Valuation of Dynamic Fund Protection under Levy Processes

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ABSTRACT

This thesis investigates the valuation of discrete dynamic fund protection (DFP) under Levy processes. Specifically, the analytical solution of discrete DFP under Levy processes is obtained in terms of Fourier transform. The derivation uses Spitzer's formula and leads to a recursion on computing the characteristic function of the maximum protection-to-fund ratio using Fourier inversion. DFP can then be valued efficiently and accurately via Fast Fourier Transform (FFT). The pricing behavior of the discrete DFP is numerically examined using several Levy processes, such as geometric Brownian motion, jump-diffusion models and variance gamma process. Numerical experiments confirm that the proposed approach produces highly accurate discrete DFP values within 1 second.

摘要

本論文研究離散動態基金保障於Levy過程中的定價。離散動態基金保障的解分析定價公式可以傅立葉轉換來表示。定價公式的推導是透過Spitzer方程式，得到最大保障基金比率特徵函數的傅立葉轉換，動態基金保障的定價可透過準確及有效的數值傅利葉反轉換公式來得到。離散動態基金保障的定價性能會被數種Levy過程作出數值檢驗，例如幾何布朗運動、跳躍擴散模型及方差伽瑪過程。數值檢驗確認了本論文提出的方法能夠於一秒內準確計算出離散動態基金保障的價值。

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Contents

List of Tables

List of Figures

Chapter 1

Introduction

Dynamic Fund Protection (DFP) is a protection feature added on a fund. The concept of DFP introduced by Gerber and Shiu (1998, 1999) extends the put option concept to provide protection at multiple time points. The DFP contract guarantees that the value of the protected fund does not fall below a guaranteed floor level at all observed times before the maturity date of the contract. If it goes below the pre-specified guarantee level at any time during the life of the contract, just enough money will be added so that the fund unit value will be upgraded. The DFP feature prevents unexpected loss from downside for an insurance contract.

Gerber and Pafumi (2000) applied the concept of DFP to equity-indexed annuities (EIA) products. They consider that the price dynamics of the primary fund to be the geometric Brownian motion and no early withdrawal from the fund. A closed-form formula for this dynamic guarantee can be then obtained. Imai and Boyle (2001) related the DFP concept to a lookback payoff, and derived the mid-contract valuation under the Black-Scholes (1973) assumption. Gerber and Shiu (2003) extended the concept of DFP to incorporate the performance of a financial index, in the way that the guarantee level of a perpetual EIA with dynamic protection is proportion to a financial index. Chu and Kwok (2004) investigated the reset and withdrawal rights of DFP, when both the naked fund and the stock index follow lognormal processes. They compute the grant-date and mid-contract valuation of these protected funds.

Besides classical Black-Scholes settings, several of diffusion models are also considered for DFP valuation. Imai and Boyle (2001) provided a numerical method under the constant elasticity of variance (CEV) model, and approximated the price of discrete monitoring DFP under CEV. Wong (2007) derived the analytical solution to DFP under CEV model in terms of a Laplace transform and demonstrated the use of numerical Laplace inversion in valuing DFP. The numercial Laplace inversion method belongs to the Dubber-Abate family. Wong and Chan (2007) valued the DFP under multiscale stochastic volatility. They assumed that the volatility is driven by two stochastic processes with one persistent factor and one fast mean-reverting factor, and obtained semi-analytical pricing formulas by means of multiscale asymptotic technique.

There is a close link between continuous DFP and lookback options. Wong and Chan (2007) develop a model-independent parity relation between the price functions of DFP and quanto lookback options when these products are monitored continuously. As the closed-form solution for quanto lookback options has been obtained by Dai, Wong and Kwok (2004), the valuation of DFP with continuously monitoring can be inferred from classical results on lookback option pricing such as Gatto, Goldman and Sosin (1979), Goldman, Sosin and Shepp (1979), and Conze and Viswanathan (1991). Wong and Kwok (2003) proposed a new pricing strategy for various types of lookback options by means of a replicating portfolio approach and obtained model-independent put-call parity relations among multi-state lookback options. However, these results are only shown to be true for continuous lookback options and DFP.

In practice, it is almost impossible to continuously monitor the fund price movements, and discrete monitoring may be more appealing. In fact, policy holders usually assess fund value once a month, and a protected fund is only upgraded, if required, with the same frequency. To cope with this practical need, Imai and Boyle (2001) and Wong and Chan (2007) approximate discrete DFP by its continuous counterpart using the adjustment formula proposed by Broadie, Glasserman and Kou (1999). Fund and Li (2003) find that such an adjustment only works for lognormal price process and frequently monitored DFP. They then develop a quadrature-type numerical integration scheme to price discrete DFP under the lognormal process and CEV model. Tse, Chang, Li and Mok (2008) further ensure that the pricing and hedging of discrete DFP are very different from those of continuous DFP, and obtain an analytical valuation for discrete DFP under the Black-Scholes model.

Unfortunately, the valuation of discrete DFP beyond continuous price processes has not been investigated so far. Ohgren (2001) realized that the Spitzer's (1956) identity is an important tool in pricing discrete path-dependent options. He proposed a method to compute the characteristic function of discretely monitored maximum stock price, and used this method to price discrete lookback option at the inception of the contract and monitoring points. Borovkov and Novikov (2002) illustrated the numerical pricing of discrete monitored exotic options when the asset follows a Levy process. Petrella and Kou (2004) obtained a numerical algorithm to compute the lookback and barrier options using the Spitzer's identity and Laplace transform for jump diffusion processes. The algorithm is applicable at any time points. Atkinson and Fusai (2007) provided a close-form pricing formula for discrete monitored exotic options (lookback, barrier and quantile) under the Black-Scholes setting. They reduced the computation of the discrete extrema of the Brownian motion to a Wiener-Hopf integral equation and solved it analytically. The solution of the Wiener-Hopf integral equation can be related to the Spitzer's identity, see Spitzer (1957).

In this thesis, the DFP under Levy process with discrete monitoring is investigated. This thesis contributes to the literature in the following ways. We show that DFP can be viewed as a quanto lookback option, even though it is monitored discretely. A quanto-prewashing procedure is then carried out to identify the process under different measure when the underlying asset follows a Levy process. Then, the Fourier transform on DFP is analytically derived as a recursion of characteristic functions. This enables DFP to be efficiently valued using

Fast Fourier Transform (FFT).

The remaining part of this thesis is organized as follows. Chapter 2 discusses the properties and applications of Levy processes in finance. Chapter 3 reviews the basic structure of the DFP. We focus on discrete DFP and investigate its relationship to discrete quanto lookback option. In Chapter 4, the Spitzer's identity is applied to this insurance product, and to obtain the characteristic function of the realized maximum over discrete time points. The valuation of DFP is then decomposed into a recursive valuation using characteristic functions. Chapter 5 establishes the pricing formula of discrete DFP. The analytical formula is expressed in Fourier transform of a new characteristic function under Girsanov's Theorem. Chapter 6 presents numerical demonstration on implementing the proposed approach, and compares the accuracy and efficiency with those from simulation for different types of Levy processes. Chapter 7 concludes the thesis.

Chapter 2

Levy Processes

2.1 Definition

An adapted real-valued stochastic process X_t, with $X_0 = 0$, is called a Levy process if it has the following properties:

(i) Independent increment. If $0 \leq t_0 < \cdots < t_n$, for any choice of $n \geq 1$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

(ii) Time-homogeneous property. The distribution of the random variable, $X_{t+s} - X_s$, does not depend upon s.

(iii) A cadlag process. It is right-continuous with left limits as a function of t.

(iv) Stochastically continuous. For any $\epsilon > 0$, $\Pr[|X_{s+t} - X_s| > \epsilon] \to 0$ as $t \to 0$.

A stochastic process is said to be stationary independent increments (PIIS) if it satisfies (i) and (ii). In fact, many well-known processes used in finance are of Levy processes. For example, geometric Brownian motion is a Levy process because it satisfies all the conditions above.

For a real-valued function $f : [a, b] \to \mathbb{R}$, f is cadlag if it is everywhere right-continuous and has left limits everywhere. i.e.

(i) The left limit: $f(t_-) = \lim_{s \uparrow t} f(s)$ exists; and

(ii) The right limit: $f(t_+) = \lim_{s \downarrow t} f(s)$ exists and equal to $f(t)$

Clearly, any continuous function is cadlag. Given a cadlag function $f(t)$, $f(t_-) = f(t)$ if and only if it is continuous at t. Hence, the jump size at t is defined by

$$\Delta f(t) = f(t) - f(t_-).$$

Levy process is a stochastic process with infinitely divisible properties. If the law of X_t is infinitely divisible, then X_t can be expressed as the sum of n independent identically distributed (iid) random variables, more specifically, where $X_{t/n}$ is the common law of each random variable.

