Some Robust Optimization Methods for Inverse Problems

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Abstract

Inverse problems are notable for their broad applications in natural sciences and industries. Their mathematical study typically leads to challenging models that are *ill-posed* in the sense of Hadamard. For numerics, general regularization strategies are developed to treat inherited instability. However, due to the variety and speciality of inverse problems, problem oriented numerical schemes are more promising for efficiency and robustness. In this thesis, we will take the inverse scattering problem as an example to design and analyze several optimization methods for numerical treatments.

The inverse scattering problem is to study obstacle or medium properties by sending and measuring propagating waves. The whole process is described by partial differential equations(e.g. Helmholtz equation, Maxwell' equation) with proper boundary, initial conditions. In recent decades, continuous efforts are made to improve the mathematical models that guarantee the existence and uniqueness of solutions. These results can serve as guidelines for practical designs and numerical reconstructions. For a model frequently used in computation that reduced to a bounded domain with absorbing boundary conditions, a unique result is derived to provide some justifications for numerical analysis in this thesis.

The inverse scattering problem can be formulated into an optimization problem governed by partial differential equations(PDE), hence relatively mature optimization techniques are ready for numerical studies. By considering structural features, we mainly modify and analyze two optimization methods for efficient and robust numerical treatments. First, starting from the recursive linearization method which is advantageous for computational efficiency, we reexamine the method from an unconstrained optimization method - the steepest decent method. By exploring some properties of this method, we suggest directions for

further improvements. After that, regarding as a PDE constrained optimization problem and noticing the bilinearity of the equation, a second order method - the augmented Lagrangian method is carefully analyzed. Optimality conditions are established under some conditions. A modified algorithm is derived to save computational costs.

論文摘要

反問題以其在自然科學和工業界的廣泛用途而著稱。對於它們的數學研究常常會導致在所謂阿達馬意義下不適定的數學模型。對數值分析而言，為了克服不適定問題內在的不穩定性，一般的正則化理論被發展起來。但是，由於反問題的多樣性與獨特性，針對具體問題而設計的數值方法在提高計算效率與穩定性方面更具潛力。本論文將以反散射問題為例，分析並提出幾種基於優化理論的數值方法。

反散射問題是通過發射和測量傳播波來探測和研究散射體或散射介質的性質。這一過程可以用帶初、邊值條件的偏微分方程（如亥姆霍茲方程，麥克斯韋方程等）來描述。最近幾十年中，為了保證解的存在唯一性而改進數學模型的努力從未停止。這些結果指導著實際設計與數值重構工作。對於在數值計算中經常使用的一個簡化模型，也即有屆區域內帶吸收邊界條件的模型，本文給出了一個唯一性的結果為以後的數值分析作准備。

反散射問題一般來說可以被表述為帶偏微分方程約束的優化問題。從而對它們的數值研究可借鑒成熟的優化理論。考慮到問題的結構特征，本文具體分析並改進了兩種優化方法。首先，從一種計算上效率較高的遞歸線性化方法出發，我們分析了一種無約束優化方法——最速下降法。通過發掘這一方法的性質，我們指出進一步改進的方向。之後，從約束優化角度，並注意到方程中的雙線性項，我們分析了一種具有二階收斂性的增廣拉格朗日方法。在合理假設條件下推導出最優化條件。此外，我們提出了一種改進算法來節省計算開銷。

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Contents

Chapter 1

Introduction

1.1 Overview of the subject

This thesis is devoted to the numerical study of some inverse problems by optimization methods while demonstrates their robustness and efficiency. Inverse problems come from many branches of natural sciences, such as geophysics, computerized tomography, antenna design and optimal control etc. When casted into mathematical models, these inverse problems usually manifest a theoretical and numerical challenge - ill-posedness. In the sense of Hadamard(1923), a well-posed problem means (i) the problem is solvable in the class of possible solutions; (ii) its solution is unique in this class; (iii) its solution is stable in this class with respect to admissible perturbations of the ingredients of the problem. Otherwise, a problem is called ill-posed. From mathematical point of view, the existence of solutions depends on the definition of solution spaces. And the uniqueness depends on how large the solution space is, while the stability concerns the topology of solution spaces.

Due to practical limitations(insufficient measurements, noises etc), ill-posedness is prevalent in inverse problems. Lack of uniqueness or existence indicates incompleteness of physical models and should be corrected. There are numerous efforts on establishing sufficient conditions to guarantee the uniqueness and existence of physically meaningful solutions. For numerics and real applications, however, stability is of special value since it allows the approximation procedure to work. In this respect, the regularization theory developed by Tikhonov A N [TA] plays an unreplaceable role in the theory of ill-posed problems. The basic idea is to introduce a bounded, so-called regularization operator to approximate unbounded operators. Henceforth, finite dimension approximation follows from the well-developed bounded operator theory, while the error caused by regularization can be estimated. Even with the help of regularization to counteract the ill-posedness, there still remain difficulties for various inverse problems, such as establishing more proper models and improving computational efficiencies etc. In this thesis, we will mainly focus on one important class of inverse problem - the inverse medium scattering problem as an example. We try to put them into proper optimization models and analyze their properties. Before that, let us briefly review some developments of this problem.

The inverse scattering problem is a primary model for lots of practical applications, such as radar, sonar, medical imaging and nondestructive testing. Here we would consider the case of scattering of time-harmonic acoustic waves by either an obstacle or a penetrable inhomogeneous medium. Later in Chapter 2, the detailed mathematical setting will be illustrated. In the past twenty years, theoretical results concerning the unique determination were developing very fast, e.g. [CoK]. Among them, one representative methodology is to exploit the behavior of the fundamental solution to the Helmholtz equation [CoK1]. Within this regime, numerical methods such as the point-source method, the probe methods [P] and the linear sampling method [CoKi] are relatively mature and well behaved, particularly for the obstacle scattering case. For example, the linear sampling method(LSM) tries to find an indicator function that tends to infinity near the boundary of obstacles and stays bounded elsewhere. By plotting the indicator function, the profile of obstacles appears. Although it possesses some intrinsic drawbacks like computational burden of sampling and heuristic choices of cut-off value, the LSM is generally workable and promising for accurate recoveries. Many researchers are then spurred to justify or improve the method, including the factorization method [CoK1] as well as some fast implementations [ABP], [LLZ]. These kinds of methods may also be applied to the medium scattering case, however, only the support can be detected. As a result, the LSM could be employed to recover piecewise constant medium functions. For general media, the above methods seems not advantageous at all. Actually, the reconstruction of obstacles mainly concerns the boundary, which is of lower dimension. For medium functions, we need to know the support(boundary) and (more importantly) the value.

For the inverse medium scattering problem, there are fewer breakthroughs, especially in the numerical regime. Since the obstacle methodology could hardly be applied as pointed above, researchers are trying fresh ideas. Of existing methods, the recursive linearization method(RLM)[BL] works well for many scattering problems. Later, we would continue the effort to analyze some properties of this method from the optimization point of view. In fact, formulating the scattering problem into optimization models can relieve many difficulties, which propels our efforts towards this direction.

1.2 Motivation

Because of the variety and difficulties arising in inverse problems, we need new techniques and fresh perspectives. Actually, too general frameworks omit problem features which may play important roles for inverse problems. Therefore, our motivation comes from mathematical structures rooted in specific inverse problems.

Mathematics reveals the structure and the underlying pattern of a problem. In computational field, delicate consideration for mathematical structures often generates efficient and stable numerical schemes. For example, the symplectic schemes for Hamiltonian systems that preserves the symplectic structure and the edge element method that preserves the de Rham diagram both result in much more stable numerical methods. In this thesis, we would exploit two structural features of the inverse medium scattering problem. After that, proper optimization methods could be selected to carry out efficient and robust numerical algorithms and theoretical results.

First, the scattering problem is associated with a two parameter function - the incident wave $u^i(k, d)$, where k is the wave number and d is the incident direction. In application, they will be discretized to sufficient many. Direct applying optimization method will lead to large matrixes which are difficult to store and solve in computers. However, the incident directions and wave numbers can be ordered by a continuation method [BL]. We will make use of this feature to break down the large problem to small pieces which improves the computational efficiency.

Second, the inverse medium scattering problem is generally nonlinear in essence. However, the non-linearity comes from the low order term qu^s, which is actually bilinear with respect to q and u^s. This natural structure implies possible facilities for analysis and numerical designs. Indeed early in [N1], a linearization approach was studied to utilize this feature. In this thesis, we would first adopt a proper optimization model that suggested in [IK]. After that, the augmented Lagrangian multiplier method is analyzed carefully. In fact, the bilinear term provides what we need to verify the optimality conditions. The optimization methods can also be applied to the discretized problem like in [HAO], and discuss the numerical efficiency. Our approach focus more on the theoretical part.

In this thesis, the main contribution, as the title suggested, is to analyze the inverse scattering problem by proper optimization methods. In chapter 2, the mathematical formulation and basic theoretical background of the inverse medium scattering problem are brief reviewed. An novel uniqueness result is established for the model proposed. In Chapter 3, after briefly reviewing the recursive linearization method, we reexamine the method and suggeste some improvements. In Chapter 4, a new optimization model are employed and the augmented Lagrangian method is analyzed to give a second order algorithm. At last, Chapter 5 summarizes this thesis and points out some research directions.

Chapter 2

Inverse Medium Scattering Problem

In this chapter, we will describe the mathematical setting of the inverse medium scattering problem. A reduced model is introduced and some potential applications are illustrated. Related to the numerical analysis, some theoretical results are discussed, and the variational formulation is derived for analysis.

2.1 Mathematical Formulation

Generally speaking, the scattering problem is to study behaviors of wave propagation through mediums or obstacles, while the inverse scattering problem intends to study properties of mediums or obstacles from those wave behaviors. There are many literatures([CoK], [Is], [K], [Ih] etc) devoted to the mathematical formulation as well as backgrounds of the problem, including acoustic scattering, electromagnetic scattering, elastic scattering etc. Here we will derive an reduced model.

Let $u(x) = u_1(x) + iu_2(x) \in H^1(\mathbb{R}^N), N = 2,3$ be the total field that is governed by the Helmholtz equation

$$\Delta u(x) + k^2(1 + q(x))u(x) = 0, \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

where $k > 0$ is the wave number, $q(x)$ is the medium function. For the inhomogeneous medium, we make following assumptions that $q(x) > -1$ is a real L^∞ function, and $q(x)$ is compactly supported.

Let $u^i(x) = e^{ikx\cdot d} \in L^\infty(\mathbb{R}^N)$ be the incident field in direction $d \in D = \{x \in \mathbb{R}^N : |x| = (\sum_{i=1}^N x_i^2)^{1/2} = 1\}$. It satisfies the homogeneous Helmholtz equation in $\mathbb{R}^N$

$$\Delta u^i(x) + k^2 u^i(x) = 0. \tag{2.2}$$

From equation (2.1) and (2.2), we can derive an equation for the scattered field $u^s(x) = u(x) - u^i(x)$,

$$\Delta u^s(x) + k^2(1 + q(x))u^s(x) = -k^2 q(x) u^i(x). \tag{2.3}$$

In the free space $\mathbb{R}^N$, the scattered field u^s is required to satisfy the Sommerfeld radiation condition:

$$\lim_{r\to\infty} r^{(N-1)/2}(\frac{\partial u^s}{\partial r} - iku^s) = 0, \quad r = |x|, \tag{2.4}$$

uniformly along all directions in D. This condition describes that waves are propagating to the infinity and will not be reflected back. It guarantees the uniqueness of the solution u^s.

2.1.1 Absorbing Boundary Conditions

When solving the scattering problem by such as the finite element method, it is convenient to truncate $\mathbb{R}^N$ to a finite domain Ω. And consequently, proper boundary conditions should be imposed on the artificial boundary $\partial\Omega$. In view of the Sommerfeld radiation condition, these conditions should minimize nonphysical reflecting waves from the boundary, hence are called non-reflecting boundary conditions(NRBC) or absorbing boundary conditions(ABC). Various ABCs can be divided into non-local ABC and local ABC. For the finite element method, local ABCs are easier to implement. Following [J], we will derive some local ABCs frequently used later. The same order ABCs may appear different by other derivations like the localization of pseudodifferential operators([EM]). More detailed descriptions can be found in [Ih], [J], [G] etc.

In $\mathbb{R}^2$, starting from the famous Wilcox expansion for the solution of the Helmhotz equation

$$u^s = \frac{e^{ikr}}{\sqrt{r}} \sum_{n=0}^{\infty} \frac{a_n(\theta)}{r^n}, \tag{2.5}$$

where (r, θ) is the polar coordinates. Define a sequence of operators by the recurrence relation

$$\mathcal{B}_m = (\frac{\partial}{\partial r} - ik + \frac{4m-3}{2r})\mathcal{B}_{m-1}, \tag{2.6}$$

with $\mathcal{B}_0 = 1$. It can be verified that the operator $\mathcal{B}_m$ cancellates the first m terms in (2.5) and gives

$$\mathcal{B}_m(u^s) = \mathcal{O}(r^{-2m-1/2}). \tag{2.7}$$

Therefore, the first and second order ABC are respectively

$$\mathcal{B}_1 = \frac{\partial}{\partial r} - ik + \frac{1}{2r}, \tag{2.8}$$

$$\mathcal{B}_2 = \frac{\partial}{\partial r} - ik + \frac{1}{2r} - \frac{1}{8r(1-ikr)} - \frac{1}{2r(1-ikr)}\frac{\partial^2}{\partial \theta^2}. \tag{2.9}$$

As expected, the finite domain approximation becomes better as r grows and higher order ABCs are taken. However, for numerical discretization, larger domain will result in more unknowns, while higher order ABCs involve higher order derivatives which are complicated to implement. Here and after, we take a bounded artificial domain $\Omega = [-L_1, L_1] \times [0, L_2]$, and impose the first-order absorbing boundary condition:

$$\frac{\partial u^s}{\partial n} - iku^s = 0, \quad \text{on } \partial\Omega, \tag{2.10}$$

where n is the normal direction to the boundary $\partial\Omega$. Of course, the region Ω should be chosen large enough to cover the support of $q(x)$. Under these assumptions, we can take $q(x) \in L_0^\infty(\Omega)$ directly. And also, $u^i \in L^2(\Omega)$.

Remark 2.1.1 *For the Maxwell's equation in $\mathbb{R}^3$ to be discussed in section 2.1.2, similar procedures can carry out corresponding ABCs. Following notations there, the first two ABCs are*

$$\mathcal{B}_1 = \vec{r} \times \nabla \times E^s + ikE_t, \tag{2.11}$$

$$\mathcal{B}_2 = \vec{r} \times \nabla \times E^s + ikE_t - \frac{r}{2(1-ikr)}\nabla \times (\vec{r}(\nabla \times E^s)_r) - ik\nabla_t E_r^s \tag{2.12}$$

where the subscription t, r stands for traverse and radial components of a vector field.

With the help of ABC, the reduced model we adopt for later analysis is the following equation:

$$\Delta u^s(x) + k^2(1 + q(x))u^s(x) = -k^2 q(x)u^i(x), \quad \text{in } \Omega, \tag{2.13}$$

$$\partial_n u^s - iku^s = 0, \quad \text{on } \partial\Omega. \tag{2.14}$$

And we call such solution u^s the ABC scattered field.

Given a medium $q(x)$ and a set of plane waves $u^i(x)$, we can measure the corresponding Dirichlet boundary values of the scattered field $u^s(x)|_{\partial\Omega}$. From these values, the inverse problem is to determine $q(x)$. In the following, we always assume $q(x)$ can be uniquely identified by enough information. An inspiring unique determination result will be discussed in section 2.2.2.

2.1.2 Applications

Many important applications share the same underlying structure to our model problem. Next, we will describe two representatives. For more backgrounds and applications, please refer to [BL], [BL1] and [BL2].

Application 1: Medium scattering by electromagnetic waves

Consider the time harmonic Maxwell's equation in $\mathbb{R}^3$,

$$\nabla \times E = i\omega\mu^* H, \tag{2.15}$$

$$\nabla \times H = -i\omega\epsilon^* E. \tag{2.16}$$

Here, E and H are the total electric field and magnetic field respectively. $\omega > 0$ is the frequency. ϵ^* and μ^* are the electric permittivity and magnetic permeability respectively. Let $\epsilon_0 > 0$ and μ_0 be the permittivity and permeability of the vacuum. Assume $\mu^* = \mu_0$, rewrite $\epsilon^* = \epsilon_0\epsilon$, $\epsilon = 1 + q(x)$. Here $q(x)$ may be complex, and the imaginary part means the medium is absorbing. Also, $q(x)$ is assumed to be compactly supported and $\Re(q(x)) > -1$.

By eliminating the magnetic field H, we get

$$\nabla \times \nabla \times E - k^2\epsilon E = 0, \tag{2.17}$$

where $k = \omega\sqrt{\epsilon_0\mu_0}$ is the wave number. Suppose the medium is illuminated by the normalized plane wave

$$E^i = ik\vec{p}e^{ikx\cdot n}, \qquad n \in S^2, \vec{p} \in S^2, \vec{p}\cdot n = 0, \tag{2.18}$$

where S^2 is the unit sphere in $\mathbb{R}^3$. Such plane wave satisfies

$$\nabla \times \nabla \times E^i - k^2 E^i = 0, \quad \text{in } \mathbb{R}^3. \tag{2.19}$$

Since $E = E^i + E^s$, we have

$$\nabla \times \nabla \times E^s - k^2(1 + q(x))E^s = k^2 q(x)E^i. \tag{2.20}$$

For the electromagnetic waves, the Sommerfeld radiation condition is replaced by the Silver-Müler radiation condition,

$$\lim_{r\to\infty} r(\nabla \times E^s \times \frac{x}{r} - ikE^s) = 0. \tag{2.21}$$

By introducing an artificial surface, the first order absorbing boundary condition gives

$$\nu \times \nabla \times E^s + ik\nu \times \nu \times E^s = 0, \tag{2.22}$$

where ν is the outer normal direction.

