Stability, Boundedness, Oscillation and Periodicity in Functional Differential Equations

Wudu Lu

A thesis submitted to
the Graduate School
of
The Chinese University of Hong Kong
(Division of Mathematics)
in partial fulfillment
of the requirements for the
Degree of Doctor of Philosophy (Ph.D)

HONG KONG
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Contents

Abstract

In this thesis we study the periodicity, boundedness, stability and oscillations of solutions to neutral functional differential equations with infinite delay or finite delay. The thesis is organized as follows.

In Chapter 1, we introduce the basic theory of neutral functional differential equations with infinite delay for convenience of reference and applications.

In Chapter 2, we discuss the existence of periodic solutions and boundedness of solutions for a class of neutral functional differential equations with infinite delay. The problem is classical. Cartwright and Massera independently proved that if the solutions of two-dimensional periodic ordinary differential systems are uniformly bounded ($U.B$) and uniformly ultimately bounded ($U.U.B$), then there is an ω-periodic solution (1950). Yoshizawa advanced this result to systems of order n (1966). Hale and Lopes obtained the same result for systems of functional differential equations with finite delay (1973). Arino, Burton and Haddock extended it to retarded functional differential equations with infinite delay in 1985. We present a direct extension of the above result to the neutral equations with infinite delay which includes the retarded functional differential equatiions with infinite delay as a special case. In the meanwhile we also present two criterion theorems of $U.B$ and $U.U.B$ of solutions.

In Chapter 3, we develop a theory on uniformly asymptotic stability in neutral functional differential equations of nonlinear *D-operator* type with infinite delay. We first introduce new applicable definitions of weak-uniformly stable *D-operator* and weak-uniformly asymptotically stable *D-operator* which generalize corresponding definitions of Hale and Cruz in a nontrivial way. Some examples will be given to demonstrate that our new definitions are reasonable and that our results are applicable to a broad class of neutral equations which contains some "real" nonlinear *D-operators* with infinite delay such as

$$D(t,\psi) = \psi(0) - \int_0^\infty B(u)\psi^n(-u)du.$$

Using Liapunov functional or function and Razumikhin techniques, we establish three uniformly asymptotic stability ($U.A.S$) theorems, and apply these results to discuss $U.A.S$ for some neutral Volterra integro-differential equations with infinite delay.

In the last two chapters, we discuss oscillations and nonoscillations of first order linear neutral differential equations with variable coefficients and first order nonlinear neutral differential equations. We prove several existence theorems of nonoscillatory solutions to a class of linear and nonlinear neutral equations. We also obtain some criterion theorems of oscillations of solutions to these equations. Our conditions for the linear neutral equations are "sharp" in the sense that when all the coefficients and delay arguments of the equations are constants, the conditions become both necessary and sufficient.

Introduction

In this thesis the qualitative behavior of solutions of a class of functional differential equations of neutral type will be discussed. Functional differential equations contain ordinary differential equations, differential difference equations and integro-differential equations as special cases and have many applications in physics, biological mathematics, automatic control, economics and so on [1, 2, 3]. The history of functional differential equations can be traced back to the time of Volterra who formulated some rather general differential equations incorporating the past states of the system in his research on predator-prey models and viscoelasticity [4, 5, 1]. "In many applications , one assumes the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. If it is also assumed that the system is governed by an equation involving the state and rate of change of the state ,then generally,one is considering either ordinary or partial differential equations. However,under closer scrutiny, the principle of causality is often only a first approximation to the true situation and a more realistic model would include some of the past states of the system. Also, in some problems it is meaningless not to have dependence on the past" [2, pp. 1].

The theory of functional differential equations has been extensively developed for the last thirty years. Many excellent monographs have appeared, including the famous book "Theory of Functional Differential Equations" by Hale in 1977, which summed up the most important results obtained by then in the study of functional differential equations with finite delay. In the late seventies and eighties , fundamental theories of retarded functional differential equations and neutral functional differential equations with unbounded delay and infinite delay were also established [6, 7, 8, 11].

In this thesis we will study the periodicity, boundedness, stability and oscillations of solutions to neutral functional differential equations with infinite delay or finite delay. The thesis will be organized as follows. In Chapter 1, we will introduce the basic theory of neutral functional differential equations with infinite delay for convenience of reference and applications. In Chapter 2, we will discuss the existence of periodic solutions and boundedness of solutions for a class of neutral functional differential equations with infinite delay. The problem is classical. Cartwright and Massera independently proved that if solutions of two-dimensional periodic ordinary differential systems are uniformly bounded ($U.B$) and uniformly ultimately bounded ($U.U.B$), then there is an ω-periodic solution (1950)[12, 13]. Yoshizawa advanced this result to systems of order n (1966)[19]. Hale and Lopes obtained the same result for systems of functional differential equations with finite delay (1973)[14]. Arino, Burton and Haddock extended it to the retarded functional differential equations with infinite delay in 1985 [15]. We will present a direct extension of the above result to the neutral equations with infinite delay which includes the retarded functional differential equations with infinite delay as a special case. In the meanwhile we will also present two criterion theorems of $U.B$ and $U.U.B$ of solutions. In Chapter 3, we will develop a theory on uniformly asymptotic stability in neutral functional differential equations of nonlinear *D-operator* type with infinite delay. We will first introduce new applicable definitions of weak-uniformly stable *D-operator* and weak-uniformly asymptotically stable *D-operator* which generalize corresponding definitions of [1, 16] in a nontrival way. Some examples will be given to demonstrate that our new definitions are available and that our results are applicable to a broad class of neutral equations which contains some "real" nonlinear *D-operators* with infinite delay such as

$$D(t,\psi) = \psi(0) - \int_0^\infty B(u)\psi^n(-u)du.$$

Using Liapunov functional or function and Razumikhin techniques, we establish three uniformly asymptotic stability ($U.A.S$) theorems, and apply these results to discuss $U.A.S$ for some neutral Volterra integro-differential equations with infinite delay. In the last two chapters, we discuss oscillations and nonoscillations of first order linear neutral differential equations with variable coefficients and first order nonlinear neutral differential equations. The oscillation theory of solutions of differential equations is one of the traditional trends in the qualitative theory of differential equations. "Its essence is to establish conditions for existence of oscillatory and nonoscillatory solutions, to study the laws of distribution of the zeros, to describe the relationship between the oscillatory and other basic properties of the solutions of various classes of differential equations,etc" [21, pp. 1]. In recent years, there are a number of investigations devoted to the oscillation theory of functional differential equations including retarded and neutral equations. A few monographs on this theory appeared [20, 21]. The study of oscillations for neutral differential equations started in 1980s. However, there are much less results on both oscillations and nonoscillations for neutral differential equations than for retarded differential equations. In these two chapters, we prove several existence theorems of nonoscillatory solutions to a class of linear and nonlinear neutral equations. We also obtain some criterion theorems of oscillations of solutions to these equations. Our conditions for the linear neutral equations are "sharp" in the sense that when all the coefficients and delay arguments of the equations are constants, the conditions become both necessary and sufficient [17, 18].

Chapter 1

The Fundamental Theory of $NFDEs$ with Infinite Delay

1.1 Introduction

In this chapter we will introduce the local theory of neutral functional differential equations ($NFDEs$) with infinite delay. This class of equations is of the form

$$\frac{d}{dt}D(t, x_t) = f(t, x_t) \tag{1.1.1}$$

where D and f are functional. (1.1.1) contains the retarded functional differential equations ($FDEs$) with infinite delay

$$\dot{x}(t) = f(t, x_t) \tag{1.1.2}$$

as a special case which was discussed in many literatures [6, 7].

In order to deal with (1.1.2) on a large variety of phase spaces, Hale and Kato [6] and Schumacher [7] independently developed a general theory which has the feature of axiomatic approach—to list certain axioms for the phase space and the right-hand side functional of (1.1.2), such that any particular space and $f(t, \psi)$ verify their axioms, automatically generate existence and uniqueness of solutions.

Following their clues, authors of [8, 9, 10, 11] established the fundamental theory for (1.1.1) in recent years.

For simplicity and convenience of applications, we will state the local theory for a little simple case of (1.1.1) in the spirit of [11]. For details about the general case of (1.1.1), we refer to [8, 11].

1.2 Phase spaces and $NFDEs$ with infinite delay

Let $|\cdot|$ denote an $\mathbb{R}^n$-norm, B be a real vector space either

1. of continuous functions that map $(-\infty, 0]$ to $\mathbb{R}^n$ with $\phi = \psi$ if $\phi(s) = \psi(s)$ on $(-\infty, 0]$, or

2. of measurable functions that map $(-\infty, 0]$ to $\mathbb{R}^n$ with $\phi = \psi$ (or ϕ is equivalent to ψ) in B if $\phi(s) = \psi(s)$ almost everywhere on $(-\infty, 0]$, and $\phi(0) = \psi(0)$.

Let B be endowed with a norm $|\cdot|_B$ such that B is complete with respect to $|\cdot|_B$. Thus B equipped with norm $|\cdot|_B$ is a Banach space. We denote this space by $(B, |\cdot|_B)$ or simply by B, whenever no confusion can result.

If $x : (-\infty, A) \longmapsto \mathbb{R}^n, 0 \le A \le \infty$, then for any $t \in [0, A)$ define x_t by $x_t(s) = x(t+s)$ for $s \le 0$. Throughout this chapter, suppose that phase space B satisfies the following conditions.

Let $0 \le \alpha < A$. If $x : (-\infty, A) \longmapsto \mathbb{R}^n$ is given such that $x_\alpha \in B$ and $x : [\alpha, A) \longmapsto \mathbb{R}^n$ is continuous, then $x_t \in B$ for all $t \in [\alpha, A)$.

Definition 1.2.1 *A space B defined above is said to be an admissible phase space if there exist a constant $J > 0$ and continuous functions $K, M : [0, \infty) \longmapsto [0, \infty)$ such that the following conditions hold.*

Let $0 \leq \alpha < A$. If $x : (-\infty, A) \longmapsto \mathbb{R}^n$ is defined on $(-\infty, A)$ with $x_\alpha \in B$ and $x : [\alpha, A) \longmapsto \mathbb{R}^n$ being continuous, then for all $t \in [\alpha, A)$,

(B_1) $x_t \in B$,

(B_2) $t \in [\alpha, A) \longmapsto x_t \in B$ *is continuous with respect to* $|\cdot|_B$,

(B_3) $|x_t|_B \leq K(t-\alpha)\max_{\alpha\leq s\leq t}|x(s)| + M(t-\alpha)|x_\alpha|_B$,

(B_4) $|\phi(0)| \leq J|\phi|_B$ *for all* $\phi \in B$.

It is easy to verify that space B_r^p and space BU mentioned later are admissible spaces.

Throughout this chapter, we always assume that B is an admissible space; $D,\ f : [0, \infty) \times B \longmapsto \mathbb{R}^n$ are continuous.

Definition 1.2.2 *A function $x : (-\infty, t_0 + \delta) \longmapsto \mathbb{R}^n$ $(t_0 \in [0, \infty), \delta > 0)$ is said to be a solution of* (1.1.1) *through $(t_0, \phi) \in [0, \infty) \times B$ on $[t_0, t_0 + \delta)$, if*

(i) $x_{t_0} = \phi$,

(ii) x *is continuous on* $[t_0, t_0 + \delta)$,

(iii) $D(t, x_t)$ *is continuous on* $[t_0, t_0 + \delta)$,

(iv) (1.1.1) *holds everywhere on* $[t_0, t_0 + \delta)$.

We denote a solution x of (1.1.1) through (t_0, ϕ) by $x(t_0, \phi)(t)$.

According to this definition, to solve (1.1.1) with $x_{t_0} = \phi$ is equivalent to solve the following equation

$$D(t, x_t) = D(t_0, x_{t_0}) + \int_{t_0}^{t} f(s, x_s) ds. \tag{1.2.1}$$

1.3 Local theory

Define

$$A(t_0, \phi, \delta, \gamma) = \left\{ z : (-\infty, t_0 + \delta] \mapsto \mathbb{R}^n : \begin{array}{l} z_{t_0} = \phi, z(t) \ \textit{is continuous} \\ \textit{in} \ t \in [t_0, t_0 + \delta] \ \textit{and} \\ \sup_{t_0 \le t \le t_0 + \delta} |z(t) - \phi(0)| \le \gamma. \end{array} \right\}$$

We always assume that $D(t, \phi)$ satisfies the following conditions:

$$D(t, \phi) - D(t, \psi) = [\phi(0) - \psi(0)] + L(t, \phi, \psi)$$

where $(t, \phi, \psi) \in [0, \infty) \times B \times B, L : [0, \infty) \times B \times B \longmapsto \mathbb{R}$ is continuous and satisfies that for any $(t_0, \phi) \in [0, \infty) \times B$, there exist constants δ,$\gamma > 0$ and $k_1 \in [0, 1)$ such that for any $x, y \in A(t_0, \phi, \delta, \gamma)$,

(A_1) $\qquad |L(t, x_t, y_t)| \le k_1 \sup_{t_0 \le s \le t} |x(s) - y(s)|.$

Theorem 1.3.1 (*Existence*) *For any* $(t_0, \phi) \in [0, \infty) \times B$, (1.1.1) *has a solution* $x(t_0, \phi)(t)$.

Proof. Define

$$E(\delta, \gamma) = \left\{ z : (-\infty, \delta] \mapsto \mathbb{R}^n : \begin{array}{l} z(t) \ \textit{is continuous}, \\ z(s) = 0 \ \textit{for} \ s \le 0 \\ \textit{and} \ \|z\| \le \gamma. \end{array} \right\}$$

where $\delta, \gamma > 0$ are constants and $\|z\| = \sup_{0 \le s \le \delta} |z(s)|$. $E(\delta, \gamma)$ with the norm $\|\cdot\|$ is a Banach space. For any given $(t_0, \phi) \in B$, $\bar{\phi}$ is defined as $\bar{\phi}_{t_0} = \phi$ and

$\bar{\phi}(t) = \phi(0)$ for $t \geq t_0$. Let $R = 1+|f(t_0,\phi)|$. Since phase space B is admissible and D and f are continuous, we can choose δ and γ sufficiently small so that (A_1) holds, $|f(t,x_t)| < R$, and for all $t \in [t_0, t_0+\delta]$ and all $x \in A(t_0,\phi,\delta,\gamma)$,

$$|D(t_0+t,\bar{\phi}_{t_0+t}) - D(t_0,\phi)| < \frac{1-k_1}{2}\gamma \ \ for \ \ t \in [0,\delta]$$

and

$$\delta R \leq \frac{1-k_1}{2}\gamma.$$

Define two operators S and U on $E(\delta,\gamma)$ as follows

$$(Sz)(t) = \begin{cases} 0 & for \quad t \leq 0, \\ -D(t_0+t,\bar{\phi}_{t_0+t}+z_t) + D(t_0,\phi) + z(t) & for \quad t \in [0,\delta] \end{cases}$$

and

$$(Uz)(t) = \begin{cases} 0 & for \quad t \leq 0, \\ \int_0^t f(t_0+s,\bar{\phi}_{t_0+s}+z_s)ds & for \quad t \in [0,\delta] \end{cases}$$

where $z \in E(\delta,\gamma)$. Obviously, $(Sz)(t)$ and $(Uz)(t)$ are continuous in $t \in [0,\delta]$, and for $t \in [0,\delta]$ we have

$$|(Uz)(t)| \leq \int_0^\delta |f(t_0+s,\bar{\phi}_{t_0+s}+z_s)|ds \leq \delta R \leq \frac{1-k_1}{2}\gamma$$

and

$$\begin{aligned} |(Sz)(t)| &= |-D(t_0+t,\bar{\phi}_{t_0+t}+z_t) + D(t_0+t,\bar{\phi}_{t_0+t}) \\ &\quad - D(t_0+t,\bar{\phi}_{t_0+t}) + D(t_0,\phi) + z(t)| \\ &= |-z(t) - L(t_0+t,\bar{\phi}_{t_0+t}+z_t,\bar{\phi}_{t_0+t}) - D(t_0+t,\bar{\phi}_{t_0+t}) \\ &\quad + D(t_0,\phi) + z(t)| \\ &\leq k_1\|z\|_{[0,t]} + |D(t_0+t,\bar{\phi}_{t_0+t}) - D(t_0,\phi)| \end{aligned}$$

$$
\begin{aligned}
&\leq k_1\gamma + \frac{1-k_1}{2}\gamma \\
&= \frac{1+k_1}{2}\gamma.
\end{aligned}
$$

Then $|(Sz)(t)+(Uz)(t)| \leq \gamma$ for $t \in [0,\delta]$. This means that $S+U$ is a mapping from $E(\delta,\gamma)$ into itself. For any $z, w \in E(\delta,\gamma)$, we have

$$
\begin{aligned}
|(Sz)(t)-(Sw)(t)| &= |L(t_0+t, \bar{\phi}_{t_0+t}+w_t, \bar{\phi}_{t_0+t}+z_t)| \\
&\leq k_1 \sup_{0\leq s\leq t} |z(s)-w(s)| \\
&\leq k_1 \|z-w\|_{[0,\delta]}.
\end{aligned}
$$

Then

$$
\|Sz - Sw\|_{[0,\delta]} \leq k_1 \|z-w\|_{[0,\delta]}
$$

which implies that S is a contraction mapping on $E(\delta,\gamma)$.

For any $t_1, t_2 \in [0,\delta]$, we have

$$
\begin{aligned}
|(Uz)(t_1)-(Uz)(t_2)| &= \left|\int_{t_2}^{t_1} f(t_0+s, \bar{\phi}_{t_0+s}+z_s)ds\right| \\
&\leq R|t_1-t_2|
\end{aligned}
$$

which means that U is a completely continuous operator on $E(\delta,\gamma)$ by Ascoli-Arzela Theorem. Therefore $S+U$ is an α-contraction mapping on $E(\delta,\gamma)$. By Darbo's fixed point theorem [1, pp. 98], $S+U$ has a fixed point z in $E(\delta,\gamma)$ and $x(t)=\bar{\phi}(t)+z(t-t_0)$ is a solution of (1.1.1) on $[t_0, t_0+\delta]$ with $x_{t_0}=\phi$. The proof is complete.

Theorem 1.3.2 (*Uniqueness*) *Assume that for any* $(t_0,\phi) \in [0,\infty) \times B$, *there exist* $\delta, \gamma > 0$ *and a function* $g : [t_0, t_0+\delta] \longmapsto [0,\infty)$ *continuous at* $t = t_0$ *with* $g(t_0) = 0$ *such that for any* $x, y \in A(t_0, \phi, \delta, \gamma)$, *we have*

$$
\left|\int_{t_0}^{t} [f(s,x_s)-f(s,y_s)]ds\right| \leq g(t) \sup_{t_0\leq s\leq t} |x(s)-y(s)|, \quad t \in [t_0, t_0+\delta]
$$

Then (1.1.1) *has a unique solution through* (t_0, ϕ).

Proof. According to the argument of Theorem 1.3.1, it suffices to prove that $S + U$ has a unique fixed point on $E(\delta, \gamma)$. Choose $\delta > 0$ sufficiently small so that

$$\sup_{t_0 \le s \le t_0+\delta} |g(s)| < \frac{1-k_1}{2}.$$

If y and z are both fixed points of $S + U$ on $E(\delta, \gamma)$, then

$$|(Sz)(t) - (Sy)(t)| \le k_1 \sup_{0 \le s \le t} |z(s) - y(s)|, \quad t \in [0, \delta]$$

and

$$\begin{aligned} |(Uz)(t) - (Uy)(t)| &= \left| \int_0^t \left[f(t_0 + s, \bar{\phi}_{t_0+s} + z_s) - f(t_0 + s, \bar{\phi}_{t_0+s} + y_s) \right] ds \right| \\ &< \frac{1-k_1}{2} \sup_{0 \le s \le t} |z(s) - y(s)|, \quad t \in [0, \delta]. \end{aligned}$$

Then

$$\begin{aligned} |z(t) - y(t)| &= |(Sz)(t) - (Sy)(t) + (Uz)(t) - (Uy)(t)| \\ &< \frac{1+k_1}{2} \sup_{0 \le s \le t} |z(s) - y(s)| \\ &\le \frac{1+k_1}{2} \sup_{0 \le s \le \delta} |z(s) - y(s)|, \quad t \in [0, \delta]. \end{aligned}$$

Then

$$\sup_{0 \le s \le \delta} |z(s) - y(s)| \le \frac{1+k_1}{2} \sup_{0 \le s \le \delta} |z(s) - y(s)|$$

which is a contradiction. The proof is complete.

Theorem 1.3.3 (*Continuation*) *Assume that*

(*i*) $f(t, \phi)$ *is completely continuous,* (*ii*) $D(t,\phi)$ *is uniformly continuous on any bounded set of* $[0,\infty)\times B$,

(*iii*) *if* $z:(-\infty,t_0+\delta)\longmapsto\mathbb{R}^n$ *is continuous on* $[t_0,t_0+\delta)$, $z_{t_0}=\phi\in B$ *and* $|z(t)|\le\gamma$ *for all* $t\in[t_0,t_0+\delta)$, *then*

$$\lim_{k\to\infty}L(t_k,z_{t_k},z_{s_k})=0$$

where $\delta,\gamma>0, t_k,s_k\in[t_0,t_0+\delta), t_k\to t_0+\delta$ *and* $s_k\to t_0+\delta$ *as* $k\to\infty$.

Then each bounded solution $x(t_0,\phi)(t)$ *of* (1.1.1) *exists on* $[t_0,\infty)$.

Proof. Let $x(t)=x(t_0,\phi)(t)$ with $|x(t)|\le\gamma$ for $t\ge t_0$ be a bounded solution of (1.1.1). If $x(t)$ exists on $[t_0,t_0+\delta)$ $(0<\delta<\infty)$ and is noncontinuable, then $\lim_{t\to t_0+\delta}x(t)$ does not exist. Otherwise, define $x(t_0+\delta)=\lim_{t\to t_0+\delta}x(t)$ and thus $x_{t_0+\delta}\in B$. By Theorem 1.3.1, $x(t)$ can be continued beyond $t_0+\delta$. By

$$|x_t|_B\le K(t-t_0)\max_{t_0\le s\le t}|x(s)|+M(t-t_0)|x_{t_0}|_B,\quad t\in[t_0,t_0+\delta),$$

$|x_t|_B$ is bounded for all $t\in[t_0+\delta)$ and then $f(t,x_t)$ is bounded for all $t\in[t_0,t_0+\delta)$. Let $|f(t,x_t)|\le N$ for all $t\in[t_0,t_0+\delta)$. Since $\lim_{t\to t_0+\delta}x(t)$ does not exist, we can find an $\varepsilon>0$ and two sequences $\{t_k\}$ and $\{s_k\}$ such that $t_0\le s_k<t_k<t_0+\delta, s_k\to t_0+\delta$ as $k\to\infty, t_k\to t_0+\delta$ as $k\to\infty$ and $|x(t_k)-x(s_k)|\ge\varepsilon$. We have

$$\begin{aligned}\int_{s_k}^{t_k}f(s,x_s)ds &= D(t_k,x_{t_k})-D(s_k,x_{s_k})\\ &= D(t_k,x_{t_k})-D(t_k,x_{s_k})+D(t_k,x_{s_k})-D(s_k,x_{s_k})\\ &= [x(t_k)-x(s_k)]+L(t_k,x_{t_k},x_{s_k})+D(t_k,x_{s_k})-D(s_k,x_{s_k}).\end{aligned}$$

Then

$$|x(t_k)-x(s_k)|\le N|t_k-s_k|+|L(t_k,x_{t_k},x_{s_k})|+|D(t_k,x_{s_k})-D(s_k,x_{s_k})|.$$

From the above inequality, we have

$$\lim_{k\to\infty} |x(t_k) - x(s_k)| = 0$$

which is a contradiction. Hence each bounded solution of (1.1.1) exists on $[t_0, \infty)$. The proof is complete.

Definition 1.3.1 *Solutions of* (1.1.1) *are B-uniformly bounded* (*B-U.B*) *if for each* $B_1 > 0$ *there exists an* $N(B_1) > 0$ *such that* $[t_0 \geq 0, |\varphi|_B \leq B_1, t \geq t_0]$ *imply that* $|x(t_0, \varphi)(t)| < N(B_1)$.

Define

$$B_M(t_0, \phi, \delta, \gamma) = \left\{ z : (-\infty, t_0 + \delta] \longmapsto \mathbb{R}^n : \begin{array}{l} z \text{ is continuous on } [t_0, t_0+\delta], \\ z_{t_0} \in B, |z_{t_0} - \phi|_B < M \text{ and} \\ \|z\|_{[t_0, t_0+\delta]} \leq \gamma. \end{array} \right\}$$

where $(t_0, \phi) \in B, M, \delta$ and γ are any positive constants.

Theorem 1.3.4 (*Continuous dependence*) *Assume that*

(*i*) *conditions of Theorem* 1.3.2 *and Theorem* 1.3.3 *hold,*

(*ii*) *solutions of* (1.1.1) *are B-U.B,*

(*iii*) *for each* $B_M(t_0, \phi, \delta, \gamma)$, *there exists a function* $O : [0, \delta] \longmapsto [0, \infty)$ *with* $\lim_{u\to 0} O(u) = 0$ *such that for all* $z \in B_M(t_0, \phi, \delta, \gamma)$ *we have*

$$|L(t, z_t, z_s)| \leq O(t - s), \;\; t_0 \leq s \leq t_0 + \delta.$$

Then for any given $\varepsilon > 0$ *and* $b > 0$, *we can find a* $\sigma > 0$ *so that if* $|\phi - \psi|_B < \sigma$, *then*

$$|x(t) - y(t)| < \varepsilon \;\; \textit{for all} \;\; t \in [t_0, t_0 + b]$$

where $x(t) = x(t_0, \phi)(t)$ *and* $y(t) = y(t_0, \psi)(t)$ *are solutions of* (1.1.1).

Proof. For any given $(t_0, \phi) \in [0, \infty) \times B$ and positive number M, let $B_1 = M + |\phi|_B$. By (ii), all solutions $x(t_0, \psi)(t)$ of (1.1.1) with $|\psi - \phi|_B < M$ belong to $B_M(t_0, \phi, b, N(B_1))$. If $x(t_0, \psi)(t) \in B_M(t_0, \phi, b, N(B_1))$, then for $t_0 \le t \le t_0 + b$,

$$\begin{aligned} |x_t|_B &\le K(t - t_0) \max_{t_0 \le s \le t} |x(s)| + M(t - t_0)|x_{t_0}|_B \\ &\le \max_{0 \le u \le b} K(u) N(B_1) + \max_{0 \le u \le b} M(u) B_1. \end{aligned}$$

By (i), for all $x(t_0, \psi)(t) \in B_M(t_0, \phi, b, N(B_1))$ and $t_0 \le t \le t_0 + b$, there exists a positive R such that

$$|f(t, x_t)| \le R$$

where $x_t = x(t_0, \psi)(t + \theta)$ for $\theta \le 0$. Then for $t_0 \le s \le t \le t_0 + b$ and $x(t_0, \psi)(t) \in B_M(t_0, \phi, b, N(B_1))$, we have

$$\begin{aligned} \int_s^t f(u, x_u) du &= D(t, x_t) - D(s, x_s) \\ &= D(t, x_t) - D(t, x_s) + D(t, x_s) - D(s, x_s) \\ &= x(t) - x(s) + L(t, x_t, x_s) + D(t, x_s) - D(s, x_s). \end{aligned}$$

Then

$$\begin{aligned} |x(t) - x(s)| &\le \int_s^t |f(u, x_u)| du + |L(t, x_t, x_s)| + |D(t, x_s) - D(s, x_s)| \\ &\le R|t - s| + O(t - s) + |D(t, x_s) - D(s, x_s)|. \end{aligned}$$

By (i), (ii) and the above inequality, all solutions of (1.1.1) belonging to $B_M(t_0, \phi, b, N(B_1))$ are uniformly bounded and equicontinuous on $[t_0, t_0 + b]$. By Ascoli-Arzela Theorem, the set of all solutions of (1.1.1) belonging to $B_M(t_0, \phi, b, N(B_1))$ is a precompact set with respect to supremum norm

$\|\cdot\|_{[t_0,t_0+b]}$. If the conclusion of Theorem 1.3.4 is not true, there exist an $\varepsilon > 0$, sequences $\{t_k\} \subset [t_0, t_0 + b]$ and $\{\phi_k\} \subset B$ such that

$$|\phi_k - \phi|_B < \frac{1}{k}$$

and

$$\left|y^{(k)}(t_k) - x(t_k)\right| \geq \varepsilon$$

where $y^{(k)}(t) = y(t_0, \phi_k)(t)$ and $x(t) = x(t_0, \phi)(t)$ are solutions of (1.1.1). When k is sufficiently large, $y^{(k)} \in B_M(t_0, \phi, b, N(B_1))$. Without loss of generality, let $y^{(k)}(t)$ converge to a continuous function $y^{(0)}(t)$ uniformly on $[t_0, t_0+b]$. Since $y^{(k)}_{t_0} = \phi_k, y^{(k)}(t_0) = \phi_k(0)$. We have

$$\left|y^{(k)}(t_0) - \phi(0)\right| = |\phi_k(0) - \phi(0)| \leq J|\phi_k - \phi|_B < \frac{J}{k}.$$

Then

$$\left|y^{(0)}(t_0) - \phi(0)\right| = \lim_{k\to\infty} \left|y^{(k)}(t_0) - \phi(0)\right| = 0.$$

Define $y : (-\infty, t_0 + b] \longmapsto \mathbb{R}^n$ as follows : $y_{t_0} = \phi$ and $y(t) = y^{(0)}(t)$ for $t_0 \leq t \leq t_0 + b$. Then $y : (-\infty, t_0 + b] \longmapsto \mathbb{R}^n$ is continuous on $[t_0, t_0 + b]$ with $y_{t_0} = \phi \in B$. On the other hand, we have

$$D\left(t, y_t^{(k)}\right) - D\left(t_0, \phi_k\right) = \int_{t_0}^{t} f\left(s, y_s^{(k)}\right) ds, \quad t_0 \leq t \leq t_0 + b.$$

which means that y is a solution of (1.1.1) through (t_0, ϕ) on $[t_0, t_0 + b]$. By the uniqueness of solution with respect to initial data, $y(t) = x(t)$ for all $t_0 \leq t \leq t_0 + b$. Then

$$\left|y^{(k)}(t_k) - y(t_k)\right| = \left|y^{(k)}(t_k) - x(t_k)\right| \geq \varepsilon$$

which is a contradiction. The proof is complete.

