Spatial Competition, Product Characteristics, and Demand Uncertainty

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Abstract

This thesis contains two papers on spatial competition, which study how product domain and demand uncertainty affect equilibrium location differentiation.

Paper 1: Spatial Competition in Two-Dimensional Product Space

Irmen and Thisse [9] considered a multi-dimensional case of Hotelling's duopoly location game with the consumers distributed over a unit box of n dimensions. For the unit square, they showed that there exists no equilibrium that involves maximum location differentiation in the two dimensions. This paper considers two modifications of their model by assuming the domain of consumers to be the unit disc and the set of vertices of a unit square. In both models, maximum differentiation in the two dimensions is an equilibrium.

Paper 2: Spatial Competition with Demand Uncertainty

Meagher and Zauner [12] incorporated demand uncertainty into a Hotelling's duopoly location game by assuming that the position (end points) of the market interval is random. They showed that the uncertainty is a differentiation force in determining the two firms' locations. This paper models demand uncertainty in another way. I assume a fixed given position of the market interval, but consider uncertainty over the market density. In addition, I assume that each firm has a fixed production capacity.

I show that the existence of market density uncertainty leads to moderate differentiation and that different revelation time of the market density leads to different equilibrium. In particular, I show that if the market density is revealed to the consumers before they incur transportation costs, then the market equilibrium tends to the social optimum when the probability of capacity constraining increases. If the density is revealed after they incur the costs, then the market equilibrium tends to maximum differentiation when the probability of capacity constraining increases.

I apply my model to study location games with negative consumption externality(cf. Grilo et al. [6]). I show that if the externality enters the consumers' utility function as a multiplicative term, then there exists an equilibrium of moderate differentiation with firms located close to the social optimum.

摘要

本論文包含兩篇關於空間競爭之文章,研究產品空間與需求不確定性如何影響均衡位置差異化。

文章一:兩維空間下之空間競爭

艾曼與提斯研究多維空間、消費者分佈於單位n維盒子之荷特靈雙頭寡佔位置博弈。其結果顯示,於單位方形下,不存在涉及於兩維空間位置差異極大化之均衡。本文研究其模型之兩個變型,分別假設消費者之空間為單位圓盤及單位方形之頂點集。於兩個變型中,本文證明兩維差異極大化乃一均衡。

文章二:需求不確定下之空間競爭

米格與桑拿透過假設隨機市場地點,將需求不確定性納入荷特靈雙頭寡佔位置博弈。其結果為需求不確定性於確定公司位置中乃差異化動力。本文以不同方法考慮需求不確定性。本文假設固定之市場地點,但市場密度為隨機。另外,本文假設每間公司有固定產能。

本文證明市場密度不確定性引致中等差異化。本文進一步顯示,不同之市場密度揭示時間引致不同之市場均衡。具體言之,如果市場密度於消費者支付運輸費前揭示,則市場均衡於產能不足機會上升下趨向社會最優點。如果市場密度於消費者支付運輸費後揭示,則市場均衡於產能不足機會上升下趨向差異極大化。

本文應用此模型於研究負值消費界外效應下之位置博弈。結果顯示,如果該界外效應以倍數形式進入消費者之功用函數,則存在接近社會最優之市場均衡。

Acknowledgments

I would like to express my sincere gratitude to my supervisor Prof. Wong Kam Chau for his continuous and optimal support from the inception of my ideas to the polishing of this thesis. I am especially grateful to his enthusiasm in teaching me research skills and enlightening me with his research ideas.

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Contents

Spatial Competition in Two-Dimensional Product Space

1.1 Introduction

Hotelling [7] introduced a duopoly model to determe firms' locations. In his model, firms first choose locations on a unit market interval and then set prices simultaneously. He showed that firms have the incentive to locate themselves close to each other. This idea became known as the "Principle of Minimum Differentiation."

However, d'Aspremont et al. [5] showed that Hotelling's analysis was flawed. In the price setting subgame, if only pure strategies are allowed, then there exists no equilibrium if the firms are too close to each other. Even if mixed strategies are allowed, Osborne and Pitchik [13] showed the "Principle" is still invalid: an equilibrium exists but it is one of moderate differentiation, with firms' locations close to the quartiles of the market interval.

Indeed, location differentiation depends on transportation costs. d'Aspremont et al. [5] showed that under the assumption of quadratic transportation costs, the unique equilibrium is maximum differentiation, with firms located at different ends of the interval. The intuition is that with quadratic transportation costs, firms seek to mitigate, as far as possible, the price competition intensified by the convexity of transportation costs.

While Hotelling [7], d'Aspremont et al. [5], and Osborne and Pitchik [13] focused on a unit market interval, Irmen and Thisse [9] extended the domain to a unit box of any dimensions. They obtained results in line with the spirit of the "Principle of Minimum Differentiation." With the assumption of quadratic transportation costs, they showed that for an n-dimensional box, n local equilibria exist. In particular, for each of these n equilibria, firms are maximally differentiated in one and only one dimension and minimally differentiated in the remaining ones. Furthermore, they showed that, for the two or three dimensional cases, there exists no equilibrium with maximum differentiation in two or three dimensions.

In this paper, I study two modifications of Irmen and Thisse [9]'s model. My first model differs from theirs in that I replace their assumption of a unit box with that of a disc. The motivation for this modification is the observation that extremities in consumers' preference seem rare in some situations. For instance, one may expect very few consumers preferring a TV with a very large screen and resolution. By assuming a disc shape distribution of consumer preference, the model captures the idea of rare extremes because the neighborhoods around the four corners of the product space represent the extremes.

The main result of my first model is contrary to that of Irmen and Thisse. I show that there exists an equilibrium in which firms maximally differentiate in two dimensions (Proposition 2). When the market is a square, maximum differentiation occurs when firms are located on two vertices on a diagonal. Firms have the incentive to move towards the horizontal or vertical central line because this allows them to be closer to the majority of the consumers, the effect of which outweighs the keener price competition due to being closer to the other firm. When the market is a disc, there is no such thing as the central line. There are infinitely many of them. Thus, every pair of locations with a distance equal to the diameter will be an equilibrium if firms cannot locate themselves outside the disc, which means they are not allowed to choose a product ideal to nobody. As the setting of my model does not impose such a restriction, which is by no means reasonable, the location pairs that ensure firms to be as far apart as possible are the diagonal vertices on the square.

In my second model, I focus on the interpretation of the location space as a product space. Instead of Irmen and Thisse's continuous unit box market space, I consider the firms' location choices, which are interpreted as product choices, to be the discrete four vertices of a unit square.

The assumption of a continuous product space seems not ideal in some occasions. In Hotelling's setting, the continuum of product space and measurement of the corresponding utility loss (transportation costs) imply that the product characteristic is ordinal in the following sense: for every consumer, products varying along the dimension give strictly varying utility. This assumption seems not well justified if the product characteristics are relevant to the consumer tastes, for example, colors of T-shirts, styles of cars, genres of movies, and so on. For these, a consumer may have a private ranking on the given choices, say, preferring blue to green, green to yellow, and thus blue to yellow. However, such a ranking and the transitivity probably disappear at the aggregate level. Ordering the products in a real space thus seems problematic. Whatever the product designs, consumers may be indifferent towards firms offering the same price but non-ideal products. In other words, there may be no such thing as a location advantage, a feature indispensable to the "Principle of Minimum Differentiation."

In my second model, I show that there exists an equilibrium with maximum differentiation in two dimensions (max-max differentiation) (Proposition 3). Again, this is contrary to Irmen and Thisse's results. With max-max differentiation, it is harder for a firm to gain the whole market, so firms will resort to the highest price if they believe there is a possibility that the competitor will charge the highest price. With min-max differentiation, it is easier for a firm to gain the whole market. Firms thus resort to the moderate price if they believe there is a possibility that the competitor will charge the moderate price.

1.2 First model: Ordinal characteristics

The product space is represented by a unit square centered on the origin $(0,0)$ with vertices $(1,1)$ and $(-1,-1)$. Consumers are uniformly distributed on the disc inscribed in the square. There are no consumers outside the disc. Location of firm i is denoted by $l_i = (x_i, y_i)$ with $x_i \in [-1,1]$ and $y_i \in [-1,1]$, $i = 1,2$. Following Mazalov and Sakaguchi [11], I rotate the disc such that $y_1 = y_2$. Note that the rotation only changes the coordinate system and thus it does not affect the results. Without loss of generality, suppose $x_1 \geq x_2$. (Figure 1.1)

The motivation for the disc-shape market is the observation that extremities in consumers' preference seem rare in a lot of situations. For instance, one may expect very few consumers preferring a TV with very large screen and resolution. By assuming a disc shape distribution of consumer preference, the model captures the idea of rare extremities because the neighborhoods around the four corners of the product space represent extremes.

