Robust Output Synchronization for Complex Nonlinear Systems

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Abstract

In recent years, the synchronization problem has received increasing attention due to its applications in various areas. So far, the existing results of the synchronization problem are based on the assumption that the master and slave systems have the same dimension and similar formation. This thesis will study synchronization problem of two complex nonlinear systems with quite different formations. We consider the synchronization problem as the output regulation problem, and solve it via internal model approach. Based on the existing framework of output regulation, which transfers the robust output regulation problem into a robust stabilization problem, we handle both local and nonlocal synchronization problems.

This thesis mainly consists of two parts. In the first part, we will consider the local robust output synchronization problem of two systems with different dimensions. Chua's circuit is considered as the slave system whose output tracks the output of the Van der Pol oscillator asymptotically. The control law is designed based on the internal model approach arising from the framework of the robust output regulation. We extend the approach of the local robust output regulation with nonlinear exosystems proposed in [6], and overcome two difficulties. The first one is the solvability of a set of nonlinear partial differential equations called regulator equations, and the second one is to establish a proper internal model with nonlinear exosystems. And this approach can also be used for the synchronization of other complex nonlinear master-slave systems.

In the second part of the thesis, we will propose a method for nonlocal synchronization problem and apply it to two applications: Duffing oscillator synchronizing with Chua's circuit and the SMIB power system synchronizing with Van der Pol oscillator. The major results are summarized as follows

(i) Based on the framework in [23], the solutions of regulator equations for the two examples can be found.

(ii) The boundedness of the master system is estimated and can be covered by the effective control region.

(iii) The solvability of the synchronization problem is transformed into the solvability of a stabilization problem of a non-autonomous system.

摘 要

近年来，随着同步控制在各领域的广泛应用，同步问题已成为非线性科学中的前沿热门课题之一，并逐渐受到更多学者的关注。现有的研究结果主要解决了两个相似同维系统的同步问题，本文针对两个结构不同的复杂的非线性系统的同步问题提出新的方法，将同步问题看作输出调节问题，结合内模原理来处理局部和非局部的同步问题。

本文由两部分组成。第一部分考虑主从系统的局部鲁棒输出同步问题。这里，主从系统是两个具有不同维数的非线性复杂系统，分别为Van der Pol系统和Chua电路。基于内模原理，结合[6]中提出的外系统为非线性系统的局部鲁棒输出调节方法来处理该类同步问题。我们主要克服了两个难题：其一，找到所谓调节器的一组非线性偏微分方程的闭型解。其二，设计一个合适的针对非线性外系统的内模。该部分提出的方法，同样适用于其它复杂的非线性主从系统同步问题。

在本文的第二部分中，我们提出了一种针对非局部鲁棒输出同步问题的处理方法，并应用到两个例子中，分别为Duffing振子与Chua电路的同步问题，以及单机无穷大(SMIB)电力系统与Van der Pol系统同步问题。这部分的主要结论可以概括为：

(i) 基于[23]提出的解决输出调节问题框架，分别找到两个例子中调节器的闭型解。

(ii) 证明并估算出主系统的界，并使其包含在有效控制区间内。

(iii) 将同步问题转化为时变系统的镇定问题。

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Contents

Chapter 1

Introduction

Synchronization of two systems has various applications such as secure communications, optimization of nonlinear system performance, modeling brain activities and pattern recognition phenomena. Typically, these studies are based on the assumption that the slave system is some system either with the same dimension as the master system or with a similar formation to the master system in order to derive an error equation. However when the two systems have different dimensions, it is not possible to transfer the synchronization problem into a stabilization problem by deriving an error equation. As a result, different techniques for synchronization are needed. In this thesis, we will apply the output regulation technique to the synchronization problem with unknown parameters. This chapter provides the overviews of theoretical backgrounds of synchronization and output regulation technique.

This chapter is organized as follows. Section 1.1 gives an overview of the synchronization problem. Section 1.2 reviews the development of output regulation and its recent research direction is given. In Section 1.3, some typical nonlinear systems are introduced. And Section 1.4 closes this chapter with the organization of this thesis.

1.1 Synchronization of Master-slave Systems

Synchronous motion was first reported by Huygens(1673), where he describes an experiment of two pendulum clocks hanging on a light weighted beam, and which exhibit frequency synchronization after a short period of time. In main practical applications of synchronization theory, the synchronous motion is the result of artificial couplings. The definition of the output synchronization is as follows. The master system is

$$\begin{aligned} \dot{v} &= a(v) \\ q &= \Phi(v) \end{aligned} \tag{1.1}$$

where $v \in R^q$ are the states and $q \in R$ is the output of the master system.

And the slave system is

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= H(x) \end{aligned} \tag{1.2}$$

where $x \in R^n$ are the states, $y \in R$ is the output of the slave system, and $u \in R$ is the control input. Synchronization occurs if, no matter how (1.1) and (1.2) are initialized, their outputs will match asymptotically

$$\lim_{t \to \infty} |y(t) - q(t)| = 0.$$

In [44], Nijmeijer points out that the synchronization problem deserves to be studied from a control perspective, which provides a basis of using a wide variability of control techniques for the synchronization research. Since then, many control algorithms such as linear feedback control [15], adaptive control [36], [51] and observed-based synchronization [?] have been applied on the synchronization problem. A framework of synchronization is proposed in [50] which unifies many results of control of dynamic systems, where the main tool is Lyapunov direct method. All these results are under the assumption that the master system has a similar formation to the slave system to derive an error equation.

Synchronization of two systems with quite different dynamics becomes an important control problem and arouses great interests in recent years, in particular, when the synchronization of various chaotic systems has found many applications, e.g., in the field of secure communication [7], [40]. In fact, the synchronization problem of dynamic systems can be considered as an output regulation problem where the master system is regarded as the exosystem, and the slave system as the given plant. In this thesis we will apply the technique of output regulation to the synchronization problem and handle some applications.

1.2 Output Regulation

The output regulation problem is one of the most fundamental problems in control theory. Briefly, the output regulation problem is to design a control law for a plant, such that the closed-loop system is internally stable, and the output of the closed-loop system asymptotically tracks a reference input produced by an autonomous differential equation called exosystem.

Various versions of the output regulation problem have been extensively studied via dynamic output feedback [24], [25], [32], [33]. A key solvability condition of the output regulation problem is that the exosystem is linear and neurally stable. Recently, In [6], Chen and Huang studied the local robust output regulation problem for nonlinear systems with the exosystem described by

$$\dot{v} = a(v) = A_1 v + \sum_{k=2}^{N} A_k v a_k(v) \tag{1.3}$$

where the matrices $A_k \in R^{q\times q}$ for $k = 1, 2, \cdots, N$, and $a_k(v) : R^q \to R$ is a smooth function with $a_k(0) = 0$. In particular, equation (1.3) is reduced to $\dot{v} = A_1 v$ when $a_k(v) \equiv 0$. Then the output regulation problem with complex nonlinear exosystems could be solved and also it provides an alternative technique for the synchronization problem with a nonlinear master system which could be considered as the exosystem. In this thesis, we will apply the approach studied in [6] to the synchronization problem and extend it to nonlocal robust synchronization problem.

The idea to solve the output regulation problem is to convert it into a stabilization problem. To tell whether a nonlinear output regulation problem is solvable or not, first, we need to consider the solvability of the regulator equations. The solutions of the regulator equations in fact characterize the steady states of the system. If we could find the steady states of the system, it is possible to convert the output regulation problem into a stabilization problem. For the stabilization problem, research is case by case due to the complexity of nonlinear systems. In fact, the stabilization problem itself is a challenging topic.

A general framework for tackling the robust output regulation problem is proposed in [21]. Under this framework, the output regulation for a given plant can be systematically converted into a stabilization problem for an appropriately defined augmented system. This general framework has been successfully applied to solve the output regulation problem of nonlinear systems. An alternative framework which converts the original problem into a stabilization problem of an extended augmented system is given in [23]. In this thesis, we will apply the framework in [23] to our synchronization problem and point out why the general framework in [21] is not suitable for our problem.

1.3 Typical Nonlinear Systems

In this section, we will introduce three well-known nonlinear systems to be studied in this thesis, namely, Chua's circuit, Duffing oscillator and the SMIB power system.

Chua's circuit was originally conceived by Chua in 1983 for generating chaotic response in a nonlinear circuit [8]. Later, the circuit has been modified by various researchers leading to what is called the Chua's circuit family. Chua's circuit has been a test-bed for studying various control problems such as chaos cancellation, stabilization, and synchronization [2], [11], [12], [38]. And the circuit studied in this thesis is taken from [53].

Duffing oscillator, a well-known nonlinear system presents itself in many physical, engineering and biological problems [14]. Originally the model was introduced by the German electrical engineer Duffing in 1918. Various control methodologies have been developed to control Duffing oscillator such as the state feedback control [10], an adaptive backstepping method [29], the input-output feedback linearizable control [40], integral-observer-based chaos synchronization [28], [30].

Power systems usually involve a high degree of nonlinearity. The recent availability of inexpensive computer power and progress in nonlinear system theories allow one to understand and analyze the complex behaviors in the power system. In [34], Menikov's technique is used to analyze chaotic motions in the two-degree-of-freedom swing equations. Chaos in a single-machine-infinite-bus system (SMIB) is studied in [43] by perturbation techniques. In [1], Abed employs the Hopf bifurcation theory to explain nonlinear oscillatory behaviors in this power system. Usually, the dynamics of the SMIB is modeled as a two-dimensional differential equation of power systems of electromechanical energy devices [52]. In this thesis, the SMIB power system is considered as the slave system of the synchronization problem.

1.4 Organization

The remaining chapters of this thesis are organized as follows:

Chapter 2: The thesis starts from the local synchronization problem of Chua's circuit and Van der Pol oscillator. We first apply the output regulation technique via internal model approach to reformulate the problem as a robust output regulation problem and then solve the local robust output regulation problem.

Chapter 3: In this chapter, we will discuss solvability of the output regulation problem with nonlinear exosystems based on the framework of output regulation proposed in [23]. An alternative framework of output regulation is studied in [21]. And we will compare the pros and cons of the two different existing frameworks for solving the output regulation problems.

Chapter 4: The technique of output regulation is used to tackle two practical robust synchronization problems, Duffing oscillator synchronizing with Chua's circuit and the chaotic SMIB power system synchronizing with Van der Pol oscillator. Simulations are given to evaluate the control strategies.

Chapter 5: Finally, some concluding remarks and recommendations for the further research are given.

The thesis is accompanied by many examples with numerical simulations based on MATLAB.

□ **End of chapter.**

Chapter 2

Synchronization of Chua's Circuit and Van der Pol Oscillator via Internal Model Approach

This Chapter considers the output synchronization of Chua's circuit and Van der Pol oscillator. We first reformulate the problem as the robust output regulation problem with Chua's circuit as the plant and Van del Pol oscillator as the exosystem and then solve the robust output regulation problem via internal model approach.

This chapter is organized as follows: Section 2.1 gives an introduction to this output synchronization problem. Section 2.2 will reformulate the problem as the robust output regulation problem which accounts for the parameter uncertainty of Chua's circuit. The summary of the framework for output regulation problem with nonlinear exosystems will be given in Section 2.3. And in Section 2.4, our approach is applied to the output synchronization of Chua's circuit and Van der Pol oscillator with evaluation by computer simulation.

2.1 Introduction

In this chapter, we consider the problem of controlled output synchronization of Chua's circuit as the slave system and Van der Pol oscillator as the master system. Shown in Figure 2.1 is Chua's circuit whose controlled output synchronization with some other systems has been studied in several papers [36], [44], [51]. Typically, these papers assume

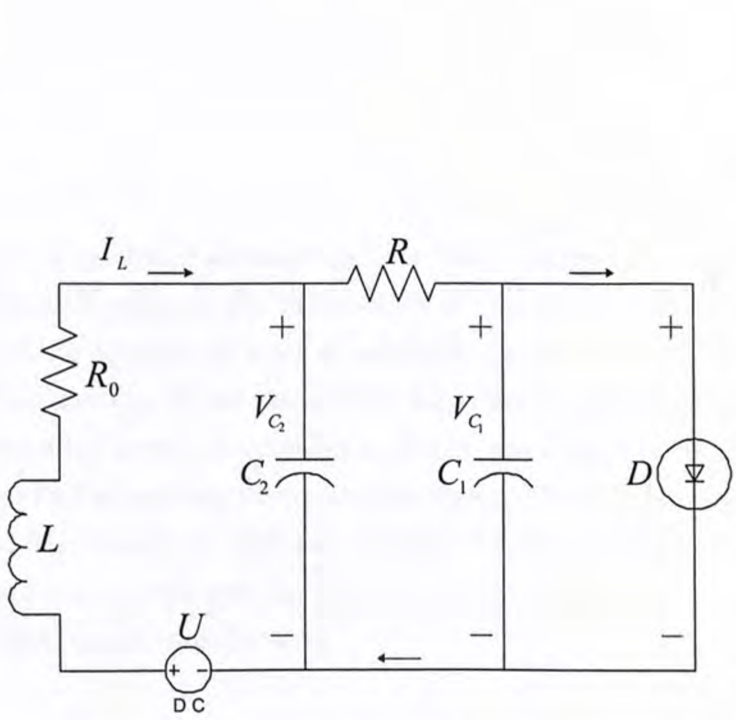

Figure 2.1: The Chua's circuit.

that the master system is either some system with the same dimension as Chua's circuit or a linearly neurally stable autonomous system. Under the first assumption, it is possible to derive an error equation whose stabilization solution leads to the solution of the synchronization of two systems. Under the second assumption, the problem can be treated by early results of the output regulation theory [26]. In this chapter, the master system is a nonlinear system with an unstable equilibrium at the origin and the dimension of the master system is different from that of the slave system. As a result, the controlled synchronization of these two systems also poses some specific difficulties. To handle our problem, we will adopt the internal model approach. The internal model approach has been developed for solving the robust output regulation problem. The approach was first developed in 1970's for solving the linear robust output regulation problem in, e.g., [9], [13], and is now in the process of being extended to solving the nonlinear robust output regulation problem [4], [21], [23]. Conceptually, an internal model associated with a plant and an exosystem is a dynamic compensator attachment of which to the given plant leads to an augmented system whose stabilization solution leads to the output regulation solution of the given plant and exosystem [23]. To date, various sufficient conditions for the existence of an internal model have been established. In particular, in [6], a set of sufficient conditions for the existence of an internal model where the exosystem is nonlinear is given. We will show, in Section 2.2, that the controlled synchronization problem of this chapter can be reformulated into a robust output regulation problem with Chua's circuit as the plant and Van del pol oscillator as the exosystem. Nevertheless, the successful application of the framework in [6] relies on the satisfaction of two key conditions. The first one is the availability of the solution of a set of nonlinear partial differential equations called regulator equations and the second one is that the solution of the regulator equations has to satisfy what is called immersion condition. When the exosystem is nonlinear, there is no systematic results for verifying these two conditions. Thus, in applying the framework of [6], we need to first overcome these two difficulties and we indeed succeed in doing so. As a result, we are able to construct an internal model and an output feedback controller to solve the problem under consideration.

2.2 Problem Formulation

The dynamic equation of the Chua's circuit is adopted from [53] and given as follows:

$$\begin{aligned} \dot{V}_{c1} &= C_1^{-1}[R^{-1}(V_{c2} - V_{c1}) - f(V_{c1})] \\ \dot{V}_{c2} &= C_2^{-1}[R^{-1}(V_{c2} - V_{c1}) + I_L] \\ \dot{I}_L &= L^{-1}[V_{c2} - R_0 I_L + u] \end{aligned} \tag{2.1}$$

where V_{c1} and V_{c2} are voltage across the capacitors C_1 and C_2, respectively, I_L is the current flowing through the inductor L, u is an independent voltage source, and $f(V_{c1})$ is the current flowing through the nonlinear resistor D_1. As in [53], we assume that $f(V_{c1})$ is a cubic function $a_1 V_{c1} + a_3 V_{c1}^3$ with $a_1 < 0,\ a_3 > 0$. Figure 2.2 shows that Chua's circuit displays chaos. The dynamic equation of Van der Pol oscillator is described as follows:

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -a v_1 + b(1 - v_1)^2 v_2. \end{aligned} \tag{2.2}$$

The phase portrait of the system with $a = 1,\ b = 1$ is shown in Figure 2.3. It is well-known that when $a > 0,\ b > 0$, the system has a stable limit cycle. Our problem is to design a feedback control law such that the solution of the closed-loop system is bounded for sufficiently small initial states and the difference of the output V_{c1} of Chua's circuit and the output v_1 of Van der Pol system approaches the origin asymptotically.

Let us first show that the above problem can be reformulated as the robust output regulation problem studied in [6]. For this purpose, letting $(x_1, x_2, x_3) = (V_{c1}, V_{c2}, I_L)$ and

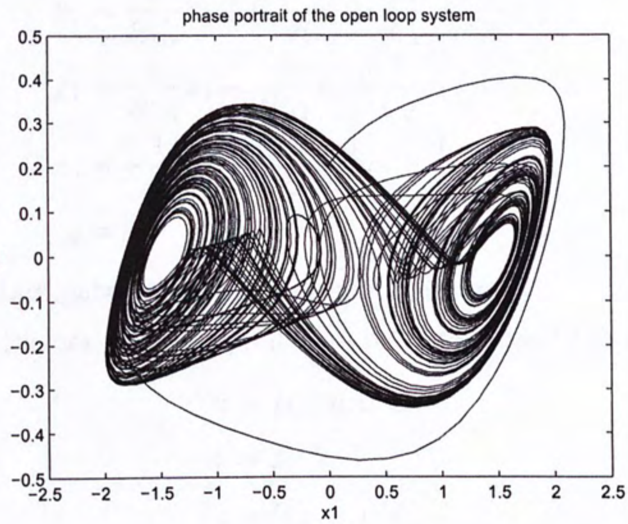

Figure 2.2: The chaotic trajectory of Chua's circuit with

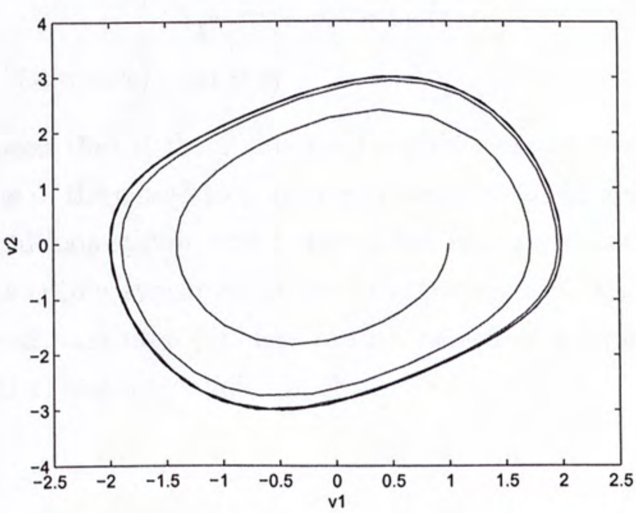

Figure 2.3: The limit cycle of the Van der Pol oscillator

$y = x_1$ put equation (2.1) in the following form:

$$\begin{aligned}\dot{x}_1 &= \frac{1}{RC_1}x_2 + (-\frac{1}{RC_1} - \frac{a_1}{C_1} - \frac{a_3}{C1}x_1^2)x_1 \\ \dot{x}_2 &= \frac{1}{RC_2}x_1 - \frac{1}{RC_2}x_2 + \frac{1}{C_2}x_3 \\ \dot{x}_3 &= -\frac{1}{L}x_2 - \frac{(R_0 + w)}{L}x_3 + \frac{1}{L}u \\ y &= x_1\end{aligned} \tag{2.3}$$

where w is an uncertain parameter with nominal value zero.

Now letting $x = [x_1,\ x_2,\ x_3]^T$, $v = [v_1,\ v_2]^T$ put equations (2.2) and (2.3) as follows

$$\begin{aligned}\dot{x} &= f(x, u, v, w) \\ \dot{v} &= a(v) \\ e &= h(x, u, v, w)\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}f(x, u, v, w) &= \begin{bmatrix}\frac{1}{RC_1}x_2 + (-\frac{1}{RC_1} - \frac{a_1}{C_1} - \frac{a_3}{C1}x_1^2)x_1 \\ \frac{1}{RC_2}x_1 - \frac{1}{RC_2}x_2 + \frac{1}{C_2}x_3 \\ -\frac{1}{L}x_2 - \frac{(R_0+w)}{L}x_3 + \frac{1}{L}u\end{bmatrix} \\ a(v) &= \begin{bmatrix}v_2 \\ -av_1 + b(1 - v_1)^2 v_2\end{bmatrix} \\ h(x, u, v, w) &= x_1 - v_1.\end{aligned}$$

Thus it can be seen that if there exists a feedback control law depending on (e, v) such that the solution of the closed-loop system is bounded for all sufficiently small initial condition of the closed-loop system and e approaches zero asymptotically, then the same control law solves the output synchronization of the two systems described above, i.e., the robust controlled synchronization problem can be viewed as a robust output regulation problem of system (2.4) which is studied in [6].

2.3 Preliminaries

In this section, we view the system (2.4) as a general nonlinear system where x is the n-dimensional plant state, u the m-dimensional plant input, e the p-dimensional plant output representing the tracking error, v the q-dimensional exogenous signal, and w the n_w-dimensional unknown constant parameter with nominal value 0. The basic idea of handling the robust output regulation problem is to convert the robust output regulation problem of a given system into a robust stabilization problem of an augmented system composed of the given plant and a dynamic compensator called internal model. The following set of sufficient conditions guarantees the existence of the augmented system.