Chapter 2

Periodicity and B_r^p-Boundedness in Neutral Systems of Nonlinear *D-operator* with Infinite Delay

2.1 Introduction

In this chapter we consider a neutral system of nonlinear *D-operator* with infinite delay

$$\frac{d}{dt}D(t, x_t) = f(t, x_t) \tag{2.1.1}$$

where $x_t = x(t+\theta), -\infty < \theta \leq 0, D$ and $f : [0, \infty) \times B \longmapsto \mathbb{R}^n$ are continuous.

We will discuss the existence of periodic solutions to (2.1.1), uniform boundedness ($U.B$) and uniform ultimate boundedness ($U.U.B$) of solutions to (2.1.1).

The problem is classical. Cartwright and Massera independently proved that if solutions of two-dimensional periodic ordinary differential systems are $U.B$ and $U.U.B$, then there is an ω-periodic solution [12, 13]. Yoshizawa advanced that result to systems of order n [19]. Horn's theorem [23] enabled Hale and Lopes to obtain the same result for systems of functional differential equations with finite delay [14]. In 1985, Arino, Burton and Haddock extended that result to the retarded functional differential equations with infinite delay

$$\dot{x}(t) = f(t, x_t) \tag{2.1.2}$$

where $x_t = x(t+\theta), -\infty < \theta \leq 0, f : \mathbb{R} \times X \longmapsto \mathbb{R}^n, X$ is a specific space of functions [15]. Recently fundamental theory for neutral functional differential equations with infinite delay has been established [8, 11]. It is natural to ask how we extend the result mentioned above for the retarded equations with infinite delay to the neutral equations with infinite delay. (2.1.1) is a more comprehensive class of equations than (2.1.2). When $D(t, \psi) = \psi(0)$, (2.1.1) becomes (2.1.2). When $f(t, \psi) = g(t)$, (2.1.1) becomes

$$D(t, x_t) = h(t) \tag{2.1.3}$$

which contains some *Volterra* integral equations as its special cases. A prototype of (2.1.1) is the equation [22]

$$\frac{d}{dt}\left[x(t) - \int_{-\infty}^{t} C(t, s, x(s))ds\right] = H(t, x(t)) + \int_{-\infty}^{t} G(t, s, x(s))ds. \tag{2.1.4}$$

The investigators of [22] studied the existence of periodic solutions to (2.1.4) by going through the limiting equations with finite delay of (2.1.4) to avoid the following two technical difficulties :

(i) an appropriate phase space should be chosen,

(ii) the uniform boundedness and uniform ultimate boundedness need to be verified,

because even for retarded equations with infinite delay, the choice of an appropriate phase space is not a trivial task, and moreover little has been done for the boundedness of solutions of neutral equations with infinite delay, especially for neutral equations of nonlinear *D-operator* with infinite delay.

In this chapter we will choose space B_r^p as the phase space and present a direct extension of the main result of [15] from retarded equation (2.1.2) to neutral equation (2.1.1), while we will provide two criterion theorems to verify B_r^p-uniform boundedness and B_r^p-uniform ultimate boundedness of solutions to (2.1.1).

As compared with (2.1.2), not only solutions of (2.1.1) are no longer differentiable, but also the qualitative behavior of solutions of (2.1.1) depend heavily upon that of the solutions of the associated functional difference equation (2.1.3). Therefore we need to place some restrictions on operator $D(t, \psi)$ before we can study the boundedness and periodicity of solutions to (2.1.1). We will introduce two classes of nonlinear *D-operators* called "B_r^p- uniformly stable *D-operator*" and "B_r^p-uniformly asymptotically stable D-operator".

In section 2, besides the two classes of nonlinear *D-operators* mentioned above we introduce the definitions of B_r^p-uniform boundedness and B_r^p-uniform ultimate boundedness for solutions and $D(t, \psi)$ of (2.1.1) and present Lemma 2.2.1 which relates the boundedness of solutions of (2.1.1) with that of $D(t, \psi)$ of (2.1.1). In section 3, we first construct a class of compact sets in space B_r^p and then, using Horn's fixed point theorem, prove the existence theorem of periodic solutions of (2.1.1).

We deal with B_r^p-$U.B$ and B_r^p-$U.U.B$ of solutions to (2.1.1) in section 4 and give two criterion theorems.

Finally in section 5 we apply the results in section 3 and 4 to a class of neutral *Volterra* integro-differential equations of nonlinear *D-operator* with infinite delay.

2.2 Preliminaries

Let

$$\|x\|_{[a,b]} = \sup\{|x(s)| : a \leq s \leq b, -\infty \leq a \leq b \leq \infty\}$$

and

$r : (-\infty, 0] \longmapsto [0.\infty)$ be a continuous, nondecreasing function with

(P_1) $\int_{-\infty}^{0} r(s)ds = \ell < \infty$ and

(P_2) $r(s_1 + s_2) \leq r(s_1)r(s_2)$ for $s_1, s_2 \leq 0$.

Define

$$B_r^p = \{\varphi : (-\infty, 0] \longmapsto \mathbb{R}^n, \text{measurable}, |\varphi|_{p,r} < \infty\},$$

$$|\varphi|_{p,r} = \left[|\varphi(0)|^p + \int_{-\infty}^{0} r(s)|\varphi(s)|^p ds\right]^{1/p}, \quad p \geq 1$$

where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $|\cdot|$ denotes a suitable norm in $\mathbb{R}^n$. B_r^p is a Banach space and is also an admissible space mentioned in [11].

The fundamental theory concerning existence, uniqueness, continuation of solutions and continuous dependence of solutions with respect to initial data for neutral functional differential equations ($NFDEs$) with infinite delay in the abstract phase spaces given in [8] and [11], including space B_r^p, has been established. We refer to [8] and [11].

By a solution of (2.1.1) through $(t_0, \varphi) \in [0, \infty) \times B_r^p$ we mean an $x : (-\infty, t_0 + \delta) \longmapsto \mathbb{R}^n$ for some $\delta > 0$ such that

(i) $x_{t_0} = \varphi$,

(ii) x is continuous on $[t_0, t_0 + \delta)$,

(iii) $D(t, x_t)$ is continuously differentiable and satisfies (2.1.1) on $[t_0, t_0 + \delta)$.

We denote a solution of (2.1.1) through (t_0, φ) by $x(t_0, \varphi)(t)$ or simply by $x(t)$.

In the following sections, we always assume that D and f satisfy certain conditions to ensure the existence,uniqueness and continuation of solutions of (2.1.1).

A strictly increasing and continuous function $W : [0, \infty) \longmapsto [0.\infty)$ is called a wedge if $W(0) = 0$ and $W(s) > 0$ as $s > 0$.

Let $V(t, \psi)$ be a continuous nonnegative functional defined on $[0, \infty) \times B_r^p$ and locally Lipschitz in ψ. The derivative of V along a solution of (2.1.1) is defined to be

$$V'(t, x_t) = \limsup_{h \to 0^-} \frac{V(t + h, x_{t+h}) - V(t, x_t)}{h}.$$

We always assume that $V'(t, x_t)$ exists.

Definition 2.2.1 *Solutions of* (2.1.1) *are B_r^p-uniformly bounded (B_r^p-U.B) if for each $B_1 > 0$ there exists $N(B_1) > 0$ such that $[t_0 \geq 0, |\varphi|_{p,r} \leq B_1, t \geq t_0]$ imply $|x(t_0, \varphi)(t)| < N(B_1)$. Solutions of* (2.1.1) *are B_r^p-uniformly ultimately bounded (B_r^p-U.U.B) for bound $B > 0$ if for each $B_3 > 0$ there exists $T(B_3) > 0$ such that $[t_0 \geq 0, |\varphi|_{p,r} \leq B_3, t \geq t_0 + T(B_3)]$ imply $|x(t_0, \varphi)(t)| < B$.*

Definition 2.2.2 *Let $D : [0.\infty) \times B_r^p \longmapsto \mathbb{R}^n$ be continuous. D is said to be B_r^p-uniformly stable if there exist constants $k > 0$ and $\sigma \geq 0$ such that* *for any* $\varphi \in B_r^p$, $\tau \in [0,\infty)$ *and* $h \in C([\tau,\infty),\mathbb{R}^n)$, *the continuous solution* $x(t) = x(\tau,\varphi,h)(t)$ *of the equation*

$$D(t,x_t) = h(t), \quad t \geq \tau \quad and \quad x_\tau = \varphi, \tag{2.2.1}$$

satisfies

$$|x(t)| \leq k|x_\tau|_{p,r} + k[\|h\|_{[\tau,t]} + \sigma], \quad t \geq \tau. \tag{2.2.2}$$

D is said to be B_r^p*-uniformly asymptotically stable if there exist constants* $k_1 > 0$ *and* $\sigma_1 \geq 0$ *and for each* $\gamma > 0$ *there exists a nonincreasing function* $g_\gamma(u) : [0,\infty) \longmapsto [0,1]$ *with* $\lim_{u\to\infty} g_\gamma(u) = 0$ *such that for any* $\tau \in [0.\infty),\varphi \in B_r^p$ *and* $h \in C([\tau,\infty),\mathbb{R}^n)$, *the continuous solution* $x(t) = x(\tau,\varphi,h)(t)$ *with* $\|x\|_{[\tau,t]} \leq \gamma$ *of* (2.2.1) *satisfies*

$$|x(t)| \leq k_1 g_\gamma(t-\tau)|x_\tau|_{p,r} + k_1(\|h\|_{[\tau,t]} + \sigma_1), \quad t \geq \tau. \tag{2.2.3}$$

Let $x(t) = x(t_0,\varphi)(t)$ be a solution of (2.1.1). Then $D(t,x_t)$ is a continuous function of t ($t \geq t_0$). Denote $D(t,x_t)$ by $H(t)$. Then $D(t,x_t) \equiv H(t)$.

Definition 2.2.3 $D(t,\psi)$ *of* (2.1.1) *is said to be* B_r^p*-uniformly bounded* (B_r^p-*U.B*) *if for each* $A_1 > 0$ *there exists* $A_2 > 0$ *such that* $[t_0 \geq 0, |\varphi|_{p,r} \leq A_1, t \geq t_0]$ *imply* $|H(t)| < A_2$. $D(t,\psi)$ *of* (2.1.1) *is said to be* B_r^p*-uniformly ultimately bounded* (B_r^p*-U.U.B*) *for bound* $A > 0$ *if for each* $A_3 > 0$ *there exists* $T^* > 0$ *such that* $[t_0 \geq 0.|\varphi|_{p,r} \leq A_3, t \geq t_0 + T^*]$ *imply* $|H(t)| < A$.

Lemma 2.2.1 *Suppose that* $D(t,\psi)$ *of* (2.1.1) *is* B_r^p*-uniformly stable and* B_r^p*-uniformly asymptotically stable. If* $D(t,\psi)$ *is* B_r^p*-U.B, then solutions of* (2.1.1) *are* B_r^p*-U.B. If* $D(t,\psi)$ *is both* B_r^p*-U.B and* B_r^p*-U.U.B, then solutions of* (2.1.1) *are also* B_r^p*-U.B and* B_r^p*-U.U.B.*

Proof. For given $A_1 > 0$ and $x(t) = x(t_0, \varphi)(t)(t \geq t_0)$ with $|\varphi|_{p,r} \leq A_1$, there exists $A_2 > 0$ so that $|H(t)| < A_2$ for $t \geq t_0$. Since $D(t, \psi)$ is B_r^p-uniformly stable, we have

$$|x(t)| \leq k|x_{t_0}|_{p,r} + k[\|H\|_{[t_0,t]} + \sigma] < k(A_1 + A_2 + \sigma), \ \ t \geq t_0.$$

This proves that solutions of (2.1.1) are B_r^p-*U.B*. Assume that $D(t, \psi)$ is B_r^p-*U.U.B* for bound $A > 0$ and set $B = 1 + k_1(A + \sigma_1)$. For given $B_3 > 0$, there exists $B_4 > 0$ so that $x(t) = x(t_0, \varphi)(t)$ with $|\varphi|_{p,r} \leq B_3$ satisfies

$$|x(t)| < B_4 \ \ , \ \ t \geq t_0.$$

Set $\gamma = \max\{B_3, B_4\}$. Then $|x_{t_0}|_{p,r} \leq \gamma$ and $|x(t)| < \gamma$ for $t \geq t_0$. We have

$$\begin{aligned} |x_t|_{p,r}^p &= |x(t)|^p + \int_{-\infty}^t r(u-t)|x(u)|^p du \\ &\leq \gamma^p + \int_{t_0}^t r(u-t)|x(u)|^p du + \int_{-\infty}^{t_0} r(u-t)|x(u)|^p du \\ &\leq \gamma^p + \ell\gamma^p + \gamma^p \\ &= (2+\ell)\gamma^p, \quad t \geq t_0. \end{aligned}$$

Then

$$|x_t|_{p,r} \leq (2+\ell)^{1/p}\gamma, \quad t \geq t_0.$$

There is $T_1 > 0$ so that $k_1 g_\gamma(u)(2+\ell)^{1/p}\gamma < 1$ for $u \geq T_1$. Since $D(t, \psi)$ is B_r^p-*U.U.B* for A, there exists $T_2 > 0$ for B_3 so that $[t_0 \geq 0, |\varphi|_{p,r} \leq B_3, t \geq t_0 + T_2]$ imply $|H(t)| < A$. Set $T^* = T_1 + T_2$ and $\tau_0 = t_0 + T_2$. For $t \geq \tau_0 + T_1 = t_0 + T^*$, we have

$$|x(t)| \ \leq \ k_1 g_\gamma(t - \tau_0)|x_{\tau_0}|_{p,r} + k_1(\|H\|_{[\tau_0,t]} + \sigma_1)$$

$$\leq \; k_1 g_\gamma(T_1)(2+\ell)^{1/p}\gamma + k_1(A+\sigma_1)$$
$$< \; 1 + k_1(A+\sigma_1) = B.$$

Hence solutions of (2.1.1) are B^p_τ-$U.U.B$. The proof is complete.

Example 2.2.1 *Consider D-operator with infinite delay*

$$D(t,\psi) = \psi(0) - \int_0^\infty B(u)q(t-u,\psi(-u))du, \quad t \geq 0, \tag{2.2.4}$$

where $B \in L^1([0,\infty))$ *and* $q \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ *which satisfy the following conditions*

(*i*) $|q(s,x)| \leq b|x|$ $(b>0)$, $|q(s,x)-q(s,y)| \leq b_1|x-y|$ $(b_1>0)$ *for* $x,y \in \mathbb{R}$,

(*ii*) $b|B(u)| \leq r(-u)$ *almost everywhere for* $u \geq 0$,
where $r \in C((-\infty,0],[0.\infty))$ *is nondecreasing,* $\lim_{u\to-\infty} r(u) = 0$, $\int_{-\infty}^0 r(u)du = \ell < 1$ *and* $r(u_1+u_2) \leq r(u_1)r(u_2)$ *for* $u_1,u_2 \leq 0$.

We will prove that $D(t,\psi)$ is B^p_τ-uniformly stable and B^p_τ-uniformly asymptotically stable. Obviously, condition (i) guarantees that $D(t,\psi)$ is continuous in $[0,\infty)\times B^p_\tau$.

Proof. We just discuss the case where $p>1$ (a similar argument holds for the case where $p=1$). For $\tau\in[0,\infty), \varphi \in B^p_\tau$ and $h \in C([\tau,\infty),\mathbb{R})$, let $x(t) = x(\tau,\varphi,h)(t)$ satisfy

$$D(t,x_t) \equiv x(t) - \int_{-\infty}^t B(t-\theta)q(\theta,x(\theta))d\theta = h(t), \quad t \geq \tau. \tag{2.2.5}$$

Then for $\tau \leq s \leq t$, we have, using Hölder inequality,

$$|x(s)| \;\leq\; \int_{-\infty}^s r(\theta-s)|x(\theta)|d\theta + |h(s)|$$

$$\begin{aligned}
&= \int_{-\infty}^{\tau} r(\theta - s)|x(\theta)|d\theta + \int_{\tau}^{s} r(\theta - s)|x(\theta)|d\theta + |h(s)| \\
&\leq \int_{-\infty}^{\tau} r(\theta - \tau)|x(\theta)|d\theta + \ell\|x\|_{[\tau,t]} + \|h\|_{[\tau,t]} \\
&= \int_{-\infty}^{0} r(u)|x_\tau(u)|du + \ell\|x\|_{[\tau,t]} + \|h\|_{[\tau,t]} \\
&\leq \left[\int_{-\infty}^{0} r(u)du\right]^{p-1/p} \left[\int_{-\infty}^{0} r(u)|x_\tau(u)|^p du\right]^{1/p} \\
&\quad + \ell\|x\|_{[\tau,t]} + \|h\|_{[\tau,t]} \\
&\leq |x_\tau|_{p,r} + \ell\|x\|_{[\tau,t]} + \|h\|_{[\tau,t]}.
\end{aligned}$$

Then

$$\|x\|_{[\tau,t]} \leq |x_\tau|_{p,r} + \ell\|x\|_{[\tau,t]} + \|h\|_{[\tau,t]}, \quad t \geq \tau,$$

and

$$\|x\|_{[\tau,t]} \leq \frac{1}{1-\ell}|x_\tau|_{p,r} + \frac{1}{1-\ell}\|h\|_{[\tau,t]}, \quad t \geq \tau. \tag{2.2.6}$$

We have

$$|x(t)| \leq k|x_\tau|_{p,r} + k\|h\|_{[\tau,t]}, \quad t \geq \tau, \tag{2.2.7}$$

where $k = \frac{1}{1-\ell}$. This proves that $D(t, \psi)$ is B_r^p-uniformly stable. Let a be a constant with $0 < a < \ell$. For any given $\gamma > 0$, let $\|x\|_{[\tau,t]} \leq \gamma$. Fix $T > 0$ with $\gamma \int_{-\infty}^{-T} r(u)du < 1$ and $r(-T) < a < \ell$. For $\tau + T \leq s \leq t$,we have, using (2.2.6),

$$\begin{aligned}
|x(s)| &\leq \int_{-\infty}^{s} r(\theta - s)|x(\theta)|d\theta + |h(s)| \\
&= \int_{-\infty}^{\tau} r(\theta - s)|x(\theta)|d\theta + \int_{\tau}^{s-T} r(\theta - s)|x(\theta)|d\theta \\
&\quad + \int_{s-T}^{s} r(\theta - s)|x(\theta)|d\theta + |h(s)| \\
&\leq r(\tau - s)\int_{-\infty}^{\tau} r(\theta - \tau)|x_\tau(\theta - \tau)|d\theta + \gamma\int_{-\infty}^{-T} r(u)du \\
&\quad + \ell\|x\|_{[\tau,t]} + \|h\|_{[\tau,t]}
\end{aligned}$$

$$\begin{aligned}
&\le r(-T)\int_{-\infty}^{0} r(u)|x_\tau(u)|du + 1 + \ell\|x\|_{[\tau,t]} + \|h\|_{[\tau,t]} \\
&\le a|x_\tau|_{p,r} + 1 + \left(\frac{\ell}{1-\ell}|x_\tau|_{p,r} + \frac{\ell}{1-\ell}\|h\|_{[\tau,t]}\right) + \|h\|_{[\tau,t]} \\
&\le (k\ell + a)|x_\tau|_{p,r} + \frac{1}{1-\ell}(\|h\|_{[\tau,t]} + 1).
\end{aligned}$$

$$\begin{aligned}
\|x\|_{[\tau+T,t]} &\le (k\ell + a)|x_\tau|_{p,r} + \frac{1}{1-\ell}(\|h\|_{[\tau,t]} + 1) \\
&\le k(\ell + a)|x_\tau|_{p,r} + \frac{1}{1-\ell}(\|h\|_{[\tau,t]} + 1), \ t \ge \tau + T. \qquad (2.2.8)
\end{aligned}$$

Assume that for nonnegative integer $n \ge 0$, we have, for $t \ge \tau + nT$,

$$\begin{aligned}
\|x\|_{[\tau+nT,t]} \le\ & k\left(\ell^n + \ell^{n-1}a + \cdots + a^n\right)|x_\tau|_{p,r} \\
&+ \frac{1}{1-\ell}(\|h\|_{[\tau,t]} + 1). \qquad (2.2.9)
\end{aligned}$$

Then for $\tau + (n+1)T \le s \le t$, we have

$$\begin{aligned}
|x(s)| &\le \int_{-\infty}^{\tau} r(\theta - s)|x(\theta)|d\theta + \int_{\tau}^{s-T} r(\theta - s)|x(\theta)|d\theta \\
&\quad + \int_{s-T}^{s} r(\theta - s)|x(\theta)|d\theta + |h(s)| \\
&\le r(-(n+1)T)|x_\tau|_{p,r} + 1 + \ell\|x\|_{[\tau+nT,t]} + |h(s)| \\
&\le a^{n+1}|x_\tau|_{p,r} + k(\ell^{n+1} + \ell^n a + \cdots + \ell a^n)|x_\tau|_{p,r} \\
&\quad + \frac{\ell}{1-\ell}(\|h\|_{[\tau,t]} + 1) + \|h\|_{[\tau,t]} + 1 \\
&\le k(\ell^{n+1} + \ell^n a + \cdots + \ell a^n + a^{n+1})|x_\tau|_{p,r} \\
&\quad + \frac{1}{1-\ell}(\|h\|_{[\tau,t]} + 1).
\end{aligned}$$

Then for $t \ge \tau + (n+1)T$,

$$\begin{aligned}
\|x\|_{[\tau+(n+1)T,t]} \le\ & k(\ell^{n+1} + \ell^n a + \cdots + a^{n+1})|x_\tau|_{p,r} \\
&+ \frac{1}{1-\ell}(\|h\|_{[\tau,t]} + 1).
\end{aligned}$$

By induction, (2.2.9) holds for all $n \geq 0$. Then

$$\begin{aligned} \|x\|_{[\tau+nT,t]} &\leq \frac{k}{\ell - a}\ell^{n+1}|x_\tau|_{p,r} + \frac{1}{1-\ell}(\|h\|_{[\tau,t]} + 1) \\ &= k_1\ell^n|x_\tau|_{p,r} + k_1(\|h\|_{[\tau,t]} + 1), \quad t \geq \tau + nT, \end{aligned}$$

where $k_1 = \frac{k\ell}{\ell - a}$ and $n = 0, 1, 2, \ldots$. Define

$$g_\gamma(u) = \ell^n \text{ for } nT \leq u \leq (n+1)T, \quad n = 0, 1, 2, \ldots,$$

Then

$$|x(t)| \leq k_1 g_\gamma(t-\tau)|x_\tau|_{p,r} + k_1(\|h\|_{[\tau,t]} + 1), \quad t \geq \tau.$$

Hence, $D(t, \psi)$ is B^p_r-uniformly asymptotically stable.

2.3 Existence of periodic solutions

In this section, we assume that f and D of (2.1.1) satisfy certain conditions to ensure existence, uniqueness, continuation of solutions and continuous dependence of solutions with respect to initial function $\varphi \in B^p_r$.

Lemma 2.3.1 *(Horn[23]) Let*

1) *$S_0 \subset S_1 \subset S_2$ be convex subsets of a Banach space X ,*

2) *S_0 and S_2 be compact ,*

3) *S_1 be open relative to S_2 ,*

4) *$F : S_2 \longmapsto X$ be a continuous mapping such that for some integer $m > 0$, $F^j(S_1) \subset S_2, 1 \leq j \leq m-1$ and $F^j(S_1) \subset S_0, m \leq j \leq 2m-1$, where F^j is the j-th iterate of F.*

Then F has a fixed point in S_0.

Lemma 2.3.2 *The following set is a compact set in B_r^p.*

$$S = \left\{ \varphi \in B_r^p : \begin{array}{l} |\varphi|_{p,r} \leq \alpha, \|\varphi\|_{(-\infty,0]} \leq \beta \text{ and} \\ |\varphi(s_1) - \varphi(s_2)| \leq K|s_1 - s_2| \text{ for } s_1, s_2 \leq 0 \end{array} \right\},$$

where $\alpha > 0, \beta > 0, K > 0$ and $\|\varphi\|_{(-\infty,0]} = \sup_{s \leq 0} |\varphi(s)|$.

Proof. Let $\{\varphi_n\}, n = 1, 2, \ldots$, be any sequence in S. Since

$$\|\varphi_n\|_{(-\infty,0]} \leq \beta \tag{2.3.1}$$

and

$$|\varphi_n(s_1) - \varphi_n(s_2)| \leq K|s_1 - s_2|, \quad s_1, s_2 \leq 0, \tag{2.3.2}$$

$\{\varphi_n\}$ is uniformly bounded and equicontinuous on each interval $[-k, 0], k = 1, 2, \ldots$. According to Arzela-Ascoli Theorem, there exists a subsequence of $\{\varphi_n\}$, still denoted by $\{\varphi_n\}$, which converges uniformly to some continuous function φ_0 on each interval $[-k, 0]$, that is, $\lim_{n\to\infty} \|\varphi_n - \varphi_0\|_{[-k,0]} = 0, k = 1, 2, \ldots$. Letting $n \to \infty$ in (2.3.2), we have

$$|\varphi_0(s_1) - \varphi_0(s_2)| \leq K|s_1 - s_2|, \quad s_1, s_2 \leq 0.$$

From (2.3.1), we have $|\varphi_n(s)| \leq \beta$ for $s \leq 0$. Letting $n \to \infty$, we get $|\varphi_0(s)| \leq \beta$ for $s \leq 0$. Hence

$$\|\varphi_0\|_{(-\infty,0]} \leq \beta.$$

We shall prove $\lim_{n\to\infty} |\varphi_n - \varphi_0|_{p,r} = 0$. First, easily see that

$$\lim_{n\to\infty} |\varphi_n(0) - \varphi_0(0)|^p = 0. \tag{2.3.3}$$

Next

$$\begin{aligned}\int_{-\infty}^{0} r(s)|\varphi_n(s)-\varphi_0(s)|^p ds &= \int_{-k}^{0} r(s)|\varphi_n(s)-\varphi_0(s)|^p ds\\ &\quad+\int_{-\infty}^{-k} r(s)|\varphi_n(s)-\varphi_0(s)|^p ds\\ &\le \ell\|\varphi_n-\varphi_0\|_{[-k,0]}^p+(2\beta)^p\int_{-\infty}^{-k} r(s)ds\end{aligned}$$

where $k=1,2,\ldots$. For any given $\varepsilon>0$, there exists a sufficiently large $k>0$ such that

$$(2\beta)^p\int_{-\infty}^{-k} r(s)ds<\frac{\varepsilon}{2}$$

and there exists some $N>0$ such that when $n\ge N$,

$$\|\varphi_n-\varphi_0\|_{[-k,0]}<\left(\frac{\varepsilon}{2\ell}\right)^{1/p}.$$

Then we have, when $n\ge N$,

$$\int_{-\infty}^{0} r(s)|\varphi_n(s)-\varphi_0(s)|^p ds<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Hence

$$\lim_{n\to\infty}\int_{-\infty}^{0} r(s)|\varphi_n(s)-\varphi_0(s)|^p ds=0. \tag{2.3.4}$$

By (2.3.3) and (2.3.4), we have

$$\lim_{n\to\infty}|\varphi_n-\varphi_0|_{p,r}=0.$$

Letting $n\to\infty$ in $|\varphi_n|_{p,r}\le\alpha$, we have

$$|\varphi_0|_{p,r}\le\alpha.$$

Hence $\varphi_0\in S$ and S is a compact set in B_r^p. The proof is complete.