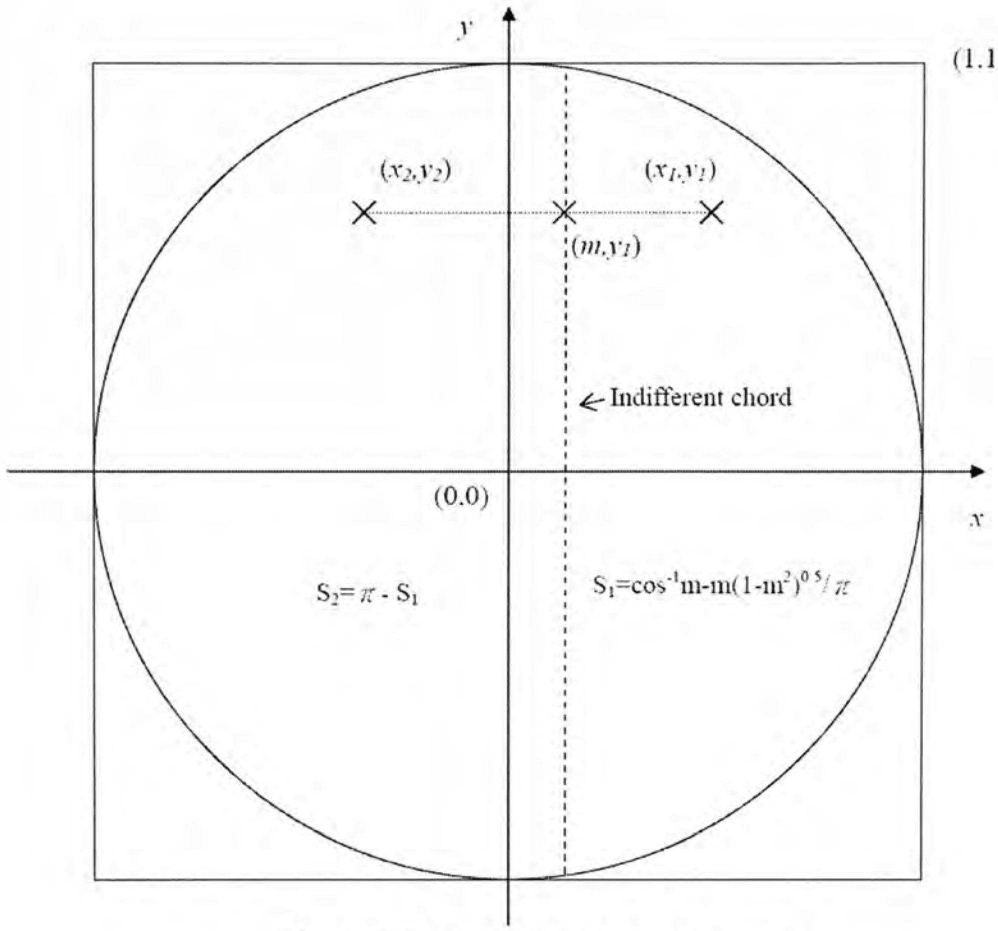

Figure 1.1: Product and market space

Sequence of the game: First, firms choose simultaneously a location $l_i = (x_i, y_i)$. Second, the firms choose prices simultaneously, $p_i \in [0, \infty]$. Finally, consumers choose a firm to patronize. Firms maximise profit while consumers maximise utility. As standard, we use the concept of subgame perfect equilibrium.

Location of each consumer z is denoted by (x_z, y_z), with the following utility if he patronizes a firm with location l_i:

$$U(z, l_i, p_i) = W - p_i - \left((x_z - x_i)^2 + (y_z - y_i)^2\right).$$

There may be some "indifferent" consumers who derive same utility from the two firms. For these consumers, the following holds:

$$-p_1 - ((x_z - x_1)^2 + (y_z - y_1)^2) = -p_2 - ((x_z - x_2)^2 + (y_z - y_2)^2) \tag{1.1}$$

Since firm 1 and firm 2 have the same y-coordinate after rotation, denote the x-coordinate of the indifferent consumers by m, one has

$$-p_1 - (m - x_1)^2 = -p_2 - (m - x_2)^2 ,$$

so

$$m = \frac{1}{2}(x_1 + x_2) + \frac{p_1 - p_2}{2(x_1 - x_2)} . \tag{1.2}$$

Equation (1.2) means that all indifferent consumers fall in a line parallel to the vertical axis. (See Figure 1.1)

Given the location pair (l_1, l_2) and price pair (p_1, p_2), denote firm i's market area by $S_i(l_1, l_2, p_1, p_2)$, which is a segment on the disc. Firm 1's profit function is

$$\pi_1 = p_1 S_1 = p_1(\arccos m - m\sqrt{1 - m^2})/\pi , \tag{1.3}$$

and for firm 2:

$$\pi_2 = p_2 S_2 = p_2(\pi - \arccos m + m\sqrt{1 - m^2})/\pi . \tag{1.4}$$

Given (l_1, l_2), firm 1 chooses p_1 and firm 2 chooses p_2 simultaneously. The first order conditions are

$$\frac{\partial \pi_1}{\partial p_1} = 0 \text{ and } \frac{\partial \pi_2}{\partial p_2} = 0 ,$$

which imply that the optimal prices are as follows:

$$\begin{aligned} p_1^* &= (x_1 - x_2)(\frac{\arccos m}{\sqrt{1-m^2}} - m) \\ p_2^* &= (x_1 - x_2)(m + \frac{\pi - \arccos m}{\sqrt{1-m^2}})\,. \end{aligned} \tag{1.5}$$

Denote firm 1's optimal profit function by $\pi_1^*(x_1, x_2) = \pi_1^*(p_1^*(x_1, x_2), x_1, x_2)$. By implicit function theorem,

$$\frac{\partial \pi_1^*}{\partial x_1} = p_1^* \frac{\partial m}{\partial x_1} + \frac{m\sqrt{1-m^2} + \arccos m}{\pi}(\frac{\partial p_1^*}{\partial x_1} + \frac{dp_1^*}{dm}\frac{\partial m}{\partial x_1})\,. \tag{1.6}$$

Denote the straight line joining the locations of the two firms by $H((x_1, y_1), (x_2, y_2))$. Denote the length of H by $T = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Denote the distance between firm 1 and the indifferent consumer on H by $T_1 = x_1 - m$. Similarly, denote the distance between firm 2 and the indifferent consumer on H by $T_2 = m - x_2 = T - T_1$.

For any l_j, T, define a *diameter location pair* (l_i^D, l_j), where $l_i^D = (x_i^D, y_i^D)$, to be the unique one satisfying: (1) (x_j, y_j), (x_i^D, y_i^D) and the origin $(0,0)$ are on a straight line, and (2) $(x_i^D, y_i^D) = \arg\min \sqrt{{x_i}^2 + {y_i}^2}$ s.t. (1) holds. Denote any location other than l_i^D by l_i^{ND}. Condition (2) ensures that the origin is between the firms, or firm 1 is closer to the origin than is firm 2. (Figure 1.2)

Lemma 1 *For fixed* $\bar{l}_2 = (\bar{x}_2, \bar{y}_2)$ *and* $\bar{T}$, $\pi_1^*(l_1^D, \bar{l}_2) > \pi_1^*(l_1^{ND}, \bar{l}_2)$.

