THEORY OF SINGULARITIES IN ALGEBRAIC GEOMETRY

by

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A thesis

Submitted to

the Graduate School of

The Chinese University of Hong Kong

(Division of Mathematics)

In partial fulfillment

of the Requirement for the Degree of

Master of Philosophy (M.Phil.)

May 1976.

The Chinese University of Hong Kong

Graduate School

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ACKNOWLEDGEMENT

I would like to thank Dr. Tzee-Char Kuo for his supervision. Thanks are also due to Miss Kicinski for her excellent typing.

A TRANSVERSALITY THEOREM FOR ALGEBRAIC VARIETIES

§1. Introduction and Results.

It is well-known that one of the most fundamental theorems in the theory of Singularities is Thom's Transversality Theorem, established some 20 years ago. These are now enough evidence showing that some analogue of Thom's theorem for algebraic varieties would have interesting applications to algebraic geometry (e.g. see [2], Cor. 2).

In this paper, we shall establish such an analogue for complex projective varieties which are complete intersections.

Let $f_1,\ldots,f_s \in \mathbb{C}_H[z_0,\ldots,z_n]$ (the ring of homogeneous polynomials in n+1 variable over $\mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers) and let $V(f_1,\ldots,f_s)$ denote the variety $f_1(z_0,\ldots,z_n) = 0,\ldots,f_s(z_0,\ldots,z_n) = 0$ in the complex projective n-space $\mathbb{C}\mathbb{P}^n$. Call $V = V(f_1,\ldots,f_s)$ a complete intersection if at every $z \in V(f_1,\ldots,f_s)$, $df_1(z),\ldots,df_s(z)$ are linearly independent. In this case, V is a manifold and the vector space generated by $df_1(z),\ldots,df_s(z)$ is the normal space of V at z, denoted by $N_z(V)$.

Let $T_a = \begin{bmatrix} a_{oo} & \cdots & a_{on} \\ \vdots & \ddots & \vdots \\ a_{no} & \cdots & a_{nn} \end{bmatrix}$ be an $(n+1) \times (n+1)$ matrix over $\mathbb{C}$. We can identify T_a with the point $a \equiv (a_{00},\ldots,a_{nn})$ in $\mathbb{C}^{(n+1)^2}$. Recall that if $\det T_a \neq 0$, then T_a defines a projective change of coordinates in $\mathbb{C}\mathbb{P}^n$ by

$$z_i' = \sum_{j=0}^{n} a_{ij} z_j, \qquad 0 \leq i \leq n,$$

and that for $\lambda \neq 0$ in $\mathbb{C}$, T_a and $T_{\lambda a}$ define the same coordinate transformation.

Thus letting [a] denote the point in $\mathbb{CP}^{(n+1)^2-1}$ represented by a, $T_{[a]}$ is a well-defined coordinate transformation of $\mathbb{CP}^n$. Note that for two representatives a_1, a_2 of [a], det $T_{a_1} \neq 0$ if and only if det $T_{a_2} \neq 0$.

Let $f_1,\ldots,f_s,g_1,\ldots,g_t$ be in $\mathbb{C}_H[z_0,\ldots,z_n]$, defining complete intersections $V_1 = V(f_1,\ldots,f_s)$ and $V_2 = V(g_1,\ldots,g_t)$. We say that V_1 intersects V_2 transversally at a point $z \in V_1 \cap V_2$ (we use the notation: $V_1 \pitchfork V_2$ at z) if $N_z(V_1) \cap N_z(V_2) = \{0\}$, and that V_1 intersects V_2 transversally ($V_1 \pitchfork V_2$) if $V_1 \pitchfork V_2$ at all $z \in V_1 \cap V_2$. We write $V_1 \not\pitchfork V_2$ if they are not transversal at at least one point.

It is clear that if $V(f_1,\ldots,f_s)$ is a complete intersection, then $T_a^{-1}(V(f_1,\ldots,f_s)) = V(f_1\circ T_a,\ldots,f_s\circ T_a)$ is also a complete intersection provided that det $T_a \neq 0$.

