A Study on Options Hedge against Purchase Cost Fluctuation in Supply Contracts

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摘要

隨著網絡時代的到來，現貨市場和基于網絡的市場漸漸擴大。現在越來越多的公司要應付巨大的進貨價格的起伏風險，這就啓發我們研究怎樣利用期權這個風險管理的工具去對衝現貨/進貨價格波動的風險。這篇論文研究能訂購一期遠期合約及承受隨機價格波動風險的多期庫存系統。現貨價格是隨機的，遠期合約的交割價格決定于現貨價格。這個系統考慮用期權去對衝現貨和期貨價格波動的風險，在每期始，系統不僅要決定買多少一期期貨還要決定怎樣對衝價格波動風險。我們通過分析發現最優的訂貨策略是經典的 (s, S) 策略及最優的對衝風險策略是要麽買與遠期合約同等量的期權，要麽不買。而且，期權合約值不值得買決定于遠期合約價格曲線，所以對衝風險決定可以獨立于遠期合約的購買決定，但是，訂貨策略同時受對衝風險策略和現貨遠期合約價格波動的影響。另外，我們在指數效用假設下得到在風險規避假設下的最優訂貨和對衝風險策略是(s, S, T)，而且我們發現在這種情況下，最優訂貨策略和最優對衝風險策略不能分開制定。

關鍵詞：遠期合約，期權，庫存訂貨策略，風險管理，隨機庫存模型

Abstract

With the advent of the internet, spot markets and internet-based market places have increased their reach. More and more companies (both buyer and sellers) have faced dramatic price variations. This motivates the research on the use of options as risk management tool to hedge risk exposure on the (spot) price and on the firm's procurement cost. This paper studies a multi-period inventory system that replenishes stock by one-period forwards contracts purchased in a spot-forward market and bears random price moves. Spot prices are random, and the forward delivery prices are contingent on spot prices. The system considers deploying options contract to hedge price fluctuation on spots and forwards and at the beginning of each period, in addition to making decision on one-period forwards, it must make a hedge decision on any forwards contract. The analysis shows that the optimal ordering decisions follow classic (s, S) ordering rules and optimal hedge decisions are either to buy the options with equal amounts of the current orders or to do nothing. Furthermore, the purchase of an options contract depends on the forwards curve and can be made independent of ordering decisions. Yet, the ordering actions are determined jointly by hedging decisions and spot-forwards price. In addition, we derive the optimal ordering and hedging policy-(s, S, T) under an expected exponential utility and find in this case that optimal ordering and hedging decisions cannot be determined separately.

Key words: Forwards and options contracts, inventory procurement policy, risk management, stochastic inventory model

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Contents

Chapter 1

Introduction

1.1 Motivation

In order to ensure future availability and to lock in prices of products during a certain period of time, distributors tend to enter into contractual agreements with suppliers, these agreements are called supply contracts. Such contracts work as lubrication between the two parties by balancing profit and signaling the information. For some products (so called spot markets or internet-based exchanges if accessible via the internet) trading communities present alternative procurement and sales channels where suppliers can sell off excess inventory and buyers can fill last minute procurement needs. With the advent of the internet, spot markets and internet-based market places have increased their reach. Thus more and more companies (both buyer and sellers) have now to deal with dramatic variations in price,no matter whether it be the selling or purchasing price (see [26]). Another new aspect of today's business world is that many big corporations have evolved into a global-supply-chain-sourcing model to keep lowering purchasing costs so as to gain a competitive edge, this causes increases in lead time and further enlarge price risks. In order to deal with the price risk in addition to the demand uncertainty, a contingent claim contract called option is widely adopted by corporations. Option as a derivative product is well studied in finance literature, however option as a supply contract applied to supply chain management is seldom studied in operations research literature.

This work is motivated by our consultatory work with an chemical distributor in Guangdong province and a cable company in Jiangshu province of China. Both companies need to purchase raw materials-chemical and copper respectively from international market to satisfy the demand in local market. Although there is positive correlation between price in local market and in international market, the long leadtime of international shipment exposes the companies to huge purchase price risk and volatile cash flow, therefore it is necessary for the management to adopt a risk management practice. A common way is to wisely use derivatives to hedge against the price risks. The international Swaps and Derivatives Association (ISDA) 2003 derivative usage survey reports that 92% of the world's 500 largest companies use derivatives for risk management. Robert Pickel, chief executive officer of ISDA, says,"The survey demonstrates that derivatives today are an integral part of corporate risk management among the world's leading companies. Across geographic regions and industry sectors, the vast majority of these corporations rely on derivatives to hedge a range of risks to which they are exposed in the normal course of business".

Although in a classical Miller-Modigliani world, a firm's hedging activities are irrelevant in determining company value and there is a lack of consensus on the economic motivation for corporate hedging, it is generally believed that hedging can increase company value by reducing a firm's cash flow volatility so as to reduce the deadweight costs caused by frictions in real financial markets which includes i) reduced corporate tax liability. ii)reduced costs of financial distress. iii) reduced risk for the firm's managers. iv) reduced cost of underinvestment due to a reduction in the agency conflict between bondholders and shareholders or to an increased facility for financing investment projects with internal funds that reduces recourse to costly external financing. (Ephraim Clark and Amrit Judge 2005), so it is interesting to consider the ordering and hedging decisions together to maximize the firm value.

In this dissertation, we study a multi-period inventory system that replenishes its stock by ordering one-period forwards from a spot-forward market.

As spot price is random, and the forwards delivery prices are contingent on the spot, the system considers a kind of options contract to hedge price fluctuation on spots and forwards. We will focus on procurement planning and hedging policy in general, and on fair pricing and hedging effects of the option in the presence of a spot market in particular.

Although academic literature on the study of supply contracts and contingent claim contract is quite rich, only a few academics consider supply contracts and contingent claim contracts in a multi-period setting and even fewer take into account the risk aversion of decision maker. A key contribution of our work is that we consider storable items and solve the problem in a multi-period setting which captures real-world needs and is missing in the existing literatures.

1.2 Literature Review

Almost all business-to-business transactions are governed by contracts, and as a consequence, academic literature on supply contracts is quite rich. Essentially there are five lines of research related to ours. We shall elaborate on the related literatures below.

1.2.1 Supply Contracts under Price Uncertainty

In general, the firms under consideration face two critical business risks: one is demand uncertainty and the other is price uncertainty. Basically, by introducing supply contracts the problem we want to solve is how to handle those uncertainties efficiently. Most of the recent operations management literature on supply contracts emphasize on the use of various supply contracts in more effective handling uncertain demand situations. In a recent paper, Li and Kouvelis (1999) develop valuation methodologies for different types of supply contracts under deterministic demand but uncertain prices that are independent of the demand. They assume the price of the material follows the geometric Brownian motion or general Ito process in the case of two suppliers to follow two correlated geometric Brownian motions. They demonstrate how time flexibility, quantity flexibility, risk-sharing feature of the contract, and supplier selection can effectively reduce the expected sourcing costs of a risk-neutral buyer in enviroments of price uncertainty. Particulary, they note that for risk sharing supply contracts, time flexibility provides substantial benefits and is the most valuable for environments with rather low holding costs, higher risk sharing factor, volatile price processes, reasonably large (i.e., neither tight nor loose) risk-sharing windows. The option studied in our paper can be viewed as a kind of risk sharing supply contracts that offer time flexibility.

1.2.2 Dual Sourcing

Many companies prefer multiple sourcing because competition drives prices down and a wider supply base mitigates the risk associated with having only a sole supplier. Especially with the advance of technology, many companies procure from spot market because of its convenience and efficiency. Naturally, there is plenty of literature that considers the procurement policy of dual sourcing or multiple sourcing. Jinxin Yi and Alan Scheller-Wolf(2003) study an inventory replenishement problem with stochastic demand and two supply cources: a regular supplier who has a long term contract with a constrained order volume with the buyer, and a spot market where prices are uncertain and the order volume is unlimited. They show that optimal sourcing decisions have a structure similar to the classic (s, S) policy.

