Some New Developments on Inverse Scattering Problems

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摘要

在电磁场的逆散射问题中, 人们想通过测量几组由入射电磁波产生的散射场来确定散射物体. 这些散射物体可以是一个障碍物, 也可以是一个衍射光栅. 逆散射问题在很多领域都有广泛的应用, 例如雷达, 非破坏性检测, 还有衍射光学. 在逆散射问题中最重要的话题之一是唯一性问题. 也就是找出能够唯一确定一个未知散射物体的入射波的最少数目. 值得注意的是迄今为止这个重要的问题在一般情况下仍然是一个开放问题. 最近二十年来, 人们通过研究一些特殊的散射体在这个问题上取得了很重要的进展. 在本文中, 我们将回顾这些进展并且给出一些新的结果. 具体来说, 我们研究了两类散射体: 一类散射体是由有限多个有界多面体组成的, 而构成这些多面体的介质是混合的全电导体和全磁导体. 另一类散射体是由全电导体构成的具有双周期结构的反射式光栅.针对第一类散射体, 我们证明了一束入射电磁波就能够保证唯一性. 针对第二类散射体, 我们找到并且刻画出了不能被一束给定入射波识别的所有的具有多面体结构的反射式光栅. 这样的话, 我们就能很容易的找出能够唯一确定一个具有多面体结构的反射式光栅的入射波的最少数目.

Abstract

In inverse electromagnetic scattering problems, one tries to determine the scatterer, which can be an obstacle or a diffractive grating, from measured data of scattered fields by some incident electromagnetic waves. The problems have found wide applications in areas such as radar, nondestructive testing and diffractive optics. One of the most important issues in inverse scattering problem is uniqueness, namely, to find the minimum number of incident waves that can uniquely determine the unknown scatterer. It is remarked that uniqueness for the general problems has been a longstanding open problem. By restricting the scatterers to some special classes, significant progress has been achieved in the past decade. In this thesis, we will review these progresses and provide some new results on this topic. More specifically, we consider two kind of scatterers: the first one is the scatterer that consists of finitely many polyhedra of mixed perfect electric conductors and perfect magnetic conductors, the other one is bi-periodic reflection grating that ruled on perfect electric conductor. We show in the first case that one incident electromagnetic wave is enough to guarantee the uniqueness. In the latter case, we are able to find and characterize all the polyhedral reflection gratings that can not be identified by a given incident wave. Using this result, the minimum number of incident waves required for the unique determination of a polyhedral reflection grating can be readily read out.

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Chapter 1

Introduction

This work is concerned with the uniqueness on the determination of scatterers by time-harmonic electromagnetic plane waves. We consider two kind of scatterers, which correspond to two types of scattering problems. Of the first kind are scatterers that consist of finitely many bounded polyhedra of mixed perfect electric conductors and perfect magnetic conductors. This kind of scatterers are generally referred to as impenetrable obstacles. Of the second kind are diffractive gratings. We assume these gratings have bi-periodic polyhedral structures and their profiles are ruled on perfect electric conductors. To the first type of scattering problem, we show that one incident wave is sufficient to uniquely determine the obstacle. While in the second type of scattering problems, it is well known that the uniqueness may not hold when only one incident wave is used. So the central issue of the problem is to find the minimum number of incident waves that can uniquely determine the gratings. Based on whether Rayleigh frequencies occur or not, we divide the analysis into two part: we first consider the simple case that excludes Rayleigh frequencies. Then we consider the general case with Rayleigh frequencies included. We find in the first case that, corresponding to each incident plane wave, there are three classes of gratings that can not be uniquely determined. We also show that any bi-periodic polyhedral grating can be uniquely determined by one incident wave if and only if it belongs to neither of the three classes. In the latter case, we find seven classes of unidentifiable gratings corresponding to each incident plane wave, where the four additional classes are related to the case when Rayleigh frequency occurs. Using these results, the minimum number of incident plane waves required for the unique determination of a bi-periodic polyhedral structure can be readily read out.

Let us first take a brief review of the first type of scattering problems, i.e. obstacle scattering problems. Let $\mathbf{D} \subset \mathbb{R}^3$ be an impenetrable obstacle that consists of finitely many disjoint bounded solid polyhedra. Let $(\mathbf{E}^i(x), \mathbf{H}^i(x))$ be an incident time-harmonic electromagnetic wave. Then the forward scattering problem is described by the following time-harmonic Maxwell's equations (see [13]):

$$\operatorname{curl}\mathbf{E} - \mathrm{i}\,k\,\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{H} + \mathrm{i}\,k\,\mathbf{E} = 0 \quad \text{in} \quad \mathbf{G} := \mathbb{R}^3 \backslash \bar{\mathbf{D}}, \tag{1.0.1}$$

$$\lim_{|x|\to\infty} (\mathbf{H}^s \times x - |x|\mathbf{E}^s) = 0, \tag{1.0.2}$$

where $(\mathbf{E}, \mathbf{H})$ are respectively the total electric and magnetic fields formed by the incident fields $\mathbf{E}^i(x)$, $\mathbf{H}^i(x)$ and scattered fields $\mathbf{E}^s(x)$ and $\mathbf{H}^s(x)$:

$$\mathbf{E}(x) = \mathbf{E}^i(x) + \mathbf{E}^s(x), \quad \mathbf{H}(x) = \mathbf{H}^i(x) + \mathbf{H}^s(x)\,.$$

For a perfect electric conducting obstacle $\mathbf{D}$, we have the boundary condition

$$\nu \times \mathbf{E} = 0 \qquad \text{on} \qquad \partial\mathbf{D},$$

where ν is the outward normal to $\partial\mathbf{D}$ directing to the exterior of $\mathbf{D}$. Similarly, the boundary condition reads as

$$\nu \times \mathbf{H} = 0 \qquad \text{on} \qquad \partial\mathbf{D}$$

for a perfect magnetic conducting obstacle. For a general impenetrable obstacle, we have the following impedance boundary condition

$$\nu \times \operatorname{curl}\mathbf{E} - i\lambda(\nu \times \mathbf{E}) \times \nu = 0 \qquad \text{on} \qquad \partial\mathbf{D}$$

where λ is a positive constant.

In this work, we deal with the mixed perfect electric conducting and perfect magnetic conducting boundary condition. To this end, we let $\partial\mathbf{D}$ have a Lipschitz dissection $\partial\mathbf{D} = \Gamma_E \cup \Sigma \cup \Gamma_H$, where Γ_E and Γ_H are disjoint, relatively open subsets of $\partial\mathbf{D}$, having Σ as their common boundary (see [34]). Then we complement the direct system (3.1.3)-(3.1.4) with the following general mixed boundary condition

$$\nu \times \mathbf{E} = 0 \qquad \text{on} \quad \Gamma_E; \qquad \nu \times \mathbf{H} = 0 \qquad \text{on} \quad \Gamma_H. \tag{1.0.3}$$

For convenience, we write $\mathcal{B}[\mathbf{E}, \mathbf{H}] = 0$ for the mixed boundary condition (1.0.3).

It is known that the scattered field $\mathbf{E}^s$ and $\mathbf{H}^s$ has the following asymptotic behaviors

$$\mathbf{E}^s(x; \mathbf{D}) = \frac{e^{ik|x|}}{|x|}\left\{\mathbf{E}_\infty(\hat{x}; \mathbf{D}) + \mathcal{O}(\frac{1}{|x|})\right\} \quad \text{as} \quad |x| \to \infty, \tag{1.0.4}$$

$$\mathbf{H}^s(x; \mathbf{D}) = \frac{e^{ik|x|}}{|x|}\left\{\mathbf{H}_\infty(\hat{x}; \mathbf{D}) + \mathcal{O}(\frac{1}{|x|})\right\} \quad \text{as} \quad |x| \to \infty, \tag{1.0.5}$$

uniformly for all $\hat{x} = x/|x| \in \mathbb{S}^2$. The functions $\mathbf{E}_\infty(\hat{x})$ and $\mathbf{H}_\infty(\hat{x})$ in (3.1.6) and (3.1.7) are called, respectively, the electric and magnetic far field patterns, and both are analytic on the unit sphere $\mathbb{S}^2$.

Given the incident field $\mathbf{E}^i$ and the obstacle $\mathbf{D}$, the direct problem is to determine the scattered field $\mathbf{E}^s$ and $\mathbf{H}^s$ or the far field pattern associated with them. The inverse problem, on the other hand, is concerned with the determination of the obstacle $\mathbf{D}$ from the knowledge of the scattered far field pattern $\mathbf{E}_\infty(\hat{x}; \mathbf{D})$. This inverse problem is of fundamental importance in exploring objects by electromagnetic waves and we refer to [13] for a detailed discussion. One of the most important issues in inverse scattering problem is the uniqueness, namely, is the correspondence between $\mathbf{E}_\infty(\hat{x}; \mathbf{D})$ (or equivalently, $\mathbf{H}_\infty(\hat{x}; \mathbf{D})$) and $\mathbf{D}$ one to one?

As far as many measurements are available, we have the following results, see [13], which is based on the idea of Isakov's point source method in acoustic obstacle scattering problem.

Theorem 1.0.1. *Assume that* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *are two perfect conductors such that for a* *fixed wave number the electric far field patterns for both scatters coincide for all incident plane waves (with all incident directions and all polarization). Then* $\mathbf{D}_1 = \mathbf{D}_2$.

Theorem 1.0.2. *Assume that* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *are two perfect conductors such that for one fixed incident direction and polarization the electric far field patterns for both scatters coincide for all wave numbers contained in some interval* $0 < k_1 < k < k_2 < \infty$. *Then* $\mathbf{D}_1 = \mathbf{D}_2$.

However, it is easy to see that the obstacle is formally determined with only one single far-field measurement, namely, the far-field pattern corresponding to a single incident wave. Hence, one may anticipate the uniqueness by using the far field data from only one or at most a finite number of incident waves. It is the similar situation as that in the inverse acoustic obstacle scattering, where one utilizes acoustic far field patterns to identify the unknown object.

The uniqueness for the inverse electromagnetic scattering problem with optimal measurement data remains an open problem (see [12]). Recently, significant progress has been achieved for the unique determination of polyhedral type scatterers in inverse acoustic scattering by means of a single or several incident waves (see [2, 10, 16, 17, 28, 31, 32, 33]). The proofs are based on various reflection principles for Helmholtz equation, in combination with suitably devised techniques, especially the path argument developed in [31]. The study for inverse acoustic scattering problems in this direction is nearly completed. Along this line, some novel reflection principles were derived in [29] and [30] for time-harmonic Maxwell's equations. They were then applied to establish some uniqueness results for the inverse electromagnetic scattering problems by using similar techniques as those developed for inverse acoustic scattering problems. Particularly, in [27] it is proved that a single incident electromagnetic wave is sufficient to uniquely recover a polyhedral obstacle with pure PEC or PMC boundary condition. The crucial idea for the path argument in [27] is to find an 'exit path' which has at most one intersection point with the "unbounded" perfect planes. We would like to remark that similar idea was also developed in [17], where uniqueness of determining a polyhedral acoustic sound-hard obstacle is proved with a single acoustic far-field measurement. The argument relies on an 'exit path' which avoids intersection with the "unbounded" Neumann planes. However, those techniques would not work for the more challenging case considered in this work, namely to recover a general polyhedral obstacle associated with the mixed boundary condition (3.1.5) by electromagnetic scattering measurement corresponding to a single incident wave, which appears to be the only problem that remains unsolved on the uniqueness in determining a general polyhedral obstacle by a single far-field measurement. We also refer to the concluding remarks in [27] for a discussion on what challenges one shall encounter when proving uniqueness in the setting posed in this paper. In realizing that the existence an "unbounded" perfect plane implies certain symmetries of the underlying scatterer, we prove the uniqueness by developing some novel and more elaborate arguments.

Now, we turn to the second type of scattering problems, or more precisely, the diffraction problems of electromagnetic wave by gratings. The grating, with profile S, we considered here has bi-periodic polyhedral structure, and is ruled on a perfect conductor. The medium above S is assumed to be homogenous with a constant dielectric coefficient $\epsilon_0 > 0$ and magnetic permeability $\mu_0 > 0$, and the corresponding region is denoted by Ω. Let $E^i(x) = se^{iq\cdot x}$ (with time dependence $e^{-i\omega t}$) be an incident time-harmonic electromagnetic wave incident to the grating structure S from above. Let E be the total field, which consisting of the incident field and the scattered field. Then the scattering problem can be modeled by the following vector-valued Helmholtz system:

$$\Delta E + k^2 E = 0 \quad in \quad \Omega, \tag{1.0.6}$$

$$\text{div} E = 0 \quad in \quad \Omega, \tag{1.0.7}$$

$$\nu \times E = 0 \quad on \quad S \tag{1.0.8}$$

where ν is the unit outward normal vector to the surface S.

The solution to above equations is assumed to be quasi-periodic and satisfies bounded out going radiation condition. From the knowledge of the fundamental solution to the periodic Helmholtz equation (cf. [7] [15]) it is known that E can be expressed in the following form:

$$E(x) = E^i(x) + \sum_{n \in \mathbf{Z}^2} A^n e^{iq^n \cdot x}, \tag{1.0.9}$$

where all A^n's are complex vectors, called the *Rayleigh coefficients*, and q^n's can be calculated by the incident field and the period.

Given the periodic structure S and the incident field E^i, the forward diffraction problem is to solve for the system (4.1.1)-(4.1.4) the total field E. The direct diffraction problem has been well studied mathematically, see, e.g., [15] [5] [6]. This work is concerned with an inverse problem associated with the system (4.1.1)-(4.1.4). For a given incident wave E^i, assume that the total field E can be measured on a plane Γ_b above the structure S, we want to find out how many measurements taken on the plane Γ_b are needed to uniquely determine the shape and position of the structure S. This inverse problem occurs in many applications, e.g., in diffractive optics; see [35].

It is well known that global uniqueness with one incident wave is generally not true. This can be seen from a simple example that when one incident wave is given, two grating profiles, both parallel to the plane $\{x_3 = 0\}$ with distance of a certain multiple of the wavelength of the incident wave, generate the same field E in the domain above Γ_b. As far as the general periodic grating profiles are concerned, the global uniqueness of this inverse problem still remains open. However, when it is confined to some special classes of periodic structures, important progress has been made in recent years, particularly in the two dimensional case. Hettlich and Kirsch showed in [21] that a finite number of incident waves is sufficient to identify a C^2-smooth periodic structure in two dimensions. In a series of works by Elschner, Schmidt and Yamamoto; see [18], [19] and [20], the global uniqueness problem for the class of periodic polygonal structures was studied, and the minimal number of incident waves to ensure the global uniqueness was obtained.

To our knowledge, there are only two uniqueness results for the aforementioned inverse problem in three dimensional case, both for gratings which can be described by graphs of C^2 $\Lambda-$periodic functions. Local uniqueness was established in [7], where a crucial step was to estimate a lower bound of the first eigenvalue to the problem (4.1.1) in a convex domain. Global uniqueness result was obtained in [3] with one incident plane wave when the medium above the perfect conductor is assumed to be lossy, i.e., k has non-zero imaginary part; while for the case when k is real, global uniqueness was established with infinite many incident waves.

In this work we will restrict ourselves to the Lipschitz polyhedral gratings, so the grating structure S is assumed to be a Lipschitz polyhedral surface which is bi-periodic of period Λ. We further divide the analysis into two part. In the first part, we deal with the case where Rayleigh frequencies are excluded, which accounts for that there is no scattered plane wave that is propagating parallel to the grating. In this case, we are able to identify, for each given incident wave, three classes of polyhedral gratings and demonstrate that these are all the possible polyhedral gratings which can not be uniquely determined by the incident wave. In the second part, we deal with the general case, that is without excluding Rayleigh frequencies. As a result, we are able to identify, for each given incident wave, seven classes of unidentifiable polyhedral gratings, where four newly appeared classes are related to the case when Rayleigh frequencies occurs. Using these results, we can determine the minimum number of incident waves to uniquely determine a polyhedral grating. A major technique used is the group and dihedral group theory, which is for the first time introduced in the study of uniqueness for inverse scattering problems, and turns out to be extremely helpful to characterize the unidentifiable periodic structures.

The thesis is organized as follows. In chapter two, we first review some basic knowledge about Maxwell equations, then we introduce the most important tool, namely the Reflection Principle for Maxwell equations, that are frequently referred to in the subsequent analysis, the fundamental concepts, perfect set and perfect plane are also presented. In chapter three we consider the obstacle scattering problems. Both direct and inverse problem are discussed, but with a focus on the uniqueness of the associated inverse problem. The discussion for scattering problems by gratings in presented in chapter four and five, where we deal with the simple case when Rayleigh frequencies are excluded in chapter four and the general case with Rayleigh frequencies included in chapter five. Again, we are mainly concerned with the uniqueness of the associated inverse problem.

Chapter 2

Preliminaries

In this chapter, we will first review, in Section 2.1, some basic knowledge about the Maxwell equations, which models the propagation of the electromagnetic wave in vacuum. We consider the time-harmonic case, where it can be deduced to space dependent form, which can further be shown to be equivalent to vector valued Helmholtz equations. This can facilitates our analysis in chapter four and five by considering only the electric fields. Then, in Section 2.2, we introduce the most important tool, Reflection Principle, more precisely, Reflection Principle for Maxwell equations, for the whole work. Since Reflection Principle can be regarded as a consequence of unique continuation, we start from Holmgren's uniqueness theorem for scalar Holmheltz equation. We then extend this result to Maxwell equations, which enables us to deduce the Reflection Principle for Maxwell equations, just as in the case of scalar Holmheltz equation. Besides, the fundamental concepts for the work, perfect plane and perfect set are also introduced.

2.1 Maxwell equations

Consider the electromagnetic wave propagation in an isotropic medium in $\mathbf{R}^3$ with space independent electric permittivity ε and magnetic permeability μ. Here we assume that the medium has vanishing conductivity. The electromagnetic wave is described by the electric field $\mathcal{E}$ and the magnetic field $\mathcal{H}$ satisfying the following Maxwell equations:

$$\begin{aligned}\operatorname{curl}\mathcal{E} + \mu\frac{\partial\mathcal{H}}{\partial t} &= 0,\\ \operatorname{curl}\mathcal{H} - \varepsilon\frac{\partial\mathcal{E}}{\partial t} &= 0.\end{aligned}$$

For the time-harmonic electromagnetic wave with frequency $\omega > 0$, we can write $\mathcal{E}$ and $\mathcal{H}$ by

$$\begin{aligned}\mathcal{E}(x,t) &= Re\{\sqrt{\varepsilon}E(x)e^{-i\omega t}\},\\ \mathcal{H}(x,t) &= Re\{\sqrt{\mu}H(x)e^{-i\omega t}\}.\end{aligned}$$

Then one can deduce that the complex valued space dependent parts E and H satisfy the reduced Maxwell equations

$$\operatorname{curl} E - ik\, H = 0, \tag{2.1.1}$$

$$\operatorname{curl} H + ik\, E = 0. \tag{2.1.2}$$

where k, referred to as wave number, is a constant given by

$$k = \omega\sqrt{\varepsilon\mu}.$$

Sometimes, it is useful to consider only the electric field E in (2.1.1) and (2.1.2). That is what we do in chapter four and five. By a simple application of the vector identities

$$\operatorname{curl}\operatorname{curl} H = -\Delta H + \operatorname{grad}\operatorname{div} H, \qquad \operatorname{div}\operatorname{curl} H = 0,$$

we can show the following result which connects Maxwell equations to vector valued Helmholtz equations.

Theorem 2.1.1. *Let E, H be a solution to the Maxwell equations (2.1.1) and (2.1.2). Then E satisfies:*

$$\Delta E + k^2 E = 0, \tag{2.1.3}$$

$$\operatorname{div} E = 0. \tag{2.1.4}$$

Conversely, let E be a solution to the equations (2.1.3), (2.1.4) above. Then E and $H := \operatorname{curl} E/ik$ satisfy the Maxwell equations (2.1.1) and (2.1.2).

2.2 Reflection principle

In this section, we will present Reflection Principle for Maxwell equations. Before we do this, we first introduce Holmgren's uniqueness theorem for scale Holmheltz equation.

Theorem 2.2.1. *Let Ω be a bound Lipschitz domain in $\mathbf{R}^3$, and let $u \in H^1(\Omega)$ be a solution to the following Helmholtz equation:*

$$\Delta u + k^2 u = 0 \qquad x \in \Omega \tag{2.2.1}$$

Assume that $u|_\Gamma = \frac{\partial u}{\partial n}|_\Gamma = 0$, where Γ is an open set of $\partial\Omega$. Then $u \equiv 0$ in Ω.

For proof and general form of Holmgren's uniqueness theorem, see e.g. [36].

Theorem 2.2.1 has an easy extension to Maxwell equations.

Lemma 2.2.1. *Let Ω be a bound Lipschitz domain in $\mathbf{R}^3$, and let (E, H) be a solution to the following Maxwell equation:*

$$\begin{aligned} \operatorname{curl} E - ikH &= 0 \qquad in \quad \Omega, \\ \operatorname{curl} H + ikE &= 0 \qquad in \quad \Omega. \end{aligned}$$

Assume both $E \times n$ and $H \times n$ vanish on Γ, an open set of $\partial\Omega$. Then E and H vanish identically in Ω.

The proof can be found in [1] and [29], where the later proof makes use of Theorem 2.2.1.

We next introduce two concepts, perfect sets and perfect planes, which are crucial for our analysis in the whole work. Their remarkable properties are stated in Reflection Principle 2.2.2.

Definition 2.2.1. *Let $F : O \to \mathbf{C}^3$ be a given analytic complex vector-valued function in a domain $O \subset \mathbf{R}^3$, $\mathcal{P}_F$ is called the perfect set of F if*

$$\begin{aligned} \mathcal{P}_F = \{x \in O;\ \ \nu \times F\,|_{\Pi \cap B_r(x) \cap O} = 0 \ for\ some\ r > 0 \\ and\ plane\ \Pi\ passing\ through\ x\}. \end{aligned}$$

The points in $\mathcal{P}_F$ are called perfect points of F. For any $x \in \mathcal{P}_F$, we denote by Π the plane involved in the definition of $\mathcal{P}_F$. Furthermore, we let $\widetilde{\Pi}$ be the connected component of $\Pi \cap O$ containing x, then by the analyticity of F and analytic continuation, we have $\nu \times F = 0$ on $\widetilde{\Pi}$. In the sequel, such $\widetilde{\Pi}$ will be referred to as a *perfect plane* of F. We also use $\mathcal{P}_F$ to denote the set of perfect planes of F whenever there is no confusion caused.

Finally, we come to the Reflection Principle for Maxwell equations; See also [29].

Lemma 2.2.2. *[Reflection Principle] Let O be a domain which is symmetric with respect to a plane Π in $\mathbf{R}^3$ and (E, H) be the electromagnetic field in O satisfying:*

$$\operatorname{curl} E - ikH = 0 \quad in \quad \Omega, \tag{2.2.2}$$

$$\operatorname{curl} H + ikE = 0 \quad in \quad \Omega. \tag{2.2.3}$$

Then $\tilde{\Pi} \subset \Pi$ is a perfect plane of E if and only if the following relations hold

$$E(x) + R'_\Pi(E(R_\Pi(x))) = 0 \quad in \quad O. \tag{2.2.4}$$

Here and throughout the thesis, we use R_Π to denote the reflection in $\mathbb{R}^3$ with respect to a plane Π and R'_Π the linear part of the affine map R_Π. In addition, if $\Gamma \subset O$ or $\Gamma \subset \partial O$ is a perfect plane of E, then $R_\Pi(\Gamma)$ is also a perfect plane of E.

Proof: We only show the part that the equality (2.2.4) holds if $\tilde{\Pi} \subset \Pi$ is a perfect plane of E, for the other parts follows easily. We first introduce a few notations. Let $s^{l,p,q}$ denote the Levi-Civita permutation symbol, that is $s^{l,p,q} = 1$ if (l, p, q) is an even permutation of $(1, 2, 3)$, $s^{l,p,q} = -1$ if (l, p, q) is n odd permutation of $(1, 2, 3)$ and $s^{l,p,q} = 0$ otherwise. Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ be two vectors in $\mathbf{R}^3$, then the cross product of a and b, denoted by $a \times b = ((a \times b)^1, (a \times b)^2, (a \times b)^3)$, is calculated by the rule

$$(a \times b)^l = s^{l,p,q} a_p b_q.$$

Here and what follows, we use Einstein summation convention. We can extend the above identity from real numbers a_i, b_j to symbols that can represent operators and functions such as $\frac{\partial}{\partial x^i}$ and F_j. As an application, the curl of a vector field $F = (F_1, F_2, F_3)$ in $\mathbf{R}^3$ can be calculated by

$$\operatorname{curl} F(x) = \nabla_x \times F,$$

with $(\nabla_x \times F)^l = s^{l,p,q} \frac{\partial}{\partial x^p} F_q = s^{l,p,q} \frac{\partial F_q}{\partial x^p}$. Here ∇_x represents for $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$.

Now, we define

$$\begin{aligned} V(x) &= R'_\Pi E(R_\Pi x), \\ W(x) &= -R'_\Pi H(R_\Pi x). \end{aligned}$$

We want to show that (V, W) satisfies the Maxwell equations (2.2.2)-(2.2.3). For this, we need to compute $\operatorname{curl} V$ and $\operatorname{curl} W$. We do this as follows.

Let $y = (y^1, y^2, y^3) = R_\Pi x = R'_\Pi x + R_\Pi 0 = Tx + R_\Pi 0$, where we write T for R'_Π for simplicity. We further write $Tx = ((Tx)^1, (Tx)^2, (Tx)^3) = (T^1_k x^k, T^2_k x^k, T^3_k x^k)$. Noting that $T = R'_\Pi$ is a symmetric matrix, we have $T^i_k = T^k_i$. Thus

$$T(E) = (T(E)_1, T(E)_2, T(E)_3) = (T^1_k E_k, T^2_k E_k, T^3_k E_k) = (T^k_1 E_k, T^k_2 E_k, T^k_3 E_k)$$

We also let $\nabla_y = (\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3})$.

After the preparation above, we can derive that

$$\begin{aligned} (\operatorname{curl} V|_x)^l &= (\nabla_x \times T(E(R_{\Pi x}))|_x)l = s^{l,p,q} \frac{\partial (T(E))_q (R_\Pi x)}{\partial x^p}|_x \\ &= s^{l,p,q} T^k_q \frac{\partial E_k(R_\Pi x)}{\partial x^p}|_x = s^{l,p,q} T^k_q \frac{\partial E_k(y)}{\partial y^i} \frac{\partial y^i}{\partial x^p}|_{y=R_\Pi x} \\ &= s^{l,p,q} T^k_q T^i_p \frac{\partial E_k(y)}{\partial y^i}|_{y=R_\Pi x} = s^{l,p,q} (T^i_p \frac{\partial}{\partial y^i})(T^k_q E_k(y))|_{y=R_\Pi x} \\ &= s^{l,p,q} (T(\nabla_y))_p (T(E(y))_q)|_{y=R_\Pi x} \\ &= (T(\nabla_y) \times T(E(y))|_{y=R_\Pi x})^l = ((\det T)\,(T^t)^{-1} (\nabla_y \times E(y))|_{y=R_\Pi x})^l \\ &= -(T(\nabla_y \times E(y))|_{y=R_\Pi x})^l. \end{aligned}$$

In the last but one step above, we used the equality that

$$Ta \times Tb = (\det T)\,(T^t)^{-1}(a \times b),$$

which has the equivalent form

$$s^{lpq}T^i_p a_i T^j_q b_j = (\det T) G^i_k s^{kpq} a_p b_q,$$

with $G = (T^t)^{-1} = (G^i_j)$. While in the last step, we used the fact that $\det T = \det R'_\Pi = -1$, $(T^t)^{-1} = T^{-1} = T$. Here T^t means the transpose of T.

Now, we have shown that $\operatorname{curl} V|_x = -T(\nabla_y \times E(y))|_{y=R_\Pi x}$. Since $\nabla_y \times E(y))|_{y=R_\Pi x} = \operatorname{curl} E|_{R_\Pi x} = ikH(R_\Pi x)$, we have

$$\operatorname{curl} V|_x = -ikR'_\Pi(H(R_\Pi x)) = kW|_x.$$

Similarly, we can derive that

$$\operatorname{curl} W|_x = -ikR'_\Pi(H(R_\Pi x)) = -kV|_x.$$

Thus (V, W) satisfies Maxwell equations (2.2.2)-(2.2.3).

Finally, we define

$$\mathfrak{E}(x) = E(x) + V(x),$$
$$\mathfrak{H}(x) = H(x) + W(x).$$

Then the pair $(\mathfrak{E}, \mathfrak{H})$ satisfies Maxwell equations (2.2.2)-(2.2.3) also. Besides, we have

$$\mathfrak{E} \times \nu_\Pi|_{\tilde{\Pi}} = (E(x) + R'_\Pi E(x)) \times \nu_\Pi|_{\tilde{\Pi}} = 0,$$
$$\mathfrak{H} \times \nu_\Pi|_{\tilde{\Pi}} = (H(x) - R'_\Pi H(x)) \times \nu_\Pi|_{\tilde{\Pi}} = 0.$$

Referring to Lemma 2.2.1, we get $\mathfrak{E} = \mathfrak{H} \equiv 0$. Our desired result follows immediately.

Remark 2.2.1. *By Theorem 2.1.1, we see that Reflection Principle in Lemma 2.2.2 holds when Maxwell equations (2.2.2)-(2.2.3) are replaced by vector valued Helmholtz equation (2.1.3)-(2.1.4). We will need this version of Reflection Principle in chapter four and five.*

Chapter 3

Scattering by General Polyhedral Obstacle

In this chapter, we consider the inverse obstacle scattering problem. We prove that a polyhedral obstacle in $\mathbb{R}^3$ consisting of finitely many polyhedra with mixed perfect electric conductor and perfect magnetic conductor boundary conditions can be uniquely determined by a single electric or magnetic far-field measurement, namely, the far-field pattern corresponding to a single incident wave.

This chapter consists of three sections. In Section 3.1, we first briefly describe the direct scattering problem. Then in Section 3.2, we introduce the associated inverse problem, and present the main result on the uniqueness of the inverse problem for polyhedral obstacles. Finally, we give the proof in Section 3.3.

3.1 Direct problem

Let $\mathbf{D} \subset \mathbb{R}^3$ be an impenetrable obstacle that consists of finitely many disjoint bounded solid polyhedra. We consider the scattering due to the obstacle corresponding to the incident normalized time-harmonic electromagnetic plane waves,

$$\mathbf{E}^i(x) := \frac{\mathrm{i}}{k} \mathrm{curl}\ \mathrm{curl}\, p\, e^{\mathrm{i}kx\cdot d} = \mathrm{i}k(d \times p) \times d e^{\mathrm{i}kx\cdot d}, \tag{3.1.1}$$

$$\mathbf{H}^i(x) := \mathrm{curl}\, p\, e^{\mathrm{i}kx\cdot d} = \mathrm{i}kd \times p e^{\mathrm{i}kx\cdot d}, \tag{3.1.2}$$

where $\mathrm{i} = \sqrt{-1}$, and $p \in \mathbb{R}^3$, $k > 0$ and $d \in \mathbb{S}^2 := \{x \in \mathbb{R}^3; |x| = 1\}$ represents respectively polarization, wave number and direction of propagation. Then the associated forward scattering problem is described by the following time-harmonic Maxwell's equations (see [13]):

$$\mathrm{curl}\,\mathbf{E} - \mathrm{i}\,k\,\mathbf{H} = 0, \quad \mathrm{curl}\,\mathbf{H} + \mathrm{i}\,k\,\mathbf{E} = 0 \quad \text{in} \quad \mathbf{G} := \mathbb{R}^3 \backslash \bar{\mathbf{D}}, \tag{3.1.3}$$

where $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{H} = (H_1, H_2, H_3)$ are respectively the total electric and magnetic fields formed by the incident fields $\mathbf{E}^i(x)$, $\mathbf{H}^i(x)$ and scattered fields $\mathbf{E}^s(x)$ and $\mathbf{H}^s(x)$:

$$\mathbf{E}(x) = \mathbf{E}^i(x) + \mathbf{E}^s(x), \quad \mathbf{H}(x) = \mathbf{H}^i(x) + \mathbf{H}^s(x).$$

The equation system (3.1.3) is complemented by the following Silver-Müller radiation condition

$$\lim_{|x|\to\infty} (\mathbf{H}^s \times x - |x|\mathbf{E}^s) = 0, \tag{3.1.4}$$

to guarantee the uniqueness of solution.

To complete the description, we need to further impose some suitable boundary conditions on $\partial\mathbf{D}$. We are interested in the following two types of boundary conditions, namely, the perfect electric conductor (PEC) boundary condition

$$\nu \times \mathbf{E} = 0 \qquad \text{on} \qquad \partial\mathbf{D},$$

and the perfect magnetic conductor (PMC) boundary condition

$$\nu \times \mathbf{H} = 0 \qquad \text{on} \qquad \partial\mathbf{D},$$

where ν is the outward normal to $\partial\mathbf{D}$ directing to the exterior of $\mathbf{D}$. To appeal for a general study, we consider the mixed PEC and PMC boundary conditions. To this end, we let $\partial\mathbf{D}$ have a Lipschitz dissection $\partial\mathbf{D} = \Gamma_E \cup \Sigma \cup \Gamma_H$, where Γ_E and Γ_H are disjoint, relatively open subsets of $\partial\mathbf{D}$, having Σ as their common boundary (see [34]). Then we complement the direct system (3.1.3)-(3.1.4) with the following general mixed boundary condition

$$\nu \times \mathbf{E} = 0 \quad \text{on} \quad \Gamma_E; \qquad \nu \times \mathbf{H} = 0 \quad \text{on} \quad \Gamma_H. \tag{3.1.5}$$

For convenience, we write $\mathcal{B}[\mathbf{E}, \mathbf{H}] = 0$ for the mixed boundary condition (3.1.5).

Given the incident field $\mathbf{E}^i$, the forward scattering problem is to solve the equation system (3.1.3)-(3.1.5). It is known that the forward scattering problem has a unique solution $(\mathbf{E}, \mathbf{H}) \in H_{loc}(\text{curl}; \mathbf{G}) \times H_{loc}(\text{curl}; \mathbf{G})$ (see [24] and [25], also [8] and [9]). The solution is regular in any neighborhood which does not meet corners and edges of $\mathbf{D}$. The singular behavior is only attached to the corners and edges (see [14]), hence both $\mathbf{E}$ and $\mathbf{H}$ are continuous up to points lying in the interior of the (open) faces of $\mathbf{D}$. Moreover, the Cartesian components of $\mathbf{E}$ and $\mathbf{H}$ are analytic in $\mathbf{G}$ and the asymptotic behaviors of the radiating fields $\mathbf{E}^s$ and $\mathbf{H}^s$ are governed by (see [13])

$$\mathbf{E}^s(x; \mathbf{D}, p, k, d) = \frac{e^{ik|x|}}{|x|}\left\{\mathbf{E}_\infty(\hat{x}; \mathbf{D}, p, k, d) + \mathcal{O}(\frac{1}{|x|})\right\} \quad \text{as} \quad |x| \to \infty, \tag{3.1.6}$$

$$\mathbf{H}^s(x; \mathbf{D}, p, k, d) = \frac{e^{ik|x|}}{|x|}\left\{\mathbf{H}_\infty(\hat{x}; \mathbf{D}, p, k, d) + \mathcal{O}(\frac{1}{|x|})\right\} \quad \text{as} \quad |x| \to \infty, \tag{3.1.7}$$

uniformly for all $\hat{x} = x/|x| \in \mathbb{S}^2$. The functions $\mathbf{E}_\infty(\hat{x})$ and $\mathbf{H}_\infty(\hat{x})$ in (3.1.6) and (3.1.7) are called, respectively, the electric and magnetic far field patterns, and both are analytic on the unit sphere $\mathbb{S}^2$. It is noted above that $\mathbf{E}^s(x; \mathbf{D}, p, k, d)$, $\mathbf{E}_\infty(\hat{x}; \mathbf{D}, p, k, d)$, etc. will be frequently used to specify their dependence on the polarization p, the wave number k and the incident direction d.