2.2 Levy-Khinchine formula

A Levy process, X_t, can be fully described by the characteristic function, which can be viewed as the Fourier-Stieltjes transform of the probability density function (pdf) of the distribution: $F(x) = P(X_t \leq x)$. Define the characteristic function as

$$\phi_X(u) = E[\exp(iuX_t)] = \int_{-\infty}^{\infty} \exp(iux)\, dF(x), \tag{2.1}$$

where $i = \sqrt{-1}$ and $u \in \mathbb{C}$. It can be shown that $E[\exp(iuX_t)] = 1$ when $u = 0$, and $|E[\exp(iuX_t)]| \leq 1$ for all $u \in \mathbb{R}$. The characteristic function always exists and is continuous. An important statistical property is that the characteristic function determines the distribution function F uniquely. The moments of X_t can also be derived from the characteristic function because it generalizes the moment-generating function to the complex domain, in which the real line is a subspace.

In particular, the characteristic function of a Levy process can be described by the Levy-Khinchine representation:

$$\begin{aligned} E[e^{iuX_t}] &= \exp\left\{aitu - \frac{1}{2}\sigma^2 tu^2 + t\int_{R\setminus\{0\}} (e^{iux} - 1 - iux\mathbb{I}_{|x|\leq 1})w(dx)\right\} \\ &= \exp\{t\psi(u)\} \ = \phi_X(u) \end{aligned} \tag{2.2}$$

where $\int_{\mathbb{R}} \min(1, x^2)w(dx) < \infty$, $\psi(u)$ is known as the characteristic exponent, w is the Levy measure of X defined on $R \setminus \{0\}$. In the formula, $a \in \mathbb{R}, \sigma^2 \geq 0$.

The notation $R \setminus \{0\}$ in the Levy-Khinchine formula indicates that zero is excluded as a possible jump amplitude. If $\int_{-1}^{1} w(dx) < \infty$, there are finite jumps in any finite time interval. In such a situation, the Levy process has finite-activity and is known as a Type I Levy process or jump-diffusion model in finance. If X_t

is a finite-activity (or jump-diffusion) Levy process, it can be described by

$$X_t = \mu t + \sigma B_t + \left(\sum_{k=1}^{N_t} J_k - t\lambda\kappa \right)$$

where B_t is a standard Brownian motion. N_t is a Poisson process with intensity λ such that $E[N_T] = \lambda t$, and $E[J] = \kappa < \infty$.

The Levy measure, w, dictates how the jump occurs. In finite-activity models, we have $\int_{\mathbb{R}} w(dx) < \infty$ and $w(dx) = \lambda dF(x)$. In the infinite-activity (Type II) case, $\int_{\mathbb{R}} w(dx) = \infty$, the Poisson intensity cannot be defined. In such a situation, the Levy measure $w(dx)$ has no mass and cannot be integrated at the origin, because there are infinite many small jumps. Fortunately, singularities (i.e. infinitely many jumps) only occur around the origin. The Levy-Khinchine representation guarantees that $w(dx)$ is always integrable near the origin. Intuitively speaking, the Levy measure describes the expected number of jumps of a certain height in a time interval of length 1.

A Levy process with a Brownian component is of unbounded variation. Pure jump Levy process (i.e. the process without Brownian component) is of infinite variation if and only if $\int_{-1}^{1} |x| v(dx) = \infty$. In such a situation, we shall focus on small jumps but the sum of the jumps after compensated by their mean does converge. Therefore, the compensator term $iux\mathbb{I}_{|x|\leq 1}$ is necessary in the Levy-Khinchine formula.

Levy-Khinchine formula provides an explicit and simple formula for all stationary Levy processes. The infinity divisible property can also be easily identified.

2.3 Applications of Levy Processes in Finance

Traditionally, stock prices are modeled by the geometric Brownian motions so that the log returns of financial assets follow a normal distribution. However, many empirical studies show that the geometric Brownian motion cannot fully describe the statistical properties of financial time series.

Skewness measures the degree to which a distribution is asymmetric. A distribution has negative skewness if it has a longer tail to the left than to the right. It has positive skewnwss if the reverse is true. For a symmetric distribution like the normal distribution, the skewness is zero. Empirical studies of daily log returns on different major indices (include S&P 500, Nasdaq, DAX, etc.) suggest a significant negative skewness (For example, Schoutens (2007)).

A way of measuring the fat tail behaviour is to look at the kurtosis of a distribution. Fat tails occur when large movements in asset price occur frequently. For the normal distribution (mesokurtic), the kurtosis is 3. If the distribution has a high peak (leptokurtic), the kurtosis is greater than 3. Otherwise, the distribution is said to have flatter top (platykurtic), the kurtosis is smaller than 3. Fama (1965) recognized that the return distribution is more leptokurtic than the normal distribution. The excess kurtosis is the main reason for considering jumps in asset prices.

Geometric Brownian motions have some palatable mathematical properties such as independent and stationary increments. We would like to have a sophisticated model with similar features which can improve the imperfections of the Brownian motions. In the late 1980s, Levy process is proposed for modelling financial data. Levy process has infinitely divisible property, it is the most direct generalization of the geometric Brownian motion. Furthermore, Levy process is general enough to account for the skew and smile effects in financial derivatives.

Table 6.1 exhibits several Levy processes and the corresponding characteristic functions. These processes are commonly used in finance. The geometric Brownian motion has been the benchmark model for the underlying asset of option contracts since the work of Black-Scholes (1973). Merton (1976) introduced the lognormal jump diffusion model. This model can generate a heavy tailed distribution and produce the volatility smile consistent with the market. Kou (2002) proposes the double exponential jump diffusion. This model can explain the shape of jump distribution by a psychological interpretation and maintain the advantages of Merton's model.

Some infinity activity Levy processes, such as variance gamma, normal inverse Gaussian and CGMY process, are recently brought to the financial market. We refer interested audiences to Shoutens (2003) for a comprehensive summary on this topic. These asset price models can be expressed as Brownian motions subject to a stochastic time change and hence called time-changed Brownian motion or subordinated Brownian motion. The generalized hyperbolic process is a generalization of some infinity activity models, which includes variance gamma process and normal inverse Gaussian process as its special cases.

2.4 Option pricing under Levy Processes

Heston (1993) is the pioneer of applying the characteristic function and Fourier analysis in option pricing. He works out the characteristic function of the diffusion process if the volatility follows an Ornstein-Uhlenbeck process, and the corresponding option pricing formula.

Let the underlying asset price at time t to be S_t such that $S_t = S_0 e^{X_t}$, where X_t is a Levy process with stationary and independent increments. The characteristic function of X_t can be represented $\phi_X(u)$ by the Levy-Khinchine formula (2.2).

2.4.1 Black-Scholes Formula with Characteristic Function

In fact, the classical Black-Scholes formula can be re-written in terms of the characteristic function of the log-asset value. Consider the T-maturity European vanilla call option which has the payoff:

$$C(S_T, K, T) = \max(S_T - K, 0),$$

where K is the strike price of the option.

Using the residue calculus, the well-known Black-Scholes (1973) formula can be alternatively expressed into the form involving the characteristic function.

Specifically,

$$C(S_0, K, T) = S_0 e^{-qT}\Pi_1 - Ke^{-rT}\Pi_2, \tag{2.3}$$

where

$$\begin{aligned}
\Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{iuk}\phi_X(u-i)}{iu}\right] du, \\
\Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{iuk}\phi_X(u)}{iu}\right] du,
\end{aligned} \tag{2.4}$$

S_0 is the current stock price, $k = \log K$, r is the constant risk-free rate, and q is the constant dividend yield. For the Black-Scholes model, the volatility has been absorbed into the characteristic function $\phi_X(u)$. The first term $e^{-qT}\Pi_1 = \dfrac{\partial C}{\partial S}$ is the delta (hedge ratio) of the option. The second term Π_2 equals to $P(S_T > K)$, the probability that the option is in-the-money at maturity.

For the moment, the formula (2.3) does not give obvious advantage of using the characteristic function because (2.4) may require a tedious numerical computation and the original Black-Scholes formula can be implemented with the normal cdf. However, (2.3) does give an insight into pricing options with characteristic function which is the only analytical component to describe the underlying Levy process. In fact, (2.4) gives an example that the required probabilities can be obtained from Fourier inversion once the characteristic function is available. Therefore, it is possible to value option under Levy process which is completely described through its characteristic function.