Application 2: Incident waves with spacial frequency

In some applications, special incident waves are preferred. Considering acoustic medium scattering problem in $\mathbb{R}^2$, the scatterer is illuminated by a one-parameter family of plane waves at fixed wave number k_0. Let $\vec{k} = (\eta, k(\eta))$ and

$$k(\eta) = \begin{cases} \sqrt{k_0^2 - \eta^2}, & k_0 \geq |\eta|, \\ i\sqrt{\eta^2 - k_0^2}, & k_0 \leq |\eta|, \end{cases}$$

where η is called the spacial frequency. Now, the incident wave adopted is $u^i = e^{i\vec{k}\cdot x}$, i.e.

$$u^i(x_1, x_2) = \begin{cases} \exp(i(\eta x_1 + \sqrt{k_0^2 - \eta^2} x_2)), & k_0 \geq |\eta|, \\ \exp(i\eta x_1 - \sqrt{\eta^2 - k_0^2} x_2), & k_0 \leq |\eta|. \end{cases}$$

It can be seen that when $|\eta| \leq k_0$, the wave is propagating. Otherwise, the wave is evanescent and can only penetrate a thin layer, as the figure shows. This particular plane wave can be generated at the interface of two media by total internal reflection, and has primally been used in near-field optics.

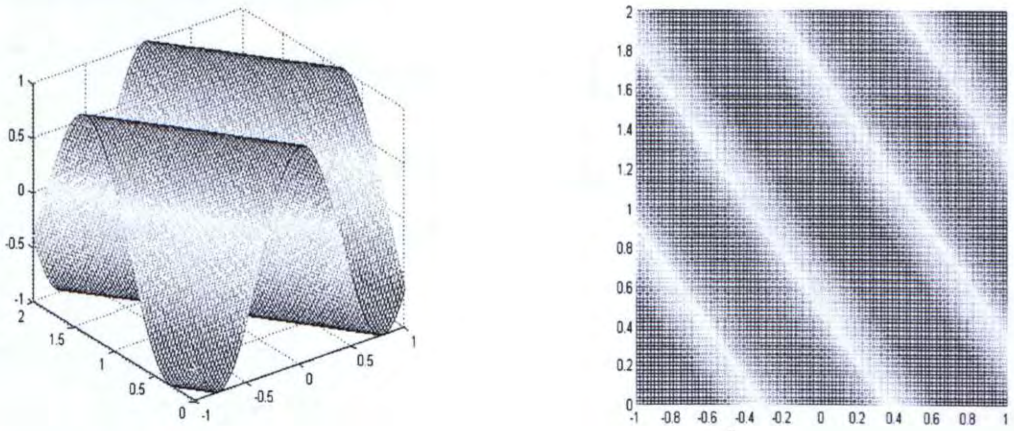


Figure 2.1: Plot of propagating wave, k = 5, $\eta = 4$

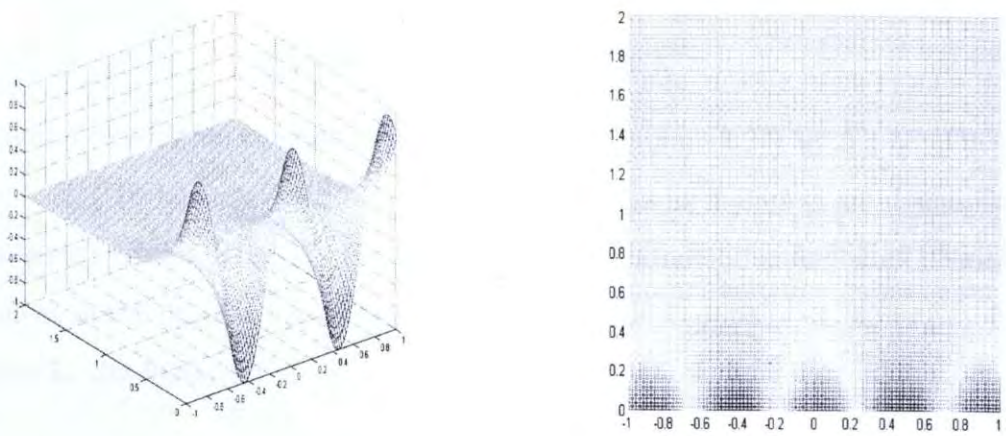


Figure 2.2: Plot of evanescent wave, k = 5, $\eta = 7$

2.2 Preliminary Results

2.2.1 Weak Formulation

We intend to base the later analysis on weak formulation, and progress systematically. By multiplying $\phi \in H^1(\Omega)$ and integrating by part, we can derive the weak formulation for (2.13), (2.14)

$$\int_{\Omega} \Delta u^s \bar{\phi} + \int_{\Omega} k^2(1+q)u^s\bar{\phi} = -k^2 \int_{\Omega} qu^i\bar{\phi} \tag{2.23}$$

$$\int_{\Omega} \nabla u^s \cdot \nabla \bar{\phi} - \int_{\Omega} k^2(1+q)u^s\bar{\phi} - \int_{\partial\Omega} iku^s\bar{\phi} = k^2 \int_{\Omega} qu^i\bar{\phi}, \tag{2.24}$$

where the boundary condition is used.

Let

$$a(u,v) = (\nabla u, \nabla v) - k^2((1+q)u, v) - ik\langle u, v\rangle, \tag{2.25}$$

and

$$b(v) = k^2(qu^i, v), \tag{2.26}$$

where $(u,v) = \int_{\Omega} u\bar{v}$ and $\langle u, v\rangle = \int_{\partial\Omega} u\bar{v}$. Then the weak formulation is to find $u^s \in H^1(\Omega)$ such that

$$a(u^s, \phi) = b(\phi), \quad \forall \phi \in H^1(\Omega). \tag{2.27}$$

Denote the solution operator to the weak formulation (2.27) by $\mathcal{S} : L^\infty(\Omega) \times L^2(\Omega) \to H^1(\Omega)$, i.e. $u^s = \mathcal{S}(q, u^i)$. Properties for this formulation are discussed in literature such as [BL]. We list several results related to further work. Since some ideas in the proof are used later, and also for the completeness, short proofs are given in the following.

Lemma 2.2.1 *Given $q \in L^\infty(\Omega)$, the variational problem admits a unique weak solution in $H^1(\Omega)$. And the estimate*

$$||\mathcal{S}(q,u^i)||_{H^1(\Omega)} \leq c||q||_{L^\infty(\Omega)}||u^i||_{L^2(\Omega)} \tag{2.28}$$

holds, where the constant c depends on k and Ω.

Proof: For the uniqueness, we only need to prove $u^s = 0$ for $u^i = 0$. By Green's formula

$$0 = \int_{\Omega} \Delta\overline{u^s}u^s - \Delta u^s\overline{u^s} = \int_{\partial\Omega} \partial_n\overline{u^s}u^s - \partial_n u^s\overline{u^s} = -2ik\int_{\partial\Omega} |u^s|^2. \tag{2.29}$$

Thus $u^s = 0$ on $\partial\Omega$, and $\partial_n u^s = 0$ on $\partial\Omega$ from the absorbing boundary condition. By the Holmgren uniqueness theorem, $u^s = 0$ in $\mathbb{R}^N \backslash \Omega$. By a unique continuation result [JK], $u^s = 0$ in Ω. Therefore, u^s is unique solvable.

For the continuity of the operator $\mathcal{S}$, we first split $a(u, v) = a_1(u, v) + k^2 a_2(u, v)$, where

$$a_1(u, v) = (\nabla u, \nabla v) - ik\langle u, v\rangle, \tag{2.30}$$

$$a_2(u, v) = -((1 + q(x))u, v). \tag{2.31}$$

Then a_1 is coercive as following

$$\begin{aligned} |a_1(u, u)| &= (||\nabla u||^4_{L^2(\Omega)} + k^2||u||^4_{H^{1/2}(\partial\Omega)})^{1/2} \\ &\geq \frac{\sqrt{2}k}{2(1+k)}(||\nabla u||^2_{L^2(\Omega)} + ||u||^2_{H^{1/2}(\partial\Omega)}) \\ &\geq c||u||^2_{H^1(\Omega)}. \end{aligned}$$

Here and after, c is a generic constant depends on k.

Next, let $\mathcal{A} : L^2(\Omega) \to H^1(\Omega)$ be such that $a_1(\mathcal{A}u, v) = a_2(u, v), \forall v \in H^1(\Omega)$. From Lax-Milgram theorem, we have

$$||\mathcal{A}u||_{H^1(\Omega)} \leq c||1 + q||_{L^\infty(\Omega)}||u||_{L^2(\Omega)}. \tag{2.32}$$

Therefore, $\mathcal{A}$ is a bounded operator form $L^2(\Omega)$ to $H^1(\Omega)$. By the compact imbedding of H^1 into L^2, $\mathcal{A}$ is compact from $L^2(\Omega)$ to $L^2(\Omega)$.

Let $\omega \in H^1(\Omega)$ be such that $a_1(\omega, v) = b(v), \forall v \in H^1(\Omega)$. Again by Lax-Milgram theorem, we have $||\omega||_{H^1(\Omega)} \leq c||q||_{L^\infty(\Omega)}||u^i||_{L^2(\Omega)}$. Using the operator $\mathcal{A}$, we see that the original variational problem (2.27) is equivalent to

$$(\mathcal{I} + k^2\mathcal{A})u^s = \omega. \tag{2.33}$$

Now, by the Fredholm alternative, we can conclude that $||u^s||_{L^2(\Omega)} \leq c||\omega||_{L^2(\Omega)} \leq c||\omega||_{H^1(\Omega)}$. From $u^s = \omega - k^2\mathcal{A}u^s$, we have

$$||u^s||_{H^1(\Omega)} \leq c||\omega||_{H^1(\Omega)} \leq c||q||_{L^\infty(\Omega)}||u^i||_{L^2(\Omega)}. \tag{2.34}$$

This finishes the proof. □

Remark 2.2.1 *From the proof we notice that the dependence of c on k is complicated by the usage of the Fredholm alternative. Actually, when k is sufficiently small, such dependence can be clarified as in [BL]. For general cases, however, this method is frustrated.*

A direct application of lemma 2.2.1 tells the continuity of operator $\mathcal{S}$ with respect to q.

Corollary 2.2.1 *Assume that* $q_1, q_2 \in L^\infty(\Omega)$, $u^i \in L^2(\Omega)$ *then*

$$||\mathcal{S}(q_1, u^i) - \mathcal{S}(q_2, u^i)||_{H^1(\Omega)} \leq c||q_1 - q_2||_{L^\infty(\Omega)}||u^i||_{L^2(\Omega)},$$

where the constant c depends on k, Ω *and* $||q_2||_{L^\infty(\Omega)}$ *(or* $||q_1||_{L^\infty(\Omega)}$*).*

Proof: Let $u_1^s = \mathcal{S}(q_1, u^i)$ and $u_2^s = \mathcal{S}(q_2, u^i)$. $\forall \phi \in H^1(\Omega)$, the weak formulation (2.27) are respectively

$$(\nabla u_1^s, \nabla \phi) - k^2((1+q_1)u_1^s, \phi) - ik\langle u_1^s, v\rangle = k^2(q_1 u^i, \phi), \qquad (2.35)$$

$$(\nabla u_2^s, \nabla \phi) - k^2((1+q_2)u_2^s, \phi) - ik\langle u_2^s, v\rangle = k^2(q_2 u^i, \phi). \qquad (2.36)$$

Let $\delta u^s = u_1^s - u_2^s$. Subtracting these two equations gives

$$(\nabla \delta u^s, \nabla \phi) - k^2((1+q_1)\delta u^s, \phi) - ik\langle \delta u^s, v\rangle = k^2((q_1 - q_2)(u^i + u_2^s), \phi). \quad (2.37)$$

Now, according to lemma 2.2.1,

$$\begin{aligned} ||\delta u^s||_{H^1(\Omega)} &\leq c||q_1 - q_2||_{L^\infty(\Omega)}(||u^i||_{L^2(\Omega)} + ||u_2^s||_{H^1(\Omega)}) \\ &\leq c||q_1 - q_2||_{L^\infty(\Omega)}||u^i||_{L^2(\Omega)}. \end{aligned}$$

This finishes the proof. □.

Later in Chapter 4, we need regularity results for another set of equation.

Lemma 2.2.2 *Given $f \in H^1(\Omega)^*$, there exists a unique weak solution $\omega \in H^1(\Omega)$ to (2.13), (2.14):*

$$-\Delta\omega + \omega = f, \quad in \quad \Omega, \tag{2.38}$$

$$\partial_n \omega = 0, \quad on \quad \partial\Omega. \tag{2.39}$$

and the estimate holds for some constant c

$$||\omega||_{H^1(\Omega)} \leq c||f||_{H^1(\Omega)^*}.$$

2.2.2 About the Unique Determination

For the inverse problem we are considering, it is of theoretical and practical importance to know the sufficient information for uniquely determining a medium. In fact, it prescribes the sufficient incident waves one should send as well as measurement point one should place. For scattering problem in $\mathbb{R}^3$ with radiation condition, such kind of results are plentiful (e.g. [Is], [CoK], [K]). The two dimensional case seems more challenging. Meanwhile, it is evident that our reduced model possesses an essential different structure: it is a bounded value problem. Therefore, we need to study the unique determination for the ABC scattered field, i.e. the solution to (2.13), (2.14). Fortunately, from recent progresses [IUY], such result is attainable under some conditions on $q(x)$.

The main theorem in [IUY] proved for a two dimensional bounded domain that the Cauchy data for the Schrödinger equation measured on an arbitrary open subset of the boundary determines uniquely the potential. Specifically, let $\tilde{\Gamma} \subset \partial\Omega$ be a non-empty open subset of the boundary. Denote $\Gamma_0 = \partial\Omega\backslash\tilde{\Gamma}$. Let $q_j \in C^{1+\alpha}(\Omega)$, $j = 1, 2$ for some $\alpha > 0$. Consider the following sets of Cauchy data on $\tilde{\Gamma}$:

$$\mathcal{C}_{q_j} = \{(u|_{\tilde{\Gamma}}, \frac{\partial u}{\partial n}|_{\tilde{\Gamma}}) \mid (\Delta + q_j)u = 0 \text{ on } \Omega, u|_{\Gamma_0} = 0, u \in H^1(\Omega)\}, \quad j = 1, 2.$$

Theorem 2.2.1 *Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. Then $q_1 = q_2$.*

This result is strong in the sense that only partial Dirichlet data are needed. However, the theorem requires the Dirichlet data to be supported on $\tilde{\Gamma}$, which is not the case we are considering. Therefore, we give up partial recovering and take $\tilde{\Gamma} = \partial\Omega$, $\Gamma_0 = \emptyset$. Also, we take the artificial domain Ω to be $B(0,r)$, which is a ball centered at 0 with radius $r > 0$.

In order to utilize this theorem, we need to work on the total field $u = u^s + u^i$, which satisfies

$$\Delta u + k^2(1 + q(x))u = 0, \quad \text{in } \Omega, \tag{2.40}$$

$$\partial_n u - iku = \partial_n u^i - iku^i, \quad \text{on } \partial\Omega. \tag{2.41}$$

As before, we call the solution u the ABC total field. Now, we prove the next density lemma.

Lemma 2.2.3 *Suppose k^2 is not an eigenvalue of $-\Delta$. For $q(x) \in L^\infty(\Omega)$, the span of ABC total field $u(k,d)$, $d \in D$ is $L^2(\Omega)$ dense in the possible solution set $Q = \{u \in H^1(\Omega) \mid (\Delta + k^2 q)u = 0 \text{ on } \Omega\}$. That is $\overline{span\{u(k,d), d \in D\}}^{L^2(\Omega)} = Q$.*

Proof: We prove the lemma by contradiction. If $u(k,d)$ is not complete in Q, there must exist some $u_0 \in Q$ such that $u_0 \notin \overline{span\{u(k,d)\}}$. According to the Hahn-Banach theorem(e.g. [Y]), there exists a $f \in L^2(\Omega)$ such that $\int_\Omega f\overline{u}(k,d) = 0, \forall d \in D$, but not for $u_0 \in Q$. Then let ω be the solution to

$$\Delta\omega + k^2(1 + q(x))\omega = f, \quad \text{in } \Omega, \tag{2.42}$$

$$\partial_n \omega - ik\omega = 0, \quad \text{on } \partial\Omega. \tag{2.43}$$

The existence of w is guaranteed by lemma 2.2.1 later. Thus by Green's formula,

$$0 \quad = \int_\Omega f\overline{u}(k,d)$$

$$
\begin{aligned}
&= \int_\Omega (\Delta\omega + k^2(1+q(x))\omega)\overline{u} \\
&= \int_\Omega (\Delta\overline{u} + k^2(1+q(x))\overline{u})\omega + \int_{\partial\Omega} \partial_n\omega\overline{u} - \partial_n\overline{u}\omega \\
&= \int_{\partial\Omega} \partial_n\omega\overline{u} - \partial_n\overline{u}\omega.
\end{aligned}
$$

Then by the boundary condition of ω and u, we further have

$$
\begin{aligned}
0 \quad &= \int_{\partial\Omega} \partial_n\omega\overline{u} - \partial_n\overline{u}\omega = \int_{\partial\Omega} ik\omega\overline{u} - \partial_n\overline{u}\omega \\
&= \int_{\partial\Omega} ik\omega\overline{u^i} - \partial_n\overline{u^i}\omega = \int_{\partial\Omega} \partial_n\omega\overline{u^i} - \partial_n\overline{u^i}\omega.
\end{aligned}
$$

Let ω_0 be the solution to the homogeneous Helmholtz equation $\Delta\omega_0 + k^2\omega_0 = 0$ in Ω with Dirichlet boundary condition $\omega_0 = \omega$ on $\partial\Omega$. Remember that u^i satisfies the homogeneous Helmholtz equation too, we have

$$
\begin{aligned}
0 \quad &= \int_\Omega (\Delta\omega_0 + k^2\omega_0)\overline{u^i} \\
&= \int_\Omega (\Delta\overline{u^i} + k^2\overline{u^i})\omega_0 + \int_{\partial\Omega} \partial_n\omega_0\overline{u^i} - \partial_n\overline{u^i}\omega_0 \\
&= \int_{\partial\Omega} \partial_n\omega_0\overline{u^i} - \partial_n\overline{u^i}\omega_0.
\end{aligned}
$$

Thus compare the above two equations and use $\omega_0 = \omega$ on $\partial\Omega$, we have $0 = \int_{\partial\Omega}(\partial_n\omega_0 - \partial_n\omega)\overline{u^i}$. By the assumption that k^2 is not an eigenvalue for $-\Delta$, we can conclude ([Is1]) that $u^i(k,d)$ are dense in $L^2(\partial\Omega)$. As a result, we get $\partial_n\omega_0 = \partial_n\omega$ on $\partial\Omega$. From the boundary condition of ω, we have $\partial_n\omega_0 - ik\omega_0 = 0$ on $\partial\Omega$.

