On the Well-posedness Theory of Compressible Navier-Stokes System and Related Topics

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Abstract

In this thesis, we study the compressible Navier-Stokes equations for quantum fluids. First, we will introduce a derivation of quantum Navier-Stokes equations from a Wigner-BGK model by a moment method and a Chapman-Enskog expansion around the quantum equilibrium. Secondly, we will prove the global-in-time existence of weak solutions to barotropic quantum Navier-Stokes equations in a two or three-dimensional torus for finite energy initial data. It is an improvement of the result by Jüngel [Global weak solutions to compressible Navier-Stokes equations for quantum fluids, SIAM J. Math. Anal., 42(2010), no.3, pp.1025–1045], where the restriction "the viscosity constant must be smaller than the scaled Planck constant" can be removed here after we get a new energy estimate. Also, the result holds for more general external forces. Finally, we will show the global existence and large time behavior of weak solutions to the compressible Navier-Stokes-Poisson equations for quantum fluids in a two-dimensional torus.

摘 要

本文討論了一類關於量子流的高維可壓Navier-Stokes方程的弱解的整體存在性以及大時間行為。首先，我們大致地給出了從Wigner-BGK模型到此方程的推導。然後，我們證明了此方程在二維環面或三維環體上允許大初值的整體弱解的存在性。基於一個更細的能量估計，我們改進了Jungel [32] 中的結果：1. 不需要限制粘性係數小於普朗克常數；2. 對更一般的外力也成立。最後，我們討論了在二維環面上關於量子流的可壓Navier-Stokes-Poisson方程的弱解的整體存在性以及大時間行為。

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Contents

Introduction

The motion of fluids can be described through a system of partial differential equations. One important system is compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = \mathrm{div}\mathbf{S}, \\ (\rho E)_t + \mathrm{div}(\rho u E + uP) = \mathrm{div}(k\nabla\theta) + \mathrm{div}(u\mathbf{S}), \end{cases} \tag{0.1}$$

which express the fundamental physical laws in continuum mechanics: the conservation of mass, momentum and energy. In (0.1), ρ and u are the density and velocity of the fluid respectively. The total energy E is given by $E = \frac{1}{2}|u|^2 + e$, where e is the internal energy. $P = P(\rho, e)$ is the pressure. θ is the temperature, $k = k(\theta) > 0$ is the thermal conductivity, and $\mathbf{S}$ is the shear stress tensor with the form

$$\mathbf{S} = \mu(\nabla u + \nabla^\top u) + \lambda(\mathrm{div}u)\mathbf{I},$$

where $\mathbf{I}$ is the $d \times d$ identity matrix, μ and λ are the Lamé viscosity coefficients satisfying the following physical constraints

$$\mu > 0, \ \ 2\mu + d\lambda \geqslant 0.$$

The relation among ρ, P, θ and e is given by the equations of state for the fluid concerned and the second law of thermodynamics.

The full Navier-Stokes equations (0.1) has been investigated by many mathematicians in a large variety of contexts, such as the earlier work by Kazhikhov and Shelukhin [36], the global existence in one-dimensional case by Hoff [20, 22] for small initial data. Chen, Hoff and Trivisa [8] gave the time-independent estimates for large discontinuous initial data. For the multi-dimensional system, Matsumura and Nishida [42] first obtained the global classical solutions for initial data close to a non-vacuum equilibrium. Later, Hoff [24] proved a global existence result with small, discontinuous initial data. In the case that the density is allowed to vanish initially, Feireisl [16, 17] showed the global existence of weak solutions in sense that the energy inequality instead of energy equality holds, under some constrains on P, provided $\gamma > \frac{d}{2}$. His proof based on the work by Lions [39], which showed the existence of weak solutions to the isentropic Navier-Stokes equations. Concerning the full Navier-Stokes equations with vacuum states, Cho and Kim [9] constructed a local strong solution, as long as a suitable compatibility condition is satisfied initially. Recently, in one-dimensional case under special pressure, viscosity and heat conductivity, Wen and Zhu [51] obtained a uniqueness global classical solution with large initial data and vacuum.

If neglecting both heat conduction and dissipation of mechanical energy, we obtain the following isentropic compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u = 0, \end{cases} \tag{0.2}$$

where $P(\rho) = A\rho^\gamma$ with $A > 0,\ \gamma > 1$.

The behavior of the solution to (0.2) is closely related to our real world, which displays an amazing range of phenomena, from ordinary patterns to turbulent states. An important feature of (0.2) is that it is a couple hyperbolic-parabolic system for non-vacuum region and maybe degenerate in the presence of vacuum. There are huge literatures to (0.2). Since it is difficult to deal with vacuum, the first results were obtained with initial data bounded away from zero. The existence of global in time solutions for Navier-Stokes equations was first addressed in dimension one for sufficient smooth initial data by Kazhikhov and Shelukhin [36], and for discontinuous initial data by Serre [45, 46] and Hoff [19]. The two-dimensional case was done by Vaǐgant and Kazhikhov [50] for large initial data and special viscosity coefficients (still in the case away from vacuum). For higher-dimensional case, the local existence and uniqueness of classical solutions are known in [44, 47] in the absence of vacuum. In 1983, Matsumura and Nishida [42] obtained the global classical solutions for initial data close to a non-vacuum equilibrium in some Sobolev space H^s with arbitrary large s. In particular, the theory requires that the solution has small oscillations from a uniform non-vacuum state so that the density is strictly away from the vacuum and the gradient of the density remains bounded uniformly in time. This result was generalized to discontinuous data by Hoff in a series of papers (See [19, 21, 23]). Later, Duanchin [12] obtained the existence and uniqueness of global solutions in a functional space invariant by the nature scaling of associated equations.

Concerning arbitrary initial data that may vanish, the major breakthrough is due to Lions [39], where he showed the existence of global in time weak solutions provided that the specific heat ratio γ is appropriately large ($\gamma \geqslant \frac{3d}{d+2}$, $d = 2, 3$). The restriction of γ is to show the existence of renormalized solutions introduced by DiPerna and Lions [14]. This result was improved later by Feireisl [16] for $\gamma > \frac{d}{2}$. Other results provide the full range $\gamma > 1$ under symmetry assumptions, see [28, 48] for instance. Recently, under the additional assumption that the viscosity coefficients μ and λ satisfy $\mu > \max\{4\lambda, -\lambda\}$, and if the far field density is away from vacuum, Hoff [25] obtained a new type of global weak solutions with small energy. Such weak solutions have extra regularity information compared with those large weak ones constructed by Lions [39] and Feireisl [16]. Note that here the weak solutions may contain vacuum though the spatial measure of the set of vacuum has to be small. For strong solutions with the initial density allowing vacuum, it was shown by Cho, Choe and Kim in [9] that the system (0.2) admits a local strong solution as long as the initial data satisfies a suitable compatibility condition. Moreover, Kim and Choe [11] obtained a local classical solution in a bounded or unbounded domain of $\mathbb{R}^3$, where the initial density does not need to be bounded away from vacuum. Very recently, Huang, Li and Xin [26] established the global existence and uniqueness of classical solutions to the three-dimensional Cauchy problem for (0.2). Note that the initial density is allowed to vanish and the spatial measure of the set of vacuum can be arbitrarily large, in particular, the initial density can even have compact support. This result generalizes previous results on classical solutions for initial densities being strictly away from vacuum, and is the first result for global classical solutions which may have large oscillations and can contain vacuum states. Later, for the two-dimensional case, Luo in her Ph.D thesis [41] showed that for spherically symmetric case, the local smooth solution $(\rho, u) \in C^1([0,T]; H^s)$ $(s > 3)$ to (0.2) has to blow up in finite time with initial density having compact support.

It is noted that, in dealing with large amplitude solutions, one has to face the possible appearance of a vacuum state. However, as observed in [21, 38, 52], the compressible Navier-Stokes equations with constant viscosity coefficients (i.e.(0.2)) behave singularly in the presence of vacuum. So in order to understand the behavior of fluids near vacuum, one can choose an alternative system of (0.2). As presented in [38], in deriving the compressible Navier-Stokes equations from the Boltzmann equations by the Chapman-Enskog expansions, the viscosity depends on the temperature, and for isentropic cases, this dependence is translated to the dependence of the density by the law of Boyle and Gay-Lussac for ideal gas. So we can modified (0.2) to the following density-dependent system:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P - 2\mathrm{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho)\mathrm{div}u) = 0. \end{cases} \tag{0.3}$$

In geophysical flows, many mathematical models correspond to (0.3) (see [1, 2, 32]). In particular, the viscous Saint-Venant system for shallow water is expressed exactly as (0.3) with $d = 2, \mu(\rho) = \rho, \lambda(\rho) = 0$, and $\gamma = 2$. Shallow water equations are to describe vertically averaged flows in three-dimensional shallow domains in term of the mean velocity u and the variation of the depth ρ due to the free surface, which is widely used in geophysical flows. Global smooth solutions for data close to equilibrium were established in [49], and related topics have been extensively studied in [1, 2], and the references therein. Nevertheless, little is known about the global existence of weak solutions for large data to the shallow water equations or more generally to the multi-dimensional compressible Navier-Stokes equations (0.3) (d = 2, 3). In fact, the system (0.3) is highly degenerate at vacuum and when dealing with vanishing viscosity coefficients on vacuum, the velocity cannot even be defined when the density vanishes, and hence we will have no uniform estimates for the velocity.

For one-dimensional compressible Navier-Stokes equations (0.3) with $\mu(\rho) = \rho^\alpha, \lambda(\rho) = 0 (\alpha \in (0, 1))$, there is much literature on the well-posedness theory of the solutions (see [29, 30, 36, 38, 53–55], and the references therein). In particular, initial boundary value problems for one-dimensional (0.3) with $\mu(\rho) = \rho^\alpha, \lambda(\rho) = 0 (\alpha > \frac{1}{2})$, were studied by Li, Li, and Xin in [37], and interesting phenomena of vacuum vanishing and blowup of solutions were found there. When it comes to multi-dimensional case, Bresch, Desjardins and Lin in [1] showed the L^1 stability of weak solutions for the Korteweg system ($\mu(\rho) = \nu\rho,\ \lambda = 0$ with the Korteweg stress tensor $k\rho\nabla\Delta\rho$), and their result was later improved in [2] to include the case of vanishing capillarity ($k = 0$) but with an additional quadratic friction term $r\rho|u|u$. Under the additional constraint on the viscosity coefficients that

$$\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho)), \tag{0.4}$$

an interesting new entropy estimate is established in [1], which provides some high regularity for the density. Adding the constraint (0.4), Mellet and Vasseur [43] proved the L^1 stability of weak solutions of the system (0.3) based on the new entropy estimate, extending the corresponding L^1 stability results of [1,2] to the case r = k = 0. Recently, Guo, Jiu and Xin [18] showed the existence of spherically symmetric solutions to (0.3) adding the constraint (0.4).

There is a related model for quantum fluids, called compressible quantum Navier-Stokes system, which reads as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - 2\varepsilon^2 \rho \nabla \left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) - 2\nu \operatorname{div}(\rho D(u)) = \rho f. \end{cases} \tag{0.5}$$

This system consists of the mass equation and the momentum equation including a third-order quantum term $-2\varepsilon^2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$ with density-dependent viscosity coefficients $\mu(\rho) = \nu\rho$, $\lambda = 0$. ρ, u are the particle density and particle velocity of the quantum fluid respectively. The function $P(\rho)$ is the pressure, and f describes external forces. The physical parameters are the (scaled) Planck constant $\varepsilon > 0$ and the viscosity constant $\nu > 0$. The system is derived from a Wigner equation, and there are many different derivations, such as [6, 34, 35]. In Chapter 2, we will introduce a derivation by using a moment method and a Chapman-Enskog expansion around the quantum equilibrium. Recently, there are some results on the existence of global solutions to the system (0.5). The existence of global-in-time classical solutions in one-dimensional case has been shown in [31] under the assumption $\varepsilon = \nu$. For multi-dimensional case, Jüngel [32] obtained the the global-in-time existence of weak solutions to (0.5) in a two or three-dimensional torus for large data. The main idea of the existence analysis is to reformulate the quantum Navier-Stokes equations (0.5) by means of a so-called effective velocity

$$w = u + \nu\nabla \log \rho,$$

leading to a viscous quantum Euler system

$$\begin{cases} \rho_t + \operatorname{div}(\rho w) = \nu \Delta \rho, \\ (\rho w)_t + \operatorname{div}(\rho w \otimes w) + \nabla P(\rho) + 2(\nu^2 - \varepsilon^2)\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \nu \Delta(\rho w) = \rho f, \end{cases} \tag{0.6}$$

The advantage of the new formulation (0.6) is that we can apply the maximum principle to the parabolic equation to deduce strict positivity of the density ρ if the initial density ρ_0 is strictly positive and the velocity w is smooth. The global existence of weak solutions to the viscous quantum Euler model (0.6) is shown by using the Faedo-Galerkin method and weak compactness techniques. However, it needs the restriction that $\varepsilon > \nu$. The case $\varepsilon = \nu$ is treated in [15] by performing the semiclassical limit $(\varepsilon - \nu) \to 0$, and Jiang [27] treated the remaining case $\varepsilon < \nu$ based on an estimate given in [15]. In Chapter 3 of this thesis, we will show the global existence of weak solutions to (0.5) for any $\varepsilon, \nu > 0$, where we get a new energy estimate, and do not need to compare ε with ν.

The thesis is organized as follows. In Chapter 1, we give some useful inequalities and fundamental lemmas which will be used in the thesis. In Chapter 2, we sketch a derivation of the compressible quantum Navier-Stokes model from a Wigner-BGK model by a moment method and a Chapman-Enskog expansion around the quantum equilibrium which is shown in [34]. In Chapter 3, we prove the global-in-time existence of weak solutions to barotropic Navier-Stokes equations in a two or three-dimensional torus for finite energy initial data. First, we reformulate the quantum Navier-Stokes equations to a viscous quantum Euler system, which has some advantages; next, we construct the approximate solutions by using the Faedo-Galerkin method and obtain an energy estimate, which is crucial to the thesis; finally, we get the weak solutions by weak compactness techniques. It is an improvement of the result in [32] since we can ignore the constraint $\varepsilon > \nu$, this is possible due to a new energy estimate which is different from the one in [32]. Chapter 4 is concerned with the compressible Navier-Stokes-Poisson equations for quantum fluids, where we show the global existence and large time asymptotic behavior of weak solutions in a two-dimensional torus. Finally, some comments about my following work are given in Chapter 5.

Chapter 1

Preliminaries

In this chapter, we will give some notations and recall some useful inequalities and fundamental lemmas to be used in the thesis.

1.1 Notations and function spaces

In this thesis, C is always an unspecified constant that may vary from line to line. If C depends on some special parameters $x_1, \cdots, x_k$, we write $C(x_1, \cdots, x_k)$.

For vector-valued functions $u = (u_1, u_2, \cdots, u_d), v = (v_1, v_2, \cdots, v_d)$ of $\mathbb{R}^d$, define

$$u \otimes v = \{u_i v_j\}_{d\times d}, \quad \nabla u : \nabla v = \sum_{i,j=1}^{d} \partial_i u_j \partial_i v_j,$$

and

$$(u \cdot \nabla)v = \sum_{i=1}^{d} u_i \partial_i v, \quad D(u) = \frac{(\nabla u) + (\nabla u)^T}{2}.$$

$L^p(\Omega), W^{k,p}(\Omega)(1 \le p \le +\infty)$ are the usual Sobolev spaces, which are equipped with the norm $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$ respectively. $[L^p(\Omega)]^d, [W^{k,p}(\Omega)]^d$ are the corresponding Sobolev spaces with elements being vector-valued functions. In many cases, we do not distinguish the vector-valued functions and scalar-valued functions very strictly. In particular, denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$. $(H^k(\Omega))^*$ stands for the dural space of $H^k(\Omega)$.

Let X denote a real Banach space, with norm $\|\cdot\|$. The space $L^p(0,T;X)$ consists of all strongly measurable functions $\mathbf{u}:[0,T]\to X$ with

$$\|\mathbf{u}\|_{L^p(0,T;X)} = \left(\int_0^T \|\mathbf{u}(\mathbf{t})\|^p dt\right)^{\frac{1}{p}} < \infty,$$

The space $C([0,T];X)$ consists of all continuous $\mathbf{u}:[0,T]\to X$ with

$$\|\mathbf{u}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\mathbf{u}(\mathbf{t})\| < \infty.$$

1.2 Some useful inequalities

We first introduce the Young's inequality.

Theorem 1.2.1 (Young's inequality) *Let* $1 < p, q < \infty$, $\dfrac{1}{p}+\dfrac{1}{q}=1$. *For any positive number a and b, it holds that*

$$ab \le \frac{a^p}{p}+\frac{b^q}{q}.$$

The Young's inequality yields immediately the following well-known Hölder's inequality.

Theorem 1.2.2 (Hölder's inequality) *Given* Ω *an arbitrary domain in* $\mathbb{R}^d$. *Assume* $1 \le p, q \le \infty$, $\dfrac{1}{p}+\dfrac{1}{q}=1$. *If* $u\in L^p(\Omega)$, $v\in L^q(\Omega)$, *then we have*

$$\int_\Omega |uv|dx \le \|u\|_{L^p(\Omega)}\cdot\|v\|_{L^q(\Omega)}.$$

Thus the interpolation inequality is shown.