2.4.2 Fast Fourier Transform

Carr and Madan (1999) advocate the Fast Fourier Transform (FFT) to compute vanilla call and put options based on the characteristic function of the log-asset value. Their approach is naturally applied to Levy processes. We rewrite the payoff for a European call option on the underlying fund and maturity T as:

$$\max(S_0 e^{X_T} - K, 0) = S_0 \max(e^{X_T} - K/S_0, 0)$$

Let k = -log(K/F0), the call value under the risk-neutral density $q_T(s)$ for a fixed time interval T is:

$$C_T(k) = \int_k^{\infty} e^{-rT}(e^s - e^{-k}) q_T(s) ds$$

$C_T(k)$ is not square integrable over $(-\infty, \infty)$, because $C_T(k)$ approaches S_0 as k tends to $-\infty$. Carr and Madan introduce a damping factor $\exp(\alpha k)$ to deal with this problem. We define the modified call price $c_T(k)$.

$$c_T(k) = \exp(\alpha k) C_T(k), \qquad \text{for some constant} \quad \alpha > 0,$$

The modification guarantees the Fourier Transform of $c_T(k)$ exist:

$$\begin{aligned} \widehat{c_T}(v) &= \int_{-\infty}^{\infty} e^{ivk} C_T(k) dk \\ &= \frac{e^{-rT}\phi_X(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v} \end{aligned} \tag{2.5}$$

Plain vanilla call option values can be obtained by the inverse Fourier Transform numerically:

$$C_T(k) = S_0 e^{-\alpha k} \mathcal{F}^{-1}_{k,v}\left[\frac{e^{-rT}\phi_X(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}\right] \tag{2.6}$$

Integration (2.6) is a direct Fourier inversion. The application of Fast Fourier Transform (FFT) can significantly improve the efficiency. Important applications of FFT include digital signal processing and solving partial differential equations.

The FFT is an efficient algorithm for computing the discrete Fourier transform:

$$C(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)}x(j), \qquad \text{for } k = 1, \dots, N. \tag{2.7}$$

Using traditional methods, evaluating these sums requires to take $O(N^2)$ arithmetical operations. FFT is an algorithm to evaluate the same summation in only $O(N \log N)$ operations.

Setting $v_j = \eta(j-1)$, the integral $C_T(k)$ can be numerical approximated by the Trapezoidal rule

$$C_T(k) \approx \frac{\eta S_0 e^{-\alpha k}}{\pi}\left[\frac{1}{2}e^{-iv_1 k}\widehat{C_T}(v_1) + \sum_{j=2}^{N-1} e^{-iv_j k}\widehat{C_T}(v_j) + \frac{1}{2}e^{-iv_N k}\widehat{C_T}(v_N)\right] \tag{2.8}$$

From (2.7), the application of FFT requires that $\lambda\eta = 2\pi/N$. The FFT returns N values of k where $log_2 N \in \mathbb{N}$. Those values of k's will have a regular spacing size of λ. Therefore, the values of k's are:

$$k_u = -b + \lambda(u-1), \qquad \text{for } u = 1, ..., N, \tag{2.9}$$

the range of the log strike prices is between $-b$ to b, where $b = \frac{N\lambda}{2}$.

Substituting (2.9) into (2.8), we have

$$C_T(k_u) \approx \frac{S_0 e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(u-1)} e^{ibv_j}\widehat{C_T}(v_j)\eta. \tag{2.10}$$

(2.10) has a similar form to (2.7). The FFT algorithm can be applied directly. The major contribution of the FFT in option pricing is that the values of call prices with N different strike levels is evaluated at once.

2.4.3 Other Payoff Functions

Besides vanilla options, Lewis (2001) shows that European-style options with any payoff functions can be priced under any Levy process with a known characteristic function.

Let $w(x_T)$ to be the payoff function, where $x = \log(x_T)$. Assume that $w(x_T)$ is bounded for $|x_T| < \infty$ and it is Fourier integrable in a strip. Its generalized Fourier transform

$$\hat{w}(u) = \int_{-\infty}^{\infty} \exp(iux_T)w(x_T)dx_T \tag{2.11}$$

exists and is regular. The current price of a European-style options $V(S_0)$ with payoff function $w(x_T)$ is given by integrating along a straight line in the complex z-plane with z within the strip of regularity. This straight line should be parallel to the real axis. When the integration contour remains regular in the strip, it can be deformed and extends to ∞ or $-\infty$ by the Cauchy's Theorem. For example, we can choose $\nu = \Im(u)$, where $\Im$ means imaginary part.

$$V(S_0) = \frac{e^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{i(-u)\log S_0} \phi_X(-u)\hat{w}(u)du, \tag{2.12}$$

For example, European call option with payoff $w(x_T) = \max(e^{x_T} - K, 0)$ at expiration. Standard integration shows that:

$$\hat{w}(u) = -\frac{K^{iu+1}}{u^2 - iu}$$

and $\hat{w}(u)$ exists in the strip $|\Im(u)| > 1$.

Option price (2.12) can be numerically obtained by Fourier inversion algorithms such as FFT once the payoff function in Fourier space exists. Compare the approaches between Carr and Madan (1999) and Lewis (2001). The former gives a spectrum of plain vanilla option values at once. The latter is flexible for different European-style payoffs. These two approaches can be applied in different situations.

Model	Characteristic Function $E[e^{iuX_t}]$
Finite-activity models	
Geometric Brownian motion	$\exp\left\{iu\mu t - \frac{1}{2}\sigma^2 tu^2\right\}$
Lognormal Jump diffusion	$\exp\left\{iu\mu t - \frac{1}{2}\sigma^2 tu^2 + \lambda t(e^{iu\mu_J - \frac{1}{2}\sigma_J^2 u^2} - 1)\right\}$
Double exponential Jump diffusion	$\exp\left\{iu\mu t - \frac{1}{2}\sigma^2 tu^2 + \lambda t\left(\frac{1-\eta^2}{1+u^2\eta^2}e^{iu\kappa} - 1\right)\right\}$
Infinite-activity models	
Variance gamma	$\exp(iu\mu t)(1 - iu\nu\theta + \frac{1}{2}\sigma^2\nu u^2)^{\frac{t}{\nu}}$
Normal inverse Gaussian	$\exp\left\{iu\mu t + \delta t\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iu)^2}\right\}$
Generalized hyperbolic	$\exp(iu\mu t)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\frac{\lambda t}{2}}\left(\frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)^t$
	where $K_\lambda(z) = \frac{\pi}{2}\frac{I_\nu(z) - I_{-\nu}(z)}{\sin(\nu\pi)}$
	and $I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty}\frac{(z^2/4)^k}{k!\Gamma(\nu+k+1)}$
Finite-moment stable	$\exp\left\{iu\mu t - t(iu\sigma)^\alpha \sec\frac{\pi\alpha}{2}\right\}$
CGMY	$\exp(C\Gamma(-Y))((M-iu)^Y - M^Y + (G+iu)^Y - G^Y)$
	where $C, G, M > 0$ *and* $Y > 2$

Table 2.1: Characteristic functions for some parametric Levy processes

Chapter 3

Dynamic Fund Protection

Consider an investor who owns one unit of the fund. Let F_t denote the value of a fund unit at time t. He would like to protect the investment against adverse fluctuations until a predetermined time T . The first suggestion is purchasing a European put option, with strike price $K > 0$ and maturity T. At the maturity, the investor will get back K if the fund value at maturity F_T is less than the strike price, and the fund value is unchanged if it is greater than K.

The put option strategy can provide a *static* protection. The major disadvantage of the strategy is: if the fund unit value suffers substantial losses, the investor has little chance of having more than K at time T. Therefore, *dynamic* protections is more attractive for the investors who concern about the shortfall of the protection from a simple put option.

Dynamic fund protection (DFP) is a protection feature added on a fund such that the fund value will be upgraded if it ever falls below a certain threshold level. DFP enhances the attractiveness of insurance policies contingent on a fund operated by the insurance company. For instance, there may be situations in which an insurance policy requires the policy holder not only to pay for loss-protection premium but that part of the premium will be invested in a fund for savings purpose. DFP can be a structure built into these insurance policies.

3.1 Discrete Dynamic Fund Protection

This continuous protection concept was first proposed by Gerber and Shiu (1998, 1999). In this thesis, we consider DFP with discrete monitoring, because it makes the DFP contract easier to administer in practice. The mechanism of discrete dynamic fund protection can be demonstrated through an example. Suppose an investor holds one unit of the underlying fund which is protected by DFP. Let K be the certain constant protection level which takes the role similar to the strike price of a put option.

The original value of a fund F_{t_n} is replaced by an upgraded value f_{t_n} if the naked fund goes below the protection level at the monitoring instant t_n, where $n = 1, \cdots N-1$ and the maturity date is T_N. Consider that the monitoring instants are equally spaced in time. (i.e. $\Delta t_j = t_j - t_{j-1}$ is the same for $j = 1, 2, \cdots, N$). The DFP guarantees that f_{t_n} does not fall below the protection floor level K. The process f_{t_n} is defined so that

(a) $f_{t_0} = F_{t_0}$.