Now consider $\Delta\omega_0 + k^2\omega_0 = 0$. Multiplying the equation by ω_0 and do integration by part, we get

$$
-(\nabla\omega_0, \nabla\omega_0) + k^2(\omega_0, \omega_0) + ik\langle\omega_0, \omega_0\rangle = 0.
$$

Since the only imaginary part is $ik\langle\omega_0, \omega_0\rangle$, we conclude that $\omega_0 = 0$ on $\partial\Omega$. Hence $\partial_n\omega = 0$ on $\partial\Omega$. Therefore, the Dirichlet value and Neumann value of ω are also zero. Thus, by the assumption that $u_0 \in Q$, we derive

$$
\int_\Omega f\overline{u}_0 \quad = \int_\Omega (\Delta\omega + k^2\omega)\overline{u}_0
$$

$$= \int_{\Omega} (\Delta \overline{u}_0 + k^2 \overline{u}_0)\omega + \int_{\partial\Omega} \partial_n \omega \overline{u}_0 - \omega \partial_n \overline{u}_0$$
$$= 0,$$

which is a contradiction. □

With the above preparation, we have

Theorem 2.2.2 *Assume* $q_1, q_2 \in H^{2+\alpha}(\Omega)$ *for* $\alpha > 0$, *and* $u_j^s(k,d)$ *are the corresponding solutions to (2.13), (2.14) with* $u^i = e^{ikx\cdot d}, \forall d \in D$ *and some fixed* $k > 0$. *Suppose* $u_1^s = u_2^s$ *on* $\partial\Omega$, *then* $q_1 = q_2$.

Proof: First, by lemma 2.2.3 and interior Schauder-type estimates([GT])it follows that the ABC total field u is $H^1(\Omega)$ dense in Q ([Is]). And from the continuity of trace operator γ, $u|_{\partial\Omega}$ is $H^{1/2}(\partial\Omega)$ dense in $Q|_{\partial\Omega}$. Since $u_1^s = u_2^s$, from the boundary condition, we have the Cauchy data $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. By theorem 2.2.1 and the embedding of $H^{2+\alpha}$ into $C^{1+\alpha}$, $q_1 = q_2$. □

Remark 2.2.2 *In the proof, we directly employed lemma in [Is1] which is applicable for* $B(0,r)$. *For general domains, a similar lemma should be established. And also, the medium should be smooth than* H^2 *to make use of the recent progress [IUY]. For lower regularities like* L^∞, *such results are not found yet.*

This theorem simply tells us that in order to uniquely determine a medium, one need all Dirichlet data on the boundary incidented by plane waves from all directions at fixed wave number k. This provides *a priori* knowledge for numerical studies. In the following analysis and numerical experiments, we always assume that media can be uniquely determined.

Chapter 3

Unconstrained Optimization: Steepest Decent Method

This chapter starts our numerical study from a simple optimization algorithm - the steepest decent method, extracted from the existing recursive linearizaion method. From optimization perspectives, the method is reexamined and some improvements are carried out.

3.1 Recursive Linearization Method Revisited

The recursive linearization method (RLM) was first proposed by Chen Yu ([Ch]) in a numerical study of inverse scattering problems by Riccati equation method. To overcome the ill-posedness and possible local minima, Chen adopted a continuation procedure along the wave number that stabilizes the algorithm. Later, Bao, G *et al* further developed the idea and applied it to scattering problems in various situations([BHL],[BL],[BL1],[BL2]). This method is easy to understand and implement despite some disadvantages which will be discussed later. We now briefly review these methods under the model of inverse medium scattering .

Let the measurements $\Psi = \{\Psi_1, \Psi_2, \cdots\} = u^s|_{\partial\Omega}$ depend on the number of incident waves. Then define the measurement map $\mathcal{M} : L^\infty(\Omega) \times L^2(\Omega) \to H^{1/2}(\partial\Omega)$ by

$$\mathcal{M}(q, u^i) = \gamma \mathcal{S}(q, u^i), \tag{3.1}$$

where $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator. The inverse problem is to solve $q(x)$ from the operator equation

$$\mathcal{M}(q, u^i) = \gamma \mathcal{S}(q, u^i) = \Psi \tag{3.2}$$

in some sense. It is easy to see that $\mathcal{S}$ is linear with respect to u^i, but nonlinear with respect to $q(x)$. The RLM first approximate the equation by a linear one, that is to solve δq for some initial value $\tilde{q}$,

$$\mathcal{DM}(\tilde{q}, u^i)\delta q = \Psi - \mathcal{M}(\tilde{q}, u^i), \tag{3.3}$$

where $\mathcal{DM}$ is the Fréchet derivative of $\mathcal{M}$ with respect to q. Then $q = \tilde{q} + \delta q$ is an approximate solution. The viability of the method depends on the following issues: Fréchet differentiability of $\mathcal{S}$, the choice of initial values and stable, efficient solver for the linearized equation.

3.1.1 Fréchet differentiability

To verify the Fréchet differentiability, [BL] employed the first order perturbation method. Here, we directly derive from definition and variational formulation, which unifies our analysis.

According to the definition, for $q \in L^\infty(\Omega), u^i \in L^2(\Omega)$, $\mathcal{S}(\cdot,\cdot) : L^\infty(\Omega) \times L^2(\Omega) \to H^1(\Omega)$ is the solution operator to

$$(\nabla u^s, \nabla v) - k^2((1+q)u^s, v) - ik\langle u^s, v\rangle = k^2(qu^i, v), \quad \forall v \in H^1(\Omega), \tag{3.4}$$

i.e. $u^s = \mathcal{S}(q, u^i)$. For, $\delta q \in L^\infty(\Omega)$, let $\mathcal{DS}(q, u^i)(\cdot) : L^\infty(\Omega) \to H^1(\Omega)$ be the solution operator to

$$(\nabla \delta u, \nabla v) - k^2((1+q)\delta u, v) - ik\langle \delta u, v\rangle = k^2(\delta q(u^i + u^s), v), \quad \forall v \in H^1(\Omega), \tag{3.5}$$

i.e. $\delta u = \mathcal{DS}(q, u^i)\delta q$. We prove that $\mathcal{DS}$ is the Fréchet derivative of $\mathcal{S}$ with respect to $q(x)$. Before that, it is easy to see that δu is the solution of the following equation

$$\Delta \delta u(x) + k^2(1 + q(x))\delta u(x) = -k^2 \delta q(x)(u^i(x) + u^s(x)), \quad \text{in } \Omega, \tag{3.6}$$

$$\partial_n \delta u - ik\delta u = 0, \quad \text{on } \partial\Omega. \tag{3.7}$$

Lemma 3.1.1 *The operator $\mathcal{DS}$ defined in (3.2) is the Fréchet derivative of the operator $\mathcal{S}$ defined in (3.1) with respect to q.*

Proof: For any $q, \tilde{q} \in L^\infty(\Omega), \delta q = \tilde{q} - q$, we have the corresponding equation

$$(\nabla u^s, \nabla v) - k^2((1+q)u^s, v) - ik\langle u^s, v\rangle = k^2(qu^i, v), \quad \forall v \in H^1(\Omega),$$

$$(\nabla \tilde{u}^s, \nabla v) - k^2((1+\tilde{q})\tilde{u}^s, v) - ik\langle \tilde{u}^s, v\rangle = k^2(\tilde{q}u^i, v), \quad \forall v \in H^1(\Omega),$$

and it follows from (3.2) that $\delta u^s = \mathcal{DS}(q, u^i)\delta q$ satisfies

$$(\nabla \delta u, \nabla v) - k^2((1+q)\delta u, v) - ik\langle \delta u, v\rangle = k^2(\delta q(u^i + u^s), v), \quad \forall v \in H^1(\Omega)$$

Denote $w = \mathcal{S}(\tilde{q}, u^i) - \mathcal{S}(q, u^i) - \mathcal{DS}(q, u^i)\delta q$, then from the above three equations, we have

$$(\nabla w, \nabla v) - k^2((1+q)w, v) - ik\langle w, v\rangle = k^2(\delta q(\tilde{u}^s - u^s), v), \quad \forall v \in H^1(\Omega).$$

By lemma 2.2.1 and corollary 2.2.2, we have

$$\begin{aligned} ||w||_{H^1(\Omega)} &\leq c||\delta q||_{L^\infty(\Omega)}||\tilde{u}^s - u^s||_{H^1(\Omega)} \\ &= c||\delta q||_{L^\infty(\Omega)}||\mathcal{S}(\tilde{q}, u^i) - \mathcal{S}(q, u^i)||_{H^1(\Omega)} \\ &\leq c||\delta q||^2_{L^\infty(\Omega)}||u^i||_{L^2(\Omega)}, \end{aligned}$$

where c is a generic constant depends on k, Ω and $||q||_{L^\infty(\Omega)}$. This finishes the proof. $\square$

As a result, for the measurement map $\mathcal{M}$, by the linearity of γ, we have

$$\mathcal{DM} = \gamma \mathcal{DS}. \tag{3.8}$$

Thus the differentiability of $\mathcal{M}$ is confirmed.

Remark 3.1.1 *Since operator $\mathcal{M}$ and $\mathcal{S}$ are linear with respect to u^i, in the following, the dependence on u^i in operator $\mathcal{DM}$ and $\mathcal{DS}$ is omitted.*

3.1.2 Initial guess

For a qualified initial value, the RLM resorts to the so-called Born approximation. Assume the wave number k is small, then the proof of lemma 2.2.1 tells that the scattered field u^s is weak in the sense of H^1 norm. Henceforth, (2.13) becomes essentially linear. The Born approximation drops the nonlinear term from (2.13) and solves u^s_B instead, which satisfies

$$\Delta u^s_B(x) + k^2 u^s_B(x) = -k^2 q(x) u^i(x), \quad \text{in } \Omega, \tag{3.9}$$

$$\partial_n u^s_B - ik u^s_B = 0, \quad \text{on } \partial\Omega. \tag{3.10}$$

To derive the Born approximation q_B for $q(x)$, we take u^s as an approximation of u^s_B. Multiply the equation (3.9) by $\tilde{u}^i = e^{ikx\cdot\tilde{d}}$ for any $\tilde{d} \in D$ and do integration by part,

$$\int_\Omega \Delta u^s \tilde{u}^i + \int_\Omega k^2 u^s \tilde{u}^i = -\int_\Omega k^2 q_B u^i \tilde{u}^i$$

$$\int_\Omega \Delta \tilde{u}^i u^s + \int_\Omega k^2 u^s \tilde{u}^i + \int_{\partial\Omega} \tilde{u}^i \partial_n u^s - u^s \partial_n \tilde{u}^i = -\int_\Omega k^2 q_B u^i \tilde{u}^i$$

$$\int_\Omega q_B u^i \tilde{u}^i = \frac{1}{k^2} \int_{\partial\Omega} (\partial_n \tilde{u}^i - ik\tilde{u}^i) u^s.$$

Due to the special form of incident waves, the last equation $Aq_B = f$ is a Fourier transform (or Fourier-Laplace transform for evanescent wave) that can be efficiently solved by FFT.

The usage of Born approximation in scattering problems has a long history, and there are numerous literatures on this subject, especially for scattering problem in $\mathbb{R}^N$. Let $\Phi(x)$ be the fundamental solution to the Helmhotz equation

$$\Phi(x) = \begin{cases} \frac{i}{4}H_0(k|x|), & N = 2, \\ \frac{e^{ik|x|}}{4\pi|x|}, & N = 3. \end{cases}$$

Then the total field u satisfies the Lipperman-Shwinger integral equation

$$u(x) = u^i(x) - k^2 \int_{\mathbb{R}^N} q(y)\Phi(x-y)u(y)dy. \tag{3.11}$$

By neglecting u^s in the integral, the Born approximation is given by

$$u_B(x) = u^i(x) - k^2 \int_{\mathbb{R}^N} q(y)\Phi(x-y)u^i(y)dy. \tag{3.12}$$

From equations (3.7), (3.8), we can derive([K]) that for $(kr)^2||q||_\infty \leq 1$,

$$||u - u_B||_\infty \leq \frac{(kr)^4}{2}||q||_\infty^2. \tag{3.13}$$

Moreover, some numerical studies (e.g. [CS]) indicate that the Born approximation is valid if

$$kr \sup_{B(0,r)} |q(x)| < 2\pi c, \tag{3.14}$$

where c is a small constant. This observation was partially verified in a recent paper [N] by F. Natterer. Following the notations in [N], if

$$M = \overline{\gamma}_n rk \sup_{B(0,r)} |q(x)| < 1, \tag{3.15}$$

then

$$||u^s - u_B^s||_\infty \leq \frac{M}{1-M}\overline{\gamma}_n rk \sup_{B(0,r)} |q(x)|, \tag{3.16}$$

where $\overline{\gamma}_n$ is an important constant.

For the approximation q_B, define q_{2k} via the finite frequency Fourier transform

$$\hat{q}_{2k}(\xi) = \begin{cases} \hat{q}(\xi), & |\xi| \leq 2k, \\ 0, & \text{otherwise;} \end{cases}$$

then for $q(x) < 1$,

$$||q_B - q_{2k}||_\infty \le c\frac{M}{1-M}\overline{\gamma}_n k^{1+N/2}||q||_\infty^2. \tag{3.17}$$

The above results tells that Born approximation is suitable for small perturbations and downgrades very fast as the wave number k increases. For more backgrounds, analysis and applications about the Born approximation, please refer to [N], [MI], [RS] etc.

To make use of the Born approximation, the RLM starts from small wave numbers and gradually pass the initial value to larger wave numbers. That is a continuation method along the wave number direction. Employing this extra continuation direction settles the initial value together with stability. Actually, this is the fundamental idea in RLM.

3.1.3 Landweber iteration

There are many candidates for the linear solver. In RLM, the author used the (projected) Landweber iteration. Although it is notorious for slow convergence and other defects, however, as an iterative method, the Landweber iteration is easy to implement and can reduce some computational costs. Moreover, the relaxation parameter can reduce the instability as well. Numerical tests also verified the efficiency and robustness of this method. By selecting a relaxation parameter, the method takes the following iteration step

$$q_n = q_{n-1} - \tau \mathcal{D}\mathcal{M}(q^{n-1})^*(\mathcal{M}(q^{n-1}) - \Psi), \quad \text{for } n = 1, 2, \cdots \tag{3.18}$$

Here, $\mathcal{M}(q) = (M(q, u_1^i), \cdots)$ stands for the vector corresponding to incident waves. In fact, we need multiple measurements corresponding to incident directions $d_i, i = 1, 2, \cdots, N$ for fixed wave number k in view of the uniqueness result. Meanwhile, the continuation requires many measurements corresponding to an increasing sequence of k. Direct discretization will lead to a very large linear system. Therefore, the method employs a projected Landweber method that breaks down the large problem into small pieces. Especially for scattering problems, the method solves only one direct problem for a single incident wave in each iterative step. Specifically, it involves two cycles. For fixed k, the inner cycle moves from one incident direction to another, that is

$$q(\tilde{d}, k) = q(d, k) - \tau \mathcal{DM}(q(d, k))^*(\mathcal{M}(q(d, k), u^i) - \Psi(d, k)). \tag{3.19}$$

And when the inner cycle finishes, for some fixed direction d, the outer cycle moves to another wave number

$$q(d, \tilde{k}) = q(d, k) - \tau \mathcal{DM}(q(d, k))^*(\mathcal{M}(q(d, k), u^i) - \Psi(d, k)). \tag{3.20}$$

Here (d, k) is corresponding to incident waves $u^i = e^{ikx \cdot d}$.

