A Multi-Period Portfolio Selection Problem

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此碩士論文著重討論三個多期投資組合選擇的模型。

第一個模型在規定了投資者的風險程度后，可制定投資策略以達到最大的終期投資回報。l_∞ 函數將做為風險水平的測度。顯式最優解以及最優投資策略會在此章節末給出。

第二個模型會達到在終期投資回報最大化同時風險程度最小化的雙目標最優。此模型將此雙目標優化問題轉換成參數優化問題。顯式最優解、有效點以及最優投資策略會在此章節給出；除此之外，數值運算會詳盡解釋此投資策略的最優性。

第三個模型在前兩個模型的基礎上添加一個定期提取現金率。故此模型會求解一個讓每期提取現金的總值與終期投資回報之和最大的優化問題。l_∞函數仍然做為風險水平的測度。在求解此問題時，筆者將用兩種方法來求解。一為用逆向動態規劃來尋求最優解：此方式可讓讀者看到每期投資資金是如何分配到不同資產中去的。另一方法為一近似解。兩種方法的數值運算都會給出，并附有比較數值。

Abstract of thesis entitled:

A Multi-Period Portfolio Selection Problem

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In this thesis, three multi-period portfolio selection models are built.

The first model is concerning the asset allocation to achieve the maximum terminal wealth while the risk level is specified. l_∞ function is used as the risk measure. An explicit optimal solution is given. A new portfolio selection rule is provided.

The second model is concerning the asset allocation to achieve two goals at the same time: the investor tries to maximize his terminal wealth while minimizing the l_∞ risk measure. It is a bicriteria portfolio optimization problem. We convert it into a parametric optimization problem. An explicit optimal solution is given for the efficient point. A numerical example is provided to give a clear picture of how good this optimal investment strategy is.

The third model allows withdraws. It still concerns the asset allocation to achieve the maximum total wealth (the sum of discounted withdraw amounts over the entire investment periods plus the terminal return of the portfolio) while the risk level is specified. l_∞ function is used as the risk measure. We have two methodologies to resolve this problem. Firstly we try to solve this problem by dynamic programming in a backward fashion. Two algorithms are provided. Dynamic programming allows us to have a clearer picture of how assets are actually reallocated period by period. Secondly we derive an approximate analytical solution. Numerical examples are provided. The comparison between the optimal solution and the approximate solution is studied as well.

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Chapter 1

Introduction

In this chapter, we will review related portfolio selection literature first. Because of some unsolved problems' existence, this thesis comes out. A physical justification, including three investment scenarios will give you a general idea of how models in this thesis work in the real life. It follows by the contribution of this thesis.

1.1 Literature Review

The portfolio selection problem receives more and more attention in recent years. It typically concerns the repositioning of funds to achieve specific goals.

Markowitz's [13] mean-variance model has been developed that allows one to determine portfolios with the highest expected returns for a given level of risk. This pioneering work has served as the basis for the development of numerous modern financial theories. But arguments have been made that the mean-variance model is appropriate only if the investor's utility is quadratic or the joint distribution of returns is normal [13]. However, these conditions are rarely satisfied in practice. Therefore, after his theory, there are works done in the literature, pursuing the development of modern portfolio theory to optimal selection of portfolios under different scenarios.

Mossin [14] extended the Markowitz model to a multi-period case, using a dynamic programming approach. The approach calls for applying the Markowitz model as the decision model at the beginning of each period. Using the cash value of the portfolio as the state variable, a dynamic programming recursive relationship is then developed. With this recursive procedure, an investor can select the optimal portfolio in the first period for a given initial wealth.

Other approaches model the stochastic nature of the problem directly as a stochastic program. For example, Mulvey [15] formulated the problem as a stochastic network problem. In his work, he describes a specialization of the primal truncated

Newton algorithm for solving nonlinear optimization problems on networks with gains. The algorithm and its implementation are able to capitalize on the special structure of the constraints. Extensive computational tests show that the algorithm is capable of solving very large problems. Besides, Mulvey and Vladimirou [17] provided another methodology, the progressive hedging algorithm for stochastic generalized network problems. In their paper, they examined the progressive hedging algorithm for solving multi-scenario generalized networks. They also presented computational results demonstrating the effect of various internal tactics on the algorithm's performance.

Most models developed so far have been single-stage or single-period models, which means the decision making process and the future events are restricted to a single time period. However, in allocating financial resources, the available options must be analyzed from various perspectives. First, the potential earnings of alternative fund positions must be evaluated in conjunction with the costs associated with the transfers of funds. Second, consumption needs must be met and anticipated future deposits and liabilities must be properly taken into account. Therefore, a multiperiod model becomes necessary.

Dynamic problems under uncertainty can be modeled as multistage stochastic programs, thereby capturing the dynamic aspects of the problem and allowing for a number of realizations of the uncertain quantities. Classic references on stochastic programs with recourse include Dantzig and Madansky [4] and Walkup and Wets [25]. In Dantzig and Madansky's paper, two-stage linear programs are studied; they considered a particular form of stochastic linear programming under uncertainty; the optimal value of the program is provided. Later in 1966, Walkup and Wets [25] studied the natural extension of the restricted model set by Dantzig: they call it a general stochastic program with recourse; solution is provided as well.

Multistage stochastic programs exhibit a multiplicative growth in size with the number of decision stages and the number of realizations admitted at each stage. Dempster [5], Ermoliev and Wets ([7], chapter 1) and Varaiya and Wets [24] address the complexity of dynamic stochastic programs. For example, in Varaiya and Wets' paper [24], they emphasized a solvable models. In their opinion, the description of stochastic dynamic optimization models is intended to exhibit some of the connections between various formulations that have appeared in the literature, and indicate some of the difficulties that must be overcome when trying to adapt solution methods that have been successfully applied to one class of problems to an apparently related but different class of problems. Therefore, they begin with the least dynamical versions of stochastic optimization models, one-stage and two-stage models, then consider discrete time models, and conclude with continuous time models. Practical applications have generally been restricted to two-stage models permitting uncertainty only in the cost coefficients and in the right-hand side vector of the constraints, as it can be told by Varaiya and Wets' paper [24].

Since Markowitz's mean-variance model, in the lasat forth years, some other measures of risk have been proposed in the financial literature. Konno and Yamazaki [11] proposed an l_1 risk function and suggested that a piecewise linear function can be used to approximate this l_1 risk function. Moreover, the portfolio optmization problem under l_1 risk measure is converted into a scalar parametric linear programming problem. So the implementation of the portfolio optimization problem with the l_1 risk function can be easily obtained. Young [27] introduced another linear program model which amounts to maximizing the minimum return over time periods. The advantage of both linear program models mentioned above are: the corresponding optimization problems are simple, and that both models perform in a similar way to that of the mean variance model for normal return data.

A minimax risk function is introduced in Cai et al. [1]. By making use of the special structure of the l_∞ risk function, a simple analytical solution scheme is obtained for the efficient frontier of the portfolio optimization problem without having to solve any optimizing problem. Moreover, Teo and Yang [23] proposed an alternative minimax risk function in portfolio optimization. Such a risk function is defined as the average of maximum individual risks over a number of past time periods. The practical meaning of this risk function is that an investor wants to minimize the average of the maximum individual risks among assets to be invested. The corresponding portfolio optimization problem is formulated as a bi-objective linear programming problem. For a given weight on the return, the scalar linear program can be easily solved. Later, Deng, Li and Wang [2] proposed another minimax principal to choose portfolio in a market without frictions. Their minimax principal is defined like this: one maximizes the worst possible expected rates of returns on portfolio.

Current portfolio selection models generally ignore the withdraws in revising an existing portfolio, like Mossin's [14] work we mentioned before. The result of this assumption is that frequent portfolio revisions may occur which are not justified relative to the resulting withdraw even if in practice sometimes we need money for emergency usage. This assumption is not true according to the real world because we usually withdraw money when we need to use it. Therefore, to fit the real world financial trading process, we have to consider withdraws.

1.2 Problem Description

In the modern financial engineering discipline, while the underlying principle of governing how the portfolios should be constructed is simple and intuitive: minimize the investment risk and maximize the expected return, the fund allocation strategies usually vary with each other regarding different real world scenarios. As a matter of fact, this is exactly what lead to the major differences among different portfolio management models.

In this thesis, we study the portfolio selection problems within three scenario settings which satisfy different demands of fund management:

Scenario I:

For a portfolio manager who manages a close-end fund with a certain duration (i.e., three years), his client may require him to estimate the expected return of the fund at the end of this investment period by setting a certain level of the maximum loss he can bear (i.e., 10%).

Scenario II:

In this scenario, other settings are similar to scenario A, but here the customer is more aggressive so he does not set the restriction on his loss. This investor just would like to know the maximum risk exposure if the fund manager tries the best to maximize the terminal wealth.

Scenario III:

In this scenario, the fund manager manages an open-end fund. But he will tell his client how much money he can withdraw so that the summation of terminal wealth and withdraws in periods will be maximized if his client sets his risk level.

The following is the general description of the three multi-period portfolio selection models:

Model I is concerning the asset allocation to achieve the maximum terminal wealth while the risk level is specified. l_∞ function is used as the risk measure. An explicit analytical solution is given. A new portfolio selection rule is provided.

The fund manager in scenario I needs Model I in this paper to find the optimal investment strategy to achieve the specific goal: maximize the terminal expected return while control the risk under a certain level.

The second model is concerning the asset allocation to achieve two goals at the same time: the investor tries to maximize his terminal wealth while minimizing the l_∞ risk measure. It is a bicriteria portfolio optimization problem. We convert it into a parametric optimization problem. An explicit analytical solution is given for the efficient point. A numerical example is provided to give a clear picture of how good this optimal investment strategy is.

Model II can help the fund manager in scenario II to compute the minimum risk exposure whilst with the maximum terminal return.

The third model allows withdraws. It still concerns the asset allocation to achieve the maximum total wealth (the sum of discounted withdraw amounts over the entire investment periods plus the terminal return of the portfolio) while the risk level is specified. l_∞ function is used as the risk measure. We have two methodologies to resolve this problem. Firstly we try to solve this problem by dynamic programming in a backward fashion. Two suggested algorithms are provided. Dynamic programming allows us to have a clearer picture of how assets are actually reallocated period by period. Secondly we derive an approximate analytical solution via the whole investment period. Numerical examples are provided. The comparison between optimal solution and approximate solution is studied as well.

Model III can be explored to resolve the fund manager's problem in scenario III.

1.3 The Main Contributions of This Thesis

In this thesis, three models are explored.

For Model I and II, we have found the optimal solution. Model I specifies a certain level of risk, and the optimal investment strategy is given. For Model II, we are able to find the optimal investment strategy and the optimal risk level at the same time. A numerical example is given. We simulate Hong Kong stock market 42 stocks in Hang Seng Index (HSI) from January 2008 to December 2008. Comparing with HSI, we find out that our optimal investment strategy can gain a profit of 241.83% while Hang Seng Index decreased by 29.16% when the investor is with a risk averse parameter.

For Model III, we have explored two dynamic programming algorithms and one approximation solution. Numerical examples are done. The optimal solution to dynamic programming and the approximate solution are compared. We can find that the gap between these two is indeed slight, especially when the investor specifies a relatively low risk level.

The optimal solutions of Model I, II and III work well especially when the market is quiet. For the approximate analytical solution, the result is very closed to the optimal ones. In fact, the approximate analytical solution works well except the market is too fluctuated. Therefore, if the investor wants to invest, he can select our model under different scenarios.

□ **End of chapter.**

Chapter 2

Model I

Model I is concerning the asset allocation to achieve the maximum terminal wealth while the risk level is specified. l_∞ function is used as the risk measure. An explicit analytical solution is given. A new portfolio selection rule is provided.

2.1 Notation

In this section, we introduce our risk measure and formulate the corresponding portfolio optimization problem with this measure. Suppose an investor has an initial wealth M to invest in n assets, $S_1, S_2, \ldots, S_n$ in T periods, from 0 to $T-1$. At time t, the total expected return after investing in period t is expressed as M_t. Then investor will continue investing M_t he gets in period t to next period $t+1$.

Let x_i^t be the allocation from M_{t-1} for investment into S_i.

Note that by assuming $x_i^t \geq 0$, we are concerned with the situation where short sale is not allowed.Thus, the feasible region for the portfolio optimization problem is:

$$\mathscr{F} = \{x^t = (x_1^t, \ldots, x_n^t) : \sum_{i=1}^{n} x_i^0 = M, \sum_{i=1}^{n} x_i^t = \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} = M_{t-1}$$
$$\text{for each t}, x_i^t \geq 0 \text{ for each i and t.}\}.$$

Assume that these n assets' return rates are independent from each other. Let R_i^t denote the return rate of asset S_i in period t, which is a random variable. For example, if the investor invests x_i^t into S_i at beginning of period t, he will get $x_i^t \cdot R_i^t$ at the end of period t.

Let $E(R_i^t)$ denote the mathematical expectation of R_i^t, and let $r_i^t = E(R_i^t)$. Therefore, r_i^t is the expected return rate of asset S_i at period t.

The expected return of a portfolio $x^t = (x_1^t, ..., x_n^t)$ in period t is given by

$$r(x_1^t, ..., x_n^t) = E[\sum_{i=1}^{n} R_i^t x_i^t] = \sum_{i=1}^{n} E[R_i^t] x_i^t = \sum_{i=1}^{n} r_i^t x_i^t.$$

Define $q_i^t = E(|R_i^t - r_i^t|)$. Therefore, q_i^t is the expected absolute deviation of R_i^t from its means.

Thus, we can define the l_∞ risk function (Cai et al. [1]) now.

Definition 2.1.1 *The l_∞ risk function is defined as*

$$w_\infty = \max_{1 \leq i \leq n,\ 0 \leq t \leq T-1} E[|\ R_i^t x_i^t - r_i^t x_i^t\ |].$$

w_∞ can be written as:

$$\begin{aligned} w_\infty &= \max_{1\le i\le n,\ 0\le t\le T-1} E[\mid R_i^t - E(R_i^t) \mid] x_i^t \\ &= \max_{1\le i\le n,\ 0\le t\le T-1} E[\mid R_i^t - r_i^t \mid] x_i^t \\ &= \max_{1\le i\le n,\ 0\le t\le T-1} q_i^t x_i^t. \end{aligned}$$

2.2 Model Formulation

In this model, our goal is that to maximize investor's expected return while specifying his risk level.

There is one assumption in our model: there is an extra risk-free security. In this case, if investor specifies the risk level to be 0, the only thing he needs to do is to invest all money into this risk-free asset. What is more, the return of every asset must be larger than 1.

Under the l_∞ risk measure as defined above, our portfolio optimization problem can be formulated as a linear program as follows, which is denoted as **(P1)**.

Definition 2.2.1 *The portfolio optimization problem* ***(P1)*** *under the l_∞ risk measure is formulated as:*

$$
\begin{aligned}
(P1)\quad \max\quad & \sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} \\
s.t.\quad & y \geq q_i^t x_i^t \ \ \textit{for all} \ \ i \in [1,n] \ \ \textit{and} \ \ t \in [0, T-1], \\
& \sum_{i=1}^{n} x_i^0 = M, \\
& \sum_{i=1}^{n} x_i^t = M_{t-1} = \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} \ \textit{for each} \ t \in [1, T-1], \\
& x_i^t \geq 0 \ \textit{for each} \ i \in [1,n] \ \textit{and} \ t \in [0, T-1].
\end{aligned}
$$

The objective function in **P1** means that investor tries to optimize the terminal value of his portfolio.

There is one assumption in this model: there exists another risk-free asset besides $i = 1, \ldots, i = n$. In this case, if the investor sets the value of y too small, he needs to invest the rest of money into the risk-free asset.

In every period t, $t \in [0, T-1]$, we number the assets according to the following order:

$$r_1^t \geq r_2^t \geq \ldots \geq r_n^t. \tag{2.1}$$

Furthermore, to avoid ambiguity, we assume that there do not exist two assets S_i and S_j, $i \neq j$, such that $r_i^t = r_j^t$ and $q_i^t = q_j^t$ (if such two assets do exist in the original problem, we may treat them as a single aggregate asset).

In addition, we consider the case where there exists one risk-free asset in this portfolio.

Proposition 2.2.2 *The optimal solution to $P2^t$ is also the optimal solution to* ***P1***:

$$\begin{aligned}
(P2^t) \quad \max \quad & \sum_{i=1}^{n} x_i^t r_i^t \\
s.t. \quad & y \geq q_i^t x_i^t, \\
& \sum_{i=1}^{n} x_i^0 = M, \\
& \sum_{i=1}^{n} x_i^t = M_{t-1}, t \neq 0 \\
& x_i^t \geq 0.
\end{aligned}$$

if we run $P2^t$ from $t = 0$ to $t = T - 1$ for T times.

Proof. $P2^t \rightarrow$ **P1**

We prove it by contradiction.

Assume x_i^{t*} is the optimal solution to $P2^t$. But for **P1**, assume its optimal solution y_i^t satisfying, for some t:

$$\sum_{i=1}^{n} y_i^t r_i^t < \sum_{i=1}^{n} x_i^{t*} r_i^t.$$

If $t = T-1$, we have

$$\sum_{i=1}^{n} y_i^{T-1} r_i^{T-1} < \sum_{i=1}^{n} x_i^{T-1*} r_i^{T-1}.$$

Then there must exist $y_i^{T-1} < x_i^{T-1}$. This means $\{y_i^{T-1}\}$ is not optimal to **P1**.

Now $t \in [1, T-1]$, we have

$$\sum_{i=1}^{n} y_i^t r_i^t < \sum_{i=1}^{n} x_i^{t*} r_i^t.$$

Then there must exist $y_i^t < x_i^t$. By mathematical induction, we can easily prove that increasing y_i^t to x_i^t, for any t, will generate a larger return to **P1**, because it is obvious that any asset i selected in the portfolio should have $r_i^t > 1$. This means $\{y_i^t\}$ is not optimal to **P1**.

It contradicts with our assumption. The proposition is proved.

□

2.3 Analytical Solution

In every period t, $t \in [0, T-1]$, we number the assets such that:

$$r_1^t \geq r_2^t \geq \ldots \geq r_n^t.$$

Theorem 2.3.1 *If $y = 0$, the investment strategy is to invest all money into risk-free asset.*

If $y \neq 0$, for each $t \in [0, T-1]$, there exists one i^*, to separate x_i^t into two types:

$$(x_i^t)^* = \begin{cases} \text{some value}, & \text{for } i \leq i^* \\ 0, & \text{for } i > i^*. \end{cases} \tag{2.2}$$

If x_i^{t*} has some value:

$$(x_i^t)^* = \begin{cases} \frac{y}{q_i^t} & \text{for } i < i^*; \\ M_{t-1} - \sum_{i=1}^{i^*-1} x_i^t & \text{for } i = i^*. \end{cases}$$

This i^* can be found by:

for $k = 1, 2, ..., n$, if $k = 1$, define

$$\sum_{j=1}^{k-1} x_j^t = 0;$$

then if

$$x_k^t = \frac{y}{q_k^t} < M_{t-1} - \sum_{j=1}^{k-1} x_j^t,$$

and

$$x_{k+1}^t = \frac{y}{q_{k+1}^t} \geq M_{t-1} - \sum_{j=1}^{k} x_j^t,$$

then we denote this $k+1$ by i^*.

