Enumerative Plane Tropical Geometry

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Abstract

Tropical geometry is the study of tropical semiring. Enumerative geometry deals with the counting of geometric objects that satisfy some given incidence conditions. Enumerative plane tropical geometry is then a combination of them, which is a tool for a better understanding of real algebraic geometry.

In this report, we will give an introduction of tropical geometry, mainly emphasis on the study of plane tropical curves. We will give some examples and deduce some properties of plane tropical curves. The main theorem of this report is Mikhalkin's Correspondence Theorem. We will give a review on it.

摘要

熱帶幾何是研究熱帶中環的一門科目。枚舉幾何是計算適合特定條件件的幾何物件數量的科目。枚舉平面熱帶幾何則是兩者的結合，是一種了解實代數幾何的工具。

在這報告中，我們會介紹熱帶幾何，並主要强調平面熱帶曲線。我們會列舉一些例子並推導它的特性。此報告的主要定理是Mikhalkin's對應定理，我們會考察它。

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Chapter 0

Introduction

Tropical geometry is a new subject in the field of algebraic geometry. The term "tropical" does not reflect any things special in this subject, but it is an honor to a mathematician and computer scientist from Brazil, Imre Simon, who resides in São Paolo. Originally, this subject is only applied to discrete mathematics and optimization. Only in the recent years that G. Mikhalkin proved the Mikhalkin's Correspondence Theorem and B. Sturmfels related those objects with polyhedral cell complexes then people realized the power of this subject.

The main idea of tropical geometry is to study algebraic varieties by piecewise linear objects, which can be studied with the help of combinatorics. It is hope that every construction in algebraic geometry has a correspondence combinatorial counterpart in tropical geometry. Thanks to the piecewise linear structure, difficult algebraic problems may be easier to handle in tropical setting.

This subject is best developed in plane tropical case. One way to study tropical geometry is via the amoebas of algebraic varieties. The idea proposed by Kontsevich and elaborated by Mikhalkin is to consider the Log image of a complex curve C in an open subset of $(\mathbb{C}^*)^2$. Precisely, they apply the map

$$\mathrm{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2 : (z, w) \mapsto (\log |z|, \log |w|)$$

to the complex curve C in the affine plane. The image is a closed and connected two-dimensional subset of $\mathbb{R}^2$, and it looks like an object with certain tentacles in some unbounded directions. When we shrink the tentacles to a certain limit, we get a graph referred as a plane tropical curve. Such a deformation also has an analogy in tropical semiring, or sometimes called the max-plus algebra, as the ring addition is the usual maximum and the ring multiplication is the usual addition. Algebraically, plane tropical curves are of the form $\overline{\mathrm{Val}(C \cap (K^*)^2)}$, where K is the field of Puiseux series and the map Val is given by

$$\mathrm{Val}: K^* \to \mathbb{R} \cup \{-\infty\} : \sum_{q \in \mathbb{Q}} a_q t^q \mapsto -\min\{q \in \mathbb{Q} \,|\, a_q \neq 0\}.$$

Alternatively, plane tropical curves are the corner locus of some piecewise linear functions. The study of polyhedral cell complexes gives the dual of Newton subdivisions and plane tropical curves, and hence plane tropical curves can also be viewed as weighted graphs with balancing conditions.

The enumerative part of this subject concerns the counting of geometric objects satisfying some incidence conditions. The so-called Mikhalkin's Correspondence Theorem relates the number of complex algebraic curves of genus g and degree $\triangle$ with that of plane tropical curves, where $\triangle$ is a given Newton polygon. It can be seen that these numbers satisfy some recursive relations and coincide with Gromov-Witten invariants of $\mathbb{P}^2$ when the Newton polygons are the standard lattice polygons with vertices at $(0,0)$, $(0,d)$ and $(d,0)$. These numbers can be computed purely combinatorially using certain lattice paths in the relevant Newton polygons. Mikhalkin has also generalized the formula for the enumerative invariants of arbitrary genus in toric surfaces.

This report aims at the "translation" of well-known facts of algebraic geometry to tropical geometry, with emphasis on plane tropical curves. In chapter 1, we will give the motivation and definitions of plane tropical curves. We will then give the duality between Newton subdivisions and plane tropical curves. This will show us how plane tropical curves should look like. In chapter 2, we will state two well-known results of enumerative plane tropical geometry, the degree-genus formula and Bézout's Theorem. They are basic properties of plane tropical curves. In chapter 3, we will study the properties of non-Archimedean amoebas. Rullgård used Ronkin functions to study spines of amoebas, where Ronkin functions are defined to be

$$N_f(x) = \frac{1}{(2\pi i)^2} \int_{\mathrm{Log}^{-1}(x)} \log|f(z)| \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}.$$

Here N_f is a strictly convex function over $A(Z_f)$, where Z is an algebraic variety defined by f. Spine of Amoeba $A(Z_f)$ is then the corner locus of N_f^∞. It can be shown that $N_f^\infty = \max_{\alpha \in J}\{\operatorname{Re}\Phi_\alpha(f) + \langle \alpha, x\rangle\}$. We will discuss some examples of $\operatorname{Re}\Phi_\alpha(f)$. We end this chapter by introducing the Patchworking method discovered by Viro, which are of uses to prove the main theorem in the next chapter. The main idea is that the degeneration of arithmetic operators in R_+ will give the ring of quantized version of R_+. This motivates why tropical semiring is defined like that. The Viro's Patchworking Theorem then states that we can use the "data" of $\mathbb{R}V_{\triangle_k} = V_{\triangle_k} \cap (\mathbb{R}^*)^n$ to recover $\mathbb{R}V_t = V_t \cap (\mathbb{R}^*)^n$ as $t \to \infty$. The last chapter is the statement and the sketch proof of the Mikhalkin's Correspondence Theorem. As mentioned before, plane tropical curves are dual to the Newton subdivisions of the relevant Newton polygons and Mikhalkin has given a way to count those curves using certain lattice paths in the Newton polygons. To prove the theorem, we need two lemmas. Instead of counting curves in $\mathbb{R}^2$ directly, we lift the complex curve C to $(\mathbb{C}^*)^2$ and define complex tropical curves. To sum up, enumerative plane tropical geometry is a new way to tackle problems in algebraic geometry. I believe that this report will be a good reference for researchers and graduate students interested in entering the field of tropical geometry.

Chapter 1

Definitions of Plane Tropical Curves

In this chapter, we give the motivation and definitions of plane tropical curves. Algebraic description of plane tropical curves makes us know how they should look like, while combinatorial description is the analogy of it and is useful to deduce more properties. Although tropical curves in $\mathbb{R}^n$ can be defined in [15], only plane tropical curves are well-studied and we will emphasis on them. For details of plane tropical curves, refer to [6], [8], [17], [18] and [20].

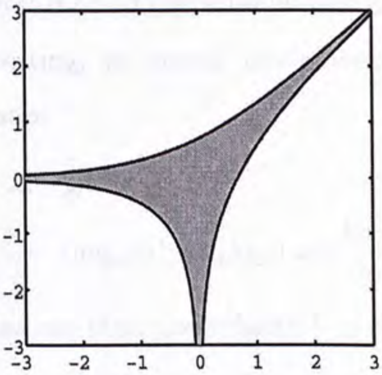
1.1 Motivation

The idea of developing tropical geometry is via the so-called amoebas of algebraic varieties. Recall that the zero locus of a polynomial in two variables is called a complex plane algebraic curve C. It is a singular surface in the 4-space $\mathbb{C}^2$ defined by an equation $f(z_1, z_2) = 0$. When C restricted to the open subset of $(\mathbb{C}^*)^2$ is mapped to $\mathbb{R}^2$ under the logarithm map, it becomes a two-dimensional image called amoeba. It has dimension two because a complex curve has real dimension two. Formally, we have the following definition.

Definition 1.1.1 (Gelfand-Kapranov-Zelevinski [7]). *The* ***amoeba of a complex curve*** *C is the subset $A = \mathrm{Log}(C \cap (\mathbb{C}^*)^2)$ of $\mathbb{R}^2$ where*

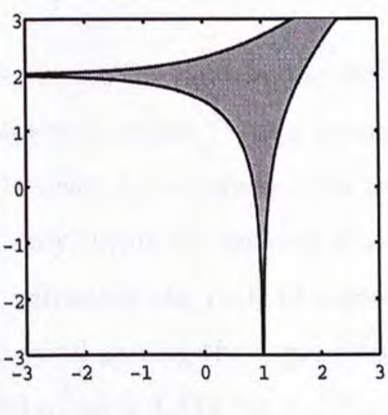
$$\begin{aligned} \mathrm{Log} : (\mathbb{C}^*)^2 &\to \mathbb{R}^2 \\ (z_1, z_2) &\mapsto (\log|z_1|, \log|z_2|). \end{aligned}$$

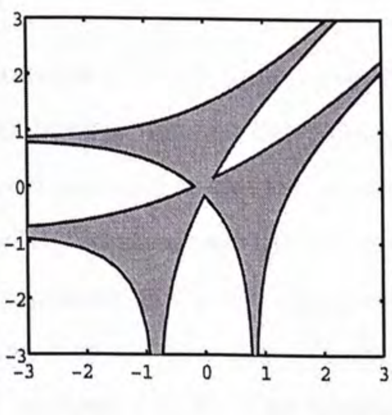
Example 1.1.1. $C_1 = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid z_1 + z_2 = 1\}$. The shaded region below shows the amoeba of C_1.



The amoeba of C_1 has three tentacles in three different directions. It can be explained as follow: C_1 contains exactly one point which the first coordinate is zero, namely $(0,1)$. When the neighborhood of this point is mapped by the Log map, it is in the neighborhood of $(-\infty, 0)$ and becomes the tentacle of the amoeba in the $(-1,0)$-direction. Similarly, the neighborhood of $(1,0)$ becomes the tentacle of the amoeba in the $(0,-1)$-direction. Finally, any points of the form $(z, 1-z)$ are mapped to the $(1,1)$-direction as $|z| \to \infty$ since $\lim\limits_{|z|\to\infty} \left|\dfrac{z}{1-z}\right| = 1$.

Example 1.1.2. $C_2 = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid e^{-1}z_1 + e^{-2}z_2 = 1\}$ and C_3 = a generic conic, i.e. a curve given by a general polynomial of degree two in two variables. Their amoebas are as shown below:





The change of coefficients in C_2 gives a amoeba similar to that of C_1. It is an additive shift of the amoeba of C_1 by shifting the origin to the point $(1,2)$. The amoeba of a generic conic has two tentacles in the same three directions as before, namely, the $(-1,0)$, $(0,-1)$ and $(1,1)$-direction. It is because the curve meets each coordinate axe in two points. It can be generalized that the amoeba determined by a general polynomial of degree d has d tentacles in each of the same three directions.

To get something interesting, we shrink the amoebas to their "skeletons" by considering the following map:

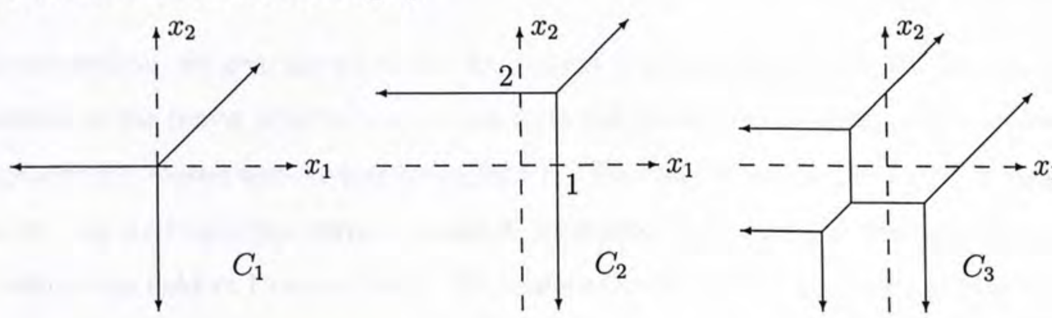
$$\begin{aligned} \mathrm{Log}_t : (\mathbb{C}^*)^2 &\rightarrow \mathbb{R}^2 \\ (z_1, z_2) &\mapsto (\log_t |z_1|, \log_t |z_2|) = (\frac{\log |z_1|}{\log t}, \frac{\log |z_2|}{\log t}). \end{aligned}$$

The limits of amoebas are then the subsets $\Gamma \subset \mathbb{R}^2$ where

$$\Gamma = \lim_{t\to\infty} A_t = \lim_{t\to\infty} \mathrm{Log}_t(C_t \cap (\mathbb{C}^*)^2)$$

for some suitable algebraic curves C_t, and the limit is taken the Hausdorff limit.

The limits of amoebas are then one-dimensional objects as the shrinking process makes them become zero width. They can be viewed as graphs in $\mathbb{R}^2$ with some bounded and semi-infinite edges. The limits of C_1, C_2 and C_3 are the following (the arrows denote semi-infinite edges):



In order to get the right limits, we have to replace the original curves with some suitable algebraic curves C_t. For instance, the limit of amoeba determined by C_1 is still C_1. However, if we choose C_t in the second example as the the same curve C_2, it will not only shrink the amoeba to zero width, but also translate its vertex to the origin. It will make the limit of amoeba determined by C_1 and C_2 have the same graph. To avoid getting the same graph in this way, we consider a family of curves $C_t = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid t^{-1}z_1 + t^{-2}z_2 = 1\}$ for large $t \in \mathbb{R}$. This family will pass through $(t, 0)$ and $(0, t^2)$ for all t, and thus $\mathrm{Log}_t(C_t \cap (\mathbb{C}^*)^2)$ will have horizontal and vertical tentacles at $x_1 = 1$ and $x_2 = 2$ respectively. Hence not only the width but also the position of the graph is kept.

Note that Log_t differs from Log by rescaling the two coordinate axes $\log t$ times. The images can also be viewed by looking at amoebas from very far away. The above examples suggest that plane tropical curves should be piecewise linear objects in some sense of the images of complex curves, which are the images under the Log map and a degeneration process. We hope that plane tropical curves still carry some properties that the original complex curves have and also are easier to handle due to its linearity.

The idea that plane tropical curves can be viewed as limit of amoebas serve as a motivation why plane tropical curves are interesting and how they look like roughly. However, it is difficult to make the notion of limit precisely. To make things easier, we will use a different approach and give the algebraic and combinatorial descriptions of plane tropical curves in the remaining sections. We will see in Chapter 3 that the objects defined by our new definitions are actually very similar to limits of amoebas.

1.2 As Varieties over the Field of Puiseux Series

In this section, we give the algebraic description of plane tropical curves. Instead of looking at the limits of amoebas, we can hide this process by looking at a complete algebraically closed non-Archimedean field K. The field K is equipped with a norm called non-Archimedean norm to make it complete. Our principal example of such a field is the field of Puiseux series. We replace the field $\mathbb{C}$ by another algebraically closed field K and hope that plane tropical curves can be defined over K. The importance about the choice of K is that we have a valuation map from K to $\mathbb{R}$ which is similar to the map Log in Definition 1.1.1.

Definition 1.2.1. *Let K be any field. A map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ satisfying*

- $|a| = 0$ *if and only if* $a = 0$,
- $|ab| = |a|\,|b|$ *and*
- $|a+b| \leq \max\{|a|, |b|\}$ *for all* $a, b \in K$

is called a ***non-Archimedean norm.***

Note that the norm can be extended to the algebraic closure of K in [3].

Definition 1.2.2. *A* ***valuation*** *is a map* $\mathrm{Val} : \mathrm{K} \to \mathbb{R} \cup \{-\infty\}$ *satisfying*

- $\mathrm{Val}(a) = -\infty$ *if and only if* $a = 0$,
- $\mathrm{Val}(ab) = \mathrm{Val}(a) + \mathrm{Val}(b)$ *and*
- $\mathrm{Val}(a + b) \leq \max\{\mathrm{Val}(a), \mathrm{Val}(b)\}$ *for all* $a, b \in K$.

Non-Archimedean norms are in bijection with valuations by $\mathrm{Val}(a) = \log|a|$. That is, K is algebraically closed and there is a map $\mathrm{Val} : K \to \mathbb{R} \cup \{-\infty\}$ such that e^{Val} defines a norm on K. In addition, K has to be complete with respect to the norm e^{Val}. Hence K becomes a **complete algebraically closed non-Archimedean field**. We are now ready to give an example of the field K and define plane tropical curves in an algebraic way.

Definition 1.2.3. ***The field of Puiseux series*** *is the field of formal power series in a variable t with complex coefficients, i.e.* $a = \sum_{q\in\mathbb{Q}} a_q t^q$, *such that the set* $\{q \in \mathbb{Q} \mid a_q \neq 0\}$ *is bounded below and has a finite set of denominators.*

The main example of a field K is the completion of the field of Puiseux series. To construct K, we consider the algebraic closure $\overline{\mathbb{C}((t))}$ of the field of Laurent series $\mathbb{C}((t))$. An element of this is of the form

$$a(t) = a_1 t^{q_1} + a_2 t^{q_2} + \cdots ,$$

where $a_i \in \mathbb{C}$ and $q_1 < q_2 < \cdots$ are rational numbers with bounded below denominators. This can let us define the valuation of a by $\mathrm{Val}(a) = -\min\{q \in \mathbb{Q} \mid a_q \neq 0\} = -q_1$. Now define K to be the completion of $\overline{\mathbb{C}((t))}$ with respect to the norm e^{Val}. The valuation extends to this completion by $\mathrm{Val}(a) = \log|a|$.

By replacing the underlaying field $\mathbb{C}^2$ by K^2, we can define algebraic curves in the affine plane over the field K. The limiting process is then hidden by the valuation map

$$\begin{aligned} \mathrm{Val} : (K^*)^2 &\to \mathbb{R}^2 \\ (z_1, z_2) &\mapsto (\mathrm{Val}(z_1), \mathrm{Val}(z_2)). \end{aligned}$$

We can now give the first definition of plane tropical curves precisely.

Definition 1.2.4. *A **plane tropical curve associated to** C is the closure of the image* $\mathrm{Val}(C \cap (K^*)^2) \subset \mathbb{R}^2$*, where* $C \subset K^2$ *is a algebraic curve and* K *is the completion of the field of Puiseux series.*

Note that this definition is purely algebraic and does not involve any limits. As K is an algebraically closed field of characteristic zero, the theory of algebraic geometry of plane curves over K is similar to that of algebraic curves over $\mathbb{C}$. Note that we replace the limit by the closure of the image of $\mathrm{Val}(C \cap (K^*)^2)$. The closure is taken because $\mathrm{Val}(C \cap (K^*)^2)$ is by definition in $\mathbb{Q}^2$ only. Let us look at an example to see if this definition match our motivation.

