factor. Like the Boston mechanism, this proposed model faces the potential problem of preference manipulation from strategic agents. However, using a game theoretical approach, we conclude that truth-telling by each participant would still be an approximate equilibrium in practical large matching market.

1.2 Background for Internet Advertising Market

Internet advertising has become one of the main sources of revenues for primary search engines nowadays. According to the newly-released report by

Interactive Advertising Bureau and PricewaterhouseCoopers [26], Internet advertising in the United States reached $22.7 billion in total revenue for the full year of 2009, where sponsored search revenue accounted for 47 percent of the total revenue.

A typical Internet search market consists of three parties: *publishers* (i.e., search engines), *advertisers* and *end users*. In the current trend of information explosion, more and more people rely on search engines to pin down their favored products or services. Whenever a query is submitted to the engines by end users, their *intents* or *interests* can be potentially captured by the engines through the inputted keywords. These intents of search users can then be sold by search engines to companies who are interested in attracting these specific users. Nowadays in major search engine operators like Google, Yahoo! and Microsoft, the advertisements for drawing users' attentions are displayed in the form of sponsored links, which appears alongside the algorithmic links (also known as organic links) in the search results pages. For each particular keyword, there are usually more than one available advertising slot in the search engine results page. How to effectively allocate these slots and charge the advertisers have been studied and discussed extensively in recent years among both academic and industrial community. Take Google's AdWords program for example. In this advertising program, advertisers could choose multiple keywords they are interested in, and for each keyword indicate the maximal willingness to pay for each click and the budget to spend over a period of time. Whenever users click on the sponsored link and are re-directed to the advertisers' site, certain payments are charged by the program until all the budgets are used up.

Most of the existing works focused on the interaction of the three parties within the scope of only *one* search engine's advertising system, and these results and suggestions from researchers did greatly improve the efficiency of mechanism held in major search engine companies. For example, the transition from generalized first price auction to generalized second price auction, from payment per impression to payment per click, from bid-based ranking to quality-based ranking and so on [27, 31]. However, considering there is usually more than one company providing search service in the market, one natural question would be how would the market evolve when there exists competition between *multiple* search engines. In particular, will all users and advertisers gradually concentrate to one leading engine or still the "inferior" companies could earn enough profits to survive when competing with the leading one? What would be the consequences if one search engine monopolizes the market? These concerns arise from the current situation of high levels of concentration in search engine market: Google has long known to possess the leading technology and obtain the largest market shares in most countries and regions, followed by Yahoo! and Microsoft Bing.

Our work aims to provide a reasonable formulation to model the competition between two search engine operators and help to address some of the intriguing problems mentioned above. We will consider a three-stage dynamic game model. In stage I, the two operators provide various services to attract end users. In stage II, the two operators simultaneously determine their prices to advertisers. In stage III, the advertisers choose the operator in which they can obtain highest utility based on the announced prices in stage II. Each operator wants to maximize its revenue subject to the competition for advertisers from the other operator.

Chapter 2

Application I: College Admissions Problem Revisited

2.1 Three Basic Mechanisms

By now we haven't mentioned any details of how the mechanism actually run. As figure 2.1 shows, the direct mechanism is like a black box with input of preference lists R from both sides and output of the matching outcome. Generally, the subjects are not constrained to men and women and can be replaced by any agents of bilateral market to adapt for different applications.[1] In this section we mainly introduce three well-known mechanisms referred most in the current research area of matching theory: respectively, the Boston mechanism, Gale-Shapley student optimal mechanism and the top trading cycles mechanism. Although all these three mechanism are introduced in the context of college admissions, we can easily adapt them for other applications like marriage problem, by setting all quotas to one.

[1] In the context of college admissions, preference list from school side is also called *priority ordering* in relevant literature.

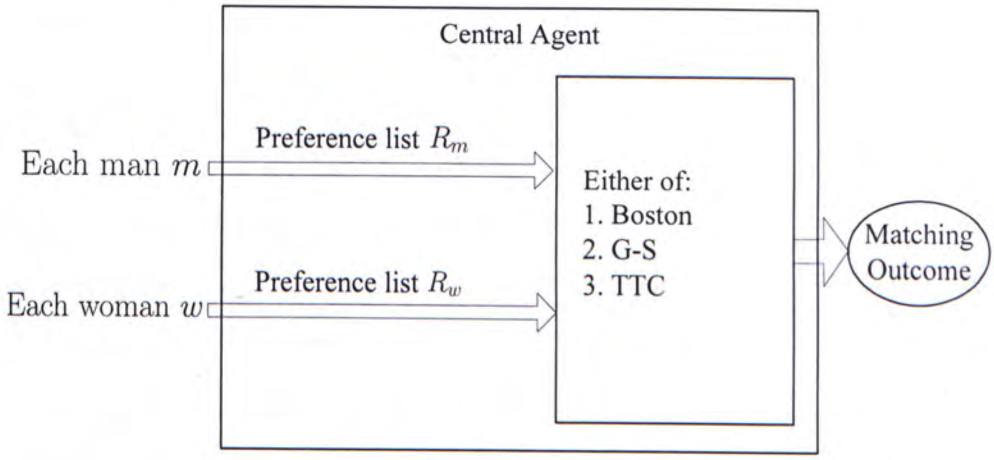


Figure 2.1: Classic Matching Mechanism

2.1.1 Boston Mechanism

One of the most widely used matching mechanism is the direct mechanism adopted by the city of Boston since July 1999 [2]. The Boston student assignment mechanism works as follows:

1. Each student submits a preference ranking of the schools.

2. Each school establishes a priority ordering of students based on certain criteria. For instance, at Boston the priority ordering is divided into four classes:

    - First priority: sibling and walk zone.
    - Second priority: sibling.
    - Third priority: walk zone.
    - Fourth priority: others.

    Students in the same priority class are further ordered by a randomly generated lottery.

3. After collecting and inputting all the preference and priority lists from both sides of agents, the mechanism computes the matching outcome as

follows:

Round 1: In Round 1 only the first choices of the students are considered. Each school considers the students who have listed it as their first choice and distribute seats/offers of the school to these students one by one following their priority ordering until either the quota of the school is filled up or all these students are already accommodated.

Round 2: Consider the remaining students who have no offers yet. In Round 2 only the second choices of these students are considered. For each school with still unfilled quota, consider the students who have listed it as their second choice and assign the remaining seats to these students one by one following their priority ordering until the quota of the school is filled up or all these students are already accommodated.

The process goes on round by round until all quota are filled up or all submitted choices are considered.

Actually the term "Boston mechanism" was originally derived from one special class of school matching problem summarized as *school choice* (SC henceforth) by Abdulkadiroğlu and Sönmez [4]. In SC setting, the schools are regarded as "objects" to be consumed by the students, and priorities at schools are determined exogenously by local regulations which do not represent school's own preferences. However, in the well-known *college admissions* (CA henceforth) problem introduced by Gale and Shapley (1962), both students and colleges are strategic agents and colleges have preferences for students too, which constitutes a complicated *two-sided* matching market. Actually, Balinski and Sönmez have defined a third class of matching problems called *student placement* (SP henceforth) in [5]: colleges are still public goods to be consumed, however, the priority ordering of students are now determined by their scores in a series of standardized tests offered by a central authority, rather than exogenous factors like proximity and siblings. However, for simplicity, in this paper we do not distinguish between CA and SP problems unless indicated explicitly. Meanwhile we use "Boston mechanism" hereafter to denote a class of matching method (in short, first choice first served) for all different applications, rather than merely in original SC setting.

One prominent property of Boston mechanism is the strategy-proofness for schools: it is a dominant strategy for any school to rank students based on its true preferences [19]. However, on the other hand, Boston mechanism would give students strong incentives to misrepresent their preferences by choosing schools which they have more chances to get in. This is because in Boston mechanism school would first consider the students who have listed it as their first choice and hold discrimination against other students no matter how excellent their academic performances are. In a typical bilateral market like SP, supply of colleges quota is much less than the huge number of high-school applicants and most colleges would be filled up by first-choice students and there are little chance for a student to get into his/her second choice school if he/she fails in the first choice unfortunately. Their first choices actually becomes their only choices in most cases! Hence, students are forced to play a very difficult preference revelation game by carefully speculating other applicants' choices and avoiding the popular and famous schools which everyone likes.

2.1.2 Gale-Shapley Student Optimal Mechanism

Gale-Shapley student optimal mechanism works as follows: after collecting preference lists from both sides, we apply the famous *student-proposing deferred acceptance (DA) algorithm* by Gale and Shapley (1962)[2] :

Step 1: Each student proposes to his/her first choice. Each school rejects the

[2]Similarly, we can also construct the Gale-Shapley school optimal mechanism by interchanging the roles of students and schools in the algorithm, or formally, by applying *school-proposing DA algorithm*. Also, in the context of marriage problem, we can apply either *man-proposing* or *woman-proposing DA algorithm*.

lowest priority students in excess of its quota and holds the remaining students tentatively.

In general, at

Step k: Each student rejected at step $k-1$ proposes to his/her next choice. Each school reconsiders the students it holds together with new proposers: it rejects the lowest priority students in excess of its quota and holds the remaining students tentatively.

The algorithm terminates when no new proposals are made and the tentative allocation becomes the final matching outcome.

Unlike the Boston mechanism, which places a heavy weight on the choice order, the GS mechanism is independent of it. The key difference is that the school does not care whether the current proposing is the student's first choice or second choice. Even the quota of certain school is already filled up by first choice students, when an excellent student with second choice arrives, the school will still enroll him/her while rejecting someone else.

A key objective in CA literature is *stability*: there should be no unmatched student-college pair (s, c) such that student s prefers college c to his/her current assignment and s has higher priority than some other student admitted by c. The *student-proposing deferred acceptance algorithm* can always produce a stable matching which is Pareto efficient among all the stable outcomes [1]. Furthermore, the GS student optimal mechanism is strategy-proof (for students) [7, 8]. Therefore, students can be relieved to truthfully reveal their preferences.

2.1.3 Top Trading Cycles Mechanism

Similarly, after collecting the preference lists of all agents, the top trading cycles (TTC henceforth) algorithm works as follows:

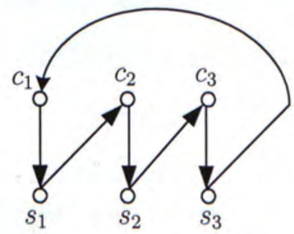


Figure 2.2: An example of cycle

Step 1: Each student *points to* his/her favorite school and each school *points to* the student with highest priority. There is at least one cycle in the form of (s_1 - c_1 - s_2 - c_2 - $\cdots$ - s_k - c_k) ($k \geq 1$) where student s_1 points to college c_1, c_1 points to s_2, ..., s_k points to c_k, c_k points to s_1. The simplest example is cycle $(s_1 - c_1)$ which denotes that student s_1 prefers college c_1 most and vice versa. Another example is shown in figure 2.2. Whenever a cycle is formed, every student in this cycle is assigned the school he/she points to and is removed. The quota of each school in the cycle is reduced by one and if it reduced to zero, the school is removed too.

In general, at

Step k: Each remaining student points to his/her favorite school among the *remaining* schools and each remaining school points to its most preferred one among the *rest* students. There is at least one cycle. Treat this cycle in the same way as in Step 1.

The TTC algorithm terminates when either all students or all schools have been removed (which infers no more cycles could be formed).

As we mentioned above, GS student optimal mechanism Pareto dominates any other *stable* mechanism. This implies that, if ignoring the requirement of stability, the efficiency may get further improvement. The TTC mechanism is one that hits the Pareto-optimal frontier but gives up the stability property: there always exists a tradeoff between efficiency and stability. Furthermore, the TTC mechanism is also strategy-proof [4]: no student can profit by misrepresenting his/her preference list unilaterally.

Example 2.1. There are two colleges c_1, c_2 and three students s_1, s_2, s_3. The preferences of students and priorities of schools are:

$$
\begin{array}{ll}
s_1 : c_2 > c_1 & c_1 : s_1 > s_3 > s_2 \\
s_2 : c_1 > c_2 & c_2 : s_2 > s_1 > s_3 \\
s_3 : c_1 > c_2 & -
\end{array}
$$

The only stable matching is:

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_2 & - \end{pmatrix}$$

There is a cycle $(c_1 - s_1 - c_2 - s_2)$, so this outcome is Pareto dominated by the TTC result as follows (which is unstable):

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ c_2 & c_1 & - \end{pmatrix}$$

2.2 College Admissions Mechanisms Around the World

In previous section we have introduced three fundamental matching algorithms/mechanisms which can be applied to all different matching markets. In this section we focus on the college admissions problem and analyze some practical mechanisms used in different countries or regions around the world: in particular, (1) serial dictatorship mechanism in Turkey, (2) JUPAS in Hong Kong, and (3) *gaokao* in China mainland. There are also several other mechanisms mentioned in relevant literature, like secondary schools enrollment in Singapore [11], higher education in Hungary [9] and Spain [10] for further reference.

2.2.1 Serial Dictatorship in Turkey

Balinski and Sönmez first presented and analyzed the college admissions mechanism in Turkey: *multi-category serial dictatorship* (MSD henceforth), which was proved to be inefficient, vulnerable to manipulation and penalizing students for improved test scores [5]. Gale-Shapley student optimal mechanism was then recommended in that paper.

When there exists one single criterion to judge the students, MSD reduces to the so-called *simple serial dictatorship*: it first sorts all the students by their scores and then assigns the first student (with highest score) his/her top choice school, the second student his/her top choice among the remaining quota, and so on, until all school quotas are filled up or all student gets processed. When there is only one category of criterion, it actually assumes all the schools reach a consensus on the priority ordering of students (according to their test scores). So no cycles like $(s_1, c_1, \ldots, s_k, c_k)(k \geq 2)$ would appear and the problem becomes much simpler.

For the general multiple categories of criteria scenario, different students may have different scores/ranking in different categories, like in Arts and Sciences respectively. Multi-category serial dictatorship considers the students in different categories separately, and in each category, simple serial dictatorship is used as before. When a student receives more than one (temporary) offer from different categories, say s_1 receives offers from c_1, c_2 and c_3 in each category, and s_1's own preference is: $c_1 > c_2 > c_3 > c_0$ (c_0 denotes the no college choice), then the preference list of s_1 would be truncated to: $c_1 > c_0(> c_2 > c_3)$. The algorithms then start over and allocate temporary offers again. If no students are assigned more than one offer, the temporary offers would become final results.

Example 2.2. Students s_1, s_2 and colleges c_1, c_2 with $q = (1,1)$, type $t(c_1) = t_1, t(c_2) = t_2$.

$$s_1 : c_1 > c_2 > c_0 \qquad \text{score } f^{s_1} = (6,8)$$

$$s_2 : c_2 > c_1 > c_0 \qquad \text{score } f^{s_2} = (8,6)$$

Student s_1 achieves a higher score in type 2 test so he/she holds higher priority in type 2 college c_2, while student s_2 is more preferred by type 1 college c_1. The temporary offer of s_1 is from c_2 and temporary offer of s_2 from c_1. Since there is no conflict (no one gets more than one offer), the offer allocation is finalized in one step. Apparently, Gale-Shapley student-proposing mechanism would be much more efficient for this example.

2.2.2 JUPAS in Hong Kong SAR

The Joint University Programmes Admissions System (JUPAS) is a central system for applying to the nine participating tertiary institutions in Hong Kong. First established in 1990, the system has evolved to be the main route of application to higher education: in 2009 admissions, 35,140 students applied for about 300 programmes from the nine member institutions [12].

In JUPAS each student can apply for at most 25 programmes in order of preference. These preferences are sub-divided and made known to the institutions in the form of five bands as follows:

The band number is made known to the institutions, however, the inner-band preferences are unrevealed to institutions. For example, in band A, the

BAND	PROGRAMME CHOICE NO.
A	1 - 3
B	4 - 6
C	7 - 10
D	11 - 14
E	15 - 25

institution has no idea whether a student lists it as his/her first, second or third choice.

After aggregating the preference lists from all the applicants, each programme will make a "*Merit order list*" for its applicants in accordance with its criterion for selection. The rating criterion is determined independently by each programme: although many programmes would adopt a Boston-like criterion which assigns band A students with highest priority, some programmes may also rate students only by their eligibility.[3] Some unpopular programmes tend to employ the latter strategy if they find most excellent students have listed it as band B or band C choices rather than band A.

Finally, after all the *merit order lists* and the applicants preference lists are sent to the JUPAS office, a central computer system will automatically match the order of preference with the position of students in each merit order list of these programmes. The matching process applies the classic *Gale-Shapley student-proposing deferred acceptance (DA) algorithm* to give the students the best offer he/she can possibly obtain.

Although *deferred acceptance algorithm* is used in the last step, the college admissions mechanism in JUPAS as a whole is not equivalent to the *Gale-Shapley student optimal mechanism* introduce in section 2.1, where students truthfully reveal their preference lists. Applicants in JUPAS face a similar problem like students in the *Boston mechanism*: their band A choices would receive higher priority than choices in other bands, although there are no

[3]Student eligibility is judged based on their academic performances, interview performances and extracurricular activities jointly. Examination score, which reflects their academic performances, is a dominant factor in determining the eligibility. [12]

discrimination over the multiple inner-band choices. Actually in JUPAS the applicants are always advised to choose appropriate programmes according to their interests as well as their qualifications [12]. It's never a dominant strategy for students to always reveal their true preferences. Therefore JUPAS may be interpreted as a hybrid of Boston mechanism regarding inter-band discrimination and GS mechanism regarding preferences within certain band. The statement of "Gale-Shapley student optimal stable mechanism is used in Hong Kong" by Abdulkadiroğlu and Sönmez in [4] is rather mis-informed and misleading: readers may falsely assumed that applicants in Hong Kong could feel free to write down their true preferences, whereas in fact there is room for students to manipulate their preference choices.