Now, for the efficient evaluation of the adjoint operator $\mathcal{DM}^*$, an adjoint equation can be introduced. Given a reconstruction $\tilde{q}$ and corresponding $\tilde{u}^s = \mathcal{S}(\tilde{q}, u^i)$, the above procedure is to find a $q = \tilde{q} + \delta q$ such that $\mathcal{M}(q, u^i) = \Psi(d, k)$. Let $\delta u^s = \mathcal{S}(\tilde{q}, u^i)$, then it satisfies (3.6)

$$\Delta \delta u^s(x) + k^2(1 + \tilde{q}(x))\delta u^s(x) = -k^2 \delta q(x)(u^i(x) + \tilde{u}^s(x)), \quad \text{in } \Omega, \tag{3.21}$$

$$\partial_n \delta u^s - ik\delta u^s = 0, \quad \text{on } \partial\Omega, \tag{3.22}$$

and $\delta q = \tau \mathcal{DM}^*(\tilde{q})\delta u^s$. Now introduce an adjoint equation

$$\Delta \delta w(x) + k^2(1 + \tilde{q}(x))\delta w(x) = 0, \quad \text{in } \Omega, \tag{3.23}$$

$$\partial_n \delta w - ik\delta w = \delta u^s, \quad \text{on } \partial\Omega, \tag{3.24}$$

Multiply equation (3.15) by $\overline{w}$ and integrate over Ω, we then derive

$$\int_\Omega \Delta \delta u^s \overline{w} + k^2(1 + \tilde{q})\delta u^s \overline{w} = -\int_\Omega k^2 \delta q(u^i + \tilde{u}^s)\overline{w},$$

$$\int_{\partial\Omega} \partial \delta u^s \overline{w} - \delta u^s \partial \overline{w} = -\int_\Omega k^2 \delta q(u^i + \tilde{u}^s)\overline{w},$$

$$\int_{\partial\Omega} \delta u^s \overline{\delta u^s} = \int_{\Omega} k^2 \delta q (u^i + \tilde{u}^s) \overline{w}.$$

Considering $\delta u^s = \mathcal{DM}(\tilde{q})\delta q$ on $\partial\Omega$, we have

$$\int_{\partial\Omega} \mathcal{DM}(\tilde{q})\delta q \overline{\delta u^s} = \int_{\Omega} k^2 \delta q (u^i + \tilde{u}^s) \overline{w},$$
$$\int_{\Omega} \delta q \overline{\mathcal{DM}^*(\tilde{q})\delta u^s} = \int_{\Omega} k^2 \delta q (u^i + \tilde{u}^s) \overline{w}.$$

Since it should hold for any $\delta q \in L^2(\Omega)$, $\mathcal{DM}^*(\tilde{q})\delta u^s = k^2\overline{(u^i + \tilde{u}^s)}w$. Therefore, δq can be evaluated by solving the adjoint equation. And in each iteration step, the RLM requires to solve a direct problem together with an adjoint problem. The adjoint equation for other situations can be derived similarly.

3.1.4 Numerical Results

At last, we give an example to demonstrate the numerical success. Here, we aim to recover Medium 1 which will be specified in section 3.3. For the wave number, we take $k = 1, 2, 3, 4, 5, 6, 7$. And for each wave number, 16 equally spaced incident directions are employed. More details on the implementation will be specified in section 3.3. The following figures show the evolution of the reconstruction. As expected, small wave number reconstruction displays fewer details of the medium. When the wave number increases, the reconstruction becomes significantly better. The last figure and table displays the relative error of the reconstruction. It deserves to notice that the error decreases slowly at small k. Actually, similar situations were observed in many numerical experiments (e.g. [BL]). Moreover, the authors observed that the convergence is not sensitive to the step size of wave number.

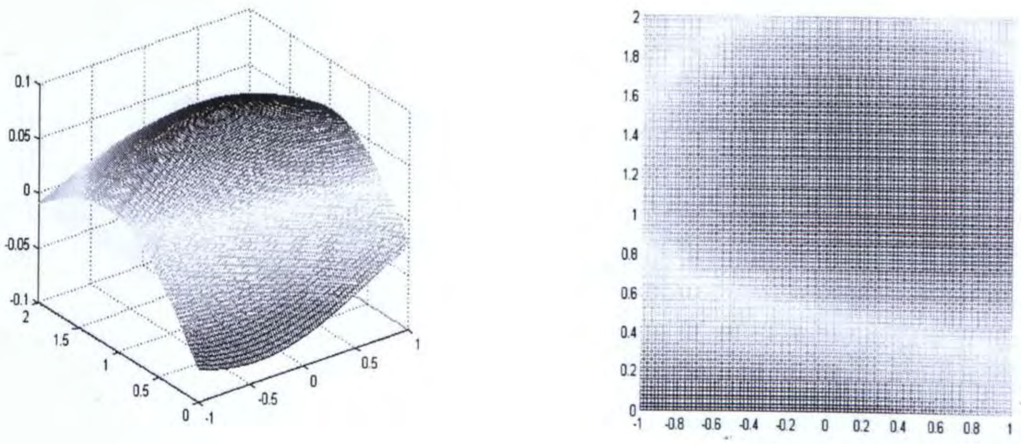


Figure 3.1: Reconstruction of q_1 at k = 1

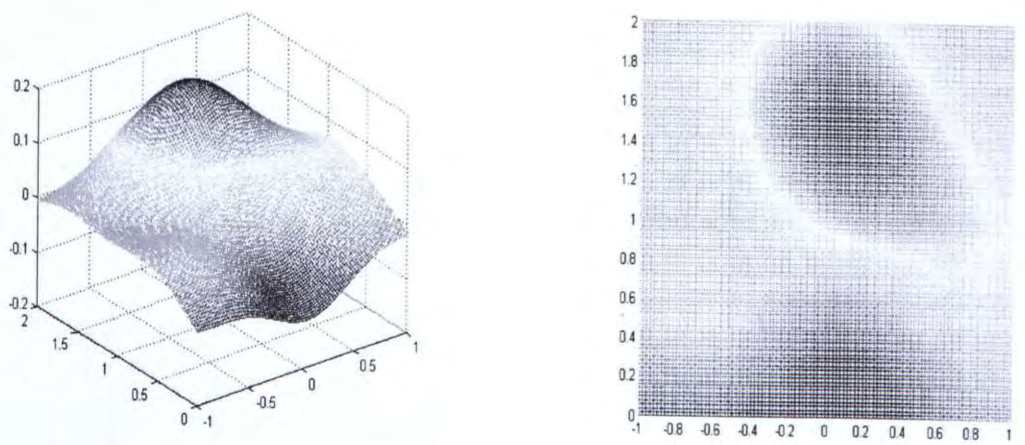


Figure 3.2: Reconstruction of q_1 at k = 3

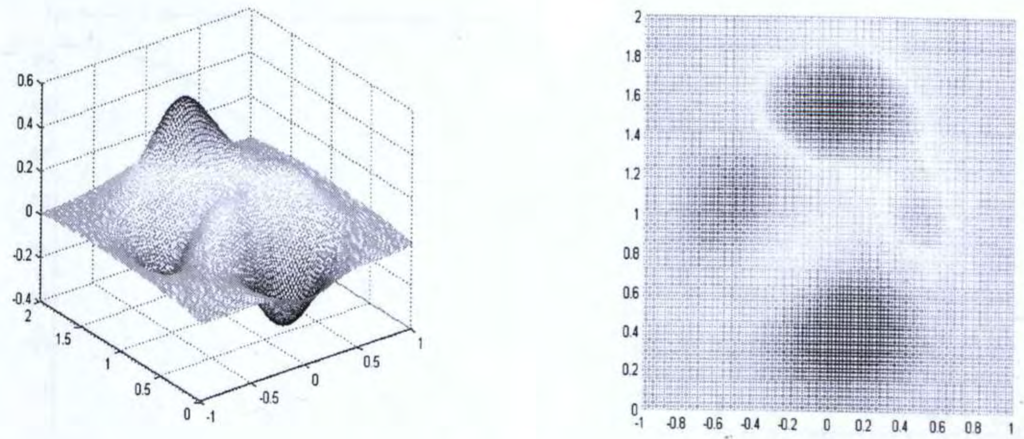


Figure 3.3: Reconstruction of q_1 at k = 5

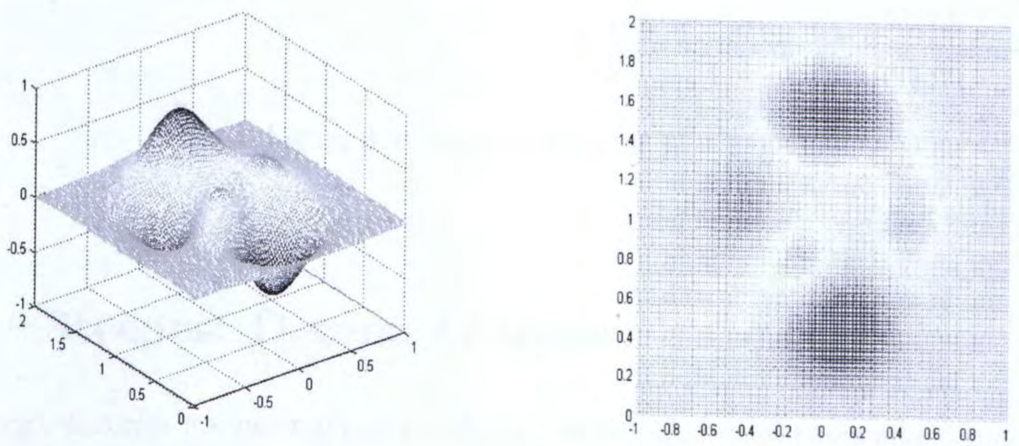
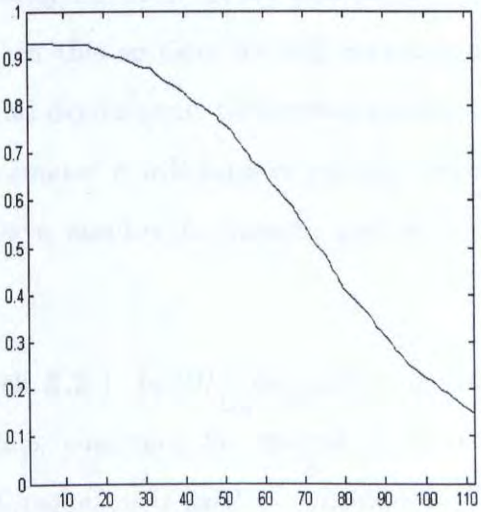


Figure 3.4: Reconstruction of q_1 at k = 7



wave number	relative error
1	0.9272
2	0.8757
3	0.7784
4	0.6143
5	0.4101
6	0.2497
7	0.1471

Figure 3.5: Relative error of q_1

3.2 Steepest Decent Analysis

Although theories are incomplete, the RLM has demonstrated tremendous successful numerical results. The major disadvantage, as can be seen, is the data redundancy. Besides the continuation of incident directions, the extra continuation needs more data(sometimes than necessary). As a result, more measurements need to be taken in real applications, and more computational costs are required for the extra data.

In view of the uniqueness result in Chapter 2, at least for smooth medium, incident waves from all directions at fixed wave number k can uniquely determine the medium. However, the RLM requires to start from small wave numbers. This discrepancy between theory and method clearly implies great room for improvements. In this section, we will reexamine the RLM from an optimization view point and derive some theoretical results. It is surprising that careful choices for the parameter τ will achieve robust convergence with much less data. Actually, fixed wave number is enough, and such numerical method matches the theory well.

Remark 3.2.1 *In [BL], the authors considered the fixed wave number case. However, they employed the special incident wave described in section 2.1.2. The spacial frequency is used as continuation direction.*

3.2.1 Single Wave Case

We first consider one incident wave case. For some incident wave $u^i = e^{ikx\cdot d}$, following notations in section 3.1, we need to solve q from

$$\mathcal{M}(q, u^i) = \Psi. \tag{3.25}$$

Then, the linearization procedure together with a Landweber iteration takes the following step

$$q_{n+1} = q_n - \tau \mathcal{D}\mathcal{M}^*(q_n)\mathcal{R}(q_n), \quad \text{for} \quad n = 0, 1, 2, \cdots \tag{3.26}$$

where $\mathcal{R}(q_n) = \mathcal{M}(q_n, u^i) - \Psi$ is the residue, $\tau > 0$.

Now, it deserves to notice that the above iteration step can be identified as a steepest decent method applied to the functional

$$\mathcal{J}(q) = \frac{1}{2}||\mathcal{M}(q, u^i) - \Psi||^2_{H^{1/2}(\partial\Omega)}, \tag{3.27}$$

with constant step size τ, as the lemma 4.2.12 modified from [K] states.

Lemma 3.2.1 *Let $\{q_n\}$ be defined by (3.26), and $\mathcal{J}$ is Fréchet differentiable such that*

$$\mathcal{J}'(q)\delta q = \Re\langle \mathcal{DM}(q)\delta q, \mathcal{M}(q, u^i) - \Psi\rangle, \tag{3.28}$$

where $\Re$ stands for the real part. Then, the linear functional $\mathcal{J}'(q)$ can be identified with $\mathcal{DM}(q)^*(\mathcal{M}(q) - \Psi)$ over the field $\mathbb{R}$. Therefore, iteration (3.26) is the steepest decent step with step size τ.

With this observation, amplitude optimization methodologies about the steepest decent method can be employed to analyze our problem. Particularly, the convergence and step size analysis are relatively easy to carry out, by comparing some results for the steepest decent method in finite dimension spaces [Be].

Theorem 3.2.1 *Let $\{q_n\}$ be the sequence generated by (3.26), and $\Gamma > 0$ be the Lipschitz constant for $\mathcal{J}' = \mathcal{DM}^*\mathcal{R}$, i.e.*

$$||\mathcal{J}'(x) - \mathcal{J}'(y)|| \leq \Gamma||x - y||, \quad \forall x, y, \tag{3.29}$$

and that there exists a scalar ϵ such that

$$0 < \epsilon < \tau \leq \frac{2-\epsilon}{\Gamma}. \tag{3.30}$$

Then every limit point $\bar{q}$ of $\{q_n\}$ is a critical point of $\mathcal{J}$, which means

$$\mathcal{J}'(\bar{q}) = 0. \tag{3.31}$$

Proof: For $n = 0, 1, 2, \cdots$, let $q_{n+1} = q_n - \tau d_n$, i.e. $d_n = \mathcal{DM}^*(q_n)\mathcal{R}(q_n)$. Then by the following equality,

$$\begin{aligned}
&\mathcal{J}(q_n - \tau d_n) - \mathcal{J}(q_n) \\
&= -\tau\Re\langle \mathcal{DM}^*(q_n)\mathcal{R}(q_n), d_n\rangle \\
&+ \int_0^\tau \Re\langle \mathcal{DM}^*(q_n)\mathcal{R}(q_n) - \mathcal{DM}^*(q_n - td_n)\mathcal{R}(q_n - td_n), d_n\rangle dt \\
&\leq -\tau||d_n||^2 + \int_0^\tau t\Gamma||d_n||^2 dt
\end{aligned}$$

$$= \tau(-1 + \frac{1}{2}\tau\Gamma)||d_n||^2.$$

By the assumption, $-1 + \frac{1}{2}\tau\Gamma \leq -\frac{1}{2}\epsilon$, therefore,

$$\mathcal{J}(q_n) - \mathcal{J}(q_{n+1}) \geq \frac{1}{2}\epsilon^2||d_n||^2.$$

For any limit point $\bar{q}$ of $\{q_n\}$, the above inequality implies $||d(\bar{q})|| = 0$. This is just

$$\mathcal{J}'(\bar{q}) = 0. \qquad \square$$

The above theorem 3.2.1 asserts that every limit point of the generated sequence is a critical point. Meanwhile, it prescribes the condition on the parameter τ in order to guarantee the validity from the viewpoint of steepest decent method. Due to the nonlinear nature, the general convergent result is difficult to get. Next, we derive some qualitative analysis towards this direction.

Since the measurement Ψ is generated by some medium, in the noise free case, we can assume there exists a q^* such that $\mathcal{M}(q^*, u^i) = \Psi$. Then,

$$\begin{aligned}
&||q_{n+1} - q^*||_{L^\infty(\Omega)} \\
&= ||q_n - q^* - \tau\mathcal{DM}^*(q_n)(\gamma\mathcal{S}(q_n, u^i) - \gamma\mathcal{S}(q^*, u^i))||_{L^\infty(\Omega)} \\
&\leq ||q_n - q^* - \tau\mathcal{DM}^*(q_n)\mathcal{DM}(q_n)(q_n - q^*)||_{L^\infty(\Omega)} + \tau||\mathcal{DM}^*||O(||q_n - q^*||^2_{L^\infty(\Omega)}) \\
&\leq ||\mathcal{I} - \tau\mathcal{DM}^*\mathcal{DM}||||(q_n - q^*)||_{L^\infty(\Omega)} + \tau||\mathcal{DM}^*||O(||q_n - q^*||^2_{L^\infty(\Omega)}).
\end{aligned}$$

To guarantee convergence, first we need to choose τ such that $\chi = ||\mathcal{I} - \tau\mathcal{DM}^*\mathcal{DM}|| < 1$. A thorough estimate of the spectral property of the operator $\mathcal{DM}$, especially the dependency on k and q_n, is a must. It is clear that previous regularity analysis in section 2.2.1 are not enough but indicate τ should be small compared to $k^2||1 + q_n||$. Next, the initial guess should be reasonably well, otherwise the nonlinear growth will dominate. Fortunately, the factor τ will relax the convergent radius.

3.2.2 Multiple Wave Case

For multiple incident waves, the analysis in section 3.2.1 works for the functional

$$\mathcal{J}(q) = \frac{1}{2}\Sigma_m ||\gamma\mathcal{S}(q, u^i_m) - \Psi_m||^2_{H^{1/2}(\partial\Omega)}. \tag{3.32}$$

However, the recursive step needs more analysis, especially for the operator $\mathcal{DM}$. By $\mathcal{DM} = \gamma\mathcal{DS}$, we turn to analyze $\mathcal{DS}$. The outer loop concerns continuation along the wave number k, while the inner loop is along the incident direction d. Therefore, we denote $\mathcal{DS}(q, u^i)$ by $\mathcal{DS}(q, k, d)$ to emphasize explicitly the dependency.

First, for the inner loop we have

Lemma 3.2.2 *For fixed k and $d, \tilde{d}$*

$$||\mathcal{DS}(q, k, \tilde{d}) - \mathcal{DS}(q, k, d)|| \le c||\tilde{d} - d||, \tag{3.33}$$

where the constant c depends on q, k and Ω.