3.2 Inverse problem and statement of main results

Now, we turn to the associated inverse scattering problem, which is the determine the obstacle $\mathbf{D}$ by using measurement data of the corresponding electric (or equivalently, magnetic) far-field patterns. This inverse problem has important applications in exploring objects by electromagnetic waves and we refer to [13] for a detailed discussion. One of the most important issues in inverse scattering problem is the uniqueness, namely, how many measurements of $\mathbf{E}_\infty(\hat{x}; \mathbf{D})$ or equivalently, $\mathbf{H}_\infty(\hat{x}; \mathbf{D})$ are needed to uniquely determine the obstacle $\mathbf{D}$? The inverse problem is nonlinear and moreover, severely ill-posed in the sense of Hadamard (see, e.g. [13]). Hence, the uniqueness is of critical importance in both theory and numerics, we refer to [23] for a general discussion. In this thesis, we establish the following result.

Theorem 3.2.1. *Let* $\mathbf{D}$ *be a polyhedral scatterer associated with the mixed boundary conditions* $\mathcal{B}[\mathbf{E}, \mathbf{H}] = 0$ *in (3.1.5). Then both* $\partial\mathbf{D}$ *and* $\mathcal{B}$ *are uniquely determined by the knowledge of* $\mathbf{E}_\infty(\hat{x}; p, k, d)$ *(or equivalently* $\mathbf{H}_\infty(\hat{x}; p, k, d)$*) for* $\hat{x} \in \mathbb{S}^2$ *and fixed* $p \in \mathbb{R}^3$, $k > 0$ *and* $d \in \mathbb{S}^2$ *(with* $p \times d \neq 0$*).*

It is remarked that the uniqueness of inverse electromagnetic scattering problem for the general obstacle with optimal measurement data is still an open problem (see [12]).

We will give a detailed proof of the result above in Section 3.3.

3.3 Proof of the main results

This entire section is devoted to prove Theorem 3.2.1. We organize the section as follows. In Subsection 3.3.1, we introduce some notation and basic concepts that are frequently referred to in the subsequent analysis, we also present the Reflection Principle for Maxwell equations. Then we begin the proof. The basic idea is that, for a given polyhedral obstacle $\mathbf{D}$ and incident electromagnetic plane wave $(\mathbf{E}^i, \mathbf{H}^i)$, we consider the set of perfect planes of the resulted total field $\mathbf{E}$ and $\mathbf{H}$. We divide these perfect planes into two groups, the "bounded" ones and "unbounded" ones. We show that all the unbounded perfect planes of $\mathbf{E}$ or $\mathbf{H}$ lie on one plane if there is any. We also observe that the bounded perfect planes of both $\mathbf{E}$ and $\mathbf{H}$ do not exist. Using this observation, we further derive that the existence of an unbounded perfect plane implies contain symmetry property of the obstacle. These properties of perfect planes were established in Subsection 3.3.2. Based on these results, our main theorem follows easily in Subsection 3.3.3.

3.3.1 Preliminaries

We start with some notations and basic concepts. We denote an open ball in $\mathbb{R}^3$ with center x and radius r by $B_r(x)$, and its boundary by $S_r(x)$. Unless specified otherwise, ν shall always denote the outward normal to the concerned domain, or the normal to a two-dimensional plane in $\mathbb{R}^3$. A curve $\gamma = \gamma(t)(t \geq 0)$ is said to be regular if it is C^1-smooth and $\frac{d}{dt}\gamma(t) \neq 0$.

Throughout the section, we let $k > 0$, $p \in \mathbb{R}^3$ and $d \in \mathbb{S}^2$ be fixed. In order for (3.1.1) and (3.1.2) to give valid incident electric and magnetic fields, we should require that $p \nparallel d$. We denote by $\mathbf{E}(x) := \mathbf{E}(x; \mathbf{D}, p, k, d)$ and $\mathbf{H}(x) := \mathbf{H}(x; \mathbf{D}, p, k, d)$ the total electric and magnetic fields to (3.1.3)-(3.1.5).

Next, we restate the definition of perfect set and perfect plane for the electromagnetic field $\mathbf{E}$ and $\mathbf{H}$.

Definition 3.3.1. *$\mathscr{P}_{\mathbf{E}}$ is called a perfect set of $\mathbf{E}$ in $\mathbf{G} := \mathbb{R}^3\backslash\bar{\mathbf{D}}$ if*

$$\mathscr{P}_{\mathbf{E}} = \{x \in \mathbf{G}; \nu \times \mathbf{E} \,|_{\Pi\cap B_r(x)\cap\mathbf{G}} = 0 \textit{ for some } r > 0 \\ \textit{and plane } \Pi \textit{ passing through } x\}.$$

Similarly, the perfect set $\mathscr{P}_{\mathbf{H}}$ of $\mathbf{H}$ is defined in $\mathbf{G}$.

For any $x \in \mathscr{P}_{\mathbf{E}}$, we denote by Π the plane involved in the definition of $\mathscr{P}_{\mathbf{E}}$. Furthermore, we let $\widetilde{\Pi}$ be the connected component of $\Pi\backslash\bar{\mathbf{D}}$ containing x, then by the analyticity of $\mathbf{E}$ in $\mathbf{G}$, we have $\nu \times \mathbf{E} = 0$ on $\widetilde{\Pi}$ by classical continuation. In the sequel, such $\widetilde{\Pi}$ will be referred to as a *perfect plane* of $\mathbf{E}$. Similarly, the perfect planes of $\mathbf{H}$ are defined. In the following, we also use $\mathscr{P}_{\mathbf{E}}$ and $\mathscr{P}_{\mathbf{H}}$ respectively to denote the sets of perfect planes of $\mathbf{E}$ and $\mathbf{H}$ whenever there is no confusion caused. We set

$\mathscr{P} = \mathscr{P}_{\mathbf{E}} \cup \mathscr{P}_{\mathbf{H}}$ and when a perfect plane is concerned, it is associated either with $\mathbf{E}$ or with $\mathbf{H}$.

For convenience, we also restate the Reflection Principle for Maxwell equations, where the remarkable property concerning perfect planes is included. (cf. [29] and [30] for details).

Theorem 3.3.1. *Let Ω be an open connected set in $\mathbf{G} := \mathbb{R}^3 \backslash \bar{\mathbf{D}}$ which is symmetric with respect to a plane Π, namely, $R_\Pi \Omega = \Omega$. Let $\widetilde{\Pi}$ be an open connected subset of Π such that $\widetilde{\Pi} \subset \Omega$. Then we have the following results:*

(i) Suppose that $\widetilde{\Pi}$ lies on some perfect plane from $\mathscr{P}_{\mathbf{E}} \cup \mathscr{P}_{\mathbf{H}}$ and $\Sigma \subset \partial\Omega$ or $\Sigma \subset \Omega$ is an open subset of a plane such that $\nu_\Sigma \times \mathbf{E} = 0$ (resp. $\nu_\Sigma \times \mathbf{H} = 0$) on Σ. Then $\nu_{\Sigma'} \times \mathbf{E} = 0$ (resp. $\nu_{\Sigma'} \times \mathbf{H} = 0$) on $\Sigma' := R_\Pi \Sigma$.

(ii) $\nu_\Pi \times \mathbf{E}|_{\widetilde{\Pi}} = 0$ (i.e., $\widetilde{\Pi}$ lies on some perfect plane from $\mathscr{P}_{\mathbf{E}}$) iff

$$\mathbf{E}(x) + R'_\Pi(\mathbf{E}(R_\Pi(x))) = 0, \quad \mathbf{H}(x) - R'_\Pi(\mathbf{H}(R_\Pi(x))) = 0, \quad x \in \Omega;$$

(iii) $\nu_\Pi \times \mathbf{H}|_{\widetilde{\Pi}} = 0$ (i.e., $\widetilde{\Pi}$ lies on some perfect plane from $\mathscr{P}_{\mathbf{H}}$) iff

$$\mathbf{E}(x) - R'_\Pi(\mathbf{E}(R_\Pi(x))) = 0, \quad \mathbf{H}(x) + R'_\Pi(\mathbf{H}(R_\Pi(x))) = 0, \quad x \in \Omega.$$

3.3.2 Properties of perfect planes

In this subsection, we will present some useful properties about the perfect planes of the electromagnetic field $\mathbf{E}$ and $\mathbf{H}$, which are crucial for the proof of Theorem 3.2.1.

For the purpose, we first fix a perfect plane $\widetilde{\Pi}$ for our subsequent discussion and let Π be the plane in $\mathbb{R}^3$ containing $\widetilde{\Pi}$. We further localize our investigation by fixing a point $x_0 \in \widetilde{\Pi} \cap \mathbf{G}$ and take a sufficiently small ball $B_{r_0}(x_0) \subset \mathbf{G}$. $B_{r_0}(x_0)$ is divided by $\widetilde{\Pi}$ into two half balls, which we respectively denote by B^+ and B^-. Let $\mathbf{G}^\pm$ be respectively the connected components of $\mathbf{G} \backslash \widetilde{\Pi}$ containing $\mathrm{B}^\pm$, and $\mathbf{\Lambda}^\pm$ respectively the connected components of $\mathbf{G}^\pm \cap R_\Pi(\mathbf{G}^\mp)$ containing $\mathrm{B}^\pm$. Finally, set $\mathbf{\Lambda} = \mathbf{\Lambda}^+ \cup \widetilde{\Pi} \cup \mathbf{\Lambda}^-$ and we see that $\mathbf{\Lambda}$ is a polyhedral domain which is symmetric with respect to Π and

$B_{\tau_0}(x_0) \subset \mathbf{\Lambda}$. By the reflection principle (Theorem 3.3.1(i)), we know $\partial\mathbf{\Lambda} \subset \partial\mathbf{D} \cup \mathscr{P}$. It is observed that the construction of $\mathbf{\Lambda}$ is irrelevant to the choice of x_0 and τ_0 and is only dependent on $\widetilde{\Pi}$. In the following, we shall always write $\mathbf{\Lambda}_{\widetilde{\Pi}}$ to denote the symmetric set constructed as above corresponding to some perfect plane $\widetilde{\Pi}$. Now, we set

$$\mathscr{P}_1 = \{\widetilde{\Pi};\ \widetilde{\Pi} \text{ is a perfect plane with bounded } \mathbf{\Lambda}_{\widetilde{\Pi}}\}, \tag{3.3.1}$$

$$\mathscr{P}_2 = \{\widetilde{\Pi};\ \widetilde{\Pi} \text{ is a perfect plane with unbounded } \mathbf{\Lambda}_{\widetilde{\Pi}}\}. \tag{3.3.2}$$

It is remarked that one always has $\widetilde{\Pi} \in \mathscr{P}_2$ if $\widetilde{\Pi}$ is an unbounded perfect plane. In fact, in such case one can verify directly that the corresponding $\mathbf{\Lambda}_{\widetilde{\Pi}}$ would contain the exterior of a sufficiently large ball containing $\mathbf{D}$. On the other hand, if $\widetilde{\Pi} \in \mathscr{P}_2$ is bounded, $\mathbf{\Lambda}_{\widetilde{\Pi}}$ would contain the exterior of a sufficiently large ball, say B_0, by noting that both $\mathbf{D}$ and $\widetilde{\Pi}$ are bounded. By the reflection principle of Theorem 3.3.1 (i), $\nu \times \mathbf{E} = 0$ or $\nu \times \mathbf{H} = 0$ on $\Pi \backslash \mathrm{B}_0$ depending on whether $\nu \times \mathbf{E} = 0$ or $\nu \times \mathbf{H} = 0$ on $\widetilde{\Pi}$. That is, a bounded $\widetilde{\Pi} \in \mathscr{P}_2$ implies the existence of some unbounded perfect planes which are coplanar to $\widetilde{\Pi}$, and in this sense, it is essentially "unbounded". Denote

$$\mathscr{P}_{2,\mathbf{E}} = \mathscr{P}_2 \cap \mathscr{P}_{\mathbf{E}} \qquad \text{and} \qquad \mathscr{P}_{2,\mathbf{H}} = \mathscr{P}_2 \cap \mathscr{P}_{\mathbf{H}}.$$

In the subsequent discussion and throughout the rest of the paper, we denote by $\widetilde{\Pi}_l$, with l being an integer, a perfect plane, and Π_l the entire plane in $\mathbb{R}^3$ containing $\widetilde{\Pi}_l$. As we did earlier, we can construct a symmetric domain $\mathbf{\Lambda}_{\widetilde{\Pi}_l}$ associated with the perfect plane $\widetilde{\Pi}_l$.

Lemma 3.3.1. *If $\mathscr{P}_{2,\mathbf{E}}$(resp. $\mathscr{P}_{2,\mathbf{H}}$) is not empty, then there exists a plane $\Pi_{\mathbf{E}}$(resp. $\Pi_{\mathbf{H}}$) in $\mathbf{R}^3$ such that $\mathscr{P}_{2,\mathbf{E}} \subset \Pi_{\mathbf{E}}$(resp. $\mathscr{P}_{2,\mathbf{H}} \subset \Pi_{\mathbf{H}}$). Moreover if both $\mathscr{P}_{2,\mathbf{E}}$ and $\mathscr{P}_{2,\mathbf{H}}$ are not empty, we have $\Pi_{\mathbf{E}} \perp \Pi_{\mathbf{H}}$.*

Proof. Assume that $\mathscr{P}_{2,\mathbf{E}}$ is not empty. First, we claim that all the perfect planes in $\mathscr{P}_{2,\mathbf{E}}$ are perpendicular to the vector $(d \times p) \times d$. To see this, let $\widetilde{\Pi}$ be one such plane in $\mathscr{P}_{2,\mathbf{E}}$. By the arguments following equation (3.3.2), we know that $\widetilde{\Pi}$ are essentially "unbounded". Hence, without loss of generality, we may assume that $\widetilde{\Pi}$ is unbounded.

By the definition of a perfect plane we know $\nu_{\widetilde{\Pi}} \times \mathbf{E}(x) = 0$ for $x \in \widetilde{\Pi}$, then one can directly verify that $\nu_{\widetilde{\Pi}} \times ((d \times p) \times d) = 0$ by noting the fact that $\mathbf{E}(x) = \mathbf{E}^i(x) + \mathbf{E}^s(x)$ and $\mathbf{E}^s(x) \to 0$ as $|x| \to \infty$, hence we have $(d \times p) \times d \perp \widetilde{\Pi}$, and our claim follows.

Similarly we can prove that all the perfect planes in $\mathscr{P}_{2,\mathbf{H}}$ are perpendicular to the vector $d \times p$.

Next, we show that all the perfect planes in $\mathscr{P}_{2,\mathbf{E}}$ lie on one plane in $\mathbb{R}^3$, which we denote by $\Pi_{\mathbf{E}}$. Assume contrarily that there are two perfect planes $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ belonging to $\mathscr{P}_{2,\mathbf{E}}$ such that $\Pi_1 \neq \Pi_2$. As we did earlier, it may be assumed again that both $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ are unbounded. By applying the reflection principle in Theorem 3.3.1 (i) repeatedly, we can get a sequence of perfect planes $\widetilde{\Pi}_l \in \mathscr{P}_{2,\mathbf{E}}, l = 1, 2, 3, \ldots,$ such that $\widetilde{\Pi}_l = R_{\Pi_{l-1}}(\widetilde{\Pi}_{l-2}) \cap \mathbf{G}, l = 3, 4, \ldots$. By the previously proved result, we know that these planes $\widetilde{\Pi}_l$ are perpendicular to the same vector $(d \times p) \times d$, so they must be all parallel to each other and equidistant. This implies the existence of a perfect plane, say $\widetilde{\Pi}_{l_0}$, from this sequence such that the scatterer $\mathbf{D}$ lies entirely on one side of Π_{l_0}. Then by the definition of a perfect plane and the fact that $\Pi_{l_0} \subset \mathbf{G}$, we know $\widetilde{\Pi}_{l_0} = \Pi_{l_0}$. Now, using the reflection principle again, we know that all the faces of $R_{\Pi_{l_0}}(\mathbf{D})$ are either in $\mathscr{P}_{2,\mathbf{E}}$ or in $\mathscr{P}_{2,\mathbf{H}}$, so they are perpendicular to the vector $d \times p$ or the vector $(d \times p) \times d$ by the results proved in the first part. This is impossible since $R_{\Pi_{l_0}}(\mathbf{D})$ is the reflection of the scatterer $\mathbf{D}$, which consists of finitely many solid polyhedra.

The case when $\mathscr{P}_{2,\mathbf{H}}$ is not empty can be treated exactly in the same manner as for $\mathscr{P}_{2,\mathbf{E}}$ above.

The result $\Pi_{\mathbf{E}} \perp \Pi_{\mathbf{H}}$ follows from the facts that $\Pi_{\mathbf{E}} \perp ((d \times p) \times d)$ and $\Pi_{\mathbf{H}} \perp (d \times p)$. □

Lemma 3.3.2. *The open sets $\mathbf{G} \backslash \bar{\mathscr{P}}_{2,\mathbf{E}}$ and $\mathbf{G} \backslash \bar{\mathscr{P}}_{2,\mathbf{H}}$ have no bounded connected components.*

Proof. By contradiction, assume that $\mathbf{G}_1$ is a bounded connected component of $\mathbf{G} \backslash \bar{\mathscr{P}}_{2,\mathbf{E}}$, then $\partial \mathbf{G}_1 \subset \partial \mathbf{D} \bigcup \mathscr{P}_{2,\mathbf{E}}$. Now we claim that $\partial \mathbf{G}_1 \not\subset \partial \mathbf{D} = \partial \mathbf{G}$, otherwise $\partial \mathbf{G}_1 \subset \partial \mathbf{G}$ which will lead to a contradiction that $\mathbf{G}_1 = \mathbf{G}$. If $\mathbf{G}_1 \neq \mathbf{G}$, we may take one point

$x \in \mathbf{G} \setminus \mathbf{G}_1$, and another point $\tilde{x} \in \mathbf{G}_1 \subset \mathbf{G}$, then we can find a path lying completely in $\mathbf{G}$ that connects x and $\tilde{x}$ by the connectedness of $\mathbf{G}$. Noting the boundedness of $\mathbf{G}_1$, the path has an intersection point with $\partial\mathbf{G}_1$, and clearly the intersection point lies in $\partial\mathbf{G}_1$ but not in $\partial\mathbf{G}$, contradicting to the relation that $\partial\mathbf{G}_1 \subset \partial\mathbf{G}$.

Clearly the above claim and the relation that $\partial\mathbf{G}_1 \subset \partial\mathbf{D} \bigcup \mathscr{P}_{2,\mathbf{E}}$ indicates the existence of an open face $\mathbf{F}$ of $\partial\mathbf{G}_1$ such that $\mathbf{F}$ lies on a perfect plane from $\mathscr{P}_{2,\mathbf{E}}$, say $\widetilde{\Pi}_0$. Associated with $\widetilde{\Pi}_0$, we can construct a symmetric domain $\mathbf{\Lambda}_{\widetilde{\Pi}_0}$; see the construction in the paragraph right after Theorem 3.3.1. We have $\mathbf{\Lambda}_{\widetilde{\Pi}_0} \subset \overline{\mathbf{G}_1 \cup R_{\Pi_0}\mathbf{G}_1}$ by its construction. Since $\mathbf{G}_1$ is bounded by the assumption, so is $\mathbf{\Lambda}_{\widetilde{\Pi}_0}$. Thus by the definition of $\mathscr{P}_{1,\mathbf{E}}$ we know $\widetilde{\Pi}_0 \in \mathscr{P}_{1,\mathbf{E}}$, which contradicts to the fact that $\widetilde{\Pi}_0 \in \mathscr{P}_{2,\mathbf{E}}$. This completes the proof. □

Lemma 3.3.3. *If $\mathscr{P}_1 \neq \emptyset$, then $\mathscr{P}_2 \neq \emptyset$.*

Proof. We assume contrarily that $\mathscr{P}_1 \neq \emptyset$ while $\mathscr{P}_2 = \emptyset$. Let $\widetilde{\Pi}_1 \in \mathscr{P}_1$ and $\gamma(t)(t \geq 0)$ be a regular curve such that $\gamma(t_1) \in \widetilde{\Pi}_1$ with $t_1 = 0$, $\gamma(t > 0) \subset \mathbf{G}\backslash\widetilde{\Pi}_1$ and γ connects to infinity. Noting that $\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ is bounded, we have $\gamma \cap \partial\mathbf{\Lambda}_{\widetilde{\Pi}_1} \neq \emptyset$. Let $x_2 = \gamma(t_2)$ be the 'last' intersection point of γ with $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$; namely, $t_2 = \max\{t > 0; \gamma(t) \in \partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}\}$, and this implies the existence of a perfect plane $\widetilde{\Pi}_2$ passing though x_2 which is extended from an open face of $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ in $\mathbf{G}$. It is clear that $B_{\tau_0}(\gamma(t_1)) \subset \mathbf{G}$, where $\tau_0 := dist(\gamma, \mathbf{D})/2 > 0$. Therefore, one can verify directly that $B_{\tau_0}(\gamma(t_1)) \subset \mathbf{\Lambda}_{\widetilde{\Pi}_1}$, which then implies $|\gamma(t_1 \leq t \leq t_2)| > \tau_0$. By further noting $\mathscr{P}_2 = \emptyset$, we have $\widetilde{\Pi}_2 \in \mathscr{P}_1$. Continuing with the above arguments, we can construct a sequence of perfect planes $\widetilde{\Pi}_n, n = 2, 3, \ldots$, and a strictly increasing sequence $t_n, n = 1, 2, \ldots$, such that $\widetilde{\Pi}_n \in \mathscr{P}_1$, $\gamma(t_n) \in \widetilde{\Pi}_n$ and $|\gamma(t_n \leq t \leq t_{n+1})| > \tau_0$. Since $\mathscr{P}_1 \subset \overline{ch(\mathbf{D})}$, where $ch(\mathbf{D})$ is the convex hull of $\mathbf{D}$ and is obviously bounded. Hence there exits a constant $T < \infty$ such that $\lim_{l\to\infty} t_{n_l} = T$. In turn, we have $|\gamma(t_n \leq t \leq t_{n+1})| = \int_{t_n}^{t_{n+1}} |\gamma'(t)|dt \to 0$ as $n \to \infty$, contradicting to our construction. The proof is completed.

□

Lemma 3.3.4. *We have $\mathscr{P}_1 = \emptyset$.*

Proof. Clearly, we need only to show that $\mathscr{P}_1 = \emptyset$ if $\mathscr{P}_2 \neq \emptyset$ by using Lemma 3.3.3. By contradiction, we assume both $\mathscr{P}_1$ and $\mathscr{P}_2$ are not empty and $\tilde{\Pi}_1 \in \mathscr{P}_1$ is a bounded perfect plane. Noting $\mathscr{P}_2 = \mathscr{P}_{2,\mathbf{E}} \cup \mathscr{P}_{2,\mathbf{H}}$, we have $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$ or $\mathscr{P}_{2,\mathbf{H}} \neq \emptyset$. We first consider the case that both $\mathscr{P}_{2,\mathbf{E}}$ and $\mathscr{P}_{2,\mathbf{H}}$ are not empty, and at the end of the proof we would indicate that the case $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$ and $\mathscr{P}_{2,\mathbf{H}} = \emptyset$ (or, $\mathscr{P}_{2,\mathbf{E}} = \emptyset$ and $\mathscr{P}_{2,\mathbf{H}} \neq \emptyset$) can be proved similarly. In the following, we let $\Pi_{\mathbf{E}}$ and $\Pi_{\mathbf{H}}$ be the two planes stated in Lemma 3.3.1.

We first construct a regular $\gamma(t)(t \geq 0)$ such that $x_1 := \gamma(t_1 = 0) \in \widetilde{\Pi}_1$ and the following 4 conditions are satisfied:

(i) $\gamma(t > 0) \subset \mathbf{G}$ and $\lim_{t\to\infty} |\gamma(t)| = \infty$;

(ii) γ has at most one intersection point with $\mathscr{P}_{2,\mathbf{E}}$;

(iii) γ does not meet the common part between two planes $\Pi_{\mathbf{E}}$ and $\Pi_{\mathbf{H}}$.

(iv) $\gamma(t)$ does not meet the planes $\Pi_{\mathbf{E}}$ and $\Pi_{\mathbf{H}}$ for sufficiently large t.

Note that condition (iii) and (iv) can be easily satisfied by making proper small deformation of the curve, so we need only to find a curve satisfying condition (i) and (ii). This can be done as follows: First note that $\mathbf{G}$ is connected and unbounded, we can find a regular curve $\gamma_1(t)(t \geq 0)$ such that $x_1 := \gamma_1(t_1 = 0) \in \widetilde{\Pi}_1$, $\gamma_1(t > 0) \subset \mathbf{G}$ and $\lim_{t\to\infty} |\gamma_1(t)| = \infty$. If $\gamma_1(t)(t \geq 0)$ does not intersect $\mathscr{P}_{2,\mathbf{E}}$ more than once, then γ_1 is our desired curve. On the other hand, if it intersects $\mathscr{P}_{2,\mathbf{E}}$ more that once, let $\gamma_1(t^*) \in \Pi^* \subset \mathscr{P}_{2,\mathbf{E}}$ be the first intersection point. By Lemma 3.3.2, Π^* belongs to some unbounded component of $\mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{E}}$, thus we can construct another regular curve $\gamma_2(t)(t \geq 0)$ such that $\gamma_2(0) = \gamma_1(t^*)$, $\gamma_2(t > 0) \subset \mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{E}}$ and $\lim_{t\to\infty} |\gamma(t)| = \infty$. Now we can connect the part of the curve $\gamma_1(t)$ for $0 \leq t \leq t^*$ with the one $\gamma_2(t)$ $(t \geq 0)$ and modify the connected curve around the connecting point so that the resulting curve, denoted by $\gamma(t)$, is regular. Clearly $\gamma(t)$ satisfies the requirements. In the sequel, we set $d_0 = dist(\gamma, \mathbf{D}) > 0$ and $r_0 = d_0/2$.

This regular curve γ will act as the 'exit path' for our subsequent *path argument* to prove the lemma. More specifically, we shall construct a sequence of pairs $(\gamma(t_n), \widetilde{\Pi}_n)$, $n = 2, 3, 4....$ such that $t_n < t_{n+1}$, $\widetilde{\Pi}_n \subset \mathscr{P}_1 \cup \mathscr{P}_2$ and $\gamma(t_n) \in \widetilde{\Pi}_n$. Moreover, each $\widetilde{\Pi}_n$ is extended from an open face of the boundary of a bounded domain in $\mathbf{G}$ whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$. Using this sequence a contradiction can be derived. We carry out the construction of the sequence now by induction.

We first construct $(\gamma(t_2), \widetilde{\Pi}_2)$. Since $\widetilde{\Pi}_1 \in \mathscr{P}_1$, we know that the corresponding symmetric set $\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ is bounded and hence $\gamma \cap \partial\mathbf{\Lambda}_{\widetilde{\Pi}_1} \neq \emptyset$. Let $x_2 = \gamma(t_2)$ be the 'last' intersection point of γ with $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$, and this then implies the existence of a perfect plane $\widetilde{\Pi}_2$ passing though x_2 which is extended from an open face of $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ in $\mathbf{G}$. Furthermore, it is clear that $|\gamma(t_1 \leq t \leq t_2)| > r_0$. So we have determined $(\gamma(t_2), \widetilde{\Pi}_2)$.

Next, we assume that we have constructed the pair $(\gamma(t_n), \widetilde{\Pi}_n)$ for $n \geq 2$ such that $\widetilde{\Pi}_n \subset \mathscr{P}_1 \cup \mathscr{P}_2$, $\gamma(t_n) \in \widetilde{\Pi}_n$ and $\widetilde{\Pi}_n$ is extended from an open face of the boundary of a bounded domain in $\mathbf{G}$, denoted by $\mathbf{\Upsilon}_n$, whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$. We proceed to construct $(\gamma(t_{n+1}), \widetilde{\Pi}_{n+1})$. Obviously we have either $\widetilde{\Pi}_n \in \mathscr{P}_1$ or $\widetilde{\Pi}_n \in \mathscr{P}_2$. If $\widetilde{\Pi}_n \in \mathscr{P}_1$, we repeat the above argument to find a perfect plane $\widetilde{\Pi}_{n+1}$ which is extended from some open face of $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_n}$ and $x_{n+1} := \gamma(t_{n+1}) \in \widetilde{\Pi}_{n+1}$ such that $|\gamma(t_n \leq t \leq t_{n+1})| > r_0$. On the other hand, if $\widetilde{\Pi}_n \in \mathscr{P}_n$ we have either $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{E}}$ or $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{H}}$. Then we can show

(1) If $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{E}}$, then there exists either an $x_{n+1} := \gamma(t_{n+1}) \in \widetilde{\Pi}_{n+1} \in \mathscr{P}_1 \cup \mathscr{P}_{2,\mathbf{H}}$ such that $|\gamma(t_n \leq t \leq t_{n+1})| > 0$, or an $x_{n+1} := \gamma(t_{n+1}) \in \widetilde{\Pi}_{n+1} \in \mathscr{P}_1 \cup \mathscr{P}_2$ such that $|\gamma(t_n \leq t \leq t_{n+1})| > r_0$. In both cases, $\widetilde{\Pi}_{n+1}$ is extended from an open face of the boundary of a bounded domain in $\mathbf{G}$ whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$.

(2) If $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{H}}$, then the same result as in (1) holds with "$\mathscr{P}_{2,\mathbf{H}}$" being replaced by $\mathscr{P}_{2,\mathbf{E}}$.

We argue first for the case (1) of $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{E}}$: Noting $\widetilde{\Pi}_n$ is extended from an open face, say $\mathbf{F}_n$, of the bounded domain $\mathbf{\Upsilon}_n$ in $\mathbf{G}$ whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$, we let Θ_0 be the connected component of $(\mathbf{\Upsilon}_n \cup \widetilde{\Pi}_n \cup R_{\Pi_n}(\mathbf{\Upsilon}_n)) \cap \mathbf{\Lambda}_{\widetilde{\Pi}_n}$ containing $\mathbf{F}_n$. Since $\mathbf{\Upsilon}_n$ is bounded, we know Θ_0 is bounded. Then by the reflection principle, we get

$\partial\Theta_0 \subset \partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$. Let us further specify two cases: (i) $x_n \in \Theta_0$; (ii) $x_n \in \partial\Theta_0$.

In case (i), we define $x_{n+1} := \gamma(t_{n+1})$ to be the 'last' intersection point of γ with $\partial\Theta_0$. Clearly we have a perfect plane $\widetilde{\Pi}_{n+1}$ which is extended from an open face of $\partial\Theta_0$ in $\mathbf{G}$. It is remarked that we would only have $|\gamma(t_n \le t \le t_{n+1})| > 0$ but not necessarily have $|\gamma(t_n \le t \le t_{n+1})| > r_0$ in this case. Noting $\widetilde{\Pi}_n$ lies on Π_n, which is exactly $\Pi_{\mathbf{E}}$ in this case, and Θ_0 is symmetric with respect to Π_n, we have $\widetilde{\Pi}_{n+1} \not\subset \Pi_{\mathbf{E}}$. Then by Lemma 3.3.1, we have that $\widetilde{\Pi}_{n+1} \in \mathscr{P}_1$ or $\widetilde{\Pi}_{n+1} \in \mathscr{P}_{2,\mathbf{H}}$.

In case (ii) with $x_n \in \partial\Theta_0$, we know there is some perfect plane $\widetilde{\Pi}'_n$ other than $\widetilde{\Pi}_n$ which contains x_n and is extended from a face of $\partial\Theta_0$. Using Lemma 3.3.1, we see that $\widetilde{\Pi}'_n \not\subset \mathscr{P}_{2,\mathbf{E}}$. Besides it holds that $\widetilde{\Pi}'_n \not\subset \mathscr{P}_{2,\mathbf{H}}$, otherwise $x_n \in \widetilde{\Pi}'_n \cap \widetilde{\Pi}_n \subset \Pi_{\mathbf{E}} \cap \Pi_{\mathbf{H}} = L$, contradicting to our construction of γ. Hence we have $\widetilde{\Pi}'_n \in \mathscr{P}_1$. Using this fact, we can easily see that $x_{n+1} := \gamma(t_{n+1})$, the 'last' intersection point between γ and $\mathbf{\Lambda}_{\widetilde{\Pi}'_n}$, satisfies $|\gamma(t_n \le t \le t_{n+1})| > r_0$, and the existence of a perfect plane $\widetilde{\Pi}_{n+1} \in \mathscr{P}_1 \cup \mathscr{P}_2$ passing through x_{n+1}.

The argument for the case (2) with $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{H}}$ is exactly the same as for the case (1) with $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{E}}$.

In the above we have constructed the pair $(\gamma(t_{n+1}), \widetilde{\Pi}_{n+1})$ in every possible cases. It is clear from our construction that $\widetilde{\Pi}_{n+1}$ is extended from a face of the boundary of a bounded domain in $\mathbf{G}$ whose boundary belongs to $\partial\mathbf{D} \cup \mathscr{P}_1 \cup \mathscr{P}_2$.

Now, we have finished the construction of the desired sequence of pairs $(\gamma(t_n), \widetilde{\Pi}_n)$, $n = 2, 3, 4, \ldots$. We claim that $\gamma(t_n) \to \infty$ as $n \to \infty$. Indeed, according to our construction of $\gamma(t)$, we know that it has at most one intersection point with $\mathscr{P}_{2,\mathbf{E}}$. So we may assume that $\widetilde{\Pi}_{n_0} \in \mathscr{P}_{2,\mathbf{E}}$ for some integer n_0, where it may happen that $n_0 = 0$, i.e., all the $\gamma(t_n)$'s with $n = 1, 2, \ldots$ do not belong to $\mathscr{P}_{2,\mathbf{E}}$. Then for all $n > n_0$, we have $\widetilde{\Pi}_n \in \mathscr{P}_1 \cup \mathscr{P}_{2,\mathbf{H}}$. i.e. either $\widetilde{\Pi}_n \in \mathscr{P}_1$ or $\widetilde{\Pi}_n \in \mathscr{P}_{2,\mathbf{H}}$. In the former case, we have $|\gamma(t_n \le t \le t_{n+1})| > r_0$ from the previous construction. While in the latter case, we know from case (2) above that either $|\gamma(t_n \le t \le t_{n+1})| > r_0$ or $|\gamma(t_n \le t \le t_{n+1})| > 0$ with $\gamma(t_{n+1}) \in \widetilde{\Pi}_{n+1} \subset \mathscr{P}_1$, in that case we can apply the same argument as in the former case to show that $|\gamma(t_{n+1} \leq t \leq t_{n+2})| > r_0$. Thus we can conclude $|\gamma(t_n \leq t \leq t_{n+2})| > r_0$ for all $n > n_0$, from which our claim follows immediately.