2.2.3 College Admissions in Mainland China

The national higher education entrance examination, known as *gaokao* in Mandarin, is the official examination held annually in mainland China as a prerequisite for entrance into tertiary education institutions. First resumed in 1977 after the Cultural Revolution, *gaokao* has become one of the most competitive examinations all over the world: in the first year of 1977 there were 5.7 million students taking *gaokao* while only 270 thousand students got admitted, i.e., 1 in 29 students could finally make it! The situation is getting much better nowadays. In the year of 2008 there were still 10.5 million students competing for about 6 million seats offered by the colleges, which means more than 4 million students would lose their chances for higher education and have to wait for another year. Since the highly competitiveness for college entrance, the examination is essentially the only criterion for tertiary education admissions in mainland [13]. This single criterion approximation can actually ease our analysis to a great extent.

In most places, students list their university preferences by filling in the "*wish form*" prior to the exam and universities put different quota in different provinces, municipalities, and autonomous regions. The admission mechanism is actually very similar to the *Boston mechanism*: each university only considers students who list it as their first "*wish*" in the first round. Only if there are remaining quota after the first round, which is very rare considering the large gap between "supply" and "demand" of education resources in China, universities would consider enrolling students who list it as second or third "wish". This mechanism remains unchanged since it was first launched in 1977 until recently. Considering the complexity of manual matching for millions of students in 1980s and 1990s when computers were scarce in China, this "*first wish first served*" principle did ease the operational burden for the college admission across the whole nation.

With gradual adoption of computers in China, computation complexity is no longer a main problem and the mechanism begins to face a lot of criticism. Considering the supply-demand relationship in domestic tertiary eduction, there is little chance for a student to attend his/her second wish college if he/she fails to enter the first choice school. The first choice is essentially the only choice in most situations. This brings heavy pressure to students and their parents when deciding their first choices. Each year there are numerous number of cases reported that students with very high score fail to receive any offers or regret choosing an unpopular school for security and miss their ideal universities.

While the Boston-like mechanism still prevails in many provinces (municipalities, or autonomous regions), some provinces began to implement a revised version and provide applicants more choices than before. We refer to this improved mechanism as "*Hunan scheme*" in this paper since it was first adopted in Hunan province in 2003.[4] In *Hunan scheme*, each student can fill in up

[4]Some latest development: the same mechanism was carried out in Jiangsu province since 2005 and in Zhejiang province since 2007. In 2008, Shanghai, Anhui and Liaoning province moved to the new scheme too. Until 2009, a total of 16 out of 31 provinces (municipalities,

to three "*parallel wishes*" as wish A, B and C respectively. By "parallel", it means that these three wishes would not be discriminated by the colleges and can be regarded as three first wishes. The matching algorithm then works as follows:

Step 0: Sort all the students by their examination scores.

Step 1: Retrieve the *wishes* of the first student (with highest score), if there is still available quota in his/her wish A college, the student-college pair gets matched and quota of this college would be reduced by one. Otherwise, search for the wish B college of this student and see if any quota still remains, and so on. If all three "parallel wishes" fails to match, the student ends up with no offers. The algorithm then moves to the next student.

Generally, in

Step k: Retrieve the *wishes* of the k-th student and try to match his/her most preferred *wish* as in step 1.

The matching procedure terminates when all the students have been processed one by one. Three "*parallel wishes*" here are similar to three choices in band A in JUPAS. By removing the ceiling on the number of parallel wishes, the mechanism *Hunan scheme* becomes the *simple serial dictatorship mechanism*, which is proved to be Pareto efficient and strategy-proof.

and autonomous regions) in China mainland have adopted the "*parallel wish*" mechanism.

2.3 Generalized Model for College Admissions: JUPAS Revisited

Consider this situation: *Bob* is the director of a tiny programme E in the department of education with quota of exactly one student. Only two students, *Jack* and *Ivy*, have applied for this programme. Jack achieved a total test score of 90 and listed programme E as his 25th choice (the last choice in JUPAS), while Ivy achieved 89 but listed programme E as her first choice (assuming both Jack and Ivy are truth-telling). We further assume that Bob has already known that Jack failed in his other 24 choices. From previous theory on college admissions, Bob should adopt the G-S mechanism without hesitation and enroll the student with the highest score, which is Pareto efficient over all stable matching outcomes. However, Bob has different concerns: he believes that Ivy is really interested in education and teaching since she ranked it as the first choice. In the contrary, ranking programme E as the last choice indicates that Jack has little interest in the programme and may have listed it for insurance. From experience, Bob believes interest plays an important role in academic achievement and feels Ivy is more suitable for the quota. The theory contradicts practice here, which infers the suggestion of transition from Boston mechanism to G-S mechanism in previous literature like [4, 6] is not completely justified and may need further consideration. Moreover, we can find numerous research proofs in the fields of educational sociology and psychology to support Bob's case. For instance, Côté and Levine, both professors of sociology at the University of Western Ontario, indicated in their study [15] that:

> ...the input intelligence quotient was negatively related to output human capital skills...In contrast, a measure of input motivation for personal and intellectual development best predicted output

skills acquisition and academic achievement, independent of intelligence quotient.

Counterintuitively, these findings suggest that motivation is indeed more important than intelligence in the context of higher education. Now Bob may have more confidence on his choice of Ivy, instead of Jack.

We have mentioned in the last section that programmes under JUPAS in Hong Kong have full right to determine how to rate students. Two factors are the most important: student eligibility and band order. In practice, most programmes put heavy weight on academic performance in determining students' eligibility which makes the examination scores a very decisive factor in admission. To simplify the analysis, we assume each applicant would attend a standard examination and gain a total score which ranges from zero to the maximum mark.

Let N denote the set of applicants, M denote the set of programmes. Each student $i \in N$ achieves a total score $f_i \in [0, f_M]$ in the standard examination where $f_M > 0$ is the full mark of the examination. Each programme $j \in M$ has a quota q_j. When student i applies programme j as his/her k-th choice, i will obtain a bonus score which would promote his/her position in programme j's *merit order list*. Generally, the bonus score should be a strictly decreasing function over *preference order*. For simplicity, we apply the linear form in this paper as follows,

$$po_j(k) = x_j - y_j \cdot k, \quad k \in \mathbb{N} \tag{2.1}$$

where x_j, y_j are positive constant determined by the individual programme and the integer k denotes the *preference order* from the student. In practice, the programme director could make a corresponding table mapping each preference order to a certain bonus score for easy reference. All applicants are then sorted by their *merit scores* in each programme, where student i's *merit score* in programme j is computed according to the following equation:

$$\begin{aligned} mrt_j(i) &= (1-\alpha_j)\cdot f_i + \alpha_j \cdot po_j(k) \\ &= (1-\alpha_j)\cdot f_i + \alpha_j \cdot (x_j - y_j k), \quad \alpha_j \in [0,1] \end{aligned} \tag{2.2}$$

The first term denotes the original score achieved by i and the second term is the bonus score from the *preference order*. In case of tie when students share the same merit score, f_i serves as the tie-breaker and student with higher f_i has higher priority. Finally, if all terms equal, we break the tie by a random lottery.

α_j is *reciprocating factor* (RF), a constant determined independently by each programme j, reflecting its sensitivity towards applicant's *preference order*. Programmes with larger RF place more weight on applicants' personal interests: other things being equal, students whose interests match with the programme are more favored. In extreme case when all α of different programmes equals to zero, it reduces to exactly the Gale-Shapley student optimal mechanism: wish order would not affect students' positions in programmes; in the contrary, when all α is set to one, it works in the same way as Boston mechanism: first wish would get served first. For a general $\alpha_j \in (0,1)$, say α_j equals to 0.2, it means that programme j would count 80% of original score and 20% of interest factor when making its merit order list.

RF partly explains why it is NOT costless to switch from Boston mechanism to G-S mechanism in practical systems. JUPAS, as a hybrid mechanism, combines the applicants' qualification with their own interests, and offers more freedom for programme to cater for its own evaluation criterion. To illustrate it, we give a simplified example conducted under JUPAS.

Example 2.3. Assuming maximum mark $f_M = 100$, for programme j, set constants $x_j = 120$ and $y_j = 20$ and there are totally five different bands, therefore the corresponding table can be easily computed by $wo_j(k) = x_j - y_j \cdot k = 120 - 20k$:

Preference Order	Bonus Score
1 (Band A)	100
2 (Band B)	80
3 (Band C)	60
4 (Band D)	40
5 (Band E)	20

Table 2.1: Corresponding Table Mapping Preference Order to Bonus Score

And α_j is set to 0.5. Assuming student i has achieved full score ($f_i = 100$) in the examination, we list i's *merit scores* respectively when he/she places programme in different bands, shown in table 2.2.

Merit Score	Preference Order
100	1st
90	2nd
80	3rd
70	4st
60	5st

Table 2.2: Merit Scores in Different Preference Order

The bonus for ranking the programme in every higher priority level is 10 marks for applicants, or equivalently, the penalty for each lower scale of interests in programme j is 10 marks. Unlike G-S student optimal mechanism, this designed discrimination in band order provides extra chances for students who really appreciate and suit for this programme to get admitted. In the other hand, compared with pure Boston mechanism, this mixed mechanism still shows sufficient respect to the efforts made by excellent and eligible students.

Recall the situation Bob faces in the beginning of this section, now Bob could feel relieved to set a non-zero RF in his programme, say setting RF equals 50%, to achieve a tradeoff between students' qualification and their personal interests in this particular programme. Unfortunately, advices from the prevailing college admissions literature, which almost exclusively advocate setting RF to zero (i.e., G-S mechanism) rather than $\alpha = 1$ (i.e., Boston mechanism) in order to achieve properties like strategy-proofness and stability, show no respects for students' interest and may not receive full support from education community, especially programme directors, like Bob. The reciprocating factor we propose in this paper helps connect these two distinctive matching mechanisms and gives the programmes/colleges more flexibility in choosing a "reasonable" enrollment mechanism:

- For programmes/colleges which hope stick to the traditional Boston-like scheme, there will be no need for any change since by default α is set to one;

- For programmes/colleges whose sole objective is to raise the average score of new-admitted students, setting α to zero would be their dominant strategy;

- For other elastic programmes/colleges concerning the students' interest as well, a suitable α between zero and one needs to be determined according to each programme/college's own admission favor and policy in each academic year.

2.4 Extension to Marriage Problem

The extension of college admissions mechanism may further apply to the classic marriage problem by Gale and Shapley (1962) [1] too. Consider the situation when a girl faces two boys' proposals and has no clear idea which one she strictly prefers. Technically, we call there exists a *tie* in the girl's preference list.[5] Roughly speaking, the existing literature mainly provides two solutions to deal with marriage problem with tie. A quick solution is just requiring the girl to flip a coin to produce a strict preference list so that the previous mechanism could be applied immediately. The other solution concerns how to find the optimal matching outcome among all these artificial tie-breaking possibilities, for instance, the polynomial-time stable improvement cycles algorithm raised in [14].

However, what if the girl further indicates: "I'll choose the one who loves me most!" And the fact is that the first boy has listed her as the first choice while the second boy listed this girl as the last choice and was rejected by every other girls in the previous rounds, assuming men-proposing deferred acceptance algorithm [1] used here. Obviously a "reasonable" matching mechanism should respect each participating agent' wish and therefore always match the first boy with the girl in this particular case. However, the existing matching mechanism provides no channel for agents to express such kind of "correlated" preferences, although mutual appreciation is a very natural and common factor in determining marriage mates.

To tackle it, we can propose a similar extended model to the classic marriage problem in this section. Formally, assuming there are n men and n women in a certain community. Rather than just giving the ordinal preference list, we require that each man m rates each woman w by a score $f_m(w) \in [0, f_{max}]$ and vice versa, where f_{max} is the maximum score each agent could rate. $f_m(w) > f_m(w')$[6] denotes that woman w is more favored than w' in man m's preference list. A simple way to convert the preference list in the classic marriage model [1] to rating scores in our extended model is letting

[5] Readers could refer to this comprehensive survey for recent development on the marriage problem in [16], especially section on "incomplete preference lists with ties".

[6] Since in this model the men and women sides are symmetric, all the equations and definitions followed can naturally apply to women side by exchanging the sign of m and w.

$f_m(w) = n + 1 - k$ if w is m's k-th choice. However, generally, the agent is allowed to express the intensity of preference by assigning an exact rating score. Furthermore, we set an additional term $h(k)$ called "bonus score", which is a strictly decreasing function over the preference order k. The *merit score* of woman w to man m is then computed as follows,

$$mrt_m(w) = (1 - \alpha_m) \cdot f_m(w) + \alpha_m \cdot h(k), \quad \alpha_m \in [0, 1] \tag{2.3}$$

where woman w has listed man m as her k-th choice[7] . For ease of analysis, we assume no man could give *exactly* the same rating score to two or more women (no matter how small the scores difference may be), i.e., no tie is allowed in the original scores so that choice order k can be uniquely determined. Similarly, α_m is called *reciprocating factor* (RF), denoting man m's sensitivity to other women's evaluation to himself. By default α_m is set to zero, which is equivalent to the classic model where man only believes his own feeling and judgement. Conversely, in the case when α_m equals to one, man m is extremely sensitive to women' opinions on him and hopes to match with the one who loves him most. In general a man may set α_m between zero and one to strike a balance between his feeling to women and women's appraisal to him. Choosing a mate with mutual appreciation seems more "reasonable" and natural in practical marriage. The parameter of *reciprocating factor* provides an opportunity for agents to more fully express their wishes than in the previous model. Finally by comparing the *merit scores* of m for different women, we can reproduce the preference list of m. In case of tie in merit scores, we resort to original score for tie breaking.

To sum up, the whole *direct* mechanism works like this (see figure 2.3): each woman and man indicates their rating scores over the opposite sex[8] and their own *reciprocating factors* as well to a central agent. In case that someone

[7]The choice order is obtained indirectly by comparing woman w's rating scores over all men.

[8]The original preference lists can be easily achieved by comparing these rating scores.

has not submitted his/her *reciprocating factor*, it will be set as zero by default, to be consistent with the classic model. The central agent then computes all merit scores according to equation 2.3. By sorting the merit scores of each man and woman, it produces the "*reciprocating preference list*" for every agent. This constitutes the *preprocess phase*. The following *matching phase* is the same with the classic model by inputting these new preference lists into the system and outputting the matching outcome. The matching algorithm is left open since all classic matching algorithms mentioned before can still be applied here for the extended model. By substituting the agents with students and programmes/schools, extended model in figure 2.3 can also be applied for college admissions problems. Actually, JUPAS in last section can be interpreted as a special case where departments have the rights to determine their specific *reciprocating factors* while students possess no such option.

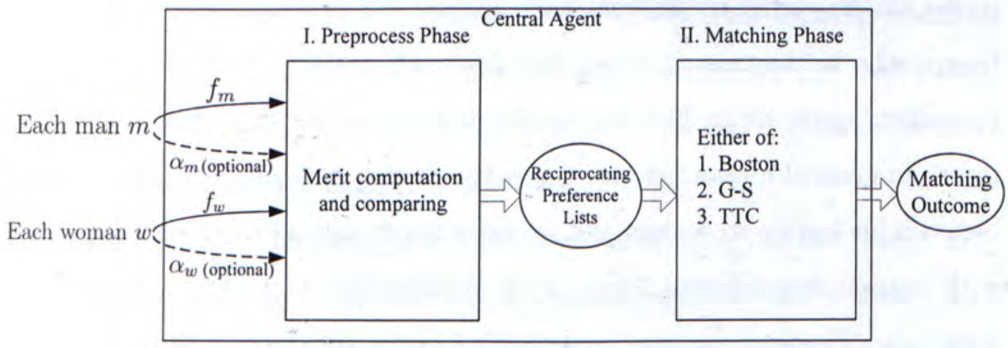


Figure 2.3: Extended Matching Mechanism

The extended model, if regarded as an updated system for the classic one, is totally downward compatible: if agents of both sides are not informed the update of RF options, the preprocess phase would produce exactly the same *reciprocating preference lists* as their original preference lists inputted, therefore would have no effect on the final matching outcome. Conversely, the classic model may face severe problems to accommodate some specific situations. For example, it's not uncommon to hear that women tend to prefer men who love and care them in choosing their mates.

2.5 Strategy Analysis in Extended Marriage Problem

In game theory terminology, the extended mechanism may be interpreted as a Bayesian game model. The agents from both sides of market constitute the players set $\mathcal{I}$. *Reciprocating factor* α_i can be regards as *types* $t_i \in \mathcal{T} = [0, 1]$ for each agent $i \in \mathcal{I}$, which depicts the personality and character of individual agent/player i. Player i only knows his/her own type and estimates other players' types drawn from certain probability distribution. The strategy for each player i is how to determine the *submitted* rating score f'_i and type t'_i, provided only player i knows his/her true preference f_i and type t_i. The objective of each agent is to match as favorable "mate" as possible in his/her *induced preference list.* In terms of game theory, each player i would maximize his/her utility function $u_i(k)$, which is strictly decreasing with the order number k of matched mate in the *reciprocating preference list.* We can now show some fundamental properties of the extended model for the marriage problem.

Proposition 2.4. *The extended mechanism is NOT strategy-proof for both men and women in general. The only exception is when all the women's reported* reciprocating factors *are zero and men-proposing deferred acceptance (DA) algorithm is applied in Phase II, it is the dominant strategy for men to act truthfully. The conclusion would still hold by interchanging the roles of men and women above.*

Remark. This conclusion is easy to see since generally (i.e., $\alpha \in (0, 1)$) the extended mechanism can be interpreted as a hybrid of G-S mechanism ($\alpha = 0$) and Boston mechanism ($\alpha = 1$). And in Boston mechanism, agents tend to shield their first choice from these highly competitive objects and choose these less popular counterparts instead.