Theorem 1.2.3 (Interpolation inequality) *Assume* $1 \le s, r, t \le \infty$ *and*

$$\frac{1}{r}=\frac{\theta}{s}+\frac{1-\theta}{t}.$$

Suppose $u\in L^s(\Omega)\cap L^t(\Omega)$. *Then* $u\in L^r(\Omega)$, *and*

$$\|u\|_{L^r(\Omega)} \le \|u\|^{\theta}_{L^s(\Omega)}\cdot\|u\|^{1-\theta}_{L^t(\Omega)}. \tag{1.2.1}$$

One more general of the interpolation inequality is the following one.

Theorem 1.2.4 (Gagliardo-Nirenberg inequality) *Let Ω be a $C^{0,1}$ domain, $m \in \mathbb{N}$, $1 \leqslant p, q, r \leqslant \infty$. Then there exists a constant $C > 0$ such that for all $u \in W^{m,q}(\Omega) \cap L^r(\Omega)$, it holds that*

$$\|D^\beta u\|_{L^p(\Omega)} \le C\|u\|^\theta_{W^{m,q}(\Omega)} \cdot \|u\|^{1-\theta}_{L^r(\Omega)},$$

where $0 \leqslant |\beta| < m$, $\theta \in [|\beta|/m, 1)$ and

$$\frac{|\beta|}{d} - \frac{1}{p} = \theta\left(\frac{m}{d} - \frac{1}{q}\right) - (1-\theta)\frac{1}{r}$$

The interpolation inequality is closely related to the Sobolev embedding theorem.

Theorem 1.2.5 (Sobolev embedding theorem) *Let Ω be a $C^{0,1}$ bounded domain in $\mathbb{R}^d$. Then,*

(1) *if $kp < d$, the space $W^{k,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, $p^* = dp/(d-kp)$, and compactly embedded in $L^q(\Omega)$ for any $q < p^*$;*

(2) *if $0 \le m < k - \dfrac{d}{p} < m+1$, the space $W^{k,p}(\Omega)$ is continuously embedded in $C^{m,\beta}(\overline{\Omega})$ for any $\beta < \alpha$.*

For functions in $W^{1,p}(\Omega)$ with some special homogeneous properties, there are Poincaré's inequalities.

Theorem 1.2.6 (Poincaré's inequalities) *Let Ω be a bounded, connected open subset of $\mathbb{R}^d$ with a C^1 boundary $\partial\Omega$. Assume $1 \le p \le \infty$. Then for each function $u \in W^{1,p}(\Omega)$, then there exists a constant C, depending only on d, p, Ω, such that*

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)},$$

where $(u)_\Omega$ =average of u over Ω.

For each $u \in W^{1,p}_0(\Omega)$, there exists a constant $\tilde{C}$, depending only on d, p, Ω, such that

$$\|u\|_{L^p(\Omega)} \le \tilde{C}\|\nabla u\|_{L^p(\Omega)}.$$

The following famous Gronwall's Lemma will be used frequently in this thesis.

Theorem 1.2.7 (Gronwall's Lemma)

a) (Differential Version) Let us assume h, r are integrable on (a, b) and nonnegative a.e. in (a, b). Further assume that $y \in C([a, b])$ and $y' \leq L^1(a, b)$ and that the following inequality is satisfied:

$$y'(t) \leq h(t) + r(t)y(t) \text{ for a.a.} t \in (a, b).$$

Then

$$y(t) \leq \left[y(a) + \int_a^t h(s) \exp\left(-\int_a^s r(\tau)\mathrm{d}\tau\right) \mathrm{d}s\right] \exp\left(\int_a^t r(s)\mathrm{d}s\right), \quad t \in [a, b].$$

b) (Integral Form) Let us assume h is continuous on $[a, b]$, r is integrable on (a, b) and nonnegative a.e. in (a, b). Further assume that $y \in C([a, b])$ satisfies the following inequality:

$$y(t) \leq h(t) + \int_a^t r(s)y(s)\mathrm{d}s \text{ for a.a.} t \in (a, b).$$

Then

$$y(t) \leq h(t) + \int_a^t h(s)r(s) \exp\left(\int_s^t r(\tau)\mathrm{d}\tau\right) \mathrm{d}s, \quad t \in [a, b].$$

c) (Local Version) Let $T, \alpha, c_0 > 0$ be given constants and let $h \in L(0, T)$ with $h \geq 0$ a.e. in $[0, T]$, for nonnegative function $y \in C^1([0, T])$ satisfy

$$y'(t) \leq h(t) + c_0 y(t)^{1+\alpha} \text{ for a.a. } t \in (0, T).$$

Let $t_0 \in [0, T]$ be such that $\alpha c_0 H(t_0)^\alpha t_0 < 1$, where

$$H(t) = f(0) + \int_0^t h(s)\mathrm{d}s.$$

Then for all $t \in [0, t_0]$ there holds

$$f(t) \leq H(t) + H(t)\left((1 - \alpha c_0 H(t)^\alpha t)^{-\frac{1}{\alpha}} - 1\right).$$

Next, we will show two inequalities (see [32, 33]), which are important in the thesis.

Theorem 1.2.8 *Let f be a strictly positive function on $\mathbb{T}^d$ $(d \geqslant 1)$ such that $\sqrt{f} \in H^2(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$, then the following inequalities hold:*

$$\int_{\mathbb{T}^d} f|\nabla^2 \log f|^2 dx \geqslant \frac{16d-4}{d^2+2d} \int_{\mathbb{T}^d} |\nabla^2 \sqrt{f}|^2 dx \tag{1.2.2}$$

$$\int_{\mathbb{T}^d} f^2|\nabla^2 \log f|^2 dx \geqslant \frac{64d-16}{(d+2)^2} \int_{\mathbb{T}^d} |\nabla \sqrt{f}|^4 dx \tag{1.2.3}$$

1.3 Fundamental lemmas

Finally, we introduce some lemmas which will be used in the thesis.

Theorem 1.3.1 (Banach fixed point theorem) *Let U be a complete subset of a normed space X, and let $A : U \to U$ be a contraction operator. Then A has a unique fixed point.*

Theorem 1.3.2 (Aubin-Lions lemma) *Assume $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$, where X, Z are reflexive Banach spaces, X is dense in Y. Set*

$$W = \{u \in L^p(0,T;X), u_t \in L^q(0,T;Z),\ 1 < p, q < \infty\}.$$

Then $W \hookrightarrow\hookrightarrow L^p(0,T;Y)$.

Chapter 2

Compressible Navier-Stokes Equations for Quantum Fluids

In this chapter, we will sketch a derivation of the compressible quantum Navier-Stokes equations from a Wigner-BGK model by a moment method and a Chapman-Enskog expansion around the quantum equilibrium which is shown in [34].

2.1 Background

A quantum fluid is a many-particle system in whose behavior not only the effects of quantum mechanics, but also those of quantum statistics, are important. It is known that it is essential to use quantum mechanics to describe the actual structure of atoms or molecules; a classical description fails to account for even the qualitative properties. However, if we consider the atoms or molecules as themselves simple entities and ask about their dynamics, we find that classical mechanics is a good approximation.

Quantum fluid modeling has become very attractive due to Bose-Einstein condensation and quantum fluid models are used to describe superfluids (such as helium-4 at low temperatures) and quantum semiconductors. Recently, two interesting dissipative quantum fluid models have been established: the viscous quantum Euler system and the quantum Navier-Stokes equations. In the following, we will introduce a derivation of the quantum Navier-Stokes equations.

2.2 Derivation of model

There are some derivations of quantum Navier-Stokes system, one can derive it from the Wigner-Fokker-Planck equation using a moment method (such as [35]), or you can derive it from a Wigner-BGK model by a moment method and a Chapman-Enskog expansion (see [34]). Here, we sketch the latter.

We start from the Wigner-BGK equation:

$$w_t + p \cdot \nabla_x w + \theta[V]w = \frac{1}{\nu}(M[w] - w) + \frac{\nu}{\tau_0}(\Delta_p w + \mathrm{div}_p(pw)), \tag{2.2.1}$$

where $w(x, p, t)$ is the Wigner function in the phase-space variables $(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3$ and time $t > 0$. $\nu > 0$ is the scaled mean free path and $\tau_0 > 0$ is a relaxation time. The potential operator $\theta[V]$ is a pseudo-differential operator

$$(\theta[V]w)(x, p, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\delta V)(x, \eta, t) w(x, p', t) e^{i(p-p')\cdot\eta} dp' d\eta$$

modeling the influence of the electric potential $V = V(x, t)$ with

$$(\delta V)(x, \eta, t) = \frac{i}{\varepsilon}(V(x + \frac{\varepsilon}{2}\eta, t) - V(x - \frac{\varepsilon}{2}\eta, t)).$$

Here $\varepsilon > 0$ denotes the scaled Planck constant. The first term on the right-hand side of (2.2.1) describes a relaxation process towards the quantum equilibrium state $M[w]$, which has been introduced by Degond and Ringhofer [13]. It is the formal maximizer of the quantum free energy subject to the constraints of given mass, momentum, and energy. More precisely, let

$$Exp\ w = W(\exp W^{-1}(w)), \qquad Log\ w = W(\log W^{-1}(w)),$$

where W is the Wigner transform and W^{-1} is its inverse. The quantum free energy is given by

$$S(w) = -\frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} w(x, p, \cdot) \left((Log\ w)(x, p, \cdot) - 1 + \frac{|p|^2}{2} - V(x, \cdot) \right) dx dp.$$

For a given Wigner function w, let $M[w]$ be the formal maximizer of $S(g)$, where g satisfies

$$\int_{\mathbb{R}^3} w\mathbf{K}(p) dp = \int_{\mathbb{R}^3} g\mathbf{K}(p) dp, \qquad \mathbf{K}(p) = (1, p, \frac{1}{2}|p|^2).$$

If such a solution exists, then it has the form

$$M[w](x, p, t) = Exp\ \left(A(x, t) - \frac{|p - U(x, t)|^2}{2\mathrm{T}(x, t)} \right),$$

where A, U, T are Lagrange multipliers.

To simplify the notations, we define for functions f,

$$\langle f(p)\rangle = \frac{1}{(2\pi\varepsilon)^3}\int_{\mathbb{R}^3} f(p)dp.$$

It is shown that the collision operator $\frac{1}{\nu}(M[w]-w)$ conserve mass, momentum and energy, i.e.

$$\langle \frac{1}{\nu}(M[w]-w)\mathbf{K}(p)\rangle = 0.$$

Next, multiplying the Wigner-BGK equation (2.2.1) by $\mathbf{K}(p)$ (i.e. $1, p, |p|^2/2$ respectively), and then integrating over $p \in \mathbb{R}^3$, we obtain the moment equations

$$\begin{cases} \langle w\rangle_t + \mathrm{div}_x\langle pw\rangle + \langle \theta[V]w\rangle = 0, \\ \langle pw\rangle_t + \mathrm{div}_x\langle wp\otimes p\rangle + \langle p\theta[V]w\rangle = -\dfrac{\nu}{\tau_0}\langle pw\rangle, \\ \langle \dfrac{1}{2}|p|^2 w\rangle_t + \mathrm{div}_x\langle \dfrac{1}{2}|p|^2 pw\rangle + \langle \dfrac{1}{2}|p|^2\theta[V]w\rangle = -\dfrac{\nu}{\tau_0}\langle |p|^2 w - 3w\rangle, \end{cases} \tag{2.2.2}$$

where $p\otimes p$ denotes the matrix with components $p_j p_k$. The particle density ρ, the momentum ρu and the energy density ρe are defined by

$$\rho = \langle w\rangle,\quad \rho u = \langle pw\rangle,\quad \rho e = \langle \frac{1}{2}|p|^2 w\rangle.$$

The variable $u = (\rho u)/\rho$ and $e = (\rho e)/\rho$ are the macroscopic velocity and the macroscopic energy respectively.

It is shown in [34] that the potential operator $\theta[V]$ can be simplified in terms of the moments $\rho, \rho u$ and ρe :

Lemma 2.2.1 **([34])** *The moments of* $\theta[V]$ *can be expressed as*

$$\begin{aligned} \langle \theta[V]\rangle &= 0, \\ \langle p\theta[V]w\rangle &= -\rho\nabla_x V, \\ \langle \frac{1}{2}|p|^2\theta[V]w\rangle &= -\rho u\nabla_x V, \\ \langle p\otimes p\theta[V]w\rangle &= -\rho u\otimes\nabla_x V - \rho\nabla_x V\otimes u, \\ \langle \frac{1}{2}p|p|^2\theta[V]w\rangle &= -(\langle p\otimes pw\rangle + \rho e\mathbf{I})\nabla_x V + \frac{\varepsilon^2}{8}\rho\nabla_x\Delta_x V. \end{aligned}$$

It remains to calculate $\langle wp \otimes p\rangle$ and $\langle \frac{1}{2}|p|^2 pw\rangle$. To this end, we employ the Chapman-Enskog expansion

$$w = M[w] + \nu g$$

and introduce the quantum stress tensor $P = \langle (p-u)\otimes(p-u)M[w]\rangle$ and quantum heat flux $q = \langle \frac{1}{2}(p-u)|p-u|^2 M[w]\rangle$. Since $\langle M[w]\rangle = \langle w\rangle = \rho$, $\langle pM[w]\rangle = \rho u$ and $\langle \frac{1}{2}|p|^2 M[w]\rangle = \rho e$, a straight calculation gives

$$\begin{aligned} \langle wp \otimes p\rangle &= P + \rho u \otimes u + \nu \langle p \otimes pg\rangle, \\ \langle \frac{1}{2}|p|^2 pw\rangle &= (P + \rho e\mathbf{I})u + q + \nu\langle \frac{1}{2}p|p|^2 g\rangle. \end{aligned}$$

Hence (2.2.2) can be rewritten as

$$\begin{cases} \rho_t + \mathrm{div}_x(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}_x(P + \rho u \otimes u) - \rho \nabla_x V = -\nu \mathrm{div}_x \langle p \otimes pg\rangle - \dfrac{\rho u}{\tau}, \\ (\rho e)_t + \mathrm{div}_x((P + \rho e\mathbf{I})u) + \mathrm{div}_x q - \rho u \cdot \nabla_x V = -\nu \mathrm{div}_x \langle \frac{1}{2}p|p|^2 g\rangle - \dfrac{2}{\tau}(\rho e - \dfrac{3}{2}\rho), \end{cases}$$

where $\tau = \tau_0/\nu$.

Inserting the Chapman-Enskog expansion in (2.2.1), we get

$$g = \frac{1}{\nu}(M[w] - w) = -M[w]_t - p \cdot \nabla_x M[w] - \theta[V]M[w] + O(\nu),$$

where $O(\nu)$ contains terms of order ν. These equations can be interpreted as a nonlocal quantum Navier-Stokes system. By expanding the quantum Maxwellian $M[w]$ in powers of the squared scaled Planck constant ε^2, we derive a local version of this system. Under the assumptions that the temperature varies slowly and the vorticity tensor $A(u) = \frac{1}{2}(\nabla u - \nabla^\top u)$ is small (i.e. $\nabla_x \log \mathrm{T} = O(\varepsilon^2)$, $A(u) = O(\varepsilon^2)$), the quantum heat flux becomes $q = -\frac{\varepsilon^2}{24}\rho(\Delta_x u + 2\nabla_x \mathrm{div}_x u) + O(\varepsilon^4)$, the quantum stress tensor $P = \rho \mathrm{T}\mathbf{I} - \frac{\varepsilon^2}{12}\rho\nabla_x^2 \log\rho + O(\varepsilon^4)$ and $\rho e = \frac{3}{2}\rho\mathrm{T} + \frac{1}{2}\rho|u|^2 -$

$\frac{\varepsilon^2}{24}\rho\Delta_x \log\rho + O(\varepsilon^4)$. Furthermore, a tedious calculation shows that

$$-\nu \mathrm{div}_x \langle p\otimes pg\rangle = \nu \mathrm{div}_x S_1,$$
$$-\nu \mathrm{div}_x \langle \frac{1}{2}p|p|^2 g\rangle = \nu \mathrm{div}_x S_2,$$

where

$$S_1 = 2\rho \mathrm{T} D(u) - \frac{2}{3}\rho \mathrm{T}\mathrm{div}_x u\mathbf{I} + O(\varepsilon^2+\nu),$$
$$S_2 = 2\rho \mathrm{T} D(u)u - \frac{2}{3}\rho \mathrm{T} u\mathrm{div}_x u + \frac{5}{2}\rho \mathrm{T}\nabla_x \mathrm{T} + O(\varepsilon^2+\nu).$$

Therefore, we obtain the following result:

Theorem 2.2.2 ([34]) *Assume that $A(u) = O(\varepsilon^2)$ and $\nabla \log \mathrm{T} = O(\varepsilon^2)$. Then, up to terms of order $O(\nu^2+\nu\varepsilon^2+\varepsilon^4)$, the moment equations of the Wigner equation read as*

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u\otimes u) + \nabla(\rho \mathrm{T}) - \frac{\varepsilon^2}{12}\mathrm{div}(\rho\nabla^2\log\rho) - \rho\nabla V = \nu \mathrm{div} S - \frac{\rho u}{\tau}, \\ (\rho e)_t + \mathrm{div}((\rho e + \rho \mathrm{T})u) - \frac{\varepsilon^2}{12}\mathrm{div}(\rho(\nabla^2\log\rho)u) \\ \qquad + \mathrm{div}(q + \frac{5}{2}\rho \mathrm{T}\nabla \mathrm{T}) - \rho u\cdot\nabla V = \nu\mathrm{div}(Su) - \frac{2}{\tau}(\rho e - \frac{3}{2}\rho), \end{cases}$$

where $S = 2\rho \mathrm{T} D(u) - \frac{2}{3}\rho \mathrm{T}\mathrm{div}_x u\mathbf{I}$.