Assumption 1 There exist sufficiently smooth functions $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$ with $\mathbf{x}(0,0) = 0$ and $\mathbf{u}(0,0) = 0$ satisfying the following regulator equations for all $v \in V$, and $w \in W$ with V and W open neighborhoods of the origins of the respective Euclidean spaces

$$\begin{aligned} \frac{\partial \mathbf{x}(v,w)}{\partial w} a(v) &= f(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w) \\ 0 &= h(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w). \end{aligned} \tag{2.5}$$

∎

Assumption 2 For some integer s, there exists a triple (θ, α, β), where $\theta : R^q \times R^{n_w} \longmapsto R^s$, $\alpha : R^s \times R^q \longmapsto R^s$, and $\beta : R^s \times R^q \longmapsto R^m$ are sufficiently smooth functions vanishing at the origin, such that, for all $v \in V$, and all $w \in W$

$$\begin{aligned} \frac{\partial \theta(v,w)}{\partial v} a(v) &= \alpha(\theta(v,w), v) \\ \mathbf{u}(v,w) &= \beta(\theta(v,w), v). \end{aligned} \tag{2.6}$$

∎

Remark 2.3.1 Equations (2.5) are called regulator equations and the solvability of these equations is a necessary condition for the solvability of the output regulation problem [26]. Since equations (2.5) are nonlinear partial differential equations, obtaining the solution of the regulator equations (2.5) has been one of the major difficulties in the applicability of the output regulation theory. Later on, we will give an explicit solution of the regulator equations associated with the Chua's circuit and Van del Pol system. The triple (θ, α, β) defines what is called a steady-state generator for (2.4) in [6]. Assumption 2 further requires that the solution of the regulator equations can be produced by some dynamic system (2.6), which can also be called immersion condition [4] . When the exosystem is nonlinear, finding such a steady-state generator can also be a challenge. ∎

Satisfaction of Assumptions 1 and 2 guarantees the existence of an internal model defined as follows.

Definition 2.3.1 : Assume system (2.4) satisfies Assumptions 1 and 2. Let $\gamma : R^s \times R^m \times R^q \longmapsto R^s$ be some sufficiently smooth function vanishing at the origin. An internal model of system (2.4) is a dynamic compensator of the following form:

$$\dot{\eta} = \gamma(\eta, u, v) \tag{2.7}$$

with the property that for all $v \in R^q$ and all $w \in R^{n_w}$,

$$\alpha(\theta(v, w), v) = \gamma(\theta(v, w), \mathbf{u}(v, w), v). \tag{2.8}$$

∎

Attaching the internal model (2.7) to the given system (2.4) leads to an augmented system as follows

$$\begin{aligned} \dot{\eta} &= \gamma(\eta, u, v) \\ \dot{x} &= f(x, u, v, w) \\ \dot{e} &= h(x, u, v, w). \end{aligned} \tag{2.9}$$

Performing the coordinate and input transformation $z_i = x_i - \mathbf{x}_i(v, w)$, for $i = 1, \cdots, n$, $\bar{\eta} = \eta - \theta(v, w), \bar{u} = u - \beta(\eta, v)$ gives a new system denoted by

$$\begin{aligned} z &= \bar{f}(z, \bar{\eta}, \bar{u}, v, w) \\ \bar{\eta} &= \bar{\gamma}(\bar{\eta}, \bar{u}, v, w) \\ \dot{e} &= \bar{h}(z, \bar{\eta}, \bar{u}, v, w). \end{aligned} \tag{2.10}$$

The augmented system (2.10) has the following property

$$\begin{aligned} \bar{f}(0, 0, 0, v(t), w) &= 0 \\ \bar{\gamma}(0, 0, v(t), w) &= 0 \\ \bar{h}(0, 0, 0, v(t), w) &= 0 \end{aligned}$$

for all trajectories $v(t) \in V$ of the exosystem and all sufficiently small w. Thus it can be seen that, if there exists a controller of the form

$$\begin{aligned} \bar{u} &= k(\xi, e) \\ \dot{\xi} &= \zeta(\xi, e) \end{aligned} \tag{2.11}$$

where $\xi \in \Re^z$, and k, ζ are sufficiently smooth functions vanishing at their origins, that exponentially stabilizes the equilibrium $(\bar{\eta}, \bar{x}) = (0,0)$ of the augmented system (2.10), then the following controller

$$\begin{aligned} u &= \beta_u(\eta, v) + k(\xi, e) \\ \dot{\eta} &= \gamma(\eta, u, e, v) \\ \dot{\xi} &= \zeta(\xi, e) \end{aligned} \tag{2.12}$$

solves the robust output regulation problem of the given plant (2.4) [6].

Remark 2.3.2 : Assumption 2 and Definition 2.3.1 show that both the steady state generator and the internal model are allowed to depend on the exogenous signal v. As a result, the exogenous signal v is allowed to appear in the control law, too. Such a control scheme can handle the case where the exogenous signal v is a reference input or a measurable disturbance.

∎

2.4 Solvability of the Problem

In this section, we will apply the framework summarized in the last section to convert the regulation problem into a stabilization problem. For this purpose , we need to perform three tasks. First, solve the regulator equations of the system (2.4); Second, determine the existence of the steady-state generator (2.6); Third, find a particular internal model of system (2.4) such that the equilibrium of system (2.10) is at least locally stabilizable. Finally, we will design a controller to achieve the synchronization of the output of Chua's circuit and Van der Pol oscillator with the uncertain parameter w in Chua's circuit.

2.4.1 The solution of the regulator equations

Now we will solve the regulator equation.

$$\dot{x}_1 = \frac{1}{RC_1}x_2 + (-\frac{1}{RC_1} - \frac{a_1}{C_1} - \frac{a_3}{C1}x_1^2)x_1 \tag{2.13}$$
$$\dot{x}_2 = \frac{1}{RC_2}x_1 - \frac{1}{RC_2}x_2 + \frac{1}{C_2}x_3 \tag{2.14}$$
$$\dot{x}_3 = -\frac{1}{L}x_2 - \frac{(R_0+w)}{L}x_3 + \frac{1}{L}u \tag{2.15}$$
$$\dot{v}_1 = v_2 \tag{2.16}$$
$$\dot{v}_2 = -v_1 + (1-v_1)^2 v_2. \tag{2.17}$$

For $h(x,u,v,w) = x_1 - v_1$, we obtain $\mathbf{x}_1(v,w) = v_1$, and $\dot{\mathbf{x}}_1(v,w) = \dot{v}_1 = v_2$. Substituting $\mathbf{x}_1(v,w)$ and $\dot{\mathbf{x}}_1(v,w)$ into equation (2.13) gives

$$\begin{aligned}\mathbf{x}_2(v,w) &= RC_1[\dot{\mathbf{x}}_1(v,w) + (\frac{1}{RC_1} + \frac{a_1}{C_1} + \frac{a_3}{C_1}\mathbf{x}_1(v,w)^2)\mathbf{x}_1(v,w)] \\ &= RC_1 v_2 + (1+a_1 R)v_1 + a_3 R v_1^3.\end{aligned} \tag{2.18}$$

We can obtain $\mathbf{x}_3(v,w) = C_2\dot{\mathbf{x}}_2(v,w) - \frac{1}{R}\mathbf{x}_1(v,w) + \frac{1}{R}\mathbf{x}_2(v,w)$ from the equation (2.14) and $\mathbf{u}(v,w) = L\dot{\mathbf{x}}_3(v,w) + \mathbf{x}_2(v,w) + (R_0+w)\mathbf{x}_3(v,w)$ from the equation (2.15).

The solution of the regulator equations takes the following form

$$\begin{aligned}\mathbf{x}_1(v,w) &= v_1 \\ \mathbf{x}_2(v,w) &= c_{21}v_1 + c_{22}v_2 + c_{230}v_1^3 \\ \mathbf{x}_3(v,w) &= c_{31}v_1 + c_{32}v_2 + c_{330}v_1^3 + c_{312}v_1v_2^2 \\ \mathbf{u}(v,w) &= m_1v_1 + m_2v_2 + m_{30}v_1^3 + m_{21}v_1^2v_2 \\ &\quad + m_{12}v_1v_2^2 + m_{41}v_1^4v_2 + r_1(w)v_1 \\ &\quad + r_2(w)v_2 + r_{30}(w)v_1^3 + r_{21}(w)v_1^2v_2\end{aligned}$$

where c_{21}, c_{22}, c_{230}, c_{31}, c_{32}, c_{330}, c_{312}, m_1, m_2, m_{30}, m_{21}, m_{12}, m_{41} are coefficients depending on $R, C_1, C_2, L, R_0, a_1, a_3$ but independent of w, while $r_1(w), r_2(w), r_{30}(w), r_{21}(w)$ are coefficients depending on the uncertain parameter w.

It can be seen that the solution $\mathbf{u}(v,w)$ can be written as follows

$$\mathbf{u}(v,w) = \mathbf{u}_c(v) + \hat{\mathbf{u}}(v,w)$$

where $\mathbf{u}_c(v) = m_1v_1 + m_2v_2 + m_{30}v_1^3 + m_{21}v_1^2v_2 + m_{12}v_1v_2^2 + m_{41}v_1^4v_2$ vanishing at the origin, and $\hat{\mathbf{u}}(v,w) = r_1(w)v_1 + r_2(w)v_2 + r_{30}(w)v_1^3 + r_{21}(w)v_1^2v_2$.

2.4.2 Steady-state generator

To find a steady-state generator, we need to recall some results in [6]. First note that the master system is Van de Pol oscillator which can be rewritten as follows:

$$\dot{v} = \begin{bmatrix} v_2 \\ -av_1 + b(1 - v_1^2)v_2 \end{bmatrix} = A_1 v + A_2 v a_2(v) \tag{2.19}$$

where $A_1 = \begin{bmatrix} 0 & 1 \\ -a & b \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -b \end{bmatrix}$, $a_2(v) = v_1^2$ and $v = \mathrm{col}(v_1, v_2)$. Since $\hat{\mathbf{u}}(v, w)$ is a polynomial in v, for some integer s, there exist sufficiently smooth function $\tau : R^q \times R^{n_w} \longmapsto R^s$, vanishing at the origin, and matrices $\Phi_1 \in R^{s \times s}$ and $\Psi \in R^{1 \times s}$ such that, for all $v \in R^q$, and all $w \in R^{n_w}$ [23],

$$\begin{aligned} \frac{\partial \tau(v, w)}{\partial v} A_1 v &= \Phi_1 \tau(v, w), \\ \hat{\mathbf{u}}(v, w) &= \Psi \tau(v, w). \end{aligned} \tag{2.20}$$

Remark 2.4.1 Assume that (2.20) is satisfied by some τ, Φ_1, and Ψ. By a result in [6], if there exists a matrix Φ_2 such that

$$\frac{\partial \tau(v, w)}{\partial v} A_2 v \quad = \quad \Phi_2 \tau(v, w) \tag{2.21}$$

then there exists a function $\phi(v) = \Phi_1 + \Phi_2 a_2(v)$ satisfying

$$\begin{aligned} \frac{\partial \tau(v, w)}{\partial v} a(v) &= \phi(v) \tau(v, w) \\ \hat{\mathbf{u}}(v, w) &= \Psi \tau(v, w). \end{aligned} \tag{2.22}$$

Equation (2.22) can be viewed as a steady-state generator with $\theta = \tau$, $\alpha(\theta, v) = a(v)\tau$, and $\beta(\theta, v) = \Psi\tau$. ∎

We will now proceed to find τ, $\phi(v)$, and Ψ satisfying (2.22). For this purpose, define

$$\begin{aligned} v^{[1]} &= (v_1,\ v_2)^T \\ v^{[3]} &= (v_1^3,\ v_1^2 v_2,\ v_1 v_2^2,\ v_2^3)^T. \end{aligned}$$

Then it follows from a result in [21] that there exist square matrices $A^{[i]}$, $i = 1, 3$, such that

$$\frac{\partial v^{[i]}}{\partial v} A_1 v = A^{[i]} v^{[i]}. \tag{2.23}$$

In fact,

$$\begin{aligned}\frac{\partial v^{[1]}}{\partial v} A_1 v &= A_1 v \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} v \\ &= A^{[1]} v^{[1]}\end{aligned}$$

so

$$A^{[1]} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Also

$$\begin{aligned}\frac{\partial v^{[3]}}{\partial v} A_1 v &= \frac{\partial v^{[3]}}{\partial v} A_1 v \\ &= \begin{bmatrix} 3v_1^2 & 0 \\ 2v_1 v_2 & v_1^2 \\ v_2^2 & 2v_1 v_2 \\ 0 & 3v_2^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} v \\ &= \begin{bmatrix} 3v_1^2 v_2 \\ -v_1^3 + v_1^2 v_2 + 2v_1 v_2^2 \\ -2v_1^2 v_2 + 2v_1 v_2^2 + v_2^3 \\ -3v_1 v_2^2 + 3v_2^3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} v_1^3 \\ v_1^2 v_2 \\ v_1 v_2^2 \\ v_2^3 \end{bmatrix} \\ &= A^{[3]} v^{[3]},\end{aligned}$$

so

$$A^{[3]} = \begin{bmatrix} 0 & 3 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix}.$$

Define π_1, π_3 as follows,

$$\pi_1 = r_1 v_1 + r_2 v_2 = F_1 v^{[1]}$$
$$\pi_3 = r_{30} v_1^3 + r_{21} v_1^2 v_2 + r_{12} v_1 v_2^2 + r_{03} v_2^3 = F_3 v^{[3]}$$

where $F_1 = [r_1,\ r_2]$, $F_3 = [r_{30},\ r_{21},\ r_{12},\ r_{03}]$, and $\hat{\mathbf{u}}(v,w) = \pi_1 + \pi_3$. Define

$$\tau_1(v,w) = [\pi_1,\ L_{A_1 v}\pi_1]^T$$
$$\tau_3(v,w) = \left[\pi_3,\ L_{A_1 v}\pi_3,\ L^2_{A_1 v}\pi_3,\ L^3_{A_1 v}\pi_3\right]^T.$$

where $L^k_{A_1 v}\pi_i = \frac{\partial L^{k-1}_{A_1 v}\pi_i(v,w)}{\partial v} A_1 v = F_i (A^{[i]})^k v^{[i]}$.

Now we need to obtain $\tau_1(v,w)$ and $\tau_3(v,w)$. First calculating

$$\begin{aligned} L_{A_1 v}\pi_1 &= \frac{\partial \pi_1}{\partial v} A_1 v \\ &= \frac{\partial F_1 v_1}{\partial v} A_1 v \\ &= F_1 A^{[1]} v^{[1]} \end{aligned}$$

gives

$$\tau_1(v,w) = \begin{bmatrix} \pi_1 \\ L_{A_1 v}\pi_1 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_1 A^{[1]} \end{bmatrix} v^{[1]}. \tag{2.24}$$

Next calculating

$$\begin{aligned} L_{A_1 v}\pi_3 &= \frac{\partial \pi_3}{\partial v} A_1 v \\ &= \frac{\partial F_3 v^{[3]}}{\partial v} A_1 v \\ &= F_3 \frac{\partial v^{[3]}}{\partial v} A_1 v \\ &= F_3 A^{[3]} v^{[3]} \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} L^{(2)}_{A_1 v}\pi_3 &= F_3 (A^{[3]})^2 v^{[3]} \\ L^{(3)}_{A_1 v}\pi_3 &= F_3 (A^{[3]})^3 v^{[3]}, \end{aligned} \tag{2.26}$$

gives

$$\tau_3(v,w) = \begin{bmatrix} \pi_3 \\ L_{A_1 v}\pi_3 \\ L^{(2)}_{A_1 v}\pi_3 \\ L^{(3)}_{A_1 v}\pi_3 \end{bmatrix} = \begin{bmatrix} F_3 \\ F_3 A^{[3]} \\ F_3 (A^{[3]})^2 \\ F_3 (A^{[3]})^3 \end{bmatrix} v^{[3]} \tag{2.27}$$

Using the method in [23] shows the existence of matrices $\Phi_1^{[i]}$ and $\Psi^{[i]}$, $i = 1, 3$, such that,

$$\begin{aligned} \frac{\partial \tau_i(v,w)}{\partial v} A_1 v &= \Phi_1^{[i]} \tau_i(v,w) \\ \pi_i(v,w) &= \Psi^{[i]} \tau_i(v,w). \end{aligned}$$

In fact, we have

$$\Phi_1^{[1]} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \ \Phi_1^{[3]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -27 & 36 & -21 & 6 \end{bmatrix}.$$

and $\Psi^{[1]} = [1\ 0]$, $\Psi^{[3]} = [1\ 0\ 0\ 0]$.

Now letting $\tau(v,w) = \left[\tau_1^T(v,w), \tau_3^T(v,w)\right]^T$, $\Phi_1 = \text{diag}[\Phi_1^{[1]},\ \Phi_1^{[3]}]$, $\Psi = [1\ 0\ 1\ 0\ 0\ 0]$ verifies equations (2.20).

Next, we need to find Φ_2 to satisfy (2.21). For this purpose, it suffices to find matrices $\Phi_2^{[i]}$, $i = 1, 3$, such that

$$\frac{\partial \tau_i(v,w)}{\partial v} A_2 v = \Phi_2^{[i]} \tau_i(v,w) \tag{2.28}$$

In fact, note that, for $i = 1, 3$,

$$\frac{\partial v^{[i]}}{\partial v} A_2 v = P^{[i]} v^{[i]} \tag{2.29}$$

where

$$P^{[1]} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \ P^{[3]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Substituting τ_1 in (2.28) with $i = 1$ gives

$$\begin{bmatrix} F_1 \\ F_1 A^{[1]} \end{bmatrix} A_2 v^{[1]} = \Phi_2^{[1]} \begin{bmatrix} F_1 \\ F_1 A^{[1]} \end{bmatrix} v^{[1]}.$$

Thus,

$$\Phi_2^{[1]} = \begin{bmatrix} F_1 \\ F_1 A^{[1]} \end{bmatrix} P^{[1]} \begin{bmatrix} F_1 \\ F_1 A^{[1]} \end{bmatrix}^{-1}.$$

Similarly, we can obtain $\Phi_2^{[3]}$ as follows:

$$\Phi_2^{[3]} = \begin{bmatrix} F_3 \\ F_3A^{[3]} \\ F_3(A^{[3]})^2 \\ F_3(A^{[3]})^3 \end{bmatrix} P^{[3]} \begin{bmatrix} F_3 \\ F_3A^{[3]} \\ F_3(A^{[3]})^2 \\ F_3(A^{[3]})^3 \end{bmatrix}^{-1}.$$

Letting $\Phi_2 = \begin{bmatrix} \Phi_2^{[1]} & 0 \\ 0 & \Phi_2^{[3]} \end{bmatrix}$ verifies (2.21). It follows from Remark 2.4.1 that τ, $\phi(v)$, and Ψ satisfy (2.22).

Remark 2.4.2 Once (2.22) is established, we can further find a family of steady-state generators. In fact, for $i = 1, 3$, let $\theta_i = T_i\tau_i(v, w)$ with $T_i \in \mathbb{R}^{i\times i}$ any nonsingular matrix. Then

$$\begin{aligned} \frac{\partial\theta_i(v,w)}{\partial v}\dot{v} &= \frac{\theta_i(v,w)}{\partial v}(A_1v + A_2va_2(v)) \\ &= T_i\phi_i(v)T_i^{-1}\theta_i(v,w) \\ &\triangleq \alpha_i(\theta_i(v,w), v) \end{aligned}$$

where $\phi_i(v) = \Phi_1^{[i]} + \varphi_i(v)$ and $\varphi_i(v) = \Phi_2^{[i]}a_2(v)$.

$$\pi_i(v,w) = \Psi_iT_i^{-1}\tau_i(v,w) \triangleq \beta_i(\theta_i(v,w), v)$$

and $\theta = \begin{bmatrix} \theta_1 \\ \theta_3 \end{bmatrix}$, $\alpha(\theta, v) = \begin{bmatrix} \alpha_1(\theta_1, v) \\ \alpha_3(\theta_3, v) \end{bmatrix}$ $\beta(\theta, v) = \begin{bmatrix} \beta_1(\theta_1, v) + \beta_3(\theta_3, v) \end{bmatrix}$.

Then, it is ready to verify that the triple

$$(\theta,\ \alpha(\theta, v),\ \beta(\theta, v))$$

is a family of steady state generators of system (2.4) parameterized by T. In the next section, it will be seen that, by appropriate choosing T, we can obtain a particular internal model. ∎

2.4.3 Internal model

Once the steady-state generator is available, it is possible to construct an internal model of the form (2.7) such that the linearization of the augmented system (2.10) at the origin is stabilizable and detectable [6]. For this purpose, pick any controllable pairs (M_i, N_i), for $i = 1, 3$, as follows:

$$M_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Then, there exists a unique nonsingular matrix T_i, such that the following Sylvester equation holds

$$T_i \Phi_1^{[i]} - M_i T_i = N_i \Psi^{[i]} \tag{2.30}$$

and the internal model with the output u is as follows

$$\dot{\eta} = \gamma(\eta, u, v) = M\eta + T\varphi(v)T^{-1}\eta + Nu. \tag{2.31}$$

with

$$\begin{aligned} \eta &= col(\eta_1,\ \eta_3) \\ T &= diag(T_1,\ T_3) \\ M &= diag(M_1,\ M_3) \\ N &= col(N_1,\ N_3) \end{aligned}$$

The augmented system is the combination of (2.3) and the internal model (2.31).