Next, by the construction of γ and Lemma 3.3.1, we see that $\gamma(t)$ does not intersect $\mathscr{P}_2 = \mathscr{P}_{2,\mathbf{E}} \cup \mathscr{P}_{2,\mathbf{H}}$ for sufficiently large t. Therefore $\gamma(t_n) \in \mathscr{P}_1$ for all n sufficiently large. But from the definition of $\mathscr{P}_1$, we clearly have $\mathscr{P}_1 \subset \overline{ch(\mathbf{D})}$, which is bounded. This contradicts to the claim that $\gamma(t_n) \to \infty$ as $n \to \infty$, thus completes the proof of the case when both $\mathscr{P}_{2,\mathbf{E}}$ and $\mathscr{P}_{2,\mathbf{H}}$ are not empty.

Finally we come to the remaining two cases: (a) $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$ and $\mathscr{P}_{2,\mathbf{H}} = \emptyset$; and (b) $\mathscr{P}_{2,\mathbf{H}} \neq \emptyset$ and $\mathscr{P}_{2,\mathbf{E}} = \emptyset$. For the case (a), we apply the same arguments as in the case when both $\mathscr{P}_{2,\mathbf{E}}$ and $\mathscr{P}_{2,\mathbf{H}}$ are not empty. It is remarked that the subcase when the perfect plane $\widetilde{\Pi}_n$'s belong to $\mathscr{P}_{2,\mathbf{H}}$ can not happen now in the construction of the pair $(\gamma(t_{n+1}), \widetilde{\Pi}_{n+1})$ since we have $\mathscr{P}_{2,\mathbf{H}} = \emptyset$. This fact leads to the same contradiction, but with much easier deductions. Similar arguments also hold for the case (b). □

We are in a position to present the symmetry properties of the scatterer **D**.

Lemma 3.3.5. *If $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$, then $\mathbf{D}$ is symmetric with respect to $\Pi_\mathbf{E}$. Similarly, if $\mathscr{P}_{2,\mathbf{H}} \neq \emptyset$, then $\mathbf{D}$ is symmetric with respect to $\Pi_\mathbf{H}$. Here $\Pi_\mathbf{E}$ and $\Pi_\mathbf{H}$ are the two planes introduced in Lemma 3.3.1.*

Proof. We consider only the case $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$, and the other case can be treated similarly. Let $\mathbf{G}^*$ be the unique unbounded open connected component of $\mathbf{G} \cap R_{\Pi_\mathbf{E}}(\mathbf{G})$ and set $\mathbf{D}^* := \mathbb{R}^3 \backslash \overline{\mathbf{G}^*}$. We see that both $\mathbf{G}^*$ and $\mathbf{D}^*$ are symmetric with respect to $\Pi_\mathbf{E}$. It is also clear that $\mathbf{G}^* \subset \mathbf{G}$ and $\mathbf{D}^*$ is bounded. Moreover, the reflection principle in Theorem 3.3.1 (i) implies that $\partial\mathbf{G}^* \subset \partial\mathbf{G} \cup (\mathscr{P}_1 \cup \mathscr{P}_2)$. Clearly, if $\mathbf{G}\backslash\overline{\mathbf{G}^*} = \emptyset$, or equivalently $\mathbf{D}^*\backslash\bar{\mathbf{D}} = \emptyset$, then one has $\mathbf{D} = \mathbf{D}^*$ and the lemma follows immediately. Thus we need only to show that $\mathbf{G}\backslash\overline{\mathbf{G}^*} = \emptyset$. By contradiction, assume that $\mathbf{G}\backslash\overline{\mathbf{G}^*} \neq \emptyset$ or equivalently $\mathbf{D}^*\backslash\bar{\mathbf{D}} \neq \emptyset$. Then we can find some nonempty open connected component of $\mathbf{D}^*\backslash\bar{\mathbf{D}}$, which we denote by $\mathbf{D}^{**}$. We see that $\partial\mathbf{D}^{**} \subset \partial\mathbf{D}^* \cup \partial\mathbf{D} = \partial\mathbf{G}^* \cup \partial\mathbf{G} \subset \partial\mathbf{G} \cup (\mathscr{P}_1 \cup \mathscr{P}_2)$.

Now we have two cases: $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} = \emptyset$ and $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} \neq \emptyset$. In the first case, using Lemma 3.3.2 we see that $\mathbf{D}^{**}$ lies entirely in one unbounded connected component of $\mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{E}}$, which we denote by $\mathbf{W}$. Whereas in the latter case when $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} \neq \emptyset$, we take $x_0 \in \mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}}$ to be an arbitrary point, and it also follows from Lemma 3.3.2 that x_0 belongs to the boundary of some unbounded connected component of $\mathbf{G}\backslash\bar{\mathscr{P}}_{2,\mathbf{E}}$, which we still denote by $\mathbf{W}$. Let $\gamma(t)(t \geq 0)$ be a regular curve such that:

i) $\gamma(0) \in \mathbf{D}^{**}$, $\gamma(t > 0) \subset \mathbf{W}$ and $\lim_{t\to\infty} |\gamma(t)| = \infty$ in the case $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} = \emptyset$;

ii) $\gamma(0) = x_0$, $\gamma(t > 0) \subset \mathbf{W}$ and $\lim_{t\to\infty} |\gamma(t)| = \infty$ in the case $\mathbf{D}^{**} \cap \mathscr{P}_{2,\mathbf{E}} \neq \emptyset$.

By the fundamental property of a connected set, we know that $\gamma \cap \partial\mathbf{D}^{**} \neq \emptyset$ for both cases. Let $\gamma(t_1)$ with $t_1 > 0$ be the 'last' intersection point between γ and $\partial\mathbf{D}^{**}$. By analytical continuation, this implies the existence a perfect plane $\widetilde{\Pi}_1$ extended from an open part of $\partial\mathbf{D}^{**}\backslash\partial\mathbf{D}$ in $\mathbf{G}$ such that $\gamma(t_1) \in \widetilde{\Pi}_1$.

We claim that $\gamma(t_1)$ lies in the interior of some face of $\partial\mathbf{D}^{**}$, which is contained in $\widetilde{\Pi}_1$ and that $\widetilde{\Pi}_1 \subset \mathscr{P}_{2,\mathbf{H}}$. Indeed, according to our previous construction of $\gamma(t)$, we know $\gamma(t) \cap \mathscr{P}_{2,\mathbf{E}} = \emptyset$ for $t > 0$, and this, along with Lemma 3.3.4 which ensures $\mathscr{P}_1 = \emptyset$, concludes that only perfect planes in $\mathscr{P}_{2,\mathbf{H}}$ have intersection points with $\gamma(t)$. Clearly our claim follows immediately by noting that $\gamma(t_1)$ lies on the perfect plane $\widetilde{\Pi}_1$ and that all the perfect planes in $\mathscr{P}_{2,\mathbf{H}}$ lie on $\Pi_{\mathbf{H}}$ due to Lemma 3.3.1.

Now, let Θ be the connected component of $(\mathbf{D}^{**} \cup \widetilde{\Pi}_1 \cup R_{\Pi_1}\mathbf{D}^{**}) \cap \mathbf{\Lambda}_{\widetilde{\Pi}_2}$ containing $\widetilde{\Pi}_1$. Then we have from our previous claim that $\gamma(t_1) \in \Theta$. Using the same construction as in the proof of case (i) of assertion (2) in Lemma 3.3.4, one can show that there exists a $t_2 > t_1$ such that $\gamma(t_2) \in \widetilde{\Pi}_2 \subset \mathscr{P}_1 \cup \mathscr{P}_{2,\mathbf{E}}$. This contradicts to the previous result that only perfect planes in $\mathscr{P}_{2,\mathbf{H}}$ have intersection points with $\gamma(t)$, thus completes the proof of the lemma.

$\square$

Lemma 3.3.6. *Let the scatterer* $\mathbf{D}$ *be associated with the mixed boundary condition (3.1.5), then it holds that* $R_{\Pi_\mathbf{E}}(\Gamma_\mathbf{E}) = \Gamma_\mathbf{E}$ *and* $R_{\Pi_\mathbf{E}}(\Gamma_\mathbf{H}) = \Gamma_\mathbf{H}$ *if* $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$; *and*

$R_{\Pi_{\mathbf{H}}}(\Gamma_{\mathbf{E}}) = \Gamma_{\mathbf{E}}$ and $R_{\Pi_{\mathbf{H}}}(\Gamma_{\mathbf{H}}) = \Gamma_{\mathbf{H}}$ if $\mathscr{P}_{2,\mathbf{H}} \neq \emptyset$.

Proof. It suffices to consider the case $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$. Assume contrarily that $R_{\Pi_{\mathbf{E}}}(\Gamma_{\mathbf{E}}) \neq \Gamma_{\mathbf{E}}$. By Lemma 3.3.5, we know that $\mathbf{D}$ is symmetric with respect to $\Pi_{\mathbf{E}}$. With the help of the reflection principle in Theorem 3.3.1 (i), it is straightforward to show that on an open subset of $\partial\mathbf{D}$, one has both $\nu\times\mathbf{E} = 0$ and $\nu\times\mathbf{H} = 0$. Hence by the unique continuation (see, e.g., Lemma 3.2 in [1]), we have $\mathbf{E} = \mathbf{H} = 0$ in $\mathbf{G}$. Recalling the asymptotic behavior of the scattered field $\mathbf{E}^s$ in (3.1.6), we derive that $\lim_{|x|\to\infty} |\mathbf{E}^i| = 0$, which is not true since $|\mathbf{E}^i|$ is a non-zero constant by (3.1.1). Therefore we have shown $R_{\Pi_{\mathbf{E}}}(\Gamma_{\mathbf{E}}) = \Gamma_{\mathbf{E}}$, which implies also $R_{\Pi_{\mathbf{E}}}(\Gamma_{\mathbf{H}}) = \Gamma_{\mathbf{H}}$. □

Remark 3.3.1. *From Lemma 3.3.5 and 3.3.6, we know that if $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$, then both the domain $\mathbf{G} = \mathbb{R}^3\backslash\bar{\mathbf{D}}$ and the boundary condition (3.1.5) are symmetric with respect to the plane $\Pi_{\mathbf{E}}$. Appealing to Theorem 3.3.1, we see that the total fields $\mathbf{E}$ and $\mathbf{H}$ are also symmetric with respect to the plane $\Pi_{\mathbf{E}}$ in the sense that*

$$\mathbf{E}(x) + R'_{\Pi}(\mathbf{E}(R_{\Pi}(x))) = 0, \quad \mathbf{H}(x) - R'_{\Pi}(\mathbf{H}(R_{\Pi}(x))) = 0, \quad x \in \mathbf{G}. \tag{3.3.3}$$

On the other hand, if there exists a plane, denoted by Π, such that $\Pi \perp (d\times p)\times d$ and that both the obstacle $\mathbf{D}$ and the boundary condition (3.1.5) are symmetric with respect to this plane, we can also show that the symmetry relations (3.3.3) hold with $\Pi_{\mathbf{E}}$ being replaced by Π, by using Theorem 3.3.1 and the uniqueness of the forward scattering system (3.1.3)-(3.1.5). This indicates that $\mathscr{P}_{2,\mathbf{E}} \neq \emptyset$, and $\mathscr{P}_{2,\mathbf{E}} \subset \Pi$ by the reflection principle in Theorem 3.3.1.

Similar symmetry results also hold for $\mathscr{P}_{2,\mathbf{H}}$.

3.3.3 Proofs

So far, we have established the properties of perfect planes the electromagnetic field $\mathbf{E}$ and $\mathbf{H}$ in Subsection 3.3.2, we are ready to prove the main result on the uniqueness in determining a general polyhedral obstacle that consists of finitely many pairwise disjoint bounded polyhedra by a single far-field measurement.

Proof of Theorem 3.2.1.

Let $\widetilde{\mathbf{D}}$ be a polyhedral scatterer associated with a boundary operator $\tilde{\mathscr{B}}$. Assume that $\mathbf{D} \neq \widetilde{\mathbf{D}}$ and

$$\mathbf{E}_\infty(\hat{x}; \mathbf{D}, p, k, d) = \mathbf{E}_\infty(\hat{x}; \widetilde{\mathbf{D}}, p, k, d) \quad \text{for} \quad x \in \mathbb{S}^2. \tag{3.3.4}$$

Let Ω be the unique unbounded connected component of $\mathbb{R}^3 \backslash (\bar{\mathbf{D}} \cup \bar{\widetilde{\mathbf{D}}})$. By Rellich's theorem (see Theorem 6.9, [13]), we infer from (3.3.4) that

$$\mathbf{E}(x; \mathbf{D}) = \mathbf{E}(x; \widetilde{\mathbf{D}}) \qquad \text{for } x \in \Omega. \tag{3.3.5}$$

Next, noting $\mathbf{D} \neq \widetilde{\mathbf{D}}$, we see that either $(\mathbb{R}^3 \backslash \bar{\Omega}) \backslash \bar{\mathbf{D}} \neq \emptyset$ or $(\mathbb{R}^3 \backslash \bar{\Omega}) \backslash \bar{\widetilde{\mathbf{D}}} \neq \emptyset$. Without loss of generality, we assume the former case and let $\mathbf{D}^*$ be a connected component of $(\mathbb{R}^3 \backslash \bar{\Omega}) \backslash \bar{\mathbf{D}} \neq \emptyset$. Clearly, $\mathbf{D}^*$ is a bounded polyhedral domain in $\mathbf{G} = \mathbb{R}^3 \backslash \bar{\mathbf{D}}$ and $\mathbf{E}(x; \mathbf{D})$ is defined over D^*. Noting $\partial \mathbf{D}^* \subset \partial \Omega \cup \partial \mathbf{D} \subset \partial \mathbf{D} \cup \partial \widetilde{\mathbf{D}}$ and using (3.3.5), we have perfect boundary conditions on $\partial \mathbf{D}^*$. It is obvious that $\partial \mathbf{D}^* \backslash \partial \mathbf{D} \neq \emptyset$, hence there must be an open face, say Σ_0, on $\partial \mathbf{D}^*$ that can be extended in $\mathbf{G}$ to form a perfect plane $\widetilde{\Pi}_0$. By Lemma 3.3.4, $\widetilde{\Pi}_0 \in \mathscr{P}_2(\mathbf{D})$. Here $\mathscr{P}_2(\mathbf{D})$ and $\mathscr{P}_2(\widetilde{\mathbf{D}})$ below represent respectively the set defined in (3.3.2) corresponding to the scatterer $\mathbf{D}$ and $\widetilde{\mathbf{D}}$. According to our discussion prior to Lemma 3.3.1, $\widetilde{\Pi}_0$ is essentially "unbounded". Without loss of generality, we may assume that $\widetilde{\Pi}_0$ is unbounded. Again by using (3.3.5) and the previous result that $\widetilde{\Pi}_0 \in \mathscr{P}_2(\mathbf{D})$, we see $\widetilde{\Pi}_0 \in \mathscr{P}_2(\widetilde{\mathbf{D}})$. By Lemma 3.3.5, $\widetilde{\mathbf{D}}$ is symmetric with respect to Π_0. But this is impossible since one of the faces of $\widetilde{\mathbf{D}}$ lies on Π_0. Hence, $\mathbf{D} = \widetilde{\mathbf{D}}$. Finally, if $\mathscr{B} \neq \tilde{\mathscr{B}}$, one can show that there is an open subset of $\partial \mathbf{D} = \partial \widetilde{\mathbf{D}}$ on which both $\mathbf{E}$ and $\mathbf{H}$ assume perfect boundary conditions, leading to a same contradiction as that in the proof of Lemma 3.3.6.

Chapter 4

Scattering by Bi-periodic Polyhedral Grating (I)

In this chapter, we consider the unique determination of a bi-periodic diffraction grating in three dimensions by scattered electromagnetic fields measured somewhere above the grating. We restrict to gratings that are of polyhedral type and assume that the Relaygh frequencies do not occur (the general case when Rayleigh frequencies are not excluded is studied in chapter five). We show that corresponding to each incident plane wave, there are three classes of unidentifiable gratings, and any periodic polyhedral structure can be uniquely determined by one incident field if and only if it belongs to neither of the three classes. Consequently, the minimum number of incident waves required for the unique determination of a periodic polyhedral structure can be readily read out from the results.

The chapter is organized as follows. In Section 4.1, we describe briefly the direct problem of scattering of electromagnetic wave by gratings. In Section 4.2, we present the associated inverse problem, we also state our main result, Theorem 4.2.1 on the uniqueness of the inverse problem. The rest sections are devoted to the proof the the main result. In Section 4.3, we present some technical tools and observations that are starting points for the subsequent analysis. In section 4.4, we discuss how to start from one or two perfect planes of the total field to find all the possible grating profiles to which the global uniqueness fail. Finally, we prove the main result about the uniqueness of our scattering problem in section 4.5.

4.1 Direct problem

We consider the scattering problem of time-harmonic electromagnetic plane wave by a diffraction grating in three dimensions. The grating, with profile denoted by S, is assumed to be bi-periodic of period $\Lambda = (\Lambda_1, \Lambda_2)$, namely, if $x = (x_1, x_2, x_3) \in S$, the point $(x_1 + n_1\Lambda_1, x_2 + n_2\Lambda_2, x_3)$ also belongs to S for all integers n_1 and n_2. It is ruled on a perfect conductor and the medium above S is assumed to be homogenous with a constant dielectric coefficient $\epsilon_0 > 0$ and magnetic permeability $\mu_0 > 0$. The corresponding region is denoted by Ω. Let $E^i(x) = se^{iq\cdot x}$ (with time dependence $e^{-i\omega t}$) be an incident time-harmonic electromagnetic wave incident to the grating structure S from above. If we write the incident direction q as $q = (\alpha_1, \alpha_2, -\beta)$, then we have $\beta > 0$, and the vectors s and q are orthogonal due to the solenoidal feature of E^i, and the corresponding wave number k and frequency ω are given by

$$k = |q|, \quad \omega = k/\sqrt{\epsilon_0\mu_0}.$$

We will also frequently use the vector $\alpha = (\alpha_1, \alpha_2, 0)$, that is parallel to q and passes through the origin. Let E be the total field E, consisting of both the incident field and the scattered field, then E satisfies the following vector-valued Helmholtz system:

$$\Delta E + k^2 E = 0 \quad in \quad \Omega, \tag{4.1.1}$$

$$\mathrm{div} E = 0 \quad in \quad \Omega, \tag{4.1.2}$$

$$\nu \times E = 0 \quad on \quad S \tag{4.1.3}$$

where ν is the unit outward normal vector to the surface S.

In view of the bi-periodic structure of the grating S, we are only interested in the quasi-periodic solutions E to the system (4.1.1)-(4.1.3), i.e. $e^{-i\alpha\cdot x}E$ is periodic respectively with period Λ_1 in the x_1 direction and Λ_2 in the x_2 direction; see [7]

[15]. We also impose a radiation condition in the x_3 direction by assuming that E is composed of bounded outgoing plane waves plus the incident wave E^i. Then it follows from the knowledge of the fundamental solution to the periodic Helmholtz equation (cf. [7] [15]) that E can be expressed in the following form:

$$E(x) = E^i(x) + \sum_{n\in\mathbf{Z}^2} A^n e^{iq^n\cdot x} \quad \text{for all } x \text{ in } \mathbf{R}^3 \text{ above the highest point on } S, \tag{4.1.4}$$

where all A^n's are complex vectors, called the *Rayleigh coefficients*, and the indices q^n's are given by $q^n = \alpha^n + \alpha + (0, 0, \beta^n)$ with $\alpha^n = (2\pi n_1/\Lambda_1, 2\pi n_2/\Lambda_2, 0)$ and

$$\beta^n = \begin{cases} \sqrt{k^2 - |\alpha^n + \alpha|^2}, & k^2 \geq |\alpha^n + \alpha|^2, \\ i\sqrt{-k^2 + |\alpha^n + \alpha|^2}, & k^2 < |\alpha^n + \alpha|^2. \end{cases} \tag{4.1.5}$$

One can see from (4.1.5) that there are only finitely many n's for which β^n are real, i.e., there are only a finite number of propagating plane waves in the scattered field while the remaining modes decay exponentially along the x_3 direction.

Throughout this chapter we assume that

$$k^2 \neq |\alpha^n + \alpha|^2 \quad \text{for all } n \in \mathbf{Z}^2. \tag{4.1.6}$$

Note that for fixed incident wave E^i and bi-period Λ, condition (4.1.6) is violated for a discrete set of frequencies ω_j, $\omega_j \to \infty$, referred to as Rayleigh frequencies.

The assumption (4.1.6) is usually the condition to ensure the uniqueness of the forward scattering problem (4.1.1)-(4.1.4) (cf. [7] [15]). When assumption (4.1.6) is violated, the existence and unique of solution to the forward problem still remains open, see [35]. In that case, we assume that the forward problem (4.1.1)-(4.1.4) has a solution, though the solution may not be unique. This assumption is made throughout chapter five.

For the sake of convenience, we introduce the index set

$$\Xi = \{n \in \mathbf{Z}^2;\ \beta^n > 0\},$$

and denote by E_p the *propagating field*, namely, the total field of E in (4.1.4) with those exponentially decaying modes removed:

$$E_p(x) = E^i(x) + \sum_{n\in\Xi} A^n e^{iq^n\cdot x}. \tag{4.1.7}$$

Unlike the total field E in the expression (4.1.4), the complex vector-valued function E_p can be extended to the whole space $\mathbf{R}^3$ naturally.

4.2 Inverse problem and statement of main results

Given the periodic structure S and the incident field E^i, the forward diffraction problem is to solve for the system (4.1.1)-(4.1.4) the total field E. In this thesis, we are concerned with the inverse problem, more specifically, the unique determination of the diffraction grating by the scattered electromagnetic fields measured above from the grating. In general, it is well known that global uniqueness may not be true when the measurement is only taken for one incident field. However, one can easily see that this inverse problem is formally determined with a single measurement (corresponding to one incident field). Then the question becomes to what extent can one incident field determine the grating structure. As far as gratings of polyhedral type are considered, we are able to classify, corresponding to each incident plane wave, all unidentifiable structures into three classes, and show that any periodic polyhedral structure can be uniquely determined by one incident field if and only if it belongs to neither of the three classes. We remark that the result is obtained under the assumption that the Rayleigh frequencies are excluded, i.e. condition (4.1.6). When this condition is relaxed, we are able to find four more classes of unidentifiable structures corresponding to each incident plane wave. That will be our objective in chapter five.

The result is presented in the following theorem.

Theorem 4.2.1. *Let $E^i = se^{iq\cdot x}$ be a given incident electric field, S_1 and S_2 be two periodic polyhedral gratings with bi-period Λ, and E_{S_1} and E_{S_2} be respectively the solutions to the forward scattering problem (4.1.1)-(4.1.4) associated with S_1 and S_2. If $\Gamma_b = \{x_3 = b\}$ is a plane located above both S_1 and S_2, then under Condition (4.1.6),* *the information*

$$e_3 \times E_{S_1} = e_3 \times E_{S_2} \quad on \ \Gamma_b \tag{4.2.1}$$

implies one of the following three cases for some point $r \in \mathbf{R}^3$*:*

(a) $S_1,\ S_2 \in \mathcal{S}_1(q,r)$*;* *(b)* $S_1, S_2 \in \mathcal{S}_2(s,q,\Lambda,r)$*;* *(c)* $S_1, S_2 \in \mathcal{S}_3(s,q,\Lambda,r)$*.*

Here the grating classes $\mathcal{S}_1(q,r)$, $\mathcal{S}_2(s,q,\Lambda,r)$ and $\mathcal{S}_3(s,q,\Lambda,r)$ will be defined later.

The basic idea for the proof is as follows: if S is an identifiable polyhedral grating profile to the incident field E^i, then one can show that the set of perfect planes of the total field E is not empty, so is the set of perfect planes of the propagating field E_p, which further implies by Reflection Principle certain symmetric property of the set of wave vectors q^n's appearing in E_p. The surprising fact is that this symmetric property combined with the fact that all these q^n's appearing in E_p have positive x_3 components makes one be able to determine E_p and the structure S as well.

We will establish Theorem 4.2.1 in Section 4.4, 4.5. In Section 4.3, we will present some preliminary conventions and results that are needed for subsequent analysis.

4.3 Preliminaries

We begin with introducing the following conventions and notations.

1. For any vector $b \in \mathbf{R}^3$, its norm is denoted by $\|b\|$. For convenience, we may often view a point $r \in \mathbf{R}^3$ also as the vector originating from the origin which directs to the point r.

2. A vector $r \in \mathbf{R}^3$ is said to be parallel to a line l in $\mathbf{R}^3$ with a tangential unit vector ν, if $r \parallel \nu$. For a plane Π in $\mathbf{R}^3$, we denote by ν_Π the unit normal vector to Π. A vector r is said to be parallel to a plane Π in $\mathbf{R}^3$, if $r \perp \nu_\Pi$.

3. For any $c \in \mathbf{C}^3$ and $r \in \mathbf{R}^3$, the dot product $c \cdot r = 0$ means $\mathrm{Re}(c) \cdot r = 0$ and $\mathrm{Im}(c) \cdot r = 0$. The same conventions will be made for the relations $c \,||\, r$, $c \times r$ and $c \perp \Pi$ for any plane Π in $\mathbf{R}^3$.

4. Let Π be a plane in $\mathbf{R}^3$, we denote by R_Π the reflection with respect to plane Π in $\mathbf{R}^3$. The reflection R_Π is always understood to act on a point in $\mathbf{R}^3$.

   Let Π' be the plane that passes through the origin and is parallel to Π, and R'_Π be the derivative of R_Π, namely the linear part of R_Π. One can see that R'_Π is the reflection with respect to the plane Π'. For a point $r \in \mathbf{R}^3$, $R'_\Pi r$ can be also viewed as the reflection of the vector, that initiates from the origin and points to the point r, with respect to the plane Π'. By natural extension, we apply R'_Π to complex vectors in $\mathbf{C}^3$ as well.

5. For a set $\mathcal{A}$, we denote by $|\mathcal{A}|$ the number of elements in $\mathcal{A}$.

6. Let G be a group which acts on a set $\mathcal{A}$, and $a \in \mathcal{A}$, then $G\{a\}$ means the orbit of a under the action of the group G. By the group property, we know that for any two elements $a, b \in \mathcal{A}$, either $G\{a\} = G\{b\}$ or $G\{a\} \cap G\{b\} = \emptyset$.

We recall the Reflection Principle for Maxwell equations.

Lemma 4.3.1. *Let O be a domain in $\mathbf{R}^3$ which is symmetric with respect to a plane Π, and E be an electric field in O satisfying the vector-valued Helmholtz equations (4.1.1)-(4.1.2). Assume that $\tilde{\Pi}$ is a connected open subset in $\Pi \cap O$, then $\tilde{\Pi}$ is a perfect plane of E if and only if the following relation holds*

$$E(x) + R'_\Pi(E(R_\Pi(x))) = 0 \quad in \;\; O. \tag{4.3.1}$$

Moreover, if $\Gamma \subset O$ or $\Gamma \subset \partial O$ is a perfect plane of E, then $R_\Pi(\Gamma)$ is also a perfect plane of E.

Next we present two more lemmas whose results will be used repeatedly in the subsequent sections.

Lemma 4.3.2. *Let $n \in \mathbf{N}$, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n real numbers, and the following relation holds for n complex vectors $a_1, \ldots, a_n \in \mathbf{C}^m$ for $m \in \mathbf{N}$:*

$$\lim_{t\to\infty} \sum_{j=1}^{n} a_j e^{i\lambda_j t} = 0 \quad \forall t \in \mathbf{R}. \tag{4.3.2}$$

Then

$$\sum_{j=1}^{n} a_j e^{i\lambda_j t} = 0 \quad \forall t \in \mathbf{R}.$$

Moreover, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are distinct, then* $a_1 = a_2 = \cdots = a_n = 0$.

Lemma 4.3.3. *For any* $n, m \in \mathbf{N}$, *let* $q_1, q_2, \ldots, q_n$ *be* n *vectors in* $\mathbf{R}^3$, *and* $a_1, a_2, \ldots, a_n$ *be* n *vectors in* $\mathbf{C}^m$ *such that*

$$\sum_{j=1}^{n} a_j e^{iq_j \cdot x} = 0 \quad \text{for all } x \in \mathbf{R}^3,$$

then it holds that

$$\sum_{i:\ q_i = q_j} a_i = 0 \quad \text{for each } j.$$

If $q_1, q_2, \ldots, q_n$ *are all different from each other, then* $a_1 = a_2 = \cdots = a_n = 0$.

4.4 Classification of unidentifiable periodic structures

The aim of this section is to determine and classify all the unidentifiable periodic structures corresponding to a given incident field.

4.4.1 Observations and auxiliary tools

We start with recalling some basic notations from Section 4.1:

$E^i(x) = se^{iq\cdot x}$: the incident electric wave;

$E(x)$: the total field;

S: the bi-periodic grating profile of period Λ;

We also write:

$$\Xi_0 = \{n \in \Xi;\ A^n \neq 0\}, \quad \mathcal{Q} = \{q\} \bigcup \{q^n\}_{n \in \Xi_0}, \tag{4.4.1}$$

$$E_p(x) = se^{iq\cdot x} + \sum_{n \in \Xi_0} A^n e^{iq^n \cdot x}, \tag{4.4.2}$$

$$\mathcal{P} = \{\Pi;\ \Pi \text{ is a perfect plane of } E_p\}. \tag{4.4.3}$$

We will denote by Γ_b the plane $\{x_3 = b\}$ above the grating profile S, on which the measurement of the electric field E is made. The domain above the plane Γ_b is denoted by Ω_b.

Definition 4.4.1. *For a given plane Π in $\mathbf{R}^3$, a periodic part S_l of the entire grating structure S is called a Π-reflecting periodic part if each face of S_l can be reflected by Π into the domain Ω_b.*

The following lemma presents a crucial observation.

Lemma 4.4.1. *If Π is a perfect plane of E, then Π is also a perfect plane of E_p.*

Proof. We separate the proof into two cases: $\Pi \nparallel \{x_3 = 0\}$ and $\Pi \parallel \{x_3 = 0\}$.

Case 1: Π is not parallel to the plane $\{x_3 = 0\}$. We choose $x_0 \in \Pi$, and two linearly independent vectors v_1, v_2 in $\mathbf{R}^3$ such that their components in the x_3 direction are positive and that $x_0 + \lambda v_1 + \mu v_2$ belongs to Π for all $\lambda > 0, \mu > 0$. Note that for $x_3 \geq b$, we have the expansion

$$E(x) = E^i(x) + \sum_{n \in \mathbf{Z}^2} A^n e^{iq^n \cdot x} := E_p(x) + E_s(x),$$

which along with the property that E_s decays exponentially as $x_3 \to \infty$, and $E \times \nu_\Pi = 0$ on Π yield that

$$\lim_{\lambda \to \infty} \nu_\Pi \times E_p(x_0 + \lambda v_1 + \mu v_2) = 0 \quad \text{for all } \mu > 0,$$

i.e.,

$$\lim_{\lambda \to \infty} \{s e^{iq\cdot(x_0+\lambda v_1+\mu v_2)} + \sum_{n \in \Xi} A^n e^{iq^n\cdot(x_0+\lambda v_1+\mu v_2)}\} \times \nu_\Pi = 0 \text{ for all } \mu > 0.$$

Then it follows from Lemma 4.3.2 that

$$\{s e^{iq\cdot(x_0+\lambda v_1+\mu v_2)} + \sum_{n \in \Xi} A^n e^{iq^n\cdot(x_0+\lambda v_1+\mu v_2)}\} \times \nu_\Pi = 0 \quad \forall \mu > 0, \lambda \in \mathbf{R}.$$

Considering that all exponential functions involved above are analytic, we know immediately that the above equality holds for all $\mu \in \mathbf{R}$ by analytic continuation. So the desired result is proved for Case 1.

Case 2: Π is parallel to the plane $\{x_3 = 0\}$. Let $\Pi = \{x_3 = c\}$ with $c > b$. Since

Π is a perfect plane of E, we have for all $x \in \Pi$ that

$$\begin{aligned} e^{-i\alpha \cdot x} E(x) \times e_3 &= e^{-i\alpha \cdot x}\{E^i(x) + \sum_{n \in \mathbf{Z}^2} A^n e^{iq^n \cdot x}\} \times e_3 \\ &= (se^{-\beta c} + A^0 e^{\beta c}) \times e_3 + \sum_{n \in \mathbf{Z}^2 \setminus \{0\}} A^n e^{i\beta^n c} \times e_3 e^{i\alpha^n \cdot x} \\ &= 0. \end{aligned}$$

Note that $\{e^{i\alpha^n \cdot x}\}_{n \in \mathbf{Z}^2}$ is an orthogonal family in $L^2((0, \Lambda_1) \times (0, \Lambda_2))$ of variables x_1 and x_2, we obtain

$$\begin{aligned} (se^{-\beta c} + A^0 e^{\beta c}) \times e_3 = 0, \\ A^n e^{i\beta^n c} \times e_3 = 0 \quad \text{for } n \in \mathbf{Z}^2 \setminus \{0\}. \end{aligned}$$

Then for all $x \in \Pi$ we derive

$$e^{-i\alpha x} E_p(x) \times \nu_\Pi = (se^{-\beta c} + A^0 e^{\beta c}) \times e_3 + \sum_{n \in \Xi} A^n e^{i\beta^n c} \times e_3 e^{i\alpha^n \cdot x} = 0,$$

so we have $E_p(x) \times \nu_\Pi = 0$ for all $x \in \Pi$. This completes the proof of Case 2. ♯

The subsequent analysis is mainly based on the study of the perfect planes of the total field E. But a perfect plane of E is usually not a true plane in $\mathbf{R}^3$. By means of Lemma 4.4.1, it would be much more convenient for us to investigate the perfect planes of the propagating field E_p as those perfect planes are truly two-dimensional plane in $\mathbf{R}^3$. To see this, we may first observe that E_p is analytic in the whole space $\mathbf{R}^3$ since E_p contains only a finite number of exponential functions, and each of the functions is analytic in $\mathbf{R}^3$. Then by the analytic continuation, each perfect plane of E_p, namely each element in $\mathcal{P}$, is a truly two-dimensional plane.

Now for each incident field E^i, we are going to find all the grating structures which can not be identified by the incident field. We begin with the following assumption, which will be the first fundamental fact to be established in the demonstration of our main result on the global uniqueness in Section 4.5.

Assumption 4.4.1. *There exists a perfect plane of E, denoted by $\tilde{\Pi}_0$, such that $\tilde{\Pi}_0 \cap \Omega_b \neq \emptyset$ and Π_0 is not perpendicular to the plane $\{x_3 = 0\}$.*

Next we present four important observations. The first one is a direct consequence of the definition of $\mathcal{Q}$, while the second follows directly from the Reflection Principle (Lemma 4.3.1) and Lemma 4.4.1.