Proof.

We apply the Karush-Kuhn-Tucker (KKT) conditions.

First, let us introduce the Lagrangian function:

$$f(x,\mu,\lambda,v,b) = -\sum_{i=1}^{n} x_i^t r_i^t + \sum_{i=1}^{n} \mu_i^t (q_j^t x_j^t - y) - \sum_{i=1}^{n} v_i^t x_i^t + b^t (\sum_{i=1}^{n} x_i^t - M_{t-1}) \tag{2.3}$$

Then the KKT conditions (see, for example, [30]) that an optimal solution $\mathbf{x}$ must satisfy can be written as follows:

$$\frac{\partial f}{\partial x_i^t} = -r_i^t + \mu_i^t q_i^t - v_i^t + b^t = 0, \tag{2.4}$$

$$\mu_i^t (q_i^t x_i^t - y) = 0, \tag{2.5}$$

$$v_i^t x_i^t = 0, \tag{2.6}$$

$$\sum_{i=0}^{n} x_i^t = M_{t-1}, \tag{2.7}$$

$$\mu_i^t \geq 0, \tag{2.8}$$

$$v_i^t \geq 0. \tag{2.9}$$

We will establish the following conjecture:

1.
$$x_i^t = \frac{y}{q_i^t}, \text{ and}$$
$$\mu_i^t = \frac{r_i^t - r_{i^*}^t}{q_i^t}, \text{ and}$$
$$v_i^t = 0, \text{ when}$$
$$i < i^*;$$

2.
$$x_{i^*}^t = M_{t-1} - \sum_{i=1}^{i^*-1} x_i^t, \text{ and}$$
$$\mu_{i^*}^t = 0, \text{ and}$$
$$v_{i^*}^t = r_{i^*}^t - r_i^t, \text{ if}$$
$$x_{i^*-1}^t = \frac{y}{q_{i^*-1}^t} < M_{t-1} - \sum_{j=1}^{i^*-2} x_j^t, \text{ but}$$
$$x_{i^*}^t = \frac{y}{q_{i^*}^t} \geq M_{t-1} - \sum_{j=1}^{i^*-1} x_j^t.$$

3.
$$x_i^t = 0, \text{ and}$$
$$\mu_i^t = 0, \text{ and}$$
$$v_i^t = r_{i^*}^t - r_i^t, \text{ when}$$
$$i > i^*.$$

We shall show in the following that the above conjecture is in fact correct in terms of satisfying all KKT conditions.

First, (2.7) is satisfied according to the constraint of $\{x_i^t\}^*$.

Now let us consider the first conjecture point.

When $i < i^*$, we have

$$q_i^t x_i^t - y = 0, \text{ or simply } x_i^t = \frac{y}{q_i^t},$$

which satisfies (2.5).

It is easy to show that $\mu_i^t \geq 0$. This together with $v_i^t = 0$ satisfies (2.6), (2.8), and (2.9).

In this case, (2.4) reduces to:

$$-r_i^t + \mu_i^t q_i^t + b^t = 0. \tag{2.10}$$

We will prove later that this KKT condition is actually satisfied, after deriving the analytical expression for b^t.

Thus, the first conjecture has been proved that four KKT conditions are satisfied : (2.5), (2.6), (2.8), and (ref8).

Now let us consider the second conjecture point.

When $i = i^*$, $\mu_{i^*}^t = 0$, and

$$x_{i^*}^t = M_{t-1} - \sum_{i=0}^{i^*-1} x_i^t \geq 0,$$

according to the constraint of $\{x_i^t\}^*$.

Thus, (2.5) - (2.9) are satisfied. In this case, we can let $v_{i^*}^t = 0$.

Since $\mu_{i^*}^t = 0$, and $v_{i^*}^t = 0$, (2.4) can be rewritten as

$$-r_{i^*}^t + b^t = 0, \text{ or}$$

$$b^t = r_{i^*}^t. \tag{2.11}$$

Thus, under the second conjecture point, five KKT conditions are all satisfied: (2.4) - (2.6), (2.8) - (2.9).

Note that b^t has no relationship with i. It relates to t only. So for each period t, there will be only one b^t, whose value is $r_{i^*}^t$.

Substituting (2.11) into (2.10), we get:

$$-r_i^t + \mu_i^t q_i^t + r_{i^*}^t = 0.$$

Therefore, for any $i < i^*$, (2.4) or equivalently, (2.11) is satisfied.

So far, for the first and the second conjecture points, we have proved that five KKT conditions are satisfied.

Now let us consider the third conjecture point.

When $i > i^*$, $\mu_i^t = 0$.

KKT conditions (2.5) and (2.8) are satisfied.

Since $x_i^t = 0$, and $v_i^t = r_{i^*}^t - r_i^t > 0$ for $i > i^*$, we have (2.6) and (2.9) satisfied.

Since $\mu_i^t = 0$, (2.4) can be rewritten as

$$-r_i^t - v_i^t + b^t = 0.$$

To summarize, the KKT conditions are satisfied for all i. Noting that our problem is a convex problem, the theorem is proved.

□

□ **End of chapter.**

Chapter 3

Model II

Model II is concerning the asset allocation to achieve two goals at the same time. It is a bicriteria portfolio optimization problem. We convert it into a parametric optimization problem. An explicit analytical solution is given for the efficient point. A numerical example is provided to give a clear picture of how good this optimal investment strategy is.

3.1 Model Formulation

In this section, we will formulate Model II, where our goal is to maximize the expected return while minimizing his risk level. This is an optimization problem with two criteria in conflict. We will keep all notation same as that in the previous section.

Under the l_∞ risk measure as defined above, our portfolio optimization problem can be formulated as a bicriteria piecewise linear program as follows, which is denoted as POL_∞ (the Portfolio Optimization problem with the l_∞ risk measure).

Definition 3.1.1 *The bicriteria portfolio optimization problem PHL_∞ under the l_∞ risk measure is formulated as* [1]*:*

$$
\begin{aligned}
(P1)\quad \min\quad & \left\{ \max_{1\le i\le n,\ 0\le t\le T-1} q_i^t x_i^t,\ -\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} \right\} \\
s.t.\quad & \sum_{i=1}^{n} x_i^0 = M, \\
& \sum_{i=1}^{n} x_i^t = \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} \text{ for each } t \in [1, T-1], \\
& x_i^t \ge 0 \text{ for each } i \in [1, n] \text{ and } t \in [0, T-1],
\end{aligned}
$$

where a feasible portfolio $x = (x_1^t, ..., x_n^t) \in \mathscr{F}$ is said to be ***efficient*** *if there exists no $y = (y_1^t, ..., y_n^t) \in \mathscr{F}$ such that*

$$
\max_{1\le i\le n,\ 0\le t\le T-1} q_i^t y_i^t \le \max_{1\le i\le n,\ 0\le t\le T-1} q_i^t x_i^t, \text{ and}
$$

$$
\sum_{i=1}^{n} r_i^{T-1} y_i^{T-1} \ge \sum_{i=1}^{n} r_i^{T-1} x_i^{T-1},
$$

and at least one of the inequalities holds strictly. Accordingly, the function value

$$
\left(\max_{1\le i\le n,\ 0\le t\le T-1} q_i^t x_i^t,\ -\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} \right)
$$

is said to be an ***efficient point.***

---

[1] $\min(A, B)$ indicates that A and B are the two criteria to be minimized.

In words, an efficient point is such that there exists no solution better than it with respect to both criteria.

The **efficient frontier** is the collection of all efficient points.

The first objective function in POL_∞ means the investor tries to minimize the maximum risk among S_i over T periods. The second objective function means that the investor tries to optimize the terminal value of his portfolio.

By a simple transformation, one can show that POL_∞ is equivalent to the following Bicriteria Linear Programming **(BLP)** problem:

$$(P2)\quad \min \quad \left\{y, -\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1}\right\}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i^0 = M,$$

$$y \geq q_i^t x_i^t \quad \text{for all} \quad i \in [1,n] \quad \text{and} \quad t \in [0, T-1],$$

$$\sum_{i=1}^{n} x_i^t = \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} \text{ for each } t \in [1, T-1],$$

$$x_i^t \geq 0 \text{ for each } i \in [1,n] \text{ and } t \in [0, T-1].$$

Now we can convert the bicriteria linear programming problem **BLP** into a parametric optimization problem with a single criterion.

For a fixed λ, where $\lambda \in (0,1)$, the efficient point of **BLP** is the optimal solution of the following Parametric Optimization (denoted by $\mathbf{PO(\lambda)}$) problem:

$$
\begin{aligned}
(P3) \quad \min \quad & \lambda y - (1-\lambda)\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} \\
\text{s.t.} \quad & y \geq q_i^t x_i^t \quad \text{for all} \quad i \in [1,n] \quad \text{and} \quad t \in [0, T-1], \\
& \sum_{i=1}^{n} x_i^0 = M, \\
& \sum_{i=1}^{n} x_i^t = \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} \text{ for each } t \in [1, T-1], \\
& x_i^t \geq 0 \text{ for each } i \in [1,n] \text{ and } t \in [0, T-1].
\end{aligned}
$$

The equivalence relation between **BLP** and $\mathbf{PO(\lambda)}$ is given below (please refer proof to [29]).

Proposition 3.1.2 *Consider the problems **BLP** and **PO(λ)**. The pair $(\boldsymbol{x}, y)$ is an **efficient point** of **BLP** if and only if there exists a $\lambda \in (0,1)$ such that $(\boldsymbol{x}, y)$ is an optimal solution of **PO(λ)**.*

One can think of λ as an investor's risk tolerance parameter — the larger the λ, the more risk the investor is to tolerate. Because of the equivalence between $\mathbf{POL_\infty}$ and $\mathbf{PO(\lambda)}$, an optimal solution for $\mathbf{PO(\lambda)}$ gives, accordingly, an efficient point for $\mathbf{POL_\infty}$. To obtain the efficient frontier of $\mathbf{POL_\infty}$, one has to know the optimal solutions of $\mathbf{PO(\lambda)}$ for all $\lambda \in (0,1)$.

We will derive the optimal solution for the problem $\mathbf{PO(\lambda)}$

with a given $\lambda \in (0,1)$. Note that the parameters $r_i^t = E(R_i^t)$, and $q_i^t = E(|R_i^t - E(R_i^t)|)$, $i = 1, 2, ..., n$, are constants in $\mathbf{PO}(\lambda)$.

3.2 Analytical Solution

In every period t, $t \in [0, T-1]$, we number the assets such that:

$$r_1^t \geq r_2^t \geq \ldots \geq r_n^t. \tag{3.1}$$

Furthermore, to avoid ambiguity, we assume that there do not exist two assets S_i and S_j, $i \neq j$, such that $r_i^t = r_j^t$ and $q_i^t = q_j^t$ (if such two assets do exist in the original problem, we may treat them as a single aggregate asset).

Theorem 3.2.1 *For each $t \in [0, T-1]$, there exists one i^* to separate x_i^t into two types:*

$$(x_i^t)^* = \begin{cases} \textit{some value}, & \textit{for } i \leq i^* \\ 0, & \textit{for } i > i^*. \end{cases} \tag{3.2}$$

If x_i^{t*} has some value:

$$(x_i^t)^* = \begin{cases} \frac{y^*}{q_i^t} & \textit{for } i < i^*; \\ M_{t-1} - \sum_{i=1}^{i^*-1} x_i^t & \textit{for } i = i^*, \end{cases} \tag{3.3}$$

where y^* satisfies:

$$f(y) = y\lambda - (1-\lambda)\sum_{i=1}^{i^*} x_i^{T-1} r_i^{T-1} + b^0 M = 0, \tag{3.4}$$

$$b^0 = (\prod_{t=0}^{T-1} r_{i^*}^t)(1-\lambda), \tag{3.5}$$

and i^* can be determined as follows:

if

$$x_k^t = \frac{y}{q_k^t} < M_{t-1} - \sum_{j=1}^{k-1} x_j^t, \tag{3.6}$$

and

$$x_{k+1}^t = \frac{y}{q_{k+1}^t} \geq M_{t-1} - \sum_{j=1}^{k} x_j^t, \tag{3.7}$$

then $i^* = k + 1$.

Proof.

We apply the Karush-Kuhn-Tucker (KKT) conditions to $\mathbf{PO}(\lambda)$.

First, let us introduce the Lagrangian function of $\mathbf{PO}(\lambda)$:

$$\begin{aligned} f(x, y, \mu, \lambda, v, b) = \;& \lambda y + (1-\lambda)(-\textstyle\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1}) + \sum_{i=1}^{n}\sum_{t=0}^{T-1} \mu_i^t (q_j^t x_j^t - y) \\ & - \textstyle\sum_{i=1}^{n}\sum_{t=0}^{T-1} v_i^t x_i^t + \sum_{t=1}^{T-1} b^t (\sum_{i=1}^{n} x_i^t - \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1}) \\ & + b^0 (\textstyle\sum_{i=1}^{n} x_i^0 - M) \end{aligned}$$

Then the KKT conditions (see, for example, [30]) that an optimal solution $(\mathbf{x}, y)$ must satisfy can be written as follows:

$$\frac{\partial f}{\partial y} = \lambda - \sum_{i=1}^{n}\sum_{t=0}^{T-1} \mu_i^t = 0 \tag{3.8}$$

$$\frac{\partial f}{\partial x_i^t} = \mu_i^t q_i^t - v_i^t + b^t - b^{t+1} r_i^t = 0, \; t = 0, 1, ..., T-2 \tag{3.9}$$

$$\frac{\partial f}{\partial x_i^{T-1}} = -(1-\lambda)r_i^{T-1} + \mu_i^{T-1}q_i^{T-1} - v_i^{T-1} + b^{T-1} = 0 \quad (3.10)$$

$$\mu_i^t(q_i^t x_i^t - y) = 0 \quad (3.11)$$

$$v_i^t x_i^t = 0 \quad (3.12)$$

$$\sum_{i=0}^{n} x_i^t - \sum_{i=0}^{n} x_i^{t-1} r_i^{t-1} = 0, \text{for } i = 1, 2, ..., T-1 \quad (3.13)$$

$$\sum_{i=0}^{n} x_i^0 = M \quad (3.14)$$

$$\mu_i^t \geq 0 \quad (3.15)$$

$$v_i^t \geq 0 \quad (3.16)$$

We will prove that the following solution satisfies the KKT conditions above:

1.

$$x_i^t = \frac{y}{q_i^t}, \text{ and}$$
$$\mu_i^t = \frac{b^t(r_i^t - r_{i^*}^t)}{q_i^t}, \text{ and}$$
$$v_i^t = 0, \text{ where}$$

$$b^t = (\prod_{j=t}^{T-1} r_{i^*}^j)(1-\lambda), \text{ when}$$

$$i < i^*;$$

2.

$$x_{i^*}^t = M_{t-1} - \sum_{i=0}^{i^*-1} x_i^t, \text{ and}$$

$$\mu_{i^*}^t = 0, \text{ and}$$

$$v_{i^*}^t = 0, \text{ where}$$

$$b^t = (\prod_{j=t}^{T-1} r_{i^*}^j)(1-\lambda);$$

if

$$x_{i^*-1}^t = \frac{y}{q_{i^*-1}^t} < M_{t-1} - \sum_{j=1}^{i^*-2} x_j^t,$$

but

$$x_{i^*}^t = \frac{y}{q_{i^*}^t} \geq M_{t-1} - \sum_{j=1}^{i^*-1} x_j^t.$$

3.

$$x_i^t = 0, \text{ and}$$

$$\mu_i^t = 0, \text{ and}$$

$$v_i^t = b^{t+1}(r_{i^*}^t - r_i^t), \text{ when}$$

$$i > i^*.$$

Let us consider the first point.

When $i < i^*$, we have

$$q_i^t x_i^t - y = 0, \text{ or } \quad x_i^t = \frac{y}{q_i^t}.$$

which satisfies (3.11).

Since $x_i^t > 0$ and $v_i^t = 0$, (3.12) and (3.16) are met.

We can also easily show that (3.9) is met and $\mu_i^t \geq 0$ because $r_i^t \geq r_{i^*}^t$.

Now let us consider the second point.

When $i = i^*$, $\mu_{i^*}^t = 0$, and

$$x_{i^*}^t = M_{t-1} - \sum_{i=0}^{i^*-1} x_i^t \geq 0.$$

Since $\mu_{i^*}^t = 0$, and $v_{i^*}^t = 0$, it follows from the expression of b^t that (3.9) is satisfied.

Thus, (3.9) - (3.12), (3.15) - (3.16) are satisfied.

Now let us consider the third conjecture point.

Firstly, when $i > i^*$, $\mu_i^t = 0$.

KKT conditions (3.11) and (3.15) are satisfied.

If $x_i^t = 0$, v_i^t can be any number. So KKT conditions (3.12) and (3.16) are satisfied.

Since $\mu_i^t = 0$, Equation 3.9 can be rewritten as

$$-v_i^t + b^t - b^{t+1} r_i^t = 0.$$

This is satisfied according to v_i^t.

Applying the similar argument, we know that KKT condition (3.10) is satisfied as well.

Moreover, (3.13) and (3.14) are satisfied according to the constraint of $\{x_i^t\}^*$.

In order to satisfy (3.8), we can let $\lambda = \sum_{t=1}^{T-1} \sum_{i=1}^{n} \mu_i^t$.

Therefore, we have proved that all KKT conditions are satisfied.

Now we show (3.4).

