Control of Multi-agent Systems by Nonlinear Techniques

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Over the past decade, the coordinated control problems for multi-agent systems have attracted extensive attention. Both centralized and distributed control protocols have been developed to study such multi-agent coordinated control problems as consensus, formation, swarming, flocking, rendezvous and so on. However, most papers employ standard linear control techniques. The results are mainly limited to linear multi-agent systems. In this thesis, we will study some coordinated control problems of both linear and nonlinear multi-agent systems by some advanced nonlinear techniques.

This thesis has mainly studied two problems.

i) *The leader-following rendezvous with connectivity preservation.* We have studied this problem for both single integrator and double integrator multi-agent systems by nonlinear control laws utilizing bounded potential function. Although the model of multi-agent system is linear, the closed-loop system is nonlinear due to the employment of nonlinear control laws. We have developed a Lyapunov-based method to analyze the performance of the closed-loop system, and conducted extensive simulations to evaluate the effectiveness of our control schemes. The specific results are summarized as follows.

- We have studied the case where the leader system is a linear autonomous system and the follower system is a multiple single-integrator system. The existing results can only handle the case where the leader signal is a constant signal or ramp signal and the control law is discontinuous. By introducing an exosystem, we have proposed a distributed state feedback smooth control law. For a class of reference signals such as step, ramp, and sinusoidal signals, our control law is able to maintain the connectivity of the system and, at the same time, achieve asymptotic tracking of all followers to the output of the leader system.

- We have also studied a leader-following rendezvous problem for a double integrator multi-agent system subject to external disturbances. Both the leader signal and disturbance signal can be a combination of step signal, ramp signal and sinusoidal signal with arbitrary amplitudes and initial phases. Motivated by some techniques in output regulation theory, we have developed both distributed state feedback control

protocol and distributed output feedback control protocol which utilizes a distributed observer. Both of our control laws are able to maintain the connectivity of an initially connected communication network, and, at the same time, achieve the objective of the asymptotic tracking of all followers to the leader regardless of external disturbances.

It is noted that even though we have only studied the rendezvous problem, the techniques of this thesis can also be used to handle other similar problems such as formation, flocking, swarming, etc.

ii) *Cooperative output regulation problem of nonlinear multi-agent systems.* We have formulated the cooperative output regulation problem for nonlinear multi-agent systems. The problem can be viewed as a generalization of the leader-following consensus/synchronization problem in that the leader signals are a class of signals generated by an exosystem, each follower subsystem can be subject to a class of external disturbances, and individual follower subsystems and the leader system have different dynamics. We first show that the problem can be converted into the global stabilization problem of a class of multi-input, multi-output nonlinear systems called augmented system via a set of distributed internal models. Then we further show that, under a set of standard assumptions, the augmented system can be globally stabilized by a distributed output feedback control law. We have solved the cooperative output regulation problem of uncertain nonlinear multi-agent systems in output feedback form. The main result can be summarized as follows: assuming the communication graph is connected, then the problem can be solved by a distributed output feedback control law if the global robust output regulation problem for each subsystem can be solved by an output feedback control law. We have also applied our approach to solve a leader-following synchronization problem for a group of Lorenz multi-agent systems.

摘 要

在过去的十年间，多智能体系统的协作控制问题引起了广泛的关注。为了解决趋同、编队、蜂拥、群聚等多智能体的协作控制问题，许多研究者提出了各种各样的集中式和分布式控制器。但是这些结果大多是针对线性的多智能体系统的，本论文将利用一些非线性技术去研究线性和非线性的多智能体系统的协作控制问题。

1. 有领导者的保持连接的群聚问题: 这类问题的研究主要是针对单点积分器和二重积分器的多智能体系统。为了保持网络的原始链接，我们引入了有界的势能函数，基于这样的势能函数，我们提出了非线性的控制器，所以尽管这样的多智能体系统本身是线性的，但闭环系统是非性的。因此我们利用李雅普诺夫定理来分析闭环系统的性能,并进行了大量的仿真实验来衡量我们的控制器的有效性。具体的结果列如下:

   - 我们首先研究的系统是带领导的单点积分器的多智能体系统，其中领导是由线性自治系统生成。现有的结果只能处理领导者信号是恒定的或者是斜波信号。而我们提出了一个分布式的状态反馈的控制器，不管领导者的信号是阶跃，斜波还是正弦信号，我们提出的这一控制器都能保持整个系统的原始连接，并且同时能实现各个子系统对领导者的渐近跟踪。
   - 我们并进一步研究了二重积分器的多智能体系统，而且这样的系统受到外部信号的干扰。领导者的信号和干扰信号可以是阶跃信号，斜波信号以及具有任意振幅和初始相位的正弦信号的组合。受到一些输出调节理论的启发，我们同时提出了分布式的全状态反馈控制器和带有分布式观测器的输出反馈控制器。尽管存在外部干扰信号，这两种控制器都能保持整个系统的初始连接，同时能实现各个子系统对领导者的渐近跟踪的目标。

   值得注意的是尽管我们研究是多智能体系统的群聚问题，这种技术同时能用来解决其他类似的编队、蜂拥等协作控制问题。

2. 非线性多智能体系统的合作输出调节问题: 我们首先明确地提出了什么是非线性多智能体系统的合作输出调节问题。这个问题可以看作是有领导者的趋同问题的一般化。这个非线性多智能体系统包含了一个领导者和各个子系统，其中领导者的信号由一外部线性自治系统产生，而每个子系统是含有不确定参数的非线性系统，并且这些子系统受到外部信号的干扰。首先我们引入分布式的内模，然后通过坐标变换，得到了一个多输入多输出的增广系统，之后我们把非线性多智能体系统的合作输出调节问题转化成了这个增广系统的全局镇定问题，最后一系列标准的假设下，我们提出了一分布式输出反馈控制器解决了镇定问题，从而解决了输出调节问题。具体来说，假设通信图是连接的，如果我们能解决每个子系统的

输出调节问题，那我们提出的分布式输出反馈调节器就能解决这个多智能体系统的合作输出调节问题。我们也把提出的这一控制器应用于洛伦兹多智能体系统的合作输出调节问题。

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Chapter 1

Introduction

1.1 Literature Review

Coordinated control of multi-agent systems has been receiving significant attention over the past decade, and much effort has been made toward the consensus, flocking, swarming, rendezvous, formation problems of multi-agent systems with both linear and nonlinear dynamics. The coordinated control problems of multi-agent systems appear to have broad applications in cooperative control of unmanned vehicle formations[1, 2], flocking[9, 65, 99], distributed sensor networks[4, 7], attitude alignment of clusters of satellites, and congestion control in communication networks[51] according to surveys[58, 60] on network communication.

The connectivity control of the communication network is always the key issue for distributed consensus and state agreement. There are several commonly used assumptions about the communication graph.

1. the graph is fixed and connected;
2. the graph is dynamic and assumed to be uniformly connected or uniformly jointly connected;
3. the graph is assumed to be initially connected, and in such case, it is defined dynamically and state-dependent.

Consensus problem, as the basic coordinated control problem for multi-agent systems, has been widely studied under assumptions 1 or 2. [23, 33] studied the leader-following consensus problem of single integrator multi-agent systems while [32, 54] proposed discrete and continuous update schemes for both leader-following and leaderless consensus problems under the assumption that the union of the graphs contains a spanning tree.

And [60, 62] further considered such consensus problem with time delays. The consensus problem for double-integrator systems has also been studied in [53, 55] under fixed and connected graph, while [24, 25, 52] further considered the consensus problem under the switching graph. Besides considering single-integrator and double-integrator multi-agent systems, [56] considered the consensus problem for harmonic oscillators and [86, 87, 88, 89] studied the same problem for a group of special linear multi-agent systems, by which harmonic oscillators could also be expressed. Moreover, many researchers have provided solutions for consensus problem of general linear multi-agent systems. For example, [40, 74] solved the leader-following consensus problem under fixed graph by full state feedback control laws; [46, 63, 79, 92] proposed control laws only using the partial information of the state of the system; and [90] provided both state feedback and output feedback control protocols regardless the uncertainty in the multi-agent systems. [49, 76, 77] extended the results into the switching communication networks. Furthermore, [48, 75] provided the stability analysis of the closed-loop system under the distributed consensus algorithms. For now, most papers focused on the consensus problem for linear multi-agent systems and a few considered the nonlinear multi-agent systems[42, 64, 100, 105], including some for Lagrangian multi-agent systems[6, 16, 57].

Flocking and rendezvous problems have been mainly studied under the third assumption: the graph is defined dynamically and always state-dependent. There was a survey on the connectivity control of such state-dependent graph given by [102], mainly focusing on flocking and rendezvous problems. The flocking problem has been widely studied and many kinds of control laws have been introduced to solve this problem. [9, 59, 84, 85] have considered flocking without a leader in both fixed and switching networks while [61, 65, 66, 71, 99] have considered the leader-following flocking control for multi-agent dynamical systems. [61] analyzed the flocking behavior of a large class of agents by proposing a decentralized flocking algorithm with collective potential functions introduced and [71] further studied the same problem by removing the two critical assumptions in [61]: all agents being informed and the virtual leader traveling at a constant velocity. Different from control laws in [9, 61, 71], some control laws can ensure the connectivity of the initial communication network. For example, by introducing potential functions, [11, 103] proposed an unbounded control law to solve the flocking problem under the assumption that the communication network was initially connected, and under the same assumption as [103], [104] further explored this problem by proposing a hybrid distributed topology control that decided on deletion and creation of links between agents.

Different from flocking, rendezvous with connectivity preservation problem usually requires all the agents move to the same location or track a leader, and some centralized or distributed control law is proposed to ensure the the connectivity of the communication graph. We will focus on this part in section 1.1.1.

1.1.1 Leader-following rendezvous with connectivity preservation problem

The connectivity preservation problem of single integrator multi-agent systems can be classified into two groups: leaderless and leader-following. So far, the study of this problem is mainly focused on the leaderless case. [98] and [101] proposed centralized and distributed control laws to control the global connectivity of the network by studying the dynamics of Laplacian matrix and its spectral properties, respectively. Potential functions have also been used to maintain the connectivity of the communication network. [10, 13, 38, 39] used the distributed navigation-like potential function, while [15, 34, 35] adopted the unbounded potential function. For the leader-following case, [5] considered the rendezvous problem with connectivity preservation for both single integrator and double integrator multi-agent systems by employing discontinuous controllers. In order to preserve the connectivity of the network, it also proposed two different kinds of potential functions. It is worth mentioning that some papers such as [22, 83] explored the sufficient conditions for the maintenance of the connectivity of the communication network.

For the connectivity preservation problem of double integrator multi-agent systems, both the leaderless and leader-following cases have been studied by [5, 73]. [73] introduced the bounded potential function, which was also used by [17, 19]. And [17, 19] further gave the exact value for the parameter in the potential function and made this parameter independent of the initial conditions of the closed-loop system by introducing some closed balls centered at the origin of the respective spaces, which also revealed the close relation of the rendezvous problem to the semi-global stabilization problem. For the leader-following case considered in [73], it focused on the situation that the leader was also a double integrator and the velocity was constant. As for the case that the velocity was time-varying, it proposed a centralized control law in the sense that the derivative of the velocity of the leader was informed by each agent. [5] also considered the case of leader's velocity being time-varying and proposed a distributed control law utilizing a distributed observer for the leader's velocity under the assumption that the derivative of leader's velocity was bounded and the upper bound was known by all agents.

1.1.2 Cooperative output regulation problem of nonlinear multi-agent systems

The output regulation problem for nonlinear systems has been widely studied, especially for the lower triangular systems[43, 44] and output feedback systems[36, 93, 94, 95]. The cooperative output regulation, viewed as a generalization of the leader-following consensus/synchronization problem, is to design a distributed control law to achieve the objectives of asymptotic tracking and disturbance rejection in the closed-loop system. Thus, the cooperative output regulation problem of nonlinear multi-agent systems is more challenging since the control law is limited to making use of the information of itself and its neighbors due to the communication constraints and one has to develop techniques to globally stabilize the multi-input, multi-output augmented nonlinear system. [45] provided a local solution for a class of nonlinear systems and [68] further studied leader-following consensus problems of nonlinear multi-agent systems with second-order dynamics assuming the leader has the same dynamic as followers' under the Lipschitz-like condition. After removing the Lipschitz-like condition, [81] first introduced a type of distributed internal model to convert the cooperative global robust output regulation problem of a class of strict feedback nonlinear uncertain multi-agent systems into a global robust stabilization problem of an augmented multi-agent system in block lower triangular form. [96, 97] solved the output regulation problem for a class of nonlinearly coupled multi-agent systems with an input-to-state stability property by proposing an internal model based controller. Furthermore, the same problem for a class of heterogeneous uncertain multi-agent systems in output feedback form with unity relative degree has been studied in [82].

1.2 Thesis Contributions

In this thesis, two kinds of problems are considered: the leader-following rendezvous with connectivity preservation problem of linear multi-agent systems and cooperative output regulation problem of nonlinear multi-agent systems. And the main contributions of this thesis are summarized as follows:

1. We study the leader-following rendezvous with connectivity preservation problem of single integrator multi-agent systems with leader having different dynamics from followers'. The leader system can be any linear autonomous system, which can generate a large class of functions, including a combination of step functions of arbitrary magnitudes, ramp functions of arbitrary slopes, and sinusoidal functions of arbitrary

amplitudes and initial phases. It proposes a simple continuous distributed control law without the information of the upper bound of the derivative of the leader signal. It is noted that the whole closed-loop system is nonlinear due to the employment of a nonlinear control law, thus Lyapunov-like function is introduced to analyze the solvability of the leader-following rendezvous with connectivity preservation problem of single integrator multi-agent systems. That is, we show that under the distributed state feedback control law, the connectivity of the initial communication network is maintained as well as the asymptotic tracking of all followers to the output of the leader system is achieved. Some examples are used to illustrate the main theories.

2. We further study the leader-following rendezvous with connectivity preservation problem of double integrator multi-agent systems where the leader system can generate a class of signals such as ramp signal and sinusoidal signals with arbitrary amplitudes and initial phases. That is, we don't require the leader system to be a double integrator system and it contains double integrator system and harmonic system as special cases. Inspired by the output regulation theory, we first solve this problem by full information feedback control law. Then we generalize this problem by allowing the external disturbances to various followers to be different and a distributed position feedback control law, depending neither on the velocity of the system nor on the external disturbances, is proposed to solve this generalized problem. Both the full information control law and position feedback control law can maintain the connectivity of an initially connected communication network, and at the same time, achieve the objective of the asymptotic tracking of all followers to the leader regardless of external disturbances.

3. We study the cooperative output regulation problem of nonlinear multi-agent systems in output feedback form under the assumption that the communication graph is connected all the time. The output feedback nonlinear system with unity relative degree is first considered and a dynamic compensator, also called internal model, is introduced to convert cooperative global robust output regulation problem into the global stabilization problem of a class of multi-input, multi-output augmented systems. Then we also consider the nonlinear multi-agent systems in output feedback form with relative degree greater than unity by further introducing a distributed observer. We can show that assuming the communication graph is connected, the cooperative output regulation problem of nonlinear multi-agent systems in output feedback form can be solved by a distributed output feedback control law if the global robust output regulation problem for each subsystem can be solved by an

output feedback control law. Our approaches are applied to solve the synchronization problem for a group of Lorenz multi-agent systems.

1.3 Thesis Organization

The rest of the thesis is organized in the following way.

Chapter 2 reviews the the fundamental definitions, lemmas and theorems that are useful for deriving the main results of the thesis. It includes the review of graph theory notation, matrix theory notation, linear output regulation and nonlinear output regulation.

Chapter 3 studies the problem of leader-following rendezvous with connectivity preservation for single integrator multi-agent systems where the leader system can be any linear autonomous system. A very simple continuous control law is proposed to maintain the connectivity of the initial connected communication network, and achieve asymptotic tracking of all the followers to the leader.

Chapters 4 considers the rendezvous problem for a double integrator multi-agent system where the leader system can generate a class of signals such as ramp signal and sinusoidal signals with arbitrary amplitudes and initial phases. This problem is solved by the distributed full information state feedback control law, which can maintain connectivity of the initial communication network as well as, achieve asymptotic tracking and disturbance rejection for a class of leader systems.

Chapters 5 further proposes a position feedback control law to solve the rendezvous problem for double integrator multi-agent systems. Additionally, the formulation of the problem in this chapter is more general than the one in Chapter 4 in that the disturbances to various followers can be different and the position feedback control law depends neither on the velocity of the system nor on the external disturbances.

Chapter 6 turns to the cooperative global robust output regulation problem for a class of nonlinear multi-agent systems in output feedback form with unity relative degree. A distributed output feedback control law is proposed to solve such problem, and the main theorem is applied to solve a leader-following synchronization problem for a group of Lorenz multi-agent systems.

Chapter 7 further considers the cooperative global robust output regulation problem for a class of nonlinear multi-agent systems in output feedback form with relative degree greater than unity. An example is also used to illustrate the effectiveness of the distributed output feedback control law.

Finally, some concluding remarks and future work are given in chapter 8.

The examples in the thesis were conducted by MATLAB. The thesis was typeset using LATEX.

Notation

Symbol	Meaning
$\mathbb{R}$	The set of all real number
$\mathbb{R}^+$	The set of all nonnegative real number
$\mathbb{R}^n$	The $n-$dimensional real Euclidean space
$\mathbb{R}^{m\times n}$	The set of all $m \times n$ real matrix
$\|\|x\|\|$	The Euclidean norm of vector x
$\|\|A\|\|$	The induced norm of matrix A by the Euclidean norm
A^T	The transpose of a matrix A
A^{-1}	The inverse of a matrix A
$0_{m\times n}$	The $m \times n$ zero matrix
I_n	The $n-$dimensional identity matrix
$\mathbb{1}_N$	The N dimensional column vector with all elements 1
$\otimes$	The Kronecker product of matrices. Some properties of Kronecker product are useful in this thesis: $(A \otimes B)(C \otimes D) = (AC) \otimes (CD)$, $(A + B) \otimes C = A \otimes C + B \otimes C$, $A \otimes (B + C) = A \otimes B + A \otimes C$
$\mathrm{col}(v_1, v_2)$	The compound column vector $[v_1^T, v_2^T]^T$ for any column vectors v_1 and v_2
$\lambda(A)$	The spectrum of a square matrix A
C^1	The class of continuously differentiable functions
$\mathcal{K}$	The class of strictly increasing positive definite functions f: $\mathbb{R}^n \mapsto \mathbb{R}^+$
$\mathcal{K}_\infty$	The class of unbounded class $\mathcal{K}$ functions

Chapter 2

Fundamentals

In this chapter, we will first present a brief overview of the graph theory notations and matrix theory notations which can be found in [21]. Then we will summarize linear output regulation and nonlinear output regulation [28].

2.1 Review of Graph Theory Notation

We first introduce some graph notation which can be found in [21]. A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set of nodes $\mathcal{V} = \{1, ..., N\}$ and an edge set $\mathcal{E} = \{(i, j), i, j \in \mathcal{V}, i \neq j\}$. A node i is called a neighbor of a node j if the edge $(i, j) \in \mathcal{E}$. $\mathcal{N}_i$ denotes the subset of $\mathcal{V}$ that consists of all the neighbors of the node i. If the graph $\mathcal{G}$ contains a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_{k+1})$, then the set $\{(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_{k+1})\}$ is called a path of $\mathcal{G}$ from i_1 to i_{k+1}, and node i_{k+1} is said to be reachable from node i_1. The edge (i, j) is called undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. The graph is called undirected if every edge in $\mathcal{E}$ is undirected. A graph is called connected if there exists a node i such that any other nodes are reachable from node i. The node i is called a root of the graph. A digraph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ is a subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if $\mathcal{V}_s \subseteq \mathcal{V}$ and $\mathcal{E}_s \subseteq \mathcal{E} \cap (\mathcal{V}_s \times \mathcal{V}_s)$.

The weighted adjacency matrix of a digraph $\mathcal{G}$ is a nonnegative matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ii} = 0$ and $a_{ij} > 0 \Leftrightarrow (j, i) \in \mathcal{E}$. A matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ with zero row sum is said to be a Laplacian matrix of a graph $\mathcal{G}$ if, for $i, j = 1, \cdots, N, i \neq j$, $l_{ij} < 0$ iff $\Leftrightarrow (j, i) \in \mathcal{E}$, and $l_{ij} = l_{ji}$ if (j, i) is a bidirected edge of $\mathcal{E}$. Clearly, $\mathcal{L}\mathbb{1}_N = 0$. Moreover, $\mathcal{L}$ is symmetric and positive semi-definite if and only if the graph $\mathcal{G}$ is undirected and connected [21].

Given any matrix $M = [m_{ij}] \in \mathbb{R}^{N \times N}$, one can define a graph denoted by $\Gamma(M) = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, ..., N\}$, and $\mathcal{E} = \{(i, j) \mid m_{ji} \neq 0, i \neq j, i, j = 1, \cdots, N\}$. Clearly, if

$\mathcal{L}$ is any Laplacian matrix of a graph $\mathcal{G}$, then $\Gamma(\mathcal{L}) = \mathcal{G}$. A matrix $M = [m_{ij}]_{N\times N}$ with nonnegative off-diagonal elements and zero row sums is called Metzler matrix. If $\mathcal{L}$ is the Laplacian of some graph $\mathcal{G}$, then $-\mathcal{L}$ is a Metzler matrix.

Remark 2.1 *It is shown in [41] that a Metzler matrix has at least one zero eigenvalue and all the nonzero eigenvalues have negative real parts. Furthermore, a Metzler matrix has exactly one zero eigenvalue and its null space is span$\{\mathbf{1}\}$ if and only if the associated graph is connected. A symmetric Metzler matrix is negative semi-definite. Let M be a symmetric Metzler matrix whose graph is connected. Let $\Delta = diag\{a_{10}, \cdots, a_{N0}\}$ where $a_{i0} \geq 0$ for $i = 1, \cdots, N$. Then for any nonzero, nonnegative diagonal matrix Δ, $-M+\Delta$ is positive definite.*

Then we list Gersgorin Theory from [26], which will be used to analyze the key property of Metzler matrix.

Theorem 2.1 (*Gersgorin Theory*[*26*]) *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, and let*

$$R_i'(A) \equiv \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 1, \ldots, n,$$

denote the deleted absolute row sums of A. Then all eigenvalues of A are located in the union of n discs

$$\bigcup_{i=1}^{n} z \in \mathbb{C} : |z - a_{ii}| \leq R_i'(A) \equiv G(A).$$

Furthermore, if a union of k of these n discs forms a connected region that is disjoint from all of the remaining $n-k$ discs, then there are precisely k eigenvalues of A in this region.

2.2 Review of Linear Output Regulation

The linear output regulation problem, as one of the central topics in linear control theory in the 1970s, aims to design a feedback control law to asymptotically track a class of reference inputs and reject external disturbances. The following materials are mainly extracted from Chapter 1 in [28].

Consider the linear time-invariant system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ev(t), \quad x(0) = x_0, \quad t \geq 0 \\ e(t) &= Cx(t) + Du(t) + Fv(t) \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the plant state, $u(t) \in \mathbb{R}^m$ is the plant input, $e(t) \in \mathbb{R}^p$ is the plant output representing the tracking error, and $v(t) \in \mathbb{R}^q = \begin{bmatrix} r \\ d \end{bmatrix}$ is the exogenous signal representing the reference inputs and the disturbances. The exogenous signal is generated by an exosystem of the form

$$\dot{v}(t) = Sv(t), \quad v(0) = v_0, \quad t \geq 0 \tag{2.2}$$

The reference inputs r and the disturbances d are both assumed to be generated by linear autonomous differential equations as follows:

$$\begin{aligned} \dot{r} &= A_{1r}r, \quad r(0) = r_0 \\ \dot{d} &= A_{1d}d, \quad d(0) = d_0 \end{aligned} \tag{2.3}$$

with $S = \begin{bmatrix} A_{1r} & 0 \\ 0 & A_{1d} \end{bmatrix}$. The above autonomous equations can generate a large class of functions, for example, a combination of step functions of arbitrary magnitudes, ramp functions of arbitrary slopes, and sinusoidal functions of arbitrary amplitudes and initial phases.

Thus in fact, the linear plant subject to disturbance $d(t)$ can be modeled as follows:

$$\begin{aligned} \dot{x} &= Ax + Bu + E_d d \\ y &= Cx + Du + F_d d \\ e &= Cx + Du + F_d d - r \end{aligned} \tag{2.4}$$

with $\begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} 0 & E_d \\ -I & F_d \end{bmatrix}$.

We list some standard assumptions needed for solving the linear output regulation problem in the sense of Definition 1.1 in [28].

Assumption 2.1 *S has no eigenvalues with negative real parts.*

Assumption 2.2 *The pair (A, B) is stabilizable.*

Assumption 2.3 *The pair $\left(\begin{bmatrix} C & F \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}\right)$ is detectable.*

2.2.1 Regulator equations

The regulator equations associated with (2.1) and (2.2) are:

$$\begin{aligned} XS &= AX + BU + E \\ 0 &= CX + DU + F \end{aligned} \tag{2.5}$$

where (X, U) is a solution pair of (2.5).

To see the role of (2.5), performing on (2.1) the following coordinate transformation

$$\begin{aligned} \bar{x} &= x - Xv \\ \bar{u} &= u - Uv \end{aligned} \tag{2.6}$$

can convert the original system into a system without external disturbances:

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + B\bar{u} \\ e &= C\bar{x} + D\bar{u} \end{aligned} \tag{2.7}$$

2.2.2 Linear feedback control laws

In Chapter 1.2 in [28], two kinds of linear control laws are proposed to solve the linear output regulation problem: state feedback and dynamic measurement output feedback, and these two classes of feedback control laws will be utilized in Chapters 3-5.

1. Static state feedback

$$u = K_x x + K_v v \tag{2.8}$$

where $K_x \in \mathbb{R}^{m\times n}$ and $K_v \in \mathbb{R}^{m\times q}$ are constant matrices. Theorem 1.7 in [28] can be used to construct K_x and K_v as follows.

Theorem 2.2 *Under Assumptions 2.1 and 2.2, let the feedback gain K_x be such that $(A + BK_x)$ is exponentially stable. Then, the linear output regulation problem is solvable by a static state feedback control of the form*

$$u = K_x x + K_v v$$

if and only if there exist two matrices X and U that satisfy the linear matrix equations (2.5), with the feedforward gain K_v being given by

$$K_v = U - K_x X$$

2. Dynamic measurement output feedback

$$u = Kz, \quad \dot{z} = g_1 z + g_2 y_m \tag{2.9}$$

where $z \in \mathbb{R}^{n_z}$, $y_m \in \mathbb{R}^{p_m}$ for some positive integer p_m is the measurement output, and $K \in \mathbb{R}^{m\times n_z}$, $g_1 \in \mathbb{R}^{n_z\times n_z}$, $g_2 \in \mathbb{R}^{n_z\times p_m}$ are constant matrices. It is assumed that y_m takes the following form:

$$y_m = C_m x(t) + D_m u(t) + F_m v(t)$$

where $C_m \in \mathbb{R}^{p_m \times n}$, $D_m \in \mathbb{R}^{p_m \times m}$, and $F_m \in \mathbb{R}^{p_m \times q}$.