Theorem 2.3.1 *Assume that*

1) *$D(t+\omega,\psi) = D(t,\psi)$ and $f(t+\omega,\psi) = f(t,\psi)$ for some $\omega > 0$, any $t \geq t_0$ and any $\psi \in B_r^p$,*

2) *Let $x(t) = x(\tau,\varphi,h)(t)$ $(t \geq \tau)$ be a continuous solution of (2.2.1). For each $\Delta > 0$ there exists a $k^*(\Delta) \geq 0$ such that for any $\ell^* \geq k^*(\Delta)$, $[\|x_t\|_{(-\infty,0]} \leq \Delta$ for all $t \geq \tau, |\varphi(\theta_1)-\varphi(\theta_2)| \leq \ell^*|\theta_1-\theta_2|$ for $\theta_1,\theta_2 \leq 0]$ imply*

$$|x(t_1) - x(t_2)| \leq \ell^*|t_1 - t_2| + |h(t_1) - h(t_2)|,$$

where $t_1, t_2 \geq \tau$ and $\|x_t\|_{(-\infty,0]} = \sup_{-\infty<\theta\leq 0} |x(t+\theta)|$,

3) *For each $\alpha > 0$, there exists an $L(t,\alpha)$ such that $|\psi|_{p,r} \leq \alpha$ implies*

$$|f(t,\psi)| \leq L(t,\alpha),$$

where $L(t,\alpha)$ is continuous with respect to t,

4) *Solutions of (2.1.1) are B_r^p-U.B and B_r^p-U.U.B for $b > 0$.*

Then (2.1.1) has an ω-periodic solution.

Proof. Without loss of generality, let $N((\ell+3)b) \geq (\ell+3)b$ where $\ell = \int_{-\infty}^{0} r(s)ds$ and N is the function defined for B_r^p-U.B in Definition 2.2.1. Define

$$S_0 = \left\{ \varphi \in B_r^p : \begin{array}{l} |\varphi|_{p,r} \leq (\ell+2)b, \|\varphi\|_{(-\infty,0]} \leq N((\ell+3)b) \\ \text{and } |\varphi(\theta_1) - \varphi(\theta_2)| \leq L^*|\theta_1 - \theta_2| \end{array} \right\},$$

$$S_1 = \left\{ \varphi \in B_r^p : \begin{array}{l} |\varphi|_{p,r} < (\ell+3)b, \|\varphi\|_{(-\infty,0]} \leq N((\ell+3)b) \\ \text{and } |\varphi(\theta_1) - \varphi(\theta_2)| \leq L^*|\theta_1 - \theta_2| \end{array} \right\},$$

$$S_2 = \left\{ \varphi \in B_r^p : \begin{array}{l} |\varphi|_{p,r} \leq (\ell+3)N((\ell+3)b), \|\varphi\|_{(-\infty,0]} \leq N((\ell+3)b) \\ \text{and } |\varphi(\theta_1) - \varphi(\theta_2)| \leq L^*|\theta_1 - \theta_2| \end{array} \right\}$$

where $L^* = \max\{k^*(N((\ell+3)b)),\ \max_{t_0 \le t \le t_0+\omega} L(t, (\ell+3)N((\ell+3)b))\}$. Obviously, S_0, S_1 and S_2 are convex subsets of B_r^p and $S_0 \subset S_1 \subset S_2$. Because $S_1 = S_2 \cap \{\varphi \in B_r^p : |\varphi|_{p,r} < (\ell+3)b\}$, therefore S_1 is open relative to S_2. By Lemma 2.3.2, S_0 and S_2 are compact.

For $\varphi \in S_2$, by the uniform boundedness of solutions, we have, for $t \ge t_0$,

$$\begin{aligned} |x_t|_{p,r} &= \left[|x(t)|^p + \int_{-\infty}^0 r(s)|x_t(s)|^p ds\right]^{1/p} \\ &\le \{(\ell+1)[N((\ell+3)N((\ell+3)b))]^p + r(0)[(\ell+3)N((\ell+3)b)]^p\}^{1/p} \\ &< \infty \end{aligned} \tag{2.3.5}$$

where $x(t) = x(t_0, \varphi)(t)$ is a solution of (2.1.1) . Define the mapping $F : S_2 \longmapsto B_r^p$ as follows

$$F(\varphi) = x_{t_0+\omega}(t_0, \varphi).$$

By the continuous dependence of solutions with respect to initial function φ, F is continuous. When $\varphi \in S_1$, by $\|\varphi\|_{(-\infty,0]} \le N((\ell+3)b)$ and B_r^p-uniform boundedness of solutions, we have

$$\|x_t\|_{(-\infty,0]} \le N((\ell+3)b), \quad t \ge t_0, \tag{2.3.6}$$

and

$$\begin{aligned} (|x_t|_{p,r})^p &= |x(t)|^p + \int_{-\infty}^0 r(s)|x_t(s)|^p ds \\ &\le [N((\ell+3)b)]^p + \ell[N((\ell+3)b)]^p \\ &= (\ell+1)[N((\ell+3)b)]^p, \quad t \ge t_0. \end{aligned}$$

Then

$$|x_t|_{p,r} \le (\ell+1)N((\ell+3)b). \tag{2.3.7}$$

By (2.3.7), 1) and 3), we have

$$\left|\frac{d}{dt}D(t,x_t)\right| = |f(t,x_t)| \le L^*, \quad t \ge t_0. \tag{2.3.8}$$

Let

$$D(t,x_t) = H(t), \quad t \ge t_0. \tag{2.3.9}$$

By 2) and (2.3.8), we have, for $t_1, t_2 \ge t_0$,

$$\begin{aligned} |x(t_1) - x(t_2)| &\le L^*|t_1 - t_2| + |H(t_1) - H(t_2)| \\ &\le L^*|t_1 - t_2|. \end{aligned}$$

Then, for $t \ge t_0$,

$$|x_t(\theta_1) - x_t(\theta_2)| \le L^*|\theta_1 - \theta_2|, \quad \theta_1, \theta_2 \le 0. \tag{2.3.10}$$

By (2.3.6), (2.3.7) and (2.3.10), we have, for each positive integer j, $F^j(S_1) \subset S_2$. Furthermore, by B^p_r-uniform ultimate boundedness of solutions, there exists a $T^* = T^*((\ell+3)b) > 0$ such that for $\varphi \in S_1$ and $t \ge t_0 + T^*$.

$$|x(t_0,\varphi)(t)| \le b.$$

Let $T_1 > \max\{T^*, \omega\}$. For $\varphi \in S_1$, we have

$$\begin{aligned} |x_{t_0+T_1}|^p_{p,r} &= |x(t_0+T_1)|^p + \int_{-\infty}^0 r(s)|x_{t_0+T_1}(s)|^p ds \\ &\le b^p + \int_{-\infty}^0 r(s)|x(t_0+T_1+s)|^p ds \\ &= b^p + \int_{t_0+T^*}^{t_0+T_1} r(s-t_0-T_1)|x(s)|^p ds \\ &\quad + \int_{-\infty}^{t_0+T^*} r(s-t_0-T_1)|x(s)|^p ds \\ &\le (1+\ell)b^p + [N((\ell+3)b)]^p \int_{-\infty}^{-(T_1-T^*)} r(s)ds. \end{aligned} \tag{2.3.11}$$

Letting T_1 sufficiently large such that

$$[N((\ell+3)b)]^p \int_{-\infty}^{-(T_1-T^*)} r(s)ds < b^p,$$

we have, by (2.3.11),

$$|x_{t_0+T_1}|_{p,r} < (\ell+2)b. \tag{2.3.12}$$

For $j\omega > T_1$, we have

$$|x_{t_0+j\omega}|_{p,r} \leq (\ell+2)b. \tag{2.3.13}$$

By (2.3.6), (2.3.10) and (2.3.13), we have $F^j(S_1) \subset S_0$ for all integers $j > T_1/\omega$. By Lemma 2.3.1, F has a fixed point φ^* in S_0, that is

$$x_{t_0+\omega}(t_0, \varphi^*) = \varphi^*.$$

We have

$$\frac{d}{dt}D(t+\omega, x_{t+\omega}) = f(t+\omega, x_{t+\omega}), \quad t \geq t_0.$$

Then

$$\frac{d}{dt}D(t, x_{t+\omega}) = f(t, x_{t+\omega}), \quad t \geq t_0.$$

Let $y_t = x_{t+\omega}(t_0, \varphi^*)$. Then

$$\frac{d}{dt}D(t, y_t) = f(t, y_t), \quad t \geq t_0.$$

Note that $y_{t_0} = x_{t_0+\omega}(t_0, \varphi^*) = \varphi^*$. By uniqueness of solutions with respect to initial data, we have

$$y_t = x_t \quad for \ t \geq t_0,$$

that is

$$x_{t+\omega} = x_t \quad for \ t \geq t_0.$$

Hence $x(t)$ is an ω-periodic solution of (2.1.1). The proof is complete.

2.4 B_r^p-$U.B$ and B_r^p-$U.U.B$ of solutions

Theorem 2.4.1 *Let $r \in C^1((-\infty,0],[0,\infty))$ be nondecreasing and satisfy (P_1) and (P_2), and let $D(t,\psi)$ of (2.1.1) be B_r^p-uniformly stable and B_r^p-uniformly asymptotically stable. Suppose that there are wedges W_i $(i=1,2,3)$ and positive constants M and c such that*

$$W_1(|D(t,\psi)|) \le V(t,\psi) \le W_2(|\psi(0)|) + W_3\left(\int_{-\infty}^{0} r(s)|\psi(s)|^p ds\right), \qquad (2.4.1)$$

$$V'(t,x_t) \le -c|x(t)|^p + M. \qquad (2.4.2)$$

Then solutions of (2.1.1) are B_r^p-$U.B$ and B_r^p-$U.U.B$.

Proof. Let $x(t) = x(t_0,\varphi)(t), V(t) = V(t,x_t)$ and $H(t) = D(t,x_t)$. According to Lemma 2.2.1, suffice it to prove that $D(t,\psi)$ is B_r^p-$U.B$ and B_r^p-$U.U.B$. Given $A_1 > 0$, we must find $A_2 > 0$ such that $[t_0 \ge 0, |\varphi|_{p,r} \le A_1, t \ge t_0]$ imply $|H(t)| < A_2$. Let $\bar{t} \in [t_0,t]$ and $V(\bar{t}) = \max_{t_0 \le s \le t} V(s)$. By (2.4.2) we have

$$\begin{aligned} c\int_{t_0}^{t} r(s-t)|x(s)|^p ds &< \ell M - \int_{t_0}^{t} r(s-t)V'(s)ds \\ &\le \ell M - r(0)V(t) + r(t_0-t)V(t_0) \\ &\quad + \int_{t_0}^{t} V(s)r'(s-t)ds \\ &\le \ell M + [V(\bar{t}) - V(t)]r(0). \end{aligned} \qquad (2.4.3)$$

For $|\varphi|_{p,r} \le A_1$, we have, by (2.4.1),

$$\begin{aligned} V(t_0) &\le W_2(|x(t_0)|) + W_3\left(\int_{-\infty}^{0} r(s)|x_{t_0}(s)|^p ds\right) \\ &\le W_2(A_1) + W_3(A_1^p). \end{aligned} \qquad (2.4.4)$$

If there exists $t > t_0$ such that $V(t) \geq V(s)$ for $t_0 \leq s \leq t$, then $V'(t) \geq 0, |x(t)| \leq (\frac{M}{c})^{1/p}$ and by (2.4.3) we have

$$\int_{t_0}^{t} r(s-t)|x(s)|^p ds < \frac{\ell M}{c}. \tag{2.4.5}$$

Then by (2.4.1) we have

$$\begin{aligned} V(t) &\leq W_2(|x(t)|) + W_3\left(\int_{-\infty}^{t_0} r(s-t)|x(s)|^p ds + \int_{t_0}^{t} r(s-t)|x(s)|^p ds\right) \\ &< W_2(U) + W_3\left(A_1^p + \frac{\ell M}{c}\right), \end{aligned} \tag{2.4.6}$$

where $U = (\frac{1+M}{c})^{1/p} > (\frac{M}{c})^{1/p}$. Then for any $t \geq t_0$ with $V(t) \geq V(s)$ for $t_0 \leq s \leq t$, we have, by (2.4.4) and (2.4.6),

$$\begin{aligned} |H(t)| &< \max\left\{W_1^{-1}[W_2(A_1) + W_3(A_1^p)], W_1^{-1}\left[W_2(U) + W_3\left(A_1^p + \frac{\ell M}{c}\right)\right]\right\} \\ &\equiv A_2. \end{aligned}$$

Hence

$$|H(t)| < A_2 \ \ for \ \ all \ \ t \geq t_0.$$

By Lemma 2.2.1, solutions of (2.1.1) are B_r^p-$U.B$. We now prove the B_r^p-$U.U.B$. For given $A_3 > 0$, there exists $A_4 > 0$ such that $[t_0 \geq 0, |\varphi|_{p,r} \leq A_3, t \geq t_0]$ imply $|x(t)| < A_4$. Then we have

$$V(t) \leq W_2(A_4) + W_3(A_3^p + \ell A_4^p), \ \ t \geq t_0. \tag{2.4.7}$$

For any $T > 0$, we have, by (2.4.2),

$$\int_{t-T}^{t} r(s-t)|x(s)|^p ds < \frac{\ell M}{c} + \frac{1}{c}[V(t') - V(t)]r(0), \ \ t \geq t_0 + T, \tag{2.4.8}$$

where $t' \in [t-T, t]$ and $V(t') = \max_{t-T \leq s \leq t} V(s)$. Fix $T > 0$ with $r(-T)(A_3^p + \ell A_4^p) < 1$ and

$$W_2(A_4) + W_3(A_3^p + \ell A_4^p) < T. \tag{2.4.9}$$

For $t \geq t_0 + T$, we have

$$\begin{aligned}
\int_{-\infty}^{t} r(s-t)|x(s)|^p ds &= \int_{-\infty}^{t_0} r(s-t)|x(s)|^p ds + \int_{t_0}^{t-T} r(s-t)|x(s)|^p ds \\
&\quad + \int_{t-T}^{t} r(s-t)|x(s)|^p ds \\
&\leq r(-T)(A_3^p + \ell A_4^p) + \int_{t-T}^{t} r(s-t)|x(s)|^p ds \\
&< 1 + \int_{t-T}^{t} r(s-t)|x(s)|^p ds \\
&< 1 + \frac{\ell M}{c} + \frac{r(0)}{c}[V(t') - V(t)], \qquad (2.4.10)
\end{aligned}$$

where $t' \in [t-T, t]$ and $V(t') = \max_{t-T \leq s \leq t} V(s)$. Define

$$\hat{I}_i = [t_0 + (i-1)T, t_0 + iT], \quad i = 1, 2, \ldots. \qquad (2.4.11)$$

From (2.4.2), (2.4.7) and (2.4.9), there must be a $t \in \hat{I}_i$ so that $|x(t)| \leq U = (\frac{1+M}{c})^{1/p}$. Choose an integer $N > 1$ with

$$W_2(A_4) + W_3(A_3^p + \ell A_4^p) - (N-1) < 0. \qquad (2.4.12)$$

If there is a $t \in (t_0 + (i-1)T, t_0 + iT]$ such that $V(t) \geq V(s)$ for all $s \in \hat{I}_i$, then take $I_i = \hat{I}_i$. If no such t exists, then find the first $\hat{t}_i \in \hat{I}_i$ such that $|x(\hat{t}_i)| \leq U$ and then take $I_i = [\hat{t}_i, t_0 + iT]$. Find $t_i \in I_i$ with $V(t_i) = \max_{s \in I_i} V(s)$. This construction will then satisfy

$$|x(t_i)| \leq U,$$

$$V(s) \leq V(t_0 + (i-1)T) \leq V(t_{i-1}) \quad for \quad s \in \hat{I}_i - I_i$$

and

$$V(t_i) = \max_{s \in I_i} V(s). \qquad (2.4.13)$$

We claim that

$$V(t_i) < W_2(U) + W_3\left(1 + \frac{\ell M + r(0)}{c}\right) \equiv P_0, \quad i \geq 2N. \tag{2.4.14}$$

Indeed, for $j \geq 3$, either

1) $V(t_j) + 1 \geq V(s)$ *for all* $s \in [t_j - T, t_j]$, or

2) there is some $s_j \in [t_j - T, t_j]$ so that $V(t_j) + 1 < V(s_j)$.

If 1) holds, then $\max_{t_j - T \leq s \leq t_j} V(s) - V(t_j) \leq 1$. From (2.4.1) and (2.4.10) we have

$$\begin{aligned} V(t_j) &\leq W_2(U) + W_3\left(\int_{-\infty}^{t_j} r(s - t_j)|x(s)|^p ds\right) \\ &< W_2(U) + W_3\left(1 + \frac{\ell M + r(0)}{c}\right) \equiv P_0. \end{aligned} \tag{2.4.15}$$

If 2) holds, by (2.4.13) it must be that $s_j \notin I_j$. Then we have

$$V(s_j) \leq V(t_{j-1}) \quad or \quad V(s_j) \leq V(t_{j-2}).$$

Then

$$V(t_j) < V(t_{j-1}) - 1 \quad (or \quad < V(t_{j-2}) - 1). \tag{2.4.16}$$

By (2.4.7) we have

$$V(t_j) < W_2(A_4) + W_3(A_3^p + \ell A_4^p) - 1. \tag{2.4.17}$$

According to the above argument, for $j \geq 3$ either (2.4.15) or (2.4.16) holds. Furthermore (2.4.16) must hold if (2.4.15) doesn't hold. Thus for $j \geq 2N$ we have that if $V(t_{j-1}) < P_0$ (or $V(t_{j-2}) < P_0$), then $V(t_j) < P_0$ and the proof is complete. Otherwise

$$V(t_{j-1}) < V(t_{j-2}) - 1 \ \ (or \quad < V(t_{j-3}) - 1)$$

or

$$V(t_{j-2}) < V(t_{j-3}) - 1 \ \ (or \ \ < V(t_{j-4}) - 1).$$

Then

$$V(t_j) < V(t_{j-2}) - 2 \ \ (or \ \ < V(t_{j-3}) - 2) \tag{2.4.18}$$

or

$$V(t_j) < V(t_{j-3}) - 2 \ \ (or \ \ < V(t_{j-4}) - 2). \tag{2.4.19}$$

By (2.4.7) we have

$$V(t_j) < W_2(A_4) + W_3(A_3^p + \ell A_4^p) - 2. \tag{2.4.20}$$

If we can repeat this argument for n consecutive times, then we have

$$V(t_j) < W_2(A_4) + W_3(A_3^p + \ell A_4^p) - n. \tag{2.4.21}$$

But $n < N - 1$ since (2.4.12). Therefore this argument can be repeated consecutively for no more than $(N-2)$ times. Hence $V(t_j) < P_0$ for $j \geq 2N$ and our claim is true . Now let $s > t_0 + 2NT$. Then $s \in \hat{I}_i$ with $i \geq 2N+1$ and then either $s \in I_i$ or $s \in \hat{I}_i - I_i$. Thus

$$V(s) \leq V(t_i) < P_0$$

or

$$V(s) \leq V(t_0 + (i-1)T) < P_0.$$

Then

$$W_1(|H(s)|) \leq V(s) < P_0, \quad s \geq t_0 + 2NT.$$

Hence

$$|H(s)| < W_1^{-1}(P_0) \ \ for \ \ all \ \ s \geq t_0 + 2NT.$$

By Lemma 2.2.1, solutions of (2.1.1) are B_r^p-*U.U.B*. The proof is complete.

Theorem 2.4.2 *Let $r \in C((-\infty, 0], [0, \infty))$ be nondecreasing and satisfy (P_1) and (P_2), and let $D(t, \psi)$ of (2.1.1) be B_r^p-uniformly stable and B_r^p-uniformly asymptotically stable. Suppose that there are wedges W_i $(i = 1, 2, 3, 4)$ and positive constants M and U such that*

$$W_1(|D(t,\psi)|) \leq V(t,\psi) \leq W_2(|\psi(0)|) + W_3\left(\int_{-\infty}^{0} r(s)|\psi(s)|^p ds\right), \quad (2.4.22)$$

$$V'(t, x_t) \leq -W_4(|x(t)|) + M, \quad (2.4.23)$$

$$W_4(U) > M \quad \text{and} \quad W_1(u) \to \infty \text{ as } u \to \infty, \quad (2.4.24)$$

for any given $\lambda > 0$ there exists $J_\lambda > 0$ such that when $u \geq J_\lambda$,

$$W_1(u) > W_2(U) + 1 + W_3[\lambda^p + \ell k_2^p(\lambda + u)^p] \quad (2.4.25)$$

where $k_2 = \max\{k, k_1\}$, k and k_1 satisfy (2.2.2) and (2.2.3) respectively. Then solutions of (2.1.1) are B_r^p-U.B and B_r^p-U.U.B.

Proof. According to Lemma 2.2.1, suffice it to prove that $D(t, \psi)$ is B_r^p-*U.B* and B_r^p-*U.U.B*. Given $A_1 > 0$ with $A_1 > U$, we must find $A_2 > 0$ such that $[t_0 \geq 0, |\varphi|_{p,r} \leq A_1, t \geq t_0]$ imply $|D(t, x_t)| < A_2$. Denote $D(t, x_t)$ and $V(t, x_t)$ by $H(t)$ and $V(t)$ respectively. Fix $t_0 \geq 0$ and $|\varphi|_{p,r} \leq A_1$. Because D is B_r^p-uniformly stable, we have, for any $\bar{t} \geq t_0$ and $t_0 \leq s \leq \bar{t}$,

$$\begin{aligned} |x(s)| &\leq k|x_{t_0}|_{p,r} + k(\|H\|_{[t_0,s]} + \sigma) \\ &\leq kA_1 + k(\|H\|_{[t_0,\bar{t}]} + \sigma). \end{aligned}$$

$$\|x\|_{[t_0,\bar{t}]} \leq kA_1 + k(\|H\|_{[t_0,\bar{t}]} + \sigma) \quad , \bar{t} \geq t_0. \quad (2.4.26)$$

If there is $t > t_0$ with $V(t) \geq V(s)$ for all $s \in [t_0, t]$, then

$$|x(t)| \leq U$$

since $V'(t) \geq 0$. Choose $t^* \in [t_0, t]$ so that $|H(t^*)| = \|H\|_{[t_0,t]}$. Then

$$\begin{aligned} W_1(|H(t^*)|) &\leq V(t^*) \\ &\leq V(t) \\ &\leq W_2(U) + W_3\left(\int_{-\infty}^{t_0} r(u-t)|x(u)|^p du + \int_{t_0}^{t} r(u-t)|x(u)|^p du\right) \\ &\leq W_2(U) + W_3\left(A_1^p + \ell\|x\|_{[t_0,t]}^p\right) \\ &\leq W_2(U) + 1 + W_3\left[(A_1+\sigma)^p + \ell k_2^p(A_1 + \sigma + |H(t^*)|)^p\right]. \end{aligned}$$

From (2.4.25) we have

$$\|H\|_{[t_0,t]} = |H(t^*)| < J_{A_1+\sigma}. \tag{2.4.27}$$

If there is a sequence $t_n \to \infty$ as $n \to \infty$ such that $V(t_n) \geq V(s)$ for $s \in [t_0, t_n]$, then easily see that

$$|H(s)| < J_{A+\sigma} \quad for \quad all \quad s \in [t_0, \infty)$$

since $J_{A+\sigma}$ is independent of t_n. Otherwise, there exists some $t' > t_0$ such that $V(t') \geq V(s)$ for all $s \in [t_0, \infty)$. Then

$$\|H\|_{[t_0.t']} < J_{A_1+\sigma}. \tag{2.4.28}$$

For $s > t'$ with $|H(s)| > \|H\|_{[t_0,t']}$, we have

$$\begin{aligned} W_1(|H(s)|) &\leq V(s) \\ &\leq V(t') \\ &\leq W_2(U) + W_3[(A_1^p + \ell k^p(A_1 + \sigma + \|H\|_{[t_0,t']})^p] \\ &\leq W_2(U) + 1 + W_3[(A_1+\sigma)^p + \ell k_2^p(A_1 + \sigma + |H(s)|)^p]. \end{aligned} \tag{2.4.29}$$

By (2.4.25) and (2.4.28) we have

$$|H(s)| < J_{A_1+\sigma} \quad for \ all \quad s \geq t_0. \tag{2.4.30}$$

If $V(t_0) \geq V(s)$ for all $s \geq t_0$, then

$$\begin{aligned} W_1(|H(s)|) &\leq V(s) \\ &\leq V(t_0) \\ &\leq W_2(|x(t_0)| + W_3\left[\int_{-\infty}^{t_0} r(u-t_0)|x(u)|^p du\right] \\ &\leq W_2(|\varphi|_{p,r}) + W_3(|\varphi|_{p,r}^p) \\ &\leq W_2(A_1) + W_3(A_1^p). \end{aligned}$$

Then

$$|H(s)| \leq W_1^{-1}[W_2(A_1) + W_3(A_1^p)] \quad for \quad all \quad s \geq t_0.$$

Set

$$A_2 = \max\left\{J_{A_1+\sigma}, W_1^{-1}[W_2(A_1) + W_3(A_1^p)]\right\}.$$

$$|H(t)| < A_2 \quad for \quad all \quad t \geq t_0. \tag{2.4.31}$$

This proves B_r^p-*U.B*. We now prove the B_r^p-*U.U.B*. We must show that for each $A_3 > 0$ there is $K > 0$ so that $[t_0 \geq 0, |\varphi|_{p,r} \leq A_3, t \geq t_0 + K]$ imply that

$$|H(t)| < A \equiv W_1^{-1}\left\{W_2(U) + W_3\left[1 + \ell k_1^p(1 + \sigma_1 + J_{1+\sigma_1})^p\right]\right\}.$$

Given $A_3 > 0$, there is $A_4 > 0$ so that $[t \geq t_0 \geq 0 \ \ and \ \ |\varphi|_{p,r} \leq A_3]$ imply that $|H(t)| < A_4$ and $|x(t)| < A_4$. Set $\gamma = \max\{A_3, A_4\}$. Then $|x_{t_0}|_{p,r} < \gamma$, $|x(t)| < \gamma$ $(t \geq t_0)$ and $|H(t)| < \gamma$ $(t \geq t_0)$. From (2.4.22) we have

$$V(t) \leq W_2(|x(t)|) + W_3\left[\int_{-\infty}^{t} r(u-t)|x(u)|^p du\right]$$

$$\begin{aligned}
&\leq W_2(\gamma) + W_3\left[\int_{-\infty}^{t_0} r(u-t)|x(u)|^p du + \int_{t_0}^{t} r(u-t)|x(u)|^p du\right] \\
&\leq W_2(\gamma) + W_3[|\varphi|_{p,r}^p + \ell\gamma^p] \\
&\leq W_2(\gamma) + W_3\left[(1+\ell)\gamma^p\right], \quad t \geq t_0. \qquad (2.4.32)
\end{aligned}$$

We have, for $t \geq t_0$,

$$\begin{aligned}
|x_t|_{p,r}^p &= |x(t)|^p + \int_{-\infty}^{0} r(s)|x_t(s)|^p ds \\
&\leq \gamma^p + \int_{-\infty}^{t} r(u-t)|x(u)|^p du \\
&= \gamma^p + \int_{-\infty}^{t_0} r(u-t)|x(u)|^p du + \int_{t_0}^{t} r(u-t)|x(u)|^p du \\
&\leq \gamma^p + \gamma^p + \ell\gamma^p \\
&= (2+\ell)\gamma^p.
\end{aligned}$$

Then

$$|x_t|_{p,r} \leq (2+\ell)^{1/p}\gamma \quad for \quad t \geq t_0. \qquad (2.4.33)$$

Fix $T > 0$ with

$$g_\gamma(T)(2+\ell)^{1/p}\gamma < 1,$$

$$r(-T)(2+\ell)\gamma^p < 1$$

and

$$W_2(\gamma) + W_3[(1+\ell)\gamma^p] - [W_4(U) - M]T < 0. \qquad (2.4.34)$$

For $t \geq t_0 + T$,

$$\begin{aligned}
V(t) &\leq W_2(|x(t)|) + W_3\left[\int_{-\infty}^{t-T} r(u-t)|x(u)|^p du + \int_{t-T}^{t} r(u-t)|x(u)|^p du\right] \\
&\leq W_2(|x(t)|) + W_3\left[r(-T)\int_{-\infty}^{t-T} r(u-(t-T))|x(u)|^p du + \ell\|x\|_{[t-T,t]}^p\right] \\
&\leq W_2(|x(t)|) + W_3\left[r(-T)|x_{t-T}|_{p,r}^p + \ell\|x\|_{[t-T,t]}^p\right]
\end{aligned}$$

$$\begin{aligned}
&\leq W_2(|x(t)|) + W_3\left[r(-T)(2+\ell)\gamma^p + \ell\|x\|^p_{[t-T,t]}\right] \\
&\leq W_2(|x(t)|) + W_3\left[1 + \ell\|x\|^p_{[t-T,t]}\right]. \qquad (2.4.35)
\end{aligned}$$

For $s \geq t_0 + T$,

$$\begin{aligned}
|x(s)| &\leq k_1 g_\gamma(T)|x_{s-T}|_{p,r} + k_1(\|H\|_{[s-T,s]} + \sigma_1) \\
&\leq k_1 g_\gamma(T)(2+\ell)^{1/p}\gamma + k_1(\|H\|_{[s-T,s]} + \sigma_1) \\
&\leq k_1(1+\sigma_1) + k_1\|H\|_{[s-T,s]}. \qquad (2.4.36)
\end{aligned}$$

For $t \geq t_0 + 2T$ and $t \geq s \geq t - T \geq t_0 + T$, we have, by (2.4.36) ,

$$|x(s)| \leq k_1(1+\sigma_1) + k_1\|H\|_{[t-2T,t]}.$$

Then

$$\|x\|_{[t-T,t]} \leq k_1(1+\sigma_1) + k_1\|H\|_{[t-2T,t]}, \quad t \geq t_0 + 2T. \qquad (2.4.37)$$

By (2.4.35) and (2.4.37), we have

$$V(t) \leq W_2(|x(t)|) + W_3\left\{1 + \ell k_1^p\left[(1+\sigma_1) + \|H\|_{[t-2T,t]}\right]^p\right\}, t \geq t_0 + 2T. \qquad (2.4.38)$$

From (2.4.23), (2.4.32) and (2.4.34), we easily see that if $b - a \geq T$, there must be a $t \in [a,b] \subset [t_0, \infty)$ so that $|x(t)| \leq U$. Choose an integer $N > 1$ with

$$W_2(\gamma) + W_3[(1+\ell)\gamma^p] - (N-1) < 0. \qquad (2.4.39)$$

Define

$$\hat{I}_i = [t_0 + 2(i-1)T, t_0 + 2iT], \quad i = 1, 2, \cdots. \qquad (2.4.40)$$

If there is a $t \in (t_0+2(i-1)T, t_0+2iT]$ such that $V(t) \geq V(s)$ for all $s \in \hat{I}_i$, then take $I_i = \hat{I}_i$. If no such t exists, then find the first $\hat{t}_i \in \hat{I}_i$ such that $|x(\hat{t}_i)| \leq U$

and then take $I_i = [\hat{t}_i, t_0 + 2iT]$. Find $t_i \in I_i$ with $V(t_i) = \max V(s)$ for $s \in I_i$. This construction will then satisfy

$$|x(t_i)| \leq U,$$

$$V(s) \leq V(t_0 + 2(i-1)T) \leq V(t_{i-1}) \quad for \quad s \in \hat{I}_i - I_i$$

and

$$V(t_i) = \max_{s \in I_i} V(s). \tag{2.4.41}$$

We claim that

$$V(t_i) < W_2(U) + W_3[1 + \ell k_1^p (1 + \sigma_1 + J_{1+\sigma_1})^p], \quad i \geq 2N. \tag{2.4.42}$$

Indeed, for $j \geq 3$, either

1) $V(t_j) + 1 \geq V(s) \quad for \quad all \quad s \in [t_j - 2T, t_j]$, or

2) there is some $s_j \in [t_j - 2T, t_j]$ so that $V(t_j) + 1 < V(s_j)$.