Example 1.2.1. Consider an algebraic curve C over K given by $C = \{z = (z_1, z_2) \in K^2 \mid tz_1 + t^2 z_2 = 1\}$. If $(z_1, z_2) \in C \cap (K^*)^2$ then $\mathrm{Val}(C \cap (K^*)^2)$ has the following three possible cases:

- If $\mathrm{Val}(z_1) < 1$ then $\mathrm{Val}(z_2) = 2$ since $z_2 = t^{-2} - t^{-1} z_1$ has a lower leading t^{-2} term. It corresponds to the semi-infinite horizontal edge starting at $(1, 2)$.

- If $\mathrm{Val}(z_2) < 2$ then $\mathrm{Val}(z_1) = 1$ since $z_1 = t^{-1} - tz_2$ has a lower leading t term. It corresponds to the semi-infinite vertical edge starting at $(1, 2)$.

- If $\mathrm{Val}(z_1) \geq 1$ and $\mathrm{Val}(z_2) \geq 2$ then $tz_1 + t^2 z_2 = 1$ shows that the lower leading term of tz_1 and $t^2 z_2$ are equal, thus $\mathrm{Val}(z_1) + 1 = \mathrm{Val}(z_2)$. Hence we recover the limit of amoeba in Example 1.1.2.

Example 1.2.2. Consider another algebraic curve given by $C = \{z = (z_1, z_2) \in K^2 \mid t^p z_1 + t^q z_2 = 1\}$ where $(p, q) \in \mathbb{R}^2$, then the plane tropical curve associated to C has the same shape of the limit of amoeba in Example 1.1.2 but shifting the vertex $(1, 2)$ to (p, q).

In the next section, we will give another definition of plane tropical curves. We will see that plane tropical curves of the form $\overline{\mathrm{Val}(C \cap (K^*)^2)}$ are actually one-dimensional polyhedral cell complexes. We will consider the ground field $(K, +, \cdot)$ and see what the map valuation does to the field structure.

1.3 As Varieties over the Tropical Semiring

We would like to describe plane tropical curves algebraically similar to zero sets of polynomials. The previous section gives an idea what happen to the operations "+" and "·" in the ground field $(K, +, \cdot)$. We begin with the following observation.

Let $C \subset K^2$ be the curve given by

$$C = \{(z_1, z_2) \in K^2 \,|\, f(z_1, z_2) = \sum_{i,j \in \mathbb{N}} a_{ij} z_1^i z_2^j = 0\}.$$

Note that

$$\begin{aligned} \mathrm{Val}(a_{ij} z_1^i z_2^j) &= \mathrm{Val}(a_{ij}) + i\mathrm{Val}(z_1) + j\mathrm{Val}(z_2) \\ &= \mathrm{Val}(a_{ij}) + ix_1 + jx_2 \end{aligned}$$

by denoting $\mathrm{Val}(z_i) = x_i \in \mathbb{R}$. Since $\sum_{i,j \in \mathbb{N}} a_{ij} z_1^i z_2^j = 0$, if

$$\mathrm{Val}(a_{i_0 j_0} z_1^{i_0} z_2^{j_0}) = \min_{i,j \in \mathbb{N}} \{\mathrm{Val}(a_{ij} z_1^i z_2^j)\}$$

for some $(i_0, j_0) \in \mathbb{N}^2$ then there must exist another $(i_0^{'}, j_0^{'}) \in \mathbb{N}^2$ such that

$$\mathrm{Val}(a_{i_0 j_0} z_1^{i_0} z_2^{j_0}) = \mathrm{Val}(a_{i_0^{'} j_0^{'}} z_1^{i_0^{'}} z_2^{j_0^{'}}).$$

Thus the lowest valuation of all summands must be attained at least twice, i.e. $\max_{i,j \in \mathbb{N}} \{ix_1 + jx_2 + \mathrm{Val}(a_{ij})\}$ must be attained at least twice. This observation makes us to define the tropical semiring and see what does the ground field looks like.

Definition 1.3.1. *The* ***tropical semiring*** $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ *is the semiring with underlying set* $\mathbb{R} \cup \{-\infty\}$ *and operations* ***tropical addition*** *and* ***tropical multiplication*** *defined as follow:*

$$x_1 \oplus x_2 = \max\{x_1, x_2\} \text{ and } x_1 \odot x_2 = x_1 + x_2.$$

Note that the tropical addition is idempotent, i.e. $x \oplus x = x$, and the $-\infty$ element is a neutral element in addition. Since there is no additive inverse for addition, this is only a semiring. The tropical multiplication is just the usual addition.

With the arithmetic operators defined, we can define tropical polynomials.

Definition 1.3.2. *A* ***tropical polynomial in two variables*** *is a finite sum of the form*

$$F = a_1 \odot x^{\odot b_1} \odot y^{\odot c_1} \oplus \cdots \oplus a_n \odot x^{\odot b_n} \odot y^{\odot c_n}, \text{ where } a_i \in \mathbb{Q} \text{ and } b_i, c_i \in \mathbb{Z}.$$

Note that F is actually given by the map

$$F : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \max\{a_1 + b_1 x + c_1 y, \ldots, a_n + b_n x + c_n y\}.$$

It is a piecewise linear convex function. Please refer to [24] for Tropical Mathematics and [11] for Tropical Arithmetic.

We then relate the definition of plane tropical curves in the previous section with the tropical semiring. Since the tropical semiring does not have an inverse operation for addition, it would not make sense to look at zero sets of tropical polynomials. What we will actually consider are the corner loci of tropical polynomials. We first get a polynomial in the ground field.

Definition 1.3.3. *Let* $f = \sum_{i \in \mathbb{N}} a_i(t) z_1^{b_i} z_2^{c_i} \in K[z_1, z_2]$, *where* $K[z_1, z_2]$ *is a Laurent polynomial of two variables over a complete algebraically closed non-Archimedean field* K. *We define* ***the tropicalization of*** f *to be the tropical polynomial*

$$\begin{aligned} F = trop(f) &= \bigoplus_{i \in \mathbb{N}} \mathrm{Val}(a_i(t)) \odot x^{\odot b_i} \odot y^{\odot c_i} \\ &= \max\{\mathrm{Val}(a_1(t)) + b_1 x + c_1 y, \ldots, \mathrm{Val}(a_n(t)) + b_n x + c_n y\}. \end{aligned}$$

Example 1.3.1. Let $f = tz_1 + t^2 z^2 - 1$ as in Example 1.2.1, the tropicalization of f is $F = -1 \odot x \oplus -2 \odot y \oplus 0$. Note that the addition of 0 is important since 0 is not a neutral element for tropical addition.

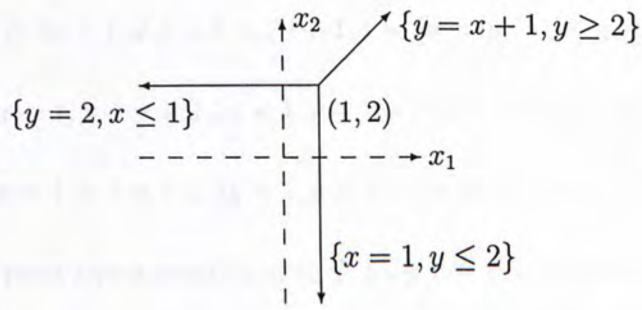
When K is taken to be the field of Puiseux series, by our observation, trop(f) corresponds to the set of which the maximum of f is attained at least twice. This motivates the following definition.

Definition 1.3.4. *A plane tropical curve is a subset of* $\mathbb{R}^2$ *that is the* ***corner locus*** *of a tropical polynomial* F *with rational coefficients, i.e. the set of points* $(x, y) \in \mathbb{R}^2$ *such that the maximum* $F(x, y)$ *is attained at least twice. The* ***plane tropical curve associated to*** F *is given by the tropicalization of* f.

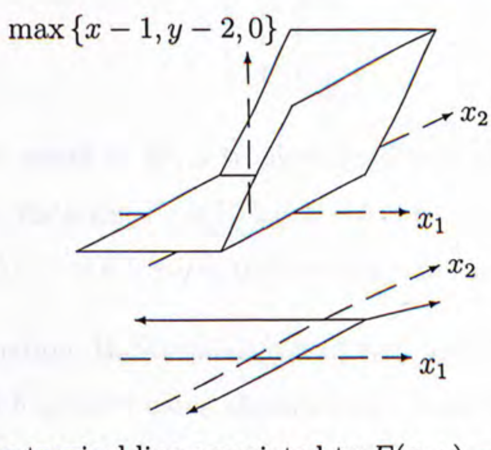
Equivalently, a plane tropical curve associated to F is given by the set of points where the piecewise linear map F is not linear, or not differentiable.

We will see what the **tropical line** associated to $F(x,y) = a \odot x \oplus b \odot y \oplus c$ and the **tropical quadratic** associated to $F(x,y) = a \odot x^2 \oplus b \odot x \odot y \oplus c \odot y^2 \oplus d \odot x \oplus e \odot y \oplus f$ look like by the following examples.

Example 1.3.2. Let $F = -1 \odot x \oplus -2 \odot y \oplus 0 = \max\{x-1, y-2, 0\}$. There are three cases where the maximum is attained at least twice:
Either $x - 1 = y - 2 \geq 0$, $x - 1 = 0 \geq y - 2$ or $y - 2 = 0 \geq x - 1$.
This give the lines $\{y = x+1, y \geq 2\}$, $\{x = 1, y \leq 2\}$ and $\{y = 2, x \leq 1\}$. All lines are semi-infinite and starting at $(1,2)$. The tropical line associated to F is as shown below:



Note that the graph is the same as Example 1.1.2 and 1.2.1. The point $(1,2)$ is which the maximum of the function F is attained triple, i.e. $x - 1 = y - 2 = 0$. The following graph shows how the corner locus of this function gives back the plane tropical curve. Note that the nonlinear parts give the semi-infinite edges in each of the three directions.

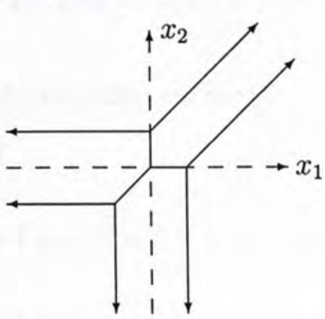


For a generic plane tropical line associated to $F(x,y) = a \odot x \oplus b \odot y \oplus c$, it has the same graph as above but centered at $(c-a, c-b)$.

Example 1.3.3. Let $F(x,y) = 1 \odot x^2 \oplus 2 \odot x \odot y \oplus 1 \odot y^2 \oplus 2 \odot x \oplus 2 \odot y \oplus 1$. We have $C_2^6 = 15$ possibilities and there is only nine possible cases as shown below:

- $2x+1 = x+y+2 \geq 2y+1, x+2, y+2, 1 - \{x = y+1, y \geq 0\}$;
- $2x+1 = x+2 \geq x+y+2, 2y+2, y+2, 1 - \{x = 1, y \leq 0\}$;
- $x+y+2 = 2y+1 \geq 2x+1, x+2, y+2, 1 - \{y = x+1, x \geq 0\}$;
- $x+y+2 = x+2 \geq 2x+1, 2y+1, y+2, 1 - \{y = 0, 0 \leq x \leq 1\}$;
- $x+y+2 = y+2 \geq 2x+1, 2y+1, x+2, 1 - \{x = 0, 1 \leq y \leq 0\}$;
- $2y+1 = 1 \geq 2x+1, x+y+2, x+2, y+2 - \{y = 1, x \leq 0\}$;
- $x+2 = y+2 \geq 2x+1, x+y+2, 2y+1, 1 - \{x = y, -1 \leq x \leq 0\}$;
- $x+2 = 1 \geq 2x+1, x+y+2, 2y+1, y+2 - \{x = -1, y \leq -1\}$ and
- $y+2 = 1 \geq 2x+1, x+y+2, 2y+1, x+2 - \{y = -1, x \leq -1\}$.

Hence the plane tropical curve associated to F look like the following with vertices at $(0,0)$, $(0,1)$, $(1,0)$ and $(-1,-1)$:



Remark 1.3.1. As a graph in $\mathbb{R}^2$, a tropical quadratic may have the same graph as a tropical line. For instance, $F_1 = 1.75 \odot x \oplus 5 \odot y \oplus 0$ and $F_2 = 0 \odot x^2 \oplus 4.5 \odot x \odot y \oplus 6.5 \odot y^2 \oplus 5.5 \odot x \oplus 8.5 \odot y \oplus 10$ have the same graph.

In the general situation, B. Sturmfels has showed in [20] that tropical quadratic can be classified into 5 general cases algebracially. Note that in [20], the tropical semiring is not defined as the same of our. The tropical addition is taken the minimum instead of the maximum, which is more widely used in Computer Science.

Since minimum is actually in bijection with maximum, the graph is simply reflected along the line $x + y = 0$, thus the combinational types of plane tropical curves are not changed. Up to symmetry, it can be classified into five general cases.

(i) F defines a tropical line of multiplicity 2. This happens if and only if

$$2b \geq a + c \text{ and } 2d \geq c + e \text{ and } 2f \geq a + e.$$

This is the degeneration of a tropical quadratic.

(ii) F defines a tropical quadratic with two double semi-infinite edges. There are three possibilities (depending on which expression is chosen as the strict inequality). It happens if and only if

$$2b \geq a + c \text{ and } 2d \geq c + e \text{ and } 2f < a + e.$$

(iii) F defines a tropical quadratic with one double semi-infinite edges. There are three possibilities (depending on which two expressions are chosen as the strict inequalities). It happens if and only if

$$2b < a + c \text{ and } 2d < c + e \text{ and } 2f \geq a + e.$$

(iv) F defines a tropical quadratic with one vertex not on any semi-infinite edges. It happens if and only if

$$b + d < c + f \text{ and } b + f < a + d \text{ and } d + f < a + e.$$

If one of these inequalities becomes equality, F defines a union of two tropical lines.

(v) F defines a tropical quadratic with each vertex on some semi-finite edges. It happens if and only if

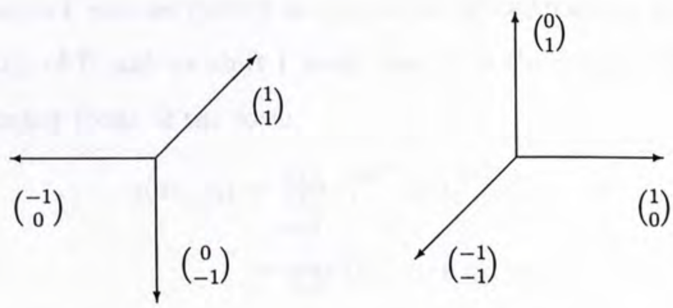
$$2b < a + c \text{ and } 2d < c + e \text{ and } 2f < a + e$$

$$\text{and } (b + d > c + f \text{ or } b + f > a + d \text{ or } d + f > a + e).$$

Note that in Example 1.3.2, which is a tropical line, it has 3 semi-infinite edges in each direction. The primitive integral vectors around the vertex sums up to 0, i.e.

$$\begin{pmatrix}0\\-1\end{pmatrix}+\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

In Example 1.3.3, which is a general tropical quadratic, it has six semi-infinite edges in each direction and each direction has exactly two edges. Also the primitive integral vectors around each vertices sum up to 0. We will see in the next section that there is actually a general property of plane tropical curves when they are thought as graphs with balancing conditions hold. Below shows the primitive integral vectors of the plane tropical curves around the vertex $(0,0)$ in Example 1.3.2 and 1.3.3 respectively.



We now give two definitions of plane tropical curves, one with the image of complex curves over the completion of the field of Puiseux series. This is motivated by the definition of amoebas and it helps us to understand why plane tropical curves also carry some similar properties of plane algebraic curves. The second is the corner loci of tropical polynomials, which is more computable and easier to give examples. Actually, these two definitions are the same, which is the following Kapranov's Theorem. For a proof, see [3], [20] or [25].

Theorem 1.3.1 (Kapranov's Theorem). *If $C \subset K^2$ is a curve given by the equation $\{f = 0\}$ and F is the tropical polynomial $F = \mathrm{trop}(f)$, then the plane tropical curve associated to C and associated to F coincides.*

1.4 A Combinatorial Description of Plane Tropical Curves

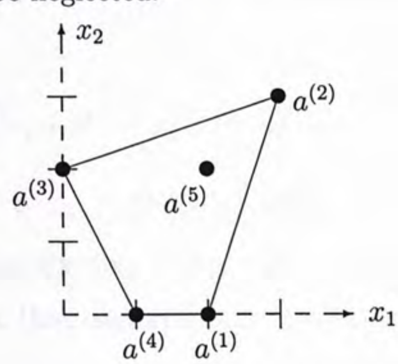
We have defined plane tropical curves as images of algebraic curves, and we have an equivalent definition by means of tropical semiring. However, these definitions are purely algebraic and do not give much information about plane tropical curves combinatorially except they are piecewise linear with rational slopes. In the last section, we expect plane tropical curves should have more properties. We will give a combinatorial description of tropical curves so that we can deal with them more easily.

We start by the following observation. We have seen in the last section that a plane tropical curve Γ can be viewed as a graph in $\mathbb{R}^2$ with edges of rational slopes. Let V be a vertex of Γ and we shift Γ such that V is the origin. Then Γ is locally around V the corner locus of the form

$$\begin{aligned} g(x_1, x_2) &= \bigoplus_{i\in\mathbb{N}} x_1^{\odot a_1^{(i)}} \odot x_2^{\odot a_2^{(i)}} \\ &= \max_{i\in\mathbb{N}} \{a_1^{(i)}x_1 + a_2^{(i)}x_2\} \end{aligned}$$

for some $a^{(i)} = (a_1^{(i)}, a_2^{(i)}) \in \mathbb{N}^2$.

Let $\triangle$ be the convex hull of these $a^{(i)}$'s. Notice that if any point $a^{(i)}$ is not a vertex of $\triangle$, then it is irrelevant to the plane tropical curve. For instance, in the following figure, $a^{(5)}$ can be neglected.