Proof. *Consider the case where firm 1's market share is less than that of firm 2 when firm 1 is on* l_1^{ND}. *The other case can be proven in a similar fashion.*

Note that the market area S_i of firm i is determined by L, the length of the indifferent chord (on which consumers are indifferent between the firms), which is determined by m, the location of the indifferent consumer who is on the same horizontal level with the firms (see Figure 1.2). For non-diameter location pairs, a price equilibrium between the firms implies

$$\begin{aligned}\frac{\partial \pi_1^{ND}}{\partial p_1} &= 0 \Longleftrightarrow p_1^{ND*}(\frac{\partial S_1}{\partial L}\frac{\partial L}{\partial m}\frac{\partial m}{\partial p_1}) + S_1(l_1^{ND}, \bar{l}_2, p_1^{ND*}, p_2^{ND*}) = 0 \quad (1.7)\\ \frac{\partial \pi_2^{ND}}{\partial p_2} &= 0 \Longleftrightarrow p_2^{ND*}(\frac{\partial S_2}{\partial L}\frac{\partial L}{\partial m}\frac{\partial m}{\partial p_2}) + S_2(l_1^{ND}, \bar{l}_2, p_1^{ND*}, p_2^{ND*}) = 0\,.\end{aligned}$$

We now show that if the firms keep the same price at (p_1^{ND*}, p_2^{ND*}) but firm 1 moves to a diameter location l_1^D, then firm 1 will have a larger market share. In particular, we want to prove:

$$S_1(l_1^D, \bar{l}_2, p_1^{ND*}, p_2^{ND*}) > S_1(l_1^{ND}, \bar{l}_2, p_1^{ND*}, p_2^{ND*})\,. \quad (1.8)$$

Note that by rotation $y_1^{ND} = \bar{y}_2$ and $H^{ND} = H((x_1^{ND}, y_1^{ND}), (\bar{x}_2, \bar{y}_2))$ is parallel to the x-axis. As indicated in Figure 1.2, firm 1's market segment area is determined by the distance between m^{ND} and $(0, \bar{y}_2)$, where m^{ND} is the intersection of H^{ND} and the the indifference chord; a shorter distance implies larger area. With (x_1^{ND}, y_1^{ND}), this distance equals $x_1^{ND} - T_1^{ND}$, where T_1^{ND}, as defined earlier, is the distance the points m^{ND} and (x_1^{ND}, y_1^{ND}). By using a standard formula for calculating segment area, firm

1's market share is

$$S_1 = \arccos(x_1 - T_1) - (x_1 - T_1)\sqrt{1 - (x_1 - T_1)^2} \tag{1.9}$$

where $x_1 = x^{ND}$, *and* $T_1 = T_1^{ND}$.

Consider the isosceles triangle formed by joining $(\bar{x}_2, \bar{y}_2)$, $(x_1^{ND}, \bar{y}_2)$ *and* (x_1^D, y_1^D) *in Figure1.2. On* H^{ND}, *the ratio of the distance between* $(x_1^{ND}, \bar{y}_2)$ *and* $(0, \bar{y}_2)$ *to the distance between* $(0, \bar{y}_2)$ *and* $(\bar{x}_2, \bar{y}_2)$ *is* $|x_1^{ND}/x_2|$. *On* H^D, *where* $H^D = H((x_1^D, y_1^D), (\bar{x}_2, \bar{y}_2))$, *the ratio of the distance between* (x_1^D, y_1^D) *and* $(0,0)$ *to the distance between* $(0,0)$ *and* $(\bar{x}_2, \bar{y}_2)$ *is* $|x_1^D/\bar{x}_2|$. *As* $x_1^D < x_1^{ND}$, $|x_1^D/\bar{x}_2| < |x_1^{ND}/\bar{x}_2|$. *As* T *is fixed and* $T_1^D = T_1^{ND}$, $|x_1^D/\bar{x}_2| < |x_1^{ND}/\bar{x}_2|$ *implies the distance between the indifferent consumer on* H^D *and* $(0,0)$ *is smaller than the distance between the indifferent consumer on* H^{ND} *and* $(0, \bar{y}_2)$. *Using equation (*1.9*) with* $x_1 = x^D$ *and* $T_1 = T^D$, *we can evaluate* S_1 *of firm* 1 *at* l_1^D *and we have* $S_1(l_1^D, \bar{l}_2, p_1^{ND*}, p_2^{ND*}) > S_1(l_1^{ND}, \bar{l}_2, p_1^{ND*}, p_2^{ND*})$. *This establishes equation (*1.8*).*

Since firm 1 *has a larger share with* p_1^{ND*}, *firm* 1 *can increase its price to* $\tilde{p}_1$ *such that the market shares of the two firms are the same as before, i.e.*

$$S_2(l_1^D, \bar{l}_2, \tilde{p}_1, p_2^{ND*}) = S_2(l_1^{ND}, \bar{l}_2, p_1^{ND*}, p_2^{ND*})\,. \tag{1.10}$$

It is obvious that

$$\left.\frac{\partial S_2}{\partial L}\frac{\partial L}{\partial m}\right|_{(l_1^D, \bar{l}_2, \tilde{p}_1, p_2^{ND*})} = \left.\frac{\partial S_2}{\partial L}\frac{\partial L}{\partial m}\right|_{(l_1^{ND}, \bar{l}_2, p_1^{ND*}, p_2^{ND*})}. \tag{1.11}$$

We now show that at $(l_1^D, \bar{l}_2, p_1^{ND*}, p_2^{ND*})$, $\frac{\partial \pi_2^D}{\partial p_2} = 0$. *given* $(l_1^D, \bar{l}_2, p_1^{ND*}, p_2^{ND*})$. *First we have*

$$\frac{\partial \pi_2^D}{\partial p_2} = p_2^{ND*}(\frac{\partial S_2}{\partial L}\frac{\partial L}{\partial m}\frac{\partial m}{\partial p_2}) + S_2(l_1^D, \bar{l}_2, p_1^{ND*}, p_2^{ND*})\,.$$

*By equations (*1.10*), (*1.11*), and (*1.7*), it suffices to show that*

$$\frac{\partial m(l_1^{ND}, \bar{l}_2, p_1^{ND*}, p_2^{ND*})}{\partial p_2} = \frac{1}{2\bar{T}} = \frac{\partial m(l_1^D, \bar{l}_2, \tilde{p}_1, p_2^{ND*})}{\partial p_2}\,.$$

As the distance between $(\bar{x}_2, \bar{y}_2)$ *and* $(x_1^{ND}, \bar{y}_2)$ *and that between* $(\bar{x}_2, \bar{y}_2)$ *and* (x_1^D, y_1^D) *are the same and equal to* T*, by equation (*1.2*), we have*

$$\frac{\partial m(l_1^{ND}, \bar{l}_2, p_1^{ND*}, p_2^{ND*})}{\partial p_2} = \frac{1}{2\bar{T}} = \frac{\partial m(l_1^D, \bar{l}_2, \tilde{p}_1, p_2^{ND*})}{\partial p_2}.$$

Hence,

$$\frac{\partial \pi_2^D(l_1^D, \bar{l}_2, \tilde{p}_1, p_2^{ND*})}{\partial p_2} = \frac{\partial \pi_2^{ND}}{\partial p_2} = 0\,.$$

Thus, with l_1^D*, firm* 1 *has the option to charge a higher price than the equilibrium price at* l_1^{ND} *while securing the same market area. Hence, firm 1 will be strictly better off by choosing* l_1^D. ■