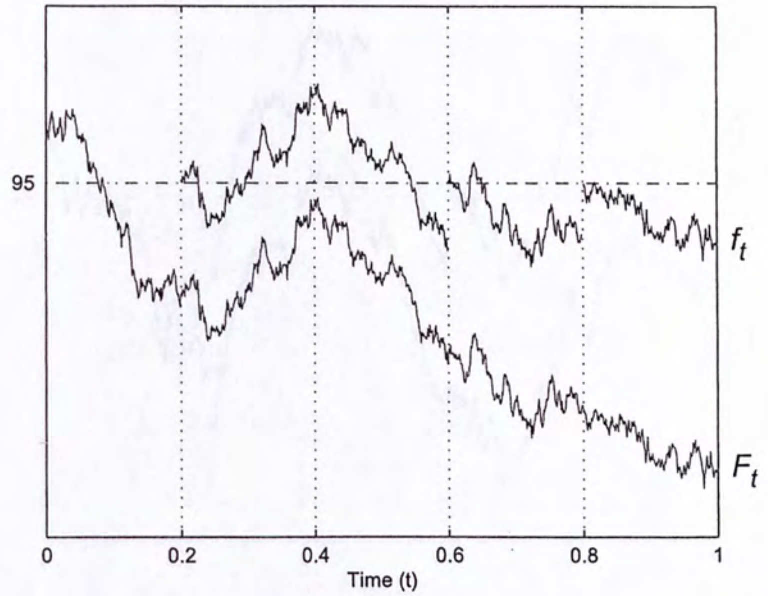

Figure 3.1: A Sample Path of the Fund Unit Values

(b) if $f_{t_n} > K$ at the monitoring instants, f_{t_n} does not need to be upgraded. It is identical to the the naked one F_{t_n}.

(c) whenever f_{t_n} drops to K, just enough money will be added so that the upgraded fund unit value does not fall below K.

Figure 3.1 illustrates a sample path of the upgraded fund which has a continuous process, protection level $K = 95$ and equally spaced monitoring instants $\Delta t_j = 0.2$. The lower curve presents the movement of the naked fund; whereas, the upper one demonstrates that of the protected fund. Unlike continuous DFP, even if the fund falls below the protection level in some periods of time, it will be upgraded at the monitoring instants only. As this thesis concentrates on the

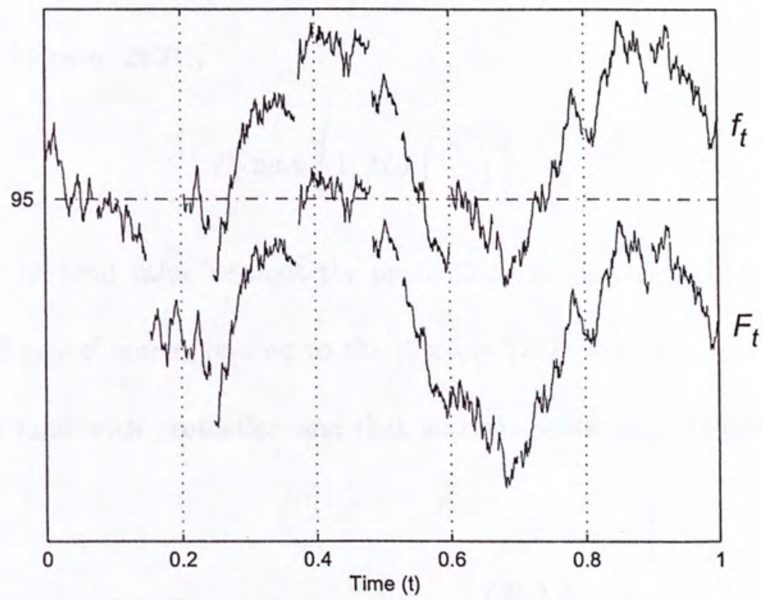

Figure 3.2: A Sample Path of the Fund Unit Values with Jumps

valuation of discrete DFP under the Levy models, Figure 3.2 shows a pair of illustrative sample paths in which the naked fund follows a jump-diffusion process.

3.2 Link DFP to Discrete Lookback Options

Let

$$M_N(S_t) = \max\{S_{t_1}, S_{t_2}, \cdots, S_{t_N}\}$$

be the maximum value of the stochastic process $\{S_t\}$ over N monitoring instants, namely $t_1 < t_2 < \cdots < t_N$. Hence, we have the maximum value of the protection level to fund value ratio:

$$M_N\left(\frac{K}{F_t}\right) = \max\left\{\frac{K}{F_{t_1}}, \frac{K}{F_{t_2}}, \cdots, \frac{K}{F_{t_N}}\right\}.$$

The payoff of a policy holder engaging into the discrete DFP is then given by, [see Imai and Boyle (2001)]

$$F_T \max\left\{1, M_N\left(\frac{K}{F}\right)\right\},$$

where F_t is the fund value without the protection (the naked fund). Therefore, the terminal payoff corresponding to the discrete DFP should be the difference between the fund with protection and that without protection. Hence, the DFP payoff reads,

$$\text{DFP}(T) = F_T \max\left\{1, M_N\left(\frac{K}{F}\right)\right\} - F_T. \tag{3.1}$$

Our objective is to determine the fair present value of this DFP.

It can be seen that the DFP payoff resembles a quanto lookback option. Lookback options are common financial instruments that allow investors having attractive gains for substantial price movement of underlying assets over a given period. The payoff of a lookback option involves extreme values(s) of the underlying asset price during the life of the option.

The term "quanto" is an abbreviation for "quantity adjusted", and it refers to the feature where the payoff of an option is determined by the financial price or index in one currency but the actual payout is realized in another currency. For details of quanto lookback options, we refer to the paper of Dai, Wong and Kwok (2004).

Proposition 3.2.1. The payoff of discrete DFP is identical to that of the quanto fixed strike lookback call option monitored discretely over the same set of monitoring instants. Specifically,

$$\mathrm{DFP}(T) \;=\; F_T C_{fix}(T, S_T^F, M_N(S_t^F)), \tag{3.2}$$

where $S_t^F = K/F_t$ and $C_{fix}(T, S_T^F, M_N(S_t^F))$ is the payoff function of fixed discrete strike lookback call option in the foreign currency world.

Proof: Let

$$S_t^F \;=\; K/F_t. \tag{3.3}$$

The payoff (3.1) then becomes:

$$\begin{aligned}\mathrm{DFP}(T) &= F_T \max(1, M_N(S_t^F)) - F_T \\ &= F_T \max(M_N(S_t^F) - 1, 0).\end{aligned} \tag{3.4}$$

The payoff of the discrete fixed strike lookback call on the underlying asset S_t^F is given by

$$C_{fix}(T, S_T^F, M_N(S_t^F)) = \max(M_N(S_t^F) - K, 0).$$

If we view F_t as an exchange rate at time t, S^F can be considered as an asset trading in the foreign currency world. The payoff of DFP is equivalent to a fixed strike lookback call on S^F, with a unity strike price trading in the foreign currency world and is then translated back to the domestic currency by the exchange rate F_T. Hence, this option can simply be valued as the fixed strike lookback call in the foreign currency world followed by multiplying the exchange rate F_0.

□

Chapter 4

Spitzer's Identity

In the last chapter, Proposition 3.2.1 asserts that the pricing of DFP can be connected to that of a quanto lookback option. This chapter discusses how discrete lookback option can be valued using Spitzer's identity. A lookback call (put) gives the option holder the right to buy (sell) an underlying asset at its lowest (highest) price over a period. Broadie, Glasserman and Kou (1999) constructed an approximation that connects continuous and discrete lookback options when the underlying asset price follows the Black-Scholes dynamics. Imai and Boyle (2001) applied this result to DFP under the geometric Brownian model.

4.1 Applications of Spitzer's Identity

For the moment, let us focus on fixed strike lookback call. Consider an asset takes the form: $S_t = S_0 e^{Y_t}$. Define the maxima of the asset process during the option life to be

$$M_N(S_t) = S_0 \exp(M_N(Y_t)),\ Y_0 = 0$$

and the payoff of the lookback option becomes

$$\begin{aligned} \max(M_N(S_t) - K, 0) &= \max(S_0 e^{M_N(Y_t)} - K, 0) \\ &= S_0 \max(e^{M_N(Y_t)} - K/S_0, 0). \end{aligned} \tag{4.1}$$

Let $Z_j = Y_j - Y_{j-1}$ and $Z_1 = Y_1$. Then we have

$$M_n(Y_t) = \max\left(0, Z_1, Z_1 + Z_2, \cdots, \sum_{j=1}^{n} Z_j\right),$$

where Z_1, Z_2, $\cdots$, Z_n are independent and identically distributed (iid) random variables when the discrete DFP is monitored with equal time spaced instants.