If 1) holds, by (2.4.38) we have, for $s \in [t_j - 2T, t_j]$,

$$\begin{aligned} W_1(|H(s)|) &\leq V(s) \\ &\leq V(t_j) + 1 \\ &\leq W_2(U) + 1 + W_3 \left\{1 + \ell k_1^p [(1 + \sigma_1) + \|H\|_{[t_j - 2T, t_j]}]^p\right\}. \end{aligned} \tag{2.4.43}$$

Then

$$\begin{aligned} W_1(\|H\|_{[t_j - 2T, t_j]}) \leq \; & W_2(U) + 1 \\ & + W_3 \left\{1 + \ell k_1^p [(1 + \sigma_1) + \|H\|_{[t_j - 2T, t_j]}]^p\right\}. \end{aligned} \tag{2.4.44}$$

By (2.4.25) we have

$$\|H\|_{[t_j-2T,t_j]} < J_{1+\sigma_1}.$$

By (2.4.43) we have

$$V(t_j) < W_2(U) + W_3\left\{1 + \ell k_1^p[(1+\sigma_1) + J_{1+\sigma_1}]^p\right\} \equiv P_0. \tag{2.4.45}$$

If 2) holds, by (2.4.41) it must be that $s_j \notin I_j$. Then for $I_j = \hat{I}_{j-1}$. By (2.4.41) we have

$$V(s_j) \le V(t_{j-1}) \quad or \quad V(s_j) \le V(t_{j-2}).$$

By an analogous argument we have the same result for $I_j \neq \hat{I}_j$. Thus we always have

$$V(t_j) \le V(t_{j-1}) - 1 \quad (or \quad < V(t_{j-2}) - 1). \tag{2.4.46}$$

By (2.4.32) we have

$$V(t_j) \le W_2(\gamma) + W_3[(1+\ell)\gamma^p] - 1. \tag{2.4.47}$$

According to the above argument, for $j \ge 3$, either (2.4.45) or (2.4.46) holds. Furthermore (2.4.46) must hold if (2.4.45) doesn't hold. Thus for $j \ge 2N$ we have that if $V(t_{j-1}) < P_0$ (or $V(t_{j-2}) < P_0$), then $V(t_j) < P_0$ and the proof is complete. Otherwise

$$V(t_{j-1}) < V(t_{j-2}) - 1 \quad (or \quad < V(t_{j-3}) - 1)$$

or

$$V(t_{j-2}) < V(t_{j-3}) - 1 \quad (or \quad < V(t_{j-4}) - 1).$$

Then

$$V(t_j) < V(t_{j-2}) - 2 \quad (or \quad < V(t_{j-3}) - 2) \tag{2.4.48}$$

or

$$V(t_j) < V(t_{j-1}) - 2 \quad (or \quad < V(t_{j-4}) - 2). \tag{2.4.49}$$

By (2.4.32) we have

$$V(t_j) < W_2(\gamma) + W_3[(1+\ell)\gamma^p] - 2. \tag{2.4.50}$$

If we can repeat this argument for n consecutive times, then we have

$$V(t_j) < W_2(\gamma) + W_3[(1+\ell)\gamma^p] - n. \tag{2.4.51}$$

But $n < N - 1$ since (2.4.39). Therefore this argument can be repeated consecutively for no more than $(N-2)$ times. Hence $V(t_j) < P_0$ for $j \geq 2N$ and our claim is true. Now let $s \geq t_0 + 4NT$. Thus $s \in \hat{I}_i$ with $i \geq 2N+1$ and then either $s \in I_i$ or $s \in \hat{I}_i - I_i$. Thus

$$V(s) \leq V(t_i) < P_0$$

or

$$V(s) \leq V(t_0 + 2(i-1)T) < P_0.$$

Then

$$W_1(|H(s)|) \leq V(s) < P_0.$$

Hence

$$|H(s)| < W_1^{-1}(P_0) \equiv A \quad for \;\; all \quad s \geq t_0 + 4NT.$$

The proof is complete.

2.5 Applications

Consider the scalar equation

$$\frac{d}{dt}\left[x(t) - \int_{-\infty}^{t} B(t-s)q(x(s))ds\right] =$$
$$-ax^m(t) + \int_{-\infty}^{t} C(t-s)x^n(s)ds + E(t), \quad t \geq t_0 \geq 0, \qquad (2.5.1)$$

where $B, C \in L^1([0,\infty)); q, E \in C(\mathbb{R},\mathbb{R})$; m, n are positive integers, m is odd and $a > 0$ is a constant. We assume

(C_1) there exist positive constants b and b_1 such that $|q(x)| \leq b|x|$ and $|q(x) - q(y)| \leq b_1|x-y|$ for all $x, y \in \mathbb{R}$.

(C_2) $b|B(u)| \leq r(-u)$ and $|C(u)| \leq r(-u)$ almost everywhere for $u \geq 0$, where $r \in C^1((-\infty,0],[0,\infty))$ is nondecreasing, $\ell = \int_{-\infty}^{0} r(u)du < 1$, $r(u_1+u_2) \leq r(u_1)r(u_2)$ for $u_1, u_2 \leq 0$ and $\int_{-\infty}^{u} r(s)ds \leq J_0 r(u)(J_0 > 0)$.

(C_3) $|E(t)| \leq N$ where $N > 0$ is a constant.

(C_4) $E(t+\omega) = E(t)$ for all $t \in \mathbb{R}$, where $\omega > 0$ is a constant.

(C_5) there exists $A(u) \geq 0$ with $\int_0^\infty A(u)du = A_0 < \infty$ such that

$$|B(\lambda+u) - B(u)| \leq A(u)\lambda \quad for \quad \lambda, u \geq 0.$$

Using Hölder inequality and Lebesgue dominated convergence theorem, it is easy to prove that condition (C_1)–(C_3) guarantee that (2.5.1) has a unique solution through any $(t_0,\varphi)\in [0,\infty) \times B_r^{m+1}$ and bounded solutions of (2.5.1) exist on $[t_0,\infty)$. Furthermore, if solutions of (2.5.1) are B_r^{m+1}-$U.B$, then $x(t_0,\varphi)(t)$ is continuous dependent on φ.

Proposition 2.5.1 *Assume that* (C_1)*-*(C_3) *hold and*

$$a > \frac{1}{1-\ell}\left[\ell + \ell^{3m+2-n/m+1} + N(1+\ell^{2m+1/m+1})\right]. \qquad (2.5.2)$$

Then solutions of (2.5.1) *are* B_r^{m+1}*-U.B and* B_r^{m+1}*-U.U.B.*

Proof. By Example 2.2.1 ,

$$D(t,\psi) = \psi(0) - \int_{-\infty}^{0} B(-u)q(\psi(u))du$$

is B_r^{m+1}-uniformly stable and B_r^{m+1}-uniformly asymptotically stable. Define

$$V(t,\psi) = \frac{1}{2}D^2(t,\psi) + K\int_{-\infty}^{0}\int_{-\infty}^{s} r(u)du|\psi(s)|^{m+1}ds.$$

where $K = \frac{a}{m+1} + \frac{n}{n+1} + \ell^{2m+1-n/m+1} + N\ell^{m/m+1}$. Using Hölder inequality, we have

$$\begin{aligned} V(t,x_t) &= \frac{1}{2}D^2(t,x_t) + K\int_{-\infty}^{t}\int_{-\infty}^{s-t} r(u)du|x(s)|^{m+1}ds \\ &\leq |x(t)|^2 + \left[\int_{-\infty}^{t} r(s-t)|x(s)|ds\right]^2 \\ &\quad + KJ_0\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds \\ &\leq |x(t)|^2 + \ell^{2m/m+1}\left[\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds\right]^{2/m+1} \\ &\quad + J_0K\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds \\ &\leq |x(t)|^2 + \ell\left[\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds\right]^{2/m+1} \\ &\quad + J_0K\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds. \end{aligned}$$

Then

$$W_1(|D(t,x_t)|) \leq V(t,x_t) \leq W_2(|x(t)|) + W_3\left(\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds\right)$$

where $W_1(z) = \frac{1}{2}z^2, W_2(z) = z^2$ and $W_3(z) = \ell z^{2/m+1} + J_0 K z$.

$$\begin{aligned}
V'(t, x_t) &= D(t, x_t)\frac{d}{dt}D(t, x_t) + K\ell|x(t)|^{m+1} - K\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds \\
&= \left[x(t) - \int_{-\infty}^{t} B(t-s)q(x(s))ds\right] \\
&\quad \times \left[-ax^m(t) + \int_{-\infty}^{t} C(t-s)x^n(s)ds + E(t)\right] \\
&\quad + K\ell|x(t)|^{m+1} - K\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds \\
&\le -a|x(t)|^{m+1} \\
&\quad + a|x(t)|^m\int_{-\infty}^{t} r(s-t)|x(s)|ds + |x(t)|\int_{-\infty}^{t} r(s-t)|x(s)|^n ds \\
&\quad + \int_{-\infty}^{t} r(s-t)|x(s)|ds\int_{-\infty}^{t} r(s-t)|x(s)|^n ds + N|x(t)| \\
&\quad + N\int_{-\infty}^{t} r(s-t)|x(s)|ds + K\ell|x(t)|^{m+1} \\
&\quad - K\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds.
\end{aligned}$$

Using Hölder inequality below

$$y_1^{\alpha_1}y_2^{\alpha_2} \le \alpha_1 y_1 + \alpha_2 y_2$$

where $y_i > 0, \alpha_i > 0 \ (i = 1, 2)$ and $\alpha_1 + \alpha_2 = 1$, we have

$$\begin{aligned}
& a|x(t)|^m\int_{-\infty}^{t} r(s-t)|x(s)|ds \\
&= a\int_{-\infty}^{t} r(s-t)|x(t)|^m|x(s)|ds \\
&\le a\int_{-\infty}^{t} r(s-t)\left(\frac{m}{m+1}|x(t)|^{m+1} + \frac{1}{m+1}|x(s)|^{m+1}\right)ds \\
&= \frac{am\ell}{m+1}|x(t)|^{m+1} + \frac{a}{m+1}\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds,
\end{aligned}$$

$$\begin{aligned}
&|x(t)|\int_{-\infty}^{t} r(s-t)|x(s)|^n ds \\
&\quad \le \int_{-\infty}^{t} r(s-t)\left(\frac{1}{n+1}|x(t)|^{n+1}+\frac{n}{n+1}|x(s)|^{n+1}\right)ds \\
&\quad \le \int_{-\infty}^{t} r(s-t)\left(\frac{1}{n+1}|x(t)|^{m+1}+\frac{n}{n+1}|x(s)|^{m+1}+1\right)ds \\
&\quad \le \frac{\ell}{n+1}|x(t)|^{m+1}+\frac{n}{n+1}\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds+\ell,
\end{aligned}$$

$$\begin{aligned}
&\int_{-\infty}^{t} r(s-t)|x(s)|ds\int_{-\infty}^{t} r(s-t)|x(s)|^n ds \\
&\quad \le \ell^{m/m+1}\left(\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds\right)^{1/m+1} \\
&\quad \times \ell^{m+1-n/m+1}\left(\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds\right)^{n/m+1} \\
&\quad \le \ell^{2m+1-n/m+1}\left(\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds\right)^{n+1/m+1} \\
&\quad \le \ell^{2m+1-n/m+1}\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds+\ell^{2m+1-n/m+1},
\end{aligned}$$

$$N|x(t)| \le N(|x(t)|^{m+1}+1) = N|x(t)|^{m+1}+N$$

and

$$\begin{aligned}
N\int_{-\infty}^{t} r(s-t)|x(s)|ds \;&\le\; N\ell^{m/m+1}\int_{-\infty}^{t} r(s-t)|x(s)|^{m+1}ds \\
&\quad + N\ell^{m/m+1}.
\end{aligned}$$

Then

$$V'(t,x_t) \le -\left[a-\left(\frac{am\ell}{m+1}+\frac{\ell}{n+1}+N+K\ell\right)\right]|x(t)|^{m+1}+2(N+1).$$

By (2.5.2) ,

$$a - \left(\frac{am\ell}{m+1} + \frac{\ell}{n+1} + N + K\ell \right) > 0.$$

By Theorem 2.4.1, solutions of (2.5.1) are B_r^{m+1}-$U.B$ and B_r^{m+1}-$U.U.B$. The proof is complete.

From Theorem 2.3.1, we have

Proposition 2.5.2 *Assume that* (C_1)*-*(C_5) *and* (2.5.2) *hold. Then* (2.5.1) *has an* ω*-periodic solution.*

Proof. We just verify 2) of Theorem 2.3.1. Let $x(t) = x(\tau, \varphi, h)(t)(t \geq \tau)$ be a continuous solution of (2.2.1). For each $\Delta > 0$, let $k^*(\Delta) = (r(0) + bA_0)\Delta$. For any $\ell^* \geq k^*(\Delta)$, if $\|x_t\|_{(-\infty,0]} \leq \Delta$, then, for $t_1 \geq t_2 \geq \tau$,

$$\begin{aligned}
|x(t_1) - x(t_2)| &\leq \int_{t_2}^{t_1} |B(t_1 - s)||q(x(s))|ds \\
&\quad + \int_{-\infty}^{t_2} |B(t_1 - s) - B(t_2 - s)||q(x(s))|ds \\
&\quad + |h(t_1) - h(t_2)| \\
&\leq \Delta \int_{t_2}^{t_1} r(s - t_1)ds + b\Delta \int_{-\infty}^{t_2} |B(t_1 - s) - B(t_2 - s)|ds \\
&\quad + |h(t_1) - h(t_2)| \\
&\leq r(0)\Delta|t_1 - t_2| + b\Delta \int_0^{\infty} |B(t_1 - t_2 + u) - B(u)|du \\
&\quad + |h(t_1) - h(t_2)| \\
&\leq (r(0) + bA_0)\Delta|t_1 - t_2| + |h(t_1) - h(t_2)| \\
&\leq \ell^*|t_1 - t_2| + |h(t_1) - h(t_2)|.
\end{aligned}$$

We can easily verify the other conditions of Theorem 2.3.1. By Theorem 2.3.1, (2.5.1) has an ω-periodic solution. The proof is complete.

Chapter 3

Stability in Neutral Differential Equations of Nonlinear *D-operator* with Infinite Delay

3.1 Introduction

M.A.Cruz and J.K.Hale introduced uniformly stable *D-operator* with finite delay in 1970 [16] (or stable *D-operator* [1]) and studied the stability of neutral differential equations with linear uniformly stable *D-operator*. A linear and continuous *D-operator* (atomic at zero) is uniformly stable (or stable [1]) if and only if there are constants $a > 0$ and $b > 0$ such that for any $h \in C([0,\infty), \mathbb{R}^n)$, any solution y of the nonhomogeneous equation

$$Dy_t = h(t), \quad t \geq 0$$

satisfies

$$\|y_t\| \leq be^{-at}\|y_0\| + b \sup_{0 \leq u \leq t} |h(u)|, \quad t \geq 0 \tag{3.1.1}$$

where $\|y_t\| = \sup_{-r \leq u \leq 0} |y_t(u)|$ for some $r \geq 0$ [1]. (For details, cf. [1])

In this chapter, we will develop a theory on uniformly asymptotic stability in neutral functional differential equations ($NFDE$) of nonlinear *D-operator* type with infinite delay. In section 2 , we introduce new applicable definitions of weak-uniformly stable and weak-uniformly asymptotically stable *D-operators* which generalize corresponding definitions of [16, 1] in a nontrivial way. Some examples will be given to demonstrate that our new definitions are available and that our results are applicable to a broad class of neutral equations which contain some "real" nonlinear *D-operators* with infinite delay such as

$$D(t,\psi) = \psi(0) - \int_0^\infty B(u)\psi^n(-u)du.$$

We observe that when operator $D(t,\psi)$ is weak-uniformly stable and weak-uniformly asymptotically stable, the stability of zero solution of $NFDE$ can be determined by asymptotic behavior of $D(t,x_t)$. We establish Lemma 3.2.1 to formulate this important fact and use it to prove the main theorems of section 3. Lemma 3.2.2 is built exclusively for Theorem 3.3.1 (in section 3).

Using Liapunov functional or function and Razumikhin techniques, we establish three uniformly asymptotic stability theorems in section 3. Theorem 3.3.1 is an extension of Burton's theorem for retarded equation with unbounded delay ((*d*) of Theorem 8 of [25]) to $NFDE$ of nonlinear *D-operator* type with infinite delay . Theorem 3.3.2 and Theorem 3.3.3 are also extensions of corresponding results for neutral equations with finite delay respectively due to Cruz and Hale [16, 1] and Lopes [26] to $NFDE$ of nonlinear *D-operator* type with infinite delay.

We apply our theorems to discuss $U.A.S$ for some neutral *Volterra* integro-differential equations in the last section.

3.2 Preliminaries

Let

$$BC = \{\psi : (-\infty, 0] \longmapsto \mathbb{R}^n : \psi \; is \; continuous \; and \; bounded \; on \; (-\infty, 0]\},$$

$$BU = \{\psi \in BC : \psi \; is \; uniformly \; continuous \; on \; (-\infty, 0]\},$$

$$\|\psi\| = \sup\{|\psi(\theta)| : -\infty < \theta \leq 0 \; for \; \psi \in BC\}$$

$$\|x\|_{[a,b]} = \sup\{|x(\theta)| : a \leq \theta \leq b, -\infty < a \leq b < \infty\},$$

$$\|h\|_{[\tau,\infty)} = \sup\{|h(t)| : \tau \leq t < \infty \; for \; h \in C(\mathbb{R}, \mathbb{R}^n)\}$$

and

$$C_\gamma = \{\psi \in BU : \|\psi\| < \gamma, \gamma > 0\}.$$

Space BC and BU with the above supremum norm are Banach space. BU satisfies all axioms for the phase space mentioned in [8] and is also an admissible phase space [11, 6, 7].

Consider the $NFDE$ with infinite delay of the form

$$\frac{d}{dt}D(t, x_t) = f(t, x_t), \quad t \geq t_0 \geq 0 \tag{3.2.1}$$

where $x_t = x(t+\theta), -\infty < \theta \leq 0, D$ and $f : [0, \infty) \times C_H \longmapsto \mathbb{R}^n$ are continuous $(H > 0)$.

By a solution of (3.2.1) we mean an $x \in C((-\infty, t_0 + A^*], \mathbb{R}^n)$ for some $A^* > 0$ and $t_0 \geq 0$ such that (i) $(t, x_t) \in [0, \infty) \times C_H$ for $t \in [t_0, t_0 + A^*]$; (ii) $D(t, x_t)$ is continuously differential and satisfies (3.2.1) on $[t_0, t_0 + A^*]$. If, in addition, $x_{t_0} = \varphi \in C_H$, then we say x is a solution of (3.2.1) through (t_0, φ) and we denote it by $x(t_0, \varphi)(t)$.

The general fundamental theory concerning existence, uniqueness, continuation of solutions in the abstract phase space for $NFDE$ with infinite delay has been established. We refer to [8, 11].

We always assume that D and f satisfy certain conditions to ensure the existence, uniqueness and continuation of solutions of (3.2.1), and that

$$D(t,0) = f(t,0) = 0.$$

Then (3.2.1) has the zero solution $x(t) = 0$.

Definition 3.2.1 *The zero solution $x(t) = 0$ of (3.2.1) is said to be uniformly stable (U.S) if for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $[t_0 \in [0,\infty), \|\varphi\| < \delta, t \geq t_0]$ imply $|x(t_0,\varphi)(t)| < \varepsilon$.*

Definition 3.2.2 *The zero solution $x(t) = 0$ of (3.2.1) is said to be uniformly asymptotically stable (U.A.S) if it is U.S and if there is a $\delta_0 > 0$ and for any $\eta > 0$ there exists a $T = T(\eta) > 0$ such that $[t_0 \in [0,\infty), \|\varphi\| < \delta_0, t \geq t_0 + T]$ imply $|x(t_0,\varphi)(t)| < \eta$.*

Definition 3.2.3 *Let operator $D : [0,\infty) \times C_H \ (H > 0) \longmapsto \mathbb{R}^n$ be continuous. D is said to be weak-uniformly stable if there exist constants $k > 0$ and $B > 0 \ (B \leq H)$ such that for any $\varphi \in C_B, \tau \in [0,\infty)$ and $h \in C([0,\infty),\mathbb{R}^n)$, the continuous solution $x(t) = x(\tau,\varphi,h)(t)$ of the functional difference equation*

$$D(t,x_t) = h(t), \quad t \geq \tau, x_\tau = \varphi \tag{3.2.2}$$

satisfies the estimate

$$|x(t)| \leq k\|x_\tau\| + k\|h\|_{[\tau,t]} \ \ whenever \ \ \|h\|_{[\tau,t]} < B \ \ for \ \ t \geq \tau. \tag{3.2.3}$$

D is said to be weak-uniformly asymptotically stable if there exists constant $k_1 > 0$ and $\gamma > 0$ ($\gamma \leq H$) and for any given $\sigma > 0$ there exists a nonincreasing function $g_{\gamma,\sigma}(u) : [0,\infty) \longmapsto [0,1]$ with $\lim_{u\to\infty} g_{\gamma,\sigma}(u) = 0$ such that for any $\tau \in [0.\infty)$ and $h \in C([0,\infty),\mathbb{R}^n)$ with $\|h\|_{[\tau,\infty)} < \gamma$, the solution $x(t)$ of (3.2.2) with $x_t \in C_\gamma$ for all $t \geq \tau$ satisfies

$$|x(t)| \leq k_1 g_{\gamma,\sigma}(t-\tau)\|x_\tau\| + k_1(\|h\|_{[\tau,t]} + \sigma), t \geq \tau. \tag{3.2.4}$$

Example 3.2.1 *Consider the scalar nonlinear D-operator with infinite delay*

$$D(t,\psi) = \psi(0) - \int_0^\infty B(t,u)Q(t-u,\psi(-u))du, \quad t \geq 0.$$

where $B(t,u) \in L^1([0,\infty))$ for each $t \in [0,\infty), Q(s,x)$ is continuous function, $|Q(s,x)| \leq b|x|^n, n$ is a positive integer, $b > 0$, $\int_0^\infty |B(t,u)|du$ converges uniformly for all $t \in [0,\infty)$, $\int_0^\infty |B(t,u)|du \leq P_1$ for all $t \in [0,\infty)$ and $bP_1 < 1$.

We will prove that the above $D(t,\psi)$ is weak-uniformly stable and weak-uniformly asymptotically stable. Indeed, choose $B = (1-bP_1)/2 < 1$. For any $\varphi \in C_B, \tau \in [0,\infty)$ and $h \in C([0,\infty),\mathbb{R})$, we claim that when $\|h\|_{[\tau,t_1]} < B$ for $t_1 > \tau, |x(t)| = |x(\tau,\varphi,h)(t) < 1$ for all $t \in [\tau,t_1]$. Otherwise, let $t^* = \inf\{t \in [\tau,t_1] : |x(t) \geq 1\}$. Obviously $\tau < t^* \leq t_1$ and $|x(t^*)| = 1$. For $\tau \leq s \leq t^*$, we have

$$\begin{aligned} |x(s)| &\leq \int_{-\infty}^{s} |B(s,s-\theta)|b|x(\theta)|^n d\theta + |h(s)| \\ &\leq \int_{-\infty}^{s} |B(s,s-\theta)|b|x(\theta)|d\theta + |h(s)| \\ &= b\int_{-\infty}^{\tau} |B(s,s-\theta)||x(\theta)|d\theta + b\int_{\tau}^{s} |B(s,s-\theta)||x(\theta)|d\theta + |h(s)| \\ &\leq bP_1\|x_\tau\| + bP_1\|x\|_{[\tau,t^*]} + \|h\|_{[\tau,t^*]}. \end{aligned}$$

Then

$$\|x\|_{[\tau,t^*]} \leq bP_1\|x_\tau\| + bP_1\|x\|_{[\tau,t^*]} + \|h\|_{[\tau,t^*]},$$

$$|x(t^*)| \leq \|x\|_{[\tau,t^*]} \leq \frac{bP_1}{1-bP_1}\|x_\tau\| + \frac{1}{1-bP_1}\|h\|_{[\tau,t^*]} < \frac{2B}{1-bP_1} = 1.$$

This contradiction implies that our claim is true. Then by an argument similar to the above, we easily have

$$\begin{aligned} |x(t)| &\leq \|x\|_{[\tau,t]} \\ &\leq \frac{bP_1}{1-bP_1}\|x_\tau\| + \frac{1}{1-bP_1}\|h\|_{[\tau,t]} \\ &\leq k\|x_\tau\| + k\|h\|_{[\tau,t]}, whenever \|h\|_{[\tau,t]} < B \; for \; t \geq \tau. \end{aligned} \quad (3.2.5)$$

where $k = 1/1 - bP_1$. This proves that $D(t, \psi)$ is weak-uniformly stable .