For instance, assuming men-proposing DA algorithm is used, and there are three men indexed as $1, 2, 3$ and two women indexed as $4, 5$ respectively. All three men have same scores over two women $f_m = (10, 8)$ and share the same $\alpha_m = 0.1$, $\forall m \in \{1, 2, 3\}$. The first component in vector f_m denotes man m's rating score over woman 4 and second component corresponds to woman 5. Both women have same scores over three men $f_w = (10, 8, 6)$ and share the same $\alpha_w = 0.9$ (i.e., resemble Boston mechanism to a large extent), $\forall w \in \{4, 5\}$. If all agents act truthfully, then the induced preference lists would be: woman 4 is the first choice and 5 the second choice for each man, and both women put man 1 in the first rank, man 2 the second and man 3 the last. Therefore the matching outcome by men-proposing DA algorithm would be $(1, 4), (2, 5), (3, \varnothing)$. However, if man 3 misrepresents his preference by raising the score of less "popular" woman 5, i.e., submitted score of man 3 is $f_3' = (8, 10)$, and further assuming bonus score function satisfies $h(1) = 10$ and $h(2) = 8$ for all agents, by equation (2.3), we have

$$mrt_w(m) = (1 - \alpha_w) \cdot f_w(m) + \alpha_w \cdot h(k) = 0.1 f_w(m) + 0.9 h(k)$$

Therefore, woman 5's merit score over man 2: $mrt_5(2) = 0.1 \times 8 + 0.9 \times 8 = 8.0$; woman 5's merit score over 3: $mrt_5(3) = 0.1 \times 6 + 0.9 \times 10 = 9.6$.

So man 3 gets promoted in woman 5's (induced) preference list and the matching outcome would be $(1, 4), (3, 5), (2, \varnothing)$, which is beneficial for man 3.

Only in the special case when all women's reported reciprocating factors are zero, would men's rating scores over women exert no effects on women's preference lists. And since G-S men-optimal mechanism is strategy-proof for all men in the classic model [7, 8], by telling their true rating scores and reciprocating factors, the true *induced preference lists* would have already generated the best possible outcome for each man. Any misrepresentation of scores or reciprocating factor by man m would only cause the change of his own *induced preference list* while preference list of each woman and every other man remains unchanged, hence would generate no better outcome for himself.

The strategic implication of theorem 1 to participating agents is rather intuitive. For certain man m, comparing to his original preference, he should try to place relatively higher scores on women who report higher RF (to promote his position in their merit lists), and place lower scores on women with small or even zero RF (which would cause no severe degradation of position in their merit lists). However, since the characteristic (i.e., the value of RF) of each woman is usually private information and unrevealed to public even after being gathered and processed in the central agent, in practice it may be very difficult for man m to manipulate his submitted preference and benefit from it for sure.

Recall that a strategic agent may not only manipulate the preference but also misrepresent his/her reciprocating factor. The following theorem can help ease our concern on the latter kind of strategy and refocus on the "traditional" preference manipulation problem[9] , such as strategic behavior analysis under Boston mechanism in [19] and under G-S student-optimal mechanism in [20].

Theorem 2.5. *Assuming the rating scores are revealed and only* reciprocating factors *can be manipulated by agents in the extended model, it's the dominant strategy for each man to reveal his true reciprocating factor when men-proposing deferred acceptance (DA) algorithm is applied in Phase II. The conclusion would still hold by interchanging the roles of men and women above.*

Proof. Assuming rating scores are predetermined, by equation (2.3) the merit scores of each woman over men would remain unchanged, so does the induced preference list of each woman. In the same way, induced preference lists of any other men except m would never be affected by m's strategy. Therefore, by misrepresenting his reciprocating factor, man m can only alter his own (induced) preference list and since in the classic model G-S men-optimal mechanism is strategy-proof for men, it will never generate a better matching

[9]Actually, Sönmez has studied another type of manipulation via underreporting capacities/quotas in [17]. Interesting readers can refer to it for more details.

outcome for himself. Due to symmetry of men and women, the conclusion still holds by exchanging their roles in the mechanism. □

Theorem 2.5 implies that manipulation on reciprocating factor alone would generate no better matching outcome for agents since it can never influence preference lists of other agents. An "effective" strategy for certain man m must be accomplished by the manipulation of his preference scores, in order to alter his position in other women's merit lists.

2.6 Strategy Analysis in JUPAS

We now analyze the strategic behaviors in the context of college admissions. Recall that under JUPAS, each department has full rights to determine its own reciprocating factor after it collects the application information from students side whereas students have no such options. Equivalently, we can interpret in the extended model that all students' reciprocating factors are zero and their induced preference lists would not be affected by any programmes. After the preprocess phase, student-proposing DA algorithm is applied in JUPAS to generate the matching outcome. By analogy to proposition 2.4, we have

Proposition 2.6. *Extended college admissions mechanism like JUPAS is NOT strategy-proof in general. The only exception is when all the programmes'* reciprocating factors *are zero, it is the dominant strategy for students to reveal their true preference lists*[10] *. However, no stable matching mechanism exists which makes it a dominant strategy for all programmes to state their true preferences.*

Remark. The statement above is similar to proposition 2.4 except the last part. This is a direct derivation from theorem 11 in [3] by Roth, which indicated that

[10] *In practice, if considering the constraint on the length of submitted preference lists, the mechanism may still be manipulable. For simplicity, we have ignored the constraint here. See more in Haeringer and Klijn's work on constrained school choice [18].*

colleges would always have incentive to manipulate even under G-S college-proposing algorithm.

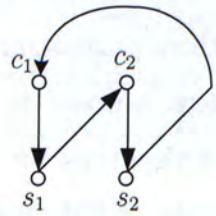
Generally, when most programmes have positive reciprocating factors, students would act strategically. For programmes with high reciprocating factors, applicants would tend to raise their positions in the preference lists, in order to obtain more bonus scores in these programmes; for other programmes which care less about the choice orders, it would do less harm to students' position in the merit order lists even if they rank the programmes in lower ordering. However, in practical systems like JUPAS, it is only after students have submitted their preferences that programmes would determine their reciprocating factors, which makes it very difficult for students to strategize in advance.

On the other hand, from the perspective of programme directors, they face the problem of determining the merit order lists after collecting the application information, mainly the band order, from the students. To be consistent with previous notation, we treat college and programme/deparment indifferently and still let c denote an arbitrary programme and s denote certain student. The applicants to programme c are initially ordered by their eligibility, which is mainly determined by their examination scores plus some subsidiary factors like interview performances and extracurricular activities. We further assume that each programme c has a predetermined parameter of reciprocating factor α_c, reflecting its admission policy on the tradeoff between student eligibility and interest. By equation (2.2), the programme would generate a merit score for each applicant and induce a merit order list R_c by comparing all these merit scores, which reflects the true preference of the programme. After all the programmes report their merit lists to the JUPAS office[11] , G-S student-proposing algorithm is run to obtain the matching result. As theorem 2.6 mentioned, programme c may falsify the merit order list R_c and report a

[11]The programme choices of students, including both inter-band and inner-band wish orders, have already been stored in the JUPAS office since the beginning of the admission process.

different list R'_c instead to get better off. We use a simple example to illustrate this possibility in JUPAS.

Example 2.7. There are two students (s_1, s_2) and two programmes (c_1, c_2) with quota $q_1 = q_2 = 1$. Their preferences constitute exactly a cycle as follows,



which means s_1 lists c_2 as the first choice and c_1 the second choice while s_2 prefers c_1 to c_2; s_1 is ranked the first in c_1's merit order list and s_2 is ranked the first in c_2's merit order list. One reason may be that s_1 performs much better than s_2 in c_1's interview but performs really poorly in c_2's.

If both programmes report their true merit lists to JUPAS office, the G-S student-proposing algorithm would allocate s_1 to c_2 and s_2 to c_1 in just one step. However, if c_1 misrepresents its merit list by indicating s_2 is unacceptable, the matching results would be s_1 with c_1 and s_2 with c_2 instead, which makes c_1 better off by manipulation.

Since telling truth is never a dominant strategy for programmes, a natural question is how likely a programme can successfully manipulate the matching result in practice when the number of participants is large and each participant has only incomplete information about others' preferences. Recent work by Kojima and Pathak [20] has derived some exciting theoretical results in large two-sided matching markets. They proved that with some *regularity* and *thickness* conditions, truthful reporting by every participant is an approximate equilibrium under G-S student-optimal mechanism in large markets. That is to say, even under G-S student-optimal mechanism, colleges still have strong incentive to reveal their true preferences in practical market.

In the generalized model, besides the common strategy of preference manipulation, programmes may also misrepresent their reciprocating factors. Let R denote the true (induced) preference list of certain programme and R' is the submitted one. By misrepresenting the reciprocating factors, programmes are in essence manipulating their induced preference list R. Therefore if in the classic model programmes have strong incentives to reveal their ture preferences, in generalized model they would still tend to act truthfully on reciprocating factors to induce the true R. This corollary could be summarized as follows:

Theorem 2.8. *Suppose the markets satisfy the condition of* regularity *and* sufficient thickness. *Then for any $\varepsilon > 0$, there exists n such that truth-telling by every programme/college is an ε-Nash equilibrium for any market in the sequence with more than n programmes/colleges.*

where ε-*Nash equilibrium* means a strategy profile such that no player could gain more than ε by unilateral deviation. This theorem is an instant deduction of theorem 2 in [20] and interesting readers could refer to it for more details on proof.

2.7 Efficiency Investigation via Simulation

2.7.1 Efficiency Definition

To define and compare social welfare under different matching outcomes, we need to quantify the utility of each participant (agent) in the mechanism first.

Denoted by S the set of students and C the set of colleges. $I = S \cup C$ is the set of all participants and $\mathcal{O}$ is the set of all possible matching outcomes. For any agent $i \in I$, let P_i be the *induced* preference list of i and $o(i)$ be the set of participants matched to i under certain outcome $o \in \mathcal{O}$.

For each agent $j \in o(i)$, denote integer $k(j, P_i)$ as the order agent j appears in i's preference list P_i. For example, $k(j, P_i) = 1$ means that j is the first choice in i's preference list.[12]

We assume that agent i's utility is only determined by the orders of the matching set in the preference list, which can be written as,

$$u_i(o) = \sum_{j \in o(i), j \in I} U_i(k(j, P_i)) \qquad \forall i \in I, o \in \mathcal{O} \tag{2.4}$$

where $u_i(k)$ is non-increasing as integer k increases. Intuitively, it means that higher order (smaller k) would generate higher degree of satisfaction for agent i.

The aggregate utility of students (or colleges) under outcome o is then:

$$\begin{aligned} \pi_S(o) &= \sum_{s \in S} u_s(o), \qquad \forall o \in \mathcal{O}; \\ \pi_C(o) &= \sum_{c \in C} u_c(o), \qquad \forall o \in \mathcal{O}. \end{aligned}$$

The *social welfare* is defined as the aggregate utility of all participants in the mechanism, which can be written as follows:

$$\Pi(o) = \sum_{i \in I} u_i(o) = \pi_S(o) + \pi_C(o), \qquad \forall o \in \mathcal{O}.$$

We say matching outcome o_1 is more *efficient* than o_2 if:

$$\Pi(o_1) > \Pi(o_2) \qquad o_1, o_2 \in \mathcal{O}.$$

We further say mechanism $\mathcal{M}_1$ is more *efficient* than $\mathcal{M}_2$ if $\mathcal{M}_1$ can always induce a more efficient matching outcome than $\mathcal{M}_2$ under any possible preference lists of agents. Generally speaking, an outcome would be more efficient if it induces more high-ranked matching. In the context of college admissions, a mechanism which generates more first-choice matching for students is likely to be more efficient.

Next we will show a simple example to better illustrate this efficiency issue.

[12] For simplicity, we assume there are no ties in the preference list. Otherwise, we can break the tie by a random lottery first.

Example 2.9. There are three students denoted by $S = \{s_1, s_2, s_3\}$ and two colleges denoted by $C = \{c_1, c_2\}$ with only one quota in each college. The utility of each agent is as follows:

$$u_s(1) = 10, u_s(2) = 8, u_s(\varnothing) = 0 \qquad \forall s \in S$$
$$u_c(1) = 10, u_c(2) = 8, u_c(3) = 6, u_s(\varnothing) = 0 \qquad \forall c \in C$$

where $u_s(\varnothing) = 0$ implies agents would receive zero utility if left unmatched.

The preferences of students are as follows:

$$P_{s_1}, P_{s_2} : c_1 > c_2 > \varnothing \qquad P_{s_3} : c_2 > c_1 > \varnothing$$

The examination score of each student is $(f_{s_1}, f_{s_2}, f_{s_3}) = (100, 90, 80)$ respectively. The bonus score for k-th order is $po_c(k) = 120 - 20k$ and reciprocating factor is $\alpha_c = 0.9$ for each $c \in C$.

We have pointed out that in JUPAS-like mechanism, it's very difficult for students to successfully manipulate the outcome and it's an approximate Nash equilibrium for colleges to act truthfully. Therefore here we assume that each agent would reveal its true preference.

Then if we apply the pure Gale-Shapley mechanism, the preferences of colleges would only be determined by the scores of students and not affected by the preference orders of students. Thus we have:

$$P_c : s_1 > s_2 > s_3 > \varnothing, \qquad \forall c \in C$$

The matching results would be $o_1 = ((s_1, c_1), (s_2, c_2), (s_3, \varnothing))$. Both s_1 and c_1 would receive their first-ranked choices while s_2 and c_2 receive their second-ranked ones. Thus it's easy to calculate the social welfare:

$$\begin{aligned}
\pi_S(o_1) &= u_{s_1}(1) + u_{s_2}(2) + u_{s_3}(\varnothing) = 10 + 8 + 0 = 18; \\
\pi_C(o_1) &= u_{c_1}(1) + u_{c_2}(2) = 10 + 8 = 18; \\
\Pi(o_1) &= \pi_S(o_1) + \pi_C(o_1) = 18 + 18 = 36.
\end{aligned}$$

On the other hand, in the JUPAS-like mechanism, the preferences of colleges are jointly determined by the scores and preference orders of students. By equation (2), we get:

$$P'_{c_1} : s_1 > s_2 > s_3 > \varnothing, \qquad P'_{c_2} : s_3 > s_1 > s_2 > \varnothing.$$

The matching results would be $o_2 = ((s_1, c_1), (s_2, \varnothing), (s_3, c_2))$. All agents except s_2 have received their first-ranked choices under o_2. The social welfare is then:

$$\begin{aligned}
\pi_S(o_2) &= u_{s_1}(1) + u_{s_2}(\varnothing) + u_{s_3}(1) = 10 + 0 + 10 = 20; \\
\pi_C(o_2) &= u_{c_1}(1) + u_{c_2}(1) = 10 + 10 = 20; \\
\Pi(o_2) &= \pi_S(o_2) + \pi_C(o_2) = 20 + 20 = 40.
\end{aligned}$$

The hybrid mechanism can generate higher degree of overall satisfactory for both students and colleges in this example.

2.7.2 Simulation Design

In previous section we showed an example where the hybrid mechanism can achieve higher social welfare than the pure GS mechanism. However, it's still unknown whether this conjecture would still be true under more general settings. To tackle this issue, we designed a simulation environment to evaluate the expected efficiency under different mechanisms.

Assuming there are 10 students and 5 colleges with just one quota in each college. We set this ratio of applicants to offers in order to emulate the current situation of college admissions in areas like mainland China where there are approximately two students competing for one offer in average. In addition, our setting can be easily adapted for the practical situation where colleges have more than one quota. From equation (2.4), we can see that a college with quota n can be decomposed into n identical colleges with quota one.

We assume the utility of each agent is as follows,

$$\begin{aligned} u_s(k) &= 11-k, \qquad k \in \{1,2,\ldots,5\} \\ u_c(k) &= 11-k, \qquad k \in \{1,2,\ldots,10\} \\ u_s(\varnothing) &= u_c(\varnothing) = 0. \end{aligned}$$

Thus a first-ranked matching would bring in utility of 10 for either students or colleges. Notice that the efficiency upper bound is 100 since there are at most five pairs of students and colleges matched with each other.

The preference lists of students are generated as follows:

Student s evaluates each college c by this formulae,

$$g_s^c = \beta g^c + (1-\beta) g_s(c), \qquad \beta \in [0,1]$$

where $(g^{c_1}, g^{c_2}, g^{c_3}, g^{c_4}, g^{c_5}) = (100, 90, 80, 70, 60)$ denotes the social reputation of each college and $g_s(c)$ denotes the individual preference of student s, which is independently drawn from uniform distribution over $[0,100]$. The factor β denotes the degree of correlation for students' preferences. The preference list P_s can therefore be deduced by comparing the value of g_s^c for different c, i.e., s prefers c_1 to c_2 if $g_s^{c_1} > g_s^{c_2}$.

The preference lists of colleges are generated as follows:

College c evaluates each student s by this formulae,

$$mrt_c(s) = \alpha_c \cdot po_c(k) + (1-\alpha_c) \cdot f_s$$

where $po_c(k) = 110 - 10k$ if c is the k-th choice in P_s. f_s is the exam score of student s, which is independently drawn from uniform distribution over $[0,100]$. We generate the reciprocating factor α_c for each college c by the following distribution:

$$\alpha_c = \begin{cases} 0 & p = 1/2 \\ 1 & p = 1/2 \end{cases}$$

In JUPAS, since students remain unknown to the reciprocating factors of colleges before submitting their preferences, we argue that by applying such distributions on α, it would be very difficult for students to form any effective strategy. Hence in the following simulation we assume students would act truthfully and analyze the efficiency based on this assumption.

The preference list P_c can then be inferred by comparing the value of $mrt_c(s)$ for different s. In case of ties, namely, $mrt_c(s_1) = mrt_c(s_2)$, s_1 is favored over s_2 if $f_{s_1} > f_{s_2}$.

2.7.3 Simulation Results

After generating the reciprocating preference lists for both sides, the matching outcome can be obtained by applying the Gale-Shapley student-proposing algorithm. We then calculate the social welfare under the matching outcome.

In the simulation, we use $\beta \in [0, 1]$, with a step size of 0.01. For each particular β, we repeat the process of preference generation for 1000 times and compute the average values of aggregate utility and social welfare. For comparison, we also calculate the average social welfare under pure GS mechanism, which can be easily implemented by just setting $\alpha_c \equiv 0$ for each college c in the distribution of reciprocating factors.

Figure 2.4 presents the simulation results for aggregate utility of students over different degrees of preference correlation. As we can see, the expected aggregate utilities under both mechanisms decrease as β increases from zero to one. The upper bound of π_S is 50 since there are at most five students who can receive their first-choice offers from colleges. When β is small (less than around 0.4), we can achieve about 94% and 92% of the upper bound under the JUPAS-like hybrid mechanism and the pure GS mechanism respectively. As β rises, the preference list of each student becomes more and more similar and there are more collision between students' interest in colleges. When β is large enough (greater than 0.91), the pre-determined social reputation of each college becomes the dominant factor in forming the preference lists of students. That's to say, P_s would be $c_1 > c_2 > c_3 > c_4 > c_5$ for all students. Thus college c_1 would always bring utility of 10 to the student community, c_2 brings 9 and so on, which forms this lower bound of $\pi_S^{low} = 10 + 9 + 8 + 7 + 6 = 40$.