Luenburger observer theory suggests a way to construct the triple (K, g_1, g_2).

$$\begin{aligned} u &= \begin{bmatrix} K_x & K_v \end{bmatrix} z \\ \dot{z} &= \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u + L(y_m - \begin{bmatrix} C_m & F_m \end{bmatrix} z - D_m u) \end{aligned} \tag{2.10}$$

where $L \in \mathbb{R}^{(n+q)\times p_m}$ is an observer gain matrix, and

$$\begin{aligned} K &= \begin{bmatrix} K_x & K_v \end{bmatrix} \\ g_1 &= \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K - L(\begin{bmatrix} C_m & F_m \end{bmatrix} z + D_m K), \quad g_2 = L \end{aligned} \tag{2.11}$$

2.2.3 Barbalat's Lemma

Barbalat's Lemma[37] is an effective tool for adaptive control, and will be used to analyze the convergence property for the rendezvous problem defined in Chapters 3-5.

Lemma 2.1 *Suppose $f(t)$ is continuously differentiable for $t \geq t_0$ for some t_0, $f(t)$ has a limit as $t \to \infty$, and $\dot{f}(t)$ is uniformly continuous. Then $\dot{f}(t) \to 0$ as $t \to \infty$.*

2.3 Review of Nonlinear Output Regulation

We will also give a brief review for nonlinear output regulation as the fundamental theories for Chapters 6 and 7, and more details can be found in [28].

We consider the nonlinear control system in the following form:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), v(t), w) \\ e(t) &= h(x(t), u(t), v(t), w) \\ y(t) &= h_m(x(t), u(t), v(t), w), \quad t \geq 0 \end{aligned} \tag{2.12}$$

where $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^m$ the plant input, $e \in \mathbb{R}^p$ the regulated error, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$, $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$, $w \in \mathbb{R}^{n_w}$ represents the unknown plant parameter, and $v \in \mathbb{R}^q$ is the exogenous signal representing the disturbance and/or the reference input, assumed to be generated by the following so called exosystem:

$$\dot{v}(t) = a(v(t)), \quad t \geq 0 \tag{2.13}$$

Then we list some standard assumptions for nonlinear output regulation problem.

Assumption 2.4 *There exist sufficiently smooth functions* $\boldsymbol{x}(v,w)$ *and* $\boldsymbol{u}(v,w)$ *with* $\boldsymbol{x}(0,0) = 0$ *and* $\boldsymbol{u}(0,0) = 0$ *that satisfy, for all* $v \in V$ *and* $w \in W$*, the following equations*

$$\begin{aligned} \frac{\partial \boldsymbol{x}(v,w)}{\partial v} a(v) &= f(\boldsymbol{x}(v,w), \boldsymbol{u}(v,w), v, w) \\ 0 &= h(\boldsymbol{x}(v,w), \boldsymbol{u}(v,w), v, w) \end{aligned} \tag{2.14}$$

Equations (2.14) are called regulator equations [31] associated with system (2.12) and exosystem (2.13).

Assumption 2.5 *The exosystem is linear and neutrally stable in the sense that* $a(v) = Sv$ *for some constant matrix* S *whose eigenvalues are all semi-simple with zero real parts.*

Assumption 2.6 *The solution of the regulator equations is a polynomial in* v.

2.3.1 From nonlinear output regulation to stabilization

One of the most important ways to solve the global robust output regulation problem for nonlinear systems is to first design a dynamic compensator, called internal model, and then convert the output regulation problem of the original system into the global stabilization problem of an augmented system. Thus let us introduce the concept of internal model in the sense of Definition 6.1 in [28].

Definition 2.1 *Under Assumption 2.4, let* s *be some positive integer and* $\gamma : \mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^m \mapsto \mathbb{R}^s$ *and* $\beta : \mathbb{R}^s \times R^q \mapsto \mathbb{R}^m$ *be two sufficiently smooth function vanishing at the origin. An internal model candidate of the plant composed of (2.12) and (2.13) is a dynamic compensator of the following form:*

$$\begin{aligned} \dot{\eta} &= \gamma(\eta, y, u) = \gamma(\eta, h_m(x,u,v,w), u) \\ u &= \beta(\eta, v) \end{aligned} \tag{2.15}$$

with the property that there exists a globally defined sufficiently smooth function $\theta : \mathbb{R}^q \times \mathbb{R}^{n_w} \mapsto \mathbb{R}^s$ *such that, for all* $v \in \mathbb{R}^q$ *and all* $w \in \mathbb{R}^{n_w}$,

$$\begin{aligned} \dot{\theta}(v,w) &= \gamma(\theta(v,w), h_m(\boldsymbol{x}(v,w), \boldsymbol{u}(v,w), v, w), \boldsymbol{u}(v,w)) \\ \boldsymbol{u}(v,w) &= \beta(\theta(v,w), v) \end{aligned} \tag{2.16}$$

where $\dot{\theta}(v,w) = \frac{\partial \theta(v,w)}{\partial v} a(v)$.

The composition of the original system and internal model constitutes the augmented system as follows:

$$
\begin{aligned}
\dot{\eta} &= \gamma(\eta, y, u) \\
\dot{x} &= f(x, u, v, w) \\
\dot{v} &= a(v) \\
e &= h(x, u, v, w) \\
y &= h_m(x, u, v, w)
\end{aligned}
\tag{2.17}
$$

The internal model candidate is defined such that the augmented system has an output zeroing invariant manifold $\mathcal{M}\{(\eta, x, v)|\eta = \theta(v, w), x = \mathbf{x}(v, w), v \in \mathbb{R}^q\}$ under the control $\mathbf{u}(v, w)$ in the sense that

$$
\begin{aligned}
\frac{\partial \theta(v, w)}{\partial v} a(v) &= \gamma(\theta(v, w), h_m(\mathbf{x}(v, w), \mathbf{u}(v, w), v, w), \mathbf{u}(v, w)) \\
\frac{\partial \mathbf{x}(v, w)}{\partial v} a(v) &= f(\mathbf{x}(v, w), \mathbf{u}(v, w), v, w) \\
0 &= h(\mathbf{x}(v, w), \mathbf{u}(v, w), v, w)
\end{aligned}
\tag{2.18}
$$

Let $\alpha(\eta, v) = \gamma(\eta, 0, v, \beta(\eta, v))$. Performing the following coordinate and input transformation on (2.17)

$$
\begin{aligned}
\bar{\eta} &= \eta - \theta(v, w) \\
\bar{x} &= x - \alpha(\eta, v) \\
\bar{u} &= u - \beta(\eta, v)
\end{aligned}
\tag{2.19}
$$

gives a system denoted by

$$
\begin{aligned}
\dot{\bar{\eta}} &= \bar{\gamma}(\bar{\eta}, \bar{x}, \bar{u}, v, w) \\
\dot{\bar{x}} &= \bar{f}(\bar{\eta}, \bar{x}, \bar{u}, v, w) \\
e &= \bar{h}(\bar{\eta}, \bar{x}, \bar{u}, v, w)
\end{aligned}
\tag{2.20}
$$

which has the property that, for all trajectories $v(t)$ of the exosystem, and all $w \in \mathbb{R}^{n_w}$,

$$
\begin{aligned}
\bar{\gamma}(0, 0, 0, v(t), w) &= 0 \\
\bar{f}(0, 0, 0, v(t), w) &= 0 \\
\bar{h}(0, 0, 0, v(t), w) &= 0
\end{aligned}
\tag{2.21}
$$

Consider an output feedback control law of the form

$$
\begin{aligned}
\bar{u} &= k(\xi) \\
\dot{\xi} &= g_{\xi}(\xi, e)
\end{aligned}
\tag{2.22}
$$

where $\xi \in \mathbb{R}^{n_\xi}$ for some integer n_ξ, and k and g_ξ are sufficiently smooth functions vanishing at their respective origins. If the control law (2.22) globally stabilizes the equilibrium of the augmented system (2.20) at the origin, then the following control law

$$\begin{aligned} u &= \beta(\eta, v) + k(e, \xi) \\ \dot{\eta} &= \gamma(\eta, e, v, u) \\ \dot{\xi} &= g_\xi(\eta, \xi) \end{aligned} \tag{2.23}$$

solves the robust output regulation problem for the original plant (2.12) globally.

2.3.2 Construction of internal model

As we can see in the last section, the key for the solvability of global output regulation problem is to find a suitable internal model which can lead to a stabilizable augmented system. Here we introduce one internal model candidate from [30]. For this purpose, we list another assumption.

Assumption 2.7 *There exist a sufficiently smooth function* $\pi : \mathbb{R}^q \times \mathbb{R}^{n_w} \mapsto \mathbb{R}$, *an integer* r *and real numbers* $a_0, \cdots, a_{r-1}$ *such that*

$$\frac{d^r \pi}{dt^r} = a_0 \pi + a_1 \frac{d\pi}{dt} + \cdots + a_{r-1} \frac{d^{r-1}\pi}{dt^{r-1}} \tag{2.24}$$

and a sufficiently smooth function $\beta : \mathbb{R}^r \mapsto \mathbb{R}$ *vanishing at the origin such that, for all* $v \in \mathbb{R}^q$ *, and* $w \in \mathbb{R}^{n_w}$,

$$\boldsymbol{u}(v, w) = \beta(\pi(v, w), \dot{\pi}(v, w), \cdots, \pi^{(r-1)}(v, w)) \tag{2.25}$$

Under Assumption 2.7, let

$$\tau(v, w) = \begin{bmatrix} \pi \\ \frac{d\pi}{dt} \\ \vdots \\ \frac{d^{r-1}\pi}{dt^{r-1}} \end{bmatrix} \tag{2.26}$$

and let $\Phi = \begin{bmatrix} 0 & I_{r-1} \\ a_0 & a_1, \cdots, a_{r-1} \end{bmatrix}$. Then it is ready to verify that

$$\dot{\tau} = \Phi \tau, \quad \mathbf{u}(v, w) = \beta(\tau) \tag{2.27}$$

Let Ψ be the gradient of β at the origin and assume the pair (Ψ, Φ) is observable. Then, for any $M \in \mathbb{R}^{r \times r}$ and $N \in \mathbb{R}^{r \times 1}$ such that (M, N) is controllable, the spectra of the matrices Φ and M are disjoint, and M is Hurwitz, there exists a unique, nonsingular matrix $T \in \mathbb{R}^{r \times r}$ that satisfies the Sylvester equation [50]

$$T\Phi - MT = N\Psi \tag{2.28}$$

Then the following dynamic compensator

$$\begin{aligned}\dot{\eta} &= M\eta + N(u - \beta(T^{-1}\eta) + \Psi T^{-1}\eta)\\ u &= \beta(\eta)\end{aligned} \tag{2.29}$$

is a nonlinear internal model candidate [30].

In this thesis, we will focus on a group of nonlinear control systems in output feedback form with unity relative degree shown in Eq. (2.30) and relative degree greater than unity shown in Eq. (2.31).

$$\begin{aligned}\dot{z} &= f(z, y, v, w)\\ \dot{y} &= b(v, w)u + g(z, y, v, w)\\ e &= y - q(v, w)\end{aligned} \tag{2.30}$$

where $(z, y) \in \mathbb{R}^n \times \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the input, $e \in \mathbb{R}$ is the error output, $w \in \mathbb{W} \subset \mathbb{R}^{n_w}$ is an uncertain parameter vector with $\mathbb{W}$ an arbitrarily prescribed subset of $\mathbb{R}^{n_w}$. It is assumed that all functions in (2.30) are globally defined, sufficiently smooth, and satisfy $f(0,0,0,w) = 0$, $g(0,0,0,w) = 0$, and $q(0,w) = 0$ for all $w \in \mathbb{W}$.

$$\begin{aligned}\dot{z} &= f(z, y, v, w)\\ \dot{x}_s &= x_{(s+1)} + g_s(z, y, v, w), \quad s = 1, \cdots, r-1\\ \dot{x}_r &= b(w)u + g_r(z, y, v, w)\\ y &= x_1\\ e &= y - q(v, w)\end{aligned} \tag{2.31}$$

where $r \geq 2$, $(z, x) \in \mathbb{R}^n \times \mathbb{R}^r$ with $x = \mathrm{col}(x_1, \cdots, x_r) \in \mathbb{R}^r$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $e \in \mathbb{R}$ is the regulated error, $w \in \mathbb{W} \subset \mathbb{R}^{n_w}$ is an uncertain parameter vector with $\mathbb{W}$ an arbitrarily prescribed compact subset of $\mathbb{R}^{n_w}$ such that $0 \in \mathbb{W}$. All functions in (2.31) are supposed to be globally defined, sufficiently smooth, and satisfy $q(0,w) = 0$, $f(0,0,0,w) = 0$, $g_s(0,0,0,w) = 0$, $s = 1, \cdots, r$ for all $w \in \mathbb{W}$.

$v(t) \in \mathbb{R}^{n_v}$ is an exogenous signal presenting both reference input and disturbance. It is assumed that $v(t)$ is generated by a linear system of the following form

$$\dot{v} = Sv \tag{2.32}$$

For such output feedback nonlinear systems, we can design linear internal model by assuming the function $\mathbf{u}(v, w)$ itself satisfies an equation of the form (2.24). Then $\beta(\tau) = \Psi T^{-1}\tau$ where $\Psi = [1, 0, \cdots, 0]$. Then (2.29) reduces to the following linear internal model candidate:

$$\dot{\eta} = M\eta + Nu, \quad u = \Psi T^{-1}\eta \tag{2.33}$$

2.3.3 Some theories

We also list some important theories and techniques, which are necessary for the solvability of the cooperative global robust output regulation problem considered in Chapters 6 and 7.

1. Lemma 7.8 in [28]

   Lemma 2.2 *Let* $f : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ *be* C^1 *function satisfying* $f(0, 0, \mu) = 0$ *for all* $\mu \in \Sigma$ *with* Σ *being a compact subset of* $\mathbb{R}^p$*. Then there exist smooth functions* $F_1 : \mathbb{R}^m \to \mathbb{R}$ *and* $F_2 : \mathbb{R}^n \to \mathbb{R}$ *satisfying* $F_1(0) = 0$ *and* $F_2(0) = 0$ *such that*

   $$|f(x, y, \mu)| \leq F_1(x) + F_2(y), \quad \forall x \in \mathbb{R}^m, \quad y \in \mathbb{R}^n, \quad \mu \in \Sigma \tag{2.34}$$

2. Changing supply rate technique
   We list Theorem 2 in [69] which will be useful in Chapters 6 and 7.

   Theorem 2.3 *Assume that* (γ, α) *is a supply pair. Suppose that* $\tilde{\alpha}$ *is a* $\mathcal{K}_\infty$ *function so that* $\tilde{\alpha}(r) = \mathcal{O}[\alpha(r)]$ *as* $r \to 0^+$*. Then there exists a* $\tilde{\gamma}$ *so that* $(\tilde{\gamma}, \tilde{\alpha})$ *is a supply pair.*

   To be specific, suppose system (2.12) satisfies the following assumption:

   Assumption 2.8 *There exists a* C^1 *function* $V(x, t)$ *satisfying*

   $$\underline{\gamma}(||x||) \leq V(x, t) \leq \bar{\gamma}(||x||)$$

   for some class $\mathcal{K}_\infty$ *functions* $\underline{\gamma}(\cdot)$ *and* $\bar{\gamma}(\cdot)$*, such that, along the trajectory of the system (2.12),*

   $$\frac{dV(x, t)}{dt} \leq -\gamma(||x||) + \omega(u)$$

   for some smooth positive definite function $\omega(u)$*, and some locally quadratic class* $\mathcal{K}_\infty$ *function* $\gamma(\cdot)$*.*

   Then, for any sufficiently smooth function $\Delta(x)$, there exists a C^1 function $W(x, t)$ satisfying

   $$\underline{\alpha}(||x||) \leq W(x, t) \leq \bar{\alpha}(||x||)$$

   for some class $\mathcal{K}_\infty$ function $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$, and

   $$\frac{dW(x, t)}{dt} \leq -\Delta(x)||x||^2 + \pi(u)||u||^2 \tag{2.35}$$

   for some smooth positive definite function $\pi(u)$.

3. Backstepping

In order to globally stabilize the augmented system, we need recursively use the following Lemma which also can be found in [28].

Lemma 2.3 *Consider the system*

$$\begin{aligned} \dot{z} &= \varphi(z, x, \mu(t)) \\ \dot{x} &= \phi(z, x, \mu(t)) + \psi(\mu(t))u, \quad t \geq t_0 \geq 0 \end{aligned} \tag{2.36}$$

in which $(z, x) \in \mathbb{R}^m \times \mathbb{R}$*,* $u \in \mathbb{R}$*,* $\mu : [t_0, \infty) \to \Sigma \subset \mathbb{R}^{n_\mu}$ *is piecewise continuous with* Σ *a prescribed compact set of* $\mathbb{R}^{n_\mu}$*,* $\varphi(z, x, \mu)$ *and* $\phi(z, x, \mu)$ *are* C^1 *functions satisfying* $\varphi(0, 0, \mu) = 0$*,* $\phi(0, 0, \mu) = 0$ *for* $\mu \in \Sigma \subset \mathbb{R}^{n_\mu}$*. Suppose the following:*

(a) The upper subsystem in (2.36) is RISS with respect to μ *with state* z *and input* x*, and has a known* C^1 *gain function* $\kappa(\cdot)$*.*

(b) For all $\mu \in \mathbb{R}^{n_\mu}$*,* $\psi(\mu) > 0$*.*

Then, there exists a positive smooth function $\rho : \mathbb{R} \to [0, \infty)$ *such that, under the controller*

$$u = -x\rho(x) + \hat{u} \tag{2.37}$$

the closed-loop system (2.36) and (2.37) is RISS with respect to μ *with state* $Z = col(z, x)$ *and input* $\hat{u}$*, and has a known* C^1 *RISS gain function* $\tilde{\kappa}(\cdot)$*.*

Chapter 3

Leader-following Rendezvous with Connectivity Preservation of Single-integrator Multi-agent Systems

3.1 Introduction

Maintaining connectivity in a multi-agent system is necessary in such problems as rendezvous, flocking, swarming and so on. These problems have been studied for both single-integrator multi-agent systems [3, 5, 12, 34, 35, 101] and double-integrator multi-agent systems [5, 17, 73]. There are two types of rendezvous problems, namely, leaderless rendezvous problem and leader-following rendezvous problem. While the leaderless rendezvous problem requires the outputs of all agents asymptotically approach a same location, the leader-following rendezvous problem further requires the outputs of all agents approach a given trajectory asymptotically. So far, the study of the problem of rendezvous with connectivity preservation is mainly focused on the leaderless case [3, 12, 34, 35, 101] with a few exceptions where the problem of leader-following rendezvous with connectivity preservation is studied [5, 17, 73]. Reference [5] studied the leader following rendezvous with connectivity preservation for both single-integrator and double-integrator systems using a discontinuous control law. Reference [73] studied the leader following rendezvous with connectivity preservation for double-integrator systems where the leader is also a double integrator system. Reference [17] also studied the leader following rendezvous with connectivity preservation for double-integrator systems under a class of leader systems including double-integrator and harmonic systems.

In this chapter, we will consider the problem of leader-following rendezvous with connectivity preservation of single-integrator multi-agent systems. In comparison with [5], we employ a continuous control law, and our control law does not need to know the upper bound of the derivative of the leader signal.

The problem formulation in this chapter is, in spirit, similar to that of [17]. However, the approach in this chapter is different from that of [17] in that our control law here does not contain a distributed observer, and is thus much simpler than the one in [17]. We need to establish Lemma 3.1 to lay the foundation of the main result of this chapter.

The rest of this chapter is organized as follows. In section 3.2, we give a precise formulation of the leader-following problem of rendezvous with connectivity preservation for single-integrator multi-agent systems. In Section 3.3, we present our main result. An example is presented in Section 3.4. Finally, some concluding remarks are given in Section 3.5.

3.2 Problem Formulation

Consider a group of single-integrator systems:

$$\dot{x}_i = u_i, \quad i = 1, \cdots, N \tag{3.1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^n$ are the state and the input of agent i. Also consider an autonomous linear system

$$\dot{x}_0 = Sx_0 \tag{3.2}$$

where $x_0 \in \mathbb{R}^n$ and S is a constant matrix.

The system composed of (3.1) and (3.2) can be viewed as a multi-agent system of $(N+1)$ agents with (3.2) as the leader and the N subsystems of (3.1) as N followers. With respect to the system composed of (3.1) and (3.2), we can define a digraph[1] $\bar{\mathcal{G}}(t) = (\bar{\mathcal{V}}, \bar{\mathcal{E}}(t))$ where $\bar{\mathcal{V}} = \{0, 1, ..., N\}$ with 0 associated with the leader system and $i = 1, \cdots, N$, associated with the ith subsystem of (3.1), and $\bar{\mathcal{E}}(t) \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$. The set $\bar{\mathcal{V}}$ is called the node set of $\bar{\mathcal{G}}(t)$ and the set $\bar{\mathcal{E}}(t)$ is called the edge set of $\bar{\mathcal{G}}(t)$. We use the notation $\bar{\mathcal{N}}_i(t)$ to denote the neighbor set of the node i for $i = 0, 1, \cdots, N$. The definition of $\bar{\mathcal{E}}(t)$ associated with the system composed of (3.1) and (3.2) is as follows.

Given any $r > 0$ and $\epsilon \in (0, r)$, for any $t \geq 0$, $\bar{\mathcal{E}}(t) = \{(i,j) \mid i, j \in \bar{\mathcal{V}}\}$ is defined such that

1. $\bar{\mathcal{E}}(0) = \{(i,j) \mid ||x_i(0) - x_j(0)|| < (r-\epsilon), i, j = 1, \cdots, N\} \cup \{(0,j) \mid ||x_0(0) - x_j(0)|| < (r-\epsilon), j = 1, \cdots, N\}$;

[1] See [74] for a summary of digraph.

2. if $||x_i(t) - x_j(t)|| \geq r$, then $(i,j) \notin \bar{\mathcal{E}}(t)$;

3. $(i,0) \notin \bar{\mathcal{E}}(t)$, for $i = 0, 1, \cdots, N$;

4. for $i = 0, 1, \cdots, N, j = 1, \cdots, N$, if $(i,j) \notin \bar{\mathcal{E}}(t^-)$ and $||x_i(t) - x_j(t)|| < (r - \epsilon)$, then $(i,j) \in \bar{\mathcal{E}}(t)$;

5. for $i = 0, 1, \cdots, N, j = 1, \cdots, N$, if $(i,j) \in \bar{\mathcal{E}}(t^-)$ and $||x_i(t) - x_j(t)|| < r$, then $(i,j) \in \bar{\mathcal{E}}(t)$.

Remark 3.1 *The definition here is the same as that in [17]. If $\epsilon = 0$, then the above definition is similar to that given in [35]. Thus the physical interpretation of r is the sensing radius of the distance sensor of each follower. The number ϵ is to introduce the effect of hysteresis. It is noted that the leader does not have a control and, therefore, there is no edge from a follower to the leader.*

We will consider the following distributed state feedback control law

$$u_i = Sx_i + h_i(x_i - x_j, j \in \bar{\mathcal{N}}_i(t)), \quad i = 1, ..., N \tag{3.3}$$

where h_i is a sufficiently smooth function to be specified later.
The control law (3.3) is called a distributed state feedback control law because, at any time instant t, the control u_i can take x_j, $j \neq i$, for feedback control if and only if the node j is a neighbor of the node i.

Define a subgraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ of $\bar{\mathcal{G}}(t)$ where $\mathcal{V} = \{1, \cdots, N\}$, and $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ is obtained from $\bar{\mathcal{E}}(t)$ by removing all edges from the node 0 to the nodes in $\mathcal{V}$. Clearly, $\mathcal{G}(t)$ is an undirected graph. For $i = 1, \cdots, N$, let $\mathcal{N}_i(t) = \bar{\mathcal{N}}_i(t) \cap \mathcal{V}$. It can be seen that, for $i = 1, \cdots, N$, $\mathcal{N}_i(t)$ is the neighbor set of the node i with respect to $\mathcal{G}(t)$.

The problem of leader-following rendezvous with connectivity preservation is described as follows.

Definition 3.1 *Given the multi-agent system composed of (3.1) and (3.2), find a control law of the form (3.3) such that, for all initial conditions $x_i(0)$, $i = 0, 1, \cdots, N$, that make $\bar{\mathcal{G}}(0)$ connected, the closed-loop system has the following properties:*

1. $\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$;

2. $\lim_{t\to\infty}(x_i - x_0) = 0$, $i = 1, \cdots, N$.

Remark 3.2 *The leader-following problem of rendezvous with connectivity preservation defined here is similar in spirit to the leader-following problem of rendezvous with connectivity preservation defined for double-integrator system in [17].*

3.3 Solvability of Problem

Like in [17, 73], we will adopt the potential function approach to design our control law. Thus, we will first introduce a bounded potential function introduced in [73] as follows

$$\psi(s) = \frac{s^2}{r - s + \frac{r^2}{Q}}, \quad 0 \le s \le r \tag{3.4}$$

where Q is some positive number. The function ψ is nonnegative and bounded over $[0, r]$, and its derivative $\frac{d\psi(s)}{ds} = \frac{s(2r-s+\frac{2r^2}{Q})}{(r-s+\frac{r^2}{Q})^2}$ is positive for all $s \in (0, r]$. Moreover, it is shown in [17] that the function has the property that, for any $\alpha > 0$, and $\beta \ge 0$, any $\epsilon \in (0, r)$, if $Q > (\frac{\alpha(r-\epsilon)^2}{\epsilon} + \beta)$, then

$$\psi(r) = Q > \alpha\psi(r - \epsilon) + \beta \tag{3.5}$$

Now we are ready to introduce our distributed state feedback control law as follows:

$$\begin{aligned} u_i &= Sx_i - \gamma \sum_{j \in \bar{\mathcal{N}}_i(t)} \frac{\partial\psi(||x_i - x_j||)}{\partial x_i} \\ &= Sx_i - \gamma \sum_{j \in \bar{\mathcal{N}}_i(t)} w_{ij}(t)(x_i - x_j), \quad i = 1, \cdots, N \end{aligned} \tag{3.6}$$

where γ is a sufficiently large positive number and

$$w_{ij}(t) = \begin{cases} \frac{2r-||x_i-x_j||+\frac{2r^2}{Q}}{(r-||x_i-x_j||+\frac{r^2}{Q})^2}, & (j, i) \in \bar{\mathcal{E}}(t) \\ 0, & \text{otherwise} \end{cases} \tag{3.7}$$

with $i = 1, \cdots, N$, $j = 0, 1, \cdots, N$, $i \ne j$.

Let $\phi(s) = \frac{2r-s+\frac{2r^2}{Q}}{(r-s+\frac{r^2}{Q})^2}$ for $0 \le s \le r$. Then $0 < \frac{2r+\frac{2r^2}{Q}}{(r+\frac{r^2}{Q})^2} = \phi(0) \le \phi(s) \le \phi(r) = \frac{r+\frac{2r^2}{Q}}{(\frac{r^2}{Q})^2}$ over $s \in [0, r]$. Let $a = \frac{2r+\frac{2r^2}{Q}}{(r+\frac{r^2}{Q})^2}$ and $b = \frac{r+\frac{2r^2}{Q}}{(\frac{r^2}{Q})^2}$. Then, for all $(j, i) \in \bar{\mathcal{E}}(t)$, $0 < a \le w_{ij}(t) \le b$.