Choose $k_1 = k = 1/1 - bP_1$ and $\gamma = B = (1 - bP_1)/2$. Let $\|h\|_{[\tau,\infty)} < \gamma$ and $x_t \in C_\gamma$ for all $t \geq \tau$. Then for any given $\sigma > 0$, fix $T > 0$ with $b\gamma \int_T^\infty |B(t,u)|du < \sigma$ for all $t \in [0, \infty)$. For $\tau + T \leq s \leq t$, we have, using (3.2.5),

$$\begin{aligned} |x(s)| &\leq b\int_{-\infty}^{s-T} |B(s,s-\theta)||x(\theta)|d\theta \\ &\quad + b\int_{s-T}^{s} |B(s,s-\theta)||x(\theta)|d\theta + \|h\|_{[\tau,t]} \\ &< bP_1\|x\|_{[\tau,t]} + (\|h\|_{[\tau,t]} + \sigma) \\ &\leq bP_1\left(\frac{bP_1}{1-bP_1}\|x_\tau\| + \frac{1}{1-bP_1}\|h\|_{[\tau,t]}\right) + \left(\|h\|_{[\tau,t]} + \sigma\right) \\ &\leq \frac{(bP_1)^2}{1-bP_1}\|x_\tau\| + \frac{1}{1-bP_1}\left(\|h\|_{[\tau,t]} + \sigma\right). \end{aligned}$$

Then

$$\|x\|_{[\tau+T,t]} \leq \frac{(bP_1)^2}{1-bP_1}\|x_\tau\| + \frac{1}{1-bP_1}\left(\|h\|_{[\tau,t]} + \sigma\right), t \geq \tau + T.$$

By induction, we have

$$\begin{aligned}\|x\|_{[\tau+(m-1)T,t]} &\leq \frac{(bP_1)^m}{1-bP_1}\|x_\tau\| + \frac{1}{1-bP_1}\left(\|h\|_{[\tau,t]}+\sigma\right) \\ &\leq k_1(bP_1)^{m-1}\|x_\tau\| + k_1\left(\|h\|_{[\tau,t]}+\sigma\right), \\ &\qquad\qquad for\ t \geq \tau+(m-1)T.\end{aligned}$$

Define

$$g_{\gamma,\sigma}(u) = (bP_1)^{m-1}\ \ for\ \ (m-1)T \leq u \leq mT, m = 1, 2, \ldots.$$

We have

$$|x(t)| \leq k_1 g_{\gamma,\sigma}(t-\tau)\|x_\tau\| + k_1\left(\|h\|_{[\tau,t]}+\sigma\right),\ \ t \geq \tau.$$

This proves that $D(t,\psi)$ is weak-uniformly asymptotically stable.

We give below a nonlinear *D-operator* with infinite delay which is weak-uniformly stable and weak-uniformly asymptotically stable, but is not "uniformly stable" (cf. (3.1.1)).

Example 3.2.2 *Consider the scalar nonlinear D-operator*

$$D(t,\psi) = \psi(0) - \frac{3t^2}{1+t^2}e^{-t}\int_0^\infty e^{-s}\psi^2(-s)ds, \quad t \geq 0.$$

For all $t \in [0,\infty)$, we have

$$\int_T^\infty \frac{3t^2}{1+t^2}e^{-t}e^{-s}ds \leq \frac{3}{2e}\int_T^\infty e^{-s}ds \leq \frac{3}{2e}, \quad T \geq 0.$$

By Example 3.2.1. $D(t,\psi)$ is weak-uniformly stable and weak-uniformly asymptotically stable. But we claim that $D(t,\psi)$ does not satisfy the conditions of "uniformly stable" (cf. (3.1.1)). Indeed , for any given $a > 0$ *and* $b > 0, x(t) = e^t$ is a continuous solution of

$$D(t,x_t) = h(t),\ \ t \geq 0,\ \ x(s) = \varphi(s)\ \ for\ \ -\infty < s \leq 0,$$

where $h(t) = e^t/1+t^2, \varphi(s) = e^s$. It is easy to see that for sufficiently large $t \geq 0$, we have

$$\begin{aligned} be^{-at}\|\varphi\|_{(-\infty,0]} + b \sup_{0\leq u\leq t} |h(u)| &\leq b + b\frac{e^t}{1+t^2} \\ &< e^t = |x(t)|. \end{aligned}$$

This proves our claim. Similarly it is easy to verify that the above $D(t,\psi)$ does not satisfy the conditions of "uniformly stable" introduced in [24] (cf. Definition 4 of [24]).

Let $x(t) = x(t_0, \varphi)(t)$ be a solution of (3.2.1) with $x_{t_0} = \varphi$. Then $D(t, x_t)$ is a continuous function of t. Denote $D(t, x_t)$ by $H(t)$. Then $D(t, x_t) \equiv H(t)$.

Lemma 3.2.1 *Let $D(t,\psi)$ of (3.2.1) be weak-uniformly stable and weak-uniformly asymptotically stable. Assume*

(A_1) *for each $\varepsilon' > 0$ there is a $\delta' = \delta'(\varepsilon')$ such that $[t_0 \in [0,\infty), \|\varphi\| < \delta', t \geq t_0]$ imply $|H(t)| < \varepsilon'$.*

Then the zero solution of (3.2.1) is U.S . Furthermore assume

(A_2) *there is a $\delta_0' > 0$ and for any $\eta' > 0$ there exists a $T' = T'(\eta') > 0$ such that $[t_0 \in [0,\infty), \|\varphi\| < \delta_0', t \geq t_0 + T']$ imply $|H(t)| < \eta'$.*

Then the zero solution of (3.2.1) is U.A.S .

Proof. For any given $\varepsilon > 0$, let $\varepsilon' = \min\{B, \varepsilon/2k\}$. Choose $\delta' = \delta'(\varepsilon')$ such that $[t_0 \in [0,\infty), \|\varphi\| < \delta', t \geq t_0]$ imply $|H(t)| < \varepsilon'$. Let $\delta = \min\{B, \delta', \varepsilon/2k\}$. Then $[t_0 \in [0,\infty), \|\varphi\| < \delta, t \geq t_0]$ imply

$$|x(t)| \leq k\|x_{t_0}\| + k\|H\|_{[t_0,t]} < \varepsilon.$$

Hence the zero solution of (3.2.1) is *U.S.* Next we will prove that the zero solution of (3.2.1) is *U.A.S.* For $\eta > 0$, choose $\sigma = \eta/3k_1$. Let $0 < \gamma_0 < \min\{\gamma, \delta_0'\}$ and find $\delta < \gamma_0$ such that $[t_0 \in [0,\infty), \|\varphi\| < \delta, t \geq t_0]$ imply $|x(t)| < \gamma_0$ and $|H(t)| < \gamma_0$. We have

$$|x(t)| \leq k_1 g_{\gamma,\sigma}(t-\tau)\|x_\tau\| + k_1(\|H\|_{[\tau,t]} + \sigma), \quad t \geq \tau \geq t_0.$$

On the other hand, by (A_2), for $\eta' = \eta/3k_1$ there exists a $T' = T'(\eta')$ such that $[t_0 \in [0,\infty), \|\varphi\| < \delta, t \geq t_0 + T']$ imply $|H(t)| < \eta'$. Choose $T" > 0$ such that $k_1 g_{\gamma,\sigma}(u)\gamma < \eta/3$ for $u \geq T"$. Then $[t_0 \in [0,\infty), \|\varphi\| < \delta, t \geq t_0 + T \ \ where \ \ T = T' + T"]$ imply

$$\begin{aligned} |x(t)| &\leq k_1 g_{\gamma,\sigma}(t - t_0 - T')\|x_{t_0+T'}\| + k_1\|H\|_{t_0+T',t]} + k_1\sigma \\ &< k_1 g_{\gamma,\sigma}(T")\gamma + k_1\eta' + k_1\sigma \\ &< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta. \end{aligned}$$

Hence the zero solution of (3.2.1) is *U.A.S.* The proof is complete.

Lemma 3.2.2 *Let $\phi \in L^1([0.\infty))$ with $\phi(s) \geq 0, x : \mathbb{R} \longmapsto [-M, M](M > 0)$ and $u_i : [0,\infty) \longmapsto [0,\infty)(i = 1,2)$ be increasing and continuous functions with $u_i(t) > 0$ as $t > 0$. If there exists a constant $\lambda > 0$ such that*

$$\int_{t-r}^{t} \phi(t-s)u_1(|x(s)|)ds \geq \lambda, \quad t \in [t_0 + r, \infty), \quad r > 0, \tag{3.2.6}$$

then there exists a constant $\mu > 0$ which is dependent only on r, λ and M such that

$$\int_{t-r}^{t} u_2(|x(s)|)ds \geq \mu.$$

Proof. Let

$$\int_0^r \phi(s)ds = R > 0,$$

$$E(t) = \left\{s : |x(t-s)| \geq u_1^{-1}(\lambda/2R), 0 \leq s \leq r\right\} \ where\ t\ satisfies\ (3.2.6)$$

and $m(E(t))$ be the Lebesgue measure of $E(t)$. If

$$\int_{E(t)} \phi(s)ds < \frac{\lambda}{2u_1(M)},$$

then

$$\begin{aligned}
\lambda &\leq \int_0^r \phi(s)u_1(|x(t-s)|)ds \\
&= \int_{E(t)} \phi(s)u_1(|x(t-s)|)ds + \int_{[0,r]\setminus E(t)} \phi(s)u_1(|x(t-s)|)ds \\
&\leq u_1(M)\int_{E(t)} \phi(s)ds + u_1\left(u_1^{-1}(\lambda/2R)\right)\int_{[0,r]\setminus E(t)} \phi(s)ds \\
&< \frac{\lambda}{2} + \frac{\lambda R}{2R} = \lambda.
\end{aligned}$$

This is a contradiction. Hence $\int_{E(t)} \phi(s)ds \geq \lambda/2u_1(M)$. Since $\phi \in L^1([0,\infty))$, there exists $\delta > 0$ for $\lambda/4u_1(M)$ such that $\int_E \phi(s)ds < \lambda/4u_1(M)$ for each $E \subset [0,\infty)$ with $m(E) < \delta$. We claim that there exists a constant $\mu' > 0$ such that $m(E(t)) \geq \mu'$ for all t which satisfy (3.2.6). Otherwise, there exists some t_1 which satisfies (3.2.6) such that $m(E(t_1)) < \delta$. Then

$$\frac{\lambda}{2u_1(M)} \leq \int_{E(t_1)} \phi(s)ds < \frac{\lambda}{4u_1(M)}$$

is a contradiction. Then for all t which satisfy (3.2.6), we have

$$\begin{aligned}
\int_{t-r}^t u_2(|x(s)|)ds &\geq \int_{E(t)} u_2(|x(t-s)|)ds \\
&\geq u_2\left(u_1^{-1}(\lambda/2R)\right)\mu' \\
&= \mu \\
&> 0.
\end{aligned}$$

The proof is complete.

3.3 Uniformly Asymptotic Stability

In this section, we assume that $D(t,\psi)$ of (3.2.1) is weak-uniformly stable and weak-uniformly asymptotically stable.

An increasing and continuous function $W : [0,\infty) \longmapsto [0,\infty)$ is called a wedge if $W(0) = 0$ and $W(s) > 0$ as $s > 0$.

Let $P, q : [0,\infty) \longmapsto [0,\infty)$ be continuous, $P(s) > s, q(s) > 0$ as $s > 0$, and $q(s)$ be nonincreasing.

Let $V(t,\psi)$ be a continuous nonnegative functional defined in $\mathbb{R}\times C_A$ where $0 < A \leq H$. The upper right-hand derivative of V along a solution of (3.2.1) is defined to be

$$V'(t,x_t) = \limsup_{\delta\to 0^+} \frac{V(t+\delta, x_{t+\delta}) - V(t,x_t)}{\delta}.$$

We always assume that $V'(t,x_t)$ exists.

Theorem 3.3.1 *Let $\phi : [0,\infty) \longmapsto [0,\infty)$ and $\int_0^\infty \phi(s)ds = \ell < \infty$. Suppose that there are $V(t,\psi)$ and wedges $W_i, i = 1,2,3,4,5,$ which satisfy the following conditions*

(*i*) $W_1(|D(t,\psi)|) \leq V(t,\psi) \leq W_2(|\psi(0)|) + W_3\left[\int_0^\infty \phi(s)W_4(|\psi(-s)|)ds\right],$

(*ii*) $V'(t,x_t) \leq -W_5(|x(t)|)$ *whenever* $P(V(t,x_t)) > V(s,x_s)$ *for* $t - q(V(t,x_t)) \leq s \leq t.$

Then the zero solution of (3.2.1) *is U.A.S.*

Proof. Let $x(t) = x(t_0, \varphi)(t)$ be a solution of (3.2.1) with $x_{t_0} = \varphi$. According to Lemma 3.2.1, suffice it to prove that condition (A_1) and (A_2) in Lemma 3.2.1 are satisfied. For any given $\varepsilon > 0$ ($\varepsilon < \min\{A/4k, A/2, B\}$), choose $\delta > 0$ such that $\delta < \varepsilon, W_2(\delta) < W_1(\varepsilon)/2$ and $W_3[W_4(\delta)\ell] < W_1(\varepsilon)/2$. Let $\|\varphi\| < \delta$ and denote $V(t) = V(t, x_t), V'(t) = V'(t, x_t)$. We first prove that $|x(t)| < A/2$ for all $t \geq t_0$ and then $x_t \in C_A$ for all $t \geq t_0$. Let $\hat{t} = \inf\{t \geq t_0 : |x(t)| \geq A/2\}$. Suffice it to consider the case where $\{t \geq t_0 : |x(t)| \geq A/2\}$ is not empty. Obviously $\hat{t} > t_0$. If $\hat{t} < \infty$, then $x_t \in C_A$ for $t \in [t_0, \hat{t}]$ and $|x(\hat{t})| = A/2$. We have

$$V(t) \leq W_2(\delta) + W_3[W_4(\delta)\ell] < W_1(\varepsilon) \ for \ -\infty < t \leq t_0. \tag{3.3.2}$$

We claim that

$$V(t) < W_1(\varepsilon) \ for \ all \ t \in [t_0, \hat{t}]. \tag{3.3.3}$$

Otherwise, there exists a $t' \in (t_0, \hat{t}]$ such that

$$V(s) \leq V(t') \ for \ s \leq t' \ and \ V'(t') > 0. \tag{3.3.4}$$

Then

$$P(V(t')) > V(t') \geq V(s) \ for \ t' - q(V(t')) \leq s \leq t'.$$

By (ii), we have $V'(t') \leq 0$. This contradiction proves our claim. Then

$$W_1(|H(t)|) \leq V(t) < W_1(\varepsilon) \ for \ all \ t \in [t_0, \hat{t}] \tag{3.3.5}$$

and

$$|H(t)| < \varepsilon \ for \ all \ t \in [t_0, \hat{t}]. \tag{3.3.6}$$

By (3.2.3), we have

$$A/2 = |x(\hat{t})| \leq k\|x_{t_0}\| + k\|H\|_{[t_0.\hat{t}]} < 2k\varepsilon < A/2.$$

This contradiction implies that $\hat{t} = \infty$. Hence

$$|x(t)| < A/2 < A \ \ for \ \ all \ \ t \geq t_0$$

and then

$$|H(t)| < \varepsilon \ \ for \ \ all \ \ t \geq t_0.$$

Condition (A_1) of Lemma 3.2.1 is satisfied and the zero solution of (3.2.1) is $U.S.$ Next we will prove that condition (A_2) in Lemma 3.2.1 is satisfied. For $\varepsilon = \min\{A, B\}$, choose $\delta_0 > 0$ ($\delta_0 < \min\{A, B\}$) such that $[t_0 \in [0, \infty), \|\varphi\| < \delta_0, t \geq t_0]$ imply $|x(t)| < \min\{A, B\}$. If $\|\varphi\| < \delta_0$, then $V(t) < W_2(A) + W_3[\ell W_4(A)] \equiv a$ for $t \geq t_0$. For any given $\eta' > 0$ ($\eta' < \min\{A, B\}$ and $W_1(\eta') < a$), choose $0 < \eta < \eta'$ and $r > 0$ such that

$$\begin{aligned} \int_{-\infty}^{t-r} \phi(t-s) W_4(|x(s)|) ds \ &\leq \ W_1(A) \int_r^{\infty} \phi(s) ds \\ &\leq \ b \\ &\leq \ \frac{1}{2} W_3^{-1} \left[\frac{1}{2} W_1(\eta) \right] \end{aligned} \tag{3.3.7}$$

where $b > 0$ is some constant. Let $h = \max\{r, q(W_1(\eta))\}, 0 < d < \inf\{P(u) - u : W_1(\eta) \leq u \leq a\}$ and N be a positive integer satisfying

$$W_1(\eta) + (N-1)d < a \leq W_1(\eta) + Nd. \tag{3.3.8}$$

If

$$\int_{t-r}^{t} \phi(t-s) W_4(|x(s)|) ds \geq \frac{1}{2} W_3^{-1} \left[\frac{1}{2} W_1(\eta) \right] \ \ t \in [t_0 + r, \infty), \tag{3.3.9}$$

then, by Lemma 3.2.2, there exists a constant $\mu > 0$ which is dependent only on η (note that r is dependent on η) such that

$$\int_{t-r}^{t} W_5(|x(s)|) ds \geq \mu. \tag{3.3.10}$$

Let K be the positive integer satisfying $(K-1)\mu \leq a < K\mu, T = (K+1)h + \frac{2a}{W_5(\beta)}$ where $\beta = W_2^{-1}\left(\frac{1}{2}W_1(\eta)\right)$ and $T_1 = t_0 + h + T$. We claim that there must exist one point $t \in [t_0 + h, T_1]$ such that

$$V(t) < W_1(\eta) + (N-1)d. \tag{3.3.11}$$

Otherwise,

$$V(t) \geq W_1(\eta) + (N-1)d \geq W_1(\eta) \ for \ t_0 + h \leq t \leq T_1. \tag{3.3.12}$$

Then

$$P(V(t)) \geq V(t) + d \geq W_1(\eta) + Nd > V(s) \ for \ t_0 \leq s \leq t. \tag{3.3.13}$$

From (ii) we have

$$V'(t) \leq -W_5(|x(t)|) \ for \ t_0 + h \leq t \leq T_1. \tag{3.3.14}$$

Then

$$V(T_1) \leq V(t_0 + h) - \int_{t_0+h}^{T_1} W_5(|x(s)|)ds < a - \int_{t_0+h}^{T_1} W_5(|x(s)|)ds. \tag{3.3.15}$$

On the other hand, we have

$$\begin{aligned} W_1(\eta) &\leq V(t) \\ &\leq W_2(|x(t)|) + W_3\left[b + \int_{t-r}^{t} \phi(t-s)W_4(|x(s)|)ds\right] \\ &\qquad t_0 + h \leq t \leq T_1. \end{aligned} \tag{3.3.16}$$

Then either

$$W_2(|x(t)|) \geq \frac{W_1(\eta)}{2} \tag{3.3.17}$$

or

$$W_3\left[b+\int_{t-r}^{t}\phi(t-s)W_4(|x(s)|)ds\right]\geq\frac{W_1(\eta)}{2}. \tag{3.3.18}$$

Let

$$E_1=\left\{t\in[t_0+h,T_1]:W_3\left[b+\int_{t-r}^{t}\phi(t-s)W_4(|x(s)|)ds\right]\geq W_1(\eta)/2\right\}$$

and

$$E_2=[t_0+h,T_1]\setminus E_1.$$

If $t\in E_1$, then

$$\int_{t-r}^{t}\phi(t-s)W_4(|x(s)|)ds\geq\frac{1}{2}W_3^{-1}\left(\frac{1}{2}W_1(\eta)\right). \tag{3.3.19}$$

If $t\in E_2$, then

$$W_2(|x(t)|)\geq\frac{1}{2}W_1(\eta). \tag{3.3.20}$$

It follows that

$$\int_{t-r}^{t}W_5(|x(s)|)ds\geq\mu \;\; for \;\; t\in E_1 \tag{3.3.21}$$

and

$$|x(t)|\geq\beta \;\; for \;\; t\in E_2. \tag{3.3.22}$$

We have either $m(E_1)\geq(K+1)h$ or $m(E_2)\geq 2a/W_5(\beta)$. If $m(E_1)\geq(K+1)h$, there must exist K points $\{t_1,t_2,\ldots,t_K\}\subset E_1$ with $t_{i+1}-t_i\geq h$ $(i=1,2,\ldots,K-1)$ and $t_1\geq t_0+2h$. Then

$$\begin{aligned}V(T_1) &\leq a-\int_{t_0+h}^{T_1}W_5(|x(s)|)ds\\ &\leq a-\sum_{i=1}^{K}\int_{t_i-r}^{t_i}W_5(|x(s)|)ds\\ &\leq a-K\mu\\ &< 0.\end{aligned} \tag{3.3.23}$$

If $m(E_2) \geq 2a/W_5(\beta)$, then

$$\begin{aligned} V(T_1) &< a - \int_{E_2} W_5(|x(s)|)ds \\ &\leq a - m(E_2)W_5(\beta) \\ &< 0. \end{aligned} \tag{3.3.24}$$

Hence there exists some $\bar{t} \in [t_0 + h, T_1]$ such that

$$V(\bar{t}) < W_1(\eta) + (N-1)d. \tag{3.3.25}$$

We claim that

$$V(t) \leq W_1(\eta) + (N-1)d \ \ for \ \ t \geq \bar{t}. \tag{3.3.26}$$

If not, there exists some $t^* > \bar{t}$ such that

$$V(t^*) > W_1(\eta) + (N-1)d \quad and \quad V'(t^*) > 0. \tag{3.3.27}$$

Then

$$P(V(t^*)) \geq V(t^*) + d > W_1(\eta) + Nd > V(s) \ \ for \ \ t_0 \leq s \leq t^*.$$

By (ii), we have $V'(t^*) \leq 0$, which is a contradiction. Hence

$$V(t) \leq W_1(\eta) + (N-1)d \ \ for \ \ t \geq T_1 = t_0 + h + T. \tag{3.3.28}$$

By induction, we can prove that

$$V(t) \leq W_1(\eta) + (N-n)d \ \ for \ \ t \geq T_n, n = 1, 2, \ldots, N. \tag{3.3.29}$$

where $T_n = T_{n-1} + (h+T) = t_0 + n(h+T)$ and $T_0 = t_0$. Then

$$W_1(|H(t)|) \leq V(t) \leq W_1(\eta) \ \ for \ \ t \geq T_N = t_0 + N(h+T) \tag{3.3.30}$$

and

$$|H(t)| \leq \eta < \eta' \ for \ t \geq t_0 + N(h+T). \tag{3.3.31}$$

Hence condition (A_2) in Lemma 3.2.1 is satisfied. The proof is complete.

Theorem 3.3.2 *Suppose that there are V, P, q and wedges $W_i (i = 1, 2, \dots, 6)$ satisfying the following conditions*

(*i*) $W_1(|D(t,\psi)|) \leq V(t,\psi) \leq W_2(\|\psi\|)$,

(*ii*) $V'(t,x_t) \leq -W_3(|D(t,x_t)|)$ *whenever* $P(V(t,x_t)) > V(s,x_s)$ *for* $\max\{0, t - q(V(t,x_t))\} \leq s \leq t$,

(*iii*) $f : [0,\infty) \times (bounded\ sets\ of\ C_A) \longmapsto bounded\ sets\ of\ \mathbb{R}^n$,

(*iv*) *for any* $\sigma_1 > 0$ *and* $B' > 0$ $(B' < \min\{A, H\})$ *there exists an* $r_1 > 0$ *such that* $[\tau_1 \in [0,\infty), \|\psi\|_{(-\infty,\tau_1 - t]} \leq B', t \geq \tau_1 + r_1]$ *imply*

$$V(t,\psi) \leq W_4(|D(t,\psi)|) + W_5(\|\psi\|_{[\tau_1 - t, 0]}) + W_6(\sigma_1).$$

Then the zero solution of (3.2.1) *is U.A.S.*

Proof. Let $x(t) = x(t_0,\varphi)(t)$ be a solution of (3.2.1) with $x_{t_0} = \varphi$. For any given $\varepsilon > 0$ $(\varepsilon < \min\{A/4k, A/2, B\})$, choose $\delta > 0$ such that $\delta < \varepsilon$ and $W_2(\delta) < W_1(\varepsilon)$. Let $\|\varphi\| < \delta$ and denote $V(t) = V(t,x_t), V'(t) = V'(t,x_t)$. Using an argument similar to Theorem 3.3.1, we easily prove $x_t \in C_A$ for all $t \geq t_0$. We have

$$V(t) \leq W_2(\delta) < W_1(\varepsilon) \ for \ t \leq t_0. \tag{3.3.32}$$

We claim that

$$V(t) < W_1(\varepsilon) \ for \ all \ t \geq t_0. \tag{3.3.33}$$

If not, there exists $t' > t_0$ such that

$$V(t') \geq V(s) \ for \ s \leq t' \ and \ V'(t') > 0. \tag{3.3.34}$$

Then

$$P(V(t')) > V(t') \geq V(s) \ for \ \max\{0, t' - q(V(t'))\} \leq s \leq t'.$$

By (ii) we have $V'(t') \leq 0$. This contradiction proves our claim.

Then

$$W_1(|H(t)|) \leq V(t) < W_1(\varepsilon) \ for \ all \ t \geq t_0 \tag{3.3.35}$$

and

$$|H(t)| < \varepsilon \ for \ all \ t \geq t_0. \tag{3.3.36}$$

By Lemma 3.2.1 , the zero solution of (3.2.1) is *U.S.*

We next show that the zero solution of (3.2.1) is *U.A.S.* For some $0 < b < \min\{A, B, \gamma\}$, choose $0 < \delta_0 < b$ such that $[t_0 \geq 0, \|\varphi\| < \delta_0, t \geq t_0]$ imply $|x(t)| < b$ and $|H(t)| < b$. For any $0 < \eta' < b$ with $W_1(\eta') < W_2(A)$, choose $0 < \eta < \eta'$ and let $0 < d < \inf\{P(u) - u : W_1(\eta) \leq u \leq W_2(A)\}$ and N be a positive integer satisfying

$$W_1(\eta) + (N-1)d < W_2(A) \leq W_1(\eta) + Nd. \tag{3.3.37}$$

For $\sigma = (1/4k_1)W_5^{-1}(W_1(\eta)/3)$ (W.L.O.G assume $k_1 \geq 1$) , we have , using (3.2.4),

$$\begin{aligned} |x(t)| &\leq k_1 g_{\gamma,\sigma}(t-\tau)\|x_\tau\| + k_1\|H\|_{[\tau,t]} + k_1\sigma \\ &< k_1 g_{\gamma,\sigma}(t-\tau)A + k_1\|H\|_{[\tau,t]} + \frac{1}{4}W_5^{-1}(W_1(\eta)/3), \\ &\qquad t \geq \tau \geq t_0. \end{aligned} \tag{3.3.38}$$

Choose $r > 0$ such that $k_1 g_{\gamma,\sigma}(r)A < \frac{1}{4}W_5^{-1}(W_1(\eta)/3)$. Then

$$|x(t)| < k_1\|H\|_{[\tau,t]} + \frac{1}{2}W_5^{-1}(W_1(\eta)/3), \quad t \geq \tau + r. \tag{3.3.39}$$

On the other hand, by (iv), for $\sigma_1 = W_6^{-1}(W_1(\eta)/3)$ and $B' = \min\{A, B\}/2$ there exists an $r_1 > 0$ such that for any $\tau_1 \in [t_0, \infty)$,

$$V(t) \leq W_4(|H(t)|) + W_5(\|x\|_{[\tau_1,t]}) + \frac{1}{3}W_1(\eta), \quad t \geq \tau_1 + r_1. \tag{3.3.40}$$

When $t \geq t_0 + r + r_1$, by (3.3.39), we have, for $t - r_1 \leq s \leq t$,

$$|x(s)| < k_1\|H\|_{[t-r-r_1,s]} + \frac{1}{2}W_5^{-1}(W_1(\eta)/3), \tag{3.3.41}$$

$$\|x\|_{[t-r_1,t]} \leq k_1\|H\|_{[t-r-r_1,t]} + \frac{1}{2}W_5^{-1}(W_1(\eta)/3), \quad t \geq t_0 + r + r_1. \tag{3.3.42}$$

Let $h = \max\{r+r_1, q(W_1(\eta))\}$ and $m = \min\{W_4^{-1}(W_1(\eta)/3), \frac{1}{2}W_5^{-1}(W_1(\eta)/3)\}$. By (iii), there exists a constant $L > 0$ such that $|H'(t)| < L$ for all $t \geq t_0$. Let K be the smallest integer $\geq Lk_1W_2(A)/mW_3(m/2k_1)$ and $T_1 = t_0 + (2K+1)h$. We claim that there exists a point $t \in [t_0 + h, T_1]$ such that

$$V(t) < W_1(\eta) + (N-1)d. \tag{3.3.43}$$

Otherwise,

$$V(t) \geq W_1(\eta) + (N-1)d \geq W_1(\eta) \ \ for \ \ t_0 + h \leq t \leq T_1. \tag{3.3.44}$$

Noting that $V(t) \leq W_2(A) \leq W_1(\eta) + Nd$ for all $t \geq t_0$, we have

$$P(V(t)) > V(t) + d \geq W_1(\eta) + Nd \geq V(s) \ \ for \ \ t_0 \leq s \leq t$$

where $t_0 + h \leq t \leq T_1$. Then

$$P(V(t)) > V(s) \ \ for \ \ t - q(V(t)) \leq s \leq t \ \ where \ \ t_0 + h \leq t \leq T_1.$$

From (ii), we have

$$V'(t) \leq -W_3(|D(t,x_t)|) \ for \ t_0 + h \leq t \leq T_1.$$

Then

$$V(T_1) \leq W_2(A) - \int_{t_0+h}^{T_1} W_3(|H(s)|)ds. \tag{3.3.45}$$

Suppose that $\|H\|_{[t-h,t]} \geq m/k_1$ for all $t \in [t_0+h, T_1]$. Since each interval of length h contains an s with $|H(s)| \geq m/k_1$, there exist K points $t_j \in [t_0+h, T_1]$ satisfying

$$t_0 + 2jh \leq t_j \leq t_0 + (2j+1)h, \ \ j = 1, 2, \ldots, K.$$

and

$$|H(t_j)| \geq m/k_1, \ \ j = 1, 2, \ldots, K. \tag{3.3.46}$$

Therefore

$$V'(t) < -W_3(m/2k_1) \ for \ t_j - \frac{m}{2Lk_1} \leq t \leq t_j + \frac{m}{2Lk_1}. \tag{3.3.47}$$

By taking a large L, if necessary, we can assume that these intervals do not overlap and $t_1 - \frac{m}{2Lk_1} \geq t_0 + h$. Hence

$$V(T_1) < W_2(A) - \frac{m}{Lk_1} W_3(m/2k_1)K \leq 0.$$

This contradiction leads to the conclusion that there exists a $\hat{t} \in [t_0+h, T_1]$ such that $\|H\|_{[\hat{t}-h,\hat{t}]} < \frac{m}{k_1} \leq m$. From (3.3.42), we have

$$\begin{aligned} \|x\|_{[\hat{t}-r_1,\hat{t}]} &\leq k_1\|H\|_{[\hat{t}-r-r_1,\hat{t}]} + \frac{1}{2}W_5^{-1}(W_1(\eta)/3) \\ &< m + \frac{1}{2}W_5^{-1}(W_1(\eta)/3) \\ &\leq W_5^{-1}(W_1(\eta)/3) \end{aligned}$$

From (3.3.40), we have

$$V(\hat{t}) \leq W_4(|H(\hat{t})|) + W_5(\|x\|_{[\hat{t}-r_1,\hat{t}]}) + \frac{1}{3}W_1(\eta) < W_1(\eta). \tag{3.3.48}$$

which contradicts (3.3.44). Hence there exists a $t^* \in [t_0 + h, T_1]$ such that

$$V(t^*) < W_1(\eta) + (N-1)d. \tag{3.3.49}$$

We claim that

$$V(t) \leq W_1(\eta) + (N-1)d \ \ for \ \ all \ \ t \geq t^*. \tag{3.3.50}$$

Otherwise, there exists a $\bar{t} > t^*$ such that $V(\bar{t}) > W_1(\eta) + (N-1)d$ and $V'(\bar{t}) > 0$. Then

$$P(V(\bar{t})) > V(\bar{t}) + d \geq W_1(\eta) + Nd \geq W_2(A) \geq V(s) \ \ for \ \ t_0 \leq s \leq \bar{t}.$$

By (ii), we have $V'(\bar{t}) \leq 0$. This contradiction implies that

$$V(t) \leq W_1(\eta) + (N-1)d \ \ for \ \ all \ \ t \geq t^*$$

and then

$$V(t) \leq W_1(\eta) + (N-1)d \ \ for \ \ all \ \ t \geq T_1 = t_0 + (2K+1)h.$$

By induction, using an argument similar to the above, we have

$$V(t) \leq W_1(\eta) + (N-n)d$$

for all $t \geq T_n = t_0 + n(2K+1)h, n = 1, 2, \ldots, N$. Then

$$W_1(|H(t)|) \leq V(t) \leq W_1(\eta) \ \ for \ \ all \ \ t \geq T_N = t_0 + (2K+1)Nh.$$

Hence

$$|H(t)| \leq \eta < \eta' \ \ for \ \ all \ \ t \geq t_0 + T' \ \ where \ \ T' = (2K+1)Nh.$$

From Lemma 3.2.1, the zero solution of (3.2.1) is $U.A.S$. The proof is complete.