The main challenge of pricing a lookback option stems on the difficult of having the distributional property of $M_N(Y_t)$. Spitzer (1956) produced a useful formula to calculate the joint distribution of the pair $(M_N(Y_t), Y_N)$ if $\{Y_i\}$ is a sequence of the sum of iid random variables. Discrete sampled Levy process in equal time spaced satisfies this property because the process is in fact an infinity divisible random walk. Spitzer proves that, for $s \leq 1$, $u, v \in \mathbb{C}$, $Im(u) \geq 0$ and $Im(v) \geq 0$, the following theorem holds.

Theorem 4.1.1. (Spitzer's Identity)

$$\sum_{j=0}^{\infty} s^j E[e^{iuM_j(Y_t)+ivY_j}] = \exp\left[\sum_{j=1}^{\infty} \frac{s^j}{j}\left(E[e^{i(u+v)Y_j^+}] + E[e^{i(-v)Y_j^-}] - 1\right)\right], \tag{4.2}$$

where $Y_j^+ = \max(Y_j, 0)$ and $Y_j^- = \min(Y_j, 0)$.

The proof can be found in Spitzer (1956). It can also be proved using Fourier Transform or probabilistic arguments, see Wendel (1958).

Spitzer identities first appeared in fluctuation theory, and plays an important role in the applications of algebra, combinatorics, queueing theory and engineering because it provides an approach to deal with the discrete maximum (minimum) of i.i.d random variables. Specifically, the identity asserts that the z-transform of the characteristic function of successive maxima (minima), i.e. the z-transform of $E[e^{iuM_j(Y_t)}]$, can be decomposed into z-transforms of characteristic functions of the positive part and negative part of the sequence of random variables.

However, the original Spitzer's identity is not readily useful for lookback option pricing, because it involves the z-transform and hence the infinite sum on the left-hand side of (4.2). Taking $v = 0$ in (4.2) and using Leibniz's formula at $s = 0$, Ohgren (2001) provides an alternative form for the Spitzer formula displayed as follows.

Theorem 4.1.2.

$$E[e^{iuM_N(Y_t)}] = \frac{1}{N}\sum_{j=0}^{N-1} E[e^{iuY^+_{N-j}}]E[e^{iuM_j(Y_t)}]. \tag{4.3}$$

Theorem 4.1.2 asserts that the characteristic function of $M_N(Y_t)$ can be decomposed into a sum involving the characteristic functions of Y_j^+ and that of $M_j(Y_t)$, where $j = 0, 1, \cdots, N-1$. Thus, the characteristic function of $M_N(Y_t)$ can be obtained through a recursion if the characteristic function of Y_j^+ is known. Notice that $M_0(Y_t) = 0$.

The importance of Spitzer's identity in pricing discrete monitoring financial products has been recently shown by Ohgren (2001), Borovkov and Novikov (2002) and Petrella and Kou (2004).

Atkinson and Fusai (2007) express the exotic options pricing problems under Black-Scholes setting into a Wiener-Hopf integral equation. The solution of the Wiener-Hopf equation requires that the transformed kernel of the integral equation to be decomposed into a product of two functions, one being analytic in the upper half-plane and the other being analytic in an overlapping lower half transform plane. Spitzer's identity can also show that these two functions are actually the characteristic function of the maximum and minimum of the process stopped at a random time with geometric distribution. It provides the contribution in giving a different probabilistic interpretation in terms of characteristic function of the maximum and minimum of a geometrically stopped random walk. Formal proof relating the Wiener-Hopf equation and combinatorial argument in Spitzer's identity can be found in Spitzer (1957) or Wendel (1958).

However, this method has not been applied to value DFP and insurance products to the best of our knowledge. Thus, this thesis brings this important tool to the insurance mathematics literature.

4.2 Discrete Lookback Options

We are now going to investigate the characteristic functions of Y_k^+

$$\begin{aligned} E[e^{iuY_k^+}] &= E[e^{iuY_k}\mathbb{I}_{\{Y_k>0\}}] + E[e^0\mathbb{I}_{\{Y_k\leq 0\}}] \\ &= E[e^{iuY_k}\mathbb{I}_{\{Y_k>0\}}] + \Pr(Y_k < 0) \\ &= E[e^{iuY_k}\mathbb{I}_{\{Y_k>0\}}] - \Pr(Y_k \geq 0) + 1. \end{aligned} \tag{4.4}$$

Consider the following function of l,

$$g(l) = E[e^{iuY_k}\mathbb{I}_{\{Y_k>l\}}] - \Pr(Y_k \geq l). \tag{4.5}$$

Proposition 4.2.1.

$$\begin{aligned} g(l) &= \mathcal{F}_{l,v}^{-1}\left[\frac{\phi_Y(u+v)-\phi_Y(v)}{iv}\right] \\ \Rightarrow E[e^{iuY_k^+}] &= \mathcal{F}_{0,v}^{-1}\left[\frac{\phi_Y(u+v)-\phi_Y(v)}{iv}\right] + 1 \end{aligned} \tag{4.6}$$

where $\mathcal{F}_{l,v}^{-1}$ represent the Fourier inversion with respect to v.

Proof: Let $p(y)$ is the risk-neutral transition density of Y_k. We now derive the Fourier Transform of $g(l)$ with respect to v.

$$\widehat{g}(v) = \int_{-\infty}^{\infty} e^{ivl}\int_l^{\infty} e^{iuy}p(y)dydl - \int_{-\infty}^{\infty} e^{ivl}\int_l^{\infty} p(y)dydl$$

We can interchange the integration order by applying Fubini's Theorem:

$$\begin{aligned} \widehat{g}(v) &= \int_{-\infty}^{\infty} e^{iuy}p(y)\int_{-\infty}^{y} e^{ivl}dldy - \int_{-\infty}^{\infty} p(y)\int_{-\infty}^{y} e^{ivl}dldy \\ &= \int_{-\infty}^{\infty} e^{iuy}p(y)\frac{e^{ivy}}{iv}dy - \int_{-\infty}^{\infty} p(y)\frac{e^{ivy}}{iv}dy \\ &= \frac{\phi_Y(u+v)-\phi_Y(v)}{iv} \end{aligned} \tag{4.7}$$

The result follows after the inverse Fourier Transform.

□

Petrella and Kou (2004) link (4.4) to a European call (put) option when iu is a positive (negative) real number for valuing lookback option under jump-diffusion models. Their approach uses the Laplace transform instead of Fourier transform. However, we would like to use the characteristic function of Y_k directly. This is more convenient for option pricing under Levy processes which are fully described through the Levy-Khinchine representation. When iu is an imaginary number, the quantity (4.4) is no longer related to call or put option. Fortunately, it can be linked back to the characteristic function of Y_k.

Once the characteristic function of the maximum is available, the lookback option can be valued by the approach of Carr and Madan (1999) as what we reviewed in Chapter 2. Consider

$$c_M(k) = \exp(\alpha k) E[\max(e^{M_N(Y_t)} - e^{-k}, 0)],$$

where $k = -\log(K/S_0)$. Using (2.6), the Fourier transform on this damped lookback price is

$$\mathcal{F}_{k,u}(c_M(k)) = \frac{\phi_M^N(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u}, \quad (4.8)$$

where $\phi_M^N(u)$ is the characterisitc function of $M_N(Y_t)$. Hence,

$$\begin{aligned} & C_{fix}(0, S_0, M_0(S_t)) \\ = \;& S_0 e^{-\alpha k} \mathcal{F}_{k,u}^{-1}\left[\frac{\phi_M^N(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u}\right], \end{aligned} \quad (4.9)$$

where $M_0(S_t)$ is the realized maximum on the contract initiating date and hence $M_0(S_t) = S_0$. In (4.9), we assume a zero interest rate because the asset S_t in the numeraire market has no drift for DFP pricing. This will be shown in a later section.

Chapter 5

Pricing Discrete DFP

This chapter is divided into three parts. In the first part, we review a well-established technique in risk-neutral pricing called the Girsanov's Theorem. In the second part, we demonstrate the use of Girsanov's Theorem in finding an equivalent martingale measure for DFP valuation. In the last section, we extend the pricing algorithm of discrete DFP to any time point between two monitoring instants.

5.1 Girsanov's Theorem

Girsanov's Theorem is an important tool in financial mathematics and actuarial sciences. It is a fundamental theorem in option pricing and premium determination. It describes how stochastic processes change under changes in probability measure. The concept of change of measure is related to a rigorous treatment of probability. This relies on use of measure theory. This technique allows us to modify the probability measure of the process, so that the process under the new measure is martingale (driftless).