Remark 2.2.3 *When we only consider the conservation of mass and momentum, the quantum equilibrium becomes*

$$M[w](x,p,t) = Exp\left(A(x,t) - \frac{|p-U(x,t)|^2}{2}\right).$$

In this case, a Chapman-Enskog expansion has been carried out in [6], where a barotropic Navier-Stokes system is obtained with

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u\otimes u) + \nabla\rho - \frac{\varepsilon^2}{6}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \rho\nabla V = 2\nu\mathrm{div}(\rho D(u)). \end{cases}$$

Chapter 3

Global Weak Solutions to Barotropic Navier-Stokes Equations for Quantum Fluids

In this chapter, we prove the global-in-time existence of weak solutions to the barotropic compressible Navier-Stokes for quantum fluids with large initial data, which improves the result by Jüngel [Global weak solutions to compressible Navier-Stokes equations for quantum fluids, SIAM J. Math. Anal., 42(2010), no.3, pp.1025–1045], where the restriction $\varepsilon > \nu$ can be removed and for more general external forces. The key is that we get a new energy estimate.

3.1 Reformulation and main results

In this chapter, we study the barotropic quantum Navier-Stokes equations, which consist of the mass conservation equation and a momentum balance equation, including a nonlinear third-order differential operator, with the quantum Bohm potential and a density-dependent viscosity. The barotropic quantum Navier-Stokes equations can be written as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, & (x,t) \in \mathbb{T}^d \times (0,T), \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - 2\varepsilon^2 \rho \nabla \left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) - 2\nu \operatorname{div}(\rho D(u)) = \rho f, \end{cases} \tag{3.1.1}$$

with the initial condition

$$\rho|_{t=0} = \rho_0(x), \quad (\rho u)|_{t=0} = \rho_0 u_0 \ \text{ in } \mathbb{T}^d. \tag{3.1.2}$$

The unknowns in this system are the particle density $\rho = \rho(x,t) : \mathbb{T}^d \times [0,+\infty) \to \mathbb{R}^+ \cup \{0\}$, and the particle velocity $u = u(x,t) : \mathbb{T}^d \times [0,+\infty) \to \mathbb{R}^d$. $u \otimes u$ is the matrix with components $u_i u_j$, $D(u) = \frac{1}{2}(\nabla u + \nabla^\top u)$ is the symmetric part of the velocity gradient, and $\mathbb{T}^d$ is the d-dimensional torus. The function $P(\rho) = \rho^\gamma$ with $\gamma \geqslant 1$ is the pressure, and f describes external forces. The physical parameters are the (scaled) Planck constant $\varepsilon > 0$ and the viscosity constant $\nu > 0$. The nonlinear dispersive term $-2\varepsilon^2\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$ is produced by the quantum Bohm potential $Q(\rho) = 2\varepsilon^2 \dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$.

We introduce an auxiliary velocity

$$w = u + \nu \nabla \log \rho,$$

then the system (3.1.1) can be rewritten as

$$\begin{cases} \rho_t + \operatorname{div}(\rho w) = \nu \Delta \rho, & (x,t) \in \mathbb{T}^d \times (0,T), \\ (\rho w)_t + \operatorname{div}(\rho w \otimes w) + \nabla P(\rho) + 2(\nu^2 - \varepsilon^2)\rho \nabla \left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) - \nu \Delta(\rho w) = \rho f, \end{cases} \tag{3.1.3}$$

and the initial condition is changed to

$$\rho|_{t=0} = \rho_0(x), \quad (\rho w)|_{t=0} = \rho_0 w_0, \text{ in } \mathbb{T}^d, \tag{3.1.4}$$

where $w_0 = u_0 + \nu \nabla \log \rho_0$.

Lemma 3.1.1 *The initial value problems (3.1.1)-(3.1.2) and (3.1.3)-(3.1.4) are equivalent if ρ_0 satisfies $\int_{\mathbb{T}^d} |\nabla \sqrt{\rho_0}|^2 dx < \infty$.*

Proof: See lemma 2.1 in [32]. □

Lemma 3.1.2 *Let $T > 0$, and (ρ, u) be a (smooth) solution to the initial value problem (3.1.1)-(3.1.2), then (ρ, u) satisfies*

$$\begin{aligned}
&\int_{\mathbb{T}^d} \rho_0^2 u_0 \cdot \phi(\cdot, 0) dx + \int_0^T \int_{\mathbb{T}^d} [\rho^2 u \cdot \phi_t - \rho^2 (\mathrm{div} u) u \cdot \phi + (\rho u \otimes \rho u) : \nabla \phi \\
&\quad + \frac{\gamma}{\gamma+1} \rho^{\gamma+1} \mathrm{div}\phi - 2\varepsilon^2 \Delta \sqrt{\rho} (\sqrt{\rho}^3 \mathrm{div}\phi + 2\sqrt{\rho} \nabla \rho \cdot \phi) \\
&\quad - 2\nu \rho D(u) : (\nabla \rho \otimes \phi + \rho \nabla \phi) + \rho^2 f \cdot \phi] dx dt = 0,
\end{aligned} \tag{3.1.5}$$

for all $\phi \in (C^1(\mathbb{T}^d \times (0, T)))^d$ with $\phi(\cdot, T) = 0$.

Equivalently, if (ρ, w) is a (smooth) solution to the initial value problem (3.1.3)-(3.1.4), then (ρ, w) satisfies

$$\begin{aligned}
&\int_{\mathbb{T}^d} \rho_0^2 w_0 \cdot \phi(\cdot, 0) dx + \int_0^T \int_{\mathbb{T}^d} [\rho^2 w \cdot \phi_t - \rho^2 (\mathrm{div} w) w \cdot \phi + (\rho w \otimes \rho w) : \nabla \phi \\
&\quad - \nu (\rho w \otimes \nabla \rho) : \nabla \phi + \frac{\gamma}{\gamma+1} \rho^{\gamma+1} \mathrm{div}\phi + 2(\nu^2 - \varepsilon^2) \Delta \sqrt{\rho} (\sqrt{\rho}^3 \mathrm{div}\phi + 2\sqrt{\rho} \nabla \rho \cdot \phi) \\
&\quad - \nu \nabla (\rho w) : (2\nabla \rho \otimes \phi + \rho \nabla \phi) + \rho^2 f \cdot \phi] dx dt = 0,
\end{aligned} \tag{3.1.6}$$

for all $\phi \in (C^1(\mathbb{T}^d \times (0, T)))^d$ with $\phi(\cdot, T) = 0$.

Proof: Let $\phi \in (C^1(\mathbb{T}^d \times (0, T)))^d$ with $\phi(\cdot, T) = 0$.

Multiplying $(3.1.1)_2$ by $\rho\phi$ and integrating over $\mathbb{T}^d \times (0,T)$, we obtain

$$\begin{aligned}
-\int_{\mathbb{T}^d} \rho_0^2 u_0 \cdot \phi(\cdot,0)dx &= \int_0^T \int_{\mathbb{T}^d} (\rho^2 u \cdot \phi)_t dxdt \\
&= \int_0^T \int_{\mathbb{T}^d} [\rho^2 u \cdot \phi_t + \rho\rho_t u \cdot \phi + \rho(\rho u)_t \cdot \phi] dxdt \\
&= \int_0^T \int_{\mathbb{T}^d} [\rho^2 u \cdot \phi_t + \rho u \cdot \phi(-\rho \mathrm{div} u - u\nabla\rho) \\
&\quad + \rho\phi(-\mathrm{div}(\rho u \otimes u) - \nabla P(\rho) + 2\varepsilon^2 \rho \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) + 2\nu \mathrm{div}(\rho D(u)) + \rho f)] dxdt \\
&= \int_0^T \int_{\mathbb{T}^d} [\rho^2 u \cdot \phi_t - \rho^2 (\mathrm{div} u) u \cdot \phi - \rho(u\cdot\phi)(u\cdot\nabla\rho) \\
&\quad + (\rho u \otimes u) : \nabla\phi + \rho(u\otimes u) : (\nabla\rho \otimes \phi) - \frac{\gamma}{\gamma+1}\nabla(\rho^{\gamma+1}) \cdot \phi + \rho^2 f \cdot \phi \\
&\quad - 2\varepsilon^2 \Delta\sqrt{\rho}(\sqrt{\rho}^3 \mathrm{div}\phi + 2\sqrt{\rho}\nabla\rho \cdot \phi) - 2\nu\rho D(u) : (\nabla\rho\otimes\phi + \rho\nabla\phi)] dxdt = 0,
\end{aligned}$$

Observing that $\rho(u\cdot\phi)(u\cdot\nabla\rho) = \rho(u\otimes u) : (\nabla\rho\otimes\phi)$, we obtain (3.1.5). Similarly, if we multiply $(3.1.3)_2$ by $\rho\phi$, integrate over $\mathbb{T}^d \times (0,T)$, and use the following elementary identities

$$\rho(w\cdot\phi)(w\cdot\nabla\rho) = \rho(w\otimes w) : (\nabla\rho\otimes\phi),$$

$$\int_{\mathbb{T}^d} \rho\Delta\rho w \cdot \phi dx = \int_{\mathbb{T}^d} [\nabla(\rho w) : (\nabla\rho\otimes\phi) + (\rho w \otimes \nabla\rho) : \nabla\phi] dx,$$

then (3.1.6) holds. □

Next, we will give the definition of weak solutions to the initial value problems (3.1.1)-(3.1.2) and (3.1.3)-(3.1.4).

Definition 3.1.3 *A pair (ρ,u) is said to be a weak solution to the initial value problems (3.1.1)-(3.1.2) if and only if (ρ,u) satisfies $(3.1.1)_1$ pointwise in $\mathbb{T}^d \times (0,T)$ and satisfies (3.1.5) for any $\phi \in (C^1(\mathbb{T}^d \times (0,T)))^d$ with $\phi(\cdot,T)=0$. Equivalently, we say (ρ,w) is a weak solution to (3.1.3)-(3.1.4) if and only if (ρ,w) satisfies $(3.1.3)_1$ pointwise in $\mathbb{T}^d \times (0,T)$ and satisfies (3.1.6) for any $\phi \in (C^1(\mathbb{T}^d \times (0,T)))^d$ with $\phi(\cdot,T)=0$.*

Remark 3.1.4 *The reason why we define the weak solutions like above is that we can not get the compactness for the convection term $\rho w \otimes w$ (or $\rho u \otimes u$). However, we are able to obtain the compactness for $\rho w \otimes \rho w$ (or $\rho u \otimes \rho u$) thanks to the third-order quantum term $-2\varepsilon^2 \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right)$.*

Now, we state the main results.

Theorem 3.1.5 *Let $d = 2, 3, T > 0, \varepsilon, \nu > 0, P(\rho) = \rho^\gamma$ with $\gamma \geqslant 1$ if $d = 2; \gamma > 3$ if $d = 3$, and $f \in L^\infty(0, T; L^p(\mathbb{T}^d))$, where $p = \begin{cases} \frac{2\gamma}{\gamma - 1}, & \gamma > 1 \\ +\infty, & \gamma = 1 \end{cases}$. Assume that the initial data (ρ_0, u_0) satisfies*

$$\begin{cases} \rho_0(x) \geqslant 0, & in \quad \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \left(\frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right) dx < \infty, \\ \int_{\mathbb{T}^d} |\nabla \sqrt{\rho_0}|^2 dx < \infty, \end{cases} \tag{3.1.7}$$

where

$$G(\rho) = \begin{cases} \frac{\rho^\gamma}{\gamma - 1}, & \gamma > 1, \\ \rho(\log \rho - 1), & \gamma = 1. \end{cases}$$

Then there exists a weak solution (ρ, u) to the initial value problem (3.1.1)-(3.1.2) with $\rho \geqslant 0$ in $\mathbb{T}^d$, and the regularity

$$\begin{cases} \sqrt{\rho} \in L^\infty(0, T; H^1(\mathbb{T}^d)) \cap L^2(0, T; H^2(\mathbb{T}^d)), \\ \rho \in H^1(0, T; L^2(\mathbb{T}^d)) \cap L^\infty(0, T; L^\gamma(\mathbb{T}^d)) \cap L^2(0, T; W^{1,3}(\mathbb{T}^d)), \\ \sqrt{\rho} u \in L^\infty(0, T; L^2(\mathbb{T}^d)), \quad \rho u \in L^2(0, T; W^{1,3/2}(\mathbb{T}^d)), \\ \rho |\nabla u| \in L^2(0, T; L^2(\mathbb{T}^d)). \end{cases} \tag{3.1.8}$$

Theorem 3.1.6 *Let the assumptions in Theorem 3.1.5 above hold. Assume that* *the initial data* (ρ_0, w_0) *satisfies*

$$\begin{cases} \rho_0(x) \geqslant 0, & in \quad \mathbb{T}^d, \\ \displaystyle\int_{\mathbb{T}^d} \left(\frac{1}{2}\rho_0|w_0|^2 + G(\rho_0)\right) dx < \infty, \\ \displaystyle\int_{\mathbb{T}^d} |\nabla\sqrt{\rho_0}|^2 dx < \infty. \end{cases} \tag{3.1.9}$$

Then there exists a weak solution (ρ, w) *to the initial value problem (3.1.3)-(3.1.4) with* $\rho \geqslant 0$ *in* $\mathbb{T}^d$, *and the regularity*

$$\begin{cases} \sqrt{\rho} \in L^\infty(0,T;H^1(\mathbb{T}^d)) \cap L^2(0,T;H^2(\mathbb{T}^d)), \\ \rho \in H^1(0,T;L^2(\mathbb{T}^d)) \cap L^\infty(0,T;L^\gamma(\mathbb{T}^d)) \cap L^2(0,T;W^{1,3}(\mathbb{T}^d)), \\ \sqrt{\rho}w \in L^\infty(0,T;L^2(\mathbb{T}^d)), \quad \rho w \in L^2(0,T;W^{1,3/2}(\mathbb{T}^d)), \\ \rho|\nabla w| \in L^2(0,T;L^2(\mathbb{T}^d)). \end{cases} \tag{3.1.10}$$

Remark 3.1.7 *In Theorem 3.1.6,* $w_0 = u_0 + \nu\nabla\log\rho_0$. *It is easy to check that if* (ρ_0, w_0) *satisfies (3.1.9), then* (ρ_0, u_0) *satisfies (4.1.4), and the regularity (3.1.10) of* (ρ, w) *implies the regularity (3.1.8) of* (ρ, u). *Hence Theorem 3.1.5 is an immediately consequence of Theorem 3.1.6.*

3.2 Construction of approximate solutions

We introduce the finite-dimensional space $X_N \triangleq span\{\psi_j\}_{j=1}^N$, where $\psi_j \in C^\infty(\mathbb{T}^d)$ and is an orthonormal basis of $L^2(\mathbb{T}^d)$ which is also an orthogonal basis of $H^1(\mathbb{T}^d)$.

We construct the approximate solutions as follows: Let the initial data $(\rho_0, w_0) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ with $\rho_0(x) \geqslant \delta > 0$ in $\mathbb{T}^d$. Consider the following system as an approximate system of (3.1.3):

$$\begin{cases} \rho_t + \operatorname{div}(\rho w) = \nu\Delta\rho, & (x,t) \in \mathbb{T}^d \times (0,T), \\ (\rho w)_t + \operatorname{div}(\rho w \otimes w) + \nabla P(\rho) + 2(\nu^2 - \varepsilon^2)\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) & \\ \qquad\qquad - \nu\Delta(\rho w) - \delta\Delta w + \delta w = \rho f, \quad in \ \ X_N^*, & \end{cases} \tag{3.2.1}$$

where X_N^* stands for the dual space of X_N.