The reason is that

$$\begin{aligned} (x_1, x_2) \in \Gamma \Leftrightarrow \exists\ i, j \text{ such that } a_1^{(i)}x_1 + a_2^{(i)}x_2 &\overset{(*)}{=} a_1^{(j)}x_1 + a_2^{(j)}x_2 \\ &= g(x_1, x_2) \\ &\geq a_1^{(k)}x_1 + a_2^{(k)}x_2,\ \forall k = 1, 2, \ldots, \end{aligned}$$

and $(*)$ only holds for adjacent $a^{(i)}$ and $a^{(j)}$.

There is another condition around the vertex V. If $(a^{(1)}, \ldots, a^{(n)})$ are the vertices of $\triangle$ in clockwise direction, then $v^{(i)} = (a_2^{(i)} - a_2^{(i+1)}, a_1^{(i+1)} - a_1^{(i)})$ is an outward normal vector of the edge joining $a^{(i)}$ and $a^{(i+1)}$. It follows that

$$\sum_{i=1}^{n} v^{(i)} = \sum_{i=1}^{n} (a_2^{(i)} - a_2^{(i+1)}, a_1^{(i+1)} - a_1^{(i)}) = 0,$$

where we denote $a_k^{(i+1)} = a_k^{(1)}$, $\forall k = 1, 2, \ldots$. Now write $v^{(i)} = w^{(i)} \cdot u^{(i)}$, where $w^{(i)} \in \mathbb{Z}_{>0}$ such that the two components of $u^{(i)}$ are coprime. We called $w^{(i)}$ the **weight** of the corresponding edge of Γ and $u^{(i)}$ the **primitive integral vector** in the direction of $v^{(i)}$. Now,

$$\sum_{i=1}^{n} w^{(i)} \cdot u^{(i)} = \sum_{i=1}^{n} v^{(i)} = 0$$

is called the **balancing condition.**

Let $\Gamma \subset \mathbb{R}^2$ be given by the corner locus of

$$g(x_1, x_2) = \max_{i \in \{1, \cdots, n\}} \{a_1^{(i)} x_1 + a_2^{(i)} x_2 + b^{(i)}\},$$

where g is of degree $d = \max_{i \in \{1, \cdots, n\}} \{a_1^{(i)} + a_2^{(i)}\}$. Then

$$a^{(i)} = (a_1^{(i)}, a_2^{(i)}) \in \triangle_d \cap \mathbb{Z}^2,$$

where $\triangle_d = \{(\xi, \eta) \in \mathbb{R}^2 \,|\, \xi, \eta \geq 0, \xi + \eta \leq d\}$. Consider $a^{(i)} \neq a^{(j)}$, if there exists $(x_1, x_2) \in \Gamma$ such that

$$a_1^{(i)} x_1 + a_2^{(i)} x_2 + b^{(i)} = a_1^{(j)} x_1 + a_2^{(j)} x_1 + b^{(j)} = g(x_1, x_2)$$

then we connect $a^{(i)}, a^{(j)} \in \triangle_d \cap \mathbb{Z}^2$ by an edge. This gives us a subdivision of $\triangle_d$, called the Newton subdivision of $\triangle_d$. Thus each edge of the plane tropical curve is dual to an edge in the Newton subdivision. Formally, we have the following definitions.

Definition 1.4.1. *Let $f = \sum a_i z_1^{b_i} z_2^{c_i} \in K[z_1, z_2]$ be a polynomial (where K denotes any field, not necessarily the completion of the field of Puiseux series). Then the convex hull of the set $\{(b_i, c_i) \in \mathbb{Z}^2 \mid a_i \neq 0\}$ is called the* ***Newton polygon of*** *f.*

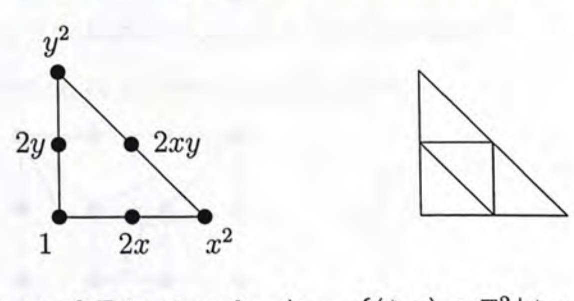
Definition 1.4.2. *Let $f = \sum a_i z_1^{b_i} z_2^{c_i} \in K[z_1, z_2]$ be a polynomial. Let $D \subset \mathbb{R}^2 \times \mathbb{R}$ be the convex hull of the set $\{(b_i, c_i, a_i) \mid a_i \neq 0\}$. Project the edges which can be seen from above to the first factor $\mathbb{R}^2$. The image will be a convex subdivision of the Newton polygon, called the* ***Newton subdivision of*** f.

Suppose F is the tropicalization of f, then we called the Newton polygon and Newton subdivision of F be the corresponding Newton polygon and Newton subdivision of f respectively.

From the above observation, we have the duality between plane tropical curves and their Newton polygons. We state the follow theorem, a proof can be found in [17].

Theorem 1.4.1. *The plane tropical curve Γ associated to the tropical polynomial F is dual to the Newton subdivision of f, where F is the tropicalization of f, in the sense that every vertex V of Γ corresponds to a 2-dimensional polytope of the subdivision and every edge of Γ is orthogonal to a 1-dimensional polytope.*
Furthermore, if a vertex V is adjacent to an edge E, then the 1-dimensional polytope dual to E is in the boundary of the 2-dimensional dual of V.

Example 1.4.1. Let $F(x, y) = 1 \odot x^2 \oplus 2 \odot x \odot y \oplus 1 \odot y^2 \oplus 2 \odot x \oplus 2 \odot y \oplus 1$ as in Example 1.3.3. The Newton polygon and Newton subdivision of F are as shown below:

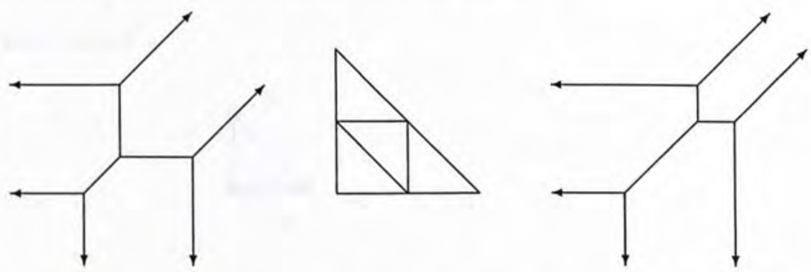


The Newton polygon of F is given by $\triangle_3 = \{(\xi, \eta) \in \mathbb{Z}^2 \,|\, \xi, \eta \geq 0, \xi + \eta \leq 3\}$. The vertices $(0, 1)$ and $(1, 0)$ is connected by an edge because the condition

$$g(x_1, x_2) = 0 \cdot x_1 + 1 \cdot x_2 + 2 = 1 \cdot x_1 + 0 \cdot x_2 + 2$$

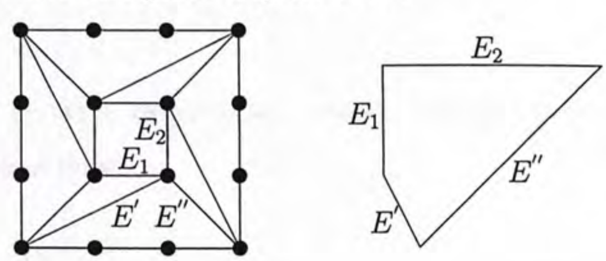
is possible. Similarly we connect $(1, 0)$ with $(1, 1)$ and $(0, 1)$ with $(1, 1)$.

Remark 1.4.1. Note that the duality between plane tropical curves and their Newton subdivision is not a 1 to 1 correspondence. In fact, many plane tropical curves can have the same Newton subdivision. The Newton subdivisions only fix the directions in which the edges of the plane tropical curves point, but not the lengths of the dual edges. The following picture shows two plane tropical curves which have the same Newton subdivision but different shapes.



Remark 1.4.2. Note that there also have several polynomials which define the same plane tropical curve. The idea of finding the Newton subdivision only gives a way of determining the plane tropical curves. The vertex which have valence 3 is where the maximum is attained triple. In general, the vertex which have valence n is where the maximum is attained $n+1$ times.

Remark 1.4.3. Note that not every Newton subdivision gives rise to a plane tropical curve. It is obvious by our construction that we need subdivisions of a Newton polygon into convex polygons in order to have a plane tropical curve associated to it. However, we may happen that there is no plane tropical curves associated to it even this condition is satisfied. Consider the following figure, which shows part of the Newton subdivision of a plane tropical curve:



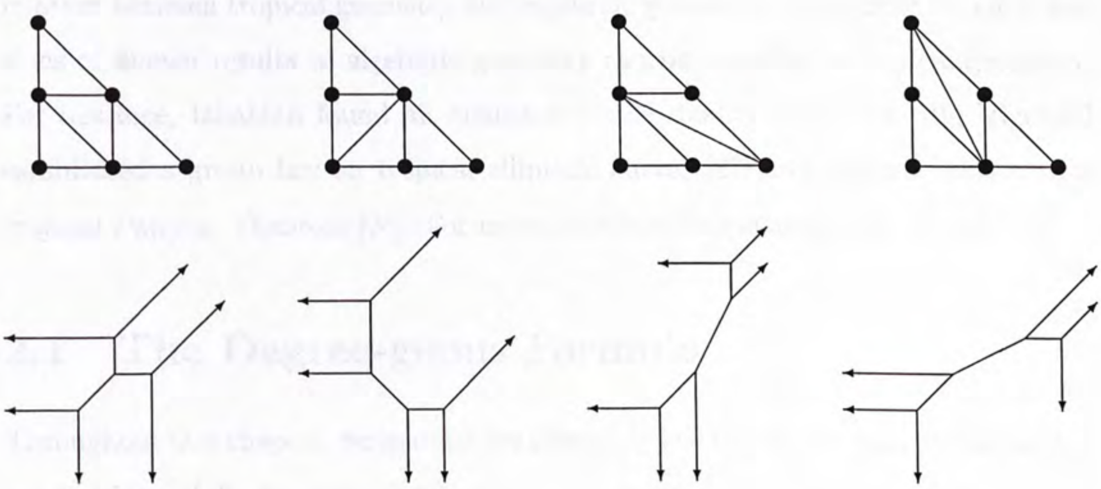
The right hand side shows part of the plane tropical curve dual to the edges E_1, E_2, E' and E''. It can be shown that E_2 is longer than E_1 as the edges of the plane tropical curves is orthogonal to the corresponding edges of the Newton subdivision. By the same argument used around each edge in the little square, each edge around the central vertex must be longer than the previous one. This is a contradiction.

Definition 1.4.3. *A Newton subdivision of a Newton polygon is called a* ***regular Newton subdivision*** *if it is dual to a plane tropical curve.*

We end this chapter by listing the combinatorial types of $\triangle_d$ for $d = 1$ and 2. When $d = 1$, there is only the trivial subdivision of $\triangle_1$. This gives only one type of plane tropical curve.

$\triangle_1$

When $d = 2$, the following shows all **non-degenerate plane tropical curves.** Here non-degenerate means that the subdivision is maximal, i.e. each subdivision polygon is of area $\frac{1}{2}$.



The following shows a **degenerate plane tropical curve**, note that it is a union of two tropical lines.

In general, an algorithm to find all such subdivisions has been studied in geometric combinatorics, see [9].

Chapter 2

Properties of Plane Tropical Curves

In this chapter, we give the tropical versions of classical results in algebraic geometry, the degree-genus formula and Bézout's theorem. As we want to find the relation between tropical geometry and algebraic geometry, there must be some versions of known results in algebraic geometry can be transfer to tropical geometry. For instance, Izhakian found an analogue to the duality of curves [10], Vigeland established a group law on tropical elliptical curves [27] and Tabera dealt with a tropical Pappus' Theorem [26]. For more details of this chapter, see [6] and [20].

2.1 The Degree-genus Formula

Throughout this chapter, we assume the plane tropical curves are dual to the Newton subdivisions of $\triangle_d$ for some $d \in \mathbb{N}$.

Definition 2.1.1. *A plane tropical curve is of degree d if it is dual to the Newton subdivisions of* $\triangle_d$.

If $C \subset \mathbb{P}^2$ is a smooth complex algebraic curve of degree d, then it is well known that it has genus $g = \dfrac{1}{2}(d-1)(d-2)$. This formula is called the degree-genus formula and counts the number of "holes" in the real surface C. If C is not smooth, the genus would not greater than the above number. We would like to ask the same question in tropical geometry.

Consider a plane tropical curve $\Gamma \subset \mathbb{R}^2$, then it is natural to define its genus to be the number of loops in Γ if we think of it as a connected graph in $\mathbb{R}^2$.

Definition 2.1.2. *The genus of a plane tropical curve $\Gamma \subset \mathbb{R}^2$ is defined to be $g(\Gamma) = \dim H_1(\Gamma, \mathbb{R})$, i.e. its first Betti number.*

Denote

$$\Gamma_0 = \text{set of vertices of } \Gamma,$$

$$\Gamma_1 = \text{set of bounded edges of } \Gamma$$

and for each $V \in \Gamma_0$, we define its valence to be

$$\mathrm{val}(V) = \text{no of edges attached to } V.$$

Since Γ has $3d$ unbounded and $|\Gamma_1|$ bounded edges. It follows that

$$3d + 2|\Gamma_1| = \sum_{V \in \Gamma_0} \mathrm{val}(V).$$

By Euler's formula,

$$|\Gamma_0| - |\Gamma_1| + g(\Gamma) + 1 = 2,$$

we have

$$g(\Gamma) = 1 + |\Gamma_1| - |\Gamma_0|.$$

Substitute it back to the original equation, we get

$$\begin{aligned} g(\Gamma) &= 1 + \frac{1}{2}\sum_{V \in \Gamma_0} \mathrm{val}(V) - \frac{3}{2}d - |\Gamma_0| \\ &= \frac{1}{2}(d-1)(d-2) - \underbrace{\Big(\frac{1}{2}d^2 - \sum_{V \in \Gamma_0} \frac{1}{2}(\mathrm{val}(V) - 2)\Big)}_{(*)}. \end{aligned}$$

Recall that each $V \in \Gamma_0$ corresponds to a convex polygon $\triangle_V \subset \triangle_d$ with $\mathrm{val}(V)$ vertices. It follows that

$$\mathrm{Area}(\triangle_V) = \sum_{i=1}^{\mathrm{val}(V)-2} \mathrm{Area}(\Gamma_i) \geq \frac{1}{2}(\mathrm{val}(V) - 2).$$

Finally, we get

$$\sum_{V \in \Gamma_0} \mathrm{Area}(\triangle_V) \geq \sum_{V \in \Gamma_0} \frac{1}{2}(\mathrm{val}(V) - 2),$$

$$\text{i.e. } \frac{1}{2}d^2 \geq \sum_{V \in \Gamma_0} \frac{1}{2}(\mathrm{val}(V) - 2).$$

Hence $(*)$ is greater or equal to 0, therefore

$$g(\Gamma) \leq \frac{1}{2}(d-1)(d-2).$$

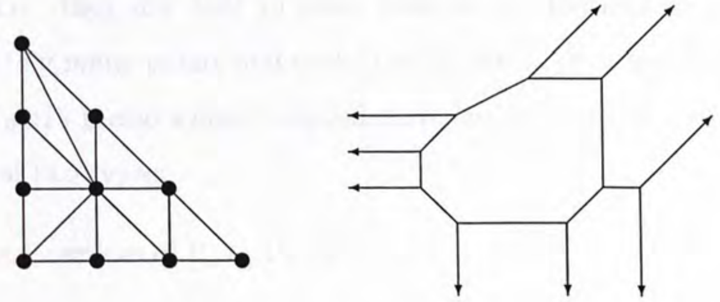
The equality holds if and only if all polygons in the Newton subdivision have minimal areas for its number of vertices. This is the degree-genus formula for plane tropical curves, see [6].

Definition 2.1.3. *A plane tropical curve is smooth if its Newton subdivision is maximal, i.e. it consists of d^2 triangles of area $\frac{1}{2}$ each.*

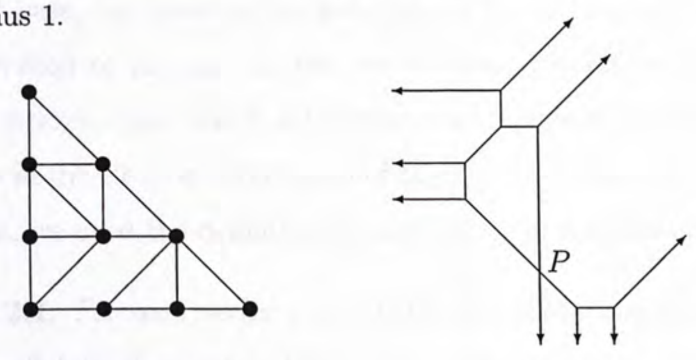
Remark 2.1.1. Equivalently, a a plane tropical curve Γ is smooth if each vertex of Γ has valence 3, all weights of the edges are 1 and the primitive integral vectors along the edges adjacent to any vertices generate the lattice $\mathbb{Z}^2$.

Remark 2.1.2. A smooth plane tropical curve has genus $\frac{1}{2}(d-1)(d-2)$.

Example 2.1.1. The following shows a smooth plane tropical curve dual to a Newton subdivision of $\triangle_3$ and of genus 1.



Example 2.1.2. The following shows a plane tropical curve Γ which is not smooth but still of genus 1.



Note that the the parallelogram in the Newton subdivision gives rise to a point P which two lines intersect. We can also think of Γ as the plane image of a graph of genus 0 that has a "crossing" at P. This corresponds to a normal crossing singularity in the classical case, i.e. to a complex curve C with a point $P \in C$ where two smooth branches meet transversely.

2.2 Bézout's Theorem

In classical geometry, Bézout theorem states that if C_1 and C_2 are two distinct smooth algebraic curves of degree d_1 and d_2 respectively, then C_1 intersects C_2 in $d_1 d_2$ points, counted with the local intersection multiplicities of C_1 and C_2.

We have seen in the previous section that there may be an analogous statement in tropical geometry. However, we cannot apply it directly here, as the intersections of two distinct smooth plane tropical curves may not be always finite. It may happen that they share some common line segments as their intersections. Let us consider a simpler case first.

Consider Γ_1 and Γ_2 are two plane tropical curves of degree d_1 and d_2 that intersect transversally, i.e. they are dual to some Newton subdivisions of $\triangle_{d_1}$ and $\triangle_{d_2}$, intersect in finitely many points and each is not a vertex of either Γ_1 or Γ_2.