2.4.4 Stabilization

Applying the following coordinate and input transformation $z = x - \mathbf{x}(v, w), \overline{\eta} = \eta - \theta(v, w)$, $\overline{u} = u - \beta(\eta)$ on the augmented system which is the combination of systems (2.3) and (2.31) gives

$$\begin{aligned}
\dot{z}_1 &= \frac{1}{RC_1}z_2 + (-\frac{1}{RC_1} - \frac{a_1}{C_1})z_1 \\
&\quad - \frac{a_3}{C1}(z_1^3 + 3z_1^2 x_1(v,w) + 3z_1 x_1(v,w)^2) \\
\dot{z}_2 &= \frac{1}{RC_2}z_1 - \frac{1}{RC_2}z_2 + \frac{1}{C_2}z_3 \\
\dot{z}_3 &= -\frac{1}{L}z_2 - \frac{R_0 + w}{L}z_3 + \frac{1}{L}(\overline{u} + \Psi T^{-1}\overline{\eta}) \\
\dot{\overline{\eta}} &= (M + N\Psi T^{-1})\overline{\eta} + N\overline{u} \\
e &= Cz.
\end{aligned} \tag{2.32}$$

The linearization of the augmented system at the origin ($\overline{\eta} = 0,\ \bar{x} = 0,\ \overline{u} = 0$) with v and w being set to zero is

$$\begin{aligned}
\dot{z} &= Az + B\overline{u} + B\Psi T^{-1}\overline{\eta} \\
\dot{\overline{\eta}} &= (M + N\Psi T^{-1})\overline{\eta} + N\overline{u} \\
e &= C_m \begin{bmatrix} z \\ \overline{\eta} \end{bmatrix}
\end{aligned}$$

where

$$A = \begin{bmatrix} -\frac{1}{RC_1} - \frac{a_1}{C_1} & \frac{1}{RC_1} & 0 \\ \frac{1}{RC_2} & -\frac{1}{RC_1} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & -\frac{R_0}{L} \end{bmatrix},\ B \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix},$$

$$C_m = [\,1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\,].$$

Find matrices K and P such that

$$\begin{bmatrix} A & B\Psi T^{-1} \\ 0 & M + N\Psi T^{-1} \end{bmatrix} + \begin{bmatrix} B \\ N \end{bmatrix} K \tag{2.33}$$

and

$$\begin{bmatrix} A & B\Psi T^{-1} \\ 0 & M + N\Psi T^{-1} \end{bmatrix} - PC_m \tag{2.34}$$

are Hurwitz. Then a linear output feedback controller of the system (2.32) can be given as follows

$$\overline{u} = -K\xi$$

$$\dot{\xi} = \begin{bmatrix} A & B\Psi T^{-1} \\ 0 & M + N\Psi T^{-1} \end{bmatrix} \xi + \begin{bmatrix} B \\ N \end{bmatrix} K\xi + P[x_1 - \xi_1]$$

The output feedback control law solves the robust output regulation problem for system (2.32)

$$
\begin{aligned}
\hat{u} &= \Psi T^{-1}\eta - K\xi \\
\dot{\eta} &= M\eta + T\phi(v)T^{-1}\eta + N(\Psi T^{-1}\eta - K\xi) \\
\dot{\xi} &= \begin{bmatrix} A & B\Psi T^{-1} \\ 0 & M + N\Psi T^{-1} \end{bmatrix} \xi + \begin{bmatrix} B \\ N \end{bmatrix} K\xi + P[x_1 - \xi_1].
\end{aligned}
$$

Consequently, the following output feedback control law

$$
\begin{aligned}
u &= u_c + \Psi T^{-1}\eta - K\xi \\
\dot{\eta} &= M\eta + T\phi(v)T^{-1}\eta + N(\Psi T^{-1}\eta - K\xi) \\
\dot{\xi} &= \begin{bmatrix} A & B\Psi T^{-1} \\ 0 & M + N\Psi T^{-1} \end{bmatrix} \xi + \begin{bmatrix} B \\ N \end{bmatrix} K\xi + P[x_1 - \xi_1].
\end{aligned}
$$

solves the original synchronization problem.

2.4.5 Simulation

Computer simulation has been conducted for showing the synchronization performance of Chua's circuit and Van der Pol oscillator with the eigenvalues of the matrix (2.33) as

$$[-28.5 \ -30 \ -18 \ -24 \ -2 \ -28 \ -.9 \ -10 \ -20.9].$$

The eigenvalues of the matrix (2.34) are

$$[-38 \ -24.5 \ -8 \ -1.2 \ -7 \ -26.5 \ -0.6 \ -20 \ -22].$$

Parameters of the system are $R = 1,\ R_0 = 0.1,\ C_1 = 1/9.5,\ C_2 = 1,\ L = 0.07,\ a_1 = -8/7,\ a_3 = 4/63$. Fig 2.4 to 2.7 show the simulation results when initial states of the closed-loop system and exosystem are $x(0) = [0.5,\ 0.3,\ 1],\ \xi(0) = 0,\ \eta(0) = 0,\ v(0) = [1,\ 0]$ with $w = 0.5R_0$. And Fig 2.8 to 2.11 show the simulation results when initial states of the closed-loop system and exosystem are $x(0) = [2,\ -1,\ 0.5]$, and $\xi(0) = 0,\ \eta(0) = 0,\ v(0) = [1,\ 0]$ with $w = -0.5R_0$.

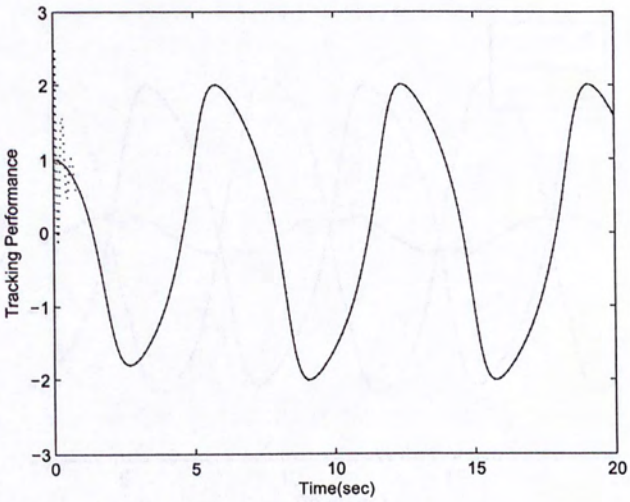

Figure 2.4: Tracking performance of the controlled circuit when $w = 0$

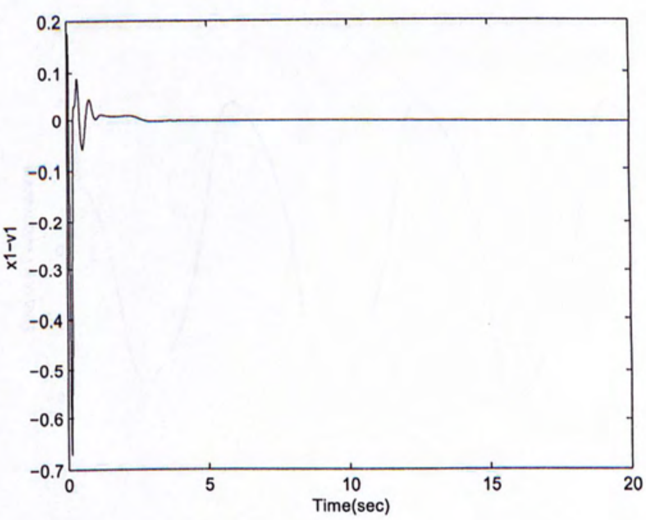

Figure 2.5: Tracking error when $w = 0$

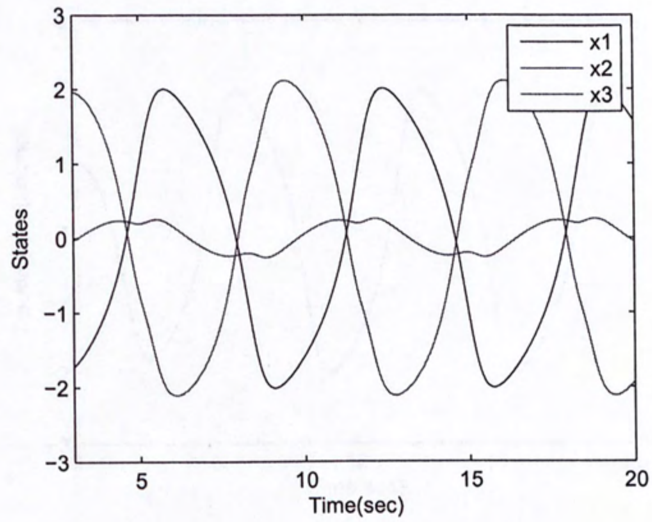

Figure 2.6: Profile of the state variables $x_1,\ x_2,\ x_3$

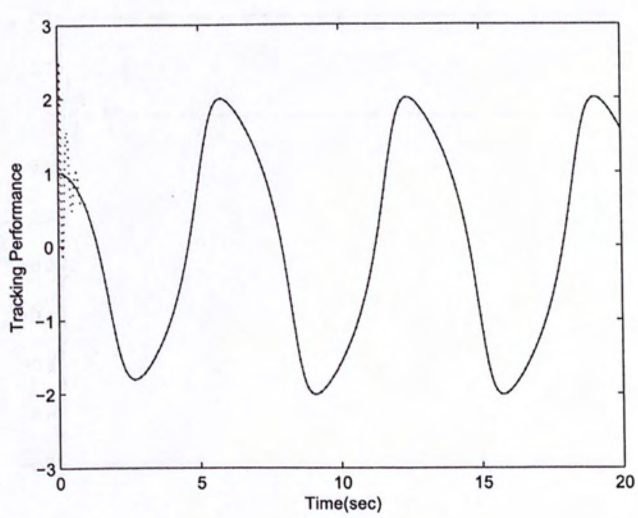

Figure 2.7: Tracking performance of the controlled circuit when $w = 0.5R_0$

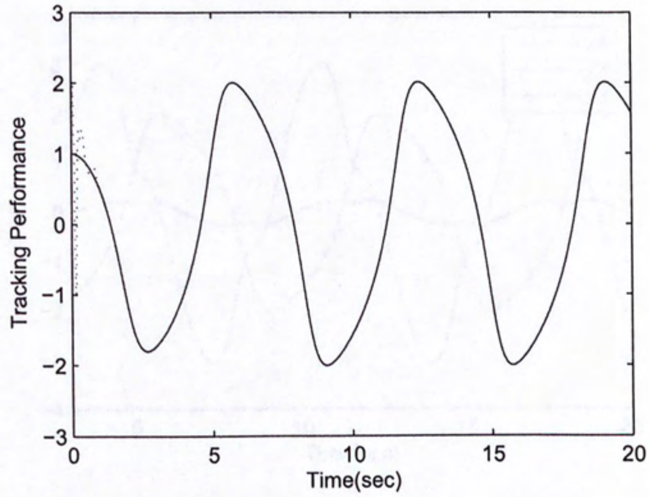

Figure 2.8: Tracking performance of the controlled circuit when $w = 0$

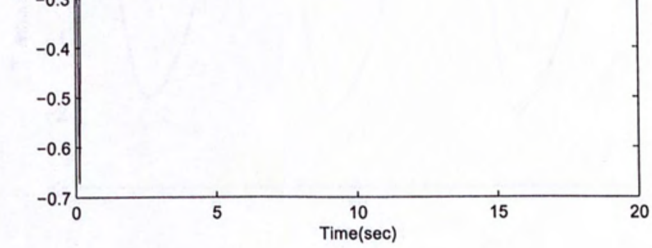

Figure 2.9: Tracking error when $w = 0$

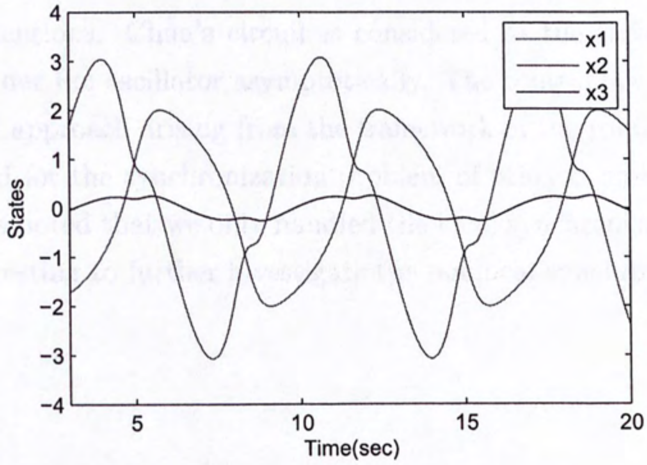

Figure 2.10: Profile of the state variables $x_1,\ x_2,\ x_3$

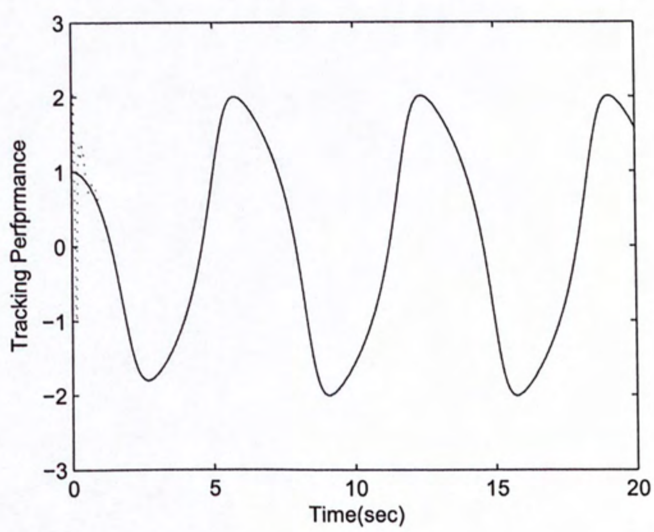

Figure 2.11: Tracking performance of the controlled circuit when $w = -0.5R_0$

2.5 Conclusions

In this chapter we have studied the problem of the output synchronization of two systems with different dimensions. Chua's circuit is considered as the slave plant which tracks the output of Van der Pol oscillator asymptotically. The control law is designed based on the internal model approach arising from the framework of the robust output regulation. It can also be used for the synchronization problem of other complex nonlinear master-slave systems. It is noted that we only handled the local synchronization problem in this chapter. It is interesting to further investigate the nonlocal synchronization problem.

□ **End of chapter.**

Chapter 3

Robust Output Regulation of Output Feedback Systems with Nonlinear Exosystems

For over a decade, the solvability of the nonlinear robust output regulation problem relies on the assumption that the exosystem is linear and neurally stable. In this chapter, we will discuss the output regulation problem with nonlinear exosystems, and compare the two existing frameworks of the output regulation with nonlinear exosystems.

This chapter is organized as follows: In Section 3.1 the output regulation problem is introduced. In Section 3.2 and 3.3 the framework of the output regulation problem will be given. In Section 3.4, we will introduce a different framework of the output regulation problem and then analyze the pros and cons of the two output regulation frameworks respectively.

3.1 Introduction

In this chapter, we study the output regulation problem for output feedback systems with nonlinear exosystems. Consider the class of nonlinear systems with uncertain parameters described in the following form

$$
\begin{aligned}
\dot{x} &= F(w)x + G(y,v,w)y + g(w)u + D_1(v,w) \\
\dot{y} &= H(w)x + K(y,v,w)y + D_2(v,w) \\
e &= y - q(v,w)
\end{aligned}
\tag{3.1}
$$

where $col(x,y) \in R^n$ are the states, $y \in R$ is the output, $u \in R$ is the control input, $q(v,w) \in R$ is the output of the master system, and $e \in R$ is the tracking error. The system contains an unknown parameter vector $w \in R^{n_w}$, and the state of the exosystem v is assumed to be generated by a class of nonlinear exosystems of the form

$$\dot{v} = a(v(t)),\ v(0) = v_0. \tag{3.2}$$

We base on the approach studied in [6], which solves the local robust output regulation problem for nonlinear exosystems, and extend it to solve the nonlocal robust output regulation problem.

As studied in [6], the nonlinear exosystems can be decomposed into the following form

$$\dot{v} = a(v) = A_1 v + \sum_{k=2}^{N} A_k v a_k(v) \tag{3.3}$$

where the matrices $A_k \in R^{q\times q}$ for $k = 1, 2, \cdots, N$, and $a_k(v) : R_q \to R$ is a smooth function with $a_k(0) = 0$.

The nonlocal output regulation problem is defined as: given any $V \subset \overline{V} \subset R^q$ and $W \subset R^{n_w}$ where $\overline{V}$ is some subset of R^q containing the origin, V and W are known compact subsets of R^q and R^{n_w}, respectively, design a feedback control law such that, for any $v(t) \in V,\ t \geq 0$ of the exosystem and any uncertain parameter $w \in W$, the solution of the closed-loop system, starting from any initial state in V exists and is bounded for all $t > 0$, and the tracking error e approaches zeros asymptotically. Comparing with the local robust output regulation problem, we allow the trajectories of the master system to belong to some known subset $\overline{V}$ of R^q which does not have to be sufficiently small.

3.2 Assumptions and Preliminaries

Under some assumptions, the robust output regulation problem for a given plant can be converted into a robust stabilization problem of an augmented system composed of the given plant and the internal model. To introduce this conversion, let us list the following standard assumptions.

Assumption 3 System (3.1) has a uniform relative degree $r \geq 2$, i.e., for all $w \in R^{n_w}$,

$$H(w)g(w) = H(w)F(w)g(w) = \cdots = H(w)F^{r-3}(w)g(w) = 0,$$

$$H(w)F^{r-2}(w)g(w) \neq 0.$$

∎

Assumption 4 System (3.1) is minimum phase with y as the output, i.e., for all $w \in R^{n_w}$, the linear system

$$\dot{x} = F(w)x + g(w)u, \quad \dot{y} = H(w)x \tag{3.4}$$

with y as the output is a minimum phase system. ∎

Assumption 5 There exist sufficiently smooth functions $\mathbf{x}(v, w)$ and $\mathbf{u}(v, w)$ with $\mathbf{x}(0, 0) = 0$ and $\mathbf{u}(0, 0) = 0$, such that, for all $v \in R^m, w \in R^{n_w}$,

$$\begin{aligned} \frac{\partial \mathbf{x}(v, w)}{\partial v} A_1 v &= F(w)\mathbf{x}(v, w) + G(q(v, w), v, w)q(v, w) + g(w)\mathbf{u}(v, w) + D_1(v, w), \\ \frac{\partial q(v, w)}{\partial v} A_1 v &= H(w)\mathbf{x}(v, w) + K(q(v, w), v, w)q(v, w) + D_2(v, w). \end{aligned} \tag{3.5}$$

∎

The solvability of the regulator equations is a necessary condition for the solvability of the output regulation problem [26]. For the purpose of dealing with the problem of robust output regulation, various conditions have to be imposed on the solution of regulator equations. One of the most common conditions is that the solution of the regulator equations is polynomial in v or trigonometric polynomial in t [4], [19], [20]. When the exosystem is linear and neutrally stable, some milder conditions were proposed in reference [21]. Since the exosystem considered in this chapter is nonlinear, an additional assumption is given as follows.

Assumption 6 The solution $\mathbf{u}(v(t), w)$ of the regulator equations is polynomial in $v(t)$ with coefficients depending on $w \in R$. ∎

Remark 3.2.1 In light of reference [4] and [20], under Assumption 6, there exist some real numbers $a_1,\ a_2, \cdots, a_r$ with r some positive integer, such that

$$L^r_{A_1 v}\mathbf{u}(v,w) = a_1\mathbf{u}(v,w) + a_2 L_{A_1 v}\mathbf{u}(v,w) + \cdots + a_r L^{r-1}_{A_1 v}\mathbf{u}(v,w)$$

where $L_{A_1 v}\mathbf{u}(v,w) = \frac{\partial \mathbf{u}(v,w)}{\partial v} A_1 v$, and $L^k_{A_1 v}\mathbf{u}(v,w) = \frac{\partial L^{k-1}_{A_1 v}\mathbf{u}(v,w)}{\partial v} A_1 v,\ k = 2, 3, \cdots, r$. ∎

Next, we will introduce the concepts of the state-steady generator and internal model for system (3.1) and (3.3).

Definition 3.2.1 Let $g_0 : R^{n+m} \mapsto R^l$ be a mapping for some positive integer $1 \leq l \leq n+m$. Under Assumption 3, the nonlinear systems (3.1) and (3.3) are said to have a steady state generator with output $g_0(z, y, u)$ if there exists a triple (θ, α, β), where $\theta : R^{q+p} \mapsto R^l, \alpha : R^{s+p} \mapsto R^s$, and $\beta : R^{s+p} \mapsto R^s$ for some integer s are sufficiently smooth functions vanishing at the origin, such that, for all trajectories $v(t)$ of system (3.3), and all w

$$\begin{aligned} \frac{\partial \theta(v,w)}{\partial v} a(v) &= \alpha(\theta(v,w), v) \\ g_0(\mathbf{z}(v,w), \mathbf{y}(v,w), \mathbf{u}(v,w)) &= \beta(\theta(v,w), v). \end{aligned} \tag{3.6}$$

Furthermore, if the pair $(\frac{\partial \theta(v,w)}{\partial v}|_{v=0,w=0}, \frac{\partial \alpha(v,w)}{\partial v}|_{v=0,w=0})$ is observable, (θ, α, β) is called a linearly observable steady state generator with output $g_0(z, y, u)$. ∎

Definition 3.2.2 Assume the nonlinear systems (3.1) and (3.3) have a steady-state generator with output $g_0(z, y, u)$. Let $\beta : R^{s+p+m+q} \mapsto R^s$ be a sufficiently smooth function vanishing at the origin. The following system

$$\dot{\eta} = \gamma(\eta, z, y, u, v) \tag{3.7}$$

is called an internal model with output $g_0(z, y, u)$ if, for all $v(t)$ of system (3.3) and all w

$$\gamma(\theta, \mathbf{z}(v,w), \mathbf{x}(v,w), \mathbf{u}(v,w), v) = \alpha(\theta(v,w), v).$$

∎

Now we denote

$$\tau(v,w) = col\left(\mathbf{u}(v,w),\ L_{A_1 v}\mathbf{u}(v,w), \cdots,\ L_{A_1 v}^{r-1}\mathbf{u}(v,w)\right),$$

and then there exist matrices

$$\Phi_1 = \begin{bmatrix} 0_{(r-1)\times 1} & I_{r-1} \\ a_1 & [a_2, \cdots, a_r] \end{bmatrix}$$

and $\Psi = [1\ 0\ \cdots\ 0]$ such that

$$\begin{aligned} \frac{\partial \tau(v,w)}{\partial v} A_1 v &= \Phi_1 \tau(v,w), \\ \mathbf{u}(v,w) &= \Psi \tau(v,w). \end{aligned} \tag{3.8}$$

It is easy to verify that the pair (Ψ, Φ_1) is observable.