1.2.3 Risk Aversion in Inventory Management

The literature on risk-averse inventory models is quite limited and mainly focuses on single period problems. Recently, some papers explore risk-averse inventory models in a multi-period setting. Bouakiz and Sobel(1992) [4]characterize the inventory replenishment strategy in a multi-period Newsvendor setting so as to minimize the expected utility of the net present value of costs over a finite planning horizon or an infinite horizon. Assuming linear ordering cost, they prove that a base stock policy is optimal.

Xin Chen (2006) propose a framework for incorporating risk aversion in multi-period inventory models as well as multi-period models that coordinate inventory and pricing strategies. They show that the structure of the optimal policy for a decision maker with exponential utility functions is almost identical to the structure of the optimal risk-neutral inventory (and pricing) policies.

Our paper is different from the above papers in that we incorporate positive leadtime and spot market in our analysis and in addition to the inventory planning strategies, we derive the hedging strategies against spot price fluctuations.

1.2.4 Hedging Operational Risk Using Financial Instruments

As the financial market gradually becomes mature, access to the trading platform is growing, operational risk is more volatile and widely recognized as well. We have seen a growing interest in hedging operational risk using financial instruments. As far as we know, many of this literature focus on single period models (Newsvendor) with demand distribution that is correlated with the return of the financial market. Recently, Gaur and Seshadri [15]address the problem of hedging inventory risk for a short lifecycle or seasonal item when its demand is correlated with the price of a financial asset in the Newsvendor setting. They show that hedging reduces the variance of profit and the investment in inventory, increases the expected utility of a risk-averse decision maker, and increases the optimal inventory level for a broad class of utility functions and the more volatile the price of the underlying asset is or the longer the lead time is, the more beneficial is hedging. Rene and Martin (2006) derive the dynamic hedging strategy to hedge against operating profits of a risk averse firm under the assumption that the firm has mean-variance risk preference. An unique feature of their paper is that they compare the dynamic hedging strategies under different information assumptions regarding whether or not the operational state variables were observable and derive the value of information. However, in our paper, we use supply contract to hedge price risks not hedged in financial market, we show that there are common features between these two hedging methods.

1.2.5 Financial Literature

Since Black and Scholes derived a simple option pricing model, the pricing of options or contingent claims has become a common and important practice in the financial community and it has become a basic theoretical construct in financial economics. In general, the valuation of financial options relies heavily on "no arbitrage" arguments; in other words, strict assumptions about the financial market. Our approach permits valuing the option contracts from the perspective of the buyer without explicitly using any arbitrage arguments.

In our work, we apply discounted cash flow analysis to evaluate future uncertain cash flows as it is probably the most common approach for problems similar to ours. This method is based on several assumptions such as a perfect market allowing unlimited borrowing and lending at a risk-free rate and usually some form of capital asset pricing model to obtain a rate that is based only on the cash flow's contribution to overall market risk which has both market assumptions and an assumption about the ability to determine the risk of the cash flow before making the decision. On the one hand, the market is not perfect, corporations do have financial constraints, these financial constraints could have a big impact on the optimal inventory or hedging policies. However, the operational problems and the financial planning problems are considered separately in most literature. On the other hand, the risk exposure of a company may increase in the company's inventory levels (see for example Singhal(1988)). The ways to alleviate this problem with discounted cash flow analysis are to use capital asset pricing model directly or to construct a utility function that assesses the utility of all stakeholders. Since our focus is to jointly derive optimal inventory and hedging policies, we don't address the issue of discounted cash flow in our model, however, incorporating one of the above methods would be a possible future work.

Another shortcoming of our work is that we don't consider the exchange rate fluctuation risk. Instead, we assume all the money is denominated by the local currency and the exchange rate stays constant during the T periods. There exists many financial products designed to help investors to hedge against exchange rate fluctuation risk and many a literature that considers how to trade these financial products in financial markets to hedge against the interest rate risk from the corporations' point of view.

1.3 Organization of the Thesis

In chapter 2, we present how we can use option as a risk management tool for hedging against price uncertainty in procurement decisions and how the use of option can affect the inventory policy. More specifically, we jointly determine the optimal ordering and hedging policy for a buyer (which we will refer to as the distributor) who can order via forwards and hedge the price uncertainty using options contract which is contingent on the forwards. To complete our study, we consider risk aversion of decision makers in chapter 3, in this setting, we assume that the buyer has exponential utility over the stochastic profits. In chapter 4, we end the paper with conclusion and future research. Finally, unless noted otherwise, we collect proofs and review of the basic theory in the appendix.

Chapter 2

A Risk-Neutral Model

2.1 Framework and Assumptions

We consider a finite horizon, periodic review, stochastic inventory system which purchases an item from a supplier one period ahead of the delivery to replenish stock and meet internal demand. The price of the item fluctuates from period to period according to a continuous-time Markov process. The discounted net revenue of the system is assessed in N planning periods that are indexed forwardly. At the beginning of a period, the firm makes two decisions: decision of placing order by entering a one-period forward whose delivery price is determined by the market spot price under no-arbitrage assumption and a decision of purchasing a number of options to hedge against a loss to be incurred by price drop. We suppose that either forward or option can be issued by the supplier in conjunction with the mechanism of the spot market. It follows therefore that the ordering costs of the forwards and the premiums for the options are determined by the market, out of the control of the supplier. The orders through forwards look like those of one-period leadtime in the inventory literature. However, as the cost of ordering is determined by the spot and forward market, this sort of orderings has not been considered in the common assumption of the literature. The firm has to satisfy all the demand and backlog what cannot be fulfilled by available stock. Retailing price acceptable to the demand is the current spot price plus a premium set in the beginning of the period according to the spot price. Holding excessive stock and backlogging incurs cost which will be counted at the end of the period.

The retailing price in the period i is $P_i + \beta_i$, where β_i is a fixed premium for the period i. This additional premium addresses the cost caused by a price gap between the international market and the local market. It may contribute to various cost involved by the system such as handling, reworking and transportation. It is common knowledge that the price gap between a referenced international spot market and a local market exists correlatively. This linear relation is a simply form to capture this gap.

But how they are related may be more complicated than this assumption assumes, since it is not the key issue about this problem, such a simplification is reasonable and feasible. The delivery price of any shipment is the forward price that was determined in the last period. The firm wants to stablize the purchase price by buying options which prevail in the market and which also are issued by the supplier. The strike price of the option is the forward price and following the assumption that there are no arbitrage opportunities in the market, the option price can be calculated by risk-neutral pricing formulae. We focus on the problem of maximizing the net revenue of the firm by optimal ordering and hedging decisions assume that option prices is given in this chapter.

2.2 Price, Forward and Convenience Yield

2.2.1 Stochastic Model of Price

In this dissertation, we will use the model introduced in Gibson and Schwartz (1990) and Schwartz (1997) to model the dynamics of spot price and instantaneous convenience yield. Specifically, We assume that $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}; P)$ is a filtered probability space,and we consider a bivariate state process comprising the spot commodity asset P_t and the spot instantaneous convenience yield δt. The dynamics of the state are given under the risk-neutral measure

Q by a system of Ito stochastic differential equations of the form:

$$dP_t = (r - \delta_t)P_t dt + \sigma_1 P_t dZ_t, \tag{2.1}$$

$$d\delta_t = [k(m + h - \delta_t) - \lambda]dt + \sigma_2 dW_t, \tag{2.2}$$

$$dZ_t dW_t = \rho dt \tag{2.3}$$

where P_t is the spot price, δ_t is the measure of instantaneous convenience yield; dZ_t and dW_t are correlated increments to standard Brownian processes and ρ denotes the correlation coefficient between the two Brownian motions; σ_1 and σ_2 are the measure of volatility associated with Brownian motions Z and W respectively; $\alpha' = m + h$ is the mean convenience yield, h is the holding cost, we explicitly regard h as a component of α' because the cumulative convenience yield is increasing with holding cost, we will elaborate the relationship between convenience yield and holding cost in the next subsection, while k is the speed of adjustment of the mean reverting process.