Propsition 4.4.1. *Each of the vectors in $\mathcal{Q}$ except q has a positive x_3 component.*

Propsition 4.4.2. *For each Π_0-reflecting periodic part S_l of grating S, each face of S_l can be reflected with respect to Π_0 to a perfect plane in $\mathcal{P}$.*

Propsition 4.4.3. *For each perfect plane Π in $\mathcal{P}$, we have $R'_\Pi \mathcal{Q} = \mathcal{Q}$, $R_\Pi \mathcal{P} = \mathcal{P}$. Moreover, both maps R'_Π: $\mathcal{Q} \to \mathcal{Q}$ and R_Π: $\mathcal{P} \to \mathcal{P}$ are bijective.*

Proof. By the Reflection Principle (Lemma 4.3.1) the following relation holds for each perfect plane $\Pi \in \mathcal{P}$ that

$$E_p(x) + R'_\Pi(E_p(R_\Pi x)) = 0 \quad \forall x \in \mathbf{R}^3 ,$$

which implies by the definition of E_p that

$$se^{iq\cdot x} + \sum_{n\in\Xi_0} A^n e^{iq^n\cdot x} + R'_\Pi s e^{iq\cdot R_\Pi x} + \sum_{n\in\Xi_0} R'_\Pi A^n e^{iq^n\cdot R_\Pi x} = 0 \quad \forall x \in \mathbf{R}^3.$$

It is easy to check that $R_\Pi x = R'_\Pi x + R_\Pi 0$, and that R'_Π is symmetric, so we can rewrite the above equation to

$$se^{iq\cdot x} + \sum_{n\in\Xi_0} A^n e^{iq^n\cdot x} + (R'_\Pi s e^{iq\cdot R_\Pi 0}) e^{iR'_\Pi q\cdot x} + \sum_{n\in\Xi_0} (R'_\Pi A^n e^{iq^n\cdot R_\Pi 0}) e^{iR'_\Pi q^n\cdot x} = 0 \quad \forall x \in \mathbf{R}^3. \tag{4.4.4}$$

From this relation and Lemma 4.3.3, we know immediately that each vector of $\mathcal{Q}$ must be reflected by R'_Π to one and only one vector in $\mathcal{Q}$. Thus $R'_\Pi \mathcal{Q} \subseteq \mathcal{Q}$. Also by noting that R'_Π is a bijective transformation in $\mathbf{R}^3$, we know $R'_\Pi \mathcal{Q} = \mathcal{Q}$ and that the map R'_Π: $\mathcal{Q} \to \mathcal{Q}$ is also a bijection.

The rest of the lemma can be shown directly by using the Reflection Principle (Lemma 4.3.1) and the bijectiveness of R_Π in $\mathbf{R}^3$. ♯

Propsition 4.4.4. *For a Π_0-reflecting periodic part S_l of grating S, each face of S_l must lie on some perfect plane in $\mathcal{P}$.*

Proof. By Proposition 4.4.2 we have $R_{\Pi_0}S_l \subset \mathcal{P}$. On the other hand, it follows from Lemma 4.4.1 that $\Pi_0 \in \mathcal{P}$. Hence we know $S_l = R_{\Pi_0}(R_{\Pi_0}S_l) \subset \mathcal{P}$ by Proposition 4.4.3. $\sharp$

The following lemma presents some useful properties about perfect planes.

Lemma 4.4.2. *Let $F = te^{ip\cdot x}$ be one of the Fourier modes of E_p in (4.4.2),*

(a) if Π is a perfect plane of the field $F = te^{ip\cdot x}$, then $t \perp \Pi$;

(b) if Π is a perfect plane in $\mathcal{P}$ such that $R'_{\Pi}p = p$, then Π is also a perfect plane of the field F.

(c)if Π and Π^* are two perfect planes in $\mathcal{P}$ such that $R'_{\Pi}p = R'_{\Pi^*}p$, then $\Pi \parallel \Pi^*$.

Proof. (a) is a direct consequence of the definition of a perfect plane. To see (b), we apply Lemma 4.3.3 to equality (4.4.4) with Π to obtain

$$te^{ip\cdot x} + R'_{\Pi}te^{ip\cdot R_{\Pi}x} = 0,$$

which shows that the field $F = te^{ip\cdot x}$ satisfies the symmetric relation (5.1.4) as in Lemma 4.3.1 with respect to the plane Π, so Π is a perfect plane of F.

Finally, we prove (c). We do it in two cases: $R'_{\Pi}p = R'_{\Pi^*}p = p$ or $R'_{\Pi}p = R'_{\Pi^*}p \neq p$. The assertion for the former case follows readily from (a) and (b), while the proof of the latter follows from the fact that $\nu_{\Pi} \parallel (p - R'_{\Pi}p) \parallel \nu_{\Pi^*}$. $\sharp$

4.4.2 First class of unidentifiable gratings

We are now ready to start our process to find all the unidentifiable gratings corresponding to the incident field $E^i = se^{iq\cdot x}$. We shall determine and classify these structures into three classes. The first unidentifiable class corresponds to the special case when all the planes in $\mathcal{P}$ are parallel to Π_0, as stated in the following lemma.

Lemma 4.4.3. *If all the planes in $\mathcal{P}$ are parallel to Π_0, then both Π_0 and S are parallel to $\{x_3 = 0\}$, and the distance between any two neighboring perfect planes in $\mathcal{P}$ is the same, equal to π/β.*

On the other hand, if Π is a plane that is parallel to Π_0 and the distance between Π and Π_0 is some integer multiple of π/β, then Π is a perfect plane in $\mathcal{P}$.

Proof. By Proposition 4.4.4 we know that each face of any Π_0-reflecting periodic part S_l of S can be reflected by Π_0 into a plane in $\mathcal{P}$, so they should be parallel to the plane Π_0. Due to the periodic structure of S, all the faces of S must be parallel to Π_0 as well. But this is possible only if S is a plane in $\mathbf{R}^3$. Noting that S is bounded in the x_3 direction, both S and Π_0 are parallel to the plane $\{x_3 = 0\}$.

To see that the distance between any two neighboring perfect planes in $\mathcal{P}$ is the same, let Π be a plane in $\mathcal{P}$, then $\Pi \parallel \Pi_0 \parallel \{x_3 = 0\}$. Let $q^* = R'_\Pi q$, then $q^* \neq q$ since otherwise $q \parallel \Pi \parallel \{x_3 = 0\}$. By Proposition 4.4.3, we have $\{q, q^*\} \subset \mathcal{Q}$. We now claim that $\mathcal{Q} = \{q, q^*\}$. If it is not true, there is some $\tilde{q} \in \mathcal{Q}$ such that $\tilde{q} \neq q, q^*$. By Proposition 4.4.3, $R'_\Pi \tilde{q} \in \mathcal{Q}$. As $\Pi \parallel \{x_3 = 0\}$, either $\tilde{q}$ or $R'_\Pi \tilde{q}$ has a non-positive x_3 component, which is in contradiction to Proposition 4.4.1.

Now, we have $q^* = R'_{\Pi_0} q$ and $\mathcal{Q} = \{q, q^*\}$. One can see easily that $q^* = (\alpha_1, \alpha_2, \beta) = q^{(0,0)}$. We further write the propagating field E_p in (4.4.2) as $E_p = s e^{iq\cdot x} + A^* e^{iq^*\cdot x}$. Applying equality (4.4.4) respectively to the planes Π_0 and Π, we get

$$s + R'_{\Pi_0} A^* e^{iq^* \cdot R_{\Pi_0} 0} = 0, \tag{4.4.5}$$

$$s + R'_{\Pi} A^* e^{iq^* \cdot R_{\Pi} 0} = 0. \tag{4.4.6}$$

From equation (4.4.5), we see that $A^* = e^{iq^* \cdot R_{\Pi_0} 0} R'_{\Pi_0} s$. Thus E_p is totally determined by the incident field E^i and the plane Π_0. Since $\Pi \parallel \Pi_0$, $R'_{\Pi_0} A^* = R'_\Pi A^*$, equation (4.4.5) minus equation (4.4.6) yields

$$R'_\Pi A^* (e^{iq^* \cdot R_{\Pi_0} 0} - e^{iq^* \cdot R_\Pi 0}) = 0\,,$$

which implies

$$e^{iq^* \cdot (R_{\Pi_0} 0 - R_\Pi 0)} = 1\,. \tag{4.4.7}$$

Let $\mathrm{dist}(\Pi, \Pi_0)$ be the distance between the planes Π_0 and Π, then we see that $R_{\Pi_0} 0 - R_\Pi 0 = \pm 2\ \mathrm{dist}(\Pi, \Pi_0) e_3$. By substituting this equality into the above equation (4.4.7) and using the fact that $q^* \cdot e_3 = -q \cdot e_3 = \beta$, we get $\text{dist}(\Pi, \Pi_0) = \frac{m\pi}{\beta}$ for some integer m.

Finally, let Π be an plane in $\mathbb{R}^3$ that is parallel to Π_0 with a distance of some integer multiple of π/β from Π_0. Then one can easily deduce using the above derivations that the relation (4.4.4) holds for Π, therefore Π is also a perfect plane in $\mathcal{P}$ by the Reflection Principle(Lemma 4.3.1). This completes the proof of Lemma 4.4.3. $\sharp$

Let r be an arbitrary point in $\mathbf{R}^3$, then Lemma 4.4.3 yields the first class of unidentifiable gratings corresponding to the incident field $E^i(x) = se^{iq\cdot x}$:

$$\mathcal{S}_1(q,r) = \Big\{\text{all planes which are parallel to } \{x_3 = 0\} \text{ and have equal distance}$$
$$\pi/\beta \text{ between each other, with } r \text{ lying on one of the planes}\Big\}.$$

Using the class $\mathcal{S}_1(q,r)$, we can conclude that any two gratings belonging to the class $\mathcal{S}_1(q,r)$ can not be distinguished by the incident wave E^i. To see this, let S_1 and S_2 be two planes in $\mathcal{S}_1(q,r)$, then E_p determined in the proof of Lemma 4.4.3 is the solution to the system (4.1.1)-(4.1.4) with the boundary S replaced respectively by S_1 and S_2. Therefore two different planes S_1 and S_2 correspond to the same total field generated by the incident wave E^i in the domain above S_1 and S_2.

4.4.3 Preparation for finding other classes of unidentifiable gratings

We considered the special case in Subsection 4.4.2 where all perfect planes are parallel to the same plane Π_0. We now proceed to study the more general case when there is some plane in $\mathcal{P}$ which is not parallel to Π_0. As we shall see, this case leads to two other classes of unidentifiable gratings. For the purpose, let Π_1 be a plane in $\mathcal{P}$ which is not parallel to the plane Π_0. Then we introduce the following notations:

L: the line of intersection between Π_0 and Π_1;

r: an arbitrary but fixed point on L;

ν: a unit tangential vector along L, with nonnegative x_3 component;

Γ: the plane in $\mathbf{R}^3$, which passes through the origin and has normal ν. For the convenience, we assign an orientation to the plane Γ: the normal ν and Γ form a right-handed coordinate system.

P_Γ: the projection from $\mathbf{R}^3$ onto Γ.

T_θ: the rotation on the plane Γ about the origin by angle θ. Clearly T_θ can be also viewed as a rotation in the whole space $\mathbf{R}^3$ about the axis fixed to be a line passing through the origin and parallel to direction ν. In both cases, the rotation is understood to be anticlockwise with respect to the assigned orientation on the plane Γ.

We remark that $\nu \nparallel e_3$, otherwise one gets $e_3 \parallel \Pi_0$ by noting that $\nu \parallel \Pi_0$, contradicting the assumption that Π_0 is not perpendicular to the plane $\{x_3 = 0\}$. As a result, Γ does not coincide with the plane $\{x_3 = 0\}$.

Viewing (ν, Γ) as a coordinate system, we can split all vectors in $\mathcal{Q}$ as follows:

$$q = \tau\nu + P_\Gamma q, \quad q^n = \tau_n \nu + P_\Gamma q^n \quad \text{for } n \in \Xi_0 \tag{4.4.8}$$

where τ and τ_n are constants. It is important to observe that

Lemma 4.4.4. *For any $p \in \mathcal{Q}$, $P_\Gamma p \neq 0$.*

Proof. We show by contradiction. If $P_\Gamma p = 0$, then p is parallel to both Π_0 and Π_1, thus $p = R'_{\Pi_0} p$ and $p = R'_{\Pi_1} p$. By Lemma 4.4.2 (3), we know $\Pi_0 \parallel \Pi_1$, which contradicts to the choice of Π_0 and Π_1. $\sharp$

Now we define

$$\begin{aligned} \mathcal{P}_0 &= \{\Pi;\ \Pi \text{ is a perfect plane of } E_p \text{ and passes through the line } L\}, \\ G &= \text{the group generated by the reflections } \{R'_\Pi;\ \Pi \in \mathcal{P}_0\}. \end{aligned}$$

Lemma 4.4.5. *The following properties are valid for the set $\mathcal{P}_0$ and group G:*

(1) The number of perfect planes in $\mathcal{P}_0$ is finite, in fact $|\mathcal{P}_0| \leq |\mathcal{Q}|$.

(2) G consists of $|\mathcal{P}_0|$ reflections and $|\mathcal{P}_0|$ rotations, so it has the structure of a dihedral group of order $2|\mathcal{P}_0|$; and the angles formed by any two neighboring planes in $\mathcal{P}_0$ are all equal.

Proof. To see (1), we take any two planes Π^* and Π^{**} in $\mathcal{P}_0$. One can easily derive by using (4.4.8) and $P_\Gamma q \neq 0$ (Lemma 4.4.4) that the following three relations are equivalent:

(a) $R'_{\Pi^*}q = R'_{\Pi^{**}}q$; (b) $R'_{\Pi^*}(P_\Gamma q) = R'_{\Pi^{**}}(P_\Gamma q)$; (c) $\Pi^* = \Pi^{**}$.

Thus $|\{R'_\Pi q\}_{\Pi\in\mathcal{P}_0}| = |\mathcal{P}_0|$. Moreover, we have $\{R'_\Pi q;\ \Pi \in \mathcal{P}_0\} \subseteq \mathcal{Q}$ by Proposition 4.4.3, hence $|\mathcal{P}_0| \leq |\mathcal{Q}| < +\infty$.

To show (2), let Π^* and Π^{**} be two planes in $\mathcal{P}_0$ such that the angle, say θ, formed by Π^* and Π^{**} is the smallest among all the angles formed by any two planes in $\mathcal{P}_0$. Then we have $\theta \leq \frac{\pi}{|\mathcal{P}_0|}$. By Proposition 4.4.3, we see that $R_\Pi \mathcal{P}_0 = \mathcal{P}_0$ for $\Pi = \Pi^*$ or Π^{**}. Thus the planes generated by rotating the plane Π^* about the axis L with angles of integral multiples of θ belong to $\mathcal{P}_0$. We can get at least $|\mathcal{P}_0|$ such planes since $\theta \leq \frac{\pi}{|\mathcal{P}_0|}$. From this we can conclude that all the planes in $\mathcal{P}_0$ can be generated in this manner, and that the angles formed by any two neighboring planes in $\mathcal{P}_0$ are all equal. Then it is easy to check that G consists of $|\mathcal{P}_0|$ reflections and $|\mathcal{P}_0|$ rotations, hence G has the structure of a *dihedral group* of order 2 $|\mathcal{P}_0|$. ♯

In the sequel, we denote by G^* the subgroup of G which consists of all its rotations. Clearly, we know $|G^*| = |\mathcal{P}_0|$. Note that the identity element, denoted by Id, of both the group G and G^* is the rotation by the angle 2π. When the domain which the transformations in G act on is restricted to the plane Γ, G reduces to the *dihedral group* which acts only on the vectors lying on the plane Γ.

Next, we present a few more useful properties about the group G and the set $\mathcal{P}_0$.

Lemma 4.4.6. *The following properties hold for the group G and the set $\mathcal{P}_0$:*

1. *For any $T \in G$, $T\mathcal{Q} = \mathcal{Q}$ and $T\nu = \nu$; For any $q^n \in G\{q\}$, $\tau_n = \tau$.*

2. $|\{R'_\Pi q;\ \Pi \in \mathcal{P}_0\}| = |G^*\{q\}| = |\mathcal{P}_0|$.

3. $|G\{q\}| = 2|\mathcal{P}_0|$ *or* $|G\{q\}| = |\mathcal{P}_0|$. *If* $|G\{q\}| = 2|\mathcal{P}_0|$, *there exists some* $q^1 \in G\{q\}$ *such that* $G\{q\} = G^*\{q\} \bigcup G^*\{q^1\}$; *if* $|G\{q\}| = |\mathcal{P}_0|$, *then* $G\{q\} = G^*\{q\}$ *and there exists a plane* Π *in* $\mathcal{P}_0$ *such that* $R'_\Pi q = q$.

4. $|G\{q\}| \geq 2$ *and there exists at least one element in* $G\{P_\Gamma q\}$ *whose* x_3 *component is non-positive. Furthermore, if* $|G\{q\}| \geq 4$*, then there are at least two elements in* $G\{P_\Gamma q\}$*, that have a non-positive* x_3 *component.*

Proof. To see (1), noting the fact that $\Pi \parallel \nu$ for all $\Pi \in \mathcal{P}_0$, we have $R'_\Pi \nu = \nu$ for all $\Pi \in \mathcal{P}_0$. On the other hand, it follows from Proposition 4.4.3 that $R'_\Pi \mathcal{Q} = \mathcal{Q}$ for all $\Pi \in \mathcal{P}_0$. Then by the definition of the group G, we know $T\mathcal{Q} = \mathcal{Q}$ and $T\nu = \nu$ for all $T \in G$. To see $\tau_n = \tau$ for any $q^n \in G\{q\}$, we have $G\{q\} = G\{\tau\nu + P_\Gamma q\} = \tau\nu + G\{P_\Gamma q\}$ by the decomposition (4.4.8). Since all the elements in $G\{P_\Gamma q\}$ lie on the plane Γ, the desired result follows immediately.

Next we consider (2). We note that $R'_\Pi q = R'_{\Pi^*} q$ if and only if $\Pi = \Pi^*$ for any two given planes Π and Π^* in $\mathcal{P}_0$. Therefore we have $|\{R'_\Pi q; \Pi \in \mathcal{P}_0\}| = |\mathcal{P}_0|$. In addition, we know $P_\Gamma q \neq 0$ by Lemma 4.4.4, so we derive $|G^*\{q\}| = |G^*\{P_\Gamma q\}| = |G^*| = |\mathcal{P}_0|$.

To show (3), we consider the stabilizer subgroup G_q of q, i.e., $G_q = \{T \in G;\ Tq = q\}$. It follows that $P_\Gamma q \neq 0$ by Lemma 4.4.4, so we know the only rotation in G which maps q into itself is the identity. On the other hand, there is at most one reflection in G which transforms q into itself. Hence we know $G_q = \{Id\}$ or $G_q = \{Id, R'_\Pi\}$ for some $\Pi \in \mathcal{P}_0$, then $|G_q| = 1$ or $|G_q| = 2$. By the orbit-stabilizer theorem and Lagrange's theorem, $|G\{q\}| = \frac{|G|}{|G_q|}$, which implies $|G\{q\}| = 2|\mathcal{P}_0|$ or $|G\{q\}| = |\mathcal{P}_0|$.

If $|G\{q\}| = |\mathcal{P}_0|$, we see from the previous analysis that $|G_q| = 2$, hence there is a reflection in G which maps q into itself. That is, there exists a plane Π in $\mathcal{P}_0$ such that $R'_\Pi q = q$. For the case with $|G\{q\}| = 2|\mathcal{P}_0| = 2|G^*|$, we first note from the group's property that either $G^*\{q^*\} = G^*\{q\}$ or $G^*\{q^*\} \bigcap G^*\{q\} = \emptyset$ for all $q^* \in G\{q\}$. Therefore there exists some $q^1 \in G\{q\} \setminus G^*\{q\}$. Observing that $|G^*\{q\}| + |G^*\{q^1\}| = 2|\mathcal{P}_0| = |G\{q\}|$, we have $G\{q\} = G^*\{q\} \bigcup G^*\{q^1\}$.

Finally we consider (4). Clearly it follows from (3) that $|G\{q\}| \geq |\mathcal{P}_0| \geq 2$. On the other hand, we can easily verify that the sum of the vectors in $G^*\{P_\Gamma q\}$ is zero as the set $G^*\{P_\Gamma q\}$ is actually formed by the vertices of a regular n-sided polygon ($n = |G^*\{P_\Gamma q\}|$) centered at the origin. Therefore there exists at least one element in

$G^*\{P_\Gamma q\} \subset G\{P_\Gamma q\}$ whose x_3 component is non-positive.

We are left with the case when $|G\{q\}| \geq 4$. Clearly we have $|G^*\{q\}| \geq 4$ or $|G^*\{q\}| = 2$. For the former case, there exist at least two elements in $G^*\{P_\Gamma q\} \subseteq G\{P_\Gamma q\}$ which have a non-positive x_3 component, by using again the fact that the set $G^*\{P_\Gamma q\}$ consists of the vertices of a regular polygon centered at the origin. For the case with $|G^*\{q\}| = 2$, we know $|G\{q\}| = 4$, then it follows from the previously proved result (3) that

$$G\{P_\Gamma q\} = G^*\{P_\Gamma q\} \bigcup G^*\{P_\Gamma q^1\}.$$

In either of the sets $G^*\{P_\Gamma q\}$ and $G^*\{P_\Gamma q^1\}$, one can find at least one element whose x_3 component is non-positive. This completes the proof of Lemma 4.4.6. $\sharp$

Recalling that the line L is the intersection line between Π_0 and Π_1, L is either parallel to the plane $\{x_3 = 0\}$ or unparallel. When L is parallel to the plane $\{x_3 = 0\}$, we can show

Lemma 4.4.7. *If $L \parallel \{x_3 = 0\}$, then $\mathcal{Q} = G\{q\}$, and $|\mathcal{P}_0| \in \{2, 3\}$.*

Proof. To see $\mathcal{Q} = G\{q\}$, it suffices to show that $\mathcal{Q} \setminus G\{q\} = \emptyset$. If this is not true, there is some $q^m \in \mathcal{Q} \setminus G\{q\}$. Then we can write $G\{q^m\} = \tau_m \nu + G\{P_\Gamma q^m\}$. By Lemma 4.4.6, at least one element from $G\{P_\Gamma q^m\}$ has a non-positive x_3 component, so does at least one element from $G\{q^m\}$ by noting that the x_3 component of ν is zero, leading to a contradiction with Proposition 4.4.1. Therefore we have $\mathcal{Q} = G\{q\}$.

As the x_3 component of ν is zero, we see from the decomposition (4.4.8) that the sign of the x_3 components of the vectors in $G\{q\}$ are determined by those in $G\{P_\Gamma q\}$. If $|\mathcal{P}_0| > 3$, then $|G\{q\}| \geq |\mathcal{P}_0| > 3$. Thus there are at least two elements in $G\{P_\Gamma q\}$ whose x_3 components are non-positive by Lemma 4.4.6(3), so does $G\{q\}$. This contradicts to Proposition 4.4.1. So we have proved that $|\mathcal{P}_0| \in \{2, 3\}$. $\sharp$

Based on Lemma 4.4.7, we will separate our subsequent analysis into two cases: $L \parallel \{x_3 = 0\}$ or $L \nparallel \{x_3 = 0\}$. The former is considered in Subsection 4.4.5 and 4.4.6, while the latter is studied in Subsection 4.4.7. As we shall see, the former case leads us to two classes of grating structures which can not be identified by the incident field E^i, while the latter case will be shown not to happen at all.

4.4.4 A simple transformation

In the subsequent analysis, we will frequently use a simple transformation by change of variables, that can significantly simplify many derivations. To do so, we fix one point r on the line L, then introduce the following simple transformation by change of variables:

$$\hat{x} = x - r. \tag{4.4.9}$$

In terms of $\hat{x}$-variable, the propagating field $E_p(x)$ in (4.4.2) takes the form

$$E_p(x) = E_p(\hat{x}+r) =: \hat{E}_p(\hat{x}) = se^{iq\cdot(\hat{x}+r)} + \sum_{n\in\Xi_0} A^n e^{iq^n\cdot(\hat{x}+r)}.$$

By setting $\hat{s} = se^{iq\cdot r}$ and $\hat{A}^n = A^n e^{iq^n\cdot r}$, we can write $\hat{E}_p(\hat{x})$ into

$$\hat{E}_p(\hat{x}) = \hat{s}e^{iq\cdot\hat{x}} + \sum_{n\in\Xi_0} \hat{A}^n e^{iq^n\cdot\hat{x}}. \tag{4.4.10}$$

A significant advantage of using the $\hat{x}$-variable, instead of the original x-variable, can be seen from the following lemma, where we write $\hat{s}$ for $\hat{A}^0$ and q for q^0.

Lemma 4.4.8. *Let Π be a plane passing through the line L, the intersection line between Π_0 and Π_1 (see the beginning of Section 4.4.3), then $\Pi \in \mathcal{P}_0$ if and only if $R'_\Pi \mathcal{Q} \subset \mathcal{Q}$ and the relation $R'_\Pi \hat{A}^l + \hat{A}^m = 0$ holds whenever $R'_\Pi q^l - q^m = 0$ for $q^l, q^m \in \mathcal{Q}$.*

Proof. We first note that $R_\Pi r = r$ since Π passing through the line L. Then it follows that

$$R_\Pi x = R_\Pi(\hat{x}+r) = R'_\Pi(\hat{x}+r) + R_\Pi 0 = R'_\Pi\hat{x} + R'_\Pi r + R_\Pi 0 = R'_\Pi\hat{x} + R_\Pi r = R'_\Pi\hat{x} + r,$$

and consequently,

$$E_p(R_\Pi x) = \hat{E}_p(R'_\Pi\hat{x}).$$

As a result, the Reflection Principle (Lemma 4.3.1) can be written in terms of $\hat{E}_p(\hat{x})$ as follows: $\Pi \in \mathcal{P}_0$ if and only if the following equation holds:

$$\hat{E}_p(\hat{x}) + R'_\Pi(\hat{E}_p(R'_\Pi(\hat{x}))) = 0 \quad \forall\, \hat{x} \in \mathbf{R}^3. \tag{4.4.11}$$

Now, by substituting the expression (4.4.10) into the equation (4.4.11) and making use of Lemma 4.3.3, we obtain the desired results by comparing the coefficients of the Fourier modes. ♯

4.4.5 Second class of unidentifiable gratings

We now start to consider the case when $L \parallel \{x_3 = 0\}$. By Lemma 4.4.7, we know $|\mathcal{P}_0| \in \{2, 3\}$. We will study the case $|\mathcal{P}_0| = 2$ in this subsection, and the case $|\mathcal{P}_0| = 3$ in the next subsection. These two cases will lead to two new classes of unidentifiable grating structures corresponding to the incident field $E^i(x) = se^{iq\cdot x}$.

We first derive a few properties corresponding to the case $|\mathcal{P}_0| = 2$.

Lemma 4.4.9. *If $L \parallel \{x_3 = 0\}$ and $|\mathcal{P}_0| = 2$, then it holds that*

1. *$\mathcal{Q} = G\{q\}$ and $|G\{q\}| = 2$.*
2. *$L \parallel (s \times e_3)$.*
3. *$\mathcal{P}_0$ consists of exactly two planes, namely Π_0 and Π_1, one is perpendicular to s, and the other perpendicular to $s \times (s \times e_3)$. For ease of subsequent exposition, we shall assume (possibly after relabeling) that $\Pi_0 \perp s$ and $\Pi_1 \perp s \times (s \times e_3)$.*
4. *Let $q^1 = R'_{\Pi_1} q = T_\pi q$, then the propagating field $E_p(x)$ in (4.4.2) or $\hat{E}_p(\hat{x})$ in (4.4.10) can be written as*

$$E_p(x) = s(e^{iq\cdot x} - e^{iq^1\cdot x + i(q-q^1)\cdot r}) \quad or \quad \hat{E}_p(\hat{x}) = \hat{s}(e^{iq\cdot\hat{x}} - e^{iq^1\cdot\hat{x}}). \tag{4.4.12}$$

Proof. It follows from Lemma 4.4.7 that $\mathcal{Q} = G\{q\}$. So we have $|G\{q\}| = 2$ or $|G\{q\}| = 4$ by Lemma 4.4.6 (3). But one concludes from Lemma 4.4.6 (4) that it is only possible to have $|G\{q\}| = 2$, leading to (1).

By (1) and Lemma 4.4.5 we know $\mathcal{P}_0 = \{\Pi_0, \Pi_1\}$, and $\Pi_0 \perp \Pi_1$. We know from Lemma 4.4.6 (3) that there exists a plane in $\mathcal{P}_0$, which we may assume to be Π_0 (possibly after relabeling the subscripts) such that $R'_{\Pi_0} q = q$. Let $q^1 = R'_{\Pi_1} q$, we have by Lemma 4.4.6 (3) that $\{q, q^1\} = G\{q\} = \mathcal{Q}$. Using this fact and the transformation (4.4.9), we can write $E_p(x) = \hat{E}_p(\hat{x})$ in (4.4.10) as

$$\hat{E}_p(\hat{x}) = \hat{s} e^{iq\cdot\hat{x}} + \hat{A}^1 e^{iq^1\cdot\hat{x}}.$$

Then we derive with the help of Lemma 4.4.8 and the relations $R'_{\Pi_0} q = q$ and $q^1 = R'_{\Pi_1} q = T_\pi q$ that

$$\hat{s} = -R'_{\Pi_0}\hat{s}, \quad \hat{A}^1 = -R'_{\Pi_1}\hat{s}. \tag{4.4.13}$$

The first relation in (4.4.13) implies $\hat{s} \perp \Pi_0$. Noting that L lies on the plane Π_0, we have $\hat{s} \perp L$. But we know $L \perp e_3$ from the assumption, so $L \parallel (\hat{s} \times e_3)$. Observing that vector s differs from $\hat{s}$ only by a scalar, we obtain immediately that $L \parallel (s \times e_3)$, hence we have proved (2) and (3). Finally, directly using the relations $\hat{s} \perp \Pi_0$ and $\Pi_0 \perp \Pi_1$ we know $\hat{s} \parallel \Pi_1$, which implies $\hat{A}^1 = -R'_{\Pi_1}\hat{s} = -\hat{s}$, hence proves (4). ♯

The next lemma indicates that all the perfect planes of the propagating field E_p are determined by the incident field $E^i = s e^{iq\cdot x}$.

Lemma 4.4.10. *If the line L is parallel to the plane $\{x_3 = 0\}$ and $|\mathcal{P}_0| = 2$, then*

1. *All the perfect planes of E_p, i.e. the set $\mathcal{P}$, are determined by the incident field $E^i = s e^{iq\cdot x}$. More specifically, $\mathcal{P}$ consists of only two sets of planes: the first set contains only all the planes that are parallel to Π_0; while the second set contains only all the planes that are parallel to Π_1 and have the distance $\frac{\pi}{\|P_\Gamma q\|}$ between each two neighboring planes.*

2. *Each face of the grating structure S lies on a plane in $\mathcal{P}$.*

Proof. We first show that all the planes in $\mathcal{P}$ are parallel to either Π_0 or Π_1. To see this, we consider a plane $\Pi \in \mathcal{P}$. By Proposition 4.4.3 we have $R'_\Pi \mathcal{Q} = \mathcal{Q}$, i.e.

$R'_\Pi\{q,q^1\}=\{q,q^1\}$. Using Lemma 4.4.8, we know that $R'_\Pi q=q$ implies $\Pi\parallel\Pi_0$, and $R'_\Pi q=q^1$ implies $\Pi\parallel\Pi_1$.

Next, it is easy to see by the definition of a perfect plane and the expression (4.4.12) that all the planes parallel to Π_0 belong to $\mathcal{P}$. So it suffices to consider only the planes that are parallel to Π_1. Let Π be such a plane in $\mathcal{P}$. By applying the relation (5.1.4) (Reflection Principle) to the plane Π_1 and Π, we obtain the following equations

$$E_p(x)+R'_{\Pi_1}(E_p(R_{\Pi_1}(x)))=0,\quad E_p(x)+R'_\Pi(E_p(R_\Pi(x)))=0.$$

Subtracting the second equation from the first one yields

$$R'_{\Pi_1}(E_p(R_{\Pi_1}(x)))-R'_\Pi(E_p(R_\Pi(x)))=0.$$

Noting that $R'_{\Pi_1}=R'_\Pi$, the above relation reduces to

$$E_p(R_{\Pi_1}(x))-E_p(R_\Pi(x))=0.$$

Hence If we substitute the expression (4.4.12) into this equation and make use of the equality $R_\Pi x=R'_\Pi x+R_\Pi 0=R'_{\Pi_1}x+R_\Pi 0$ and the symmetry of the reflection transformation R'_{Π_1}, we further deduce

$$e^{i(R'_{\Pi_1}q\cdot x+q\cdot R_{\Pi_1}0)}-e^{i(R'_{\Pi_1}q^1\cdot x+q^1\cdot R_{\Pi_1}0+(q-q^1)\cdot r)}=e^{i(R'_\Pi q\cdot x+q\cdot R_\Pi 0)}-e^{i(R'_\Pi q^1\cdot x+q^1\cdot R_\Pi 0+(q-q^1)\cdot r)},$$

which implies by using Lemma 4.3.3 that

$$e^{iq\cdot(R_\Pi 0-R_{\Pi_1}0)}=1,\quad e^{iq^1\cdot(R_\Pi 0-R_{\Pi_1}0)}=1. \tag{4.4.14}$$

Now, we claim that $|q\cdot(R_\Pi 0-R_{\Pi_1}0)|=|q^1\cdot(R_\Pi 0-R_{\Pi_1}0)|=2\mathrm{d}(\Pi_1,\Pi)\|P_\Gamma q\|$, where $\mathrm{d}(\Pi_1,\Pi)$ is the distance between Π_1 and Π. Indeed, by using the relation $\nu\cdot\nu_{\Pi_1}=0$ and $R_\Pi 0-R_{\Pi_1}0=\pm 2\mathrm{d}(\Pi_1,\Pi)\nu_{\Pi_1}$, we get $|q\cdot(R_\Pi 0-R_{\Pi_1}0)|=2\mathrm{d}(\Pi_1,\Pi)|P_\Gamma q\cdot\nu_{\Pi_1}|$ and $|q^1\cdot(R_\Pi 0-R_{\Pi_1}0)|=2\mathrm{d}(\Pi_1,\Pi)|P_\Gamma q^1\cdot\nu_{\Pi_1}|$. But we know from Lemma 4.4.9 that $P_\Gamma q$ and ν_{Π_1} are parallel to each other since they both are perpendicular to the vector s and ν. Thus $|P_\Gamma q\cdot\nu_{\Pi_1}|=\|P_\Gamma q\|$ and our claim follows readily from the observation that $P_\Gamma q^1=T_\pi P_\Gamma q=-P_\Gamma q$. By the claim we know that the two equalities in (4.4.14) hold if and only if the distance between Π and Π_1 is some multiple of $\frac{\pi}{\|P_\Gamma q\|}$. Therefore, we have shown that if $\Pi \parallel \Pi_1$ is a plane in $\mathcal{P}$ then $\mathrm{d}(\Pi_0, \Pi)$ equals some multiple of $\frac{\pi}{\|P_\Gamma q\|}$. On the other hand, if Π is a plane such that $\Pi \parallel \Pi_1$ and that $\mathrm{d}(\Pi_0, \Pi)$ equals some multiple of $\frac{\pi}{\|P_\Gamma q\|}$, then one can reverse the above deduction to show that the equation (5.1.4) holds for all $x \in \mathbf{R}^3$. Then by the Reflection Principle, we conclude that $\Pi \in \mathcal{P}$. This completes the proof of the first part of the lemma.