Assumption 7 For $k = 2, \cdots, N$, there exists some matrix Φ_k satisfying

$$\frac{\partial \tau(v,w)}{\partial v} A_k v = \Phi_k \tau(v,w)$$

∎

Remark 3.2.2 Next we will construct the steady-state generator. Let $\theta = T\tau(v,w)$ with $T \in R^{s\times s}$ any nonsingular matrix. Under Assumptions 6, 7, the Lie derivative of $\theta(v,w)$ along the master system (3.3) satisfies

$$\begin{aligned} \frac{\partial \theta(v,w)}{\partial v} a(v) &= \frac{\partial \theta(v,w)}{\partial v}(A_1 v + \sum_{k=2}^{N} A_k v a_k(v)) \\ &= T\Phi_1 T^{-1}\theta(v,w) + \sum_{k=2}^{N} T\Phi_k a_k(v) T^{-1}\theta(v,w) \\ &= T[\Phi_1 + \sum_{k=2}^{N} \Phi_k a_k(v)]T^{-1}\theta(v,w) \\ &= T\phi(v)T^{-1}\theta(v,w) \triangleq \alpha(\theta(v,w),v) \\ \mathbf{u}(v,w) &= \Psi T^{-1}\theta(v,w) \triangleq \beta(\theta(v,w),v) \end{aligned} \tag{3.9}$$

where $\phi(v) = \Phi_1 + \varphi(v)$, $\varphi(v) = \sum_{k=2}^{N} \Phi_k a_k(v)$ and the pair $(\Psi T^{-1}, T\Phi_1 T^{-1})$ is observable. Thus, $\{\theta(v,w),\ \alpha(\theta(v,w),v),\ \beta(\theta(v,w),v)\}$ is the steady-state generator of systems (3.1) and (3.3) with the output $g_0(z,y,u) = u$. ∎

It is ready to design the internal model for the master-slave systems. Choose $M = diag(-\mu_1, -\mu_2, \cdots - \mu_s)$, $(\mu_i > 0,\ \forall i = 1, \cdots, s)$ with μ_i not an eigenvalue of Φ_1, and vector $N \in R^s$, such that the pair (M, N) is controllable. Then, there exists a unique nonsingular matrix T, such that the following Sylvester equation holds

$$T\Phi_1 - MT = N\Psi$$

Then, the dynamic system

$$\dot{\eta} = \gamma(\eta, u, w) = M\eta + T\varphi(v)T^{-1}\eta + Nu \tag{3.10}$$

is the internal model of the systems (3.1) and (3.3) , since

$$\begin{aligned}\gamma(\theta, u, v) &= M\theta + T\varphi(v)T^{-1}\theta + Nu \\ &= T\Phi_1 T^{-1}\theta + T\varphi(v)T^{-1}\theta \\ &= T\phi(v)T^{-1}\theta \\ &= \alpha(\theta, v).\end{aligned}$$

3.3 Solvability of the Synchronization Problem

Attaching (3.1) to (3.10) leads to what is called the augmented system. Performing the following coordinate and input transformation as in [23]

$$\begin{aligned}z &= x - \mathbf{x}(v, w), \\ e &= y - q(v, w), \\ \overline{\eta} &= \eta - \theta(v, w), \\ \overline{u} &= u - \beta(\eta)\end{aligned} \tag{3.11}$$

on (3.1) and (3.10) to obtain a system of the form

$$\begin{aligned}\dot{z} &= F(w)z + \widetilde{G}(e, v, w)e + g(w)(\overline{u} + \beta(\overline{\eta})) \\ \dot{e} &= H(w)z + \widetilde{K}(e, v, w)e \\ \dot{\overline{\eta}} &= (M + N\Psi T^{-1} + T\varphi(v)T^{-1})\overline{\eta} + N\overline{u}\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}\widetilde{G}(e, v, w)e &= G(q + e, v, w)(q + e) - G(q, v, w)q \\ \widetilde{K}(e, v, w)e &= K(q + e, v, w)(q + e) - K(q, v, w)q\end{aligned}$$

To solve the regional robust stabilization problem for system (3.11), we further perform a coordinate transformation as follows:

$$\widetilde{\eta} = \overline{\eta} - NP(w)z, \tag{3.13}$$

where

$$P(w) = \frac{H(w)F^{r-2}(w)}{b(w)}, \quad b(w) = H(w)F^{r-2}(w)g(w). \tag{3.14}$$

As a result, the first equation of system (3.12) becomes

$$\begin{aligned}
\dot{z} &= F(w)z + \widetilde{G}(e,v,w)e + g(w)(\overline{u} + \Psi T^{-1}\overline{\eta}) \\
&= F(w)z + \widetilde{G}(e,v,w)e + g(w)[\overline{u} + \Psi T^{-1}(\widetilde{\eta} + NP(w)z)] \\
&= [F(w) + g(w)\Psi T^{-1}NP(w)]z + g(w)\Psi T^{-1}\widetilde{\eta} + \widetilde{G}(e,v,w)e + g(w)\overline{u} \\
&= F_{11}(w)z + F_{12}(w)\widetilde{\eta} + \widetilde{G}(e,v,w)e + g(w)\overline{u}.
\end{aligned} \tag{3.15}$$

From (3.14) we have $P(w)g(w) = 1$, and we calculate the derivative of $\widetilde{\eta}$ in equation (3.13)

$$\begin{aligned}
\dot{\widetilde{\eta}} &= \dot{\overline{\eta}} - NP(w)\dot{z} \\
&= (M + N\Psi T^{-1} + T\varphi(v)T^{-1})(\widetilde{\eta} + NP(w)z) + N\overline{u} \\
&\quad - NP(w)[(F(w) + g(w)\Psi T^{-1}NP)z + g(w)\Psi T^{-1}\widetilde{\eta} + \widetilde{G}(e,v,w)e + g(w)\overline{u}] \\
&= [MNP(w) + T\varphi(v)T^{-1}NP(w) - NP(w)F(w)]z + (M + F_{11})\widetilde{\eta} \\
&\quad - NP(w)\widetilde{G}(e,v,w)e \\
&= [MNP(w) + T\varphi(v)T^{-1}NP(w) - NP(w)F(w)]z \\
&\quad + (M + F_{11})(\widetilde{\eta} + NP(w)z) - NP(w)\widetilde{G}(e,v,w)e \\
&= F_{21}(v,w)z + (M + T\varphi(v)T^{-1})\widetilde{\eta} - NP(w)\widetilde{G}(e,v,w)e
\end{aligned} \tag{3.16}$$

where $F_{11}(w) = F(w) + g(w)\Psi T^{-1}NP$, $F_{12}(w) = g(w)\Psi T^{-1}$ and $F_{21}(v,w) = MNP + T\varphi(v)T^{-1}NP$.

Now let

$$\zeta = \begin{bmatrix} z \\ \widetilde{\eta} \end{bmatrix},$$

$$F_a(v,w) = \begin{bmatrix} F_{11}(w) & F_{12}(w) \\ F_{21}(v,w) & M + T\varphi(v)T^{-1} \end{bmatrix},$$

$$\widetilde{G}(e,v,w)e = \begin{bmatrix} \widetilde{G}_e(e,v,w)e \\ -NP(w)\widetilde{G}(e,v,w)e \end{bmatrix},$$

$$g_a(w) = \begin{bmatrix} g(w) \\ 0 \end{bmatrix}, \quad H_a(w) = \begin{bmatrix} \overline{H}(w) & 0 \end{bmatrix}.$$

Then, in the coordinates of ζ, e, system (3.12) can be put in the following form

$$\begin{aligned} \dot{\zeta} &= F_a(v,w)\zeta + \widetilde{G}_e(e,v,w)e + g_a(w)\overline{u}, \\ \dot{e} &= H_a(w)\zeta + \widetilde{K}(e,v,w)e, \end{aligned} \tag{3.17}$$

Comparing with the system studied in [21] where the $F_a(v,w)$ is not a function on v, in our system (3.17) $F_a(v,w)$ depends on v and thus depends on time t.

We first cite the technique of output regulation studied in [21], where $F_a(v,w) = F_a(w)$ does not depend on v, and then apply the approach to our case where $F_a(v,w)$ depends on v.

Remark 3.3.1 Under Assumptions 3 and 4, system (3.17) has the following two properties

i) It has a uniform relative degree $r \geq 2$, i.e., for all $w \in R^{n_w}$,

$$H_a(w)g_a(w) = H_a(w)F_a(v,w)g_a(w) = \cdots = H_a(w)F_a^{r-3}(v,w)g_a(w) = 0,$$

and

$$H_a(w)F_a^{r-2}(v,w)g_a(w) = H(w)F^{r-2}(v,w)g(w) \neq 0.$$

ii) It is a minimum phase system with e as the output. ∎

Now define the following dynamic extension as in [21]

$$\dot{\xi} = F_e\xi + G_e\overline{u} \tag{3.18}$$

where $\xi = col(\xi_1, \xi_2, \cdots, \xi_{r-1})$ with $\xi_i \in R$ for $i = 1, 2, \cdots, r-1$,

$$F_e = \begin{bmatrix} -\lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{r-1} \end{bmatrix}, \quad G_e = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

with $\lambda_i, i = 1, 2, \cdots, r-1$, being positive numbers.

Call the system composed of (3.17) and (3.18) as the extended augmented system. It is known from [42] that if $F_a(v, w)$ does not rely on v, then there exists a coordinate transformation as follows

$$\overline{\zeta} = \zeta - D(w)\xi - h(w)e \tag{3.19}$$

which turns the extended augmented system into a lower triangular form.

However for our case, where $F_a(v, w)$ depends on v, the transformation (3.19) is not effective to turn the extended augmented system into a lower triangular form with the reason to be explained next.

First we also extend system (3.17) with the dynamic system (3.18) as in [21], and perform the following transformation

$$\overline{\zeta} = \zeta - D(v, w)\xi - h(v, w)e \tag{3.20}$$

where $D(v, w)$ and $h(v, w)$ are functions of v to be determined later. Calculate the derivative of $\overline{\zeta}$ in equation (3.20) to obtain

$$\begin{aligned}
\dot{\overline{\zeta}} =& \dot{\zeta} - D(v, w)\dot{\xi} - \dot{D}(v, w)\xi - h(v, w)\dot{e} - \dot{h}(v, w)e \\
=& F_a(v, w)\zeta + \widetilde{G}_e(e, v, w)e + g_a(w)\overline{u} - D(v, w)(-\xi + \overline{u}) - \dot{D}(v, w)\xi \\
& - h(v, w)[H_a(w)\zeta + \widetilde{K}(e, v, w)e] - \dot{h}(v, w)e \\
=& F_a(v, w)[\overline{\zeta} + D(v, w)\xi + h(v, w)e] + \widetilde{G}_e(e, v, w) + g_a(w)\overline{u} \\
& - D(v, w)(-\xi + \overline{u}) - \dot{D}(v, w)\xi - h(v, w)H_a(w)[\overline{\zeta} + D(v, w)\xi + h(v, w)e] \\
& - h(v, w)\widetilde{K}(e, v, w)e - \dot{h}(v, w)e \\
=& [F_a(v, w) - h(v, w)H_a(w)]\overline{\zeta} \\
& + [F_a(v, w)D(v, w) + D(v, w) - h(v, w)H_a(w)D(v, w) - \dot{D}(v, w)]\xi \\
& [F_a(v, w)h(v, w) + \widetilde{G}_e(e, v, w) - h(v, w)H_a(w)h(v, w) \\
& - h(v, w)\widetilde{K}(e, v, w) - \dot{h}(v, w)]e
\end{aligned} \tag{3.21}$$

And the transportation (3.20) turns the extended augmented system (3.17) and (3.18) into the following form

$$
\begin{aligned}
\dot{\overline{\zeta}} =&[F_a(v,w) - h(v,w)H_a(w)]\overline{\zeta} \\
&+ [F_a(v,w)D(v,w) + D(v,w) - h(v,w)H_a(w)D(v,w) - \dot{D}(v,w)]\xi \\
&[F_a(v,w)h(v,w) + \widetilde{G}_e(e,v,w) - h(v,w)H_a(w)h(v,w) - h(v,w)\widetilde{K}(e,v,w) - \dot{h}(v,w)]e \\
\dot{e} =&H_a(w)\overline{\zeta} + [H_a(w)\frac{d(v,w)}{b(v,w)} + \widetilde{K}(e,v,w)]e + H_a(w)D(v,w)\xi \\
\dot{\xi} =&F_e\xi + G_e\overline{u}
\end{aligned}
\tag{3.22}
$$

According to the approach in [22] which renders the system (3.22) a lower triangular form, we choose $D(v,w)$ and $h(v,w)$ such that, for some function $b(v,w)$

$$
\begin{aligned}
F_a(v,w)D(v,w) - D(v,w)F_e =&h(v,w)H_a(w)D(v,s) \\
g(w) =&D(v,w)G_e \\
H_a(w)D(v,w)\xi =&b(v,w)\xi_1
\end{aligned}
\tag{3.23}
$$

or, equivalently, for some function $b(v,w)$

$$
\begin{aligned}
F_a(v,w)D(v,w) - D(v,w)F_e =&h(v,w)[b(v,w),0,\cdots,0] \\
g(w) =&D(v,w)[0,\cdots,0,1]^T \\
H_a(w)D(v,w)\xi =&[b(v,w),0,\cdots,0].
\end{aligned}
\tag{3.24}
$$

Assume

$$
D(v,w) = [d_1(v,w), d_2(v,w), \cdots, d_{r-1}(v,w)]. \tag{3.25}
$$

Substituting (3.25) into the first equation of (3.22) gives

$$
d_{i-1} = (\lambda_i I + F_a(v,w))d_i(w), \quad i = r-1, \cdots, 2, \tag{3.26}
$$

with $h(v,w)$ and $b(v,w)$ satisfying

$$
(\lambda_1 I + F_a(v,w))d_1(w) = h(v,w)b(v,w). \tag{3.27}
$$

Substituting (3.25) into the second equation of (3.22) gives

$$
d_{r-1}(v,w) = d_{r-1}(w) = g(w). \tag{3.28}
$$

Substituting (3.28) into (3.26) gives

$$\begin{aligned}
d_{r-1}(w) &= g(w) \\
d_{r-2}(v,w) &= (F_a(v,w) + \lambda_{r-1} I) g(w) \\
&\vdots \\
d_1(v,w) &= (F_a(v,w)(w) + \lambda_2 I) \cdots (F_a(v,w) + \lambda_{r-1} I) g(w).
\end{aligned} \tag{3.29}$$

It is noted that, when $r = 2$, the last equation of (3.29) should be understood as $d_1(v,w) = g(w)$.

Letting

$$b(v,w) = H_a(w) F_a(v,w)^{r-2} g(w),$$

based on the Remark 3.3.1, $D(v,w)$ defined as (3.25) and (3.29) satisfies the third equation of (3.24).

Finally, substituting $d_1(v,w)$ into (3.27) gives

$$h(v,w) = \frac{d(v,w)}{b(v,w)}, \tag{3.30}$$

where

$$d(v,w) = (F_a(v,w) + \lambda_1 I)(F_a(v,w)(w) + \lambda_2 I) \cdots (F_a(v,w) + \lambda_{r-1} I) g(w).$$

Consider the coefficient matrix of ξ in (3.21)

$$\begin{aligned}
& F_a(v,w) D(v,w) + D(v,w) - h(v,w) H_a(w) D(v,w) - \dot{D}(v,w) \\
=& F_a(v,w) g_a(w) + g_a(w) - \frac{(F_a(v,w) + I) g_a(w)}{H_a(w) g_a(w)} H_a(w) g_a(w) - \dot{D}(v,w) \\
=& 0 - \dot{D}(v,w) \\
=& - \dot{D}(v,w).
\end{aligned} \tag{3.31}$$

Let

$$\begin{aligned}
\overline{F}(v,w) &= F_a(v,w) - \frac{d(v,w)}{b(v,w)} H_a(w), \\
\overline{G}(e,v,w) &= -\dot{h}(v,w) + \left(F_a(v,w) - \frac{d(v,w)}{b(v,w)} H_a(w) \right) \frac{d(v,w)}{b(v,w)} \\
&\quad + \widetilde{G}_e(e,v,w) - \frac{d(v,w)}{b(v,w)} \widetilde{K}(e,v,w), \\
\overline{H}(w) &= H_a(w), \\
\overline{K}(e,v,w) &= H_a(w) \frac{d(v,w)}{b(v,w)} + \widetilde{K}(e,v,w),
\end{aligned} \tag{3.32}$$

and then the extended augmented system turns into the following form:

$$\begin{aligned}
\dot{\overline{\zeta}} &= \overline{F}(v,w)\overline{\zeta} + \overline{G}(e,v,w)e - \dot{D}(v,w)\xi \\
\dot{e} &= \overline{H}(w)\overline{\zeta} + \overline{K}(e,v,w)e + b(v,w)\xi \\
\dot{\xi}_i &= -\lambda_i \xi_i + \xi_{i+1}, \quad i = 1,2,\cdots,r-2, \\
\dot{\xi}_{r-1} &= -\lambda_{r-1}\xi_{r-1} + \overline{u}.
\end{aligned} \tag{3.33}$$

The above system is not in a standard lower triangular form due to the existence of $\dot{D}(v,w)\xi$ in the first equation of (3.33). And it will be difficult to settle the stabilization problem of a non-lower triangular form system. However, when the relative degree $r = 2$, the equations (3.25) and (3.29) show that

$$D(v,w) = d_1(v,w) = g(w). \tag{3.34}$$

where $D(v,w)$ does not depend on v and thus does not depend on time t. So $\dot{D}(v,w) = 0$, and the extended augmented system (3.33) is in a standard lower triangular form.

On the other hand, when the relative degree $r \geq 3$, $D(v,w)$ is a function of time with $\dot{D}(v,w) \neq 0$, and the extended augmented system cannot be transformed into a standard lower triangular form. So only when the relative degree is $r = 2$, can we apply the above approach and stabilizing techniques for lower triangular systems to our output regulation problem with nonlinear exosystems.

Assumption 8 The original system (3.1) has a uniform relative degree $r = 2$, i.e., for all $w \in R^{n_w}$,

$$H(w)g(w) \neq 0.$$

∎

Under Assumption 8 perform the modified transformation as follows

$$\overline{\zeta} = \zeta - D(w)\xi_1 - h(v,w)e \tag{3.35}$$

where

$$D(w) = d_1(w) = g(w) \tag{3.36}$$

and

$$\begin{aligned}
d(v,w) &= (F_a(v,w) + \lambda_1 I)g(w) \\
b(w) &= H_a(w)g(w) \\
h(v,w) &= \frac{d(v,w)}{b(w)}
\end{aligned} \tag{3.37}$$

Then we have

$$
\begin{aligned}
\dot{\overline{\zeta}} &= \overline{F}(v,w)\overline{\zeta} + \overline{G}(e,v,w)e \\
\dot{e} &= \overline{H}(w)\overline{\zeta} + \overline{K}(e,v,w)e + b(w)\xi_1 \\
\dot{\xi}_1 &= -\xi_1 + \overline{u}.
\end{aligned} \tag{3.38}
$$

System (3.38) is in a lower triangular form as discussed in [42], and we have already converted the original output regulation problem into a stabilization problem of system (3.38). However, the stabilization problem itself is not an easy one. First system (3.38) is a non-autonomous system since the matrix $\overline{F}(v,w)$ is a function of $v(t)$ varying with time t. Second, the state $\overline{\zeta}$ is obtained from a complex transformation performing on the combination of the original states of the slave system and the states of the internal model. So the matrices $\overline{F}(v,w)$ and $\overline{G}(e,v,w)$ usually have large dimensions and complex forms.