Proof: For any $\delta q \in L^\infty(\Omega)$, let $u = \mathcal{DS}(q, k, d)\delta q$ and $w = \mathcal{DS}(q, k, \tilde{d})\delta q$, we have

$$(\nabla u, \nabla v) - k^2((1+q)u, v) - ik\langle u, v\rangle = k^2(\delta q(u^i + u^s), v), \quad \forall v \in H^1(\Omega),$$

$$(\nabla w, \nabla v) - k^2((1+q)w, v) - ik\langle w, v\rangle = k^2(\delta q(\tilde{u}^i + w^s), v), \quad \forall v \in H^1(\Omega),$$

where u^s, w^s correspond to $u^i(k, d), u^i(k, \tilde{d})$ respectively. Then we get

$$\begin{aligned}&(\nabla(u-w), \nabla v) - k^2((1+q)(u-w), v) - ik\langle u-w, v\rangle\\ &= k^2(\delta q(u^i - \tilde{u}^i + u^s - w^s), v), \quad \forall v \in H^1(\Omega)\end{aligned}$$

According to lemma 2.2.1, we have the estimate

$$\begin{aligned}||u-w||_{H^1(\Omega)} &\lesssim (||u^i - \tilde{u}^i||_{L^2(\Omega)} + ||u^s - w^s||_{L^2(\Omega)})||\delta q||_{L^\infty(\Omega)}\\ &\le (||u^i - \tilde{u}^i||_{L^2(\Omega)} + ||u^s - w^s||_{H^1(\Omega)})||\delta q||_{L^\infty(\Omega)}.\end{aligned}$$

The symbol $\lesssim$ means if $x \lesssim y$ then $x \leq cy$ for some constant c. Then corollary 2.2.1 gives

$$||u^s - w^s||_{H^1(\Omega)} \leq c||u^i - \tilde{u}^i||_{L^2(\Omega)}||q||_{L^\infty(\Omega)}.$$

Besides, a direct calculation tells that

$$\begin{aligned}
||u^i - \tilde{u}^i||^2_{L^2(\Omega)} &= \int_\Omega |e^{ikx\cdot d} - e^{ikx\cdot\tilde{d}}|^2 dx \\
&= \int_\Omega (\cos(kx\cdot d) - \cos(kx\cdot\tilde{d}))^2 + (\sin(kx\cdot d) - \sin(kx\cdot\tilde{d}))^2 dx \\
&\leq \int_\Omega 2k^2|x\cdot(d-\tilde{d})|^2 dx \\
&\lesssim ||d-\tilde{d}||^2,
\end{aligned}$$

where the last norm is the Euclidean norm, and we used $|\cos a - \cos b| \leq |a-b|$. Then combining the above estimates gives

$$\begin{aligned}
||u - w||_{H^1(\Omega)} &\lesssim ||u^i - \tilde{u}^i||_{L^2(\Omega)}||\delta q||_{L^\infty(\Omega)} \\
&\lesssim ||d-\tilde{d}||||\delta q||_{L^\infty(\Omega)}.
\end{aligned}$$

This finishes the proof. □

Next, the outer loop has a similar result but involves more work, since the underlying equation changes according to k.

Lemma 3.2.3 *For fixed d and $k, \tilde{k}$*

$$||\mathcal{DS}(q,\tilde{k},d) - \mathcal{DS}(q,k,d)|| \leq c|\tilde{k}-k|, \tag{3.34}$$

where the constant c depends on $q, k, \tilde{k}$ and Ω.

Proof: For any $\delta q \in L^\infty(\Omega)$, let $u = \mathcal{DS}(q,k,d)\delta q$ and $w = \mathcal{DS}(q,\tilde{k},d)\delta q$, we have

$$(\nabla u, \nabla v) - k^2((1+q)u, v) - ik\langle u, v\rangle = k^2(\delta q(u^i + u^s), v), \quad \forall v \in H^1(\Omega),$$

$$(\nabla w, \nabla v) - \tilde{k}^2((1+q)w, v) - i\tilde{k}\langle w, v\rangle = \tilde{k}^2(\delta q(\tilde{u}^i + w^s), v), \quad \forall v \in H^1(\Omega),$$

where u^s, w^s correspond to $u^i(k,d), u^i(\tilde{k},d)$ respectively.

Subtracting these two equations, we have

$$\begin{aligned}
&(\nabla(u-w),\nabla v) - k^2((1+q)(u-w),v) - ik\langle(u-w),v\rangle \\
&= -(k^2-\tilde{k}^2)((1+q)w,v) - i(k-\tilde{k})\langle w,v\rangle \\
&+k^2(\delta q(\tilde{u}^i+u^s),v) - \tilde{k}^2(\delta q(\tilde{u}^i+w^s),v), \quad \forall v \in H^1(\Omega).
\end{aligned}$$

From the proof of lemma 2.2.1, we have

$$\begin{aligned}
||u-w||_{H^1(\Omega)} \quad &\lesssim |k^2-\tilde{k}^2|||1+q||_{L^\infty(\Omega)}||w||_{H^1(\Omega)} + |k-\tilde{k}|||w||_{H^1(\Omega)} \\
&+|k^2-\tilde{k}^2|||\tilde{u}^i||_{L^\infty(\Omega)}||\delta q||_{L^\infty(\Omega)} + ||k^2u^s-\tilde{k}^2w^s||_{H^1(\Omega)}||\delta q||_{L^\infty(\Omega)}.
\end{aligned}$$

Now, estimates on w, u^s, w^s together with lemma 2.2.1, corollary 2.2.1 give

$$||u-w||_{H^1(\Omega)} \lesssim |k-\tilde{k}|||\delta q||_{L^\infty(\Omega)},$$

which proves the lemma. □

With above preparations, we turn to analyze the recursive linearization step. For any inner loop, the step is

$$\begin{aligned}
\tilde{q} \quad &= q - \tau\mathcal{DM}^*(q,k,d)(\mathcal{M}(q,k,\tilde{d}) - \Psi(k,\tilde{d})) \\
&= q + \tau\mathcal{DM}^*(q,k,d)\mathcal{DM}(q,k,\tilde{d})(q-q^*) + O(||q-q^*||^2),
\end{aligned}$$

and then

$$\begin{aligned}
&||\tilde{q}-q^*||_{L^\infty(\Omega)} \\
&\leq ||q-q^* - \tau\mathcal{DM}^*(q,k,d)(\mathcal{M}(q,k,\tilde{d}) - \Psi(k,\tilde{d}))||_{L^\infty(\Omega)} \\
&\leq ||q-q^* + \tau\mathcal{DM}^*(q,k,d)\mathcal{DM}(q,k,\tilde{d})(q-q^*)||_{L^\infty(\Omega)} + O(||q-q^*||^2) \\
&\leq ||\mathcal{I} - \tau\mathcal{DM}^*(q,k,d)\mathcal{DM}(q,k,d)||||(q-q^*)||_{L^\infty(\Omega)} \\
&+c\tau||\tilde{d}-d||||\mathcal{DM}^*(q,k,d)||||q-q^*||_{L^\infty(\Omega)} + O(||q-q^*||^2).
\end{aligned}$$

The outer loop admits a similar estimate but changing $||\tilde{d} - d||$ to $|\tilde{k} - k|$. From the above analysis we can see that the recursive linearization method is to minimize the functionals

$$\mathcal{J}(q) = \frac{1}{2}||\gamma\mathcal{S}(q, u_m^i) - \Psi_m||^2_{H^{1/2}(\partial\Omega)}, \tag{3.35}$$

sequentially by a steepest decent method. It can be estimated (heuristically) that for fixed τ, $\chi = ||\mathcal{I} - \tau\mathcal{DM}^*(q,k,d)\mathcal{DM}(q,k,d)||$ grows as k increases. Therefore, low wave number k admits larger convergence radius. As the approximation becomes better, the wave number can be increased safely. That is the mechanism and remarkable feature of the recursive linearization method. The strategy delicately avoids difficulties arising from estimates on χ. Despite more measurements data, the method has wide applicability and adaptability.

However, when few data dominates, it should be noted that the choice of τ can simplify the procedure. First, the discrepancy term $c\tau|\tilde{k} - k|||\mathcal{DM}^*(q,k,d)||$ can be negligible if τ is small, which admits larger steps in the wave number direction. This phenomena has been observed in many numerical experiments. Second, τ can be selected such that $\chi = ||\mathcal{I} - \tau\mathcal{DM}^*(q,k,d)\mathcal{DM}(q,k,d)|| < 1$. Henceforth, we can directly start from one larger wave number, or at most recover the Born approximation at some small wave number. Unfortunately, the explicit τ relies on a thorough study of the operator $\mathcal{S}, \mathcal{DS}$, especially the explicit dependency on k. Although this may be quite difficult in general, the reliability and efficiency of the method are prescribed by such results, and meanwhile reveal the limitation of the method. Finally, as k increases, it is expected harder to reduce χ. Consequently, the convergence will become slower. In next section, we will give several numerical experiments to verify our idea.

3.3 Numerical Experiments and Discussions

In this section, we will conduct several numerical experiments to verify the theoretical analysis. According to the proof of lemma 2.2.1 and the analysis in section 3.2.2, we set the parameter τ heuristically by $\mathcal{O}(k^{-2})$. Regardless of the convergent speed, this choice guarantees convergence for all the following numerical experiments. Although these improvements may arouse other difficulties, the numerical results are as accurate as the original recursive linearization method. And more importantly, we only need incident waves at one fixed wave number k.

The computational region is taken to be $\Omega = [-1,1]\times[0,2]$. All the scattering data are generated by a finite element solver(FEM) with step size $h = 0.01$. We use 16 incident directions that equally spaced in the unit circle. That is $d_i = (\cos(\alpha_i), \sin(\alpha_i)), \alpha_i = 2\pi i/16, i = 0, 2, \cdots, 15$. And the data are measured on all boundary nodes. Actually, the incident waves are not so sufficient theoretically. However, they are enough for our experiments. For the implementation of the recovery, we used the finite element method with step size $h = 0.02$ to solve the direct and adjoint equations. The relative error is computed by

$$e_n = \frac{(\sum_{i,j} |q^n_{i,j} - q^*_{i,j}|^2)^{1/2}}{(\sum_{i,j} |q^*_{i,j}|^2)^{1/2}}, \tag{3.36}$$

where q^* is the true scatterer and q^n is the reconstruction at step n. To test the robustness, some relative random noise is added to the data, i. e. the measurements takes the form

$$\Psi = (1 + \sigma \text{rand}) u^s|_{\partial\Omega}. \tag{3.37}$$

Here *rand* is uniformly distributed numbers in $[-1,1]$ and σ is a noise level parameter taken to be 0.05 in all numerical experiments.

In the following, we will mainly consider recovering two media.

Medium 1: Peaks

Let $q_1 = q(3x_1, 3(x_2 - 1))$, where

$$q(x_1, x_2) = 0.3(1 - x_1)^2 \exp(-x_1^2 - (x_2 + 1)^2) - (\frac{x_1}{5} - x_1^3 - x_2^5)\exp(-(x_1^2 + x_2^2)) \\ -\frac{1}{30}\exp(-(x_1 + 1)^2 - x_2^2)$$

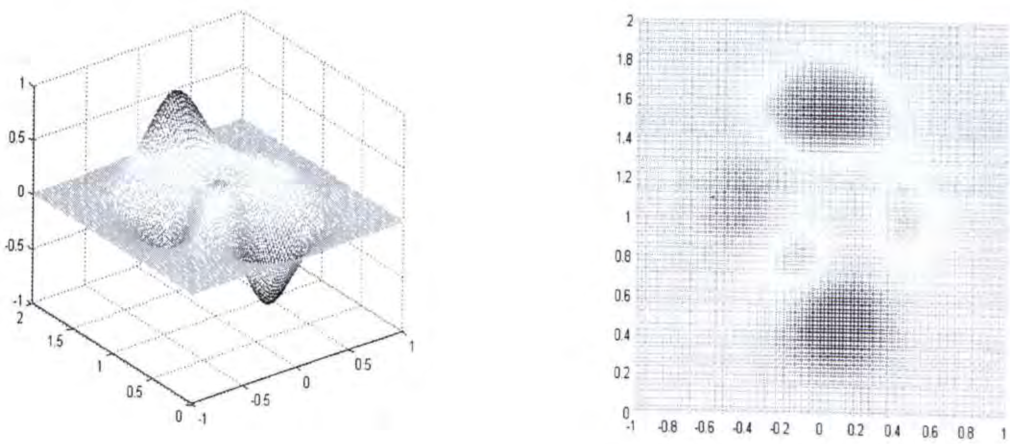


Figure 3.6: Plot of medium 1

Medium 2: Two bumps

Let $q_2 = \exp(-30(x_1^2 + (x_2 - 1.5)^2)) + \exp(-30(x_1^2 + (x_2 - 0.5)^2))$.

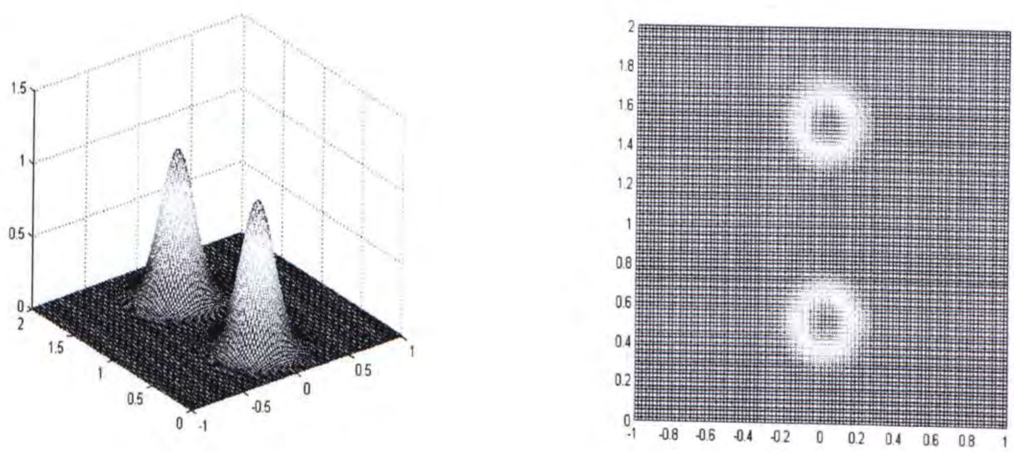


Figure 3.7: Plot of medium 2

It is worthy to notice that the above two media are smooth and not very large in the sense of L^2 or H^2 norm. Therefore, $q_0 = 0$ is already a good initial value. In the numerical experiments, we even dropped the Born approximation procedure. The parameter τ is chosen to be $1/k^2$ as stated before.

For the first experiment, we use incident waves at $k = 10$ to recover medium 1. This example was tested in [BL] at $k = 10$ with spacial frequency η from 12 to 0. By the choices of τ and q_0, the convergence can be achieved without aids from spacial frequency. Fig 3.8, Fig 3.9, Fig 3.10 show the evolution of convergence. And Fig 3.11 illustrates the relative error.

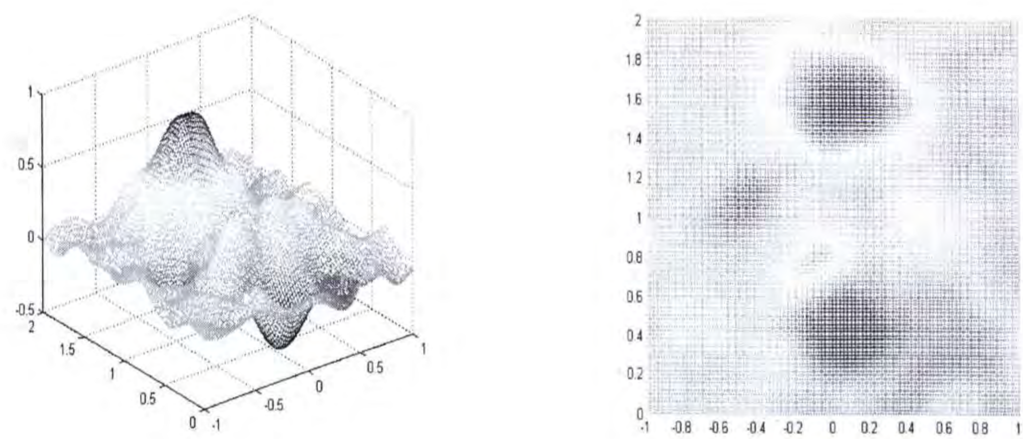


Figure 3.8: Reconstruction of q_1 at n = 16

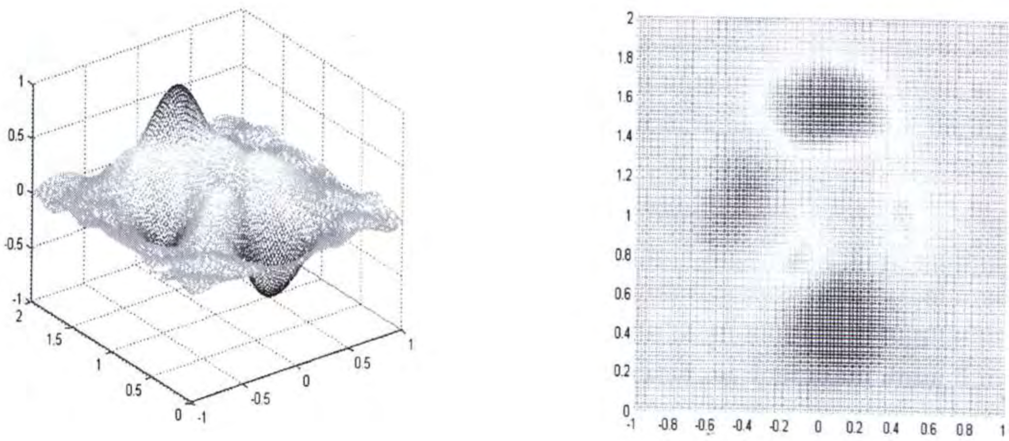


Figure 3.9: Reconstruction of q_1 at n = 32

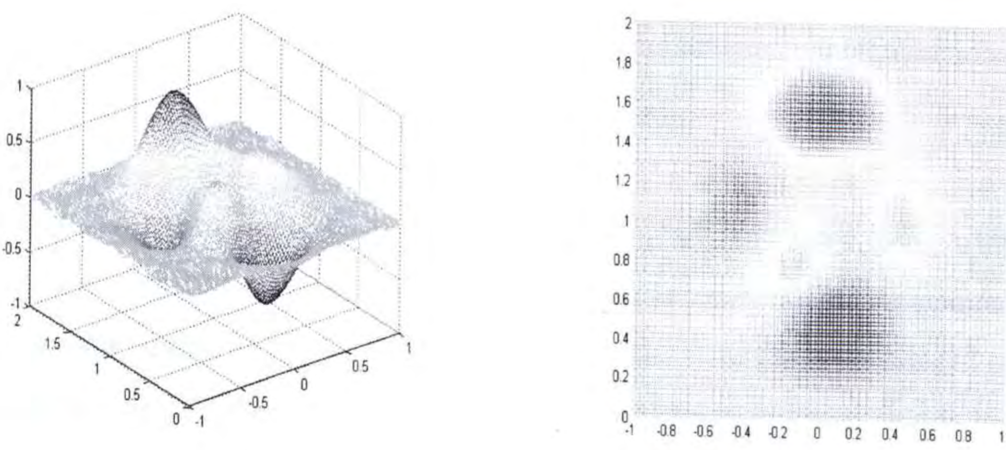
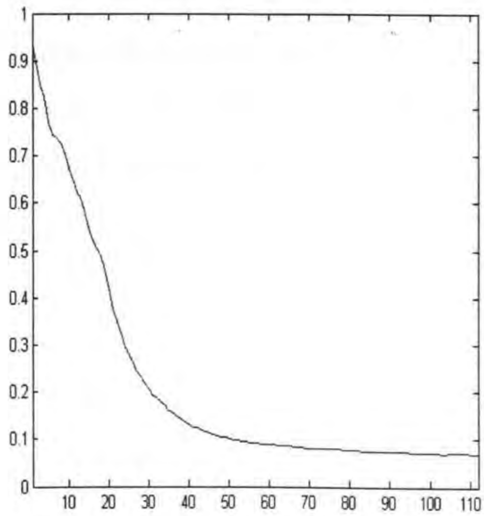