Let $\bar{x}_i = x_i - x_0$, $i = 0, 1, \cdots, N$. Then, in terms of $\bar{x}_i$, the closed-loop system is

$$\dot{\bar{x}}_i = S\bar{x}_i - \gamma \sum_{j \in \bar{\mathcal{N}}_i(t)} w_{ij}(t)(\bar{x}_i - \bar{x}_j), \quad i = 1, \cdots, N \tag{3.8}$$

Let $\bar{x} = \begin{bmatrix} \bar{x}_1^T & \bar{x}_2^T & \cdots & \bar{x}_N^T \end{bmatrix}^T$. Then the closed-loop system can be put in the following compact form

$$\dot{\bar{x}} = (I_N \otimes S - \gamma H(t) \otimes I_n)\bar{x} \tag{3.9}$$

where $H(t) = \begin{bmatrix} \beta_1(t) & -w_{12}(t) & \cdots & -w_{1N}(t) \\ -w_{21}(t) & \beta_2(t) & \cdots & -w_{2N}(t) \\ \vdots & \vdots & \vdots & \vdots \\ -w_{N1}(t) & -w_{N2}(t) & \cdots & \beta_N(t) \end{bmatrix}$ with $\beta_i(t) = \sum_{j=0,j\neq i}^{N} w_{ij}(t)$, $i = 1, \cdots, N$.

Before presenting our main result, we will establish the following Lemma.

Lemma 3.1 *Assume the graph $\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$, and $\bar{\mathcal{G}}(t_1) \subset \bar{\mathcal{G}}(t_2)$ whenever $t_2 \geq t_1 \geq 0$. Then*

1. *There exist constant positive definite matrices H_1 and H_2 such that $0 < H_1 \leq H(t) \leq H_2$ for all $t \geq 0$.*
2. *Let $\gamma_0 = \frac{\delta \lambda_M(H_2)}{\lambda_m^2(H_1)}$ where $\lambda_M(H)$ and $\lambda_m(H)$ are the largest and smallest eigenvalues of a positive definite matrix H and $\delta \geq 0$. Then, for all $\gamma > \gamma_0$, $\gamma\lambda_m^2(H_1) - \delta\lambda_M(H_2) > 0$ and $\gamma H^2(t) \otimes I_n - \delta H(t) \otimes I_n \geq (\gamma\lambda_m^2(H_1) - \delta\lambda_M(H_2))I_{Nn}$ for all $t \geq 0$.*

Proof: Part 1) Let $\bar{\mathcal{G}}_0 = (\bar{\mathcal{V}}, \bar{\mathcal{E}}_0)$ be any connected graph such that $\bar{\mathcal{G}}_0 \subset \bar{\mathcal{G}}(0)$. Let

$$H_1 = \begin{bmatrix} \delta_1 & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & \delta_2 & \cdots & -a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{N1} & -a_{N2} & \cdots & \delta_N \end{bmatrix} \tag{3.10}$$

where

$$a_{ij} = \begin{cases} a, & (j,i) \in \bar{\mathcal{E}}_0 \\ 0, & \text{otherwise} \end{cases} \tag{3.11}$$

and $\delta_i = \sum_{j=0,j\neq i}^{N} a_{ij}$, $i = 1, \cdots, N$, and let

$$H_2 = \begin{bmatrix} Nb & -b & \cdots & -b \\ -b & Nb & \cdots & -b \\ \vdots & \vdots & \vdots & \vdots \\ -b & -b & \cdots & Nb \end{bmatrix}. \tag{3.12}$$

Note that $H_1 = -M_1 + \Delta_1$ where M_1 is a constant Metzler matrix with zero row sum whose nonzero off-diagonal entries are equal to a, and Δ_1 is a nonnegative diagonal matrix with at least one positive diagonal entry. It can be seen that $\Gamma(M_1) = \mathcal{G}_0$ where $\mathcal{G}_0$ is the subgraph of $\bar{\mathcal{G}}_0$ obtained from $\bar{\mathcal{G}}_0$ by removing the node 0 and all edges adjacent to the node 0. By Remark 2.1, H_1 is positive definite.

Let $H - H_1$

$$= \begin{bmatrix} \beta_1 - \delta_1 & (a_{12} - w_{12}) & \cdots & (a_{1N} - w_{1N}) \\ (a_{21} - w_{21}) & \beta_2 - \delta_2 & \cdots & (a_{2N} - w_{2N}) \\ \vdots & \vdots & \vdots & \vdots \\ (a_{N1} - w_{N1}) & (a_{N2} - w_{N2}) & \cdots & \beta_N - \delta_N \end{bmatrix}$$

Since, for $i = 1, \cdots, N$, $(\beta_i - \delta_i) = \sum_{j=0, j\neq i}^{N} (w_{ij} - a_{ij}) \geq \sum_{j=1, j\neq i}^{N} (w_{ij} - a_{ij}) \geq 0$, $H - H_1$ is positive semi-definite by Gersgorin Theorem.

Next, let $H_2 - H$

$$= \begin{bmatrix} Nb - \beta_1 & -(b - w_{12}) & \cdots & -(b - w_{1N}) \\ -(b - w_{21}) & Nb - \beta_2 & \cdots & -(b - w_{2N}) \\ \vdots & \vdots & \vdots & \vdots \\ -(b - w_{N1}) & -(b - w_{N2}) & \cdots & Nb - \beta_N \end{bmatrix}$$

Then, for $i = 1, \cdots, N$, $(Nb - \beta_i) = \sum_{j=0, j\neq i}^{N} (b - w_{ij}) \geq 0$. Thus, $H_2 - H$ is positive semi-definite by Gersgorin Theorem. Thus we have $0 < H_1 \leq H(t) \leq H_2$ for all $t \geq 0$.

Part 2) Let $\lambda_i(t)$ and $\alpha_i(t)$, $i = 1, \cdots, N$, be the eigenvalues and the eigenvectors of $H(t)$, respectively. Let e_i, $i = 1, \cdots, n$, be the i^{th} column of I_n. Then it can be verified that, for $i = 1, \cdots, N, j = 1, \cdots, n$, $(\gamma H^2(t) \otimes I_n - \delta H(t) \otimes I_n)(\alpha_i \otimes e_j) = (\gamma \lambda_i^2 - \delta \lambda_i)(\alpha_i \otimes e_j)$, and $(\alpha_i \otimes e_j)$ are linearly independent. Thus, the eigenvalues of $(\gamma H^2(t) \otimes I_n - \delta H(t) \otimes I_n)$ belong to the set $\{(\gamma \lambda_i^2 - \delta \lambda_i) \mid i = 1, \cdots, N\}$ which has a lower bound $\gamma \lambda_m^2(H_1) - \delta \lambda_M(H_2)$ since $H_1 \leq H(t) \leq H_2$. Since, for all $\gamma > \gamma_0$, $\gamma \lambda_m^2(H_1) - \delta \lambda_M(H_2) > 0$. Thus, for all $\gamma > \gamma_0$, $\gamma H^2(t) \otimes I_n - \delta H(t) \otimes I_n \geq (\gamma \lambda_m^2(H_1) - \delta \lambda_M(H_2)) I_{Nn}$.

Now we present our main result.

Theorem 3.1 *Given any $r > 0$ and $\epsilon \in (0, r)$, the leader-following problem of rendezvous with connectivity preservation of the system composed of (3.1) and (3.2) is solvable by a control protocol of the form (3.6).*

Proof: Given $r > 0$ and $\epsilon \in (0, r)$, the control law is determined by two design parameters Q and γ. Let $\alpha = \frac{N(N-1)}{2} + N$, $\beta = 0$, and

$$Q_{\max} = \frac{\alpha (r - \epsilon)^2}{\epsilon} \tag{3.13}$$

Pick any finite Q such that $Q > Q_{\max}$.

To determine γ, note that, there are only finitely many connected graphs with N edges (an undirected edge is counted as one edge). Denote these graphs by $\bar{\mathcal{G}}_i$, $i =$

$1, \cdots, p$, for some positive integer p. For each $\bar{\mathcal{G}}_i$, we can define a matrix $\bar{H}_i$ in the same way as we define H_1 in (3.10). Denote the minimal eigenvalue of $\bar{H}_i$ by $\lambda_m(\bar{H}_i)$. Let $\lambda_m = \min\{\lambda_m(\bar{H}_1), \cdots, \lambda_m(\bar{H}_p)\}$. Let $\gamma_0 = \frac{||S||\lambda_M(H_2)}{\lambda_m^2}$ where H_2 is as defined in (3.12). For any $t \geq 0$, if $\bar{\mathcal{G}}(t)$ is connected, there exists some $1 \leq i \leq p$ such that $\bar{\mathcal{G}}_i \subset \bar{\mathcal{G}}(t)$. Thus, by Lemma 3.1, for all $\gamma > \gamma_0$, $\gamma\lambda_m^2 - ||S||\lambda_M(H_2) > 0$, and for any $t \geq 0$, $\gamma H^2(t) \otimes I_n - ||S||H(t) \otimes I_n \geq (\gamma\lambda_m^2 - ||S||\lambda_M(H_2))I_{Nn}$ if $\bar{\mathcal{G}}(t)$ is connected. Fix this γ and let $\lambda_\gamma = \gamma\lambda_m^2 - ||S||\lambda_M(H_2)$ and $P(t) = \gamma H^2(t) \otimes I_n - ||S||H(t) \otimes I_n$.

Now we will first show that the above control law is such that the graph $\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$. For this purpose, let

$$V = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i(t)} \psi(||\bar{x}_i - \bar{x}_j||) + \sum_{i=1}^{N} b_i(t)\psi(||\bar{x}_i||) \tag{3.14}$$

where $b_i(t) = 1$ if $(0, i) \in \bar{\mathcal{E}}(t)$, and $b_i(t) = 0$ if otherwise.
We call V as an energy function for (3.9). It can be seen that for all initial conditions $x_i(0)$, $i = 0, 1, \cdots, N$, that make $\bar{\mathcal{G}}(0)$ connected,

$$V(0) \leq Q_{\max} \tag{3.15}$$

From (3.8), we have

$$\sum_{j\in\bar{\mathcal{N}}_i(t)} w_{ij}(t)(\bar{x}_i - \bar{x}_j) = \frac{1}{\gamma}(S\bar{x}_i - \dot{\bar{x}}_i),\ i = 1, \cdots, N \tag{3.16}$$

Then the derivative of the energy function (3.14) along (3.9) satisfies

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i(t)} \dot{\psi}(||\bar{x}_i - \bar{x}_j||) + \sum_{i=1}^{N} b_i(t)\dot{\psi}(||\bar{x}_i||) \\
&= \sum_{i=1}^{N} \dot{\bar{x}}_i^T \sum_{j\in\mathcal{N}_i(t)} w_{ij}(t)(\bar{x}_i - \bar{x}_j) + \sum_{i=1}^{N} \dot{\bar{x}}_i^T w_{i0}(t)\bar{x}_i \\
&= \sum_{i=1}^{N} \dot{\bar{x}}_i^T \sum_{j\in\bar{\mathcal{N}}_i(t)} w_{ij}(t)(\bar{x}_i - \bar{x}_j) \\
&= \sum_{i=1}^{N} \dot{\bar{x}}_i^T \frac{1}{\gamma}(S\bar{x}_i - \dot{\bar{x}}_i) \\
&= \frac{1}{\gamma}\sum_{i=1}^{N} \dot{\bar{x}}_i^T S\bar{x}_i - \frac{1}{\gamma}\sum_{i=1}^{N} \dot{\bar{x}}_i^T \dot{\bar{x}}_i \\
&= \frac{1}{\gamma}\dot{\bar{x}}^T (I_N \otimes S)\bar{x} - \frac{1}{\gamma}\dot{\bar{x}}^T \dot{\bar{x}} \\
&= \frac{1}{\gamma}\bar{x}^T (I_N \otimes S - \gamma H(t) \otimes I_n)^T (I_N \otimes S)\bar{x}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\gamma}\bar{x}^T(I_N \otimes S - \gamma H(t) \otimes I_n)^T(I_N \otimes S - \gamma H(t) \otimes I_n)\bar{x} \\
&= \frac{1}{\gamma}\bar{x}^T(I_N \otimes S^T S - \gamma H(t) \otimes S - I_N \otimes S^T S + \gamma H(t) \otimes S^T \\
&+ \gamma H \otimes S - \gamma^2 HH \otimes I_n)\bar{x} \\
&= -\bar{x}^T(\gamma(H(t)H(t) \otimes I_n - H(t) \otimes S^T)\bar{x} \\
&= -\bar{x}^T(\gamma H(t)H(t) \otimes I_n)\bar{x} + \sum_{i=1}^{N} w_{i0}\bar{x}_i^T S^T \bar{x}_i \\
&+ \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i(t)} w_{ij}(\bar{x}_i - \bar{x}_j)^T S^T(\bar{x}_i - \bar{x}_j) \\
&\leq -\bar{x}^T(\gamma H(t)H(t) \otimes I_n)\bar{x} + \sum_{i=1}^{N} w_{i0}||S||\bar{x}_i^T\bar{x}_i \\
&+ \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i(t)} w_{ij}||S||(\bar{x}_i - \bar{x}_j)^T(\bar{x}_i - \bar{x}_j) \\
&= -\bar{x}^T((\gamma H(t)H(t) - ||S||H(t)) \otimes I_n)\bar{x} \\
&= -\bar{x}^T P(t)\bar{x}
\end{aligned}
\tag{3.17}
$$

By the continuity of the solution of (3.9), there exists $0 < t_1 \leq \infty$ such that $\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(0)$ for all $t \in [0, t_1)$. If $t_1 = \infty$, then the graph is connected for all $t \geq 0$. Thus, $P(t) \geq \lambda_\gamma I_{Nn}$ for all $t \geq 0$. Therefore, for all $t \geq 0$,

$$\dot{V}(t) \leq -\lambda_\gamma ||\bar{x}||^2 \tag{3.18}$$

Then (3.18) implies

$$V(t) \leq V(0) \leq Q_{\max} < Q = \psi(r), \quad \forall\, t \geq 0. \tag{3.19}$$

If $\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(0)$ does not hold for all $t \geq 0$, then there must exist some finite $t_1 \geq 0$ such that

$$
\begin{aligned}
\bar{\mathcal{G}}(t) &= \bar{\mathcal{G}}(0), \quad t \in [0, t_1) \\
\bar{\mathcal{G}}(t_1) &\neq \bar{\mathcal{G}}(0)
\end{aligned}
\tag{3.20}
$$

We now claim $\bar{\mathcal{G}}(t_1) \supset \bar{\mathcal{G}}(0)$. In fact, since $\bar{\mathcal{G}}(0)$ is connected, our choice of γ guarantees $P(t) \geq \lambda_\gamma I_{Nn}$ for all $0 \leq t < t_1$, therefore, (3.17) implies

$$V(t) \leq V(0) \leq Q_{\max} < Q = \psi(r), \quad \forall\, t \in [0, t_1) \tag{3.21}$$

Assume, for some $(i, j) \in \bar{\mathcal{E}}(0)$, $(i, j) \notin \bar{\mathcal{E}}(t_1)$. Then $\lim_{t\to t_1^-} ||x_i - x_j|| = r$ which implies $V(t_1) \geq Q$ thus contradicting (3.21). Thus, the graph will not lost edges at time t_1. Therefore, $\bar{\mathcal{G}}(t_1) \supset \bar{\mathcal{G}}(0)$.

Since $\bar{\mathcal{G}}(t)$ can only have finitely many edges, there exists a finite integer $k > 0$ such that

$$\begin{aligned}
&\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(0), \quad t \in [0, t_1) \\
&\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(t_i) \supset \bar{\mathcal{G}}(t_{i-1}), \quad t \in [t_i, t_{i+1}),\ i = 1, \cdots, k-1 \\
&\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(t_k) \supset \bar{\mathcal{G}}(t_{k-1}), \quad t \in [t_k, \infty)
\end{aligned}$$

Thus, along any trajectory of the closed-loop system, we have

$$\dot{V}(t) \leq -\lambda_\gamma ||\bar{x}||^2,\ t \geq t_k \tag{3.22}$$

Thus,

$$V(t) \leq V(0) \leq Q_{\max} < Q = \psi(r), \quad \forall\, t \geq 0. \tag{3.23}$$

Therefore, for all $t \geq 0$, the graph $\bar{\mathcal{G}}(t)$ is connected.

Next, we will show $\lim_{t\to\infty} \bar{x}(t) = 0$. Since the graph $\bar{\mathcal{G}}(t)$ is connected for all $t \geq t_k$, and $H(t)$ is continuous and bounded for all $t \geq t_k$, by (3.14) and (3.22), $\bar{x}_i - \bar{x}_j$ with $j \in \bar{\mathcal{N}}_i(t_k)$ is bounded for all $t \geq t_k$, and $\lim_{t\to\infty} V(t)$ exists. From (3.9), $\dot{\bar{x}}$ is bounded over $t \geq t_k$. Thus $\dot{V}(t)$ is uniformly continuous for all $t \geq t_k$. Thus by Barbalat's Lemma [67], $\lim_{t\to\infty} \dot{V}(t) = 0$. Thus, (3.22) implies

$$\lim_{t\to\infty} \lambda_\gamma ||\bar{x}||^2 = 0 \tag{3.24}$$

Therefore, $\lim_{t\to\infty} \bar{x}(t) = 0$, i.e., $\lim_{t\to\infty}(x_i - x_0) = 0$, $i = 1, \cdots, N$. The proof is thus completed.

Remark 3.3 *Since we allow our leader system to be a class of autonomous linear systems of the form (3.2), our control law relies on the matrix S defining the leader system. This feature is a reminiscence of the classical internal model design method which can be found in [20, 28]. It is this feature that enables a control law to track, instead of a single trajectory, but a class of trajectories. It is noted that knowing the information of the matrix S is less demanding than knowing the trajectory of the leader. For example, a sinusoidal signal $A\sin(\omega t + \phi)$ is defined by three parameters A, ω, and ϕ. The matrix S is solely defined by ω. By using S in the control law, we can handle a sinusoidal signal with frequency ω, arbitrary amplitude A, and arbitrary initial phase ϕ. In the special case where the leader signal is step function or ramp function, we simply take $S = 0$, or $S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Therefore, in these two special cases, the control law does not really depend on any parameter of the leader signal.*

3.4 Example

Consider the multi-agent system composed of (3.1) and (3.2) in the two dimensional space with $N = 4$ and $S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Assume the sensing range is $r = 1$ and $\epsilon = 0.1$. To obtain the design parameter Q in the potential function, using (3.5) with $\alpha = \frac{N(N-1)}{2} + N = 10$ gives $\frac{\alpha(r-\epsilon)^2}{\epsilon} = 81$. Then taking $Q = 82$ makes (3.5) satisfied. Thus the potential function is

$$\psi(s) = \frac{s^2}{1 - s + \frac{1}{82}}, \quad 0 \leq s \leq r \tag{3.25}$$

Choose $\gamma = 2600$ according to the method in the proof of Theorem 3.1. For the purpose of simulation, let the initial values of various variables be

$$x_0(0) = \begin{bmatrix} 1 & 2 \end{bmatrix}^T,$$
$$x_1(0) = \begin{bmatrix} 1 & 1.3 \end{bmatrix}^T, \quad x_2(0) = \begin{bmatrix} 1.75 & 1.3 \end{bmatrix}^T,$$
$$x_3(0) = \begin{bmatrix} 1.75 & 0.6 \end{bmatrix}^T, \quad x_4(0) = \begin{bmatrix} 1.75 & -0.1 \end{bmatrix}^T$$

It can be verified that these initial values are such that
$\bar{\mathcal{E}}(0) = \{(0,1),\ (1,2),\ (2,3),\ (3,4)\}$ which forms a connected graph.

With these parameters, we have simulated the performance of the control law (3.6). Some of the simulation results are shown in Figures 3.1 and 3.2. Figure 3.1 shows the distances of the edges $\{(0,1),\ (1,2),\ (2,3),\ (3,4)\}$ which constitute the initial edge set. It can be seen that the network is connected. Figure 3.2 further shows that all the followers asymptotically approach the leader.

3.5 Conclusion

In this chapter, we have considered the leader-following problem of rendezvous with connectivity preservation for a multiple single-integrator system where the leader system can be any linear autonomous system. We have proposed a distributed state feedback control protocol that is able to maintain the connectivity of the system, and, at the same time, achieve asymptotic tracking of all followers to the output of the leader system.

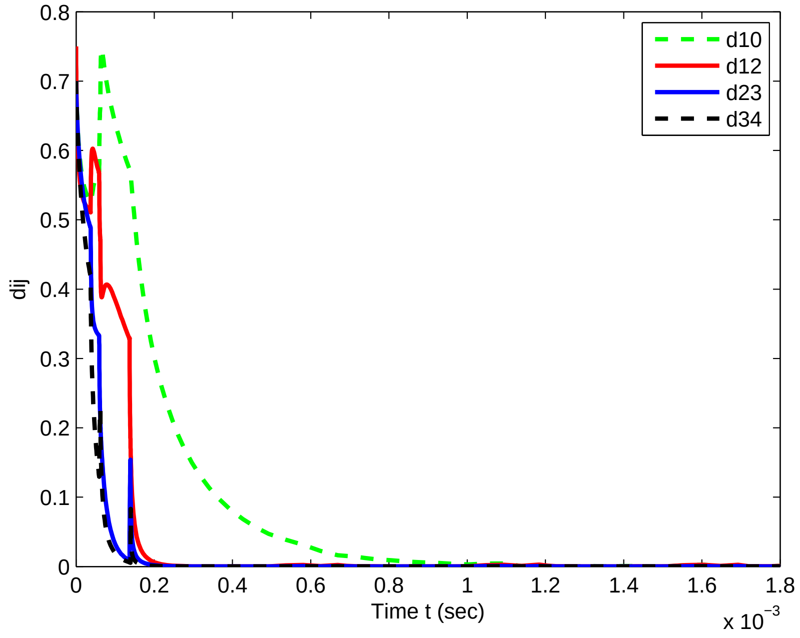

Figure 3.1: Distances among initially connected agents

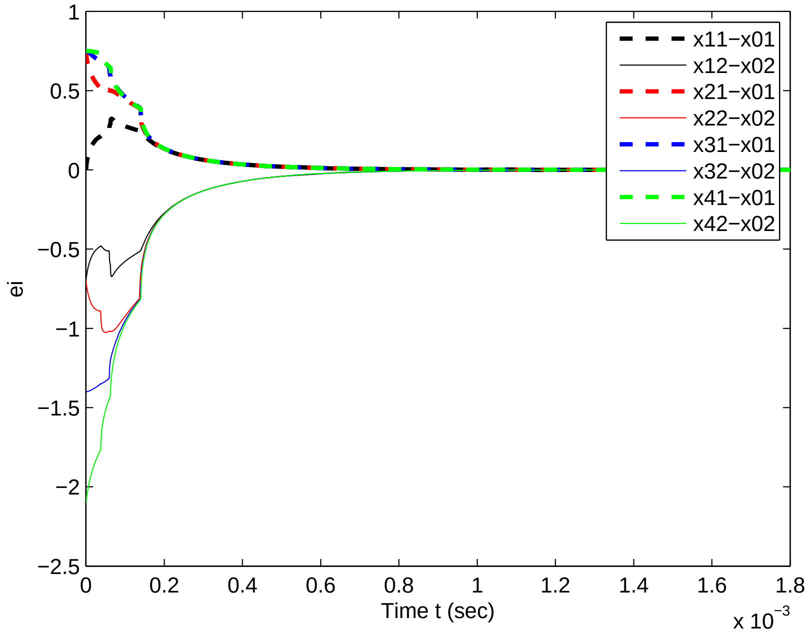

Figure 3.2: Position difference between follower and leader

Chapter 4

A Leader-following Rendezvous Problem of Double Integrator Multi-agent Systems

In this chapter, we will consider the leader-following rendezvous problem of double integrator multi-agent systems, and will solve this problem by a distributed full information state feedback control law.

4.1 Introduction

Consider a collection of double integrator systems of the following form

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + Ex_0 \\ e_i &= Cx_i + Fx_0 \qquad i = 1, \cdots, N \end{aligned} \tag{4.1}$$

where, for $i = 1, \cdots, N$, $x_i = \begin{bmatrix} q_i \\ p_i \end{bmatrix}$ with $q_i \in \mathbb{R}^n$ denoting the position and $p_i \in \mathbb{R}^n$ denoting the velocity, $u_i \in \mathbb{R}^n$, and $e_i \in \mathbb{R}^{n_e}$ are the input, and regulated output of agent i. $x_0 = \begin{bmatrix} q_0 \\ p_0 \end{bmatrix} \in \mathbb{R}^{2n}$ with $q_0, p_0 \in \mathbb{R}^n$ is some exogenous signal generated by a so-called exosystem as follows

$$\dot{x}_0 = Sx_0 \tag{4.2}$$

In general, various matrices in (4.1) and (4.2) can be arbitrarily given. However, in this chapter, we assume $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes I_n$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes I_n$, $E = \begin{bmatrix} 0_{n\times 2n} \\ D \end{bmatrix}$ with $D \in \mathbb{R}^{n\times 2n}$,

$C = I_{2n}$, $F = -I_{2n}$, and $S = \begin{bmatrix} 0_{n\times n} & I_n \\ S_1 & S_2 \end{bmatrix}$ with $S_1, S_2 \in \mathbb{R}^{n\times n}$. It can be seen that the first equation of (4.1) can be written as

$$\ddot{q}_i = u_i + Dx_0, \quad i = 1, \cdots, N. \tag{4.3}$$

Thus, when $D = 0$, (4.1) is a double integrator system. When $S_1 = S_2 = 0$, (4.2) is also a double integrator system. However, in this chapter, we don't require (4.2) to be a double integrator system and it can be seen that system (4.2) contains double integrator system and harmonic system as special cases.

The system composed of (4.1) and (4.2) can be viewed as a multi-agent system of $(N+1)$ agents with (4.2) as the leader and the N subsystems of (4.1) as N followers. With respect to the system composed of (4.1) and (4.2), we can define a digraph $\bar{\mathcal{G}}(t) = (\bar{\mathcal{V}}, \bar{\mathcal{E}}(t))$ where $\bar{\mathcal{V}} = \{0, 1, ..., N\}$ with 0 associated with the leader system and $i = 1, \cdots, N$, associated with the ith subsystem of (4.1), and $\bar{\mathcal{E}}(t) \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$. The set $\bar{\mathcal{V}}$ is called the node set of $\bar{\mathcal{G}}(t)$ and the set $\bar{\mathcal{E}}(t)$ is called the edge set of $\bar{\mathcal{G}}(t)$. We use the notation $\bar{\mathcal{N}}_i(t)$ to denote the neighbor set of the node i for $i = 0, 1, \cdots, N$.