Theorem 5.1.1. (Girsanov's Theorem)

The probability measure $\widetilde{\mathbb{Q}}$ is absolutely continuous with respect to $\mathbb{Q}$ if there exists a positive function L_t, called the Radon-Nikodym derivative, such that, for any event A,

$$\begin{aligned} \widetilde{\mathbb{Q}}(A) &= \int_A L_t(\omega)\mathbb{Q}(d\omega) \\ \text{or} \quad \left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} &= L_t. \end{aligned} \tag{5.1}$$

The modified probability measure is equivalent to the original probability measure under Girsanov's Theorem. That is, both probability measures have the same null sets. In finance, it is used to begin a modeling with a risk-neutral measure $\mathbb{Q}$, which is equivalent to the physical probability measure $\mathbb{P}$, such that $e^{-rt}F_t$ is a $\mathbb{Q}$-martingale. Once the Radon-Nikodym derivative L_t is identified, the following expectations can be calculated as

(a) $E^{\mathbb{Q}}[X_t] = E^{\mathbb{P}}[X_t L_t]$, or

(b) $E^{\mathbb{Q}}[X_t|\mathcal{F}_s] = E^{\mathbb{P}}\left[X_t \frac{L_t}{L_s}\middle| \mathcal{F}_s\right]$, where $\mathcal{F}_s$ is the information accumulated up to time s.

5.2 Equivalent Martingale Measure in DFP

Let X_T be the log-return of F so that $F_T = F_0 e^{X_T}$. The fair present value of DFP is given by

$$\begin{aligned} \text{DFP}(0) &= e^{-rT} E^{\mathbb{Q}} \left\{ F_0 e^{X_T} \max(M_N(S_t^F) - 1, 0) \right\} \\ &= e^{-rT} E^{\mathbb{Q}} \left\{ F_0 \frac{e^{X_T}}{E^{\mathbb{Q}}[e^{X_T}]} E^{\mathbb{Q}}[e^{X_T}] \max(M_N(S_t^F) - 1, 0) \right\}. \end{aligned} \tag{5.2}$$

We now regard the process $\dfrac{e^{X_T}}{E^{\mathbb{Q}}[e^{X_T}]}$ as the Radon-Nikodym derivative that defines an equivalent probability measure $\widetilde{\mathbb{Q}}$ using the Girsanov's Theorem. This measure has been used by Huang and Hung (2005) in pricing foreign equity options under Levy processes.

$$\text{DFP}(0) = e^{-rT} E^{\mathbb{Q}} \left\{ F_0 e^{X_T} \right\} E^{\tilde{\mathbb{Q}}} \left\{ \max(M_N(S_t^F) - 1, 0) \right\}.$$

Using the matingale property: $E^{\mathbb{Q}} \left\{ F_0 e^{X_T} \right\} = F_0 e^{rT}$, we obtain

$$\begin{aligned} \text{DFP}(0) &= e^{-rT} F_0 e^{rT} E^{\tilde{\mathbb{Q}}} \left\{ \max(M_N(S_t^F) - 1, 0) \right\} \\ &= F_0 E^{\tilde{\mathbb{Q}}} \left\{ \max(M_N(S_t^F) - 1, 0) \right\}. \end{aligned} \tag{5.3}$$

Let $S_t^F = S_0^F e^{Y_t}$, where S_t^F has been defined in (3.3). The following result holds.

$$\begin{aligned} S_0^F e^{Y_t} &= \frac{K}{F_0 e^{X_t}} \\ \frac{K}{F_0} e^{Y_t} &= \frac{K}{F_0 e^{X_t}} \\ Y_t &= -X_t, \end{aligned} \tag{5.4}$$

where X_t is the log-return of the fund, and X_t is modeled as Levy process.

In order to determine the law of process Y_t under the new measure $\widetilde{\mathbb{Q}}$, consider the characteristic function of the process X_t given in (2.2) in which the process is defined under the risk neutral measure $\mathbb{Q}$. The following proposition links the characteristic function of X_t under $\widetilde{\mathbb{Q}}$ to that under $\mathbb{Q}$.

Proposition 5.2.1. The characteristic function of Y_t in the new measure $\widetilde{\mathbb{Q}}$ is given by:

$$E^{\widetilde{\mathbb{Q}}}[e^{iuY_t}] = E^{\mathbb{Q}}\left[e^{i(-i-u)X_t}\right]e^{-rt} \tag{5.5}$$

Alternatively, we can write:

$$\phi_Y^{\widetilde{\mathbb{Q}}}(u) = \phi_X^{\mathbb{Q}}(-i-u)e^{-rt} \tag{5.6}$$

where $\phi_X^{\mathbb{Q}}(u)$ is the characteristic function of X_t defined in the Levy-Khinchine formula (2.2) under the $\mathbb{Q}$-measure.

Proof:

$$\begin{aligned} E^{\widetilde{\mathbb{Q}}}[e^{iuY_t}] &= E^{\mathbb{Q}}\left[\frac{e^{X_t}}{E^{\mathbb{Q}}[e^{X_t}]}e^{iuY_t}\right] \\ &= E^{\mathbb{Q}}\left[\frac{e^{X_t}}{E^{\mathbb{Q}}[e^{X_t}]}e^{iu(-X_t)}\right] \\ &= E^{\mathbb{Q}}\left[e^{X_t}e^{-iuX_t}e^{-rt}\right] \\ &= E^{\mathbb{Q}}\left[e^{(1-iu)X_t}e^{-rt}\right] \\ &= E^{\mathbb{Q}}\left[e^{i(-i-u)X_t}\right]e^{-rt} \end{aligned} \tag{5.7}$$

In the above calculation, $E^{\mathbb{Q}}[e^{X_t}]$ is equal to e^{rt} because X_t follows the martingale property in the measure $\mathbb{Q}$. The modified characteristic function (5.5) under $\widetilde{\mathbb{Q}}$ can help us to obtain call prices numerically using the Fourier transform.

5.3 Pricing DFP at any Time Points

The above results show how to compute the price of discrete monitoring DFP using the Spitzer's formula, given that the characteristic function of the underlying movement is known. The limitation of this algorithm is the price can only be calculated at the inception of the contract or at monitoring points (if the achieved maximum or minimum can be ignored).

Petrella and Kou (2004) introduce a method based on Laplace Transform which can compute the price and hedging parameters of discretely monitored barrier and lookback options at any point in time, even if the previous achieved maximum or minimum cannot be ignored. In this thesis, the DFP price can also be computed at any point in time with this methodology. First we consider the DFP price in (5.3) under a given risk-neutral measure.

Consider at time τ, where τ is any time between two monitoring instant. That is $\tau \in [t_{(l-1)}, t_{(l)})$, with $l \geq 1$. The observed discrete maximum at time τ is denoted by $M_{l-1}(S_t^F)$. We are interested with the discrete maximum asset value during the whole contract life, that is $M_N(S_t^F) = \max(M_{l-1}(S_t^F), M_{\tau,N}(S_t^F))$, where $M_{\tau,N}(S_t^F)) = \max(S_{t_l}, \cdots, S_{t_N})$. For the fixed strike lookback call whose payoff is $\max(M_N(S_t) - K, 0)$, we face the following two situations at time τ.

1. $M_{l-1}(S_t^F) \geq K$. In this case the observed maximum has been greater than the strike price. The option is guaranteed to expire in-the-money and hence the payoff should become $\max(M_{l-1}(S_t^F), M_{\tau,N}(S_t^F)) - K$, which can

be alternatively expressed as

$$\max(0, M_{\tau,N}(S_t^F) - M_{l-1}(S_t^F)) + M_{l-1}(S_t^F) - K.$$

Ignoring the interest rate effect and the known cash payment $M_{l-1}(S_t^F) - K$, the task is to determine $E^{\tilde{\mathbb{Q}}}[\max(M_{\tau,N}(S_t^F) - M_{l-1}(S_t^F)), 0)]$, which is identical to the lookback call option with another fixed strike $M_{l-1}(S_t^F)$.

2. $M_{l-1}(S_t^F) < K$: The option will expire in-the-money if and only if the future discrete maximum is larger than the strike price K. Due to the Markovian nature of Levy processes, the lookback call option value solely depends on the current asset value S_τ and is independent of the realized maximum in the past. Hence, ignoring the interest rate effect, the lookback option is $E^{\tilde{\mathbb{Q}}}[\max(M_{\tau,N}(S_t^F) - K), 0)]$.

Computing the expectations in the two situations requires the determination of the characteristic function $E^{\tilde{\mathbb{Q}}}[e^{iuM_{\tau,N}(Y_t)}]$.