Finally, we show the second part of the lemma. Let $\tilde{F}$ be an open face of the grating S and F be the plane such that $\tilde{F} \subset F$. It suffices to prove that $F \subset \mathcal{P}$. For this purpose, consider the four sequences of open faces $\{\tilde{F} + m\Lambda_1 e_1\}_{m<0}$, $\{\tilde{F} + m\Lambda_1 e_1\}_{m>0}$, $\{\tilde{F} + m\Lambda_2 e_2\}_{m<0}$ and $\{\tilde{F} + m\Lambda_2 e_2\}_{m>0}$, all of which lying on S due to the periodicity of the grating S. By the definition of a Π_0-reflecting periodic part, we see that for sufficiently large $m_0 > 0$, one of the sequences $\{\tilde{F} + m\Lambda_1 e_1\}_{m<-m_0}$, $\{\tilde{F} + m\Lambda_1 e_1\}_{m>m_0}$, $\{\tilde{F} + m\Lambda_2 e_2\}_{m<-m_0}$ and $\{\tilde{F} + m\Lambda_2 e_2\}_{m>m_0}$, belongs to the Π_0-reflecting periodic part of S. Let $\{\tilde{F} + m\Lambda_1 e_1\}_{m<-m_0}$ be such a sequence. Then Proposition 4.4.4 implies that all the open faces $\{\tilde{F} + m\Lambda_1 e_1\}$ for $m < -m_0$ lie on planes in $\mathcal{P}$. Choose $n_0 > m_0$, then $\tilde{F}$ is the reflection of $\tilde{F} - 2n_0\Lambda_1 e_1$ with respect to $\tilde{F} - n_0\Lambda_1 e_1$. By Reflection Principal (Lemma 4.3.1) again, we know $\tilde{F}$ is a perfect plane of E_p, so is the plane F, namely $F \in \mathcal{P}$. This completes the proof of Lemma 4.4.10. ♯

Lemma 4.4.10 enables us to find a new class of unidentifiable grating profiles. To describe the class explicitly, we first clarify some notations:

r: a position vector, viewed as a point in $\mathbf{R}^3$;

Γ: a plane which passes through the origin with normal $s \times e_3$;

Π_0: a plane which passes through r with normal s;

Π_1: a plane which passes through r with normal $(s \times e_3) \times s$.

Then by Lemma 4.4.10 all the perfect planes of E_p can be described by

$$\mathcal{P} = \Big\{\text{plane } \Pi;\ \Pi \parallel \Pi_0\Big\} \bigcup \Big\{\text{plane } \Pi;\ \Pi \parallel \Pi_1,\ \operatorname{dist}(\Pi, \Pi_1) = \frac{m\pi}{\|P_\Gamma q\|} \text{ for some } m \in \mathbf{N}\Big\}.$$

This suggests a new class of unidentifiable gratings corresponding to the incident field

$E^i = s\, e^{iq\cdot x}$:

$$\mathcal{S}_2(s,q,\Lambda,r) \;=\; \Big\{\text{gratings with profile } S\text{, which are }\Lambda\text{-periodic polyhedral structures such that faces of } S \text{ lie on planes in } \mathcal{P}\Big\}.$$

One can see that each class $\mathcal{S}_2(s,q,\Lambda,r)$ corresponds to a unique electric field E_p, which solves the direct scattering problem (4.1.1)-(4.1.4) for any gratings in $\mathcal{S}_2(s,q,\Lambda,r)$. So any two grating in $\mathcal{S}_2(s,q,\Lambda,r)$ can not be identified by the incident field $E^i(x) = se^{iq\cdot x}$. This proves

Lemma 4.4.11. *If the line L is parallel to $\{x_3 = 0\}$ and $|\mathcal{P}_0| = 2$, then the grating profile S belongs to $\mathcal{S}_2(s,q,\Lambda,r)$ for some point $r \in \mathbf{R}^3$. Furthermore, all gratings in $\mathcal{S}_2(s,q,\Lambda,r)$ generate the same total field.*

Next we give a concrete example which has a non-empty class $\mathcal{S}_2(s,q,\Lambda,r)$.

Example 4.4.1. *Let $\Lambda = (\Lambda_1,\Lambda_2) = (\frac{2\pi}{\beta},0)$, $q = (\alpha_1,\alpha_2,-\beta) = (-\beta,\alpha_2,-\beta)$, $s = e_1 - e_3 = (1,0,-1)$ and r be the origin. Then we can check by direct computations that $(s\times e_3) \parallel e_2$, and*

$$q^1 = T_\pi q = (\beta,\alpha_2,-\beta) = q^{(1,0)} = (\alpha^{(1,0)} + \alpha) + (0,0,\beta^{(1,0)}).$$

In this case $\mathcal{P}$ consists of two sets of planes: the first set contains planes that are perpendicular to the vector $s = e_1 - e_3$; and the second set contains planes that are perpendicular to the vector $e_1 + e_3$ and have the distance $\frac{\pi}{\|P_\Gamma q\|} = \frac{\sqrt{2}\pi}{\beta}$ between each two neighboring planes.

Next we try to find some Λ-periodic structures in $\mathcal{P}$. For each $l \in \mathbf{Z}$, we denote by L_{2l} and L_{2l+1} the line of the form $\{(\frac{2l\pi}{\beta},\lambda,0);\ \lambda\in\mathbf{R}\}$ and $\{(\frac{(2l+1)\pi}{\beta},\lambda,\frac{\pi}{\beta});\ \lambda\in\mathbf{R}\}$, respectively. For any $m \in \mathbf{Z}$, let Π_m be the plane that passes through the line L_m and L_{m+1}, and F_m be the part on Π_m which lies between the line L_m and L_{m+1}. Then it is clear that Π_m belongs to $\mathcal{P}$ for all $m\in\mathbf{Z}$ and that $\bigcup_{m\in\mathbf{Z}} F_m$ forms a $\Lambda-$periodic structure in $\mathcal{S}_2(s,q,\Lambda,r)$. By appropriate translations, we see that there are infinitely many Λ-periodic structures in $\mathcal{S}_2(s,q,\Lambda,r)$.

4.4.6 Third class of unidentifiable gratings

Continuing the discussion in the previous subsection, we now consider the case when $L \parallel \{x_3 = 0\}$ and $|\mathcal{P}_0| = 3$. This case will lead to the third class of unidentifiable gratings corresponding to the given incident field E^i. We first present some helpful properties.

Lemma 4.4.12. *If the line L is parallel to the plane $\{x_3 = 0\}$ and $|\mathcal{P}_0| = 3$, then we have*

1. $\mathcal{Q} = G\{q\}$ *and* $|G\{q\}| = 3$.

2. $L \parallel (s \times e_3)$.

3. $\mathcal{P}_0$ *consists of only three planes: one is perpendicular to s, which we denote by Π_0, and the other two are generated by rotating the plane Π_0 about the axis L by angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ respectively, which are labeled as Π_1 and Π_2.*

4. *The propagating field E_p in (4.4.2) or $\hat{E}_p$ in (4.4.10) can be written as*

$$E_p(x) = se^{iq\cdot x} - (T_{\frac{2\pi}{3}}s)e^{iq^1\cdot x+(q-q^1)\cdot r} + (T_{\frac{4\pi}{3}}s)e^{iq^2\cdot x+(q-q^2)\cdot r}, \quad (4.4.15)$$

$$\hat{E}_p(\hat{x}) = \hat{s}e^{iq\cdot\hat{x}} - (T_{\frac{2\pi}{3}}\hat{s})e^{iq^1\cdot\hat{x}} + (T_{\frac{4\pi}{3}}\hat{s})e^{iq^2\cdot\hat{x}}, \quad (4.4.16)$$

where $q^j = R'_{\Pi_j}q = T_{\frac{2j\pi}{3}}q$ for $j = 1, 2$.

Proof. The proof of (1) is the same as that of Lemma 4.4.9 (1). To prove the rest, we may write $\mathcal{P}_0 = \{\Pi_0, \Pi_1, \Pi_2\}$ due to the fact that $|\mathcal{P}_0| = 3$. And we know that each two neighboring planes in $\mathcal{P}_0$ form an angle of $\pi/3$. By Lemma 4.4.6 (3), there exists a plane, say Π, in $\mathcal{P}_0$ such that $R'_{\Pi}q = q$. Without loss of generality, we may assume that this plane is Π_0. For $j = 1, 2$, let

$$q^j = R'_{\Pi_j}q = R'_{\Pi_j}(R'_{\Pi_0}q) = (R'_{\Pi_j} \circ R'_{\Pi_0})q = T_{\frac{2j\pi}{3}}q,$$

then it follows from Part (1) that $\{q, q^1, q^2\} = G\{q\} = \mathcal{Q}$, and we can write $\hat{E}_p$ in (4.4.10) as

$$\hat{E}_p(\hat{x}) = \hat{s}e^{iq\cdot\hat{x}} + \hat{A}^1e^{iq^1\cdot\hat{x}} + \hat{A}^2e^{iq^2\cdot\hat{x}}. \quad (4.4.17)$$

Now using Lemma 4.4.8 and the relation $q^j = R'_{\Pi_j} q$ we deduce

$$\hat{s} = -R'_{\Pi_0}\hat{s}, \quad \hat{A}^1 = -R'_{\Pi_1}\hat{s}, \quad \hat{A}^2 = -R'_{\Pi_2}\hat{s}. \tag{4.4.18}$$

The first equation in (4.4.18) implies $\hat{s} \perp \Pi_0$, and further yields

$$\begin{aligned} \hat{A}^1 &= -R'_{\Pi_1}\hat{s} = (R'_{\Pi_1} \circ R'_{\Pi_0})\hat{s} = T_{2\pi/3}\hat{s}, \\ \hat{A}^2 &= -R'_{\Pi_2}\hat{s} = (R'_{\Pi_2} \circ R'_{\Pi_0})\hat{s} = T_{4\pi/3}\hat{s}. \end{aligned}$$

Finally a similar argument to that of Lemma 4.4.9 (2) leads to the relation $L \parallel (s \times e_3)$. This proves Lemma 4.4.12. ♯

Similarly to the proof of Lemma 4.4.10, we can derive

Lemma 4.4.13. *If the line L is parallel to the plane $\{x_3 = 0\}$ and $|\mathcal{P}_0| = 3$, then*

1. *all the perfect planes of E_p, namely the set $\mathcal{P}$, are determined by the incident field $E^i = se^{iq\cdot x}$. More specifically, $\mathcal{P}$ consists of only three sets of parallel planes, where every two neighboring planes in each set have equal distance $\frac{2\pi}{\sqrt{3}\|P_\Gamma q\|}$: the first set contains only the planes that are parallel to Π_0, the second set contains only the planes that are parallel to Π_1, while the third set contains only the planes that are parallel to Π_2.*

2. *Each face of the grating structure S lies on a plane in $\mathcal{P}$.*

Lemma 4.4.13 leads us to a new class of unidentifiable grating profiles corresponding to the incident field $E^i = s\,e^{iq\cdot x}$. To describe this class explicitly, we first clarify a few notations:

r: a position vector, viewed as a point in $\mathbf{R}^3$;

Γ: a plane which passes through the origin with normal $s \times e_3$;

Π_0: a plane which passes through r with normal s;

Π_1, Π_2: planes which pass through r and form a angle of $\pi/3$ and $2\pi/3$ with Π_0, respectively.

Then by Lemma 4.4.13, we can describe all the perfect planes of E_p in (4.4.2) by

$$\mathcal{P} = \left\{ \text{plane}\,\Pi;\ \exists j \in \{0,1,2\} \text{ such that } \Pi \parallel \Pi_j,\ \text{dist}(\Pi, \Pi_j) = \frac{2m\pi}{\sqrt{3}\|P_\Gamma q\|} \text{ for some } m \in \mathbf{N} \right\},$$

which suggests a new class of unidentifiable gratings corresponding to the incident field E^i:

$$\begin{aligned} \mathcal{S}_3(s,q,\Lambda,r) \;&=\; \Big\{ \text{gratings with profile } S, \text{ which are } \Lambda\text{-periodic polyhedral structures} \\ &\qquad \text{such that faces of } S \text{ lie on planes in } \mathcal{P} \Big\}. \end{aligned}$$

One can see that each class $\mathcal{S}_3(s,q,\Lambda,r)$ corresponds to a unique propagating field E_p, which solves the direct scattering problem (4.1.1)-(4.1.4) for any grating in $\mathcal{S}_2(s,q,\Lambda,r)$. Thus any two gratings in $\mathcal{S}_3(s,q,\Lambda,r)$ can not be identified by the incident field $E^i(x) = se^{iq\cdot x}$. This leads to the following lemma.

Lemma 4.4.14. *If the line L is parallel to the plane $\{x_3 = 0\}$ and $|\mathcal{P}_0| = 3$, then the grating S belongs to $\mathcal{S}_3(s,q,\Lambda,r)$ for some point $r \in \mathbf{R}^3$. Furthermore, all the gratings in $\mathcal{S}_3(s,q,\Lambda,r)$ generate the same total field.*

Next we give a concrete example which has a non-empty class $\mathcal{S}_3(s,q,\Lambda,r)$.

Example 4.4.2. *Let $\Lambda = (\Lambda_1, \Lambda_2) = (\frac{4\pi}{\sqrt{3}}, 0)$, $q = (\alpha_1, \alpha_2, -\beta) = (0,0,-1)$, $s = e_1$ and r be the origin. Then we can check by direct computations that $(s \times e_3) \parallel e_2$, and*

$$\begin{aligned} q^1 &= (\frac{\sqrt{3}}{2}, 0, \frac{1}{2}) = q^{(1,0)} = (\alpha^{(1,0)} + \alpha) + (0,0,\beta^{(1,0)}), \\ q^2 &= (-\frac{\sqrt{3}}{2}, 0, \frac{1}{2}) = q^{(-1,0)} = (\alpha^{(-1,0)} + \alpha) + (0,0,\beta^{(-1,0)}). \end{aligned}$$

Moreover, $\mathcal{P}$ consists of three sets of parallel planes, respectively with e_1, $-e_1 + \sqrt{3}e_3$ and $-e_1 + \sqrt{3}e_3$ as the normal directions, and the distance between any two neighboring parallel planes in each set is $\frac{2\pi}{\sqrt{3}\|P_\Gamma q\|} = \frac{2\pi}{\sqrt{3}}$.

Next we try to find some Λ-periodic structures in $\mathcal{P}$. For each $l \in \mathbf{Z}$, we denote by L_{2l} and L_{2l+1} the line of the form $\{(\frac{4l\pi}{\sqrt{3}}, \lambda, 0);\ \lambda \in \mathbf{R}\}$ and $\{(\frac{2(2l+1)\pi}{\sqrt{3}}, \lambda, \frac{2\pi}{\sqrt{3}});\ \lambda \in \mathbf{R}\}$, respectively. For any $m \in \mathbf{Z}$, let Π_m be the plane that passes through the line L_m and

L_{m+1}, and F_m be the part of Π_m which lies between the line L_m and L_{m+1}. Then it is clear that Π_m belongs to $\mathcal{P}$ for all $m \in \mathbf{Z}$ and that $\bigcup_{m\in\mathbf{Z}} F_m$ forms a Λ-periodic structure in $S_3(s,q,\Lambda,r)$. By appropriate translations, we can find infinitely many Λ-periodic structures in $S_3(s,q,\Lambda,r)$. All these structure can not be distinguished by the incident field E^i.

4.4.7 Excluding the case with $L \nparallel \{x_3 = 0\}$

We have discussed in the subsections 4.4.5-4.4.6 the case that the intersection line L between two perfect planes Π_0 and Π_1 is parallel to the plane $\{x_3 = 0\}$, which has led to two non-trivial classes of unidentifiable grating structures. In this subsection we study the case when the line L is not parallel to the plane $\{x_3 = 0\}$. As we will see, this case can not happen. Recall that the x_3 component of ν is positive when $L \nparallel \{x_3 = 0\}$. We start with the following result that is the foundation of the analysis in this subsection.

Lemma 4.4.15. *If $L \nparallel \{x_3 = 0\}$, then there exists a plane $\Pi^* \in \mathcal{P}$ such that $\Pi^* \nparallel L$.*

Proof. Assume the lemma does not hold, then all the perfect planes of E_p are parallel to the line L. But by Proposition 4.4.4 each face of a Π_0-reflecting periodic part S_l of grating S lies on some perfect plane of E_p. Hence all the faces of S_l are parallel to the line L, so do all the faces of S due to the periodic structure of S. Since $L \nparallel \{x_3 = 0\}$, then the grating profile S can not be bounded in the x_3 direction, which is a contradiction. $\sharp$

Next, we show some useful relations for the subsequent analysis.

Lemma 4.4.16. *If line L is not parallel to the plane $\{x_3 = 0\}$, then it holds that*

$$q + \sum_{n\in\Xi_0} q^n = 0\,, \tag{4.4.19}$$

$$\tau + \sum_{n\in\Xi_0} \tau_n = 0\,, \tag{4.4.20}$$

$$\tau_n > 0 \quad \forall q^n \in \mathcal{Q} \setminus G\{q\}\,. \tag{4.4.21}$$

Proof. We know from Lemma 4.4.15 that there exists a plane $\Pi^* \in \mathcal{P}$ such that $\Pi^* \nparallel L$. To see (4.4.19), we set $Q = q + \sum_{n \in \Xi_0} q^n$. Recall that for any $\Pi \in \mathcal{P}$, $R'_\Pi \mathcal{Q} = \mathcal{Q}$, so $R'_\Pi Q = Q$ as R'_Π is bijective. Therefore, $Q \parallel \Pi$ for all $\Pi \in \mathcal{P}$. Especially, we have $Q \parallel \Pi_0$, $Q \parallel \Pi_1$, and $Q \parallel \Pi^*$, hence $Q = 0$ by noting that $L \parallel \Pi_0$, $L \parallel \Pi_1$, but $L \nparallel \Pi^*$. It follows immediately that

$$Q \cdot \nu = \tau + \sum_{n \in \Xi_0} \tau_n = 0.$$

.

To see (4.4.21), consider the orbit $G\{q^n\}$ of $q^n \in \mathcal{Q} \setminus G\{q\}$ under the action of G, we have $G\{q^n\} = \tau_n \nu + G\{P_\Gamma q^n\}$. Referring to Lemma 4.4.6 (4), there exists at least one element of $G\{P_\Gamma q^n\}$ with non-positive x_3 component, so does at least one element in $G\{q^n\}$ if $\tau_n \leq 0$. This is in contradiction to Proposition 4.4.1. Therefore we have $\tau_n > 0$. $\sharp$

Based on the following decomposition from (4.4.8),

$$q = \tau \nu + P_\Gamma q \,, \tag{4.4.22}$$

we are now going to separate the remaining arguments into two cases: $\tau = 0$ and $\tau \neq 0$, and show that both cases can not happen (Lemmas 4.4.17, 4.4.18 and 4.4.19), thus concluding that the case when the line L is not parallel to the plane $\{x_3 = 0\}$ can not occur.

Lemma 4.4.17. *If $L \nparallel \{x_3 = 0\}$, then q has a non-zero projection on ν, i.e., $\tau \neq 0$ (see (4.4.22)).*

Proof. We prove the lemma by contradiction. Assume $\tau = 0$. Using (4.4.20) and (4.4.21), we know $\mathcal{Q} \setminus G\{q\} = \emptyset$, so we have $\mathcal{Q} = G\{q\}$. We now turn to exclude each possibility based on $|G\{q\}|$.

1. $|G\{q\}| > 3$. By Lemma 4.4.6 (4), there are at least two elements in $G\{q\}$, whose x_3 components are not positive. This yields a contradiction to Proposition 4.4.1.

2. $|G\{q\}| = 3$. By Lemma 4.4.6 (3), we should have $|G\{q\}| = |\mathcal{P}_0| = 3$, which further yields that $G\{q\} = \{R'_\Pi q : \Pi \in \mathcal{P}_0\}$. Let Π^* be the plane mentioned in Lemma 4.4.15, then it follows from Proposition 4.4.3 that $R'_{\Pi^*} q \in \mathcal{Q} = G\{q\}$, thus there exists a plane $\Pi \in \mathcal{P}_0$ such that $R'_{\Pi^*} q = R'_\Pi q$. Using Lemma 4.4.2, we know $\Pi \parallel \Pi^*$, giving a contradiction to the facts that $\Pi \parallel L$ but $\Pi^* \nparallel L$.

3. $|G\{q\}| = 2$. It can be shown in the same argument as that for the case $|G\{q\}| = 3$.

   ♯

Using Lemma 4.4.17, we need only to consider the case with $\tau \neq 0$.

Lemma 4.4.18. *If $L \nparallel \{x_3 = 0\}$, it can not happen that $|G\{q\}| \geq 3$ and $\tau \neq 0$.*

Proof. We first exclude the case with $|G\{q\}| > 3$ and $\tau \neq 0$. Noting $\tau_n = \tau$ for any $q^n \in G\{q\}$ (Lemma 4.4.6 (1)) and the equalities (4.4.20) and (4.4.21), we know $\tau_n \neq 0$, so is the x_3 component of the vector $\tau\nu$. Now consider the set $G\{q\} = \tau\nu + G\{P_\Gamma q\}$. By Lemma 4.4.6 (4), there exist at least two elements in $G\{P_\Gamma q\}$, whose x_3 components are non-positive. Hence there are at least two elements in $G\{q\}$, which have non-positive x_3 components, yielding a contradiction to Proposition 4.4.1.

We next exclude the case with $|G\{q\}| = 3$ and $\tau \neq 0$. By equality (4.4.20) and that $\tau \neq 0$, we know $\mathcal{Q} \setminus G\{q\} \neq \emptyset$. So there exists some $q^m \in \mathcal{Q} \setminus G\{q\}$ such that $\tau_m > 0$ by (4.4.21). As $|G\{q\}| = 3$, we have by Lemma 4.4.6 (3) that $|\mathcal{P}_0| = 3$ and $|G\{q^m\}| \geq 3$; in fact $|G\{q^m\}| = 3$ or $= 6$. Now, noting $\tau_n = \tau$ for all $q^n \in G\{q\}$, we have

$$3\tau + 3\tau_m \leq \tau + \sum_{q^n \in G\{q\}} \tau_n + \sum_{q^n \in G\{q^m\}} \tau_n \leq \tau + \sum_{q^n \in \mathcal{Q}} \tau_n = \tau + \sum_{n \in \Xi_0} \tau_n = 0.$$

Define

$$d_0 = \min\{x_3;\ x = (x_1, x_2, x_3) \in \Gamma \text{ and } \|x\| \leq \|P_\Gamma q\|\,\} < 0.$$

We can find $q^{n_1} \in G\{q\} \setminus \{q\}$ such that the x_3 component of $P_\Gamma q^{n_1}$ is less than or equal to $-\frac{1}{2}d_0$. Since $\|q^m\| = \|q\| = k$ and $\tau + \tau_m \leq 0$, we deduce $\|P_\Gamma q^m\| \geq \|P_\Gamma q\|$. We can find $q^{n_2} \in G\{q^m\}$ such that the x_3 component of $P_\Gamma q^{n_2}$ is less than or equal to $\frac{1}{2}d_0$.

Then the x_3 component of

$$q^{n_1} + q^{n_2} = (\tau + \tau_m)\nu + P_\Gamma q^{n_1} + P_\Gamma q^{n_2}$$

is non-positive, which is in contradiction to Proposition 4.4.1. $\sharp$

Lemma 4.4.19. *If $L \nparallel \{x_3 = 0\}$, then it can not happen that $|G\{q\}| \leq 2$ and $\tau \neq 0$.*

Proof. First by Lemma 4.4.6 (3) we know that $G\{q\} \geq 2$, so it suffices to show that the case with $|G\{q\}| = 2$ and $\tau \neq 0$ can not occur. We will prove this by contradiction. By Lemma 4.4.15, there exists a perfect plane $\Pi^* \in \mathcal{P}$ such that $\Pi^* \nparallel L$. We shall derive contradictions for each possibility specified by the geometric relations among the planes Π^* and Π_0, Π_1.

Case 1: $\Pi^* \perp \Pi_0$, $\Pi^* \perp \Pi_1$. Since $|G\{q\}| = 2$, we may write $G\{q\} = \{q, q^*\}$. Clearly there are only two reflections in G, and we know that $q^* = \tau\nu - P_\Gamma q$ if $q = \tau\nu + P_\Gamma q$. As $\Pi^* \perp \Pi_0$ and Π_1, we derive $R'_{\Pi^*} G\{q\} = \{-\tau\nu + P_\Gamma q, -\tau\nu - P_\Gamma q\} \subset \mathcal{Q}$. This shows both q^* and $-q^*$ belong to $\mathcal{Q}$, as a result one of them must have a non-positive x_3 component, that is in contradiction to Proposition 4.4.1.

Case 2: $\Pi^* \not\perp \Pi_0$. Let $\tilde{L}$, $\tilde{\mathcal{P}}_0$, $\tilde{G}$ denote the line of intersection between Π^* and Π_0, perfect planes in $\mathcal{P}$ which passes through the line $\tilde{L}$, and the group generated by the reflections $\{R'_\Pi : \Pi \in \tilde{\mathcal{P}}_0\}$. Since $\Pi^* \not\perp \Pi_0$, we see that $|\tilde{\mathcal{P}}_0| \geq 3$, and hence $|\tilde{G}\{q\}| \geq 3$ by following the proof of Lemma 4.4.6 (3).

We first consider the case when $\tilde{L}$ is not parallel to the plane $\{x_3 = 0\}$. In this case we have $|\tilde{G}\{q\}| \geq 3$. Then one can deduce a contradiction by following the same arguments as those in Lemmas 4.4.17 and 4.4.18.

We then consider the case when $\tilde{L} \parallel \{x_3 = 0\}$. This is the same situation as we have addressed in Lemma 4.4.7, Subsection 4.4.3 and the entire Subsection 4.4.6. We see that the current case can occur only when $|\tilde{G}\{q\}| = |\tilde{G}| = 3$. Following the steps in Subsection 4.4.6, we can work out all the perfect planes in $\mathcal{P}$. As shown in the case considered in Subsection 4.4.6, the intersection lines of the planes in $\mathcal{P}$ are all parallel to $\tilde{L}$. In particular, we have $L \parallel \tilde{L}$, thus $L \parallel \{x_3 = 0\}$, which is in contradiction to the assumption of Lemma 4.4.19.

Case 3: $\Pi^* \not\perp \Pi_1$. Same argument as that for Case 2 above leads to a contradiction.

This completes the proof of Lemma 4.4.18. $\sharp$

One can conclude from Lemmas 4.4.17, 4.4.18 and 4.4.19 that the case considered in this subsection when the line L is not parallel to the plane $\{x_3 = 0\}$ can not occur.

4.4.8 Summary on all unidentifiable gratings

Summing up the results in Subsection 4.4.2, 4.4.3 4.4.5, 4.4.6 and 4.4.7, especially Lemma 4.4.3, 4.4.7, 4.4.11, 4.4.14 and Lemma 4.4.17-4.4.19 we obtain the following conclusion.

Theorem 4.4.1. *Let S be a polyhedral grating with bi-period Λ, $E^i(x) = se^{iq\cdot x}$ be an incident electric field, and E be a solution to the direct scattering problems (4.1.1)-(4.1.4). Then under Condition (4.1.6) and Assumption 4.4.1, we have*

$$S \in \mathcal{S}_1(q,r) \quad or \quad S \in \mathcal{S}_1(s,q,\Lambda,r) \quad or \quad S \in \mathcal{S}_2(s,q,\Lambda,r).$$

4.5 Proof of Main results

With the results developed in the previous section on the classification of unidentifiable periodic grating structures in correspondence to one incident field, we are ready to prove our main theorem 4.2.1 for the unique determination of a given periodic polyhedral grating profile by the scattered field.

Proof of Theorem 4.2.1. Assume that (4.2.1) is true for two different S_1 and S_2. We first show that $E_{S_1} = E_{S_2}$ in the domain above the measurement plane $\{x_3 = b\}$. By Assumption 1, we have the expansions

$$\begin{aligned} E_{S_1}(x) &= E^i(x) + \textstyle\sum_{n\in\mathbf{Z}^2} A_1^n e^{iq^n\cdot x}, \\ E_{S_2}(x) &= E^i(x) + \textstyle\sum_{n\in\mathbf{Z}^2} A_2^n e^{iq^n\cdot x}. \end{aligned}$$

It suffices to show that $A_1^n = A_2^n$ for all $n \in \mathbf{Z}^2$. To see this, we have by (4.2.1) that

$$(E_{S_1} - E_{S_2}) \times e_3|_{x_3=b} = \sum_{n\in\mathbf{Z}^2} (A_1^n - A_2^n) \times e_3 e^{i\beta^n \cdot b} e^{i\alpha^n \cdot x} = 0. \tag{4.5.1}$$

Noting that $\{e^{i\alpha^n \cdot x}\}_{n\in\mathbf{Z}^2}$ is an orthogonal family in $L^2((0,\Lambda_1)\times(0,\Lambda_2))$ of variables x_1 and x_2, we derive from (5.3.2) that

$$(A_1^n - A_2^n) \times e_3 = 0 \quad \forall n \in \mathbf{Z}^2. \tag{4.5.2}$$

In addition, as $A_1^n \cdot q^n = 0$ and $A_2^n \cdot q^n = 0$, which can be deduced from the fact that both the fields E_{S_1} and E_{S_2} are divergence free, we have

$$(A_1^n - A_2^n) \cdot q^n = 0 \quad \forall n \in \mathbf{Z}^2. \tag{4.5.3}$$

Now, if $A_1^n - A_2^n \neq 0$ for some $n \in \mathbf{Z}^2$ then we can conclude from (4.5.2) and (4.5.3) that

$$q^n \cdot e_3 = \beta^n = 0,$$

which is in contradiction to the assumption (4.1.6). Thus we have shown that $E_{S_1} = E_{S_2}$ in the domain above the plane $\{x_3 = b\}$. Since both $E_{S_1}(x)$ and $E_{S_2}(x)$ are analytic functions, we see that $E_{S_1}(x) = E_{S_2}(x)$ in their common domain $S_1^+ \cap S_2^+$, where S_i^+ is the domain in $\mathbb{R}^3$ above S_i for $i = 1, 2$. As a result, faces of the grating profile S_2 which lie above S_1 are perfect planes of E_{S_1}, and faces of the grating profile S_1 which lie above S_2 are perfect planes of E_{S_2}. Now we may assume without loss of generality that the grating profile S_2 has some part above S_1. Then the Lipschitz condition we imposed on S_2 allows us to find a face (referring to an open part of a plane) on S_2 which is above S_1 and is not parallel to e_3. By the previous discussions, this face is a perfect plane of E_{S_1}, which is denoted by Π.

One may notice that the perfect plane Π may not necessarily extend above the measurement plane $\{x_3 = b\}$. In order to apply Theorem 4.4.1, we need to construct a perfect plane which extends above the plane Γ_b and which is not parallel to e_3. This is done in the following.

First we choose a perfect point $x_0 \in \Pi$, let $\gamma(t)$ be the ray emitted from x_0 and directed along e_3, more precisely, $\gamma(t) = x_0 + te_3$ for $t \geq 0$. Let $d_0 > 0$ be a number less than half of the distance between the ray $\gamma(t)$ and the grating profile S_1. Consider the ray $R_\Pi\gamma(t)$: if it does not intersect S_1, then it directs upward or horizontally and lies entirely above the grating profile S_1, so does its projection on the plane Π, which is then our desired perfect plane. Otherwise, the ray $R_\Pi\gamma(t)$ intersects S_1 at some point, let

$$t_1 = \min\{t \geq 0;\ R_\Pi\gamma(t) \text{ belongs to } S_1\}.$$

Clearly $t_1 > d_0$ and $R_\Pi\gamma(t_1)$ belongs to a (closed) face of S_1, which is not parallel to $R'_\Pi e_3$. By the Reflection Principle (Lemma 4.3.1), $\gamma(t_1)$ is a perfect point of E_{S_1}, and has a perfect plane, say Π_1, which passes through $\gamma(t_1)$ but $\Pi_1 \not\parallel e_3$. We can do the same procedure for $\gamma(t_1)$ and Π_1 as we did for $\gamma(t_0) = \gamma(0)$ and Π above, then either Π_1 is our desired perfect plane or we can find another perfect point $\gamma(t_2)$ and perfect plane Π_2 of E_{S_1} such that Π_2 passes through $\gamma(t_2)$ and $\Pi_2 \not\parallel e_3$. Clearly we see that $t_2 - t_1 > d_0$. Repeating this procedure, we can get a sequence of perfect points $\{\gamma(t_n)\}$ and perfect planes $\{\Pi_n\}$ of E_{S_1} such that Π_n passes through $\gamma(t_n)$ and $\Pi_n \not\parallel e_3$, and $t_n - t_{n-1} > d_0$. So after a finite number of steps we can get a desired perfect plane of E_{S_1}, denoted as Π^*, that extends above Γ_b. Since E_{S_1} and E_{S_2} coincide in $S_1^+ \cap S_2^+$, Π^* is a perfect plane for both E_{S_1} and E_{S_2}. Now Theorem 4.2.1 follows from Theorem 4.4.1. $\sharp$

Theorem 4.2.1 leads immediately to the following two corollaries.

Corollary 4.5.1. *Let S be a polyhedral grating of bi-period Λ. Assume that condition (4.1.6) holds for the period Λ and some incident field E^i. Consider a plane $\Gamma_b = \{x_3 = b\}$ located above both S. Then the measurement of the total field $e_3 \times E_S$ on Γ_b corresponds to the incident field E^i determines S uniquely if the following condition holds:*

There are two faces of S which do not form of an angle of $\pi/3$, $\pi/2$ or $2\pi/3$.

Corollary 4.5.2. *Let S be a polyhedral grating of bi-period Λ which is not an entire plane in R^3, $E_{i,1} = s_1 e^{iq_1\cdot x}$ and $E_{i,2} = s_2 e^{iq_2\cdot x}$ be two incident fields. Assume that condition (4.1.6) holds in both cases. Let $\Gamma_b = \{x_3 = b\}$ be a plane located above S. Then the measurement of the total fields $e_3 \times E_S^1$ and $e_3 \times E_S^2$ on Γ_b correspond respectively to the incident waves $E^{i,1}$ and $E^{i,2}$ determines S uniquely if the following condition holds:*

(a) The vectors s_1 and s_2 are not parallel, orthogonal or form a angle of $\pi/3$ or $2\pi/3$.

Proof. For the incident waves $E_{i,1}$ and $E_{i,2}$, we define

$$\mathcal{S}_1 = (\bigcup_{r\in\mathbf{R}^3} \mathcal{S}_1(s_1, q_1, \Lambda, r))\bigcup(\bigcup_{r\in\mathbf{R}^3} \mathcal{S}_2(s_1, q_1, \Lambda, r))\,, \tag{4.5.4}$$

$$\mathcal{S}_2 = (\bigcup_{r\in\mathbf{R}^3} \mathcal{S}_1(s_2, q_2, \Lambda, r))\bigcup(\bigcup_{r\in\mathbf{R}^3} \mathcal{S}_2(s_2, q_2, \Lambda, r))\,, \tag{4.5.5}$$

then by Theorem 4.2.1 a grating profile S which is not parallel to $\{x_3 = 0\}$ can not be identified by both $E_{i,1}$ and $E_{i,2}$ if and only if $S \in \mathcal{S}_1 \bigcap \mathcal{S}_2$. We see clearly that if the condition (a) holds, then $\mathcal{S}_1 \bigcap \mathcal{S}_2 = \emptyset$, which leads to the corollary immediately. ♯

One can see from Corollaries 4.5.1 and 4.5.2 that a general polyhedral grating structure can be uniquely determined by one or two incident plane waves. Only gratings of very special structures may require more incident waves for their unique determination.