The cooperative output regulation problem of the system composed of (4.1) and (4.2) has been extensively studied recently in several papers [29, 74, 91] where a distributed linear control law is designed such that the overall closed-loop system is asymptotically stable when x_0 is set to zero and the error output approaches zero asymptotically. Roughly, by a distributed linear control law, we mean a control law whose component u_i can only make use of the states of its neighbor subsystems. It is shown in [29, 74, 91] that the cooperative output regulation problem of (4.1) and (4.2) is solvable only if the graph $\bar{\mathcal{G}}(t)$ associated with (4.1) and (4.2) is connected. In the problem formulation of [29, 74, 91], the graph $\bar{\mathcal{G}}(t)$ is assumed to be connected. However, in many real applications such as rendezvous, flocking, and swarming, the graph $\bar{\mathcal{G}}(t)$ is not given but defined dynamically. For example, when (4.1) is a single or double integrator system, the edge set $\bar{\mathcal{E}}(t)$ is defined such that $(i, j) \in \bar{\mathcal{E}}(t)$ if and only if $||q_i(t) - q_j(t)|| < r$ for some real number $r > 0$ [22, 35, 84, 102]. The rational of this definition is based on the assumption that each agent is equipped with a distance sensor whose sensing radius is r. In [72] and [73], the definition of the edge set is further modified by employing hysteresis technique.

Suppose $\bar{\mathcal{G}}(0)$ is connected. Then the problem of designing a distributed control law such that the graph $\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$ is called connectivity preservation problem. If, in addition, the control law is such that, for all $i = 1, \cdots, N$, $\lim_{t\to\infty} e_i(t) = 0$, then the problem is called leader-following rendezvous problem. A special case of the above problem where the disturbance is absent and the leader is a known ramp signal was recently studied in [73]. More recently, a leader-following rendezvous problem for a class of second order nonlinear multi-agent systems was further studied in [70]. Our problem here is different from [70] and [73] in at least two aspects. First, the dynamics of the leader do not have to be the same as those of the followers when the control is set to zero. In fact, our leader system as described by the exosystem (4.2) can not only generate ramp signal, but also sinusoidal signals with arbitrary amplitudes and initial phases. Second, the plant is allowed to be subject to an external disturbance.

The rest of this chapter is organized as follows. In Section 4.2, we will formulate our problem precisely. In Sections 4.3 and 4.4, we will present our main result together with illustrative examples. Finally, we will close this chapter in Section 4.5.

4.2 Problem Formulation

To introduce our problem, we need to first define the edge set $\bar{\mathcal{E}}(t)$ for the system composed of (4.1) and (4.2) as follows.
Given any $r > 0$ and $\epsilon \in (0, r)$, for any $t \geq 0$, $\bar{\mathcal{E}}(t) = \{(i, j) \mid i, j \in \bar{\mathcal{V}}\}$ is defined such that

1. $\bar{\mathcal{E}}(0) = \{(i, j) \mid ||q_i(0) - q_j(0)|| < (r - \epsilon), i, j = 1, \cdots, N\} \cup \{(0, j) \mid ||q_0(0) - q_j(0)|| < (r - \epsilon), j = 1, \cdots, N\}$;
2. if $||q_i(t) - q_j(t)|| \geq r$, then $(i, j) \notin \bar{\mathcal{E}}(t)$;
3. $(i, 0) \notin \bar{\mathcal{E}}(t)$, for $i = 0, 1, \cdots, N$;
4. for $i = 0, 1, \cdots, N, j = 1, \cdots, N$, if $(i, j) \notin \bar{\mathcal{E}}(t^-)$ and $||q_i(t) - q_j(t)|| < (r - \epsilon)$, then $(i, j) \in \bar{\mathcal{E}}(t)$.
5. for $i = 0, 1, \cdots, N, j = 1, \cdots, N$, if $(i, j) \in \bar{\mathcal{E}}(t^-)$ and $||q_i(t) - q_j(t)|| < r$, then $(i, j) \in \bar{\mathcal{E}}(t)$.

Remark 4.1 *The definition is somehow different from that in literature mainly in that the node* 0 *associated with the leader as well as the edges adjacent to the node* 0 *is part of the graph. It is noted that the leader does not have a control and, therefore, there is no edge from a follower to the leader. If* $\epsilon = 0$*, then the above definition is similar to that given in [35]. Thus the physical interpretation of* r *is the sensing radius of the distance sensor of each follower. The number* ϵ *is to introduce the effect of hysteresis.*

Remark 4.2 *We can define a subgraph* $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ *of* $\bar{\mathcal{G}}(t)$*, where* $\mathcal{V} = \{1, \cdots, N\}$*, and* $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ *is obtained from* $\bar{\mathcal{E}}(t)$ *by removing all edges between the node 0 and the nodes in* $\mathcal{V}$*. Clearly,* $\mathcal{G}(t)$ *is an undirected graph. For* $i = 1, \cdots, N$*, let* $\mathcal{N}_i(t) = \bar{\mathcal{N}}_i(t) \cap \mathcal{V}$*.*

It can be seen that, for $i = 1, \cdots, N$, $\mathcal{N}_i(t)$ is the neighbor set of the node i with respect to $\mathcal{V}$.

We will consider the following distributed dynamic state feedback control law

$$\begin{aligned} u_i &= h_i(\eta_i, \eta_j, x_i - x_j, j \in \bar{\mathcal{N}_i}(t)) \\ \dot{\eta}_i &= g_i(\eta_i, \eta_j, x_i - x_j, j \in \bar{\mathcal{N}_i}(t)),\ i = 1, ..., N \end{aligned} \tag{4.4}$$

where $\eta = \text{col}(\eta_1, \cdots, \eta_N)$ with $\eta_i \in \mathbb{R}^{2n}$, and h_i and g_i are sufficiently smooth functions to be specified later.

The control law (4.4) includes the static state feedback control law as a special case by allowing the dimension of η to be zero.

The leader-following rendezvous problem is described as follows.

Definition 4.1 *Given the multi-agent system composed of* (4.1) *and* (4.2)*, $r > 0$ and $\epsilon \in (0, r)$, and arbitrary positive real numbers P_i, N_i, $i = 1, \cdots, N$, find a distributed control law of the form* (4.4) *such that, for all initial conditions $x_0(0)$, $q_i(0)$, $p_i(0)$, $\eta_i(0)$, $i = 1, \cdots, N$, that make $\bar{\mathcal{G}}(0)$ connected, and satisfy $||p_i(0) - p_0(0)|| \leq P_i$, and $||\eta_i(0) - x_0(0)|| \leq N_i$, the closed-loop system has the following properties:*

1. *$\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$;*

2. $\lim_{t\to\infty}(x_i - x_0) = 0$*, $i = 1, \cdots, N$.*

Remark 4.3 *Two different rendezvous problems, i.e., the leaderless and leader-following rendezvous problems have been studied in [5, 22, 35, 70, 73]. The definition given here has generalized the leader-following rendezvous problems given in literature in at least two aspects. First, the leader system does not have to be the same as the follower system when the control is set to zero. In particular, the leader system can generate sinusoidal signals with arbitrary amplitudes and initial phases. Second, the plant is allowed to be subject to an external disturbance. In order to accomplish these generalizations, instead of employing a static state feedback control law as in the literature, we will employ a dynamic state feedback control law to deal with our problem.*

Remark 4.4 *The leader-following rendezvous problem described here is also related to the cooperative output regulation problem studied recently in [29, 74, 91]. In fact, if, instead of achieving properties 1) and 2), we only need to achieve property 2) while assuming the graph $\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$, then the leader-following rendezvous problem described here would reduce to the cooperative output regulation problem.*

Remark 4.5 *The positive real numbers P_i and N_i in the above definition are to define some closed balls in which the initial states of the system are allowed to stay. They are introduced so that we can make use of a bounded potential function in our control law as will be seen in the next section. With these positive real numbers given, the control law can be made independent of the initial condition of the closed-loop system. Thus, the definition here reveals the close relation of the rendezvous problem to the semi-global stabilization problem.*

4.3 Main Result

One of the main objectives here is to deal with the external disturbance Dx_0. For this purpose, let

$$X = I_{2n}, \quad U = \begin{bmatrix} S_1 & S_2 \end{bmatrix} - D. \tag{4.5}$$

Then it can be verified that performing on (4.1) the following coordinate transformation

$$\begin{aligned} \bar{x}_i &= \begin{bmatrix} \bar{q}_i \\ \bar{p}_i \end{bmatrix} = x_i - Xx_0, \quad i = 0, 1, \cdots, N \\ \bar{u}_i &= u_i - Ux_0, \quad i = 1, \cdots, N \end{aligned} \tag{4.6}$$

converts system (4.1) to the following double-integrator system without disturbance

$$\dot{\bar{q}}_i = \bar{p}_i, \quad \dot{\bar{p}}_i = \bar{u}_i, \quad i = 1, \cdots, N \tag{4.7}$$

Remark 4.6 *The transformation (4.6) is inspired by the output regulation theory [20]. In fact, associated with (4.1) is the following linear matrix equations*

$$XS = AX + BU + E, \quad 0 = CX + F \tag{4.8}$$

is called regulator equations [28]. It can be verified that (4.5) is a solution pair of (4.8). The transformation (4.6) is a standard technique for converting an output regulation problem to a stabilization problem.

Both the leaderless and the leader-following rendezvous problems for (4.7) have been well studied in the literature. In particular, it is known that the leader-following rendezvous problem can be solved by a distributed control law of the form

$$\bar{u}_i = k_i(\bar{x}_i - \bar{x}_j, j \in \bar{\mathcal{N}}_i(t)), \quad i = 1, \cdots, N \tag{4.9}$$

where the specific expressions for the functions $k_i(\bar{x}_i - \bar{x}_j, j \in \bar{\mathcal{N}}_i(t))$ will be given later in (4.20). By the output regulation theory [20, 28], it can be concluded that if the control law

(4.9) solves the leader-following rendezvous problem for (4.7), then the following control law

$$u_i = k_i(x_i - x_j, j \in \bar{\mathcal{N}}_i(t)) + Ux_0, \quad i = 1, \cdots, N$$

solves the leader-following rendezvous problem for (4.1) provided that $0 \in \bar{\mathcal{N}}_i(t)$ for $i = 1, \cdots, N$. Nevertheless, since the state of the leader is not always available for all control u_i, motivated by [74], we will consider a control law of the following form:

$$\begin{aligned} \bar{u}_i &= k_i(\bar{x}_i - \bar{x}_j, j \in \bar{\mathcal{N}}_i(t)) + U\bar{\eta}_i, \qquad i = 1, \cdots, N \\ \dot{\bar{\eta}}_i &= S\bar{\eta}_i + \gamma(\sum_{j=1}^{N} a_{ij}(t)(\bar{\eta}_j - \bar{\eta}_i) - a_{i0}(t)\bar{\eta}_i) \end{aligned} \tag{4.10}$$

where γ is a sufficiently large positive number, and

$$a_{ij}(t) = \begin{cases} 1, & (j,i) \in \bar{\mathcal{E}}(t) \\ 0, & \text{otherwise} \end{cases} \tag{4.11}$$

with $i = 1, \cdots, N$, $j = 0, \cdots, N$.

If (4.10) solves the rendezvous problem of the system (4.7), then, letting $\eta_i = \bar{\eta}_i + x_0$, $i = 1, \cdots, N$, shows that the following control law

$$\begin{aligned} u_i &= k_i(x_i - x_j, j \in \bar{\mathcal{N}}_i(t)) + U\eta_i, \qquad i = 1, \cdots, N \\ \dot{\eta}_i &= S\eta_i + \gamma(\sum_{j=1}^{N} a_{ij}(t)(\eta_j - \eta_i) + a_{i0}(t)(x_0 - \eta_i)) \end{aligned} \tag{4.12}$$

solves the rendezvous problem for (4.1). It was shown in [74] that the second equation of (4.12) is such that $\lim_{t\to\infty}(\eta_i - x_0) = 0$ if $\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$, and therefore can be viewed as a distributed asymptotic observer of the leader system.

Under the control law (4.10), the closed-loop system of each agent becomes

$$\begin{aligned} \dot{\bar{q}}_i &= \bar{p}_i, \qquad i = 1, \cdots, N \\ \dot{\bar{p}}_i &= \bar{u}_i = k_i(\bar{x}_i - \bar{x}_j, j \in \bar{\mathcal{N}}_i(t)) + U\bar{\eta}_i \\ \dot{\bar{\eta}}_i &= S\bar{\eta}_i + \gamma(\sum_{j=1}^{N} a_{ij}(t)(\bar{\eta}_j - \bar{\eta}_i) - a_{i0}(t)\bar{\eta}_i) \end{aligned} \tag{4.13}$$

The functions $k_i(\bar{x}_i - \bar{x}_j, j \in \bar{\mathcal{N}}_i(t))$ in (4.9) can be expressed in terms of potential functions as can be found in [12, 35, 73]. To be specific, here we adopt the bounded potential function proposed in [73] as follows

$$\psi(s) = \frac{s^2}{r - s + \frac{r^2}{Q}}, \ 0 \leq s \leq r \tag{4.14}$$

where Q is some positive number. The function is nonnegative and bounded over $[0, r]$, and its derivative $\frac{d\psi(s)}{ds} = \frac{s(2r-s+\frac{2r^2}{Q})}{(r-s+\frac{r^2}{Q})^2}$ is positive for all $s \in (0, r]$. Moreover, the function has the property that, for any $\alpha > 0$, $\beta \geq 0$, and any $\epsilon \in (0, r)$, there exists some $Q > 0$ such that

$$\psi(r) = Q \geq \alpha\psi(r - \epsilon) + \beta \tag{4.15}$$

To show this property, we note that, for $Q > 0$, the function $\alpha\frac{(r-\epsilon)^2}{\epsilon+r^2/Q} + \beta$ is monotonously increasing with respect to Q, and has a finite upper bound $\frac{\alpha(r-\epsilon)^2}{\epsilon} + \beta$. Therefore, (4.15) holds whenever $Q > (\frac{\alpha(r-\epsilon)^2}{\epsilon} + \beta)$.

For $i = 1, \cdots, N$ and $j = 0, 1, \cdots, N$, let

$$w_{ij}(t) = \begin{cases} \frac{2r-||q_i-q_j||+\frac{2r^2}{Q}}{(r-||q_i-q_j||+\frac{r^2}{Q})^2}, & (j,i) \in \bar{\mathcal{E}}(t) \\ 0, & \text{otherwise} \end{cases} \tag{4.16}$$

We call the following function

$$V(\bar{q}, \bar{p}, t) = \frac{1}{2}\sum_{i=1}^{N}(\sum_{j\in\mathcal{N}_i(t)} \psi(||\bar{q}_i - \bar{q}_j||) + \bar{p}_i^T\bar{p}_i) \tag{4.17}$$

an energy function for system (4.7). It has been shown that the following control law

$$\bar{u}_i = -\sum_{j\in\mathcal{N}_i(t)} (w_{ij}(t)(\bar{q}_i - \bar{q}_j) + a_{ij}(t)(\bar{p}_i - \bar{p}_j)) \tag{4.18}$$

solves the leaderless rendezvous problem for system (4.7) [73].

To solve the leader-following rendezvous problem for system (4.1) and (4.2), we define the following energy function for system (4.13).

$$V(\bar{q}, \bar{p}, \bar{\eta}, t) = \frac{1}{2}\sum_{i=1}^{N}(\sum_{j\in\mathcal{N}_i(t)} \psi(||\bar{q}_i - \bar{q}_j||) + 2a_{i0}(t)\psi(||\bar{q}_i||) + \bar{p}_i^T\bar{p}_i + \bar{\eta}_i^T\bar{\eta}_i) \tag{4.19}$$

Correspondingly, we define the function $k_i(\bar{x}_i - \bar{x}_j, j \in \bar{\mathcal{N}}_i(t))$ in (4.13) as follows

$$k_i(\bar{x}_i - \bar{x}_j, j \in \bar{\mathcal{N}}_i(t)) = -\sum_{j\in\bar{\mathcal{N}}_i(t)} (w_{ij}(t)(\bar{q}_i - \bar{q}_j) + a_{ij}(t)(\bar{p}_i - \bar{p}_j)) \tag{4.20}$$

Before stating our main result, let us first introduce a lemma as follows.

Lemma 4.1 *Consider the following symmetric matrix*

$$P = \begin{bmatrix} H_{11} & -Z \\ -Z^T & \gamma H_{12} + M_1 \end{bmatrix} \tag{4.21}$$

where $\gamma \in \mathbb{R}$, $H_{11} \in \mathbb{R}^{n_1\times n_1}$ and $H_{12} \in \mathbb{R}^{n_2\times n_2}$ are symmetric and positive definite, and $Z \in \mathbb{R}^{n_1\times n_2}$ and $M_1 \in \mathbb{R}^{n_2\times n_2}$ are any constant matrices with M_1 symmetric. Then, there exists $\gamma_0 > 0$ such that P is positive definite for all $\gamma > \gamma_0$.

Proof: From the following identity

$$\begin{bmatrix} A_1 & C_1 \\ C_1^T & B_1 \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ C_1^T A_1^{-1} & I \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & B_1 - C_1^T A_1^{-1} C_1 \end{bmatrix} \begin{bmatrix} I & A_1^{-1} C_1 \\ 0 & I \end{bmatrix}$$

where A_1 and B_1 are symmetric with A_1 nonsingular, P is positive definite if and only if H_{11} and $(\gamma H_{12} + M_1) - Z^T H_{11}^{-1} Z$ are positive definite. Let $\lambda_1(A_1)$ be the minimal eigenvalue of a symmetric matrix A_1. Then P is positive definite when

$$\gamma > \gamma_0 = \frac{-\lambda_1(M_1 - Z^T H_{11}^{-1} Z)}{\lambda_1(H_{12})}. \tag{4.22}$$

We can now state our main result as follows.

Theorem 4.1 *The leader-following rendezvous problem for system (4.7) is solvable by a control law composed of (4.10) and* (4.20) *where γ is a sufficiently large positive constant. As a result, the same problem for system composed of (4.1) and (4.2) is solvable by a control law composed of (4.12) and (4.20).*

Proof: Given $r > 0$, $\epsilon \in (0, r)$, and arbitrary positive real numbers P_i, N_i, $i = 1, \cdots, N$, the control law is determined by two design parameters Q and γ. Let $\bar{\eta} = \mathrm{col}(\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_N)$, $\bar{q} = \mathrm{col}(\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_N)$, and $\bar{p} = \mathrm{col}(\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_N)$. Let

$$Q_{\max} = \frac{\alpha(r-\epsilon)^2}{\epsilon} + \beta \tag{4.23}$$

where $\alpha = \frac{N(N-1)}{2} + N$ and

$$\beta = \frac{1}{2} \sum_{i=1}^{N} (P_i^2 + N_i^2) \tag{4.24}$$

Pick any finite Q such that $Q > Q_{\max}$.

To determine γ, let

$$P(t) = \begin{bmatrix} H(t) \otimes I_n & -I_N \otimes \frac{U}{2} \\ -I_N \otimes \frac{U^T}{2} & \gamma H(t) \otimes I_{2n} - I_N \otimes \frac{S+S^T}{2} \end{bmatrix} \tag{4.25}$$

where

$$H(t) = \begin{bmatrix} \sum_{j=0,j\neq 1}^{N} a_{1j}(t) & -a_{12}(t) & \cdots & -a_{1N}(t) \\ -a_{21}(t) & \sum_{j=0,j\neq 2}^{N} a_{2j}(t) & \cdots & -a_{2N}(t) \\ \vdots & \vdots & \vdots & \vdots \\ -a_{N1}(t) & -a_{N2}(t) & \cdots & \sum_{j=0,j\neq N}^{N} a_{Nj}(t) \end{bmatrix}$$

where $a_{ij}(t)$ is given by (4.11). Note that $H(0) = -M + \Delta$ where M is a constant Metzler matrix and $\Delta = \text{diag}[a_{10}(0)\cdots, a_{N0}(0)]$. By [41], $H(0)$ is positive definite since the graph $\bar{\mathcal{G}}(0)$ is connected. Since $H(0)$ and hence $P(0)$ are uniquely determined by $\bar{\mathcal{G}}(0)$, and there are only finitely many different connected $\bar{\mathcal{G}}(0)$ with N edges, by Lemma 4.1, there is a $\gamma > 0$ such that $P(0)$ is positive definite for all possible connected $\bar{\mathcal{G}}(0)$ with N edges. Fix this γ. We now claim if $\bar{\mathcal{G}}(0) \subset \bar{\mathcal{G}}(t)$ for any $t \geq 0$, then $P(t) \geq P(0) > 0$. In fact, $H(t)$ can be written as $H(0) + \Delta H$ for some positive semi-definite symmetric matrix ΔH. Thus, $P(t)$ can be written as $P(0) + \Delta P$ for some positive semi-definite symmetric matrix ΔP. Thus, our choice of γ guarantees $P(t) \geq P(0) > 0$.

Now, we will first show that the above control law is such that the graph $\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$. Let the energy function be given by (4.19). Then it can be seen that for all initial conditions $x_0(0), q_i(0), p_i(0), \eta_i(0)$ that make $\bar{\mathcal{G}}(0)$ connected and satisfy $||p_i(0) - p_0(0)|| \leq P_i$, $||\eta_i(0) - x_0(0)|| \leq N_i$, our choice of Q is such that

$$V(0) = V(\bar{q}(0), \bar{p}(0), \bar{\eta}(0), 0) \leq Q_{\max} \tag{4.26}$$

The time derivative of the function (4.19) along the closed-loop system (4.13) is

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{N} \sum_{j\in\bar{\mathcal{N}}_i(t)} \dot{\psi}(||\bar{q}_i - \bar{q}_j||) + \bar{p}^T\dot{\bar{p}} + \bar{\eta}^T\dot{\bar{\eta}} \\ &= \sum_{i=1}^{N} \bar{p}_i^T \sum_{j\in\bar{\mathcal{N}}_i(t)} w_{ij}(t)(\bar{q}_i - \bar{q}_j) - \sum_{i=1}^{N} \bar{p}_i^T \sum_{j\in\bar{\mathcal{N}}_i(t)} w_{ij}(t)(\bar{q}_i - \bar{q}_j) \\ &- \sum_{i=1}^{N} \bar{p}_i^T (\sum_{j\in\bar{\mathcal{N}}_i(t)} a_{ij}(\bar{p}_i - \bar{p}_j)) + \sum_{i=1}^{N} \bar{p}_i^T U\bar{\eta}_i + \bar{\eta}^T(I_N \otimes S - \gamma H(t) \otimes I_{2n})\bar{\eta} \\ &= -\bar{p}^T(H(t)\otimes I_n)\bar{p} + \bar{p}^T(I_N \otimes U)\bar{\eta} + \bar{\eta}^T(I_N \otimes \frac{S+S^T}{2} - \gamma H(t)\otimes I_{2n})\bar{\eta} \\ &= -\begin{bmatrix} \bar{p} \\ \bar{\eta} \end{bmatrix}^T P(t) \begin{bmatrix} \bar{p} \\ \bar{\eta} \end{bmatrix} \end{aligned} \tag{4.27}$$

By the continuity of the solution of (4.13), there exists $0 < t_1 \leq \infty$ such that $\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(0)$ for all $t \in [0, t_1)$. If $t_1 = \infty$, then the graph is connected for all $t \geq 0$. Thus, our choice of γ guarantees $P(t) = P(0)$ is positive definite for all $t \geq 0$. Therefore, (4.26) and (4.27) imply

$$V(t) \leq V(0) \leq Q_{\max} < Q = \psi(r), \quad \forall\, t \geq 0. \tag{4.28}$$

If $\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(0)$ does not hold for all $t \geq 0$, then there must exist some $t_1 \geq 0$ such that

$$\begin{aligned} \bar{\mathcal{G}}(t) &= \bar{\mathcal{G}}(0), \quad t \in [0, t_1) \\ \bar{\mathcal{G}}(t_1) &\neq \bar{\mathcal{G}}(0) \end{aligned} \tag{4.29}$$

We now claim $\bar{\mathcal{G}}(t_1) \supset \bar{\mathcal{G}}(0)$. In fact, since $\bar{\mathcal{G}}(0)$ is connected, our choice of γ guarantees that $P(t) = P(0)$ is positive definite for all $0 \leq t < t_1$. Therefore, (4.26) and (4.27) imply

$$V(t) \leq V(0) \leq Q_{\max} < Q = \psi(r), \quad \forall\, t \in [0, t_1) \tag{4.30}$$

Assume, for some $(i, j) \in \bar{\mathcal{E}}(0)$, $(i, j) \notin \bar{\mathcal{E}}(t_1)$. Then $\lim_{t \to t_1^-} ||q_i - q_j|| = r$ which implies $V(t_1) \geq Q$ thus contradicting (4.30). Thus, the graph will not lose edges at time t_1. Therefore, $\bar{\mathcal{G}}(t_1) \supset \bar{\mathcal{G}}(0)$. Thus $P(t_1)$ is positive definite.

Since $\bar{\mathcal{G}}(t)$ can only have finitely many edges, there exists a finite integer $k > 0$ such that

$$\begin{aligned} &\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(0), \quad t \in [0, t_1) \\ &\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(t_i) \supset \bar{\mathcal{G}}(t_{i-1}), \quad t \in [t_i, t_{i+1}),\ i = 1, \cdots, k-1 \\ &\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(t_k) \supset \bar{\mathcal{G}}(t_{k-1}), \quad t \in [t_k, \infty) \end{aligned}$$

Thus, for all $t \geq t_k$, along any trajectory of the closed-loop system, we have

$$\dot{V}(t) = - \begin{bmatrix} \bar{p} \\ \bar{\eta} \end{bmatrix}^T P(t_k) \begin{bmatrix} \bar{p} \\ \bar{\eta} \end{bmatrix}, \ t \geq t_k. \tag{4.31}$$

Thus,

$$V(t) \leq V(0) \leq Q_{\max} < Q, \quad \forall\, t \geq 0. \tag{4.32}$$

Therefore, for all $t \geq 0$, the graph $\bar{\mathcal{G}}(t)$ is connected.

Next we will show that $\lim_{t \to \infty}(x_i - x_0) = 0$. Since, for all $t \geq t_k$, V is positive semi-definite and $\dot{V}$ is nonincreasing, for $i = 1, \cdots, N$, $\bar{p}_i, \bar{\eta}_i$ as well as $\bar{q}_i - \bar{q}_j$ with $j \in \bar{\mathcal{N}}_i(t_k)$ are bounded. Thus, $\lim_{t \to \infty} V(t)$ exists. From (4.13), $\dot{\bar{p}}$ and $\dot{\bar{\eta}}$ are bounded over $[t_k, \infty)$. Thus $\dot{V}(t)$ is uniformly continuous for all $t \geq t_k$. So, by Barbalat's lemma [67], $\lim_{t \to \infty} \dot{V}(t) = 0$. Therefore, we have $\lim_{t \to \infty} \bar{p} = 0$, i.e., for $i = 1, \cdots, N$, $\lim_{t \to \infty}(p_i - p_0) = 0$.

It remains to show that, for $i = 1, \cdots, N$, $\lim_{t \to \infty}(q_i - q_0) = 0$.