To solve the stabilization problem, we need another assumption:

Assumption 9 There exists a positive definite matrix $Q(w)$, such that for all $v(t) \in V$, $w \in W$

$$
\overline{F}(v,w)^T Q(w) + Q(w)\overline{F}(v,w) < -I_{n-1+s}. \tag{3.39}
$$

∎

Next let us introduce some inequalities to be used for the stabilization of system (3.38). Since $\overline{G}(e,v,w)$ and $\overline{K}(e,v,w)$ are real valued continuous functions, there exist smooth real valued functions $q_i(v,w)$, $a_i(e)$, $i = 1,2$, such that for each $v \in V,\ w \in W,\ e \in R$,

$$
\begin{aligned}
|\overline{G}(e,v,w)e|^2 &\leq q_1(v,w)a_1(e)e^2 \\
|\overline{K}(e,v,w)e|^2 &\leq q_2(v,w)a_2(e)e^2
\end{aligned}
$$

Define

$$
V_0(\overline{\zeta},t) = l\overline{\zeta}^T Q(w)\overline{\zeta}
$$

where $Q(w)$ satisfies Assumption 9, and l is a positive constant to be determined later. Thus

$$
\begin{aligned}
\dot{V}_0 &= l\overline{\zeta}^T[\overline{F}(v,w)^T Q(w) + Q(w)\overline{F}(v,w)]\overline{\zeta} + 2l\overline{\zeta}^T Q(w)\overline{G}(e,v,w)e \\
&\leq -l|\overline{\zeta}|^2 + \epsilon l|\overline{\zeta}|^2 + \frac{1}{\epsilon}l|Q(w)|^2 q_1(v,w)a_1(e)e^2 \\
&= -(1-\epsilon)l|\overline{\zeta}|^2 + \frac{1}{\epsilon}l|Q(w)|^2 q_1(v,w)a_1(e)e^2
\end{aligned}
$$

where $0 < \epsilon < 1$.

Let

$$\begin{aligned} V &= V_0 + \frac{1}{2}b(w)(k - \bar{k}(w))^2 + \frac{1}{2}e^2 + \frac{1}{2}\bar{\xi}_1^2 \\ &= l\bar{\zeta}^T Q(w)\bar{\zeta} + \frac{1}{2}b(w)(k - \bar{k}(w))^2 + \frac{1}{2}e^2 + \frac{1}{2}\bar{\xi}_1^2, \end{aligned} \tag{3.40}$$

where k is generated by

$$\dot{k} = \rho(e)e^2, \tag{3.41}$$

$\rho(\cdot)$ is some smooth positive function, and

$$\bar{\xi}_1 = \xi_1 - \alpha_1(e, k),$$

$$\alpha_1(e, k) = -k\rho(e)e.$$

The derivative of V along the trajectory of the system composed of (3.38) and (3.41) satisfies

$$\dot{V} \le -|\bar{\zeta}|^2 - \bar{e}^2 - \bar{\xi}_1^2 + \bar{\xi}_1(\bar{u} - \alpha_1(e, k)) \tag{3.42}$$

Let

$$\bar{u} = \alpha_1(e, k), \quad \dot{k} = \rho(e)e^2, \tag{3.43}$$

Then (3.42) becomes

$$\dot{V} \le -|\bar{\zeta}|^2 - \bar{e}^2 - \bar{\xi}_1^2. \tag{3.44}$$

With the inequality (3.44) ready, it is possible to make use of the Lyapunov function candidate V to conclude that the following dynamic output feedback control law

$$\begin{aligned} u &= \alpha_1(e, k) + \beta(\eta) \\ \dot{k} &= \rho(e)e^2 \\ \dot{\xi} &= F_e\xi + G_e\bar{u} \\ \dot{\eta} &= M\eta + N(\alpha_1(e, \xi, k) + \beta(\eta)) \end{aligned} \tag{3.45}$$

solves the nonlocal robust output regulation problem of the original systems (3.1) and (3.3).

3.4 Comparing Two Approaches for Output Regulation

In Section 3.3, we consider the synchronization problem as the output regulation problem and extend the approach studied in [23] to deal with it. Another approach for output regulation has been extensively studied in [6] and [49]. In this section, we will compare pros and cons of the two approaches for output regulation.

3.4.1 Differences between the two approaches for the output regulation problem

The approach we proposed in Section 3.2 and 3.3 to deal with the output regulation problem involves the following steps. First find the solution of the regulator equations, obtain the augmented system by augmenting the given system an internal model, and then performing certain coordinate and input transformation on the given system and the internal model to convert the output regulation problem into a stabilization problem. Second, extend the augmented system by a dynamic filter and call this system an extended augmented system. Finally, solve the robust stabilization problem of the extended augmented system by a dynamic output feedback controller.

Another approach to deal with the same output regulation problem described in (3.1) has been first studied in [6]. The approach involves the following steps. First attach the same dynamic filter as that in the second step of the first approach to the given system to form an extended system which is in a lower triangular form. Second, convert the output regulation problem of the extended system into a stabilization problem of a so-called augmented extended system. The conversion needs the solution of regulator equations of the extended system. The augmented extended system is obtained by augmenting the extended system by an internal model and then performing certain coordinate and input transformation on the extended system and the internal model. Finally, solve the robust stabilization problem for the augmented extended system which leads to the solution of the robust output regulation problem of the original systems.

It is noted that the major difference between these two approaches is the order of the conversion and extension. And the difference will give the first approach some advantages and disadvantages comparing with the second approach. One advantage of the first approach is that it is more natural and directly follows from the second approach for handling the output regulation problem developed in [21]. Another advantage is that the core assumption of the solvability of the output regulation problem by both the two approaches is the solvability of the regulator equations. For the second approach, we can only solve the regulator equations for some special examples, e.g. the exosystems are linear systems. However for the first approach discussed in Section 3.3, the solution of the regulator equations for more general nonlinear exosystems, e.g. the exosystems are nonlinear systems, can be obtained. So the first approach can be applied on more general output regulation problems.

On the other hand, the second approach has its own advantages. First the relative degree r of the slave system needs not to be equal to 2. Second, after both approaches boil down the output regulation problem to the stabilization problem in a standard lower triangular form, the extended augmented system resulting from the first approach is somehow more complex than the augmented extended system resulting from the second approach. Next we will compare the differences of the two approaches in detail.

3.4.2 Solvability of the regulator equations

We cite the class of uncertain nonlinear systems (3.1) described as follows

$$\begin{aligned}
\dot{x} &= F(w)x + G(y, v, w)y + g(w)u + D_1(v, w) \\
\dot{y} &= H(w)x + K(y, v, w)y + D_2(v, w) \\
e &= y - q(v, w).
\end{aligned} \tag{3.46}$$

Consider it as the plant of the regulation problem, and exosystem is

$$\dot{v} = a(v) = A_1 v + \sum_{k=2}^{N} A_k v a_k(v) \tag{3.47}$$

Based on the first approach discussed in Section 3.3, the solution of the regulator equations of systems (3.46) and (3.47) can be calculated directly. First we can get $\mathbf{y}(v, w) = q(v, w)$ by observation. And then we could obtain $\mathbf{x}(v, w)$ from the first equation of system (3.46)

$$\mathbf{x}(v, w) = \frac{\dot{\mathbf{y}} - K(\mathbf{y}, v, w)\mathbf{y} - D_2(v, w)}{H(w)} \tag{3.48}$$

and $\mathbf{u}(v, w)$ from the second equation of system (3.46)

$$\mathbf{u}(v, w) = \frac{\dot{\mathbf{x}} - F(w)\mathbf{x} - G(\mathbf{y}, v, w)\mathbf{y} - D_1(v, w)}{g(w)} \tag{3.49}$$

We can see that if the given plant of the output regulation problem can be written in the form of (3.46), we could find the solution of the regulator equations for both linear and nonlinear exosystems. The process is direct and shown in the example in Section 2.4.1 where Chua's circuit is the given plant and Van der Pol oscillator is the nonlinear exosystem.

However, the first approach needs the relative degree r of the slave system equal to 2 in order to transfer the augmented system into the extended augmented system in a lower triangular form.

Next we discuss the solvability of the regulator equations for the second approach proposed in [6]. Under Assumption 3, system (3.46) has a uniform relative degree r which could be larger than or equal to 2, and extend system (3.46) by attaching the following filter

$$\begin{aligned} \dot{\xi}_i &= -\lambda_i \xi_i + \xi_{i+1}, \quad i = 1, \cdots, r-2 \\ \dot{\xi}_{r-1} &= -\lambda_{r-1}\xi_{r-1} + u \end{aligned} \tag{3.50}$$

and performing certain coordinate transformation to get the extended system

$$\begin{aligned} \dot{z} &= \widehat{F}(w)z + \widehat{G}(y, v, w)y + \widehat{D}_1(v, w) \\ \dot{y} &= \widehat{H}(w)z + \widehat{K}y + b(w)\xi_1 + \widehat{D}_2(v, w) \\ \dot{\xi}_i &= -\lambda_i \xi_i + \xi_{i+1}, \quad i = 1, \cdots, r-2 \\ \dot{\xi}_{r-1} &= -\lambda_{r-1}\xi_{r-1} + u\xi \\ e &= y - q(v, w). \end{aligned} \tag{3.51}$$

To solve the regulator equations of the extended system, similarly, first we obtain $\mathbf{y}(v, w) = q(v, w)$, and substitute it into the last equation of (3.51). Now we have

$$\dot{\mathbf{z}} = \widehat{F}(w)\mathbf{z} + \widehat{G}(\mathbf{y}, v, w)\mathbf{y} + \widehat{D}_1(v, w). \tag{3.52}$$

When the exosystem system is linear, we can use method of undetermined coefficients to achieve $\mathbf{z}(v, w)$. Because the order of $\mathbf{z}(v, w)$ in v is the same with the order of $\dot{\mathbf{z}}(v, w)$ in v. However, when the exosystem is a polynomial with the order higher than one which can be written in the form of (3.3), the order of $\dot{\mathbf{z}}(v, w)$ in v will be larger than that of $\mathbf{z}(v, w)$ in v, and for most cases, we cannot find the solution $\mathbf{z}(v, w)$ in polynomial.

Now we use the example studied in Chapter 2 that Chua's system is the given plant and Van der Pol oscillator is the exosystem to proof that we cannot find the solution of the regulator equations in polynomial via the second approach.

The dynamic equation for Chua's circuit is as follows

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{RC_2}x_1 + \frac{1}{C_2}x_2 + \frac{1}{RC_2}y \\ \dot{x}_2 &= -\frac{1}{L}x_1 - \frac{R_0}{L} + \frac{1}{L}u \\ \dot{y} &= \frac{1}{RC_1}x_1 + (-\frac{1}{RC_1} - \frac{a_1}{C_1} - \frac{a_3}{C1}y^2)y \\ e &= y - v_1.\end{aligned} \tag{3.53}$$

which can be rewritten in the form of (3.46) with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $F(w) = \begin{bmatrix} -\frac{1}{RC_2} & 1/C_2 \\ -\frac{1}{L} & 0 \end{bmatrix}$, $G(y,v,w) = \begin{bmatrix} \frac{1}{RC_2} \\ 0 \end{bmatrix}$, $g(w) = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$, $H(w) = \begin{bmatrix} \frac{1}{RC_1} & 0 \end{bmatrix}$, $K(y,v,w) = -\frac{a_1}{C_1} - \frac{1}{RC_1} - \frac{a_3}{C_1}y^2$, $b(w) = \frac{1}{RLC_1C_2}$, $D_1 = 0$, $D_2 = 0$. Since system (3.53) has a uniform relative degree $r = 3$, we can extend the system by the following filter [22]

$$\begin{aligned}\dot{\xi}_1 &= -\xi_1 + \xi_2 \\ \dot{\xi}_2 &= -\xi_2 + u\end{aligned} \tag{3.54}$$

Performing the coordinate transformation

$$z = x - D(w)\xi - h(w)y \tag{3.55}$$

where $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$, on the extended system (3.53) and (3.54) with

$$\begin{aligned} d_2(w) &= g(w) \\ d_1(w) &= (F(w) + I)g(w) \\ D(w) &= [d_1(w),\ d_2(w)] \\ d(w) &= (F(w) + I)(F(w) + I)g(w) \\ h(w) &= \frac{d(w)}{b(w)} \end{aligned} \tag{3.56}$$

gives an extended system in the form of (3.51) as follows:

$$\begin{aligned} \dot{z} &= \widehat{F}(w)z + \widehat{G}(y,v,w)y \\ \dot{y} &= \widehat{H}(w)z + \widehat{K}(y,v,w)y + b(w)\xi_1 \\ \dot{\xi}_1 &= -\xi_1 + \xi_2 \\ \dot{\xi}_2 &= -\xi_2 + u \\ e &= y - v_1 \end{aligned} \tag{3.57}$$

where $\widehat{F}(w) = \begin{bmatrix} -2 & \frac{1}{C_2} \\ -C_2 & 0 \end{bmatrix}$, $\widehat{G}(y,v,w) = \begin{bmatrix} c_{11} + c_{12}a_3y^2 \\ c_{21} + c_{22}a_3y^2 \end{bmatrix}$, $\widehat{H}(w) = \begin{bmatrix} \frac{1}{RC_1} & 0 \end{bmatrix}$, $\widehat{K}(y,v,w) = \frac{1}{R}c_{12} - \frac{1}{RC_1} - \frac{a_1}{c_1} - \frac{a_3}{c_1}y^2$, $b(w) = \frac{1}{RLC_1C_2}$, and $c_{11} = \frac{2C_1}{c_2} - \frac{RC_1}{LC_2} - 3RC_1 + \frac{1}{RC_2} + c_{12}(\frac{1}{R} + a_1)$, $c_{12} = -\frac{1}{C_2} + 2R$, $c_{21} = C_1 - 2RC_1C_2 + C_{22}(\frac{1}{R} + a_1)$, $c_{22} = -RC_2(\frac{1}{LC_2} - 1)$ and $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. The parameters of Chua's circuit are $R = 1$, $R_0 = 0.1$, $C_1 = 1/9.5$, $C_2 = 1$, $L = 0.07$, $a_1 = -8/7$, $a_3 = 0.2$.

The exosystem is Van der Pol oscillator

$$\dot{v} = \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ -av_1 + b(1 - v_1^2)v_2 \end{bmatrix} = A_1v + A_2va_2(v) \tag{3.58}$$

where $v = \text{col}(v_1, v_2)$, $A_1 = \begin{bmatrix} 0 & 1 \\ -a & b \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -b \end{bmatrix}$ and $a_2(v) = v_1^2$. And in the following analysis, we use $a = 1, b = 1$.

By inspection, we can obtain

$$\mathbf{y}(v) = v_1. \tag{3.59}$$

Then

$$\begin{aligned} G(\mathbf{y}, v, w)\mathbf{y} &= \begin{bmatrix} c_{11}v_1 + c_{12}v_1^3 \\ c_{21}v_1 + c_{22}v_1^3 \end{bmatrix} \\ &= \begin{bmatrix} c_{11} & 0 \\ c_{21} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} c_{12} & 0 \\ c_{22} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} v_1^2 \\ &= G_1v + G_2vv_1^2 \end{aligned}$$

where

$$G_1 = \begin{bmatrix} c_{11} & 0 \\ c_{21} & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} c_{12} & 0 \\ c_{22} & 0 \end{bmatrix}.$$

Our purpose is to prove that there exists no polynomial function $\mathbf{z}(v)$ in v, with $\mathbf{z}(0,0) = 0$, such that the equation

$$\frac{\partial \mathbf{z}(v)}{\partial v}\dot{v} = F(w)\mathbf{z}(v) + G(\mathbf{y}(v), v, w) \tag{3.60}$$

holds.

First we suppose $\mathbf{z}(v)$ is linear in v and can be written as

$$\mathbf{z}(v) = \begin{bmatrix} m_1v_1 + m_2v_2 \\ n_1v_1 + n_2v_2 \end{bmatrix} = Z_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

where $Z_1 = \begin{bmatrix} m_1 & m_2 \\ n_1 & n_2 \end{bmatrix}$ is the coefficient matrix of $\mathbf{z}(v)$ in v, and $\frac{\partial \mathbf{z}}{\partial v} = Z_1$. Thus the first equation of (3.57) becomes

$$Z_1(A_1 v + A_2 v a_2(v)) = F(w) Z_1 v + G_1 v + G_2 v v_1^2 \tag{3.61}$$

The first order coefficients of v on the left hand side and the right hand side of equation (3.61) should be equal. So

$$Z_1 A_1 = F(w) Z_1 + G_1 \tag{3.62}$$

Similarly, the third order coefficients of v on the left hand side and the right hand side of equation (3.61) should also be equal. So

$$Z_1 A_2 v_1^2 = \begin{bmatrix} c_{12} & 0 \\ c_{22} & 0 \end{bmatrix} v_1^2 \tag{3.63}$$

From equation (3.62) we can obtain

$$Z_1 = \begin{bmatrix} -\frac{3}{133} \cdot \frac{33a(w)^2 - 106a(w) + 95}{5a(w)^2 - 4a(w) + 3} & \frac{1}{133} \cdot \frac{33a(w)^2 - 95 * a(w) - 62}{5a(w)^2 - 4a(w) + 3} \\ -\frac{1}{133} \cdot \frac{409a(w) - 533}{5a(w)^2 - 4a(w) + 3} & \frac{3}{133} \cdot \frac{11a(w) - 157}{5a(w)^2 - 4a(w) + 3} \end{bmatrix}$$

where $a(w) = 0.5 + w$. However this Z_1 does not satisfy equation (3.63). So there exists no linear solution of $\mathbf{z}(v)$.

Next we suppose the solution $\mathbf{z}(v)$ is a n-order polynomial of v with $n \geq 2$. Considering the left hand side of the equation (3.61), the order of $\frac{\partial \mathbf{z}(v)}{\partial v}$ is $n-1$ and the order of $\dot{v}$ is 3. So the order of $\frac{\partial \mathbf{z}(v)}{\partial v}\dot{v}$ on the left hand side of the equation (3.61) is $r_{left} = n + 2$.

Consider the order of the right hand side of equation (3.61) that the order of $F(w)\mathbf{z}(v)$ in v is n, and the order of $G(\mathbf{y}(v), v, w)$ is 3. So the order of the right hand side of equation (3.61) in v is $r_{right} = \max(n, 3) \neq n + 2$ for $n \geq 2$. So when $n \geq 2$, $r_{left} \neq r_{right}$, and there exists no n-order polynomial solution of the equation (3.61).

Now we have proved that there exists no the solution of the regulator equations in polynomial based on the second approach.

3.4.3 Solvability of stabilization

If the solutions of the regulator equations for both approaches exist, then the final step for both of the two approaches is to solve the robust stabilization problem of the extended augmented system or the augmented extended system by a dynamic output feedback controller. Though both the extended augmented system and the augmented extended system are in the lower triangular form, the complexity of stabilization for the two systems are not the same. It is more difficult to solve the stabilization problem of the extended augmented system resulting from the first approach.

The extended augmented system resulting from the first approach is

$$\begin{aligned}\dot{\overline{\zeta}} &= \overline{F}(v,w)\overline{\zeta} + \overline{G}(e,v,w)e \\ \dot{e} &= \overline{H}(w)\overline{\zeta} + \overline{K}(e,v,w)e + b(w)\xi_1 \\ \dot{\xi}_1 &= -\xi_1 + \overline{u}\end{aligned} \tag{3.64}$$

where

$$\overline{\zeta} = \zeta - D(w)\xi_1 - h(v,w)e, \quad \zeta = \begin{bmatrix} z \\ \widetilde{\eta} \end{bmatrix} \tag{3.65}$$

and

$$\begin{aligned}\overline{F}(v,w) = &\begin{bmatrix} F(w) + g(w)\Psi T^{-1}NP & g(w)\Psi T^{-1} \\ MNP + T\varphi(v)T^{-1}NP - NPF(w) & M + T\varphi(v)T^{-1} \end{bmatrix} \\ &- \frac{\left(\begin{bmatrix} F(w) + g(w)\Psi T^{-1}NP + \lambda I & g(w)\Psi T^{-1} \\ MNP + T\varphi(v)T^{-1}NP - NPF(w) & M + T\varphi(v)T^{-1} + \lambda I \end{bmatrix}\right)^2}{b(w)} \begin{bmatrix} g(w)\overline{H}(w) & 0 \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{3.66}$$

The difficulties of the stabilization are due to the following reasons: First $\overline{F}(v,w)$ is a time-varying matrix. Second, z and $\widetilde{\eta}$ have been coupled together and have to be considered as a whole state which gives the $\overline{F}(v,w)$ a large dimension. Third it is hard to make the stabilization easier by choosing proper M and N which are the only matrices we can choose, because we cannot predict the features of $\overline{F}(v,w)$ by analyzing its complex expression (3.66).

The augmented extended system for the second approach is

$$\begin{aligned}\dot{z} &= \widehat{F}(w)z + \widehat{G}(e,v,w)y \\ \dot{\overline{\eta}} &= (M + T\varphi(v)T^{-1})\overline{\eta} + \chi(z,e,v,w) \\ \dot{e} &= \widehat{H}(w) + (\widehat{K}(e,v,w) + \Psi T^{-1}N)e + b(w)\Psi T^{-1}\overline{\eta} + b(w)\xi_1 \\ \dot{\xi}_i &= -\lambda_i\xi_i + \xi_{i+1}, \quad i = 1,\cdots,r-2 \\ \dot{\xi}_{r-1} &= -\lambda_{r-1}\xi_{r-1} + u\end{aligned} \tag{3.67}$$

where z and $\overline{\eta}$ are not coupled together and the stabilization is easier.

3.5 Conclusions

In this Chapter we have extended the major results of the local robust output regulation problem to the nonlocal robust output regulation problem with nonlinear exosystems. We need two steps to settle the output regulation problem. The first step is to convert the robust output regulation problem of the given plant into a robust stabilization problem, and the second step is to solve the robust stabilization problem. In addition, we have compared two existing frameworks of output regulation that one framework has the advantage for solving the solution of the regulator equations and the other can make the stabilization problem easier.

□ **End of chapter.**

Chapter 4

Applications of Robust Regional Synchronization via Output Regulation Techniques

It is noted that in Chapter 2 we have handled the local synchronization problem where the initial states of the master, slave systems and the controller as well as the uncertain parameter w are sufficiently small. In practice, it is desirable to design controllers for the synchronization problem with master signals and uncertain parameter w large enough. The topic of this chapter is to apply the output regulation techniques discussed in Chapter 3 to our nonlocal robust synchronization problem.