We also notice in figure 2.4 that in general π_S is slightly larger under the hybrid mechanism than under the GS mechanism. This result helps ease the concern that the JUPAS-like mechanism would hurt the interest of student community as a whole. The intuition is that while some students with higher exam scores may get worse in the hybrid mechanism, other students with slightly lower scores would have more chances to enter the programmes/colleges in which they are really interested.

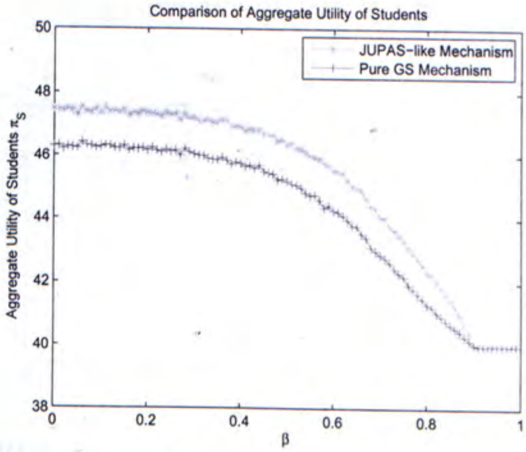


Figure 2.4: Expected Aggregate Utility of Students over Different Degrees of Preference Correlation

Figure 2.5 shows the result for aggregate utility of colleges. In the pure GS mechanism, since preferences of colleges are only determined by the exam scores of students, all colleges would share exactly the same preference list over students. Therefore the student with the highest score would always bring utility of 10 to the college side, the student with the second highest score brings 9 and so on. That's why π_C would be always equal to $10+9+8+7+6 = 40$ under the GS mechanism. The upper bound of π_C is also 50, which occurs only if all five colleges realize their first choices. As shown in the figure, we can achieve about 93% of the upper bound under the JUPAS-like hybrid mechanism when $\beta \in [0, 0.8]$. As students' preferences become more similar, colleges tend to have similar reciprocating preference, which means more conflict would occur among different colleges. Thus as β continues increasing from about 0.8, the aggregate utility of colleges would decrease rapidly. When β is large enough (greater than 0.91), all colleges would share the same preference over students. Thus student with the highest score would always bring utility of 10 to colleges, student with the second highest score brings 9 and so on, which forms the lower bound of $\pi_C^{low} = 10 + 9 + 8 + 7 + 6 = 40$.

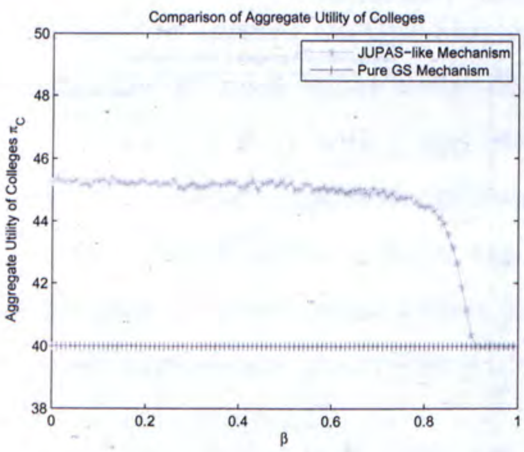


Figure 2.5: Expected Aggregate Utility of Colleges over Different Degrees of Preference Correlation

The expected social welfare under different values of β is shown in figure 2.6. In the same way, we obtain the upper bound of social welfare as $\Pi^{up} = 100$. When β is small (less than 0.5) and students have various preferences over colleges, we achieve about 93% and 86% of the upper bound under the hybrid mechanism and the GS mechanism respectively. When β approaches to one and students share common opinion on colleges, the ratio would both decrease to 80%. This comparative result of social welfare helps justify the implementation of JUPAS-like mechanism in college admissions. The transfer from the hybrid mechanism to the GS mechanism can only achieve the well-known incentive compatible property at the cost of potentially significant loss of efficiency, especially when students have independent opinions on different colleges.

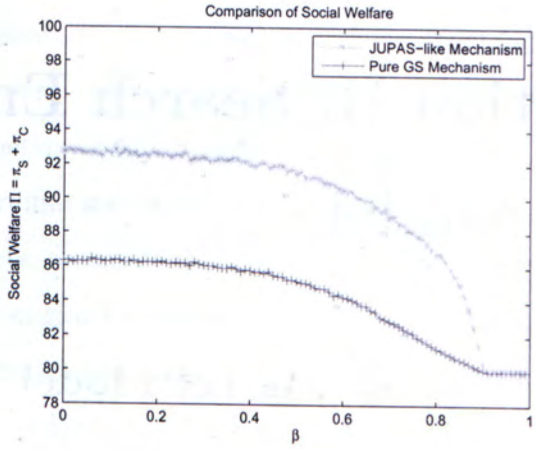


Figure 2.6: The Expected Social Welfare over Different Degrees of Preference Correlation

Chapter 3

Application II: Search Engines Market Model

3.1 The Monopoly Market Model

In this section we consider the monopoly market first, which serves as a starting point for analyzing the more general competition market. Suppose there is only one search engine in the market servicing for a fixed set of end users and provides advertising opportunity for a set of advertisers denoted by $\mathcal{I}$ ($|\mathcal{I}| = m$). Assuming all users are homogeneous and each of them tends to generate the same number of impressions (or clicks) for a particular keyword (query). Since we assumes that the search engine owns a fixed number of users, it would be able to supply a fixed number of attentions (in the form of impressions or clicks) for advertisers.

Suppose that the engine has a limited supply S of attentions for a particular keyword in a given time interval. Each advertiser $i \in \mathcal{I}$ has two private parameters: *value* v_i denoting i's maximal willingness to pay for each attention and *budget* B_i in a given time interval (could be daily, weekly, monthly budget and so on). The search engine needs to determine the optimal price per attention to maximize its revenue[1] :

$$R = p \cdot \min(S, D(p)) = \min(p \cdot S, pD(p))$$

where $D(p)$ is the demand function over price p.

In the following analysis we consider this revenue maximization problem in two different perspectives: the *ex ante* perspective where the search engine only has an rough estimation to the parameters of participating advertisers, and the *ex post* perspective where the engine just needs to make decision based on the *submitted* parameters of advertisers. Although in practice, the advertising systems do determine the prices only *after* advertisers have submitted their values and budgets, we assert that the ex ante view of revenue to be a natural fit for the search engine's objective. This is because typically the interaction between search engine and advertisers is not one-shot and would usually last for many rounds. Advertisers can actually adjust their submitted parameters at any time to achieve better payoff. Thus the ex ante result could provide valuable prediction of the *long-term* revenue for the search engine, rather than the *short-term* profit from one particular instance of the ex post case.

3.1.1 The Ex Ante Case

Assuming the search engine can have a rough estimation of the distribution of advertisers' values and budgets. For simplicity, we only consider the scenario when the parameters are independent and identically-distributed random variables. To be specific, suppose values are drawn from a distribution with density function $f(v)$ and CDF $F(v)$ over the range of $[\underline{v}, \overline{v}]$, and budgets are drawn from distriubtion with density function $g(B)$ and CDF $G(B)$.

[1] In practice, the optimal price is usually determined automatically by an auction mechanism. Specifically, this automation process can be imagined as an ascending-bid auction [28] where the auctioneer (i.e., the search engine) iteratively raise the price until there is no excessive demand than supply. Considering strategic issues, [42] propose an asymptotically revenue-maximizing truthful mechanism. For simplicity of analysis, we ignore the detailed implementation of auctions and assume the search engine can solve the revenue-maximizing problem instantaneously.

After search engine announce the uniform price p, advertiser i would make the deal if only the value v_i is larger than p. The quantity advertiser i could purchase is constrained by the budget B_i.

Therefore the expected aggregate demand under price p from all advertisers would be:

$$D(p) = \sum_{i \in \mathcal{I}} \frac{E(B_i)}{p} \cdot \text{Prob}\{v_i > p\} = m \cdot \frac{E(B)}{p}[1 - F(p)]$$

Rewrite it as:

$$p \cdot D(p) = m \cdot E(B) \cdot [1 - F(p)] \tag{3.1}$$

which is a non-increasing function over p.

We can use figure 3.1 to illustrate the revenue of search engine over price p.

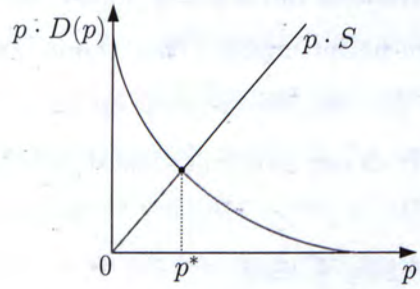


Figure 3.1: Search Engine Revenue Over Prices (Ex Ante)

Proposition 3.1. *The revenue R is maximized when $S = D(p)$, i.e., when the demand equals the supply.*

Proposition 3.1 can be proved by contradiction. If supply exceeds demand under the current price, the search engine will cut down the price to achieve higher revenue (since $R = p \cdot D(p)$ is non-increasing over price p); if demand exceeds supply, the search engine can raise the price and reach a higher revenue (since $R = p \cdot S$ is monotonically increasing over price p).

Example 3.2. Assuming v_i is drawn from uniform distribution with positive support on the interval $[\underline{v}, \overline{v}]$, where $0 \leq \underline{v} < \overline{v}$ and $\Delta v = \overline{v} - \underline{v}$. Then $1 - F(p) = \frac{\overline{v}-p}{\Delta v}$. From $S = D(p)$ we have:

$$p^* = \frac{m \cdot E(B) \cdot \overline{v}}{m \cdot E(B) + S \cdot \Delta v}$$

The intuition is that the more demand (larger m) there is, the higher market clearing price would be; and the more supply (larger S) there is, the lower market clearing price is.

3.1.2 The Ex Post Case

In practical search engine advertising system, advertisers need to submit their values and budgets to the advertising system. The search engine therefore could determine the optimal price based on the ex post variables.

Reorder the index of advertisers such that $v_j \leq v_{j+1}, j = 1, \ldots, m-1$. Then the aggregate demand can be written as:

$$D(p) = \sum_{i \in \mathcal{I}^+(p)} \frac{B_i}{p}$$

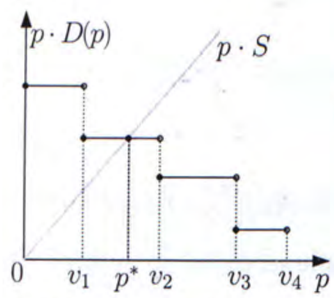
where we define the set:

$$\mathcal{I}^+(p) \triangleq \{i \in \mathcal{I} : v_i > p\}$$

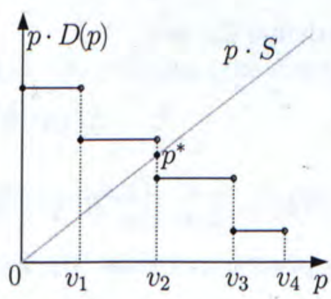
Thus $p \cdot D(p) = \sum_{i \in \mathcal{I}^+(p)} B_i$ is a non-increasing function over p since $\mathcal{I}^+(p)$ shrinks as price p increases. By letting demand equal to supply, we have

$$p(\mathcal{I}) = \frac{\sum_{i \in \mathcal{I}^+(p)} B_i}{S}$$

Notice that the term of price appears in both sides of the equation. Thus in general we cannot derive the closed-form solution for optimal price. Since $pD(p)$ is piece-wise constant and (weakly) decreasing over p, we can illustrate the search engine revenue through examples in figure 3.2. Here we assume there are four advertisers ordered such that $v_1 \leq v_2 \leq v_3 \leq v_4$ and initially when the price is zero, $\mathcal{I}^+(p) = \mathcal{I} = \{1, 2, 3, 4\}$. As the price exceeds v_1, advertiser 1 would have no incentive to stay and $\mathcal{I}^+(p)$ becomes $\{2, 3, 4\}$. The crossing point of demand and supply shows that the optimal price p^* is located in $[v_1, v_2]$. To be more exact, $p^*(\mathcal{I}) = (B_2 + B_3 + B_4)/S$. In figure 3.2b we also show the other case when there is one advertiser who is indifferent between participating and quitting the ad campaign since the optimal price is equal to its value. In typical search engine systems like Google AdWords, after advertisers input their maximal willingness to pay (i.e., their values) and budgets, the ad system would automatically allocate attentions to advertisers as long as the current price doesn't exceed their values and the budgets have not been exhausted yet. Thus here for ease of expression we can assume that the indifferent advertiser would continue participating the ad campaign under the budget constraint. For example, as shown in figure 3.2b, the optimal price is equal to v_2 (satisfying $B_3 + B_4 < v_2 S < B_2 + B_3 + B_4$), advertiser 2 would consume the remaining supply of $S - \frac{B_3+B_4}{v_2}$ and only spent $v_2 S - B_3 - B_4$ which is less than its budget B_2.



(a) Determined Advertisers



(b) Undetermined Advertiser

Figure 3.2: Search Engine Revenue Over Prices (Ex Post)

We can now show a polynomial step algorithm for search engine to compute the optimal price. By inputting the parameters of advertisers (assuming the indexes of advertisers are re-ordered such that $v_i \leq v_{i+1}$, $i \in \{1, \ldots, m-1\}$), algorithm 1 would return the value of optimal price. The time complexity of the algorithm is $O(m^2)$ where m is the number of advertisers[2].

Algorithm 1 Calculate Optimal Price $p^*(\mathcal{I})$

Begin

$v_0 = 0;$
for $i = 1 : m$
    $sum = 0;$
    for $j = i : m$
        $sum += B_j;$
    end for;
    $p = sum/S;$
    if $(p \leq v_i)$
        return $\max(p, v_{i-1});$
    end if;
end for;
return $v_m;$

End

After determining the optimal price p^*, the quantity of attentions allocated to each advertiser i, denoted by q_i, can be easily computed. The search engine would first find *the least index* of advertiser whose value is larger than or equal to p^*, which is denoted by $j \in \{i \in \mathcal{I} : v_i \geq p^*, v_{i-1} < p^*\}$ (we define $v_0 = 0$). If there are no undetermined advertisers, which implies $v_j > p^*$, the quantity allocated to advertiser i would be $q_i = B_i/p^*$ for $i \geq j$ and $q_i = 0$ otherwise. If undetermined advertiser does exist, which implies $v_j = p^*$, we have $q_i = B_i/p^*$ for $i > j$, $q_j = S - \sum_{i=j+1}^{m} q_i$, and $q_i = 0$ for $i < j$. In both cases, the demand equals the supply, i.e., $\sum_{i \in \mathcal{I}} q_i = S$. This can be summarized in algorithm 2.

The revenue of search engine is:

$$R = p^* \cdot S \tag{3.2}$$

[2]The time complexity of the algorithm may be further reduced to $O(m)$ by computing and saving the value of $sum_i \triangleq \sum_{j \in \{i,\ldots,m\}} B_j$, $i \in \{1, \ldots, m\}$ first, which can be finished in $O(m)$ steps. Then the inner "for" loop in algorithm 1 can be substituted by the stored value of sum_i and the complexity of the algorithm is reduced to $O(m)$. Here for simplicity of exposition, we just show the $O(m^2)$ algorithm.

Algorithm 2 Calculate Allocation q_i, $i \in \mathcal{I}$

```
Begin
sum = 0;
p = p*;
for j = 1 : m
    if (v_j >= p)
        break;
    end if;
end for;
for i = 1 : (j - 1)
    q_i = 0;
end for;
for i = (j + 1) : m
    q_i = B_i/p;
    sum+ = q_i;
end for;
q_j = S - sum;
End
```

The aggregate utility of advertisers is:

$$U_A = \sum_{i \in \mathcal{I}^+(p^*)} (v_i - p^*) \frac{B_i}{p^*} \tag{3.3}$$

Notice that the indifferent advertiser, if exists, would always achieve zero utility since the current price equals its value, thus we don't need to consider it in the expression.

The *social welfare* of the advertising system[3] is:

$$SW = R + U_A \tag{3.4}$$

Lemma 3.3. *The optimal price is non-decreasing over the set of participating advertisers given fixed supply S. That is to say, for any advertisers set $\mathcal{I}_1$ and $\mathcal{I}_2$, if $\mathcal{I}_1 \subseteq \mathcal{I}_2$, we have $p^*(\mathcal{I}_1) \leq p^*(\mathcal{I}_2)$, where $p^*(\mathcal{I})$ is obtained according to algorithm 1.*

Proof. We can prove the above lemma by contradiction. For simplicity of notation, we write $p_1 \triangleq p^*(\mathcal{I}_1)$ and $p_2 \triangleq p^*(\mathcal{I}_2)$ and assume that $p_1 > p_2$.

[3] We don't consider search users' utility in the expression here.

Since under optimal price, supply must be equal to demand, we have:

$$S = \sum_{i \in \mathcal{I}_1^+(p_1)} \frac{B_i}{p_1} + \frac{\alpha B_l}{p_1} \quad \text{and} \quad S = \sum_{i \in \mathcal{I}_2^+(p_2)} \frac{B_i}{p_2} + \frac{\beta B_{l'}}{p_2}$$

where $\alpha \in [0, 1]$, and $\alpha > 0$ if and only if there exists an indifferent advertiser l whose value v_l equals p_1; Similarly, $\beta \in [0, 1]$, and $\beta > 0$ if and only if there exists an advertiser l' such that $v_{l'} = p_2$.