Chapter 5

Scattering by Bi-periodic Polyhedral Grating (II)

This chapter studies the uniqueness of determine a bi-periodic diffractive gratings by electromagnetic wave under the general setting without excluding Rayleigh frequencies. It is a continuation of the work in chapter four. Our objective is to show that in this case, there are only seven classes of unidentifiable polyhedral gratings corresponding to each incident plane wave, where the four newly appeared classes are related to the case when Rayleigh frequencies occur. The approach basically follows from that in chapter four. However, the discussion is much more complicated. Since we have describe the direct as well as its associated inverse problem in chapter four, we will go directly to the problem of finding all the gratings that the uniqueness for one incident wave fails.

The chapter is organized as follows. In Section 5.1 we recall some preliminary knowledge from chapter four, that includes basic notations, some useful lemmas and observations that are the starting point for the subsequent analysis. In Section 5.2, we discuss how to start from one or two perfect planes of the total field to find all the possible gratings to which the uniqueness fail. The discussion here is much parallel to that in chapter four, though it is more complicated. Some cases considered in this chapter have already been studied in chapter four and some others are not. In order to make the chapter more readable, we will need to recite some results and proofs in chapter four frequently. Finally, we establish the main result about the uniqueness for the inverse scattering problem in Section 5.3.

5.1 Preliminaries

We start with recalling some basic notations from Section 4.1 in chapter four:

S: the profile of a bi-periodic polyhedral grating with period Λ;

$E^i(x) = se^{iq\cdot x}$: the incident electric wave;

$E(x) = E^i(x) + \sum_{n\in\mathbf{Z}^2} A^n e^{iq^n\cdot x}$: the total field;

$\Xi = \{n \in \mathbf{Z}^2;\ \beta^n \geq 0\}$: the index set of non-evanescent modes in the scattered field;

and

$$\Xi_0 = \{n \in \Xi;\ A^n \neq 0\}, \quad \mathcal{Q} = \{q\}\bigcup\{q^n\}_{n\in\Xi_0}, \tag{5.1.1}$$

$$E_p(x) = se^{iq\cdot x} + \sum_{n\in\Xi_0} A^n e^{iq^n\cdot x}, \tag{5.1.2}$$

$$\mathcal{P} = \{\Pi;\ \Pi \text{ is a perfect plane of } E_p\}. \tag{5.1.3}$$

For later use, we also let

$$\Xi^* = \{n \in \mathbf{Z}^2;\ \beta^n = 0\}$$

be the index set of non-evanescent modes in the scattered field that propagate parallel to the grating.

Throughout this chapter, we do the analysis under the general setting without excluding Rayleigh frequencies. So the set Ξ^* may not be empty. We also assume that the direct problem has a solution though the solution may not be unique.

We will denote by Γ_b the plane $\{x_3 = b\}$ above the grating profile S, on which the measurement of the electric field E is made. The domain above the plane Γ_b is denoted by Ω_b.

We recall the following Reflection Principle.

Lemma 5.1.1. *Let O be a domain in $\mathbf{R}^3$ which is symmetric with respect to a plane Π, and E be an electric field in O satisfying the vector-valued Helmholtz equations (4.1.1)-(4.1.2). Assume that $\tilde{\Pi}$ is a connected open subset in $\Pi \cap O$, then $\tilde{\Pi}$ is a perfect plane of E if and only if the following relation holds*

$$E(x) + R'_\Pi(E(R_\Pi(x))) = 0 \quad in \;\; O. \tag{5.1.4}$$

Moreover, if $\Gamma \subset O$ or $\Gamma \subset \partial O$ is a perfect plane of E,then $R_\Pi(\Gamma)$ is also a perfect plane of E.

Definition 5.1.1. *For a given plane Π in $\mathbf{R}^3$, a periodic part S_l of the entire grating structure S is called a Π-reflecting periodic part if each face of S_l can be reflected by Π into the domain Ω_b.*

The following lemma presents a crucial observation.

Lemma 5.1.2. *If Π is a perfect plane of E, then Π is also a perfect plane of E_p.*

Now for each incident field E^i, we are going to find all the grating structures which can not be identified by the incident field. We begin with the following assumption, which will be the first fundamental fact to be established in the demonstration of our main result on the global uniqueness in Section 5.3.

Assumption 5.1.1. *There exists a perfect plane of E, denoted by $\tilde{\Pi}_0$, such that $\tilde{\Pi}_0 \cap \Omega_b \neq \emptyset$ and Π_0 is not perpendicular to the plane $\{x_3 = 0\}$.*

Next we present four important observations as those in chapter four.

Propsition 5.1.1. *Each of the vectors in $\mathcal{Q}$ except q has nonnegative x_3 component.*

Propsition 5.1.2. *For each Π_0-reflecting periodic part S_l of grating S, each face of S_l can be reflected with respect to Π_0 to a perfect plane in $\mathcal{P}$.*

Propsition 5.1.3. *For each perfect plane Π in $\mathcal{P}$, we have $R'_\Pi \mathcal{Q} = \mathcal{Q}$, $R_\Pi \mathcal{P} = \mathcal{P}$. Moreover, both maps R'_Π: $\mathcal{Q} \to \mathcal{Q}$ and R_Π: $\mathcal{P} \to \mathcal{P}$ are bijective.*

Propsition 5.1.4. *For a Π_0-reflecting periodic part S_l of grating S, each face of S_l must lie on some perfect plane in $\mathcal{P}$.*

The following lemma provides some useful properties about perfect planes. See chapter four for the proof.

Lemma 5.1.3. *Let $F = te^{ip\cdot x}$ be one of the Fourier modes of E_p in (5.1.2),*

1. *If Π is a perfect plane of the field $F = te^{ip\cdot x}$, then $t \perp \Pi$;*

2. *If Π and Π^* are two perfect planes in $\mathcal{P}$ such that $R'_{\Pi}p = R'_{\Pi^*}p$, then $\Pi \parallel \Pi^*$.*

5.2 Classification of unidentifiable periodic structures

After the preparations in the previous section, we may start our process to find all the unidentifiable polyhedral gratings corresponding to the incident field $E^i = se^{iq\cdot x}$ in this section.

5.2.1 First class of unidentifiable gratings

The first unidentifiable class corresponds to the special case when all the planes in $\mathcal{P}$ are parallel to Π_0, as stated in the following lemma. See chapter four for the proof.

Lemma 5.2.1. *If all the planes in $\mathcal{P}$ are parallel to Π_0, then both Π_0 and S are parallel to $\{x_3 = 0\}$, and the distance between any two neighboring perfect planes in $\mathcal{P}$ is the same, equal to π/β.*

On the other hand, if Π is a plane that is parallel to Π_0 and the distance between Π and Π_0 is some integer multiple of π/β, then Π is a perfect plane in $\mathcal{P}$.

Let r be an arbitrary point in $\mathbf{R}^3$, then Lemma 5.2.1 yields the first class of unidentifiable gratings corresponding to the incident field $E^i(x) = se^{iq\cdot x}$:

$$\mathcal{S}_1(q,r) = \Big\{\text{all planes which are parallel to } \{x_3 = 0\} \text{ and have equal distance } \pi/\beta \text{ between each other, with } r \text{ lying on one of the planes}\Big\}.$$

Using the class $\mathcal{S}_1(q, r)$, we can conclude that any two gratings belonging to the class $\mathcal{S}_1(q, r)$ can not be distinguished by the incident wave E^i.

5.2.2 Preparation for finding other classes of unidentifiable gratings

We have considered the special case, where all perfect planes are parallel to the same plane Π_0, in Subsection 5.2.1. We now proceed to study the more general case when there is some plane in $\mathcal{P}$ which is not parallel to Π_0. As we shall see, this case leads to six other classes of unidentifiable gratings. For the purpose, let Π_1 be a plane in $\mathcal{P}$ which is not parallel to the plane Π_0. Then we introduce the following notations:

L: the line of intersection between Π_0 and Π_1;

r: an arbitrary but fixed point on L;

ν: a unit tangential vector along L, with nonnegative x_3 component;

Γ: the plane in $\mathbf{R}^3$, which passes through the origin and has normal ν. For the convenience, we assign an orientation to the plane Γ: the normal ν and Γ form a right-handed coordinate system.

P_Γ: the projection from $\mathbf{R}^3$ onto Γ.

T_θ: the rotation on the plane Γ about the origin by angle θ. Clearly T_θ can be also viewed as a rotation in the whole space $\mathbf{R}^3$ about the axis fixed to be a line passing through the origin and parallel to direction ν. In both cases, the rotation is understood to be anticlockwise with respect to the assigned orientation on the plane Γ.

We remark that $\nu \nparallel e_3$, otherwise one gets $e_3 \parallel \Pi_0$ by noting that $\nu \parallel \Pi_0$, contradicting the assumption that Π_0 is not perpendicular to the plane $\{x_3 = 0\}$. As a result, Γ does not coincide with the plane $\{x_3 = 0\}$.

Viewing (ν, Γ) as a coordinate system, we can split all vectors in $\mathcal{Q}$ as follows:

$$q = \tau\nu + P_\Gamma q, \quad q^n = \tau_n \nu + P_\Gamma q^n \quad \text{for} \quad n \in \Xi_0 \tag{5.2.1}$$

where τ and τ_n are constants. It is important to observe that

Lemma 5.2.2. *For any $p \in \mathcal{Q}$, $P_\Gamma p \neq 0$.*

Now we define

$$\begin{aligned}\mathcal{P}_0 &= \{\Pi;\ \Pi \text{ is a perfect plane of } E_p \text{ and passes through the line } L\},\\ G &= \text{the group generated by the reflections } \{R'_\Pi;\ \Pi \in \mathcal{P}_0\}.\end{aligned}$$

Then we have the following properties about the set $\mathcal{P}_0$ and the group G. See chapter four for the proof.

Lemma 5.2.3. *The number of perfect planes in $\mathcal{P}_0$ is finite and the angles formed by any two neighboring planes in $\mathcal{P}_0$ are all equal; G consists of $|\mathcal{P}_0|$ reflections and $|\mathcal{P}_0|$ rotations, and it has the structure of a dihedral group of order $2|\mathcal{P}_0|$.*

In the sequel, we denote by G^* the subgroup of G which consists of all its rotations. Clearly, we know $|G^*| = |\mathcal{P}_0|$. Note that the identity element, denoted by Id, of both the group G and G^* is the rotation by the angle 2π. When the domain which the transformations in G act on is restricted to the plane Γ, G reduces to the *dihedral group* which acts only on the vectors lying on the plane Γ.

Next, we present a few more useful properties about the group G and the set $\mathcal{P}_0$, similar to those in chapter four.

Lemma 5.2.4. *The following properties hold for the group G and the set $\mathcal{P}_0$:*

1. *For any $T \in G$, $T\mathcal{Q} = \mathcal{Q}$ and $T\nu = \nu$; For any $q^n \in G\{q\}$, $\tau_n = \tau$.*

2. $|\{R'_\Pi q;\ \Pi \in \mathcal{P}_0\}| = |G^*\{q\}| = |\mathcal{P}_0|$.

3. $|G\{q\}| = 2|\mathcal{P}_0|$ *or* $|G\{q\}| = |\mathcal{P}_0|$. *If $|G\{q\}| = 2|\mathcal{P}_0|$, there exists some $q^1 \in G\{q\}$ such that $G\{q\} = G^*\{q\} \bigcup G^*\{q^1\}$; if $|G\{q\}| = |\mathcal{P}_0|$, then $G\{q\} = G^*\{q\}$ and there exists a plane Π in $\mathcal{P}_0$ such that $R'_\Pi q = q$. Moreover, the same results hold for q being replaced by any element in $G\{q\}$.*

4. *If $|G\{q\}| = 3$, there exists at least one element in $G\{P_\Gamma q\}$ whose x_3 components are negative.*

5. *If $|G\{q\}| = 4$, there exists at least two elements in $G\{P_\Gamma q\}$ whose x_3 components are non-positive.*

6. *If $|G\{q\}| > 4$, there exists at least two elements in $G\{P_\Gamma q\}$ whose x_3 components are negative.*

Recalling that the line L is the intersection line between Π_0 and Π_1, L is either parallel to the plane $\{x_3 = 0\}$ or unparallel. Then the following three cases may happen:

1. Line L is parallel to the plane $\{x_3 = 0\}$, and $\mathcal{Q} = G\{q\}$;

2. Line L is parallel to the plane $\{x_3 = 0\}$, and $\mathcal{Q} \neq G\{q\}$;

3. Line L is not parallel to the plane $\{x_3 = 0\}$.

We will study these three cases in Subsection 5.2.3 5.2.4, 5.2.5, respectively.

Before we go to the study of these cases, we first introduce a simple transformation that can significantly simplify our derivations. To do so, we fix one point r on the line L, then introduce the following simple transformation by change of variables:

$$\hat{x} = x - r. \tag{5.2.2}$$

In terms of $\hat{x}$-variable, the propagating field $E_p(x)$ in (5.1.2) takes the form

$$E_p(x) = E_p(\hat{x} + r) =: \hat{E}_p(\hat{x}) = s e^{iq\cdot(\hat{x}+r)} + \sum_{n\in\Xi_0} A^n e^{iq^n\cdot(\hat{x}+r)}.$$

By setting $\hat{s} = s e^{iq\cdot r}$ and $\hat{A}^n = A^n e^{iq^n\cdot r}$, we can write $\hat{E}_p(\hat{x})$ into

$$\hat{E}_p(\hat{x}) = \hat{s} e^{iq\cdot\hat{x}} + \sum_{n\in\Xi_0} \hat{A}^n e^{iq^n\cdot\hat{x}}. \tag{5.2.3}$$

A significant advantage of using the $\hat{x}$-variable, instead of the original x-variable, can be seen from the following lemma, where we write $\hat{s}$ for $\hat{A}^0$ and q for q^0 in the lemma. See chapter four for the proof.

Lemma 5.2.5. *Let Π be a plane passing through the line L, the intersection line between Π_0 and Π_1, then $\Pi \in \mathcal{P}_0$ if and only if $R'_\Pi \mathcal{Q} \subset \mathcal{Q}$ and the relation $R'_\Pi \hat{A}^l + \hat{A}^m = 0$ holds whenever $R'_\Pi q^l - q^m = 0$ for $q^l, q^m \in \mathcal{Q}$.*

5.2.3 Studying of the case $L \parallel \{x_3 = 0\}$ with $\mathcal{Q} = G\{q\}$

After the preparation in Subsection 5.2.2, we may now start to consider the case when there are some plane in $\mathcal{P}$ that are not parallel to the plane Π_0. As indicated in Subsection 5.2.2, we consider in the subsection the case with $L \parallel \{x_3 = 0\}$ and $\mathcal{Q} = G\{q\}$. We shall that that the study of this case reveals to four different classes of unidentifiable grating corresponding to the incident field E^i.

Preliminary result

We first present the following lemma which will facilitate us to divide the analysis under different situations that may happen under the assumption $L \parallel \{x_3 = 0\}$ with $\mathcal{Q} = G\{q\}$.

Lemma 5.2.6. *If line L is parallel to $\{x_3 = 0\}$ and $\mathcal{Q} = G\{q\}$, then only the following four cases are possible to happen:*

(1) $|\mathcal{P}_0| = 4$ *and* $|G\{q\}| = 4$, *(2)* $|\mathcal{P}_0| = 2$ *and* $|G\{q\}| = 4$,

(3) $|\mathcal{P}_0| = 3$ *and* $|G\{q\}| = 3$, *(4)* $|\mathcal{P}_0| = 2$ *and* $|G\{q\}| = 2$.

Proof. Since $L \parallel \{x_3 = 0\}$, the x_3 component of ν is zero. Using the decomposition (5.2.1) and Proposition 5.1.1, we see that all the vectors in $G\{P_\Gamma q\}$, except $P_\Gamma q$, have nonnegative x_3-components. Then our lemma follows directly from Lemma 5.2.4 (3), (4), (5. ♯

We shall deal with case (1), (2), (3), (4) in the lemma above in Subsection 5.2.3, 5.2.3, 5.2.3, 5.2.3, respectively.

Second class of unidentifiable gratings

Now we study in this subsection case (1) in Lemma 5.2.6. This case will lead to the second class of unidentifiable gratings corresponding to the given incident field E^i. The following Lemma shows that both E_p and $\mathcal{P}_0$ are determined by the incident field $E^i = se^{iq\cdot x}$. See chapter four for the proof.

Lemma 5.2.7. *If $L \parallel \{x_3 = 0\}$ and $|\mathcal{P}_0| = |G\{q\}| = 2$, then it holds that*

1. *$L \parallel (s \times e_3)$.*

2. *$\mathcal{P}_0$ consists of two planes, namely Π_0 and Π_1, one is perpendicular to s, and the other perpendicular to $s \times (s \times e_3)$. For ease of subsequent exposition, we shall assume that $\Pi_0 \perp s$ and $\Pi_1 \perp s \times (s \times e_3)$, possibly after relabeling.*

3. *Let $q^1 = R'_{\Pi_1} q = T_\pi q$, then the propagating field $E_p(x)$ in (5.1.2) or $\hat{E}_p(\hat{x})$ in (5.2.3) can be written as*

$$E_p(x) = s(e^{iq\cdot x} - e^{iq^1\cdot x + (q-q^1)\cdot r}) \quad or \quad \hat{E}_p(\hat{x}) = \hat{s}(e^{iq\cdot\hat{x}} - e^{iq^1\cdot\hat{x}}). \tag{5.2.4}$$

The next lemma indicates that all the perfect planes of the propagating field E_p are determined by the incident field $E^i = se^{iq\cdot x}$ also. See chapter four for the proof.

Lemma 5.2.8. *If the line L is parallel to the plane $\{x_3 = 0\}$ and $|\mathcal{P}_0| = 2$, then*

1. *The set of perfect planes of E_p, i.e. $\mathcal{P}$, is determined by the incident field $E^i = se^{iq\cdot x}$. More specifically, $\mathcal{P}$ consists of only two sets of planes: the first set contains only all the planes that are parallel to Π_0, while the second set contains only all the planes that are parallel to Π_1 and have the distance $\frac{\pi}{\|P_\Gamma q\|}$ between each two neighboring planes.*

2. *Each face of the grating structure S lies on a plane in $\mathcal{P}$.*

Lemma 5.2.8 enables us to find a new class of unidentifiable grating profiles. To describe the class explicitly, we first clarify some notations:

r: a position vector, viewed as a point in $\mathbf{R}^3$;

Γ: a plane which passes through origin with normal $s \times e_3$;

Π_0: a plane which passes through r with normal s;

Π_1: a plane which passes through r with normal $(s \times e_3) \times s$.

Then by Lemma 4.4.10) all the perfect planes of E_p can be described by

$$\mathcal{P} = \Big\{\text{plane } \Pi;\ \Pi \parallel \Pi_0\Big\} \bigcup \Big\{\text{plane } \Pi;\ \Pi \parallel \Pi_1,\ \text{dist}(\Pi, \Pi_1) = \frac{m\pi}{\|P_\Gamma q\|} \text{ for some } m \in \mathbf{N}\Big\}.$$

This suggests a new class of unidentifiable gratings corresponding to the incident field $E^i = s\,e^{iq\cdot x}$:

$$\mathcal{S}_2(s,q,\Lambda,r) = \Big\{\text{gratings with profile } S\text{, which are } \Lambda\text{-periodic polyhedral structures such that faces of } S \text{ lie on planes in } \mathcal{P}\Big\}.$$

One can see that each class $\mathcal{S}_2(s,q,\Lambda,r)$ corresponds to an unique electric field E_p, which solves the direct scattering problem for any gratings in $\mathcal{S}_2(s,q,\Lambda,r)$. So any two grating in $\mathcal{S}_2(s,q,\Lambda,r)$ can not be identified by the incident field $E^i(x) = se^{iq\cdot x}$. This proves

Lemma 5.2.9. *If the line L is parallel to $\{x_3 = 0\}$ and $|\mathcal{P}_0| = 2$, then the grating profile S belongs to $\mathcal{S}_2(s,q,\Lambda,r)$ for some point $r \in \mathbf{R}^3$. Furthermore, any grating in $\mathcal{S}_2(s,q,\Lambda,r)$ can generate the same total field.*

Third class of unidentifiable gratings

In this subsection, we study case (2) in Lemma 5.2.6. This case will lead to the third class of unidentifiable gratings corresponding to the given incident field E^i. We show first in the following lemma that both the propagating field E_p and and its set of perfect planes $\mathcal{P}_0$ are determined by the incident field E^i. See chapter four for the proof.

Lemma 5.2.10. *If $L \parallel \{x_3 = 0\}$ and $|\mathcal{P}_0| = |G\{q\}| = |\mathcal{Q}| = 3$, then:*

1. *$L \parallel (s \times e_3)$.*
2. *$\mathcal{P}_0$ consists of only three planes: one is perpendicular to s, which we denote by Π_0, and the other two are generated by rotating the plane Π_0 about the axis L by angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ respectively, which are labeled as Π_1 and Π_2.*
3. *The propagating field E_p in (5.1.2) or $\hat{E}_p$ in (5.2.3) can be written as*

$$E_p(x) = se^{iq\cdot x} - (T_{\frac{2\pi}{3}}s)e^{iq^1\cdot x+(q-q^1)\cdot r} + (T_{\frac{4\pi}{3}}s)e^{iq^2\cdot x+(q-q^2)\cdot r}, \quad (5.2.5)$$

$$\hat{E}_p(\hat{x}) = \hat{s}e^{iq\cdot\hat{x}} - (T_{\frac{2\pi}{3}}\hat{s})e^{iq^1\cdot\hat{x}} + (T_{\frac{4\pi}{3}}\hat{s})e^{iq^2\cdot\hat{x}}, \quad (5.2.6)$$

where $q^j = R'_{\Pi_j}q = T_{\frac{2j\pi}{3}}q$ for $j = 1,2$.

Similarly to Lemma 5.2.8, we have

Lemma 5.2.11. *If $L \parallel \{x_3 = 0\}$ and $|\mathcal{P}_0| = |G\{q\}| = |\mathcal{Q}| = 3$, then*

1. *The set of perfect planes of E_p, i.e. $\mathcal{P}$ is determined. More specifically, $\mathcal{P}$ consists of only three sets of parallel planes, where every two neighboring planes in each set have equal distance $\frac{2\pi}{\sqrt{3}\|P_\Gamma q\|}$: the first set contains only the planes that are parallel to Π_0, the second set contains only the planes that are parallel to Π_1, while the third set contains only the planes that are parallel to Π_2.*

2. *Each face of the grating structure S lies on a plane in $\mathcal{P}$.*

Lemma 5.2.11 leads to another class of unidentifiable grating profiles corresponding to the incident field $E^i = s\, e^{iq\cdot x}$. To describe this class explicitly, we first clarify a few notations:

r: a position vector, viewed as a point in $\mathbf{R}^3$;

Γ: a plane which passes through the origin with normal $s \times e_3$;

Π_0: a plane which passes through r with normal s;

Π_1, Π_2: planes which pass through r and form a angle of $\pi/3$ and $2\pi/3$ with Π_0, respectively.

Then by Lemma 5.2.11, we can describe all the perfect planes of E_p in (5.1.2) by

$$\mathcal{P} = \Big\{\text{plane}\, \Pi;\ \exists j \in \{0,1,2\} \text{ such that } \Pi \parallel \Pi_j,\ \operatorname{dist}(\Pi, \Pi_j) = \frac{2m\pi}{\sqrt{3}\|P_\Gamma q\|} \text{ for some } m \in \mathbf{N}\Big\},$$

which suggests a new class of unidentifiable gratings corresponding to the incident field E^i:

$$\begin{aligned}\mathcal{S}_3(s,q,\Lambda,r) \quad = \quad & \Big\{\text{gratings with profile } S, \text{ which are } \Lambda\text{-periodic polyhedral structures} \\ & \quad \text{such that faces of } S \text{ lie on planes in } \mathcal{P}\Big\}.\end{aligned}$$

One can see that each class $\mathcal{S}_3(s,q,\Lambda,r)$ corresponds to a unique propagating field E_p, which solves the direct scattering problem for any grating in $\mathcal{S}_3(s,q,\Lambda,r)$. Thus any two gratings in $\mathcal{S}_3(s,q,\Lambda,r)$ can not be identified by the incident field $E^i(x) = se^{iq\cdot x}$. This leads to the following lemma.

Lemma 5.2.12. *If the line L is parallel to the plane $\{x_3 = 0\}$ and $|\mathcal{P}_0| = 3$, then the grating S belongs to $\mathcal{S}_3(s, q, \Lambda, r)$ for some point $r \in \mathbf{R}^3$. Furthermore, all the gratings in $\mathcal{S}_3(s, q, \Lambda, r)$ generate the same total field.*

Fourth and Fifth classes of unidentifiable grating structures

We continue to study in this subsection case (3) in Lemma 5.2.6. This case will lead to two classes of unidentifiable gratings corresponding to the given incident field E^i, with one class being a special case in the other class. We show first in the following lemma that both the propagating field E_p and and its set of perfect planes $\mathcal{P}_0$ are determined by the incident field E^i and the vector ν, which is the direction vector of the line L.

Lemma 5.2.13. *If $L \parallel \{x_3 = 0\}$, $\mathcal{Q} = G\{q\}$, $|\mathcal{P}_0| = 2$ and $|G\{q\}| = 4$, then we have:*

1. *$\mathcal{P}_0$ consists of two planes: Π_0 and Π_1, which are perpendicular to the vectors $P_\Gamma q + \sqrt{k^2 - \tau^2}(e_3 \times \nu)$, $P_\Gamma q - \sqrt{k^2 - \tau^2}(e_3 \times \nu)$, respectively.*

2. *The propagating field E_p in (5.1.2) or $\hat{E}_p$ in (5.2.3) can be written as*

$$E_p(x) = se^{iq\cdot x} - R'_{\Pi_0} se^{iq^1\cdot x + (q-q^1)\cdot r} + R'_{\Pi_1}(R'_{\Pi_0}s)e^{iq^2\cdot x+(q-q^2)\cdot r} - R'_{\Pi_1} se^{iq^3\cdot x+(q-q^3)\cdot r}$$
$$\hat{E}_p(\hat{x}) = \hat{s}e^{iq\cdot\hat{x}} - R'_{\Pi_0}\hat{s}e^{iq^1\cdot\hat{x}} + R'_{\Pi_1}(R'_{\Pi_0}\hat{s})e^{iq^2\cdot\hat{x}} - R'_{\Pi_1}\hat{s}e^{iq^3\cdot\hat{x}}.$$

where $q^1 = \tau\nu + \sqrt{k^2 - \tau^2}(e_3 \times \nu)$, $q^3 = \tau\nu - \sqrt{k^2-\tau^2}(e_3 \times \nu)$ and $q^2 = \tau\nu - P_\Gamma q$. Moreover, $q^1, q^3 \in \Xi^*$.

Proof. Step 1. Note that $|G| = |G\{q\}| = 4$. Referring to Lemma 5.2.4, we can find $q^1 \in G\{q\}$ such that

$$G\{q\} = G^*\{q\} \bigcup G^*\{q^1\}.$$

But $G^* = \{T_\pi, T_{2\pi}\}$, so we can further write

$$G^*\{q\} = \{\tau\nu + P_\Gamma q, \tau\nu - P_\Gamma q\}, \quad G^*\{q^1\} = \{\tau\nu + P_\Gamma q^1, \tau\nu - P_\Gamma q^1\}.$$

By Proposition 5.1.1 and the assumption that $\nu \parallel \{x_3 = 0\}$, the x_3 component of $P_\Gamma q^1$ should be zero. Therefore, we have $P_\Gamma q^1 \parallel \{x_3 = 0\}$. This yields $P_\Gamma q^1 \parallel (e_3 \times \nu)$ since

$P_\Gamma q^1 \perp \nu$ also. Besides, recalling that $\|q^1\|^2 = k^2 = \tau^2 + \|P_\Gamma q^1\|^2 = \tau^2 + \|P_\Gamma q^1\|^2$, we get $P_\Gamma q^1 = \pm\sqrt{k^2 - |q\cdot\nu|^2}(e_3 \times \nu)$. We then fix q^1 by letting

$$P_\Gamma q^1 = \sqrt{k^2 - |q\cdot\nu|^2}(e_3 \times \nu).$$

We also write $q^2 = \tau\nu - P_\Gamma q$, $q^3 = \tau\nu - P_\Gamma q^1$.

Step 2. It is clear that $\mathcal{P}_0 = \{\Pi_0, \Pi_1\}$ since $|\mathcal{P}_0| = 2$. Let $\Pi \in \{\Pi_0, \Pi_1\}$, we claim that $R'_\Pi q \in \{q^1, q^3\}$. Indeed, by Proposition 5.1.3, $R'_\Pi\{q, q^1, q^2, q^3\} = \{q, q^1, q^2, q^3\}$. Since $|\mathcal{P}_0| = 2$ and $|G\{q\}| = |G\{q^1\}| = |G\{q^3\}| = 4$, Lemma 5.2.4 (3) implies that $R'_\Pi q \neq q$, $R'_\Pi q^1 \neq q^1$ and $R'_\Pi q^3 \neq q^3$. So we need only show that the case: $R'_\Pi q = q^2$ and $R'_\Pi q^1 = q^3$ can not happen. Indeed, if $R'_\Pi q = q^2$ and $R'_\Pi q^1 = q^3$, then $R'_\Pi P_\Gamma q = -P_\Gamma q$ and $R'_\Pi P_\Gamma q^1 = -P_\Gamma q^1$ by recalling that $R'_\Pi \nu = \nu$. Hence both $P_\Gamma q$ and $P_\Gamma q^1$ are perpendicular to the plane Π and it follows that $P_\Gamma q \parallel P_\Gamma q^1$, which is obviously not true by the result in step 1. This contradiction proves our claim.

Step 3. By the result in step 2, we can fix Π_0 by letting

$$R'_{\Pi_0} q = q^1.$$

After this, one gets

$$R'_{\Pi_0} q^2 = q^3, \quad R'_{\Pi_1} q = q^3, \quad R'_{\Pi_1} q^1 = q^2.$$

Using $R'_\Pi \nu = \nu$ for $\Pi = \Pi_0$, Π_1 again, we have $R'_{\Pi_0} P_\Gamma q = P_\Gamma q^1$ and $R'_{\Pi_1} P_\Gamma q = P_\Gamma q^3$. Thus

$$\nu_{\Pi_0} \perp (P_\Gamma q + P_\Gamma q^1), \quad \nu_{\Pi_1} \perp (P_\Gamma q + P_\Gamma q^3).$$

Recalling that $P_\Gamma q^1 = \sqrt{k^2 - \tau^2}(e_3 \times \nu) = -P_\Gamma q^3$, the first part of our lemma follows immediately.

Step 4. We write

$$\hat{E}_p(\hat{x}) = \hat{s}e^{iq\cdot\hat{x}} + \hat{A}^1 e^{iq^1\cdot\hat{x}} + \hat{A}^2 e^{iq^2\cdot\hat{x}} + \hat{A}^3 e^{iq^3\cdot\hat{x}}.$$

Then direct application of Lemma 5.2.5 and the relations $R'_{\Pi_0} q = q^1$, $R'_{\Pi_0} q^2 = q^3$ and $R'_{\Pi_1} q = q^3$ yields

$$\hat{A}^1 = -R'_{\Pi_0}\hat{s}, \quad \hat{A}^3 = -R'_{\Pi_1}\hat{s}, \quad \hat{A}^2 = -R'_{\Pi_0}\hat{A}^3 = R'_{\Pi_0}(R'_{\Pi_1}\hat{s}).$$

Our lemma is proved. $\sharp$

Now, we want to find all the perfect planes of E_p. It will be little complicated this time. We first present the following two lemmas.

Lemma 5.2.14. *Let E_p be determined in Lemma 5.2.13 above, then there is no perfect plane Π of E_p such that $R'_\Pi q = q^2$ or $R'_\Pi q^1 = q^3$.*

Proof. By contradiction, assume $R'_\Pi q = q^2$ for some $\Pi \in \mathcal{P}$. Then $\nu_\Pi \parallel (q^2 - q)$. Recalling from Lemma 5.2.13 that $q^2 = \tau\nu - P_\Gamma q$, we have $\nu_\Pi \parallel P_\Gamma q$. On the other hand, since $|G\{q^1\}| = |G\{q\}| = |G| = 4$, we get by Lemma 5.2.4 (3) that $R'_\Pi q^1 \neq q^1$. Then it follows from Proposition 5.1.3 that we should have $R'_\Pi q^1 = q^3$, from which one can further deduce that $\nu_\Pi \parallel (q^3 - q^1) \parallel P_\Gamma q^1$. However, this is impossible since we know from Lemma 5.2.13 that $P_\Gamma q^1 \nparallel P_\Gamma q$. This contradiction shows that the case $R'_\Pi q = q^2$ can not happen. A same argument shows that the case $R'_\Pi q^1 = q^3$ can not happen either.

Lemma 5.2.15. *Let E_p be determined in Lemma 5.2.13, then there is a perfect plane Π of E_p such that $R'_\Pi q = q$ if and only if $\nu \parallel s \parallel \{x_3 = 0\}$. In such a case, one has*

$$\hat{E}_p(\hat{x}) = \hat{s}(e^{iq\cdot\hat{x}} - e^{iq^1\cdot\hat{x}} + e^{-iq\cdot\hat{x}} - e^{iq^1\cdot\hat{x}}), \tag{5.2.7}$$

where $q^1 = P_\Gamma q^1 = ke_3 \times \nu$.

Proof. We first prove the sufficient part: assume that $s \parallel \nu$, then $R'_{\Pi_0}\hat{s} = R'_{\Pi_1}\hat{s} = \hat{s}$. As a result,

$$\hat{A}^1 = -R'_{\Pi_0}\hat{s} = -\hat{s}, \quad \hat{A}^3 = -R'_{\Pi_1}\hat{s} = -\hat{s}, \quad \hat{A}^2 = -R'_{\Pi_0}\hat{A}^3 = \hat{s}.$$

Besides, noting that $s \cdot q = 0$, we have $\tau = q \cdot \nu = 0$. Consequently, $q^1 = P_\Gamma q^1 = \sqrt{k^2 - \tau^2}(e_3 \times \nu) = ke_3 \times \nu$ and $q^2 = -q$, $q^3 = -q^1$. Therefore,

$$\hat{E}_p(\hat{x}) = \hat{s}(e^{iq\cdot\hat{x}} - e^{iq^1\cdot\hat{x}} + e^{-iq\cdot\hat{x}} - e^{iq^1\cdot\hat{x}}).$$

From the expression above, it is easy to see that all the planes perpendicular to s are perfect planes of $\hat{E}_p$. Our sufficient part is proved.

Next, we show the necessary part: let $\Pi \in \mathcal{P}$ such that $R'_\Pi q = q$. By Proposition 5.1.3, we have $R'_\Pi\{q, q^1, q^2, q^3\} = \{q, q^1, q^2, q^3\}$. We claim $R'_\Pi q^2 = q^2$. Indeed, if $R'_\Pi q^2 \neq q^2$, then $R'_\Pi q^2 = q^1$ or $R'_\Pi q^2 = q^3$. We consider the case $R'_\Pi q^2 = q^1$ first. Clear we have $\nu_\Pi \parallel (q^1 - q^2)$. Noting that $q^2 - q^1 = P_\Gamma q + \sqrt{k^2 - \tau^2}(e_3 \times \nu)$ by Lemma 5.2.13, we see that $\Pi \parallel \Pi_0$. But then $R'_\Pi q = R'_{\Pi_0} q = q^1$ by Lemma 5.2.13 also. This contradicts to our assumption that $R'_\Pi q = q$. Thus we have shown $R'_\Pi q^2 \neq q^1$. Similarly, we can show that $R'_\Pi q^2 \neq q^1$, and $R'_\Pi q^2 = q^2$ follows.