From (4.13), $\ddot{\bar{p}}_i$ is bounded over $[t_k, \infty)$. Thus $\dot{\bar{p}}_i$ is uniformly continuous over $[t_k, \infty)$. By Barbalat's lemma again, we have

$$\begin{aligned}
\lim_{t\to\infty} \dot{\bar{p}}_i &= -\lim_{t\to\infty} \sum_{j\in\bar{\mathcal{N}}_i(t)} w_{ij}(t)(\bar{q}_i - \bar{q}_j) \\
&= -\lim_{t\to\infty} \sum_{j\in\bar{\mathcal{N}}_i(t)} \frac{2r - ||\bar{q}_i - \bar{q}_j|| + \frac{2r^2}{Q}}{(r - ||\bar{q}_i - \bar{q}_j|| + \frac{r^2}{Q})^2}(\bar{q}_i - \bar{q}_j) \\
&= 0
\end{aligned} \tag{4.33}$$

Let $\phi(s) = \frac{2r-s+\frac{2r^2}{Q}}{(r-s+\frac{r^2}{Q})^2}$ for $0 \leq s \leq r$. Then $0 < \frac{2r+\frac{2r^2}{Q}}{(r+\frac{r^2}{Q})^2} = \phi(0) \leq \phi(s) \leq \phi(r) = \frac{r+\frac{2r^2}{Q}}{(\frac{r^2}{Q})^2}$ over $s \in [0, r]$. let

$$H_1(t) = \begin{bmatrix} \beta_1(t) & -w_{12}(t) & \cdots & -w_{1N}(t) \\ -w_{21}(t) & \beta_2(t) & \cdots & -w_{2N}(t) \\ \vdots & \vdots & \vdots & \vdots \\ -w_{N1}(t) & -w_{N2}(t) & \cdots & \beta_N(t) \end{bmatrix} \tag{4.34}$$

where $\beta_i(t) = \sum_{j=0,j\neq i}^{N} w_{ij}(t)$. It is noted that for all $t \geq t_k$, $w_{ij}(t)$ is nonnegative and bounded.

It can be seen that (4.33) can be put in the following form

$$\lim_{t\to\infty} (H_1(t) \otimes I_n)\bar{q} = 0 \tag{4.35}$$

We now show $H_1(t) \geq H_0$ for some constant positive definite matrix H_0. In fact, let $\bar{\mathcal{G}}_0 = (\bar{\mathcal{V}}, \bar{\mathcal{E}}_0)$ be any connected graph such that $\bar{\mathcal{G}}_0 \subset \bar{\mathcal{G}}(0)$. Let

$$H_0 = \begin{bmatrix} \delta_1 & -m_{12} & \cdots & -m_{1N} \\ -m_{21} & \delta_2 & \cdots & -m_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ -m_{N1} & -m_{N2} & \cdots & \delta_N \end{bmatrix} \tag{4.36}$$

where

$$m_{ij} = \begin{cases} \phi(0), & (j, i) \in \bar{\mathcal{E}}_0 \\ 0, & \text{otherwise} \end{cases} \tag{4.37}$$

and $\delta_i = \sum_{j=0,j\neq i}^{N} m_{ij}$. Note that $H_0 = -M_0 + \Delta_0$ where M_0 is a constant Metzler matrix with zero row sum whose nonzero off-diagonal entries are equal to $\phi(0)$, and Δ_0 is a nonnegative diagonal matrix with at least one diagonal entry equaling to $\phi(0)$. It can be seen that $\Gamma(M_0) = \mathcal{G}_0$ where $\mathcal{G}_0$ is a subgraph of $\bar{\mathcal{G}}_0$ and is obtained from $\bar{\mathcal{G}}_0$ by removing the node 0 and all edges incident to the node 0. By [41], H_0 is positive definite. Note that $H_1 - H_0$ takes the following form:

$$\begin{bmatrix} \beta_1-\delta_1 & -(w_{12}-m_{12}) & \cdots & -(w_{1N}-m_{1N}) \\ -(w_{21}-m_{21}) & \beta_2-\delta_2 & \cdots & -(w_{2N}-m_{2N}) \\ \vdots & \vdots & \vdots & \vdots \\ -(w_{N1}-m_{N1}) & -(w_{N2}-m_{N2}) & \cdots & \beta_N-\delta_N \end{bmatrix}$$

Since, for $i = 1, \cdots, N$, $(\beta_i - \delta_i) = \sum_{j=0, j\neq i}^{N}(w_{ij} - m_{ij}) \geq \sum_{j=1, j\neq i}^{N}(w_{ij} - m_{ij}) \geq 0$, $H_1 - H_0$ is positive semi-definite by Gersgorin Theorem, i.e., $(H_1(t) \otimes I_n) \geq (H_0 \otimes I_n)$. Thus (4.35) implies

$$0 = \lim_{t\to\infty} \bar{q}^T(H_1(t) \otimes I_n)\bar{q} \geq \lim_{t\to\infty} \bar{q}^T(H_0 \otimes I_n)\bar{q} \geq 0 \tag{4.38}$$

Thus, $\bar{q}$ converges to the origin asymptotically, i.e., $\lim_{t\to\infty}(q_i - q_0) = 0$, $i = 1, \cdots, N$. The proof is thus completed.

4.4 Illustrative Examples

4.4.1 Example 1

Consider the system (4.1) with $N = 4$ and $D = \begin{bmatrix} 2 & 3 \end{bmatrix} \otimes I_2$, and the leader system (4.2) with $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_2$. Assume $r = 8$, and $P_i = 22, N_i = 25$, $i = 1, \cdots, 4$. To obtain the design parameter Q in the potential function, we first obtain $\beta = 2218$ using (4.24). Take $\epsilon = 0.5$. Using (4.23) with $\alpha = \frac{N(N-1)}{2} + N = 10$ and $\beta = 2218$ gives $(\frac{\alpha(r-\epsilon)^2}{\epsilon} + \beta) = 3343$. Then taking $Q = 4000$ makes (4.15) satisfied. Thus the potential function is

$$\psi(s) = \frac{s^2}{8 - s + \frac{64}{4000}}, \quad 0 \leq s \leq r \tag{4.39}$$

To complete the design of the control law (4.12), let $\gamma = 150$ which is such that $P(0)$ is positive definite.

Let the initial values of various variables be

$$x_1(0) = \begin{bmatrix} 1 & 7 & 9 & 8 \end{bmatrix}^T, \quad x_2(0) = \begin{bmatrix} 1 & 1 & 1 & 3 \end{bmatrix}^T$$

$$x_3(0) = \begin{bmatrix} 5 & 4 & 6 & 7 \end{bmatrix}^T, \quad x_4(0) = \begin{bmatrix} 1 & -5 & 6 & 4 \end{bmatrix}^T$$

$$\eta_1(0) = \begin{bmatrix} 12 & 9 & 8 & 6 \end{bmatrix}^T; \quad \eta_2(0) = \begin{bmatrix} 12 & 2 & -5 & -5 \end{bmatrix}^T$$

$$\eta_3(0) = \begin{bmatrix} 3 & 1 & 4 & 6 \end{bmatrix}^T; \quad \eta_4(0) = \begin{bmatrix} 8 & 2 & 1 & 6 \end{bmatrix}^T$$

$$x_0(0) = \begin{bmatrix} 1 & 12 & 2 & 6 \end{bmatrix}^T$$

It can be verified that these initial values are such that $||p_i(0) - p_0(0)|| \leq P_i$, $||\eta_i(0) - x_0(0)|| \leq N_i$, and $\bar{\mathcal{E}}(0) = \{(0,1),\ (1,2),\ (1,3),\ (2,3),\ (2,4)\}$ which forms a connected graph. The performance of the control law (4.12) are shown in Figures 4.1 to 4.3. Figure 4.1 shows the distances of the edges $\{(0,1),\ (1,2),\ (1,3),\ (2,3),\ (2,4)\}$ which constitute the initial edge set. It can be seen that, for all $t \geq 0$, these distances are smaller than the sensing range $r = 8$. Thus, the connectivity of the network is maintained. Figures 4.2 and 4.3 further show that both the position and the velocity of all the followers asymptotically approach the position and the velocity of the leader, respectively.

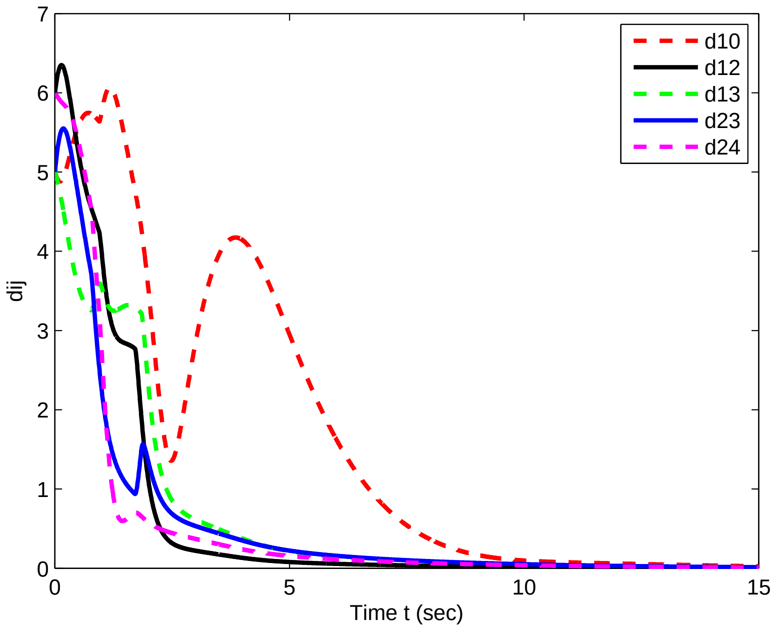

Figure 4.1: Distances among agents

4.4.2 Example 2

Consider the following double integrator systems with disturbance with $N = 4$ and $n = 2$

$$\ddot{q}_i = u_i + Dx_0,\ \ i = 1, 2, 3, 4 \tag{4.40}$$

where $D = \begin{bmatrix} 2 & 3 \end{bmatrix} \otimes I_2$, and the leader system

$$\dot{x}_0 = Sx_0 \tag{4.41}$$

where $S = \begin{bmatrix} 0_{2\times 2} & I_2 \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}$.

The simulation results are shown in Figures 4.4-4.6. And the satisfactory performance is also observed.

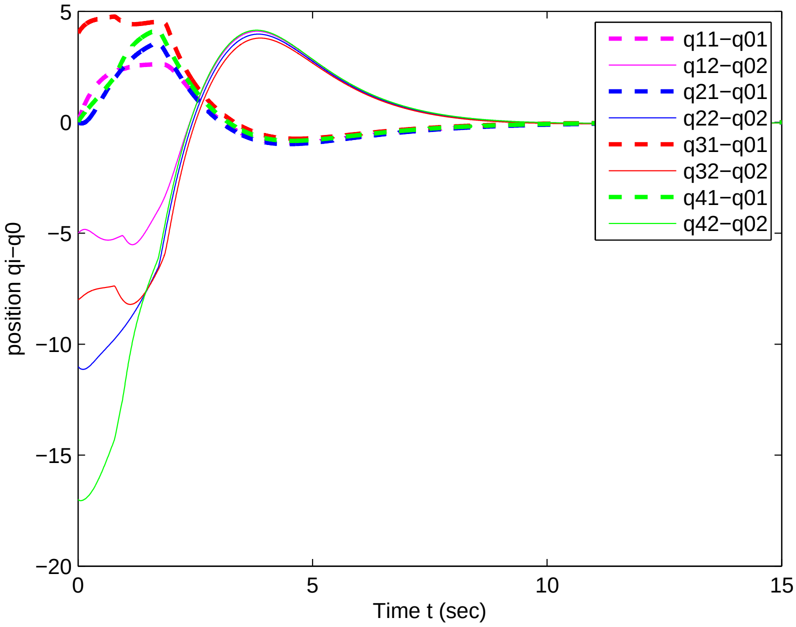

Figure 4.2: Position difference between followers and the leader

4.5 Conclusion

We have designed a dynamic state feedback control law to solve the leader-following rendezvous problem of a set of double integrator systems subject to a class of external disturbances. This control law can not only maintain the connectivity of the network graph, but also achieve asymptotic tracking and disturbance rejection for a class of leader system.

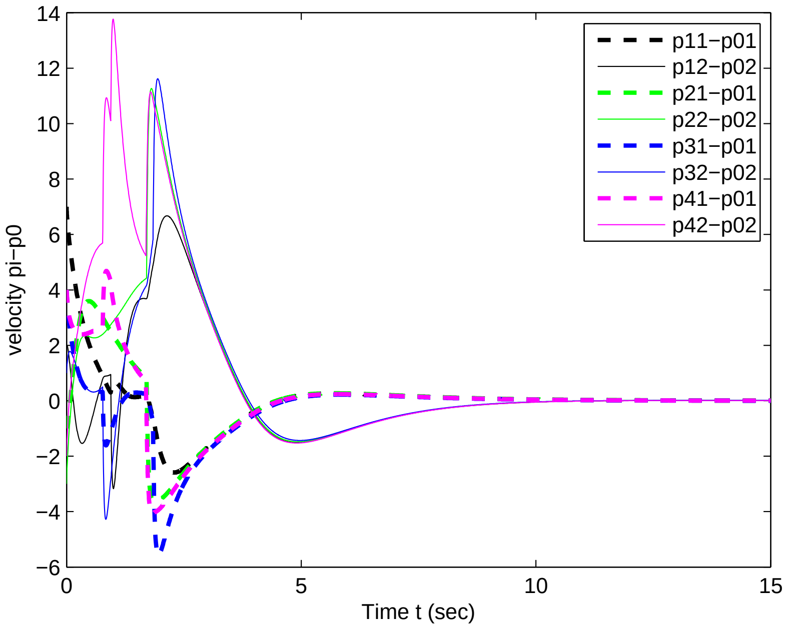

Figure 4.3: Velocity difference between followers and the leader

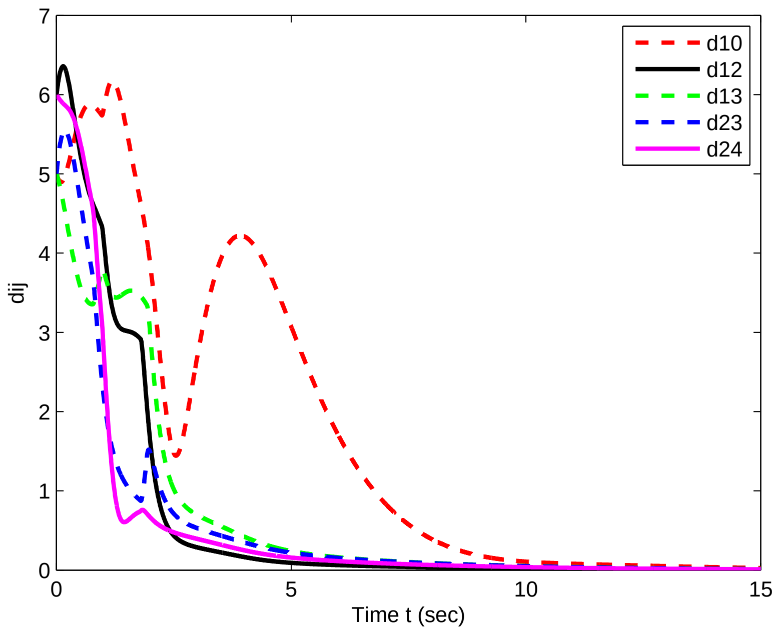

Figure 4.4: Distances between initially connected agents

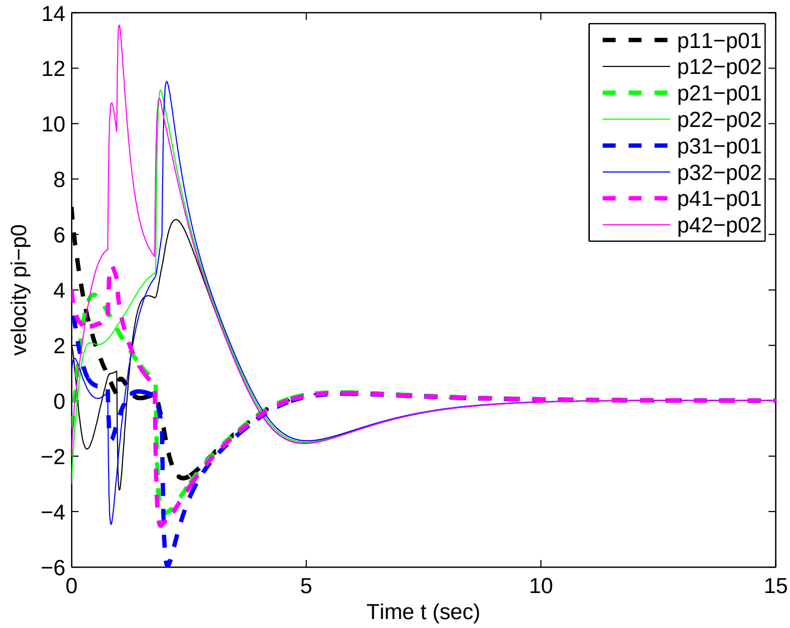

Figure 4.5: Position difference between each agent and the leader

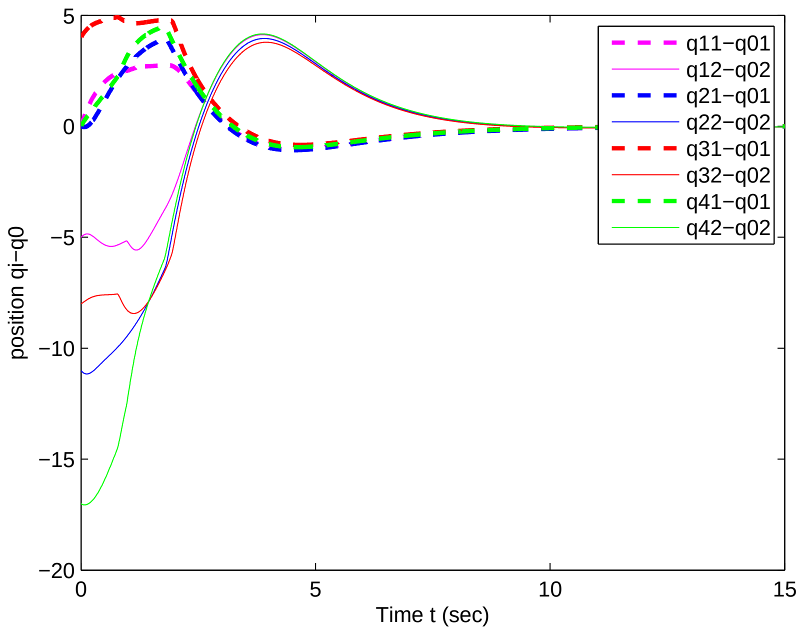

Figure 4.6: Velocity difference between each agent and the leader

Chapter 5

Leader-following Connectivity Preservation Rendezvous of Multi-agent Systems Based Only Position Measurements

In this chapter, we will further consider the leader-following rendezvous problem of double integrator multi-agent systems by position feedback control law, which is independent of the velocity of the system and the external disturbances.

5.1 Introduction

Consider a collection of double integrator systems of the following form

$$\ddot{q}_i = u_i + d_i, \quad i = 1, \cdots, N. \tag{5.1}$$

where $q_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}^n$ are the position, input, and the external disturbance of the subsystem i of (5.1). It is assumed that, for $i = 1, \cdots, N$, d_i is generated by an exosystem as follows

$$\dot{w}_i = S_i w_i, \quad d_i = D_i w_i \tag{5.2}$$

where $w_i \in \mathbb{R}^{s_i}$, $S_i \in \mathbb{R}^{s_i \times s_i}$ and $D_i \in \mathbb{R}^{n \times s_i}$ are constant matrices. Without loss of generality, we assume the pair (D_i, S_i) is detectable.

Also, let $q_0 \in \mathbb{R}^n$ be a reference trajectory generated by a system as follows

$$\ddot{q}_0 = S_{01} q_0 + S_{02} \dot{q}_0 \tag{5.3}$$

where $S_{01}, S_{02} \in \mathbb{R}^{n \times n}$ are arbitrary constant matrices.

It is noted that, when $D_i = 0_{n\times s_i}$, (5.1) is a double integrator system, and when $S_{01} = S_{02} = 0_{n\times n}$, (5.3) is also a double integrator system. However, we don't require (5.3) to be a double integrator system and it can be seen that system (5.3) contains double integrator system and harmonic system as special cases.

Like [17], we view the system composed of (5.1) and (5.3) as a multi-agent system of $(N+1)$ agents with (5.3) as the leader and the N subsystems of (5.1) as N followers. With respect to the system composed of (5.1) and (5.3), we can define a digraph $\bar{\mathcal{G}}(t) = (\bar{\mathcal{V}}, \bar{\mathcal{E}}(t))$ where $\bar{\mathcal{V}} = \{0, 1, ..., N\}$ with 0 associated with the leader system and $i = 1, \cdots, N$, associated with the ith subsystem of (5.1), and $\bar{\mathcal{E}}(t) \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$. The set $\bar{\mathcal{V}}$ is called the node set of $\bar{\mathcal{G}}(t)$ and the set $\bar{\mathcal{E}}(t)$ is called the edge set of $\bar{\mathcal{G}}(t)$.

The rendezvous problem with connectivity preservation of the double integrator multi-agent system was studied recently in [5], [17], [72] and [73]. In particular, the problem was studied in [5, 73] via full state feedback control assuming the leader system was also a double integrator and the follower was not subject to external disturbances. The problem was further studied in [72] via position feedback control only. Recently, the problem in [73] was generalized in [17] to the case where the leader system can be a linear autonomous system described in (5.3) and all follower subsystems are allowed to be subject to a disturbance generated by (5.3). In this chapter, we will further generalize the result of [17] in two aspects. First, we allow the disturbances to various followers to be different. In particular, they can be different from the leader signal. Second, we will solve the problem by a position feedback control law as described in (5.4). This control law depends neither on the velocity of the system nor on the external disturbances, is thus more practical and economic than the one in [17]. It is noted that since the closed-loop system is nonlinear, the validity of the output feedback control law cannot be directly established by the result of the state feedback control and linear observer theory. We have to derive our result using a rigorous Lyapunov-like analysis.

The rest of this chapter is organized as follows. In Section 5.2, we will formulate our problem precisely. In Section 5.3, we will present our main result, which will be illustrated by an example in Section 5.4. Finally, we close this chapter in Section 5.5 with some concluding remarks.

5.2 Problem Formulation

Let us first characterize the edge set $\bar{\mathcal{E}}(t)$ introduced in [17] as follows.
Given any $r > 0$ and $\epsilon \in (0, r)$, for any $t \geq 0$, $\bar{\mathcal{E}}(t) = \{(i, j) \mid i, j \in \bar{\mathcal{V}}\}$ is defined such that

1. $\bar{\mathcal{E}}(0) = \{(i, j) \mid ||q_i(0) - q_j(0)|| < (r - \epsilon), i, j = 1, \cdots, N\} \cup \{(0, j) \mid ||q_0(0) - q_j(0)|| <$

$(r-\epsilon), j = 1, \cdots, N\}$;

2. if $||q_i(t) - q_j(t)|| \geq r$, then $(i,j) \notin \bar{\mathcal{E}}(t)$;

3. $(i, 0) \notin \bar{\mathcal{E}}(t)$, for $i = 0, 1, \cdots, N$;

4. for $i = 0, 1, \cdots, N, j = 1, \cdots, N$, if $(i,j) \notin \bar{\mathcal{E}}(t^-)$ and $||q_i(t) - q_j(t)|| < (r - \epsilon)$, then $(i,j) \in \bar{\mathcal{E}}(t)$.

5. for $i = 0, 1, \cdots, N, j = 1, \cdots, N$, if $(i,j) \in \bar{\mathcal{E}}(t^-)$ and $||q_i(t) - q_j(t)|| < r$, then $(i,j) \in \bar{\mathcal{E}}(t)$.

As pointed out in [17], the definition of edge is somehow different from that in literature mainly in that the node 0 associated with the leader as well as the edges adjacent to the node 0 is part of the graph. Since the leader does not have a control, there is no edge from a follower to the leader. If $\epsilon = 0$, then the above definition is similar to that given in [34, 35]. Thus the physical interpretation of r is the sensing radius of the distance sensor of each follower. The number ϵ is to introduce the effect of hysteresis.

To describe our control law, we use the notation $\bar{\mathcal{N}_i}(t)$ to denote the neighbor set of the node i for $i = 0, 1, \cdots, N$. Define a subgraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ of $\bar{\mathcal{G}}(t)$, where $\mathcal{V} = \{1, \cdots, N\}$, $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ is obtained from $\bar{\mathcal{E}}(t)$ by removing all edges between the node 0 and the nodes in $\mathcal{V}$. Clearly, $\mathcal{G}(t)$ is an undirected graph. For $i = 1, \cdots, N$, let $\mathcal{N}_i(t) = \bar{\mathcal{N}_i}(t) \cap \mathcal{V}$. It can be seen that, for $i = 1, \cdots, N$, $\mathcal{N}_i(t)$ is the neighbor set of the node i with respect to $\mathcal{V}$.

Our control law takes the the following form:

$$\begin{aligned} u_i &= h_i(q_i - q_j, \zeta_i, \zeta_j, j \in \bar{\mathcal{N}_i}(t)), \quad i = 1, ..., N \\ \dot{\zeta}_i &= g_i(\dot{q}_0, \zeta_i, \zeta_j, q_i, q_j, j \in \bar{\mathcal{N}_i}(t)) \end{aligned} \tag{5.4}$$

where h_i, g_i are sufficiently smooth functions to be specified later, and $\zeta_i \in R^{(2n+s_i+2n)}$ is used to estimate $\text{col}(q_i, \dot{q}_i, w_i, q_0, \dot{q}_0)$.

In contrast with the control law in [17], the control law (5.4) only depends on the position information of the neighboring subsystems. Thus it is called the distributed position feedback control law.

The leader-following rendezvous problem with connectivity preservation is described as follows.