For any advertiser $i \in \mathcal{I}_1^+(p_1)$, we have $i \in \mathcal{I}_1 \subseteq \mathcal{I}_2$ and $v_i > p_1 > p_2$, thus $i \in \mathcal{I}_2^+(p_2)$, which infers that $\mathcal{I}_1^+(p_1) \subseteq \mathcal{I}_2^+(p_2)$; since $v_l = p_1 > p_2$, we also have $l \in \mathcal{I}_2^+(p_2)$. Therefore,

$$S = \sum_{i \in \mathcal{I}_1^+(p_1)} \frac{B_i}{p_1} + \frac{\alpha B_l}{p_1} \le \sum_{i \in \mathcal{I}_1^+(p_1)} \frac{B_i}{p_1} + \frac{B_l}{p_1} \le \sum_{i \in \mathcal{I}_2^+(p_2)} \frac{B_i}{p_1} < \sum_{i \in \mathcal{I}_2^+(p_2)} \frac{B_i}{p_2} \le \sum_{i \in \mathcal{I}_2^+(p_2)} \frac{B_i}{p_2} + \frac{\beta B_{l'}}{p_2}$$

Contradiction to the conclusion that supply should equal demand under optimal price p_2. □

Lemma 3.4. *The revenue of search engine is non-decreasing over the set of participating advertisers given fixed supply S. That is to say, for any advertisers set $\mathcal{I}_1$ and $\mathcal{I}_2$, if $\mathcal{I}_1 \subseteq \mathcal{I}_2$, we have $R(\mathcal{I}_1) \le R(\mathcal{I}_2)$.*

Proof. This conclusion can be deducted from lemma 3.3 immediately:

$$R(\mathcal{I}_1) = p^*(\mathcal{I}_1) \cdot S \le p^*(\mathcal{I}_2) \cdot S \le R(\mathcal{I}_2).$$

□

Lemma 3.5. *The optimal price is non-increasing over the supply given the set of participating advertisers $\mathcal{I}$. That is to say, for any supply $S_1, S_2 \in [0, \infty)$, if $S_1 > S_2$, we have $p^*(S_1) \le p^*(S_2)$.*

Proof. We prove the lemma by contradiction. For simplicity, we write $p_1 \triangleq p^*(S_1)$ and $p_2 \triangleq p^*(S_2)$ and assume that $p_1 > p_2$.

Since under optimal price, supply equals demand, we get:

$$S_1 = \sum_{i \in \mathcal{I}^+(p_1)} \frac{B_i}{p_1} + \frac{\alpha B_l}{p_1} \quad \text{and} \quad S_2 = \sum_{i \in \mathcal{I}^+(p_2)} \frac{B_i}{p_2} + \frac{\beta B_{l'}}{p_2}$$

where $\alpha \in [0,1]$, and $\alpha > 0$ if and only if there exists an indifferent advertiser l whose value v_l equals p_1; Similarly, $\beta \in [0,1]$, and $\beta > 0$ if and only if there exists an advertiser l' such that $v_{l'} = p_2$.

For any advertiser $i \in \mathcal{I}^+(p_1)$, we have $v_i > p_1 > p_2$, thus $i \in \mathcal{I}^+(p_2)$, which infers that $\mathcal{I}^+(p_1) \subseteq \mathcal{I}^+(p_2)$; since $v_l = p_1 > p_2$, we also have $l \in \mathcal{I}^+(p_2)$. Therefore,

$$S_1 = \sum_{i\in\mathcal{I}^+(p_1)} \frac{B_i}{p_1} + \frac{\alpha B_l}{p_1} \le \sum_{i\in\mathcal{I}^+(p_1)} \frac{B_i}{p_1} + \frac{B_l}{p_1} \le \sum_{i\in\mathcal{I}^+(p_2)} \frac{B_i}{p_1} < \sum_{i\in\mathcal{I}^+(p_2)} \frac{B_i}{p_2} \le \sum_{i\in\mathcal{I}^+(p_2)} \frac{B_i}{p_2} + \frac{\beta B_{l'}}{p_2} = S_2$$

Contradiction to our assumption of $S_1 > S_2$. □

Lemma 3.6. *The revenue of search engine is non-decreasing over the supply given the set of participating advertisers $\mathcal{I}$. That is to say, for any supply $S_1, S_2 \in [0,\infty)$, if $S_1 > S_2$, we have $R(S_1) \ge R(S_2)$.*

Proof. For simplicity, we write $p_1 \triangleq p^*(S_1)$ and $p_2 \triangleq p^*(S_2)$ and from lemma 3.5 we know that $p_1 < p_2$.

Since under optimal price, supply equals demand, we get:

$$S_1 = \sum_{i\in\mathcal{I}^+(p_1)} \frac{B_i}{p_1} + \frac{\alpha B_l}{p_1} \quad \text{and} \quad S_2 = \sum_{i\in\mathcal{I}^+(p_2)} \frac{B_i}{p_2} + \frac{\beta B_{l'}}{p_2}$$

where $\alpha \in [0,1]$, and $\alpha > 0$ if and only if there exists an indifferent advertiser l whose value v_l equals p_1; Similarly, $\beta \in [0,1]$, and $\beta > 0$ if and only if there exists an advertiser l' such that $v_{l'} = p_2$.

For any advertiser $i \in \mathcal{I}^+(p_2)$, we have $v_i > p_2 > p_1$, thus $i \in \mathcal{I}^+(p_1)$, which infers that $\mathcal{I}^+(p_2) \subseteq \mathcal{I}^+(p_1)$; since $v_{l'} = p_2 > p_1$, we also have $l' \in \mathcal{I}^+(p_1)$. Therefore,

$$R(S_2) = p_2 \cdot S_2 = \sum_{i\in\mathcal{I}^+(p_2)} B_i + \beta B_{l'} \le \sum_{i\in\mathcal{I}^+(p_2)} B_i + B_{l'} \le \sum_{i\in\mathcal{I}^+(p_1)} B_i \le \sum_{i\in\mathcal{I}^+(p_1)} B_i + \alpha B_l = R(S_1)$$

□

Lemmas 3.3 - 3.6 present the relationship between price/revenue and supply/advertisers set. These conclusions conform to the general economic law that the price of goods would be lower if there are more supply (or less demand from advertisers), and the revenue of provider would increase if there are more supply (or more demand from advertisers).

3.1.3 Formulated As An Optimization Problem

The revenue maximization problem confronting the monopolistic search engine could also be interpreted as an optimization problem as follows:

$$\begin{aligned}
\text{maximize} \quad & p \cdot \sum_i q_i && (3.5)\\
\text{subject to} \quad & \forall i \in \mathcal{I} : p \cdot q_i \leq B_i && (3.6)\\
& \forall i \in \mathcal{I} : (v_i - p) \cdot q_i \geq 0 && (3.7)\\
& \sum_i q_i \leq S && (3.8)\\
& \forall i \in \mathcal{I} : p, q_i \geq 0 && (3.9)
\end{aligned}$$

where the search engine needs to determine its optimal price p and allocation of supply q_i to each advertiser i in objective function 3.5. Constraint (3.6) means that each advertiser could not spend more than its budget. Constraint (3.7) shows that the utility of advertiser must be non-negative, i.e., when the price p exceeds the value v_i, which implies that q_i must be zero, advertiser i would just quit the ad campaign. The total supply to all advertisers is limited by S, which is shown in constraint (3.8). The last constraint states that all variables (p and q_i, $i \in \mathcal{I}$) should be non-negative.

As we have shown in proposition 3.1 that the maximal revenue can only be obtained when supply equals demand, therefore the above formulation may be further reduced to:

$$\begin{aligned} \text{maximize} \quad & p \cdot S & (3.10) \\ \text{subject to} \quad & \forall i \in \mathcal{I} : p \cdot q_i \leq B_i & (3.11) \\ & \forall i \in \mathcal{I} : (v_i - p) \cdot q_i \geq 0 & (3.12) \\ & \sum_i q_i = S & (3.13) \\ & \forall i \in \mathcal{I} : p, q_i \geq 0 & (3.14) \end{aligned}$$

where constraint (3.13) becomes tight and the objective function is simplified to maximize variable p only. Thus it's easy to see that the optimal solution for p is unique. Otherwise, if both p^* and $p^{*\prime}$ maximize the objective function, it must be $p^* \cdot S = p^{*\prime} \cdot S$, so $p^* = p^{*\prime}$. It remains to be inspected whether the optimal allocation vector of $\vec{q} \triangleq (q_1, q_2, \ldots, q_m)$ for the optimization problem is unique too. We summarize our conclusions in the following proposition.

Proposition 3.7. *In general, for revenue maximization problem (3.10), the optimal price p is unique, however, there may be multiple optimal solutions for allocation vector $\vec{p}$.*

This can be shown by constructing a simple example as follows: there are two advertisers with $v_1 = 1, v_2 = 2$ and $B_1 = 2, B_2 = 1$, search engine's supply is $S = 2$. Assuming the optimal price p^* is larger than 1, q_1 must be zero since $v_1 < p^*$, then q_2 must be 2 according to constraint (3.13). This would lead to $p \cdot q_2 > 2$, which contradicts with the budget constraint of $B_2 = 1$. Now assuming the optimal price $p^* = 1$, the constraints of the optimization problem would be reduced to $q_1 + q_2 = 2$ and $q_2 \leq 1$, thus the optimal allocation vector could be $\vec{q^*} = (q_1^*, q_2^*) = (2 - q, q)$, $\forall q \in [0, 1]$.

We now turn to investigate the effect of different optimal solutions to the social welfare. Since in general $R = p \cdot \sum_{i \in \mathcal{I}} q_i$ and $U_A = \sum_{i \in \mathcal{I}} (v_i - p) q_i$, from equation (3.4) we get $SW = \sum_{i \in \mathcal{I}} v_i \cdot q_i$, where the payments between search engine and advertiser are crossed off. If we examine the original *social welfare* *maximization* (SWM) problem under the same constraints (3.11)-(3.14), we can immediately present a trivial solution as follows: $p = 0$, $q_m = S$, $q_i = 0$ for $i \in \mathcal{I} \backslash \{m\}$, i.e., letting the advertiser with the highest value acquire all the supply exclusively. Now the maximal social welfare would be $SW_{max} = v_m \cdot S$. However, this solution is infeasible since it induces zero profit to the search engine. An alternative problem the search engine may be interested in is to maximize the social welfare while maintaining the optimal revenue it has achieved in (3.10). In other words, we need to pick out one among the multiple optimal allocations of (3.10) to maximize the social welfare. We call it as the *constrained social welfare maximization* (C-SWM) problem henceforth. This following theorem gives the solution to the C-SWM problem.

Theorem 3.8. *Among all optimal solutions to the profit maximization problem (3.10), algorithm 1 and 2 yield the one which maximizes the social welfare, i.e., the solution to the constrained social welfare maximization (C-SWM) problem.*

Proof. Denote the optimal price and allocation induced by algorithms as p^* and $q_1^*, \ldots, q_m^*$, and assuming there is another allocation vector $\hat{q}_1, \ldots, \hat{q}_m$ satisfying the constraints (3.11)-(3.14). Assuming advertiser j has the cutting-off value such that $v_i \geq p^*$ for $i \geq j$ and $v_i < p^*$ for $i < j$. Therefore from constraint (3.12), we must have $q_i^* = \hat{q}_i = 0$ for all $i < j$. According to algorithm 2, $q_i^* = \frac{B_i}{p^*}$ for all $i > j$ and $q_j^* = S - \sum_{i=j+1}^{m} q_i$. Due to the constraint (3.12), for all $i > j$, it must be $\hat{q}_i \leq \frac{B_i}{p^*} = q_i^*$. Hence we can assume that $\hat{q}_i = q_i^* - \delta_i$, where $\delta_i \geq 0$ for all $i > j$. Then the social welfare under either allocation is:

$$\begin{aligned}
SW_1 &= \sum_{i \in \mathcal{I}} v_i \cdot q_i^* \\
&= \sum_{i=j+1}^{m} v_i \cdot q_i^* + v_j \cdot (S - \sum_{i=j+1}^{m} q_i^*) \\
SW_2 &= \sum_{i \in \mathcal{I}} v_i \cdot \hat{q}_i \\
&= \sum_{i=j+1}^{m} v_i \cdot (q_i^* - \delta_i) + v_j \cdot [S - \sum_{i=j+1}^{m} (q_i^* - \delta_i)] \\
&= SW_1 + \sum_{i=j+1}^{m} \delta_i \cdot (v_j - v_i)
\end{aligned}$$

Since $\delta_i \geq 0$ and $v_j \leq v_i$ for $i > j$, we have $SW_2 \leq SW_1$. Therefore $q_1^*, \ldots, q_m^*$ maximizes the social welfare over all possible optimal allocations of (3.10). □

3.2 The Duopoly Market Model

In this section we switch from the monopoly model to the more practical competitive model. Considering the common situation where there are usually one leading search company and one major competitor in the market (for example, Google and Yahoo! in the United States), we describe a duopoly model where one search engine has an advantage over the other. We formulate their competition as a three-stage dynamic game and solve it from the *ex post* perspective as follows.

3.2.1 Competition for End Users in Stage I

In Stage I search engines would choose different strategies for attracting end users with different tastes. The user bases they attract in this Stage would be the decisive factor for determining their supply of user attentions to advertisers in subsequent stages.

We assume that there are two *horizontally* and *vertically* differentiated search engines $\mathcal{J} = \{1, 2\}$ providing search results to users and selling ad opportunity to advertisers.

Here *horizontal difference* means the different design of their home pages and diversity of extra services such as email, news and other applications. Different users may have different tastes and preferences and hence be attracted by different search engines.

Vertical difference means the quality of searching results. The higher the quality is, the better users and advertisers would feel. We assume that search engine 1 possesses the leading technology to match ads to search queries and can provide better service for both users and advertisers than search engine 2.

In terms of horizontal difference, the canonical Hotelling's model of spatial competition [32] provides an appealing framework to address the equilibrium in characteristic space. The behaviors of providers could then be rationalized as the best-response strategies of players in a location game. The dynamics of the game can be described as follows: each provider chooses a location in the characteristic space which denotes the specific feature of service it provides to users. And each user is characterized by an address reflecting his individual preference of ideal features search engines should provide. Searching at engine $j \in \mathcal{J}$ involves quadratic transportation cost[4] for a user if engine j is not located in his ideal position. Users would choose search engine which provides better search results and also induces as low transportation cost as possible.

Assuming users are uniformly distributed on the circumference of a unit circle. The address of user is denoted by $t \in [0, 1)$. Without loss of generality, let search engine 1 locate at $x_1 = 0$ and search engine 2 $x_2 \in [0, 1)$, as shown

[4]Actually in the seminar paper of Hotelling [32] the author assumed the linear transportation cost, which resulted in no equilibrium results. Later literatures on Hotelling's model usually modified this assumption to the quadratic transportation cost which ensures existence of equilibrium. Here we followed this line of revised model as applied in recent papers such as [33, 34]. Interesting readers may further refer to the excellent survey of [35] for a comprehensive discussion and review of different variants of the Hotelling's model.

in figure 3.3.

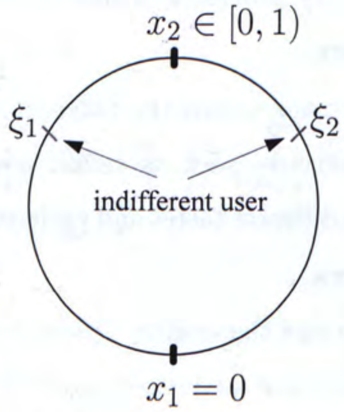


Figure 3.3: Users in Circular Domain

Assuming search engine 1 can provide higher quality results for users than search engine 2. Then the utility of the user searching in either engine would be as follows:

$$u_1(t) = \zeta_1 q - C(t, x_1) = q - \min\{t^2, (1-t)^2\} \tag{3.15}$$

$$u_2(t) = \zeta_2 q - C(t, x_2) = \zeta q - (t - x_2)^2 \tag{3.16}$$

where $\zeta \in [0,1]$ denotes the comparative "disability" of search engine 2 to provide the best search result to users; q is the positive payoff users perceive when certain information is returned by the search engine for a particular query; $C(t, x_j)$ is the transportation cost incurred when there is some distance between user's address t and search engine j's location x_j.

Let $u_1(\xi) = u_2(\xi)$ we can find the location of users who are indifferent between searching in two engines:

$$\xi_1 = \frac{(1-\zeta)q + x_2^2}{2x_2}$$

$$\xi_2 = \frac{1 - x_2^2 - (1-\zeta)q}{2(1-x_2)}$$

Then the market share of search engine 2 is

$$n_2(x_2) = \xi_2 - \xi_1 = \frac{1}{2}[1 - \frac{(1-\zeta)q}{x_2(1-x_2)}]$$

and search engine 1 obtains the remaining market share: $n_1 = 1 - n_2$. By applying the first-order condition $\frac{dn_2}{dx_2} = 0$, we have $x_2^* = \frac{1}{2}$, i.e., the maximum differentiation.

Letting $x_2 = \frac{1}{2}$ we have

$$\begin{aligned} n_1 &= \frac{1}{2} + 2(1-\zeta)q \\ n_2 &= \frac{1}{2} - 2(1-\zeta)q \end{aligned}$$

As we can see, when two search engines provide the same quality of service ($\zeta = 1$), they will divide the market share equally. The less quality search engine 2 provides, the less market share it can hold.

Since the impression number for a particular keyword in a search engine is proportional to the users it attracts: the more users see the advertisement, the more impressions the ad would receive in general. To be aligned with the monopoly case in previous section, here we assume the total supply is still S and the supply of each search engine is denoted by:

$$S_1 = S \cdot \frac{n_1}{n_1 + n_2} = S \cdot n_1$$
$$S_2 = S \cdot \frac{n_2}{n_1 + n_2} = S \cdot n_2$$

Since $n_1 \geq n_2$, we have also $S_1 \geq S_2$.

3.2.2 Competition for Advertisers in Stage II and III

Search engines compete for advertisers in the last two stages to maximize their revenues *subject to* the supply constraint (S_1, S_2) determined in Stage I. In Stage II, search engines determine their optimal prices (p_1, p_2) for charging advertisers; and consequently in Stage III, advertisers would choose their favorite search engine for advertisements based on the previously announced prices. Facing the new advertiser sets in Stage III, search engines may want to revert to the second stage and revise their optimal prices, and consequently, advertisers would make necessary adjustment in the third stage. Therefore, Stage II and III would *alternate* dynamically until it reaches certain stable state. we will discuss this dynamic process in details in the following section.