Now, we have $R'_\Pi\{q^1, q^3\} = \{q^1, q^3\}$, we proceed to show $R'_\Pi q^1 = q^1$ and $R'_\Pi q^3 = q^3$. Indeed, if $R'_\Pi q^1 = q^3$, then $\nu_\Pi \parallel (q^1 - q^3)$, which further implies that $\nu_\Pi \parallel (e_3 \times \nu)$ by using Lemma 5.2.13. It follows that the vector $R'_\Pi q$ has negative x_3-component and hence does not belong to $\mathcal{Q}$. This clearly contradicts to Proposition 5.1.3. As a result, we see that all the vectors q, q^1, q^2, q^3 are invariant under the mapping R'_Π, so they must all lie on the plane that is parallel to Π and passing through the origin. But $\{q, q^1, q^2, q^3\}$ lie also on the plane $\tau\nu + \Gamma$, this implies that $\tau = 0$ and $\Gamma \parallel \Pi$. Then we see $R'_\Pi q = R'_\Gamma q = R'_\Gamma P_\Gamma q = P_\Gamma q = q$, from which $s \perp \Pi$ follows by referring to Lemma 5.1.3. Since $\Gamma \parallel \Pi$, this clearly shows that $s \perp \Gamma$ and hence $s \parallel \nu$. Our lemma is proved. $\sharp$

With the help of Lemma 5.2.14, 5.2.15 above, we are able to use the same method as in Lemma 5.2.8 to determine the set of perfect planes of E_p in Lemma 5.2.13. We have

Lemma 5.2.16. *Let E_p, Π_0, Π_1 be determined in Lemma 5.2.13. Then*

1. *The set $\mathcal{P}$ is determined by the incident field $E^i = se^{iq\cdot x}$ and the vector ν. More specifically, $\mathcal{P}$ consists of only two sets of planes: the first set contains only all the planes that are parallel to Π_0 with neighboring distance $\frac{\pi}{|q\cdot\nu_{\Pi_0}|}$, while the second set contains only all the planes that are parallel to Π_1 with neighboring distance $\frac{\pi}{|q\cdot\nu_{\Pi_1}|}$. In the special case when the condition*

$$s \parallel \nu \parallel \{x_3 = 0\}$$

holds, $\mathcal{P}$ contains an additional set of planes-planes that are planes perpendicular to the vector s.

2. *Each face of the grating structure S lies on a plane in $\mathcal{P}$.*

Lemma 5.2.16 enables us to find a new class of unidentifiable grating profiles. To describe the class explicitly, we first clarify some notations:

r: a position vector, viewed as a point in $\mathbf{R}^3$;

ν: a nonzero vector in $\mathbf{R}^3$ which is parallel to the plane $\{x_3 = 0\}$;

Γ: the plane in $\mathbf{R}^3$, which passes through the origin and takes ν as its normal;

L: a line in $\mathbf{R}^3$ passing through r and has direction ν;

Π_0: a plane in $\mathbf{R}^3$ which passes through L with $P_\Gamma q + \sqrt{k^2 - \tau^2}(e_3 \times \nu)$ as its normal;

Π_1: a plane in $\mathbf{R}^3$ which passes through L and is perpendicular to Π_0;

With these notations, the set of perfect planes of E_p is given by

$$
\begin{aligned}
\mathcal{P}_1 \quad = \quad & \Big\{ \text{plane } \Pi;\ \exists j \in \{0,1\} \text{ such that } \Pi \parallel \Pi_j, \quad \text{and for} \\
& \qquad j = 0, \quad \operatorname{dist}(\Pi, \Pi_0) = \frac{m\pi}{|q \cdot \nu_{\Pi_0}|} \quad \text{for some } m \in \mathbf{N} \\
& \qquad \text{for } \ j = 1, \quad \operatorname{dist}(\Pi, \Pi_1) = \frac{m}{\pi |q \cdot \nu_{\Pi_1}|} \quad \text{for some } m \in \mathbf{N} \Big\}
\end{aligned}
$$

in the general case and by

$$
\mathcal{P}_2 = \Big\{ \text{planes perpendicular to } s \Big\} \bigcup \mathcal{P}_1
$$

in the special case.

Now, we can define our classes of unidentifiable grating profiles. For the incident field $E^i = s e^{iq\cdot x}$, a point $r \in \mathbf{R}^3$ and a nonzero vector $\nu \parallel \{x_3 = 0\}$, if the following condition

$$
s \parallel \nu \parallel \{x_3 = 0\},
$$

does not hold, then we define the fourth class of unidentifiable grating profiles of S by

$$
\begin{aligned}
\mathcal{S}_4(s, q, \Lambda, r, \nu) \quad = \quad & \Big\{ \text{gratings with profile } S, \text{ which are } \Lambda\text{-periodic polyhedral structures} \\
& \qquad \text{such that faces of } S \text{ lie on planes in } \mathcal{P}_1 \Big\}.
\end{aligned}
$$

On the other hand, if the condition holds, then we define the fifth class of unidentifiable grating profiles by

$$\mathcal{S}_5(s,q,\Lambda,r) = \Big\{\text{gratings with profile } S\text{, which are } \Lambda\text{-periodic polyhedral structures such that faces of } S \text{ lie on planes in } \mathcal{P}_2\Big\}.$$

Our final result for this subsection is

Lemma 5.2.17. *If $L \parallel \{x_3 = 0\}$, $\mathcal{Q} = G\{q\}$, $|\mathcal{P}_0| = 2$ and $|G\{q\}| = 4$, then the grating profile S belongs to $\mathcal{S}_4(s,q,\Lambda,r)$ or $\mathcal{S}_5(s,q,\Lambda,r,\nu)$ for some point $r \in \mathbf{R}^3$ and direction vector $\nu \parallel \{x_3 = 0\}$.*

We make a reminder here that we shall extend our definition of the fifth class of unidentifiable grating profiles $\mathcal{S}_5(s,q,\Lambda,r)$ by removing the condition that $s \parallel \{x_3 = 0\}$.

Finally, we give a concrete example which has a non-empty class $\mathcal{S}_5(s,q,\Lambda,r)$.

Example 5.2.1. *Let $\Lambda = (\Lambda_1,\Lambda_2) = (0,4\pi)$, $q = (\alpha_1,\alpha_2,-\beta) = (\alpha_1,\frac{1}{2},-\frac{\sqrt{3}}{2})$, $s = \nu = e_1 = (1,0,0)$ and r be the origin. Then we can check by direct computations that*

$$\begin{aligned}
q^1 &= (\alpha_1,1,0) = q^{(0,3)} = (\alpha^{(0,3)}+\alpha)+(0,0,\beta^{(0,3)}),\\
q^2 &= (\alpha_1,\frac{1}{2},\frac{\sqrt{3}}{2}) = q^{(0,2)} = (\alpha^{(0,2)}+\alpha)+(0,0,\beta^{(0,2)}),\\
q^3 &= (\alpha_1,-1,0) = q^{(0,1)} = (\alpha^{(0,3)}+\alpha)+(0,0,\beta^{(0,3)}),
\end{aligned}$$

In this case $\nu_{\Pi_0} = (0,\frac{\sqrt{3}}{2},\frac{1}{2})$, $\nu_{\Pi_1} = (0,\frac{1}{2},-\frac{\sqrt{3}}{2})$, and $\mathcal{P}$ consists of two sets of planes: the first set contains all the planes that are parallel to Π_0 with neighboring distance $\frac{\pi}{|q\cdot\nu_{\Pi_0}|} = \frac{2\pi}{\sqrt{3}}$; the second set contains all the planes that are parallel to Π_1 with neighboring distance $\frac{\pi}{|q\cdot\nu_{\Pi_1}|} = 2\pi$.

Next we try to find some Λ-periodic structures in $\mathcal{S}_5(s,q,\Lambda,r,\nu)$. For each $l \in \mathbf{Z}$, we denote by L_{3l}, L_{3l+1}, L_{3l+2} the line of the form $\{(\lambda,4l\pi,0) : \lambda \in \mathbf{R}\}$, $\{(\lambda,4l\pi+\pi,\frac{\sqrt{3}\pi}{3}) : \lambda \in \mathbf{R}\}$, $\{(\lambda,4l\pi+2\pi,-\frac{2\sqrt{3}\pi}{3}) : \lambda \in \mathbf{R}\}$, respectively. For each $m \in \mathbf{Z}$, let Π_m be the plane that passes through the line L_m and L_{m+1}, and F_m be the part on Π_m which is *between the line L_m and L_{m+1}. Then it is clear that Π_m belongs to $\mathcal{P}$ for all $m \in \mathbf{Z}$ and that $\bigcup_{m\in\mathbf{Z}} F_m$ forms a Λ-periodic structure in $\mathcal{S}_5(s,q,\Lambda,r,\nu)$. By making proper translations, we see that there are infinite Λ-periodic structure in $\mathcal{S}_5(s,q,\Lambda,r,\nu)$. All these structure can not be distinguished by the incident field E^i.*

Sixth class of unidentifiable grating structures

We now consider in this subsection the last case, case (4) in Lemma 5.2.6. This case will lead to the sixth class of unidentifiable gratings corresponding to the given incident field E^i. We show first in the following lemma that both the propagating field E_p and and its set of perfect planes $\mathcal{P}_0$ are determined by E^i.

Lemma 5.2.18. *If $L \parallel \{x_3 = 0\}$, $\mathcal{Q} = G\{q\}$ and $|\mathcal{P}_0| = |G\{q\}| = 4$, then we have*

1. *$s \parallel \{x_3 = 0\}$, $L \parallel (s \times e_3)$.*
2. *$\mathcal{P}_0$ contains a plane which is perpendicular to s and which we denote it by Π_0. The other three planes in $\mathcal{P}_0$ are $\Pi_1 = T_{\pi/4}\Pi_0$, $\Pi_2 = T_{\pi/2}\Pi_0$, $\Pi_3 = T_{3\pi/4}\Pi_0$.*
3. *The propagating field $E_p(x)$ in (5.1.2) or $\hat{E}_p(\hat{x})$ in (5.2.3) can be written as*

$$\begin{aligned} E_p(x) &= s(e^{iq\cdot x} - e^{iq^2\cdot x + (q-q^2)\cdot r}) + T_{\pi/2}s(e^{iq^1\cdot x+(q-q^1)\cdot r} - e^{iq^3\cdot x+(q-q^3)\cdot r}), \\ \hat{E}_p(\hat{x}) &= \hat{s}(e^{iq\cdot\hat{x}} - e^{iq^2\cdot\hat{x}}) + T_{\pi/2}\hat{s}(e^{iq^1\cdot\hat{x}} - e^{iq^3\cdot\hat{x}}). \end{aligned}$$

where $q^j = R'_{\Pi_j} q = T_{j\pi/2}q$, $j = 1,2,3$. Moreover, $q^1, q^3 \in \Xi^*$.

Proof. Step 1. We show that $P_\Gamma q \parallel e_3$. Indeed, since $|G\{q\}| = |\mathcal{P}_0|$, we have by Lemma 5.2.4 (3) that

$$\begin{aligned} \mathcal{Q} = G\{q\} = G^*\{q\} = \tau\nu + G^*\{P_\Gamma q\} &= \tau\nu + \{P_\Gamma q, T_{\pi/2}P_\Gamma q, T_\pi P_\Gamma q, T_{3\pi/2}P_\Gamma q\} \\ &= \tau\nu + \{P_\Gamma q, T_{\pi/2}P_\Gamma q, -P_\Gamma q, -T_{\pi/2}P_\Gamma q\}. \end{aligned}$$

Noting that $\nu \parallel \{x_3 = 0\}$, we should have $T_{\pi/2}P_\Gamma q \parallel \{x_3 = 0\}$ by Proposition 5.1.1. Then it follows that $P_\Gamma q \parallel e_3$ since $P_\Gamma q \perp T_{\pi/2}P_\Gamma q$.

Step 2. We proceed to determine $\mathcal{P}_0$. By Lemma 5.2.4 (3), there is a plane in $\mathcal{P}_0$, which we denote by Π_0, such that $R'_{\Pi_0}q = q$. Since $R'_{\Pi_0}\nu = \nu$, we have $R'_{\Pi_0}P_\Gamma q = P_\Gamma q$. Thus $P_\Gamma q \parallel \Pi_0$. Therefore we see that $\Pi_0 \perp \{x_3 = 0\}$ by using the result $P_\Gamma q \parallel e_3$ in the previous paragraph. But according to Lemma 5.1.3, $s \perp \Pi_0$ also, so we can conclude that $s \parallel \{x_3 = 0\}$. The vector ν can also be determined. Indeed, since $s \perp \Pi_0$ and $\nu \parallel \Pi_0$, we have $\nu \perp e_3$. Besides, noting that $\nu \perp e_3$, we have $\nu \parallel (s \times e_3)$. By now, we can see that the planes in $\mathcal{P}_0$ are totally determined: the other three planes are formed by rotating Π_0 with angles $\pi/4$, $\pi/2$ and $3\pi/4$ respectively on the axis L.

Step 3. By Lemma 5.2.4 (2), we have

$$\mathcal{Q} = \{R'_\Pi q : \Pi \in \mathcal{P}_0\} = \{R'_{\Pi_0}q, R'_{\Pi_1}q, R'_{\Pi_2}q, R'_{\Pi_3}q\}$$

Let $q^j = R'_{\Pi_j}q$, $j = 1, 2, 3$. Then

$$q^j = \tau\nu + T_{j\pi/2}P_\Gamma q = T_{j\pi/2}q, \quad j = 1, 2, 3.$$

We write $\hat{E}_p$ as

$$\hat{E}_p(\hat{x}) = \hat{s}e^{iq\cdot\hat{x}} + \hat{A}^1 e^{iq^1\cdot\hat{x}} + \hat{A}^2 e^{iq^2\cdot\hat{x}} + \hat{A}^3 e^{iq^3\cdot\hat{x}}. \tag{5.2.8}$$

Using Lemma 5.2.5 and the relation $q^j = R'_{\Pi_j}q$, for $j = 1, 2, 3$. we get

$$\hat{s} = -R'_{\Pi_0}\hat{s}, \;\; \hat{A}^1 = -R'_{\Pi_1}\hat{s}, \;\; \hat{A}^2 = -R'_{\Pi_2}\hat{s}, \;\; \hat{A}^3 = -R'_{\Pi_3}\hat{s},$$

which further yields

$$\hat{A}^1 = (R'_{\Pi_1}\circ R'_{\Pi_0})\hat{s} = T_{\pi/2}\hat{s}, \;\; \hat{A}^2 = (R'_{\Pi_2}\circ R'_{\Pi_0})\hat{s} = T_\pi\hat{s} = -s, \;\; \hat{A}^3 = (R'_{\Pi_3}\circ R'_{\Pi_0})\hat{s} = T_{3\pi/2}\hat{s} = -\hat{A}^1.$$

Our lemma is proved. ♯

The set of perfect planes of $\hat{E}_p$ is also determined.

Lemma 5.2.19. *Let $\hat{E}_p$ and Π_j, $j = 1, 2, 3, 4$ be determined in Lemma 5.2.18, then*

1. *The set $\mathcal{P}$ is determined by the incident field $E^i = se^{iq\cdot x}$; Specifically, a plane Π belongs to $\mathcal{P}$ if and only if there exists $j \in \{0, 1, 2, 3\}$ such that $\Pi \parallel \Pi_j$ and the distance between Π and Π_j, denoted by $d(\Pi_j, \Pi)$ equals $\frac{m\pi}{|q\cdot\nu_{\Pi_j}|} = m\frac{\pi}{\|P_\Gamma q\|}$ for $j = 0, 2$ and $\frac{m\sqrt{2}\pi}{\|P_\Gamma q\|}$ for $j = 1, 3$, where m is some integer.*

2. Each face of the grating structure S lies on a plane in $\mathcal{P}$. $\sharp$

Now, we are ready to define the class of unidentifiable grating profiles. Let's first clarify the notations in this subsection:

r: a position vector, viewed as a point in $\mathbf{R}^3$;

L: a line in $\mathbf{R}^3$, which passes through the point r and has a direction $s \times e_3$;

Π_0: a plane in $\mathbf{R}^3$ which passes through L and perpendicular to the plane $\{x_3 = 0\}$;

Π_1, Π_2 and Π_3: planes in $\mathbf{R}^3$ which passes through L and form angles of $\pi/4$, $\pi/2$, $3\pi/4$ with Π_0 respectively.

With the notations, the set of perfect planes of E_p can be described by

$$\begin{aligned}\mathcal{P} \;=\; \Big\{ &\text{plane } \Pi;\ \exists j \in \{0,1,2,3\} \text{ such that } \Pi \parallel \Pi_j, \quad \text{and for} \\ &j = 0, 2, \quad \operatorname{dist}(\Pi, \Pi_j) = \frac{m\pi}{|q \cdot \nu_{\Pi_0}|} \quad \text{for some } m \in \mathbf{N}, \\ &\text{for } \ j = 1, 3, \quad \operatorname{dist}(\Pi, \Pi_j) = \frac{m\pi}{|q \cdot \nu_{\Pi_1}|} \quad \text{for some } m \in \mathbf{N} \Big\}.\end{aligned}$$

The corresponding class of unidentifiable grating profiles is defined for the incident field $E^i = s e^{iq\cdot x}$ and the point $r \in \mathbf{R}^3$ by

$$\begin{aligned}\mathcal{S}_6(s, q, \Lambda, r) \;=\; \Big\{ &\text{gratings with profile } S, \text{ which are } \Lambda\text{-periodic polyhedral structures} \\ &\text{such that faces of } S \text{ lie on planes in } \mathcal{P} \Big\}.\end{aligned}$$

We summarize the results in this subsection by the following lemma.

Lemma 5.2.20. *If $L \parallel \{x_3 = 0\}$, $\mathcal{Q} = G\{q\}$ and $|\mathcal{P}_0| = |G\{q\}| = 4$, then the grating profile S belongs to $\mathcal{S}_6(s, q, \Lambda, r)$ for some $r \in \mathbf{R}^3$. Further more, any grating profiles in $\mathcal{S}_6(s, q, \Lambda, r)$ can generate the same total field(which is $\hat{E}_p$) as S.*

Example 5.2.2. *Let $\Lambda = (\Lambda_1, \Lambda_2) = (0, 2\pi)$, $q = (\alpha_1, \alpha_2, -\beta) = (0, 0, -1)$, $s = e_2 =$*

$(0,1,0)$ *and* r *be the origin. Then we can check by direct computations that*

$$q^1 = (0,1,0) = q^{(0,1)} = (\alpha^{(0,1)} + \alpha) + (0,0,\beta^{(0,1)}),$$
$$q^2 = (0,0,1) = q^{(0,0)} = (\alpha^{(0,0)} + \alpha) + (0,0,\beta^{(0,0)}),$$
$$q^3 = (0,-1,0) = q^{(0,-1)} = (\alpha^{(0,-1)} + \alpha) + (0,0,\beta^{(0,-1)}),$$

In this case $\nu \parallel (s \times e_3) \parallel e_1$, $\nu_{\Pi_0} = (0,1,0)$, $\nu_{\Pi_1} = (0,\frac{1}{2},\frac{1}{2})$, $\nu_{\Pi_2} = (0,0,1)$, $\nu_{\Pi_3} = (0,-\frac{1}{2},\frac{1}{2})$ *and* $\mathcal{P}$ *consists of four sets of planes: the first two sets contains all the planes that are parallel to* Π_0 *or* Π_2 *with neighboring distance* $\frac{\pi}{|q\cdot\nu_{\Pi_0}|} = \pi$*; the last two sets contains all the planes that are parallel to* Π_1 *or* Π_3 *with neighboring distance* $\frac{\pi}{|q\cdot\nu_{\Pi_1}|} = \sqrt{2}\pi$*;*

We next try to find some Λ*-periodic structures in* $\mathcal{S}_6(s,q,\Lambda,r)$*. For each* $l \in \mathbf{Z}$*, we denote by* L_{3l}, L_{3l+1}, L_{3l+2} *the line of the form* $\{(\lambda, 2l\pi, 0) : \lambda \in \mathbf{R}\}$, $\{(\lambda, 2l\pi + \pi, \pi) : \lambda \in \mathbf{R}\}$, $\{(\lambda, 2l\pi + \pi, 0) : \lambda \in \mathbf{R}\}$*, respectively. For each* $m \in \mathbf{Z}$*, let* Π_m *be the plane that passes through the line* L_m *and* L_{m+1}*, and* F_m *be the part on* Π_m *which is between the line* L_m *and* L_{m+1}*. Then it is clear that* Π_m *belongs to* $\mathcal{P}$ *for all* $m \in \mathbf{Z}$ *and that* $\bigcup_{m\in\mathbf{Z}} F_m$ *forms a* $\Lambda-$*periodic structure in* $\mathcal{S}_6(s,q,\Lambda,r)$*. By making proper translations, we see that there are infinite* $\Lambda-$*periodic structure in* $\mathcal{S}_6(s,q,\Lambda,r)$*. All these structure can not be distinguished by the incident field* E^i*.*

5.2.4 Study of the case with $L \parallel \{x_3 = 0\}$ and $\mathcal{Q} \neq G\{q\}$

We have studied the case with $L \parallel \{x_3 = 0\}$ and $\mathcal{Q} \neq G\{q\}$ in the previous subsection, we now continue to study the case with $L \parallel \{x_3 = 0\}$ and $\mathcal{Q} \neq G\{q\}$ in this subsection. We shall see that this case leads to the seventh unidentifiable grating class corresponding to the incident electric field E^i.

Preliminary results

We first present some useful properties about the propagating field E_p and the set perfect planes that passes through the line L, i.e. $\mathcal{P}_0$, which facilitate our subsequent analysis.

Lemma 5.2.21. *If $L \parallel \{x_3 = 0\}$ and $\mathcal{Q} \neq G\{q\}$, then we have:*

1. $s \parallel \{x_3 = 0\}$, $L \parallel (s \times e_3)$;

2. $\mathcal{P}_0$ *consists of two planes: one is perpendicular to the vector s and is denoted by Π_0; the other is perpendicular to e_3 and is denoted by Π_1;*

3. $R'_{\Pi_0} q = q$;

4. $\mathcal{Q} \setminus G\{q\} \subset \Xi^*$.

Proof. Step 1. For each $q^* \in \mathcal{Q} \setminus G\{q\}$, consider the set $G^*\{q^*\} \subset \mathcal{Q}$, we see that $|G^*\{q^*\}| \geq 2$. According to Proposition 5.1.1, all its elements have non-negative x_3-components. However, this is possible only if $G^* = \{Id, T_\pi\}$ and $P_\Gamma q^* \parallel \{x_3 = 0\}$ by the result in Lemma 5.2.4 and the assumption that the x_3-component of ν is zero. Hence $|\mathcal{P}_0| = 2$ and (4) follows also.

Step 2. We write $\mathcal{P}_0 = \{\Pi_0, \Pi_1\}$ since $|\mathcal{P}_0| = 2$. Let $q^* \in \mathcal{Q} \setminus G\{q\}$, then $\{R'_\Pi q^* : \Pi = \Pi_0, \Pi_1\} \subset G\{q^*\} \subset \mathcal{Q} \setminus G\{q\} \subset \Xi^*$, we see all the vectors $R'_{\Pi_0} q^*$, $R'_{\Pi_1} q^*$ and q^* lie on the plane $\{x_3 = 0\}$. But this is possible only if both Π_0 and Π_1 are either parallel or perpendicular to the plane $\{x_3 = 0\}$. Since Π_0 and Π_1 are perpendicular to each other, thus one of them is parallel to $\{x_3 = 0\}$ and other is perpendicular to $\{x_3 = 0\}$. For this, we let the plane which is perpendicular to the plane $\{x_3 = 0\}$ be Π_0, and the other be Π_1.

Step 3. We claim that $R'_{\Pi_0} q = q$. Indeed, since $\Pi_0 \perp \{x_3 = 0\}$, we see that the x_3-component of $R'_{\Pi_0} q = \tau\nu + R'_{\Pi_0} P_\Gamma q$ is negative. But q is the only element in $\mathcal{Q}$ which has negative x_3 component by Proposition 5.1.1, so we have $R'_{\Pi_0} q = q$. Then Lemma 5.1.3 implies that $s \perp \Pi_0$. Noting that $\Pi_0 \perp \{x_3 = 0\}$, we have $s \parallel \{x_3 = 0\}$. Besides, since $\nu \perp \Pi_0$ and $\nu \perp e_3$, we see that $\nu \parallel (s \times e_3)$. This proves (1).

Step 4. We have $\nu_{\Pi_1} \perp s$ since $\Pi_1 \perp \Pi_0$ and $s \parallel \Pi_0$. Besides, noting that $\Pi_1 \parallel \nu \parallel (s \times e_3)$, so we have $\nu_{\Pi_1} \perp (s \times e_3)$ also. Then it follows that $\nu_{\Pi_1} \perp s \times (s \times e_3)$. But (1) implies that $s \perp e_3$, thus we have $\nu_{\Pi_1} \parallel e_3$. Our lemma is proved. $\sharp$

Seventh class of unidentifiable grating structures

By now, we have shown some useful information about E_p and $\mathcal{P}_0$ in Lemma 5.2.21. We continue to determine E_p and $\mathcal{P}$ as we do before. For this, we consider first the case with $|\mathcal{Q}| = 4$.

Lemma 5.2.22. *If $L \parallel \{x_3 = 0\}$, $|G| = 4$, $|G\{q\}| = 2$ and $|\mathcal{Q}| = 4$, then the propagating field E_p in (5.1.2) or $\hat{E}_p$ in (5.2.3) can be written as*

$$E_p(x) = s(e^{iq\cdot x} - e^{iq^2\cdot x + (q-q^2)\cdot r}) + A^1(e^{iq^1\cdot x} - e^{iq^3\cdot x + (q^1-q^3)\cdot r}); \qquad (5.2.9)$$

$$\hat{E}_p(\hat{x}) = \hat{s}(e^{iq\cdot\hat{x}} - e^{iq^2\cdot\hat{x}}) + \hat{A}^1(e^{iq^1\cdot\hat{x}} - e^{iq^3\cdot\hat{x}}); \qquad (5.2.10)$$

where $q^2 = R'_{\Pi_1}q = T_\pi q$, $q^1, q^3 \in \Xi^*$ with $R'_{\Pi_0}q^1 = q^3$, and $\hat{A}^1$ is non-zero vector which is parallel to e_3.

Proof. Since $|G\{q\}| = 2$ and $|\mathcal{Q}| = 4$, we can write $\mathcal{Q} = G\{q\} \bigcup G\{q^1\}$, with $G\{q\} = \{q, q^2\}$ and $G\{q^1\} = \{q^1, q^3\}$. By Lemma 5.2.21, $q^1, q^3 \in \Xi^*$. It is also easy to see that the following relations hold:

$$R'_{\Pi_0}q = q, \quad R'_{\Pi_1}q = q^2; \quad R'_{\Pi_1}q^1 = q^1, \quad R'_{\Pi_0}q^1 = q^3.$$

We write $\hat{E}_p$ in the form

$$\hat{E}_p(\hat{x}) = \hat{s}e^{iq\cdot\hat{x}} + \hat{A}^1e^{iq^1\cdot\hat{x}} + \hat{A}^2e^{iq^2\cdot\hat{x}} + \hat{A}^3e^{iq^3\cdot\hat{x}}.$$

Using Lemma 5.2.5, Lemma 5.1.3 and the relations $R'_{\Pi_1}q = q^2$, $R'_{\Pi_1}q^1 = q^1$, $R'_{\Pi_0}q^1 = q^3$, we get

$$\hat{A}^2 = -R'_{\Pi_1}s, \quad \hat{A}^1 \perp \Pi_1, \quad \hat{A}^3 = -R'_{\Pi_0}\hat{A}^1,$$

Note that $s \parallel \Pi_1$ and $e_3 \parallel \Pi_0$ by the result in Lemma 5.2.21. We get immediately that $\hat{A}^2 = -\hat{s}$, $\hat{A}^1 \parallel e_3$ and $\hat{A}^3 = -\hat{A}^1$. Consequently

$$\hat{E}_p(\hat{x}) = \hat{s}(e^{iq\cdot\hat{x}} - e^{iq^2\cdot\hat{x}}) + \hat{A}^1(e^{iq^1\cdot\hat{x}} - e^{iq^3\cdot\hat{x}}).$$

This end our proof. $\sharp$

By Lemma 5.2.22 above, we see that the electric field E_p is not totally determined under the assumption that $L \parallel \{x_3 = 0\}$, $|G| = 4$, $|G\{q\}| = 2$ and $|\mathcal{Q}| = 4$. In fact (q^1, q^3) can be any pair in Ξ^* such that $R'_{\Pi_0} q^1 = q^3$, and $\hat{A}^1$ can be any non-zero vector that is parallel to e_3.

Next, we want to find the set of perfect planes of E_p in Lemma 5.2.22. We have the following two results.

Lemma 5.2.23. *Let E_p be determined in Lemma 5.2.22, if there is a perfect plane Π of E_p such that $R'_\Pi q = q^1$, then $q^1 = T_{j\pi/2} q$, $q^3 = T_{3\pi/2} q$ and E_p can be written as*

$$E_p(x) = s(e^{iq\cdot x} - e^{iq^2\cdot x + (q-q^2)\cdot \tilde{r}}) + T_{\pi/2} s(e^{iq^1\cdot x + (q-q^1)\cdot \tilde{r}} - e^{iq^3\cdot x + (q-q^3)\cdot \tilde{r}})$$

for some $\tilde{r} \in \mathbf{R}^3$. In such a special case, the set of perfect planes of E_p has the same structure as that considered in Subsection 5.2.3. Moreover, by the symmetry of q^1 and q^3, similar result holds if we have $R'_\Pi q = q^3$.

Proof. Since $R'_\Pi q = q^1$, we have by Lemma 5.2.5 that $R'_\Pi \hat{s} + \hat{A}^1 = 0$. Thus $\nu_\Pi \parallel (\hat{s} - \hat{A}^1)$. But we know from Lemma 5.2.21 and Lemma 5.2.22 that both $\hat{s}$ and $\hat{A}^1$ are perpendicular to ν, so we get $\nu_\Pi \perp \nu$ and hence it follows that $\Pi \parallel L$. Besides, it is clear that $\Pi \nparallel \Pi_0$ and $\Pi \nparallel \Pi_1$ for otherwise either $R'_\Pi q = R'_{\Pi_0} q = q$ or $R'_\Pi q = R'_{\Pi_1} q = q^2$, both of which contradicts to the assumption that $R'_\Pi q = q^1$. Now, let $\tilde{L}$, $\tilde{\mathcal{P}_0}$, $\tilde{G}$ denote the line of intersection between Π and Π_0, the set of perfect planes in $\mathcal{P}$ which passes through the line $\tilde{L}$, and the group generated by the reflections $\{R'_{\Pi^*} : \Pi^* \in \tilde{\mathcal{P}_0}\}$, respectively. We also chose some $\tilde{r}$ on the line $\tilde{L}$. Clearly we have $\tilde{L} \parallel L$ and $|\tilde{\mathcal{P}_0}| \geq 3$. Noting that $|\mathcal{Q}| = 4$, we should have $|\tilde{\mathcal{P}_0}| = 4$. Then we come to the situation that $\tilde{L} \parallel \{x_3 = 0\}$ and $|\tilde{\mathcal{P}_0}| = |\mathcal{Q}| = 4$, which we have studied in subsection 5.2.3. By the results there (Lemma 5.2.18, 5.2.19), we get the desired result immediately.

Lemma 5.2.24. *Let E_p be in Lemma 5.2.22 but not the special case considered in Lemma 5.2.23, then*

1. *$\mathcal{P}$ is determined. Specifically, $\mathcal{P}$ consists of two set of planes: the first set contains planes that parallel to Π_0 with distance some multiple of $\frac{\pi}{|q^1\cdot \nu_{\Pi_0}|} = \frac{\pi}{\|P_\Gamma q^1\|}$; the*

second set contains planes that are parallel to Π_1 *with distance some multiple of* $\frac{\pi}{|q\cdot\nu_{\Pi_1}|} = \frac{\pi}{\|P_\Gamma q\|}$.

2. *Each face of the grating structure* S *lies on a plane in* $\mathcal{P}$.

Proof. Following Proposition 5.1.3, we see that $R'_\Pi\{q, q^1, q^2, q^3\} = \{q, q^1, q^2, q^3\}$ if $\Pi \in \mathcal{P}$. Then four cases may happen:

(i) $R'_\Pi q = q$; (ii) $R'_\Pi q = q^1$; (iii) $R'_\Pi q = q^2$; (iv) $R'_\Pi q = q^3$.

By the same method as in Lemma 5.2.8, we can show that case (i) happens if and only if $\Pi \parallel \Pi_0$ and the distance between Π and Π_0 is some multiple of $\frac{\pi}{|q^1\cdot\nu_{\Pi_0}|} = \frac{\pi}{\|P_\Gamma q^1\|}$, and similar result for case (iii). However, case (ii) and (iv) will leads to the special case we considered in Lemma 5.2.23. This finishes our proof.

We have studied the case with $|\mathcal{Q}| = 4$ above, now we turn to the case with $|\mathcal{Q}| > 4$.

Consider the set $\mathcal{Q}$, it can be divided into pairwise disjoint orbits under the group G's action, that is:

$$\mathcal{Q} = G\{q\} \bigcup \Big(\bigcup_{1\le j\le m_0-1} G\{q^j\}\Big)$$

for some integer $m_0 > 2$. By Lemma 5.2.21, we know that $\Pi_1 \parallel \{x_3 = 0\}$ and q^j lie on the plane $\{x_3 = 0\}$ for $1 \le j \le m_0 - 1$. Thus $R'_{\Pi_1} q^j = q^j$ and it follows that $G\{q^j\} = G^*\{q^j\} = \{q^j, (R'_{\Pi_0} \circ R'_{\Pi_1})q^j\} = \{q^j, R'_{\Pi_0} q^j\}$ by using Lemma 5.2.4 (3).

Now, it is clear that $|\mathcal{Q}| = 2m_0$. Following the same steps as in Lemma 5.2.22, Lemma 5.2.24, (where $m_0 = 2$), we can work out E_p and $\mathcal{P}$.

Lemma 5.2.25. *If* $L \parallel \{x_3 = 0\}$, $|G| = 4$, $|G\{q\}| = 2$ *and* $|\mathcal{Q}| = 2m_0$ *for some integer* $m_0 > 2$, *then*

1. $\hat{E}_p$ *has the expression*

$$\hat{E}_p(\hat{x}) = \hat{s}(e^{iq\cdot\hat{x}} - e^{iq^{m_0}\cdot\hat{x}}) + \sum_{1\le j\le m_0-1} \hat{A}^j(e^{iq^j\cdot\hat{x}} - e^{iq^{j+m_0}\cdot\hat{x}})$$

where each q^j *is a vector in* $\Xi^* \bigcap \Xi_0$ *such that* $q^{j+m_0} =: R'_{\Pi_1} q^j$ *belongs to* $\Xi^* \bigcap \Xi_0$ *also. Here each* $\hat{A}^j$ *is a non-zero vector that is parallel to* e_3.