Let $X_t = \ln P_t$, after a straightforward discretization of the time, the equations become(see [6]):

$$X_{n+1} = X_n + (r - \delta_n - \sigma_1^2/2)\Delta_n + \xi_n, \quad \xi_n \sim \mathcal{N}(0, \sigma_1^2 \Delta_n) \tag{2.4}$$

$$\delta_{n+1} = e^{-k\Delta_t}\delta_n + (1 - e^{-k\Delta_n})(m + h - \lambda/k) + \eta_n, \quad \eta_n \sim \mathcal{N}(0, \frac{\sigma_2^2}{2k}(1 - e^{-2k\Delta_n})) \tag{2.5}$$

here Δ_n is the length of n-th peirod. Given the spot price P_t and convenience yield δ_t in period t, the instantaneous convenience yield in period $t+1$ follows normal distribution, the spot price in period $t+1$ follows log-normal distribution and the joint conditional distribution of spot price and instantaneous convenience yield can be easily derived from the above equations. Let the joint conditional distribution be denoted by $\psi(p, \delta|\mathcal{F}_t)$, since X_n and δ_n are Markovian processes and only dependent on the values in last period, so we can redefine the distribution as $\psi(p, \delta|P_t, \delta_t)$. About forward/future pricing under this model and no-arbitrage argument, please refer to Gibson and Schwartz (1990) and Petter Bjerksund (1991).

2.2.2 Marginal Convenience Yield

A commodity's price is a combination of future asset and current consumption values. However, unlike financial derivatives, storage of commodity is costly, meanwhile, physical ownership of the commodity offers the agent the option of flexibility with regards to consumption (no risk of commodity shortage). Thus while a forward contract guarantees its owner the possession of a commodity at a pre-specified time in the future, say $t+1$, the physical ownership of a commodity at t, together with the ability to store it, not only guarantee at least the benefit of owning it at time $t+1$, but also give its owner the benefit of direct access between time t and t+1. It implies that to prevent arbitrage, the spot price of the commodity plus the cost of storing it must be in excess of the present value of its futures price. This price differential has led researchers to interpret marginal convenience yield as uncompensated cost of carrying (Fama French (1988)) or as the cost borrowing the commodity over the intervenning time interval. (Williams (1986)). For example, time delays, lumpy replenishment costs, or high costs of short-term changes in output can lead to a convenience yield on inventory held to meet customer demand for spot delivery.(Fama French (1988)) Marginal convenience yield is therefore a random variable and for traded commodity is determined by the stochastic evolution of the term structure of prices in the commodity's market. In this paper, we interpret the marginal convenience yield as a stochastic holding cost imposed on the firm by the market and consistent with the theory of storage, we define the relationship between spot price, marginal convenience yield and forward price as following:

$$P_t + h - \delta_t = \gamma F_{t+1}.$$

2.3 Optimality Equations

At the beginning of period t, the firm reviews its inventory level, observes the spot price of the product and contracts the supplier to place an order in a one-period forwards. The one-period forwards is a contract that delivers the ordered quantity to the firm at the futures price of the product. Then, the demand is fully fulfilled with instant delivery if the inventory is sufficient or with delayed delivery if some demand is backlogged because of insufficient inventory. Suppose the inventory of the period before ordering is x units hold on stock. If the firm places an order of q units by entering a one-period forward contract with the supplier that specifies the order quantity to be q units and delivery price to be F_{t+1}, the futures price of period t that is to be delivered in the following period. For simplicity's sake, let us suppose that as soon as the contract is signed the procurement cost is paid, resulting in qF_{t+1} and F_{t+1} is contingent on the spot price P_t and is determined by the forward market. Then, the firm receives the shipment of the forward that the firm entered in the last period, if any, and adds the shipment into available stock.

In addition, the firm is risk-averse and is concerned about the price discrepancy between the delivery price of the one-period forward and the spot price P_{t+1} that will be available next period. To hedge this timing spread of the procurement cost, the firm wants to hedge the purchase cost risk by purchasing α_t units óf the put options. For simplicity, we suppose that the strike of the call options equals F_{t+1} the forward delivery price. With options, the firm will be able to exercise the right of selling a unit of the product at the forward price F_{t+1} and will have to settle with the writer of the options financially by getting $(F_{t+1} - P_{t+1})^+$ from the writer. It is obvious that if $F_{t+1} \geq P_{t+1}$, then the firm would like to exercise the options and get the difference $F_{t+1} - P_{t+1}$, and otherwise, nothing will happen. The unit price of the put options is U_t and the discussion about how it is determined will be postponed.

The state of this Markov decision process is $(x, \alpha, F_t, F_{t+1}, U_t, P_t, \delta_t)$ in the period t before any decision to be made. Here, α is the hedge amount made in period $t-1$ against a loss due to the fall of the price, and F_t is the forward price at time $t-1$ for the forward which is to be delivered at time t. Both α and F_t were determined in period $t-1$. As F_t is a function of P_{t-1}, it becomes a component of the state in period t. However, P_t is the spot price, δ_t is the convenience yield, F_{t+1} is the forward price and U_t is the put option price which is observed in period t. Let $f_t(x, \alpha, F_t, F_{t+1}, U_t, P_t, \delta_t)$ denote the expected optimal present value of the discount profit over periods t through N. The underlying Markov decision process satisfies the following optimality equations:

$$
\begin{aligned}
& f_t(x, \alpha_{t-1}, F_t, F_{t+1}, U_t, P_t, \delta_t) \qquad (2.6) \\
= & \max_{y \geq x, y-x \geq \alpha \geq 0} \Big\{ (P_t + \beta_t) E[D_t] - K\delta(y-x) - \gamma F_{t+1} \times (y-x) - E[\mathcal{L}(x - D_t)] \\
- & \alpha U_t + \alpha_{t-1}(F_t - P_t)^+ + \gamma E[f_{t+1}(y - D_t, \alpha, F_{t+1}, F_{t+2}, U_{t+1}, P_{t+1}, \delta_{t+1}) | P_t, \delta_t] \Big\}
\end{aligned}
$$

and,

$$f_T(x, \alpha_{T-1}, F_T, F_{T+1}, U_T, P_T, \delta_T) = P_T x + \alpha_{T-1}(F_T - P_T)^+$$

where

$$E[f_{t+1}(y - D_t, \alpha, F_{t+1}, F_{t+2}, U_{t+1}, P_{t+1}, \delta_{t+1})|P_t, \delta_t] \tag{2.7}$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty f_{t+1}(y - \xi, \alpha, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta)\phi_t(\xi)\psi(p, \delta|P_t, \delta_t)d\xi dp d\delta,$$

and the triple integral in (2.7) is to be interpreted as the expected value over the demand distribution $\phi_t(\xi)$, (which we assume is a normal distribution) and the Riemann integral over conditional joint distribution of $(p_{t+\Delta t}, \delta_{t+\Delta t})$ given $\mathcal{F}_t$ (the information about p and δ at time t).