For advertiser $i \in \mathcal{I}$, the utility of participating in the ad campaign in either search engine is:

$$\pi_1^i = \max\{(v_i - p_1)\frac{B_i}{p_1}, 0\} \tag{3.17}$$

$$\pi_2^i = \max\{(v_i\rho_i - p_2)\frac{B_i}{p_2}, 0\} \tag{3.18}$$

where $\rho_i \in [0,1]$ is called *discount factor* denoting advertiser i's perceived "disability" of search engine 2 to convert the impressions to clicks (or sales of products). We assume that search engine 1 owns better technology and is able to match users' interest with the most suitable ads, hence can generate a higher *click-through rate* (users' probability of clicking after seeing the ads) or *conversion rate* (users' probability of purchase the product or service after clicking the ads) than search engine 2. So in general advertisers would evaluate each impression in search engine 1 higher than in engine 2. For simplicity of notation, we have normalized the discount factor of per-impression value in search engine 1 as unity. In practical market, advertisers can be roughly classified into two categories: *branding advertisers* and *sales advertisers* [36]. Branding advertisers usually have higher ρ since they aim to promote the brand awareness among users and hence the relative technology disadvantage in search engine 2 would have less effect on their values for each attention/impression. However, for sales advertisers who care more on the click-through rate or conversion rate, the technology disadvantage would affect their values for each impression more and therefore result in lower values of ρ. To be more exact, we let the expectation $E(\rho)$ of discount factor serve as the cutting-off value for two types of advertisers, i.e., advertisers with higher ρ than $E(\rho)$ is defined as branding advertisers and the others are sales advertisers in our model.

By letting $\pi_1^i \geq \pi_2^i$ we can derive the condition under which advertiser i

would choose search engine 1:

$$\rho_i \leq \frac{p_2}{p_1}$$

Assuming that advertisers are re-ordered according to ρ_i. Then the division of advertisers can be depicted in figure 3.4 where $\mathcal{I}_1(p_1, p_2) = \{i \in \mathcal{I} : \rho_i \leq \frac{p_2}{p_1}\}$ denotes the set of advertisers who prefer search engine 1 and $\mathcal{I}_2(p_1, p_2) = \{i \in \mathcal{I} : \rho_i > \frac{p_2}{p_1}\}$ the set of advertisers preferring engine 2.

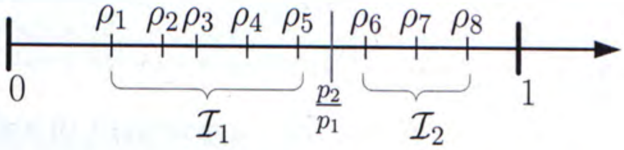


Figure 3.4: The Division of Advertisers

After initial price p_1 and p_2 are set in the market, the advertisers set is divided into $\mathcal{I}_1$ and $\mathcal{I}_2$. Then each search engine can compute its optimal price $p_1^*(\mathcal{I}_1)$ and $p_2^*(\mathcal{I}_2)$ independently as the monopoly case and price ratio $\frac{p_2^*(\mathcal{I}_2)}{p_1^*(\mathcal{I}_1)}$ gets updated. If it happens that the new price ratio divides the advertisers set into $\mathcal{I}_1$ and $\mathcal{I}_2$, we say this is a *Nash equilibrium (NE) price pair* as (p_1^{NE}, p_2^{NE}) and neither search engine has incentive to deviate unilaterally. Otherwise, the process will iterate until the prices become stable.

Defining first the set of advertisers who participate the advertising campaign as follows.

$$\mathcal{I}_1^+(p_1, p_2) \triangleq \{i \in \mathcal{I} : \rho_i \leq \frac{p_2}{p_1}, v_i \geq p_1\} \tag{3.19}$$

$$\mathcal{I}_2^+(p_1, p_2) \triangleq \{i \in \mathcal{I} : \rho_i > \frac{p_2}{p_1}, \rho_i v_i \geq p_2\} \tag{3.20}$$

We now give the formal definition of NE price pair.

Definition 3.9. *A price pair of* (p_1, p_2) *is called a* Nash equilibrium price pair *if* $p_1 = p^*(\mathcal{I}_1^+(p_1, p_2))$ *and* $p_2 = p^*(\mathcal{I}_2^+(p_1, p_2))$ *where* $p^*(\mathcal{I})$ *is computed according to algorithm 1.*

It's easy to see that under the NE price pair (p_1, p_2), for any advertiser $i \in \mathcal{I}_1^+$ or $i \in \mathcal{I}_2^+$, it would have no incentive to switch to the other search engine; for advertiser $i \in \mathcal{I}_1 \backslash \mathcal{I}_1^+$, since $\rho_i \leq \frac{p_2}{p_1}$ and $v_i < p_1$, it holds that $\rho_i v_i < p_2$, thus i would not switch to engine 2 which generates zero utility according to equation (3.17); for advertiser $i \in \mathcal{I}_2 \backslash \mathcal{I}_2^+$, since $\rho_i > \frac{p_2}{p_1}$ and $\rho_i v_i < p_2$, we have $v_i < p_1$ so i have no incentive to switch to engine 1. Thus the NE price pair would induce a stable state to the competition system.

Proposition 3.10. *None-zero NE price pairs may not exist.*

A simple counter-example to illustrate proposition 3.10 is when there is only one advertiser in the system. No matter which search engine this advertiser chooses, the price in *the other* search engine would be zero since it attracts no advertisers. Then the advertiser would have incentive to join *the other* search engine due to the zero price. However, once the advertiser switches, the price in the other search engine would become positive and price in the original engine decreases to zero. Thus the advertiser would keep switching between two search engines and no stable prices can be reached.

Proposition 3.11. *If NE price pair (p_1^{NE}, p_2^{NE}) exists, it must be $p_1^{NE} \geq p_2^{NE}$.*

Proof. Assuming $p_1^{NE} < p_2^{NE}$, then since $\frac{p_2^{NE}}{p_1^{NE}} > 1$, we have $\mathcal{I}_2^+(p_1, p_2) = \varnothing$. Therefore $p_2^{NE} = 0$ and $p_1^{NE} < 0$. However, it cannot be the case since rational search engine would never set negative prices. □

Proposition 3.12. *In the stable state, search engine 2 cannot make higher revenue than engine 1.*

Proof. Since $R_1 = p_1^{NE} \cdot S_1$ and $R_2 = p_2^{NE} \cdot S_2$, from proposition 3.11 and $S_1 \geq S_2$, it's easy to see that $R_1 \geq R_2$. □

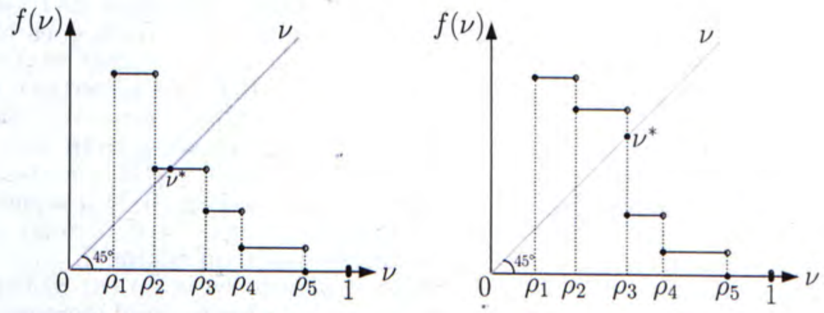
Denote ν as the price ratio $\frac{p_2}{p_1}$ which determines advertisers' preferences, we define the optimal price ratio as:

$$f(\nu) = \frac{p_2^*(\mathcal{I}_2(\nu))}{p_1^*(\mathcal{I}_1(\nu))}$$

where $p^*(\mathcal{I})$ is obtained according to algorithm 1. If the advertiser partitions generated by ν are the same as those generated by the optimal price ratio $f(\nu)$, then the partitions are stable and the optimal prices become NE price pair. The problem reduces to find the *fixed points* which satisfy $f(\nu^*) = \nu^*$. Notice that $f(\nu)$ is piece-wise constant and its value changes at $\nu = \rho_i$, $i \in \{1, \ldots, m\}$.

From the definitions of $\mathcal{I}_1$ and $\mathcal{I}_2$ we see that as ν increases, the preferred set of $\mathcal{I}_1$ would expand while $\mathcal{I}_2$ shrinks. According to lemma 3.3, we know that p_1^* would increase while p_2^* decreases. Therefore $f(\nu)$ should be a non-increasing function of ν.

We can now show the dynamics of function $f(\nu)$ in figure 3.5. In this example we assume there are five advertisers re-ordered by their values of ρ such that $\rho_i \leq \rho_{i+1}$, $i \in \{1, 2, 3, 4\}$. Similarly, there may be two different scenarios for the location of fixed point v^*: (a) $v^* \in (\rho_i, \rho_{i+1})$, $i \in \{1, \ldots, m-1\}$ as shown in figure 3.5a, and (b) $v^* = \rho_i$, $i \in \{1, \ldots, m\}$ as shown in figure 3.5b.



(a) Determined Division (b) Undetermined Division

Figure 3.5: Division of Advertisers Set by their Preferences

For case (a), the optimal ν^* divides the advertisers set into exactly two subsets, and all advertisers with strict preferences over certain search engine are aggregated to one of the subsets. The market would become stable after each search engine sets their optimal price. For case (b), however, there is one special advertiser who would keep switching from one search engine to the other. To illustrate it, assuming the index of this special advertiser is l which satisfies the following condition:

$$\frac{p_2^*(\{l, l+1, \ldots, m\})}{p_1^*(\{1, 2, \ldots, l-1\})} > \rho_l > \frac{p_2^*(\{l+1, \ldots, m\})}{p_1^*(\{1, 2, \ldots, l\})}$$

The first inequality above implies that advertiser l prefers search engine 1 if l has already joined the system of search engine 2, while the second inequality implies that he would prefer engine 2 if he is associated with engine 1. Therefore, advertiser l would keep switching between two search engines. In this case, we call advertiser l as the *undetermined* advertiser.

The undetermined advertiser problem arises from our assumption that advertisers can only purchase service from one engine. If we relax this assumption and allow advertisers to split their budgets into both engines, then the market could still reach the stable state. Now the dynamics of undetermined advertiser l's strategic behavior could be interpreted as follows: assuming starting from the initial state where advertiser l has joined engine 1, and is facing a lower price ratio ($\frac{p_2}{p_1} < \rho_l$) which indicates him to invest more on engine 2; then l would try to split his budget into two parts: $(1-\alpha)B_l$ goes to engine 1 and the rest of αB_l goes to engine 2, with $\alpha \in [0, 1]$. As advertiser l invests more and more budgets on engine 2, i.e., α keeps growing from zero to one, the price ratio $\frac{p_2}{p_1}$ would keep rising until at certain $\alpha^* \in (0, 1)$ it equals ρ_l and advertiser l would have no incentive to invest more on engine 2.

It remains to be shown whether the price ratio above would increase "smoothly"[5] as α increases and whether there always exists α^* for the undetermined advertiser to divide his budget. To prove this property of continuity, we first present lemma 3.13 as follows.

Lemma 3.13. *Given a monopolistic search engine and the advertisers set $\mathcal{I}$. Assuming all parameters of the ad system, including engine's supply, advertisers' values and budgets, are definite except the budget B_i of certain advertiser*

[5]To be more exact, we need to guarantee that there are no discontinuous points. Otherwise, the optimal α^* may not exist.

$i \in \mathcal{I}$, then the optimal price $p^*(\mathcal{I})$ can be regarded as a function over the variable $B_i \geq 0$. Furthermore, the function is continuous and non-decreasing as B_i increases.

Proof. Since from algorithm 1, for each realization of B_i, we can compute a definite value of optimal price, the optimal price can therefore be regarded as a function of B_i.

The property of non-decreasing (or weakly increasing) is easy to see by contradiction. Assuming for B_i the optimal price is p_1 and for $B_i + \delta$ $(\delta > 0)$ the optimal price is p_2 with $p_2 < p_1$. Then p_1 would be the optimal solution for formulation in (3.10)-(3.14) and we let $\vec{q^*}$ be one of the binding optimal allocations. After we increase advertiser i's budget to $B_i + \delta$, the previous solution combination $(p_1, \vec{q^*})$ would still satisfy the constraints of the revised optimization problem since $p_1 \cdot q_i^* \leq B_i \leq B_i + \delta$ and all other conditions remain unchanged. Since p_2 is the optimal solution for the revised problem which maximizes the objective function of $p \cdot S$, we have $p_2 \cdot S \geq p_1 \cdot S$, thus, $p_2 \geq p_1$. This contradicts with our previous assumption of $p_2 < p_1$.

We now turn to prove the property of continuity. Let $p^*(B_i)$ denote the optimal price under budget B_i and $p^*(B_i + \varepsilon)$ the optimal price under budget $B_i + \varepsilon$ where ε is any small real number. As shown in figure 3.2, for arbitrary budget $B_i \in [0, \infty)$, there exist two different scenarios for computing the optimal price: (a) all advertisers are determined; (b) there is one undetermined advertiser. We will discuss these two cases separately as follows.

Case (a): $v_l < p^* < v_{l+1}$, $l \in \{0, 1, \ldots, m-1\}$ (let $v_0 = 0$). We further consider two cases for the index of advertiser i whose budget B_i is the variable: (i) $i \leq l$; (ii) $i > l$. For type (i), since $p^*(B_i) = (B_{l+1} + B_{l+2} + \cdots + B_m)/S$, the change of B_i would not affect the value of optimal price, therefore, we have $\lim_{\varepsilon \to 0} p^*(B_i + \varepsilon) = p^*(B_i)$; For type (ii), since advertiser i is in the participating set, the change of B_i does affect the optimal price. Assuming ε is small enough such that $|\varepsilon| < \min\{(p^* - v_l) \cdot S, (v_{l+1} - p^*) \cdot S\}$. This condition guarantees that $p^*(B_i+\varepsilon) = p^*(B_i) + \varepsilon/S$ is still in the interval of (v_l, v_{l+1}). Therefore, $\lim_{\varepsilon\to 0} p^*(B_i+\varepsilon) = p^*(B_i) + \lim_{\varepsilon\to 0} \varepsilon/S = p^*(B_i)$.

Case (b): $p^* = v_l$, $l \in \{1, \ldots, m\}$ and $B_{l+1}+\cdots+B_m \leq v_l \cdot S \leq B_l + B_{l+1} + \cdots + B_m$ where advertiser l would only consume part of his budget under price v_l. For $i < l$, the change of B_i would not affect p^*, so we only need to consider the case for $i \geq l$. We now consider three possible scenarios for $v_l \cdot S$:

(i) $B_{l+1}+\cdots+B_m < v_l \cdot S < B_l + B_{l+1}+\cdots+B_m$. Assuming ε is small enough such that $|\varepsilon| < \min\{v_l \cdot S - (B_{l+1}+\cdots+B_m), B_l+B_{l+1}+\cdots+B_m - v_l \cdot S\}$. This condition guarantees that after B_i has changed ε, $v_l \cdot S$ is still in the interval of $(B_{l+1}+\cdots+B_m+\varepsilon, B_l+B_{l+1}+\cdots+B_m+\varepsilon)$, which means that $\lim_{\varepsilon\to 0} p^*(B_i+\varepsilon) = \lim_{\varepsilon\to 0} v_l = v_l = p^*(B_i)$;

(ii) $v_l \cdot S = B_{l+1}+\cdots+B_m$. When ε is negative, it will be equivalent to the above case (i) and therefore we have $\lim_{\varepsilon\to 0^-} p^*(B_i+\varepsilon) = \lim_{\varepsilon\to 0^-} v_l = p^*(B_i)$; when ε is positive, it will be equivalent to case (a) and therefore $\lim_{\varepsilon\to 0^+} p^*(B_i+\varepsilon) = p^*(B_i) + \lim_{\varepsilon\to 0^+} \varepsilon/S = p^*(B_i)$;

(iii) $v_l \cdot S = B_l + B_{l+1}+\cdots+B_m$. When ε is negative, it will be equivalent to case (a) and therefore $\lim_{\varepsilon\to 0^-} p^*(B_i+\varepsilon) = p^*(B_i) + \lim_{\varepsilon\to 0^-} \varepsilon/S = p^*(B_i)$; when ε is positive, it will be equivalent to the above case (i) and therefore we have $\lim_{\varepsilon\to 0^+} p^*(B_i+\varepsilon) = \lim_{\varepsilon\to 0^+} v_l = p^*(B_i)$.

Therefore now we can conclude that for any $B_i \in [0,\infty)$, it always holds that $\lim_{\varepsilon\to 0} p^*(B_i+\varepsilon) = p^*(B_i)$. So $p^*(B_i)$ is a continuous function over $B_i \in [0,\infty)$. □

We then summarize our main result in the following theorem.

Theorem 3.14. *Assuming that there exists an undetermined advertiser l and this advertiser can purchase service from both search engines. In particular, advertiser l can arbitrarily split his budget into $(1-\alpha)B_l$ and αB_l with $\alpha \in$*

$[0, 1]$, *where the former is invested to engine 1 and the latter to engine 2. Then there must exist* p_1^*, p_2^* *and* $\alpha^* \in [0, 1]$ *such that* $\frac{p_2^*}{p_1^*} = \rho_l$.

Proof. Denote $\mathcal{I}_1 \triangleq \{1, 2, \ldots, l-1\}$ and $\mathcal{I}_2 \triangleq \{l+1, l+2, \ldots, m\}$. By previous analysis we know that advertisers in $\mathcal{I}_1$ would always be associated with engine 1 and advertisers in $\mathcal{I}_2$ be associated with engine 2. It remains to be shown the effect of advertiser l's splitting decision on the price ratio. Define first that $p_1^0 = p_1^*(\mathcal{I}_1)$, $p_1^1 = p_1^*(\mathcal{I}_1 \cup \{l\})$ and $p_1^\alpha = p_1^*(\mathcal{I}_1 \cup \alpha\{l\})$ where $\alpha\{l\}$ denotes that advertiser l participates in search engine 1, but with fractional budget of αB_i; similarly, we define $p_2^0 = p_2^*(\mathcal{I}_2)$, $p_2^1 = p_2^*(\mathcal{I}_2 \cup \{l\})$ and $p_2^\alpha = p_2^*(\mathcal{I}_2 \cup \alpha\{l\})$.