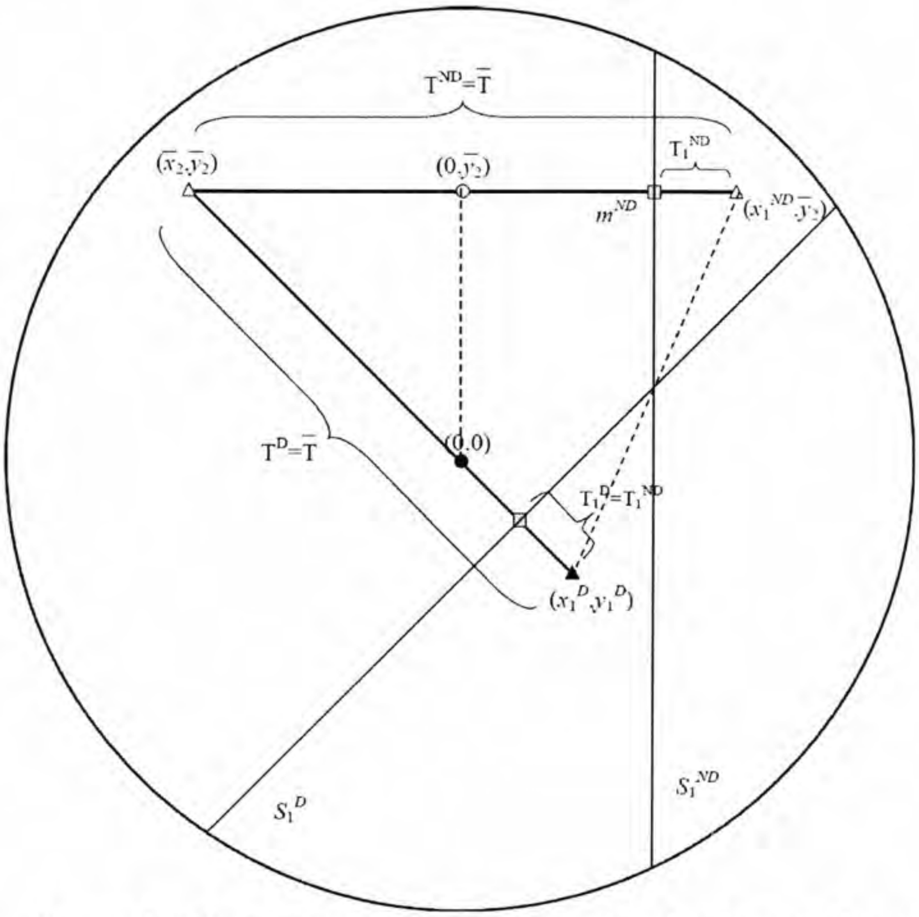

Figure 1.2: Larger market share for firm 1 under diameter locations

Proposition 2 *Maximum differentiation* $((-1,1),(1,-1))$ *and* $((1,1),(-1,-1))$ *are equilibria.*

Proof. *Note that* $((-1,1),(1,-1))$ *are equivalent to* $((-\frac{\sqrt{2}}{2},y),(\frac{\sqrt{2}}{2},y))$ *after rotation of the disc. Direct computation of* $\frac{\partial \pi_1^*}{\partial x_1}$ *by equations (1.2), (1.5) and (1.6) shows that at* $m=0$, $\frac{\partial \pi_1^*}{\partial x_1}<0$ *for* $x_1=-\frac{\sqrt{2}}{2}$ *and* $x_2=\frac{\sqrt{2}}{2}$. *This shows that firm 1 has no incentive to move closer to firm 2 along the diagonal. By lemma 2, firm 1 has no incentive to* *move in other directions. Hence, the location pair* $((-1,1),(1,-1))$ *is an equilibrium, and so is the other pair, by symmetry.* ■

Irmen and Thisse ([9], Corollary 3) showed that maximum differentiation in two dimensions cannot be an equilibrium under a unit box of two or three dimensions. The intuition is as follows When the market is a square, maximum differentiation occurs when firms are located on the diagonal. Firms have the incentive to move towards the central line because this allows them to be closer to the majority of the consumers, the effect of which outweighs that of being closer to the other firm. When the market is a disc, there is no such thing as the central line. There are infinitely many of them. Thus, every pair of locations with a distance equal to the diameter will be an equilibrium if firms cannot locate themselves outside the disc, which means they are not allowed to choose a product which is ideal to nobody. As the present setting does not impose such a restriction, which is by no means reasonable, the location pairs that ensure firms are as far apart as possible are the diagonal vertices on the square. Hence, the disc-shape market model shows that, as in the one-dimensional case, there is maximum differentiation in both dimension. This implies that the welfare loss is still substantial in the two-dimensional case.

Having proven that max-max differentiation is an equilibrium under the present model, one may wonder whether the min-max combination (with the two firms each located on one end point of the central horizontal diameter of the disc) is also an equilibrium. Intuitively, the answer should be positive. The intuition goes as follows: under Irmen and Thisse's setting, in which the market is a square, the min-max is an equilibrium. This means that given its competitor's location at an end of the central horizontal line, a firm has no incentive to move upwards or downards even though this will result in a larger distance between the firms. This implies that the effect of getting closer to the majority of the market outweighs that of a smaller distance between the firms. With the present disc setting, it seems natural to expect that the effect of being closer to the majority of the market is even larger, given that the market is empty at the corners. Thus, it seems intuitively correct that the min-max combination is also an equilibrium in the present disc market space.

1.3 Second model: Categorical characteristics

The assumption of a continuous product space seems not realistic in some occasions. In Hotelling's [7] setting, the continuum of product space and measurement of the corresponding utility loss (transportation costs) imply that the product characteristic is ordinal in the following sense: for every consumer, products varying along the dimension give strictly varying utility. This assumption seems not well grounded if the product characteristics are relevant to consumers' tastes, for example, colors of T-shirts, styles of cars, genres of movies, and so on. For these, a consumer may have a private ranking on the given choices, say, preferring blue to green, green to yellow and thus blue to yellow. However, such a ranking and the transitivity probably disappear at the aggregate level. Ordering the products in a real space is thus problematic. Whatever the product design, consumers may be indifferent between firms offering the same price but non-ideal products. In other words, there may be no such thing as a location advantage, a feature indispensable to the "Principle of Minimum Differentiation".

I now discuss the second model. The product space is the set of four vertices of the unit square. Thus, the product space is $\{(0,0),(0,1),(1,1),(1,0)\}$. Two vertices located on the same side represent two products differing in one dimension but identical in the other. Each vertex contains one consumer. A consumer's transportation cost is the square of the distance between his location and the product. To simplify the analysis, each firm's price strategy is restricted to $\{0, \frac{R}{2}, R\}$.

Sequence of the game: In the first stage, the firms choose simultaneously a location l_i , $i = 1, 2$, from the four vertices. In the second stage, the firms choose prices simultaneously, $p_i \in \{0, \frac{R}{2}, R\}$. Finally, consumers patronize a firm that gives him the lowest sum of price and transportation costs.

Consider the following three possible configurations of location choices: (i) max-max differentiation: firm 1 choose $(0,0)$, firm 2 chooses $(1,1)$; (ii) min-max differentiation: firm 1 choose $(0,0)$, firm 2 chooses $(1,0)$ and (iii) min-min differentiation. Since for $R \geq 4$ the three cases result in the same equilibria, I only consider the cases for $R < 4$.

Under max-max differentiation, firm 1 has: 4 consumers if $p_1 \leq p_2 - 2$; 3 consumers if $p_2 - 2 < p_1, p_2$; 2 consumers if $p_1 = p_2$; 1 consumer if $p_2 < p_1 \leq p_2 + 2$; and 0 consumers if $p_1 > p_2 + 2$. The case for firm 2 is similar.

Under min-max differentiation, firm 1 has: 4 consumers if $p_1 \leq p_2 - 1$; 2 consumers if $p_2 - 1 < p_1 \leq p_2 + 1$; and 0 consumers if $p_1 > p_2 + 1$. The case for firm 2 is similar. The following tables show the payoff matrices for $2 < R < 4$.

		Firm 2		
		0	$\frac{R}{2}$	R
	0	$0,0$	$0,\frac{R}{2}$	$0,0$
Firm 1	$\frac{R}{2}$	$\frac{R}{2},0$	R,R	$\frac{3R}{2},R$
	R	$0,0$	$R,\frac{3R}{2}$	$2R,2R$

Table 1.1: Firms payoff matrix under max-max differentiation with $2<R<4$

		Firm 2		
		0	$\frac{R}{2}$	R
	0	$0,0$	$0,\frac{R}{2}$	$0,0$
Firm 1	$\frac{R}{2}$	$\frac{R}{2},0$	R,R	$2R,0$
	R	$0,0$	$0,2R$	$2R,2R$

Table 1.2: Firms payoff matrix under min-max differentiation with $2<R<4$

		Firm 2		
		0	$\frac{R}{2}$	R
	0	$0,0$	$0,0$	$0,0$
Firm 1	$\frac{R}{2}$	$0,0$	R,R	$2R,0$
	R	$0,0$	$0,2R$	$2R,2R$

Table 1.3: Firms payoff matrix under min-min differentiation with $2<R<4$

Proposition 3 *Let* $2 < R < 4$ *and firms rule out weakly dominated strategies in the stage of choosing price. Then a location pair is the equilibrium if and only if it is max-max differentiation.*