Multiplying x_i^t on each side of (3.9) and (3.10), we get:

$$\mu_i^t q_i^t x_i^t - v_i^t x_i^t + b^t x_i^t - b^{t+1} r_i^t x_i^t = 0,$$

$$-(1-\lambda) r_i^{T-1} x_i^{T-1} + \mu_i^{T-1} q_i^{T-1} x_i^{T-1} - v_i^{T-1} x_i^{T-1} + b^{T-1} x_i^{T-1} = 0. \tag{3.17}$$

Since $v_i^t x_i^t = 0$, these equations can be rewritten as

$$\mu_i^t q_i^t x_i^t + b^t x_i^t - b^{t+1} r_i^t x_i^t = 0,$$

$$-(1-\lambda)r_i^{T-1}x_i^{T-1} + \mu_i^{T-1}q_i^{T-1}x_i^{T-1} + b^{T-1}x_i^{T-1} = 0. \quad (3.18)$$

Taking the summation of t and i respectively on both of equations in (3.18), and then summing them together, we get:

$$\sum_{t=0}^{T-1}\sum_{i=1}^{n}\mu_i^t q_i^t x_i^t + \sum_{t=0}^{T-1}\sum_{i=1}^{n} b^t x_i^t - \sum_{t=0}^{T-2}\sum_{i=1}^{n} b^{t+1} r_i^t x_i^t - (1-\lambda)\sum_{i=1}^{n} r_i^{T-1}x_i^{T-1} = 0. \quad (3.19)$$

From (3.11), we know that

$$\sum_{t=0}^{T-1}\sum_{i=1}^{n}\mu_i^t q_i^t x_i^t = \sum_{t=0}^{T-1}\sum_{i=1}^{n}\mu_i^t y.$$

Thus

$$y\sum_{t=0}^{T-1}\sum_{i=1}^{n}\mu_i^t + \sum_{t=0}^{T-1} b^t \sum_{i=1}^{i^*} x_i^t - \sum_{t=0}^{T-2} b^{t+1}\sum_{i=1}^{i^*} r_i^t x_i^t - (1-\lambda)\sum_{i=1}^{i^*} r_i^{T-1}x_i^{T-1} = 0. \quad (3.20)$$

According to (3.8), we know that

$$\lambda = \sum_{t=0}^{T-1}\sum_{i=1}^{n}\mu_i^t.$$

Therefore, (3.20) can be rewritten as:

$$y\lambda + \sum_{t=0}^{T-1} b^t \sum_{i=1}^{i^*} x_i^t - \sum_{t=0}^{T-2} b^{t+1}\sum_{i=1}^{i^*} r_i^t x_i^t - (1-\lambda)\sum_{i=1}^{i^*} r_i^{T-1}x_i^{T-1} = 0. \quad (3.21)$$

Since

$$\sum_{i=1}^{n} x_i^t = M_{t-1} = \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1},$$

(3.21) can be rewritten as:

$$y\lambda + \sum_{t=0}^{T-1} b^t \sum_{i=1}^{i^*} x_i^t - \sum_{t=0}^{T-2} b^{t+1}\sum_{i=1}^{i^*} x_i^{t+1} - (1-\lambda)\sum_{i=1}^{i^*} r_i^{T-1}x_i^{T-1} = 0.$$

Furthermore, items of $\sum_{t=0}^{T-1} b^t \sum_{i=1}^{i^*} x_i^t$ and $\sum_{t=0}^{T-1} b^{t+1} \sum_{i=1}^{i^*} x_i^{t+1}$ can be canceled from each other. Thus, what left for us from the above equation is:

$$y\lambda - (1-\lambda)\sum_{i=1}^{i^*} x_i^{T-1} r_i^{T-1} + b^0 M = 0. \qquad (3.22)$$

To summarize, the theorem is proved.

□

3.3 How to Find y

Note that we only show in Theorem 3.2.1, that y should meet an equation (3.4):

$$f(y) = y\lambda - (1-\lambda)\sum_{i=1}^{i^*} x_i^{T-1} r_i^{T-1} + b^0 M = 0.$$

In this section, we will discuss how to compute y.

Since y is a scaler, we propose to determine the value of it by enumeration. We can set the boundary of y, and make the enumeration less complicated.

As the above theorem stated,

$$f(y) = y\lambda - (1-\lambda)\sum_{i=1}^{i^*} x_i^{T-1} r_i^{T-1} + b^0 M_0 = 0.$$

In other words,

$$y = \frac{M r_{i^*}^0 r_{i^*}^1 \ldots r_{i^*}^{T-1}}{\sum_{i=1}^{i^*} \frac{r_i^{T-1}}{q_i^{T-1}} - \frac{\lambda}{1-\lambda}}.$$

Firstly, we determine the lower bound of y.

If y is small enough, all the asset will be invested because $x_i^t = \frac{y}{q_i^t}$.

It means if y is small enough, i^* will be the last asset, which is n. In this case,

$$y = \frac{M r_n^0 r_n^1 \ldots r_n^{T-1}}{\sum_{i=1}^{i^*} \frac{r_i^{T-1}}{q_i^{T-1}} - \frac{\lambda}{1-\lambda}}.$$

r_n^t is always the smallest among all r_i^t in each period t. Therefore, $Mr_n^0 r_n^1 \dots r_n^{T-1}$ is the smallest numerator.

Meanwhile, if we want to derive the lower bound of y, the denominator should be as large as possible.

Since $-\frac{\lambda}{1-\lambda}$ is a constant, we need to make $\sum_{i=1}^{i^*} \frac{r_i^{T-1}}{q_i^{T-1}}$ as large as possible. Of course, the largest one should be $i^* = n$.

Therefore, the lower bound of y is:

$$y = \frac{Mr_n^0 r_n^1 \dots r_n^{T-1}}{\sum_{i=1}^{n} \frac{r_i^{T-1}}{q_i^{T-1}} - \frac{\lambda}{1-\lambda}}.$$

Secondly, we determine the upper bound of y.

If y is large enough, all money will be purely invested into a single asset who can give us the largest return. In this case, $i^* = 1$.

However, since

$$y \geq x_i^t q_i^t,$$

we know that $y \geq 0$. It means

$$\sum_{i=1}^{i^*} \frac{r_i^{T-1}}{q_i^{T-1}} - \frac{\lambda}{1-\lambda}$$

should be larger than 0, or

$$\sum_{i=1}^{i^*} \frac{r_i^{T-1}}{q_i^{T-1}} > \frac{\lambda}{1-\lambda}.$$

Thus, in order to let $y \geq 0$, for the final period $T-1$, we have to find an asset s, who can make

$$\sum_{i=1}^{s} \frac{r_i^{T-1}}{q_i^{T-1}} > \frac{\lambda}{1-\lambda},$$

but also make the denominator

$$\sum_{i=1}^{s} \frac{r_i^{T-1}}{q_i^{T-1}} - \frac{\lambda}{1-\lambda}$$

as small as possible (because we want to determine the upper bound of y).

Therefore, the upper bound of y is

$$y = \frac{M r_1^0 r_1^1 r_1^2 \ldots r_1^{T-2} r_s^{T-1}}{\sum_{i=1}^{s} \frac{r_i^{T-1}}{q_i^{T-1}} - \frac{\lambda}{1-\lambda}}.$$

This s can be found by:

if

$$\sum_{j=1}^{i} \frac{r_j^{T-1}}{q_j^{T-1}} < \frac{\lambda}{1-\lambda},$$

but

$$\sum_{j=1}^{i+1} \frac{r_j^{T-1}}{q_j^{T-1}} > \frac{\lambda}{1-\lambda},$$

then

$$s = i + 1.$$

In summary, to enumerate y, we can start with its lower bound, and then increase it gradually, until the upper bound is reached.

For any given value of y, we can apply Theorem 3.2.1 to determine $\{x_i^t\}$, and then check if equation (3.4) is satisfied or not. If it is a yes, the enumeration can be terminated.

3.4 Numerical Example

We consider two sets of Hong Kong stock market: one set includes stock price from January 2007 to December 2007; the other includes stock price from January 2008 to December 2008. The time unit is month. Hence $T = 12$ respectively.

There are 40 stocks in total that constitute the Hang Seng Index during that period, which are used as our asset pool. Hence $n = 40$.

The actual monthly return can be calculated as:

$$R_i^t = \frac{P_i^t}{P_i^{t-1}},$$

where P_i^t is the stock price of S_i at period t.

The expectation of R_i^t can be derived based on historical data of 12 months before period t. For example, the expectation of R_1^1 can be derived based on historical data of S_1 from January 2007 to December 2007:

$$r_i^t = E[R_i^t] = \frac{\sum_{s=1}^{12} R_i^{t-s}}{12},$$

where s is the s^{th} month before month i.

We also need to define q_i^t. Since

$$q_i^t = E[|R_i^t - r_i^t|],$$

we use the data we just calculate for R_i^t and r_i^t to obtain the value of q_i^t.

Assume we have initial wealth of USD 100. After 12 months' investment, the efficient frontier can be plotted (as shown in Figure 3.1 and Figure 3.2), which is the collection of all efficient points (the efficient point in our model is y and $\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1}$).

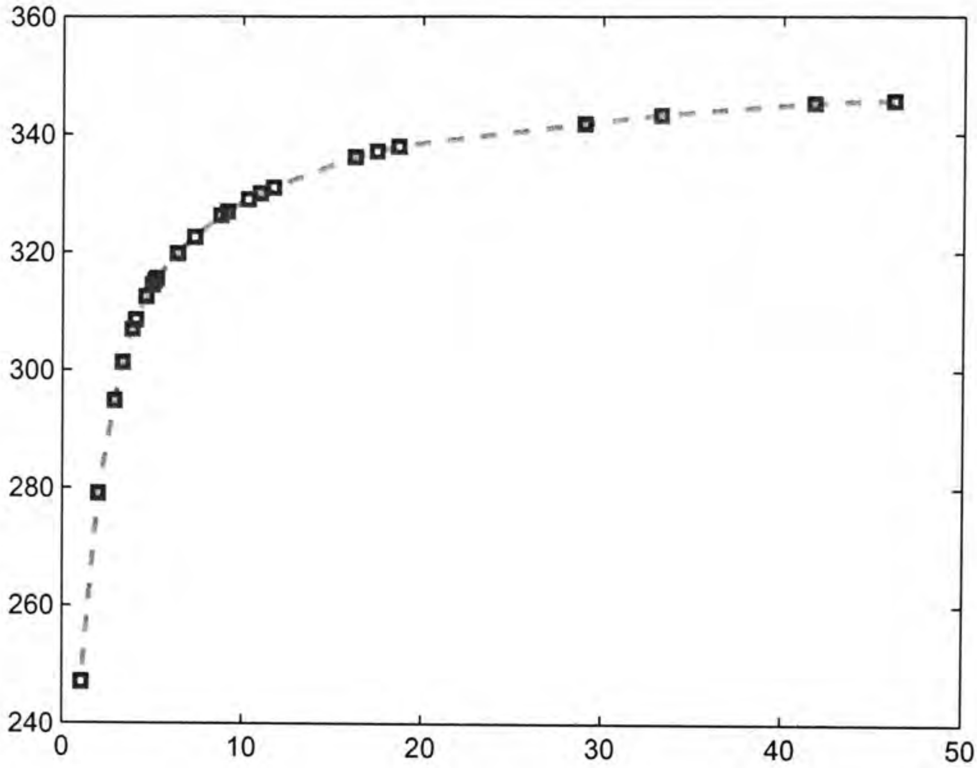

Figure 3.1: Efficient Frontier of 2007 Data

x axis is for y, while y axis is for $\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1}$. Every single point corresponds to one λ. Since $\lambda \in [0, 1]$, in this numerical example, we divide λ by 100. Thus the value of λ varies from $0.00, 0.01, 0.02, \ldots$ to $0.99, 1.00$.

From this efficient frontier graph, we can see that as λ gets

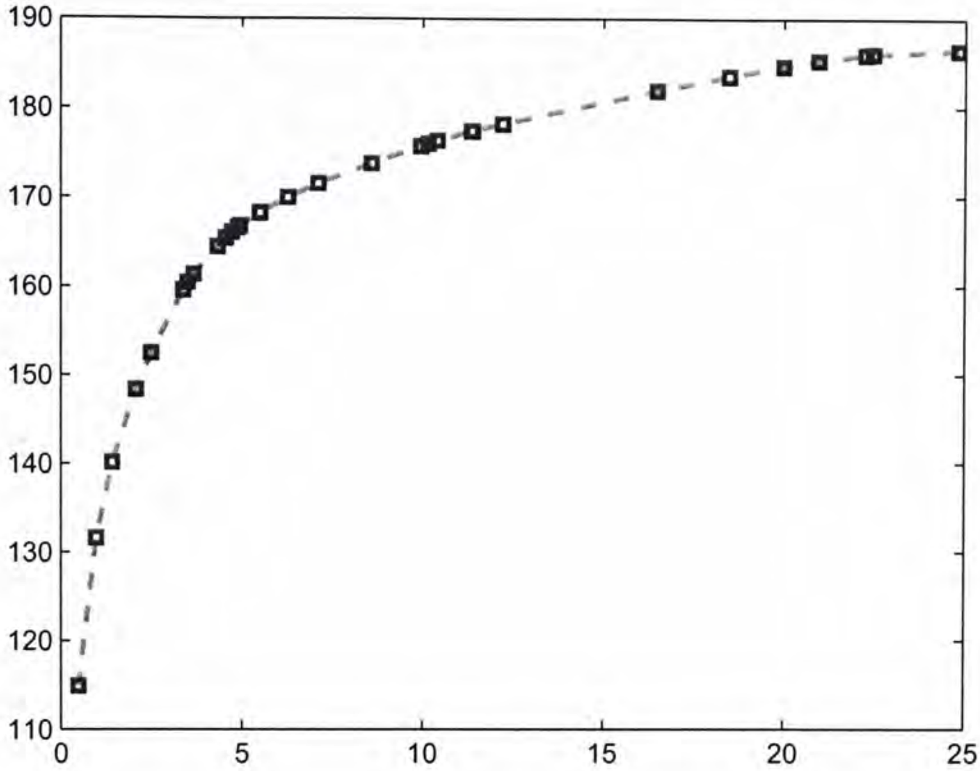

Figure 3.2: Efficient Frontier of 2008 Data

smaller and smaller, the terminal wealth is getting larger and larger. That is because our objective function is

$$\min \lambda y - (1-\lambda)\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1}.$$

If λ is smaller, it means that this investor pays less attention to risk, or the investor emphasizes more on terminal wealth. That is the reason that $\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1}$ is getting larger and larger if λ is getting smaller and smaller.

All source data of this numerical example is provided in Appendix A.1, including r_i^t and q_i^t.

Besides that, we list six tables for 2007 x_i^t and 2008 x_i^t when $\lambda = 0.3$, $\lambda = 0.6$ and $\lambda = 0.9$ in Appendix B.1 - B.3. The value of every period optimal total investment return is attached to the last line.

When the optimal investment strategy is implemented under the real case, the actual investment return can be obtained, which could be different from the optimal return, since optimal return uses the historical data, while the true return uses the real data. Therefore, we also calculate the actual return when using the optimal investment strategy. The x_i^t is proportional to the each period total wealth according to the optimal investment strategy. Please refer to B.4 for the actual investment strategy.

Here let us take 2007 actual return as examples:

- Year 2007 $\lambda = 0.3$: $M_T = 120.29$, and $y = 18.68$.

  Hang Seng Index increased by 13.45%; our model gains a profit of 20.29%.

  As you can see from Appendix B.7, when $t = 5$, model II assigns all investment money USD 100 to stock 28 and 29, because they give us an average historical return of 1.10, which is the largest among all stocks in that period. Meanwhile, its risk measure is $x_{28}^5 q_{28}^5 + x_{29}^5 q_{29}^5 = 67.82 \times 0.08 + 67.82 \times 0.08 = 10.85$, which is less than $y = 18.68$.

  Another example can be given is when $t = 11$. Model assigns 59.87 to stock 23, while assigns 85.79 to stock 33. From Appendix A.1, we can find that $r_{23}^{1}1 = 1.10$ and $r_{23}^{1}1 = 1.09$. The reason we cannot assign all investment money to stock 23 is because $M_{11}q_{23}^{11} = 0.15 \times 159.3 = 23.90$, which has reached the upper risk level bound $y = 18.68$. That is the reason we assign the rest of investment money to stock 33.

This is the way Model II assigns investment money.

As the result, it can gain the investor a return of USD 120.29 when $y = 18.68$.

- Year 2007 $\lambda = 0.6$: $M_T = 124.61$, and $y = 10.98$.

  Hang Seng Index increased by 13.45%; our model gains a profit of 24.61%.

  Please refer to Appendix A and Appendix B.2 and B.4 for detail asset allocation result.

- $\lambda = 0.9$: $M_T = 164.02$, and $y = 3.88$.

  Hang Seng Index increased by 13.45%; our model gains a profit of 64.02%.

  Please refer to Appendix A and Appendix B.3 and B.4 for detail asset allocation result.

□ **End of chapter.**

Chapter 4

Model III

Model III allows withdraws. It still concerns the asset allocation to achieve the maximum total wealth (the sum of discounted withdraw amounts over the entire investment periods plus the terminal return of the portfolio) while the risk level is specified. l_∞ function is used as the risk measure. We have two methodologies to resolve this problem. Firstly we try to solve this problem by dynamic programming in a backward fashion. Two suggested algorithms are provided. Dynamic programming allows us to have a clearer picture of how assets are actually reallocated period by period. Secondly we derive an approximate analytical solution via the whole investment period. Numerical examples are provided. The comparison between optimal solution and approximate solution is studied as well.

4.1 Model Formulation

In this model, we will add another nonnegative parameter: a withdraw rate δ_t in each period.

The instinct for such a parameter is that during investment period, sometimes investor wants to withdraw part of their investment money for emergency usage, or put it into bank, to get a risk-free return with return rate ρ.

We slightly revise the definition of M_t:

$$\sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} = M_t.$$

In our present model, we want to maximize the total return, including the sum of discounted withdraw amount over the entire investment horizon plus the terminal return of the portfolio.

Let ρ be the discounted rate.

Our model is formulated as:

$$
\begin{aligned}
(P1)\quad \max\quad & \sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} + \sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t} \\
\text{s.t.}\quad & y \geq q_i^t x_i^t \quad \text{for all} \quad i \in [1,n] \quad \text{and} \quad t \in [0, T-1], \\
& \sum_{i=1}^{n} x_i^t = (1-\delta_t) M_t, \ \text{for each } t \in [0, T-1], \\
& M_t = \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} \ \text{for each } t \in [1, T-1], \\
& x_i^t \geq 0 \ \text{for each } i \in [1,n] \text{ and } t \in [0, T-1], \\
& 0 \leq \delta_t \leq 1 \ \text{for each } t \in [0, T-1],
\end{aligned}
$$

where y, M, q_i^t, ρ and r_i^t are known, and decision variables are x_i^t and δ_t.

In what follows, we try to use two methodologies to find the optimal solution of this model.

The first methodology will view this problem via a dynamic programming perspective: two dynamic programs are explored, while the first one enumerates δ_t and the second one enumerates M_t for $t \in [0, T-1]$; optimal solution is found; numerical examples are given respectively.

The second methodology will suggest an approximate analytical solution. A numerical example will be provided as well. The comparison of the approximate solution and the optimal solution will be highlighted.

4.2 Dynamic Programming

4.2.1 DP I

Now we intend to use dynamic programming by enumerating δ_t to resolve this portfolio selection problem in a backward fashion.

The state of this dynamic programming is M_t. The return function is $f(M_t)$ which is also the value we are trying to maximize.

For the recursive relationship, firstly we explore the boundary condition of **(P1)** when $t = T - 1$, we have:

$$
\begin{aligned}
f(M_{T-1}) \; &= \; \max_{\delta_{T-1} \geq 0} \left\{ \sum_{i=0}^{n} x_i^{T-1} r_i^{T-1} + \delta_{T-1} M_{T-1} \rho^{T-(T-1)} \right\} \\
\text{s.t.} \quad & y \geq q_i^{T-1} x_i^{T-1}, \\
& \sum_{i=1}^{n} x_i^{T-1} = (1 - \delta_{T-1}) M_{T-1}, \\
& x_i^{T-1} \geq 0,
\end{aligned}
$$

where decision variables are: x_i^{T-1} and δ_{T-1}, while M_{T-1} is the state.