To simplify the expression of the optimality equations, we let

$$V_t(x, F_{t+1}, U_t, P_t, \delta_t) = f_t(x, \alpha_{t-1}, F_t, F_{t+1}, U_t, P_t, \delta_t) - \alpha_{t-1}(F_t - P_t)^+,$$

thus we have:

$$V_t(x, F_{t+1}, U_t, P_t, \delta_t) \tag{2.8}$$
$$= \max_{y \geq x, y-x \geq \alpha \geq 0} \Big\{ (P_t + \beta)E[D_t] - K\delta(y - x) - \gamma F_{t+1} \times (y - x) - E[\mathcal{L}(x - D_t)]$$
$$+ \quad \alpha(\gamma E[F_{t+1} - P_{t+1}|P_t, \delta_t]^+ - U_t) + \gamma E[V_{t+1}(y - D_t, F_{t+2}, U_{t+1}, P_{t+1}, \delta_{t+1})|P_t, \delta_t] \Big\}$$

Observe that $\alpha_{t-1}(F_t - P_t)^+$ is the earning earned by engaging the hedging activity in period $t - 1$, the transformation we made implies we can either count this earning in period $t - 1$ or in period t.

2.4 The Structure of the Optimal Policy

We analyze the optimal decisions based on the optimality equations. It is found that optimal hedge and ordering decisions can be made separately.

2.4.1 One-period Optimal Hedge Decision Rule

We suppose that the firm places an order of $y-x$ units through a forward contract in period i that lifts the inventory position from x to $y \geq x$. The firm hedges the price risk of this forward by buying options protecting price fall for either whole or part of the order. The number of options will not exceed the total amount $y-x$ in the forward contract. Suppose that the firm buys the α units of options to hedge a price drop, where $0 \leq \alpha \leq y-x$. Thus, we define the payoff function resulting from these ordering and hedge decisions as

$$\begin{aligned} G_i(y, \alpha, F_{i+1}, U_i, P_i, \delta_i) = & \quad -\gamma F_{i+1} y + \alpha(E[\gamma(F_{i+1} - P_{i+1})^+ | P_i, \delta_i] - U_i) \\ + & \quad \gamma E[V_{i+1}(y - D_i, F_{i+2}, U_{i+1}, P_{i+1}, \delta_{i+1}) | P_t, \delta_t] \end{aligned}$$

As the last term of the right-hand side of the preceding equation does not contain hedge amount α, the decision of choosing α to maximize $G_i(y, \alpha, F_{i+1}, U_i, P_i, \delta_i)$ is irrelevant to this term. Therefore, when inventory position (level) before ordering in period i is x, and a decision of placing an order by selling a forwards of $y-x$ units, where $y > x$, is made, maximizing the expected profit

$G_i(y, \alpha, F_{i+1}, U_i, P_i, \delta_i)$ through the hedge decision is equivalent to maximizing $G_i(y, \alpha, F_{i+1}, U_i, P_i, \delta_i)$ by

$$\bar{G}_i(y, \alpha, F_{i+1}, U_i, P_i, \delta_i) = \begin{cases} G_i(y, y-x, F_{i+1}, U_i, P_i, \delta_i) & \text{if } E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i] \geq \frac{U_i}{\gamma} \\ G_i(y, 0, F_{i+1}, U_i, P_i, \delta_i) & \text{otherwise.} \end{cases}$$

To put the formula in words, the amount of options, α, can be optimized by $y - x$ options if the quantity $E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i] - \frac{U_i}{\gamma}$ is non-negative, or zero otherwise.

From this expression, it is clear that optimal hedge decision is either full hedge or no hedge, and whether or not the firm wants full hedge is independent of how much the firm purchases in the forwards. The premise that the firm makes full hedge is it has to place an order first through entering a forwards.

As $\gamma E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i]$ is the expected marginal profit restored in the procurement cost by exercising options, and U_i is the marginal cost of the options, the relation $\gamma E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i] - U_i > 0$ states that the potential profit of longing an options is larger than the cost of longing the options. Thus, the relation established the principle of longing options. Observe that $\gamma E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i]$ under risk-neutral measure is the fair price of the option under non-arbitrage argument, so for risk-neutral buyer, they will only use the options as speculation tool.

2.4.2 One-period Optimal Orderings Decision Rule

From the last subsection, the optimal ordering decisions can be made independent of hedge decisions. Thus, we can optimize the expected net profit by placing an optimal order with regardless to the hedge. We subtract those terms relating to the hedge decisions from $G_i(y, \alpha, F_{i+1}, U_i, P_i, \delta_i)$, and define the resulting function as

$$H_i(y, F_{i+1}, U_i, P_i, \delta_i) = G_i(y, \alpha, F_{i+1}, U_i, P_i, \delta_i) - \alpha \left(\gamma E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i] - U_i \right).$$

When the spot price and convenience yield and inventory before ordering in period i are P_i and δ_i and x, respectively, optimizing $H_i(y, F_{i+1}, U_i, P_i, \delta_i)$ is an intermediate decisive objective over $y \geq x$. As this function does not relate to α, α is not an independent variable of it. More explicitly, as

$$\begin{aligned} & H_i(y, F_{i+1}, U_i, P_i, \delta_i) \\ = \quad & -\gamma F_{i+1} y + \gamma E[V_{i+1}(y - D_i, F_{i+2}, U_{i+1}, P_{i+1}, \delta_{i+1}) | P_t, \delta_t] \end{aligned}$$

if $\gamma V_{i+1}(y, F_{i+2}, U_{i+1}, P_{i+1}, \delta_{i+1})$ is a K-concave function of y for any given F_{i+1} and P_{i+1} and δ_{i+1}, $H_i(y, F_{i+1}, U_i, P_i, \delta_i)$ is a K-concave function of y. Following a standard approach, the maximum of $H_i(y, F_{i+1}, U_i, P_i, \delta_i)$ exists, and is attained by $S_i(P_i, \delta_i)$. In addition, define

$$s_i(P_i, \delta_i) = \sup\{y \leq S_i(P_i, \delta_i) : H_i(y, F_{i+1}, U_i, P_i, \delta_i) < H_i(S_i(P_i, \delta_i), F_{i+1}, U_i, P_i, \delta_i) - K\}$$

as the *reorder point*, which does exist finitely. Note that the definitions of pairs $(s_i(P_i, \delta_i), S_i(P_i, \delta_i))$ depend on the spot price P_i and convenience yield δ_i, and forms optimal (s, S) ordering policies that is identified to (s, S) ordering parameters in the inventory literature. Further, let

$$H_i^*(x, F_{i+1}, U_i, P_i, \delta_i) = \max\{H_i(x, F_{i+1}, U_i, P_i, \delta_i), \max_{y \geq x} H_i(y, F_{i+1}, U_i, P_i, \delta_i) - K\},$$

then

$$H_i^*(x, F_{i+1}, U_i, P_i, \delta_i) = \begin{cases} H_i(x, F_{i+1}, U_i, P_i, \delta_i) & \text{if } x \geq s_i(P_i, \delta_i) \\ H_i(S_i(P_i, \delta_i)), F_{i+1}, U_i, P_i, \delta_i)) - K & \text{otherwise} \end{cases}$$

is the maximum expected profit by optimal ordering decision for inventory x.

2.4.3 Optimal Policy

We consider function $f(x, F, U, P, \delta)$ where x is an inventory level,F is the forward price, U is the option price, P is the spot price and δ is the convenience yield. Let $\mathcal{V}$ be the collection of all $f(x, F, U, P, \delta)$ such that for each $P \geq 0$, $f(x, F, U, P, \delta)$ is a K-concave function.