<u>THEOREM</u>. Let $V_1 \equiv V(f_1,\ldots,f_s)$ and $V_2 \equiv V(g_1,\ldots,g_t)$ be two complete intersections, then the set $\{[a] \in \mathbb{CP}^{(n+1)^2-1} \mid \det T_a \neq 0,\ T_a^{-1}(V_1) \not\pitchfork V_2\}$ is an open subset of a proper subvariety of $\mathbb{CP}^{(n+1)^2-1}$.

For two matrices T_a and T_b, define $\|T_a - T_b\| \equiv \sup_{0\leq i,j\leq n} |a_{ij}-b_{ij}|$.

Let $I = (\delta_{ij})$ denote the identity matrix where $\delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases}$, $i,j = 0,\ldots,n$.

<u>COROLLARY 1</u>. For any two complete intersections $V_1 = V(f_1,\ldots,f_s)$ and $V_2 = V(g_1,\ldots,g_t)$ in $\mathbb{CP}^n$, there is a projective change of coordinates T_a such that $T_a^{-1}(V_1) \pitchfork V_2$. Morevoer, given $\varepsilon > 0$, T_a can be chosen such that $\|T_a - I\| < \varepsilon$ (that is, a suitable arbitrarily small "perturbation" applied to V_1 will make the image of V_1 intersecting V_2 transversally).

<u>COROLLARY 2</u>. If $V_1 \pitchfork V_2$, then there exists $\varepsilon > 0$ such that for all T_a with $\|T_a - I\| < \varepsilon$, we have det $T_a \neq 0$ and $T_a^{-1}(V_1) \pitchfork V_2$. (Geometrically, if V_1 and V_2 intersect transversally, then after applying any sufficiently small

"perturbation" to V_1, the image of V_1 still intersects V_2 transversally.)

§2. Proof of Theorem

First we show that $\{[a] \in \mathbb{CP}^{(n+1)^2-1} \mid \det T_a \neq 0,$ $T_{[a]}^{-1}(V_1) \pitchfork V_2\}$ is an open subset of a subvariety of $\mathbb{CP}^{(n+1)^2-1}$.

<u>LEMMA (2.1)</u>. Let $f_1(z,w),\ldots,f_r(z,w)$ be in $\mathbb{C}_H[z_0,\ldots,z_n,w_0,\ldots,w_m]$. Moreover, suppose they are also homogeneous in the w's. Then the set $B = \{w \mid \exists z \neq 0,\ f_i(z,w) = 0,\ 1 \leq i \leq r\}$ is a projective variety in $\mathbb{CP}^m$.

That is, the projection of $V(f_1,\ldots,f_r)$ to the w-space is also a variety. Note that for varieties over $\mathbb{R}$, there is no such a theorem. Consider, for example, $x^2 + y^2 = 1$ in $\mathbb{R}^2$; its projection to the x-axis is the interval $[-1,1]$, which is not a variety.

<u>Proof</u>. Let

$$f_s = \sum_{|j|=e_s} C_{sj}(w_0,\ldots,w_m) z_0^{j_0},\ldots,z_n^{j_n}$$

where $j = (j_0,\ldots,j_n)$, $|j| = j_0 + \ldots + j_n$, e_s is a constant, $\deg(f_s) = d_s$, $\deg(C_{sj}) = d_s - e_s$.

By elimination theory ([4], Vol. II, p. 8), $f_1,\ldots,f_r$ possess a resultant system of integral polynomials b_ν in $C_{sj}(w_0,\ldots,w_m)$ such that the vanishing of all resultants is necessary and sufficient in order that the equations $f_1 = 0,\ldots,f_r = 0$ has a nonzero solution $(z_0,\ldots,z_n)$. Moreover, the b_ν are homogeneous in $C_{sj}(w_0,\ldots,w_m)$ for each s.