Note that $\Gamma_0 \cup \Gamma_1$ is also a plane tropical curve and of degree $d_1 + d_2$. The vertices of $\Gamma_1 \cup \Gamma_2$ are of two types:

(1) the original vertices of Γ_1 or Γ_2, or

(2) the intersections of Γ_1 and Γ_2.

In the first case, the Newton sub-polygons of Γ_1 and Γ_2 can be found in the Newton subdivision of $\triangle_{d_1+d_2}$. In the second case, the intersections are always given by two straight lines which intersect. The intersections correspond to the parallelograms in the Newton subdivision of $\triangle_{d_1+d_2}$. To count correctly the number of intersections, we need the definition of multiplicity of a intersection point.

Definition 2.2.1. *For each vertex $p \in \Gamma_1 \cap \Gamma_2$, its multiplicity is defined to be the area of the parallelogram corresponds to p.*

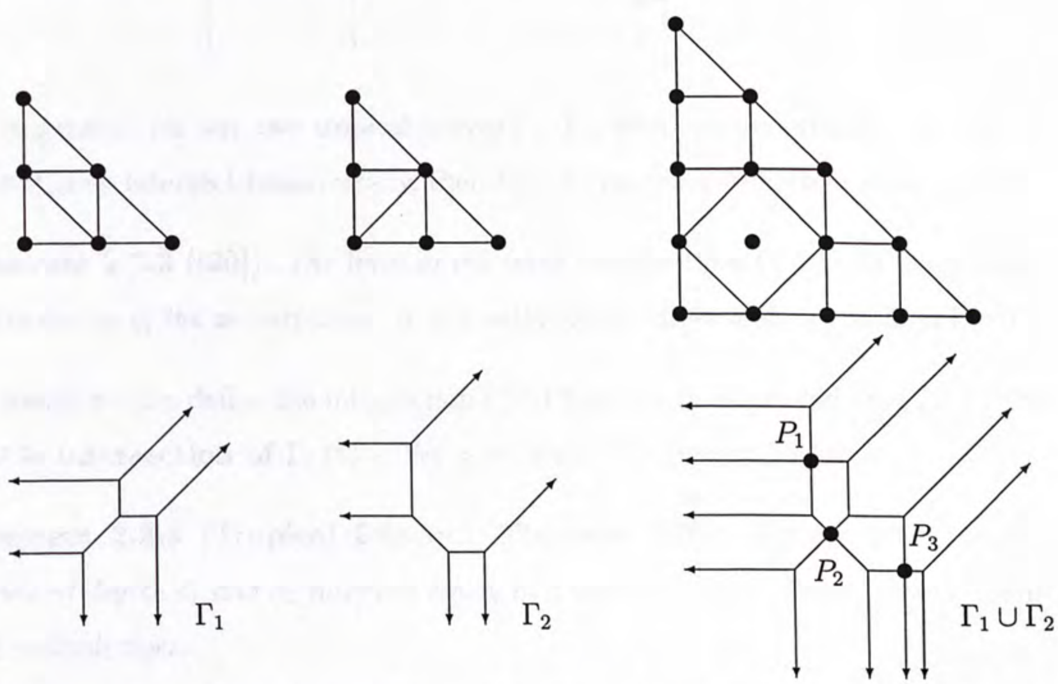
Proposition 2.2.1 (Bézout Theorem [6]). *When two plane tropical curves Γ_1 and Γ_2 intersect transversally, the number of intersection points, counted with multiplicity, is equal to $d_1 d_2$.*

Proof.

$$\begin{aligned}
&\#\Gamma_1 \cap \Gamma_2 (\text{counted with multiplicity}) \\
= &\sum_{p \in \Gamma_1 \cap \Gamma_2} \text{Area of parallelogram corresponds to vertex } p \in \triangle_{d_1+d_2} \\
= &\operatorname{Area}(\triangle_{d_1+d_2}) - \operatorname{Area}(\triangle_{d_1}) - \operatorname{Area}(\triangle_{d_2}) \\
= &\frac{1}{2}(d_1+d_2)^2 - \frac{1}{2}d_1^2 - \frac{1}{2}d_2^2 = d_1 d_2.
\end{aligned}$$

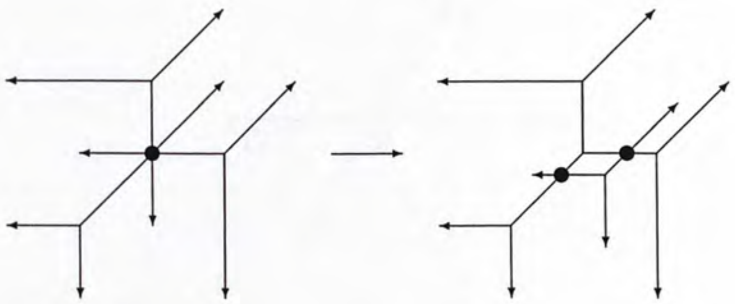
□

Example 2.2.1. The degree of both Γ_1 and Γ_2 are 2 and they give $\Gamma_1 \cup \Gamma_2$ of degree 4. Note that all the eight triangles in the Newton subdivision of $\Gamma_1 \cup \Gamma_2$ come from that of Γ_1 and Γ_2. The multiplicity of P_2 is 2, and the multiplicity of P_1 and P_3 are 1, so the total intersection number is 4.

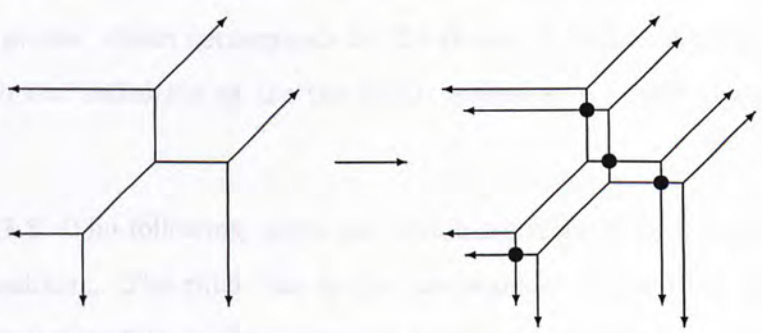


We now deal with the general case when two plane tropical curves which may not intersect transversally. In classical geometry, when C and C' are two algebraic curves in $\mathbb{P}^2$ then $C \cdot C' = C \cdot C''$ where C' is linear equivalent to C''. We observe that there is a similar situation in tropical geometry.

Example 2.2.2. When one of the intersection point is a vertex of either curves, we may perturb one of the curve and consider the intersection number accordingly.



Example 2.2.3. An even more extreme case occurs when the plane tropical curve intersects itself.



In general, for any two tropical curves Γ_1, Γ_2, take any perturbation Γ_1^ϵ and Γ_2^ϵ so that they intersect transversally, then $\Gamma_1^\epsilon \cap \Gamma_2^\epsilon$ intersect in finitely many points.

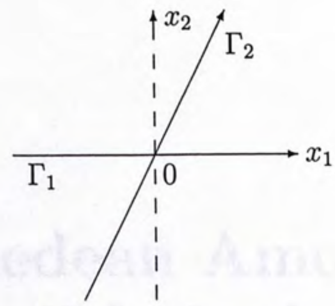
Theorem 2.2.2 ([20]). *The limit of the point configuration $\Gamma_1^\epsilon \cap \Gamma_2^\epsilon$ is independent of the choice of the perturbation. It is a well-defined subset of $d_1 d_2$ points in $\Gamma_1 \cap \Gamma_2$.*

Hence we can define the intersection $\Gamma_1^\epsilon \cap \Gamma_2^\epsilon$ as $\epsilon \to 0$. We called $\lim\limits_{\epsilon \to 0} \Gamma_1^\epsilon \cap \Gamma_2^\epsilon$ **the stable intersection of $\Gamma_1 \cap \Gamma_2$**. We now restate the Bézout Theorem.

Theorem 2.2.3 (Tropical Bézout's Theorem [20]). *Any two plane tropical curves of degree d_1 and d_2 intersect stably in a well defined set of $d_1 d_2$ points, counting multiplicities.*

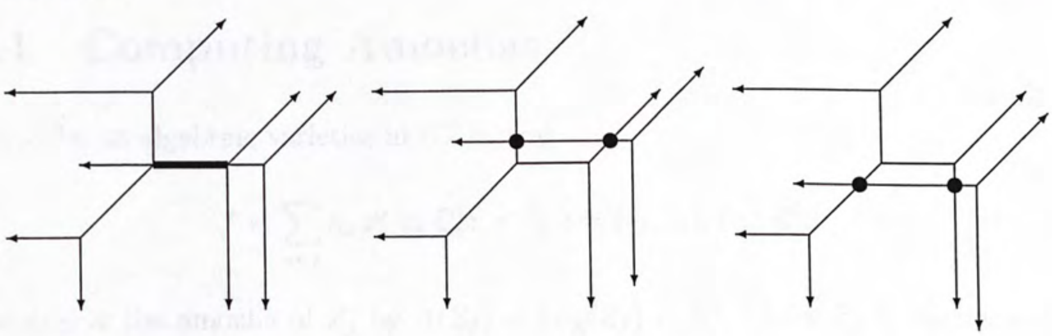
Example 2.2.4. We would like to compare the definition of tropical multiplicity with that of the classical case. Consider the following local graph of two tropical curves with local equations $\Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 = 0\}$ and $\Gamma_2 = \{(x_2, x_2) \in \mathbb{R}^2 \,|$

$x_2 = nx_1\}$ for some $n \in \mathbb{N}_{>0}$.



The intersection point is shifted to the origin for convenience. The corresponding complex curves which map to them are $C_1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = 1\}$ and $C_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = z_1^n\}$ respectively. Observe that the intersection of C_1 and C_2 consists of n points, which corresponds to the choice of n-th root unity of z_1. This coincides with our definition as the the origin is dual to a parallelogram of base 1 and height n.

Example 2.2.5. The following shows the stable intersection of a tropical line and a tropical quadratic. The thick line is the line segment where they intersect. We can use either the middle or the rightmost figure to find out that the intersection number is 2.



Chapter 3

Non-Archimedean Amoebas and Patchworking Method

In this chapter, we study amoebas more deeply. As the spines of amoebas (refer to Definition 3.1.2) are not always equal to the associated plane tropical curves, we want to know when they will equal. In the first part, we use Ronkin functions to study spines of amoebas. In the second part, we introduce the Patchworking method. The dequantization of arithmetic operators in $\mathbb{R}_+$ brings up the associated operations in tropical semiring. This technique is helpful to prove the main theorem in the next chapter.

3.1 Computing Amoebas

Let Z be an algebraic varieties in $(\mathbb{C}^*)^2$, and

$$f = \sum_{\alpha \in I} a_\alpha z^\alpha \text{ in } \mathbb{C}[z, z^{-1}], z = (z_1, z_2), I \subset \mathbb{Z}^2.$$

We denote the amoeba of Z_f by $A(Z_f) = \operatorname{Log}(Z_f) \subset \mathbb{R}^2$, where Z_f is the zeros of f. It is remarked in [7] that $A(Z_f)$ is closed with non-empty complements and its Lebesgue area is well defined. Also,

Theorem 3.1.1 ([7]). *Any connected component of $\mathbb{R}^2 \backslash A(Z_f)$ is open and convex.*

In [19], [22] and [23], Passare and Rullgård use Ronkin functions to study spines of amoebas. Recall that a function f in a domain $\Omega \subset \mathbb{C}^2$ is called pluriharmonic if its restriction to any complex line is subharmonic. Since f is a holomorphic function,

$\log|f| : (\mathbb{C}^*)^2 \backslash V \to \mathbb{R}$ is a pluriharmonic function, where V is a hypersurface of $(\mathbb{C}^*)^2$. Furthermore, if we set $\log 0 = -\infty$, then we have a pluriharmonic function

$$\log|f| : (\mathbb{C}^*)^2 \to \mathbb{R} \cup \{-\infty\},$$

which is strictly pluriharmonic over V.

Definition 3.1.1. *Let $N_f : \mathbb{R}^2 \to \mathbb{R}$ be the push forward of $\log|f|$ under the logarithm map $\mathrm{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$, i.e.*

$$N_f(x_1, x_2) = \frac{1}{(2\pi i)^2} \int_{\mathrm{Log}^{-1}(x_1,x_2)} \log|f(z)| \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}.$$

*This function is called the **Ronkin function of** f.*

Theorem 3.1.2 (Ronkin-Passare-Rullgård [19],[21]). *The function $N_f : \mathbb{R}^2 \to \mathbb{R}$ is convex. Moreover, it is strictly convex over $A(Z_f)$ and linear over each component of $\mathbb{R}^2 \backslash A(Z_f)$.*

N_f is affine on each component of $\mathbb{R}^2 \backslash A(Z_f)$ by the following two propositions:

Proposition 3.1.3. *The derivative of N_f with respect to x_j is the real part of*

$$v_j(x) = \frac{1}{(2\pi i)^2} \int_{\mathrm{Log}^{-1}(x)} \frac{\partial f}{\partial z_j} \frac{z_j}{f(z)} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}.$$

Proof. Write the coordinates in polar coordinates $z_k = e^{x_k + i\theta_k}$. Then for fixed x_k, $dz_k = iz_k d\theta_k$. We have

$$(2\pi i)^2 N_f(x) = \int_0^{2\pi} \int_0^{2\pi} \log\left|f(e^{x_1+i\theta_1}, e^{x_2+i\theta_2})\right| i^2 d\theta_1 \wedge d\theta_2.$$

Differentiating with respect to x_j, we get

$$\begin{aligned}(2\pi i)^2 \frac{\partial N_f}{\partial x_j}(x) &= \int_0^{2\pi} \int_0^{2\pi} \mathrm{Re}\left(\frac{\partial f}{\partial z_j} \frac{e^{x_j+i\theta_j}}{f(z)}\right) i^2 d\theta_1 \wedge d\theta_2 \\ &= \int_{\mathrm{Log}^{-1}(x)} \mathrm{Re}\left(\frac{\partial f}{\partial z_j} \frac{z_j}{f(z)}\right) d\theta_1 \wedge d\theta_2.\end{aligned}$$

□

For x in a connected component $\mathcal{F}$ of $\mathbb{R}^2 \backslash A(Z_f)$, this is a constant and was defined in [4] to be **the order of the component** $\mathcal{F}$. The next proposition was proved, based on Residue formula, refer to [23].

Proposition 3.1.4. *Let x be a point in a connected component $\mathcal{F}$ of $\mathbb{R}^2 \backslash A(Z_f)$, then v_j is an integer.*

Proof. Consider for fixed θ_k, $k \neq j$, the integral

$$\frac{1}{2\pi i}\int_{|z_j|=e^{x_j}} \frac{\partial f}{\partial z_j}\frac{1}{f(z)}dz_j.$$

By the residue formula this is an integer and it counts the number of zeros of the function $z_j \mapsto f(z_1, z_2)$ minus the number of poles in the disk of the boundaries $|z_j| = e^{x_j}$. Since it depends continuously on θ_k, it is independent of them. The integral is equal to v_j because

$$(2\pi i)v_j(x) = \int_0^{2\pi}\Big(\frac{1}{2\pi i}\int_{|z_j|=e^{x_j}} \frac{\partial f}{\partial z_j}\frac{1}{f(z)}dz_1\Big)d\theta_2.$$

□

Note that the fact that v_j is constant over any connected component of the complement implies that the partial derivatives of N_f in each such connected component are constant, thus N_f is affine there.

Remark 3.1.1. Note that just the existence of a convex function N_f, which is strictly convex over $A(Z_f)$ and linear over each components of $\mathbb{R}^2 \backslash A(Z_f)$ implies that each component of $\mathbb{R}^2 \backslash A(Z_f)$ is convex.

The following two propositions relate amoebas and their Newton polygons.

Proposition 3.1.5 ([4]). *$v = (v_1(x), \dots, v_n(x))$ is a lattice point of the Newton polygon $\triangle$ of f, i.e. the convex hull of the element ω of I for which $a_\omega \neq 0$.*

Proof. The vector v is in $\triangle$ if and only if for any vector $u \in \mathbb{Z}^2 \backslash \{0\}$,

$$\langle u, v\rangle \leq \max_{\omega\in\triangle} \langle u, \omega\rangle.$$

Indeed, v is in $\triangle$ if and only if for any line l passing through 0, its orthogonal projection on l belongs to the projection of $\triangle$ on l. By density we can assume that l has a rational slope. The vector u appearing here represents the slope of l, and the scalar product can be seen as the projection on l.

Claim: $\langle u, v\rangle$ is the number of zeroes minus the order of the poles at the origin of the one-variable Laurent polynomial $\omega \mapsto f((c\omega)^u)$ inside the unit circle $\{|\omega| = 1\}$, where c is any point of $\mathrm{Log}^{-1}(x)$ and x being the point where v is computed. But this polynomial has top degree which equal to $\max_{\omega\in\triangle} \langle u, \omega\rangle$. Hence we are done.

It remains to prove the claim. The numbers of zeroes minus the number of the poles of the function $\omega \mapsto f((c\omega)^u)$ in the disk is given by the formula

$$\frac{1}{2\pi i}\int_{|w|=1} \partial f((c\omega)^u).$$

By a change of variable formula $\omega \mapsto (c\omega)^u$, the image of the circle $\{|\omega| = 1\}$ is then a loop in $\mathrm{Log}^{-1}(x)$, homologous to the sum $u_1\gamma_1 + u_2\gamma_2$, where γ_1 is the circle $t \mapsto (c_1 e^{2\pi t}, c_2)$, $t \in [0, 1)$ (respectively for γ_2). Hence

$$\begin{aligned} \frac{1}{2\pi i}\int_{|w|=1} \partial f((c\omega)^u) &= \sum_j u_j \int_{\gamma_j} \partial \log f(w) \\ &= \sum_j u_j \int_{|z_j|=e^{x_j}} \frac{\partial f}{\partial z_j}\frac{1}{f(z)} dz_j = 2\pi i\langle u, v\rangle. \end{aligned}$$

□

Proposition 3.1.6 ([4]). *The map*

$$\begin{aligned} \mathrm{ind} : \mathbb{R}^2\backslash A(Z_f) &\rightarrow \triangle \cap \mathbb{Z}^2 \\ x &\mapsto (v_1(x), \ldots, v_n(x)) \end{aligned}$$

sends two different connected components to two different points.