Therefore, when we are in period t, $t \in [0, T-2]$, we have the following recursive relationship:

$$
\begin{aligned}
(P2)\quad f(M_t) &= \max_{\delta_t \geq 0} \left\{ f(M_{t+1}) + \delta_t M_t \rho^{T-t} \right\} \\
\text{s.t.} \quad & y \geq q_i^t x_i^t, \\
& \sum_{i=1}^{n} x_i^t = (1-\delta_t) M_t, \\
& x_i^t \geq 0,
\end{aligned}
$$

where decision variables are: x_i^t and δ_t, while M_t is the state.

For any given M_t, for each M_t we will enumerate δ_t. After doing that, the second part of objective function of **P2** becomes a constant $(\delta_t M_t \rho^{T-t})$. In this case, in order to maximize $f(M_t) = \max\{f(M_{t+1}) + \delta_t M_t \rho^{T-t}\}$, it is equivalent to maximize $f(M_t) = f(M_{t+1})$, which is equivalent to maximize M_{t+1}:

$$
\begin{aligned}
(P3)\quad \max \quad & M_{t+1} = \sum_{i=1}^{n} r_i^t x_i^t \\
\text{s.t.} \quad & y \geq q_i^t x_i^t, \\
& \sum_{i=1}^{n} x_i^t = (1-\delta_t) M_t, \\
& x_i^t \geq 0,
\end{aligned}
$$

where decision variables is x_i^t.

From the formality of **P3**, we can see that the analytical solution of Model I can be used to find its optimal solution. The only difference is that in each period, the initial wealth is $(1-\delta_t)M_t$

instead of M_t.

As stated above, we enumerate δ_t for any given M_t. Since δ_t is the withdraw rate, the range should be $[0, 1]$. Therefore, we divide δ_t into 51 pieces, from 0.00, 0.02, to 0.98, 1.00.

The algorithm is described in page 69.

From the algorithm, we can see that the complexity of this algorithm is $O(T^2ab)$, where a and b are the maximum numbers of M_t and δ_t enumerated in each period $t \in [0, T-1]$ respectively.

4.2.2 DP II

Now we develop another dynamic program to resolve the problem.

The state of this dynamic programming is M_t. The return function is $f(M_t)$ which is also the value we are trying to maximize.

For the recursive relationship, firstly we explore the boundary condition of **(P1)** when $t = T - 1$ is:

$$\begin{aligned} f(M_{T-1}) &= \max_{M_T} \left\{ \sum_{i=0}^{n} x_i^{T-1} r_i^{T-1} + \delta_{T-1} M_{T-1} \rho^{(T-(T-1))} \right\} \\ \text{s.t.} \quad & y \geq q_i^{T-1} x_i^{T-1}, \\ & \sum_{i=1}^{n} x_i^{T-1} = (1 - \delta_{T-1}) M_{T-1}, \\ & x_i^{T-1} \geq 0, \\ & 0 \leq \delta_{T-1} \leq 1, \end{aligned}$$

where the decision variables are x_i^{T-1} and δ_{T-1}, while M_{T-1} is the state.

For period t, $t \in [0, T - 2]$, we have the following recursive relationship:

$$\begin{aligned}
f(M_t) &= \max_{M_{t+1}} \left\{ f(M_{t+1}) + \delta_t M_t \rho^{(T-t)} \right\} \\
\text{s.t.} \quad & y \geq q_i^t x_i^t, \\
& \sum_{i=1}^{n} x_i^t = (1-\delta_t) M_t, \\
& x_i^t \geq 0, \\
& 0 \leq \delta_{T-1} \leq 1.
\end{aligned}$$

We can see that given M_t and M_{t+1}, the above is a linear problem where x_i^t and δ_t are unknown. Therefore, a linear problem algorithm can help us to find the optimal solution.

As stated above, we have to enumerate M_t and M_{t+1} in the dynamic programming. We will enumerate all the possible values for M_t from $t = 1$. Since $M_{t+1} = \sum_{i=0}^{n} x_i^t r_i^t$, for $t \in [0, T-1]$, and since $M_t = \sum_{i=0}^{n} x_i^t$ for $t \in [0, T-1]$, and also because for each period t, we have arranged the order of r_i^t into the following way:

$$r_1^t \geq r_2^t \geq \ldots \geq r_n^t, \text{for all } t \in [0, T-1],$$

the largest value of M_{t+1} is $M_t r_1^t$; and the smallest value M_{t+1} is $M_t r_n^t$. That is how we will determine the upper bound and lower bound of M_{t+1}:

$$M_{t+1} \in [M_t r_n^t, M_t r_1^t], \text{ or}$$

$$M_{t+1} \in [M_0 \prod_{i=0}^{t} r_n^i, M_0 \prod_{i=0}^{t} r_1^i].$$

Since we have already known M_0 beforehand, all bounds of M_t where $t \in [1, T-1]$ can be determined.

The algorithm is described in page 70.

4.3 Approximate Analytical Solution

The original problem in Model III is trying to find the optimal solution. However, what if the investor wants to know the approximate investment result is a more efficient manner. In this section, an approximate analysis will be given.

Recall the original problem:

$$
\begin{aligned}
(P1)\quad \max\quad & \sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} + \sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t} \\
\text{s.t.}\quad & y \geq q_i^t x_i^t \quad \text{for all} \quad i \in [1,n] \quad \text{and} \quad t \in [0, T-1], \\
& \sum_{i=1}^{n} x_i^t = (1-\delta_t) M_t, \quad \text{for each } t \in [0, T-1], \\
& M_t = \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} \text{ for each } t \in [1, T-1], \\
& x_i^t \geq 0 \text{ for each } i \in [1,n] \text{ and } t \in [0, T-1], \\
& 0 \leq \delta_t \leq 1 \text{ for each } t \in [0, T-1].
\end{aligned}
$$

Since

$$
\begin{aligned}
\delta_t M_t &= M_t - \sum_{i=1}^{n} x_i^t \\
&= \sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} - \sum_{i=1}^{n} x_i^t \text{ for each } t,
\end{aligned}
$$

equation

$$
\sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t}
$$

can be rewritten as

$$
\begin{aligned}
& \sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t} \\
= & \sum_{t=0}^{T-1} [(\sum_{i=1}^{n} x_i^{t-1} r_i^{t-1} - \sum_{i=1}^{n} x_i^t) \rho^{T-t}] \\
= & [\sum_{i=1}^{n} x_i^{T-2} r_i^{T-2} - \sum_{i=1}^{n} x_i^{T-1}] \rho^{T-(T-1)} + [\sum_{i=1}^{n} x_i^{T-3} r_i^{T-3} - \sum_{i=1}^{n} x_i^{T-2}] \rho^{T-(T-2)} + \\
& \ldots\ldots + [\sum_{i=1}^{n} x_i^0 r_i^0 - \sum_{i=1}^{n} x_i^1] \rho^{T-1} + [M_0 - \sum_{i=1}^{n} x_i^0] \rho^{T-0} \\
= & [\sum_{i=1}^{n} x_i^{T-2} r_i^{T-2} - \sum_{i=1}^{n} x_i^{T-1}] \rho^1 + [\sum_{i=1}^{n} x_i^{T-3} r_i^{T-3} - \sum_{i=1}^{n} x_i^{T-2}] \rho^2 + \\
& \ldots\ldots + [\sum_{i=1}^{n} x_i^0 r_i^0 - \sum_{i=1}^{n} x_i^1] \rho^{T-1} + [M_0 - \sum_{i=1}^{n} x_i^0] \rho^T.
\end{aligned}
$$

Furthermore, the above equation can be simplified to

$$
\begin{aligned}
& \sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t} \\
= & -\sum_{i=1}^{n} x_i^{T-1} \rho + \sum_{i=1}^{n} x_i^{T-2} (r_i^{T-2} \rho^1 - \rho^2) + \\
& \ldots\ldots + \sum_{i=1}^{n} x_i^0 (r_i^0 \rho^{T-1} - \rho^T) + M_0 \rho^T.
\end{aligned}
$$

Continuously, the above equation can be simplified to

$$
\begin{aligned}
&\sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t} \\
= &\ -\sum_{i=1}^{n} x_i^{T-1}\rho + \rho \times \sum_{i=1}^{n} x_i^{T-2}(r_i^{T-2} - \rho) + \\
&\ \ldots\ldots + \rho^{T-1} \times \sum_{i=1}^{n} x_i^{0}(r_i^{0} - \rho) + M_0\rho^T.
\end{aligned}
$$

We define

$$\tilde{\alpha}_i^t = r_i^t - \rho.$$

Then the above equation can be converted to

$$\sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t} = -\sum_{i=1}^{n} x_i^{T-1}\rho + \sum_{t=0}^{T-2}[\rho^{T-t-1} \times \sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t] + M_0\rho^T. \tag{4.1}$$

Taking Equation (4.1) into **(P1)** objective function, we can get:

$$
\begin{aligned}
& \sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} + \sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t} \\
= & \sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} - \sum_{i=1}^{n} x_i^{T-1} \rho + \sum_{t=0}^{T-2} [\rho^{T-t-1} \times \sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t] + M_0 \rho^T \\
= & \rho^0 \sum_{i=1}^{n} x_i^{T-1} (r_i^{T-1} - r) + \sum_{t=0}^{T-2} [\rho^{T-t-1} \times \sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t] + M_0 \rho^T \\
= & \rho^0 \sum_{i=1}^{n} x_i^{T-1} \tilde{\alpha}_i^{T-1} + \sum_{t=0}^{T-2} [\rho^{T-t-1} \times \sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t] + M_0 \rho^T \\
= & \sum_{t=0}^{T-1} [\rho^{T-t-1} \times \sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t] + M_0 \rho^T
\end{aligned}
$$

Therefore, if we are trying to find the optimal solution to

$$
\max \left\{ \sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} + \sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t} \right\},
$$

it is equivalent to find the optimal solution to

$$
\max \left\{ \sum_{t=0}^{T-1} [\rho^{T-t-1} \times \sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t] + M_0 \rho^T \right\}.
$$

Since

$$
M_0 \rho^T
$$

is a constant, if we are trying to find the optimal solution to

$$\max\left\{\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} + \sum_{t=0}^{T-1} \delta_t M_t \rho^{T-t}\right\},$$

it is equivalent to find the optimal solution to

$$\max\left\{\sum_{t=0}^{T-1} [\rho^{T-t-1} \times \sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t]\right\}.$$

Therefore, the approximate model can be developed into this way:

$$\begin{aligned}
(P4) \quad \max \quad & \left\{\rho^{T-t} \times \sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t\right\} \\
\text{s.t.} \quad & y \geq q_i^t x_i^t \quad \text{for all} \quad i \in [1,n] \quad \text{and} \quad t \in [0, T-1], \\
& x_i^t \geq 0 \text{ for each } i \in [1,n] \text{ and } t \in [0, T-1], \\
& 0 \leq \delta_t \leq 1 \text{ for each } t \in [0, T-1],
\end{aligned}$$

where $\tilde{\alpha}_i^t = r_i^t - \rho$.

Now we can treat **P1** via a different point of view: instead of thinking of the withdraw rate δ_t, now we have a new portfolio selection model **P4**, where $\tilde{\alpha}_i^t$ is the return rate.

Although this approximate approach can only give us a sub-optimal solution, it is very efficient. And we show in the following numerical study that actually the result gives us a very closed approximation for the original problem.

Theorem 4.3.1 *The optimal solution to **P4** is:*

Arrange $\tilde{\alpha}_i^t$ according to the following order for each period $t \in [0, T-1]$:

$$\tilde{\alpha}_1^t \geq \tilde{\alpha}_2^t \geq \ldots \geq \tilde{\alpha}_n^t, \text{ for all } t \in [0, T-1].$$

From $i = 1$ to $i = n$, we do:

We initialize every x_i^t with:

$$x_i^t = \frac{y}{q_i^t}.$$

If $\tilde{\alpha}_i^t > 0$ and $x_i^t < M_t - \sum_{j=1}^{i-1} x_i^t$ (but $x_i^t \neq 0$), then

$$x_i^t = \frac{y}{q_i^t};$$

if $\tilde{\alpha}_i^t > 0$, $x_{i-1}^t < M_t - \sum_{j=1}^{i-2} x_i^t$ but $x_i^t \geq M_t - \sum_{j=1}^{i-1} x_i^t$, we denote this i by i^* and

$$x_{i^*}^t = M_t - \sum_{j=1}^{i^*-1} x_j^t;$$

otherwise,

$$x_i^t = 0.$$

If $\tilde{\alpha}_i^t \leq 0$,

$$x_i^t = 0.$$

For the last $\tilde{\alpha}_i^t$ who is larger than 0 in each period t, we denote this i by k. If $M_t - \sum_{j=1}^{k} x_j^t > 0$, we make δ_t to

$$\delta_t = 1 - \frac{\sum_{i=1}^{k} x_i^t}{M_t}.$$

Otherwise,

$$\delta_t = 0.$$

Proof.

In the objective function of **P4**, r^{T-t} is a constant for each period t, so if we want to maximize $r^{T-t} \times \sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t$, it is equivalent to maximize $\sum_{i=0}^{n} x_i^t \tilde{\alpha}_i^t$.

It is obvious that the objective function of **P4** is a linear combination regarding to the nonnegative value x_i^t. We treat $\tilde{\alpha}_i^t$ as the weight of x_i^t. Thus, in order to find the maximum value of the objective function, we can assign the value of x_i^t as large as possible, according to the descending order of its weight $\tilde{\alpha}_i^t$, under the condition that the total assigned values will not exceed M_t. Since x_i^t is nonnegative, we will only assign value to x_i^t whose weight is a positive number.

Since we have already arranged $\tilde{\alpha}_i^t$ according to the following order for each period $t \in [0, T-1]$: $\tilde{\alpha}_1^t \geq \tilde{\alpha}_2^t \geq \ldots \geq \tilde{\alpha}_n^t$, we will assign the value of x_1^t as large as possible. Since $y \geq x_i^t q_i^t$, the largest value we can assign to x_1^t is $x_1^t = \frac{y}{q_1^t}$, if $x_1^t \leq M_t$.

If both constraints, $y \geq x_1^t q_1^t$ and $M_t - x_1^t > 0$ are satisfied, and $x_1^t \geq 0$, we will continue assigning value to x_2^t, applying the same methodology we used to assign x_1^t.

This assignment task will continue if $\tilde{\alpha}_i^t > 0$, until we find there is an i that if we assign $x_i^t = \frac{y}{q_i^t}$, we will have $\sum_{j=1}^{i} x_j^t > M_t$.

It means that there is no enough money for this x_i^t's assignment.

In this case, we will denote this i by i^*, and allocate all rest of money to this $x_{i^*}^t$; in other words,

$$x_{i^*}^t = M_t - \sum_{j=1}^{i^*-1} x_j^t,$$

and $\delta_t = 0$ for **P4** because all money M_t are allocated.

If $i > i^*$, despite $\tilde{\alpha}_i^t > 0$ or $\tilde{\alpha}_i^t \leq 0$, we set $x_i^t = 0$.

However, we might have the case that for the very last $\tilde{\alpha}_i^t$ whose value is larger than 0,

$$\sum_{j=1}^{i} x_j^t < M_t.$$

If this happens, it means that there are some money left, and we do not plan to invest any more because continue investing, it will give investor a negative return ($\tilde{\alpha}_i^t \leq 0$). We denote this i by k. We withdraw this amount of money $M_t - \sum_{j=1}^{k} x_j^t$.

In other words, we have obtained the value of δ_t for **P4**:

$$\delta_t = 1 - \frac{\sum_{j=1}^{k} x_j^t}{M_t}.$$

For $\tilde{\alpha}_i^t < 0$, x_i^t always has the value of 0 because we try to maximize the sum of the product of $\tilde{\alpha}_i^t$ and x_i^t. Since $\tilde{\alpha}_i^t$ is the weight, and x_i^t is always nonnegative, we must set all $x_i^t = 0$ if the corresponding $\tilde{\alpha}_i^t \leq 0$; otherwise, it will decrease the value of the objective function.

Therefore, the theorem is proved.

□

4.4 Computational Result Comparison

In this section, three numerical examples are given, according to three different ways to resolve Model III.

We use the same data as in Model II:

We consider two sets of Hong Kong stock market: one set includes stock price from January 2007 to December 2007; the other includes stock price from January 2008 to December 2008. The time unit is month. Hence $T = 12$ respectively.

There are 40 stocks in total that constitute the Hang Seng Index during that period, which are used as our asset pool. Hence $n = 40$.

The actual monthly return can be calculated as:

$$R_i^t = \frac{P_i^t}{P_i^{t-1}},$$

where P_i^t is the stock price of S_i at period t.

The expectation of R_i^t can be derived based on historical data. For example, the expectation of R_1^1 can be derived based on historical data of S_1 from January 2008 to December 2008:

$$r_i^t = E[R_i^t] = \frac{\sum_{s=1}^{12} R_i^{t-s}}{12},$$

where s is the s^{th} month before month i.

We also need to define q_i^t. Since $q_i^t = E[|R_i^t - r_i^t|]$, we use the data we just calculate for R_i^t and r_i^t to obtain the value of q_i^t.

There is one more parameter we need to specify: ρ. The investor withdraws money and could either spend it or save it. In any case, we set ρ as the risk-free interest rate so that we could calculate the future value of this withdraw amount when $t = T$. We take reference of Hang Seng risk-free interest rate, and set it as $\rho = 1.04$.

In all models, we have to specify the risk level. Here we set it as $y = 4$. Thus this is a relatively conservative investor.

Here is the computational result for these three models:

1. **DP II**

   Firstly we enumerate all possible value of M_t. Since we have determined the upper bound and lower bound of M_t:

$$M_{t+1} \in [M_0 \prod_{i=0}^{t} r_n^i, M_0 \prod_{i=0}^{t} r_1^i],$$

   for each period t, we enumerate 100 M_t in each period by adding an equal amount.

   Please refer to Appendix C.1 for detailed M_t enumeration.

   Secondly, we implement our Algorithm II in a backward fashion. We note down every last optimal point number so that we can track down the whole optimal investment strategy after finishing the whole algorithm.

Please refer to Appendix C.2 for details.

Last but not least, according to the tracking numbers in Appendix C.2, we mark the optimal value of $f(M_t)$.

Please refer to Appendix C.3.

As a conclusion, the optimal sum of discounted withdraw amount over the entire investment period plus the terminal wealth we can get is USD $297,120$, if the initial wealth is USD $100,000$, and risk level is 4. Hang Seng Index increased by 13.45% during that period, while our model gains a profit of 197.12%.