Theorem 1. *Suppose that for* $i = 1, \cdots, N-1$,

$$f_{i+1}(y, \alpha, F, F_{i+2}, U, P, \delta) = \alpha(F - P)^+ + V_{i+1}(y, F_{i+2}, U, P, \delta)$$

where $V_{i+1}(y, F_{i+2}, U, P, \delta) \in \mathcal{V}$, *and* $V_N(y, F_{i+2}, U, P, \delta) = Py$. *Then,*

$$f_i(x, \alpha, F, F_{i+1}, U, P, \delta) - \gamma F_{i+1}x - \alpha(F - P)^+ - (P + \beta)E[D_i] + E[\mathcal{L}(x - D_i)]$$

$$= \begin{cases} H_i^*(x, F, U, P, \delta) + U_i[S_i(P_i, \delta_i) - x] & \text{if } x < s_i(P, \delta) \\ & \text{and } E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i] > \frac{U_i}{\gamma} \\ H_i^*(x, F, U, P, \delta) & \text{otherwise.} \end{cases}$$

In addition, $(s_i(P, \delta), S_i(P, \delta))$ *optimizes the ordering decisions. That is, when* $x < s_i(P, \delta)$, *enter a one-period forwards of* $S_i(P, \delta) - x$ *units and hedge a price drop by purchasing options that equal order quantity if and only if* $E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i] > \frac{U_i}{\gamma}$. *Furthermore, when* α, F *and* P, δ *are fixed,* $f_i(x, \alpha, F, P, \delta)$ *is a member of* $\mathcal{V}$.

Proof: Let x denote the inventory position in the beginning of period i before any decision. Let F, U, P and δ denote the delivery price of the forwards contract that is delivered in the period, the put option price, the spot price and the convenience yield in the period. By assumption, we have:

$$f_{i+1}(y, \alpha, F, F_{i+2}, U, P, \delta) = \alpha(F - P)^+ + V_{i+1}(y, F_{i+2}, U, P, \delta)$$

By assumption, $V_i(y, F_{i+1}, U, P, \delta)$ is a K-concave function of y so that function $H_i(x, F_{i+1}, U, P, \delta)$ is a K-concave function of x for each P and δ, and thus can be optimized by a maximizer $S_i(P, \delta)$. Analogously, we define $s_i(P, \delta)$ in accordance with the ordering optimization analysis. It is known from a standard approach of (s, S) stochastic inventory models that objective function

$$\max_{y \geq x}\{-K\delta(y-x) + H_i(y, F_{i+1}, U, P, \delta)\} = H_i^*(x, F_{i+1}, U, P, \delta),$$

and is attained by optimal $(s_i(P,\delta), S_i(P,\delta))$ policy. The workings of the policy are as follows: If $x < s_i(P,\delta)$, we place an order of $S_i(P,\delta) - x$ to increase the inventory position up to $S_i(P,\delta)$ by the forwards contract, and thus to attain $H_i^*(x, F_{i+1}, U, P, \delta) = -K + H_i(S_i(P,\delta), F_{i+1}, U, P, \delta)$. Otherwise, it is optimal not to place any order to attain $H_i^*(x, F_{i+1}, U, P, \delta) = H_i(x, F_{i+1}, U, P, \delta)$. Thus, the optimal hedge is either $S_i(P,\delta) - x$ or zero according to whether or not $E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i] > U_i$ takes effect. Therefore, we establish (2.9).

Furthermore, because $H_i(x, F_{i+1}, U, P, \delta) \in \mathcal{V}$, for any $P > 0, \delta > 0$, optimal function

$$
\begin{aligned}
& H_i^*(x, F_{i+1}, U, P, \delta) \\
= & \begin{cases}
H_i(S_i(P,\delta), F_{i+1}, U, P, \delta) - K + U_i[S_i(P) - x] & \text{if } x < s_i(P,\delta) \\
 & \text{and } E[(F_{i+1} - P_{i+1})^+ | P_i, \delta_i] > U_i \\
H_i(x, F_{i+1}, U, P, \delta) & \text{otherwise}
\end{cases}
\end{aligned}
$$

is K-concave, and $H_i^*(x, F_{i+1}, U, P, \delta) \in \mathcal{V}$. As a consequence, when α and F are fixed, optimal value function $f_i(x, \alpha, F, F_{i+1}, U, P, \delta) \in \mathcal{V}$. Therefore, the lemma follows. □

Theorem 2. *For each i, α_i decreases in the option price U_i.*

Proof. In view of the submodularity of $H_i(y^*, F_{i+1}, U_i, P_i, \delta_i)$ and the definition of α_i, it is implied that α_i is a decreasing function of U_i.

Chapter 3

A Risk-Averse Model

In the last chapter, it is found that under risk-neutral assumption, the hedge becomes a speculation activity. It is natural to investigate the optimal hedging policy and in particular how the price risk affects the decision process for the risk-averse manager. To this end, we begin to introduce the framework of a risk-averse model.

3.1 Risk Aversion Modeling and Utility Function

Economic decisions are usually not based on total or average return alone but rather on the utility of the return as viewed by the decision maker. Although decision analysts have divergent views and approaches to the problem of defining a utility function involving consumption or income over time, we choose the net present value (NPV) modeling method and exponential utility function here.

Specifically, let the income stream over T periods be represented by $X = (f_1, f_2, \cdots, f_T)$, where f_t is the income obtained in period t. Let $U(X)$ be the utility value of the income stream. Then, the decision maker's objective is to maximize her expected utility as follows.

$$EU(f_1, f_2, \cdots, f_T)$$

over the planning horizon of T periods. We call this approach as the net present value method.

We determine the optimal investment strategy by assessing an utility function on NPV and use it to determine certainty equivalents for each possible strategy. In calculating NPV, we take the interest rate for risk-free borrowing and lending as the discount factor, reflecting the fact that the decision maker could borrow and lend over time and convert any deterministic cash flow into its NPV. Bouakiz and Sobel [4] employed this method to solve the risk-aversion multi-period inventory replenishment problem.

The basis of modern axiomatic utility theory was developed by von Neumann and Morgenstern. Basic axioms of behavior for von Neumann-Morgenstern (vNM) utility functions require $U(X)$ to be nondecreasing and concave in each f_i separately and require preferences based on $U(X)$ to be invariant under any positive linear transformation of $U(\cdot)$.

Later, axiomatic approaches were also employed to derive certain types of utility functions for multi-period problems. In particular, the so called "additive independence axiom" implies additive utility functions of the following form,

$$U(f_1, f_2, \cdots, f_T) = \sum_{t=1}^{T} u_t(f_t)$$

That is, the utility of the income flow is the summation of the utility from the income in each time period, where function u_t is increasing and concave.

The exponential utility function is typical of von Neumann-Morgenstern expected utility function, and widely used in economics and decision analysis. It has the form of $U(X) = E[w(X)] = E[-e^{-AX}]$, where X is the amount of random revenue resulted from running a policy and $w(x) = -e^{-Ax}$ with $A > 0$ as the reference on the profit and risk. The Arrow-Pratt coefficient of absolute risk aversion of $w(x)$ at any x is $r_A(x) = A$ and a constant, and so indifferent at x, this advantage grants the applicability of dynamic programming.

For a risk tolerate parameter A, we denote the "certainty equivalent" operator with respect to a random variable X to be

$$\mathcal{CE}^A(X) = -\frac{1}{A} \ln E[e^{-AX}]$$

For a decision maker with risk tolerance A and an exponential utility function, the above certainty equivalent represents the amount of money she feels indifferent to a gamble with random payoff X. Observe that

$$\begin{aligned} -\frac{1}{A}\ln E[e^{-AX}] = & -\frac{1}{A}\ln\left[1 - AE(X) + \frac{A^2}{2}E(X^2) + O(A^3)\right] \\ = & \quad E[X] - \frac{A}{2}E(X - EX)^2 + O(A^3) \end{aligned}$$

as $A \to 0$. As $E[X - E(X)]^2$ is the variance for X, if A is small, the above certainty equivalent of $U(X)$ is close to its mean-variance value, thus when A is small, the maximization of the expected exponential utility helps select the optimal policy that maximizes expected return along with a constant increase in variance.