Proof. Take any two points x and x' in $\mathbb{Q}^2\backslash A(Z_f)$, and let $v = \mathrm{ind}(x)$ and $v' = \mathrm{ind}(x')$. Let $u \in \mathbb{Z}^2\backslash\{0\}$ such that $x' = x + ru$ for some $r > 0$. The claim in the last proposition implies that $\langle u, v\rangle$ and $\langle u, v'\rangle$ are the number of zeros inside $\{|\omega| = 1\}$ of two polynomials $\omega \mapsto f((c\omega)^u)$ and $\omega \mapsto f((c'\omega)^u)$, where $\mathrm{Log}(c) = x$ and $\mathrm{Log}(c') = x'$. We choose c' such that $\frac{c_j'}{c_j} = e^{ru_j}$, i.e. they have the same argument. Hence $(c'\omega)^u = (e^r c\omega)^u$. Thus $\langle u, v'\rangle$ is the number of zeros $\omega \mapsto f((c\omega)^u)$ inside the circle $\{|\omega| = e^r\}$.

If $v = v'$, this means that $\omega \mapsto f((c\omega)^u)$ has no zeros in $\{1 < |\omega| < e^r\}$. Hence there is no points of the amoeba on the segment $[x, x']$. This implies that x and x' are in the same component. □

Hence, each component of order v in $\mathbb{R}^2 \backslash A(Z_f)$ is uniquely determined by v. Indeed, it is determined by the lattice points of the Newton polygon of f.

Example 3.1.1. In example 1.1.1, the order of the three complement components are $(0,0)$, $(1,0)$ and $(0,1)$.

We are now ready to define the spine of an amoeba. Recall that in Theorem 3.1.2, N_f is piecewise linear on $\mathbb{R}^2 \backslash A(Z_f)$ and convex in $\mathbb{R}^2$. We then define a convex linear function N_f^∞ by letting

$$N_f^\infty = \max_{\mathcal{F}} N_{\mathcal{F}},$$

where $\mathcal{F}$ runs through all connected components of $\mathbb{R}^2 \backslash A(Z_f)$, and $N_f : \mathbb{R}^2 \to \mathbb{R}$ is the extension of $N_f|_{\mathcal{F}}$ to $\mathbb{R}^2$ by linearity.

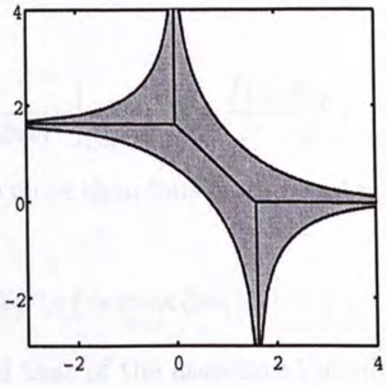
Definition 3.1.2 (Passare-Rullgård [19]). *The* ***spine*** $\mathcal{S}$ ***of an amoeba*** $A(Z_f)$ *is the corner locus of* N_f^∞, *i.e. the set of points in* $\mathbb{R}^2$ *which is not linear.*

Note that $\mathcal{S} \subset A(Z_f)$ and that $\mathcal{S}$ is a piecewise linear polyhedral cell complex. The following theorem shows that $\mathcal{S}$ is indeed a spine of $A(Z_f)$ in the topological sense.

Theorem 3.1.7 (Passare-Rullgård [19],[23]). *The spine of* $A(Z_f)$ *is a strong deformational retract of* $A(Z_f)$.

Thus each component of $\mathbb{R}^2 \backslash \mathcal{S}$ (i.e. each maximal domain where N_f^∞ is linear) contains a unique component of $\mathbb{R}^2 \backslash A(Z_f)$.

Example 3.1.2. An amoeba and its spine given by $f = -5 + z_1 + z_2 + z_1 z_2$.



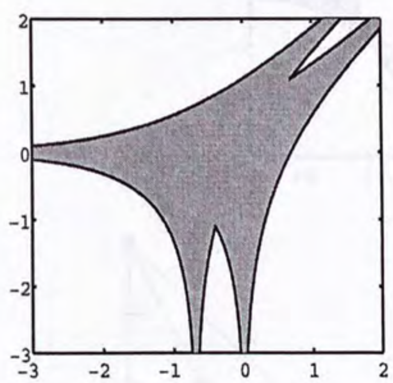
Example 3.1.3. The spine of an amoeba and the associated plane tropical curve are not always equal. Consider an amoeba given by

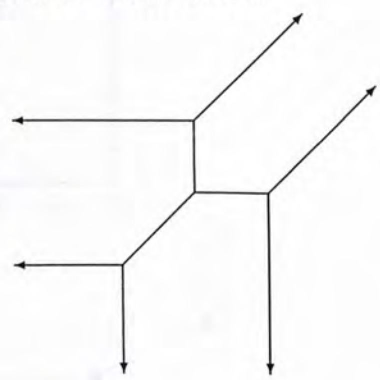
$$f = 3z_1 + 3z_1z_2 + 2z_1^2 + z_2^2 + 2z_2 + 1 = 3z_1 + 3z_1z_2 + 2z_2^2 + (z_2+1)^2.$$

The associated plane tropical curve is given by

$$F = \log 3 \odot x \oplus \log 3 \odot x \odot y \oplus \log 2 \odot x^{\odot 2} \oplus y^{\odot 2} \oplus \log 2 \odot y \oplus 0.$$

The amoeba $A(Z_f)$ and the plane tropical curve is as shown below:





Since f has only a point of contact 2 with the line $\{z_1 = 0\}$, it leads to only one tentacle of the amoeba in the $(-1,0)$-direction. Then the spine of amoeba has only one edge in the $(-1,0)$-direction as the spine is contained in the amoeba.

In [19] and [23], the non-obvious relation between the coefficients of f and the coefficients of the "tropical polynomial" N_f^∞ is studied.

Theorem 3.1.8.

$$N_f^\infty(x) = \max_{v \in J} \{\mathrm{Re}\, \Phi_v(f) + \langle v, x\rangle\},$$

where J is the subset of I of v for which there exists a connected component $\mathcal{F}_v$ of order v, and

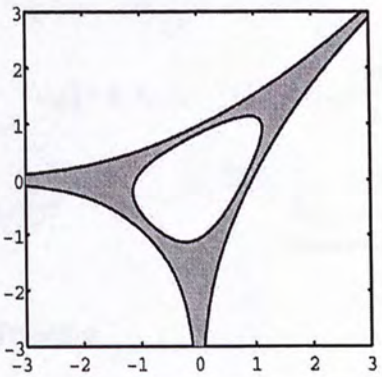
$$\Phi_v(f) = \frac{1}{(2\pi i)^2} \int_{\mathrm{Log}^{-1}(x)} \log \frac{f(z)}{z^v} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}, x \in \mathcal{F}_v.$$

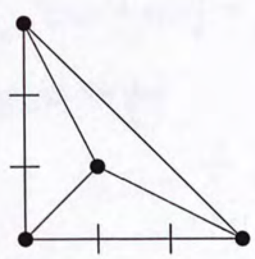
Remark 3.1.2. If I has no more than four points, and no three of these are collinear. Then

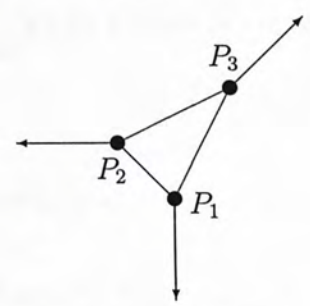
$$N_f^\infty(x) = \max_{v \in I} (\log |a_v| + \langle v, x\rangle).$$

In this case, the spines and that of the associated plane tropical curves coincides.

Example 3.1.4. Let $f = 1+z_1^3+z_2^3-6z_1z_2$, then $I = \{(0,0),(3,0),(0,3),(1,1)\}$ and the associated plane tropical curves are given by $F = \log 6 \odot x \odot y \oplus 0 \odot x^{\odot 3} \oplus 0 \odot y^{\odot 3} \oplus 0$. The amoeba, Newton subdivision and the associated plane tropical curves (i.e. the spine) are as shown below:







By the graph of the associated plane tropical curve, the vertices of the spine can be computed explicitly, namely $P_1 = (-\log 6, 0)$, $P_2 = (0, -\log 6)$ and $P_3 = (\log 6, \log 6)$. Note that the set for which the component has order $(1,1)$ is empty, so the plane tropical curve has genus 1.

Example 3.1.5. Consider

$$\begin{aligned} f(z) &= f_0 + f_1 z + \cdots + f_{m-1}z^{m-1} + z^m \\ &= (z+a_1)\cdots(z+a_m), \end{aligned}$$

where $|a_1| \leq \cdots \leq |a_m|$. Then $A(Z_f) = \{\log|a_1|, \ldots, \log|a_m|\}$, which is a discrete point set. Each bounded complement component of f is of the form $(\log|a_v|, \log|a_{v+1}|)$, $1 \leq v \leq m-1$, which is of order v. The other two unbounded components $(-\infty, \log|a_1|)$ and $(\log|a_m|, +\infty)$ are of order 0 and m, respectively.

Suppose a complement component of $A(Z_f)$ has order v and x is a point inside that component. We have

$$\begin{aligned}
\Phi_v(f) &= \frac{1}{2\pi i}\int_{\mathrm{Log}^{-1}(x)} \log\frac{f(z)}{z^v}\frac{dz}{z} \\
&= \frac{1}{2\pi i}\int_{\mathrm{Log}^{-1}(x)} \log\frac{(z+a_1)\cdots(z+a_v)}{z^v}\frac{dz}{z} \\
&\quad + \frac{1}{2\pi i}\int_{\mathrm{Log}^{-1}(x)} \log\,(z+a_{v+1})\cdots(z+a_m)\frac{dz}{z} \\
&= \frac{1}{2\pi i}\Big(\int_{\log|z|=x}\sum_{j=1}^{v}\log\,(\frac{z+a_j}{z})\frac{dz}{z} + \underbrace{\int_{\log|z|=x}\sum_{j=v+1}^{m}\log\,(z+a_j)\frac{dz}{z}}_{(*)}\Big).
\end{aligned}$$

By Jensen's formula, $(*)$ becomes

$$\int_{|z|=e^x}\log\,(z+a_j)\frac{dz}{z} = \frac{1}{2\pi}\int_{|z|=e^x}\log\,(z+a_j)d\theta = \log|a_j|.$$

By Cauchy's theorem, we get

$$\Phi_v(f) = \sum_{j=v+1}^{m}\log a_j = \log\,(a_{v+1}\cdots a_m).$$

Since for any Laurent polynomial f, we have $N_f(x) \geq \max_{0\leq v\leq m}\{\mathrm{Re}\,\Phi_v(f) + \langle v,x\rangle\}$ with equality in the closure of $A(Z_f)$. If f is a polynomial in one variable, then $\mathbb{R}\backslash A(Z_f)$ is dense in $\mathbb{R}$. Hence,

$$\begin{aligned}
N_f(x) &= \max_{0\leq v\leq m}\{\log|a_{v+1}\cdots a_m| + \langle v,x\rangle\} \\
&= \max\{\log|a_1\cdots a_m|, x+\log|a_2\cdots a_m|, \ldots, mx+\log|a_m|\}.
\end{aligned}$$

Example 3.1.6. Next we consider polynomials in two variables of the form $f(z) = a + z_1 + z_2 + z_1z_2$, where a is an arbitrary complex constant. It can be shown that the amoeba of f is the set of points satisfying

$$\begin{aligned}
&|a|^4 - 2|a|^2e^{2x_1} + e^{4x_1} - 2|a|^2e^{2x_2} - (2-8|a|+2|a|^2)e^{2x_1+2x_2} \\
&\qquad\qquad -2e^{4x_1+2x_2} + e^{4x_2} - 2e^{2x_1+4x_2} + e^{4x_1+4x_2} \leq 0.
\end{aligned}$$

We now focus on the case when a is real. Then the amoeba depends on the sign of a. Suppose $a < 0$, the above inequality can be written as

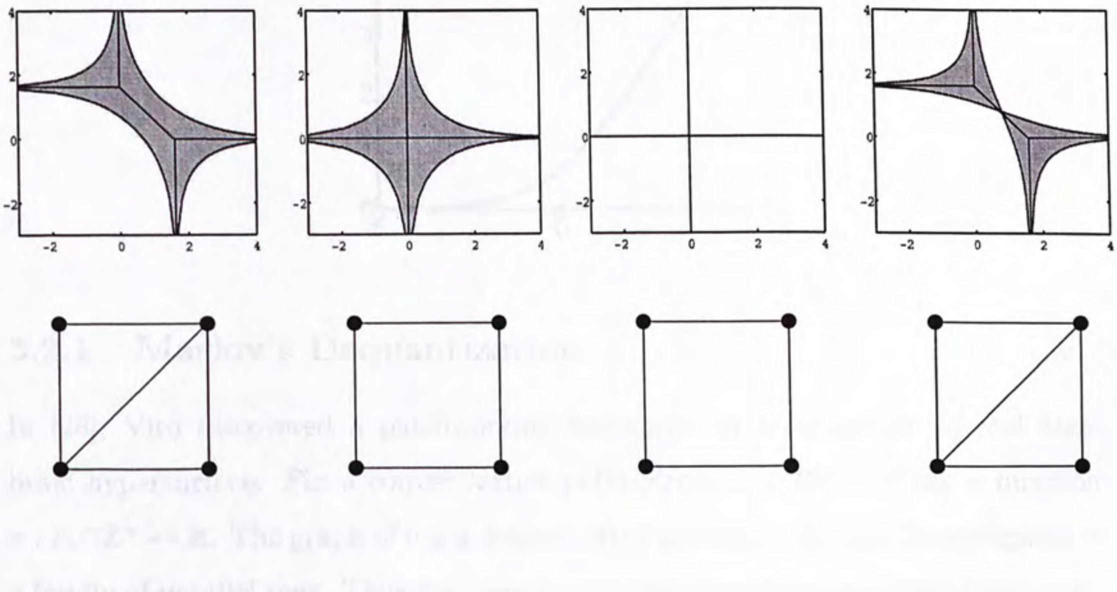
$$(e^{x_1+x_2} - e^{x_1} - e^{x_2} - |a|)(e^{x_1+x_2} - e^{x_1} + e^{x_2} + |a|)$$
$$\times(e^{x_1+x_2} + e^{x_1} - e^{x_2} + |a|)(e^{x_1+x_2} + e^{x_1} + e^{x_2} - |a|) \leq 0.$$

Note that each factor vanishes on the boundary of one of the complement component of the amoeba. Suppose $a > 0$, we have a similar factorization

$$(e^{x_1+x_2} - e^{x_1} - e^{x_2} + |a|)(e^{x_1+x_2} - e^{x_1} + e^{x_2} - |a|)$$
$$\times(e^{x_1+x_2} + e^{x_1} - e^{x_2} - |a|)(e^{x_1+x_2} + e^{x_1} + e^{x_2} + |a|) \leq 0.$$

Note that the fourth factor is always positive while the first factor vanishes on the boundaries of two complement components, namely, those of order $(0,0)$ and $(1,1)$ if $a < 1$ and those of order $(0,1)$ and $(1,0)$ if $a > 1$. The remaining two factors define two curves, each of which constitutes part of the boundary of two different complement components. The curves intersect at their common point of inflection $(\log\frac{a}{2}, \log\frac{a}{2})$.

The following shows the amoeba of f when $a = -5$, -1, 1 and 5 together with their spines and their Newton subdivisions. Note that the amoebas of f when $a = -5$ and 5 are different (respectively for $a = -1$ and 1), but their spines are equal, which can also be deduced from Theorem 3.1.8.

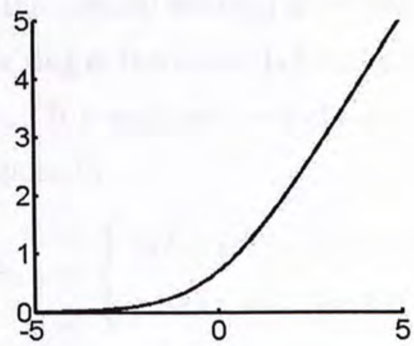


3.2 Patchworking Method

In this section, we describe the Patchworking method. The motivation comes from the graph of real algebraic curves on a logarithm paper, see [29]. A logarithm paper is a graph paper which the usual Cartesian coordinates are replaced by the logarithm scales. It corresponds to a change of coordinates $u = \log x$ and $v = \log y$. On a logarithm paper, real algebraic curves look like smooth broken lines which can be obtained from the limits of those curves. The corresponding deformation can be viewed as a quantization, in which the broken line is a classical object and the curves are quantum. This generalizes to a new connection between algebraic geometry and the geometry of polyhedra.

Example 3.2.1. On a logarithm paper, the real curve $y = ax^k$, $a > 0$ is given by $v = \log y = \log a + k \log x = \log a + ku$, which is a straight line.

Example 3.2.2. Consider $y = 1 + x$, then $v = \log y = \log(1 + x) = \log(1 + e^u)$. When the graph is viewed from very far, it looks like the broken line $v = \max\{0, u\}$ with a smooth corner around the origin. The following shows the graph of $v = \log(1 + e^u)$.



3.2.1 Maslov's Dequantization

In [28], Viro discovered a patchworking technique for construction of real algebraic hypersurface. Fix a convex lattice polyhedron $\triangle \in \mathbb{R}^n$. Choose a function $v : \triangle \cap \mathbb{Z}^n \to \mathbb{R}$. The graph of v is a discrete set of points in $\mathbb{R}^n \times \mathbb{R}$. The overgraph is a family of parallel rays. Thus the convex hull of the overgraph is a semi-finite polyhedron $\widetilde{\triangle}$. The facets of $\widetilde{\triangle}$ which project isomorphically to $\mathbb{R}^n$ define a subdivision of $\triangle$ into smaller convex lattice polyhedra $\triangle_k$.

Let $F(z) = \sum_{j \in \triangle} a_j z^j$ be a generic polynomial in the class of polynomial whose Newton polyhedron is $\triangle$. The truncation of F to $\triangle_k$ is $F_{\triangle_k} = \sum_{j \in \triangle_k} a_j z^j$. The patchworking polynomial f is defined by formula $f_t^v(z) = \sum_j a_j t^{v(j)} z^j, z \in \mathbb{R}^n, t > 1$ and $j \in \mathbb{R}^n$.