Let $b_\nu = \sum_{t=1}^{k_\nu} d_t \prod_{s=1}^{r} \prod_{|j|=e_s} C_{sj}(w)^{\ell_{sj}^t}$, where k_ν is a constant depending on ν. Then

$$\sum_{|j|=e_s} \ell^t_{sj} = q_s$$

for all t, where q_s is a constant depending on s.

Therefore each term of b_ν is of degree

$$\sum_{s=1}^{r} (d_s - e_s) \Big(\sum_{|j|=e_s} \ell^t_{sj} \Big) = \sum_{s=1}^{r} (d_s - e_s) q_s$$

in $w_0, \ldots, w_m$; note that this last number is a constant, hence b_ν is homogeneous in $w_0, \ldots, w_m$.

Now $w \in B$ if and only if $b_\nu(w) = 0$ for all ν; therefore B is the projective variety $V(b_\nu)$ in $\mathbb{CP}^m$.

Assume $\det T_a \neq 0$. Observe that $T_a^{-1}(V_1) \not\pitchfork V_2$ if and only if there exists $z \neq 0$ such that $(z,a) \equiv (z_0, \ldots, z_n, a_{00}, \ldots, a_{nn})$ satisfies

$$F_\ell \equiv f_\ell \circ T_a = f_\ell\Big(\sum_{\lambda=0}^{n} a_{0\lambda} z_\lambda, \ldots, \sum_{\lambda=0}^{n} a_{n\lambda} z_\lambda \Big) = 0, \qquad 1 \leq \ell \leq s, \tag{2.2}$$

$$g_k = g_k(z_0, \ldots, z_n) = 0, \qquad 1 \leq k \leq t, \tag{2.3}$$

and

$$\begin{bmatrix} \frac{\partial F_1}{\partial z_0} & \cdots & \frac{\partial F_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial F_s}{\partial z_0} & \cdots & \frac{\partial F_s}{\partial z_n} \\ \frac{\partial g_1}{\partial z_0} & \cdots & \frac{\partial g_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial g_t}{\partial z_0} & \cdots & \frac{\partial g_t}{\partial z_n} \end{bmatrix} \text{ has rank} < s + t. \tag{2.4}$$

Since (2.2), (2.3), (2.4) are polynomials homogeneous both in (z,a) and in z, the set

$$A_1 = \{[a] \in \mathbb{CP}^{(n+1)^2-1} \mid \exists z \neq 0,\ (z,a) \text{ satisfies } (2.2), (2.3), 2.4)\}$$

is a projective variety in $\mathbb{CP}^{(n+1)^2-1}$, by lemma (2.1).

Now, since $A_2 = \{[a] \mid \det T_a = 0\}$ is a projective variety in $\mathbb{CP}^{(n+1)^2-1}$, we have the following,

<u>LEMMA (2.5)</u>. The set $A = \{[a] \in \mathbb{CP}^{(n+1)^2-1} \mid \det T_a \neq 0,\ T_a^{-1}(V_1) \pitchfork V_2\}$ is an open subset of the projective variety A_1 defined above. In fact $A = A_1 - A_2$.

It remains to show that A_1 is a proper variety in $\mathbb{CP}^{(n+1)^2-1}$.

Remark. When $s + t > n + 1$, (2.4) is always true. Hence $A_1 = \{[a] \in \mathbb{CP}^{(n+1)^2-1} \mid \exists z \neq 0,\ (z,a) \text{ satisfies } (2.2), (2.3)\}$, which is a proper subset of $\mathbb{CP}^{(n+1)^2-1}$ ([1] Vol 2. P. 157, Theorem 1). Therefore our theorem holds for $s + t > n + 1$.

In the following, we shall always assume $s + t \leq n + 1$.

Let $W = \{(z,a) \in \mathbb{C}^{(n+1)+(n+1)^2} \mid F_\ell(z,a) = 0,\ g_k(z) = 0,\ 1 \leq \ell \leq s,\ 1 \leq k \leq t\}$. A point (z,a) in W is called a simple point if $dF_1(z,a),\dots,dF_s(z,a),\ dg_1(z,a),\dots,dg_t(z,a)$ are linearly independent, otherwise it is called a singular point.