Let the velocity $v \in C([0,T]; X_N)$ be given, then v can be written as

$$v(x,t) = \sum_{j=1}^{N} a_j(t)\psi_j(x), \qquad (x,t) \in \mathbb{T}^d \times [0,T],$$

for some $a_j(t) \in C[0,T]$, and the norm of v in $C([0,T]; X_N)$ can be formulated as

$$\|v\|_{C([0,T];X_N)} = \max_{t\in[0,T]} \sum_{j=1}^{N} |a_j(t)|.$$

Linearize (3.2.1) by

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = \nu\Delta\rho, & (x,t) \in \mathbb{T}^d \times (0,T), \\ (\rho w)_t + \operatorname{div}(\rho v \otimes w) + \nabla P(\rho) + 2(\nu^2 - \varepsilon^2)\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) & \\ \qquad\qquad - \nu\Delta(\rho w) - \delta\Delta w + \delta w = \rho f, \quad in \ \ X_N^*. & \end{cases} \tag{3.2.2}$$

Lemma 3.2.1 *Assume $\rho_0(x) \in C^\infty(\mathbb{T}^d), 0 < \delta \leqslant \rho_0(x) \leqslant M < \infty, v \in C([0,T]; X_N)$, then there exists an operator $S : C([0,T]; X_N) \to C^1([0,T]; C^3(\mathbb{T}^d))$ satisfying*

(1) *$\rho = S(v)$ is a unique classical solution to the initial value problem*

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = \nu\Delta\rho, & (x,t) \in \mathbb{T}^d \times (0,T), \\ \rho|_{t=0} = \rho_0. \end{cases}$$

(2) *$\rho = S(v)$ is strictly positive and bounded from below and above, i.e. for any $(x,t) \in \mathbb{T}^d \times (0,T)$, it holds that*

$$0 < \delta e^{-\int_0^t \|\operatorname{div} v\|_{L^\infty(\mathbb{T}^d)} ds} \leqslant S(v) \leqslant M e^{\int_0^t \|\operatorname{div} v\|_{L^\infty(\mathbb{T}^d)} ds}. \tag{3.2.3}$$

(3) *For any $w_1, w_2 \in C([0,T]; X_N)$, there exists constant $C > 0$ such that*

$$\|S(w_1) - S(w_2)\|^2_{H^1(\mathbb{T}^d)} \leqslant Ct\|w_1 - w_2\|^2_{C([0,T];X_N)}, \quad \forall\, t \in (0,T), \qquad (3.2.4)$$

where $C = C'e^{C'T(\|w_1\|^2_{C([0,T];X_N)} + \|w_2\|_{C([0,T];X_N)})}(1 + T\|w_2\|^2_{C([0,T];X_N)})$,

C' *is a constant.*

Proof:

(1) See [40].

(2) Denote $Lf \triangleq f_t + \text{div}(fv) - \nu\Delta f$, then a computation shows that

$$\begin{aligned}
&L\left(\delta e^{-\int_0^t \|\text{div}v\|_{L^\infty(\mathbb{T}^d)} ds}\right) = \delta e^{-\int_0^t \|\text{div}v\|_{L^\infty(\mathbb{T}^d)} ds}(\text{div}v - \|\text{div}v\|_{L^\infty(\mathbb{T}^d)}) \leqslant 0,\\
&L\rho = 0,\\
&L\left(Me^{\int_0^t \|\text{div}v\|_{L^\infty(\mathbb{T}^d)} ds}\right) = Me^{\int_0^t \|\text{div}v\|_{L^\infty(\mathbb{T}^d)} ds}(\text{div}v + \|\text{div}v\|_{L^\infty(\mathbb{T}^d)}) \geqslant 0.
\end{aligned}$$

By maximum principle, we have

$$\delta e^{-\int_0^t \|\text{div}v\|_{L^\infty(\mathbb{T}^d)} ds} \leqslant \rho(x,t) \leqslant Me^{\int_0^t \|\text{div}v\|_{L^\infty(\mathbb{T}^d)} ds}.$$

(3) For any $w_1, w_2 \in C([0,T]; X_N)$, $(\rho_1 = S(w_1), \rho_2 = S(w_2)) \in (C^1([0,T]; C^3(\mathbb{T}^d)))^2$ satisfy

$$\partial_t\rho_1 + \text{div}(\rho_1 w_1) = \nu\Delta\rho_1, \qquad (3.2.5)$$

$$\partial_t\rho_2 + \text{div}(\rho_2 w_2) = \nu\Delta\rho_2, \qquad (3.2.6)$$

and $(\rho_1 - \rho_2)$ satisfies

$$\partial_t(\rho_1 - \rho_2) + \text{div}(\rho_1 w_1 - \rho_2 w_2) = \nu\Delta(\rho_1 - \rho_2). \qquad (3.2.7)$$

Since

$$\int_{\mathbb{T}^d} \rho_1(x,t)dx = \int_{\mathbb{T}^d} \rho_0(x)dx = \int_{\mathbb{T}^d} \rho_2(x,t)dx,$$

we have for any $t \in (0,T)$,

$$\int_{\mathbb{T}^d} (\rho_1 - \rho_2)(x,t)dx = 0.$$

So by Poincaré's inequality, we get

$$\|\rho_1 - \rho_2\|_{L^2(\mathbb{T}^d)} \leqslant C\|\nabla(\rho_1 - \rho_2)\|_{L^2(\mathbb{T}^d)}, \quad \forall\, t \in (0,T). \tag{3.2.8}$$

Multiplying (3.2.6) by $-\Delta\rho_2$ and integrating over $\mathbb{T}^d$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d}\frac{1}{2}|\nabla\rho_2|^2dx + \nu\int_{\mathbb{T}^d}|\Delta\rho_2|^2dx &\leqslant \int_{\mathbb{T}^d}(|w_2||\nabla\rho_2||\Delta\rho_2| + |\mathrm{div}w_2||\rho_2||\Delta\rho_2|)dx\\ &\leqslant \frac{\nu}{2}\int_{\mathbb{T}^d}|\Delta\rho_2|^2dx + C\|w_2\|^2_{C([0,T];X_N)}\int_{\mathbb{T}^d}(|\nabla\rho_2|^2 + |\rho_2|^2)dx.\end{aligned}$$

In view of (3.2.3), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^d}|\nabla\rho_2|^2dx + \nu\int_{\mathbb{T}^d}|\Delta\rho_2|^2dx\\ &\quad\leqslant C\|w_2\|^2_{C([0,T];X_N)}\int_{\mathbb{T}^d}|\nabla\rho_2|^2dx + C\|w_2\|^2_{C([0,T];X_N)}e^{CT\|w_2\|_{C([0,T];X_N)}},\end{aligned}$$

Grownwall's inequality yields that

$$\sup_{t\in[0,T]}\|\nabla\rho_2\|^2_{L^2(\mathbb{T}^d)} \leqslant C(1 + T\|w_2\|^2_{C([0,T];X_N)})e^{CT\|w_2\|_{C([0,T];X_N)}}. \tag{3.2.9}$$

Next, multiplying (3.2.7) by $-\Delta(\rho_1 - \rho_2)$ and integrating over $\mathbb{T}^d$ leads to

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^d}\frac{1}{2}|\nabla(\rho_1-\rho_2)|^2dx + \nu\int_{\mathbb{T}^d}|\Delta(\rho_1-\rho_2)|^2dx\\ &\leqslant \int_{\mathbb{T}^d}(|w_1||\nabla(\rho_1-\rho_2)||\Delta(\rho_1-\rho_2)| + |\mathrm{div}w_1||\rho_1-\rho_2||\Delta(\rho_1-\rho_2)|\\ &\quad+ |\rho_2||\mathrm{div}(w_1-w_2)||\Delta(\rho_1-\rho_2)| + |\nabla\rho_2||w_1-w_2||\Delta(\rho_1-\rho_2)|)dx\\ &\leqslant \frac{\nu}{2}\int_{\mathbb{T}^d}|\Delta(\rho_1-\rho_2)|^2dx + C\|w_1\|^2_{C([0,T];X_N)}\int_{\mathbb{T}^d}(|\nabla(\rho_1-\rho_2)|^2 + |\rho_1-\rho_2|^2)dx\\ &\quad+ \frac{2}{\nu}(\|\rho_2\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\rho_2\|^2_{L^2(\mathbb{T}^d)})\|w_1-w_2\|^2_{C([0,T];X_N)}.\end{aligned}$$

It follows from (3.2.3), (3.2.8) and (3.2.9) that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^d}|\nabla(\rho_1-\rho_2)|^2dx + \nu\int_{\mathbb{T}^d}|\Delta(\rho_1-\rho_2)|^2dx\\ &\leqslant C\|w_1\|^2_{C([0,T];X_N)}\int_{\mathbb{T}^d}|\nabla(\rho_1-\rho_2)|^2dx\\ &\quad+ C(1 + T\|w_2\|^2_{C([0,T];X_N)})e^{T\|w_2\|_{C([0,T];X_N)}}\|w_1-w_2\|^2_{C([0,T];X_N)}.\end{aligned}$$

Then (3.2.4) holds after we apply Grownwall's inequality.

□

Next, we wish to solve the equation $(3.2.2)_2$ in X_N^*. In other words, for given $\rho = S(v)$, we are looking for a function $w \in C([0,T]; X_N)$ such that

$$\begin{aligned}\int_{\mathbb{T}^d} \rho_0 w_0 \cdot \phi(\cdot, 0) dx + \int_0^T \int_{\mathbb{T}^d} [\rho w \cdot \phi_t + (\rho v \otimes w) : \nabla \phi + P(\rho) \mathrm{div} \phi + \rho f \cdot \phi \\ + 2(\nu^2 - \varepsilon^2) \Delta \sqrt{\rho} (\sqrt{\rho} \mathrm{div} \phi + 2 \nabla \sqrt{\rho} \cdot \phi) - \nu \nabla (\rho w) : \nabla \phi \\ - \delta (\nabla w : \nabla \phi + w \cdot \phi)] dx dt = 0, \qquad (3.2.10)\end{aligned}$$

for all $\phi \in (C^1([0,T]; X_N))^d$ with $\phi(\cdot, T) = 0$.

In order to deal with ρ in (3.2.10), we introduce a family of operators (See [16]): Given a function $\rho \in L^1(\mathbb{T}^d)$, with $\rho \geqslant \underline{\rho} > 0$, define

$$\begin{aligned} &M[\rho]: \ X_N \to X_N^*, \\ &\langle M[\rho] u, w \rangle = \int_{\mathbb{T}^d} \rho u \cdot w dx, \quad \forall \ u, w \in X_N. \end{aligned}$$

As stated in [16], $M[\rho]$ has the following properties:

- $M[\rho]$ is invertible with

$$\|M^{-1}[\rho]\|_{\mathcal{L}(X_N^*, X_N)} \leqslant \underline{\rho}^{-1},$$

here $\mathcal{L}(X_N^*, X_N)$ is the set of bounded linear mapping from X_N^* to X_N.

- Moreover, $M^{-1}[\rho]$ is Lipschitz continuous in the following sense:

For $\forall \ \rho_1, \rho_2 \in L^1(\mathbb{T}^d)$, with $\rho_1, \rho_2 \geqslant \underline{\rho} > 0$, there exists a constant $C = C(N, \underline{\rho}) > 0$ such that

$$\|M^{-1}[\rho_1] - M^{-1}[\rho_2]\|_{\mathcal{L}(X_N^*, X_N)} \leqslant C \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)}. \qquad (3.2.11)$$

With the preparation above, now we can rephrase $(3.2.2)_2$ as an ordinary differential system:

$$\begin{cases} \langle \frac{d}{dt}(M[\rho(t)] w(t)), \psi_i \rangle = \langle N[v, w(t)], \psi_i \rangle, \quad t \in [0,T], \\ \langle M[\rho(t)] w(t)|_{t=0}, \psi_i \rangle = \langle M[\rho_0] w_0, \psi_i \rangle, \end{cases} \qquad (3.2.12)$$

where ψ_i $(1 \leqslant i \leqslant N)$ is the orthonormal basis of X_N, $\rho = S(v)$ and

$$\begin{aligned}\langle N[v,w(t)],\psi_i\rangle = \int_{\mathbb{T}^d} [(\rho v\otimes w):\nabla\psi_i + P(\rho)\mathrm{div}\psi_i + \rho f\cdot\psi_i \\ + 2(\nu^2-\varepsilon^2)\Delta\sqrt{\rho}(\sqrt{\rho}\mathrm{div}\psi_i + 2\nabla\sqrt{\rho}\cdot\psi_i) \\ - \nu\nabla(\rho w):\nabla\psi_i - \delta(\nabla w:\nabla\psi_i + w\cdot\psi_i)]dx.\end{aligned}$$

Recall that $\rho = S(v) \in C^1([0,T];C^3(\mathbb{T}^d))$ is bounded from below and above, so the above integral is well defined. $N[v,\cdot]$ is an operator from X_N to X_N^* and is continuous in time. Then standard theory for finite dimensional ODE system provides the existence of a unique classical solution to (3.2.12). In other words, for a given $v \in C([0,T];X_N)$, $\rho = S(v)$, there exists a unique solution $w \in C^1([0,T];X_N)$ to $(3.2.2)_2$.

Lemma 3.2.2 (Local existence) *Assume* $(\rho_0, w_0) \in C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)$, $\rho_0(x) \geqslant \eta > 0(\ \forall\, \eta > 0)$ *in* $\mathbb{T}^d$ *and (3.1.9) holds. There is a time interval* $[0,T']$ $(0 < T' \leqslant T)$ *such that there exists a solution* $(\rho, w) \in C^1([0,T'];C^3(\mathbb{T}^d)) \times C^1([0,T'];X_N)$ *to the approximate system (3.2.1) on* $\mathbb{T}^d \times [0,T']$ *with initial condition (3.1.4).*

Proof: Let $R > 0$ large enough, and $T' \in (0,T]$ to be fixed.
Consider a bounded ball $\mathcal{B}_R$ in $C([0,T'];X_N)$,

$$\mathcal{B}_R \triangleq \{v \in C([0,T'];X_N) \mid \|v\|_{C([0,T'];X_N)} \leqslant R\}.$$

Define a mapping

$$\begin{aligned}&\mathcal{T}:\ \mathcal{B}_R \to C([0,T'];X_N), \\ &\mathcal{T}(w) = M^{-1}[S(w)(t)]\left(M[\rho_0]w_0 + \int_0^t N[w,w(s)]ds\right),\quad where\ \ \rho = S(w).\end{aligned}$$

- For $\forall\ w \in \mathcal{B}_R$,

$$\begin{aligned}\|\mathcal{T}(w)\|_{X_N} &\leqslant \|M^{-1}[S(w)(t)]\|_{\mathcal{L}(X_N^*,X_N)}(\|M[\rho_0]w_0\|_{X_N^*} + t\|N[w,w(s)]\|_{X_N^*}) \\ &\leqslant \left(\eta e^{-tR}\right)^{-1}(\|M[\rho_0]w_0\|_{X_N^*} + t\|N[w,w(s)]\|_{X_N^*}).\end{aligned}$$

Observe that

$$\|M[\rho_0]w_0\|_{X_N^*} = \sup_{\|\phi\|_{X_N}=1} \mid \int_{\mathbb{T}^d} \rho_0 w_0 \cdot \phi dx \mid \leqslant \left(\int_{\mathbb{T}^d} \rho_0^2 |w_0|^2 dx \right)^{1/2} \triangleq C_0,$$

$$\|N[w, w(s)]\|_{X_N^*} \leqslant CR + C(1 + R + R^2)e^{CR} \triangleq F(R).$$

Hence, for any $t \in [0, T']$,

$$\|\mathcal{T}(w)\|_{X_N} \leqslant \eta^{-1} e^{tR}(C_0 + tF(R)) \leqslant R,$$

provided $T' = \min\{R^{-1}, \frac{\eta R - C_0}{F(R)}\} > 0$, here we assume R is large enough. So $\mathcal{T}$ maps $\mathcal{B}_R$ to itself.

- Next, for $\forall\, w_1, w_2 \in \mathcal{B}_R$,

$$\begin{aligned}
\|\mathcal{T}(w_1) - \mathcal{T}(w_2)\|_{X_N} &\leqslant \big(\|M[\rho_0]w_0\|_{X_N^*} + t(\|N[w_1, w_1(s)]\|_{X_N^*} + \|N[w_2, w_2(s)]\|_{X_N^*})\big) \\
&\quad \cdot \|M^{-1}[S(w_1)] - M^{-1}[S(w_2)]\|_{\mathcal{L}(X_N^*, X_N)} \\
&\leqslant (C_0 + 2TF(R))\|M^{-1}[S(w_1)] - M^{-1}[S(w_2)]\|_{\mathcal{L}(X_N^*, X_N)} \\
&\leqslant C(C_0 + 2TF(R))\|S(w_1) - S(w_2)\|_{L^1(\mathbb{T}^d)} \\
&\leqslant C(C_0 + 2TF(R))\|S(w_1) - S(w_2)\|_{H^1(\mathbb{T}^d)} \\
&\leqslant C\sqrt{t}\|w_1 - w_2\|_{C([0,T'];X_N)} \\
&\leqslant \theta\|w_1 - w_2\|_{C([0,T'];X_N)}, \quad (\theta < 1)
\end{aligned}$$

provided $t \leqslant \frac{1}{C^2 + 1}$.

Hence, let $T' = \min\{R^{-1}, \frac{\eta R - C_0}{F(R)}, \frac{1}{C^2+1}\}$, then $\mathcal{T}$ maps $\mathcal{B}_R$ to itself and is a contraction mapping. By Banach fixed point theorem, there exists a unique function $w \in C([0, T']; X_N)$, such that $\mathcal{T}(w) = w$. In other words, there exists a unique solution $w \in C([0, T']; X_N)$ to the equation $(3.2.2)_2$ in X_N^*, furthermore, $w \in C^1([0, T']; X_N)$. Let $\rho = S(w)$, then $\rho \in C^1([0, T']; C^3(\mathbb{T}^d))$.

Hence, there exists a unique local-in-time solution (ρ, w) to (3.2.1) with initial condition (3.1.4). □

We have the following key energy estimate.