Consider the hypersurface $V_{\triangle_k}$ and V_t in $(\mathbb{C}^*)^n$ defined by $F_{\triangle_k}$ and f_t^v. If F has real coefficients then we denote $\mathbb{R}V_{\triangle_k} = V_{\triangle_k} \cap (\mathbb{R}^*)^n$ and $\mathbb{R}V_t = V_t \cap (\mathbb{R}^*)^n$. Viro's patchoworking theorem [28] then state that for large value of t the hypersurface $\mathbb{R}V_t$ can be obtained from $\mathbb{R}V_{\triangle_k}$ by a certain patchworking procedure. The same also holds for amoebas of the hypersurface V_t and $\mathbb{R}V_{\triangle_k}$. In fact the patchworking of hypersurfaces can be intercepted as the real version of patchworking of amoebas, see [13]. It was noted by Viro in [29] that patchworking is related to the so-called Maslov's dequantization of positive real numbers.

Recall that a **quantization** of a semiring R is a family of semirings R_h, $h \geq 0$ such that $R_0 = R$ and $R_t \approx R_s$ as long as $s, t > 0$, but R_0 is not isomorphic to R_t. The semiring R_h with $h > 0$ is called a **quantized version** of R_0.

Maslov observed that the classical semiring $\mathbb{R}_+$ of real positive number is quantized version of some other ring in this sense. Let R_h be the ring of positive number of quantized version of $\mathbb{R}_+$. It is equipped with the usual multiplication and with the addition operation defined by

$$z \oplus_h w = \begin{cases} (z^{\frac{1}{h}} + w^{\frac{1}{h}})^h & \text{for } h > 0 \\ \max\{z, w\} & \text{for } h = 0. \end{cases}$$

Note that

$$\lim_{h \to 0} (z^{\frac{1}{h}} + w^{\frac{1}{h}})^h = \max\{z, w\}.$$

Thus this is a continuous family of arithmetic operations.

The semiring R_1 coincides with the standard semiring $\mathbb{R}_+$. We have the following isomorphism between $\mathbb{R}_+$ and R_h for $h > 0$.

$$\begin{aligned} \varphi_h : \mathbb{R}_+ &\to R_h \\ z &\mapsto z^h \end{aligned}$$

It can be checked that $\varphi_h(z+w) = \varphi_h(z) \oplus \varphi_h(w)$ and $\varphi_h(zw) = \varphi_h(z)\varphi_h(w)$. On the other hand the semiring R_0 is not isomorphic to $\mathbb{R}_+$ since $z \oplus_0 z = z$.

Alternatively, we may define the dequantization deformation with the help of the logarithm. The logarithm $\log_t$, $t > 1$ induces a semiring structure on $\mathbb{R}$ from $\mathbb{R}_+$, the map $\log_{t>1} : \mathbb{R}_+ \to \mathbb{R}$ is defined as follow:

$$z \oplus_t w = \log_t (t^z + t^w) \text{ and } z \otimes_t w = z + w.$$

Similarly we have $z \oplus_\infty w = \max\{z, w\}$. Let R_t^{log} be the resulting semiring. The following proposition shows that R_h and R_t^{log} are indeed isomorphic.

Proposition 3.2.1. *The map* $\log : R_h \to R_t^{log}$, *where* $t = e^{\frac{1}{h}}$, *is an isomorphism.*

3.2.2 Patchworking Method

As we seen in the previous section, the patchworking polynomial can be viewed as a deformation of the polynomial f_1^v. We define a similar deformation with the help of Maslov's dequantization. Instead of deforming the coefficients, we keep the coefficients but deform the arithmetic operations.

Choose any coefficients $\alpha_j, j \in \triangle$. Let $\phi_t : (R_t^{\log})^n \to R_t^{\log}, t \geq e$, be a polynomial whose coefficients are α, i.e.

$$\phi_t(x) = \bigoplus_t (\alpha_j + jx), x \in \mathbb{R}^n.$$

Let $\mathrm{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}$ be defined as before.

Proposition 3.2.2 (Viro [29]). *The function* $f_t = (\log_t)^{-1} \circ \phi_t \circ \mathrm{Log}_t : (\mathbb{R}_+)^n \to \mathbb{R}_+$ *is a polynomial with respect to the standard arithmetic operations in* $\mathbb{R}_+$, *namely we have*

$$f_t(z) = \sum_j t^{\alpha_j} z^j.$$

This is a special case of the patchworking polynomial. The coefficients α_j define the function $v : \triangle \cap \mathbb{Z}^n \to \mathbb{R}$.

Suppose $V_K \subset (K^*)^2$ be an algebraic variety, its non-Archimedean amoeba is $A_K = \mathrm{Val}(V_K) \subset \mathbb{R}^2$. Denote $A_t = \mathrm{Log}_t(V_t)$. We have a uniform convergence if the addition operation in R_t^{log} to the addition operation in R_∞^{log}. Observe that

$$\begin{aligned}\max\{x,y\} &\leq x \oplus_t y = \log_t(t^x + t^y)\\ &\leq \max\{x,y\} + \log_t 2.\end{aligned}$$

It can be generalized to the following lemma.

Lemma 3.2.3. $\max\limits_{j\in\triangle}\{\alpha_j + jx\} \leq \phi_t(x) \leq \max\limits_{j\in\triangle}\{\alpha_j + jx\} + \log N$, *where N is the number of lattice points in $\triangle$.*

Recall that the Hausdorff metric of two sets A and B are defined to be

$$d_{Hausdorff}(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}.$$

Theorem 3.2.4 (Mikhalkin [16], Rullgård [23]). *The subset $A_t \subset \mathbb{R}^2$ tends in the Hausdroff metric to A_K when $t \to \infty$, i.e. for any compact $K' \subset \mathbb{R}^2$,*

$$\lim_{t\to\infty} d_{\text{Hausdorff}}(A_t \cap K', A_K \cap K') = 0.$$

Proof. By lemma, A_t converges to a subset in A_K. Actually, by rewriting $\left|a_j t^{v(j)} z^j\right| = |t^{c_j} z^j|$ where $c_j = v(j) + \log_t |a_j|$, the monomial induces a linear function $c_j + jx$ and the inequality $c_k + kx \leq \max\limits_{j\neq k}\{c_j + jx\} + \log_t N$, and cuts out a uniformly bounded neighborhood of A_K. It is because

$$v(k) \odot x^{\odot k} \leq \bigoplus_{j\neq k} v(j) \odot x^{\odot j}$$

by the triangle inequality in R_t. It follows that

$$v(k) + kx \leq \max_{j\neq k}\{v(j) + jx\} + \log_t N, \text{ where } N = \text{number of monomials in } f_t - 1.$$

Thus $\lim A_t \subset A_K$ as A_K is the corner locus of $N_f^K(x) = \max\limits_j \{v(j) + jx\}$.

Conversely, each complement component is given by

$$\max_{j\neq k}\{c_k + jx\} + \log_t N < c_k + kx.$$

This will contain in $\mathbb{R}^2 \backslash A_t$ corresponding to the index k, so different complements of the set are in different components of $\mathbb{R}^2 \backslash A_t$. $\square$

Chapter 4

Mikhalkin's Correspondence Theorem

In this chapter, we give the main theorem of this report. This theorem gives the invariant numbers which count the number of plane tropical curves through a number of points. It is an analogy to the algebraic case. In the first part, we give the main statement and count these numbers via lattice paths. In the second part, we sketch a proof of the main theorem, following [17], which is divided into two lemmas.

4.1 Parameterized Plane Tropical Curves

Following [17], we define parameterized plane tropical curves in the abstract way.

Definition 4.1.1. *Let $\bar{\Gamma}$ be a weighted finite graph and $\Gamma = \bar{\Gamma}\backslash\mathcal{V}$, where $\mathcal{V}$ is the set of all 1-valent vertices. A proper map $h : \Gamma \to \mathbb{R}^2$ is called a* ***parameterized plane tropical curve*** *if it satisfied the following two conditions:*

- *For every edge $E \subset \Gamma$, the restriction $h|_E$ is either an embedding or a constant map. The image $h(E)$ is contained in a line $l \subset \mathbb{R}^2$ such that the slope of l is rational.*
- *For every vertex $V \in \Gamma$, the balancing condition holds.*

Definition 4.1.2. *Two parameterized plane tropical curves $h : \Gamma \to \mathbb{R}^2$ and $h' : \Gamma' \to \mathbb{R}^2$ are equivalent if there exists a homeomorphism $\Phi : \Gamma \to \Gamma'$ which respect the weights of the edges and such that $h = h' \circ \Phi$.*

Equivalent plane parameterized tropical curves are not distinguished. The image $h(\Gamma) \subset \mathbb{R}^2$ is called the unparameterized plane tropical curve or just the plane tropical curve.

Remark 4.1.1. The notion of plane tropical curve coincides with the notion of (p,q)-webs introduced by Aharony, Hanany and Kol in [1].

Definition 4.1.3. *A plane tropical curve $h : \Gamma \to \mathbb{R}^2$ is called* ***reducible*** *if Γ is disconnected , and is called* ***reducible*** *if it can be presented as a union of two distinct plane tropical curves.*

Definition 4.1.4. *The genus of a parameterized plane tropical curve $\Gamma \to \mathbb{R}^2$ is* $\dim H_1(\Gamma) - \dim H_0(\Gamma) + 1$. *In particular, for irreducible parameterized plane tropical curves the genus is the first Betti number of Γ. The genus of a plane tropical curve Γ is the minimum genus among all parameterizations of Γ.*

Remark 4.1.2. Note that the genus can be negative in Definition 4.1.4.

Remark 4.1.3. If Γ is an embedded 3-valent graph, then the parameterization is unique. However, in general, there might be several parameterizations of different genus and taking the minimum value is essential.

To begin the next section, we need a larger class of plane tropical curves whose behavior is as simple as that of smooth curves.

Definition 4.1.5. *A parameterized plane tropical curve $h : \Gamma \to \mathbb{R}^2$ is called* ***simple*** *if it satisfies all of the following conditions.*

- *The graph is 3-valent.*
- *The map h is an immersion.*
- *For any $y \in \mathbb{R}^2$, the inverse image $h^{-1}(y)$ consists of at most two points.*
- *If $a, b \in \Gamma$, $a \neq b$, are such that $h(a) = h(b)$, then either a nor b can be a vertex of Γ.*

Definition 4.1.6. *A plane tropical curve is called* ***simple*** *if it admits a simple parameterization.*

The next proposition tells us that simple plane tropical curves can be easily identified.

Proposition 4.1.1. *A plane tropical curve is simple if and only if it is the variety of a tropical polynomial F such that the Newton subdivision of F is a subdivision into triangles and parallelograms.*

The next definition concerns the tropical general positions of points in $\mathbb{R}^2$, we will use it in the next section.

Definition 4.1.7. *Points $p_1, \ldots, p_n \in \mathbb{R}^2$ are said to be in* ***tropical general position*** *if for any parameterized plane tropical curve $h : \Gamma \to \mathbb{R}^2$ of genus g and with x ends such that $k \geq g + x - 1$ and $p_1, \ldots, p_n \in h(\Gamma)$, the following conditions hold.*

- *The curve $h : \Gamma \to \mathbb{R}^2$ is simple.*
- *Inverse images $h^{-1}(p_1), \ldots, h^{-1}(p_n)$ are disjoint from the vertices of Γ.*
- *$k = g + x - 1$.*

Example 4.1.1. Two distinct points $p_1, p_2 \in \mathbb{R}^2$ are in tropical general position if and only if the slope of the line in $\mathbb{R}^2$ passing through p_1 and p_2 is irrational.

4.2 Statement of the Main Theorem

We set up an enumerative problem in $(\mathbb{C}^*)^2$ first. Fix a number $g \in \mathbb{Z}_{\geq 0}$ and a convex polygon $\triangle \subset \mathbb{R}^2$. Let $s = \#(\partial\triangle \cap \mathbb{Z}^2)$. Let $\mathcal{Q} = \{q_1, \ldots, q_{s+g-1}\} \subset (\mathbb{C}^*)^2$ be a configuration of $s + g - 1$ points in general position.

Definition 4.2.1. *Suppose a complex algebraic curve $C \subset (\mathbb{C}^*)^2$ is defined by a polynomial $f : (\mathbb{C}^*)^2 \to \mathbb{C}$ with complex coefficients.* ***The degree of*** *C is defined to be the Newton polygon of f.*

Definition 4.2.2. *The number $N^{irr}(g, \triangle)$ is defined to be the number of irreducible complex algebraic curves of genus g and degree $\triangle$ passing through $\mathcal{Q}$. Similarly we define the number $N(g, \triangle)$ be the number of complex algebraic curves of genus g and degree $\triangle$ passing through $\mathcal{Q}$.*

Proposition 4.2.1. *For a generic choice of $\mathcal{Q}$, the numbers $N^{irr}(g, \triangle)$ and $N(g, \triangle)$ are finite and do not depend on $\mathcal{Q}$.*

This proposition is well-known, see [2].

To set up a similar problem in tropical case. Let us define the degree of a plane tropical curve.

Definition 4.2.3. *Let $\triangle$ be the Newton polygon of f defining a plane tropical curve Γ as in Theorem 1.4.1. Then $\triangle$ is called* ***the degree of*** *C.*

Remark 4.2.1. Let $l = \#(\mathrm{Int}\triangle \cap \mathbb{Z}^2)$. Then the number s counts the number of unbounded edges of the curve if edges are counted with multiplicities. If $\triangle = \triangle_d$, the number l is the genus of a smooth plane tropical curve of degree d.

To set up an enumerative problem in $\mathbb{R}^2$, we fix the degree $\triangle \subset \mathbb{R}^2$ with $s = \#(\partial\triangle\cap\mathbb{Z}^2)$, and the genus $g \in \mathbb{Z}_{\geq 0}$. Consider a configuration $\mathcal{P} = \{p_1, \cdots, p_{s+g-1}\} \subset \mathbb{R}^2$ of $s+g-1$ points in $\mathbb{R}^2$ in tropical general position. We would like to count the number of plane tropical curves passing through $\mathcal{P}$ with certain multiplicities.

For a plane tropical curve Γ passing through generic points, it is a 3-valence graph. We define the multiplicity of a 3-valent vertex as follow:

Definition 4.2.4. *For each vertex $V \in \Gamma$, its multiplicity is defined to be*

$$m(V) = w_1 w_2 \left|\det(u_1, u_2)\right|,$$

where w_1, w_2, w_3 are weights of the edges adjacent to V and u_1, u_2, u_3 are the primitive integral vectors in the directions of the edges.

Remark 4.2.2. The above definition is well-defined because of the balancing condition $\sum_{i=1}^{3} w_i u_i = 0$. We have

$$\begin{aligned} w_1 w_2 \left|\det(u_1, u_2)\right| &= w_1 \left|\det(u_1, w_2 u_2)\right| \\ &= w_1 \left|\det(u_1, w_1 u_1 + w_3 u_3)\right| \\ &= w_1 w_3 \left|\det(u_1, u_3)\right| \\ &= w_2 w_3 \left|\det(u_2, u_3)\right|. \end{aligned}$$

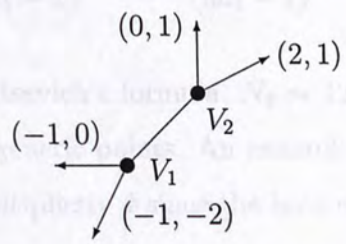
Note that it also coincides with Definition 2.2.1.

Definition 4.2.5. ***The multiplicity of a plane tropical curve*** Γ, *$m(\Gamma)$ is defined to be the product of the multiplicities of all the 3-valent vertices of* Γ, *i.e.*

$$m(\Gamma) = \prod_{V \in \Gamma_0'} m(V),$$

where Γ_0' is the set of all 3-valence vertices.

Example 4.2.1. Let Γ be a plane tropical curve with vertices $V_1 = (1,1)$ and $V_2 = (2,2)$ with the directions of unbounded edges as follow:



Note that the bounded edge has weight 2, so $m(V_1) = m(V_2) = 2$ and $m(\Gamma) = 4$.

Definition 4.2.6. *We define the number $N^{irr}_{trop}(g, \triangle)$ to be the number of irreducible plane tropical curves of degree $\triangle$ and genus g passing through $\mathcal{P}$, counted with multiplicities. Similarly we define the number $N_{trop}(g, \triangle)$ be the number of plane tropical curves of degree $\triangle$ and genus g passing through $\mathcal{P}$, counted with multiplicities.*

Proposition 4.2.2. *The numbers $N^{irr}_{trop}(g, \triangle)$ and $N_{trop}(g, \triangle)$ are finite and do not depend on the choice of $\mathcal{P}$.*

Example 4.2.2. Note that $N_{trop}(0, \triangle_1) = 1$ since there is only one combinational type of plane tropical curves dual to $\triangle_1$. Different choices of two generic points have the following cases.

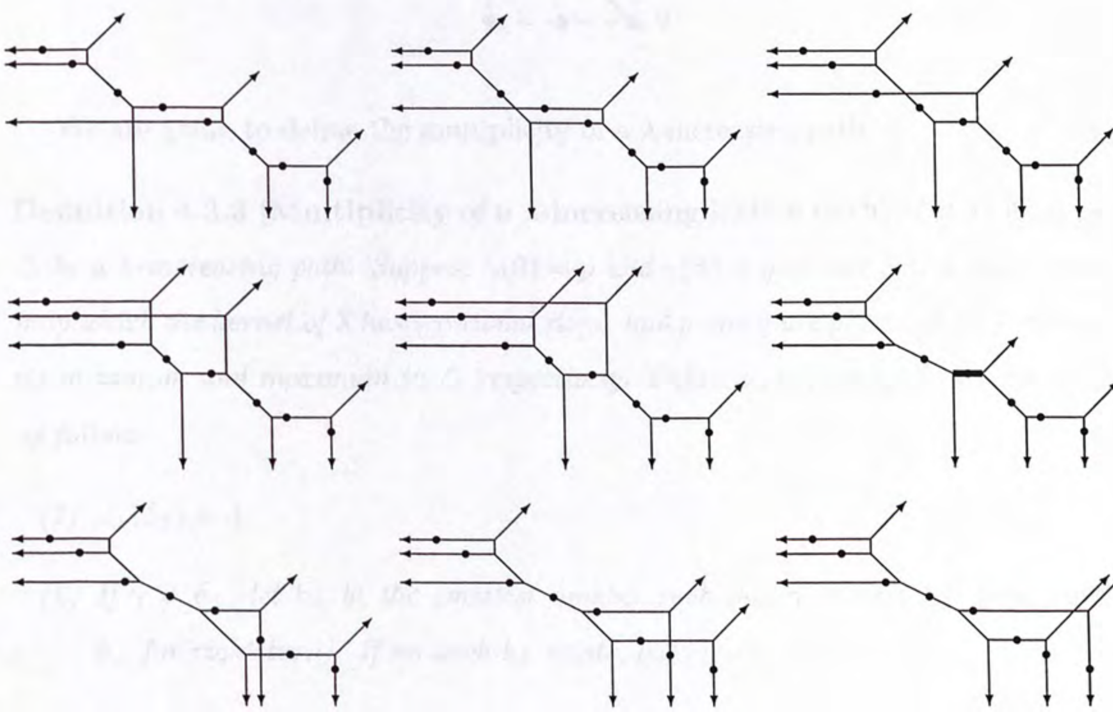
We can now state the main theorem.