In the following lemma, every differential or partial derivative is evaluated at (z^*,a^*) or z^*.

<u>LEMMA (2.6)</u>. Assume $\det T_{a^*} \neq 0$, then each (z^*,a^*) in W with $z^* \neq 0$ is a simple point.

Proof. Note that

$$dF_\ell = (\frac{\partial F_\ell}{\partial z_0},\dots,\frac{\partial F_\ell}{\partial z_n}, \frac{\partial F_\ell}{\partial a_{00}},\dots,\frac{\partial F_\ell}{\partial a_{nn}}),\ 1 \le \ell \le s,$$

$$dg_k = (\frac{\partial g_k}{\partial z_0},\dots,\frac{\partial g_k}{\partial z_n}, 0,\dots,0), \qquad 1 \le k \le t. \tag{2.7}$$

First we show that $(\frac{\partial F_1}{\partial a_{00}},\dots,\frac{\partial F_1}{\partial a_{nn}}),\dots,(\frac{\partial F_s}{\partial a_{00}},\dots,\frac{\partial F_s}{\partial a_{nn}})$ are linearly independent.

By the chain rule

$$\frac{\partial F_\ell}{\partial a_{ij}} = \frac{\partial f_\ell}{\partial z_i}(\Sigma a_{0\lambda}{}^*z_\lambda{}^*,\dots,\Sigma a_{n\lambda}{}^*z_\lambda{}^*)z_j{}^* \tag{2.8}$$

Since $\det T_{a*} \ne 0$ and $z^* \ne 0$, $b^* = (\Sigma a_{0\lambda}{}^*z_\lambda{}^*,\dots,\Sigma a_{n\lambda}{}^*z_\lambda{}^*)$ is nonzero. Now $(z^*,a^*) \in W$ implies $F_\ell(z^*,a^*) = f_\ell(\Sigma a_{0\lambda}{}^*z_\lambda{}^*,\dots,\Sigma a_{n\lambda}{}^*z_\lambda{}^*) = 0$, $1 \le \ell \le s$, so $b^* \in V(f_1,\dots,f_s)$. Since $V(f_1,\dots,f_s)$ is a complete intersection and b^* is a set of homogeneous coordinates of a point in $\mathbb{CP}^n$, $df_1(b^*),\dots,df_s(b^*)$ are linearly independent.

Assume $\sum_{\ell=1}^{s} c_\ell(\frac{\partial F_\ell}{\partial a_{00}},\dots,\frac{\partial F_\ell}{\partial a_{nn}}) = 0$, then by (2.8),

$$\sum_{\ell=1}^{s} c_\ell(\frac{\partial F_\ell}{\partial a_{00}},\dots,\frac{\partial F_\ell}{\partial a_{nn}})$$

$$= \sum_{\ell=1}^{s} c_\ell(\frac{\partial f_\ell}{\partial z_0}(b^*)z_0{}^*,\dots,\frac{\partial f_\ell}{\partial z_0}(b^*)z_n{}^*,\dots,\frac{\partial f_\ell}{\partial z_n}(b^*)z_0{}^*,\dots,\frac{\partial f_\ell}{\partial z_n}(b^*)z_n{}^*)$$

$$= 0$$

This implies

$$0 = \sum_{\ell=1}^{s} c_\ell(\frac{\partial f_\ell}{\partial z_0}(b^*)z_j{}^*,\dots,\frac{\partial f_\ell}{\partial z_n}(b^*)z_j{}^*)$$

$$= \left[\sum_{\ell=1}^{s} c_\ell(\frac{\partial f_\ell}{\partial z_0}(b^*),\dots,\frac{\partial f_\ell}{\partial z_n}(b^*))\right] z_j{}^*, \quad 0 \le j \le n.$$