Proposition 3.2.3 *Let $T' \leqslant T$, $(\rho, w) \in C^1([0,T']; C^3(\mathbb{T}^d)) \times C^1([0,T']; X_N)$ be a local-in-time solution to (3.2.1) with smooth initial condition (3.1.4). Then it holds that*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d}\left(\frac{1}{2}\rho|u|^2 + \frac{1}{2}\rho|u+\nabla\varphi|^2 + 2G(\rho) + 4\varepsilon^2|\nabla\sqrt{\rho}|^2\right)dx \\
&\quad + 2\nu\int_{\mathbb{T}^d}\left(\rho|D(u)|^2 + \rho|A(u)|^2\right)dx + 2\nu\int_{\mathbb{T}^d}\gamma\rho^{\gamma-2}|\nabla\rho|^2dx \\
&\quad + 2\nu\varepsilon^2\int_{\mathbb{T}^d}\rho|\nabla^2\log\rho|^2dx + 2\delta\int_{\mathbb{T}^d}(|\nabla w|^2 + |w|^2)dx \\
&\leqslant \begin{cases} \nu\displaystyle\int_{\mathbb{T}^d}\left(\frac{1}{2}\rho|u|^2 + \frac{1}{2}\rho|u+\nabla\varphi|^2 + 2G(\rho)\right)dx + C\|f\|^{\frac{2\gamma}{\gamma-1}}_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^d)}, (\gamma>1) \\ \displaystyle\frac{\nu}{2}\int_{\mathbb{T}^d}(\rho|u|^2 + \rho|u+\nabla\varphi|^2)dx + \frac{1}{\nu}\|f\|^2_{L^\infty(\mathbb{T}^d\times(0,T))}\|\rho_0\|_{L^1(\mathbb{T}^d)}, (\gamma=1) \end{cases}
\end{aligned} \tag{3.2.13}$$

where $\nabla\varphi = 2\nu\nabla\log\rho$, $u = w - \frac{1}{2}\nabla\varphi$ *and* $A(u) = \dfrac{\nabla u - \nabla^\top u}{2}$.

Proof: Let $\varphi = 2\nu\log\rho$, $u = w - \frac{1}{2}\nabla\varphi$, then we can rewrite (3.2.1) as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u\otimes u) + \nabla P(\rho) - 2\varepsilon^2\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \\ \qquad\qquad - 2\nu\operatorname{div}(\rho D(u)) - \delta\Delta w + \delta w = \rho f, \quad in \ X_N^*. \end{cases} \tag{3.2.14}$$

Multiplying $(3.2.14)_1$ by $\varphi'(\rho)$, and operating $\rho\partial_i$ to both sides gives

$$\rho(\nabla\varphi)_t + 2\nu\rho\nabla\operatorname{div}u + \rho\nabla(u\cdot\nabla)\varphi + \rho(u\cdot\nabla)\nabla\varphi = 0.$$

Using $(3.2.14)_1$ once more, we get that

$$(\rho\nabla\varphi)_t + \operatorname{div}(\nabla\varphi\otimes\rho u) + 2\nu\rho\nabla\operatorname{div}u + \rho\nabla(u\cdot\nabla)\varphi = 0. \tag{3.2.15}$$

Observing that

$$\begin{aligned}
\operatorname{div}(\rho D(u)) &= \operatorname{div}(\rho A(u)) + \operatorname{div}(\rho\nabla^\top u) \\
&= \operatorname{div}(\rho A(u)) + \rho\nabla\operatorname{div}u + \nabla(u\cdot\nabla)\rho,
\end{aligned}$$

Multiplying (3.2.15) with $(u+\nabla\varphi)$, and integrating over $\mathbb{T}^d$, one obtains

$$\int_{\mathbb{T}^d}[(\rho\nabla\varphi)_t+\mathrm{div}(\nabla\varphi\otimes\rho u)+2\nu\mathrm{div}(\rho D(u))-2\nu\mathrm{div}(\rho A(u))]\cdot(u+\nabla\varphi)dx=0. \tag{3.2.16}$$

Multiplying $(3.2.14)_2$ by $2w=(u+\nabla\varphi)+u$, and integrating by parts, we get

$$\begin{aligned}&\int_{\mathbb{T}^d}[(\rho u)_t+\mathrm{div}(u\otimes\rho u)+\nabla P(\rho)-2\nu\mathrm{div}(\rho D(u))]\cdot(u+\nabla\varphi+u)dx\\&\quad-4\varepsilon^2\int_{\mathbb{T}^d}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\cdot wdx+2\delta\int_{\mathbb{T}^d}(|\nabla w|^2+|w|^2)dx=2\int_{\mathbb{T}^d}\rho f\cdot wdx.\end{aligned} \tag{3.2.17}$$

Combining (3.2.16) and (3.2.17) gives

$$\begin{aligned}&\int_{\mathbb{T}^d}[(\rho(u+\nabla\varphi))_t+\mathrm{div}((u+\nabla\varphi)\otimes\rho u)+\nabla P(\rho)-2\nu\mathrm{div}(\rho A(u))]\cdot(u+\nabla\varphi)dx\\&\quad+\int_{\mathbb{T}^d}[(\rho u)_t+\mathrm{div}(u\otimes\rho u)+\nabla P(\rho)-2\nu\mathrm{div}(\rho D(u))]\cdot udx\\&\quad-4\varepsilon^2\int_{\mathbb{T}^d}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\cdot wdx+2\delta\int_{\mathbb{T}^d}(|\nabla w|^2+|w|^2)dx=2\int_{\mathbb{T}^d}\rho f\cdot wdx.\end{aligned} \tag{3.2.18}$$

Integration by parts and the identity $2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)=\mathrm{div}(\rho\nabla^2\log\rho)$ yield

$$\begin{aligned}-4\varepsilon^2\int_{\mathbb{T}^d}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\cdot wdx&=4\varepsilon^2\int_{\mathbb{T}^d}\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)(-\rho_t+\nu\Delta\rho)dx\\&=-4\varepsilon^2\int_{\mathbb{T}^d}(\Delta\sqrt{\rho})(\sqrt{\rho})_tdx-4\nu\varepsilon^2\int_{\mathbb{T}^d}\rho\nabla\log\rho\cdot\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)dx\\&=4\varepsilon^2\frac{d}{dt}\int_{\mathbb{T}^d}|\nabla\sqrt{\rho}|^2dx-2\nu\varepsilon^2\int_{\mathbb{T}^d}\nabla\log\rho\cdot\mathrm{div}(\rho\nabla^2\log\rho)dx\\&=4\varepsilon^2\frac{d}{dt}\int_{\mathbb{T}^d}|\nabla\sqrt{\rho}|^2dx+2\nu\varepsilon^2\int_{\mathbb{T}^d}\rho|\nabla^2\log\rho|^2dx.\end{aligned}$$

Since $A(u) = \dfrac{\nabla u - \nabla^\top u}{2}$, we have

$$\begin{aligned}
\int_{\mathbb{T}^d} \mathrm{div}(\rho A(u)) \cdot \nabla\varphi dx &= \int_{\mathbb{T}^d} \rho \mathrm{div}(A(u)) \cdot \nabla\varphi dx + \int_{\mathbb{T}^d} (\nabla\rho)^\top A(u) \nabla\varphi dx \\
&= 2\nu \int_{\mathbb{T}^d} \mathrm{div}(A(u)) \cdot \nabla\rho dx + 2\nu \int_{\mathbb{T}^d} \rho^{-1} (\nabla\rho)^\top A(u) \nabla\rho dx \\
&= -2\nu \int_{\mathbb{T}^d} \rho \mathrm{div}(\mathrm{div}(A(u))) dx \\
&= 0.
\end{aligned}$$

Hence, (3.2.18) leads to

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{T}^d} \left(\frac{1}{2}\rho|u|^2 + \frac{1}{2}\rho|u+\nabla\varphi|^2 + 2G(\rho) + 4\varepsilon^2 |\nabla\sqrt{\rho}|^2 \right) dx \\
&\quad + 2\nu \int_{\mathbb{T}^d} \left(\rho|D(u)|^2 + \rho|A(u)|^2 \right) dx + 2\nu \int_{\mathbb{T}^d} \gamma\rho^{\gamma-2} |\nabla\rho|^2 dx \\
&\quad + 2\nu\varepsilon^2 \int_{\mathbb{T}^d} \rho |\nabla^2 \log\rho|^2 dx + 2\delta \int_{\mathbb{T}^d} (|\nabla w|^2 + |w|^2) dx = 2\int_{\mathbb{T}^d} \rho f \cdot w dx.
\end{aligned}$$

Finally, using Young's inequality, the right-hand side is bounded by

if $\gamma > 1$,

$$\begin{aligned}
2\int_{\mathbb{T}^d} \rho f \cdot w dx &= \int_{\mathbb{T}^d} \rho f \cdot u dx + \int_{\mathbb{T}^d} \rho f \cdot (u + \nabla\varphi) dx \\
&\leqslant \frac{\nu}{2} \int_{\mathbb{T}^d} \rho|u|^2 + \rho|u+\nabla\varphi|^2 dx + \frac{1}{\nu} \int_{\mathbb{T}^d} \rho|f|^2 dx \\
&\leqslant \nu \int_{\mathbb{T}^d} \frac{1}{2}\rho|u|^2 + \frac{1}{2}\rho|u+\nabla\varphi|^2 + 2G(\rho) dx + C(\gamma,\nu) \|f\|^{\frac{2\gamma}{\gamma-1}}_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^d)},
\end{aligned}$$

if $\gamma = 1$,

$$\begin{aligned}
2\int_{\mathbb{T}^d} \rho f \cdot w dx &= \int_{\mathbb{T}^d} \rho f \cdot u dx + \int_{\mathbb{T}^d} \rho f \cdot (u + \nabla\varphi) dx \\
&\leqslant \frac{\nu}{2} \int_{\mathbb{T}^d} \rho|u|^2 + \rho|u+\nabla\varphi|^2 dx + \frac{1}{\nu} \int_{\mathbb{T}^d} \rho|f|^2 dx \\
&\leqslant \frac{\nu}{2} \int_{\mathbb{T}^d} (\rho|u|^2 + \rho|u+\nabla\varphi|^2) dx + \frac{1}{\nu} \|f\|^2_{L^\infty(\mathbb{T}^d \times (0,T))} \|\rho_0\|_{L^1(\mathbb{T}^d)},
\end{aligned}$$

since $\|\rho\|_{L^\infty(0,T';L^1(\mathbb{T}^d))} = \|\rho_0\|_{L^1(\mathbb{T}^d)}$. □

After we get the energy estimate, we can obtain the global existence of the approximate solution, which will be stated in the following lemma.

Lemma 3.2.4 (Global existence) *Let $T > 0$, and the assumptions in Lemma 3.2.3 hold. Then There exists a pair of functions, denoted by $(\rho_{N,\delta}, w_{N,\delta})$, in $C^1([0,T]; C^3(\mathbb{T}^d)) \times C^1([0,T]; X_N)$, which is a solution to the approximate system (3.2.1) with smooth initial condition (3.1.4).*
Furthermore, it satisfies the following estimates:

$$\|\sqrt{\rho_{N,\delta}}\|_{L^\infty(0,T;H^1(\mathbb{T}^d))} \leqslant C, \tag{3.2.19}$$

$$\|\rho_{N,\delta}\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^d))} \leqslant C, \tag{3.2.20}$$

$$\|\sqrt{\rho_{N,\delta}}\nabla^2 \log \rho_{N,\delta}\|_{L^2(0,T;L^2(\mathbb{T}^d))} \leqslant C, \tag{3.2.21}$$

$$\|\sqrt{\rho_{N,\delta}}\|_{L^2(0,T;H^2(\mathbb{T}^d))} + \|\sqrt[4]{\rho_{N,\delta}}\|_{L^4(0,T;W^{1,4}(\mathbb{T}^d))} \leqslant C, \tag{3.2.22}$$

$$\|\sqrt{\rho_{N,\delta}} w_{N,\delta}\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} + \|\sqrt{\rho_{N,\delta}}\nabla w_{N,\delta}\|_{L^2(0,T;L^2(\mathbb{T}^d))} \leqslant C, \tag{3.2.23}$$

$$\sqrt{\delta}\|w_{N,\delta}\|_{L^2(0,T;H^1(\mathbb{T}^d))} \leqslant C, \tag{3.2.24}$$

where the constant $C > 0$ is independent of N and δ.

Proof: By Lemma 3.2.2, there exists a $T' > 0$ such that the approximate system (3.2.1) with initial condition (3.1.4) has a solution (denote it by $(\rho_{N,\delta}, w_{N,\delta})$) on $\mathbb{T}^d \times [0, T']$. Let

$$T^* = \{\sup T' | (\rho_{N,\delta}, w_{N,\delta}) \ \textit{exists on} \ \ \mathbb{T}^d \times [0, T']\}. \tag{3.2.25}$$

To prove $T^* = T$, we only need to show that

$$\sup_{t\in[0,T^*)} \|w_{N,\delta}\|_{X_N} \leqslant C < \infty, \tag{3.2.26}$$

where C is independent of T^*.
In fact, if (3.2.26) holds but $T^* < T$. We consider

$$\begin{cases} w_{N,\delta}(T^*) = \lim\limits_{t\to T^*} w_{N,\delta}(t), \\ \rho_{N,\delta}(T^*) = \lim\limits_{t\to T^*} S(w_{N,\delta})(t), \end{cases}$$

as the initial data of the approximate system (3.2.1). By the similar arguments like above, we can extend T^* to a larger time $T^{**} > T^*$, which is a contradiction to (3.2.25). Hence $T^* = T$, furthermore, $(\rho_{N,\delta}, w_{N,\delta})$ exists on the closed interval $[0, T]$.

Next, we show that (3.2.26) holds.

Applying Grownwall's inequality in (3.2.13) gives

$$\sup_{t\in[0,T^*)}\int_{\mathbb{T}^d}\left(\frac{1}{2}\rho_{N,\delta}|u_{N,\delta}|^2+\frac{1}{2}\rho_{N,\delta}|u_{N,\delta}+\nabla\varphi_{N,\delta}|^2+2G(\rho_{N,\delta})+4\varepsilon^2|\nabla\sqrt{\rho_{N,\delta}}|^2\right)dx$$
$$+2\nu\int_0^{T^*}\int_{\mathbb{T}^d}\left(\rho_{N,\delta}|D(u_{N,\delta})|^2+\rho_{N,\delta}|A(u_{N,\delta})|^2\right)dx+2\nu\int_0^{T^*}\int_{\mathbb{T}^d}\gamma\rho_{N,\delta}^{\gamma-2}|\nabla\rho_{N,\delta}|^2dx$$
$$+2\nu\varepsilon^2\int_0^{T^*}\int_{\mathbb{T}^d}\rho_{N,\delta}|\nabla^2\log\rho_{N,\delta}|^2dx+2\delta\int_0^{T^*}\int_{\mathbb{T}^d}(|\nabla w_{N,\delta}|^2+|w_{N,\delta}|^2)dx\leqslant C,$$

where $w_{N,\delta} = u_{N,\delta} + \frac{1}{2}\nabla\varphi_{N,\delta} = u + \nu\nabla\log\rho_{N,\delta}$, and $C > 0$ is a constant independent of T^*, N, and δ. It follows that

$$\|\nabla\sqrt{\rho_{N,\delta}}\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\leqslant C, \tag{3.2.27}$$

$$\|\rho_{N,\delta}\|_{L^\infty(0,T^*;L^\gamma(\mathbb{T}^d))}\leqslant C, \tag{3.2.28}$$

$$\begin{aligned}\|\sqrt{\rho_{N,\delta}}w_{N,\delta}\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}&\leqslant\frac{1}{2}\|\sqrt{\rho_{N,\delta}}(u_{N,\delta}+\nabla\varphi_{N,\delta})\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\\&\quad+\frac{1}{2}\|\sqrt{\rho_{N,\delta}}u_{N,\delta}\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\\&\leqslant C,\end{aligned} \tag{3.2.29}$$

$$\begin{aligned}\|\sqrt{\rho_{N,\delta}}\nabla(\nabla\varphi_{N,\delta})\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}&=\|\sqrt{\rho_{N,\delta}}\nabla^2\log\rho_{N,\delta}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\\&\leqslant C,\end{aligned} \tag{3.2.30}$$

$$\begin{aligned}\|\sqrt{\rho_{N,\delta}}\nabla w_{N,\delta}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}&\leqslant\|\sqrt{\rho_{N,\delta}}\nabla u_{N,\delta}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\\&\quad+\|\sqrt{\rho_{N,\delta}}\nabla(\nabla\varphi_{N,\delta})\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\\&\leqslant\|\sqrt{\rho_{N,\delta}}D(u_{N,\delta})\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\\&\quad+\|\sqrt{\rho_{N,\delta}}A(u_{N,\delta})\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\\&\quad+\|\sqrt{\rho_{N,\delta}}\nabla^2\log\rho_{N,\delta}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\\&\leqslant C,\end{aligned} \tag{3.2.31}$$

$$\sqrt{\delta}\|w_{N,\delta}\|_{L^2(0,T^*;H^1(\mathbb{T}^d))}\leqslant C. \tag{3.2.32}$$

In view of (3.2.3), we have

$$\begin{aligned}\rho_{N,\delta} &\geqslant \delta e^{-\int_0^t \|\mathrm{div} w_{N,\delta}\|_{L^\infty(\mathbb{T}^d)}\,ds} \\ &\geqslant \delta e^{-C\int_0^t \|\mathrm{div} w_{N,\delta}\|^2_{H^1(\mathbb{T}^d)}\,ds} \\ &\geqslant C > 0, \end{aligned} \tag{3.2.33}$$

here C is independent of T^*.