Theorem 4.2.3 (Mikhalkin's Correspondence Theorem). *For any generic choice $\mathcal{P}$, we have $N^{irr}(g, \triangle) = N^{irr}_{trop}(g, \triangle)$ and $N(g, \triangle) = N_{trop}(g, \triangle)$. Furthermore, there exists a configuration $\mathcal{Q} \subset (\mathbb{C}^*)^2$ of $s + g - 1$ points in tropical general* *position such that for every tropical curve Γ of genus g and degree $\triangle$ passing through $\mathcal{P}$, we have $m(\Gamma)$ distinct complex curves of genus g and degree $\triangle$ passing through $\mathcal{Q}$. These curves are distinct for distinct Γ and are irreducible if Γ is irreducible.*

Remark 4.2.3. In [12], $N(g, \triangle_d)$ is known as the Gromov-Witten invariant of $\mathbb{CP}^2$. When $g = 0$, a recursive relation was given by Kontsevich's formula [5], which came from the associativity of the quantum cohomology.

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2} \right), d > 1 \text{ with } N_1 = 1.$$

Example 4.2.3. By Kontsevich's formula, $N_3 = 12$. There are 12 plane tropical curves passing through 8 generic points. An example is as shown below. Note that one of the curves have multiplicity 4 since the bold edge has multiplicity 2.



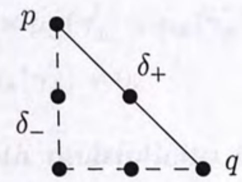
4.3 Lattice Paths

In this section, we introduce a method to count the number $N_{trop}(g, \triangle)$ in the previous section via lattice paths. Lattice paths are paths in the Newton polygons which can be found by a certain recursive relation. Refer to [14] for a short review.

Definition 4.3.1. *A path $\gamma : [0,n] \to \mathbb{R}^2$ is called a* **lattice path** *if $\gamma|_{[j-1,j]}$, $j = 1,\dots,n$ is an affine linear map and $\gamma(j) \in \mathbb{Z}^2$ for any $j = 0,\dots,n$.*

Definition 4.3.2. *Let $\lambda : \mathbb{R}^2 \to \mathbb{R}$ be a fixed linear map which the kernel of λ has irrational slope. A lattice path γ is called* ***λ-increasing*** *if $\lambda \circ \gamma$ is strictly increasing.*

Example 4.3.1. Suppose $\lambda = x - \varepsilon y$, $\varepsilon > 0$ is a small irrational number. Let p and q be the points where $\triangle_2$ attains its minimum and maximum respectively. Then $\delta_+ : [0,2] \to \partial\triangle_2$ and $\delta_- : [0,4] \to \partial\triangle_2$ divide $\partial\triangle_2$ into 2 λ-increasing paths. The image of of the path δ_+ is drawn as a line while that of the path δ_- is drawn as a dotted line.



We are going to define the multiplicity of a λ-increasing path.

Definition 4.3.3 (Multiplicity of a λ-increasing lattice path). *Let $\gamma : [0,n] \to \triangle$ be a λ-increasing path. Suppose $\gamma(0) = p$ and $\gamma(n) = q$, where λ is a fixed linear map which the kernel of λ has irrational slope, and p and q are points where λ attains its minimum and maximum in $\triangle$ respectively. Define $\mu_+(\gamma)$ and $\mu_-(\gamma)$ recursively as follow:*

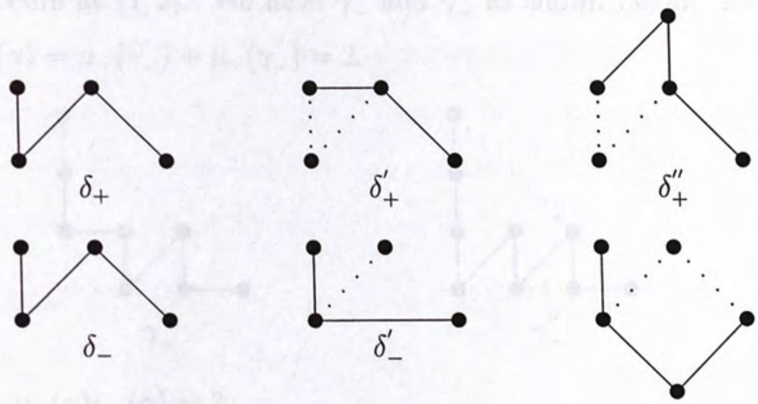
(1) $\mu_\pm(\delta_\pm) = 1$.

(2) If $\gamma \neq \delta_\pm$, let k_+ be the smallest number such that γ takes a left turn (resp. k_- for right turn). If no such $k_\pm$ exists, $\mu_\pm(\delta) = 0$.

(3) Define $\gamma'_\pm$ and $\gamma''_\pm$ as follow:

$$\gamma'_\pm(j) = \begin{cases} \gamma(j), j < k_\pm, \\ \gamma(j+1), j \geq k_\pm. \end{cases}$$

$$\text{and} \quad \gamma''_\pm(j) = \begin{cases} \gamma(j), j \neq k_\pm, \\ \gamma(k_\pm - 1) + \gamma(k_\pm + 1) - \gamma(k_\pm), j = k_\pm. \end{cases}$$



δ_+ δ'_+ δ''_+

δ_- δ'_- δ''_-

Let T be the triangle with vertices $\gamma(k_{\pm 1}), \gamma(k_\pm)$ and $\gamma(k_{\pm+1})$.

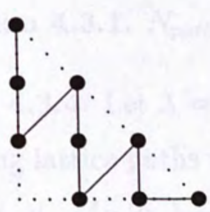
Set $\mu_\pm(\gamma) = 2 \times \text{Area}(T) \times \mu_\pm(\gamma'_\pm) + \mu_\pm(\gamma''_\pm)$.

If $\gamma''_\pm$ do not map to $\triangle$, $\mu_\pm(\gamma''_\pm) = 0$.

(4) Finally, we define the ***path multiplicity of a λ-increasing path*** *γ to be* $\mu(\gamma) = \mu_+(\gamma)\mu_-(\gamma)$.

Remark 4.3.1. μ is well-defined since γ'_+ and γ''_+ bounded less area with δ_+, respectively for γ'_- and γ''_- with δ_-.

Example 4.3.2. Let $\gamma : [0, 8] \to \triangle_3$ be a λ-increasing lattice path as follow. We would like to compute the path multiplicity of γ.



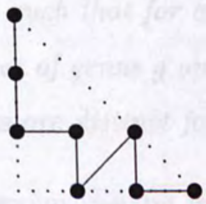
γ

$$\gamma = \begin{cases} (0, 3-t) & \text{if } 0 \le t \le 2, \\ (t-1, t-1) & \text{if } 2 \le t \le 3, \\ (1, 5-t) & \text{if } 3 \le t \le 5, \\ (t-4, t-5) & \text{if } 5 \le t \le 6, \\ (2, 7-t) & \text{if } 6 \le t \le 7, \\ (t-5, 0) & \text{if } 7 \le t \le 8. \end{cases}$$

k_+ occurs at $(0, 1)$. We have γ'_+ and γ''_+ as shown below. Since γ''_+ does not map to $\triangle_3$, $\mu_+(\gamma''_+) = 0$. Hence $\mu_+(\gamma) = \mu_+(\gamma'_+) = \mu_+((\gamma'_+)'_+) = 1$.

γ'_+ γ''_+ $(\gamma'_+)'_+$

Similarly, k_- occurs at $(1,2)$. We have γ'_- and γ''_- as shown below. By a similar calculation, $\mu_-(\gamma) = \mu_-(\gamma'_-) + \mu_-(\gamma''_-) = 2$.

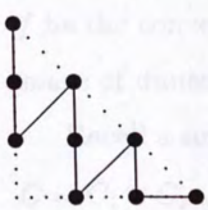


γ'_-

γ''_-

Finally, $\mu(\gamma) = \mu_+(\gamma)\mu_-(\gamma) = 2$.

Example 4.3.3. The following shows all λ-increasing paths in 8 steps in $\triangle_3$ with $\lambda = x - \epsilon y$ and their path multiplicities. This shows that $N_{path}(0, \triangle_3) = 12$.

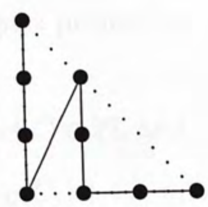


$\mu = 2$

$\mu = 3$

$\mu = 4$

$\mu = 1$



$\mu = 2$

Definition 4.3.4. *Let $s = \#(\partial\triangle \cap \mathbb{Z}^2)$. $N_{path}(g,d)$ is defined to be the number of λ-increasing lattice paths $\gamma : [0, s+g-1] \to \triangle$ with $\gamma(0) = p$ and $\gamma(s+g-1) = q$, counted with path multiplicity.*

Proposition 4.3.1. *$N_{path}(g, \triangle)$ is independent of the choice of λ.*

Example 4.3.4. Let $\lambda = y - (1+\epsilon x)$ with a small $\epsilon > 0$. The following shows all λ-increasing lattice paths and their path multiplicities with 8 steps in $\triangle_3$. This also shows that $N_{path}(g, \triangle_3) = 12$.

$\mu = 3$

$\mu = 2$

$\mu = 4$

$\mu = 1$

$\mu = 2$

The follow theorem is an efficient way to compute $N_{trop}(g, \triangle)$ and respectively $N(g, \triangle)$.

Theorem 4.3.2. *The number $N_{trop}(g, \triangle)$ is equal to $N_{path}(g, \triangle)$, counted with path multiplicity. Furthermore, there exists a configuration $\mathcal{P} \subset \mathbb{R}^2$ of $s + g - 1$ points in general position such that for each λ-increasing lattice path corresponds to some plane tropical curves of genus g and degree $\triangle$ passing through $\mathcal{P}$ of total multiplicity $\mu(\gamma)$. These curves are distinct for distinct paths.*

The above theorem can be intercepted as follow. Let $\triangle \subset \mathbb{R}^2$ be a convex polygon with integer vertices. Suppose

$$L = \Big\{ f = \sum_{j,k \in \triangle \cap \mathbb{Z}^2} a_{jk} z^j w^k \subset (\mathbb{C}^*)^2 \Big\}$$

as before. Then L is a vector space with linear structure. Let the Newton polygon of f be the convex hull of $\{(j,k) \in \triangle \cap \mathbb{Z}^2 \,|\, a_{jk} \neq 0\}$. Let $\mathbb{PL}$ be the complex projective space of dimension $\#(\triangle \cap \mathbb{Z}^2) - 1$.

Recall a smooth generic curve in $\mathbb{L}$ is of genus $l = \#(\mathrm{Int}\triangle \cap \mathbb{Z}^2)$. Let $C \in \mathbb{PL}$ and $C = C_1 \cup C_2 \cup \ldots \cup C_n$, where C_j are irreducible. Define $g(C) = \sum_{j=1}^{n} g(C_j) + 1 - n$, note that $g(C)$ may be negative.

Note that curves of genus $l - \delta$ with Newton polygon $\triangle$ form a subvariety $\sum_\delta^\circ$, it is called a Severi variety. Let $\sum_\delta$ be the projective closure of $\sum_\delta^\circ$. Define $N(\triangle, \delta)$ to be the degree of $(m - \delta)$-dimensional subvariety $\sum_\delta$ in $\mathbb{L}$, where degree is the intersection number with projective space of codimension $m - \delta$.

We have the following interception. Let $z_1, \ldots, z_{m-\delta} \in (\mathbb{C}^*)^2$, then $N(\triangle, \delta)$ is the number of algebraic curves of Newton polygon $\triangle$ and genus $l - \delta$ passing through $z_1, \ldots, z_{m-\delta}$.

Another way to look at it is the following. Consider the compactification of $(\mathbb{C}^*)^2$, then $\triangle$ induces a compact toric surface $\mathbb{C}T_\triangle$. Recall that $\mathbb{P}^2$ can be given by $\{|z_1|^2 + |z_2|^2 + |z_3|^3 = 1\}$ identified by the $U(1)$-action $(z_1, z_2, z_3) \sim e^{i\theta}(z_1, z_2, z_3)$. If we write $(x, y, z) = (|z_1|^2, |z_2|^2, |z_3|^2)$. The toric base is given by $z = 1 - x - y$. Then $N(\triangle, \delta)$ is the number of holomorphic curves $\bar{C} \subset \mathbb{C}T_\triangle$ such that $z_1, \ldots, z_{m-\delta} \in \bar{C}$, with homology class $[\bar{C}]$ Poincáre dual to $c_1(H)$, where H is the holomorphic linear bundle over $\mathbb{C}T_\triangle$.

Now, we can restate Theorem 4.3.2 as follow.

Theorem 4.3.3. *$N(\triangle, \delta)$ is the number of λ-increasing lattice paths $\gamma : [0, m-\delta] \to \triangle$ connecting p and q.*

Remark 4.3.2. Let $\triangle = \triangle_d$, then $\mathbb{C}T_\triangle = \mathbb{P}^2$. We have

$$\begin{aligned} m &= \#(\delta \cap \mathbb{Z}^2) - 1 = \frac{(d+1)(d+2)}{2} - 1 = \frac{d^2+3d}{2}, \\ \text{and} \qquad 3d + g - 1 &= 3d + \binom{d-1}{2} - \delta - 1 = m - \delta. \end{aligned}$$

4.4 Complex Tropical Curves

To prove Mikhalkin's Correspondence Theorem, instead of considering the plane tropical curves themselves, we lift the curves from $\mathbb{R}^2$ to $(\mathbb{C}^*)^2$ and consider complex tropical curves.

Recall the shrinking of amoebas is the map $(z, w) \mapsto (\log_t |z|, \log_t |w|)$. We now shrink the curves in $(\mathbb{C}^*)^2$ and apply the Log map, this induces a new complex structure on $(\mathbb{C}^*)^2$.

Definition 4.4.1. *Let $t > 1$ be real number. Define*

$$\begin{aligned} H_t : (\mathbb{C}^*)^2 &\to (\mathbb{C}^*)^2 \\ (z, w) &\mapsto (|z|^{\frac{1}{\log t}} \frac{z}{|z|}, |w|^{\frac{1}{\log t}} \frac{w}{|w|}). \end{aligned}$$

*A **J_t-holomorphic curve** V_t is the image $V_t = H_t(V)$ of a holomorphic curve V. A J_t-holomorphic curve V_t is called irreducible if its preimage V is irreducible, and reducible else.*

We can also define J_∞-holomorphic curve algebraically over a non-Archimedean field. Let K be the field of Puiseux series. The multiplicative homomorphism $\mathrm{val} : K^* \to \mathbb{R}$ can be complexified to $\omega : K^* \to \mathbb{C}^* \approx \mathbb{R} \times S^1$ by setting

$$\omega(a) = e^{\mathrm{val}(a) + i\arg(a_{\mathrm{val}(a)})}.$$

Applying this map coordinatewise, we get the map

$$W : (K^*)^2 \to \mathbb{R}^2 \times (S^1)^2 \approx (\mathbb{C}^*)^2.$$

Applying the map val coordinatewise, we get the map Val : $(K^*)^2 \to \mathbb{R}^2$, Val $=$ Log $\circ\, W$. The image of an algebraic curve V_K under W turns out to be a J_∞-holomorphic curve.

Observe that

$$\begin{aligned}\operatorname{Log} \circ H_t(z,w) &= \operatorname{Log}(|z|^{\frac{1}{\log t}}\frac{z}{|z|}, |w|^{\frac{1}{\log t}}\frac{w}{|w|}) \\ &= (|z|^{\frac{1}{\log t}}, |w|^{\frac{1}{\log t}}) = (\log_t |z|, \log_t |w|).\end{aligned}$$

So H_t corresponds to a $\log t$-contraction $(x,y) \mapsto (\frac{x}{\log t}, \frac{y}{\log t})$ under Log.

Let V_k be a sequence of J_{t_k}-holomorphic curves, where $t_k \to \infty$ as $k \to \infty$. Assume the sequence converges in the Hausdorff metric to V_∞. Then V_∞ is called a **complex tropical curve**. There are other equivalent definitions, refer to [17].

Definition 4.4.2. *Let $V_\infty \subset (\mathbb{C}^*)^2$. The following conditions are equivalent.*

- *$V_\infty = W(V_K)$, where $V_K \subset (K^*)^2$ is an algebraic curve.*
- *V_∞ is the limit when $k \to \infty$ in the Hausdorff metric of a sequence of J_{t_k}-holomorphic curves V_{t_k} with $\lim_{k\to\infty} t_k = \infty$.*

Proposition 4.4.1. *Let $V_\infty = W(V_K)$, where $V_K \subset (K^*)^2$ is an algebraic curve with the Newton polygon Δ. Then* $\operatorname{Log}(V_\infty) \subset \mathbb{R}^2$ *is a graph. Furthermore, it is possible to equip the edges of* $\operatorname{Log}(V_\infty)$ *with natural weight so that the result is a plane tropical curve of degree Δ in $\mathbb{R}^2$.*

This proposition follows from Kapranov's Theorem [3].