Choose j such that $z_j^* \neq 0$, then by linear independency of $df_1(b^*),\dots,df_s(b^*)$, $C_1 = \dots = C_s = 0$. Hence

$$\left(\frac{\partial F_1}{\partial a_{00}},\dots,\frac{\partial F_1}{\partial a_{nn}}\right),\dots,\left(\frac{\partial F_s}{\partial a_{00}},\dots,\frac{\partial F_s}{\partial a_{nn}}\right) \text{ are linearly independent.}$$

Now we complete the proof of the lemma. If

$\sum_{\ell=1}^{s} d_\ell dF_\ell + \sum_{k=1}^{t} e_k dg_k = 0$, then by (2.7) and above, $d_1 = \dots = d_s = 0$. Since $dg_1,\dots,dg_t$ are linearly independent, $e_1 = \dots = e_t = 0$ also.

Let p be an analytic (resp. differential) map of M_1 to M_2, where M_1, M_2 are analytic (resp., differential) manifolds. A point q of M_1 is called a critical point of p if for some coordinate charts (U,ϕ) about q and (V,ψ) about $p(q)$, the Jacobian matrix of $\psi \circ p \circ \phi^{-1}$ at $\phi(q)$ is of rank less than the dimension of M_2. A point $q \in M_2$ such that $p^{-1}(q)$ contains at least one critical point is called a critical value. Note that the above definitions are independent of the choice of (U,ϕ) and (V,ψ).

Let $V = V(h_1,\dots,h_r) \subset K^N$, where $K = \mathbb{C}$ or $\mathbb{R}$, with $r \leq N$, $q \in V$. Suppose $dh_1(q),\dots,dh_r(q)$ are linearly independent at q (q is then a simple point of V), then V is a manifold near q, of dimension $N-r$. Let p be an analytic function from K^N to K^m, let p' denote the restriction of p to V, and let $Tq(V) = \{z \in K^N \mid z.dh_i(q) = 0,\ i = 1,\dots,r\}$, where "." is the standard inner product in K^N, then q is a critical point of p' if and only if $dp(q)\,(TqV)$ is a proper subset of K^m.

Let $W' = \{(z,a) \in W \mid \det T_a \neq 0,\ z \neq 0\}$, then W' consists entirely of simple points by lemma (2.6), so it is a manifold. Let P be the projection of $\mathbb{C}^{(n+1)+(n+1)^2}$ to $\mathbb{C}^{(n+1)^2}$ which maps (z,a) to a. Let $\bar{P}$ denote the restriction of P to W'. Since the differential of a projection is itself, we have $dP(z,a) = P$.

Now we prove our crucial lemma, which gives us a characterization of transversality in Differential Topology.

LEMMA (2.9). Assume $\det T_{a*} \neq 0$, $z* \neq 0$, then $(z*,a*)$ is a critical point of $\bar{P}$ if and only if $(z*,a*)$ satisfies (2.2), (2.3) and (2.4).

Proof. In this proof, all differentials and partial derivatives are evaluated at $(z*,a*)$ or $z*$. Also, (z,a) stands for points in $\mathbb{C}^{(n+1)+(n+1)^2}$.

($\Rightarrow$) Since $dP(z*,a*)(T_{(z*,a*)}W')$

$$= P(\{(z,a)\,|\,(z,a).dF_{\ell} = 0,\ (z,a).dg_k = 0,\ 1 \leq \ell \leq s;\ 1 \leq k \leq t\})$$

$$= \{a\,|\,\exists z, (z,a).dF_{\ell} = (z,a).dg_k = 0,\ 1 \leq \ell \leq s,\ 1 \leq k \leq t\},$$