Then it follows from (3.2.29) that

$$\|w_{N,\delta}\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))} \leqslant C,$$

for some $C > 0$ which is independent of T^*.

So (3.2.26) holds.

Finally, since we have shown $T = T^*$, (3.2.19)-(3.2.21),(3.2.23),(3.2.24) follows from (3.2.27)-(3.2.32); and (3.2.22) is a consequence of (3.2.21),(1.2.2) and (1.2.3). □

3.3 A priori estimates

In this section, we will conclude some estimates from the energy estimate of Proposition 3.2.3, which are useful in the proof of the main results. In the following, we always assume that $\gamma \geqslant 1$ if $d = 2$; $\gamma > 3$ if $d = 3$.

Lemma 3.3.1 *The following estimates hold for some $C > 0$ which is independent of N and δ:*

$$\|\rho_{N,\delta}\|_{L^{\frac{4}{d}\gamma+1}(0,T;L^{\frac{4}{d}\gamma+1}(\mathbb{T}^d))} \leqslant C, \tag{3.3.1}$$

$$\|\rho_{N,\delta}\|_{L^2(0,T;W^{2,p}(\mathbb{T}^d))} \leqslant C, \tag{3.3.2}$$

$$\|\rho_{N,\delta} w_{N,\delta}\|_{L^2(0,T;W^{1,\frac{3}{2}}(\mathbb{T}^d))} \leqslant C, \tag{3.3.3}$$

where $p = \dfrac{2\gamma}{\gamma+1}$ *if* $d = 3$; *and* $p < 2$ *if* $d = 2$.

Proof: See Lemma 4.3 in [32]. □

Lemma 3.3.2 *The following estimates hold for some* $C > 0$ *which is independent of* N *and* δ*:*

$$\|\partial_t \rho_{N,\delta}\|_{L^2(0,T;L^{\frac{3}{2}}(\mathbb{T}^d))} \leqslant C, \tag{3.3.4}$$

$$\|\partial_t \sqrt{\rho_{N,\delta}}\|_{L^2(0,T;(H^1(\mathbb{T}^d))^*)} \leqslant C, \tag{3.3.5}$$

$$\|\partial_t(\rho_{N,\delta} w_{N,\delta})\|_{L^{\frac{4}{3}}(0,T;(H^s(\mathbb{T}^d))^*)} \leqslant C, \tag{3.3.6}$$

where $s > \dfrac{d}{2} + 1$.

Proof: See Lemma 4.4 in [32].

□

3.4 Proof of Theorem 3.1.6

In this section, we proceed similarly as in [32], and divide the proof into 3 steps. For step 1 and step 2, we consider the two limits "$N \to \infty$" and "$\delta \to 0$" separately to prove that there exists a weak solution (ρ, w) to the initial value problem (3.1.3)-(3.1.4) if the initial data is smooth; finally, in step 3, we will show that it holds for any finite-energy initial data which only satisfies (3.1.9).

STEP 1 First, we fix $\delta > 0$, let $N \to \infty$, and have the following lemma:

Lemma 3.4.1 *Let $\delta > 0$ be fixed, there exists a pair (ρ_δ, w_δ) such that*

$$\partial_t(\rho_\delta)+\mathrm{div}(\rho_\delta w_\delta) = \nu\Delta\rho_\delta, \quad pointwise \;\; in \;\; \mathbb{T}^d \times (0,T),$$

$$\begin{aligned}\int_{\mathbb{T}^d} \rho_0 w_0 \cdot \phi(\cdot,0)dx + \int_0^T \int_{\mathbb{T}^d} [\rho_\delta w_\delta \cdot \phi_t + (\rho_\delta w_\delta \otimes w_\delta) : \nabla\phi + P(\rho_\delta)\mathrm{div}\phi + \rho_\delta f \cdot \phi \\ + 2(\nu^2-\varepsilon^2)\Delta\sqrt{\rho_\delta}(\sqrt{\rho_\delta}\mathrm{div}\phi + 2\nabla\sqrt{\rho_\delta}\cdot\phi) - \nu\nabla(\rho_\delta w_\delta) : \nabla\phi \\ - \delta(\nabla w_\delta : \nabla\phi + w_\delta \cdot \phi)]dxdt = 0,\end{aligned} \tag{3.4.1}$$

for all test functions ϕ such that the integrals above are well defined.

Proof: For approximate solution $(\rho_{N,\delta}, w_{N,\delta})$, we have shown that for any $\phi \in (C^1([0,T]; X_N))^d$ with $\phi(\cdot, T) = 0$,

$$\begin{aligned}&\partial_t(\rho_{N,\delta}) + \mathrm{div}(\rho_{N,\delta} w_{N,\delta}) = \nu\Delta\rho_{N,\delta}, \quad in \;\; \mathbb{T}^d \times (0,T), \\ &\int_{\mathbb{T}^d} \rho_0 w_0 \cdot \phi(\cdot,0)dx + \int_0^T \int_{\mathbb{T}^d} [\rho_{N,\delta} w_{N,\delta} \cdot \phi_t + (\rho_{N,\delta} w_{N,\delta} \otimes w_{N,\delta}) : \nabla\phi + P(\rho_{N,\delta})\mathrm{div}\phi \\ &\quad + 2(\nu^2-\varepsilon^2)\Delta\sqrt{\rho_{N,\delta}}(\sqrt{\rho_{N,\delta}}\mathrm{div}\phi + 2\nabla\sqrt{\rho_{N,\delta}}\cdot\phi) - \nu\nabla(\rho_{N,\delta} w_{N,\delta}) : \nabla\phi \\ &\quad - \delta(\nabla w_{N,\delta} : \nabla\phi + w_{N,\delta}\cdot\phi) + \rho_{N,\delta} f \cdot \phi]dxdt = 0,\end{aligned}$$

We are now let $N \to +\infty$.

Since

$$W^{2,p}(\mathbb{T}^d) \hookrightarrow\hookrightarrow L^\infty(\mathbb{T}^d), \; H^2(\mathbb{T}^d) \hookrightarrow\hookrightarrow H^1(\mathbb{T}^d), \; W^{1,\frac{3}{2}} \hookrightarrow\hookrightarrow L^2(\mathbb{T}^d),$$

by Aubin-Lions lemma, it follows from (3.3.2)&(3.3.4), (3.2.22)&(3.3.5) and (3.3.3)&(3.3.6) that there exist subsequences $\{\rho_{N,\delta}\}$, $\{\sqrt{\rho_{N,\delta}}\}$, $\{\rho_{N,\delta} w_{N,\delta}\}$ (not relabeled) such that for some functions ρ_δ and j_δ, it holds that

$$\begin{aligned}\rho_{N,\delta} \to \rho_\delta \quad strongly \quad in \quad L^2(0,T; L^\infty(\mathbb{T}^d)) \;\; as \;\; N \to +\infty, \\ \sqrt{\rho_{N,\delta}} \to \sqrt{\rho_\delta} \quad strongly \quad in \quad L^2(0,T; H^1(\mathbb{T}^d)) \;\; as \;\; N \to +\infty, \\ \rho_{N,\delta} w_{N,\delta} \to j_\delta \quad strongly \quad in \quad L^2(0,T; L^2(\mathbb{T}^d)) \;\; as \;\; N \to +\infty.\end{aligned}$$

Also, we have the following weak convergence:

$$\sqrt{\rho_{N,\delta}} \rightharpoonup \sqrt{\rho_\delta} \quad weakly \quad in \quad L^2(0,T;H^2(\mathbb{T}^d)) \quad as \quad N \to +\infty,$$
$$\nabla(\rho_{N,\delta} w_{N,\delta}) \rightharpoonup \nabla(\rho_\delta w_\delta) \quad weakly \quad in \quad L^2(0,T;L^{\frac{3}{2}}(\mathbb{T}^d)) \quad as \quad N \to +\infty,$$
$$w_{N,\delta} \rightharpoonup w_\delta \quad weakly \quad in \quad L^2(0,T;L^6(\mathbb{T}^d)) \quad as \quad N \to +\infty.$$

Since it is easy to check that

$$\rho_{N,\delta} w_{N,\delta} \rightharpoonup \rho_\delta w_\delta \quad weakly \quad in \quad L^1(0,T;L^6(\mathbb{T}^d)) \quad as \quad N \to +\infty,$$

we infer that $j_\delta = \rho_\delta w_\delta$.

With the convergence results above, we then can finish the proof immediately. □

STEP 2 Next, let the test function in (3.4.1) above be $\rho_\delta \phi$, according to Lemma 3.1.2, we get

$$\begin{aligned}
&\int_{\mathbb{T}^d} \rho_0^2 w_0 \cdot \phi(\cdot,0)dx + \int_0^T \int_{\mathbb{T}^d} [\rho_\delta^2 w_\delta \cdot \phi_t - \rho_\delta^2 w_\delta \mathrm{div} w_\delta \cdot \phi + (\rho_\delta w_\delta \otimes \rho_\delta w_\delta) : \nabla \phi \\
&\quad - \nu(\rho_\delta w_\delta \otimes \nabla \rho_\delta) : \nabla \phi + 2(\nu^2 - \varepsilon^2)\Delta\sqrt{\rho_\delta}(\sqrt{\rho_\delta}^3 \mathrm{div}\phi + 2\sqrt{\rho_\delta}\nabla\rho_\delta \cdot \phi) \\
&\quad + \frac{\gamma}{\gamma+1}\rho_\delta^{\gamma+1}\mathrm{div}\phi - \nu\nabla(\rho_\delta w_\delta) : (2\nabla\rho_\delta \otimes \phi + \rho_\delta \nabla\phi) + \rho_\delta^2 f \cdot \phi] dxdt = 0.
\end{aligned}$$

Now, we want to pass the limit $\delta \to 0$ term by term.

Using Aubin-Lions lemma, we have for some functions ρ and j,

$$\rho_\delta \to \rho \quad strongly \quad in \quad L^2(0,T;W^{1,m}(\mathbb{T}^d)) \quad as \quad \delta \to 0, \tag{3.4.2}$$
$$\sqrt{\rho_\delta} \to \sqrt{\rho} \quad strongly \quad in \quad L^\infty(0,T;L^r(\mathbb{T}^d)) \quad as \quad \delta \to 0, \tag{3.4.3}$$
$$\rho_\delta w_\delta \to j \quad strongly \quad in \quad L^2(0,T;L^q(\mathbb{T}^d)) \quad as \quad \delta \to 0, \tag{3.4.4}$$

where $m \in (1, \frac{6\gamma}{\gamma+3})$, $r \in [1,6)$, $q \in [1,3)$.

Since $\frac{\rho_\delta w_\delta}{\sqrt{\rho_\delta}}$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$, Fatou's lemma yields

$$\int_{\mathbb{T}^d} \liminf_{\delta \to 0} \frac{|\rho_\delta w_\delta|^2}{\rho_\delta} dx < +\infty.$$

In particular, we have $j = 0$ a.e. in $\{\rho = 0\}$. So if we define the limit velocity w by setting

$$w \triangleq \begin{cases} \dfrac{j}{\rho}, & when \ \ \rho \neq 0, \\ 0, & when \ \ \rho = 0, \end{cases}$$

then we have $j = \rho w$.

Lemma 3.4.2 *Up to subsequences, for some functions ρ and w, it holds that*

$$\rho_\delta^2 w_\delta \to \rho^2 w \quad strongly \ \ in \ \ L^1(0,T;L^q(\mathbb{T}^d)) \ \ as \ \ \delta \to 0, \tag{3.4.5}$$

$$\rho_\delta w_\delta \otimes \nabla\rho_\delta \to \rho w \otimes \nabla\rho \quad strongly \ \ in \ \ L^1(0,T;L^{\frac{3}{2}}(\mathbb{T}^d)) \ \ as \ \ \delta \to 0, \tag{3.4.6}$$

$$\rho_\delta w_\delta \otimes \rho_\delta w_\delta \to \rho w \otimes \rho w \quad strongly \ \ in \ \ L^1(0,T;L^{\frac{q}{2}}(\mathbb{T}^d)) \ \ as \ \ \delta \to 0, \tag{3.4.7}$$

$$\rho_\delta^{\gamma+1} \to \rho^{\gamma+1} \quad strongly \ \ in \ \ L^1(0,T;L^1(\mathbb{T}^d)) \ \ as \ \ \delta \to 0, \tag{3.4.8}$$

$$\Delta\sqrt{\rho_\delta}\sqrt{\rho_\delta}\nabla\rho_\delta \rightharpoonup \Delta\sqrt{\rho}\sqrt{\rho}\nabla\rho \quad weakly \ \ in \ \ L^1(0,T;L^1(\mathbb{T}^d)) \ \ as \ \ \delta \to 0, \tag{3.4.9}$$

$$\nabla(\rho_\delta w_\delta)\nabla\rho_\delta \rightharpoonup \nabla(\rho w)\nabla\rho \quad weakly \ \ in \ \ L^1(0,T;L^1(\mathbb{T}^d)) \ \ as \ \ \delta \to 0, \tag{3.4.10}$$

$$\nabla(\rho_\delta w_\delta)\rho_\delta \rightharpoonup \nabla(\rho w)\rho \quad weakly \ \ in \ \ L^1(0,T;L^1(\mathbb{T}^d)) \ \ as \ \ \delta \to 0, \tag{3.4.11}$$

$$\rho_\delta^2 w_\delta \mathrm{div} w_\delta \rightharpoonup \rho^2 w \mathrm{div} w \quad weakly \ \ in \ \ L^1(0,T;L^1(\mathbb{T}^d)) \ \ as \ \ \delta \to 0, \tag{3.4.12}$$

$$\rho_\delta^2 f \rightharpoonup \rho^2 f \quad weakly \ \ in \ \ L^1(0,T;L^1(\mathbb{T}^d)) \ \ as \ \ \delta \to 0, \tag{3.4.13}$$

where $q \in [1,3)$.

Proof: (3.4.5)-(3.4.12) have been shown in [32], and (3.4.13) can be easily got due to (3.4.2). □

Finally, in view of the estimate (3.2.24) for $\sqrt{\delta}w_\delta$, we have for smooth test functions ϕ, as $\delta \to 0$,

$$
\begin{aligned}
&\delta \int_{\mathbb{T}^d} \rho_\delta \nabla w_\delta : \nabla \phi dx \\
&\qquad \leqslant \sqrt{\delta} \|\sqrt{\delta}\nabla w_\delta\|_{L^2(0,T;L^2(\mathbb{T}^d))} \|\rho_\delta\|_{L^2(0,T;L^\infty(\mathbb{T}^d))} \|\phi\|_{L^\infty(0,T;H^1(\mathbb{T}^d))} \to 0,
\end{aligned}
$$

$$
\begin{aligned}
&\delta \int_{\mathbb{T}^d} \nabla w_\delta : (\nabla \rho_\delta \otimes \phi) dx \\
&\qquad \leqslant \sqrt{\delta} \|\sqrt{\delta}\nabla w_\delta\|_{L^2(0,T;L^2(\mathbb{T}^d))} \|\rho_\delta\|_{L^2(0,T;W^{1,3}(\mathbb{T}^d))} \|\phi\|_{L^\infty(0,T;L^6(\mathbb{T}^d))} \to 0,
\end{aligned}
$$

$$
\delta \int_{\mathbb{T}^d} \rho_\delta w_\delta \cdot \phi dx \leqslant \delta \|\rho_\delta w_\delta\|_{L^2(0,T;L^{\frac{3}{2}}(\mathbb{T}^d))} \|\phi\|_{L^2(0,T;L^3(\mathbb{T}^d))} \to 0.
$$

Therefore, we now are able to pass the limit $\delta \to 0$ term by term, and obtain that (ρ, w) (defined above) is a weak solution to (3.1.3)-(3.1.4) for smooth initial data.

Remark 3.4.3 *The restriction $\gamma > 3$ when $d = 3$ is crucial in the proof of (3.4.9)-(3.4.12). For example, we have the convergence*

$$
\begin{aligned}
\sqrt{\rho_\delta} \to \sqrt{\rho} \quad &\textit{strongly in} \quad L^\infty(0,T;L^r(\mathbb{T}^d)) \quad (r < 6), \\
\Delta\sqrt{\rho_\delta} \rightharpoonup \Delta\sqrt{\rho} \quad &\textit{weakly in} \quad L^2(0,T;L^2(\mathbb{T}^d)), \\
\nabla \rho_\delta \to \nabla \rho \quad &\textit{strongly in} \quad L^2(0,T;L^m(\mathbb{T}^d)),
\end{aligned}
$$

then

$$
\Delta\sqrt{\rho_\delta}\sqrt{\rho_\delta}\nabla\rho_\delta \rightharpoonup \Delta\sqrt{\rho}\sqrt{\rho}\nabla\rho \quad \textit{weakly in} \quad L^1(0,T;L^1(\mathbb{T}^d)),
$$

provided $m > 3$.
Since $m < \dfrac{6\gamma}{\gamma+3}$, so we need $\dfrac{6\gamma}{\gamma+3} > 3$, i.e. $\gamma > 3$.