Proof. *Without loss of generality, consider the max-max differentiation* $((0,0),(1,1))$*, the min-max differentiation* $((0,0),(1,0))$*, and the min-min differentiation* $((0,0),(0,0))$*. From Table 1.1, under max-max differentiation, by elimination of weakly dominated strategies, the unique equilibrium is* (R,R)*, giving payoffs* $(2R,2R)$*. From Table 1.2, under min-max differentiation, by elimination of weakly dominated strategies, the only equilibrium is* $(\frac{R}{2},\frac{R}{2})$*, giving payoffs* (R,R)*. From Table 1.3, under min-min differentiation, by elimination of weakly dominated strategies, the only equilibrium is* $(\frac{R}{2},\frac{R}{2})$*, giving payoffs* (R,R)*. Thus, given firm* 1*'s location of* $(0,0)$*, firm* 2*'s payoff of choosing* $(1,1)$ *is higher than that of choosing* $(1,0)$ *(hence also* $(0,1)$*) or* $(0,0)$*. By symmetry, given firm* 2*'s location* $(1,1)$*, firm 1's unique best response is* $(0,0)$*. Hence, max-max differentiation is the equilibrium.* ■

The above results are in contrast to those of Irmen and Thisse ([9], Corollary 3) that maximum differentiation in two dimensions cannot be an equilibrium under a unit box of two or three dimensions. The intuition behind Proposition 3 is as follows. With max-max differentiation, it is harder for a firm to gain the whole market. Thus, firms will resort to the highest price if they believe there is a possibility that the competitor will charge the highest price. This makes (R,R) the unique equilibrium. With min-max differentiation, it is easier for a firm to gain the whole market. Thus, firms will resort to the moderate price if they believe there is a possibility that the competitor will charge the moderate price.

1.4 Conclusion

In the first model, the disc shape market assumption resembles reality in that extremities in consumer preference are rare. The model shows that max-max differentiation is an equilibrium. This result is contrary to that of Irmen and Thisse, whose model assumes a square market. Be it a disc or square, however, there is a boundary on the consumer preference. It remains to be investigated what kind of equiblirium would emerge if there is no such boundary. A natural case of such is that consumer preference follows normal distribution.

In the second model, consumers are assumed to be located over the vertices of a unit square. This assumption should be applicable to cases where consumer preference is taste rather than function related. By assuming the firms' price strategies to be a finite set of three choices, I have shown that maximum differentiation in the two dimensions is an equilibrium. Expanding the price strategy set is left for future research.

The results of the two models presented in this paper suggest that Irmen and Thisse's results are not robust against alterations in their assumption. The same can be said about the present models. This warns against any careless application of these location models. These results altogether suggest that in applying any of these models, one must not only know about the dimensions of the product space-but also whether the product space is discrete or continuous, as different models lead to different results.

Models	Consumer distribution	Transport costs	Special assumptions	Results
Hotelling [7]	Uniform in [0,1]	Linear	-	Min. differentiation
Irmen and Thisse [9]	Uniform in R^n	Quadratic	-	Min. differentiation in n-1 dimensions
Paper 1-1st model	Uniform in a disc	Quadratic	-	Max. differentiation in 2 dimensions
Paper 1-2nd model	Discrete over square vertices	Quadratic	Finite strategies in prices	Max. differentiation in 2 dimensions

Table 1.4: Summary

Spatial Competition with Demand Uncertainty

2.1 Introduction

d'Aspremont et al. [5] showed that in Hotelling's [7] duopoly location game, if transportation costs are quadratic, then maximum differentiation is an equilibrium. Maintaining the quadratic costs assumption, the existing literature has obtained the result of maximum differentiation in different settings. For example, Grilo et al. [6] considered the game in the presence of consumption externality. The maximum differentiation result remains valid as long as the agglomeration force due to the positive externality does not outweigh the differentiation force due to the negative externality. Meagher and Zauner [12] and Casado-Izaga [3] incorporated demand uncertainty into their models. They showed that demand uncertainty exacerbates differentiation.

Following Meagher and Zauner [12], this paper incorporates demand uncertainty, but in a different sense, into the duopoly location game. In addition, firms are assumed to have a fixed capacity.

Meagher and Zauner considered the game with uncertain demand location in the sense that the position of the market interval is random, while the length and market density are both normalized to one. In this paper, the position and length of the market interval is fixed at $[0, 1]$, while the market density is a random variable with uniform distribution over an interval. As the market length is fixed, the random density implies a random market mass (market size). The justification for this assumption is that the whole market may be affected by some common factors like an economic downturn, epidemic outbreaks, and others. These factors may not have any effect over consumers' location (preference) distribution, but they determine how many consumers can be found in each location. This issue has not been well explored in the existing literature. This is not surprising given that the random size alone does not give any new insights. Whatever the market sizes turns out to be, as long as consumers are uniformly distributed over the fixed unit interval, maximum differentiation is always the equilibrium.

If one pursues the analysis further, however, one should take note of the possibility that the firms cannot serve the entirety of a large market if we are to accept that the firms are somehow capacity constrained. This motivates the assumption of fixed capacity. To the best of my review, there has been no previous study incorporating fixed capacity into the Hotelling model with quadratic transportation costs. Wauthy [15] and Boccard and Wauthy [2] considered capacity constraints in the location model with linear transportation costs. Wauthy relied on the assumption of low reservation price to establish the existence of an equilibrium when two firms are close. Boccard and Wauthy showed that capacity precommitment softens price competition, but they did not consider the location choice problem. This paper differs from those studies in several ways. First, my model studies the location game with the assumption of quadratic costs. Second, the model does not require a low reservation price. Third, unlike Boccard and Wauthy's model, my model considers the location choice problem. Finally, their models are under a deterministic setting, while my model incorporates uncertainty over demand.

The main results of this paper are as follows. When both firms and consumers are uncertain about the market density when they take action, a positive probability, caused by the fixed capacity, that a firm cannot serve all of its clientele invalidates maximum differentiation under quadratic transportation costs invalidates the "Principle of Maximum Differentiation." (Proposition 5). There exists an equilibrium location pair with both firms located between the first and third quartiles of the unit market interval (Proposition 7). Furthermore, different revelation time of the market density leads to different equilibrium. In particular, if the density is revealed to consumers before they incur transportation costs, then increasing probability of capacity constraining implies smaller distance between the equilibrium location pair and the social optimum (Proposition 7). If the density is realized after transportation, then the equilibrium location pair tends to maximum differentiation as the probability of capacity constraining becomes large (Proposition 11). Lastly, if consumers are perfectly informed, or if they can switch among the firms costlessly, then maximum differentiation will be an equilibrium (Proposition 12). To the best of my knowledge, these results are new and are in contrast with some existing findings in the literature (e.g., Boccard and Wauthy [2], Casado-Izaga [3], Meagher and Zauner [12], Wang and Yang [14]). (See Table 2 for a summary)

In Meagher and Zauner's [12] model, uncertainty over demand locations weakens market share loss due to moving away from the competitor. This weakening of the loss leads to larger differentiation force. At first impression, such weakening of the effect carries over to the present model. A firm will not lose any share if the market size turns out to be large. If this leads one to conclude that the equilibrium under random market size should be maximum differentiation, then this paper clears the illusion.

Intuition for some results of this paper is as follows. The possibility of large market weakens price competition in that consumers take into account the probability that a firm may be capacity constrained. This weakened price competition allows for the equilibrium proximity of the firms. If consumers need to incur transportation costs when they cannot purchase the good, they will opt for the firm that is more likely to have free capacity, which will be a firm residing at one end of the market. Lastly, if consumers have perfect information, uncertainty over the market size does not change the result of maximum differentiation because of the weaker incentive for the firms to win a larger market share: a firm trying to profit from a larger share may find some of its unserved customers switching to its competitor.

The technique of the models in this paper can be used to model negative externality. The effect of the capacity constraints is similar to that of negative externality. With fixed capacity, a larger clientele is undesirable to a consumer: the higher the chance a consumer's selected firm is capacity constrained, the lower his expected utility.

Tackling externality and location problems together, Ahlin and Ahlin [1] showed that negative externality in the form of congestion costs can restore the existence of pure strategy equilibrium in a linear transportation costs setting, and the higher the costs, the longer the distance between firms. Grilo et al. [6] considered the location game with quadratic transportation costs by investigating the effect of externality, which they dubbed vanity or conformity, on price competition and market share. Grilo et al argued that negative externality leads to maximum differentiation. Common to Grilo et al. and Ahlin, the externality enters consumers' utility function as an additive term. The application of the present model considers the externality as a multiplier. Contrary to additive externality models, I show that moderate differentiation is an equilibrium under multiplicative externality.