Proposition 5.3.1.

$$\begin{aligned} E^{\tilde{\mathbb{Q}}}[e^{iuM_{c,N}(Y_t)}|\mathcal{F}_c] &= E^{\tilde{\mathbb{Q}}}[e^{iu(Y_{t_l}-Y_c)}|\mathcal{F}_c] \times E^{\tilde{\mathbb{Q}}}[e^{iu[M_{c,N}(Y_t)-(Y_{t_l}-Y_c)]}|\mathcal{F}_c] \\ &= \phi^{\tilde{\mathbb{Q}}}_{Y_{t_l}-c}(u)\phi^{\tilde{\mathbb{Q}},N-l}_{c,M}(u) \end{aligned} \tag{5.8}$$

The former characteristic function $\phi^{\tilde{\mathbb{Q}}}_{Y_{t_l}-c}(u)$ is determined by the model for the naked fund and will be linked to the characteristic function of X_t in the previous subsection. The latter characteristic function can be determined by the Spitzer's formula using Theorem 4.1.2 and Proposition 4.2.1.

Proof: We can write $M_{\tau,N}(Y_t)$ in the form of:

$$M_{\tau,N}(Y_t) = Y_{t_l} - Y_\tau + \max\left(0, Z_{l+1}, Z_{l+1} + Z_{l+2}, \cdots, \sum_{j+l+1}^{N} Z_j\right),$$

and $Y_{t_l} - Y_\tau$ is independent of $M_{\tau,N}(Y_t) - (Y_{t_l} - Y_\tau)$ because of the independent increment property. Now, it is clear that the proposition holds.

□

Using the same logic as for deriving (4.9), we can conclude a general formula for the lookback option at any time $\tau < T$. Specifically, for $\tau \in [t_{n-1}, t_n), n \geq 1$,

$$C_{fix}(\tau, S_\tau, M_{n-1}(S_t)) = \begin{cases} S_\tau e^{-\alpha k_1} \mathcal{F}^{-1}_{k_1,u}\left[\dfrac{\phi^{N-n}_{\tau,M}(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}\right] + M_{n-1}(S_t) - K, & \text{if } M_{n-1}(S_t) \geq K, \\ S_\tau e^{-\alpha k_2} \mathcal{F}^{-1}_{k_2,u}\left[\dfrac{\phi^{N-n}_{\tau,M}(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}\right], & \text{if } M_{n-1}(S_t) < K. \end{cases} \tag{5.9}$$

5.4 The Main Algorithm

We now summarize the results in the previous sections for computing the price of discrete DFP, assuming that the log return of naked fund process is a Levy process.

(i) Select a Levy process and its parameters for the naked fund process F_t. Hence takes the corresponding characteristic function.

(ii) Use Proposition 5.2.1 to obtain the characteristic function for the new process $S_t^F = K/F_t$. This process is martingale under the $\widetilde{\mathbb{Q}}$-measure. To do this, simply replace the original characteristic function $\phi_{Y_t}^{\widetilde{\mathbb{Q}}}(u)$ by $\phi_{X_t}^{\mathbb{Q}}(-i-u)e^{-rt}$.

(iii) Calculate the characteristic function ϕ_M^{N-n} using the Spitzer's formula (Theorem 4.1.2).

(iv) If the valuation time τ is not the inception of the contract, compute the characteristic function $\phi_{\tau,M}^{N-n}$ using Proposition 5.3.1.

(v) Calculate the DFP value using Proposition 3.2.1, (5.9) and FFT.

Chapter 6

Numerical Results

6.1 Simulation of Discrete DFP

In this section, we discuss the simulation procedure of discrete DFP under Levy processes. Some basic Levy processes, such as geometric Brownian motion, can be simulated easily using the random number generators which can provide us with standard normal random numbers. Imai and Boyle (2001) obtained the simulated value of discrete DFP under geometric Brownian motion.

Finite-activity Levy processes can be simulated by making use the fact that the inter-arrival times of the jumps follow an exponential distribution. Some infinite-activity Levy processes such as variance gamma and normal inverse Gaussian can be classified as a subordinated (time-changed) Brownian motion. The sampling path can be done by pursuing the time change and then the simulation of standard Brownian motion. Simulation techniques for other common Levy processes can be found in Schoutens (2007).

The simulation of a general Levy process can be helped by the characteristic functions defined by the Levy-Khinchine formula (2.2). Recall from (2.1) that characteristic function is the Fourier transform of the probability density function $f(x)$. We can write down the inverse transform:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi(u)du \tag{6.1}$$

Consider the relationship between the probability density function and cumulative distribution function (c.d.f.):

$$\begin{aligned} \text{CDF}(y) &= 1 - \int_{y}^{\infty} f(x)dx \\ &= 1 - \int_{y}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi(u)dudx \\ &= 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iuy}}{iu}\phi(u)du \\ &= 1 - \left[\frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} Re\left[\frac{e^{-iuy}}{iu}\phi(u)\right] du\right] \\ &= \frac{1}{2} - \frac{1}{\pi}\int_{0}^{\infty} Re\left[\frac{e^{-iuy}}{iu}\phi(u)\right] du \end{aligned} \tag{6.2}$$

This distribution function allows us to obtain the Levy random variables using the inverse transformation algorithm. First obtain the c.d.f. value for a large spectrum of real numbers with small partition. Then generate a Uniform(0,1) random number. Finally compare the random number to the c.d.f. value to obtain the Levy random variable.

6.2 Numerical Implementation

In this section, numerical examples illustrate the use of the analytical formulas. Three processes are considered. The first one is the classical geometric Brownian motion process. And the second model is the double exponential jump diffusion model, the finite-activity Levy model proposed by Kou (2002). The third model is the variance-gamma model, a popular infinite-activity Levy Process. We assume that under the chosen martingale measure, this martingale measure may not unique generally. We consider the following parameters value for the diffusion component if it exists in the model:

$$F_0 = 100, T = 1, r = 0.05, \sigma = 0.2,$$

In addition, $\theta = 0, \nu = 1$ are used in the variance-gamma model. $\lambda = 2.3, \eta_1 = 10, \eta_2 = 5, p = 0.6$ is chosen for the double exponential jump diffusion model.

The characteristic function used in simulation is under the original $\mathbb{Q}$ measure. However, the characteristic function used in the Fourier inversion is under the new $\widetilde{\mathbb{Q}}$ measure given by Proposition 5.3.1.

Table 6.1 and 6.2 shows the results of numerical evaluations of discrete DFP with guarantee level 110 under geometric Brownian motion (GBM) and double exponential jump diffusion model respectively. For comparison, we also gives the prices estimated using Monte Carlo method with 10^6 simulation runs. The reported time is the CPU time on a Pentium 1.8 GHz to compute the DFP price using the Fourier method. It can be seen that the execution times are all less than 1 second. We can observe that the computation time is proportional to the monitoring points, because the number of summation terms in the Spitzer's identity is equal to the monitoring points. The simulation, however, requires several minutes to produce one DFP value.

Tables 6.1 and 6.2 show that the DFP value increases when the number of monitoring points increases, because the more the monitoring points the higher the probability of upgrading the fund. Thus, a higer premium should be charged to policyholders. When jumps are added to the model, the DFP becomes more expensive, as we demonstrate in Table 6.2.

The damping coefficient α in (2.6) should be set appropriately for different processes. Carr and Mandan (1999) show that the damping coefficient for variance gamma model should satisfy:

$$\alpha < \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu}} - \frac{\theta}{\sigma^2} - 1 \tag{6.3}$$

The upper bound is 6.07 for this set of parameters. A value of α above unity and well below the upper bound should be chosen. We set α to be 2 for demonstration. The Monte Carlo simulation for variance gamma process is obtained by the inverse transformation algorithm in the previous section. Table 6.3 demonstrate the result. In this case, the MC simulation takes a significantly longer time than geometric Brownian motion. Tables 6.4 - 6.6 show the results for GBM, double exponential jump diffusion and variance-gamma process using the same settings, except that the strike price is 120. It can be seen that the FFT approach works

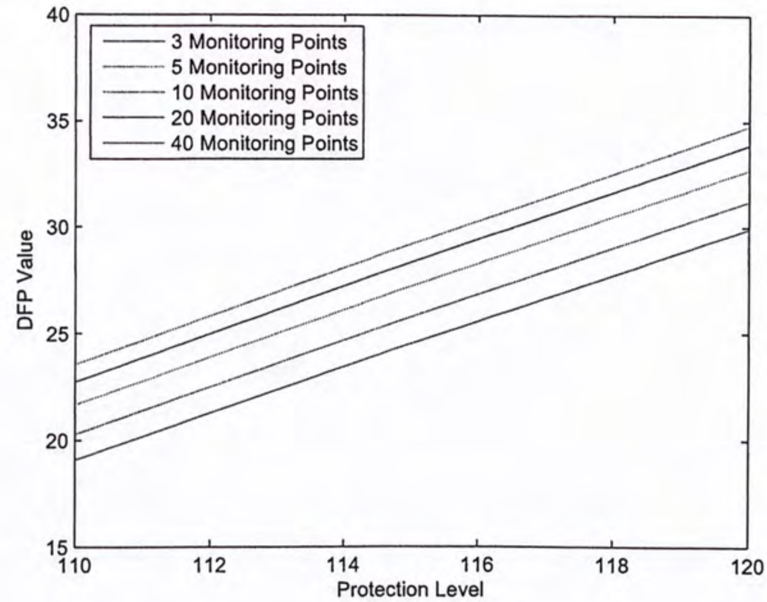

Figure 6.1: DFP value under GBM

equally accurately and efficiently. Thus, this method is robust to the choice of protection level.