Corollary 3.3.1 *Suppose that there are V, P, q and wedges W_i $(i = 1, 2, \ldots, 5)$ satisfying the following conditions*

(*i*) $W_1(|\psi(0)|) \leq V(t, \psi) \leq W_2(\|\psi\|)$,

(*ii*) $V'(t, x_t) \leq -W_3(|x(t)|)$ *whenever* $P(V(t, x_t)) > V(s, x_s)$ *for* $\max\{0, t - q(V(t, x_t)) \leq s \leq t$,

(*iii*) $f : [0, \infty) \times (\textit{bounded sets of } C_A) \longmapsto \textit{bounded sets of } \mathbb{R}^n$,

(*iv*) *for any* $\sigma_1 > 0$ *and any* $B' > 0$ *there exists an* $r_1 > 0$ *such that* $\left[\tau_1 \in [0, \infty), \|\psi\|_{[-\infty, \tau_1 - t]} < B', t \geq \tau_1 + r_1\right]$ *imply*

$$V(t, \psi) \leq W_4(\|\psi\|_{[\tau_1 - t, 0]}) + W_5(\sigma_1).$$

Then the zero solution of the equation with infinite delay

$$x'(t) = f(t, x_t)$$

is U.A.S.

Theorem 3.3.3 *Suppose that there are* $V : \mathbb{R} \times \{x \in \mathbb{R}^n : |x| < A\} \longmapsto [0, \infty)$, *continuous nondecreasing function P, continuous nonincreasing function q and wedges $W_i (i = 0, 1, 2, 3)$ and there exists a strictly increasing and continuous function $a(s)$ with $a(0) = 0$ satisfying $W_2(W_0(s)) \leq W_1(a(s))$ and $P(W_2(W_0(s))) > W_2(k_2 s + k_2 a(s))$ for small $s > 0$ where $k_2 = \max\{k, k_1, 1\}$ and k, k_1 are the constants in Definition 3.2.3. If*

(*i*) $|D(t, \psi)| \leq W_0(\|\psi\|)$,

(*ii*) $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$, (*iii*) $V'(t, D(t, x_t)) \leq -W_3(|D(t, x_t)|)$ *whenever* $P(V(t, D(t, x_t))) >$ $V(s, x(s))$ *for* $\max\{0, t - q(V(t, D(t, x_t)))\} \leq s \leq t$, *where*

$$V'(t, D(t, x_t)) = \limsup_{\delta \to 0^+} \frac{V(t+\delta, D(t+\delta, x_{t+\delta})) - V(t, D(t, x_t))}{\delta},$$

then the zero solution of (3.2.1) *is U.A.S.*

Proof. Let $x(t) = x(t_0, \varphi)(t)$ be a solution of (3.2.1) with $x_{t_0} = \varphi$. For any given $\varepsilon > 0$ ($\varepsilon < \min\{A/4k_2, B/4k_2\}$), choose small $\delta > 0$ such that $\delta < \varepsilon, W_0(\delta) < \varepsilon$ and $a(\delta) < \varepsilon$. Let $\|\varphi\| < \delta$ and $S = \{t \geq t_0 : |x(t)| \geq b \; or \; |H(t)| \geq b\}$ where $b = \min\{A/2, B/2\}$ and $\hat{t} = \inf\{t : t \in S\}$. If S is not empty, obviously $\hat{t} > t_0$. If $\hat{t} < \infty$, then either $|x(\hat{t})| \geq b$ or $|H(\hat{t})| \geq b$. We have

$$V(t, H(t)) \leq W_2(|H(t)|) < W_2(W_0(\delta)), \;\; 0 \leq t \leq t_0.$$

We claim that

$$V(t, H(t)) \leq W_2(W_0(\delta)) \;\; for \;\; all \;\; t \in [t_0, \hat{t}]. \tag{3.3.51}$$

If not, there exists a $t' \in (t_0, \hat{t}]$ such that $W_2(W_0(\delta)) = V(t', H(t')) \geq V(t, H(t))$ for all $0 \leq t \leq t'$ and $V'(t', H(t')) \geq 0$. We have

$$W_1(|H(t)|) \leq V(t, H(t)) \leq W_2(W_0(\delta)) \leq W_1(a(\delta)) \; for \; 0 \leq t \leq t'.$$

Then

$$|H(t)| \leq a(\delta) < \varepsilon \;\; for \;\; 0 \leq t \leq t'.$$

By (3.2.3),

$$\begin{aligned} |x(t)| &\leq k\|x_{t_0}\| + k\|H\|_{[t_0, t']} \\ &\leq k_2\delta + k_2 a(\delta) \\ &< 2k_2\varepsilon \;\; for \;\; 0 \leq t \leq t'. \end{aligned}$$

Then

$$\begin{aligned} P(V(t', H(t'))) &= P(W_2(W_0(\delta))) \\ &> W_2(k_2\delta + k_2 a(\delta)) \\ &\geq W_2(|x(t)|) \\ &\geq V(t, x(t)) \ for \ 0 \leq t \leq t'. \end{aligned}$$

By (iii),

$$V'(t', H(t')) \leq -W_3(|H(t')|) < 0$$

since $0 < W_2(W_0(\delta)) = V(t', H(t')) \leq W_2(|H(t')|)$ and $|H(t')| > 0$. This contradiction implies that (3.3.51) holds. Then we easily have

$$|x(t)| \leq k_2\delta + k_2 a(\delta) < 2k_2\varepsilon < b \ for \ 0 \leq t \leq \hat{t}$$

and

$$|H(t)| \leq a(\delta) < \varepsilon < b \ for \ 0 \leq t \leq \hat{t}.$$

Then

$$b \leq |x(\hat{t})| < b \ \ and \ \ b \leq |H(\hat{t})| < b$$

which imply that $\hat{t} = \infty$. Similarly, we can prove

$$|H(t)| < \varepsilon \ for \ t \geq t_0. \tag{3.3.52}$$

If S is empty, then $\hat{t} = \infty$. We can easily prove (3.3.52) holds also . From Lemma 3.2.1, the zero solution of (3.2.1) is *U.S.*

We next prove the *U.A.S.* For $\min\{A, B, \gamma\}$, choose small $\delta_0, 0 < \delta_0 < \min\{A, B, \gamma\}$, such that $\|\varphi\| < \delta_0$ implies $|x(t)| < \min\{A, B, \gamma\}$ for $t \geq t_0$, $|H(t)| < \min\{A, B, \gamma\}$ for $t \geq t_0$ and $V(t, H(t)) \leq W_2(W_0(\delta_0))$ for $t \geq t_0$. For any given $\eta' > 0$, choose $0 < \eta < \min\{\eta', \delta_0, a(\delta_0), W_1^{-1}(W_2(W_0(\delta_0)))\}$ and choose a sufficiently small $d > 0$ satisfying $d < W_1(\eta), P(W_2(W_0(s)) - d) > W_2(k_2 s + k_2 a(s))$ for $a^{-1}(\eta)/2 \leq s \leq \delta_0$ and $W_2(W_0(a^{-1}(\eta)/2)) < W_1(\eta) - d$ such that δ_i defined below are greater than $a^{-1}(\eta)/2$. Let N be the positive integer satisfying

$$W_1(\eta) + (N-1)d < W_2(W_0(\delta_0)) \leq W_1(\eta) + Nd \tag{3.3.53}$$

and define $\delta_i > 0$ $(i = 1, 2, \ldots, N)$ as follows

$$W_2(W_0(\delta_i)) = W_2(W_0(\delta_0)) - id, \;\; i = 1, 2, \ldots, N. \tag{3.3.54}$$

Obviously $\delta_0 > \delta_1 > \ldots > \delta_N$. Using (3.2.4), it is easy to prove that there exists a $T(\delta_i) > 0$ for each $\delta_i (i = 1, 2, \ldots, N)$ such that for any $\tau \geq 0$, when $|H(t)| \leq a(\delta_i)$ for $t \geq \tau$,

$$|x(t)| \leq k_2 \delta_i + k_2 a(\delta_i) \;\; for \;\; t \geq \tau + T(\delta_i) \tag{3.3.55}$$

Let

$$h = q(W_1(\eta) - d), \quad T' = \sum_{i=1}^{N} T(\delta_i)$$

and

$$T^* = T' + h + (W_2(W_0(\delta_0)) + 1)/W_3(W_2^{-1}(W_1(\eta) - d)).$$

We claim that there exists some $t'' \in [t_0 + h + T', t_0 + T^*]$ such that

$$V(t'', H(t'')) \leq W_2(W_0(\delta_0)) - d. \tag{3.3.56}$$

Otherwise,

$$W_1(\eta) - d \leq W_2(W_0(\delta_0)) - d < V(t, H(t)) \leq W_2(|H(t)|),$$

where $t_0 + h + T' \le t \le t_0 + T^*$. Then

$$W_2^{-1}(W_1(\eta) - d) < |H(t)| \ for \ t_0 + h + T' \le t \le t_0 + T^*.$$

We have

$$W_1(|H(t)|) \le V(t, H(t)) \le W_2(W_0(\delta_0)) \le W_1(a(\delta_0)), \ t \ge t_0,$$

and then

$$|H(t)| \le a(\delta_0) \ for \ t \ge t_0.$$

By (3.2.3), we have

$$|x(t)| \le k_2\delta_0 + k_2 a(\delta_0) \ for \ t \ge t_0. \tag{3.3.57}$$

Then for $t \in [t_0 + h + T', t_0 + T^*]$, we have, noting that $h = q(W_1(\eta) - d) \ge q(V(t, H(t)))$,

$$\begin{aligned} P(V(t, H(t))) &\ge P(W_2(W_0(\delta_0)) - d) \\ &> W_2(k_2\delta_0 + k_2 a(\delta_0)) \\ &\ge W_2(|x(s)|) \\ &\ge V(s, x(s)) \ for \ t - q(V(t, H(t))) \le s \le t. \end{aligned}$$

By (iii), we have

$$V(t_0 + T^*, H(t_0 + T^*)) \le W_2(W_0(\delta_0)) - [W_2(W_0(\delta_0)) + 1] < 0.$$

This contradiction implies that our claim is true. We claim that

$$V(t, H(t)) \le W_2(W_0(\delta_0)) - d \ for \ all \ t \ge t''.$$

If not, there exists a $t_1 > t''$ such that $V'(t_1, H(t_1)) > 0$ and $V(t_1, H(t_1)) > W_2(W_0(\delta_0)) - d$. We have

$$\begin{aligned} P(V(t_1, H(t_1))) &\geq P(W_2(W_0(\delta_0)) - d) \\ &> W_2(k_2\delta_0 + k_2 a(\delta_0)) \\ &\geq W_2(|x(s)|) \\ &\geq V(s, x(s)) \ \ for \ \ t_1 - h \leq s \leq t_1. \end{aligned}$$

By (iii),

$$V'(t_1, H(t_1)) \leq 0.$$

This contradiction implies that

$$\begin{aligned} V(t, H(t)) &\leq W_2(W_0(\delta_0)) - d \\ &= W_2(W_0(\delta_1)) \\ &\leq W_1(a(\delta_1)) \ \ for \ \ all \ \ t \geq t_0 + T^*. \end{aligned} \tag{3.3.58}$$

By (ii), we easily have

$$|H(t)| \leq a(\delta_1) \ \ for \ \ all \ \ t \geq t_0 + T^*$$

and then by (3.3.55),

$$|x(t)| \leq k_2\delta_1 + k_2 a(\delta_1) \ \ for \ \ all \ \ t \geq t_0 + T^* + T(\delta_1).$$

Similarly, we can prove

$$\begin{aligned} V(t, H(t)) &< W_2(W_0(\delta_0)) - id \\ &= W_2(W_0(\delta_i)) \\ &\leq W_1(a(\delta_i)), \ \ t \geq t_0 + iT^*, \ \ i = 0, 1, 2, \ldots, N. \end{aligned} \tag{3.3.59}$$

Then

$$W_1(|H(t)|) \leq V(t, H(t)) < W_2(W_0(\delta_0)) - Nd \leq W_1(\eta), \ \ t \geq t_0 + NT^*.$$

hence

$$|H(t)| \leq \eta < \eta', \ \ t \geq t_0 + NT^*.$$

By Lemma 3.2.1, the zero solution of (3.2.1) is $U.A.S.$ The proof is complete.

3.4 Applications

We review the proofs of Theorem 3.3.1–Theorem 3.3.3. $V(t, \psi)$ or $V(t, x)$ is defined in $\mathbb{R} \times C_A$ or $\mathbb{R} \times \{x \in \mathbb{R}^n : |x| < A\}$ where A is some positive number and it doesn't matter how much A is. In applications, we can choose a small $A < 1$, if necessary. It will be convenient for us to construct the Liapunov functionals or functions.

Example 3.4.1 *Consider the scalar equation*

$$\begin{aligned} \frac{d}{dt}\left[x(t) - \int_{-\infty}^{t} B(t-s)x^n(s)ds\right] &= \\ -ax^k(t) + \int_{-\infty}^{t} C(t-s)x^m(s)ds, \ \ t \geq t_0 \geq 0, \qquad (3.4.2) \end{aligned}$$

where m and n are positive integers, k is a positive odd integer. If the following conditions are satisfied

(i) $a > 0$, $\int_0^\infty |B(s)|ds = P_1 < 1$, $\int_0^\infty |C(s)|ds = P_2 < \infty$,
$\int_t^\infty |B(u)|du \in L^1([0, \infty))$ *and* $\int_t^\infty |C(u)|ds \in L^1([0, \infty))$,

(ii) $m, n \geq k + 2$,

then the zero solution of (3.4.2) *is* $U.A.S.$

Proof. By example 3.2.1,

$$D(t,\psi) = \psi(0) - \int_0^\infty B(u)\psi^n(-u)du$$

is weak-uniformly stable and weak-uniformly asymptotically stable. Let

$$V(t,\psi) = \frac{1}{2}D^2(t,\psi) + \int_0^\infty \int_s^\infty [|B(u)| + |C(u)|]du|\psi(-s)|^{k+2}ds.$$

We can consider $V(t,\psi)$ defined in $[0,\infty) \times C_A$, where $0 < A < 1$ satisfies the following conditions : $a/2 > A(P_1 + P_2), 1 \geq aA^k + \frac{1}{2}A^m P_2$ and $1 \geq A + \frac{1}{2}A^n P_1$. It is easy to verify

$$\begin{aligned}
W_1(|D(t,\psi)|) &\leq V(t,\psi) \\
&\leq |\psi(0)|^2 + \left(\int_0^\infty |B(s)||\psi(-s)|^n ds\right)^2 \\
&\quad + \int_0^\infty \int_s^\infty [|B(u)| + |C(u)|]du|\psi(-s)|^{k+2}ds \\
&\leq W_2(|\psi(0)|) + W_3\left[\int_0^\infty \phi(s)W_4(|\psi(-s)|)ds\right]
\end{aligned}$$

where $W_1(z) = \frac{1}{2}z^2$, $W_2(z) = z^2$, $W_3(z) = z^2 + z$, $W_4(z) = z^n + z^{k+2}$ and $\phi(s) = |B(s)| + \int_s^\infty [|B(u)| + |C(u)|)]du$. When $\|x_t\| < A < 1$, we have

$$\begin{aligned}
V'(t,x_t) &= \left[x(t) - \int_{-\infty}^t B(t-s)x^n(s)ds\right]\left[-ax^k(t) + \int_{-\infty}^t C(t-s)x^m(s)ds\right] \\
&\quad + (P_1 + P_2)|x(t)|^{k+2} - \int_{-\infty}^t [|B(t-s)| + |C(t-s)|]|x(s)|^{k+2}ds \\
&\leq -a|x(t)|^{k+1} + aA^k \int_{-\infty}^t |B(t-s)||x(s)|^{k+2}ds \\
&\quad + A\int_{-\infty}^t |C(t-s)||x(s)|^{k+2}ds + \frac{1}{2}A^m P_2 \int_{-\infty}^t |B(t-s)||x(s)|^{k+2}ds \\
&\quad + \frac{1}{2}A^n P_1 \int_{-\infty}^t |C(t-s)||x(s)|^{k+2}ds + A(P_1 + P_2)|x(t)|^{k+1} \\
&\quad - \int_{-\infty}^t [|B(t-s)| + |C(t-s)|]|x(s)|^{k+2}ds \\
&\leq -[a - A(P_1 + P_2)]|x(t)|^{k+1}
\end{aligned}$$

$$
\begin{aligned}
& -\int_{-\infty}^{t}\left(1-aA^{k}-\frac{1}{2}A^{m}P_{2}\right]|B(t-s)||x(s)|^{k+2}ds \\
& -\int_{-\infty}^{t}\left(1-A-\frac{1}{2}A^{n}P_{1}\right)|C(t-s)||x(s)|^{k+2}ds \\
\leq & -\frac{a}{2}|x(t)|^{k+1} \\
= & -W_5(|x(t)|),
\end{aligned}
$$

where $W_5(z) = (a/2)z^{k+1}$. By Theorem 3.3.1, the zero solution of (3.4.2) is $U.A.S.$

We can also use Theorem 3.3.2 to prove that the zero solution of (3.4.2) is $U.A.S.$ For the sake of simplicity, let $k = 1$. Here we consider $V(t,\psi)$ defined in $[0,\infty)\times C_A$, where $0 < A < 1$ satisfies the following conditions: $a \geq 2A(P_1+P_2), 2 \geq a(2A+P_1A^n)$ and $1 \geq A + P_1A^m$. It is easy to verify

$$
\frac{1}{2}|D(t,\psi)|^2 \leq V(t,\psi) \leq \frac{1}{2}\left(\|\psi\| + P_1\|\psi\|^n\right)^2 + L\|\psi\|^3
$$

Where $L = \int_0^\infty \int_s^\infty (|B(u)| + |C(u)|)duds$. When $\|x_t\| < A < 1$, we have

$$
\begin{aligned}
V'(t,x_t) = & -\frac{a}{2}D^2(t,x_t) - \frac{a}{2}x(t)D(t,x_t) \\
& -\frac{a}{2}D(t,x_t)\int_{-\infty}^{t} B(t-s)x^n(s)ds \\
& + D(t,x_t)\int_{-\infty}^{t} C(t-s)x^m(s)ds + (P_1+P_2)|x(t)|^3 \\
& -\int_{-\infty}^{t}(|B(t-s)| + |C(t-s)|)|x(s)|^3 ds \\
\leq & -\frac{a}{2}|D(t,x_t)|^2 - \frac{a}{2}x^2(t) + \frac{a}{2}(2A+P_1A^n)\int_{-\infty}^{t}|B(t-s)||x(s)|^3 ds \\
& + (A+P_1A^m)\int_{-\infty}^{t}|C(t-s)||x(s)|^3 ds + A(P_1+P_2)|x(t)|^2 \\
& -\int_{-\infty}^{t}[|B(t-s)| + |C(t-s)|]|x(s)|^3 ds \\
\leq & -\frac{a}{2}|D(t,x_t)|^2 - \left[\frac{a}{2} - A(P_1+P_2)\right]x^2(t) \\
& -\left[1-\frac{a}{2}(2A+P_1A^n)\right]\int_{-\infty}^{t}|B(t-s)||x(s)|^3 ds
\end{aligned}
$$

$$
\begin{aligned}
& -[1-(A+P_1A^m)]\int_{-\infty}^{t}|C(t-s)||x(s)|^3ds \\
\leq & -\frac{a}{2}|D(t,x_t)|^2.
\end{aligned}
$$

We easily see that (iii) of Theorem 3.3.2 is also satisfied. Finally, we verify (iv) of Theorem 3.3.2. For any given $\sigma_1 > 0$ and $B' > 0$, choose $r_1 > 0$ such that

$$
(B')^2\int_{r_1}^{\infty}\int_{s}^{\infty}(|B(u)|+|C(u)|)duds \leq \sigma_1.
$$

Then when $\tau_1 \in [0,\infty)$, $\|\psi\|_{(-\infty,\tau_1-t]} \leq B'$ and $t \geq \tau_1 + r_1$, we have

$$
\begin{aligned}
V(t,\psi) &= \frac{1}{2}D^2(t,\psi) + \int_{0}^{t-\tau_1}\int_{s}^{\infty}[|B(u)|+|C(u)|]du|\psi(-s)|^3ds \\
&\quad + \int_{t-\tau_1}^{\infty}\int_{s}^{\infty}[|B(u)|+|C(u)|]du|\psi(-s)|^3ds \\
&\leq \frac{1}{2}|D(t,\psi)|^2 + L\|\psi\|^3_{[\tau_1-t,0]} \\
&\quad + \|\psi\|^3_{(-\infty,\tau_1+t]}\int_{\tau_1}^{\infty}\int_{s}^{\infty}[|B(u)|+|C(u)|]duds \\
&\leq \frac{1}{2}|D(t,\psi)|^2 + L\|\psi\|^3_{[\tau_1-t,0]} + \sigma_1.
\end{aligned}
$$

By Theorem 3.3.2, the zero solution of (3.4.2) with $k = 1$ is *U.A.S.*

Example 3.4.2 *Consider the scalar equation*

$$
\begin{aligned}
&\frac{d}{dt}\left[x(t) - \int_{-\infty}^{t}B(t-s)x(s)ds\right] = \\
&\quad -x(t) + cx(t-r) + \int_{-\infty}^{t-r_1}B(t-s)x(s)ds, \;\; t \geq t_0 \geq 0, \qquad (3.4.3)
\end{aligned}
$$

where $r > 0$ *and* $r_1 \geq 0$. *If*

(i) $\int_0^\infty |B(s)|ds = P_1 < 1$ *and*

(ii) *there is some* $c' > 1$ *such that*

$$
|c|c' + \int_0^r |B(u)|du < 1 \text{ and } c' > \frac{2+P_1}{1-P_1^2},
$$

then the zero solution of (3.4.3) *is U.A.S.*

Proof. By Example 3.2.1, $D(t,\psi)$ is weak-uniformly stable and weak-uniformly asymptotically stable. We have

$$|D(t,\psi)| \le (1+P_1)\|\psi\| = W_0(\|\psi\|)$$

where $W_0(s) = (1+P_1)s$. Let $V(t,x) = |x|, W_1(s) = W_2(s) = s, P(s) = c's, q(s) = \max\{r, r_1\}$ and $a(s) = (1+P_1)s$. Then we can verify that $W_2(W_0(s)) \le W_1(a(s)), P(W_2(W_0(s))) > W_2(k_2 s + k_2 a(s))$ for $s > 0, W_1(|x|) \le V(t,x) \le W_2(|x|)$. We have

$$\begin{aligned} V'(t, D(t,x_t)) &\le -|D(t,x_t)| + |c||x(t-r)| + \int_{t-r_1}^{t} |B(t-s)||x(s)|ds \\ &\le -\left[1 - \left(|c|c' + \int_0^{r_1} |B(u)|du\right)\right] |D(t,x_t)| \\ &= -W_3(|D(t,x_t)|) \\ &\quad \textit{where } W_3(s) = \left[1 - \left(|c|c' + \int_0^{r_1} |B(u)|du\right)\right] s, \end{aligned}$$

whenever $P(|D(t,x_t)|) > |x(s)|$ for $t - \max\{r, r_1\} \le s \le t$. By Theorem 3.3.3, the zero solution of (3.4.3) is $U.A.S.$

Chapter 4

Nonoscillation and Oscillation of First Order Linear Neutral Equations

4.1 Introduction

We study nonoscillations and oscillations of the equation

$$\frac{d}{dt}\left[x(t) - \sum_{i=1}^{k} c_i(t)x(t-\gamma_i(t))\right] + Q(t)x(t-\sigma) = 0, \quad t \geq t_0, \tag{4.1.1}$$

where $Q(t), c_i(t) \in C[t_0, \infty), Q(t) > 0, c_i(t) \geq 0, 0 < \gamma_0 < \gamma_i(t) \leq \gamma$, and $\sigma \geq 0, i \in I_k = \{1, 2, \ldots, k\}$.

There are only a few results for the existence of nonoscillatory solutions of first order neutral equations with variable coefficients [28, 31, 33]. In Section 2, we obtain several new existence theorems of nonoscillatory solutions. Theorem 4.2.2 is an extension of a well-known result for delay differential equations to neutral equations. Theorem 4.2.3 presents another sufficient condition for (4.1.1) to have nonoscillatory solutions which is "sharp" in the sense that when all the coefficients and delay arguments of (4.1.1) are constants, the condition is also necessary. Theorem 4.2.4 is a comparison theorem for neutral equation (4.1.1) to have nonoscillatory solutions.

In Section 3, we study oscillations of (4.1.1) (Theorem 4.3.1). There have been a lot of activities [27, 28, 29, 31, 32] in the study of oscillations of first order neutral equations with variable coefficients. Our Theorem 4.3.1 generalizes and improves a main result of [31] under weaker conditions. When the coefficients and delay arguments of (4.1.1) are constants, the conditions of Theorem 4.3.1 are both necessary and sufficient.

In Remark 1, we point out some shortcomings appearing in the proofs of some theorems in [31, 36].

A solution of (4.1.1) is called oscillatory if it has arbitrary large zeros and nonoscillatory if it is eventually positive or eventually negative.