From lemma 3.13 we see that p_2^α is continuous and weakly increasing function of α and $p_1^{1-\alpha}$ is continuous and weakly decreasing function of α, thus the price ratio $\frac{p_2^\alpha}{p_1^{1-\alpha}}$ is continuous and weakly increasing from $\frac{p_2^0}{p_1^1}$ to $\frac{p_2^1}{p_1^0}$ as α rises from zero to one. Thus there must exist an optimal $\alpha^* \in [0, 1]$ such that $\frac{p_2^{\alpha^*}}{p_1^{1-\alpha^*}} = \rho_l \in (\frac{p_2^0}{p_1^1}, \frac{p_2^1}{p_1^0})$. □

3.2.3 Comparison of Competition and Monopoly

After showing the existence of Nash equilibrium prices under a relaxed assumption, we can apply this NE outcome to predict the revenue and social welfare in the duopoly environment, and compare them with the corresponding results when one search engine monopolize the market. These comparative results would be instructive in practice considering the attempt of cooperation among large search companies such as Google and Yahoo!.[6]

We now turn to compare the prices under competition and monopoly. The main results are given in the following theorem.

[6]In June 2008, Google and Yahoo! announced an advertising cooperation agreement which was later on forced to be abandoned due to antitrust concern of government regulators. See the article "Antitrust Concerns Kill Yahoo-Google Ad Deal," CNET, November 5, 2008 (`http://news.cnet.com/8301-1023_3-10082800-93.html`).

Theorem 3.15. *The equilibrium price in search engine 1 (or engine 2) under competition is no less (or larger) than the optimal price when engine 1 monopolizes the market.*

Proof. Assuming all discount factors are randomly drawn in the range of $(\overline{\rho}, \underline{\rho})$. According to equation (3.1) and proposition 3.1, the monopoly price p^* satisfies the following condition:

$$p^* \cdot S = m \cdot E(B) \cdot [1 - F(p^*)] \tag{3.21}$$

Now we divide the total supply arbitrarily into $S_1 = \alpha S$ and $S_2 = (1-\alpha)S$, $\alpha \in [0,1]$ for search engine 1 and 2. Suppose the optimal prices are p_1 and p_2 respectively and there are m_1 advertisers attracted by engine 1 and m_2 advertisers by engine 2 where $m_1 + m_2 = m$, by applying equation (3.1) and proposition 3.1 we get the following equations for both search engines,

$$\begin{aligned} p_1 \cdot \alpha S &= m_1 \cdot E(B) \cdot [1 - F(p_1)] && (3.22) \\ p_2 \cdot (1-\alpha) S &= \sum_{i \in \mathcal{I}_2} E(B) \cdot \text{Prob}\{\rho_i v_i > p_2\} && (3.23) \end{aligned}$$

since in equilibrium $p_2/p_1 = \nu^*$ and for each advertiser $i \in \mathcal{I}_2$ it holds that $\rho_i \geq \nu^*$, we can derive that:

$$\text{Prob}\{\rho_i v_i > p_2\} = \text{Prob}\{\rho_i v_i > \nu^* p_1\} \geq \text{Prob}\{v_i > p_1\} = 1 - F(p_1)$$

Now the equation (3.23) would become:

$$\nu^*(1-\alpha) \cdot p_1 S \geq m_2 \cdot E(B) \cdot [1 - F(p_1)] \tag{3.24}$$

Summing over conditions (3.22) and (3.24), we have

$$[\alpha + \nu^*(1-\alpha)] \cdot p_1 S \geq m \cdot E(B) \cdot [1 - F(p_1)] \tag{3.25}$$

Since $\nu^* \leq 1$, we have $\alpha + \nu^*(1-\alpha) \leq \alpha + (1-\alpha) = 1$. Defining the following function first,

$$h(p) \triangleq \frac{1-F(p)}{p}$$

which is strictly decreasing over p. Then by comparing conditions (3.21) and (3.25), we get

$$h(p_1) \leq \frac{[\alpha + \nu^*(1-\alpha)] \cdot S}{m \cdot E(B)} \leq \frac{S}{m \cdot E(B)} = h(p^*)$$

we can infer that $p_1 \geq p^*$.

Since from proposition 3.11 we have $p_1 \geq p_2$ and $1 - F(p)$ is a monotonic decreasing function of p, equation (3.22) would become:

$$p_2 \cdot \alpha S \leq p_1 \cdot \alpha S \leq m_1 \cdot E(B) \cdot [1 - F(p_2)] \tag{3.26}$$

And since $\rho_i \leq 1$ for any $i \in \mathcal{I}_2$, we know that:

$$\text{Prob}\{\rho_i v_i > p_2\} \leq \text{Prob}\{v_i > p_2\} = 1 - F(p_2)$$

thus equation (3.23) would become:

$$p_2 \cdot (1-\alpha)S \leq m_2 \cdot E(B) \cdot [1 - F(p_2)] \tag{3.27}$$

Summing over inequalities (3.26) and (3.27), we get

$$p_2 S \leq m \cdot E(B) \cdot [1 - F(p_2)]$$

So we have:

$$h(p_2) \geq \frac{S}{m \cdot E(B)} = h(p^*)$$

which infers that $p_2 \leq p^*$.

So in general, we have that $p_2 \leq p^* \leq p_1$. □

One natural question to the duopoly market is that whether the company acting as a follower would merge with the leading company in the market. To answer it, we follow the conventional way of analyzing the total revenue and social welfare under competition and monopoly. At first glance, it seems that allowing the search engine with better technology to monopolize the market would generate higher total revenue and social welfare since it can provide better service for both advertisers and end users. However, it turns out the answer depends on the specific parameters of participants in the market. We summarize the comparison results of total revenue and social welfare under competition and monopoly in the following theorem.

Theorem 3.16. *Whether monopoly would bring in higher total revenue and social welfare than competition* depends on *the specific parameters of advertisers in the advertising systems.*

We can prove this theorem by constructing the counter-examples 3.17 and 3.18 as follows.

Example 3.17. Suppose there are two advertisers $\{1, 2\}$ participating in the advertising system. The value, budget and discount factor of each advertiser are as follows: $v_1 = 1$, $B_1 = 2$, $\rho_1 = 1$ and $v_2 = 4$, $B_2 = 2$, $\rho_2 = 0$. The total supply of advertising opportunity is $S = 1$.

Under monopoly, the optimal price $p^* = 2$ and the maximal revenue is $R(p^*) = p^* \cdot S = 2$. For any $p < p^*$, the corresponding revenue would be $R(p) = p \cdot S < p^* \cdot S = 2$; for any $p > p^*$, since $v_1 < p$ which means advertiser 1 would not attend the system, $R(p)$ is upper bounded by the budget of advertiser 2, i.e., $R(p) \leq B_2 = 2$. This analysis proves the optimality of p^* and $R(p^*)$.

Under competition, advertiser 1 would choose engine 2 since $\rho_1 = 1$ and advertiser 2 would choose the other engine since $\rho_2 = 0$. We equally divide the supply into two parts: $S_1 = S_2 = 0.5$. Now in engine 1, the optimal price is $p_1 = 4$ and $R_1 = p_1 \cdot S_1 = 2$; in engine 2, the optimal price is $p_2 = 1$ and $R_2 = p_2 \cdot S_2 = 0.5$. And the price ratio $p_2/p_1 = 0.25$ is less than ρ_1 and greater than ρ_2.

Therefore in this example, the competition would bring in even higher total revenue ($R_1 + R_2 = 2.5$) than the monopoly ($R(p^*) = 2$).

Example 3.18. There are still two advertisers in the system, with the following parameters: $v_1 = 2$, $B_1 = 0.75$, $\rho_1 = 0$ and $v_2 = 4$, $B_2 = 0.25$, $\rho_2 = 1$. The total supply of advertising opportunity is still $S = 1$.

Under monopoly, the optimal price $p^* = 1$ and the allocation vector is $(q_1, q_2) = (0.75, 0.25)$. Thus the social welfare would be $SW = v_1 q_1 + v_2 q_2 = 2.5$.

Under competition, advertiser 1 would choose engine 1 since $\rho_1 = 0$ and advertiser 2 would choose the other engine since $\rho_2 = 1$. We still divide the supply into $S_1 = S_2 = 0.5$. Now in engine 1, the optimal price is $p_1 = 1.5$ and the social welfare $SW_1 = v_1 q_1 = v_1 S_1 = 0.75$; in engine 2, the optimal price is $p_2 = 0.5$ and $SW_2 = v_2 q_2 = v_2 S_2 = 2$. And the price ratio $p_2/p_1 = 1/3$ is greater than ρ_1 and less than ρ_2.

Therefore in this example, the competition would bring in even higher social welfare ($SW_1 + SW_2 = 2.75$) than the monopoly ($SW = 2.5$).

Theorem 3.16 shows that there is no common conclusion on whether the existence of an inferior company (or product) in the market would raise or drive down the social welfare (or total revenue). Our observation here based on the particular search engine competition model coincides with the finding in the recent paper [37] that the viability of *differentiated services* scheme depends on the specific characteristics of users in the system. The services provided by search engine 1 and engine 2 can be regarded as the 1st and 2nd class services in [37] where the 1st class is usually charged higher price than the 2nd (analogous to our proposition that $p_1^{NE} \geq p_2^{NE}$).

Recall that our conclusions above are based on the *ex post* perspective which includes all possible instances of the competitive market. To show the more general *ex ante* results under common parameter setting of participants, we conduct the simulation in the next section.

3.3 Numerical Results and Observations

In this section we present some numerical results showing the effects of different parameters in our model. There are four major criteria we would like to explore in the model:

(a.1) *Prices*: We would like to compare the equilibrium prices of both engines with the monopoly price if there is only one search engine dominates the market. In the following section we denote (p_1, p_2) as the duopoly prices and p_M as the monopoly price.

(a.2) *Revenues*: It would be intriguing to study the comparative results of total revenues under competition and monopoly. The gap between revenues under competition and monopoly would serve as a signal of whether the leading company would like to propose a merger or acquisition to its competitor. A huge gap would infer that reaching certain cooperation agreement between the two competitors would significantly promote the revenues for both companies.

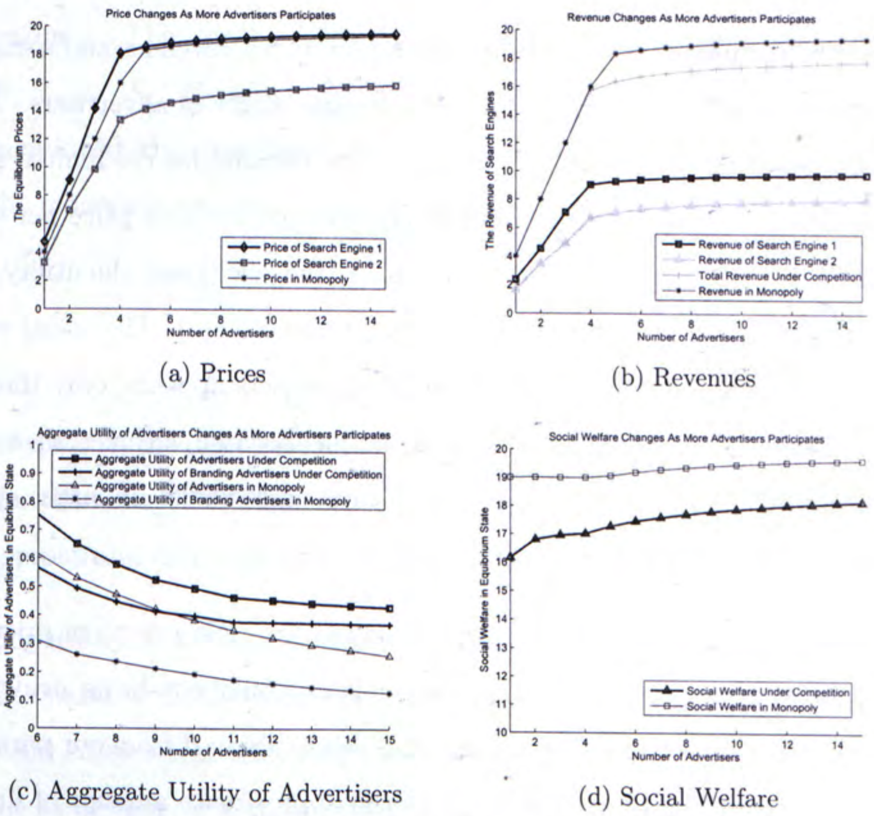
(a.3) *Aggregate Utility of Advertisers*: We compare the aggregate utility of advertisers to see whether the monopoly would be detrimental to the interest of advertisers, and if so, how severe the loss would be. In particular, we examine the aggregate utility for branding advertisers who benefits from the relatively lower price of the inferior search engine in the duopoly market.

(a.4) *Social Welfare*: Social welfare can be regarded as the *realized* value of advertisers and is the benchmark for addressing the interest of the community as a whole. Under competition, the social welfare is computed according to the following equation:

$$SW = \sum_{i \in \mathcal{I}_1} v_i q_i + \sum_{i \in \mathcal{I}_2} \rho_i v_i q_i. \tag{3.28}$$

where q_i is amount of supply allocated to advertiser i.

In the following, we carry out a set of simulation to investigate the comparative results under different parameter settings. For each simulation setting, we randomly generate 5000 instances of parameters and calculate the average value of each criterion. The expected values from *ex ante* perspective can then be approximated by the average values of large amounts of *ex post* instances.



(a) Prices

(b) Revenues

(c) Aggregate Utility of Advertisers

(d) Social Welfare

Figure 3.6: Baseline Setting

3.3.1 Baseline Setting

We consider two search engines equally dividing the market and the total supply is normalized to unity. Thus the supply of either search engine is $S_1 = S_2 = 0.5$. Advertisers' values are uniformly distributed over $(18, 20)$, and their budgets are also drawn from uniform distribution with expectation $E(B) = 4$. Discount factors of advertisers are uniformly distributed over $(0.5, 0.9)$. Therefore there would be expectedly one half of advertisers with discount factors larger than the average value $E(\rho) = 0.7$, which we define as the *branding advertisers.*

The simulation results under baseline setting are presented in figure 3.6. We can make the following observations from figure 3.6(a)-(d):

1) As the number of advertisers increases, the prices, revenues and social welfare would all get raised except the aggregate utility of advertisers. This is because as more advertisers participate, the demand for the limited supply would get boosted, which would finally drive up the unit price per supply and raise the revenue of search engines. As the price rises, the utility of advertisers would keep decreasing as seen in figure 3.6(c). The social welfare can still be improved since when more advertisers appears, only those advertisers with higher values can stay and be allocated with certain amount of supply. Thus the *realized* values of advertisers would be larger and the social welfare get enhanced.

2) After the number of advertisers reach about five, the growth of prices and revenues seems saturated: more advertisers would not bring evident enhancement in prices and revenues. This can be derived from our parameter setting: $E(B)/E(v) = 4/19 \approx 0.2$ is the approximate amount of demand for each advertiser, and since the total supply is one, in expectation it would be sufficient for five advertisers to consume all the supply.

3) Figure 3.6(a) corresponds with theorem 3.15 that the monopoly price is smaller than the duopoly price p_1 of engine 1 and larger than price p_2 of engine 2. We further notice that the monopoly price is actually very close to p_1 but p_2 is much smaller than p_1. This is because the monopoly engine and engine 1 in competition face advertisers with the same distribution of

values. Recall that value is the maximal willingness to pay for advertisers, thus when there are too many advertisers competing with each other, the price would approach to the maximal possible value, which is 20 according to the distribution range. However, for search engine 2, the *actual values* of advertisers are the *original values* discounted by ρ. The lower ρ is, the larger the gap between p_1 and p_2 would be.

4) Figure 3.6(b) shows that revenue of search engine 1 is larger than that of engine 2. This can be easily deducted since the revenue of each engine is $R_1 = p_1 \cdot S_1$, $R_2 = p_2 \cdot S_2$ and we have $p_1 > p_2$, $S_1 = S_2$. As we have mentioned, the monopoly price p_M is approximately equal to p_1. Therefore the monopoly revenue can be denoted as follows:

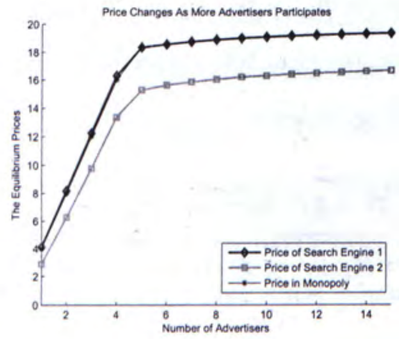
$$\begin{aligned} R_M &= p_M \cdot S \approx p_1 \cdot S = p_1 \cdot S_1 + p_1 \cdot S_2 \\ &= R_1 + \frac{p_1}{p_2} R_2 = R_1 + \frac{R_2}{\rho^*} > R_1 + R_2 \end{aligned} \tag{3.29}$$

where ρ^* is the discount factor of the indifferent advertisers which is always less than one. This inequality explains the gap between total revenue under competition and monopoly in figure 3.6(b).

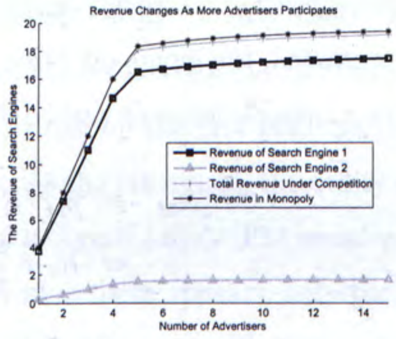
5) The utility of advertisers depends on two factors: the value and the price. Compared with the monopoly, under competition a portion of advertisers could enjoy a relatively lower price which would result in higher utility; at the same time, due to the effect of ρ, the values of advertisers in engine 2 get discounted which would cause lower utility. When the positive factor of lower price dominate the negative factor of lower value, the utility under competition would be greater and vice versa. Since in our baseline setting we set a relatively large ρ, the negative factor would be small and advertisers in engine 2 can benefit from the lower price. This conjecture can be verified in figure 3.6(c). Since in average *branding advertisers* account for half of all advertisers, in monopoly the utility of branding advertisers is always

half of the utility of all advertisers as shown in figure 3.6(c). However, under competition the branding advertisers would benefit more than the rest advertisers since they have higher discount factors which means lower negative effect on values but confronting a lower price in engine 2.