2. **Approximate Analytical Solution**

Please refer to the asset allocation result in Appendix C.4 when $y = 4$. The approximate total wealth (the sum of discounted withdraw amounts over the entire investment periods plus the terminal return of the portfolio) is USD $299,094$. Compare to the optimal solution in DP II (terminal wealth of USD $297,120$), this approximate result only has an error rate of 0.66% (an error rate means the absolute difference between the optimal value and the approximate value over the optimal value), or USD $1,974$.

We also extend our numerical study to different values of y

in 2007, the comparison result tells us that on average, the total wealth error rate is 0.52%, or the total wealth gap is USD 1,860, and the standard deviation is 0.22%. Please refer to the following table. Note that all return should time 1,000.

y	Approximate Total Return \$,000	Optimal Total Return \$,000
1	243.54	243.12
1.2	252.93	252.39
1.4	261.00	260.34
1.6	267.79	267.04
1.8	273.73	272.75
2	278.78	277.30
2.2	282.93	281.82
2.4	286.52	284.81
2.6	289.87	288.39
2.8	293.08	292.16
3	296.20	294.90
3.2	299.11	298.12
3.4	301.78	299.55
3.6	303.95	302.17
3.8	306.07	304.35
4	307.83	305.57
4.2	309.32	307.20
4.4	310.74	307.69
4.6	312.10	309.59
4.8	313.33	311.03
5	314.44	312.25

From the above data, we can conclude that the approximate analytical solution is very closed to the optimal solution, say the DP II. In fact, the approximate analytical solution works well except the market is too fluctuated. However, under some extreme cases, for example, y is a huge number, this solution does not work well.

□ **End of chapter.**

Algorithm 1 Multi-Period Portfolio Management Algorithm I

for each M_{T-1} **do**

  for each δ_t **do**

    We take the analytical solution of Model I to resolve **P2** to get the value of x_i^{T-1}.

$$\begin{aligned} (P2) \quad f(M_{T-1}) &= \max\left\{\sum_{i=0}^{n} x_i^{T-1} r_i^{T-1}\right\} \\ \text{s.t.} \quad & y \geq q_i^{T-1} x_i^{T-1}, \\ & \sum_{i=1}^{n} x_i^{T-1} = (1-\delta_{T-1}) M_{T-1}, \\ & x_i^{T-1} \geq 0, \end{aligned}$$

  end for

  The optimal δ_{T-1} is such that $\sum_{i=1}^{n} x_i^{T-1} r_i^{T-1} + \delta_{T-1} M_{T-1} \rho$ is maximized.

end for

for every t = T-1, ..., 0 **do**

  for each M_t **do**

    for each δ_t **do**

$$\begin{aligned} f(M_t) &= \max\left\{\sum_{i=1}^{n} x_i^t r_i^t\right\} \\ \text{s.t.} \quad & y \geq q_i^t x_i^t, \\ & \sum_{i=1}^{n} x_i^t = (1-\delta_t) M_t, \\ & x_i^t \geq 0, \end{aligned}$$

      to get the value of x_i^t.

    end for

    The optimal δ_t is such that $\sum_{i=1}^{n} x_i^t r_i^t + \delta_t M_t \rho^{T-t}$ is maximized.

  end for

end for

Algorithm 2 Multi-Period Portfolio Management Algorithm II

Enumerate M_{T-1}

for each M_{T-1} **do**

We resolve **P3** to get the value of x_i^{T-1} and δ_{T-1}.

$$
\begin{aligned}
(P3) \quad \max \quad & \left\{ \sum_{i=0}^{n} x_i^{T-1} r_i^{T-1} + \delta_{T-1} M_{T-1} \rho^{(T-(T-1))} \right\} \\
\text{s.t.} \quad & y \geq q_i^{T-1} x_i^{T-1}, \\
& \sum_{i=1}^{n} x_i^{T-1} = (1 - \delta_{T-1}) M_{T-1}, \\
& x_i^{T-1} \geq 0, \\
& \delta_{T-1} \geq 0.
\end{aligned}
$$

end for

for every t = T-1, ..., 0 **do**

for each M_t and M_{t+1} **do**

We resolve **P4** to get the value of x_i^t and δ_t.

$$
\begin{aligned}
(P4) \quad \max \quad & \left\{ f(M_{t+1}) + \delta_t M_t \rho^{T-t} \right\} \\
\text{s.t.} \quad & y \geq q_i^t x_i^t, \\
& \sum_{i=1}^{n} x_i^t = (1 - \delta_t) M_t, \\
& x_i^t \geq 0, \\
& \delta_t \geq 0.
\end{aligned}
$$

end for

The optimal x_i^t and δ_t are such that $f(M_{t+1}) + \delta_t M_t \rho^{T-t}$ is maximized.

end for

Chapter 5

Conclusions

Three multi-period portfolio selection models are built in this thesis. They are all models under minimax rule with l_∞ risk measure.

The first model is concerning the asset allocation to achieve the maximum terminal wealth while the risk level is specified. An explicit optimal analytical solution is given.

The second model is concerning the asset allocation to achieve two goals at the same time: investor tries to maximize his terminal wealth while minimize the l_∞ risk measure. It is a bicriteria portfolio optimization problem. We convert it into a parametric optimization problem. An explicit optimal analytical solution is given for the efficient point. A numerical example is provided, and efficient frontier is drawn.

The third model allows withdraws. It still concerns the asset allocation to achieve the maximum total wealth (the sum of discounted withdraw amounts over the entire investment periods plus the terminal return of the portfolio) while the risk level is specified. l_∞ function is used as the risk measure. We have two methodologies to resolve this problem. Firstly we try to solve this problem by dynamic programming in a backward fashion. Two suggested algorithms are provided. Dynamic programming allows us to have a clearer picture of how assets are actually reallocated period by period. Secondly we derive an approximate analytical solution via the whole investment period. Numerical examples are provided. The comparison between optimal solution and approximate solution is studied as well.

The optimal solutions of Model I, II and III work well especially when the market is quiet. For the approximate analytical solution, the result is very closed to the optimal ones. In fact, the approximate analytical solution works well except the market is too fluctuated. However, under some extreme cases, for example, y is a huge number, this solution does not work well.

There are still some more future works can be done. For example, dynamic programming is an elegant methodology to show people step by step how the whole theory works. However, the algorithm could be improved, in order to run more efficiently.

Besides that transaction cost is a practical parameter in daily trading. It should be added into the model as well.

Last but not least, there is one constraint in our models: $x_i^t \geq 0$, which means short sale is not allowed. Nevertheless, it is not the real case in financial world. It would be interesting for further research to study how investment strategy would change if short sale is permitted.

Portfolio selection problem has both theoretical and practical meanings. Along with the development of world financial market, this topic will gain more and more attentions in future.

□ **End of chapter.**

Appendix A

Source Data

A.1 r_i^t

Table A.1: Table of 2007 r_i^t

r_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	1.02	1.02	1.02	1.02	1.02	1.02	1.03	1.03	1.04	1.05	1.04	1.04
n = 2	1.03	1.03	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.01	1.01	1.00
n = 3	1.00	0.99	1.00	1.00	1.00	1.00	1.01	1.00	1.00	1.02	1.03	1.03
n = 4	1.00	1.00	1.00	1.00	1.01	1.01	1.01	1.02	1.03	1.05	1.05	1.04
n = 5	1.01	1.00	1.00	1.01	1.01	1.00	1.00	1.00	1.00	1.00	0.99	1.00
n = 6	1.01	1.01	1.01	1.01	1.02	1.01	1.01	1.01	1.01	1.01	1.01	1.02
n = 7	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.03	1.04	1.04	1.04
n = 8	1.01	1.01	1.01	1.01	1.03	1.03	1.03	1.02	1.03	1.04	1.04	1.05
n = 9	1.00	1.00	1.01	1.00	1.01	1.01	1.02	1.01	1.02	1.03	1.02	1.01
n = 10	1.02	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.04	1.05	1.06	1.06

Cont.

r_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 11	1.04	1.03	1.02	1.03	1.04	1.04	1.03	1.03	1.04	1.07	1.07	1.05
n = 12	1.02	1.02	1.02	1.01	1.02	1.01	1.01	1.00	1.02	1.03	1.03	1.03
n = 13	1.06	1.05	1.04	1.04	1.04	1.03	1.04	1.02	1.02	1.03	1.01	1.02
n = 14	1.02	1.02	1.01	1.00	1.00	1.00	1.00	1.00	1.02	1.03	1.03	1.04
n = 15	1.05	1.04	1.04	1.03	1.04	1.03	1.03	1.03	1.03	1.06	1.06	1.04
n = 16	1.03	1.04	1.04	1.04	1.05	1.06	1.06	1.05	1.07	1.07	1.07	1.06
n = 17	1.04	1.03	1.04	1.03	1.04	1.04	1.05	1.05	1.07	1.08	1.06	1.04
n = 18	1.02	1.02	1.03	1.01	1.03	1.05	1.06	1.05	1.07	1.06	1.05	1.05
n = 19	1.04	1.03	1.05	1.05	1.06	1.06	1.06	1.06	1.07	1.06	1.05	1.04
n = 20	1.03	1.03	1.04	1.04	1.05	1.04	1.04	1.03	1.03	1.03	1.01	1.01
n = 21	1.02	1.03	1.04	1.04	1.04	1.04	1.05	1.05	1.05	1.05	1.03	1.03
n = 22	1.03	1.03	1.04	1.03	1.06	1.06	1.06	1.06	1.07	1.07	1.06	1.05
n = 23	1.07	1.07	1.05	1.03	1.05	1.08	1.09	1.09	1.14	1.14	1.13	1.10
n = 24	1.05	1.04	1.03	1.03	1.05	1.05	1.05	1.04	1.05	1.05	1.03	1.03
n = 25	1.08	1.06	1.06	1.07	1.08	1.09	1.13	1.11	1.11	1.09	1.07	1.05
n = 26	1.10	1.09	1.06	1.05	1.07	1.06	1.08	1.08	1.10	1.12	1.10	1.08
n = 27	1.05	1.05	1.06	1.06	1.05	1.07	1.07	1.07	1.07	1.08	1.07	1.05
n = 28	1.08	1.07	1.07	1.08	1.09	1.10	1.11	1.11	1.10	1.10	1.09	1.08
n = 29	1.08	1.07	1.07	1.08	1.09	1.10	1.11	1.11	1.10	1.10	1.09	1.08
n = 30	1.01	1.00	1.02	1.01	1.02	1.04	1.04	1.03	1.07	1.09	1.07	1.06
n = 31	1.04	1.02	1.02	1.03	1.03	1.04	1.05	1.06	1.07	1.09	1.06	1.03
n = 32	1.06	1.06	1.05	1.04	1.05	1.06	1.05	1.06	1.07	1.08	1.07	1.07
n = 33	1.06	1.05	1.03	1.03	1.05	1.06	1.07	1.08	1.12	1.12	1.11	1.09
n = 34	1.02	1.03	1.02	1.01	1.02	1.02	1.02	1.02	1.04	1.04	1.02	1.01
n = 35	1.05	1.05	1.05	1.04	1.02	1.03	1.02	1.00	0.99	0.99	0.99	0.97
n = 36	1.08	1.07	1.06	1.07	1.07	1.08	1.09	1.11	1.13	1.13	1.11	1.07
n = 37	1.03	1.02	1.02	1.02	1.02	1.02	1.03	1.01	1.02	1.02	1.01	1.01
n = 38	1.01	1.01	1.01	1.02	1.05	1.08	1.11	1.13	1.14	1.13	1.10	1.08
n = 39	1.10	1.08	1.08	1.08	1.07	1.08	1.09	1.10	1.10	1.11	1.08	1.04
n = 40	1.06	1.05	1.05	1.05	1.05	1.05	1.05	1.06	1.05	1.09	1.06	1.02

Table A.2: Table of 2008 r_i^t

r_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	1.02	1.02	1.01	1.02	1.02	1.01	1.01	1.00	0.98	0.95	0.95	0.95
n = 2	1.01	1.01	1.01	1.01	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.01
n = 3	1.02	1.03	1.03	1.02	1.02	1.02	1.00	1.00	1.00	0.97	0.96	0.95
n = 4	1.04	1.04	1.03	1.03	1.02	1.01	1.01	1.00	0.96	0.92	0.94	0.96
n = 5	0.99	0.99	1.00	1.00	0.99	0.99	0.99	0.99	0.99	0.96	0.97	0.96
n = 6	1.02	1.01	1.02	1.02	1.02	1.02	1.02	1.03	1.02	1.01	1.01	1.00
n = 7	1.04	1.03	1.03	1.04	1.04	1.04	1.02	1.03	1.01	0.97	0.98	0.97
n = 8	1.04	1.04	1.02	1.03	1.01	1.00	1.00	1.00	0.96	0.94	0.93	0.93
n = 9	1.00	1.00	1.00	1.01	1.02	1.01	0.99	1.00	0.98	0.94	0.94	0.94
n = 10	1.05	1.04	1.03	1.04	1.03	1.02	1.02	1.01	0.97	0.94	0.93	0.93
n = 11	1.03	1.02	1.01	1.02	1.01	0.99	0.99	0.97	0.94	0.89	0.89	0.92
n = 12	1.02	1.01	1.01	1.01	1.01	1.00	1.00	1.00	0.98	0.95	0.95	0.95
n = 13	1.01	1.00	1.00	1.00	1.01	1.01	0.99	0.98	0.96	0.92	0.93	0.92
n = 14	1.04	1.03	1.03	1.04	1.03	1.03	1.03	1.03	1.00	0.97	0.97	0.97
n = 15	1.03	1.02	1.01	1.02	1.02	1.01	1.00	0.99	0.95	0.91	0.89	0.93
n = 16	1.04	1.03	1.03	1.03	1.02	1.00	0.99	1.00	0.95	0.95	0.95	0.95
n = 17	1.03	1.04	1.02	1.02	1.01	0.99	0.99	0.98	0.95	0.92	0.91	0.92
n = 18	1.03	1.05	1.02	1.03	1.01	0.98	0.98	0.97	0.94	0.89	0.88	0.93
n = 19	1.01	1.02	1.00	1.01	1.00	0.99	0.97	0.97	0.96	0.94	0.93	0.94
n = 20	0.99	0.99	0.98	0.99	0.99	0.98	0.98	0.98	0.97	0.94	0.93	0.94

r_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 21	1.02	1.02	1.01	1.00	1.00	0.99	0.99	0.96	0.93	0.92	0.92	0.93
n = 22	1.03	1.04	1.02	1.03	1.01	1.00	1.01	1.01	0.98	0.95	0.95	0.94
n = 23	1.08	1.08	1.07	1.09	1.06	1.03	1.02	1.00	0.94	0.91	0.90	0.93
n = 24	1.02	1.02	1.02	1.03	1.02	0.99	1.01	0.99	0.96	0.94	0.95	0.94
n = 25	1.05	1.07	1.04	1.06	1.04	1.01	0.99	0.99	0.96	0.95	0.97	0.98
n = 26	1.05	1.07	1.06	1.07	1.07	1.07	1.07	1.06	1.02	1.00	0.99	1.00
n = 27	1.05	1.06	1.04	1.04	1.05	1.02	1.02	1.00	0.98	0.97	0.96	0.96
n = 28	1.05	1.06	1.04	1.05	1.05	1.02	1.00	1.01	0.99	0.96	0.97	0.97
n = 29	1.05	1.06	1.04	1.05	1.05	1.02	1.00	1.01	0.99	0.96	0.97	0.97
n = 30	1.06	1.08	1.06	1.08	1.07	1.05	1.04	1.04	0.98	0.94	0.95	0.97
n = 31	1.02	1.04	1.03	1.04	1.04	1.02	1.02	1.01	0.98	0.94	0.96	0.98
n = 32	1.05	1.05	1.05	1.06	1.05	1.03	1.02	0.99	0.97	0.94	0.95	0.96
n = 33	1.07	1.07	1.06	1.07	1.05	1.02	1.01	0.99	0.94	0.91	0.92	0.93
n = 34	0.99	0.99	0.99	0.99	0.98	0.97	0.97	0.96	0.93	0.90	0.91	0.94
n = 35	0.96	0.96	0.94	0.95	0.94	0.92	0.92	0.91	0.87	0.86	0.84	0.87
n = 36	1.05	1.06	1.05	1.07	1.05	1.03	1.00	0.99	0.94	0.92	0.93	0.96
n = 37	1.00	1.01	1.01	1.01	1.01	1.02	1.00	1.00	0.98	0.94	0.95	0.94
n = 38	1.06	1.09	1.06	1.06	1.05	1.00	0.97	0.93	0.90	0.87	0.91	0.93
n = 39	1.03	1.05	1.03	1.04	1.04	1.01	1.00	1.00	0.98	0.94	0.95	0.97
n = 40	1.01	1.03	1.02	1.04	1.03	1.02	1.03	1.02	0.99	0.92	0.95	0.96