2. *there are two set of perfect planes in $\mathcal{P}$. The first set contains planes that are parallel to Π_0 with a distance some multiple of each of the numbers $\frac{\pi}{|q^j \cdot \nu_{\Pi_0}|}$ for $1 \le j \le m_0 - 1$; The second sets contains planes that are parallel to Π_1 with a distance some multiple of $\frac{\pi}{|q \cdot \nu_{\Pi_1}|}$.*

3. *Each face of the grating structure S lies on a plane in $\mathcal{P}$.*

Now, we are ready to define the class of unidentifiable grating profiles. Since the special case considered in Lemma 5.2.23 has the same nature as that considered in Subsection 5.2.3, we concern only with the general case under the assumption $L \parallel \{x_3 = 0\}$ and $\mathcal{Q} \ne G\{q\}$ in what follows.

We first recall the following notations:

r: a position vector, viewed as a point in $\mathbf{R}^3$;

L: a line in $\mathbf{R}^3$, which passes through the point r and has a direction $s \times e_3$;

Π_0: a plane in $\mathbf{R}^3$ which passes through L with $s/\|s\|$ as its normal;

Π_1: a plane in $\mathbf{R}^3$ which passes through L and is perpendicular to Π_0.

With these notations, we have

$$\begin{aligned}\mathcal{P} \;=\; \Big\{ & \text{plane } \Pi;\ \exists j \in \{0,1\} \text{ such that } \Pi \parallel \Pi_j, \quad \text{and for } j = 0, \\ & \operatorname{dist}(\Pi, \Pi_j) = \frac{m\pi}{|q^n \cdot \nu_{\Pi_0}|} \quad \text{for some } m \in \mathbf{N} \text{ and all } q^j \in \Xi^* \bigcap \Xi_0 \text{ satisfying } R'_{\Pi_1} q^j \in \Xi^*; \\ & \text{for } j = 1, \quad \operatorname{dist}(\Pi, \Pi_j) = \frac{m\pi}{|q \cdot \nu_{\Pi_1}|} \quad \text{for some } m \in \mathbf{N} \Big\}.\end{aligned}$$

To sum up the results we considered for both cases, we define the seventh class of unidentifiable grating profiles for each incident field $E^i = s e^{iq\cdot x}$ with $s \parallel \{x_3 = 0\}$ and each point $r \in \mathbf{R}^3$ by

$$\begin{aligned}\mathcal{S}_7(s, q, \Lambda, r) \;=\; \Big\{ & \text{gratings with profile } S, \text{ which are } \Lambda\text{-periodic polyhedral structures} \\ & \text{such that faces of } S \text{ lie on planes in } \mathcal{P} \Big\}.\end{aligned}$$

Our result can be included in the following lemma:

Lemma 5.2.26. *If $L \parallel \{x_3 = 0\}$ and $\mathcal{Q} \neq G\{q\}$, the grating profile S belongs to $\mathcal{S}_7(s, q, \Lambda, r)$ or $\mathcal{S}_6(s, q, \Lambda, r)$ for some $r \in \mathbf{R}^3$. Further more, any grating profiles in $\mathcal{S}_7(s, q, \Lambda, r)$ can generate the same total field as S.*

Example 5.2.3. *Let $\Lambda = (\Lambda_1, \Lambda_2) = (0, 2\pi)$, $q = (\alpha_1, \alpha_2, -\beta) = (0, 0, -1)$, $s = e_2 = (0, 1, 0)$ and r be the origin. Then we can check by direct computations that*

$$q^1 = (0,1,0) = q^{(0,1)} = (\alpha^{(0,1)} + \alpha) + (0, 0, \beta^{(0,1)}),$$
$$q^2 = (0,0,1) = q^{(0,0)} = (\alpha^{(0,0)} + \alpha) + (0, 0, \beta^{(0,0)}),$$
$$q^3 = (0,-1,0) = q^{(0,-1)} = (\alpha^{(0,-1)} + \alpha) + (0, 0, \beta^{(0,-1)}),$$

In this case $\nu \parallel (s \times e_3) \parallel e_1$, $\nu_{\Pi_0} = (0, 1, 0)$, $\nu_{\Pi_1} = (0, 0, 1)$ and $\mathcal{P}$ consists of two sets of planes: the first set contains all the planes that are parallel to Π_0 with neighboring distance $\frac{m\pi}{|q \cdot \nu_{\Pi_0}|} = m\pi$; the second set contains all the planes that are parallel to Π_1 with neighboring distance $\frac{m\pi}{|q \cdot \nu_{\Pi_1}|} = m\pi$;

We next try to find some Λ-periodic structures in $\mathcal{S}_7(s, q, \Lambda, r)$. For each $l \in \mathbf{Z}$, we denote by L_{4l}, L_{4l+1}, L_{4l+2}, L_{4l+3} the line of the form $\{(\lambda, 2l\pi, 0) : \lambda \in \mathbf{R}\}$, $\{(\lambda, 2l\pi + \pi, 0) : \lambda \in \mathbf{R}\}$ $\{(\lambda, 2l\pi + \pi, \pi) : \lambda \in \mathbf{R}\}$, $\{(\lambda, 2l\pi + 2\pi, 0) : \lambda \in \mathbf{R}\}$, respectively. For each $m \in \mathbf{Z}$, Let Π_m be the plane that passes through the line L_m and L_{m+1}, and F_m be the part on Π_m which is between the line L_m and L_{m+1}. Then it is clear that Π_m belongs to $\mathcal{P}$ for all $m \in \mathbf{Z}$ and that $\bigcup_{m \in \mathbf{Z}} F_m$ forms a $\Lambda-$periodic structure in $\mathcal{S}_7(s, q, \Lambda, r)$. By making proper translations, we see that there are infinite $\Lambda-$periodic structure in $\mathcal{S}_7(s, q, \Lambda, r)$. All these structure can not be distinguished by the incident field E^i.

5.2.5 Study of the case with $L \nparallel \{x_3 = 0\}$

We have studied the case with $L \parallel \{x_3 = 0\}$ in Subsection 5.2.3, 5.2.4, which leads to six classes of unidentifiable grating structures corresponding to the incident field $E^i(x) = se^{iq \cdot x}$. Now, we continue the discussion to study the last possible case with $L \nparallel \{x_3 = 0\}$ in this subsection. We shall show that this case leads to the extended fifth class of unidentifiable grating structures, as remarked at the end of Subsection 5.2.3.

Preliminary results

We first present two lemmas that are fundamental to the subsequent analysis.

Lemma 5.2.27. *If $L \nparallel \{x_3 = 0\}$, then there exists a plane $\Pi^* \in \mathcal{P}$ such that $\Pi^* \nparallel L$. As a result, $\Pi^* \nparallel \Pi_0$ and $\Pi^* \nparallel \Pi_1$.*

Lemma 5.2.28. *If $L \nparallel \{x_3 = 0\}$, then it holds that*

$$q + \sum_{n \in \Xi_0} q^n = 0\,, \tag{5.2.11}$$

$$\tau + \sum_{n \in \Xi_0} \tau_n = 0\,, \tag{5.2.12}$$

$$\tau_n \geq 0 \quad \forall q^n \in \mathcal{Q} \setminus G\{q\}\,. \tag{5.2.13}$$

It is clear that $\tau = 0$ or $\tau \neq 0$. Based on this, We further divide our following analysis into two subsections, Subsection 5.2.5, 5.2.5, which deal with the case $\tau = 0$ and the case $\tau \neq 0$ respectively. We shall show that the former case may provide us with a class of unidentifiable grating structures while the later one can be excluded.

Study of the case with $L \nparallel \{x_3 = 0\}$ and $\tau = 0$

First of all, we consider the case with $L \nparallel \{x_3 = 0\}$ and $\tau = 0$. We have

Lemma 5.2.29. *If $L \nparallel \{x_3 = 0\}$ and $\tau = 0$, then only the case with $|\mathcal{Q}| = |G| = 4$ is possible to happen.*

Proof. We prove by showing that all the other cases can not happen. First, using Lemma 5.2.28 and Lemma 5.2.4 (1), we see that $\tau_n = 0$ for all $q^n \in \mathcal{Q}$, whence it follows that all the elements in $\mathcal{Q}$ lie on the plane Γ.

Secondly, we exclude the case $|\mathcal{Q}| = 2$. If $|\mathcal{Q}| = 2$, then it is easy to see that $\mathcal{Q} = G\{q\}$ and $|\mathcal{P}_0| = 2$. Thus we have $\mathcal{P}_0 = \{\Pi_0, \Pi_1\}$. We write $\mathcal{Q} = G\{q\} = \{q, q^1\}$. Using Lemma 5.2.4 (3), we can assume (after relabeling if necessary) that $R'_{\Pi_0} q = q$ and $R'_{\Pi_1} q = q^1$. Let Π^* be the plane in Lemma 5.2.27, By Proposition 5.1.3, $R'_{\Pi^*}\{q, q^1\} =$

$\{q, q^1\}$. So either $R'_{\Pi^*}q = q$ or $R'_{\Pi^*}q = q^1$. Then one can deduce from Lemma 5.1.3 that $\Pi^* \parallel \Pi_0$ or $\Pi^* \parallel \Pi_1$. But this contradiction to Lemma 5.2.27.

Finally, we consider the case $|\mathcal{Q}| \geq 3$, we show first that $\Pi^* \parallel \Gamma$. To see this, we note that the vectors in $\mathcal{Q}$ span the plane Γ. By Proposition 5.1.3, we have $R'_{\Pi^*}\Gamma = \Gamma$. As a result, either $\Pi^* \parallel \Gamma$ or $\Pi^* \perp \Gamma$. However, $\Pi^* \perp \Gamma$ implies that $\Pi^* \parallel L$, which contradicts to Lemma 5.2.27. This proves that $\Pi^* \parallel \Gamma$. Now, we have $\Pi^* \parallel \Gamma$ and $|\mathcal{Q}| \geq 3$. Using Lemma 5.2.4 (6), (3), we can conclude that only the following three cases are possible:

(1) $|\mathcal{Q}| = 3$ and $|G| = 6$; (2) $|\mathcal{Q}| = 4$ and $|G|{=}8$; (3) $|\mathcal{Q}| = |G| = 4$.

However, case (1) and (2) above can not occur. Indeed, in both cases, we can find a plane Π in $\mathcal{P}_0$ such that $R'_{\Pi}q = q$ by Lemma 5.2.4 (3). Besides, we have $R'_{\Pi^*}q = q$ since $\Pi^* \parallel \Gamma$ and $\tau = 0$. Then Lemma 5.1.3 implies that $\Pi \parallel \Pi^*$ and hence contradicts to Lemma 5.2.27. This finishes our proof. $\sharp$

We know from Lemma 5.2.29 that there is only one possible case that may happen under the assumption $L \nparallel \{x_3 = 0\}$, $\tau = 0$. We will study the case in what follows. We shall see that it leads to the extended fifth class of unidentifiable grating structures corresponding to the incident field $E^i(x) = se^{iq\cdot x}$.

Lemma 5.2.30. *If $L \nparallel \{x_3 = 0\}$, $\tau = 0$ and $|\mathcal{Q}| = |G| = 4$, then*

1. *$s \parallel \nu$.*

2. *Let $q^1 = k\frac{s\times e_3}{\|s\times e_3\|}$. Then the propagating field E_p in (5.1.2) or $\hat{E}_p$ in (5.2.3) can be written as*

$$E_p(x) = s(e^{iq\cdot x} - e^{iq^1\cdot x+(q-q^1)\cdot r} + e^{-iq\cdot x+2q\cdot r} - e^{-iq^1\cdot x+(q+q^1)\cdot r}); \quad (5.2.14)$$

$$\hat{E}_p(\hat{x}) = \hat{s}(e^{iq\cdot\hat{x}} - e^{iq^1\cdot\hat{x}} + e^{-iq\cdot\hat{x}} - e^{-iq^1\cdot\hat{x}}). \quad (5.2.15)$$

3. *$\mathcal{P}_0$ consists of two planes, one is perpendicular to the vector $q+q^1$ and is denoted by Π_0; the other is perpendicular to the vector $q-q^1$ and is denoted by Π_1.*

Proof. Step 1. Let Π^* be as in Lemma 5.2.27, as shown in the proof of Lemma 5.2.29, we have that $\Pi^* \parallel \Gamma$ and all the elements in $\mathcal{Q}$ lie on the plane Γ.

Step 2. We show $\mathcal{Q} = G\{q\}$. Indeed, if $\mathcal{Q} \neq G\{q\}$, then $|G\{q\}| = 2$. By Lemma 5.2.4 (3), we can find Π in $\mathcal{P}_0$ such that $R'_\Pi q = q$. On the other hand, we have $R'_{\Pi^*} q = q$ too since $\tau = 0$ and $\Pi^* \parallel \Gamma$. Then we see by Lemma 5.1.3 that $\Pi \parallel \Pi^*$, which contradicts to Lemma 5.2.27. Thus we have $\mathcal{Q} = G\{q\}$.

Step 3. Since $|G\{q\}| = |\mathcal{Q}| = |G| = 4$, Lemma 5.2.4 (3) implies that

$$G\{q\} = G^*\{q\} \bigcup G^*\{q^1\}$$

for some $q^1 \in G\{q\}$. By the result in step 1, all the elements in $\mathcal{Q}$ lie on the plane Γ, so we have $q = P_\Gamma q$, $q^1 = P_\Gamma q^1$, thus

$$G\{q\} = G^*\{P_\Gamma q\} \bigcup G^*\{P_\Gamma q^1\} = \{P_\Gamma q, -P_\Gamma q\} \bigcup \{P_\Gamma q^1, -P_\Gamma q^1\} = \{q, -q\} \bigcup \{q^1, -q^1\}.$$

Then it follows from Proposition 5.1.1 that $q^1 \parallel \{x_3 = 0\}$, or equivalently $q^1 \in \Xi^*$.

Step 4. Clearly, $\mathcal{P}_0 = \{\Pi_0, \Pi_1\}$ since $|\mathcal{P}_0| = 2$. Let $q^2 = -q$, $q^3 = -q^1$, we claim that $R'_\Pi q \in \{q^1, q^3\}$ for $\Pi \in \mathcal{P}_0$. Indeed, by Proposition 5.1.3, we have $R'_\Pi\{q, q^1, q^2, q^3\} = \{q, q^1, q^2, q^3\}$. Noting that $|G| = |G\{q\}| = |G\{q^1\}| = |G\{q^3\}| = 4$, we have $R'_\Pi q \neq q$, $R'_\Pi q^1 \neq q^1$ and $R'_\Pi q^3 \neq q^3$ due to Lemma 5.2.4 (3). So we need only show that the case with $R'_\Pi q = q^2$ and $R'_\Pi q^1 = q^3 = -q^1$ can not occur. This follows easily by observing that $R'_\Pi q = -q$ and $R'_\Pi q^1 = -q^1$ implies both q and q^1 are perpendicular to the plane Π and hence are parallel to each other, which is impossible since $q^1 \parallel \{x_3 = 0\}$ and $q \nparallel \{x_3 = 0\}$. This contradiction proves our claim.

Step 5. By the result in step 4, we can fix Π_0 by letting

$$R'_{\Pi_0} q = q^1, \quad R'_{\Pi_0} q^2 = q^3.$$

It follows that $R'_{\Pi_1} q = q^3$, $R'_{\Pi_1} q^1 = q^2$. We write

$$\hat{E}_p(\hat{x}) = \hat{s} e^{iq\cdot\hat{x}} + \hat{A}^1 e^{iq^1\cdot\hat{x}} + \hat{A}^2 e^{-iq\cdot\hat{x}} + \hat{A}^3 e^{-iq^1\cdot\hat{x}}.$$

A simple application of Lemma 5.2.5 yields that

$$\hat{A}^1 = -R'_{\Pi_0}\hat{s}, \ \hat{A}^3 = -R'_{\Pi_1}\hat{s}, \ \hat{A}^2 = -R'_{\Pi_1}\hat{A}^1 = -R'_{\Pi_1}(-R'_{\Pi_0}\hat{s}). \tag{5.2.16}$$

Now, we show $\hat{s} \parallel \nu$. Indeed, by the result in step 1, we have $R'_{\Pi^*}q = q$. Referring to Lemma 5.1.3, this implies that $s \perp \Pi^*$ and thus $s \perp \Gamma$. Therefore we have $s \parallel \nu$, and $\hat{s} \parallel \nu$ follows immediately. As a result, we further derive from the relations in (5.2.16) that

$$\hat{A}^1 = \hat{A}^3 = -\hat{s}, \ \ \hat{A}^2 = \hat{s}.$$

Besides, the relation $s \parallel \nu$ also implies that $q^1 \parallel (s \times e_3)$ since $q^1 \perp e_3$ and $q^1 \perp \nu$. Recall that $\|q\| = k$. Then we see $q^1 = \pm k\frac{s\times e_3}{\|s\times e_3\|}$. We now can fix q^1 by letting $q^1 = k\frac{s\times e_3}{\|s\times e_3\|}$. The rest of the lemma follows. Our lemma is proved. $\sharp$

So far, we have determined the electric field E_p in terms of the vector s, q. The set of perfect planes of E_p can be determined also. By following the same method as in Lemma 5.2.8, we can show that it contains three sets of planes: the first set consists planes that are parallel to Π_0 with a distance some multiple of $\frac{\pi}{|q\cdot\nu_{\Pi_0}|}$; the second set consists planes that are parallel to Π_1 with a distance some multiple of $\frac{\pi}{|q\cdot\nu_{\Pi_1}|}$; the last set consists planes that are perpendicular to s.

We note that the electric field E_p and its set of perfect planes $\mathcal{P}$ determined above are nearly the same as those corresponding to the third unidentifiable classes of grating except that we have $s \nparallel \{x_3 = 0\}$ here other than $s \parallel \{x_3 = 0\}$. For this reason, we extend our definition of the fifth class of unidentifiable grating profiles $\mathcal{S}_5(s, q, \Lambda, r)$ by removing the condition that $s \parallel \{x_3 = 0\}$, and denote it still by $\mathcal{S}_5(s, q, \Lambda, r)$.

To sum up the analysis above, we have the following lemma.

Lemma 5.2.31. *If $L \nparallel \{x_3 = 0\}$, $\tau = 0$ and $|\mathcal{Q}| = |G| = 4$, then the grating profile S belongs to $\mathcal{S}_5(s, q, \Lambda, r)$ for some point $r \in \mathbf{R}^3$. Further more, any grating profiles in $\mathcal{S}_5(s, q, \Lambda, r)$ can generate the same total field as S.*

Excluding of the remaining cases

Finally, we investigate the case with $L \nparallel \{x_3 = 0\}$ and $\tau \neq 0$. We shall show that this case can not happen in the following three lemmas, where we consider different cases according to the number of elements in the set $G\{q\}$.

Lemma 5.2.32. *If $L \nparallel \{x_3 = 0\}$, then the case with $\tau \neq 0$ and $|G\{q\}| = 3$ can not happen.*

Proof. We prove by contradiction.

Step 1. Note that $\tau_n = \tau$ for all $q^n \in G\{q\}$. Then we have by (5.2.12) in Lemma 5.2.28 that:

$$3\tau + \sum_{q^n \in \mathcal{Q}\setminus G\{q\}} \tau_n = 0. \tag{5.2.17}$$

Since $\tau_n \geq 0$ for all $q^n \in \mathcal{Q} \setminus G\{q\}$ ((5.2.13) in Lemma 5.2.28), we see that $\tau < 0$ and there exists some $q^m \in \mathcal{Q} \setminus G\{q\}$ such that $\tau_m > 0$.

Step 2. We claim $\mathcal{Q} = G\{q\} \bigcup G\{q^m\}$ and $|G\{q^m\}| = 3$. Indeed, since $|G\{q\}| = 3$, we have by Lemma 5.2.4 (3) that $|G| = 6$ and hence either $|G\{q^m\}| = 3$ or $|G\{q^m\}| = 6$ according to Lemma 5.2.4 (3) again. Then the equation (5.2.17) above further reads:

$$3\tau + 3\tau_m \leq \tau + \sum_{q^n \in G\{q\}} \tau_n + \sum_{q^n \in G\{q^m\}} \tau_n \leq \tau + \sum_{q^n \in \mathcal{Q}} \tau_n = \tau + \sum_{n \in \Xi_0} \tau_n = 0.$$

Thus we have $\tau + \tau_m \leq 0$. Define

$$d_0 = \min\{x_3;\ x = (x_1, x_2, x_3) \in \Gamma \text{ and } \|x\| \leq \|P_\Gamma q\|\ \} < 0.$$

We can find $q^{n_1} \in G\{q\} \setminus \{q\}$ such that the x_3 component of $P_\Gamma q^{n_1}$ is less than or equal to $-\frac{1}{2}d_0$. Since $\|q^m\| = \|q\| = k$ and $\tau + \tau_m \leq 0$, we have $\|P_\Gamma q^m\| \geq \|P_\Gamma q\|$. Therefore we can find $q^{n_2} \in G\{q^m\}$ on the plane Γ such that its x_3-component is less than or equal to $\frac{1}{2}d_0$. Now, consider the x_3 component of of the vector

$$q^{n_1} + q^{n_2} = (\tau + \tau_m)\nu + P_\Gamma q^{n_1} + P_\Gamma q^{n_2},$$

it is non-negative by Proposition 5.1.1. However, this is possible only if the follow three conditions hold:

(1) $\tau + \tau_m = 0$, $\|P_\Gamma q\| = \|P_\Gamma q^m\|$;

(2) The x_3 component of $P_\Gamma q^{n_1}$ is equal to $-\frac{1}{2}d_0$;

(3) The x_3 component of $P_\Gamma q^{n_2}$ is equal to $\frac{1}{2}d_0$.

It follows from condition (1) and the equation (5.2.12)-(5.2.13) in Lemma 5.2.28 that $|G\{q^m\}| = 3$ and $\tau_n = 0$ for all $q^n \in \mathcal{Q} \setminus (G\{q\} \bigcup G\{q^m\})$. Now, take $q^n \in \mathcal{Q} \setminus (G\{q\} \bigcup G\{q^m\}$. Similar to Lemma 5.2.4 (4), we have that at least one elements from the set

$$G^*\{q^n\} = \tau_n \nu + \{P_\Gamma q^n, T_{\pi/3} P_\Gamma q^n, T_{2\pi/3} P_\Gamma q^n\} = \{P_\Gamma q^n, T_{\pi/3} P_\Gamma q^n, T_{2\pi/3} P_\Gamma q^n\}.$$

has negative x_3-component. This is impossible according to Proposition 5.1.1. Thus $\mathcal{Q} \setminus (G\{q\} \bigcup G\{q^m\} = \emptyset$ and our claim is proved.

Step 3. We show that $G\{q^m\} = -G\{q\}$. Indeed, let V_1 and V_2 be the two vectors on the plane Γ such that $\|V_1\| = \|V_2\| = \|P_\Gamma q\|$, $V_1 \parallel \{x_3 = 0\}$ and the x_3 component of V_2 is d_0. It is clear that $V_1 \perp V_2$ and the set $\{\nu, V_1, V_2\}$ forms an orthogonal basis in $\mathbf{R}^3$. By using condition (2) and (3) in step 2, we see that the sets $G\{q\}$, $G\{q^m\}$ can be written as:

$$\begin{aligned} G\{q\} &= \tau\nu + \{V_2, T_{\pi/3}V_2, T_{\pi/3}V_2\}, \\ G\{q^m\} &= -\tau_m\nu + \{-V_2, -T_{\pi/3}V_2, -T_{\pi/3}V_2\} = -\tau\nu + \{-V_2, -T_{\pi/3}V_2, -T_{\pi/3}V_2\}, \end{aligned}$$

whence the conclusion $G\{q^m\} = -G\{q\}$ follows.

Step 4. Let $G\{q\} = \{q, q^1, q^2\}$, then $\mathcal{Q} = \{q, q^1, q^2, -q, -q^1, -q^2\}$. Clearly both q^1 and q^2 lie on the plane $\{x_3 = 0\}$ by Proposition 5.1.1. Since $|G\{q\}| = 3$, we can use Lemma 5.2.4 (3) to deduce that $|\mathcal{P}_0| = 3$. Let $\mathcal{P}_0 = \{\Pi_0, \Pi_1, \Pi_2\}$. Lemma 5.2.4 (3) further implies that there is a plane from $\mathcal{P}_0$, say, Π_0 such that $q = R'_{\Pi_0} q$. Referring to Lemma 5.2.4 (2), we see that $\{R'_{\Pi_1} q, R'_{\Pi_2} q\} = \{q^1, q^2\}$. Now, we fix q^1, q^2 by letting $q^1 = R'_{\Pi_2} q$, $q^2 = R'_{\Pi_1} q$. Then it is easy to check that $q^1 = R'_{\Pi_1} q^1$ and $q^2 = R'_{\Pi_2} q^2$.

Let Π^* be the perfect plane in Lemma 5.2.27. By Proposition 5.1.3, we have

$$R'_{\Pi^*}\{q, q^1, q^2, -q, -q^1, -q^2\} = \{q, q^1, q^2, -q, -q^1, -q^2\}.$$

Then six cases may happen. We shall finish our proof by rule out all of them.

1. $R'_{\Pi^*}q \in \{q, q^1, q^2\} = \{R'_{\Pi} : \Pi \in \mathcal{P}_0\}$. These cases can easily be excluded by using Lemma 5.1.3, Lemma 5.2.27 and the relations that $q = R'_{\Pi_0}q$, $q^1 = R'_{\Pi_2}q$, $q^2 = R'_{\Pi_1}q$.

2. $R'_{\Pi^*}q = -q$. In this case, $R'_{\Pi^*}\{q^1, q^2, -q^1, -q^2\} = \{q^1, q^2, -q^1, -q^2\}$. So R'_{Π^*} maps the plane $\{x_3 = 0\}$ to itself. Therefore, we see that either $\Pi^* \perp \{x_3 = 0\}$ or $\Pi^* \parallel \{x_3 = 0\}$. But $R'_{\Pi^*}q = -q$ also implies that $q \perp \Pi^*$, so we should have $\Pi^* \parallel \{x_3 = 0\}$. Then it follows that $R'_{\Pi^*}q^1 = q^1$ since q^1 lies on the plane $\{x_3 = 0\}$. This together with the relation $R'_{\Pi_1}q^1 = q^1$ will yield a contradiction by using Lemma 5.1.3 and Lemma 5.2.27.

3. $R'_{\Pi^*}q = -q^1$. In this case, $R'_{\Pi^*}\{q^2, -q^2\} = \{q^2, -q^2\}$. Since $R'_{\Pi^*}q^2 = q^2$ implies that $\Pi^* \parallel \Pi_2$ which can not happen, we should have $R'_{\Pi^*}q^2 = -q^2$. As a result, $q^2 \perp \Pi^*$. Noting that q^2 lies on the plane $\{x_3 = 0\}$, we have $\Pi^* \perp \{x_3 = 0\}$. But q^1 also lies on the plane $\{x_3 = 0\}$, so it is impossible to have $R'_{\Pi^*}q = q^1$ since q does not lie on the plane $\{x_3 = 0\}$. This contradiction shows that the case $R'_{\Pi^*}q = -q^1$ can not happen.

4. $R'_{\Pi^*}q = -q^2$. This cases can be excluded by the same method as we do for $R'_{\Pi^*}q = -q^1$.

Our lemma is proved. $\sharp$

Lemma 5.2.33. *If $L \nparallel \{x_3 = 0\}$, then the case with $\tau \neq 0$ and $|G\{q\}| > 3$ can not happen.*

Proof. We prove by contradiction. Assume that $|G\{q\}| > 3$. We consider the set $G\{P_\Gamma q\}$. By Lemma 5.2.4, there exist at least two elements in $G\{P_\Gamma q\}$ whose x_3 components are non-positive. So we should have $\tau > 0$, for otherwise $G\{q\}$ would contain at least two elements whose x_3 components are negative, which contradicts to Proposition 5.1.1. But then the relation (5.2.17) and (5.2.13) in Lemma 5.2.28 can not be satisfied. This contradiction proves our lemma.

Lemma 5.2.34. *If $L \nparallel \{x_3 = 0\}$, then the case with $\tau \neq 0$ and $|G\{q\}| \leq 2$ can not happen.*

Proof. It is clear $|G\{q\}| > 1$ since $|\mathcal{P}_0| > 1|$ and $|G\{q\}| = |\mathcal{P}_0|$ or $|G\{q\}| = 2|\mathcal{P}_0|$ (Lemma 5.2.4 (3)). So we need only show that the case with $|G\{q\}| \leq 2$ can not happen. We do this by contradiction. Assume that $\tau \neq 0$ and $|G\{q\}|$. Let Π^* be the perfect plane in Lemma 5.2.27. We shall derive contradictions by considering all possibilities based on whether Π^* is perpendicular to planes Π_0 and Π_1.

Case 1: $\Pi^* \perp \Pi_0$, $\Pi^* \perp \Pi_1$. As $|G\{q\}| = 2$, we may write $G\{q\} = \{q, q^*\}$. Clearly there are only two reflections in G, and we know that $q^* = \tau\nu - P_\Gamma q$ if $q = \tau\nu + P_\Gamma q$. As $\Pi^* \perp \Pi_0$ and Π_1, we derive $R'_{\Pi^*}G\{q\} = \{-\tau\nu + P_\Gamma q, -\tau\nu - P_\Gamma q\} \subset \mathcal{Q}$. This shows both q^* and $-q^*$ belong to $\mathcal{Q}$, then one of them must have a negative x_3 component, which contradicts to Proposition 5.1.1.

Case 2: $\Pi^* \not\perp \Pi_0$. Let $\tilde{L}$, $\tilde{\mathcal{P}_0}$, $\tilde{G}$ denote the line of intersection between Π^* and Π_0, the set of perfect planes in $\mathcal{P}$ which passes through the line $\tilde{L}$, and the group generated by the reflections $\{R'_\Pi : \Pi \in \tilde{\mathcal{P}_0}\}$, respectively. As $\Pi^* \not\perp \Pi_0$, we see that $|\tilde{\mathcal{P}_0}| \geq 3$, and $|\tilde{G}| \geq 6$.

First, we consider the case when $\tilde{L}$ is not parallel to the plane $\{x_3 = 0\}$. In this case we have $| \geq \tilde{G}\{q\} \geq 3$ and $|\tilde{G}| \geq 6$. Similar to the results in lemmas 5.2.29, 5.2.33, 5.2.32, we can deduce that this case can not happen.

Second, we consider the case when $\tilde{L} \parallel \{x_3 = 0\}$. Similar to the discussion in Section 5.2.3, 5.2.4, we see that only two cases: (i) $|\tilde{G}| = 8$, $|\tilde{G}\{q\}| = 4$; (ii) $|\tilde{G}| = 6$, $|\tilde{G}\{q\}| = 3$ are possible to happen. Following the same steps as in Subsection 5.2.3, 5.2.3, we can work out all the perfect planes in $\mathcal{P}$ and find out that the intersection lines of the planes in $\mathcal{P}$ are all parallel to $\tilde{L}$. In particular, we have $L \parallel \tilde{L}$, thus $L \parallel \{x_3 = 0\}$, which contradicts to the assumption of this subsection that L is not parallel to $\{x_3 = 0\}$.

Case 3: $\Pi^* \not\perp \Pi_0$. This can be dealt with in exactly the same way as we did for Case 2.

This completes the proof of our lemma. $\sharp$

Now, we can conclude from Lemmas 5.2.32, 5.2.33, 5.2.34, that the case considered in this subsection with $L \nparallel \{x_3 = 0\}$ and $\tau \neq 0$ can not occur.

5.2.6 Summary on all unidentifiable gratings

Summing up the results in Subsection 5.2.1, 5.2.3, 5.2.4, 5.2.5, especially Lemma 5.2.1, 5.2.6, 5.2.9, 5.2.12, 5.2.17, 5.2.20, 5.2.26, 5.2.29, 5.2.31, 5.2.32, 5.2.33, 5.2.34, we obtain the following conclusion.

Theorem 5.2.1. *Let S be a polyhedral grating with bi-period Λ, $E^i(x) = se^{iq\cdot x}$ be an incident electric field, and E be a solution to the direct scattering problems (4.1.1)-(4.1.4). Then we have under Assumption 5.1.1 that seven cases may happen: $S \in \mathcal{S}_1(q,r)$, $S \in \mathcal{S}_2(s,q,r,\Lambda)$, $S \in \mathcal{S}_3(s,q,r,\nu,\Lambda)$, $S \in \mathcal{S}_4(s,q,r,\Lambda)$, $S \in \mathcal{S}_5(s,q,r,\Lambda)$, $S \in \mathcal{S}_6(s,q,r,\Lambda)$ or $S \in \mathcal{S}_7(s,q,r,\Lambda)$.*

5.3 Unique determination of bi-periodic polyhedral grating

In this section we apply the results developed in the previous section on the classification of unidentifiable periodic grating structures in correspondence to one incident field for the unique determination of a given periodic polyhedral grating profile by the scattered field.

Theorem 5.3.1. *Let $E^i = se^{iq\cdot x}$ be a given incident electric field, S_1 and S_2 be two periodic polyhedral gratings with bi-period Λ, and E_1 and E_2 be respectively the solutions to the forward scattering problem (1)-(4) associated with S_1 and S_2. If $\Gamma_b = \{x_3 = b\}$ is a plane located above both S_1 and S_2, then*

$$E_1 = E_2 \quad on \ \ \Gamma_b \tag{5.3.1}$$

implies that both S_1 and S_2 belong to the grating classes $\mathcal{S}_1(q,r)$, $\mathcal{S}_2(s,q,r,\Lambda)$, $\mathcal{S}_3(s,q,r,\nu,\Lambda)$, $\mathcal{S}_4(s,q,r,\Lambda)$, $\mathcal{S}_5(s,q,r,\Lambda)$, $\mathcal{S}_6(s,q,r,\Lambda)$ or $\mathcal{S}_7(s,q,r,\Lambda)$.

Proof. Assume that (5.3.1) is true for two different S_1 and S_2. We first show that $E_1 = E_2$ in the domain above the measurement plane $\{x_3 = b\}$. Using the expansions

$$\begin{aligned} E_1(x) &= E^i(x) + \textstyle\sum_{n\in\mathbf{Z}^2} A_1^n e^{iq^n\cdot x}, \\ E_2(x) &= E^i(x) + \textstyle\sum_{n\in\mathbf{Z}^2} A_2^n e^{iq^n\cdot x}. \end{aligned}$$

it suffices to show that $A_1^n = A_2^n$ for all $n \in \mathbf{Z}^2$. To see this, we have by (5.3.1) that

$$(E_1 - E_2)|_{x_3=b} = \sum_{n\in\mathbf{Z}^2} (A_1^n - A_2^n) e^{i\beta^n\cdot b} e^{i\alpha^n\cdot x} = 0. \tag{5.3.2}$$

Noting that $\{e^{i\alpha^n\cdot x}\}_{n\in\mathbf{Z}^2}$ is an orthogonal family in $L^2((0,\Lambda_1)\times(0,\Lambda_2))$ of variables x_1 and x_2, we get the desired result immediately.

Now, we have shown that $E_1 = E_2$ in the domain above the plane $\{x_3 = b\}$. Following the same method as in Theorem 4.2.1 and using Theorem 5.2.1, we get the desired result immediately.

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