$(z*,a*)$ is a critical point of $\bar{P}$ if and only if there exists $a \in \mathbb{C}^{(n+1)^2}$ such that the system

$$\begin{cases} z.\left(\dfrac{\partial F_{\ell}}{\partial z_0},\ldots,\dfrac{\partial F_{\ell}}{\partial z_n}\right) = -a.\left(\dfrac{\partial F_{\ell}}{\partial a_{00}},\ldots,\dfrac{\partial F_{\ell}}{\partial a_{nn}}\right), & 1 \leq \ell \leq s \quad (2.10) \\ z.\left(\dfrac{\partial g_k}{\partial z_0},\ldots,\dfrac{\partial g_k}{\partial z_n}\right) = 0\,, & 1 \leq k \leq t \quad (2.11) \end{cases}$$

has no solution in $\mathbb{C}^{n+1}$. Now if $(z*,a*)$ is a critical point, then there exists a such that (2.10) and (2.11) have no common solution, therefore $\left(\dfrac{\partial F_1}{\partial z_0},\ldots,\dfrac{\partial F_1}{\partial z_n}\right),\ldots,\left(\dfrac{\partial g_t}{\partial z_n},\ldots,\dfrac{\partial g_t}{\partial z_n}\right)$ must be linearly dependent at $(z*,a*)$. Therefore, $(z*,a*)$ satisfies (2.4). Since $(z*,a*) \in W' \subset W$, $(z*,a*)$ also satisfies (2.2) and (2.3).

(⇐) Conversely, assume (z^*,a^*) satisfies (2.2), (2.3) and (2.4). since $\det T_{a^*} \neq 0$, $z^* \neq 0$, $(z^*,a^*) \in W'$. As (z^*,a^*) satisfies (2.4), $\alpha_1 = (\frac{\partial F_1}{\partial z_0},\ldots,\frac{\partial F_1}{\partial z_n}),\ldots,\ \alpha_s = (\frac{\partial F_s}{\partial z_0},\ldots,\frac{\partial F_s}{\partial z_n})$, $dg_1,\ldots,dg_t$ are linearly dependent, so there exist d_ℓ, e_k, not all zero, such that

$$\sum_{\ell=1}^{s} d_\ell \alpha_\ell + \sum_{k=1}^{t} e_k dg_k = 0. \tag{2.12}$$

Note that since $dg_1,\ldots,dg_t$ are linearly independent, some d_ℓ must be nonzero.

Since $(\frac{\partial F_1}{\partial a_{00}},\ldots,\frac{\partial F_1}{\partial a_{nn}}),\ldots,(\frac{\partial F_s}{\partial a_{00}},\ldots,\frac{\partial F_s}{\partial a_{nn}})$ are linearly independent (see the proof of lemma (2.6)), there exists $b \in \mathbb{C}^{(n+1)^2}$ such that $-b\cdot(\frac{\partial F_\ell}{\partial a_{00}},\ldots,\frac{\partial F_\ell}{\partial a_{nn}}) = d_\ell$, $1 \leq \ell \leq s$.

Now, if for this b, (2.10), (2.11) have a solution, say z, then $z.\alpha_\ell = d_\ell$ and $z.dg_k = 0$, hence (2.12) implies

$$\begin{aligned} 0 &= \sum_{\ell=1}^{s} d_\ell(z.\alpha_\ell) + \sum_{k=1}^{t} e_k(z.dg_k) \\ &= \sum_{\ell=1}^{s} d_\ell^2. \end{aligned}$$

But $\sum_{\ell=1}^{s} d_\ell^2 \neq 0$, we have a contradiction. Therefore (z^*,a^*) is a critical point of $\bar{P}$.

In order to apply Sard's Theorem (which is stated for real differential manifolds) to complete the proof, we need the following:

<u>LEMMA (2.13)</u>. Let $V = V(h_1,\dots,h_r) \subset \mathbb{C}^N$, $r \leq N$, and let $z_j = x_j + iy_j$ $(x_j, y_j \in \mathbb{R})$, $1 \leq j \leq N$, then $z = x + iy$ where $x = (x_1,\dots,x_N)$, $y = (y_1,\dots,y_N)$. Let $(x,y) = (x_1,y_1,\dots,x_N,y_N)$, then $h_\ell(z) = u_\ell(x,y) + iv_\ell(x,y)$, $1 \leq \ell \leq r$, where u_ℓ, v_ℓ are the real and imaginary parts of h_ℓ. It is clear that $z^* = x^* + iy^* \in V$ if and only if $(x^*,y^*) \in V(u_1,v_1,\dots,u_\ell,v_\ell) \equiv V' \subset \mathbb{R}^{2N}$.