A.2 q_i^t

□ **End of chapter.**

Table A.3: Table of 2007 q_i^t

q_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.05	0.05	0.05
n = 2	0.03	0.03	0.03	0.03	0.04	0.04	0.04	0.03	0.03	0.03	0.03	0.02
n = 3	0.03	0.02	0.02	0.03	0.03	0.03	0.04	0.04	0.04	0.05	0.05	0.05
n = 4	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.03	0.04	0.06	0.06	0.07
n = 5	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.03	0.03
n = 6	0.03	0.02	0.02	0.02	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02
n = 7	0.01	0.02	0.02	0.02	0.02	0.02	0.04	0.04	0.05	0.06	0.06	0.06
n = 8	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.05	0.05	0.05
n = 9	0.04	0.03	0.03	0.03	0.02	0.02	0.03	0.04	0.04	0.05	0.05	0.05
n = 10	0.04	0.04	0.04	0.03	0.02	0.02	0.03	0.03	0.05	0.05	0.06	0.05
n = 11	0.05	0.05	0.04	0.04	0.03	0.03	0.03	0.04	0.04	0.06	0.06	0.07
n = 12	0.03	0.03	0.04	0.03	0.03	0.02	0.02	0.02	0.03	0.04	0.04	0.04
n = 13	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.04	0.05	0.06	0.06	0.06
n = 14	0.05	0.05	0.05	0.03	0.03	0.03	0.03	0.03	0.05	0.06	0.06	0.07
n = 15	0.07	0.07	0.07	0.06	0.05	0.06	0.06	0.06	0.06	0.08	0.08	0.06
n = 16	0.05	0.05	0.05	0.05	0.04	0.04	0.03	0.04	0.05	0.04	0.05	0.05
n = 17	0.08	0.08	0.08	0.07	0.06	0.06	0.06	0.05	0.06	0.06	0.07	0.06
n = 18	0.05	0.06	0.06	0.05	0.04	0.06	0.05	0.05	0.06	0.06	0.07	0.07
n = 19	0.06	0.07	0.06	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06	0.05
n = 20	0.04	0.05	0.04	0.04	0.03	0.03	0.04	0.04	0.03	0.03	0.04	0.04
n = 21	0.06	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.05	0.05
n = 22	0.07	0.08	0.07	0.07	0.08	0.07	0.08	0.08	0.09	0.09	0.09	0.08
n = 23	0.08	0.08	0.08	0.07	0.07	0.08	0.09	0.08	0.14	0.14	0.15	0.15
n = 24	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.04	0.05	0.06	0.06
n = 25	0.12	0.13	0.13	0.13	0.12	0.11	0.11	0.12	0.13	0.13	0.13	0.12
n = 26	0.09	0.10	0.10	0.08	0.10	0.10	0.10	0.10	0.10	0.11	0.13	0.12
n = 27	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.07	0.08	0.08	0.07
n = 28	0.07	0.08	0.08	0.08	0.07	0.08	0.08	0.08	0.07	0.07	0.07	0.07
n = 29	0.07	0.08	0.08	0.08	0.07	0.08	0.08	0.08	0.07	0.07	0.07	0.07
n = 30	0.05	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.08	0.10	0.11	0.12
n = 31	0.08	0.07	0.06	0.06	0.06	0.07	0.07	0.07	0.07	0.08	0.09	0.09
n = 32	0.05	0.04	0.05	0.05	0.04	0.04	0.04	0.05	0.06	0.07	0.08	0.09
n = 33	0.08	0.07	0.07	0.07	0.07	0.08	0.08	0.08	0.09	0.09	0.10	0.09
n = 34	0.06	0.07	0.07	0.06	0.05	0.05	0.05	0.05	0.04	0.05	0.06	0.06
n = 35	0.10	0.11	0.12	0.12	0.10	0.09	0.09	0.09	0.08	0.07	0.07	0.07
n = 36	0.08	0.09	0.09	0.09	0.09	0.09	0.10	0.11	0.13	0.12	0.14	0.13
n = 37	0.04	0.05	0.04	0.04	0.04	0.04	0.05	0.05	0.05	0.06	0.06	0.06
n = 38	0.09	0.09	0.09	0.10	0.08	0.08	0.07	0.09	0.08	0.09	0.11	0.12
n = 39	0.07	0.08	0.08	0.08	0.08	0.09	0.10	0.10	0.10	0.10	0.12	0.11
n = 40	0.08	0.09	0.09	0.08	0.09	0.09	0.08	0.08	0.08	0.13	0.12	0.09

Table A.4: Table of 2008 q_i^t

q_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.06	0.06	0.06	0.07	0.07	0.08	0.08	0.07	0.08	0.07	0.08	0.08
n = 2	0.03	0.03	0.03	0.03	0.05	0.05	0.05	0.05	0.05	0.06	0.06	0.06
n = 3	0.06	0.06	0.06	0.06	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
n = 4	0.07	0.07	0.08	0.08	0.07	0.08	0.09	0.10	0.10	0.10	0.12	0.13
n = 5	0.04	0.04	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.07	0.07	0.07
n = 6	0.02	0.03	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.05	0.05	0.04
n = 7	0.07	0.07	0.07	0.08	0.07	0.07	0.06	0.06	0.06	0.07	0.07	0.07
n = 8	0.06	0.06	0.07	0.08	0.07	0.08	0.08	0.08	0.08	0.08	0.08	0.08
n = 9	0.06	0.06	0.06	0.06	0.07	0.08	0.07	0.07	0.07	0.08	0.08	0.08
n = 10	0.06	0.07	0.07	0.08	0.09	0.10	0.11	0.11	0.11	0.10	0.08	0.09
n = 11	0.07	0.08	0.09	0.09	0.09	0.10	0.11	0.11	0.11	0.09	0.09	0.12
n = 12	0.04	0.05	0.05	0.05	0.06	0.06	0.07	0.07	0.07	0.07	0.07	0.07
n = 13	0.08	0.08	0.08	0.09	0.09	0.09	0.10	0.10	0.11	0.11	0.12	0.11
n = 14	0.07	0.07	0.07	0.07	0.07	0.08	0.07	0.07	0.06	0.07	0.07	0.07
n = 15	0.07	0.09	0.09	0.10	0.10	0.12	0.11	0.12	0.15	0.13	0.11	0.15
n = 16	0.07	0.07	0.07	0.08	0.09	0.09	0.08	0.08	0.08	0.08	0.09	0.08
n = 17	0.07	0.08	0.09	0.09	0.11	0.11	0.11	0.10	0.09	0.09	0.10	0.10
n = 18	0.08	0.08	0.10	0.11	0.11	0.10	0.10	0.10	0.09	0.14	0.14	0.20
n = 19	0.06	0.06	0.06	0.07	0.07	0.09	0.08	0.09	0.09	0.10	0.10	0.11
n = 20	0.05	0.06	0.06	0.06	0.06	0.07	0.06	0.06	0.06	0.07	0.08	0.09
n = 21	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.07	0.08	0.07	0.08	0.09
n = 22	0.09	0.09	0.11	0.13	0.11	0.12	0.12	0.13	0.13	0.12	0.12	0.12
n = 23	0.16	0.17	0.17	0.18	0.19	0.18	0.17	0.17	0.09	0.08	0.09	0.11
n = 24	0.06	0.06	0.06	0.07	0.07	0.08	0.09	0.10	0.10	0.10	0.09	0.08
n = 25	0.12	0.12	0.12	0.12	0.13	0.13	0.11	0.11	0.11	0.11	0.13	0.14
n = 26	0.14	0.13	0.13	0.14	0.14	0.14	0.14	0.15	0.15	0.12	0.13	0.14
n = 27	0.06	0.06	0.06	0.06	0.07	0.07	0.08	0.10	0.09	0.08	0.08	0.08
n = 28	0.11	0.09	0.12	0.13	0.13	0.13	0.14	0.14	0.14	0.13	0.14	0.14
n = 29	0.11	0.09	0.12	0.13	0.13	0.13	0.14	0.14	0.14	0.13	0.14	0.14
n = 30	0.12	0.13	0.15	0.15	0.15	0.15	0.15	0.15	0.13	0.13	0.13	0.14
n = 31	0.10	0.10	0.11	0.12	0.12	0.12	0.12	0.12	0.12	0.11	0.12	0.12
n = 32	0.10	0.10	0.10	0.11	0.12	0.12	0.12	0.11	0.10	0.08	0.08	0.09
n = 33	0.10	0.10	0.12	0.12	0.12	0.12	0.11	0.11	0.10	0.10	0.11	0.12
n = 34	0.07	0.07	0.08	0.08	0.07	0.08	0.08	0.08	0.09	0.10	0.11	0.15
n = 35	0.08	0.08	0.07	0.08	0.08	0.10	0.09	0.10	0.12	0.11	0.11	0.15
n = 36	0.15	0.14	0.15	0.17	0.19	0.18	0.16	0.15	0.13	0.14	0.13	0.16
n = 37	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.07	0.08	0.10	0.10	0.09
n = 38	0.14	0.17	0.19	0.19	0.18	0.19	0.18	0.15	0.15	0.17	0.19	0.21
n = 39	0.13	0.12	0.14	0.15	0.16	0.16	0.15	0.14	0.13	0.12	0.12	0.14
n = 40	0.10	0.10	0.11	0.13	0.13	0.14	0.14	0.15	0.16	0.11	0.13	0.14

Appendix B

Model II Numerical Example and Result

B.1 Value of x_i^t when $\lambda = 0.3$

Table B.1: Table x_i^t when $\lambda = 0.3$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Cont.

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	137.30	137.00	124.71	127.15
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	0.00	0.00	167.66	0.00	0.00	0.00	0.00	0.00
n = 26	100.00	110.24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	64.67	69.89	76.18	0.00	0.00	0.00	0.00	0.00	0.00
n = 29	0.00	0.00	0.00	64.67	69.89	76.18	0.00	0.00	0.00	0.00	0.00	0.00
n = 30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	23.38	182.21
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	106.62	129.19	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	0.00	0.00	0.00	0.00	0.00	0.00	188.90	76.08	0.00	0.00	0.00
n = 39	0.00	0.00	119.93	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_t	110.24	119.93	129.33	139.79	152.36	167.66	188.90	213.38	243.62	277.27	309.36	**337.92**

B.2 Value of x_i^t when $\lambda = 0.6$

Table B.2: Table 2008 x_i^t when $\lambda = 0.3$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	184.91
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	25.06	181.21	182.84	0.00
n = 7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	100.00	0.00	117.31	125.87	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 26	0.00	0.00	0.00	0.00	137.33	147.03	155.84	150.81	152.24	0.00	0.00	0.00
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 30	0.00	0.00	0.00	0.00	0.00	0.00	1.39	17.32	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	107.90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 39	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_t	107.90	117.31	125.87	137.33	147.03	157.23	168.13	177.30	181.21	182.84	184.91	186.01

Table B.3: Table 2007 x_i^t when $\lambda = 0.6$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	80.70	80.53	73.30	74.74
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	0.00	0.00	98.55	0.00	0.00	0.00	0.00	0.00
n = 26	100.00	110.24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	64.67	69.89	76.18	34.55	0.00	0.00	0.00	0.00	8.36
n = 29	0.00	0.00	0.00	64.67	69.89	76.18	34.55	0.00	0.00	0.00	0.00	8.36
n = 30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	107.45	119.22
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	0.00	0.00	0.00	0.00	60.17	0.00	88.84	75.93	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	0.00	0.00	0.00	0.00	0.00	0.00	127.40	130.04	71.03	16.18	92.66
n = 39	0.00	0.00	119.93	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_t	110.24	119.93	129.33	139.79	152.36	167.66	187.56	210.75	240.40	272.87	303.33	329.94

Table B.4: Table 2008 x_i^t when $\lambda = 0.6$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	11.34	133.32
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	107.23	169.42	159.60	39.54
n = 7	0.00	0.00	0.00	0.00	0.00	24.55	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	45.14	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	52.06	0.00	50.39	48.56	17.98	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 26	0.00	0.00	0.00	0.00	60.83	60.19	60.12	58.18	58.73	0.00	0.00	0.00
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 30	0.00	57.52	22.06	56.77	55.78	59.14	55.74	55.98	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	47.94	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	19.09	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	50.10	44.13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 39	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	36.27	0.00	0.00	0.00	0.00	0.00
M_t	107.62	116.58	124.41	134.59	143.88	152.13	159.30	165.96	169.42	170.95	172.87	173.77

B.3 Value of x_i^t when $\lambda = 0.9$

Table B.5: Table 2007 x_i^t when $\lambda = 0.9$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	28.54	28.48	25.92	26.43
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	33.10	33.98	34.85	31.09	31.00	0.00	0.00	0.00
n = 26	45.08	38.99	0.00	0.00	0.00	0.00	0.00	0.00	0.00	33.98	29.70	32.20
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	11.27	34.16	48.98	52.45	45.95	50.10	35.16	0.00	5.36	51.25	56.45
n = 29	0.00	11.27	34.16	48.98	52.45	45.95	50.10	35.16	0.00	5.36	51.25	56.45
n = 30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	6.71
n = 33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	43.28	42.03	38.00	42.16
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	0.00	0.25	24.56	0.00	36.67	30.86	31.42	26.85	31.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	0.00	0.00	0.00	0.00	0.00	29.75	45.05	50.67	45.32	35.03	32.77
n = 39	54.92	48.61	50.85	29.98	0.00	0.00	0.00	0.00	19.64	37.75	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_t	110.14	119.17	127.94	138.25	150.44	164.81	183.14	204.00	229.69	258.01	284.17	306.89

Table B.6: Table 2008 x_i^t when $\lambda = 0.9$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	50.22	43.05	40.22	38.73
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	33.81	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	38.05	57.55	49.22	46.36	58.23
n = 7	0.00	0.00	0.00	0.00	0.00	36.48	39.79	40.72	26.57	7.20	33.33	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	32.82	33.69	34.88	0.00	0.00	14.16	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	15.12	15.06	14.64	14.11	13.15	13.51	0.00	0.00	0.00	0.00	0.00	0.00
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	19.98	0.00	0.83	0.00	0.00	0.00	0.00	0.00	0.00	0.00	18.06
n = 26	0.00	13.17	18.71	17.83	17.67	17.48	17.46	16.90	17.06	20.80	19.38	17.50
n = 27	22.77	0.00	0.00	0.00	13.71	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 30	20.11	19.22	17.19	16.49	16.20	17.18	16.19	16.26	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	17.61	0.00	0.00	0.00	0.00	20.40
n = 32	0.00	0.00	13.13	23.67	20.88	19.23	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	24.70	24.36	21.56	20.26	20.82	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	16.28	14.54	13.39	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	17.30	14.55	12.82	13.29	13.61	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 39	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	17.52	0.00	0.00	0.00	0.00	0.00
M_t	106.34	114.32	121.01	129.44	136.71	142.27	146.81	151.39	154.08	153.46	152.93	152.48

B.4 True Value of x_i^t

□ **End of chapter.**

Table B.7: Table 2007 x_i^t when $\lambda = 0.3$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	101.12	94.09	66.95	59.87
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	0.00	0.00	149.98	0.00	0.00	0.00	0.00	0.00
n = 26	1000.00	89.48	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	48.35	54.68	67.82	0.00	0.00	0.00	0.00	0.00	0.00
n = 29	0.00	0.00	0.00	48.35	54.68	67.82	0.00	0.00	0.00	0.00	0.00	0.00
n = 30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	12.55	85.79
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	73.23	69.35	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	0.00	0.00	0.00	0.00	0.00	0.00	148.52	56.04	0.00	0.00	0.00
n = 39	0.00	0.00	87.73	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_t	89.48	87.73	96.70	109.36	135.64	149.98	148.52	157.16	167.32	148.84	145.65	120.29

Table B.8: Table 2007 x_i^t when $\lambda = 0.6$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	68.47	62.41	43.10	39.25
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	0.00	0.00	88.15	0.00	0.00	0.00	0.00	0.00
n = 26	100.00	89.48	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	48.35	54.68	67.82	30.91	0.00	0.00	0.00	0.00	4.39
n = 29	0.00	0.00	0.00	48.35	54.68	67.82	30.91	0.00	0.00	0.00	0.00	4.39
n = 30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	63.18	62.61
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	0.00	0.00	0.00	0.00	49.89	0.00	68.85	44.65	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	0.00	0.00	0.00	0.00	0.00	0.00	105.65	110.34	55.05	9.51	48.66
n = 39	0.00	0.00	87.73	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_t	89.48	87.73	96.70	109.36	135.64	149.98	155.54	178.81	186.31	160.44	159.30	124.61

Table B.9: Table 2007 x_i^t when $\lambda = 0.9$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	32.49	30.45	21.89	20.56
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	29.44	33.86	37.95	34.38	35.29	0.00	0.00	0.00
n = 26	45.08	32.53	0.00	0.00	0.00	0.00	0.00	0.00	0.00	36.33	25.08	25.05
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	9.40	27.80	42.68	46.65	45.78	54.56	38.89	0.00	5.73	43.29	43.91
n = 29	0.00	9.40	27.80	42.68	46.65	45.78	54.56	38.89	0.00	5.73	43.29	43.91
n = 30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	5.22
n = 33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	49.28	44.94	32.09	32.80
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	0.00	0.23	24.47	0.00	40.55	35.14	33.59	22.68	24.11
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	0.00	0.00	0.00	0.00	0.00	32.40	49.83	57.69	48.46	29.58	25.49
n = 39	54.92	40.57	41.39	26.13	0.00	0.00	0.00	0.00	22.36	40.37	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_t	91.91	97.00	111.49	122.97	149.88	179.46	202.54	232.25	245.59	217.91	221.05	164.02

Table B.10: Table 2008 x_i^t when $\lambda = 0.3$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	127.67
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	13.89	88.36	87.53	0.00
n = 7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	100.00	0.00	95.18	129.22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 26	0.00	0.00	0.00	0.00	91.06	92.08	108.05	86.92	84.39	0.00	0.00	0.00
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 30	0.00	0.00	0.00	0.00	0.00	0.00	0.96	9.98	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	101.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 39	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_t	101.10	95.18	129.22	91.06	92.08	109.02	96.90	98.28	88.36	87.53	127.67	99.39

Table B.11: Table 2008 x_i^t when $\lambda = 0.6$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	6.54	95.67
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	56.45	85.65	92.03	28.37
n = 7	0.00	0.00	0.00	0.00	0.00	15.94	0.00	0.00	0.00	0.00	0.00	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	27.80	0.00	0.00	0.00	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	52.06	0.00	38.64	49.33	13.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 26	0.00	0.00	0.00	0.00	44.00	39.09	41.83	35.84	30.92	0.00	0.00	0.00
n = 27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 30	0.00	56.06	16.92	57.67	40.35	38.41	38.79	34.48	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	47.94	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	0.00	19.39	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	0.00	48.82	33.84	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 39	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	25.24	0.00	0.00	0.00	0.00	0.00
M_t	104.88	89.39	126.40	97.36	93.45	105.86	98.12	87.37	85.65	98.58	124.04	104.77

Table B.12: Table 2008 x_i^t when $\lambda = 0.9$

x_i^t	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
n = 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	27.73	23.09	27.25	29.52
n = 3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	18.14	0.00	0.00
n = 4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	24.52	31.78	26.40	31.41	44.38
n = 7	0.00	0.00	0.00	0.00	0.00	23.69	28.66	26.23	14.67	3.86	22.58	0.00
n = 8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 14	0.00	0.00	0.00	0.00	0.00	21.31	24.26	22.48	0.00	0.00	9.59	0.00
n = 15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 23	15.12	16.40	11.50	13.77	10.74	8.77	0.00	0.00	0.00	0.00	0.00	0.00
n = 24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 25	0.00	21.76	0.00	0.81	0.00	0.00	0.00	0.00	0.00	0.00	0.00	13.77
n = 26	0.00	14.35	14.70	17.40	14.43	11.35	12.58	10.89	9.42	11.16	13.13	13.34
n = 27	22.77	0.00	0.00	0.00	11.20	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 28	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 30	20.11	20.93	13.50	16.09	13.23	11.15	11.66	10.48	0.00	0.00	0.00	0.00
n = 31	0.00	0.00	0.00	0.00	0.00	0.00	12.68	0.00	0.00	0.00	0.00	15.55
n = 32	0.00	0.00	10.31	23.10	17.05	12.48	0.00	0.00	0.00	0.00	0.00	0.00
n = 33	24.70	26.53	16.94	19.77	17.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 36	0.00	0.00	12.78	14.19	10.94	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 37	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 38	17.30	15.85	10.07	12.97	11.11	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 39	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
n = 40	0.00	0.00	0.00	0.00	0.00	0.00	12.62	0.00	0.00	0.00	0.00	0.00
M_t	115.83	89.79	118.09	105.71	88.76	102.46	94.60	83.60	82.65	103.97	116.57	96.65