Definition 5.1 *Given the multi-agent system composed of* (5.1), (5.2) *and (5.3),* $r > 0$ *and* $\epsilon \in (0, r)$, *and arbitrary positive real numbers* P_i, κ_i, $i = 1, \cdots, N$, *find a distributed control law of the form (5.4) such that, for all initial conditions* $q_0(0)$, $\dot{q}_0(0)$, $w_i(0)$, $q_i(0)$,

$\dot{q}_i(0)$, $\zeta_i(0)$, $i = 1, \cdots, N$, that make $\bar{\mathcal{G}}(0)$ connected, and satisfy $||\dot{q}_i(0) - \dot{q}_0(0)|| \leq P_i$, and $||\zeta_i(0) - col(q_i(0), \dot{q}_i(0), w_i(0), q_0(0), \dot{q}_0(0))|| \leq \kappa_i$, the closed-loop system has the following properties:

1. *$\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$;*
2. *$\lim_{t\to\infty}(q_i - q_0) = 0$ and $\lim_{t\to\infty}(\dot{q}_i - \dot{q}_0) = 0$, $i = 1, \cdots, N$.*

Remark 5.1 *If, for $i = 1, \cdots, N$, $D_i = D$ for some matrix D, $w_i = col(q_0, \dot{q}_0)$ and $y_i = col(q_i, \dot{q}_i)$, then the above problem is reduced to the problem studied in [17]. What makes our current problem challenging and thus interesting is that our control law will be independent of not only $\dot{q}_i$ but also w_i.*

5.3 Construction of Distributed Controller

We will make use of some techniques in output regulation problem to deal with our problem. For this purpose, we can convert our system in state space form as follows

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + E_i w_i \\ y_i &= C_m x_i \\ e_i &= x_i - x_0 \qquad i = 1, \cdots, N \end{aligned} \tag{5.5}$$

where, for $i = 1, \cdots, N$, $x_i = \begin{bmatrix} q_i \\ p_i \end{bmatrix}$ with $p_i = \dot{q}_i$, $y_i \in \mathbb{R}^n$ $e_i \in \mathbb{R}^{2n}$ are the state, measurement output, and regulated output of agent i, respectively. Also, let $x_0 = \begin{bmatrix} q_0 \\ p_0 \end{bmatrix} \in \mathbb{R}^{2n}$ with $p_0 = \dot{q}_0$. Then,

$$\dot{x}_0 = S_0 x_0 \tag{5.6}$$

where $S_0 = \begin{bmatrix} 0_{n\times n} & I_n \\ S_{01} & S_{02} \end{bmatrix}$. Various matrices in (5.5) are as follows: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes I_n$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes I_n$, $E_i = \begin{bmatrix} 0_{n\times s_i} \\ D_i \end{bmatrix}$, $C_m = \begin{bmatrix} 1 & 0 \end{bmatrix} \otimes I_n$.

Remark 5.2 *Let $\hat{A}_i = \begin{bmatrix} A & E_i \\ 0_{s_i\times 2n} & S_i \end{bmatrix}$ and $\hat{C}_{mi} = \begin{bmatrix} C_m & 0_{n\times s_i} \end{bmatrix}$. Then, noting that the pair (D_i, S_i) can always be assumed to be detectable, it can be verified that the pair $(\hat{C}_{mi}, \hat{A}_i)$ is also detectable. Thus, there exists $L_i = \begin{bmatrix} L_{i1} \\ L_{i2} \end{bmatrix}$ with $L_{i1} \in \mathbb{R}^{2n\times n}$ and $L_{i2} \in \mathbb{R}^{s_i\times n}$ such that $\hat{A}_i + L_i\hat{C}_{mi}$ is Hurwitz.*

One of the main objectives here is to deal with the external disturbance $E_i w_i$. For this purpose, let

$$X_i = \begin{bmatrix} I_{2n} & 0_{2n\times s_i} \end{bmatrix}, \quad U_i = \begin{bmatrix} S_{01} & S_{02} & -D_i \end{bmatrix}. \tag{5.7}$$

Then it can be verified that performing on (5.5) the following coordinate transformation

$$\begin{aligned} \bar{x}_i &= \begin{bmatrix} \bar{q}_i \\ \bar{p}_i \end{bmatrix} = x_i - x_0, \quad i = 0, 1, \cdots, N \\ \bar{u}_i &= u_i - U_i v_i, \quad i = 1, \cdots, N \end{aligned} \tag{5.8}$$

with $v_i = \begin{bmatrix} x_0 \\ w_i \end{bmatrix}$, $i = 1, \cdots, N$, converts system (5.5) to the following double-integrator system without disturbance

$$\begin{aligned} \dot{\bar{q}}_i &= \bar{p}_i \\ \dot{\bar{p}}_i &= \bar{u}_i, \quad i = 1, \cdots, N \end{aligned} \tag{5.9}$$

Remark 5.3 *The transformation (5.8) is inspired by the output regulation theory [20]. In fact, associated with (5.5) are the following linear matrix equations*

$$\begin{aligned} X_i \bar{S}_i &= A X_i + B U_i + \bar{E}_i \\ 0 &= X_i + \begin{bmatrix} -I_{2n} & 0_{2n\times s_i} \end{bmatrix} \end{aligned} \tag{5.10}$$

with $\bar{S}_i = \begin{bmatrix} S_0 & 0_{2n\times s_i} \\ 0_{s_i\times 2n} & S_i \end{bmatrix}$, $\bar{E}_i = \begin{bmatrix} 0_{2n\times 2n} & E_i \end{bmatrix}$. *(5.10) is called regulator equations associated with the ith follower [28]. It can be verified that (5.7) is a solution pair of (5.10). The transformation (5.8) is a standard technique for converting an output regulation problem to a stabilization problem.*

As in [17], our control law will utilize the bounded potential function $\psi(\cdot)$ introduced in [73] as follows.

$$\psi(s) = \frac{s^2}{r - s + \frac{r^2}{Q}}, \quad 0 \le s \le r \tag{5.11}$$

where Q is some positive number. The function is nonnegative and bounded over $[0, r]$, and its derivative $\frac{d\psi(s)}{ds} = \frac{s(2r - s + \frac{2r^2}{Q})}{(r - s + \frac{r^2}{Q})^2}$ is positive for all $s \in (0, r]$. Moreover, the function has the property that, for any $\alpha > 0$, $\beta \ge 0$, and any $\epsilon \in (0, r)$, there exists some $Q > 0$ such that

$$\psi(r) = Q \ge \alpha\psi(r - \epsilon) + \beta \tag{5.12}$$

In fact, as pointed out in [17], (5.12) holds whenever $Q > (\frac{\alpha(r-\epsilon)^2}{\epsilon} + \beta)$.

Now we can propose the dynamic distributed position feedback control law as follows:

$$
\begin{aligned}
u_i &= -\sum_{j\in\bar{\mathcal{N}}_i(t)} \nabla_{\bar{q}_i}\psi(||\bar{q}_i-\bar{q}_j||) - \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\xi_{2i}-\xi_{2j}) \\
&\quad - a_{i0}(\xi_{2i}-p_0) + U_i\mathrm{col}(\eta_i,\hat{w}_i) \qquad i=1,\cdots,N \\
\dot{\xi}_i &= A\xi_i + Bu_i + E_i\hat{w}_i + L_{i1}(C_m\xi_i - y_i) \\
\dot{\hat{w}}_i &= S_i\hat{w}_i + L_{i2}(C_m\xi_i - y_i) \\
\dot{\eta}_i &= S_0\eta_i + \gamma(\sum_{j=1}^{N} a_{ij}(t)(\eta_j-\eta_i) - a_{i0}(t)(\eta_i - x_0))
\end{aligned}
\tag{5.13}
$$

where, for $i=1,\cdots,N$, $j=0,\cdots,N$,

$$
a_{ij}(t) = \begin{cases} 1, & (j,i)\in\bar{\mathcal{E}}(t) \\ 0, & \text{otherwise} \end{cases}
\tag{5.14}
$$

$\xi_i = \begin{bmatrix} \xi_{1i} \\ \xi_{2i} \end{bmatrix}$ with $\xi_{1i}\in\mathbb{R}^n$ and $\xi_{2i}\in\mathbb{R}^n$, γ is a sufficiently large positive number, and L_i is as described in Remark 5.2. Since $\hat{A}_i + L_i\hat{C}_{mi}$ is Hurwitz, there exist positive definite matrices $\bar{P}_i$, $i=1,\cdots,N$, such that $(\hat{A}_i+L_i\hat{C}_{mi})^T\bar{P}_i + \bar{P}_i(\hat{A}_i+L_i\hat{C}_{mi}) = -I_{2n+s_i}$. It can be seen that the control law is in the form of (5.4) with $\zeta_i = \mathrm{col}(\xi_i,\hat{w}_i,\eta_i)$.

Let $\bar{\xi}_i = \xi_i - x_i$, $\bar{w}_i = \hat{w}_i - w_i$ and $\bar{\eta}_i = \eta_i - x_0$, $i=1,\cdots,N$. Then, under the control law (5.13), the closed-loop system of each agent becomes

$$
\begin{aligned}
\dot{\bar{q}}_i &= \bar{p}_i, \qquad i=1,\cdots,N \\
\dot{\bar{p}}_i &= -\sum_{j\in\bar{\mathcal{N}}_i(t)} \nabla_{\bar{q}_i}\psi(||\bar{q}_i-\bar{q}_j||) - \sum_{j\in\bar{\mathcal{N}}_i(t)} a_{ij}(t)(\bar{p}_i-\bar{p}_j) \\
&\quad -(\sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\bar{\xi}_{2i}-\bar{\xi}_{2j}) + a_{i0}\bar{\xi}_{2i} + D_i\bar{w}_i) + \begin{bmatrix} S_{01} & S_{02} \end{bmatrix}\bar{\eta}_i \\
\begin{bmatrix} \dot{\bar{\xi}}_i \\ \dot{\bar{w}}_i \end{bmatrix} &= (\hat{A}_i + L_i\hat{C}_{mi}) \begin{bmatrix} \bar{\xi}_i \\ \bar{w}_i \end{bmatrix} \\
\dot{\bar{\eta}}_i &= S_0\bar{\eta}_i + \gamma(\sum_{j=1}^{N} a_{ij}(t)(\bar{\eta}_j - \bar{\eta}_i) - a_{i0}(t)\bar{\eta}_i)
\end{aligned}
\tag{5.15}
$$

Before establishing our main result, we note that, associated with the graph $\bar{\mathcal{G}}(t)$, $t\geq 0$, we can define matrices

$$
H(t) = \begin{bmatrix} \bar{a}_1(t) & -a_{12}(t) & \cdots & -a_{1N}(t) \\ -a_{21}(t) & \bar{a}_2(t) & \cdots & -a_{2N}(t) \\ \vdots & \vdots & \vdots & \vdots \\ -a_{N1}(t) & -a_{N2}(t) & \cdots & \bar{a}_N(t) \end{bmatrix}
\tag{5.16}
$$

where $\bar{a}_i(t) = \sum_{j=0, j\neq i}^{N} a_{ij}(t)$, $i = 1, \cdots, N$,

$$P_0(t) = \begin{bmatrix} H(t) \otimes I_n & \frac{\Lambda(t)}{2} \\ \frac{\Lambda^T(t)}{2} & \theta I_\iota \end{bmatrix} \tag{5.17}$$

where θ is some real number, $\iota = 2Nn + s_1 + \cdots + s_N$, $\Lambda(t) = \begin{bmatrix} 0_{Nn\times Nn} & H(t) \otimes I_n & D \end{bmatrix}$ with $D = \text{diag}(D_1, \cdots, D_N)$, and $P(t) =$

$$\begin{bmatrix} H(t) \otimes I_n & \frac{\Lambda(t)}{2} & Z_1 \\ \frac{\Lambda^T(t)}{2} & \theta I_\iota & 0_{\iota\times 2Nn} \\ Z_1^T & 0_{2Nn\times\iota} & Y(t) \end{bmatrix} \tag{5.18}$$

where $Z_1 = -\frac{1}{2} I_N \otimes \begin{bmatrix} S_{01} & S_{02} \end{bmatrix}$ and $Y(t) = \gamma H(t) \otimes I_{2n} - I_N \otimes \frac{S_0 + S_0^T}{2}$ with γ some real number. We have the following lemma.

Lemma 5.1 *Assume the graph $\bar{\mathcal{G}}(t)$ is connected for all $t \geq 0$. Then*

1. *there exists positive number θ such that $P_0(t)$ is positive definite for all $t \geq 0$;*

2. *there exists positive number γ such that $P(t)$ is positive definite for all $t \geq 0$.*

Proof: Part 1). Note that, for any $t \geq 0$, $H(t) = -M + \Delta$ where M is a Metzler matrix and $\Delta = \text{diag}[a_{10}(t) \cdots, a_{N0}(t)]$. By Remark 2.1, $H(t)$ is positive definite since the graph $\bar{\mathcal{G}}(t)$ is connected. By Lemma 3.1 in [17], if there exists finite number $\theta > 0$ such that, for all $t \geq 0$, $\theta > \lambda_M(\frac{\Lambda^T(t)}{2}(H^{-1}(t) \otimes I_n)\frac{\Lambda(t)}{2})$ where $\lambda_M(A)$ denotes the largest eigenvalue of a square matrix A, then $P_0(t)$ is positive definite for all $t \geq 0$.

It is noted that

$$\Lambda^T(t)(H^{-1}(t) \otimes I_n)\Lambda(t) = \begin{bmatrix} 0_{Nn\times Nn} \\ H(t) \otimes I_n \\ D^T \end{bmatrix} \times (H^{-1}(t) \otimes I_n) \begin{bmatrix} 0_{Nn\times Nn} & H(t) \otimes I_n & D \end{bmatrix}$$

$$= \begin{bmatrix} 0_{Nn\times Nn} & 0_{Nn\times Nn} & 0_{Nn\times s} \\ 0_{Nn\times Nn} & H(t) \otimes I_n & D \\ 0_{s\times Nn} & D^T & D^T(H^{-1}(t) \otimes I_n)D \end{bmatrix} \tag{5.19}$$

with $s = s_1 + \cdots + s_N$. Thus, $P_0(t)$ is positive definite for all $t \geq 0$ if, for all $t \geq 0$,

$$\theta > \lambda_M \left(\frac{1}{4} \begin{bmatrix} H(t) \otimes I_n & D \\ D^T & D^T(H^{-1}(t) \otimes I_n)D \end{bmatrix} \right) \tag{5.20}$$

Since $H(t)$ is uniquely determined by $\bar{\mathcal{G}}(t)$, and there are only finitely many different connected graphs with $N + 1$ nodes, such a finite number θ always exists. Fix θ.

Part 2). Let $P(t) = \begin{bmatrix} P_0(t) & Z \\ Z^T & Y(t) \end{bmatrix}$ where $Z = \begin{bmatrix} -\frac{1}{2} I_N \otimes \begin{bmatrix} S_{01} & S_{02} \end{bmatrix} \\ 0_{\iota \times 2Nn} \end{bmatrix}$. Then by Lemma 3.1 in [17], if there exists finite real number γ such that, for all $t \geq 0$,

$$\gamma > \frac{\lambda_M(I_N \otimes \frac{S_0 + S_0^T}{2} + Z^T P_0^{-1}(t) Z)}{\lambda_m(H(t))} \tag{5.21}$$

with $\lambda_m(A)$ denotes the smallest eigenvalue of a square matrix A, then $P(t)$ is positive definite for all $t \geq 0$.

Since $H(t)$ is uniquely determined by $\bar{\mathcal{G}}(t)$, and there are only finitely many different connected graphs with $N + 1$ nodes, such a finite constant always exists.

We can now state our main result as follows.

Theorem 5.1 *The leader-following connectivity preservation rendezvous problem for system composed of (5.1), (5.2) and (5.3) is solvable by the control law (5.13) where γ is a sufficiently large positive constant.*

Proof: Let $\bar{\eta} = \mathrm{col}(\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_N)$, $\bar{q} = \mathrm{col}(\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_N)$, $\bar{p} = \mathrm{col}(\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_N)$, $\bar{\xi} = \mathrm{col}(\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_N)$, $\mu_i = \begin{bmatrix} \bar{\xi}_i \\ \bar{w}_i \end{bmatrix}$, $i = 1, \cdots, N$, $\mu = \mathrm{col}(\mu_1, \mu_2, \cdots, \mu_N)$. Let $\bar{\mu} = \mathrm{col}(\bar{\xi}_{11}, \cdots, \bar{\xi}_{1N}, \bar{\xi}_{21}, \cdots, \bar{\xi}_{2N}, \bar{w}_1, \cdots, \bar{w}_2) = T\mu$ with $T =$
$$\begin{bmatrix} I_n & 0_{n\times n} & 0_{n\times s_1} & \cdots & 0_{n\times n} & 0_{n\times n} & 0_{n\times s_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times s_1} & \cdots & I_n & 0_{n\times n} & 0_{n\times s_N} \\ 0_{n\times n} & I_n & 0_{n\times s_1} & \cdots & 0_{n\times n} & 0_{n\times n} & 0_{n\times s_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times s_1} & \cdots & 0_{n\times n} & I_n & 0_{n\times s_N} \\ 0_{s_1\times n} & 0_{s_1\times n} & I_{s_1} & \cdots & 0_{s_1\times n} & 0_{s_1\times n} & 0_{s_1\times s_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{s_N\times n} & 0_{s_N\times n} & 0_{s_N\times s_1} & \cdots & 0_{s_N\times n} & 0_{s_N\times n} & I_{s_N} \end{bmatrix}$$
It is noted that $T^{-T}T^{-1} = I_\iota$.

Given $r > 0$, $\epsilon \in (0, r)$, and arbitrary positive real numbers P_i, κ_i, $i = 1, \cdots, N$, the control law is determined by two design parameters Q and γ. Let us first determine γ. By Lemma 5.1, there are $\theta > 0$ and $\gamma > 0$ such that $P(t)$ is positive definite for all possible connected $\bar{\mathcal{G}}(t)$ with $N + 1$ nodes. Fix θ and γ.

To determine Q, we introduce the following energy function for system (5.15).

$$
\begin{aligned}
V(\bar{q},\bar{p},\mu,\bar{\eta},t) &= \frac{1}{2}\sum_{i=1}^{N}(\sum_{j\in\mathcal{N}_i(t)}\psi(||\bar{q}_i-\bar{q}_j||)+2a_{i0}\psi(||\bar{q}_i||)+\bar{p}_i^T\bar{p}_i+2\theta\mu_i^T\bar{P}_i\mu_i+\bar{\eta}_i^T\bar{\eta}_i)\\
&= \frac{1}{2}\sum_{i=1}^{N}(\sum_{j\in\mathcal{N}_i(t)}\psi(||\bar{q}_i-\bar{q}_j||)+2a_{i0}\psi(||\bar{q}_i||)+\bar{p}_i^T\bar{p}_i+\bar{\eta}_i^T\bar{\eta}_i)+\theta\bar{\mu}^TT^{-T}\bar{P}T^{-1}\bar{\mu}
\end{aligned}
\tag{5.22}
$$

with $\bar{P}=\text{diag}(\bar{P}_1,\cdots,\bar{P}_N)$. Let

$$
Q_{\max}=\frac{\alpha(r-\epsilon)^2}{\epsilon}+\beta \tag{5.23}
$$

where $\alpha=\frac{N(N-1)}{2}+N$ and

$$
\beta=\frac{1}{2}\sum_{i=1}^{N}(P_i^2+\delta_i\kappa_i^2) \tag{5.24}
$$

where $\delta_i=\max\{1,2\theta\lambda_M(\bar{P}_i)\}$. Then pick any $Q>Q_{\max}$.

Now, we will show that the above control law is such that the graph $\bar{\mathcal{G}}(t)$ is connected for all $t\geq 0$. Let the energy function be given by (5.22). Then it can be seen that for all initial conditions $x_0(0),w_i(0),q_i(0),p_i(0),\xi_i(0),\hat{w}_i(0),\eta_i(0)$ that make $\bar{\mathcal{G}}(0)$ connected and satisfy $||p_i(0)-p_0(0)||\leq P_i$, $||\text{col}(\xi_i(0),\hat{w}_i(0),\eta_i(0))-\text{col}(x_i(0),w_i(0),x_0(0))||\leq\kappa_i$, our choice of Q is such that

$$
V(0)=V(\bar{q}(0),\bar{p}(0),\mu(0),\bar{\eta}(0),0)\leq Q_{\max} \tag{5.25}
$$

It can be verified that the time derivative of the function (5.22) along the closed-loop system (5.15) satisfies

$$
\begin{aligned}
\dot{V} &= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i(t)}\dot{\psi}(||\bar{q}_i-\bar{q}_j||)+\sum_{i=1}^{N}\bar{p}_i^T\dot{\bar{p}}_i+\sum_{i=1}^{N}\theta(\dot{\mu}_i^T\bar{P}_i\mu_i+\mu_i^T\bar{P}_i\dot{\mu}_i)+\bar{\eta}^T\dot{\bar{\eta}}\\
&= -\sum_{i=1}^{N}\bar{p}_i^T(\sum_{j\in\bar{\mathcal{N}}_i(t)}a_{ij}(\bar{p}_i-\bar{p}_j)+\begin{bmatrix} S_{01} & S_{02}\end{bmatrix}\bar{\eta}_i)\\
&\quad -\sum_{i=1}^{N}\bar{p}_i^T(\sum_{j\in\mathcal{N}_i(t)}a_{ij}(t)(\bar{\xi}_{2i}-\bar{\xi}_{2j})+a_{i0}\bar{\xi}_{2i}+D_i\bar{w}_i)\\
&\quad +\bar{\eta}^T(I_N\otimes S_0-\gamma H(t)\otimes I_{2n})\bar{\eta}+\theta\sum_{i=1}^{N}\mu_i^T((\hat{A}_i+L_i\hat{C}_{mi})^T\bar{P}_i\mu_i+\bar{P}_i(\hat{A}_i+L_i\hat{C}_{mi}))\mu_i\\
&= -\bar{p}^T(H(t)\otimes I_n)\bar{p}-\theta\bar{\mu}^TT^{-T}T^{-1}\bar{\mu}-\bar{p}^T\Lambda(t)\bar{\mu}+\bar{p}^T(I_N\otimes\begin{bmatrix} S_{01} & S_{02}\end{bmatrix})\bar{\eta}\\
&\quad +\bar{\eta}^T(I_N\otimes\frac{S_0+S_0^T}{2}-\gamma H(t)\otimes I_{2n})\bar{\eta}
\end{aligned}
$$

$$= -\begin{bmatrix} \bar{p} \\ \bar{\mu} \\ \bar{\eta} \end{bmatrix}^T P(t) \begin{bmatrix} \bar{p} \\ \bar{\mu} \\ \bar{\eta} \end{bmatrix} \tag{5.26}$$

By using the same argument as what is used in the proof of Theorem 3.1 in [17], we can conclude that there exists a finite integer $k > 0$ such that

$$\begin{aligned} &\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(0), \quad t \in [0, t_1) \\ &\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(t_i) \supset \bar{\mathcal{G}}(t_{i-1}), \quad t \in [t_i, t_{i+1}),\ i = 1, \cdots, k-1 \\ &\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(t_k) \supset \bar{\mathcal{G}}(t_{k-1}), \quad t \in [t_k, \infty) \end{aligned}$$

Thus, for all $t \geq t_k$, along any trajectory of the closed-loop system, we have

$$\dot{V}(t) = -\begin{bmatrix} \bar{p} \\ \bar{\mu} \\ \bar{\eta} \end{bmatrix}^T P(t_k) \begin{bmatrix} \bar{p} \\ \bar{\mu} \\ \bar{\eta} \end{bmatrix},\ t \geq t_k. \tag{5.27}$$

with $P(t_k)$ positive definite.
thus

$$V(t) \leq V(0) \leq Q_{\max} < Q, \quad \text{for } t \geq t_k \tag{5.28}$$

Therefore, for all $t \geq t_k$, the graph $\bar{\mathcal{G}}(t)$ is connected.

Moreover, by using the same argument as what is used in the proof of Theorem 3.1 in [17], we can conclude also that, $\bar{p}$, $\bar{q}$, μ and, $\bar{\eta}$ are bounded for all $t \geq 0$, and for $i = 1, \cdots, N$,

$$\begin{aligned} \lim_{t\to\infty} (p_i - p_0) &= 0 \\ \lim_{t\to\infty} (q_i - q_0) &= 0 \end{aligned}$$

5.4 Example

Consider the following double integrator systems with disturbance with $N = 4$ and $n = 2$

$$\ddot{q}_i = u_i + d_i,\ i = 1, 2, 3 \tag{5.29}$$

The leader system is

$$\dot{x}_0 = S_0 x_0 \tag{5.30}$$

where $S_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_2$.
All the followers are subject to different external disturbances generated by

$$\dot{w}_i = S_i w_i, \quad d_i = D_i w_i, \quad i = 1, 2, 3 \tag{5.31}$$

with $S_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $S_2 = 0$, $S_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $D_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The values for L_i, $i = 1, 2, 3$ can be calculated as follows:

$$L_1 = \begin{bmatrix} -11.1058 & -1.5488 \\ 0.2136 & -12.8942 \\ -30.0300 & -14.7458 \\ 7.6435 & -47.6888 \\ 7.1842 & -35.6389 \\ 33.1849 & -0.4725 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -7.5772 & 1.8537 \\ 1.6309 & -7.9228 \\ -15.5195 & 10.2289 \\ 8.9683 & -17.2211 \\ -8.0898 & 9.3967 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} -11.2883 & -2.1704 \\ -1.3852 & -9.7117 \\ -40.5053 & -17.7951 \\ -9.6792 & -27.8726 \\ -50.2439 & -40.3254 \\ -15.1637 & -22.4045 \end{bmatrix}$$

Thus, $\lambda_M(\bar{P}_1) = 4.6733$, $\lambda_M(\bar{P}_2) = 2.5605$ and $\lambda_M(\bar{P}_3) = 8.3629$.

Assume the sensing range is $r = 8$ and $\epsilon = 0.5$. For $i = 1, 2, 3$, let

$$P_i = 14, \quad \kappa_i = 20. \tag{5.32}$$

By Eq. (5.20), we can obtain $\theta > 2.4016$, then take $\theta = 2.5$. Also using (5.12) with $\alpha = \frac{N(N-1)}{2} + N = 6$ and $\beta = 19010$ gives $(\frac{\alpha(r-\epsilon)^2}{\epsilon} + \beta) = 19685$. Then taking $Q = 20000$ makes (5.12) satisfied. Thus the potential function is

$$\psi(s) = \frac{s^2}{8 - s + \frac{64}{20000}}, \quad 0 \leq s \leq r \tag{5.33}$$

By Eq. (5.21), we obtain $\gamma > 154.7491$, and let $\gamma = 155$ such that $P(t)$ is positive definite. For the purpose of simulation, let the initial values of various variables be

$$x_0(0) = \begin{bmatrix} 1 & 7 & 9 & 8 \end{bmatrix}^T$$

$$w_1(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T \quad w_2(0) = -1 \quad w_3(0) = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$$

$$x_1(0) = \begin{bmatrix} 1 & 1 & 8 & 3 \end{bmatrix}^T \quad x_2(0) = \begin{bmatrix} 1 & -6 & 9 & 7 \end{bmatrix}^T$$

$$x_3(0) = \begin{bmatrix} 6 & -2 & 5 & 2 \end{bmatrix}^T$$

$$
\begin{aligned}
&\eta_1(0) = \left[\begin{array}{cccc} 8 & 1 & 4 & 3 \end{array}\right]^T \ \eta_2(0) = \left[\begin{array}{cccc} 1 & 11 & -8 & 3 \end{array}\right]^T \\
&\eta_3(0) = \left[\begin{array}{cccc} 1 & 5 & 3 & 4 \end{array}\right]^T \\
&\xi_1(0) = \left[\begin{array}{cccc} 12 & 9 & 8 & 6 \end{array}\right]^T \ \ \xi_2(0) = \left[\begin{array}{cccc} 7 & 2 & 1 & 3 \end{array}\right]^T \\
&\xi_3(0) = \left[\begin{array}{cccc} 6 & 3 & 1 & 3 \end{array}\right]^T \\
&\hat{w}_1(0) = \left[\begin{array}{cc} 0 & 1 \end{array}\right]^T \ \hat{w}_2(0) = 2 \ \ \hat{w}_3(0) = \left[\begin{array}{cc} 1 & 1 \end{array}\right]^T
\end{aligned}
\tag{5.34}
$$

It can be verified that these initial values are such that $||p_i(0) - p_0(0)|| \leq P_i$, $||\zeta_i(0) - \text{col}(x_i(0), \eta_i(0), w_i(0))|| \leq \kappa_i$, and $\bar{\mathcal{E}}(0) = \{(0,1),\ (1,2),\ (2,3),\ (1,3)\}$ which forms a connected graph.