3.2 Multi-Period Inventory Modelling

There are three random sources in the model, which are demand, spot price and convenience yield. We assume that the probability distribution of demand ξ is i.i.d and given by $F(u) = Pr\{\xi \leq u\}$. As in last section, the spot price and convenience yield follows the stochastic process described in equations 2.1 – 2.3.

The distributors make orders via long-term forward contracts. The long-term forward commitment with the supplier specifies the total quantity commitment contracted, (in this chapter, we first assume that the total quantity is infinite, then we relax this condition in extension.) however, the distributor has the discretion over when to take delivery of, and make payment for, this quantity. Since the distributor has ordering flexibility and can use spot market as cache, the quantity risk he faces is reduced, however because the distributor faces one period leadtime and a volatile spot market, he is exposed to large price risk. Here we assume that the distributor use the same hedging tool-put option as in last section, the distributor's decision problem under such supply contract and hedging tool will be as follows:

In the beginning of period t, the firm reviews its inventory level, observes the spot price of the product, and places the order via the long-term contract. Then, demand is realized and shortage and holding cost is calculated.

In addition, because there is a one period leadtime for the order via long-term contract, the distributor is exposed to the price discrepancy between the delivery price of the long-term supply contract and the spot price P_{t+1} that will be available next period. To hedge this timing spread of the procurement cost, the firm wants to hedge the purchase cost risk by purchasing α_t units of the put options with the strike price F_{t+1}, which is the exercise price of the long-term contract.

We assume that the distributor is risk-averse, we explore the optimal ordering and hedging strategy following under the assumption that the distributor has exponential utility.

3.3 Exponential Utility Model

Given P_t and F_t, we define the reward of engaging in the business in period t as

$$R_t(x, \alpha_{t-1}, \xi) = (P_t + \beta)\xi - \mathcal{L}(x - \xi) + \alpha_{t-1}(F_t - P_t)^+$$

Assume the utility function we have is $\mathcal{U}(z) = -e^{-Az}$, then our objective is to maximize the expected utility of the present value of net income, which is stated by the dynamic programming equations below:

$$\begin{aligned}
& W_t(x, \alpha_{t-1}, F_t, F_{t+1}, U_t, P_t, \delta_t, a_t) && (3.1)\\
= & \max_{y \geq x, y-x \geq \alpha \geq 0} \Big\{ - e^{-a_t[-K\delta(y-x) - \gamma F_{t+1}y - \alpha U_t]} e^{-a_t \gamma F_{t+1} x} E\left[e^{-a_t R_t(x, \alpha_{t-1}, \xi)}\right] \\
\times & \; EW_{t+1}(y - \xi, \alpha, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta, \gamma a_t)\Big\}
\end{aligned}$$

Where W_t represents the profit-to-go in period t, p and δ are the random variables that represent P_{t+1} and δ_{t+1} respectively given that the spot price P_t and convenience yield δ_t in period t. Our model contains negative net benefits, i.e. costs; so maximizing the expected utility corresponds to minimize $Eexp^{AB(N)}$, where larger values of $A > 0$ connote greater sensitivity to risk and $B(N)$ is the present value of negative net profits incurred during an N-period planning horizon, so equation 3.1 can be equally transformed to the following one:

$$\begin{aligned} & W_t(x, \alpha_{t-1}, F_t, F_{t+1}, U_t, P_t, \delta_t, a_t) \qquad (3.2) \\ = & \min_{y\geq x, y-x\geq\alpha\geq 0} \left\{ e^{a_t[K\delta(y-x)+\gamma F_{t+1}y+\alpha U_t]} e^{-a_t\gamma F_{t+1}x} E\left[e^{-a_t R_t(x,\alpha_{t-1},\xi)}\right] \right. \\ \times & \left. EW_{t+1}(y-\xi, \alpha, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta, \gamma a_t) \right\}, \end{aligned}$$

Proposition 1. *If we define $\widetilde{\mathcal{CE}}(X) = \frac{1}{A}\ln E[e^{AX}]$ and $h(x,y)$ is k-convex in x for any y, let ξ be a random variable, y is a realization of ξ, then for any $A > 0$ the function*

$$\widetilde{g}(x) = \widetilde{\mathcal{CE}}[h(x,\xi)]$$

is also K-convex, similarly, if a function $f(x,y)$ is k-concave in x for any y, then for any $A > 0$ the function

$$g(x) = \mathcal{CE}[f(x,\xi)]$$

is also K-concave.

Proof. We only prove the case with K-convexity; the other cases can be proven by following similar steps.

Define random variable $X = h(\lambda x_0 + (1-\lambda)x_1, \xi)$, random variable $Y = \lambda h(x_0,\xi) + (1-\lambda)(h(x_1,\xi)+K)$ and $f(x) = \exp(Ax)$, $A > 0$. It suffices to prove that for any x_0, x_1 with $x_0 \leq x_1$ and any $\lambda \in [0,1]$,

$$Ef(X) \leq Ef(Y)$$

Because $h(x,y)$ is K-convex in x for any y, we have

$$h(\lambda x_0 + (1-\lambda)x_1, \xi) \le \lambda h(x_0, \xi) + (1-\lambda)(h(x_1, \xi) + K)$$

for any realization of ξ, thus we can declare that random variable X is stochastically less than random variable Y. Furthermore, notice that $f(x)$ is a increasing convex function of x, so we have

$$Ef(X) \le Ef(Y)$$

. □

Theorem 3. *The inventory decisions in the risk-averse inventory control model Eq. 3.2 can be calculated through the following dynamic programming recursion*

$$\begin{aligned} & G_t(x, \alpha_{t-1}, F_t, F_{t+1}, U_t, P_t, \delta_t) \quad (3.3) \\ = & -\gamma F_{t+1}x + \widetilde{\mathcal{CE}}[-R_t(x, \alpha_{t-1}, \xi)] \\ + & \min_{y \ge x, y-x \ge \alpha \ge 0} \left\{ K\delta(y-x) + \gamma F_{t+1}y + \alpha U_t + \gamma \widetilde{\mathcal{CE}}\left[G_{t+1}(y-\xi, \alpha, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta)\right] \right\} \end{aligned}$$

and $G_{T+1}(x, \alpha_T, F_{T+1}, F_{T+2}, U_{T+1}, P_{T+1}, \delta_{T+1}) = 0$

Proof. First consider the last period, period T.

$$\begin{aligned} & W_T(x, \alpha_{T-1}, F_T, F_{T+1}, U_T, P_T, \delta_T) \quad (3.4) \\ = & \min_{y \ge x, y-x \ge \alpha \ge 0} \left\{ e^{a_T[K\delta(y-x) + \gamma F_{T+1}y + \alpha U_T]} e^{-a_T \gamma F_{T+1}x} E\left[e^{-a_T R_T(x, \alpha_{T-1}, \xi)}\right] \right\} \\ = & e^{-a_T \gamma F_{T+1}x} E\left[e^{-a_T R_T(x, \alpha_{T-1}, \xi)}\right] \min_{y \ge x, y-x \ge \alpha \ge 0} \left\{ e^{a_T[K\delta(y-x) + \gamma F_{T+1}y + \alpha U_T]} \right\} \end{aligned}$$

Define

$$
\begin{aligned}
& G_T(x, \alpha_{T-1}, F_T, F_{T+1}, U_T, P_T, \delta_T) \\
= \quad & -\gamma F_{T+1} x + \widetilde{\mathcal{CE}}\left[-R_T(x, \alpha_{T-1}, \xi)\right] + \min_{y \geq x, y-x \geq \alpha \geq 0} \left\{ K\delta(y-x) + \gamma F_{T+1} y + \alpha U_T] \right\}
\end{aligned}
$$