By applying Cauchy-Riemann conditions, we have the following:

(i) $dh_1(z^*),\dots,dh_r(z^*)$ are linearly independent if and only if $du_1(x^*,y^*)$, $dv_1(x^*,y^*),\dots,du_r(x^*,y^*)$, $dv_r(x^*,y^*)$ are linearly independent.

(ii) $T_{(x^*,y^*)}V' = \{(x,y) \mid x+iy \in T_{z^*}V\}$.

(iii) Let f be an analytic map of $\mathbb{C}^N$ to $\mathbb{C}^m$, and let f' be the map from $\mathbb{R}^{2N}$ to $\mathbb{R}^{2m}$ defined by

$$f'(x,y) = (\mathrm{Re}[f(x+iy)], \mathrm{Im}[f(x+iy)]).$$

Clearly, f' is a differential function and

$$df'(x^*,y^*)(x,y) = (\mathrm{Re}[df(z^*)(x+iy)], \mathrm{Im}[df(z^*)(x+iy)]).$$

By (ii) and (iii), $df'(x^*,y^*)(T_{(x^*,y^*)}V')$ is a proper subset of $\mathbb{R}^{2N}$ if and only if $df(z^*)(T_{z^*}V)$ is a proper subset of $\mathbb{C}^N$.

Now, if z^* is a simple point, so is (x^*,y^*) (by (i)); z^* is a critical point of $f|_V$ implies that (x^*,y^*) is a critical point of $f'|_{V'}$ (by (iii)). The converse of the above statement is also true.

Proof of Theorem. By lemma (2.5) and (2.9), $[a] \in A$ if and only if $\det T_a \neq 0$ and there exists $z \neq 0$ such that (z,a) is a critical point of $\bar{P}$. Now in lemma (2.13), let $V = W$, $f = P$, $f|_{W'} = \bar{P}$ and $V'' = \{(x,y,\mu,\nu) | (x+iy, \mu+i\nu) \in W'\}$. Clearly, $V'' \subset V'$. Now $(z,a) \equiv (x+iy,\mu+i\nu)$ is a critical point of $\bar{P}$ if and only if $(x.y,\mu,\nu)$ is a critical point of $f'|_{V''}$.

For a subset D of $\mathbb{CP}^{(n+1)^2-1}$, let $[D] = \{z_1 \in \mathbb{C}^{(n+1)^2} | z_1 = \lambda z, \lambda \in \mathbb{C}, z \in D\}$, then clearly $[\mathbb{CP}^{(n+1)^2-1} - D] \supset \mathbb{C}^{(n+1)^2} - [D]$.

By Sard's Theorem ([3], p. 47), the set $C = \{(\mu,\nu) | (\mu,\nu)$ is a critical value of $f'|_{V''}\}$ does not contain any open subset of $\mathbb{R}^{2(n+1)^2}$. Therefore [A] contains no open subset of $\mathbb{C}^{(n+1)^2}$. If $A' = \mathbb{CP}^{(n+1)^2-1}$, $[A'] = \mathbb{C}^{(n+1)^2}$, then since [A"] is a proper closed subset of $\mathbb{C}^{(n+1)^2}$, $[A] = [A'-A''] = [\mathbb{CP}^{(n+1)^2-1} - A''] \supset \mathbb{C}^{(n+1)^2} - [A'']$, which is a nonempty open subset of $\mathbb{C}^{(n+1)^2}$. This is a contradiction. Therefore $A' \subsetneq \mathbb{CP}^{(n+1)^2-1}$. Now the theorem follows immediately from lemma (2.5).

Proof of Corollaries. Corollary 1 is obvious. Corollary 2 is also obvious by noting that A" is a proper subvariety of $\mathbb{CP}^{(n+1)^2-1}$.

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