STEP 3 After STEP 1 and STEP 2, we have proved that (ρ, w) solves (3.1.3)-(3.1.4) for smooth initial data. If (ρ_0, w_0) only satisfies (3.1.9), we can construct an approximate sequence $(\rho_0^\delta, w_0^\delta)$ such that

$$\begin{cases} (\rho_0^\delta, w_0^\delta) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d), \quad \rho_0^\delta \geqslant \delta > 0 \ \ in \ \ \mathbb{T}^d, \\ \sqrt{\rho_0^\delta} \to \sqrt{\rho_0} \quad strongly \quad in \quad H^1(\mathbb{T}^d) \ \ as \ \ \delta \to 0, \\ \sqrt{\rho_0^\delta} w_0^\delta \to \sqrt{\rho_0} w_0 \quad strongly \quad in \quad L^2(\mathbb{T}^d) \ \ as \ \ \delta \to 0. \end{cases}$$

In particular,

$$\sqrt{\rho_0^\delta} \to \sqrt{\rho_0} \quad strongly \quad in \quad L^6(\mathbb{T}^d) \ \ as \ \ \delta \to 0,$$

and therefore

$$\rho_0^\delta w_0^\delta \to \ \rho_0 w_0 \quad strongly \quad in \quad L^{\frac{3}{2}}(\mathbb{T}^d) \ \ as \ \ \delta \to 0.$$

By the above proof, there exists a weak solution (ρ_δ, w_δ) to (3.1.3)-(3.1.4) with initial data $(\rho_0^\delta, w_0^\delta)$. In particular, $(\rho_\delta, \rho_\delta w_\delta)$ converges strongly to $(\rho, \rho w)$ as $\delta \to 0$ in some space, and there exist uniform bounds for ρ_δ in $H^1(0, T; L^{\frac{3}{2}}(\mathbb{T}^d))$ and for $\rho_\delta w_\delta$ in $W^{1,\frac{4}{3}}(0, T; (H^s(\mathbb{T}^d))^*)$.

Thus, up to subsequences,

$$\begin{cases} \rho_0^\delta = \rho_\delta(\cdot, 0) \rightharpoonup \rho(\cdot, 0) \quad weakly \quad in \quad L^{\frac{3}{2}}(\mathbb{T}^d) \ \ as \ \ \delta \to 0, \\ \rho_0^\delta w_0^\delta = (\rho_\delta w_\delta)(\cdot, 0) \rightharpoonup (\rho w)(\cdot, 0) \quad weakly \quad in \quad (H^s(\mathbb{T}^d))^* \ \ as \ \ \delta \to 0. \end{cases}$$

This shows that $\rho(\cdot, 0) = \rho_0$ and $(\rho w)(\cdot, 0) = \rho_0 w_0$ in the sense of distributions. Hence, we finish the proof of Theorem 3.1.6, which gives Theorem 3.1.5.

Chapter 4

Global Existence and Large Time Behavior of Weak Solutions to Quantum Navier-Stokes-Poisson Equations

In this chapter, we prove the global existence of weak solutions to the compressible Navier-Stokes-Poisson for quantum fluids in $\mathbb{T}^2$ with large initial data, and then show the large time behavior of the weak solutions.

4.1 Global existence of weak solutions

In this chapter, we consider the quantum Navier-Stokes-Poisson system, which is the quantum Navier-Stokes equations coupled self-consistently to a Poisson equation for the electric potential. This system reads as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \qquad\qquad (x,t) \in \mathbb{T}^2 \times (0,T), \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \rho - 2\varepsilon^2 \rho \nabla \left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - 2\nu \operatorname{div}(\rho D(u)) = \rho \nabla V - \dfrac{\rho u}{\tau}, \\ \lambda^2 \Delta V = \rho - K, \end{cases} \tag{4.1.1}$$

with the initial condition

$$\rho|_{t=0} = \rho_0(x), \quad (\rho u)|_{t=0} = \rho_0 u_0, \ \text{ in } \mathbb{T}^2. \tag{4.1.2}$$

Here the unknowns are the particle density $\rho = \rho(x,t) : \mathbb{T}^2 \times [0,+\infty) \to \mathbb{R}^+ \cup \{0\}$, the particle velocity $u = u(x,t) : \mathbb{T}^2 \times [0,+\infty) \to \mathbb{R}^2$, and the electric potential $V = V(x,t) : \mathbb{T}^2 \times [0,+\infty) \to \mathbb{R}$. $u \otimes u$ is the matrix with components $u_i u_j$, $D(u) = \dfrac{1}{2}(\nabla u + \nabla^\top u)$ is the symmetric part of the velocity gradient, and $\mathbb{T}^2$ is the two-dimensional torus. The scaled physical parameters are the (scaled) Planck constant ε, the viscosity constant ν, the momentum relaxation time τ and the Debye length λ. All these constants are assumed to be positive. The nonlinear dispersive term $-2\varepsilon^2 \rho \nabla \left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$ is produced by the quantum Bohm potential $Q(\rho) = 2\varepsilon^2 \dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$. K is the doping profile of background charges.

Moreover, we assume that

$$\int_{\mathbb{T}^2} V(x,t)dx = 0, \quad t \geqslant 0,$$

and the compatibility condition

$$\int_{\mathbb{T}^2} (\rho_0(x) - K)dx = 0. \tag{4.1.3}$$

The compatibility condition (4.1.3) is necessary; otherwise the Poisson equation for V would not be solvable.

Theorem 4.1.1 (Global existence)
Let $T > 0, \varepsilon, \nu, \tau > 0$ with $\nu < \tau$. Assume that the initial data (ρ_0, u_0) satisfies the compatibility condition (4.1.3) and

$$\begin{cases} \rho_0(x) \geqslant 0, & in \quad \mathbb{T}^2, \\ \displaystyle\int_{\mathbb{T}^2} \left(\frac{1}{2}\rho_0|u_0|^2 + G(\rho_0)\right) dx < \infty, \\ \displaystyle\int_{\mathbb{T}^2} |\nabla\sqrt{\rho_0}|^2 dx < \infty, \end{cases} \tag{4.1.4}$$

where

$$G(\rho) = \rho(\log\frac{\rho}{K} - 1) + K.$$

Then there exists a weak solution (ρ, u, V) to the initial value problem (4.1.1)-(4.1.2) with $\rho \geqslant 0$ in $\mathbb{T}^2$ in the sense that (ρ, u, V) satisfies $(4.1.1)_1$, $(4.1.1)_3$ *pointwise and satisfies*

$$\begin{aligned} &\int_{\mathbb{T}^2} \rho_0^2 u_0 \cdot \phi(\cdot, 0) dx + \int_0^T \int_{\mathbb{T}^2} [\rho^2 u \cdot \phi_t - \rho^2 u \mathrm{div} u \cdot \phi + (\rho u \otimes \rho u) : \nabla\phi \\ &\quad + \frac{1}{2}\rho^2 \mathrm{div}\phi - 2\varepsilon^2 \Delta\sqrt{\rho}(\sqrt{\rho}^3 \mathrm{div}\phi + 2\sqrt{\rho}\nabla\rho \cdot \phi) \\ &\quad - 2\nu\rho D(u) : (\nabla\rho \otimes \phi + \rho\nabla\phi) + \rho^2 \nabla V \cdot \phi + \frac{\rho u}{\tau} \cdot \phi] dx dt = 0, \end{aligned} \tag{4.1.5}$$

for any $\phi \in (C^1(\mathbb{T}^2 \times (0, T)))^2$ with $\phi(\cdot, T) = 0$.

The proof of this theorem is similar to the one of the quantum Navier-Stokes equations, which has been shown in Chapter 3. Hence we only need to sketch the proof.

First, by introducing an auxiliary velocity

$$w = u + \nu\nabla\log\rho,$$

we reformulate the initial value problem (4.1.1)-(4.1.2) to

$$\begin{cases} \rho_t + \operatorname{div}(\rho w) = \nu\Delta\rho, & (x,t) \in \mathbb{T}^2 \times (0,T), \\ (\rho w)_t + \operatorname{div}(\rho w \otimes w) + k\nabla\rho + 2(\nu^2 - \varepsilon^2)\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \nu\Delta(\rho w) = \rho\nabla V - \dfrac{\rho w}{\tau}, \\ \lambda^2\Delta V = \rho - K, \end{cases} \tag{4.1.6}$$

with the initial condition

$$\rho|_{t=0} = \rho_0(x), \quad (\rho w)|_{t=0} = \rho_0 w_0, \ \text{ in } \ \mathbb{T}^2, \tag{4.1.7}$$

where $k = 1 - \dfrac{\nu}{\tau} > 0$, $\ w_0 = u_0 + \nu\nabla\log\rho_0$.

From

$$\lambda^2\Delta V = \rho - K \ \ in \ \ \mathbb{T}^2,$$

and

$$\int_{\mathbb{T}^2} V(x,t)dx = 0, \quad t \geqslant 0,$$

we have the estimate

$$\|\nabla V\|_{L^p(\mathbb{T}^2)} \leqslant \|\rho - K\|_{L^p(\mathbb{T}^2)}, \ \ p \in (1, +\infty).$$

The existence proof for the approximate solutions is similar to Section 3.2 after we replacing f by $\nabla V[\rho]$ satisfying $(4.1.1)_3$ with $\rho = S[w]$ and observing the new energy estimate

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{T}^2}\left(\frac{1}{2}\rho|u|^2 + \frac{1}{2}\rho|u+\nabla\varphi|^2 + 2kG(\rho) + 4\varepsilon^2|\nabla\sqrt{\rho}|^2 + \lambda^2|\nabla V|^2\right)dx \\ &\quad + 2\nu\int_{\mathbb{T}^2}\left(\rho|D(u)|^2 + \rho|A(u)|^2\right)dx + 2\nu\int_{\mathbb{T}^2}\gamma\rho^{\gamma-2}|\nabla\rho|^2 dx \\ &\quad + 2\nu\varepsilon^2\int_{\mathbb{T}^2}\rho|\nabla^2\log\rho|^2 dx + 2\delta\int_{\mathbb{T}^2}(|\nabla w|^2 + |w|^2)dx + \frac{2\nu}{\lambda^2}\int_{\mathbb{T}^2}\rho^2 dx \\ &\leqslant \frac{2\nu K}{\lambda^2}\|\rho_0\|_{L^1(\mathbb{T}^2)}, \end{aligned} \tag{4.1.8}$$

where $\nabla\varphi = 2\nu\nabla\log\rho$, $u = w - \dfrac{1}{2}\nabla\varphi$ and $A(u) = \dfrac{\nabla u - \nabla^\top u}{2}$.

After we get the energy estimate above, we obtain the same a priori estimates as those in Section 3.3, and can proceed in the same way as in Section 3.4 to get the existence of the weak solutions.

4.2 Large time behavior

In this section, we will show the large time behavior of the weak solutions which can be stated in the following theorem:

Theorem 4.2.1 *The weak solution (ρ, u, V) is closed to a steady state in the following sense:*
For $\forall\ \epsilon > 0$, there exists a time $T(\epsilon) > 0$ such that for $\forall t > T(\epsilon)$, it holds that

$$\|\sqrt{\rho(t,\cdot)} - \sqrt{K}\|_{L^m(\mathbb{T}^2)} < \epsilon,$$
$$\|\nabla V(t,\cdot)\|_{L^2(\mathbb{T}^2)} < \epsilon,$$

where $m \in (1, +\infty)$.

Proof: During the proof of the existence of weak solutions, we have shown that there exists a sequence $\{(\rho_\delta, w_\delta, V_\delta)\}_{\delta>0}$ satisfying

$$\begin{aligned}
&(\rho_\delta)_t + \operatorname{div}(\rho_\delta w_\delta) = \nu\Delta\rho_\delta, \quad \textit{pointwise in } \mathbb{T}^2 \times (0,T),\\
&\qquad \lambda^2 \Delta V_\delta = \rho_\delta - K, \quad \textit{pointwise in } \mathbb{T}^2 \times (0,T),\\
&\int_{\mathbb{T}^2} [(\rho_\delta u_\delta)_t + \operatorname{div}(\rho_\delta u_\delta \otimes u_\delta) + k\nabla\rho_\delta - 2\varepsilon^2 \rho_\delta \nabla\left(\frac{\Delta\sqrt{\rho_\delta}}{\sqrt{\rho_\delta}}\right)\\
&\qquad - 2\nu \operatorname{div}(\rho_\delta D(u_\delta)) - \delta\Delta w_\delta + \delta w_\delta - \rho_\delta \nabla V_\delta + \frac{\rho_\delta w_\delta}{\tau}] \cdot \phi dx = 0, \qquad (4.2.1)
\end{aligned}$$

for any smooth functions ϕ, and

$$\rho_\delta \to \rho \quad strongly \;\; in \;\; L^2(0,T;W^{1,m}(\mathbb{T}^2)) \;\; as \;\; \delta \to 0, \tag{4.2.2}$$

$$\sqrt{\rho_\delta} \to \sqrt{\rho} \quad strongly \;\; in \;\; L^\infty(0,T;L^m(\mathbb{T}^2)) \;\; as \;\; \delta \to 0, \tag{4.2.3}$$

$$\nabla V_\delta \to \nabla V \quad strongly \;\; in \;\; L^2(0,T;L^m(\mathbb{T}^2)) \;\; as \;\; \delta \to 0, \tag{4.2.4}$$

for $\forall\, m \in (1,+\infty)$.

Define

$$\begin{aligned} E_\delta(t) &= \int_{\mathbb{T}^2} \left(\frac{1}{2}\rho_\delta(|u_\delta|^2 + |u_\delta + \nabla\varphi_\delta|^2) + 2kG(\rho_\delta) + 4\varepsilon^2|\nabla\sqrt{\rho_\delta}|^2 + \lambda^2|\nabla V_\delta|^2 \right) dx \\ &\triangleq E_1 + E_2 + E_3 + E_4, \end{aligned}$$

where $u_\delta = w_\delta - \nu\nabla\log\rho_\delta$.

Following an approach of [7], we compute the time derivatives of $E_\delta(t)$ term by term.

Let $\phi = 2w_\delta$ in (4.2.1), a computation like (3.2.18) gives

$$\begin{aligned} \partial_t E_1 = &-2\nu \int_{\mathbb{T}^2} \left(\rho_\delta|D(u_\delta)|^2 + \rho_\delta|A(u_\delta)|^2\right) dx + 4\varepsilon^2 \int_{\mathbb{T}^2} \nabla B(\rho_\delta)\cdot(\rho_\delta w_\delta)dx \\ &+ 2\int_{\mathbb{T}^2} \nabla V_\delta \cdot (\rho_\delta w_\delta)dx - 2k\int_{\mathbb{T}^2} w_\delta\cdot\nabla\rho_\delta dx - \delta\int_{\mathbb{T}^2}(|\nabla w_\delta|^2 + |w_\delta|^2)dx - \frac{1}{\tau}E_1, \end{aligned} \tag{4.2.5}$$

and integration by parts yields

$$\begin{aligned} \partial_t E_2 &= 2k\int_{\mathbb{T}^2} (\rho_\delta)_t \log\rho_\delta dx \\ &= 2k\int_{\mathbb{T}^2} \log\rho_\delta(\nu\Delta\rho_\delta - \operatorname{div}(\rho_\delta w_\delta))dx \\ &= -\frac{2k\nu}{\varepsilon^2}E_3 + 2k\int_{\mathbb{T}^2} w_\delta\cdot\nabla\rho_\delta dx, \end{aligned} \tag{4.2.6}$$

$$
\begin{aligned}
\partial_t E_3 &= 8\varepsilon^2 \int_{\mathbb{T}^2} \nabla\sqrt{\rho_\delta}\cdot(\nabla\sqrt{\rho_\delta})_t dx \\
&= -8\varepsilon^2 \int_{\mathbb{T}^2} \Delta\sqrt{\rho_\delta}(\sqrt{\rho_\delta})_t dx \\
&= -4\varepsilon^2 \int_{\mathbb{T}^2} B(\rho_\delta)(\rho_\delta)_t dx \\
&= 4\varepsilon^2 \int_{\mathbb{T}^2} B(\rho_\delta)(\operatorname{div}(\rho_\delta w_\delta) - \nu\Delta\rho_\delta) dx, \\
&= 4\varepsilon^2 \int_{\mathbb{T}^2} B(\rho_\delta)\operatorname{div}(\rho_\delta w_\delta) - 2\varepsilon^2\nu \int_{\mathbb{T}^2} \rho_\delta |\nabla^2 \log\rho_\delta|^2 dx, \qquad (4.2.7)
\end{aligned}
$$

where $B(\rho) = (\Delta\sqrt{\rho})/\sqrt{\rho}$.

$$
\begin{aligned}
\partial_t E_4 &= 2\lambda^2 \int_{\mathbb{T}^2} \nabla V_\delta \cdot (\nabla V_\delta)_t dx \\
&= -2\lambda^2 \int_{\mathbb{T}^2} V_\delta (\Delta V_\delta)_t dx \\
&= -2 \int_{\mathbb{T}^2} V_\delta (\rho_\delta)_t dx \\
&= 2 \int_{\mathbb{T}^2} V_\delta (\operatorname{div}(\rho_\delta w_\delta) - \nu\Delta\rho_\delta) dx \\
&= -2 \int_{\mathbb{T}^2} \nabla V_\delta \cdot (\rho_\delta w_\delta) dx + 2\nu \int_{\mathbb{T}^2} \nabla V_\delta \cdot \nabla\rho_\delta dx. \qquad (4.2.8)
\end{aligned}
$$