We have,

$$
W_T(x, \alpha_{T-1}, F_T, F_{T+1}, U_T, P_T, \delta_T) = e^{a_T G_T(x, \alpha_{T-1}, F_T, F_{T+1}, U_T, P_T, \delta_T)}
$$

where $a_T = \gamma a_{T-1} = \gamma^{T-1} a$

Now we start induction. Assume

$$
W_{t+1}(x, \alpha_t, F_{t+1}, F_{t+2}, U_{t+1}, P_{t+1}, \delta_{t+1}) = e^{a_{t+1} G_{t+1}(x, \alpha_t, F_{t+1}, F_{t+2}, U_{t+1}, P_{t+1}, \delta_{t+1})}
$$

then we have:

$$
\begin{aligned}
& W_t(x, \alpha_{t-1}, F_t, F_{t+1}, U_t, P_t, \delta_t) \\
= \quad & \max_{y \geq x, y-x \geq \alpha \geq 0} \left\{ e^{a_t[K\delta(y-x) + \gamma F_{t+1} y + \alpha U_t]} e^{-a_t \gamma F_{t+1} x} E\left[e^{-a_t R_t(x, \alpha_{t-1}, \xi)}\right] E\left[e^{\gamma a_t G_{t+1}}\right] \right\}
\end{aligned}
$$

If we define:

$$
\begin{aligned}
& G_t(x, \alpha_{t-1}, F_t, F_{t+1}, U_t, P_t, \delta_t) \qquad (3.5) \\
= \quad & -\gamma F_{t+1} x + \widetilde{\mathcal{CE}}[-R_t(x, \alpha_{t-1}, \xi)] \\
+ \quad & \min_{y \geq x, y-x \geq \alpha \geq 0} \left\{ K\delta(y-x) + \gamma F_{t+1} y + \alpha U_t + \gamma \widetilde{\mathcal{CE}}\left[G_{t+1}(y-\xi, \alpha, F_t, F_{t+1}, U_t, p, \delta)\right] \right\}
\end{aligned}
$$

Then,

$$
W_t(x, \alpha_{t-1}, P_t, \delta_t) = e^{a_t G_t(x, \alpha_{t-1}, F_t, F_{t+1}, U_t, P_t, \delta_t)}
$$

□

3.4 Optimal Ordering and Hedging Policy for Multi-Period Problem

Following we will show that in this multi-period procurement problem with put option, a $(s(p,\delta), S(p,\delta), \mathcal{T}(p,\delta))$ policy is optimal. For simplicity, we will henceforth suppress the subscript t from notation whenever the context makes it clear.

A $(s(p,\delta), S(p,\delta), \mathcal{T}(p,\delta))$ order with put option policy specifies that the inventory level after ordering $y(x)$ when the current inventory level is x, is given by

$$y(x) = \begin{cases} S & \text{if } x < s, \\ x & \text{if } x \geq s, \end{cases}$$

and the option quantity we should buy after we place the order is given by $\alpha = \mathcal{T}$. To facilitate the exposition, we define a set by

$$\mathcal{C} := \{(y,\alpha) | y \in C, \alpha \in Q(y) = [0, y-x]\}$$

and the functions of

$$m_t(y, \alpha, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta) = \alpha U_t + \gamma \widetilde{\mathcal{CE}}\left[G_{t+1}(y-\xi, \alpha, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta)\right]$$

$$M_t(y, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta) = \gamma F_{t+1} y + \min_{\alpha \in Q(y)} \{\alpha U_t + \gamma \widetilde{\mathcal{CE}}\left[G_{t+1}(y-\xi, \alpha, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta)\right]\}$$

$$\begin{aligned} & G_t^*(x, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta) \\ = \quad & \min\left\{ M_t(x, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta), \min_{y > x} \left[M_t(y, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta) + K\right]\right\} \end{aligned}$$

It then follows that

$$G_t(x, \alpha_{t-1}, F_t, F_{t+1}, U_t, P_t, \delta_t) = -\gamma F_{t+1}x + \widetilde{\mathcal{CE}}[-R_t(x, \alpha_{t-1}, \xi)] + G_t^*(x, F_{t+1}, F_{t+2}, U_{t+1}, p, \delta)$$

Theorem 4. *In the multi-period procurement problem with put option, the following hold for all $t = 1, \cdots, T.$, there exists $s(p, \delta) := s_t(p, \delta)$, $S(p, \delta) := S_t(p, \delta)$, and $\mathcal{T}(p, \delta) := \mathcal{T}_t(p, \delta)$ such that an $(s, S, \mathcal{T})$ policy is optimal in period t.*

Proof. From the optimality equation, we can see that G_T for all $t = 1, \cdots, T.$ is a joint K-convex function of x and α_{T-1}, assume G_{t+1} is a joint K-convex function of y and α, with proposition 1, we can conclude that m_t is a joint K-convex function of y and α, so there exists an unique minimizer of m_t over $\alpha \in Q(y)$, let it be $\mathcal{T}(p, \delta)_t$. Furthermore, with lemma 3 in Appendix, we have M_t is a K-convex function of y, so following the standard approach, we can claim that G_t is a joint K-convex function of x and α_{t-1} and the optimal inventory policy is a state dependent $(s(p, \delta), S(p, \delta))$ policy. □

For each t and fixed z,p and δ, because $\frac{\partial m_t(U,\alpha)}{\partial U \partial \alpha} = 1 > 0$, $m_t(U, \alpha)$ is supermodular on its domain.

Theorem 5. *For each t, $\mathcal{T}_t$ is decreasing in U_t.*

Proof. The supermodularity of $m_t(U, \alpha)$ and the definition of $\mathcal{T}(U)$ directly implies that $\mathcal{T}(U)$ is a decreasing function of U. □

Remark 1. *If we assume that* $K = 0$*, then the optimal ordering policy is state-dependent base stock policy and the result can be easily extended to the situation that the total quantity commitment contracted is limit. To give a hint, we just need to put an uppper bound on the decision variable* y*.*

Chapter 4

Conclusion and Future Research

In this paper, we present a framework for analyzing the use of supply contracts (particularly options here)to hedge against purchase cost fluctuation risk. We structure the options contract as a series of European put options which is contingent on the Forwards order over an N-period time horizon.

Under risk-neutral assumptions, we show that the hedging decision and the ordering decision via long-term contract(particularly here is forwards) can be separated under our setting, which means that the options contract becomes a mere financial tool, i.e. the role of the options is to hedge against (spot) price uncertainty, essentially becoming financial derivatives on the spot price independent of the inventory problem. Notice that although we are considering a risk-neutral decision-maker, an additional benefit of options is that they could help the distributor to balance the cash flow so as to reduce the probability of bankruptcy, however since risk-neutral decision-maker doesn't care about risk, they only use put options as a speculation tool. Furthermore, we point out that the hedging decision and the ordering decision via long-term contract can not be separated under risk-averse assumption and there is an optimal hedging quantity for every optimal ordering quantity.

The framework proposed in this paper and the results obtained motivate a number of extensions.

- Impact of hedging: Although we know that hedging can balance the cash flow and reduce the probability of bankruptcy, we would like to figure out how to quantify the impact of hedging on risk-averse decision maker's utility.

- Fair value of option: We would like to determine the fair value of the options specific to both parties by analyzing the Stackelberg game where the supplier acts as the leader and sets the contract pricing parameters in anticipation of the distributor's response. Hopefully, we can get a demand curve from the distributor's side and a supply curve from the supplier's side and then find the intersection of those curves as the fair value of the options.