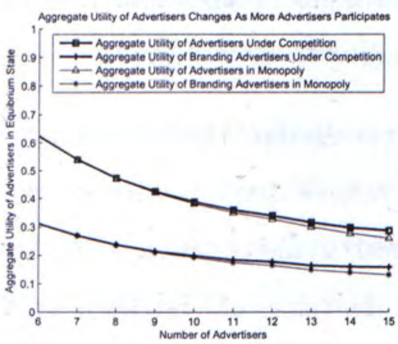
6) Figure 3.6(d) indicates that the social welfare under competition is lower than that under monopoly since the realized values in equation (3.28) get discounted due to the factor ρ.



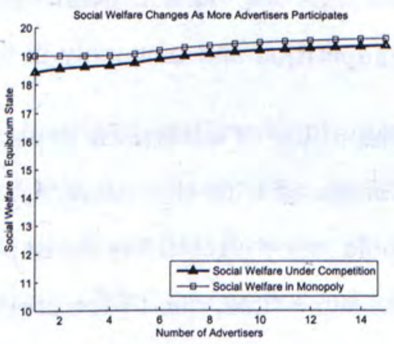
(a) Prices



(b) Revenues



(c) Aggregate Utility of Advertisers



(d) Social Welfare

Figure 3.7: When Supplies Change To $S_1 : S_2 = 9 : 1$

3.3.2 Effect of Supplies

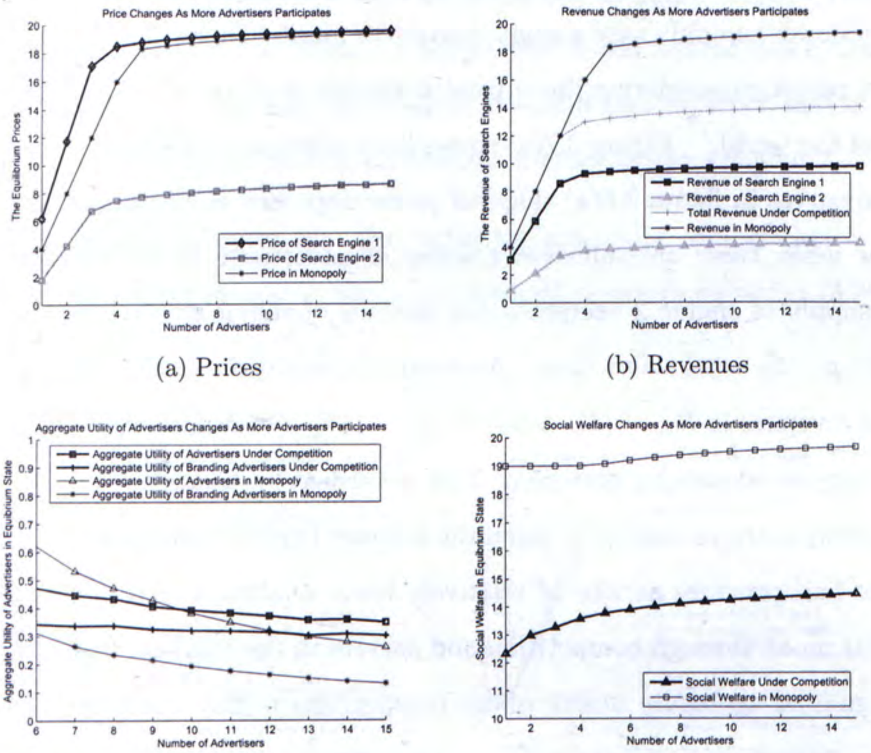
We now change the supplies to $S_1 = 0.9$ and $S_2 = 0.1$ while all other parameters remain the same. The simulation results are presented in figure 3.7.

In this setting one search engine plays the leading role in the market and the follower can only take a small fraction of market share. This assumption is more realistic considering the current dominant position of Google in most areas of the world.[7] Figure 3.7(a) shows little difference with the corresponding price curves in figure 3.6(a) since all prices approach to the maximal possible value when there are sufficient number of advertisers in the market. Since the supply of engine 2 decreases, the revenue of engine 2 which is denoted by $R_2 = p_2 \cdot S_2$ would also drop. According to equation (3.29), the gap of the total revenues is $R_M - (R_1 + R_2) \approx (\frac{1}{\rho^*} - 1)R_2$. Therefore when R_2 is small, the gap would also be negligible. This is verified by figure 3.7(b). This result also demonstrates that even when the follower takes a small portion of market share and provides service of relatively lower quality, it can still make non-trivial profit through competition and survive in the market. In figure 3.7(c), the gain of aggregate utility under competition is tiny compared with that under monopoly since engine 2 can only provide limited supply (10% of the total supply) and only very few of advertisers can take advantage of it. Figure 3.7(d) shows that the loss of social welfare under competition is very small since only 10% of the total supply is of lower quality, i.e., the second term in equation (3.28) is insignificant.

3.3.3 Effect of Discount Factors

We now turn to investigate the effect of technology gap between two search engines. Let the discount factors be drawn uniformly on $(0.1, 0.5)$ and all other parameters are the same as the baseline setting.

[7] In the United States around 72 percent of the total search volumes are conducted on Google while Yahoo and Bing jointly account for about 25 percent. Source from "Top 20 Sites & Engines," Hitwise, May 20, 2010 (`http://www.hitwise.com/us/datacenter/main/dashboard-10133.html`). In some other countries like France, UK and Germany, Google even possessed a market share of over 90 percent. Source from the article "Google's Market Share in Your Country," March 13, 2009 (`http://googlesystem.blogspot.com/2009/03/googles-market-share-in-your-country.html`).



(a) Prices (b) Revenues

(c) Aggregate Utility of Advertisers (d) Social Welfare

Figure 3.8: When ρ Belongs to $(0.1, 0.5)$

As figure 3.8(a) shows, the equilibrium price p_1 of engine 1 and the monopoly price p_M are still close to the maximal value of advertisers and p_2 approaches to the maximal discounted value ρv. As ρ decreases, the price of engine 2 also diminishes compared with p_2 in figure 3.6(a) and 3.7(a).

Since the revenue $R = p \cdot S$ and from above analysis we know p_1 has little change and p_2 diminishes, the revenue R_1 of engine 1 would stay almost the same while R_2 reduces as shown in figure 3.8(b). Since we have also mentioned that the gap between the total revenues approximates to $R_M - (R_1 + R_2) \approx (\frac{1}{\rho^*} - 1)R_2$, when ρ is small, the gap would become larger. This can be easily seen by comparing the corresponding revenue curves in figure 3.6(b) and figure 3.8(b).

The aggregate utilities of advertisers in monopoly are the same in figure

3.6(c) and figure 3.8(c); however, the aggregate utilities under competition in figure 3.8(c) is much smaller than those in figure 3.6(c) due to the negative effect of ρ on advertisers' values. In figure 3.8(c) we see that there is certain intersection between utility under competition and monopoly. When there are only a few of advertisers, the prices are still low and the main factor affecting utility is the value. Under competition the existence of ρ would drive down advertisers' values and thereby results in lower aggregate utility. As the number of advertisers increases, the monopoly price p_M would approach the maximal value $v_{max} = 20$ and the aggregate utility would gradually reduce to zero. However, even when p_1 and p_M approach 20, the rest of advertisers whose discount factors are larger than the equilibrium price ratio $\rho^* = p_2/p_1$ can still obtain nontrivial utility which can be approximated as $(\bar{\rho}v - p_2)S_2 \approx (\bar{\rho}v - \rho^* v)S_2$ ($\bar{\rho}$ denotes the average value of advertisers with discount factors larger than the price ratio ρ^*).

Figure 3.8(d) displays the hug gap between social welfare under competition and monopoly. Since half of the supply ($S_2 = 0.5$) is allocated to advertisers with discount factors less than 0.5, the realized values in equation (3.28) would become significantly smaller than social welfare under monopoly.

Chapter 4

Related Work

Some recent developments in matching theory share the same concern with ours that the transition from the Boston mechanism to the GS mechanism is not problem-free and object to the hasty rejection of the Boston mechanism. One main research direction is the analysis of efficiency in school choice (SC) setting where schools do not have *strict* preferences over students and have to largely rely on *random lotteries* to determine their preferences.

Abdulkadiroğlu, Che and Yasuda first brought the uncertainty factor of lotteries into efficiency consideration in [22]. They showed an elegant example when students share identical ordinal preference but differ in preference intensities, the Boston mechanism can dominate the GS mechanism in terms of *expected* cardinal efficiency. A new Choice-Augmented Deferred Acceptance (CADA) mechanism was proposed accordingly which supports a greater scope of efficiency than the pure GS mechanism in [22]. The same authors further generalized the single example into a "baseline model" where students have common ordinal preferences and schools have no priorities in [23]. Besides, Miralles showed in [24] that above analytical results could extend to more realistic cases like weak priorities by the simulation results.

Featherstone and Niederle then classified the efficiency issue in SC into three categories in [21]:

Ex post: Each student knows preferences of other students and lottery results in each school. The matching outcome as well as the efficiency are both deterministic.

Interim: Students know preferences of other students but remain unknown to the lottery results, i.e., we would investigate the efficiency before the lotteries are drawn. The distribution of lottery results would induce an expected, other than deterministic, value of efficiency.

Ex ante: Students only know the distribution of other students' preferences and still remain ignorant of lottery results.

The authors concluded in the same paper that, when student preferences are uniformly distributed and schools are completely symmetric, the Boston mechanism can first-order stochastically dominate the GS mechanism in terms of *ex ante* efficiency, both in theory and in the laboratory.

Following the efficiency classification above, results in [22, 23, 24] would all fall into the *interim* viewpoint with highly *correlated* student preferences, which complements the conclusion in [21] under *independent* student preferences.

Although sharing the same caution against a hasty replacement of the Boston mechanism, our work stands distinct from these above works in several aspects:

- While the related works focused on the SC setting where schools are regarded as objects to be consumed, our paper tackled with the general college admissions (CA) problem where schools can also act strategically in the two-sided market.
- One *key* assumption for the above works is the *weak* or even no priorities in schools such that lotteries are largely relied upon in schools in order to break the tie. It is this "randomness" that causes the potential *ex ante* efficiency loss of the GS mechanism. However, in practical CA context

where students' scores rather than the random lotteries play the decisive role in admissions, the above assumption would no longer hold, so would the corresponding conclusions.

- Our work follows a distinctive and unique research direction and shows that even when the priorities in schools are *strict*, Boston still exhibits some prominent properties such as respecting the interests of applicants. The sociological consideration has been largely ignored in previous research of college admissions system.

In terms of interdependent preferences, we find a recent work of [25] proposing "interdependent values" in two-sided matching which can be regarded as a complementary notion to our *reciprocating preferences.* In [25] the authors argue that a college c's evaluation to a student s could be affected by (or depend on) other colleges' evaluation to this student s, while we consider the scenario where s's value to c is dependent on c's value to s.

For the application of sponsored search, there are mainly two lines of research work. The mainstream of literature focuses on the interaction between advertisers and search engines and aims to understand and devise viable mechanism for the Internet advertising market. There is significant work on the auction mechanism held by major search engines, starting from two seminal works of [38] and [39] which independently investigated the "generalized second-price" (GSP) auction prevailing in major search engines such as Google and Yahoo!. In [43] the authors compared the "direct ranking" method by Overture with the "revenue ranking" method by Google and proposed a *truthful* mechanism named as "laddered auction". Considering the non-strategyproofness property of GSP mechanism, [44] analyzed one prevalent strategy of advertisers called "vindictive bidding" in real-world keyword auction. [49] and [50] relaxed the basic assumption of separable click-through rate in [43] and modeled the *externality effect* among advertisements which appeared in the same search page simultaneously. [51] proposed a new valuation model to absorb the adverse effect of the competing advertisements on the advertiser's value per click. There is also an abundance of works on proposing more *expressive* but still *scalable* mechanism for sponsored search such as [47, 48, 46, 52, 45]. In particular, in [52] the advertisers were allowed to submit a two-dimensional bid (b, b') where b was the bid for exclusive display and b' for sharing slots with other advertisements. In [45] the authors proposed a *truthful* hybrid auctions where advertisers can make a per-impression as well as per-click bid and showed that it can generate higher revenue for search engine compared with the pure per-click scheme.

It's worth pointing out there are also a few works considering the practical situation where similar keyword auctions are held simultaneously by *multiple* search engines. For example, in [40] the authors investigated the revenue properties of two search engines with different click-through rates which competed for the same set of advertisers. The study in [41] considered competition between two search engines which differs in their ranking rules: one applied the direct ranking method we mentioned above, and the other applied the revenue ranking method.[1] We assert that this assumption of search engine difference is unrealistic since major engines tend to use the same policy which proves to work efficiently in practice and it's unlikely that certain engine would switch back to the obsolete rules.[2] The Nash equilibrium solution in the former paper of [40] is also not so practical since it requires advertisers to adopt certain randomized strategy. It's very difficult for individual advertiser to implement such complex strategies which would incur unnecessary maintenance cost.

The other line of work is developed mainly by economists to address the broad issues of search engine competition from social welfare perspective. [29]

[1] In the original paper of [41], the authors used the terms of "price-only ranking rule" and "quality-adjusted ranking rule" which has the same meaning.

[2] One typical example is that, in May 2002 Google first introduced the revenue ranking approach which proved to be more efficient. And then in 2007, Yahoo! switched from prior direct ranking to revenue ranking rule similar to Google's [27].

introduced a quality choice game model where end users choose the search engine with highest quality of search results, and showed that no Nash equilibrium exists in this game. Based on this proposition, the author argues that the search engine market would evolve towards monopoly in the absence of necessary regulatory interventions. [30] proposed a duopoly model which shares much similarity to our formulation, however, as many of the technical details of the practical advertising system are ignored, it's doubtful whether this can serve as an accurate model to predict the outcome of search engine market. Similarly, [29] faces the same problem that the vague description of participants' utility may not be strong enough to support the predictive conclusions in the paper.

These two lines of important work have little intersections so far: the mainstream of work concentrates on the technical progress in designing "better" advertising system, and the other line usually involves less technical details (like the budgets of advertisers in practical advertising system) and targets the macro-effect of competitive market. Observing this, we believe that a *comprehensive* study of the current search engine ecosystem in a *competitive* way is vital for addressing many of the unresolved issues in this thriving market. Our work manages to narrow the gap between these two fields of research and makes some initial progress in this direction. This observation helps differentiate our work from most of the existing literature.

Chapter 5

Summary and Future Directions

We have studied two important applications of matching mechanism: one is the college admissions problem motivated by our investigations of JUPAS in Hong Kong; the other arises in the thriving Internet advertising market and focus on the competition of search engines for attracting both end users and advertisers.

In the first work we presented an extended matching mechanism which can incorporate both the Boston mechanism and the famous Gale-Shapley mechanism. Initially inspired by a practical college admissions system, i.e., JUPAS in Hong Kong, we proposed a common parameter, namely *reciprocating factor* or α, for the generalized two-sided market. This parameter serves as a bridge between Boston and GS mechanisms: when α equals to zero, the matching mechanism would become pure Boston mechanism; when α equals to one, the matching mechanism is equivalent to pure GS mechanism. Practical system like JUPAS can be regarded as a hybrid of Boston and GS mechanisms with reciprocating factor between zero and one. In the context of college admissions, reciprocating factor is of practical significance for departments to determine the tradeoff between students' eligibility and interest. Moreover, in the context of marriage problem, the classic instance for two-sided market, a similar parameter can still be applied to reflect the individual's sensitivity to other agents' appraisals of himself/herself. Finally, the possible strategic behaviors of participants in the extended mechanism were described and analyzed using the game theoretical approach. Our analysis partly explains why manipulation by colleges/programmes is not a severe problem in large admissions system like JUPAS.

Although the proposed model and strategy analysis are still incomplete and preliminary, our work paves the way for future research on the generalized matching model in various directions:

1. One main open question regarding our work is how to combine the study of traditional matching theory with relevant social subjects like educational or marital psychology. The latter aspect was largely neglected in prevalent matching theory literature.

2. Another question is whether we can design any experiments or use any practical data to substantiate our conjecture: the generalized model could provide greater social welfare for all the participants in the two-sided market than the classic one.

3. Although applicants in JUPAS have merely incomplete information on other agents' behaviors, the students may still refer to the enrollment history of individual programme to predict its subsequent strategy. How would the participants' strategies evolve every year would be an interesting issue worthy of further study.

For the second application, we propose an analytical framework to model the interaction of publishers, advertisers and users under monopoly and duopoly scenarios. For monopoly market, we can give the analytical results of price and revenue for both ex ante and ex post case. For duopoly market, we formulate a three-stage dynamic game to model the search engines' competition for both users and advertisers and prove the existence of Nash equilibrium from *ex post* perspective. To see the *long-term* effect of competition from *ex ante* perspective, we carry out computer simulations for different settings of participants' parameters. The comparative results of revenues and social welfare under competition and monopoly are then presented and discussed extensively in the paper. Our analysis could provide some insight in regulating the search engine market and protecting the interests of advertisers and end users. Although the cooperation between search engines can probably bring more total revenues, advertisers and users may be averse to such plan which eliminates their freedom to choose from diverse services provided by different search companies.

There are several possible ways to extend this work. Throughout our paper, we implicitly assume that advertisers would reveal their true parameters such as values and budgets to the search engines. Since our framework is by no means strategy-proof, how would rational advertisers' strategies affect our conclusions would be an interesting question for further investigation. Another non-trivial problem is how to associate our analytical result of revenue from *one* particular keyword with the practical revenue of an industrial search company which gathers from *numerous* keywords queried by end users everyday. Besides, to be in line with the practical advertising system nowadays, we will consider incorporating the quality factor of advertisement for the revenue-ranking rule as well as the generalized second-price auction prevailing in major search engines. Finally, it would be intriguing to extend our model for multiple search engines scenario besides the basic duopoly scenario.

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