Figure 3.10: Reconstruction of q_1 at n = 112



iteration	relative error
1	0.9384
8	0.7263
16	0.5221
32	0.1848
64	0.0858
112	0.0697

Figure 3.11: Relative error of q_1

It can be seen that the reconstruction is nearly perfect. Moreover, comparing with the original RLM, we find that the convergence is much faster. As pointed before, in many numerical experiments of the RLM, the error stays unchanged or slightly decreases for small wave numbers. That could be interpreted as τ is too small comparing with the wave number k. When k grows to match τ, convergence becomes fast. By choosing a better τ, we achieved the latter immediately.

For the second experiment, we increased k to 20 for the recovering of medium 2. In fact, smaller k such as 7 or 10 are fairly enough for such medium. By this experiments, we intend to test the robustness of our improvements. Fig 3.12, Fig 3.13, Fig 3.14, Fig 3.15, displays the convergence status. Fig 3.16 shows the step error. The convergence is also achieved but with a much slower convergent speed. This can be interpreted as in section 3.2.2 - it is harder to decrease χ as

k grows. The limitation of the steepest decent method will become cumbersome for even lager k, which inspires us to study other methods to overcome those difficulties, especially the convergent rate. Also, from these experiments, we feel that a further study of the spectral property of operator $\mathcal{S}$ is quite useful and meaningful.

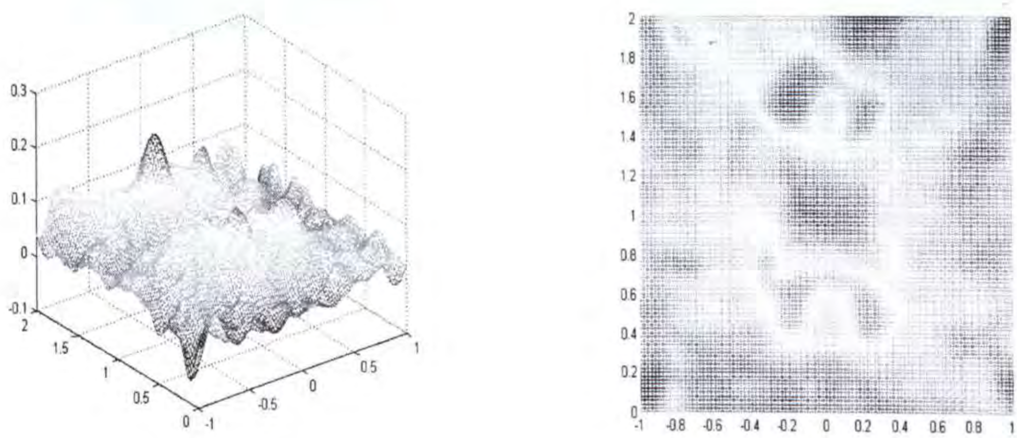


Figure 3.12: Reconstruction at of q_2 n = 16

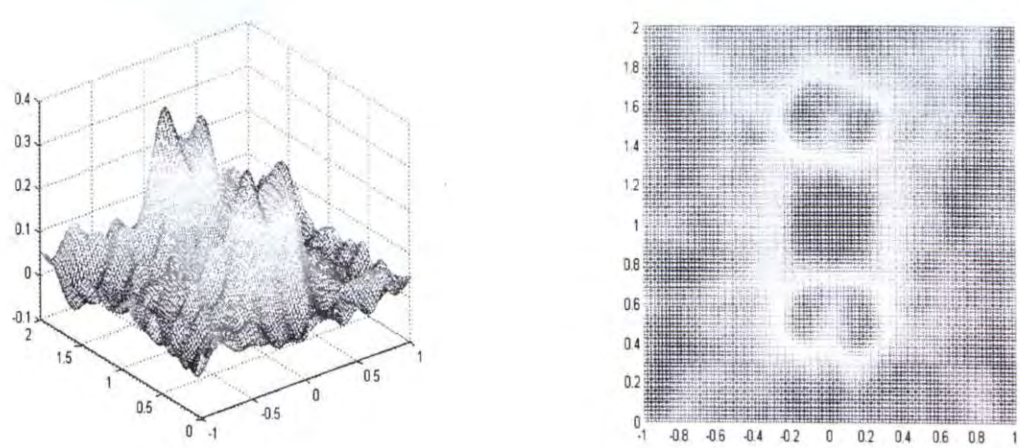


Figure 3.13: Reconstruction at of q_2 n = 32

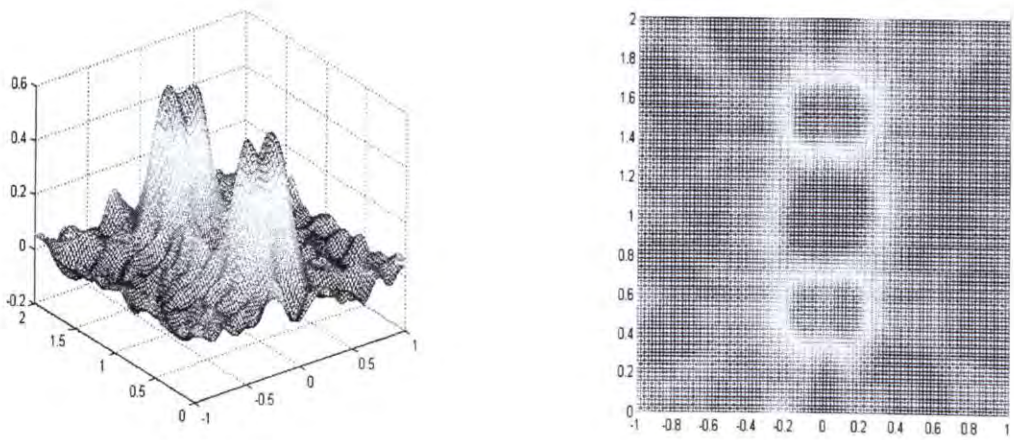


Figure 3.14: Reconstruction at of q_2 n = 64

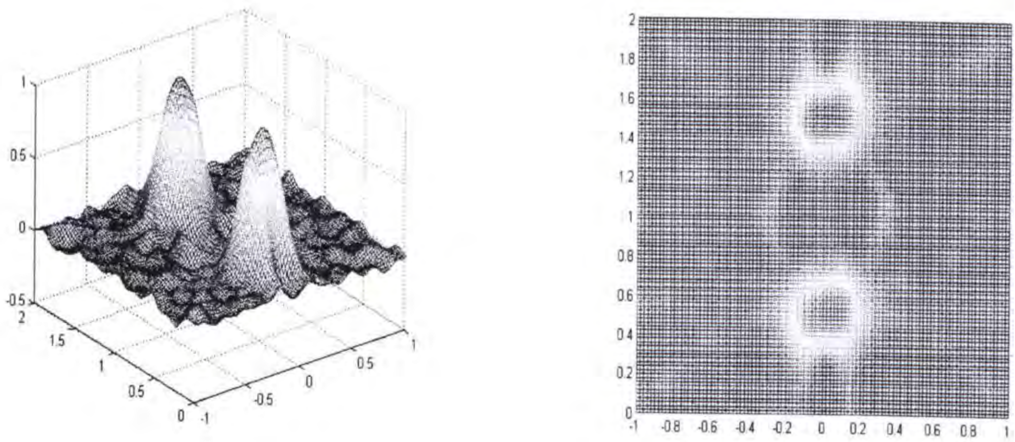
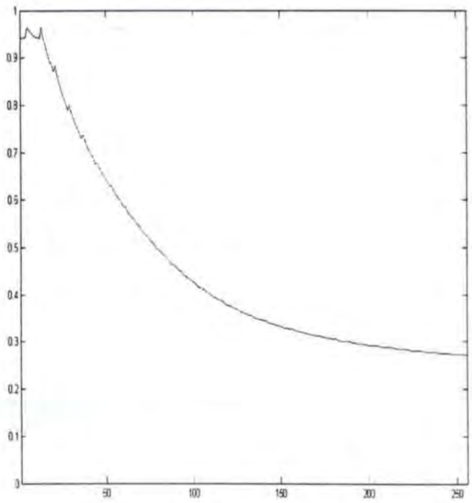


Figure 3.15: Reconstruction at of q_2 n = 256



iteration	relative error
1	0.9538
16	0.9167
32	0.7644
64	0.5660
128	0.3620
256	0.2714

Figure 3.16: Relative error of q_2

Chapter 4

Constrained Optimization: Augmented Lagrangian Method

Continuing our efforts on reconstruction techniques for inverse medium scattering problems, we focus on higher order methods in this chapter. The purpose is two folded. First, the steepest decent analysis in the previous chapter can be regarded as a global method to raise good initial values. By combining with higher order methods, the convergent rate can be improved. Second, frequently, we will consider media with *a priori* knowledge such as piecewise constant or discontinuity. The previous optimization model cannot reflect and make use of these information which may contribute to computational robustness and efficiency. Therefore, new models that incorporate the medium features should be developed to treat these situations.

4.1 Method Review

In [IK] and [IK1], the authors carefully studied some robust optimization methods, especially the augmented Lagrangian-SQP-methods, in a Hilbert space setting. In addition, they successfully applied the methods to parameter identification problems in elliptic systems and optimal control problems governed by partial differential equations. The major features of these methods are of second order convergence and better global behavior, which could benefit our problem. In this section, we briefly summarize their main results.

For the optimization problem:

$$(\mathcal{P}) \quad \min F(x) \quad \text{subject to} \quad e(x) = 0, \tag{4.1}$$

where $F : X \to \mathbb{R}$, $e : X \to Y$, with X, Y Hilbert spaces. The Lagrangian associated with $(\mathcal{P})$ is defined to be $\mathcal{L} : X \times Y \to \mathbb{R}$,

$$\mathcal{L}(x, \lambda) = F(x) + \langle \lambda, e(x) \rangle_Y, \tag{4.2}$$

where $\langle \cdot, \cdot \rangle_Y$ denotes the inner product in Y. Here and after, we do not distinguish functional and its Riesz representation since all we consider are Hilbert spaces.

An element $\lambda^* \in Y$ is called a Lagrangian multiplier if

$$\mathcal{L}'(x, \lambda) = F'(x) + e'(x)^* \lambda^* = 0. \tag{4.3}$$

Here F' denotes the Fréchet derivative and $e'(x)^*$ denotes the adjoint of $e'(x)$ in Hilbert space Y. We need several hypothesis on these functionals.

(H1) $(\mathcal{P})$ has at least a local solution x^*. $F(x)$ and $e(x)$ are twice continuously Fréchet differentiable, and their second Fréchet derivatives are Lipschitz continuous in some neighborhood $\tilde{V}(x^*)$ of x^*;

(H2) $e'(x^*)$ is surjective;

(H3) There exists $\kappa > 0$ such that $\mathcal{L}''(x^*, \lambda^*)(h, h) \geq \kappa |h|_X^2, \quad \forall h \in \mathrm{Ker} e'(x^*)$.

The augmented Lagrangian method for constrained optimization problem was developed from the penalty method and the Lagrangian method. As is well known, the penalty method suffers from ill-conditioning and slow convergence when the penalty parameter becomes large, while the original Lagrangian method cannot enforce convergence when far from a solution. The augmented Lagrangian methods moderates both disadvantages and is defined by

$$\mathcal{L}_c(x,\lambda) = \mathcal{L}(x,\lambda) + \frac{c}{2}|e(x)|_Y^2, \forall c > 0. \tag{4.4}$$

Under hypothesis (H1) - (H3), there exists a neighborhood $V(x^*) \subset \tilde{V}(x^*)$ and constants $\bar{c} > 0, \sigma > 0$ such that

$$\mathcal{L}_c(x,\lambda^*) \geq \mathcal{L}_c(x^*,\lambda^*) + \sigma|x - x^*|_X^2, \quad \text{for all} \quad x \in V(x^*) \quad \text{and} \quad c \geq \bar{c}. \tag{4.5}$$

Therefore, $\mathcal{L}_c$ is bounded below by a quadratic function. This fact is referred to as augmentability of $\mathcal{L}_c$ ([IK]).

Introduce

$$M(x,\lambda) = \begin{pmatrix} \mathcal{L}''(x,\lambda) & e'(x)^* \\ e'(x) & 0 \end{pmatrix},$$

we now give the main algorithm and convergence results.

Algorithm 1

(i) Choose $(x_0, \lambda_0) \in X \times Y$, $c \geq 0$ *and set* $n = 0$;

(ii) Set $\tilde{\lambda} = \lambda_n + ce(x_n)$;

(iii) Solve for $(\hat{x}, \hat{\lambda})$:

$$M(x_n, \tilde{\lambda}) \begin{pmatrix} \hat{x} - x_n \\ \hat{\lambda} - \tilde{\lambda} \end{pmatrix} = - \begin{pmatrix} \mathcal{L}'(x_n, \tilde{\lambda}) \\ e(x_n) \end{pmatrix} \tag{4.6}$$

(iv) Set $(x_{n+1}, \lambda_{n+1}) = (\hat{x}, \hat{\lambda})$, $n = n + 1$ *and goto (ii).*

Theorem 4.1.1 *Let* $(H1), (H2)$ *and* $(H3)$ *hold, if* $|(x_0, \lambda_0) - (x^*, \lambda^*)|_{X\times Y}$ *is sufficiently small, then Algorithm 1 is well defined and satisfies*

$$|(x_{n+1}, \lambda_{n+1}) - (x^*, \lambda^*)|_{X\times Y} \le K|(x_n, \lambda_n) - (x^*, \lambda^*)|^2_{X\times Y},$$

for some K *depending on* c, *and* $n = 0, 1, \cdots$.

Remark 4.1.1 *This algorithm is the original SQP method. It can be combined with other methods to enlarge the convergent radius as analyzed in [Be] for finite dimensional cases. In [IK], the authors also provided alternatives - second order update of the multiplier.*

4.2 Problem Formulation

With previous preparations in section 4.1, we formulate the inverse medium scattering problem into an optimization problem. Besides Tikhonov regularization to counteract the ill-posedness, we choose a mixed method that combines the output least-squares method and the equation error method as introduced in [IK]. Specifically, the method is to minimize.

$$\begin{cases} \mathcal{J}(u^s, q) = \frac{1}{2}||\gamma u^s - z||^2_X + \frac{\beta}{2}||Nq||^2_Z, \\ \text{subject to} \quad (2.13), (2.14), \end{cases}$$

where

- $X = H^{1/2}(\partial\Omega)$ and $Z = H^2(\Omega)$;
- $z \in X$ is the observation data;
- $u^s \in H^1(\Omega)$ is the scattered field;
- $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator;

- β is the regularization parameter;
- $N : H^2(\Omega) \to Z$ is the regularization operator. Here $N = Id$.

This formulation has several advantages. First the output least-squares term easily incorporates information at hand, with little modification according to various wave numbers and incident directions. This is especially convenient since medium properties would affect the sufficient number of incident fields. Second, the regularization term can easily reflect certain *a priori* knowledge about the medium $q(x)$. The above full-norm regularization operator N works for smooth media, while BV or TV regularization would be suitable for piecewise constant medium or discontinuous medium.

There is one remark about the choice of function space for q. We know that H^2 can be embedded into L^∞, and this fact will benefit our analysis for the bilinear term. Although such high regularity may cause computational difficulties, it is reasonable in view of the uniqueness result in section 2.2.2.