2.2 Model

There is a large potential pool of consumers. Each of them is in a location $z \in [0, 1]$. I assume that the distribution of consumers is uniform over $[0, 1]$. These consumers constitute the market. The market size s, defined as the total number of consumers, is a random variable. The greatest and least possible size of s are v and 0 respectively, and s is a random variable with uniform distribution over $[0, v]$. I write this as $s \sim [0, v]$.

Meagher and Zauner [12] considered the game with uncertain demand location in the sense that the location of the market interval is random, while the length and market density are both normalized to one. Their location space is $[M - \frac{1}{2}, M + \frac{1}{2}]$, where M is random. In this paper, the location and length of the market interval is fixed at $[0, 1]$, while the density is a random variable with uniform distribution $s \sim [0, v]$.

The justification for this assumption is that the whole market may be affected by some common factors like an economic downturn, epidemic outbreaks, and others. These factors may not have any effect over consumers' location (preference) distribution but they determine how many consumers can be found in each location.

One should find it reasonable to expect the possibility that firms cannot serve the entirety of a market which turns out to be large. This motivates the assumption of fixed capacity. To the best of my review, there has been no previous study incorporating fixed capacity into the Hotelling model with quadratic transportation costs. Wauthy [15] and Boccard and Wauthy [2] considered capacity constraints in the location model with linear transportation costs. Wauthy relied on the assumption of low reservation price to establish the existence of an equilibrium when two firms are close apart. Boccard and Wauthy showed that capacity precommitment softens price competition, but they did not consider the location choice problem. This paper differs from those studies in several ways. First, my model studies the location game with the assumption of quadratic costs. Second, the model does not require a low reservation price. Third, unlike the model of Boccard and Wauthy, my model considers the location choice problem. Finally, their models are under a deterministic setting, while my model incorporates uncertain demand.

In this paper, I also assume capacity constraint for firms. Define K to be the capacity. I assume that the capacity K and upper support of market density v are chosen such that K/v is not too large. Intuitively, this means that there is always a positive probability that a firm cannot serve all consumers. This assumption is the key assumption, without which the model is reduced to Hotelling's original model with quadratic transportation costs.

The effect of removing the assumption of fixed capacity or that of random market density can be considered as follows. With the assumption of random market density alone, the equilibrium will be maximum differentiation, as in the case of no uncertainty. This is because consumers will always be served, and whatever the market density turns out to be, maximum differentiation maximizes a firm's profit.

With the assumption of fixed capacity and a fixed market size normalized to 1, maximum differentiation will still be an equilibrium because firms will have no incentive to gain a larger market share in this case.

Sequence of the game: In the first stage, the firms choose locations $l_i \in [0,1]$, $i = 1,2$, simultaneously. In the second stage, the firms choose prices simultaneously, given the firms' locations. In the third stage, consumers choose a firm to patronize. As will be specified in the following sections, the revelation time of the market density will be before or after the third stage.

There are two firms with constant marginal costs normalized to 0. Each firm can only serve a mass of consumers no larger than K, which is each firm's fixed capacity.

For a consumer located at z who purchases from firm i, the total costs of purchase consist of the following: (i) transportation costs (utility loss) that depend on the squared distance between z and and firm i's location $l_i \in [0,1]$, $t(z-l_i)^2$ (without loss of generality, $t = 1$), and (ii) the price, p_i, charged by firm i. When there is no capacity constraining, a consumer at location z derives the following utility:

$$U(z, l_i, p_i) = W - p_i - (l_i - z)^2\,, \tag{2.12}$$

where W represents consumers' income. Following the standard assumption in the literature (e.g., Meagher and Zauner [12], Irmen and Thisse [9], Grilo et al. [6]), W is assumed to be bounded and large enough so that consumers everywhere on the unit interval will have positive consumer surplus.

Throughout this paper, unless otherwise specified, attention is restricted to symmetric pure-strategy subgame-perfect Nash equilibria by establishing their existence. Investigation on asymmetric or random locations equilibria is left for future research.

Section 2.3 studies the case in which the revelation time of s is just before consumers take transportation, while in Section 2.4 s is revealed after transportation. Section 2.5 considers the case in which consumers are perfectly informed about the market density and the firms' capacity when they choose a firm.

2.3 Revelation of market density before transportation

The market density s is realized to the consumers before they incur transportation costs. If the indifferent consumer finds that the firm he wants to patronize is capacity-constrained, he will not take any transportation or try to patronize the other firm, which means that the consumer derives zero utility in this case.

Denote locations of the two firms by $l_1 = x$ and $l_2 = y$. Without loss of generality, let $x < y$. For every location pair (x, y), there is a location m where a consumer is indifferent between the two firms. Consumers with locations $z < m$ will derive higher utility from firm 1, while the others will derive higher utility from firm 2. Firm 1 and 2's market shares are thus m and $1 - m$, respectively. The respective market sizes are thus ms for firm 1 and $(1 - m)s$ for firm 2, where $s \sim [0, v]$ is the market density. I assume that K/v is not too large. In particular, I assume that for any equilibrium location, $P(ms > K) = \frac{K}{mv}$ and $P((1-m)s > K) = \frac{K}{(1-m)v}$ are both less than 1. Thus for symmetric equilibrium where $m = \frac{1}{2}$, $2K < v$. Moreover, in a neighborhood of a symmetric equilibrium, $(2+\varepsilon)K < v$.

The indifferent consumer may end up not getting the good, in which case $U(m, l_i, p_i) = 0$, if market density turns out large. Considering any location pair (x, y), for the indifferent consumer, the following must hold:

$$\frac{K}{mv}(W - p_1 - (m-x)^2) = \frac{K}{(1-m)v}(W - p_2 - (y-m)^2) \,. \tag{2.13}$$

Rearranging the terms, we have

$$F(x, y, p_1, p_2, m) \equiv \frac{1}{m}[W - p_1 - (m-x)^2] - \frac{1}{(1-m)}[W - p_2 - (y-m)^2] = 0 \,, \tag{2.14}$$

which defines implicitly the function $m(x, y, p_1, p_2)$. m and $(1-m)$ represent firm 1 and 2's market share, respectively.

Firm 1's profit function is

$$\pi_1 = p_1 Q_1 \,,$$

where $Q_1 = ms$ if $K \geq ms$ and $Q_1 = K$ if $K < ms$. The expected function can be shown to be:

$$E(\pi_1) = Kp_1(1 - \frac{K}{2mv}) \,. \tag{2.15}$$

Similarly,

$$E(\pi_2) = Kp_2(1 - \frac{K}{2(1-m)v}) \,. \tag{2.16}$$

First order conditions $\frac{\partial E(\pi_1)}{\partial p_1} = 0$ and $\frac{\partial E(\pi_2)}{\partial p_2} = 0$ imply the following:

$$\begin{aligned} p_1^* &= \frac{m(K-2mv)}{K}\frac{\partial p_1}{\partial m} \\ p_2^* &= -\frac{(1-m)(K-2(1-m)v)}{K}\frac{\partial p_2}{\partial m}, \end{aligned} \tag{2.17}$$

where $\frac{\partial p_i}{\partial m}$ can be found by

$$\frac{\partial p_i}{\partial m} = -\frac{\frac{\partial(F)}{\partial m}}{\frac{\partial(F)}{\partial p_i}}. \tag{2.18}$$

Solving for p_1^* and p_2^* simultaneously, we have

$$p_1^* = \frac{(K-2mv)(-4m^3-2m^4+W-x^2-2m(W-x^2)+m^2(1+2W-x^2+2y-y^2))}{(1-m)^2(K-2v)} \tag{2.19}$$

$$p_2^* = \frac{(K-2(1-m)v)(-4m^3-2m^4+W-x^2-2m(W-x^2)+m^2(1+2W-x^2+2y-y^2))}{m^2(K-2v)}. \tag{2.20}$$

As $E(\pi_1) = 0$ for $p_1 = 0$ and $E(\pi_1) > 0$ for $p_1 = \bar{p}_2 \neq 0$, the continuity of $E(\pi_1)$ ensures that the first order conditions give the maximum.