With these parameters, we can simulate the performance of the control law (5.13), and some of the simulation results are shown in Figures 5.1 to 5.6. Figure 5.1 shows the distances of the edges $\{(0,1),\ (1,2),\ (2,3),\ (1,3)\}$ which constitute the initial edge set. It can be seen that, for all $t \geq 0$, these distances are smaller than the sensing range $r = 8$. Thus, the connectivity of the network is maintained. Figures 5.2 and 5.3 further show that both the position and the velocity of all the followers asymptotically approach the position and the velocity of the leader, respectively. Moreover, Figures 5.4, 5.5 and 5.6 show that the observers ξ_i, η_i and $\hat{w}_i$ approach x_i, x_0 and w_i, respectively.

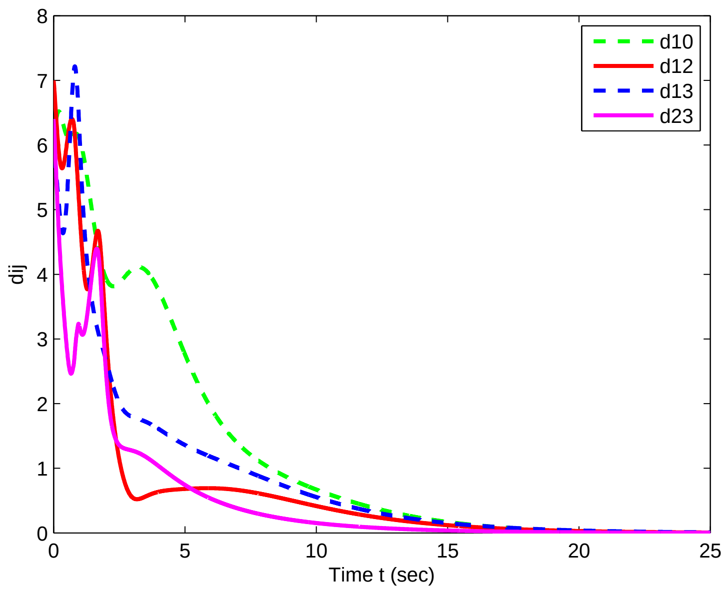

Figure 5.1: Distances between initially connected agents

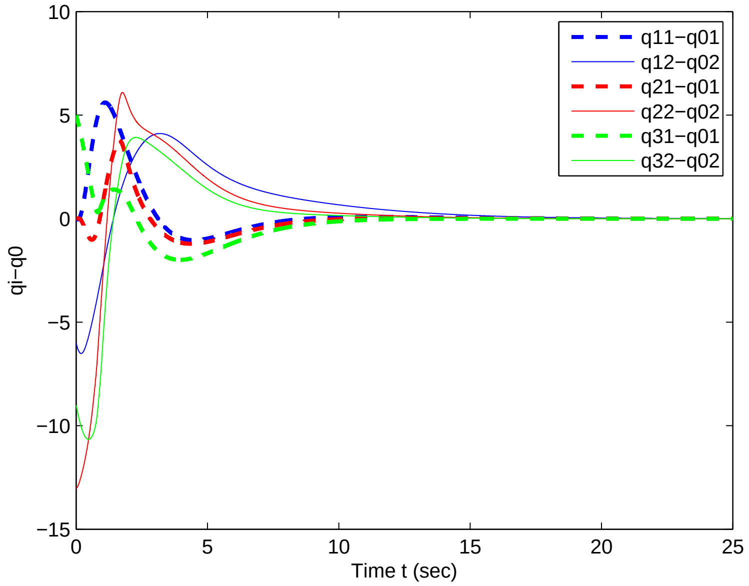

Figure 5.2: Differences of position between followers and leader

5.5 Conclusion

We have further investigated the problem of leader-following rendezvous with connectivity preservation for a double integrator multi-agent system via the position feedback control only. The formulation of the problem in this chapter is more general than that in the previous chapter in that the disturbances to different followers are different and, in particular, they are different from the leader signal. Our control law is not only independent of the velocity but also independent of the disturbance signals. Thus the result of this chapter is more practical than that of Chapter 4.

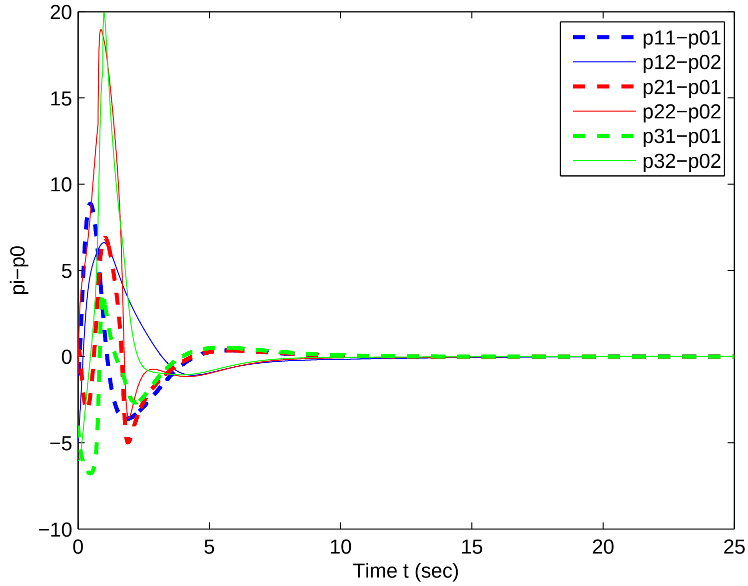

Figure 5.3: Differences of velocity between followers and leader

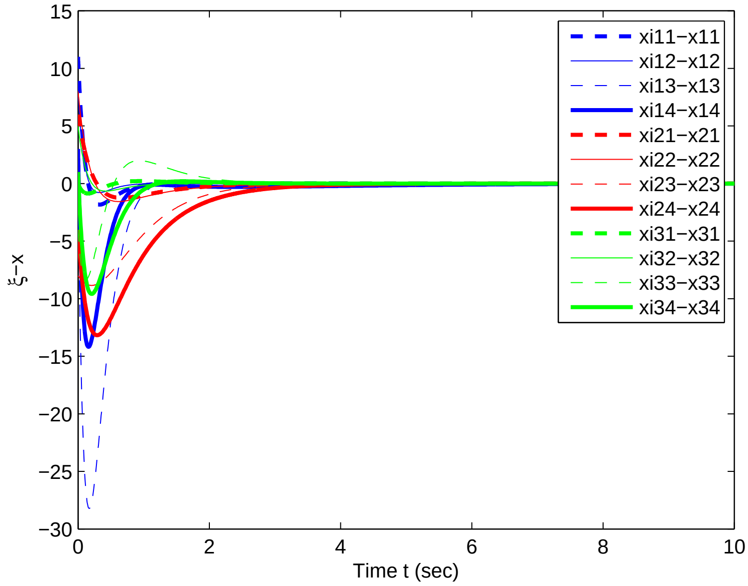

Figure 5.4: Differences of ξ_i and x_i

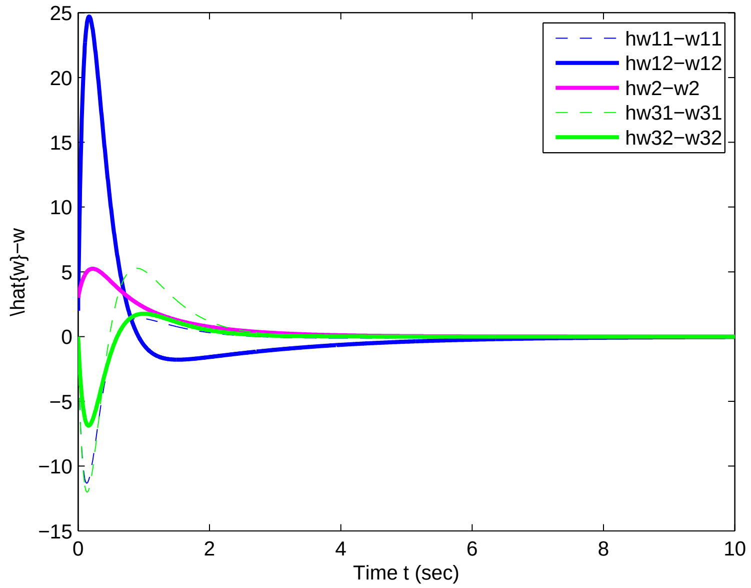

Figure 5.5: Differences of $\hat{w}_i$ and w_i

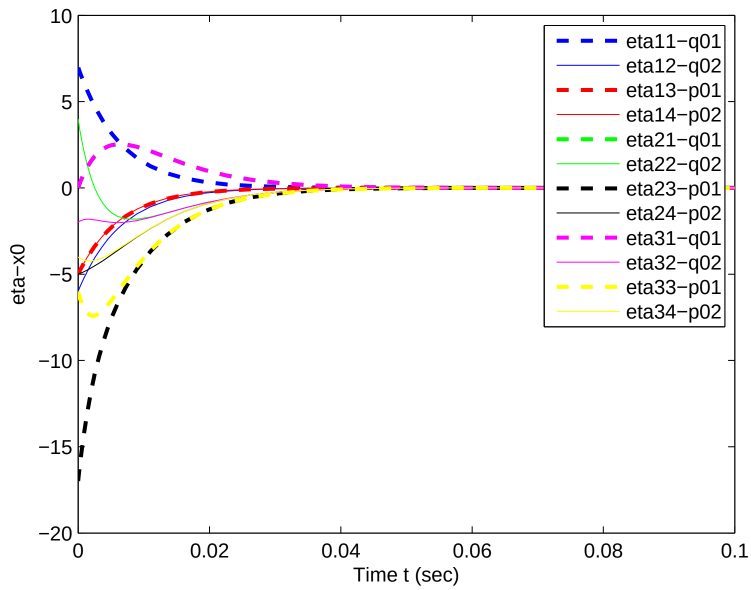

Figure 5.6: Differences of η_i and x_0

Chapter 6

Cooperative Global Robust Output Regulation for Nonlinear Multi-agent Systems in Output Feedback Form

In this chapter, we turn to consider the cooperative global robust output regulation for nonlinear multi-agent systems in output feedback form with unity relative degree.

6.1 Introduction

Recently, the cooperative robust output regulation problem for linear multi-agent systems was studied in [78, 80, 90]. The problem can be viewed as a generalization of the leader-following consensus/synchronization problem because it will not only address the issue of asymptotic tracking but also address such issues as disturbance rejection, robustness with respect to parameter uncertainties, etc. The same problem was also studied for a class of nonlinear systems in [45]. However, only a local solution was given in [45]. In this chapter, we will further consider the cooperative output regulation problem for the following class of nonlinear systems:

$$\begin{aligned} \dot{z}_i &= f_i(z_i, y_i, v, w) \\ \dot{y}_i &= b_i(v, w)u_i + g_i(z_i, y_i, v, w) \\ e_i &= y_i - q(v, w), \quad i = 1, \cdots, N \end{aligned} \tag{6.1}$$

where, for $i = 1, \cdots, N$, $(z_i, y_i) \in \mathbb{R}^n \times \mathbb{R}$ is the state, $u_i \in \mathbb{R}$ is the input, $e_i \in \mathbb{R}$ is the error output, $w \in \mathbb{W} \subset \mathbb{R}^{n_w}$ is an uncertain parameter vector with $\mathbb{W}$ an arbitrarily prescribed subset of $\mathbb{R}^{n_w}$, and $v(t) \in \mathbb{R}^{n_v}$ is an exogenous signal presenting both reference input and disturbance. It is assumed that $v(t)$ is generated by a linear system of the following form

$$\begin{aligned} \dot{v} &= Sv \\ y_0 &= q(v, w) \end{aligned} \tag{6.2}$$

It is assumed that all functions in (6.1) are globally defined, sufficiently smooth, and satisfy $f_i(0, 0, 0, w) = 0$, $g_i(0, 0, 0, w) = 0$, and $q(0, w) = 0$ for all $w \in \mathbb{W}$.

The system composed of (6.1) and (6.2) can be viewed as a multi-agent system of $(N + 1)$ agents with (6.2) as the leader and the N subsystems of (6.1) as N followers. With respect to the system composed of (6.1) and (6.2), we can define a digraph[1] $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ where $\bar{\mathcal{V}} = \{0, 1, ..., N\}$ with 0 associated with the leader system and with i, $i = 1, \cdots, N$, associated with the N followers, respectively, and $(j, i) \in \bar{\mathcal{E}}$, $j = 0, 1, \cdots, N$ and $i = 1, \cdots, N$, if and only if the control u_i can make use of y_j for feedback control. Thus our control law is of the following form:

$$\begin{aligned} u_i &= k_i(\eta_i, y_i - y_j, j \in \bar{\mathcal{N}}_i), \quad i = 1, \cdots, N \\ \dot{\eta}_i &= \hat{g}_i(\eta_i, y_i - y_j, j \in \bar{\mathcal{N}}_i) \end{aligned} \tag{6.3}$$

where $\bar{\mathcal{N}}_i$ is the neighbor set of the node i, k_i and $\hat{g}_i$ are sufficiently smooth functions vanishing at the origin, and $\eta_i \in \mathbb{R}^{n_{\eta_i}}$ with n_{η_i} to be defined later. A control law of the form (6.3) is called a distributed dynamic output feedback control law because the control of each subsystem can only take the information of its neighbors and itself for control.

Define a subgraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of $\bar{\mathcal{G}}$, where $\mathcal{V} = \{1, \cdots, N\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is obtained from $\bar{\mathcal{E}}$ by removing all edges between the node 0 and the nodes in $\mathcal{V}$. For $i = 1, \cdots, N$, let $\mathcal{N}_i = \bar{\mathcal{N}}_i(t) \cap \mathcal{V}$. It can be seen that, for $i = 1, \cdots, N$, $\mathcal{N}_i$ is the neighbor set of the node i with respect to $\mathcal{V}$.

We call the composition of (6.1) and (6.3) as the overall closed-loop system which can be put in the following form

$$\dot{x}_c = f_c(x_c, v, w) \tag{6.4}$$

where $x_c = \mathrm{col}(z_1, y_1, \eta_1, \cdots, z_N, y_N, \eta_N) \in \mathbb{R}^{n_c}$ for some integer n_c. Then f_c is sufficiently smooth satisfying $f_c(0, 0, w) = 0$ for all $w \in \mathbb{W}$. Then we can describe our problem as follows:

Definition 6.1 *Given the multi-agent system (6.1), the exosystem (6.2), the corresponding digraph $\bar{\mathcal{G}}$, and any compact subsets $\mathbb{V} \in \mathbb{R}^{n_v}$ and $\mathbb{W} \in \mathbb{R}^{n_w}$ which contain $v = 0$ and $w = 0$, respectively, find a control law of the form (6.3) such that, for any $v(0) \in \mathbb{V}$,*

[1] See [74] for a summary of digraph.

$w \in \mathbb{W}$, *the trajectory of the closed-loop system (6.4) starting from any initial state* $x_c(0)$ *exists and is bounded for all* $t \geq 0$, *and* $\lim_{t\to\infty} e(t) = 0$ *with* $e = col(e_1, \cdots, e_N)$.

It can be seen that, for the special case where $N = 1$, the above problem is the robust output regulation problem for output feedback systems as studied in [93]. The current problem is more challenging and interesting than that of [93] in at least two ways. First, the system in [93] is a single-input, single-output system. It can be converted into a global stabilization problem of an augmented system through the employment of an internal model. The augmented system is still a single-input, single-output system whose stabilization problem can be handled by established techniques for global stabilization. In contrast, for a multi-agent system, the augmented system is a multi-input, multi-output nonlinear system and we have to develop techniques that apply to multi-input, multi-output nonlinear systems. Second, due to the communication constraint described by the communication graph $\bar{\mathcal{G}}$, we have to limit ourselves to the distributed control law as described in (6.3) to solve the stabilization problem for the augmented system.

The rest of this chapter is organized as follows. In Section 5.2, we will present the preliminaries for our problem. In Section 5.3, we will present our main result. In Section 5.4, we will apply our approach to solve a leader-following synchronization problem for a group of Lorenz systems. Finally, we close this chapter in Section 5.5 with some concluding remarks.

6.2 Preliminaries

By the general framework for handling the output regulation problem for nonlinear systems described in [30], the first step of our approach is to find an internal model for (6.1) to form an augmented system. This step for the special case where $N = 1$ was conducted in [93]. Here we will generalize the procedure in [93] to the general case with any $N > 1$. For this purpose, we need to make some standard assumptions as follows:

Assumption 6.1 *The exosystem is neutrally stable, i.e., all the eigenvalues of* S *are semi-simple with zero real parts.*

Assumption 6.2 $|b_i(v, w)| > 0$, $i = 1, \cdots, N$, *for all* $v \in \mathbb{R}^{n_v}$ *and all* $w \in \mathbb{R}^{n_w}$.

Remark 6.1 *Without loss of generality, we can assume* $b_i(v, w) > 0$, $i = 1, \cdots, N$, *for all* $v \in \mathbb{R}^{n_v}$ *and all* $w \in \mathbb{R}^{n_w}$. *In this case, for any known compact subsets* $\mathbb{V}$ *and* $\mathbb{W}$, *there exist some known positive numbers* $b_{\max}$ *and* $b_{\min}$ *such that,* $i = 1, \cdots, N$, $b_{\min} \leq b_i(v, w) \leq b_{\max}$ *for all* $v \in \mathbb{R}^{n_v}$ *and all* $w \in \mathbb{R}^{n_w}$.

Assumption 6.3 *There exist globally defined smooth functions* $\boldsymbol{z}_i : \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \mapsto \mathbb{R}^n$ *with* $\boldsymbol{z}_i(0, w) = 0$ *such that*

$$\frac{\partial \boldsymbol{z}_i(v, w)}{\partial v} Sv = f_i(\boldsymbol{z}_i(v, w), q(v, w), v, w) \tag{6.5}$$

for all $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$.

Under Assumption 6.3, let $\mathbf{y}_i(v, w) = q(v, w)$ and

$$\mathbf{u}_i(v, w) = b_i^{-1}(\frac{\partial q(v, w)}{\partial v} Sv - g_i(\mathbf{z}_i(v, w), q(v, w), v, w)) \tag{6.6}$$

Then, $\mathbf{z}_i(v, w)$, $\mathbf{y}_i(v, w)$ and $\mathbf{u}_i(v, w)$ are the solutions for the regulator equations associated with Eqs. (6.1) and (6.2).

Assumption 6.4 $\boldsymbol{u}_i(v, w)$*,* $i = 1, \cdots, N$*, are polynomials in* v *with coefficients depending on* w*.*

Remark 6.2 *As remarked in [93], under Assumption 6.4, there exist integers* s_i *such that* $\boldsymbol{u}_i(v, w)$ *satisfy, for all trajectories* $v(t)$ *of the exosystem and all* $w \in \mathbb{W}$

$$\frac{d^{s_i} \boldsymbol{u}_i}{dt^{s_i}} = a_{1i} \boldsymbol{u}_i + a_{2i} \frac{d \boldsymbol{u}_i}{dt} + \cdots + a_{s_i i} \frac{d^{s_i - 1} \boldsymbol{u}_i}{dt^{s_i - 1}} \tag{6.7}$$

where $a_{1i}, a_{2i}, \cdots, a_{s_i i}$ *are real scalars such that all the roots of the polynomial* $P_i(\lambda) = \lambda^{s_i} - a_{1i} - a_{2i}\lambda - \cdots - a_{s_i i}\lambda^{s_i - 1}$ *are distinct with zero real parts [28].*

Let $\tau_i(v, w) = col(\boldsymbol{u}, \dot{\boldsymbol{u}}, \cdots, \boldsymbol{u}^{(s_i - 1)})$

$$\Phi_i = \begin{bmatrix} 0 & I_{s_i - 1} \\ a_{1i} & a_{2i}, \cdots, a_{s_i i} \end{bmatrix} \tag{6.8}$$

and $\Gamma_i = [1, 0, \cdots, 0]_{1 \times s_i}$*. Then* $\tau_i(v, w)$*,* Φ_i *and* Γ_i *satisfy the following equations:*

$$\frac{\partial \tau_i(v, w)}{\partial v} Sv = \Phi_i \tau_i(v, w), \quad \boldsymbol{u}_i(v, w) = \Gamma_i \tau_i(v, w) \tag{6.9}$$

System (6.9) can be used to generate the steady-state input $\boldsymbol{u}_i(v, w)$*, and thus it is called a steady-state generator with output* u_i *[30]. Since* (Γ_i, Φ_i) *is observable and the eigenvalues of* Φ_i *have zero real parts, for any controllable pair* (M_i, N_i)*, where* $M_i \in \mathbb{R}^{s_i \times s_i}$ *is a Hurwitz matrix and* $N_i \in \mathbb{R}^{s_i \times 1}$ *is a column vector, there is a unique nonsingular matrix* T_i *satisfying the Sylvester equation*

$$T_i \Phi_i - M_i T_i = N_i \Gamma_i \tag{6.10}$$

Let $\theta_i(v, w) = T_i \tau_i(v, w)$*, which satisfies* $\dot{\theta}_i = (M_i + N_i \Psi_i)\theta_i$ *and* $\boldsymbol{u}_i(v, w) = \Psi_i \theta_i$*, where* $\Psi_i = \Gamma_i T_i^{-1}$*. Then we can define a dynamic compensator as follows [50]:*

$$\dot{\eta}_i = M_i \eta_i + N_i u_i \tag{6.11}$$

which is called the internal model with output u_i *in the sense of Definition 3.4 in [30].*

Attaching the internal model (6.11) to (6.1) and performing the following coordinate and input transformation:

$$\begin{aligned} \bar{z}_i &= z_i - \mathbf{z}_i(v,w), \quad \tilde{\eta}_i = \eta_i - \theta_i(v,w) - N_i b_i^{-1} e_i \\ e_i &= y_i - q(v,w), \quad \bar{u}_i = u_i - \Psi_i \eta_i, \quad i = 1, \cdots, N \end{aligned} \tag{6.12}$$

give a system as follows:

$$\begin{aligned} \dot{\bar{z}}_i &= \bar{f}_i(\bar{z}_i, e_i, \mu) \\ \dot{\tilde{\eta}}_i &= M_i \tilde{\eta}_i + M_i N_i b_i^{-1} e_i - N_i b_i^{-1} \bar{g}_i(\bar{z}_i, e_i, \mu) - N_i \frac{\partial b_i^{-1}(v,w)}{\partial v} S v e_i \\ \dot{e}_i &= \bar{g}_i(\bar{z}_i, e_i, \mu) + b_i \Psi_i \tilde{\eta}_i + \Psi_i N_i e_i + b_i \bar{u}_i \end{aligned} \tag{6.13}$$

where $\mu = (v,w)$, $\bar{f}_i(\bar{z}_i, e_i, \mu) = f_i(\bar{z}_i + \mathbf{z}_i, e_i + q, v, w) - f_i(\mathbf{z}_i, q, v, w)$ and $\bar{g}_i(\bar{z}_i, e_i, \mu) = g_i(\bar{z}_i + \mathbf{z}_i, e_i + q, v, w) - g_i(\mathbf{z}_i, q, v, w)$. It can be verified that $\bar{f}_i(0,0,\mu) = 0$ and $\bar{g}_i(0,0,\mu) = 0$ for any $\mu \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$.

As pointed out in [93], an important property of the augmented system (6.13) is that, for all v and w, the origin is an equilibrium point of (6.13) and the output e_i is identically zero at the origin. As a result, if for any compact subsets $\mathbb{V} \subseteq \mathbb{R}^v$ and $\mathbb{W} \subseteq \mathbb{R}^{n_w}$ which contain $v = 0$ and $w = 0$, respectively, there is an output feedback control law of the form

$$\bar{u}_i = \bar{h}_i(e_i), \quad i = 1, \cdots, N \tag{6.14}$$

vanishing at the origin such that, for all $v(t) \in \mathbb{V} \subseteq \mathbb{R}^v$ and all $w \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$, the equilibrium point of the closed-loop system composed of (6.13) and (6.14) is globally asymptotically stable, then the following control law

$$\begin{aligned} u_i &= \bar{h}_i(e_i) + \Psi_i \eta_i \\ \dot{\eta}_i &= M_i \eta_i + N_i u_i, \quad i = 1, \cdots, N \end{aligned} \tag{6.15}$$

solves the global output regulation problem for system (6.1).

However, due to the communication constraints, the control law of (6.14) is not admissible. Thus we can only use a distributed control law of the form (6.3). To find such a control law, let $\bar{\mathcal{A}} = [c_{ij}]_{i,j=0}^N$ be any weighted adjacency matrix of $\bar{\mathcal{G}}$. For $i = 1, \cdots, N$, let

$$e_{vi} = \sum_{j=0}^{N} c_{ij}(y_i - y_j) \tag{6.16}$$

Since $\mathbf{y}_i(v,w) = \mathbf{y}_j(v,w)$ for any $i \neq j$, we have $e_{vi} = \sum_{j=0}^N c_{ij}(e_i - e_j)$ where $e_0 = 0$. Then we will consider a class of output feedback controllers as follows

$$\bar{u}_i = k_i(e_{vi}), \quad i = 1, \cdots, N \tag{6.17}$$

where $k_i(.)$ is a globally defined sufficiently smooth function that vanishes at the origin.