This chapter is organized as follows: In Section 4.1 we will reformulate the synchronization problem as a robust output regulation problem. And in Section 4.2 the approach discussed in Chapter 3 is applied to the problem of Duffing oscillator synchronizing with Chua's circuit. The effectiveness of our approach is evaluated by computer simulation. In Section 4.3, the nonlinear motion of SMIB has been investigated which exhibits chaotic behaviors and a feedback control law has been designed for the SMIB power system to synchronize with Van der Pol oscillator.

4.1 Problem Formulation

We first reformulate the robust synchronization problem as a robust output regulation problem where the given plant can be considered as the slave system and the exosystem as the master system. The dynamic equations of the slave system can be written in the form of the given plant (3.1) of the output regulation problem and we rewrite it here

$$\begin{aligned} \dot{x} &= F(w)x + G(y,v,w)y + g(w)u + D_1(v,w) \\ \dot{y} &= H(w)x + K(y,v,w)y + D_2(v,w) \end{aligned} \tag{4.1}$$

As we will discuss in Section 4.2 and 4.3, the dynamic equations of both the Duffing oscillator and the SMIB power system can be written in the form of (4.1).

The nonlinear master system can be decomposed into the following form

$$\dot{v} = a(v) = A_1 v + \sum_{k=2}^{N} A_k v a_k(v) \tag{4.2}$$

If the function $a(v)$ is sufficiently smooth and vanishes at its origin, it can always be written in the form of (4.2).

Our problem is to design a feedback control law such that the solution of the closed loop system starting from any initial state inside a known region which can be large enough is bounded for all $t \geq 0$ and the output of the slave system tracks the output of the master system asymptotically.

$$e = h(x,u,v,w) = y - q(v,w) \tag{4.3}$$

where $q(v,w)$ is the output of the master system. Then the robust synchronization problem can be viewed as the robust output regulation problem studied in Chapter 3.

4.2 Duffing Oscillator Synchronizes with Chua's Circuit

In this section, we will use the approach studied in Chapter 3 to solve the problem of the output of Duffing oscillator synchronizing with the output of Chua's Circuit.

The slave system is Duffing oscillator whose dynamic equations are given by

$$\ddot{y} + \delta\dot{y} + \alpha y + \beta y^3 = \gamma \cos \omega t + u \tag{4.4}$$

where $\alpha(w) = \alpha_0 + w$ is an uncertain parameter of the system, γ, δ and β are fixed parameters of the system.

The first step of our approach is to formulate the synchronization problem as the output regulation problem

$$
\begin{aligned}
\dot{x} &= -\delta y - \alpha(w)y - \beta y^3 + v_5 + u \\
\dot{y} &= x \\
\dot{v}_4 &= \omega v_5 \\
\dot{v}_5 &= -\omega v_4
\end{aligned}
\tag{4.5}
$$

where $x = \dot{y}$, x is the state, y is the output and u is the controller. The initial states of $v_4,\ v_5$ are $v_4(0) = 0,\ v_5(0) = \gamma$.

The master system is Chua's circuit

$$
\begin{aligned}
\dot{v}_1 &= \frac{1}{RC_1}v_2 + (-\frac{1}{RC_1} - \frac{a_1}{C_1} - \frac{a_3}{C1}v_1^2)v_1 \\
\dot{v}_2 &= \frac{1}{RC_2}v_1 - \frac{1}{RC_2}v_2 + \frac{1}{C_2}v_3 \\
\dot{v}_3 &= -\frac{1}{L}v_2
\end{aligned}
\tag{4.6}
$$

with parameters $R = 1,\ C_1 = 1/9.5,\ C_2 = 1,\ L = 0.2,\ a_1 = -8/7,\ a_3 = 1/5$. And the dynamics (4.6) can be rewritten in the form of (3.3) as

$$
\dot{v} = a(v) = \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = A_1 v + A_2 v a_2(v) \tag{4.7}
$$

where $A_1 = \begin{bmatrix} -\frac{1}{RC_1} - \frac{a_1}{C_1} & \frac{1}{RC_1} & 0 \\ \frac{1}{RC_2} & -\frac{1}{RC_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & -\frac{R_0}{L} \end{bmatrix}$, $A_2 = \begin{bmatrix} -\frac{a_3}{C_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $a_2(v) = v_1^2$.

4.2.1 Transfer the synchronization problem into the stabilization problem

Next we give the solution of regulator equations

$$
\begin{aligned}
\mathbf{y}(v,w) &= v_1 \\
\mathbf{x}(v,w) &= (-\frac{1}{RC_1} - \frac{a_1}{C_1})v_1 + \frac{1}{RC_1}v_2 - \frac{a_3}{C1}v_1^3 \\
\mathbf{u}(v,w) &= \alpha(w)v_1 R^2 C_1^2 C_2 \\
&\quad + \frac{1}{R^2C_1^2C_2}[(C_2 + 2C_2a_1R + C_2a_1^2R^2 + C_1 + R^2C_1C_2a_1 - \delta RC_1C_2 - \delta R^2C_1C_2a_1)v_1 \\
&\quad + (-C_2 - C_2a_1R - C_1 - RC_1C_2 + \delta RC_1C_2)v_2 + C_1Rv_3 \\
&\quad + (4C_2a_3R + 4C_2a_1R^2a_3 + R^2C_1C_2a_3 - RC_1C_2 - R^2C_1C_2a_1)v_1^3 \\
&\quad - 3a_3RC_2v_1^2v_2 + (3a_3^2R^2C_2 - R^2C_1C_2a_3)v_1^5] - v_5 \\
&= \alpha(w)v_1 + m_1v_1 + m_2v_2 + m_3v_3 + m_{30}v_1^3 + m_{21}v_1^2v_2 + m_{50}v_1^5 - v_5
\end{aligned}
$$

where m_1, m_2, m_3, m_{30}, m_{21}, m_{50} are coefficients depending on R, C_1, C_2, L, R_0, a_1, a_3 but independent of w. We can see that the solution $\mathbf{u}(v,w)$ is polynomial and in the form of

$$\mathbf{u}(v,w) = \mathbf{u}_c(v) + \hat{\mathbf{u}}(v,w)$$

where $\mathbf{u}_c(v) = m_1v_1 + m_2v_2 + m_3v_3 + m_{30}v_1^3 + m_{21}v_1^2v_2 + m_{50}v_1^5 - v_5$, $\hat{\mathbf{u}}(v,w) = \alpha(w)v_1$ and $\alpha(w)$ is the uncertain parameter of the slave system.

Performing an input transformation $\hat{u} = u - \mathbf{u}_c(v)$ on the Duffing system gives the following system:

$$
\begin{aligned}
\dot{x} &= -\delta\dot{y} - \alpha(w)y - \beta y^3 + v_5 + \mathbf{u}_c(v) + \hat{u} \\
\dot{y} &= x \\
v &= a(v).
\end{aligned}
\tag{4.8}
$$

And define

$$v^{[1]} = (v_1,\ v_2,\ v_3)^T.$$

We also define τ as follows,

$$
\begin{aligned}
\tau &= \hat{\mathbf{u}}(v,w) = \alpha(w)v_1 \\
&= r_1v_1 + r_2v_2 + r_3v_3 = Iv^{[1]}
\end{aligned}
\tag{4.9}
$$

where $I = [r_1,\ r_2,\ r_3] = [\alpha(w),\ 0,\ 0]$.

Using the approach in Section 2.4.2. we can obtain

$$\Phi_1 = \begin{bmatrix} I \\ IA_1 \\ IA_1^2 \end{bmatrix} A_1 \begin{bmatrix} I \\ IA_1 \\ IA_1^2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \\ 6.7857 & 5.8571 & 0.3571 \end{bmatrix} \tag{4.10}$$

and

$$\Phi_2 = \begin{bmatrix} I \\ IA_1 \\ IA_1^2 \end{bmatrix} A_2 \begin{bmatrix} I \\ IA_1 \\ IA_1^2 \end{bmatrix}^{-1} = \begin{bmatrix} -1.9 & 0 & 0 \\ -2.5786 & 0 & 0 \\ -21.5495 & 0 & 0 \end{bmatrix} \tag{4.11}$$

satisfying Assumption 7.

Pick a controllable pair (M, N)

$$M = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -72 \end{bmatrix}, \quad N = \begin{bmatrix} 3 \\ 15 \\ 36 \end{bmatrix},$$

and solve T from the sylvester equation $T\Phi_1 - MT = N\Psi$

$$T = \begin{bmatrix} 0.3699 & -0.0507 & 0.0061 \\ 0.3750 & -0.0094 & 0.0002 \\ 0.5000 & -0.0070 & 0.0001 \end{bmatrix}.$$

Finally, let $\theta = T\tau$ with $T \in R^{3\times 3}$ any nonsingular matrix. Then

$$\begin{aligned} \frac{\partial \theta(v,w)}{\partial v}\dot{v} &= \frac{\partial \theta(v,w)}{\partial v}(A_1 v + A_2 v a_2(v)) \\ &= T\phi(v)T^{-1}\theta(v,w) \\ &\triangleq \alpha(\theta(v,w), v) \end{aligned}$$

where $\phi(v) = \Phi_1 + \varphi(v)$, $\varphi(v) = \Phi_2 a_2(v)$ and

$$\hat{\mathbf{u}}(v,w) = \Psi T^{-1}\theta(v,w) \triangleq \beta(\theta(v,w), v).$$

The dynamic system

$$\dot{\eta} = \gamma(\eta, u, v) = M\eta + T\varphi(v)T^{-1}\eta + N\hat{u} \tag{4.12}$$

is the internal model with the output $\hat{u}$. The following augmented system (4.13) is the combination of (4.4) and the internal model (4.12)

$$\begin{aligned}\dot{x} &= -\delta\dot{y} - \alpha(w)y - \beta y^3 + v_5 + \mathbf{u}_c(v) + \hat{u} \\ \dot{y} &= x \\ \dot{\eta} &= M\eta + T\varphi(v)T^{-1}\eta + N\hat{u}.\end{aligned} \tag{4.13}$$

Applying the following coordinate and input transformation

$$\begin{aligned}z &= x - \mathbf{x}(v,w), \\ \overline{\eta} &= \eta - \theta(v,w), \\ e &= y - \mathbf{y}(v,w) \\ \overline{u} &= \hat{u} - \beta(\eta)\end{aligned} \tag{4.14}$$

on the augmented system (4.13) gives

$$\begin{aligned}\dot{z} &= -\alpha(w)e - \beta(3v_1^2 e + 3v_1 e^2 + e^3) - \delta z + \bar{u} + \beta(\bar{\eta}) \\ \dot{e} &= z \\ \dot{\overline{\eta}} &= (M + N\Psi T^{-1} + T\varphi(v)T^{-1})\overline{\eta} + N\overline{u}\end{aligned} \tag{4.15}$$

which can be reformed as

$$\begin{aligned}\dot{z} &= F(w)z + \widetilde{G}(e,v,w)e + \bar{u} + \beta(\bar{\eta}) \\ \dot{e} &= H(w)z + \widetilde{K}(e,v,w)e \\ \dot{\overline{\eta}} &= (M + N\Psi T^{-1} + T\varphi(v)T^{-1})\overline{\eta} + N\overline{u}\end{aligned} \tag{4.16}$$

where $F(v,w) = -\delta$, $\widetilde{G}(e,v,w)e = (3v_1^2 + 3v_1 e + e^2)e - \alpha(w)e$, $H(w) = 1$, $\widetilde{K}(e,v,w)e = 0$.

The relative degree of the above system is $r = 2$ satisfying Assumption 8. We further perform a coordinate transformation $\widetilde{\eta} = \overline{\eta} - NP(w)z$ on the augmented system (4.16) where $P(w) = \frac{H(w)}{b(w)}$, $b(w) = H(w)g(w)$, and obtain

$$\begin{aligned}\dot{z} &= [F(v,w) + g(w)\Psi T^{-1}NP]z + g(w)\Psi T^{-1}\widetilde{\eta} + \widetilde{G}(e,v,w)e + g(w)\overline{u} \\ &= F_{11}(v,w)z + F_{12}(w)\widetilde{\eta} + \widetilde{G}(e,v,w)e + g(w)\overline{u},\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\dot{\widetilde{\eta}} &= [MNP(w) + T\varphi(v)T^{-1}NP(w) - NP(w)F(w)]z + (M + F_{11})\overline{\eta} \\ &\quad - NP(w)\widetilde{G}(e,v,w)e \\ &= F_{21}(v,w)z + (M + T\varphi(v)T^{-1})\widetilde{\eta} - NP(w)\widetilde{G}(e,v,w)e\end{aligned} \tag{4.18}$$

Now let

$$\widetilde{G}_e(e,v,w) = \begin{bmatrix} \widetilde{G}(e,v,w) \\ -NP(w)\widetilde{G}(e,v,w) \end{bmatrix}, \ \zeta = \begin{bmatrix} z \\ \widetilde{\eta} \end{bmatrix},$$

$$F_a(v,w) = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & M + T\varphi(v)T^{-1} \end{bmatrix},$$

$$g_a(w) = \begin{bmatrix} g(w) \\ 0 \end{bmatrix}, \quad H_a(w) = \begin{bmatrix} \overline{H}(w) & 0 \end{bmatrix},$$

Then, in the coordinates ζ, e, system (4.16) can be put in the following form

$$\begin{aligned} \dot{\zeta} &= F_a(v,w)\zeta + \widetilde{G}_e(e,v,w)e + g_a(w)\overline{u}, \\ \dot{e} &= H_a(w)\zeta + \widetilde{K}(e,v,w)e, \end{aligned} \tag{4.19}$$

Define the following dynamic extension:

$$\dot{\xi}_1 = -\xi_1 + \overline{u}, \tag{4.20}$$

and perform the transformation as follows

$$\overline{\zeta} = \zeta - D(w)\xi_1 - h(v,w)e \tag{4.21}$$

where $D(w)$ and $h(v,w)$ can be calculated from (3.25), (3.29) and (3.30)

$$D(w) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad h(v,w) = \begin{bmatrix} 120.3 \\ -21 - 84.6v_1^2 \\ -585 - 83.4v_1^2 \\ -2556 - 112.4v_1^2 \end{bmatrix}$$

which turns the extended augmented system into the following:

$$\begin{aligned} \dot{\overline{\zeta}} &= \overline{F}(v,w)\overline{\zeta} + \overline{G}(e,v,w)e, \\ \dot{e} &= \overline{H}(w)\overline{\zeta} + \overline{K}(e,v,w)e + b(w)\xi_1, \\ \dot{\xi}_1 &= -\xi_1 + \overline{u} \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \overline{F}(v,w) &= F_a(v,w) - \frac{d(v,w)}{b(w)}H_a(w), \\ &= \begin{bmatrix} -1 & 0.0805 & -4.19 & 5.08 \\ 0 & -8 - 0.0566v_1^2 & 2.94v_1^2 & -3.57v_1^2 \\ 0 & -0.0566v_1^2 & -40 + 2.90v_1^2 & -3.52v_1^2 \\ 0 & -0.0752v_1^2 & 3.91v_1^2 & -72 - 4.74v_1^2 \end{bmatrix} \end{aligned} \tag{4.23}$$

$$
\begin{aligned}
\overline{G}(e,v,w) =& -\frac{\partial h(v,w)}{\partial v}\dot{v} + \left(F_a(v,w) - \frac{d(v,w)}{b(w)}H_a(w)\right)\frac{d(v,w)}{b(w)} \\
&+ \widetilde{G}(e,v,w) - \frac{d(v,w)}{b(w)}\widetilde{K}(e,v,w), \\
=& \begin{bmatrix} -10662 - 231.7v_1^2 - 3ev_1 - e^2 - \alpha(w) \\ 171 + 8089v_1^2 + 160v_1^4 + 9ev_1 + 3e^2 + 166v_1v_2 + 3\alpha(w) \\ 10689v_1^2 + 159v_1^4 + 23415 + 45ev_1 + 15e^2 + 167v_1v_2 + 15\alpha(w) \\ 18040v_1^2 + 213v_1^4 + 18416ev_1 + 36e^2 + 225v_1v_2 + 36\alpha(w) \end{bmatrix}
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
\overline{H}(w) &= H_a(w) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\
\overline{K}(e,v,w) &= H_a(w)\frac{d(v,w)}{b(w)} + \widetilde{K}(e,v,w) = 120.3.
\end{aligned}
\tag{4.25}
$$

Now the problem turns to be a non-autonomous stabilization problem. First we need to find the boundedness of Chua's circuit.

4.2.2 Boundedness of Chua's circuit

We rewrite Chua's circuit system (4.6) with parameters given in Section in an easier way.

$$
\begin{aligned}
\dot{v}_1 &= p[v_2 - v_1 - g(v_1)] \\
\dot{v}_2 &= v_1 - v_2 + v_3 \\
\dot{v}_3 &= -qv_2
\end{aligned}
\tag{4.26}
$$

where $p = 9.5$, $g(v_1) = a_1v_1 + a_3v_1^3 = -\frac{8}{7}v_1 + 0.2v_1^3$ and $q = 5$.

A smooth Chua's circuit's ultimate boundedness has been obtained in [37] by constructing a radially unbounded and positive definite Lyapunov function

$$
\begin{aligned}
V &= \frac{v_1^2}{p} + v_2^2 + \frac{v_3^2}{q} - \sigma v_2 v_3 \\
&= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^T G_1 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.
\end{aligned}
\tag{4.27}
$$

Differentiating V w.r.t. time t

$$
\begin{aligned}
\dot{V} &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^T G_2(v_1) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
&= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^T G_2(v_{b1}) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - 2a_3(v_1^2 - v_{b1}^2)v_1^2 \\
&\leq \frac{\mu_M(G_2(v_{b1}))}{\lambda_M(G_1)} \left[V - \frac{2a_3 v_{b1}^4}{\frac{-\mu_M(G_2(v_{b1}))}{\lambda_M(G_1)}} \right]
\end{aligned}
\tag{4.28}
$$

where

$$
G_1 = \begin{bmatrix} \frac{1}{p} & 0 & 0 \\ 0 & 1 & -\frac{\sigma}{2} \\ 0 & -\frac{\sigma}{2} & \frac{1}{q} \end{bmatrix},
$$

$$
G_2(v_1) = \begin{bmatrix} -2(a_1+1) - a_3 v_1^2 & 2 & \frac{\sigma}{2} \\ 2 & -2 + \sigma q & -\frac{\sigma}{2} \\ -\frac{\sigma}{2} & \frac{\sigma}{2} & -\sigma \end{bmatrix},
$$

and $\mu_M(G_2(v_{b1}))$ and $\lambda_M(G_1)$ are respectively the maximum eigenvalues of $G_2(v_{b1})$ and G_1.

Choosing proper σ and v_{b1}, we can have

$$
\begin{cases} G_1 > 0, \\ G_2(v_{b1}) < 0. \end{cases} \tag{4.29}
$$

And when

$$
V(v_0) > \frac{2a_3 v_{b1}^4}{\frac{-\mu_M(G_2(v_{b1}))}{\lambda_M(G_1)}} \triangleq V_b, \tag{4.30}
$$

and

$$
V(v(t, t_0, v(0))) > V_b \tag{4.31}
$$

the following inequality

$$
(V(v(t, t_0, v(0))) - V_b) \leq (V(v(0)) - V_b) e^{\frac{2a_3 v_{b1}^4}{\frac{-\mu_M(G_2(v_1(0)))}{\lambda_M(G_1)}}(t - t_0)} \tag{4.32}
$$

holds.

We can obtain the range of σ and v_{b1} from inequalities (4.29)

$$0 < \sigma < \min\{\frac{2}{p},\ \frac{2}{\sqrt{q}}\} \tag{4.33}$$

$$v_{b1}^2 > \max\left\{\frac{1}{2a_3}\left[\frac{(2-\frac{\sigma}{2})^2}{2-\sigma q-\frac{\sigma}{4}} - 2(a_1+1)+\frac{\sigma}{4}\right],\ \frac{-2(a_1+1)+\frac{\sigma}{4}}{2a_3}\right\}. \tag{4.34}$$

After calculation, we choose $\sigma = 0.15$, $v_{b1} = 2.95$ and obtain $V_b = 47.7$ from (4.30).

Let $\overline{\Omega} = \{v|V(v) = V_b\}$ and $v_{b2}^2 = \sup_{\Omega\in v^2} v_1^2$. Then when $v^2 \geq \max\{v_{b1}^2,\ v_{b2}^2\} \triangleq v_b^2$, $V(v(t,t_0,v_0))$ is exponentially decreasing and ultimately enters into the region $V(v) \leq V_b$, i.e.

$$V = \frac{v_1^2}{p} + v_2^2 + \frac{v_3^2}{q} - \sigma v_2 v_3 \tag{4.35}$$

and we have $\frac{v_{b2}^2}{p} < V_b$ which is $|v_1| < |v_{b2}| = 20$.

We have estimated the boundedness of v_1, and the boundedness of v_2 and v_3 is proved in [37]. Because we only need the estimation of the boundedness of v_1 for stabilization, here we ignore proof of the boundedness of v_2 and v_3.