4.2 Existence of Nonoscillatory Solutions

Set

$$\sum_{i=1}^{k} c_i(t) \le 1. \tag{4.2.1}$$

Lemma 4.2.1 *Assume that* (4.2.1) *holds and* $x(t)$ *is an eventually positive solution of* (4.1.1). *Then* $y(t) > 0$ *for all sufficiently large* t, *where*

$$y(t) = x(t) - \sum_{i=1}^{k} c_i(t)x(t - \gamma_i(t)). \tag{4.2.2}$$

Proof. From (4.1.1), we have $y'(t) < 0$ for all large t. Then $y(t) > 0$ or $y(t) < 0$ for all large t. We claim that $y(t) < 0$ is impossible. Otherwise, if $x(t)$ is unbounded, there is a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = +\infty$ and

$x(t_n) = \max_{t \le t_n} x(t)$. Then

$$\begin{aligned} 0 &> y(t_n) \\ &= x(t_n) - \sum_{i=1}^{k} c_i(t_n) x(t_n - \gamma_i(t_n)) \\ &\ge x(t_n)\left[1 - \sum_{i=1}^{k} c_i(t_n)\right] \\ &\ge 0. \end{aligned} \tag{4.2.3}$$

If $x(t)$ is bounded, there is a sequence $\{t_j\}$ such that $\lim_{j\to\infty} t_j = +\infty$ and $\lim_{j\to\infty} x(t_j) = \limsup_{t\to\infty} x(t)$. Without loss of generality, we assume that $\{c_i(t_j)\}$ and $\{x(t_j - \gamma_i(t_j))\}$ $(i \in I_k)$ are convergent . Then

$$\begin{aligned} 0 &> \lim_{j\to\infty} y(t_j) \\ &\ge \limsup_{t\to\infty} x(t) \lim_{j\to\infty}\left[1 - \sum_{i=1}^{k} c_i(t_j)\right] \\ &\ge 0. \end{aligned} \tag{4.2.4}$$

By (4.2.3) and (4.2.4), we conclude that $y(t) < 0$ is impossible. The proof is complete.

Lemma 4.2.2 *Assume that* (4.2.1) *holds. Then* (4.1.1) *has a nonoscillatory solution if and only if the integral equation*

$$\begin{aligned} \lambda(t) &= \sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))} \lambda(t - \gamma_i(t-\sigma)) \exp\left(\int_{t-\gamma_i(t-\sigma)}^{t} \lambda(s)ds\right) \\ &\quad + Q(t)\exp\left(\int_{t-\sigma}^{t} \lambda(s)ds\right), \quad t \ge T. \end{aligned} \tag{4.2.5}$$

has a positive continuous solution $\lambda(t) \in C[T-\tau, \infty)$ *for some sufficiently large* T, *where* $\tau = \max\{\gamma, \sigma\}$.

Proof. Necessity. Without loss of generality, we assume that $x(t) > 0$ $(t \geq T - 2\tau)$ is a nonoscillatory solution of (4.1.1). Set

$$\lambda(t) = \frac{-y'(t)}{y(t)} \quad where \ y(t) = x(t) - \sum_{i=1}^{k} c_i(t)x(t-\gamma_i(t)).$$

By Lemma4.2.1, we have $\lambda(t) > 0$ $(t \geq T - \tau)$ and

$$\frac{y(t_1)}{y(t_2)} = \exp\left(\int_{t_1}^{t_2} \lambda(s)ds\right). \tag{4.2.6}$$

From (4.1.1), we have

$$y'(t) = -Q(t)x(t-\sigma). \tag{4.2.7}$$

Then

$$\begin{aligned}
\lambda(t) &= \frac{Q(t)x(t-\sigma)}{y(t)} \\
&= \frac{Q(t)}{y(t)}\left[y(t-\sigma) + \sum_{i=1}^{k} c_i(t-\sigma)x(t-\sigma-\gamma_i(t-\sigma))\right] \\
&= Q(t)\exp\left(\int_{t-\sigma}^{t} \lambda(s)ds\right) \\
&\quad + \sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))} \frac{Q(t-\gamma_i(t-\sigma))x(t-\sigma-\gamma_i(t-\sigma))}{y(t)} \\
&= Q(t)\exp\left(\int_{t-\sigma}^{t} \lambda(s)ds\right) \\
&\quad + \sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))} \frac{-y'(t-\gamma_i(t-\sigma))}{y(t-\gamma_i(t-\sigma))} \frac{y(t-\gamma_i(t-\sigma))}{y(t)} \\
&= Q(t)\exp\left(\int_{t-\sigma}^{t} \lambda(s)ds\right) \\
&\quad + \sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))} \\
&\quad \times \lambda(t-\gamma_i(t-\sigma))\exp\left(\int_{t-\gamma_i(t-\sigma)}^{t} \lambda(s)ds\right), \quad t \geq T.
\end{aligned} \tag{4.2.8}$$

Sufficiency. Assume that (4.2.5) has a positive continuous solution $\lambda(t) \in C[T-\tau, \infty)$. Let $y(t) = \exp(-\int_{T-\tau}^{t} \lambda(s)ds)$. Multiplying both sides of (4.2.5) by $\exp(-\int_{t-\tau}^{t}\lambda(s)ds)$ and rearranging it, we have

$$y'(t)-\sum_{i=1}^{k}\frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}y'(t-\gamma_i(t-\sigma))+Q(t)y(t-\sigma) \;=\; 0,\ t\geq T. \tag{4.2.9}$$

Then

$$y(t)=\frac{-y(t+\sigma)}{Q(t+\sigma)}-\sum_{i=1}^{k}c_i(t)\frac{-y'(t-\gamma_i(t)+\sigma)}{Q(t-\gamma_i(t)+\sigma)},\ t\geq T-\sigma. \tag{4.2.10}$$

Setting $x(t)=-y'(t+\sigma)/Q(t+\sigma)$ and noting $y'(t)<0$ $(t\geq T-\tau)$, we have

$$x(t)>0,\quad t\geq T-\tau-\sigma$$

and

$$y'(t)=-Q(t)x(t-\sigma),\quad t\geq T-\tau. \tag{4.2.11}$$

From (4.2.10) and (4.2.11), we have

$$\frac{d}{dt}\left[x(t)-\sum_{i=1}^{k}c_i(t)x(t-\gamma_i(t))\right]+Q(t)x(t-\sigma)=0,\quad t\geq T-\tau.$$

The proof is complete.

Define a mapping for nonnegative $\lambda(t)\in C[T-\tau,\infty)$:

$$(P\lambda)(t)=\begin{cases}\sum_{i=1}^{k}\frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}\lambda(t-\gamma_i(t-\sigma))\\ \quad\times\exp\left(\int_{t-\gamma_i(t-\sigma)}^{t}\lambda(s)ds\right)\\ \quad+Q(t)\exp\left(\int_{t-\sigma}^{t}\lambda(s)ds\right),\quad t\geq T,\\ (P\lambda)(T),\quad T-\tau\leq t<T,\ \tau=\max\{\gamma,\sigma\}.\end{cases}$$

Define a sequence of functions:

$$\lambda_0(t)=0,\quad T-\tau\leq t<\infty$$

$$\lambda_{n+1}(t)=\begin{cases}(P\lambda_n)(t),& t\geq T\\ (P\lambda_n)(T),& T-\tau\leq t<T,\quad n\;=\;0,\ 1,\ 2,\ \ldots.\end{cases} \tag{4.2.12}$$

It is easy to prove that $\lambda_{n+1}(t)\geq\lambda_n(t)$ for $t\geq T-\tau$.

Lemma 4.2.3 *Assume that* (4.2.1) *holds and* (4.2.12) *converges to a finite limit function everywhere on* $[T-\tau,\infty)$. *Then* (4.2.5) *has a positive continuous solution* $\lambda(t)\in C[T-\tau,\infty)$.

Proof. Assume that (4.2.12) converges to a finite function $\lambda(t)$ everywhere on $[T-\tau,\infty)$. Then

$$\lambda_n(t)\le\lambda(t),\quad t\ge T-\tau.$$

Letting $n\to\infty$ in (4.2.12), we have by Lebesgue Theorem

$$\lambda(t)=\begin{cases}\sum_{i=1}^{k}\frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}\lambda(t-\gamma_i(t-\sigma))\\ \quad\times\exp\left(\int_{t-\gamma_i(t-\sigma)}^{t}\lambda(s)ds\right)\\ \quad+Q(t)\exp\left(\int_{t-\sigma}^{t}\lambda(s)ds\right),\quad t\ge T.\\ \lambda(T),\quad T-\tau\le t<T.\end{cases}\tag{4.2.13}$$

Because $\lambda(t)$ is continuous on $[T-\tau,T]$, therefore $\lambda(t-\gamma_i(t-\sigma))(i\in I_k)$ is continuous at every point in $[T,T+\gamma_0]$. From (4.2.13), $\lambda(t)$ is continuous at every point in $[T-\tau,T+\gamma_0]$. By induction, it is easy to prove that $\lambda(t)$ is continuous at every point in $[T-\tau,T+n\gamma_0](n=0,2,\ldots)$. Hence $\lambda(t)$ is a positive continuous solution of (4.2.5) . The proof is complete.

Remark 1. We would like to point out that from (4.2.13) and induction we can merely conclude that $\lambda_n(t)\le\lambda(t)$ for $t\ge T+(n-1)\tau$, but not for $t\ge T-\tau$. So we cannot prove that (4.2.12) converges to a finite limit function everywhere on $[T-\tau,\infty)$ if (4.2.5) has a positive continuous solution $\lambda(t)\in C[T-\tau,\infty)$. Hence Lemma 2 of [31] is not true, and then the proofs of Theorem 1 and 2 of [31] are incomplete. Similar mistakes also appeared in [36].

From Lemma 4.2.2 and Lemma 4.2.3, we have

Theorem 4.2.1 *Assume that* (4.2.1) *holds and* (4.2.12) *converges to finite limit function everywhere on* $[T-\tau,\infty)$ *for some sufficiently large* T. *Then* (4.1.1) *has a nonoscillatory solution.*

Theorem 4.2.2 *Assume that* (4.2.1) *holds and there exist* $0 \le \varepsilon < 1$ *and* $T \ge t_0 + \tau$ *such that*

$$\int_{t-\tau}^{t} Q(s)ds \le \frac{1-\varepsilon}{e} \quad when \quad t \ge T-\tau \tag{4.2.14}$$

and

$$e^{-\varepsilon}\left[e\sum_{i=1}^{k} c_i(t-\sigma)+1\right] \le 1. \tag{4.2.15}$$

Then (4.1.1) *has a nonoscillatory solution.*

Proof. Set $V_0(t) = -eQ(t)$, $t \ge T-\tau$.

$$V_{n+1}(t) = \begin{cases} \sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))} V_n(t-\gamma_i(t-\sigma)) \\ \quad \times \exp\left(-\int_{t-\gamma_i(t-\sigma)}^{t} V_n(s)ds\right) \\ \quad -Q(t)\exp\left(-\int_{t-\sigma}^{t} V_n(s)ds\right), \quad t \ge T, \\ \phi_{n+1}(t) = \max\{V_{n+1}(T), V_0(t)\}, \\ \quad T-\tau \le t \le T, \quad n = 0,\ 1,\ 2,\ \ldots. \end{cases} \tag{4.2.16}$$

From (4.2.14), we have

$$\int_{t-\tau}^{t} eQ(s)ds \le 1-\varepsilon.$$

When $t \ge T$,

$$\begin{aligned} V_1(t) &= \sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}(-eQ(t-\gamma_i(t-\sigma))) \\ &\quad \times \exp\left(\int_{t-\gamma_i(t-\sigma)}^{t} eQ(s)ds\right) \\ &\quad - Q(t)\exp\left(\int_{t-\sigma}^{t} eQ(s)ds\right) \end{aligned}$$

$$
\begin{aligned}
&\geq e^{-\varepsilon}\left[e\sum_{i=1}^{k}c_i(t-\sigma)+1\right](-eQ(t))\\
&\geq -eQ(t)\\
&= V_0(t).
\end{aligned}
$$

Then $V_1(T) \geq V_0(T)$ and it is easy to prove that $V_1(t)$ is continuous on $[T-\tau,\infty)$ and $V_1(t) \geq V_0(t)$ for $t \geq T-\tau$. By induction we can easily prove that $V_{n+1}(t)(n=0,1,2,\dots)$ is continuous on $[T-\tau,\infty)$ and $V_{n+1}(t) \geq V_n(t)$ for $t \geq T-\tau$. Then

$$V_0(t) \leq V_1(t) \leq \ldots \leq V_n(t)\ldots \leq 0. \quad t \geq T-\tau. \tag{4.2.17}$$

Set $\lim_{n\to\infty} V_n(t) = V(t), t \geq T-\tau$. By (4.2.16) and the Lebesgue Theorem, we have

$$
\begin{aligned}
V(t) &= \sum_{i=1}^{k}\frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}V(t-\gamma_i(t-\sigma))\exp\left(-\int_{t-\gamma_i(t-\sigma)}^{t}V(s)ds\right)\\
&\quad -Q(t)\exp\left(-\int_{t-\sigma}^{t}V(s)ds\right), \quad t \geq T.
\end{aligned} \tag{4.2.18}
$$

From (4.2.16) and (4.2.17), we have

$$\sup_{T-\tau\leq t\leq T}|V_{n+m}(t)-V_n(t)| \leq |V_{n+m}(T)-V_n(T)|.$$

Then $V_n(t)$ converges to $V(t)$ uniformly on $[T-\tau,T]$. Hence $V(t)$ is continuous on $[T-\tau,T]$. From (4.2.18) and using a method similar to that of Lemma 4.2.3, we easily prove that $V(t)$ is continuous on $[T-\tau,\infty)$ and $V(t)<0$. Set $u(t)=-V(t)$. Then $u(t)$ is a positive continuous solution of (4.2.5). By Lemma 4.2.2, (4.1.1) has a nonoscillatory solution. The proof is complete.

The following well-known result can be derived immediately from Theorem 4.2.2.

Corollary 4.2.1 *Consider the equation*

$$x'(t) + Q(t)x(t-\sigma) = 0 \tag{4.2.19}$$

where $Q(t) \in C[t_0, \infty), Q(t) > 0$ *and* $\sigma > 0$. *If*

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} Q(s)ds < \frac{1}{e},$$

then (4.2.19) *has a nonoscillatory solution.*

Theorem 4.2.3 *Assume that* (4.2.1) *holds and there exist a* $\mu > 0$ *and a sufficiently large* T *so that*

$$\sup_{t\geq T}\left[\sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))} \exp(\mu\gamma_i(t-\sigma)) + \frac{1}{\mu}Q(t)\exp(\mu\sigma)\right] \leq 1. \tag{4.2.20}$$

Then (4.1.1) *has a nonoscillatory solution.*

Proof. Set $\mu_0 = \sup_{t\geq T} Q(t)$,

$$\begin{aligned} \mu_{n+1} &= \sup_{t\geq T}\left[\sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}\mu_n \exp(\mu_n\gamma_i(t-\sigma)) + Q(t)\exp(\mu_n\sigma)\right], \\ & \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{4.2.21}$$

Comparing (4.2.12) with (4.2.21), we easily have

$$\lambda_1(t) \leq \mu_0 \qquad for \quad t \geq T - \tau.$$

When $t \geq T$,

$$\begin{aligned} \lambda_2(t) &= \sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}\lambda_1(t-\gamma_i(t-\sigma))\exp\left(\int_{t-\gamma_i(t-\sigma)}^{t} \lambda_1(s)ds\right) \\ & \quad + Q(t)\exp\left(\int_{t-\sigma}^{t} \lambda_1(s)ds\right) \\ &\leq \sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}\mu_0 \exp(\mu_0\gamma_i(t-\sigma)) + Q(t)\exp(\mu_0\sigma) \\ &\leq \mu_1. \end{aligned} \tag{4.2.22}$$

Hence $\lambda_2(t) \le \mu_1$ for $t \ge T - \tau$. By induction, we have

$$\lambda_{n+1}(t) \le \mu_n \quad for \quad t \ge T - \tau, \;\; n = 0, 1, 2, \ldots.$$

On the other hand, we have from (4.2.20)

$$\sup_{t \ge T} \left[\sum_{i=1}^{k} \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))} \mu \exp(\mu\gamma_i(t-\sigma)) + Q(t)\exp(\mu\sigma) \right] \le \mu. \tag{4.2.23}$$

We easily have

$$\mu_0 \le \mu. \tag{4.2.24}$$

By (4.2.23), (4.2.24) and induction, we have

$$\mu_n \le \mu, \qquad n = 0, 1, 2, \ldots.$$

Then

$$\lambda_{n+1}(t) \le \mu \qquad for \quad t \ge T - \tau.$$

Hence (4.2.12) converges to a finite limit function everywhere. By Theorem 4.2.1, (4.1.1) has a nonoscillatory solution. The proof is complete.

Corollary 4.2.2 *Consider the equation*

$$\frac{d}{dt}\left[x(t) - \sum_{i=1}^{k} c_i x(t-\gamma_i) \right] + q x(t-\sigma) = 0, \quad t \ge t_0, \tag{4.2.25}$$

where $c_i \ge 0, \gamma_i > 0$ $(i \in I_k), \sigma \ge 0$ *and* $q > 0$. *If there exists a* $\mu > 0$ *such that*

$$\sum_{i=1}^{k} c_i \exp(\mu\gamma_i) + \frac{1}{\mu} q \exp(\mu\sigma) \le 1, \tag{4.2.26}$$

then (4.2.25) *has a nonoscillatory solution.*

Proof. (4.2.26) implies that $\sum_{i=1}^{k} c_i < 1$. By Theorem 4.2.3, Corollary 4.2.2 is true. The proof is complete.

Corollary 4.2.3 *Assume that* $\sum_{i=1}^k c_i(t) \le 1 - \delta$ $(0 < \delta \le 1)$ *and* $Q(t)$ *monotonically tends to zero. Then* (4.1.1) *has a nonoscillatory solution.*

Proof. Choose a positive number μ such that $1 - (1-\delta)\exp(\mu\gamma) > 0$. Then choose a sufficiently large T so that

$$\sup_{t\ge T} \frac{1}{\mu} Q(t)\exp(\mu\sigma) \le 1 - (1-\delta)\exp(\mu\gamma).$$

Then we have

$$\begin{aligned}&\sup_{t\ge T}\left[\sum_{i=1}^k \frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}\exp(\mu\gamma_i(t-\sigma)) + \frac{1}{\mu}Q(t)\exp(\mu\sigma)\right]\\ &\le (1-\delta)\exp(\mu\gamma) + \sup_{t\ge T}\frac{1}{\mu}Q(t)\exp(\mu\sigma)\\ &\le 1.\end{aligned}$$

By Theorem 4.2.3, (4.1.1) has a nonoscillatory solution. The proof is complete.

Theorem 4.2.4 (*i*) *Assume that* $Q(t)$ *is nonincreasing,* $Q(t) \le q, \gamma_i(t) \le \gamma_i$ $(i \in I_k), c_i(t) \le c_i$ $(i \in I_k)$, *and* $\sum_{i=1}^k c_i(t) \le 1$. *If* (4.2.25) *has a nonoscillatory solution, then* (4.1.1) *has a nonoscillatory solution.*
(*ii*) *Assume that* $Q(t)$ *is nonincreasing,* $Q(t) \le q, \gamma_i(t) \le \gamma$ $(i \in I_k)$, *and* $\sum_{i=1}^k c_i(t) \le c \le 1$. *If the equation*

$$\frac{d}{dt}[x(t) - cx(t-\gamma)] + qx(t-\sigma) = 0, \qquad t \ge t_0, \tag{4.2.27}$$

has a nonoscillatory solution, then (4.1.1) *has a nonoscillatory solution.*

Proof. (*i*) Assume that (4.2.25) has a nonoscillatory solution. According to Corollary 4.3.2 in Section 3 (or refer to [30, 34, 35]), there exists a $\mu > 0$ such that

$$\sum_{i=1}^k c_i\exp(\mu\gamma_i) + \frac{1}{\mu}q\exp(\mu\sigma) \le 1. \tag{4.2.28}$$

Then

$$\begin{aligned}&\sup_{t\geq T}\left[\sum_{i=1}^{k}\frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}\exp(\mu\gamma_i(t-\sigma))+\frac{1}{\mu}Q(t)\exp(\mu\sigma)\right]\\&\leq\sum_{i=1}^{k}c_i\exp(\mu\gamma_i)+\frac{1}{\mu}q\exp(\mu\sigma)\\&\leq 1\end{aligned}$$

By Theorem 4.2.3, (4.1.1) has a nonoscillatory solution. Analogously, we can prove that the conclusion of (ii) is true . The proof is complete.

4.3 Oscillation

The following result generalizes and improves Theorem 2 of Grove *et al.*[31].

Theorem 4.3.1 *Assume that* (4.2.1) *holds and there is a sufficiently large* T *such that*

$$\inf_{t\geq T,\mu>0}\left[\sum_{i=1}^{k}\frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))}\exp(\mu\gamma_i(t-\sigma))+\frac{1}{\mu}Q(t)\exp(\mu\sigma)\right]>1. \quad (4.3.1)$$

Then all solutions of (4.1.1) *oscillate.*

Proof. If (4.1.1) has a nonoscillatory solution $x(t)$, we assume that $x(t)>0$ for $t\geq T-2\tau$. By Lemma 4.2.2, there exists a positive continuous function $\lambda(t)\in C[T-\tau,\infty)$ such that

$$\begin{aligned}\lambda(t) &= \sum_{i=1}^{k}E_i(t)\lambda(t-\gamma_i(t-\sigma))\exp\left(\int_{t-\gamma_i(t-\sigma)}^{t}\lambda(s)ds\right)\\&\quad+Q(t)\exp\left(\int_{t-\sigma}^{t}\lambda(s)ds\right),\quad t\geq T,\end{aligned}\qquad (4.3.2)$$

where

$$E_i(t)=\frac{c_i(t-\sigma)Q(t)}{Q(t-\gamma_i(t-\sigma))},\qquad i\in I_k.$$

Set

$$\mu_0 = 0,$$

$$\mu_n = \inf_{t\geq T}\left[\sum_{i=1}^{k} E_i(t)\mu_{n-1}\exp(\mu_{n-1}\gamma_i(t-\sigma)) + Q(t)\exp(\mu_{n-1}\sigma)\right], \quad n = 1, 2, \ldots. \tag{4.3.3}$$

By induction, it is easy to prove that

$$\mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots. \tag{4.3.4}$$

When $t \geq T - \tau$ and $\mu_0 < \lambda(t)$, using (4.3.2), (4.3.3), and induction, we easily prove that

$$\mu_n \leq \lambda(t) \quad for \quad t \geq T + (n-1)\tau, \quad n = 1, 2, \ldots.$$

Set $\lim_{n\to\infty}\mu_n = \mu^*$. If $\mu^* = +\infty$, then $\lim_{t\to\infty}\lambda(t) = +\infty$. Integrate (4.1.1) from $t - \sigma/2$ to t and then divide it by $y(t - \sigma/2)$. Noting that $y(t) \leq x(t)$ and $y(t)$ is decreasing, we have

$$\frac{y(t)}{y(t-\sigma/2)} - 1 + \frac{y(t-\sigma)}{y(t-\sigma/2)}\int_{t-\sigma/2}^{t} Q(s)ds \leq 0, \quad t \geq T.$$

Using (4.2.6), we have

$$\exp\left(\int_t^{t-\sigma/2}\lambda(s)ds\right) - 1 + \exp\left(\int_{t-\sigma}^{t-\sigma/2}\lambda(s)ds\right)\int_{t-\sigma/2}^{t} Q(s)ds \leq 0, t \geq T. \tag{4.3.5}$$

We claim that there exists some $a > 0$ such that $Q(t) \geq a$ $(t \geq T - \tau)$. Otherwise, there exists a sequence $\{t_n\}$ such that $Q(t_n) = \min_{t\leq t_n} Q(t)$ and $\lim_{n\to\infty} Q(t_n) = 0$. We have

$$\lim_{n\to\infty}\frac{1}{\mu}Q(t_n)\exp(\mu\sigma) = 0, \qquad \mu > 0, \tag{4.3.6}$$

and

$$\sum_{i=1}^{k} E_i(t_n)\exp(\mu\gamma_i(t_n-\sigma)) \le \exp(\mu\gamma), \quad \mu > 0. \tag{4.3.7}$$

For any given $\varepsilon > 0$, in view of (4.3.6) and (4.3.7) we can select a sufficiently small $\mu > 0$ and a sufficiently large n so that

$$\sum_{i=1}^{k} E_i(t_n)\exp(\mu\gamma_i(t_n-\sigma)) + \frac{1}{\mu}Q(t_n)\exp(\mu\sigma) \le 1+\varepsilon. \tag{4.3.8}$$

Hence

$$\inf_{t\ge T,\mu>0}\left[\sum_{i=1}^{k} E_i(t)\exp(\mu\gamma_i(t-\sigma)) + \frac{1}{\mu}Q(t)\exp(\mu\sigma)\right] \le 1+\varepsilon. \tag{4.3.9}$$

Letting $\varepsilon \to 0$ in (4.3.9), we have

$$\inf_{t\ge T,\mu>0}\left[\sum_{i=1}^{k} E_i(t)\exp(\mu\gamma_i(t-\sigma)) + \frac{1}{\mu}Q(t)\exp(\mu\sigma)\right] \le 1$$

which contradicts (4.3.1). Let $t \to +\infty$ in (4.3.5). Then the first term of (4.3.5) tends to zero and the third term tends to $+\infty$. This is a contradiction. Hence $\mu^* < \infty$. Set

$$\varphi_n(t) = \sum_{i=1}^{k} E_i(t)\mu_{n-1}\exp(\mu_{n-1}\gamma_i(t-\sigma)) + Q(t)\exp(\mu_{n-1}\sigma), \tag{4.3.10}$$

and

$$\varphi(t) = \sum_{i=1}^{k} E_i(t)\mu^*\exp(\mu^*\gamma_i(t-\sigma)) + Q(t)\exp(\mu^*\sigma). \tag{4.3.11}$$

For any given $\varepsilon > 0$, there exists a $t_n \ge T$ for each $\varphi_n(t)$ such that

$$\varphi_n(t_n) \le \mu_n + \varepsilon \le \mu^* + \varepsilon. \tag{4.3.12}$$

In view of (4.3.12), $\{E_i(t_n)\}$ and $\{Q(t_n)\}$ are bounded. Without loss of generality, we assume that $\lim_{n\to\infty} E_i(t_n)$ $(i \in I_k)$, $\lim_{n\to\infty} Q(t_n)$, and $\lim_{n\to\infty}\gamma_i(t_n-\sigma)$ $(i \in I_k)$ exist. Set

$$\varphi^* = \lim_{n\to\infty}\left[\sum_{i=1}^{k} E_i(t_n)\mu^*\exp(\mu^*\gamma_i(t_n-\sigma)) + Q(t_n)\exp(\mu^*\sigma)\right].$$

Then $\lim_{n\to\infty}\varphi_n(t_n)=\varphi^*$. Hence $\inf_{t\geq T}\varphi(t)\leq\varphi^*\leq\mu^*+\varepsilon$. Letting $\varepsilon\to 0$, we have that $\inf_{t\geq T}\varphi(t)\leq\mu^*$. Then

$$\inf_{t\geq T}\left[\sum_{i=1}^{k}E_i(t)\exp(\mu^*\gamma_i(t-\sigma))+\frac{1}{\mu^*}Q(t)\exp(\mu^*\sigma)\right]\leq 1$$

which contradicts (4.3.1). The proof is complete.

Remark 2. The condition $0<k_1\leq Q(t)\leq k_2$ has been assumed in Theorem 2 of [31]. Here we do not require such an assumption in Theorem 4.3.1.

Corollary 4.3.1 *Assume that* $\sum_{i=1}^{k}c_i\leq 1$. *If*

$$\sum_{i=1}^{k}c_i\exp(\mu\gamma_i)+\frac{1}{\mu}q\exp(\mu\sigma)>1 \tag{4.3.13}$$

holds for all $\mu>0$, *then all solutions of* (4.2.25) *oscillate.*

Combining Corollary 4.2.2 and Corollary 4.3.1, we have

Corollary 4.3.2 *All solutions of* (4.2.25) *oscillate if and only if* (4.3.13) *holds for all* $\mu>0$.

Chapter 5

Nonoscillation and Oscillation of First Order Nonlinear Neutral Equations

5.1 Introduction

Recently oscillations of first order linear neutral equations have been discussed in many papers [27],[28]–[41],[34, 35]. However, there are few results for oscillations of first order nonlinear neutral equations and there are only three papers [28, 31, 33] dealing with the existence of nonoscillatory solutions of first order neutral equations with variable coefficients. [28] and [31] deal with linear neutral equations and [33] discusses nonlinear neutral equations which have nonoscillatory solutions $x(t)$ with $\liminf_{t\to\infty} |x(t)| > 0$.