Substituting p_1^* and p_2^* into F, we have

$$F^*(x,y,m) \equiv \frac{1}{m}[W - p_1^* - (m-x)^2] - \frac{1}{(1-m)}[W - p_2^* - (y-m)^2] = 0\,. \tag{2.21}$$

The location tendencies of the firms are captured by:

$$\begin{aligned} \frac{\partial E(\pi_1^*)}{\partial x} &= K(\frac{\partial m}{\partial x}(\frac{\partial p_1^*}{\partial m}(1-\frac{K}{2mv}) + \frac{p_1^* K}{2vm^2}) + \frac{\partial p_1^*}{\partial x}(1-\frac{K}{2mv})) \qquad (2.22) \\ \frac{\partial E(\pi_2^*)}{\partial y} &= K(\frac{\partial m}{\partial x}(\frac{\partial p_1^*}{\partial m}(1-\frac{K}{2(1-m)v}) - \frac{p_1^* K}{2v(1-m)^2}) \\ &\quad + \frac{\partial p_1^*}{\partial x}(1-\frac{K}{2(1-m)v}))\,. \end{aligned}$$

Focusing on symmetric equilibrium in which $x = 1-y$, $p_1^* = p_2^*$, and $m = \frac{1}{2}$, equation (2.19) and (2.22) become

$$p_1^* = p_2^* = \frac{(v-K)(4W-4x^2+1)}{2(2v-K)} \tag{2.23}$$

$$\frac{\partial E(\pi_1^*)}{\partial x} = -\frac{\partial E(\pi_2^*)}{\partial y} = -\frac{K(K-v)(3v(1-4x)+K(-1+8x))}{3v(K-2v)} . \tag{2.24}$$

For equation (2.23) to be legitimate equilibrium prices, it is required that W is larger than the total cost of purchase for every consumer. A sufficient condition for this is

$$p_1^* = p_2^* \leq W - 0.25 .$$

.

I assume W to be greater than the equilibrium price by 0.25 throughout this paper.

Lemma 4 *Given K, v, W and a symmetric location pair (x, y), if $W > \frac{v}{K}$, then any pure strategy price equilibrium is unique.*

Proof. *As every given $\bar{m} = \frac{1}{2}$ corresponds to a unique price pair $p_1^*(\bar{m})$=$p_2^*(\bar{m})$ by equation (2.19), it suffices to check that $\tilde{m} \neq \frac{1}{2}$ cannot constitute an equilibrium. Without loss of generality, suppose $\tilde{m} < \frac{1}{2}$. It can be readily verified from equation (2.19) that p_1^* is strictly increasing with m if $W > \frac{v}{K}$. This implies $p_1^*(\tilde{m}) < p_1^*(\bar{m})$ and $p_2^*(\tilde{m}) > p_2^*(\bar{m})$, while $\tilde{m} < \bar{m}$, under which equation (2.13) cannot hold.* ■

Proposition 5 *Maximum differentiation with $x = 0$ and $y = 1$ is not an equilibrium.*

Proof. *With maximum differentiation, $x = 0$ and $y = 1$. From (2.22), one obtains*

$$\frac{\partial E(\pi_1^*)}{\partial x} = \frac{K(3v - K)(v - K)}{3v(2v - K)} > 0\,.$$

Thus, firm 1 can increase its expected profit by moving towards firm 2. ■

Proposition 6 *Minimum differentiation $x = y = \frac{1}{2}$ is not an equilibrium.*

Proof. *For $x = y = \frac{1}{2}$, one obtains*

$$\frac{\partial E(\pi_1^*)}{\partial x} = -\frac{K(K - v)^2}{(2v - K)} < 0\,.$$

Thus, firm 1 can increase its expected profit by moving away from firm 2. ■

Proposition 7 *There exists an equilibrium with moderate differentiation, in which $x^* > 0$ and $y^* < 1$. Furthermore, if $\frac{v}{K}$ tends to infinity, then such differentiation will tend to the social optimum $x = \frac{1}{4}$ and $y = \frac{3}{4}$.*

Proof. *Solving*

$$\begin{aligned} \frac{\partial E(\pi_1^*)}{\partial x} &= 0 \\ \frac{\partial E(\pi_2^*)}{\partial y} &= 0 \end{aligned}$$

gives one

$$\begin{aligned} x^* &= \frac{K - 3v}{4(2K - 3v)} > 0, \\ &\leq \frac{5}{16} \end{aligned}$$

For the second part of the proposition, one can show that

$$\lim_{K\to 0} x^* = \frac{K-3v}{4(2K-3v)} = \frac{1}{4}$$

and

$$\begin{aligned}
& \lim_{v\to\infty} \frac{K-3v}{4(2K-3v)} \\
= & \lim_{v\to\infty} \frac{-3}{-12} \text{ (by l'Hopital's rule)} \\
= & \frac{1}{4}.
\end{aligned}$$

■

Proposition 8 *For symmetric equilibrium locations, increasing the parameter v increases each firm's equilibrium price.*

Proof. *By lemma 4, the price equilibrium is unique for given v. Substituting the equilibrium location*

$$x^* = \frac{K-3v}{4(2K-3v)}$$

into the equilibrium price

$$p_1^* = \frac{(v-K)(4W-4x^2+1)}{2(2v-K)},$$

we have

$$\begin{aligned}
p_1^{**} &= \frac{(K-v)(\frac{3(5K-9v)(K-v)}{(2K-3v)^2} + 16W)}{8(K-2v)} \\
\frac{\partial p_1^{**}}{\partial v} &= \frac{K(\frac{3(K-v)(12K^2-43Kv+39v^2)}{(2K-3v)^3} + 16W)}{8(K-2v)^2} > 0\,.
\end{aligned}$$

■

Increasing v means that the probability of capacity constraining is increasing. Proposition 8 ensures that this increases the equilibrium prices of the firms.

The insight offered by Proposition (7) is that firms will adopt maximum differentiation only if they are 100% sure they can serve the whole market. As long as there is a chance of capacity constraining, which is guaranteed by the assumption that K/v is not too large, which means the fixed capacity is small relative to the maximum possible market size, firms will seek moderate differentiation. Thus, the demand uncertainty introduces a jump on the equilibrium locations. In the location game under certainty and quadratic transportation costs, if firms are allowed to choose outside the interval $[0, 1]$, the equilibrium location pair is $(-\frac{1}{4}, \frac{5}{4})$; thus, firms are 1.5 units apart. By introducing the possibility of capacity constraining, however small it is, the equilibrium distance between the firms is at most $\frac{1}{2}$. In this light, market size uncertainty is an immense force counteracting maximum differentiation.

This force is present because consumers need to discount the utility they obtain from patronizing a firm by the probability that the firm is able to serve them. A larger discount will be needed for a firm winning a larger market portion by lowering its price. This leads to weaker price competition, and thus sustaining the equilibrium proximity between the firms.

The market density does not affect consumers' choosing between the two firms because it is common to both firms. When the density increases, firms will be farther apart because larger market density weakens the incentive to capture a larger market share.

2.4 Revelation of market density after transportation

The market density s is realized to the consumers after they incur transportation costs. If the indifferent consumer finds that the firm he plans to patronize is capacity-constrained, he will lose the transportation costs without getting the good.

In this section, consumers derive negative utility equal to the transportation costs. Equation (2.13) becomes

$$\begin{aligned} & \frac{K}{mv}(W - p_1 - (m-x)^2) - (1 - \frac{K}{mv})(m-x)^2 \\ = \; & \frac{1}{(1-m)}(W - p_2 - (y-m)^2) - (1 - \frac{K}{(1-m)v})(y-m)^2 \,, \end{aligned} \tag{2.25}$$

where $(1 - \frac{K}{mv})$ represents the probability firm 1 will be capacity constrained, and $(m-x)^2$ is the transportation costs lost.

By following the same procedures as in the previous section, at $m = \frac{1}{2}$, we have

$$\frac{\partial E(\pi_1^*)}{\partial x} = -\frac{\partial E(\pi_2^*)}{\partial y} = \frac{(v-K)(K + Kx - 3vx)}{3(2v-K)} \tag{2.26}$$

and

$$p_1^* = p_2^* = \frac{(v-K)(v + 4KW - 2vx)}{2K(2v-K)} \,. \tag{2.27}$$

It can be readily verified that the moderate differentiation results still hold in the section. However, the case of approaching social optimum no longer holds.