Combining (4.2.5)-(4.2.8) leads to

$$
\begin{aligned}
\partial_t E_\delta(t) = &-\frac{1}{\tau}E_1 - \frac{2k\nu}{\varepsilon^2}E_3 + 2\nu\int_{\mathbb{T}^2} \nabla V_\delta\cdot\nabla\rho_\delta dx - \delta\int_{\mathbb{T}^2}(|\nabla w_\delta|^2 + |w_\delta|^2)dx \\
&- 2\varepsilon^2\nu \int_{\mathbb{T}^2} \rho_\delta|\nabla^2\log\rho_\delta|^2 dx - 2\nu\int_{\mathbb{T}^2}\left(\rho_\delta|D(u_\delta)|^2 + \rho_\delta|A(u_\delta)|^2\right)dx.
\end{aligned}
$$

We divide the third term into two parts:

$$
\begin{aligned}
2\nu\int_{\mathbb{T}^2}\nabla V_\delta\cdot\nabla\rho_\delta dx &= 2\nu\int_{\mathbb{T}^2}\nabla V_\delta\cdot\nabla(\rho_\delta - K)dx \\
&= -2\nu\int_{\mathbb{T}^2}\Delta V_\delta(\rho_\delta - K)dx \\
&= -2\nu\lambda^2\int_{\mathbb{T}^2}|\Delta V_\delta|^2 dx \\
&= -\nu\lambda^2\int_{\mathbb{T}^2}|\Delta V_\delta|^2 dx - \frac{\nu}{\lambda^2}\int_{\mathbb{T}^2}(\rho_\delta - K)^2 dx.
\end{aligned}
$$

If β denotes the first positive eigenvalue of $-\Delta$ on $\mathbb{T}^2$, then

$$\|\nabla V_\delta\|^2_{L^2(\mathbb{T}^2)} \leqslant \frac{1}{\beta}\|\Delta V_\delta\|^2_{L^2(\mathbb{T}^2)},$$

which gives

$$-\nu\lambda^2 \int_{\mathbb{T}^2} |\Delta V_\delta|^2 dx \leqslant -\nu\beta E_4.$$

Since $s(\log s - 1) + 1 \leqslant (s-1)^2$ for $s > 0$,
let $s = \frac{\rho_\delta}{K}$, we have

$$\rho_\delta(\log \frac{\rho_\delta}{K} - 1) + K \leqslant K^{-1}(\rho_\delta - K)^2,$$

which leads to

$$-\frac{\nu}{\lambda^2} \int_{\mathbb{T}^2} (\rho_\delta - K)^2 dx \leqslant -\frac{\nu K}{2k\lambda^2} E_2.$$

It then follows that

$$\begin{aligned}\partial_t E_\delta(t) \leqslant & -\frac{1}{\tau}E_1 - \frac{\nu K}{2k\lambda^2}E_2 - \frac{2k\nu}{\varepsilon^2}E_3 - \nu\beta E_4 - \delta \int_{\mathbb{T}^2} (|\nabla w_\delta|^2 + |w_\delta|^2) dx \\ & - 2\varepsilon^2\nu \int_{\mathbb{T}^2} \rho_\delta |\nabla^2 \log \rho_\delta|^2 dx - 2\nu \int_{\mathbb{T}^2} \left(\rho_\delta |D(u_\delta)|^2 + \rho_\delta |A(u_\delta)|^2\right) dx \\ \leqslant & -\sigma E_\delta(t),\end{aligned}$$

where $\sigma = \min\{\frac{1}{\tau}, \frac{\nu K}{2k\lambda^2}, \frac{2k\nu}{\varepsilon^2}, \nu\beta\} > 0$.

Therefore, we have

$$E_\delta(t) \leqslant e^{-\sigma t} E_\delta(0).$$

Since we can choose the approximate initial data sequence $(\rho_0^\delta, w_0^\delta)$ such that

$$\begin{cases} (\rho_0^\delta, w_0^\delta) \in C^\infty(\mathbb{T}^2) \times C^\infty(\mathbb{T}^2), \quad \rho_0^\delta \geqslant \delta > 0 \ \ in \ \ \mathbb{T}^2, \\ \sqrt{\rho_0^\delta} \to \sqrt{\rho_0} \ \ strongly \ \ in \ \ H^1(\mathbb{T}^2) \ \ as \ \ \delta \to 0, \\ \sqrt{\rho_0^\delta} w_0^\delta \to \sqrt{\rho_0} w_0 \ \ strongly \ \ in \ \ L^2(\mathbb{T}^2) \ \ as \ \ \delta \to 0, \end{cases}$$

so there there exists a constant $C > 0$, which is independent of δ, such that $E_\delta(0) < C$.

Therefore, we get

$$4\varepsilon^2\|\nabla(\sqrt{\rho_\delta}(t,\cdot)-\sqrt{K})\|^2_{L^2(\mathbb{T}^2)}\leqslant Ce^{-\sigma t}, \tag{4.2.9}$$

$$\lambda^2\|\nabla V_\delta(t,\cdot)\|^2_{L^2(\mathbb{T}^2)}\leqslant Ce^{-\sigma t}. \tag{4.2.10}$$

Next, it is easy to check that

$$(s-1)^2\leqslant s^2(\log s^2-1)+1,\quad s\in(0,+\infty),$$

which gives

$$(\sqrt{\rho_\delta}-\sqrt{K})^2\leqslant \rho_\delta(\log\frac{\rho_\delta}{K}-1)+K,$$

so it holds that

$$2k\|\sqrt{\rho_\delta}(t,\cdot)-\sqrt{K}\|^2_{L^2(\mathbb{T}^2)}\leqslant E_2(t)\leqslant Ce^{-\sigma t}. \tag{4.2.11}$$

The embedding $H^1(\mathbb{T}^2)\hookrightarrow L^m(\mathbb{T}^2)(1<m<+\infty)$ and (4.2.9),(4.2.11) yield

$$\|\sqrt{\rho_\delta}(t,\cdot)-\sqrt{K}\|^2_{L^m(\mathbb{T}^2)}\leqslant Ce^{-\sigma t},\quad \forall\ m\in(1,+\infty). \tag{4.2.12}$$

For any $\epsilon>0$, there exists $T(\epsilon)>0$ such that

$$\|\sqrt{\rho_\delta(t,\cdot)}-\sqrt{K}\|_{L^m(\mathbb{T}^2)}<\frac{\epsilon}{2},\quad t>T(\epsilon).$$

The strong convergence (4.2.3) implies that for this $\epsilon>0$ and any $t>T(\epsilon)$, we can always choose δ small enough such that

$$\|\sqrt{\rho_\delta(t,\cdot)}-\sqrt{\rho(t,\cdot)}\|_{L^m(\mathbb{T}^2)}<\frac{\epsilon}{2}.$$

Hence, for any $t>T(\epsilon)$,

$$\|\sqrt{\rho(t,\cdot)}-\sqrt{K}\|_{L^m(\mathbb{T}^2)}\leqslant\|\sqrt{\rho_\delta(t,\cdot)}-\sqrt{\rho(t,\cdot)}\|_{L^m(\mathbb{T}^2)}+\|\sqrt{\rho_\delta(t,\cdot)}-\sqrt{K}\|_{L^m(\mathbb{T}^2)}<\epsilon.$$

Similarly, we can obtain $\|\nabla V(t,\cdot)\|_{L^2(\mathbb{T}^2)}<\epsilon\ (t>T(\epsilon))$ from (4.2.4) and (4.2.10) by the same way. □

Chapter 5

Discussions and Future Work

In this chapter, we will discuss some problems about compressible Navier-Stokes equations that I will focus on in the following several years.

As discussed in the introduction, to my best knowledge, there is few results on the global existence of solutions to the full Navier-Stokes equations when vacuum appears except three results under special pressure, viscosity and heat conductivity assumptions (see [16] where the viscosity μ=constant and the so-called variational solutions are obtained, see [5] where the viscosity $\mu = \mu(\rho)$ degenerated when the density vanishes and the global weak solutions are got, and see [51] for global classical large solutions in one-dimensional case). I am now working on this problem in order to obtain some satisfied results.

Next, for the barotropic compressible Navier-Stokes equations with density-dependent viscosity, although an interesting new entropy estimate is established in [1], which provides some high regularity for the density, there is few results on the global existence of solutions in the multi-dimensional case except one result(see [18] where spherically symmetric solutions is obtained in 3 dimensional case). The key issue now is how to construct approximate solutions, which is another problem that I am interested in.

Finally, in studying the barotropic compressible Navier-Stokes equations with density-dependent viscosity, I find that it is easier to get the global existence of weak solutions to the system with an added term (see [1] for the Korteweg system with the Korteweg stress tensor $k\rho\nabla\Delta\rho$, see [2] with an additional quadratic friction term $r\rho|u|u$, and see [32] where the global existence of weak solutions to the barotropic compressible quantum Navier-Stokes equations ($\mu(\rho) = \nu\rho$, $\lambda = 0$ with the quantum Bohm potential $2\varepsilon^2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$) in a three-dimensional torus for large data is proved). So I think studying a special model may help us to deal with the general model that we stated above.

Bibliography

[1] D. Bresch, B. Desjardins and C.K. Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems, *Comm. Partial Differential Equations*, 28(2003), 843–868.

[2] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, *Comm. Math. Phys.*, 238(2003), 211–223.

[3] D. Bresch and B. Desjardins, On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models, *J. Math. Pures Appl.*, 86(2006), no.4, 362–368.

[4] D. Bresch, B. Desjardins and D. Gérard-Varet, On compressible Navier-Stokes equations with density dependent viscosities in bounded domains, *J. Math. Pures Appl.* , 87(2007), no.2, 227–235.

[5] D. Bresch and B. Desjardins, On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids, *J. Math. Pures Appl.*, 87(2007), 57–90.

[6] S. Brull and F. Méhats, Derivation of viscous correction terms for the isothermal quantum Euler model, *Z. Angew. Math. Mech.*, 90(2010), 219–230.

[7] L. Chen and M. Dreher, The viscous model of quantum hydrodynamics in several dimensions, *Math. Models Methods Appl. Sci.*, 17(2007), 1065–1093.

[8] G.Q. Chen, D. Hoff and K. Trivisa, Global solutions of the compressible Navier-Stokes equations with large discontinuous initial data, *Comm. Partial Diff. Eqn.*, 25(2000), 2233–2257.

[9] Y. Cho and H.J. Choe and H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, *J. Math. Pures Appl.* , 83(2004), no.2, 243–275.

[10] Y. Cho and H. Kim, Existence results for viscous polytropic fluids with vacuum, *J. Differential Equations*, 228(2006), no.2, 377–411.

[11] Y. Cho and H. Kim, On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities, *Manuscripta Math.*, 120(2006), no.1, 91–129.

[12] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, Invent. Math., 141(2000), no.3, 579-614.

[13] P. Degond and C. Ringhofer, Quantum moment hydrodynamics and the entropy principle, J. Stat. Phys., 112(2003), 587–628.

[14] R.J. DiPerna and P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math., 98(1989), 511–547.

[15] J.W. Dong, A note on barotropic compressible quantum Navier-Stokes equations, Nonlinear Anal., 73(2010), 854–856.

[16] E. Feireisl, Dynamics of viscous compressible fluid, *Oxford university Press Inc.*, 1987.

[17] E. Feireisl, On the motion of a viscous, compressible, and heat conducting fluid, *Indiana Univ. J.*, 53(2004), 1705–1783.

[18] Z.H. Guo, Q.S. Jiu and Z.P. Xin, Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, *SIAM J. Math. Anal.*, 39(2008), no.5, 1402–1427.

[19] D. Hoff, Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data, *Trans. Amer. Math. Soc.*, 303(1987), no.1, 169–181.

[20] D. Hoff, Discontinuous solutions of the Navier-Stokes equations for compressible flow, *Arch. Rational Mech. Anal.*, 114(1991), 15–46.

[21] D. Hoff and D. Serre, The failure of continuous dependence on initial data for the Navier-Stokes equations of compressible flow, *SIAM J. Appl. Math.*, 51(1991), no.4, 887–898.

[22] D. Hoff, Global well-posedness of the Cauchy problem for the Navier-Stokes equations of nonisentropic flow with discontinuous initial data, *J. Differential Equations*, 95(1992), no.1, 33–74.

[23] D. Hoff, Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data, *Arch. Rational Mech. Anal.*, 132(1995), no.1, 1–14.

[24] D. Hoff, Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids, *Arch. Rational Mech. Anal.*, 193(1997), no.4, 303–354.

[25] D. Hoff, Compressible flow in a half-space with Navier boundary conditions, *J. Math. Fluid Mech.*, 7(2005), no.3, 315–338.

[26] X.D. Huang, J. Li and Z.P. Xin, Global Well-Posedness of Classical Solutions with Large Oscillations and Vacuum to the Three-Dimensional Isentropic Compressible Navier-Stokes Equations, preprint.

[27] F. Jiang, A remark on weak solutions to the barotropic compressible quantum Navier-Stokes equations, *Nonlinear Anal.*, 12(2011), 1733–1735.

[28] S. Jiang and P. Zhang, On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations, *Comm. Math. Phys.*, 215(2001), no.3, 559–581.

[29] S. Jiang, Global smooth solutions of the equations of a viscous, heat-conducting, one-dimensional gas with density-dependent viscosity, *Math. Nachr.*, 190(1998), 169–183.

[30] S. Jiang, Z.P. Xin and P. Zhang, Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity, *Methods Appl. Anal.*, 12(2005), no.3, 239–251.

[31] A. Jüngel, Effective velocity in compressible Navier-Stokes equations with third-order derivatives, *Nonlinear Anal.*, 74(2011), 2813–2818.

[32] A. Jüngel, Global weak solutions to compressible Navier-Stokes equations for quantum fluids, *SIAM J.Math. Anal.*, 42(2010), no.3, 1025–1045.

[33] A. Jüngel and D. Matthes, The Derrida-Lebowitz-Speer-Spohn equation: Existence,non-uniqueness, and decay rates of the solutions, *SIAM J.Math. Anal.*, 39(2008), 1996–2015.

[34] A. Jüngel and J.-P. Milišić, Full compressible Navier-Stokes equations for quantum fluids: derivation and numerical solution, preprint, 2011.

[35] A. Jüngel, J.L. López and J. Montejo-Gámez, A new derivation of the quantum Navier-Stokes equations in the Wigner-Fokker-Planck approach, preprint.

[36] A. V. Kazhikhov and V. V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.*, 41(1977), no.3, 273–282.

[37] H.L. Li, J. Li and Z.P. Xin, Vanishing of Vacuum States and Blow-up Phenomena of the Compressible Navier-Stokes Equations, *Comm. Math. Phys.*, 281(2008), 401–444.

[38] T.P. Liu, Z.P. Xin and T. Yang, Vacuum states for compressible flow, *Discrete Contin. Dynam. Systems*, 4(1998), no.1, 1–32.

[39] P.L. Lions, Mathematical Topics in Fluid Mechanics, *Clarendon Press, Oxford,* Vol.1 and 2, 1998.

[40] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, 1995.

[41] Z. Luo, On the motion of viscous compressible flows, Ph.D thesis, 2010.

[42] A. Matsumura and T. Nishida, Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, *Comm. Math. Phys.*, 89(1983), no.4, 445-464.

[43] A. Mellet and A. Vasseur, On the barotropic compressible Navier-Stokes equations, *Comm. Partial Differential Equations*, 32(2007), 431–452.

[44] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, *Bull. Soc. Math. France.*, 90(1962), 487–497.

[45] D. Serre, Sur l'équation monodimensionnelle d'un fluide visqueux, compressible et conducteur de chaleur, C. R. Acad. Sci. Paris Sér. I Math., 303(1986), no.13, 703–706.

[46] D. Serre, Solutions faibles global des équations de Navier-Stokes pour un fluide compressible, C. R. Acad. Sci. Paris Sér. I Math., 303(1986), no.13, 639–642.

[47] J. Serrin, On the uniqueness of compressible fluid motion, *Arch. Rational Mech. Anal.*, 3(1959), 271–288.

[48] W.J. Sun, S. Jiang and Z.H. Guo, Helically symmetric solutions to the 3-D Navier-Stokes equations for compressible isentropic fluids, *J. Differential Equations*, 222(2006), 263–296.

[49] L. Sundbye, Global existence for the Dirichlet problem for the viscous shallow water equations, *J. Math. Anal. Appl.*, 202(1996), no.1, 236–258.

[50] V.A. Vaǐgant and A.V. Kazhikhov, On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid, Sibirsk. Mat. Zh., 36(1995), no.6, 1283–1316.

[51] H.Y. Wen and C.J. Zhu, Global classical large solutions to Navier-Stokes equations for viscous compressible and heat conducting fluids with vacuum, preprint.

[52] Xin, Zhouping, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density, *Comm. Pure Appl. Math.*, 51(1998), no.3, 229–240.

[53] T. Yang, Z.A. Yao and C.J. Zhu, Compressible Navier-Stokes equations with density-dependent viscosity and vacuum, *Comm. Partial Differential Equations*, 26(2001), 965–981.

[54] T. Yang and H.J Zhao, A vacuum problem for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity, *J. Differential Equations*, 184(2002), no.1, 163–184.

[55] T. Yang and C.J. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum, *Comm. Math. Phys.*, 230(2002), no.2, 329–363.