We first discuss the existence of nonoscillatory solutions for the first order nonlinear neutral equation

$$\frac{d}{dt}\left[x(t) - \sum_{i=1}^{K} c_i(t)x(t-\gamma_i)\right] + f(t, x(t-\sigma_1), \ldots, x(t-\sigma_n)) = 0, \qquad (5.1.1)$$

and obtain a new sufficient criterion. Next, we discuss oscillations of the nonlinear neutral equation

$$\frac{d}{dt}\left[x(t)-\sum_{i=1}^{K}c_i(t)x(t-\gamma_i)\right]+p(t)\left[\prod_{k=1}^{m}|x(t-\sigma_k)|^{\alpha_k}\right]sgnx(t)=0. \quad (5.1.2)$$

and obtain a new condition for all solutions of (5.1.2) to oscillate.

Our conditions are "sharp" in the sense that when (5.1.1) and (5.1.2) are linear neutral equations with constant coefficients, the conditions become both necessary and sufficient.

We refer to [37, 42, 44, 45, 46] for oscillations of higher order neutral equations.

A solution of (5.1.1) or (5.1.2) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

5.2 Existence of Nonoscillatory Solutions

Consider the equation

$$\frac{d}{dt}\left[x(t)-\sum_{i=1}^{K}c_i(t)x(t-\gamma_i)\right]+f(t,x(t-\sigma_1),\ldots,x(t-\sigma_n))=0, \quad t\geq t_0>0, \quad (5.2.1)$$

where $\gamma_i>0, i\in I_K=\{1,2,\ldots,K\}, \sigma_j\geq 0, j\in I_n=\{1,2,\ldots,n\}; c_i(t)$ $(i\in I_K)$ and f are continuous functions and satisfy the following conditions:

(i) $c_i(t)\geq 0,\ \sum_{i=1}^{K}c_i(t)\leq C$ $(0<C<1)$ for all sufficiently large t and there is a $c_i(t)\geq c_0>0$.

(ii) $f(t,y_1,\ldots,y_n)\geq 0$ when $y_j\geq 0$ for all $j\in I_n$;
$f(t,z_1,\ldots,z_n)\geq f(t,y_1,\ldots,y_n)$ when $z_j\geq y_j\geq 0$ for all $j\in I_n$.

Definition 5.2.1 *A family of functions is equicontinuous on $[t_0, +\infty)$ if for any given $\varepsilon > 0$, the interval $[t_0, +\infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family have oscillations less than ε.*

A set of functions in $C[t_0, +\infty)$ with $\|x\| = \sup_{t \geq t_0} |x(t)|$ is relatively compact if it is uniformly bounded and equicontinuous on $[t_0, +\infty)$ [20, 43].

Theorem 5.2.1 *Assume that (i) and (ii) hold,*

$$|c_i(t_2) - c_i(t_1)| \leq k_0 |t_2 - t_1| \tag{5.2.2}$$

where $k_0 > 0$ is a constant, and there exists a $k_1 > 0$ such that

$$\sup_{t \geq t_0} f(t, \exp(-k_1(t - \sigma_1)), \ldots, \exp(-k_1(t - \sigma_n))) = M < \infty \tag{5.2.3}$$

and

$$\sum_{i=1}^{K} c_i(t) \exp(k_1 \gamma_i) + \exp(k_1 t) \int_t^{\infty} f(s, \exp(-k_1(s - \sigma_1)), \ldots, \exp(-k_1(s - \sigma_n))) ds \leq 1 \tag{5.2.4}$$

for all sufficiently large t.

Then (5.2.1) has a nonoscillatory solution which tends to zero.

Proof. Set

$$S = \left\{ x(t) \in C[t_0, +\infty) : \begin{array}{l} \exp(-k_2 t) \leq x(t) \leq \exp(-k_1 t), \\ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|, \ t_2 \geq t_1 \geq t_0 \end{array} \right\}$$

where k_2 is sufficiently large such that $k_2 > k_1$ and $\sum_{i=1}^{K} c_i(t) \exp(k_2 \gamma_i) \geq 1$; $L \geq \max\{k_0, k_2\}$ and $C + \frac{M}{L} < 1$.

We denote C_B all bounded continuous functions in $C[t_0, +\infty)$ and define a norm $\|x\| = \sup_{t \geq t_0} |x(t)|$ in C_B. Then C_B is a Banach space and S is a bounded convex closed set in C_B.

Define a mapping as follows:

$$(Px)(t) = \begin{cases} \sum_{i=1}^{K} c_i(t)x(t-\gamma_i) + \int_t^\infty f(s, x(s-\sigma_1), \dots, x(s-\sigma_n))ds, t \geq T, \\ \exp\left(\frac{\ln(Px)(T)}{T} t\right), \quad t_0 \leq t < T. \end{cases} \tag{5.2.5}$$

where T is sufficiently large such that $T \geq t_0 + \max\{\gamma_1, \dots, \gamma_K, \sigma_1, \dots, \sigma_n\}$, (5.2.4) holds and

$$\sum_{i=1}^{K} c_i(t_2) + \sum_{i=1}^{K} \exp(-k_1(t_1 - \gamma_i)) + \frac{M}{L} \leq 1 \;\; for \;\; t_2 \geq t_1 \geq T. \tag{5.2.6}$$

We need to prove

a) $PS \subset S$. When $t \geq T$, we have for $x \in S$

$$\begin{aligned} (Px)(t) &\leq \sum_{i=1}^{K} c_i(t) \exp(-k_1(t-\gamma_i)) \\ &\quad + \int_t^\infty f(s, \exp(-k_1(s-\sigma_1)), \dots, \exp(-k_1(s-\sigma_n)))ds \\ &= \exp(-k_1 t) \left[\sum_{i=1}^{K} c_i(t) \exp(k_1\gamma_i) \right. \\ &\quad \left. + \exp(k_1 t) \int_t^\infty f(s, \exp(-k_1(s-\sigma_1)), \dots, \exp(-k_1(s-\sigma_n)))ds \right] \\ &\leq \exp(-k_1 t) \end{aligned}$$

and

$$\begin{aligned} (Px)(t) &\geq \sum_{i=1}^{K} c_i(t) \exp(-k_2(t-\gamma_i)) \\ &= \exp(-k_2 t) \sum_{i=1}^{K} c_i(t) \exp(k_2\gamma_i) \\ &\geq \exp(-k_2 t). \end{aligned}$$

Hence $\exp(-k_2T) \le (Px)(T) \le \exp(-k_1T)$. Then

$$-k_2 \le \frac{\ln(Px)(T)}{T} \le -k_1. \tag{5.2.7}$$

From (5.2.5) and (5.2.7), we have $(Px)(t) \in C[t_0, \infty)$ and

$$\exp(-k_2t) \le (Px)(t) \le \exp(-k_1t) \quad for \;\; t \ge t_0.$$

When $t_2 \ge t_1 \ge T$, we have

$$\begin{aligned}
&|(Px)(t_2) - (Px)(t_1)| \\
\le\ & \sum_{i=1}^{K} |c_i(t_2)x(t_2-\gamma_i) - c_i(t_1)x(t_1-\gamma_i)| \\
&+ \int_{t_1}^{t_2} f(s, x(s-\sigma_1), \ldots, x(s-\sigma_n))ds \\
\le\ & \sum_{i=1}^{K} [c_i(t_2)|x(t_2-\gamma_i) - x(t_1-\gamma_i)| + |c_i(t_2) - c_i(t_1)|x(t_1-\gamma_i)] \\
&+ \int_{t_1}^{t_2} f(s, \exp(-k_1(s-\sigma_1)), \ldots, \exp(-k_1(s-\sigma_n)))ds \\
\le\ & \left\{\sum_{i=1}^{K}[c_i(t_2) + \exp(-k_1(t_1-\gamma_i))]\right\} L|t_2-t_1| \\
&+ \sup_{s\ge T} f(s, \exp(-k_1(s-\sigma_1)), \ldots, \exp(-k_1(s-\sigma_n)))|t_2-t_1| \\
\le\ & \left[\sum_{i=1}^{K} c_i(t_2) + \sum_{i=1}^{K} \exp(-k_1(t_1-\gamma_i)) + \frac{M}{L}\right] L|t_2-t_1| \\
\le\ & L|t_2-t_1|.
\end{aligned}$$

When $t_0 \le t_1 \le t_2 \le T$, using the Mean Value Theorem we have

$$\begin{aligned}
|(Px)(t_2) - (Px)(t_1)| &= \left|\exp\left(\frac{ln(Px)(T)}{T}t_2\right) - \exp\left(\frac{ln(Px)(T)}{T}t_1\right)\right| \\
&\le k_2|t_2-t_1| \\
&\le L|t_2-t_1|.
\end{aligned}$$

Then

$$|(Px)(t_2) - (Px)(t_1)| \le L|t_2-t_1| \quad for \;\; t_2 \ge t_1 \ge t_0.$$

Hence $Px \in S$.

b) P is a continuous mapping. Set $x_k \in S$ and $\lim_{k\to\infty} \|x_k - x\| = 0$. Then $x \in S$. When $t \geq T$,

$$\begin{aligned}
&|(Px_k)(t) - (Px)(t)| \\
&\leq \sum_{i=1}^{K} c_i(t)|x_k(t-\gamma_i) - x(t-\gamma_i)| \\
&\quad + \int_t^\infty |f(s, x_k(s-\sigma_1), \ldots, x_k(s-\sigma_n)) \\
&\qquad\qquad -f(s, x(s-\sigma_1), \ldots, x(s-\sigma_n))|ds \\
&\leq \sum_{i=1}^{K} c_i(t)\|x_k - x\| + \int_T^\infty G_k(s)ds \\
&\leq \|x_k - x\| + \int_T^\infty G_k(s)ds
\end{aligned}$$

where $G_k(s) = |f(s, x_k(s-\sigma_1), \ldots, x_k(s-\sigma_n)) - f(s, x(s-\sigma_1), \ldots, x(s-\sigma_n))|$. Obviously, $\lim_{k\to\infty} G_k(s) = 0$ and

$$G_k(s) \leq 2f(s, \exp(-k_1(s-\sigma_1)), \ldots, \exp(-k_1(s-\sigma_n))).$$

From the Lebesgue Theorem, we have

$$\lim_{k\to\infty} \int_T^\infty G_k(s)ds = 0.$$

Hence

$$\lim_{k\to\infty} \left(\sup_{t\geq T} |(Px_k)(t) - (Px)(t)| \right) = 0. \tag{5.2.8}$$

Then

$$\lim_{k\to\infty} |(Px_k)(T) - (Px)(T)| = 0. \tag{5.2.9}$$

When $t_0 \leq t \leq T$,

$$\begin{aligned}
|(Px_k)(t) - (Px)(t)| &= \left| \frac{\ln(Px_k)(T)}{T} - \frac{\ln(Px)(T)}{T} \right| t \\
&\leq |\ln(Px_k)(T) - \ln(Px)(T)|. \qquad (5.2.10)
\end{aligned}$$

Combining (5.2.9) and (5.2.10), we have

$$\lim_{k\to\infty}\left[\sup_{t_0\le t\le T}|(Px_k)(t)-(Px)(t)|\right]=0. \tag{5.2.11}$$

From (5.2.8) and (5.2.11), it follows that

$$\lim_{k\to\infty}\|Px_k-Px\|=0.$$

c) PS is relatively compact. Obviously, PS is uniformly bounded. For any $x\in S$, we have

$$|(Px)(t)|\le\exp(-k_1t)$$

and

$$|(Px)(t_2)-(Px)(t_1)|\le L|t_2-t_1| \quad for\ \ t_2\ge t_1\ge t_0.$$

Then for any given $\varepsilon>0$, there exists a sufficiently large $T'>t_0$ such that $\exp(-k_1t)<\frac{\varepsilon}{2}$ for $t\ge T'$ and then

$$|(Px)(t_2)-(Px)(t_1)|<\varepsilon \quad for\ \ t_2\ge t_1\ge T'. \tag{5.2.12}$$

Let $\delta=\varepsilon/L$. When $t_0\le t_1\le t_2\le T'$ and $|t_2-t_1|\le\delta$,

$$|(Px)(t_2)-(Px)(t_1)|<\varepsilon. \tag{5.2.13}$$

From (5.2.12) and (5.2.13), PS is equicontinuous on $[t_0,\infty)$. Hence PS is a relatively compact set. According to Schauder's fixed point theorem, P has a fixed point $x^*(t)$ in S. Obviously, $x^*(t)$ is a nonoscillatory solution of (5.2.1) which tends to zero. The proof is complete.

From Theorem 5.2.1, we have

Corollary 5.2.1 *Assume that $c_i(t)$ and $p_j(t)$ are nonnegative continuous functions and $c_i(t)$ satisfies (i) and (3). If $c_i(t) \le c_i$, $p_j(t) \le p_j$ and there exists a positive μ such that*

$$\sum_{i=1}^{K} c_i \exp(\mu\gamma_i) + \frac{1}{\mu}\sum_{j=1}^{n} p_j \exp(\mu\sigma_j) \le 1, \tag{5.2.14}$$

then the equation

$$\frac{d}{dt}\left[x(t) - \sum_{i=1}^{K} c_i(t)x(t-\gamma_i)\right] + \sum_{j=1}^{n} p_j(t)x(t-\sigma_j) = 0, \quad t \ge t_0 \ge 0, \tag{5.2.15}$$

has a nonoscillatory solution which tends to zero.

Remark 1. When $c_i(t) \equiv c_i$ and $p_j(t) \equiv p_j$, (5.2.14) is equivalent to that the characteristic equation of (5.2.15) has no real roots. Hence (5.2.14) is a necessary and sufficient condition for (5.2.15) with constant coefficients to have a nonoscillatory solution [27, 30, 34, 35].

Remark 2. All nonoscillation theorems of [28] can be derived from Corollary 5.2.1 or Theorem 5.2.1.

Corollary 5.2.2 *Consider*

$$\frac{d}{dt}\left[x(t) - \sum_{i=1}^{K} c_i(t)x(t-\gamma_i)\right] + \sum_{j=1}^{n} p_j(t)\left[\prod_{k=1}^{m_j} |x(t-\sigma_{j_k})|^{\alpha_{j_k}}\right] sgn x(t) = 0, \tag{5.2.16}$$

where $t \ge t_0 > 0, \gamma_i > 0, \sigma_{j_k} \ge 0, \alpha_{j_k} \ge 0$ $(i \in I_K, j \in I_n, k \in I_{m_j} = \{1, 2, \ldots, m_j\})$; $c_i(t)$ and $p_j(t)$ are nonnegative continuous functions; $c_i(t)$ satisfies (i) and (3).

If there exists a positive number μ such that for some sufficiently large T,

$$\sup_{t \ge T}\left[p_j(t)\exp\left(-\mu\sum_{k=1}^{m_j} \alpha_{j_k} t\right)\right] < \infty \quad for\ all\ j \in I_n \tag{5.2.17}$$

and

$$\sup_{t\geq T}\left\{\sum_{i=1}^{K}c_i(t)\exp(\mu\gamma_i)+\sum_{j=1}^{n}\exp\left(\mu\sum_{k=1}^{m_j}\alpha_{j_k}\sigma_{j_k}\right)\right.$$
$$\left.\times\int_t^\infty p_j(s)\exp\left[-\mu\left(\sum_{k=1}^{m_j}\alpha_{j_k}s-t\right)\right]ds\right\}\leq 1. \qquad (5.2.18)$$

then (5.2.16) *has a nonoscillatory solution which tends to zero.*

5.3 Oscillation

Consider the equation

$$\frac{d}{dt}\left[x(t)-\sum_{i=1}^{K}c_i(t)x(t-\gamma_i)\right]+p(t)\prod_{k=1}^{m}|x(t-\sigma_k)|^{\alpha_k}sgnx(t)=0, t\geq t_0>0, \qquad (5.3.1)$$

where $0<\gamma_1\leq\gamma_2\leq\cdots\leq\gamma_K, 0\leq\sigma_1\leq\cdots\leq\sigma_m, \alpha_k>0$ and $\sum_{k=1}^m\alpha_k\leq 1; c_i(t)\geq 0$ $(i\in I_K)$ and $p(t)>0$ are continuous.

Lemma 5.3.1 *Assume that* $\sum_{i=1}^K c_i(t)\leq C<1$ *and* $\int^\infty p(s)ds=\infty$. *If* $x(t)$ *is an eventually positive solution of* (5.3.1), *then* $y(t)>0$ *eventually monotonically tends to zero, where* $y(t)=x(t)-\sum_{i=1}^K c_i(t)x(t-\gamma_i)$.

Proof. From (5.3.1), we have $y'(t)<0$ eventually. Then

$$\lim_{t\to\infty}y(t)=-\infty \qquad (5.3.2)$$

or

$$\lim_{t\to\infty}y(t)=a>-\infty. \qquad (5.3.3)$$

If (5.3.2) holds, then $x(t)$ is unbounded and there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty}t_k=+\infty$ and $x(t_k)=\max_{s\leq t_k}x(s)$. We have

$$y(t_k) = x(t_k)-\sum_{i=1}^{K}c_i(t_k)x(t_k-\gamma_i)$$

$$\begin{aligned} &\geq x(t_k)\left(1-\sum_{i=1}^{K} c_i(t_k)\right) \\ &\geq 0 \end{aligned} \tag{5.3.4}$$

which contradicts (5.3.2). Hence (5.3.3) holds. From (5.3.4), $x(t)$ must be bounded. Set $\lim_{k\to\infty} x(t'_k) = \limsup_{t\to\infty} x(t)$. Without loss of generality, we assume that $\lim_{k\to\infty} c_i(t'_k)$ and $\lim_{k\to\infty} x(t'_k - \gamma_i)$ exist. Then

$$\begin{aligned} a &= \lim_{k\to\infty} y(t'_k) \\ &\geq \limsup_{t\to\infty} x(t)\left[1-\lim_{k\to\infty}\sum_{i=1}^{K} c_i(t'_k)\right] \\ &\geq 0. \end{aligned}$$

If $a>0$, then from (5.3.1) and $x(t)\geq y(t)$ we have

$$a-y(T) = -\int_T^{+\infty} p(s)\prod_{k=1}^{m} |x(s-\sigma_k)|^{\alpha_k} ds = -\infty.$$

This contradiction implies that $a=0$. The proof is complete.

Theorem 5.3.1 *If* $\sum_{i=1}^{K} c_i(t) \leq C < 1$ *and there exists some sufficiently large* T *such that*

$$\inf_{t\geq T,\mu>0}\left\{D(t)\left[\frac{1}{\mu}p(t)\exp\left(\mu\sum_{k=1}^{m}\alpha_k\sigma_k\right)+\sum_{i=1}^{K}E_i(t)\exp(\mu\gamma_i)\right]\right\} > 1 \tag{5.3.5}$$

where $D(t) = \prod_{k=1}^{m}\left[1+\sum_{i=1}^{K} c_i(t-\sigma_k)\right]^{\alpha_k-1}$ *and* $E_i(t) = \frac{p(t)}{p(t-\gamma_i)}\prod_{k=1}^{m} c_i(t-\sigma_k)(i\in I_K)$, *then all solutions of* (5.3.1) *oscillate.*

Proof. By (5.3.5) we can prove that there exists some $d>0$ such that $p(t)\geq d\ (t\geq T)$. Otherwise, $\inf_{t\geq T} p(t) = 0$ and there exists a sequence $\{t_n\}$ such that $p(t_n) = \min_{t\leq t_n} p(t)$ and $\lim_{n\to\infty} p(t_n) = 0$. Then

$$\lim_{n\to\infty}\frac{1}{\mu}p(t_n)\exp\left(\mu\sum_{k=1}^{m}\alpha_k\sigma_k\right) = 0, \quad \mu>0. \tag{5.3.6}$$

and

$$\begin{aligned}\sum_{i=1}^{K} E_i(t_n)\exp(\mu\gamma_i) &\le \sum_{i=1}^{K} c_i(t_n-\sigma_1)\exp(\mu\gamma_i)\\ &\le \sum_{i=1}^{K} c_i(t_n-\sigma_1)\exp(\mu\gamma_K)\\ &\le C\exp(\mu\gamma_K). \qquad (5.3.7)\end{aligned}$$

From (5.3.6) and (5.3.7), noting that $\left(\frac{1}{2}\right)^m \le D(t) \le 1$, when $\mu > 0$ is sufficiently small and n is sufficiently large we have

$$D(t_n)\left[\frac{1}{\mu}p(t_n)\exp\left(\mu\sum_{k=1}^{m}\alpha_k\sigma_k\right)+\sum_{i=1}^{K}E_i(t_n)\exp(\mu\gamma_i)\right]\le 1$$

which contradicts (5.3.5). If (5.3.1) has a nonoscillatory solution $x(t) > 0$, then set

$$y(t) = x(t) - \sum_{i=1}^{K} c_i(t)x(t-\gamma_i). \qquad (5.3.8)$$

According to Lemma 5.3.1, there exists a T such that when $t \ge T - \gamma_K - \sigma_m, x(t) > 0, 0 < y(t) \le 1$ and $y'(t) < 0$. Set

$$u(t) = -\frac{y'(t)}{y(t)}, \quad t \ge T. \qquad (5.3.9)$$

Then

$$\frac{y(t_1)}{y(t_2)} = \exp\left(\int_{t_1}^{t_2} u(s)ds\right) \quad for \ t_1, t_2 \in [T,\infty). \qquad (5.3.10)$$

From (5.3.9) and (5.3.1), using Jensen's inequality, when $t \ge T$ we have

$$\begin{aligned}u(t) &= \frac{p(t)}{y(t)}\prod_{k=1}^{m}\left[y(t-\sigma_k)+\sum_{i=1}^{K}c_i(t-\sigma_k)x(t-\sigma_k-\gamma_i)\right]^{\alpha_k}\\ &\ge \frac{p(t)}{y(t)}\prod_{k=1}^{m}\left[1+\sum_{i=1}^{K}c_i(t-\sigma_k)\right]^{\alpha_k-1}\\ &\quad\times\prod_{k=1}^{m}\left[y^{\alpha_k}(t-\sigma_k)+\sum_{i=1}^{K}c_i(t-\sigma_k)x^{\alpha_k}(t-\sigma_k-\gamma_i)\right]\end{aligned}$$

$$\begin{aligned}
&\geq \frac{p(t)}{y(t)} \prod_{k=1}^{m} \left[1+\sum_{i=1}^{K} c_i(t-\sigma_k)\right]^{\alpha_k - 1} \\
&\quad \times \left[\prod_{k=1}^{m} y^{\alpha_k}(t-\sigma_k) + \sum_{i=1}^{K}\prod_{k=1}^{m} c_i(t-\sigma_k)\prod_{k=1}^{m} x^{\alpha_k}(t-\sigma_k-\gamma_i)\right] \\
&\geq D(t)\left[p(t)\prod_{k=1}^{m}\left(\frac{y(t-\sigma_k)}{y(t)}\right)^{\alpha_k}\right. \\
&\qquad \left. +\frac{1}{y(t)}\sum_{i=1}^{K} E_i(t)p(t-\gamma_i)\prod_{k=1}^{m} x^{\alpha_k}(t-\sigma_k-\gamma_i)\right] \\
&= D(t)\left[p(t)\prod_{k=1}^{m}\exp\left(\alpha_k\int_{t-\sigma_k}^{t} u(s)ds\right) + \sum_{i=1}^{K} E_i(t)\frac{-y'(t-\gamma_i)}{y(t-\gamma_i)}\frac{y(t-\gamma_i)}{y(t)}\right] \\
&= D(t)\left[p(t)\exp\left(\sum_{k=1}^{m}\alpha_k\int_{t-\sigma_k}^{t} u(s)ds\right)\right. \\
&\qquad \left. +\sum_{i=1}^{K} E_i(t)u(t-\gamma_i)\exp\left(\int_{t-\gamma_i}^{t} u(s)ds\right)\right]. \qquad (5.3.11)
\end{aligned}$$

Set $\lambda_0 = 0$,

$$\lambda_n = \inf_{t\geq T}\left\{D(t)\left[p(t)\exp\left(\lambda_{n-1}\sum_{k=1}^{m}\alpha_k\sigma_k\right) + \sum_{i=1}^{K} E_i(t)\lambda_{n-1}\exp(\lambda_{n-1}\gamma_i)\right]\right\}, \quad n = 1, 2, \ldots. \qquad (5.3.12)$$

By induction, it is easy to prove

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

When $t \geq T$, $\lambda_0 < u(t)$. Using (5.3.11), (5.3.12) and induction, we easily prove that $\lambda_n \leq u(t)$ for $t \geq T + n\max\{\gamma_K, \sigma_m\}$. Set

$$\lim_{n\to\infty} \lambda_n = \lambda^*.$$

If $\lambda^* = \infty$, then $\lim_{t\to\infty} u(t) = +\infty$. Integrating (5.3.1) from $t - \frac{\sigma_1}{2}$ to t, then dividing it by $y\left(t - \frac{\sigma_1}{2}\right)$ and noting that $y(t) \leq 1$ is decreasing, we easily have

$$\frac{y(t)}{y\left(t-\frac{\sigma_1}{2}\right)} - 1 + \frac{1}{y\left(t-\frac{\sigma_1}{2}\right)}\int_{t-\frac{\sigma_1}{2}}^{t} p(s)\prod_{k=1}^{m} y^{\alpha_k}(s-\sigma_k)ds \leq 0, \quad t \geq T.$$

Then

$$\frac{y(t)}{y\left(t-\frac{\sigma_1}{2}\right)}-1+\frac{y(t-\sigma_1)}{y\left(t-\frac{\sigma_1}{2}\right)}\int_{t-\frac{\sigma_1}{2}}^{t}p(s)ds\le 0,\ \ t\ge T,$$

and

$$\exp\left(\int_t^{t-\frac{\sigma_1}{2}}u(s)ds\right)-1+\frac{d\sigma_1}{2}\exp\left(\int_{t-\sigma_1}^{t-\frac{\sigma_1}{2}}u(s)ds\right)\le 0,\ \ t\ge T. \quad (5.3.13)$$

Letting $t\to\infty$, then the first term of (5.3.13) tends to zero and the third term of (5.3.13) tends to $+\infty$. This leads to a contradiction. Hence, $0<\lambda^*<+\infty$.

Set

$$\varphi_n(t)=D(t)\left[p(t)\exp\left(\lambda_{n-1}\sum_{k=1}^{m}\alpha_k\sigma_k\right)+\sum_{i=1}^{K}E_i(t)\lambda_{n-1}\exp(\lambda_{n-1}\gamma_i)\right], \quad (5.3.14)$$

and

$$\varphi(t)=D(t)\left[p(t)\exp\left(\lambda^*\sum_{k=1}^{m}\alpha_k\sigma_k\right)+\sum_{i=1}^{K}E_i(t)\lambda^*\exp(\lambda^*\gamma_i)\right]. \quad (5.3.15)$$

For any given $\varepsilon>0$, there exists a $t_n\ge T$ for each $\varphi_n(t)$ such that

$$\varphi_n(t_n)\le\lambda_n+\varepsilon\le\lambda^*+\varepsilon. \quad (5.3.16)$$

By (5.3.16), it is easy to prove that $\{p(t_n)\}$ and $\{E_i(t_n)\}$ $(i\in I_K)$ are bounded. Without loss of generality, assume that $\lim_{n\to\infty}D(t_n)$, $\lim_{n\to\infty}p(t_n)$ and $\lim_{n\to\infty}E_i(t_n)$ $(i\in I_K)$ exist. Set

$$\varphi^*=\lim_{n\to\infty}D(t_n)\left[p(t_n)\exp\left(\lambda^*\sum_{k=1}^{m}\alpha_k\sigma_k\right)+\sum_{i=1}^{K}E_i(t_n)\lambda^*\exp(\lambda^*\gamma_i)\right].$$

Then $\lim_{n\to\infty}\varphi_n(t_n)=\varphi^*$. Hence $\inf_{t\ge T}\varphi(t)\le\varphi^*\le\lambda^*+\varepsilon$. Letting $\varepsilon\to 0$, we have

$$\inf_{t\ge T}\varphi(t)\le\lambda^*.$$

Then

$$\inf_{t\geq T}\left\{D(t)\left[\frac{1}{\lambda^*}p(t)\exp\left(\lambda^*\sum_{k=1}^{m}\alpha_k\sigma_k\right)+\sum_{i=1}^{K}E_i(t)\exp(\lambda^*\gamma_i)\right]\right\}\leq 1$$

which contradicts (5.3.5). The proof is complete.

Remark 3. When $m = 1, \alpha_1 = 1, c_i(t) \equiv c_i$ and $p(t) \equiv p$, (5.3.5) becomes

$$\frac{1}{\mu}p\exp(\mu\sigma_1)+\sum_{i=1}^{K}c_i\exp(\mu\gamma_i) > 1 \quad for\ all\ \ \mu > 0. \tag{5.3.17}$$

By Corollary 5.2.2 or Corollary 5.2.1, it is easy to prove that (5.3.17) is a necessary and sufficient condition for all solutions of (5.3.1) to oscillate.

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19. WUDU LU, Periodicity and B_r^p-Boundedness in Neutral Systems of Nonlinear *D-operator* with Infinite Delay (to appear)