4.2.3 Stabilization

Now we will show that for all $v \in R^3$, $\overline{F}(v,w) < 0$. Here we find that $\overline{F}(v,w) = \overline{F}(v)$ in system (4.22) is not dependent on w

$$\overline{F}(v) = \begin{bmatrix} -1 & 0.0805 & -4.19 & 5.08 \\ 0 & -8-0.0566v_1^2 & 2.94v_1^2 & -3.57v_1^2 \\ 0 & -0.0566v_1^2 & -40+2.90v_1^2 & -3.52v_1^2 \\ 0 & -0.0752v_1^2 & 3.91v_1^2 & -72-4.74v_1^2 \end{bmatrix}$$

We can check that

$$
\begin{aligned}
&-1<0\\
&\begin{vmatrix}-1 & 0.0805\\ 0 & -8-0.0566v_1^2\end{vmatrix}>0\\
&\begin{vmatrix}-1 & 0.0805 & -4.19\\ 0 & -8-0.0566v_1^2 & 2.94v_1^2\\ 0 & -0.0566v_1^2 & -40+2.90v_1^2\end{vmatrix}<0\\
&\begin{vmatrix}-1 & 0.0805 & -4.19 & 5.08\\ 0 & -8-0.0566v_1^2 & 2.94v_1^2 & -3.57v_1^2\\ 0 & -0.0566v_1^2 & -40+2.90v_1^2 & -3.52v_1^2\\ 0 & -0.0752v_1^2 & 3.91v_1^2 & -72-4.74v_1^2\end{vmatrix}>0
\end{aligned}
$$

for all $v \in R^3$. So $\overline{F}(v,w)<0$, for all $v \in R^3$.

Define $V_0(\overline{\zeta},t)=l\overline{\zeta}^T Q\overline{\zeta}$, and we need to find $Q>0$, for all $v \in V$

$$Q\overline{F}(v)+\overline{F}(v)^T Q < -I. \tag{4.36}$$

Use LMI tools in Matlab, we can obtain a positive definite Q

$$Q=\begin{bmatrix}7.9462 & 0.0775 & -0.7968 & 0.5339\\ 0.0775 & 0.0015 & -0.0136 & 0.0089\\ -0.7968 & -0.0136 & 0.1629 & -0.1168\\ 0.5339 & 0.0089 & -0.1168 & 0.0887\end{bmatrix}\times 10^5$$

and the eigenvalues of Q are [0.3, 1.73, 64.2, 1243]. The bound of V at least needs to cover the whole trajectories of Chua's circuit for $t \geq 0$.

$$
\begin{aligned}
&Q\overline{F}(v)+\overline{F}(v)^T Q+I=\\
&\begin{bmatrix}-1589244 & -5723-6.9v_1^2 & -61560+359v_1^2 & 140220-435v_1^2\\ -5723-6.9v_1^2 & -1217+0.535v_1^2 & 26389+32.3v_1^2 & -27487-48.59v_1^2\\ -61561+359v_1^2 & 26389+32.3v_1^2 & -640631-4808v_1^2 & 686143+6322v_1^2\\ 140220-435v_1^2 & -27487-48.6v_1^2 & 686143+6322v_1^2 & -742968-8262v_1^2\end{bmatrix}<0
\end{aligned}
$$

which is equivalent to the following inequalities (4.37), (4.38), (4.39) and (4.40):

$$-1589244<0. \tag{4.37}$$

$$\begin{vmatrix} -1589244 & -5723-6.9v_1^2 \\ -5723-6.9v_1^2 & -1217+0.535v_1^2 \end{vmatrix} > 0 \tag{4.38}$$

for all $|v_1| < 43.2$.

$$\begin{vmatrix} -1589244 & -5723-6.9v_1^2 & -61560+359v_1^2 \\ -5723-6.9v_1^2 & -1217+0.535v_1^2 & 26389+32.3v_1^2 \\ -61561+359v_1^2 & 26389+32.3v_1^2 & -640631-4808v_1^2 \end{vmatrix} < 0 \tag{4.39}$$

for all $|v_1| < 31.5$.

$$\begin{vmatrix} -1589244 & -5723-6.9v_1^2 & -61560+359v_1^2 & 140220-435v_1^2 \\ -5723-6.9v_1^2 & -1217+0.535v_1^2 & 26389+32.3v_1^2 & -27487-48.59v_1^2 \\ -61561+359v_1^2 & 26389+32.3v_1^2 & -640631-4808v_1^2 & 686143+6322v_1^2 \\ 140220-435v_1^2 & -27487-48.6v_1^2 & 686143+6322v_1^2 & -742968-8262v_1^2 \end{vmatrix} > 0 \tag{4.40}$$

for all $|v_1| < 26$. The inequalities (4.37) to (4.40) prove that $Q\overline{F}(v) + \overline{F}(v)^T Q + I < 0$, for all $v \in V = \{v| \ |v_1| < 26\}$ which covers the trajectories of Chua's circuit.

To design the control law of the synchronization problem, let us introduce some inequalities . There exist smooth real valued functions $q_i(v,w), \ a_i(e), \ i = 1,2$ such that, for all $v \in V$ and $w \in W$

$$|\overline{G}(e,v,w)e|^2 \leq q_1(v,w)a_1(e)e^2 \tag{4.41}$$

$$|\overline{K}(e,v,w)e|^2 \leq q_2(v,w)a_2(e)e^2. \tag{4.42}$$

Here we can use $a_1(e) = 1 + e^4$, $a_2(e) = 1$ and $q_i(v,w), \ i = 1,2$ are bounded.

Define,

$$V_0(\overline{\zeta}, t) = l\overline{\zeta}^T Q \overline{\zeta}$$

which is positive definite. Thus

$$\begin{aligned} \dot{V}_0 &= l\overline{\zeta}^T [\overline{F}(v,w)^T Q + Q\overline{F}(v,w)]\overline{\zeta} + 2l\overline{\zeta}^T Q\overline{G}(e,v,w)e \\ &\leq -l|\overline{\zeta}|^2 + \epsilon l|\overline{\zeta}|^2 + \frac{1}{\epsilon} l|Q|^2 q_1(v,w)a_1(e)e^2 \end{aligned} \tag{4.43}$$

Next we define

$$\overline{\xi}_1 = \xi_1 - \alpha_1 \tag{4.44}$$

$$\dot{k} = \rho(e)e^2, \tag{4.45}$$

$$\alpha_1(e,k) = -k\rho(e)e \tag{4.46}$$

and

$$V_1 = l\bar{\zeta}^T Q\bar{\zeta} + \frac{1}{2}e^2 + \frac{1}{2}b(w)(k - \bar{k})^2 \tag{4.47}$$

The time derivative of V_1 along the trajectory of (4.44) to (4.46) is given by

$$\begin{aligned}
\dot{V}_1 =& \dot{V}_0 + e\dot{e} + b(w)(k - \bar{k})\dot{k} \\
\leq& - l|\bar{\zeta}|^2 + \epsilon l|\bar{\zeta}|^2 + \frac{1}{\epsilon}l|Q|^2 q_1(v,w)a_1(e)e^2 \\
&+ e[\overline{H}(w)\bar{\zeta} + \overline{K}(e,v,w)e + b(w)\xi_1] + b(w)(k - \bar{k})\rho(e)e^2 \\
\leq& - (l - \epsilon)|\bar{\zeta}|^2 + \frac{1}{\epsilon}l|Q|^2 q_1(v,w)a_1(e)e^2 + |\overline{H}|^2|\bar{\zeta}|^2 + \frac{1}{4}e^2 \\
&\frac{1}{4}e^2 + q_2(v,w)a_2(e)e^2 + b(w)e\bar{\xi}_1 - b(w)\overline{k}\rho(e)e^2 \\
=& - (l - \epsilon - |\overline{H}|^2)|\bar{\zeta}|^2 + [\frac{1}{\epsilon}l|Q|^2 q_1(v,w)a_1(e) \\
&+ \frac{1}{2} + q_2(v,w)a_2(e) - b(w)\overline{k}\rho(e)]e^2 + b(w)e\bar{\xi}_1
\end{aligned} \tag{4.48}$$

Define

$$V_2 = V_1 + \frac{1}{2}\bar{\xi}_1^2 \tag{4.49}$$

and

$$\begin{aligned}
\bar{\xi}_1\dot{\bar{\xi}}_1 =& \bar{\xi}(\dot{\xi}_1 - \dot{\alpha}_1) \\
=& \bar{\xi}_1(-\dot{\xi}_1 + \bar{u}) - \bar{\xi}_1\dot{\alpha}_1 \\
=& \bar{\xi}_1[-(\dot{\bar{\xi}}_1 + \alpha_1) + \bar{u}] - \bar{\xi}_1\frac{\partial\alpha_1}{\partial k}\rho(e)e^2 \\
&- \bar{\xi}_1\frac{\partial\alpha_1}{\partial e}(\overline{H}(w)\zeta + \overline{K}(e,v,w)e + b(w)\xi_1) \\
\leq& - \bar{\xi}_1^2 + \bar{\xi}_1(-\alpha_1 + \bar{u} - \frac{\partial\alpha_1}{\partial k}\rho(e)e^2) + \frac{1}{4}|\overline{H}|^2|\bar{\zeta}|^2 \\
&+ (\bar{\xi}_1\frac{\partial\alpha_1}{\partial e})^2 + (\bar{\xi}_1\frac{\partial\alpha_1}{\partial e})^2 + \frac{1}{4}q_2(v,w)a_2(e)e^2 - \bar{\xi}_1\frac{\partial\alpha_1}{\partial k}b(w)\xi_1 \\
=& - \bar{\xi}_1^2 + \bar{\xi}_1(-\alpha_1 + \bar{u} - \frac{\partial\alpha_1}{\partial k}\rho(e)e^2 + 2(\bar{\xi}_1\frac{\partial\alpha_1}{\partial e})^2 - \frac{\partial\alpha_1}{\partial e}\xi_1) \\
&+ \frac{1}{4}|\overline{H}|^2|\bar{\zeta}|^2 + \frac{1}{4}q_2(v,w)a_2(e)e^2
\end{aligned} \tag{4.50}$$

Then we have

$$
\begin{aligned}
\dot{V}_2 =& \dot{V}_1 + \bar{\xi}_1 \dot{\bar{\xi}}_1 \\
=& -(l - \epsilon - \frac{5}{4}|\overline{H}|^2)|\bar{\zeta}|^2 \\
&+ [\frac{1}{\epsilon} l(w)|Q(w)|^2 q_1(v,w) a_1(e) + \frac{1}{2} + \frac{5}{4} q_2(v,w) a_2(e) - b(w)\bar{k}\rho(e)]e^2 \\
&- \bar{\xi}_1^2 + \bar{\xi}_1 [b(w)e - \alpha_1 + \bar{u} - \frac{\partial \alpha_1}{\partial k}\rho(e)e^2 + 2(\bar{\xi}_1 \frac{\partial \alpha_1}{\partial e})^2]
\end{aligned}
\tag{4.51}
$$

At the end of the design, taking

$$
\bar{u} = -b(w)e + \alpha_1 + \frac{\partial \alpha_1}{\partial k}\rho(e)e^2 - 2(\bar{\xi}_1 \frac{\partial \alpha_1}{\partial e})^2 + \frac{\partial \alpha_1}{\partial e}\xi_1 \tag{4.52}
$$

gives

$$
\begin{aligned}
\dot{V}_2 =& -(l - \epsilon - \frac{5}{4}|\overline{H}|^2)|\bar{\zeta}|^2 - \bar{\xi}_1^2 \\
&+ [\frac{1}{\epsilon} l(w)|Q(w)|^2 q_1(v,w) a_1(e) + \frac{1}{2} + \frac{5}{4} q_2(v,w) a_2(e) - b(w)\bar{k}\rho(e)]e^2
\end{aligned}
\tag{4.53}
$$

Because $a_1(e) = 1 + e^4$, $a_2(e) = 1$ and $q_1,\ q_2$ are bounded, we can choose $\rho(\cdot) = 1 + e^4$, and $\epsilon > 0$, $l > 0$ satisfying $l - \epsilon - \frac{5}{4}|\overline{H}|^2 > 0$. Since $b(w) = 1 > 0$, there exists a sufficiently large positive constant $\bar{k}$, satisfying

$$
\bar{k} > \frac{\frac{1}{\epsilon} l |Q|^2 q_1(v,w) a_1(e) + \frac{1}{2} + \frac{5}{4} q_2(v,w) a_2(e)}{b(w)\rho(e)} \tag{4.54}
$$

Then applying the design above to the augmented system yields a dynamic output feedback controller given by

$$
\begin{aligned}
u &= \bar{u} + \beta(\eta) + \mathbf{u}_c(v,w) \\
\hat{u} &= \bar{u} + \beta(\eta) \\
\dot{\eta} &= \gamma(\eta, u, v) = M\eta + T\varphi(v)T^{-1}\eta + N\hat{u} \\
\dot{k} &= (1 + e^4)e^2 \\
\dot{\xi}_1 &= -\xi_1 + \bar{u}
\end{aligned}
\tag{4.55}
$$

which solves the original synchronization of Duffing oscillator (4.8) with Chua's circuit (4.6).

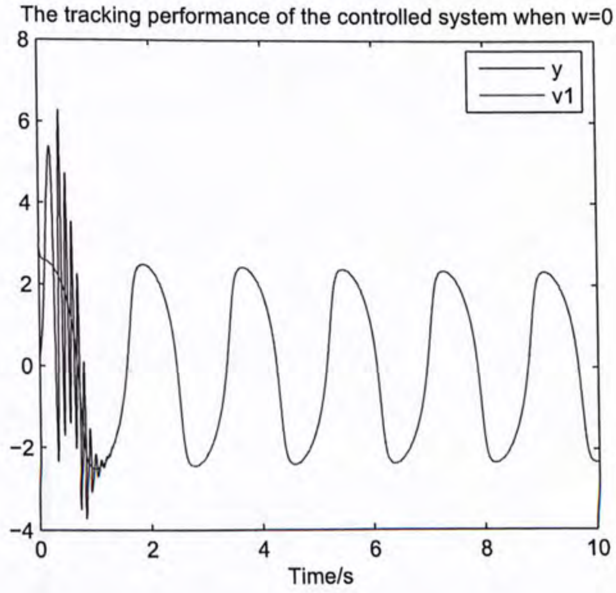

Figure 4.1: Tracking performance of the controlled circuit when $w = 0$

4.2.4 Simulation Results

The performance of our design is verified by computer simulation. Parameters of the system (4.8) are $\alpha_0 = 1,\ \beta = 1,\ \gamma = 1$ with the uncertain parameter $w = 0$ or $w = \pm 0.5$, and initial states of the closed-loop system and exosystem are $x(0) = [7,\ 7],\ \xi(0) = 7,\ \eta(0) = [1,\ 1,\ 1],\ v(0) = [1,\ 1,\ 1,\ 0,\ 1],\ k(0) = 1$. Fig 4.1, 4.2 show the tracking performance and synchronization error when the uncertain parameter $w = 0$. Fig 4.3, 4.4 show the tracking performance and synchronization error when the uncertain parameter $w = \pm 0.5$.

4.3 The Chaotic SMIB Power System Synchronizes with Van der Pol Oscillator

Electrical power systems are essentially nonlinear dynamic systems. Studies on the properties of power systems and its control [16], [18] and [17] could be of considerable importance from the point of view of avoiding undesirable behaviors such as power blackout. Synchronization in power systems is of huge importance from the management point of view of complex power systems. This section is devoted to complete synchronization of

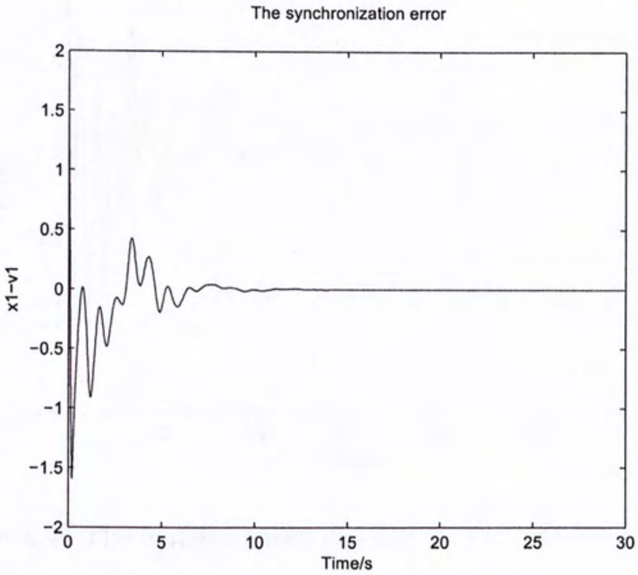

Figure 4.2: The synchronization error $x_1 - v_1$ when $w = 0$

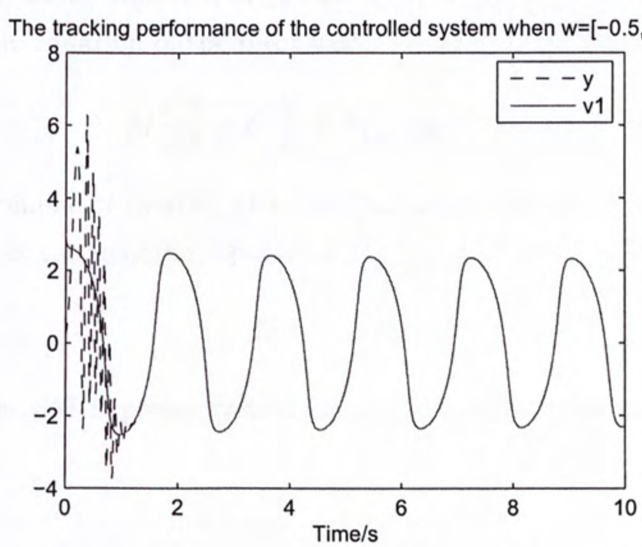

Figure 4.3: Tracking performance of the controlled circuit when $w = \pm 0.5$

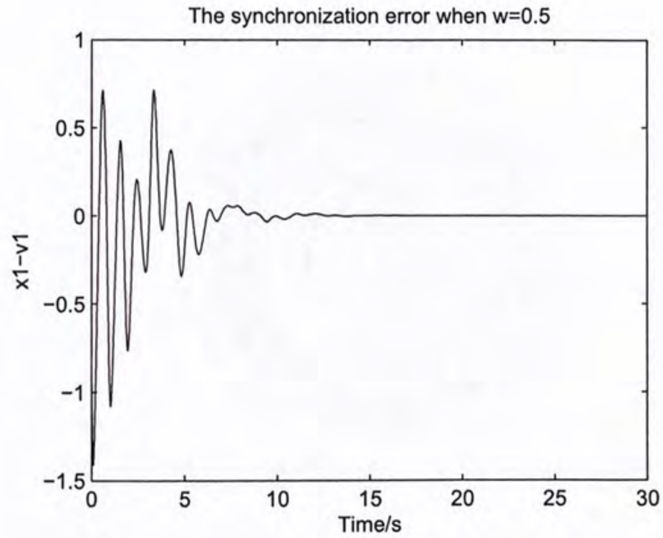

Figure 4.4: The synchronization error $x_1 - v_1$ when $w = \pm 0.5$

a single-machine-infinite-bus (SMIB) power system with Van der Pol oscillator. We use numerical simulations to estimate the control law for the synchronization to occur.

The power transmission of the synchronous generator is reflected on the dynamic behavior of the magnet rotor. So the dynamic performance of the generator could be described by the dynamic equation of the rotor. Considering the classical SMIB power system, the dynamic equation of the rotor with the rotor angle θ as the state is as follows

$$M\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + P_{max}\sin\theta = P_m \tag{4.56}$$

where M is the moment of inertia, D is the damping constant, P_m is the power of the machine, and P_{max} is the maximum power of generator. The P_m is also assumed to be as

$$P_m = A\sin\omega t \tag{4.57}$$

We note that the SMIB power system can be reformulated as an autonomous system as follows

$$\begin{aligned}
\dot{x} &= -cx - \beta\sin y + v_3 + u \\
\dot{y} &= x \\
\dot{v}_3 &= \omega v_4 \\
\dot{v}_4 &= -\omega v_3
\end{aligned} \tag{4.58}$$

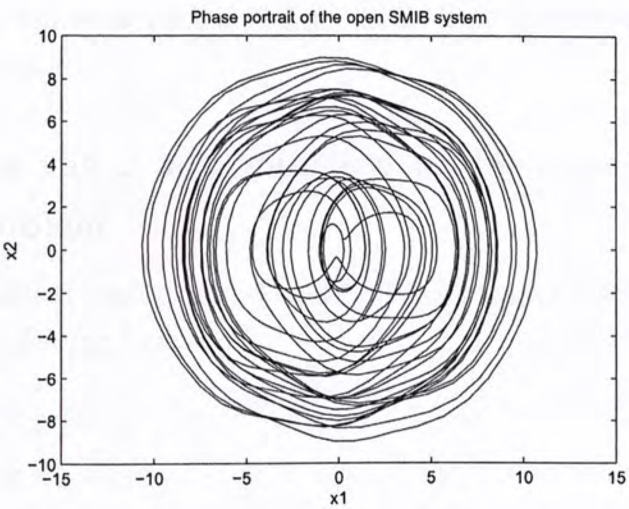

Figure 4.5: The trajectory of SMIB power system

where

$$y = \theta,\ x = \dot{\theta},\ c = \frac{D}{M},\ \beta = \frac{P_m}{M},\ f = \frac{A}{M}$$

and

$$v_3(0) = 0,\ v_4(0) = f.$$

Conclusive investigation of the dynamic behavior of the SMIB power system is conducted in [5]. Specifically the authors in [5] establish the range of parameters, when such a simple system exhibits a chaotic behavior, e.g. Fig. 4.5 shows the chaotic behavior when $\beta = 1,\ c = 0.5,\ \omega = 1$, and $f = 2.45$.