- Other types of hedging tools: We plan to consider other types of options. e.g. the supply contracts with options which combines the long-term contract and the options contract into one contract or barrier options which set a risk level that the distributor can tolerate.

- Continuous time models: Continuous time models and dynamic hedging strategies are widely studied in finance literature. Thus, it is interesting to extend our periodic review framework to models in which inventory decisions are reviewed in continuous time and financial hedging takes place in continuous time as well.

- Integration with financial planning:Both procurement decision and hedging decision are subject to financial constraints, the assumption in our model that the company can borrow unlimited money from banks at interest rate is not realistic. The more practical and accurate way is to integrate the financial planning decision together with procurement decision and hedging decision.

We also notice that the result of the paper can also be easily extended to the case that the distributor can buy and sell in the spot market. In that case, spot market can be used as a cache to handle the demand uncertainty.

Finally, we need to caution the reader about some practical challenges of our model. Firstly, the accurate calculation of the optimal policy could be complicated because of too many input variables. Secondly, although the expected utility theory is commonly used for modeling risk-averse decision making problems, it does not capture all the aspects of human beings' choice behavior under uncertainties. Our model also bears the same practical challenges as other models based on expected utility theory-for example, specifying the decision maker's utility function and determining related parameters are not easy.

Nevertheless, our model provides a framework to analysis the optimal ordering and hedging policy for a periodic review system with positive leadtime which is lacking in current literatures.

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Appendix A

Appendix

A.1 Notation

Below is a list of the notation that are employed throughout the paper.

- P_i : the price of the commodity that prevails in period i
- β_i : the fixed premium in period i
- α_i is the hedging amount of period i
- K : the order setup cost
- $\mathcal{L}(x)$: the expected holding and shortage cost function, a convex function of inventory level x
- D_i : the demand in the period i, a random variable.
- x : inventory position in a period before ordering
- y : inventory position after ordering
- $\delta(y)$: indicator function that is defined by

$$\delta(y) = \begin{cases} 1, & \text{y} > 0 \\ 0, & \text{y} = 0 \end{cases}$$

- $V_i(x, \alpha, F_i, F_{i+1}, U_i, P_i, \delta_i)$: the profit-to-go of period i where x refers to the inventory position in the period before ordering, α refers to the number of options, F_i refers to the delivery price of the forwards entered in period $i-1$, similarly, F_{i+1} refers to the delivery price of the forwards entered in current period i, U_i is the given price of options in period i, P_i is the spot price of period i and δ_i is the convenience yield of period i.

- γ : the one-period discount factor.
- Z_t, and W_t are 1-dimensional Wiener processes satisfying $d\langle Z, W\rangle_t = \rho dt$;
- r is the risk-free(constant)interest rate;
- σ_1 is the volatility of spot price;
- k is the instantaneous convenience yield's speed of mean reversion;
- $\alpha' = m + h$ is the convenience yield long-run mean, that is, the level to which δ_t reverts as t goes to infinity and h is the market holding cost per unit, we explicitly write it out as a component of the convenience yield long-run mean.
- λ denotes the market price per unit of convenience yield risk.
- σ_2 is the volatility of instantaneous convenience yield.

The following section contains proofs for important propositions in the previous chapters and basic concepts and lemmas that this work bases on.

A.2 K-Concavity

In this section, we review some important properties of K-concavity/K-convexity that are used in this thesis;see Porteus [34] for more detail.
The concept of k-convexity (the opposition of K-concavity) was introduced by Scarf to prove the optimality of an (s, S) for the traditional inventory control problem.

Definition 1. *A real-valued function f is called K-concave for $K \geq 0$, if for any $x_0 \leq x_1$ and $\lambda \in [0,1]$,*

$$f((1-\lambda)x_0 + \lambda x_1) \geq (1-\lambda)f(x_0) + \lambda(f(x_1) - K)$$

Below we summarize the properties of K-concave functions:

Lemma 1. **a** *A real-valued concave function is also 0-concave and hence k-concave for all $k \geq 0$. A k_1-concave function is also a k_2-concave function for $k_1 \leq k_2$.*

b *If $f_1(x)$ and $f_2(x)$ are k_1-concave and k_2-concave respectively, then for $\alpha, \beta \geq 0$, $\alpha f_1(x) + \beta f_2(x)$is $(\alpha k_1 + \beta k_2)$-concave.*

c *If $f(x)$ is k-concave and w is a random variable, then $Ef(x-w)$ is also k-concave, provided $E|f(x-w)| < \infty$ for all x.*

d *Assume that f is a continuous k-concave function and $f(x) \to \infty$ as $|x| \to \infty$.let S be a minimum point of g and s be any element of the set*

$$\{x|x \leq S, f(x) = gf(x) + k\}$$

Then the following results hold.

1. $f(S) - k = f(s) \geq f(x)$, *for all* $x \leq s$.
2. $f(x)$ *is a non-decreasing function on* $(-\infty, s)$.
3. $f(x) \geq f(y) - k$ *for all* x, y *with* $s \leq x \leq y$.

Further, we have some properties of joint K-concave functions that can be relayed from those of k-concave functions.

Lemma 2. *Following properties about the joint k-concave functions hold:*

1. *If h is joint k_1-concave, and g is joint k_2-concave, then $h + g$ is joint* $\max(k_1, k_2)$*-concave.*
2. *If h is joint k-concave, then for $0 \leq \theta \leq 1$, θh is joint θk-concave.*

Proof. *These properties are straightforward as they are similar to those classic one-dimension k-concave functions.* □

Lemma 3. *(K-Concavity Preservation under Maximization) If X is a convex set,$Q(x)$ is a nonempty set for every $x \in X$,the set $C := \{(x, q)|x \in X, q \in Q(x)\}$ is a convex set, $h(x, q)$ is a k-concave function on C,*

$$g(x) := \max_{q \in Q(x)} h(x, q)$$

and $g(x) < \infty$ for every $x \in X$, then g is a k-concave function on X.

Proof. *let x_1 and x_2 be arbitrary elements of X. let $0 \leq \theta \leq 1$,and let $\overline{\theta} := 1 - \theta$.Select arbitrary $\delta > 0$.By the definition of g, there must exist $q_1 \in Q(x)$ and $q_2 \in Q(x)$ such that $h(x_1, q_1) \geq g(x_1) - \delta$ and $h(x_2, q_2) \geq g(x_2) - \delta$. Then,*

$$\begin{aligned}
\theta g(x_1) + \overline{\theta}[g(x_2) - K] \quad & \leq \theta h(x_1, q_1) + \overline{\theta}[h(x_2, q_2) - K] + \delta \\
& \leq h(\theta x_1 + \overline{\theta} x_2, \theta q_1 + \overline{\theta} q_2) + \delta \\
& \leq g(\theta x_1 + \overline{\theta} x_2) + \delta
\end{aligned}$$

Because δ is arbitrary, the inequality must hold for $\delta = 0$. (Otherwise, a contradiction can be reached.) □

Lemma 4. *Suppose that h is a k-concave function of (x, q) where $q \geq 0$ and there exists a pair (x_0, q_0) such that $\lim_{x,q \to \infty} h(x, q) < -K + h(x_0, q_0)$. Then, there exists a maximizer of h.*

Proof. *It follows from Lemma 3 that* $g(x)$ *is* k*-concave. Let* x_0 *be a minimizer of* g *and* q_0 *is its auxiliary minimizer of* q *that attains* $g(x_0)$*:*

$$g(x_0) = h(x_0, q_0).$$

We claim that (x_0, q_0) *is a maximizer of* h*. As a matter of fact, for any* (x, q)*,* $h(x_0, q_0) = g(x_0) \geq g(x) \geq h(x, q)$*. The lemma holds immediately.* $\square$