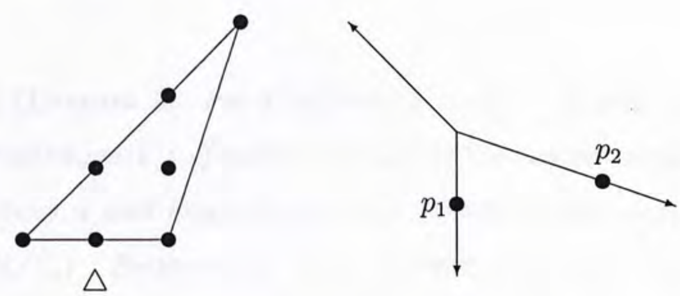
Definition 4.4.3. *We say that a complex tropical curve V_∞ with a choice of natural weights for the edges of $\Gamma = \operatorname{Log}(V_\infty)$ has degree Δ if these weights turn Γ to be a plane tropical curve of degree Δ.*

Definition 4.4.4. *A complex tropical curve $V_\infty \subset (\mathbb{C}^*)^2$ is said to have genus g if V_∞ is the limit (in the sense of the Hausdorff metric in $(\mathbb{C}^*)^2$) of a sequence of J_{t_k}-holomorphic curves in $(\mathbb{C}^*)^2$ with $t_k \to \infty$ of genus g and cannot be presented as a limit of a sequence of J_{t_k}-holomorphic curves of smaller genus.*

We start by having an elementary enumerative problem in $(\mathbb{C}^*)^2$.

Proposition 4.4.2. *Let q_1, q_2 be two points in tropical general position such that $p_i = \mathrm{Log}(q_i)$. Let $\triangle$ be a lattice polygon. Then there is exactly a plane tropical curve dual to $\triangle$ and passing through p_1 and p_2. Let the two edges that passing through p_1 and p_2 are of weight w_1 and w_2 respectively. Then there are $\frac{2 \times \mathrm{Area}(\triangle)}{w_1 w_2}$ distinct rational complex tropical curve that passing through q_1 and q_2.*

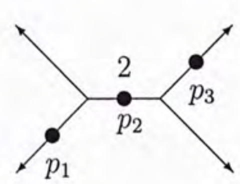
Example 4.4.1. The area of the Newton polygon $\triangle$ is 3. The weights of the two edges passing through $p_i = \mathrm{Log}(q_i)$ are 2 and 1 respectively. So the proposition claims that there are 3 rational complex tropical curve passing through q_i that project to this plane tropical curve under Log.



To state the next proposition, we define the edge multiplicity of a complex tropical curve.

Definition 4.4.5. *We define the* ***edge multiplicity*** $\mu_{edge}(\Gamma, \mathcal{P})$ *of a plane tropical curve $\Gamma \supset \mathcal{P} = \mathrm{Log}(\mathcal{Q})$ to be the product of the weights of all the edges of the parameterizing plane tropical curve $\Gamma \subset \mathbb{R}^2$ that are disjoint from $\mathcal{P}$ times the product of the squares of the weights of all the edges of Γ that are not disjoint from $\mathcal{P}$.*

Example 4.4.2. Let $\mathcal{P} = \{p_1, p_2, p_3\}$. The edge multiplicity of the following plane tropical curve is 4.



Proposition 4.4.3. *Let Γ be one of the plane tropical curves passing through $\mathcal{P}$. Then there are $\frac{m(\Gamma)}{\mu_{edge}(\Gamma, \mathcal{P})}$ complex tropical curves in $(\mathbb{C}^*)^2$ of genus g and degree $\triangle$ such that they project to Γ and pass via $\mathcal{Q}$.*

4.5 Sketch Proof of the Main Theorem

We will give a sketch proof in this section. We first have the following proposition.

Proposition 4.5.1. *For almost all $t > 1$, there are $N(g, \triangle)$ J_t-holomorphic curves of genus g and degree $\triangle$ through a fixed configuration $\mathcal{Q}$.*

The following two lemmas are important to complete the proof.

Lemma 4.5.2 (Lemma 1). *For any $\epsilon > 0$, there exists a $T > 1$ such that if $t > T$ and V is a J_t-holomorphic curve of genus g, degree $\triangle$ and passing through $\mathcal{Q}$, then its amoeba $\mathrm{Log}(V)$ is contained in the ϵ-neighborhood $N_\epsilon(\Gamma_j)$ of Γ_j for some $j = 1, \ldots, m$.*

Lemma 4.5.3 (Lemma 2). *For a sufficiently small $\epsilon > 0$ and a sufficiently large $t > 0$, the multiplicity $m(\Gamma_j)$ of each Γ_j is equal to the number of the J_t-holomorphic curves V of genus g and degree $\triangle$ passing through $\mathcal{Q}$ and such that $\mathrm{Log}(V)$ is contained in $N_\epsilon(\Gamma_j)$. Furthermore, if Γ_j is irreducible, then any J_t-holomorphic curve V of genus g and degree $\triangle$ passing through $\mathcal{Q}$ with $\mathrm{Log}(V) \subset N_\epsilon(\Gamma_j)$ is irreducible while if Γ_j is reducible, then any such V_t is reducible.*

We are going to give a sketch proof of Lemma 4.5.2 and Lemma 4.5.3. Using these two lemmas, we now give a proof of the main theorem.

Proof of the Mikhalkin's Correspondence Theorem. By Proposition 4.5.1, there are $N(g, \triangle)$ J_t-holomorphic curves of genus g and degree $\triangle$ passing through $\mathcal{Q}$. By Lemma 4.5.2, the amoebas of all such J_t-holomorphic curves lie in an ϵ-neighborhood of one of the plane tropical curves passing through $\mathcal{P}$. By Lemma 4.5.3, there are $m(\Gamma_j)$ J_t-holomorphic curves lie in the ϵ-neighborhood of Γ_j for each Γ_j. Hence the number $N(g, \triangle)$ is equal to $N_{trop}(g, \triangle)$, counted with the multiplicity $m(\Gamma)$. Now the last statement of the main theorem follows as any irreducible J_t-holomorphic curve project to an irreducible plane tropical curve. $\square$

4.5.1 Proof of Lemma 1

To prove Lemma 4.5.2, recall that if a holomorphic curve $V \subset (\mathbb{C}^*)^2$ is given by the polynomial $f(z_1, z_2) = \sum_{(j,k)\in\triangle} a_{j,k} z_1^j z_2^k$, then its tropicalization is given by the tropical polynomial

$$F^{trop}(x_1, x_2) = \max_{(j,k)\in\triangle} \{jx_1 + kx_2 + \log|a_{j,k}|\}.$$

The amoeba $\mathrm{Log}(V)$ gets thinner with larger t by the following proposition.

Proposition 4.5.4. *The amoeba* $\mathrm{Log}(V)$ *is contained in the* δ*-neighborhood of* V^{trop} *(with respect to the Euclidean metric in* $\mathbb{R}^2$*), where* $\delta = \log(\#(\triangle \cap \mathbb{Z}^2) - 1)$.

Proof. Suppose (y_1, y_2) is a point not in the δ-neighborhood of V^{trop}. Then there exists (j', k') such that

$$j'y_1 + k'y_2 + \log|a_{j',k'}| > jy_1 + ky_2 + \log|a_{j,k}| + \delta \text{ for any } (j,k) \neq (j',k').$$

Let $(z_1, z_2) \in V \subset (\mathbb{C}^*)^2$ such that $\mathrm{Log}(z_1, z_2) = (y_1, y_2)$. Since $\sum_{(j,k)\in\triangle} a_{j,k} z_1^j z_2^k = 0$ and by triangle's inequality, we have

$$\left|a_{j',k'} z_1^{j'} z_2^{k'}\right| \leq \sum_{(j,k)\neq(j',k')} \left|a_{j,k} z_1^j z_2^k\right|.$$

Apply the map log to both sides, then

$$\begin{aligned} j'y_1 + k'y_2 + \log|a_{j',k'}| &= \log\left|a_{j',k'} z_1^{j'} z_2^{k'}\right| \\ &\leq \log \sum_{(j,k)\neq(j',k')} \left|a_{j,k} z_1^j z_2^k\right| \\ &\leq \log\left((\#(\triangle \cap \mathbb{Z}^2) - 1) \times \max_{(j,k)\neq(j',k')} \left|a_{j,k} z_1^j z_2^k\right|\right) \\ &= \delta + \max_{(j,k)\neq(j',k')} \{jy_1 + ky_2 + \log|a_{j,k}|\}. \end{aligned}$$

This is a contradiction. □

Corollary 4.5.5. *The amoeba* $\mathrm{Log}(V_t)$ *of a* J_t*-holomorphic curve* $V_t = H_t(V)$ *is contained in the* δ*-neighborhood of some plane tropical curves, where*

$$\delta = \log_t(\#(\triangle \cap \mathbb{Z}^2) - 1).$$

Proposition 4.5.6. *There is a subsequence $V_{k_\alpha}, \alpha \in \mathbb{N}$, such that the sets $\mathcal{A}_{k_\alpha} \subset \mathbb{R}^2$ converge in the Hausdorff metric in $\mathbb{R}^2$ to some tropical curves Γ_j.*

Idea of Proof of Lemma 4.5.2. Let V_{k_α} be a sequence of curves of genus g and degree $\triangle$ passing through $\mathcal{Q}$ such that V_k is a J_{t_k}-holomorphic curve, where $t_k \to \infty$ as $k \to \infty$. Let A_k be the amoeba $\text{Log}(V_k)$ and S_k be the spine of amoeba $LogV_k$.

It can be shown that the corresponding sequence of spines S_{k_α} is a union of subsequences, each of which consists either of finitely many terms, or converges to one of the plane tropical curves Γ_j. By Corollary 4.5.5, the spines S_k and the amoebas A_k converge. Also the limiting amoeba is in the δ-neighborhood of another plane tropical curve, which is the tropicalization of V_k. In particular, this means that its thickness cannot be bigger than 2δ. The number δ depends on t_k and $\delta = \log_{t_k}(\#(\triangle \cap \mathbb{Z}^2) - 1)$ for J_{t_k}-holomorphic curves. As $t_k \to \infty$ when $k \to \infty$, the thickness of the amoebas A_k gets smaller for larger k. In particular, A_k is contained in a small neighborhood of Γ_j.

Note that a spine S_k which is contained in the sequence that converges to Γ_j can have more edges than Γ_j, but these edges vanish in the limit. All other edges of S_k tend to a parallel edge of Γ_j. It can be shown that, based on Proposition 4.5.6, there is a small value $\tilde{\delta}(t_k)$ depending on t_k such that all edges of the spine S_k are in a $\tilde{\delta}(t_k)$-neighborhood of the corresponding edge of Γ_j. Since the thickness of the amoeba A_k is smaller than $\delta = \log_{t_k}(\#(\triangle \cap \mathbb{Z}^2) - 1)$, we can conclude that A_k is contained in a $\tilde{\delta}(t_k) + 2\delta$-neighborhood of Γ_j. □

4.5.2 Proof of Lemma 2

Idea of Proof of Lemma 4.5.3. By Proposition 4.4.3, it remains to show that there are $\mu_{edge}(\Gamma, \mathcal{P})$ J_t-holomorphic curves of genus g and degree $\triangle$ passing through $\mathcal{Q}$ in the neighborhood of each complex tropical curve which map to C under Log.

The main idea is to prove this statement separately for each polygon in the Newton subdivision of $\triangle$ dual to Γ, and then "glue" the J_t-holomorphic curve corresponding to each polygon using the Patchworking method.

We can think of Γ given by the tropical polynomial $F = \max\{a_j + b_j x + c_j y\}$. Now let V_∞ be one of the complex tropical curves which maps to Γ under Log.

Consider

$$f_t^\zeta = \sum \arg\zeta_j + t^{\log\zeta_j} z_1^{b_j} z_2^{c_j},$$

and assume V_t is a J_t-holomorphic curve in the neighborhood of V_∞. Then $V_t = V_t^\zeta$ can be presented as $V_t^\zeta = H_t(\{f_t^\zeta = 0\})$ with the set of the coefficients ζ has to satisfy a condition given by a_j.

The aim is then to count those J_t-holomorphic curves V_t^ζ in the neighborhood of V_∞ which are of genus g, degree $\triangle$ and pass via $\mathcal{Q}$. □

Cover $\mathbb{R}^2$ with open sets $U(\triangle')$ corresponding to the polygons $\triangle'$ in the Newton subdivisions in the following way:

- If $\triangle'$ is a 2-polygon, then it is dual to a point $p_{\triangle'} \in \Gamma$. We choose $U(\triangle')$ be a small open disc centered at $p_{\triangle'}$.

- If $\triangle'$ is an edge, then it is dual to an edge $e_{\triangle'} \subset \Gamma$ connecting $p_{\triangle_1}, p_{\triangle_2} \in \Gamma$, where $\triangle_1$, $\triangle_2$ are two 2-polygons adjacent to $e_{\triangle'}$. We choose $U(\triangle')$ to be a small regular open neighborhood of $e_{\triangle'} \backslash (U(\triangle_1) \cup U(\triangle_2))$.

- Now Γ is a deformational retract of $\mathcal{U} = \sum_{\triangle'} U(\triangle')$, there is a bijection between the components of $\mathbb{R}^2 \backslash \mathcal{U}$ and the vertices of the Newton subdivision of Γ. We choose $U(\triangle')$ to be a small open neighborhood of the components of $\mathbb{R}^2 \backslash \mathcal{U}$ corresponding to $\triangle'$ if $\triangle'$ is in the vertices of the Newton subdivision of Γ.

Proposition 4.5.7. *Suppose that $\zeta \in \mathcal{D}$, $\mathcal{Q} \subset V_t^\zeta$ and t is large. The curve V_t^ζ is a curve of genus g if and only if all the following conditions hold.*

- *If $\triangle'$ is a parallelogram with vertices k_0, k_1, k_2, $k_3 \in \mathbb{Z}^2$, $k_3 - k_2 = k_1 - k_0$, then $V_t^\zeta \cap \mathrm{Log}_t^{-1}(U(\triangle'))$ is a union of two (not necessarily connected) curves, one in a small neighborhood of a complete tropical curve with the Newton polygon $[k_0, k_1]$ and one in a small neighborhood of a complex tropical curve with the Newton polygon $[k_2, k_3]$.*

- *If $\triangle'$ is an edge, then $V_t \cap \mathrm{Log}_t^{-1}(U(\triangle'))$ is homeomorphic to an immersed annulus (and, therefore, connected).*

- *If $\triangle'$ is a triangle, then $V_t \cap \mathrm{Log}_t^{-1}(U(\triangle'))$ has genus* 0.

This proposition takes care about the conditions of curves of genus g, and the next proposition is to prove the last statement of Lemma 4.5.3.

Corollary 4.5.8. *If a curve $\Gamma = \Gamma_j$ is irreducible, then any J_t-holomorphic curve V of genus g and degree $\triangle$ with large t passing through $\mathcal{Q}$ with $\mathrm{Log}(V) \subset N_\epsilon(\Gamma_j)$ is irreducible while if Γ_j is reducible, then any such curve V is reducible.*

We state the following three lemmas.

If $\Gamma \subset \mathbb{R}^2$ is a plane tropical curve passing through $\mathcal{P}$, then we can mark the k edges of the Newton subdivision of Γ dual to $p_1, \dots, p_k$. Let $\Xi \subset \triangle$ be the union of the marked k edges.

Lemma 4.5.9. *Let $[k', k'']$ be the edge of Ξ and let $q \in (\mathbb{C}^*)^2$ be any point. For any choice of $b_j \in \mathbb{C}$, $j \in [k', k'']\backslash\{k''\}$, there exists a unique choice of $b_{k''}$ such that q is a point of*

$$\Big\{z \in (\mathbb{C}^*)^2 \mid \sum_{j \in [k', k'']} b_j z^j = 0\Big\}.$$

Lemma 4.5.10. *Let $\triangle' \subset \mathbb{R}^2$ be a parallelogram with vertices k_0, k_1, k_2, $k_3 \in \mathbb{Z}^2$, $k_1 - k_0 = k_3 - k_2$. For any choice $b_j \in \mathbb{C}^*$, $j \in [k_0, k_1] \cup [k_0, k_2]$, there exists a unique choice of coefficients $\{b_j\}$, $j \in (\triangle' \cap \mathbb{Z}^2)\backslash([k_0, k_1] \cup [k_0, k_2])$, such that the curve*

$$\Big\{z \in (\mathbb{C}^*)^2 \mid \sum_{j \in \triangle' \cap \mathbb{Z}^2} b_j z^j = 0\Big\}$$

is a union of two curves with Newton polygons $[k_0, k_1]$ and $[k_0, k_2]$, respectively.

Lemma 4.5.11. *Let $\triangle' \subset \mathbb{R}^2$ be a triangle with vertices k_0, k_1, $k_2 \in \mathbb{Z}^2$. For any choice b_{k_0}, b_{k_1}, $b_{k_2} \in \mathbb{C}^2$, there exists $2 \times$ Area$(\triangle')$ distinct choice of coefficients $\{b_j\}$ such that*

$$V^b = \Big\{z \in (\mathbb{C}^*)^2 \mid \sum_{j \in \triangle' \cap \mathbb{Z}^2} b_j z^j = 0\Big\}$$

is a rational curve if degree $\triangle'$ with three ends at infinity.

Now, Lemma 4.5.9, 4.5.10 and 4.5.11 count the number of suitable ζ for each polygon $\triangle'$ of the Newton subdivision in $\triangle$, i.e. it satisfies the conditions of Proposition 4.5.7 and pass through $\mathcal{Q}$. In other words, these three lemmas count the number of suitable coefficients ζ for each polygon separately, and the glued curve V_t^{ζ} is a suitable curve.

It can be shown that there exists an order on the polygons $\triangle'$ such that we choose the suitable ζ for each $\triangle'$ separately. To show how many choices are there to choose the new coefficients such that they are compatible to the old coefficients. We have the following proposition, the above three lemmas are needed to prove it.

Proposition 4.5.12. *Suppose that t is large and $\zeta \in \mathcal{D}$ is chosen compatible with $\triangle_1', \dots, \triangle_{k-1}'$. There exists $\prod_{u=1}^{k} \mu'(\triangle_u')$ choices of $\zeta' \in \mathcal{D}$ with the following properties.*

- *The parameter ζ' is compatible with $\triangle_1', \dots, \triangle_{k-1}'$.*
- *We have $\zeta_j' = \zeta_j$ if $j \in G_{\triangle_u'}$, $u > k$.*
- *$V_t^{\zeta} \subset N(V_\infty)$.*

Hence when the Newton subdivision does not have multiple edges, i.e. no edges of weight greater than 1, we have proven that there is $\mu_{edge}(\Gamma, \mathcal{P})$ J_t-holomorphic curves of genus g, degree $\triangle$ and pass through $\mathcal{Q}$ in the neighborhood of each complex tropical curve which map to Γ.

When there is multiple edges, we need to show that the choices of coefficients ζ of $\triangle'$ is compatible with the definition of edge multiplicities. It should also guarantee the complex tropical curve is of genus g and with the right number. This proved Lemma 4.5.3.

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