Proposition 9 *Maximum differentiation with $x = 0$ and $y = 1$ is not an equilibrium if K is not zero and v is finite.*

Proof. *At $x = 0$ and $y = 1$,*

$$\frac{\partial E(\pi_1^*)}{\partial x} = \frac{(v - K)(K)}{3(2v - K)} > 0\,.$$

Thus, firm 1 has the incentive to move towards firm 2. ■

Proposition 10 *Minimum differentiation $x = y = \frac{1}{2}$ is not an equilibrium.*

Proof. *At $x = y = \frac{1}{2}$,*

$$\frac{\partial E(\pi_1^*)}{\partial x} = \frac{(K - v)^2}{2(K - 2v)} < 0\,.$$

■

Proposition 11 *If $\frac{v}{K}$ tends to infinity, then such differentiation will tend to the maximum differentiation $x = 0$ and $y = 1$.*

Proof. *Solving*

$$\begin{aligned} \frac{\partial E(\pi_1^*)}{\partial x} &= 0 \\ \frac{\partial E(\pi_2^*)}{\partial y} &= 0 \end{aligned}$$

gives one

$$x^* = \frac{K}{3v - K} > 0, v \geq 2K \textit{ and}$$

$$\lim_{K \to 0} \frac{K}{3v - K} = 0\,.$$

■

Unlike the results in the previous section, for increasing probability of capacity constraining, firms tend to be maximally differentiated. The intuition is that if there is a high chance a consumer cannot get the good, and he needs to incur the transportation costs anyway, he will opt for the firm that is more likely to have free capacity, which will be a firm residing at one end of the market.

2.5 Perfectly informed consumers

Perfectly informed consumers are aware of the market density before they decide on which firm to patronize. Unlike in the previous two sections, the consumers now maximize utility instead of expected utility. If one firm is constrained, consumers will switch to another. When both firms are constrained, some consumers derive zero utility. If there is rationing among consumers, consumers know the rationing result before incurring any costs. The equation defining the indifferent consumer becomes

$$-p_1 - (m - x)^2 = -p_2 - (y - m)^2\,. \tag{2.28}$$

Firm 1's profit function is

$$\pi_1 = p_1 Q_1\,,$$

where Q_1 is random.

$$\begin{aligned} Q_1 &= P(NoSwitch)ms + \\ & P(Switch)(P(s-K<K)(s-K)+P(s-K>K)K\,. \quad (2.29) \end{aligned}$$

Then

$$\begin{aligned} E(Q_1) &= \frac{K}{(1-m)v}\frac{mK}{2(1-m)} + \\ & (1-\frac{K}{(1-m)v})(\frac{2K-\frac{K}{1-m}}{v-\frac{K}{1-m}}(\frac{2K+\frac{K}{1-m}}{2}-K)+(1-\frac{2K-\frac{K}{1-m}}{v-\frac{K}{1-m}})K)\,. \end{aligned}$$

Thus,

$$\begin{aligned} E(\pi_1) &= \frac{1}{2}Kp_1(2+\frac{K(3-\frac{2(p_1-p_2+x^2-y^2)}{x-y})}{v(-1+\frac{p_1-p_2+x^2-y^2}{2(x-y)})}) \\ E(\pi_2) &= \frac{Kp_2(x-y)(K-\frac{2K(p_1-p_2+x^2-y^2)}{x-y}+\frac{v(p_1-p_2+x^2-y^2)}{x-y})}{v(p_1-p_2+x^2-y^2)}\,. \end{aligned}$$

After solving for p_1 and p_2 simultaneously by first order conditions and normalizing K to be 1, the expected profits become:

$$\begin{aligned} E(\pi_1^*) &= -\frac{(x-y)(9+4v^2-2x-2y+v(-12+x+y))^2}{(7-4v)^2v} \\ E(\pi_2^*) &= -\frac{(x-y)(5+4v^2+2x+2y-v(10+x+y))^2}{(7-4v)^2v}\,. \end{aligned}$$

For $x=0$ and $y=1$,

$$\frac{\partial E(\pi_1^*)}{\partial x} = \frac{-(11-13v+4v^2)(7-11v+4v^2)}{(7-4v)^2v} < 0 \text{ for } v>2$$

$$\frac{\partial E(\pi_2^*)}{\partial y} = \frac{(11 - 13v + 4v^2)(7 - 11v + 4v^2)}{(7 - 4v)^2 v} > 0 \text{ for } v > 2 \, .$$

Hence, one obtains the following proposition:

Proposition 12 *If consumers have perfect information when making purchase decision, or if they can switch costlessly among the firms, then maximum differentiation is an equilibrium.*

As consumers can switch to another firm if one firm is constrained, they do not face any disutility due to the chance that a firm of their choice can be constrained. The lack of such disutility causes the main difference between the results of this section and the previous two, where such disutility is represented by a probability term discounting the utility consumers enjoy from patronizing a frim. Thus, the model presented in this section yields the standard result of maximum differentiation under quardratic transportation costs.

2.6 Application: Negative externality

It is noteworthy that the probability of capacity constraining is very similar to the effect of negative externality. Indeed, the models in this paper give results contrary to existing literature regarding externality in location games.

In the location game context, the existing literature (Grilo et al. [6]) models externality by

$$U(z, l_i, p_i, m_i) = W - (l_i - z)^2 - p_i + H(m_i)\,, \tag{2.30}$$

where

$$H(m_i) = \alpha m_i - \beta m_i^2 \tag{2.31}$$

is the externality affecting firm i's consumers. Clearly, this formulation will be inappropriate if the effect of externality is not constant across all consumers. In particular, if the effect increases with consumers' utility, then the following formulation should be more appropriate:

$$U(z, l_i, p_i, m_i) = \frac{W - (l_i - z)^2}{m_i} - p_i\,. \tag{2.32}$$

Note that m_i divides $W - (l_i - z)^2$ rather than W. The rationale is that if the location space is interpreted as a product space, then the transportation costs represent utility loss because consumers do not get an ideal product.

Firm 1's profit function is

$$\pi_1 = p_1 m\,,$$

where m is defined by

$$\frac{W - (m - x)^2}{m} - p_1 = \frac{W - (y - m)^2}{(1 - m)} - p_2\,. \tag{2.33}$$

Following the procedures in the previous section, we have

$$\frac{\partial E(\pi_1^*)}{\partial x} = -\frac{\partial E(\pi_2^*)}{\partial y} = \frac{2}{3}(16x - 5) \text{ for } x = 1 - y. \tag{2.34}$$

Proposition 13 *In equilibrium, firms have moderate equilibrium with* $x = \frac{5}{16}$ *and* $y = \frac{11}{16}$.

Proof. $\frac{\partial E(\pi_1^*)}{\partial x} = 0$ *implies* $x = \frac{5}{16}$, *thus* $y = \frac{11}{16}$. ■

2.7 Conclusion

I have shown that the existence of market size uncertainty leads to moderate differentation. I have explored the implications of different revelation time of the uncertain market density. If the market density is revealed to consumers before they incur transportation costs, then the market equilibrium tends to the social optimum when the probability of capacity constraining increases. If the density is revealed after they incur the costs, then the market equilibrium tends to maximum differentiation when the probability of capacity constraining increases.

The main idea of this paper is that uncertainty over market size under fixed capacity counteracts maximum differentiation. In deriving the results, several assumptions have simplified the analysis. As can be shown by using the procedures in this paper, the results for Sections 2.3 and 2.4 hold against the following changes: (i) replacing the bounded reservation price assumption by the assumption that firms are risk averse with log utility functions, or (ii) assuming that the indifferent consumer located at m can buy the good with a probability equal to the firm's capacity K divided by its demand ms if the firm is capacity constrained. Future research may consider different distributions of the market density rather than the uniform distribution currently assumed.

Models	Consumer distribution	Transport costs	Special assumptions	Results
Hotelling [7]	Uniform in [0,1]	Linear	-	Min. differentiation
de Palma et al [4]	Uniform in [0,1]	Linear	Consumers have random utility	Min. differentiation
Wauthy [15]	Uniform in [0,1]	Linear	Capacity constraints, low reservation price	Moderate differentiation
Meagher and Zauner [12]	Uniform in [M-0.5, M+0.5]	Quadratic	M is random	Uncertainty of M is a differentiation force
Grilo et al [6]	Uniform in [0,1]	Quadratic	Consumption externality	Max. differentiation under negative externality
Chapter 2	Uniform in [0,1]	Quadratic	Density over [0,1] is uniformly random; fixed capacity	Moderate differentiation

Table 2: Summary

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