Appendix C

Model III Numerical Example and Result

C.1 The Value of M_t of DP II

Unit: USD $\times$ 10,000

Table C.1: Table of M_t of DP II

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
1	10.00	9.78	9.18	8.60	7.72	7.57	7.16	6.53	5.96	5.23	4.20	3.20
2	10.00	9.81	9.23	8.66	7.81	7.68	7.29	6.70	6.18	5.54	4.64	3.77
3	10.00	9.83	9.28	8.72	7.89	7.78	7.42	6.87	6.41	5.84	5.09	4.35
4	10.00	9.86	9.33	8.79	7.98	7.88	7.54	7.04	6.64	6.15	5.53	4.92
5	10.00	9.88	9.37	8.85	8.07	7.98	7.67	7.21	6.87	6.46	5.98	5.49
6	10.00	9.91	9.42	8.91	8.15	8.09	7.80	7.39	7.10	6.77	6.43	6.07
7	10.00	9.93	9.47	8.98	8.24	8.19	7.93	7.56	7.33	7.08	6.87	6.64
8	10.00	9.96	9.52	9.04	8.32	8.29	8.06	7.73	7.55	7.39	7.32	7.21
9	10.00	9.98	9.57	9.10	8.41	8.40	8.19	7.90	7.78	7.70	7.76	7.79
10	10.00	10.00	9.61	9.16	8.49	8.50	8.32	8.07	8.01	8.01	8.21	8.36

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
11	10.00	10.03	9.66	9.23	8.58	8.60	8.45	8.25	8.24	8.31	8.65	8.94
12	10.00	10.05	9.71	9.29	8.66	8.70	8.58	8.42	8.47	8.62	9.10	9.51
13	10.00	10.08	9.76	9.35	8.75	8.81	8.71	8.59	8.70	8.93	9.54	10.08
14	10.00	10.10	9.80	9.42	8.83	8.91	8.84	8.76	8.92	9.24	9.99	10.66
15	10.00	10.13	9.85	9.48	8.92	9.01	8.97	8.93	9.15	9.55	10.44	11.23
16	10.00	10.15	9.90	9.54	9.00	9.12	9.10	9.11	9.38	9.86	10.88	11.80
17	10.00	10.18	9.95	9.60	9.09	9.22	9.23	9.28	9.61	10.17	11.33	12.38
18	10.00	10.20	9.99	9.67	9.17	9.32	9.36	9.45	9.84	10.47	11.77	12.95
19	10.00	10.23	10.04	9.73	9.26	9.42	9.48	9.62	10.06	10.78	12.22	13.52
20	10.00	10.25	10.09	9.79	9.35	9.53	9.61	9.79	10.29	11.09	12.66	14.10
21	10.00	10.27	10.14	9.86	9.43	9.63	9.74	9.96	10.52	11.40	13.11	14.67
22	10.00	10.30	10.19	9.92	9.52	9.73	9.87	10.14	10.75	11.71	13.55	15.25
23	10.00	10.32	10.23	9.98	9.60	9.84	10.00	10.31	10.98	12.02	14.00	15.82
24	10.00	10.35	10.28	10.04	9.69	9.94	10.13	10.48	11.21	12.33	14.44	16.39
25	10.00	10.37	10.33	10.11	9.77	10.04	10.26	10.65	11.43	12.64	14.89	16.97
26	10.00	10.40	10.38	10.17	9.86	10.14	10.39	10.82	11.66	12.94	15.34	17.54
27	10.00	10.42	10.42	10.23	9.94	10.25	10.52	11.00	11.89	13.25	15.78	18.11
28	10.00	10.45	10.47	10.30	10.03	10.35	10.65	11.17	12.12	13.56	16.23	18.69
29	10.00	10.47	10.52	10.36	10.11	10.45	10.78	11.34	12.35	13.87	16.67	19.26
30	10.00	10.50	10.57	10.42	10.20	10.55	10.91	11.51	12.57	14.18	17.12	19.83
31	10.00	10.52	10.61	10.49	10.28	10.66	11.04	11.68	12.80	14.49	17.56	20.41
32	10.00	10.54	10.66	10.55	10.37	10.76	11.17	11.86	13.03	14.80	18.01	20.98
33	10.00	10.57	10.71	10.61	10.45	10.86	11.30	12.03	13.26	15.11	18.45	21.56
34	10.00	10.59	10.76	10.67	10.54	10.97	11.42	12.20	13.49	15.41	18.90	22.13
35	10.00	10.62	10.81	10.74	10.63	11.07	11.55	12.37	13.72	15.72	19.35	22.70
36	10.00	10.64	10.85	10.80	10.71	11.17	11.68	12.54	13.94	16.03	19.79	23.28
37	10.00	10.67	10.90	10.86	10.80	11.27	11.81	12.71	14.17	16.34	20.24	23.85
38	10.00	10.69	10.95	10.93	10.88	11.38	11.94	12.89	14.40	16.65	20.68	24.42
39	10.00	10.72	11.00	10.99	10.97	11.48	12.07	13.06	14.63	16.96	21.13	25.00
40	10.00	10.74	11.04	11.05	11.05	11.58	12.20	13.23	14.86	17.27	21.57	25.57
41	10.00	10.77	11.09	11.11	11.14	11.69	12.33	13.40	15.09	17.58	22.02	26.14
42	10.00	10.79	11.14	11.18	11.22	11.79	12.46	13.57	15.31	17.88	22.46	26.72
43	10.00	10.81	11.19	11.24	11.31	11.89	12.59	13.75	15.54	18.19	22.91	27.29
44	10.00	10.84	11.23	11.30	11.39	11.99	12.72	13.92	15.77	18.50	23.36	27.87
45	10.00	10.86	11.28	11.37	11.48	12.10	12.85	14.09	16.00	18.81	23.80	28.44
46	10.00	10.89	11.33	11.43	11.56	12.20	12.98	14.26	16.23	19.12	24.25	29.01
47	10.00	10.91	11.38	11.49	11.65	12.30	13.11	14.43	16.45	19.43	24.69	29.59
48	10.00	10.94	11.43	11.55	11.73	12.41	13.23	14.61	16.68	19.74	25.14	30.16
49	10.00	10.96	11.47	11.62	11.82	12.51	13.36	14.78	16.91	20.05	25.58	30.73
50	10.00	10.99	11.52	11.68	11.91	12.61	13.49	14.95	17.14	20.35	26.03	31.31
51	10.00	11.01	11.57	11.74	11.99	12.71	13.62	15.12	17.37	20.66	26.47	31.88
52	10.00	11.04	11.62	11.81	12.08	12.82	13.75	15.29	17.60	20.97	26.92	32.45

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
53	10.00	11.06	11.66	11.87	12.16	12.92	13.88	15.46	17.82	21.28	27.37	33.03
54	10.00	11.08	11.71	11.93	12.25	13.02	14.01	15.64	18.05	21.59	27.81	33.60
55	10.00	11.11	11.76	11.99	12.33	13.12	14.14	15.81	18.28	21.90	28.26	34.18
56	10.00	11.13	11.81	12.06	12.42	13.23	14.27	15.98	18.51	22.21	28.70	34.75
57	10.00	11.16	11.85	12.12	12.50	13.33	14.40	16.15	18.74	22.52	29.15	35.32
58	10.00	11.18	11.90	12.18	12.59	13.43	14.53	16.32	18.96	22.82	29.59	35.90
59	10.00	11.21	11.95	12.25	12.67	13.54	14.66	16.50	19.19	23.13	30.04	36.47
60	10.00	11.23	12.00	12.31	12.76	13.64	14.79	16.67	19.42	23.44	30.48	37.04
61	10.00	11.26	12.04	12.37	12.84	13.74	14.92	16.84	19.65	23.75	30.93	37.62
62	10.00	11.28	12.09	12.43	12.93	13.84	15.05	17.01	19.88	24.06	31.38	38.19
63	10.00	11.31	12.14	12.50	13.01	13.95	15.17	17.18	20.11	24.37	31.82	38.76
64	10.00	11.33	12.19	12.56	13.10	14.05	15.30	17.36	20.33	24.68	32.27	39.34
65	10.00	11.35	12.24	12.62	13.19	14.15	15.43	17.53	20.56	24.99	32.71	39.91
66	10.00	11.38	12.28	12.69	13.27	14.26	15.56	17.70	20.79	25.29	33.16	40.49
67	10.00	11.40	12.33	12.75	13.36	14.36	15.69	17.87	21.02	25.60	33.60	41.06
68	10.00	11.43	12.38	12.81	13.44	14.46	15.82	18.04	21.25	25.91	34.05	41.63
69	10.00	11.45	12.43	12.87	13.53	14.56	15.95	18.21	21.48	26.22	34.49	42.21
70	10.00	11.48	12.47	12.94	13.61	14.67	16.08	18.39	21.70	26.53	34.94	42.78
71	10.00	11.50	12.52	13.00	13.70	14.77	16.21	18.56	21.93	26.84	35.38	43.35
72	10.00	11.53	12.57	13.06	13.78	14.87	16.34	18.73	22.16	27.15	35.83	43.93
73	10.00	11.55	12.62	13.13	13.87	14.98	16.47	18.90	22.39	27.46	36.28	44.50
74	10.00	11.58	12.66	13.19	13.95	15.08	16.60	19.07	22.62	27.76	36.72	45.07
75	10.00	11.60	12.71	13.25	14.04	15.18	16.73	19.25	22.84	28.07	37.17	45.65
76	10.00	11.62	12.76	13.31	14.12	15.28	16.86	19.42	23.07	28.38	37.61	46.22
77	10.00	11.65	12.81	13.38	14.21	15.39	16.99	19.59	23.30	28.69	38.06	46.80
78	10.00	11.67	12.86	13.44	14.29	15.49	17.11	19.76	23.53	29.00	38.50	47.37
79	10.00	11.70	12.90	13.50	14.38	15.59	17.24	19.93	23.76	29.31	38.95	47.94
80	10.00	11.72	12.95	13.57	14.47	15.70	17.37	20.11	23.99	29.62	39.39	48.52
81	10.00	11.75	13.00	13.63	14.55	15.80	17.50	20.28	24.21	29.93	39.84	49.09
82	10.00	11.77	13.05	13.69	14.64	15.90	17.63	20.45	24.44	30.23	40.29	49.66
83	10.00	11.80	13.09	13.75	14.72	16.00	17.76	20.62	24.67	30.54	40.73	50.24
84	10.00	11.82	13.14	13.82	14.81	16.11	17.89	20.79	24.90	30.85	41.18	50.81
85	10.00	11.85	13.19	13.88	14.89	16.21	18.02	20.97	25.13	31.16	41.62	51.38
86	10.00	11.87	13.24	13.94	14.98	16.31	18.15	21.14	25.35	31.47	42.07	51.96
87	10.00	11.89	13.28	14.01	15.06	16.41	18.28	21.31	25.58	31.78	42.51	52.53
88	10.00	11.92	13.33	14.07	15.15	16.52	18.41	21.48	25.81	32.09	42.96	53.11
89	10.00	11.94	13.38	14.13	15.23	16.62	18.54	21.65	26.04	32.40	43.40	53.68
90	10.00	11.97	13.43	14.19	15.32	16.72	18.67	21.82	26.27	32.70	43.85	54.25
91	10.00	11.99	13.48	14.26	15.40	16.83	18.80	22.00	26.50	33.01	44.30	54.83
92	10.00	12.02	13.52	14.32	15.49	16.93	18.93	22.17	26.72	33.32	44.74	55.40
93	10.00	12.04	13.57	14.38	15.57	17.03	19.05	22.34	26.95	33.63	45.19	55.97
94	10.00	12.07	13.62	14.45	15.66	17.13	19.18	22.51	27.18	33.94	45.63	56.55
95	10.00	12.09	13.67	14.51	15.75	17.24	19.31	22.68	27.41	34.25	46.08	57.12
96	10.00	12.12	13.71	14.57	15.83	17.34	19.44	22.86	27.64	34.56	46.52	57.69
97	10.00	12.14	13.76	14.64	15.92	17.44	19.57	23.03	27.87	34.87	46.97	58.27
98	10.00	12.16	13.81	14.70	16.00	17.55	19.70	23.20	28.09	35.17	47.41	58.84
99	10.00	12.19	13.86	14.76	16.09	17.65	19.83	23.37	28.32	35.48	47.86	59.42
100	10.00	12.21	13.90	14.82	16.17	17.75	19.96	23.54	28.55	35.79	48.31	59.99

C.2 Track of Optimal Value of DP II

Table C.2: Table of Track of Optimal Value of DP II

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
1	98	45	24	16	13	11	11	11	10	11	10	0
2	98	46	25	17	14	12	12	12	11	11	10	0
3	98	47	26	17	15	13	13	13	12	12	11	0
4	98	47	27	18	15	14	13	14	13	13	12	0
5	98	48	27	19	16	15	14	14	13	14	13	0
6	98	48	28	20	17	15	15	15	14	15	14	0
7	98	49	29	21	18	16	16	16	15	16	15	0
8	98	49	30	22	19	17	17	17	16	17	16	0
9	98	50	30	22	20	18	18	18	17	18	17	0
10	98	50	31	23	21	19	19	19	18	19	18	0
11	98	51	32	24	21	20	20	20	19	19	18	0
12	98	51	33	25	22	21	20	20	20	20	19	0
13	98	52	33	26	23	22	21	21	21	21	20	0
14	98	53	34	26	24	22	22	22	21	22	21	0
15	98	53	35	27	25	23	23	23	22	23	22	0
16	98	54	35	28	26	24	24	24	23	24	23	0
17	98	54	36	29	27	25	25	25	24	25	24	0
18	98	55	37	30	27	26	26	26	25	26	25	0
19	98	55	38	31	28	27	27	26	26	27	26	0
20	98	56	38	31	29	28	28	27	27	27	26	0
21	98	56	39	32	30	29	28	28	28	28	27	0
22	98	57	40	33	31	29	29	29	28	29	28	0
23	98	57	41	34	32	30	30	30	29	30	29	0
24	98	58	41	35	32	31	31	31	30	31	30	0
25	98	59	42	36	33	32	32	32	31	32	31	0
26	98	59	43	36	34	33	33	32	32	33	32	0
27	98	60	44	37	35	34	34	33	33	34	33	0
28	98	60	44	38	36	35	35	34	34	34	34	0
29	98	61	45	39	37	36	36	35	35	35	34	0
30	98	61	46	40	38	36	36	36	35	36	35	0

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
31	98	62	46	40	38	37	37	37	36	37	36	0
32	98	62	47	41	39	38	38	38	37	38	37	0
33	98	63	48	42	40	39	39	38	38	39	38	0
34	98	63	49	43	41	40	40	39	39	40	39	0
35	98	64	49	44	42	41	41	40	40	41	40	0
36	98	64	50	45	43	42	42	41	41	41	41	0
37	98	65	51	45	44	42	43	42	42	42	42	0
38	98	66	52	46	44	43	43	43	42	43	42	0
39	98	66	52	47	45	44	44	44	43	44	43	0
40	98	67	53	48	46	45	45	44	44	45	44	0
41	98	67	54	49	47	46	46	45	45	46	45	0
42	98	68	55	50	48	47	47	46	46	47	46	0
43	98	68	55	50	49	48	48	47	47	48	47	0
44	98	69	56	51	50	49	49	48	48	49	48	0
45	98	69	57	52	50	49	50	49	49	49	49	0
46	98	70	57	53	51	50	51	50	49	50	50	0
47	98	70	58	54	52	51	51	50	50	51	50	0
48	98	71	59	54	53	52	52	51	51	52	51	0
49	98	71	60	55	54	53	53	52	52	53	52	0
50	98	72	60	56	55	54	54	53	53	54	53	0
51	98	72	61	57	56	55	55	54	54	55	54	0
52	98	73	62	58	56	56	56	55	55	56	55	0
53	98	74	63	59	57	56	57	56	56	56	56	0
54	98	74	63	59	58	57	58	56	56	57	57	0
55	98	75	64	60	59	58	58	57	57	58	58	0
56	98	75	65	61	60	59	59	58	58	59	58	0
57	98	76	66	62	61	60	60	59	59	60	59	0
58	98	76	66	63	62	61	61	60	60	61	60	0
59	98	77	67	64	62	62	62	61	61	62	61	0
60	98	77	68	64	63	63	63	62	62	62	62	0
61	98	78	68	65	64	63	64	62	62	63	63	0
62	98	78	69	66	65	64	65	63	63	64	64	0
63	98	79	70	67	66	65	66	64	64	65	65	0
64	98	79	71	68	67	66	66	65	65	66	66	0
65	98	80	71	68	68	67	67	66	66	67	66	0
66	98	80	72	69	68	68	68	67	67	68	67	0
67	98	81	73	70	69	69	69	68	68	69	68	0
68	98	82	74	71	70	70	70	68	69	69	69	0
69	98	82	74	72	71	70	71	69	69	70	70	0
70	98	83	75	73	72	71	72	70	70	71	71	0
71	98	83	76	73	73	72	73	71	71	72	72	0
72	98	84	77	74	74	73	73	72	72	73	73	0
73	98	84	77	75	74	74	74	73	73	74	73	0
74	98	85	78	76	75	75	75	74	74	75	74	0
75	98	85	79	77	76	76	76	74	75	75	75	0

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
76	98	86	80	78	77	77	77	75	76	76	76	0
77	98	86	80	78	78	77	78	76	76	77	77	0
78	98	87	81	79	79	78	79	77	77	78	78	0
79	98	87	82	80	79	79	80	78	78	79	79	0
80	98	88	82	81	80	80	81	79	79	80	80	0
81	98	88	83	82	81	81	81	80	80	81	80	0
82	98	89	84	82	82	82	82	80	81	82	81	0
83	98	90	85	83	83	83	83	81	82	82	82	0
84	98	90	85	84	84	84	84	82	83	83	83	0
85	98	91	86	85	85	84	85	83	83	84	84	0
86	98	91	87	86	85	85	86	84	84	85	85	0
87	98	92	88	87	86	86	87	85	85	86	86	0
88	98	92	88	87	87	87	88	86	86	87	87	0
89	98	93	89	88	88	88	88	86	87	88	87	0
90	98	93	90	89	89	89	89	87	88	88	88	0
91	98	94	91	90	90	90	90	88	89	89	89	0
92	98	94	91	91	91	90	91	89	89	90	90	0
93	98	95	92	92	91	91	92	90	90	91	91	0
94	98	95	93	92	92	92	93	91	91	92	92	0
95	98	96	93	93	93	93	94	92	92	93	93	0
96	98	96	94	94	94	94	95	92	93	94	93	0
97	98	97	95	95	95	95	96	93	94	95	94	0
98	98	98	96	96	96	96	96	94	95	95	95	0
99	98	98	96	96	96	97	97	95	96	96	96	0
100	98	99	97	97	97	97	98	96	96	97	97	0