Under the same set of parameters, Figure 6.1 - 6.3 shows the effect of monitoring points on discrete DFP value. Within a fixed length of maturity, DFP with more monitoring points always has higher value then those with less monitoring points. The difference of DFP value does not have significant dependence on the protection level and the choice of model. We see that even the number of monitoring point get doubled at each time, the increment of DFP value will become slower. Hence, we have evidence that the discrete DFP value will converge to continuous DFP value when the monitoring frequency is sufficiently high.

We also compare our numerical values to the simulation results from Table

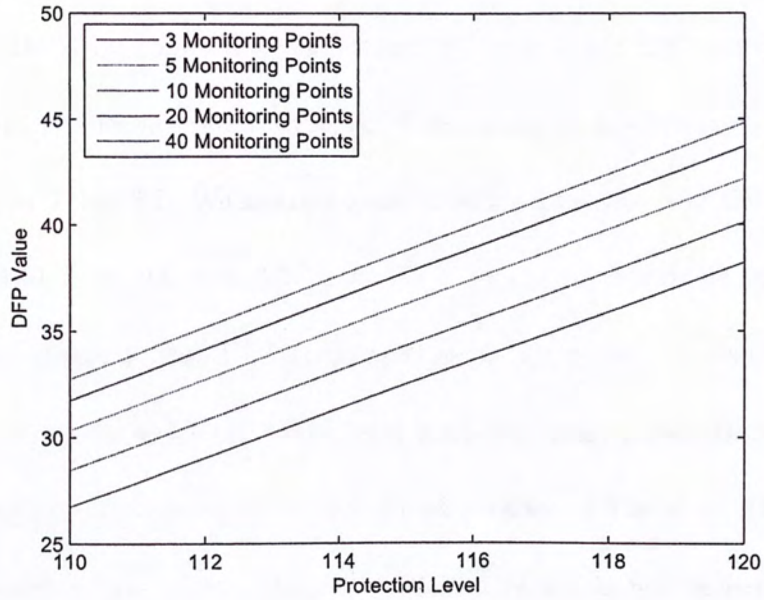

Figure 6.2: DFP value under double exponential jump diffusion model

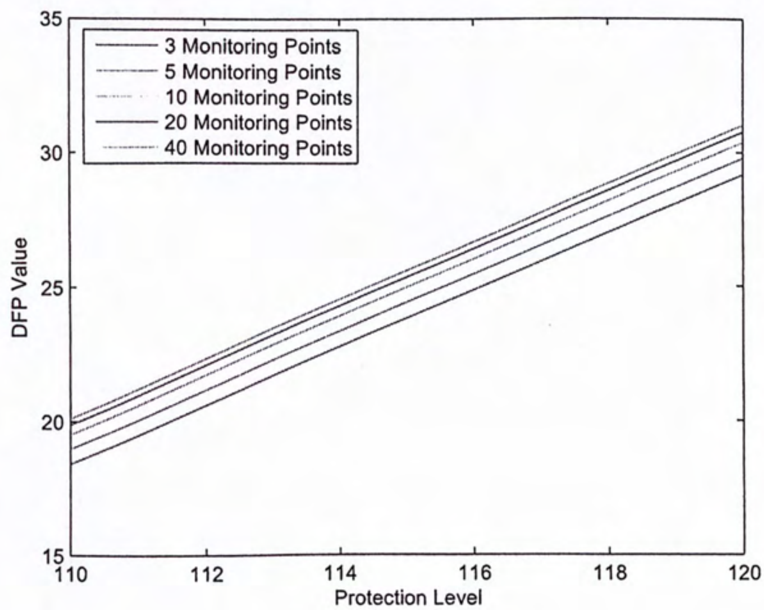

Figure 6.3: DFP value under VG model

10 of Imai and Boyle (2001). They extend the result of Broadie, Glasserman and Kou (1999) by computing the correction term using MC simulation. The discrete DFP values are reported in the form of confidence interval. The results are shown in Table 6.7. We assume lognormal fund process and the parameters are $F_0 = 100, K = 100, r = 0.04, \sigma = 0.2, T = 1, 3, 5$. It can be seen that the DFP values obtained from FFT are very close to the upper bounds of confidence intervals. Thus, the mid-point of the confidence level may underestimate the DFP value. We also compare our FFT results with those of Tse et al. (2008) under GBM, numerical values are similar to theirs, thus we do not report them here. The strength of the present FFT approach is that jumps and Levy processes can be accomplished with the same level of efficiency and accuracy.

	Geometric Brownian Motion		
Monitoring Points	Fourier method	MC Simulation	FT Time (sec.)
3	19.0829	19.0812	0.0625
5	20.2824	20.2850	0.1094
10	21.6541	21.6545	0.2188
20	22.7306	22.7239	0.4063
40	23.5478	23.5472	0.8215

Table 6.1: DFP value under GBM, Protection Level = 110.

	Double Exponential Jump Diffusion		
Monitoring Points	Fourier method	MC Simulation	FT Time (sec.)
3	26.7063	26.7044	0.0781
5	28.4469	28.4336	0.1250
10	30.3308	30.3334	0.2188
20	31.7289	31.7288	0.4688
40	32.7421	32.7244	0.9063

Table 6.2: DFP value under double exponential jump diffusion model, Protection Level = 110.

	Variance Gamma		
Monitoring Points	Fourier method	MC Simulation	FT Time (sec.)
3	18.2943	18.2836	0.0313
5	18.9173	18.7418	0.0625
10	19.5029	19.2668	0.0938
20	19.8451	19.8632	0.2188
40	20.0399	20.2085	0.3750

Table 6.3: DFP value under VG model, Protection Level = 110.

	Geometric Brownian Motion		
Monitoring Points	Fourier method	MC Simulation	FT Time (sec.)
3	29.9086	29.9086	0.0625
5	31.2172	31.2206	0.1094
10	32.7135	32.7158	0.2031
20	33.8879	33.8865	0.4219
40	34.7794	34.7769	0.7969

Table 6.4: DFP value under GBM, Protection Level = 120.

	Double Exponential Jump Diffusion		
Monitoring Points	Fourier method	MC Simulation	FT Time (sec.)
3	38.2250	38.2244	0.0781
5	40.1239	40.1364	0.1250
10	42.1791	42.1791	0.2344
20	43.7042	43.6950	0.4688
40	44.8096	44.8275	0.9531

Table 6.5: DFP value under double exponential jump diffusion model, Protection Level = 120.

	Variance Gamma		
Monitoring Points	Fourier method	MC Simulation	FT Time (sec.)
3	29.0231	29.0263	0.0313
5	29.7280	29.6949	0.0625
10	30.3667	30.2242	0.1094
20	30.7402	30.7526	0.2344
40	30.9527	31.2425	0.4531

Table 6.6: DFP value under VG model, Protection Level = 120.

Monitoring Frequency		Imai and Boyle (2001)		Fourier method
T=1	Monthly	11.096	11.375	11.3608
	Weekly	12.977	13.053	13.0389
	Daily	14.098	14.119	14.1066
T=3	Monthly	19.890	20.060	20.0089
	Weekly	21.915	21.993	21.9430
	Daily	23.124	23.177	23.1277
T=5	Monthly	25.021	25.097	25.0915
	Weekly	27.097	27.130	27.1463
	Daily	28.389	28.395	28.3916

Table 6.7: DFP value under GBM with different maturities and monitoring frequencies.

Chapter 7

Conclusion

This thesis considers the pricing of dynamic fund protection, a insurance product with discrete running maxima or minima. The pricing model can be easily extended to general Levy processes once the corresponding characteristic function is available. By the Girsanov's Theorem and Spitzer's formula, we analytically value the DFP value under fast Fourier Transform and a recursion. Numerical examples show that the proposed approach is accurate and efficient. Given that the implementation takes less than 1 second, the FFT approach can be treated as a closed-form solution in practice for valuation and calibration purposes. This thesis also contributes to the literature by producing an analytical tractable and implementable scheme to value discrete DFP under general Levy processes.

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