In order to represent the PDE constraint (2.13), (2.14) into an operator equation, we first define an operator $\tilde{e}(\cdot,\cdot) : H^1(\Omega) \times H^2(\Omega) \to H^{-1}(\Omega)$ by weak formulation:

$$\begin{aligned}(\tilde{e}(u^s, q), \phi)_{H^{-1}(\Omega),H^1(\Omega)} &= (\nabla u^s, \nabla\phi) - k^2((1+q)u^s, \phi) \\ &\quad -ik\langle u^s, \phi\rangle - k^2(qu^i, \phi), \quad \forall\phi \in H^1(\Omega)\end{aligned}$$

and $\mathcal{N} : H^{-1}(\Omega) \to H^1(\Omega)$ by $\omega = \mathcal{N}f$,

$$(\nabla\omega, \nabla\phi) + (\omega, \phi) = (f, \phi)_{H^{-1}(\Omega),H^1(\Omega)}, \quad \forall\phi \in H^1(\Omega). \tag{4.7}$$

It is easily seen that $\mathcal{N}$ is the solution operator to (2.38), (2.39). Then let $e(\cdot,\cdot) : H^1(\Omega) \times H^2(\Omega) \to H^1(\Omega)$ to be $e = \mathcal{N}\tilde{e}$, we get the constraint $e = 0$

To summarize, the optimization problem for later analysis is

$$(\mathcal{P}_\beta) \quad \begin{cases} \min \mathcal{J}(u^s, q) = \frac{1}{2}||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} + \frac{\beta}{2}||Nq||^2_{H^2(\Omega)} \\ \text{subject to} \quad e(u^s, q) = 0 \end{cases}$$

4.3 First Order Optimality Condition

Because the Hilbert norm is Fréchet differentiable and γ is continuous, it can be concluded that $\mathcal{J}(u^s, q)$ is twice continuously Fréchet differentiable. Specifically, for some $u^s \in H^1(\Omega)$ and $q \in H^2(\Omega)$, we have

$$\mathcal{J}_u^s(u^s, q)\delta u^s = \Re(\gamma u^s - z, \gamma \delta u^s), \quad \forall \delta u^s \in H^1(\Omega), \tag{4.8}$$

and

$$\mathcal{J}_q(u^s, q)\delta q = (q, \delta q), \quad \forall \delta q \in H^2(\Omega). \tag{4.9}$$

Moreover, for the second order derivative, we have $\mathcal{J}_{u^s,q}(u^s, q) = 0$, and

$$\mathcal{J}_{u^s,u^s}(u^s, q)(\delta u^s, \delta u^s) = ||\gamma \delta u^s||_{H^{1/2}(\Omega)}, \tag{4.10}$$

$$\mathcal{J}_{q,q}(u^s, q)(\delta q, \delta q) = ||\delta q||_{H^2(\Omega)}. \tag{4.11}$$

The following lemma asserts the Fréchet differentiability of e.

Lemma 4.3.1 *The operator $e : H^1(\Omega) \times H^2(\Omega) \to H^1(\Omega)$ is Fréchet differentiable with respect to (u^s, q).*

Proof: From the definition of $\mathcal{N}$ and lemma 2.2.2, $\mathcal{N}$ is a bounded linear operator. Next, for $u^s \in H^1(\Omega)$ being the solution of $u^s = \mathcal{S}(q, u^i)$, $q \in H^2(\Omega)$, let $\tilde{e}'(u^s, q)(\cdot, \cdot) : H^1(\Omega) \times H^2(\Omega) \to H^{-1}(\Omega)$ be such that

$$(\tilde{e}'(u^s, q)(\delta u^s, \delta q), \phi)_{H^{-1}(\Omega), H^1(\Omega)}$$

$$= \quad (\nabla \delta u^s, \nabla \phi) - ik\langle \delta u^s, \phi \rangle - k^2((1+q)\delta u^s, \phi) - k^2(\delta q(u^s + u^i), \phi), \quad \forall \phi \in H^1(\Omega).$$

Therefore,

$$\begin{aligned}
&||\tilde{e}'(u^s, q)(\delta u^s, \delta q)||_{H^{-1}(\Omega)} \\
&= \sup_{\phi \neq 0} \frac{|(\tilde{e}'(u^s, q)(\delta u^s, \delta q), \phi)|}{||\phi||_{H^1 \Omega}} \\
&\leq C||\delta u^s||_{H^1(\Omega)} + k^2||q||_{L^\infty(\Omega)}||\delta u^s||_{H^1(\Omega)} + k^2||\delta q||_{L^\infty(\Omega)}||u^s||_{H^1(\Omega)} \\
&+k^2||\delta q||_{H^1(\Omega)}||u^i||_{L^\infty(\Omega)},
\end{aligned}$$

where C is a generic constant. Now, by the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$, we have

$$\begin{aligned}
&||\tilde{e}'(u^s, q)(\delta u^s, \delta q)||_{H^{-1}(\Omega)} \\
&\leq C(q, k)||\delta u^s||_{H^1(\Omega)} + \tilde{C}k^2||\delta q||_{H^2(\Omega)}||u^s||_{H^1(\Omega)} + k^2||\delta q||_{H^1(\Omega)}||u^i||_{L^\infty(\Omega)} \\
&\leq C(||\delta u^s||_{H^1(\Omega)} + ||\delta q||_{H^2(\Omega)}),
\end{aligned}$$

where C depends on k, u^s, u^i, q and Ω. Hence $\tilde{e}'(u^s, q)$ is a bounded operator. It is evident that $\tilde{e}'(u^s, q)$ is linear.

From direct calculation,

$$(\tilde{e}(u^s + \delta u^s, q + \delta q) - \tilde{e}(u^s, q), \phi) = (\tilde{e}'(u^s, q)(\delta u^s, \delta q), \phi) - k^2(\delta q \delta u^s, \phi).$$

Therefore,

$$\begin{aligned}
&||\tilde{e}(u^s + \delta u^s, q + \delta q) - \tilde{e}(u^s, q) - \tilde{e}'(u^s, q)(\delta u^s, \delta q)||_{H^{-1}(\Omega)} \\
&= \sup_{\phi \neq 0} \frac{|k^2(\delta u^s \delta q, \phi)|}{||\phi||} \\
&\leq k^2||\delta u^s||_{H^1(\Omega)}||\delta q||_{L^\infty(\Omega)} \\
&\leq Ck^2||\delta u^s||_{H^1(\Omega)}||\delta q||_{H^2(\Omega)},
\end{aligned}$$

where $C > 0$ is the embedding constant from $H^2(\Omega)$ to $L^\infty(\Omega)$. Thus, $\tilde{e}'$ is the Fréchet derivative of $\tilde{e}$. By the chain rule, we find $e' = \mathcal{N}\tilde{e}'$ is the Frechét derivative for e. The second derivative can be proved similarly. □

For the later use, we calculate the representation for $\tilde{e}_{u^s}(u^s,q)(\cdot)$ and $\tilde{e}_q(u^s,q)(\cdot)$ as follows

$$(\tilde{e}_{u^s}(u^s,q)\delta u^s,\phi) = (\nabla\delta u^s,\nabla\phi) - k^2((1+q)\delta u^s,\phi) - ik\langle\delta u^s,\phi\rangle, \quad \forall\delta u^s \in H^1(\Omega)$$

$$(\tilde{e}_q(u^s,q)\delta q,\phi) = -k^2(\delta q(u^s+u^i),\phi), \quad \forall\delta q \in H^2(\Omega).$$

And also, we have $\tilde{e}_{u^s,u^s}(u^s,q) = 0, \tilde{e}_{q,q}(u^s,q) = 0$ and

$$(\tilde{e}_{q,u^s}(u^s,q)(\delta u^s,\delta q),\phi) = -k^2(\delta q\delta u^s,\phi). \tag{4.12}$$

In addition, we can characterize the kernel of $e'(u^s,q)(\cdot,\cdot)$. For $(\delta u^s,\delta q) \in$Ker $e'(u^s,q)$, we have $\tilde{e}'(u^s,q)(\delta u^s,\delta q) = 0$. Hence force $(\delta u^s,\delta q)$ satisfies

$$\Delta\delta u^s + k^2(1+q)\delta u^s = -k^2\delta q(u^s+u^i), \quad \text{in } \Omega \tag{4.13}$$

$$\partial_n\delta u^s - ik\delta u^s = 0, \quad \text{on } \partial\Omega. \tag{4.14}$$

In order to verify (H1), we need

Lemma 4.3.2 *$(\mathcal{P}_\beta)$ admits a local solution for $\beta > 0$.*

Proof: Let $(u^{s,n},q^n)$ be a minimizing sequence such that $\mathcal{J}(u^{s,n+1},q^{n+1}) \le \mathcal{J}(u^{s,n},q^n)$ and $e(u^{s,i},q^i) = 0$, $i = 1,2,\cdots$. Therefore,

$$\begin{aligned}||q^n||^2_{H^2(\Omega)} &\le ||\gamma u^{s,n} - z||^2_{H^{1/2}(\partial\Omega)} + ||q^n||^2_{H^2(\Omega)} \\ &\le ||\gamma u^{s,0} - z||^2_{H^{1/2}(\partial\Omega)} + ||q^0||^2_{H^2(\Omega)} \\ &< \infty.\end{aligned}$$

$\{q^n\}$ is bounded. Considering $e(u^{s,n},q^n) = 0$ and lemma 2.2.1, $||u^s||_{H^1(\Omega)} \le c||q^n||_{H^2(\Omega)}$, where the constant c only depends on k and $||u^i||_{L^2(\Omega)}$. Hence $\{(u^{s,n},q^n)\}$ is bounded. Now, from the Eberlein-Šmuljan theorem, there exists a weakly convergent subsequence with weak limit (u^s,q). Still denote the subsequence by

$(u^{s,n}, q^n)$. Since $\mathcal{J}$ is weakly lower semi-continuous, we have

$$\mathcal{J}(u^s, q) \leq \liminf_{n\to\infty} \mathcal{J}(u^{s,n}, q^n). \tag{4.15}$$

Since $e(u^{s,n}, q^n) = 0$, we know that $e(u^{s,n}, q^n) \rightharpoonup e(u^s, q) = 0$. Therefore (u^s, q) is a solution to $(\mathcal{P}_\beta)$, and (H1) is verified. $\square$

Towards (H2), first notice that $e_u^s(u^s, q)$ is surjective already implies (H2). According to the definition, we know that $\mathcal{N}$ is surjective. Moreover, for any $f \in H^1(\Omega)^*$ such that $\tilde{e}_u^s(u^s, q)\delta u^s = f$, we know it satisfies the following equation

$$\begin{cases} \Delta\delta u^s + k^2(1+q)\delta u^s = -f, & \text{in} \quad \Omega, \\ \partial_n \delta u^s - ik\delta u^s = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

By lemma 2.2.2, $\tilde{e}_u^s(u^s, q)$ is surjective, thus $e_{u^s} = \mathcal{N}\tilde{e}_{u^s}$ is surjective. (H2) is verified.

With (H2) holding, there exists a Lagrange multiplier $\lambda^* \in H^1(\Omega)$ such that the first order optimality condition is satisfied, i.e.

$$\mathcal{L}'(u^s, q, \lambda^*) = 0, \quad e(u^s, q) = 0.$$

In our case, $\mathcal{L}(u^s, q, \lambda) = \frac{1}{2}||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} + \frac{\beta}{2}||Nq||^2_{H^2(\Omega)} + (\lambda, e(u^s, q))_{H^1(\Omega)}$. By direct calculation, we find

$$\begin{aligned} 0 &= \mathcal{L}_q(u^s, q, \lambda)\delta q \\ &= \Re(Nq, N\delta q) - k^2(\lambda, \delta q(u^s + u^i)), \quad \forall \delta q \in H^2(\Omega) \end{aligned}$$

and

$$\begin{aligned} 0 &= \mathcal{L}_{u^s}(u^s, q, \lambda)\delta u^s \\ &= \Re(\gamma u^s - z, \gamma\delta u^s) + (\nabla\lambda, \nabla\delta u^s) - ik\langle\lambda, \delta u^s\rangle - k^2(\lambda, (1+q)\delta u^s), \quad \forall \delta u^s \in H^1(\Omega) \end{aligned}$$

From the above characterization, we can derive the equation λ satisfies. From lemma 2.2.1, we observe that

$$||\lambda||_{H^1(\Omega)} \leq c||\gamma^*(\gamma u^s - z)||_{H^1(\Omega)} \leq c||\gamma||||\gamma u^s - z||_{H^{1/2}(\partial\Omega)}.$$

4.4 Second Order Optimality Condition

To verify (H3), we need the following lemma modified from [CK].

Lemma 4.4.1 *Suppose $(\mathcal{P}_\beta)$ has a solution $(u^s_\beta, q_\beta) \in U_\beta \times Q_\beta$, which is the solution set for $\beta > 0$. And $(\mathcal{P}_0)$ admits a solution $(u^s, q) \in U \times Q$ Then we have*

(i) $\sup_{q_\beta \in Q_\beta} ||Nq_\beta||^2_{H^2(\Omega)} \leq \inf_{q\in Q} ||Nq||^2_{H^2(\Omega)}$;

(ii) $\sup_{u^s \in U} ||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} \leq \inf_{u^s_\beta \in U_\beta} ||\gamma u^s_\beta - z||^2_{H^{1/2}(\partial\Omega)}$;

(iii) $\sup_{u^s_\beta \in U_\beta} ||\gamma u^s_\beta - z||^2_{H^{1/2}(\partial\Omega)} \leq ||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} + \beta(\inf_{q\in Q} ||Nq||^2_{H^2(\Omega)} - \sup_{q_\beta \in Q_\beta} ||Nq_\beta||^2_{H^2(\Omega)})$.

Proof: Since any $(u^s, q) \in U \times Q$ is a solution for $(\mathcal{P})$, we have $||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} \leq ||\gamma u^s_\beta - z||^2_{H^{1/2}(\partial\Omega)}$, for any $(u^s_\beta, q_\beta) \in U_\beta \times Q_\beta$. Thus (ii) is proved. By adding $\beta||Nq_\beta||^2_{H^2(\Omega)}$ to both sides,

$$\begin{aligned}
&||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} + \beta||Nq_\beta||^2_{H^2(\Omega)} \\
&\leq ||\gamma u^s_\beta - z||^2_{H^{1/2}(\partial\Omega)} + \beta||Nq_\beta||^2_{H^2(\Omega)} \\
&\leq ||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} + \beta||Nq||^2_{H^2(\Omega)}.
\end{aligned}$$

Therefore, $||Nq_\beta||^2 \leq ||Nq||^2$, (i) is proved. Besides, we have

$$\sup ||\gamma u^s_\beta - z||^2_{H^{1/2}(\partial\Omega)} + \beta \sup ||Nq_\beta||^2_{H^2(\Omega)} \leq \inf ||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} + \beta \inf ||Nq||^2_{H^2(\Omega)},$$

$$\sup ||\gamma u^s_\beta - z||^2_{H^{1/2}(\partial\Omega)} \leq ||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} + \beta(\inf ||Nq||^2_{H^2(\Omega)} - \sup ||Nq_\beta||^2_{H^2(\Omega)})$$

(iii) is proved. □

Now it is easy to see that

$$
\begin{aligned}
&\mathcal{L}''(u^s_\beta, q_\beta, \lambda)(\delta u^s, \delta q)^2 \\
&= ||\gamma\delta u^s||^2_{H^{1/2}(\partial\Omega)} + \beta||N\delta q||^2_{H^2(\Omega)} + (\lambda, e''(u^s_\beta, q_\beta)(\delta u^s, \delta q)^2)_{H^1(\Omega)} \\
&= ||\gamma\delta u^s||^2_{H^{1/2}(\partial\Omega)} + \beta||\delta q||^2_{H^2(\Omega)} + \overline{(\tilde{e}''(u^s_\beta, q_\beta)(\delta u^s, \delta q)^2, \lambda)}_{H^{-1}(\Omega), H^1(\Omega)} \\
&= ||\gamma\delta u^s||^2_{H^{1/2}(\partial\Omega)} + \beta||\delta q||^2_{H^2(\Omega)} - k^2(\lambda, \delta u^s \delta q)_{H^1(\Omega)} \\
&\geq ||\gamma\delta u^s||^2_{H^{1/2}(\partial\Omega)} + \beta||\delta q||^2_{H^2(\Omega)} - k^2||\lambda||_{H^1(\Omega)}||\delta u^s||_{H^1(\Omega)}||\delta q||_{L^\infty(\Omega)} \\
&\geq ||\gamma\delta u^s||^2_{H^{1/2}(\partial\Omega)} + \beta||\delta q||^2_{H^2(\Omega)} - ck^2||\gamma||||\gamma u^s_\beta - z||_{H^{1/2}(\partial\Omega)}||\delta u^s||_{H^1(\Omega)}||\delta q||_{H^2(\Omega)}.
\end{aligned}
$$

Since $(\delta u^s, \delta q)\in$Ker $\tilde{e}'(u^s_\beta, q_\beta)$, by (4.13), (4.10) and lemma 2.2.1, $||\delta u^s|| \leq \tilde{c}||\delta q||$. Let c be a generic constant, we have

$$
\begin{aligned}
&\mathcal{L}''(u^s_\beta, q_\beta, \lambda)(\delta u^s, \delta q)^2 \\
&\geq ||\gamma\delta u^s||^2_{H^{1/2}(\Omega)} + \beta||\delta q||^2_{H^2(\Omega)} - ck^2||\gamma||||\gamma u^s_\beta - z||_{H^{1/2}(\partial\Omega)}||\delta q||^2_{H^2(\Omega)} \\
&\geq ||\gamma\delta u^s||^2_{H^{1/2}(\Omega)} + \beta||\delta q||^2_{H^2(\Omega)} - ck^2||\gamma||(||\gamma u^s - z||^2_{H^{1/2}(\partial\Omega)} \\
&+\beta(\inf_{q\in Q} ||Nq||^2_{H^2(\Omega)} - \sup_{q_\beta\in Q_\beta} ||Nq_\beta||^2_{H^2(\Omega)}))^{1/2}||\delta q||^2_{H^2(\Omega)}.
\end{aligned}
$$

Assume that

$$
||\gamma u^s - z||^2_{H^{1/2}(\Omega)} \leq (\frac{\beta}{2ck^2||\gamma||})^2 - \beta(\inf_{q\in Q} ||Nq||^2_{H^2(\Omega)} - \sup_{q_\beta\in Q_\beta} ||Nq_\beta||^2_{H^2(\Omega)})),
$$

we arrive at

$$
\begin{aligned}
\mathcal{L}''(u^s_\beta, q_\beta, \lambda)(\delta u^s, \delta q)^2 \quad &\geq ||\gamma\delta u^s||^2_{H^{1/2}(\partial\Omega)} + \frac{\beta}{2}||\delta q||^2_{H^2(\Omega)} \\
&\geq ||\gamma\delta u^s||^2_{H^{1/2}(\partial\Omega)} + c\frac{\beta}{2}(||\delta q||^2_{H^2(\Omega)} + ||\delta u^s||_{H^1(\Omega)}).
\end{aligned}
$$

Here we used $||\delta u^s||_{H^1(\Omega)} \leq c$. This verifies (H3). We summarize the result in next lemma.