C.3 The Optimal Total Wealth of DP II

Unit: USD $\times$ 1,000

Table C.3: Table of The Optimal Total Wealth of DP II

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
1	297.12	268.09	242.38	225.07	209.41	196.01	183.44	166.97	149.50	129.29	107.67	83.38
2	297.12	268.31	243.29	225.70	210.09	196.83	184.42	168.24	150.95	130.92	109.66	85.79
3	297.12	269.05	243.68	226.24	211.30	198.15	185.56	169.82	152.56	132.90	111.85	88.21
4	297.12	269.24	244.59	226.79	211.97	199.47	186.50	170.99	154.33	134.87	114.03	90.62
5	297.12	269.97	244.97	227.86	213.01	200.40	187.54	172.09	155.68	136.85	116.22	93.03
6	297.12	270.17	245.39	228.55	213.82	201.19	188.97	173.50	157.46	138.81	118.40	95.45
7	297.12	270.40	245.82	229.12	214.57	202.20	190.41	175.08	159.24	140.77	120.59	97.86
8	297.12	270.60	246.31	230.19	215.78	203.52	191.84	176.66	161.01	142.72	122.77	100.27
9	297.12	270.82	246.69	230.72	218.32	204.39	193.11	178.24	162.79	146.84	124.96	105.25
10	297.12	271.02	247.09	231.32	219.22	205.71	194.09	181.57	164.55	148.79	127.14	107.67
11	297.12	271.60	248.00	231.87	219.88	207.02	195.52	183.15	166.31	150.42	129.13	110.08
12	297.12	271.80	248.63	232.93	220.57	207.85	196.46	184.25	167.76	152.21	131.32	112.49
13	297.12	272.04	249.02	233.83	221.28	209.17	197.89	185.51	169.53	154.17	133.50	114.91
14	297.12	272.31	249.92	234.37	222.49	209.96	199.33	186.68	170.88	156.12	135.69	117.32
15	297.12	272.51	250.83	235.15	223.69	211.28	200.37	188.24	172.47	158.08	137.86	119.73
16	297.12	272.73	251.21	235.78	224.90	212.43	201.49	189.79	174.23	160.03	140.02	122.15
17	297.12	272.92	251.62	236.35	225.65	213.36	202.93	191.35	176.00	161.99	142.19	124.56
18	297.12	273.65	252.02	237.42	226.31	214.23	203.91	192.74	177.76	163.93	146.51	126.97
19	297.12	273.85	252.47	237.97	227.49	215.54	205.34	193.85	181.27	165.87	148.68	129.39
20	297.12	274.08	252.86	238.50	228.31	218.20	206.78	195.40	183.03	167.50	150.67	131.80
21	297.12	274.28	253.41	239.55	228.99	219.21	207.71	196.96	184.48	169.44	152.83	134.21
22	297.12	274.56	253.84	240.59	230.17	220.00	209.15	198.52	185.83	171.22	155.00	136.63
23	297.12	274.76	254.60	241.29	230.88	220.83	210.58	199.69	187.57	173.16	157.16	139.04
24	297.12	274.98	254.98	241.88	231.55	222.15	211.85	200.93	189.31	175.10	159.33	141.45
25	297.12	275.70	255.48	242.93	232.72	223.46	212.89	202.49	191.05	177.05	161.49	146.02
26	297.12	275.90	255.88	243.46	233.74	224.78	213.87	203.60	192.63	180.74	163.66	148.44
27	297.12	276.22	256.78	244.51	234.63	225.64	215.30	205.15	194.37	182.68	165.82	150.85
28	297.12	276.42	257.16	245.06	235.38	226.94	218.07	206.71	196.11	184.31	167.99	153.26
29	297.12	276.64	257.57	245.63	236.07	227.86	219.20	208.27	197.85	186.25	169.98	155.68
30	297.12	276.84	259.73	246.26	237.25	228.66	220.13	209.83	199.20	188.19	172.15	158.09
31	297.12	277.29	260.11	246.80	237.91	229.95	221.57	211.22	200.63	190.12	174.31	160.50
32	297.12	277.49	261.02	247.84	239.07	230.77	223.00	212.39	202.37	191.90	176.48	162.92
33	297.12	277.73	261.41	248.62	240.23	232.06	224.43	213.49	204.11	193.84	180.39	165.33
34	297.12	277.93	262.32	249.66	241.04	233.19	225.41	215.05	205.85	195.77	182.55	167.74
35	297.12	278.15	262.70	250.70	241.75	234.19	226.82	217.94	207.60	197.71	184.71	170.16

Cont.

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
36	297.12	278.35	263.34	251.26	242.91	235.06	227.86	219.19	209.34	199.34	186.88	172.57
37	297.12	278.56	263.77	251.79	244.07	235.85	229.27	220.74	210.91	201.28	189.04	174.98
38	297.12	279.28	264.22	252.39	244.74	237.15	230.20	222.30	212.26	203.22	191.03	179.15
39	297.12	279.47	264.60	253.08	245.42	238.42	231.61	223.86	214.00	205.16	193.19	181.56
40	297.12	279.85	265.01	253.65	246.17	239.69	232.85	224.96	217.08	207.10	195.35	183.97
41	297.12	280.05	265.41	254.55	247.33	240.61	233.97	226.49	218.51	209.04	197.51	186.39
42	297.12	280.30	266.31	255.18	248.22	241.44	234.95	227.66	220.25	210.81	199.67	188.80
43	297.12	280.50	266.70	255.72	249.38	242.71	236.36	229.19	221.99	212.75	201.83	191.21
44	297.12	280.71	267.19	256.76	250.54	243.98	237.74	230.72	223.73	216.02	203.99	193.63
45	297.12	280.91	267.74	257.31	251.20	244.78	239.13	232.08	225.44	217.65	206.15	196.04
46	297.12	281.62	268.13	259.60	251.91	245.64	240.17	233.33	226.79	219.59	208.32	198.45
47	297.12	281.82	269.03	260.64	252.72	246.91	241.11	234.43	228.51	221.53	210.31	200.87
48	297.12	282.52	269.92	261.18	253.41	247.91	242.49	235.96	230.23	223.46	212.47	203.28
49	297.12	282.72	270.32	262.22	254.42	249.18	243.88	237.47	231.77	225.37	215.96	205.69
50	297.12	282.94	270.71	263.00	255.17	250.45	244.86	238.98	233.21	227.29	218.13	208.11
51	297.12	283.14	271.45	263.57	256.32	251.27	246.24	240.15	234.92	229.20	220.29	210.52
52	297.12	283.35	271.86	264.16	256.99	252.20	247.36	241.66	236.62	230.95	222.45	214.27
53	297.12	283.59	272.29	264.71	259.40	252.99	248.75	243.17	238.32	232.58	224.61	216.68
54	297.12	283.79	272.68	265.25	260.56	254.12	250.13	244.27	239.67	234.49	226.77	219.09
55	297.12	284.47	273.57	266.29	261.71	254.98	251.07	245.78	241.36	236.39	228.93	221.51
56	297.12	284.67	273.96	266.92	262.60	256.25	252.11	247.03	243.06	238.28	230.92	223.92
57	297.12	284.95	274.41	267.61	263.29	258.77	253.34	248.54	244.76	240.18	233.08	226.33
58	297.12	285.14	274.80	268.65	263.99	260.04	254.32	250.05	246.19	242.07	235.23	228.75
59	297.12	285.36	275.68	269.69	264.66	261.31	255.70	251.21	247.89	243.97	237.37	231.16
60	297.12	285.55	276.17	270.22	265.82	262.31	258.34	252.57	249.58	245.60	239.51	233.57
61	297.12	285.87	276.56	271.11	266.56	263.10	259.72	253.67	250.93	247.49	241.65	235.98
62	297.12	286.07	277.19	271.66	267.37	263.92	261.11	255.18	252.48	249.39	243.79	238.40
63	297.12	286.29	277.60	272.23	268.53	265.19	262.23	257.95	254.17	251.14	245.94	240.81
64	297.12	286.49	277.99	273.26	269.68	266.05	263.16	259.46	257.12	253.03	248.08	243.22
65	297.12	286.70	278.38	273.79	270.68	266.97	264.55	260.97	258.82	256.18	250.07	245.64
66	297.12	286.90	279.27	274.38	271.34	268.24	265.52	262.21	260.51	258.07	252.22	248.05
67	297.12	287.58	279.82	275.41	272.03	269.50	266.56	263.72	261.94	259.97	255.61	250.46
68	297.12	288.15	280.24	276.03	273.17	270.61	267.95	264.82	263.64	261.60	257.75	254.13
69	297.12	288.35	280.63	276.80	273.87	271.41	269.32	265.98	264.99	263.49	259.89	256.54
70	297.12	288.60	281.52	277.35	275.01	272.67	270.55	267.49	266.68	265.39	262.04	258.96
71	297.12	288.80	282.39	277.89	275.76	273.49	271.93	269.00	268.37	267.28	264.18	261.37
72	297.12	289.01	282.79	278.92	276.64	274.75	272.87	270.35	269.91	269.02	266.32	263.78
73	297.12	289.20	283.17	279.61	277.31	275.61	274.25	271.85	271.60	270.91	268.32	266.20
74	297.12	289.44	283.58	280.17	278.45	276.61	275.22	273.36	273.29	272.79	270.46	268.61
75	297.12	289.64	284.44	281.20	279.25	277.87	276.34	274.46	274.72	274.42	272.60	271.02

Cont.

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
76	297.12	291.35	284.89	282.21	279.93	278.79	277.72	275.70	276.40	276.31	274.74	273.44
77	297.12	291.55	285.27	282.75	281.07	279.59	278.76	277.21	277.75	278.20	276.89	275.85
78	297.12	291.78	285.76	283.30	282.20	280.85	280.13	278.37	279.44	280.08	279.01	278.26
79	297.12	291.97	286.16	284.30	282.87	282.10	281.50	279.87	281.12	281.93	281.12	280.68
80	297.12	292.18	286.54	284.88	283.98	283.33	282.84	281.36	282.77	283.79	283.23	283.09
81	297.12	292.38	287.40	285.51	284.68	284.15	283.78	282.83	284.43	285.53	285.23	286.53
82	297.12	292.84	288.14	286.05	285.43	285.01	284.76	283.93	285.97	288.41	288.37	288.94
83	297.12	293.21	288.56	287.04	286.54	286.23	286.10	285.41	288.66	290.04	290.48	291.36
84	297.12	293.41	288.95	287.92	287.53	287.35	287.33	286.76	290.09	291.90	292.59	293.77
85	297.12	294.09	289.36	288.49	288.21	288.14	289.71	289.26	291.44	293.76	294.70	296.18
86	297.12	294.29	291.25	289.04	288.88	290.40	290.82	290.50	293.09	295.61	296.81	298.60
87	297.12	294.50	291.64	291.06	291.02	291.40	291.86	291.67	294.75	297.47	298.92	301.01
88	297.12	294.70	292.03	291.60	291.90	292.32	293.21	293.14	296.41	299.33	301.03	303.42
89	297.12	294.98	292.66	292.37	292.70	293.55	294.14	294.24	298.06	301.18	303.02	305.84
90	297.12	295.17	293.20	293.05	293.81	294.37	295.12	295.71	299.72	302.81	305.14	308.25
91	297.12	295.49	294.06	294.05	294.51	295.23	296.46	297.18	301.38	304.55	307.25	310.66
92	297.12	295.69	294.44	294.64	295.26	296.02	297.81	298.65	302.73	306.41	309.36	313.08
93	297.12	295.90	294.89	295.26	295.92	297.25	299.15	300.12	304.15	308.27	311.47	315.49
94	297.12	296.09	295.38	295.80	296.60	298.48	300.50	301.59	305.69	310.12	313.58	317.90
95	297.12	296.32	295.76	296.35	297.71	299.71	301.84	302.75	307.35	311.98	315.68	320.32
96	297.12	296.52	296.16	296.91	298.82	300.93	302.88	303.85	309.01	313.84	317.68	322.73
97	297.12	296.75	296.57	297.91	299.93	302.16	303.85	305.09	310.66	315.68	319.78	325.14
98	297.12	297.01	297.00	298.90	301.04	303.08	304.79	306.45	312.32	317.31	321.88	327.56
99	297.12	297.20	297.38	299.44	301.71	303.94	305.91	307.92	313.97	319.04	323.98	329.97
100	297.12	297.42	298.24	300.43	302.82	304.74	307.14	309.39	315.32	320.88	326.08	332.38

C.4 The Optimal Asset Allocation of P4

Table C.4: Table of The Optimal Asset Allocation of P4

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
23	0.00	0.00	0.00	0.00	0.00	0.00	0.00	5.44	37.07	29.40	29.34	26.71
24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
25	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
26	0.00	0.00	0.00	0.00	31.54	34.11	35.01	35.91	32.03	31.94	17.96	31.21
27	27.15	46.45	40.17	35.82	0.00	20.78	0.00	0.00	0.00	0.00	35.01	30.60
28	0.00	0.00	0.00	0.00	13.86	0.00	0.00	0.00	0.00	0.00	0.00	0.00
29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	37.01

Cont.

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9	t = 10	t = 11
31	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	12.39
33	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	44.59	43.30	39.15
34	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
36	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
37	0.00	8.94	32.84	44.72	44.09	46.16	42.23	38.39	37.78	31.80	32.37	27.67
38	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
39	0.00	0.00	0.00	0.00	0.00	0.00	50.58	55.75	46.42	52.21	46.69	36.09
40	72.85	56.58	50.09	52.39	52.74	51.19	35.51	41.47	41.95	26.52	38.90	32.54
41	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
42	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

□ **End of chapter.**

Bibliography

[1] X. Cai, K.-L. Teo, X. Yang, and X. Y. Zhou: Portfolio Optimization Under a Minimax Rule, Management Science, 46(7): 957 - 972, July, 2000.

[2] X. T. Deng, Z. F. Li, S. Y. Wang: A Minimax Portfolio Selection Strategy with Equilibrium: European Journal of Operational Research, Vol. 166, pp. 278-292, 2005.

[3] G. B. Dantzig, and P. W. Glynn: Parallel Processors for Planning under Uncertainty, Annual Operation Research, Vol. 22, pp. 1-21, 1990.

[4] G. B. Dantzig, and A. Madansky: On the Solution of Two-Stage Linear Programs under Uncertainty, Proc. 4th Berkeley Symposium in Math. Statistics and Probability, Vol. 1, University of California Press, Berkeley, CA, pp. 165-176, 1961.

[5] M. A. H. Dempster: On Stochastic Programming: II, Dynamic Problems under Risk, Stochastics, Vol. 25, No. 1, pp.

15-42, 1988.

[6] Xiao-Tie Deng, Zhong-Fei Li, Shou-Yang Wang: A Minimax Portfolio Selection Strategy with Equilibrium: European Journal of Operational Research, Vol. 155, pp. 278 - 292, 2005.

[7] Y. Ermoliev, and R. J.-B. Wets (Eds.), Nutmerical Techniquies for Stochastic Optimization, Springer-Verlag, Berlin and New York, 1988.

[8] K Fan: Minimax Theorems, Proceedings of National Academy of Science, Vol. 39, pp. 42 - 47.

[9] H. Gassmann: MSLiP: A Computer Code for the Multistage Stochastic Linear Programming Problem, Mathematical Programming 47 pp. 407-423, 1990.

[10] H. Konno: Piecewise linear risk function and portfolio optimization, J. Oper. Res. Soc. Janpan, Vol. 33, pp. 139 - 156, 1990.

[11] H. Konno and H. Yamazaki: Mean-absolute Deviation Portfolio Optimization Model and Its Application to Tokyo Stock Market, Management Science, Vol. 37, pp. 519 - 531, 1991.

[12] D. Li and W.-L. Ng: Optimal dynamic portfolio selection: Multi-period mean-variance formulation, Mathematical Finance, Vol. 10, No. 3, pp. 387-406, 2000.

[13] H. Markowitz: Portfolio Selection: Efficient Diversification of Investments, John Wiley and Sons, New York, 1959.

[14] Jan Mossin: Optimal Multiperiod Portfolio Policies, Journal of Business, Vol. 41, No. 2, pp. 221 - 223, 1968.

[15] J. M. Mulvey: Nonlinear Network Models In Finance, Advances in Mathematical Programming and Financial Planning, JAI Press, Vol. 1, pp. 253, 1987.

[16] J. M. Mulvey and H. Vladimirou: Stochastic Network Optimization Models for Investment Planning, Annals of Operations Research, Vol. 20, pp. 187-127, 1989.

[17] J. M. Mulvey and H. Vladimirou: Applying the Progressive Hedging Algorithm to Stochastic Generalized Networks, Statistics and Operations Research Series Report, SOR-89-9, Princeton University, Princeton, New Jersey, 08544.

[18] A. Nagurney: Variational inequalities in the analysis and computation of multi-sector, multi-instrument financial equilibria, J Econ Dynam Control 18, pp. 161C184, 1994.

[19] A. Nagurney, and J. Dong: Supernetworks: Decision-making for the information age, Edward Elgar, Cheltenham, UK, 2002.

[20] A. Nagurney, and S. Siokos: Financial networks: Statics and dynamics, Springer-Verlag, Berlin, Germany, 1997.

[21] Nitin R. Patel, and Marti G. Subrahmanyam: A Simple Algorithm for Optimal Portfolio Selection with Fixed Transaction Costs, Management Scient, Vol. 28, pp. 303 - 314, 1982.

[22] G. A., Pogue: An Extension of the Markowitz Portfolio Selection Model to Include Variable Transaction Costs, Short Sales, Leverage Policies and Taxes," J. Finance, Vol. 25, No. 5, December 1970, pp. 1005 - 1028, 1970.

[23] K. L. Teo, X. Q. Yang: Portfolio Selection Problem with Minimax Type Risk Function, Annals of Operations Research, Vol. 101, pp. 333 - 349, 2001.

[24] P. Varaiya, and R. J.-B. Wets: Stochastic Dynamic Optimization Approaches and Computation, Working Paper WP-88-87, IIASA, Laxenburg, Australia, 1988.

[25] D. W. Walup, and R. J.-B. Wets: Stochastic Programs with Recourse, SIAM J. Appl. Math., 14, pp. 89-105, 1966.

[26] Shou-Yang Wang, Y. Yamamoto and Mei Yu: A Minimax Rule for Portfolio Selection in Frictional Markets: Math. Meth. Oper. Res., Vol. 57, pp. 141 - 155, 2003

[27] M.R. Young: A Minimax Portfolio Selection Rule with Linear Programming Solution, Management Science, Vol. 44, pp. 673 - 683, 1998.

[28] Mei Yu, Shou-Yang Wang, Kin Keugh Lai and X. Chao: Multiperiod Portfolio Selection on a Minimax Rule, Dynamics of Continuous Discrete and Impulsive Systems Series B: Application and Algorithms, Vol. 12, pp. 565 - 587, 2005.

[29] P. L. Yu: Cone Convexity, Cone Extreme Points, and Nondominated Solutions in Decision Problems with Multiobjectives, Journal Optimizal Theory and Applition, Vol. 14, pp. 319 - 376.

[30] M. Zeleny: Multiple Criteria Decision Making, McGraw-Hill, New Yord, 1981.