In this section we will design a controller to synchronize the chaotic SMIB power system with Van der Pol oscillator based on the approach discussed in Chapter 3.

The following typical Van der Pal oscillator is considered as the master system

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -av_1 + (1 - v_1^2)v_2 \end{aligned} \tag{4.59}$$

and can be rewritten as

$$\dot{v} = \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = A_1 v + A_2 v a_2(v) \tag{4.60}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -a & b \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix},\ a_2(v) = v_1^2.$$

For convenience of discussion, let $a = 1,\ b = 1$. The Fig 2.3 shows a globally asymptotically stable limit cycle.

4.3.1 Transfer the synchronization problem into the stabilization problem

It is easy to verify that the solution of the regulator equations of system (4.58) and (4.60) exists and has the following form

$$\begin{aligned}
\mathbf{y}(v,w) &= v_1 \\
\mathbf{x}(v,w) &= v_2 \\
\mathbf{u}(v,w) &= (c_0 + w)v_2 - v_1 + (1 - v_1^2)v_2 + \beta\sin(v_1) - v_3
\end{aligned}$$

It can be seen that the solution $\mathbf{u}(v,w)$ can be written as follows

$$\mathbf{u}(v,w) = \mathbf{u}_c(v) + \hat{\mathbf{u}}(v,w)$$

where $\hat{\mathbf{u}}(v,w) = (c_0 + w)v_2$ vanishing at the origin, and $\mathbf{u}_c(v,w) = -v_1 + (1 - v_1^2)v_2 + \beta\sin(v_1) - v_3$ which does no depend on w.

Define π as

$$\pi = \hat{\mathbf{u}}(v) = (c_0 + w)v_2 = r_1 v_1 + r_2 v_2 = F_1 v^{[1]}$$

where $F_1 = [r_1,\ r_2] = [0,\ c_0 + w]$, and $\hat{\mathbf{u}}(v,w) = \pi$. Define

$$\tau(v,w) = [\pi,\ L_{A_1 v}\pi]^T,$$

and there exist matrices

$$\Phi_1 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \Psi = [1,\ 0]$$

and

$$\Phi_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

satisfying Assumptions 7.

Let $\beta(v,w) = T\tau(v,w)$, where $T \in R^{2\times 2}$ is any nonsingular matrix. The derivative of θ satisfies

$$\frac{\partial \theta(v,w)}{\partial v} a(v) = T\phi(v)T^{-1}\theta(v,w)$$

where $\phi(v) = \Phi_1 + \varphi(v)$ and $\varphi(v) = \Phi_2 a_2(v)$, with $\varphi(0) = 0$ and $\phi(0) = \Phi_1$.

Let

$$M_1 = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

There exists a unique nonsingular matrix T, such that the following Sylvester equation $T\Phi_1 - MT = N\Psi$ holds

$$T = \begin{bmatrix} 0.6667 & -0.3333 \\ 0.9677 & -0.1613 \end{bmatrix}.$$

The internal model with the output $\hat{u}$ is as follows

$$\dot{\eta} = \gamma(\eta, u, v) = M\eta + T\varphi(v)T^{-1}\eta + N\hat{u} \tag{4.61}$$

where $\hat{u} = u - \mathbf{u}_c(v, w)$. Performing the following coordinate and input transformation

$$\begin{aligned} z &= x - \mathbf{x}(v, w), \\ e &= y - q(v, w), \\ \overline{\eta} &= \eta - \theta(v, w), \\ \overline{u} &= \hat{u} - \beta(\eta) \end{aligned}$$

on the augmented system one can obtain the following system

$$\begin{aligned} \dot{z} &= -(c_0 + w)z - \beta[\sin(e + v_1) - \sin v_1] + \bar{u} + \beta(\eta) - \beta(\theta) \\ \dot{e} &= z \\ \dot{\overline{\eta}} &= (M + N\Psi T^{-1} + T\varphi(v)T^{-1})\overline{\eta} + N\overline{u}. \end{aligned} \tag{4.62}$$

We further perform the following coordinate transformation on (4.62)

$$\widetilde{\eta} = \overline{\eta} - NP(w)z$$

where $P(w) = \frac{H(w)}{b(w)} = 1$, $b(w) = H(w)g(w) = 1$. Define $\zeta = [z,\ \widetilde{\eta}]^T$, and the system (4.62) can be written in the following form

$$\begin{aligned} \dot{\zeta} &= F_a(v, w)\zeta + G_a(e, v, w) + g_a(w)\overline{u} \\ \dot{e} &= H_a(w)\zeta + K_a(e, v, w) \end{aligned} \tag{4.63}$$

where

$$
\begin{aligned}
F_a(v,w) &= \begin{bmatrix} -(c_0+w)+7 & -0.75 & \frac{31}{20} \\ -1+\frac{11v_1^2}{3}+(c_0+w) & -1-\frac{3v_1^2}{2} & \frac{31v_1^2}{30} \\ -25+\frac{50v_1^2}{31}+5(c_0+w) & -\frac{45v_1^2}{62} & -5+\frac{v_1^2}{2} \end{bmatrix}, \\
G_a(e,v,w) &= \begin{bmatrix} -\sin(v_1+e)+\sin v_1 \\ \sin(v_1+e)-\sin v_1 \\ 5\sin(v_1+e)-5\sin v_1 \end{bmatrix} \\
g_a(w) &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
H_a(w) &= [1,\ 0,\ 0] \\
K_a(e,v,w) &= 0
\end{aligned}
$$

There is a difference between the above system (4.62) and the standard system (3.17) that $G_a(e,v,w)$ can not factorize a factor e as the form $G_e(e,v,w)e$ in system (3.17).

Attach the following dynamic system to (4.63),

$$\dot{\xi}_1 = -\xi_1 + \overline{u} \tag{4.64}$$

and perform the following transformation on the extended augmented system (4.63) and (4.64)

$$\overline{\zeta} = \zeta - D(w)\xi_1 - h(v,w)e \tag{4.65}$$

where $D(w),\ h(v,w)$ can be calculated as follows

$$
\begin{aligned}
b(w) &= H_a(w)g_a(w) \\
d(v,w) &= (F_a(v,w)+I)g_a(w) \\
h(v,w) &= \frac{d(v,w)}{b(w)} \\
D(w) &= g_a(w)
\end{aligned} \tag{4.66}
$$

Then the extended augmented system turns into the following form:

$$
\begin{aligned}
\dot{\overline{\zeta}} &= \overline{F}(v,w)\overline{\zeta} + \overline{G}(e,v,w) \\
\dot{e} &= \overline{H}(w)\overline{\zeta} + \overline{K}(e,v,w) + b(w)\xi_1 \\
\dot{\xi}_1 &= -\xi_1 + \overline{u}
\end{aligned} \tag{4.67}
$$

where

$$
\begin{aligned}
\overline{F}(v,w) &= F_a(v,w) - \frac{d(v,w)}{b(w)} H_a(w), \\
\overline{G}(e,v,w) &= -\frac{\partial h(v,w)}{\partial v}\dot{v}e + \left(F_a(v,w) - \frac{d(v,w)}{b(w)} H_a(w)\right)\frac{d(v,w)}{b(w)}e \\
&\quad + G_a(e,v,w) - \frac{d(v,w)}{b(w)} K_a(e,v,w), \\
\overline{H}(w) &= H_a(w), \\
\overline{K}(e,v,w) &= H_a(w)\frac{d(v,w)}{b(w)}e + K_a(e,v,w)
\end{aligned}
\tag{4.68}
$$

Substituting the parameters of the system into (4.68), we get

$$
\begin{aligned}
\overline{F}(v,w) &= \begin{bmatrix} -1 & -0.75 & \frac{31}{20} \\ 0 & -1-\frac{3v_1^2}{2} & \frac{31v_1^2}{30} \\ 0 & -\frac{45v_1^2}{62} & -5+\frac{v_1^2}{2} \end{bmatrix} \\
\overline{G}(e,v,w) &= \begin{bmatrix} 8ce - 46e - \sin(v_1+e) + \sin v_1 \\ e - 28ev_1^2 - ce - \frac{11}{3}(ev_1^4 + ev_1^2 c) + \sin(v_1+e) - \sin v_1 - \frac{22}{3}v_1 v_2 e \\ \frac{-640}{31}ev_1^2 - \frac{55}{31}e(v_1^4 + ev_1^2 c) + 125e - 25ce + 5\sin(v_1+e) - 5\sin v_1 - \frac{110}{31}v_1 v_2 e \end{bmatrix} \\
\overline{H}(w) &= [1,\ 0,\ 0] \\
\overline{K}(e,v,w) &= -ce + 8e \\
c &= c_0 + w
\end{aligned}
$$

4.3.2 Stabilization

The original synchronization problem has been transfer into the stabilization problem of system (4.67). First we can get

$$
\overline{F}(v,w) = \begin{bmatrix} -1 & -0.75 & \frac{31}{20} \\ 0 & -1-\frac{3v_1^2}{2} & \frac{31v_1^2}{30} \\ 0 & -\frac{45v_1^2}{62} & -5+\frac{v_1^2}{2} \end{bmatrix} < 0
$$

for all $v_1 \in R$. To deal with the stabilization problem, we use the following inequality

$$
|\sin(v_1+e) - \sin v_1| = |-2\sin(\frac{2v_1+e}{2})\sin\frac{e}{2}| \leq |e|\cdot|\sin(\frac{2v_1+e}{2})| \tag{4.69}
$$

on $\overline{G}(e,v,w)$

$$|\overline{G}(e,v,w)| \leq \left| \begin{matrix} 8ce-46e \\ e-28ev_1^2-ce-\frac{11}{3}(ev_1^4+ev_1^2c)-\frac{22}{3}v_1v_2e \\ \frac{-640}{31}ev_1^2-\frac{55}{31}e(v_1^4+ev_1^2c)+125e-25ce-\frac{110}{31}v_1v_2e \end{matrix} \right| + \left| \begin{matrix} -\sin(v_1+e)+\sin v_1 \\ \sin(v_1+e)-\sin v_1 \\ 5\sin(v_1+e)-5\sin v_1 \end{matrix} \right|$$
$$\leq \left(\left| \begin{matrix} 8c-46 \\ 1-28v_1^2-c-\frac{11}{3}(v_1^4+v_1^2c)-\frac{22}{3}v_1v_2 \\ \frac{-640}{31}v_1^2-\frac{55}{31}(v_1^4+v_1^2c)+125-25c-\frac{110}{31}v_1v_2 \end{matrix} \right| + \left| \begin{matrix} 1 \\ 1 \\ 5 \end{matrix} \right| \right) |e| \tag{4.70}$$

where v and w are bounded. Because Van der Pol oscillator has a limit cycle, we know that when the initial state is inside the limit cycle, the trajectories will be contained in the limit cycle, that is $v \in V = \{v|\ |v_1| < 3\}$ for all $t \geq 0$.

Then we suppose $w \in W$ is bounded, and

$$|\overline{K}(e,v,w)|^2 = |-ce+8e|^2 \leq |-c+8|^2|e|^2.$$

So $q_2(v,w) = |-c+8|^2$ and is bounded. Now we have found the smooth real valued functions $q_i(v,w) > 0,\ i = 1,2$ for each $v \in V,\ w \in W,\ e \in R$, such that the following inequalities are satisfied

$$|\overline{G}(e,v,w)|^2 \leq q_1(v,w)e^2 \tag{4.71}$$
$$|\overline{K}(e,v,w)|^2 \leq q_2(v,w)e^2. \tag{4.72}$$

Comparing inequalities (4.71) and (4.72) with inequalities (4.41) and (4.42) in the last example, we have $a_1(e) = 1,\ a_2(e) = 1$.

Next define $V_0(\overline{\zeta},t)$ as follows

$$V_0(\overline{\zeta},t) = l\overline{\zeta}^T Q\overline{\zeta}$$

where Q is positive definite, $l > 0$, and for all $v \in V$, Q satisfying

$$QF(v) + F(v)^T Q < -I$$

We use $v \in V = \{v|\ |v_1| < 3\}$, and V covers the whole limit cycle of Van del Pol oscillator. And we choose

$$Q = \begin{bmatrix} 1.1254 & -0.3268 & 0.2867 \\ -0.3268 & 1.3938 & -0.1170 \\ 0.2867 & -0.1170 & 0.3318 \end{bmatrix}$$

which satisfies

$$QF(v) + F(v)^T Q =$$

$$\begin{bmatrix} -2.25 & -0.19 + 0.282v_1^2 & 0.024 - 0.194v_1^2 \\ -0.19 + 0.282v_1^2 & -2.30 - 4.0v_1^2 & -0.019 + 1.32v_1^2 \\ 0.024 - 0.19v_1^2 & -0.019 + 1.32v_1^2 & -2.43 + 0.09v_1^2 \end{bmatrix} < -I$$

Define

$$\overline{\xi}_1 = \xi_1 - \alpha_1 \tag{4.73}$$

$$\dot{k} = \rho(e)e^2, \tag{4.74}$$

$$\alpha_1(e,k) = -k\rho(e)e \tag{4.75}$$

and

$$V = l\overline{\zeta}^T Q\overline{\zeta} + \frac{1}{2}e^2 + \frac{1}{2}b(w)(k-\bar{k})^2 + \frac{1}{2}\bar{\xi}_1^2$$

Then

$$\begin{aligned}\dot{V} = &-(l - \epsilon - \frac{5}{4}|\overline{H}|^2)|\overline{\zeta}|^2 \\ &+ [\frac{1}{\epsilon}l(w)|Q|^2 q_1(v,w) + \frac{1}{2} + \frac{5}{4}q_2(v,w) - b(w)\overline{k}\rho(e)]e^2 \\ &- \bar{\xi}_1^2 + \bar{\xi}_1[b(w)e - \alpha_1 + \bar{u} - \frac{\partial\alpha_1}{\partial k}\rho(e)e^2 + 2(\bar{\xi}_1\frac{\partial\alpha_1}{\partial e})^2].\end{aligned}$$

At the end of the design, taking

$$\bar{u} = -b(w)e + \alpha_1 + \frac{\partial\alpha_1}{\partial k}\rho(e)e^2 - 2(\bar{\xi}_1\frac{\partial\alpha_1}{\partial e})^2 + \frac{\partial\alpha_1}{\partial e}\xi_1,$$

choosing

$$\begin{aligned}\bar{l} &= (l - \epsilon - |H|^2) > 0 \\ \bar{k} &> \frac{\frac{1}{\epsilon}l|Q|^2 q_1(v,w) + \frac{1}{2} + \frac{5}{4}q_2(v,w)}{b(w)\rho(e)} \\ \rho(\cdot) &= 1,\end{aligned}$$

gives

$$\dot{V}_2 = -\bar{l}|\overline{\zeta}|^2 - \bar{\xi}_1^2.$$

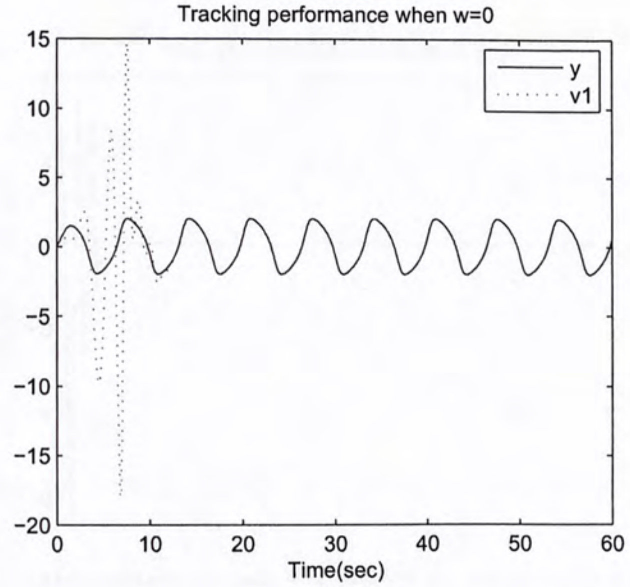

Figure 4.6: The tracking performance of the controlled system when $w = 0$

Applying the design above to the augmented system yields a dynamic output feedback controller as follows

$$
\begin{aligned}
u &= \bar{u} + \beta(\eta) + \mathbf{u}_c(v, w) \\
\dot{\eta} &= \gamma(\eta, u, v) = M\eta + T\varphi(v)T^{-1}\eta + N\hat{u} \\
\dot{k} &= e^2 \\
\dot{\xi}_1 &= -\xi_1 + \bar{u}
\end{aligned}
$$

4.3.3 Simulation Results

The performance of our design is verified by computer simulation, and Fig 4.6 and 4.8 and show the tracking performance of the closed-loop system, and Fig 4.7 and ?? show that error tends to zero when time is large enough. Parameters of the system are $b = 1,\ a = 1$ with $w = [-0.2, 0, 0.2]$, and initial states of the closed-loop system and exosystem are $x(0) = [0.2,\ 0.5]$, $\xi(0) = 1$, $\eta(0) = [1,\ 1]$, $v(0) = [0,\ 1,\ 0,\ 1]$, $k(0) = 1$.

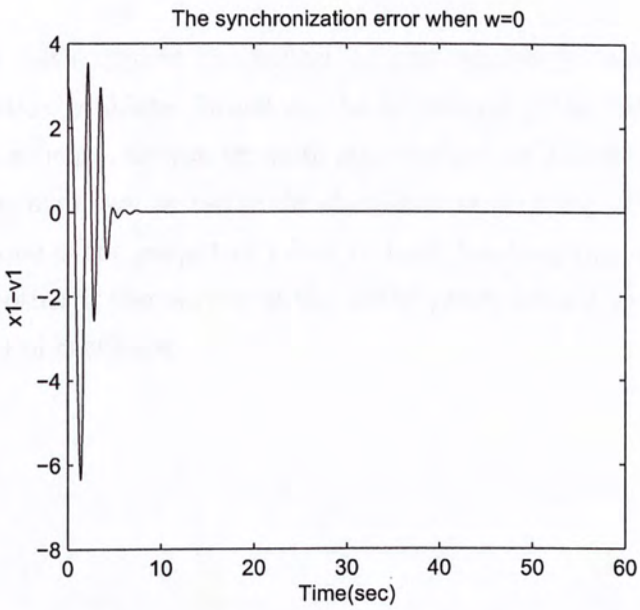

Figure 4.7: The synchronization error when $w = 0$

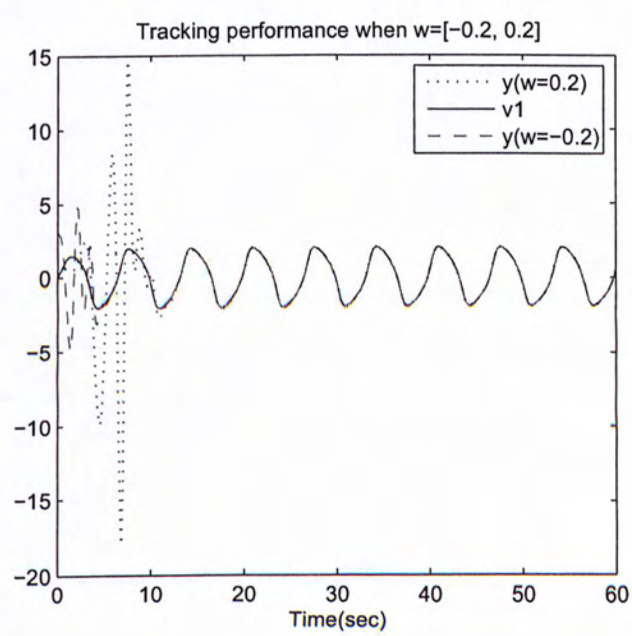

Figure 4.8: The tracking performance of the controlled circuit when $w = [-0.2, 0.2]$

4.4 Conclusions

In this chapter we have applied the robust output regulation technique to handle the robust synchronization problem. Based on the solvability of the regulator equations, we transfer the robust synchronization problem into a robust stabilization problem. Then we design a feedback control law to settle the stabilization problem. Two applications have been studied that one is the output of Chua's circuit tracking the output of Van der Pol oscillator, and the other is the output of the SMIB power system synchronizing with the output of Van der Pol oscillator.

□ **End of chapter.**

Chapter 5

Conclusions

This thesis has mainly considered the problem of applying the framework of the robust output regulation to solve the robust output synchronization problem. The main contributions of the thesis are summarized as follows:

In the first part, we solve the local robust synchronization problem of Chua's circuit synchronizing with Van der Pol oscillator by the robust output regulation method. The control law is designed based on the internal model approach arising from the framework of the robust output regulation. We have obtained a design overcoming the obstacle of devising a nonlinear internal model to account for the nonlinear master system.

In the second part, we first investigate the solvability of the nonlocal robust output problem with the effective control region large enough to cover the attractive region of the master systems. We have overcome three obstacles. The first is to choose a proper framework of output regulation to transfer the synchronization problem into a stabilization problem. The second is to add a filter to the time-varying augmented system and perform a proper coordinate transformation to obtain an extended augmented system in a lower triangular form. The third is to settle the time-varying stabilization problem. In addition, we have analyzed the differences between the two frameworks for tackling the output regulation problem and point out which framework is suitable for our problem.

In addition, we have also settled two applications, the synchronization of Duffing oscillator with Chua's circuit and the synchronization of the SMIB power system with Van der Pol oscillator.

My future work will be focus on the following problems:

- Continue to investigate the global robust synchronization problem and apply it to more emerging and significant engineering problems.

- For the nonlocal robust synchronization, remove the restriction that the relative degree of the slave system is $r = 2$ and further study the cases when the relative degree is $r \geq 3$.

- Study more nonlinear control techniques in order to obtain more methods for stabilizing various non-autonomous systems.

□ **End of chapter.**

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