Lemma 4.4.2 *Assume (u^s_β, q_β) is a solution to $(\mathcal{P}_\beta)$ and that*

$$
||\gamma u^s - z||^2_{H^{1/2}(\Omega)} \leq (\frac{\beta}{2ck^2||\gamma||})^2 - \beta(\inf_{q\in Q} ||Nq||^2_{H^2(\Omega)} - \sup_{q_\beta\in Q_\beta} ||Nq_\beta||^2_{H^2(\Omega)})). \quad (4.16)
$$

Then we have

$$\mathcal{L}''(u^s_\beta, q_\beta, \lambda)(\delta u^s, \delta q)^2 \geq c\frac{\beta}{2}(||\delta q||^2_{H^2(\Omega)} + ||\delta u^s||_{H^1(\Omega)}) \tag{4.17}$$

for all $(\delta u^s, \delta q) \in$ *Ker* $e'(u^s_\beta, q_\beta)$.

Actually, the assumption is a little stringent. However, without satisfactory regularity results about (2.13), (2.14), we have to employ full-norm regularization. Otherwise, we can gain something from $||\gamma \delta u^s||_{H^{1/2}(\Omega)}$ and relax the assumption as in [IK]. But in view of the uniqueness result in section 2.2.2, the H^2 regularity is reasonable.

4.5 Modified Algorithm

For the inverse medium scattering problem, we usually need multi-incoming waves to guarantee uniqueness of the medium. Suppose we have M incident waves coming with different directions $d \in D$ and various wave number $k > 0$. Let $u^s = (u^s_1, u^s_2, \cdots, u^s_M)$ be the corresponding scattered fields, and $e(u^s, q) = (e_1(u^s_1, q), e_2(u^s_2, q), \cdots, e_M(u^s_M, q))$ be the corresponding PDE constraints. Now the optimization problem is

$$(\tilde{\mathcal{P}}_\beta) \quad \begin{cases} \min \frac{1}{2}\sum_{i=1}^M ||\gamma u^s_i - z_i||^2_{H^{1/2}(\Omega)} + \frac{\beta}{2}||Nq||^2_{H^2(\Omega)} \\ \text{subject to} \quad e(u^s, q) = 0 \end{cases}$$

Of course, previous analysis works for this vector-valued case. However, as M becomes larger, the resulting discretized linear system becomes increasingly difficult to store and solve in computer. As in Chapter 3, we now propose a modified algorithm that breaks down the vector-valued problem to pieces of scalar ones. In each iteration, we only need to solve one optimization problem corresponding to one incident field. That is

$$(\tilde{\mathcal{P}}_{\beta,i}) \quad \begin{cases} \min \frac{1}{2}||\gamma u^s_i - z_i||^2_{H^{1/2}(\Omega)} + \frac{\beta}{2}||Nq||^2_{H^2(\Omega)} \\ \text{subject to} \quad e_i(u^s, q) = 0 \end{cases}$$

Algorithm 2 (Modified Algorithm)

(i) *Choose* $(u^{s,0}, q^0) = (u_1^{s,0}, u_2^{s,0}, \cdots, u_M^{s,0}, q^0)$, $c \geq 0$ *and set* $n = 0$;

(ii) *Set* $\tilde{\lambda} = \lambda_n + ce(u^{s,n}, q^n) = (\lambda_i + ce_i(u_i^{s,n}, q^n))_{i=1}^M$;

(iii) *Set* $q_0^n = q^n$;

(iv) *For* $i = 1 : M$, *solve for* $(\hat{u}_i^s, \hat{q}, \hat{\lambda}_i)$:

$$M_i(u_i^{s,n}, q_i^n, \tilde{\lambda}_i) \begin{pmatrix} \hat{u}_i^s - u_i^{s,n} \\ \hat{q} - q_i^n \\ \hat{\lambda}_i - \tilde{\lambda}_i \end{pmatrix} = - \begin{pmatrix} \mathcal{L}'(u_i^{s,n}, q_i^n, \tilde{\lambda}_i) \\ e_i(u_i^{s,n}, q_i^n) \end{pmatrix} \tag{4.18}$$

Set $u_i^{s,n+1} = \hat{u}_i^s, q_{i+1}^n = \hat{q}$ *and* $\lambda_i^{n+1} = \hat{\lambda}_i$;

(v) *Set* $q^{n+1} = q_M^n$, $n = n + 1$ *and goto (ii)*.

For the convergence analysis of the modified algorithm, we need the following lemma [IK].

Lemma 4.5.1 *Let* $(H2), (H3)$ *hold for some* $(x^*, \lambda^*) \in X \times Y$ *(both Hilbert spaces), then there exists a constant* $\kappa > 0$ *and a neighborhood* $U(x^*, \lambda^*)$ *such that*

$$||M^{-1}(x, \lambda)|| \leq \kappa, \quad \forall (x, \lambda) \in U(x^*, \lambda^*). \tag{4.19}$$

There exists a constant $K > 0$ *such that the solution to*

$$M(x, \lambda) \begin{pmatrix} \hat{x} - x \\ \hat{\lambda} - \lambda \end{pmatrix} = - \begin{pmatrix} \mathcal{L}'(x, \lambda) \\ e(x) \end{pmatrix} \tag{4.20}$$

satisfies

$$||(\hat{x}, \hat{\lambda}) - (x^*, \lambda^*)||_{X \times Y} \leq K ||(x, \lambda) - (x^*, \lambda^*)||_{X \times Y}^2. \tag{4.21}$$

Now we have a corresponding lemma.

Lemma 4.5.2 *Let $(u, q, \lambda) = (u_1, u_2, \cdots, u_M, q, \lambda_1, \lambda_2, \cdots, \lambda_M)$ in some neighborhood $U(u^*, q^*, \lambda^*)$ of (u^*, q^*, λ^*). Through Step (iv) of the modified algorithm, it becomes $(\hat{u}, \hat{q}, \hat{\lambda})$. Then there exists a constant $\tilde{K} > 0$ such that*

$$||(\hat{u}, \hat{q}, \hat{\lambda}) - (u^*, q^*, \lambda^*)|| \le \tilde{K}||(u, q, \lambda) - (u^*, q^*, \lambda^*)||^2, \tag{4.22}$$

where $\tilde{K}$ depends on K_i and U.

Proof: It can be deduced from $(H1), (H2)$ and $(H3)$ of $(\tilde{\mathcal{P}}_\beta)$ that (u_i^*, q^*) satisfies $(H2), (H3)$ for $(\tilde{\mathcal{P}}_{\beta,i})$. From lemma 4.5.1, we have constants $K_i > 0$ such that

$$||(\hat{u}_i, q_i, \hat{\lambda}_i) - (u_i^*, q^*, \lambda_i^*)|| \le K_i||(u_i, q_{i-1}, \lambda_i) - (u_i^*, q^*, \lambda_i^*)||^2, \tag{4.23}$$

where q_i denote the intermediate variable in Step (iv) of the modified algorithm with $q_0 = q$ and $q_M = \hat{q}$, $i = 1, 2, \cdots, M$. Therefore,

$$\begin{aligned}
&||(\hat{u}, \hat{q}, \hat{\lambda}) - (u^*, q^*, \lambda^*)||^2 \\
&= \sum_{i=1}^{M-1} ||(\hat{u}_i, \hat{\lambda}_i) - (u_i^*, \lambda_i^*)||^2 + ||(\hat{u}_M, q_M, \hat{\lambda}_M) - (u_M^*, q^*, \lambda_M^*)||^2 \\
&\le \sum_{i=1}^{M-1} ||(\hat{u}_i, \hat{\lambda}_i) - (u_i^*, \lambda_i^*)||^2 + K_M^2||(u_M, q_{M-1}, \lambda_M) - (u_M^*, q^*, \lambda_M^*)||^4 \\
&\le \sum_{i=1}^{M-1} ||(\hat{u}_i, \hat{\lambda}_i) - (u_i^*, \lambda_i^*)||^2 + 2K_M^2||q_{M-1} - q^*||^4 + 2K_M^2||(u_M, \lambda_M) - (u_M^*, \lambda_M^*)||^4 \\
&\le \sum_{i=1}^{M-2} ||(\hat{u}_i, \hat{\lambda}_i) - (u_i^*, \lambda_i^*)||^2 + ||(\hat{u}_{M-1}, \hat{\lambda}_{M-1}) - (u_{M-1}^*, \lambda_{M-1}^*)||^2 + ||q_{M-1} - q^*||^2 \\
&+K_M^4||q_{M-1} - q^*||^6 + 2K_M^2||(u_M, \lambda_M) - (u_M^*, \lambda_M^*)||^4 \\
&\le \sum_{i=1}^{M-2} ||(\hat{u}_i, \hat{\lambda}_i) - (u_i^*, \lambda_i^*)||^2 + ||(\hat{u}_{M-1}, q_{M-1}, \hat{\lambda}_{M-1}) - (u_{M-1}^*, q^*, \lambda_{M-1}^*)||^2 \\
&+K_M^4||q_{M-1} - q^*||^6 + 2K_M^2||(u_M, \lambda_M) - (u_M^*, \lambda_M^*)||^4 \\
&\le \sum_{i=1}^{M-2} ||(\hat{u}_i, \hat{\lambda}_i) - (u_i^*, \lambda_i^*)||^2 + 2K_{M-1}^2||(\hat{u}_{M-1}, \hat{\lambda}_{M-1}) - (u_{M-1}^*, \lambda_{M-1}^*)||^4
\end{aligned}$$

$$+2K_{M-1}^2||q_{M-2}-q^*||^4+2K_M^2||(u_M,\lambda_M)-(u_M^*,\lambda_M^*)||^4+K_M^4||q_{M-1}-q^*||^6$$

$$\cdots,\cdots$$

$$\cdots,\cdots$$

$$\leq \sum_{i=1}^{M} 2K_i^2||(u_i,\lambda_i)-(u_i^*,\lambda_i^*)||^4+2K_1^2||q_0-q^*||^4+\sum_{i=1}^{M-1} K_i^4||q_i-q^*||^6.$$

Let $K=\max_i 2K_i^2$, we have

$$||(\hat{u},\hat{q},\hat{\lambda})-(u^*,q^*,\lambda^*)||^2 \leq K||(u,q,\lambda)-(u^*,q^*,\lambda^*)||^4+\sum_{i=1}^{M-1} K_i^4||q_i-q^*||^6.$$

Denote the diameter of $U(u^*,q^*,\lambda^*)$ by $diam(U)$, we have

$$||(u,q,\lambda)-(u^*,q^*,\lambda^*)||/diam(U) \leq 1. \tag{4.24}$$

Since $||q_i-q^*|| \leq ||(u_i,q_i,\lambda_i)-(u^*,q^*,\lambda^*)||$, by the same procedure, we conclude that

$$||(\hat{u},\hat{q},\hat{\lambda})-(u^*,q^*,\lambda^*)|| \leq \tilde{K}||(u,q,\lambda)-(u^*,q^*,\lambda^*)||^2, \tag{4.25}$$

where $\tilde{K}$ depends on K and $diam(U)$. This finishes the proof. □

With the help of the above lemma, all analysis reduces to Algorithm 1. Therefore, all convergent results hold for the modified algorithm.

Theorem 4.5.1 *Let (H1), (H2) and (H3) hold, if* $||(u^{s,0},q^0,\lambda^0)-(u^{s,*},q^*,\lambda^*)||$ *is sufficiently small, then Algorithm 2 (Modified Algorithm) is well-defined and satisfies*

$$||(u^{s,n+1},q^{n+1},\lambda^{n+1})-(u^{s,*},q^*,\lambda^*)|| \leq K||(u^{s,n},q^n,\lambda^n)-(u^{s,*},q^*,\lambda^*)||^2 \tag{4.26}$$

for some K *depending on* c*, and* $n=1,2,\cdots$.

Proof: Let $\hat{\eta}$ be the largest radius for a ball centered at $(u^{s,*},q^*,\lambda^*)$ and contained in $U(u^{s,*},q^*,\lambda^*)$. Introduce

$$\eta=\min(\frac{\hat{\eta}}{\sqrt{a}},\frac{1}{\tilde{K}a}), \tag{4.27}$$

where $a = \max(2, 1 + 2c^2\Gamma^2)$, and $\tilde{K}$ is the constant in lemma 4.5.2, Γ is the local Lipschitz constant of e.

Assume that

$$||(u^{s,0}, q^0, \lambda^0) - (u^{s,*}, q^*, \lambda^*)|| < \eta. \tag{4.28}$$

We proceed by induction and the case $n = 0$ follows from the general arguments below. Suppose

$$||(u^{s,n}, q^n, \lambda^n) - (u^{s,*}, q^*, \lambda^*)|| < \eta, \tag{4.29}$$

then we have

$$\begin{aligned}
&||(u^{s,n}, q^n, \tilde{\lambda}) - (u^{s,*}, q^*, \lambda^*)||^2 \\
&= ||(u^{s,n}, q^n) - (u^{s,*}, q^*)||^2 + ||\tilde{\lambda} - \lambda^*||^2 \\
&\leq ||(u^{s,n}, q^n) - (u^{s,*}, q^*)||^2 + 2c^2||e(u^{s,n}, q^n) - e(u^{s,*}, q^*)||^2 + 2||\lambda^n - \lambda^*||^2 \\
&\leq ||(u^{s,n}, q^n) - (u^{s,*}, q^*)||^2 + 2c^2\Gamma^2||(u^{s,n}, q^n) - (u^{s,*}, q^*)||^2 + 2||\lambda^n - \lambda^*||^2 \\
&\leq ||(u^{s,n}, q^n, \lambda^n) - (u^{s,*}, q^*, \lambda^*)||^2 \\
&\leq \hat{\eta}^2.
\end{aligned}$$

Therefore, lemma 4.5.2 is applicable, we get

$$\begin{aligned}
&||(u^{s,n+1}, q^{n+1}, \lambda^{n+1}) - (u^{s,*}, q^*, \lambda^*)|| \\
&\leq \tilde{K}||(u^{s,n}, q^n, \tilde{\lambda}) - (u^{s,*}, q^*, \lambda^*)||^2 \\
&\leq \tilde{K}a||(u^{s,n}, q^n, \lambda^n) - (u^{s,*}, q^*, \lambda^*)||^2 \\
&\leq \eta.
\end{aligned}$$

Let $K = \tilde{K}a$, the theorem is proved. □

Comparing with the original algorithm, this theorem establishes the same order convergence result for the modified one. As pointed before, the modified algorithm solves a sub-problem $(\tilde{\mathcal{P}}_\beta)$ in each iteration, hence saves some computational resources. However, from the proof of lemma 4.5.2 and theorem 4.5.1, we know that the convergence radius is further shrunk and the constant K is larger than before, which requires a better initial value. At this point, it is advantageous to combine this second order method with the steepest decent method in Chapter 3.

Chapter 5

Conclusions and Future Work

We have mainly studied two optimization methods for the inverse medium scattering problem - the steepest decent method and the augmented Lagrangian method. Taking problem structures into account, these general methods are shown to be efficient and robust, both numerically and theoretically. Despite these advantages, there still remains unsettled difficulties for further improvements.

First of all, it is clear that our analysis heavily depends on the continuity property of the solution operator $\mathcal{M}$, especially the explicit dependency on wave number k. In fact, this bound can greatly affect the choice of step size β in the steepest decent method as well as an alleviated assumption in the augmented Lagrangian method. Unfortunately, the proof in [BL] relies on the Fredholm alternative thus cannot provide such an explicit estimate unless the wave number k is small(at least less than 1).

Secondly, to overcome the ill-posedness, we employed the Tikhonov regularization or Landweber iteration. These regularization methods are well developed for linear equations that are ill-posed in the Hadamard sense, i.e. (i) solution may not exist; (ii) solution may not be unique; (iii) solution may not be stable. For operators, (iii) means that the inverse operator is unbounded. And the idea of above regularization methods is to approximate the inverse by bounded operators. Nevertheless, as pointed out in [Ch1], there exists another new type of ill-posedness for the Helmholtz problem - the shielding effect. The paper carefully discussed the obstacle scattering case to illustrate this effect. Roughly speaking, this ill-posedness comes out as two obstacles can generate very similar eigen-systems of corresponding scattering operators. For the medium scattering problem, it is quite possible that this new ill-posedness still exits. The different mechanism calls for new techniques beyond the regime of Tikhonov regularization. On the other hand, for non-linear problems as we encountered, we should borrow many existing proper methods to improve our results further(e.g. [TLY]).

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