Clearly, if the augmented system (6.13) can be globally stabilized by the control law (6.17), then the output regulation problem of the original system will be solved by the following distributed control law:

$$\begin{aligned} u_i &= k_i(e_{vi}) + \Psi_i \eta_i \\ \dot{\eta}_i &= M_i \eta_i + N_i u_i, \quad i = 1, \cdots, N \end{aligned} \tag{6.18}$$

For convenience, let $Z_i = \mathrm{col}(\bar{z}_i, \tilde{\eta}_i)$ and $F_i(Z_i, e_i, \mu) = \mathrm{col}(\bar{f}_i(\bar{z}_i, e_i, \mu), M_i\tilde{\eta}_i + M_i N_i b_i^{-1} e_i - N_i b_i^{-1} \bar{g}_i(\bar{z}_i, e_i, \mu) - N_i \frac{\partial b_i^{-1}(v,w)}{\partial v} Sve_i)$. Then the system (6.13) can be put in the following more compact form:

$$\begin{aligned} \dot{Z}_i &= F_i(Z_i, e_i, \mu) \\ \dot{e}_i &= \bar{g}_i(\bar{z}_i, e_i, \mu) + b_i \Psi_i \tilde{\eta}_i + \Psi_i N_i e_i + b_i \bar{u}_i \end{aligned} \tag{6.19}$$

6.3 Construction of Distributed Controller

In this section, we will focus on globally stabilizing the augmented system (6.13) by a control law of the form (6.17). For this purpose, we need two more assumptions as follows

Assumption 6.5 *For any compact subset $\Omega \subset \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$, there exists a C^1 function $V_{\bar{z}_i}$ satisfying $\underline{\alpha}_i(||\bar{z}_i||) \leq V_{\bar{z}_i}(\bar{z}_i) \leq \overline{\alpha}_i(||\bar{z}_i||)$, for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_i(.)$ and $\overline{\alpha}_i(.)$ such that, for any $\mu \in \Omega$ along the trajectory of the subsystem $\dot{\bar{z}}_i = \bar{f}_i(\bar{z}_i, e_i, \mu)$, $\dot{V}_{\bar{z}_i} \leq -\alpha_i(||\bar{z}_i||) + \gamma_i(e_i)$, where $\alpha_i(.)$ is some known class $\mathcal{K}_\infty$ function satisfying $\lim_{s\to 0^+} \sup(\alpha_i^{-1}(s^2)/s) < \infty$, and $\gamma_i(.)$ is some known smooth positive definite function.*

Remark 6.3 *This assumption is quite standard in the literature of the global robust stabilization and output regulation [36], [93]. It guarantees that the subsystem $\dot{\bar{z}}_i = \bar{f}_i(\bar{z}_i, e_i, \mu)$ is input-to-state stable [69]. Under this assumption, by Lemma 3.1 in [93], there exists a C^1 function $V_i(Z_i)$ satisfying $\underline{\alpha}_{2i}(||Z_i||) \leq V_i(Z_i) \leq \overline{\alpha}_{2i}(||Z_i||)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{2i}(.)$ and $\overline{\alpha}_{2i}(.)$, such that, for all $\mu \in \Omega$, along the trajectory of Z_i-subsystem in (6.19),*

$$\dot{V}_i(Z_i) \leq -||Z_i||^2 + \pi_i(e_i) \tag{6.20}$$

for some known smooth positive definite function $\pi_i(e_i)$.

Assumption 6.6 *Every node $i = 1, \cdots, N$ is reachable from the node 0 in the digraph $\bar{\mathcal{G}}$, and $\mathcal{G}$ is an undirected graph.*

Remark 6.4 *Let Δ be an $N\times N$ nonnegative diagonal matrix whose ith diagonal element is c_{i0}, $i=1,\cdots,N$. Then* $\bar{\mathcal{L}} \triangleq \begin{bmatrix} 0 & 0_{1\times N} \\ -\Delta\mathbb{1}_N & H \end{bmatrix}$*, where* $H=[h_{ij}]_{i,j=1}^N$ *with* $h_{ii}=\sum_{i=0}^N c_{ij}$ *and* $h_{ij}=-c_{ij}$*,* $0_{m\times n}$ *denoting the zero matrix in* $\mathbb{R}^{m\times n}$*, is a Laplacian of* $\bar{\mathcal{G}}$*. So* $H\mathbb{1}_N = \Delta\mathbb{1}_N$*, where* $\mathbb{1}_N$ *denotes an* $N\times 1$ *column vector whose elements are all 1. Moreover, by Lemma 4 in [27], all the eigenvalues of* H *have positive real parts if and only if Assumption 6.6 is satisfied. Since* $\mathcal{G}$ *is an undirected graph,* H *is also symmetric.*

Lemma 6.1 *Under Assumptions 6.1-6.4, 6.5 and 6.6, the global stabilization problem of system (6.13) can be solved by the distributed output feedback control law of the form*

$$\bar{u}_i = -\rho_i(e_{vi})e_{vi}, \quad i=1,\cdots,N \tag{6.21}$$

where $\rho_i(.)$*,* $i=1,\cdots,N$*, are some sufficiently smooth positive definite functions to be specified in the proof of this Lemma.*

Proof: By the changing supply rate technique [69], given any smooth function $\bar{\vartheta}_i(Z_i)>0$, there exists a C^1 function $\bar{V}_i(Z_i)$ satisfying $\underline{\alpha}_{3i}(||Z_i||)\le \bar{V}_i(Z_i)\le \overline{\alpha}_{3i}(||Z_i||)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{3i}(.)$ and $\overline{\alpha}_{3i}(.)$, such that, for all $\mu\in\Omega$, along the trajectory of Z_i-subsystem of (6.19),

$$\dot{\bar{V}}_i(Z_i) \le -\bar{\vartheta}_i(Z_i)||Z_i||^2 + \bar{\pi}_i(e_i)|e_i|^2 \tag{6.22}$$

where $\bar{\pi}_i(.)$, $i=1,\cdots,N$, are some known smooth positive definite functions.

Next, let

$$\begin{aligned}\tilde{g}_i(Z_i,e_i,\mu) &= \bar{g}_i(\bar{z}_i,e_i,\mu) + b_i\Psi_i\tilde{\eta}_i + \Psi_i N_i e_i \\ \tilde{g}(Z,e,\mu) &= \mathrm{col}(\tilde{g}_1(Z_1,e_1,\mu),\cdots,\tilde{g}_N(Z_N,e_N,\mu))\end{aligned}$$

Since $\tilde{g}_i(Z_i,e_i,\mu)$, $i=1,\cdots,N$, are all smooth and satisfy $\tilde{g}_i(0,0,\mu)=0$, by Lemma 7.8 of [28] again, there exist some smooth functions $\delta_i(Z)\ge 1$ and $\chi_i(e_i)\ge 1$, such that, for all $Z_i\in\mathbb{R}^{n+s_i}$, $e_i\in\mathbb{R}$ and $\mu\in\Omega$,

$$|\tilde{g}_i(Z_i,e_i,\mu)|^2 \le \delta_i(Z_i)||Z_i||^2 + \chi_i(e_i)e_i^2 \tag{6.23}$$

Let $e_v = \text{col}(e_{v1}, \cdots, e_{vN})$. Then $e = H^{-1}e_v$. Let $V_e = \frac{1}{2}e^THe$. Then the derivative of V_e along the subsystems $\dot{e}_i = \tilde{g}_i(Z_i, e_v, \mu) - b_i\rho_i(e_{vi})e_{vi}$, $i = 1, \cdots, N$, satisfies

$$\begin{aligned}
\dot{V}_e &= e^TH\dot{e} = e_v^T\dot{e} = \sum_{i=1}^{N} e_{vi}\dot{e}_i \\
&= \sum_{i=1}^{N} -b_i\rho_i(e_{vi})e_{vi}^2 + e^TH\tilde{g}(Z, e, \mu) \\
&\leq \sum_{i=1}^{N} -b_i\rho_i(e_{vi})e_{vi}^2 + \frac{||H||^2}{2\epsilon}e^2 + \frac{\epsilon}{2}||\tilde{g}(Z, e, \mu)||^2 \\
&\leq \sum_{i=1}^{N} -b_i\rho_i(e_{vi})e_{vi}^2 + \frac{\epsilon}{2}\sum_{i=1}^{N}\delta_i(Z_i)||Z_i||^2 + \sum_{i=1}^{N}(\frac{||H||^2}{2\epsilon} + \frac{\epsilon}{2}\chi_i(e_i))e_i^2
\end{aligned} \tag{6.24}$$

with $\epsilon > 0$. Let $V_Z(Z) = \sum_{i=1}^{N} \bar{V}_i(Z_i)$ where $Z = \text{col}(Z_1, \cdots, Z_N)$. Finally, let $U(Z, e) = V_Z + V_e$. Then the derivative of U along the trajectory of the closed-loop system satisfies

$$\dot{U} = -\sum_{i=1}^{N} b_i\rho_i(e_{vi})e_{vi}^2 - \sum_{i=1}^{N}(\bar{\vartheta}_i(Z_i) - \frac{\epsilon}{2}\delta_i(Z_i))||Z_i||^2 + \bar{\rho}(e) \tag{6.25}$$

with $\bar{\rho}(e) = \sum_{i=1}^{N}(\frac{||H||^2}{2\epsilon} + \frac{\epsilon}{2}\chi_i(e_i) + \bar{\pi}_i(e_i))e_i^2$.
By Lemma 7.8 of [28], there exist known smooth positive definite functions $\tilde{\rho}_i(e_{vi})$, $i = 1, \cdots, N$, such that $\bar{\rho}(e) = \bar{\rho}(H^{-1}e_v) \leq \sum_{i=1}^{N} \tilde{\rho}_i(e_{vi})e_{vi}^2$.
Let $\rho_i(e_{vi}) \geq b_{\min}^{-1}(\tilde{\rho}_i(e_{vi}) + \iota_1)$ and $\bar{\vartheta}_i(Z_i) \geq \frac{\epsilon}{2}\delta_i(Z_i) + \iota_2$ with $\iota_1, \iota_2 > 0$. Then $\dot{U} \leq -\iota_1||Z||^2 - \iota_2||e_v||^2$. Thus under the output feedback control law (6.21), the equilibrium of the closed-loop system is uniformly globally asymptotically stable. Thus the proof is completed.

Lemma 6.1 leads to our main result as follows.

Theorem 6.1 *Under Assumptions 6.1-6.4, 6.5 and 6.6, the cooperative global robust output regulation problem of system (6.1) can be solved by the distributed output feedback control law of the form*

$$\begin{aligned}
u_i &= -\rho_i(e_{vi})e_{vi} + \Psi_i\eta_i, \quad i = 1, \cdots, N \\
\dot{\eta}_i &= M_i\eta_i + N_iu_i
\end{aligned} \tag{6.26}$$

6.4 Application to Lorenz Multi-agent Systems

Consider the following multi-agent system:

$$\begin{aligned} \dot{x}_{1i} &= -L_{1i}x_{1i} + L_{1i}x_{2i} \\ \dot{x}_{2i} &= b_i u_i + L_{3i}x_{1i} - x_{2i} - x_{1i}x_{3i} \\ \dot{x}_{3i} &= L_{2i}x_{3i} + x_{1i}x_{2i} \\ e_i &= x_{2i} - v_1, \quad i = 1, \cdots, N \end{aligned} \tag{6.27}$$

where $b_i = 1$, $i = 1, \cdots, N$. The exosystem is

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = S \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{6.28}$$

with $\omega = 1$. For each i, system (6.28) is called controlled Lorenz system [8], [93].

By letting $(z_{1i}, z_{2i}, y_i) = (x_{1i}, x_{3i}, x_{2i})$, we can put the system (6.27) in the standard form (6.1) as follows:

$$\begin{aligned} \dot{z}_{1i} &= -L_{1i}z_{1i} + L_{1i}y_i \\ \dot{z}_{2i} &= L_{2i}z_{2i} + z_{1i}y_i \\ \dot{y}_i &= b_i u_i + L_{3i}z_{1i} - y_i - z_{1i}z_{2i} \\ e_i &= y_i - v_1 \end{aligned} \tag{6.29}$$

For the special case where $N = 1$, the output regulation problem for this system was studied by a decentralized control law in [93]. Here we assume $N = 3$ and the interconnection among various subsystems is determined by Figure 6.1. We will design a

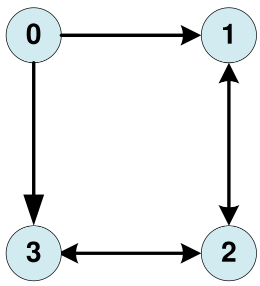

Figure 6.1: The network topology

distributed control law to solve our problem. To make our problem more interesting, as in [93], we allow the parameter vector $L_i = (L_{1i}, L_{2i}, L_{3i})$ to undergo some perturbation. To be more specific, let the normal value of L_i be $(10, -\frac{8}{3}, 28)$, $i = 1, 2, 3$. Then $L_i = (10, -\frac{8}{3}, 28) + (w_{1i}, w_{2i}, w_{3i})$ where (w_{1i}, w_{2i}, w_{3i}) represents the uncertainty of L_i for $i = 1, 2, 3$. Let $w_i = \text{col}(w_{1i}, w_{2i}, w_{3i})$, $i = 1, 2, 3$, and $w = \text{col}(w_1, w_2, w_3)$. Define $\mathbb{W} = \{w | w \in \mathbb{R}^9, ||w_i|| \leq 1, i = 1, 2, 3\}$, and $\mathbb{V} = \{v(t) | ||v(t)|| \leq 1\}$.

Using the result of [93], it can be directly inferred that, for each $i = 1, 2, 3$, the composite system (6.28) and (6.29) satisfies Assumptions 6.1-6.4. In fact, for the sake of self-containment, we can get $\mathbf{y}_i(v, w) = v_1$ from the last equation of (6.29). Substituting $\mathbf{y}_i(v, w)$ into the equations of (6.29) yields

$$\begin{aligned}
\mathbf{z}_{1i}(v, w) &= r_{11i}v_1 + r_{12i}v_2 \\
\mathbf{z}_{2i}(v, w) &= r_{21i}v_1^2 + r_{22i}v_2^2 + r_{23i}v_1v_2 \\
\mathbf{u}_i(v, w) &= r_{31i}v_1 + r_{32i}v_2 + r_{33i}v_1^3 + r_{34i}v_2^3 + r_{35i}v_1^2v_2 + r_{36i}v_1v_2^2
\end{aligned} \tag{6.30}$$

with

$$\begin{aligned}
r_{11i}(w) &= \frac{L_{1i}^2}{\omega^2 + L_{1i}^2}, \quad r_{12i}(w) = -\frac{L_{1i}\omega}{\omega^2 + L_{1i}^2} \\
r_{21i}(w) &= -\frac{\omega r_{23i} + r_{11i}}{L_{2i}}, \quad r_{22i}(w) = \frac{\omega r_{23i}}{L_{2i}} \\
r_{23i}(w) &= -\frac{r_{12i}L_{2i} + 2\omega r_{11i}}{L_{2i}^2 + 4\omega^2} \\
r_{31i}(w) &= -b_i^{-1}(L_{3i}r_{11i} - 1), \ r_{32i}(w) = b_i^{-1}(\omega - L_{3i}r_{12i}) \\
r_{33i}(w) &= b_i^{-1}r_{11i}r_{21i}, \quad r_{34i}(w) = b_i^{-1}r_{12i}r_{22i} \\
r_{35i}(w) &= b_i^{-1}(r_{12i}r_{21i} + r_{11i}r_{23i}) \\
r_{36i}(w) &= b_i^{-1}(r_{11i}r_{22i} + r_{12i}r_{23i})
\end{aligned} \tag{6.31}$$

It can be further verified that

$$\frac{d^4\mathbf{u}_i(v, w)}{dt^4} + 9\omega^4\mathbf{u}_i(v, w) + 10\omega^2\frac{d^2\mathbf{u}_i(v, w)}{dt^2} = 0 \tag{6.32}$$

From Figure 6.1, we can directly see the satisfaction of Assumption 6.6. We only need to further verify Assumption 6.5. For this purpose, note that either from (6.32) or from [93], we can obtain the steady-state generator (6.9) as follows

$$\tau_i(v, w) = \mathrm{col}(\mathbf{u}_i, \dot{\mathbf{u}}_i, \mathbf{u}_i^{(2)}, \mathbf{u}_i^{(3)}) \tag{6.33}$$

$$\Phi_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9\omega^4 & 0 & -10\omega^2 & 0 \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \tag{6.34}$$

which leads to the internal model (6.11) with

$$M_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -12 & -13 & -6 \end{bmatrix}, N_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solving the Sylvester equation $T_i\Phi_i - M_iT_i = N_i\Gamma_i$ gives $\Psi_i = \Gamma_i T_i^{-1} = [4-9\omega^4, 12, 13-10\omega^2, 6]$.

Performing the coordinate transformation (6.12) yields the augmented system (6.13) as follows:

$$\begin{aligned}
\dot{\bar{z}}_{1i} &= -L_{1i}\bar{z}_{1i} + L_{1i}e_i \\
\dot{\bar{z}}_{2i} &= L_{2i}\bar{z}_{2i} + (\bar{z}_{1i} + \mathbf{z}_{1i})(e_i + v_1) - \mathbf{z}_{1i}v_1 \\
\dot{\tilde{\eta}}_i &= M_i\tilde{\eta}_i + M_iN_ib_i^{-1}e_i - N_ib_i^{-1}\bar{g}_i(\bar{z}_{1i}, \bar{z}_{1i}, e_i, \mu) \\
\dot{e}_i &= \bar{g}_i(\bar{z}_{1i}, \bar{z}_{1i}, e_i, \mu) + b_i\Psi_i\tilde{\eta}_i + \Psi_iN_ie_i + b_i\bar{u}_i
\end{aligned} \tag{6.35}$$

with $\bar{g}_i(\bar{z}_{1i}, \bar{z}_{2i}, e_i, \mu) = L_{3i}\bar{z}_{1i} - e_i - \bar{z}_{1i}\bar{z}_{2i} - \mathbf{z}_{1i}\bar{z}_{2i} - \bar{z}_{1i}\mathbf{z}_{2i}$.

For the $(\bar{z}_{1i}, \bar{z}_{2i})-$subsystem in (6.35), let

$$V_{\bar{z}_i} = \frac{h_1}{2}\bar{z}_{1i}^2 + \frac{h_1}{4}\bar{z}_{1i}^4 + \frac{h_1}{8}\bar{z}_{1i}^8 + \frac{h_2}{2}\bar{z}_{2i}^2 + \frac{h_2}{4}\bar{z}_{2i}^4 \tag{6.36}$$

The time derivative of $(\bar{z}_{1i}, \bar{z}_{2i})-$subsystem is given by

$$\begin{aligned}
\dot{V}_{\bar{z}_i} &= h_1\bar{z}_{1i}\dot{\bar{z}}_{1i} + h_1\bar{z}_{1i}^3\dot{\bar{z}}_{1i} + h_1\bar{z}_{1i}^7\dot{\bar{z}}_{1i} + h_2\bar{z}_{2i}\dot{\bar{z}}_{2i} + h_2\bar{z}_{2i}^3\dot{\bar{z}}_{2i} \\
&= -h_1L_{1i}\bar{z}_{1i}^2 + h_1L_{1i}\bar{z}_{1i}e_i - h_1L_{1i}\bar{z}_{1i}^4 + h_1L_{1i}\bar{z}_{1i}^3e_i \\
&\quad - h_1L_{1i}\bar{z}_{1i}^8 + h_1L_{1i}\bar{z}_{1i}^7e_i + L_{2i}h_2\bar{z}_{2i}^2 + h_2\bar{z}_{1i}\bar{z}_{2i}e_i \\
&\quad + h_2\bar{z}_{2i}\mathbf{z}_{1i}e_i + h_2\bar{z}_{2i}\bar{z}_{1i}v_1 + L_{2i}h_2\bar{z}_{2i}^4 + h_2\bar{z}_{1i}\bar{z}_{2i}^3e_i \\
&\quad + h_2\bar{z}_{2i}^3\mathbf{z}_{1i}e_i + h_2\bar{z}_{2i}^3\bar{z}_{1i}v_1
\end{aligned} \tag{6.37}$$

By completing the square in (6.37), we can further obtain

$$\dot{V}_{\bar{z}_i} \le -l_{1i}\bar{z}_{1i}^2 - l_{2i}\bar{z}_{1i}^4 - l_{3i}\bar{z}_{1i}^8 - l_{4i}\bar{z}_{2i}^2 - l_{5i}\bar{z}_{2i}^4 + l_{6i}e_i^2 + l_{7i}e_i^4 + l_{8i}e_i^8 \tag{6.38}$$

where

$$\begin{aligned}
&l_{1i} = h_1L_{1i} - \frac{0.01}{2} - \frac{v_1^2h_2^2}{2}, \quad l_{2i} = h_1L_{1i} - 1 - \frac{h_2^4v_1^4}{4} \\
&l_{3i} = h_1L_{1i} - 1, \quad l_{4i} = -L_{2i}h_2 - \frac{3}{2} \\
&l_{5i} = -h_2L_{2i} - \frac{9}{4}, \quad l_{6i} = \frac{h_1^2L_{1i}^2}{0.02} + \frac{h_2^2\mathbf{z}_{1i}^2}{2} \\
&l_{7i} = \frac{h_1^4L_{1i}^4}{4} + \frac{h_2^4}{4} + \frac{h_2^4\mathbf{z}_{1i}^4}{4}, \quad l_{8i} = \frac{h_1^8L_{1i}^8}{8} + \frac{h_2^8}{8}
\end{aligned} \tag{6.39}$$

It can be seen that for proper $h_1 > 0$ and $h_2 > 0$, $l_{1i}, \cdots, l_{8i} > 0$. Thus Assumption 6.5 is also satisfied. By Theorem 6.1, there exists a distributed output feedback control law of the form (6.26) to solve the cooperative robust output regulation problem for this system. In fact, using the approach detailed in the proof of Lemma 6.1, we can first construct a control law of the form (6.21) with $\rho_i(e_{vi}) = 4.187 \times 10^4 e_{vi}^6 + 4714e_{vi}^2 + 1778$ that globally stabilizes the augmented system (6.35). Then, a control law of the form (6.26) will solve the cooperative output regulation problem for this example.

The performance of this control law is evaluated by computer simulation with the following initial conditions of the closed-loop system

$$[z_1(0), y_1(0)] = \begin{bmatrix} 0 & 2 & 0.5 \end{bmatrix}^T$$
$$[z_2(0), y_2(0)] = \begin{bmatrix} 1 & 2 & 1.5 \end{bmatrix}^T$$
$$[z_3(0), y_3(0)] = \begin{bmatrix} 1 & 2 & 0.8 \end{bmatrix}^T$$
$$v(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \quad \eta_1(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$$
$$\eta_2(0) = \begin{bmatrix} 0 & 0 & 6 & 0 \end{bmatrix}^T, \quad \eta_3(0) = \begin{bmatrix} 0 & 6 & 0 & 6 \end{bmatrix}^T$$

and the following values of the uncertain parameters $w_i = \text{col}(0.1i, 0.1i, 0.1i)$, $i = 1, 2, 3$. Figures 6.2-6.4 show the tracking error and state of each follower, respectively. It can be seen that the tracking errors of all subsystems approach the origin asymptotically.

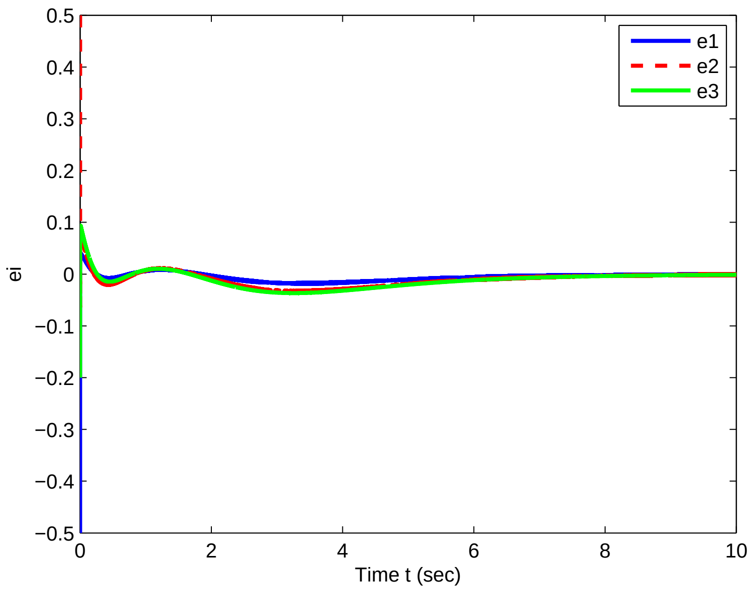

Figure 6.2: The tracking error e_i

6.5 Conclusion

In this chapter, we have studied the global robust output regulation problem for a class of nonlinear multi-agent systems in output feedback from with unity relative degree. We have

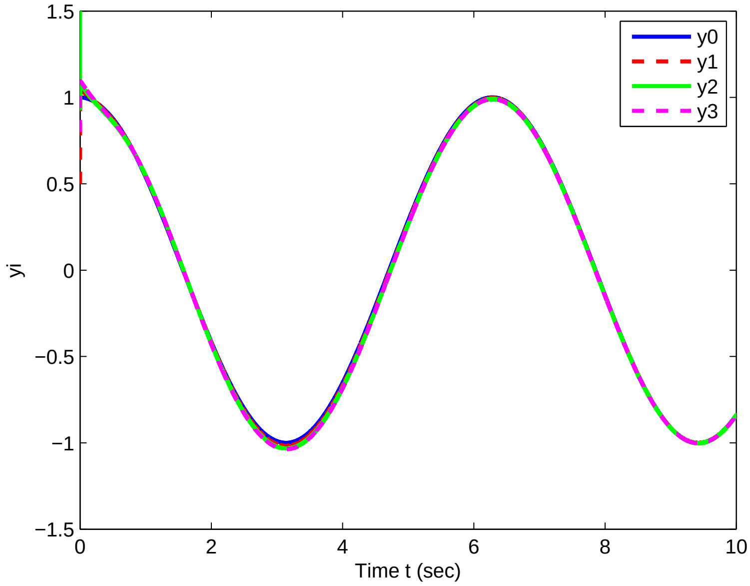

Figure 6.3: The output y_i of all agents

applied a distributed internal model to convert the problem into the global stabilization problem of an augmented system. Then we have further globally stabilized the augmented system via a distributed output feedback control law, thus leading to the solution of the the global robust output regulation problem of the original system.

Our problem is a generalization of the leader-following problem in several ways. In particular, by introducing an exosystem, we alow the leader system to have different dynamics as the follower system. As a result, our control law can handle a class of reference inputs and a class of disturbances generated the exosystem as apposed to the leader-following consensus problem where the leader system usually has the same dynamics as the follower system.

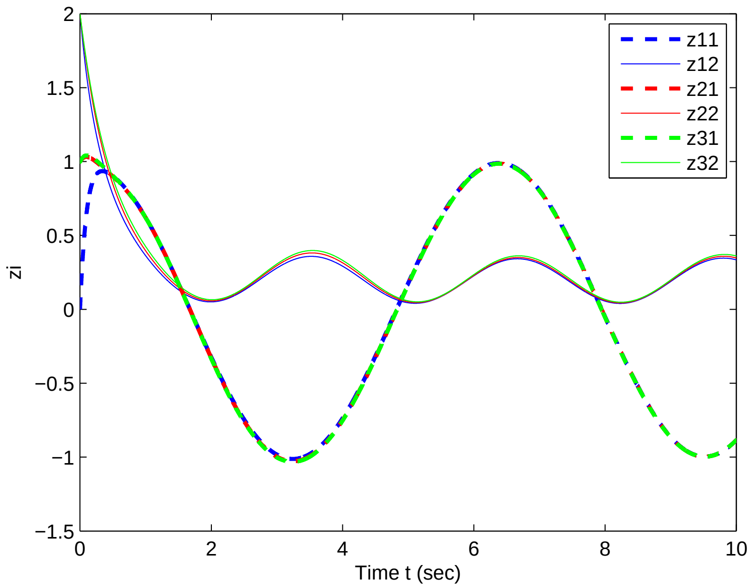

Figure 6.4: The state z_i of all agents

Chapter 7

Cooperative Global Output Regulation for a Class of Nonlinear Multi-agent Systems

7.1 Introduction

Consider a collection of nonlinear systems of the following form

$$\begin{aligned}
\dot{z}_i &= f_i(z_i, y_i, v, w) \\
\dot{x}_{si} &= x_{(s+1)i} + g_{si}(z_i, y_i, v, w), \quad s = 1, \cdots, r-1 \\
\dot{x}_{ri} &= b_i(w)u_i + g_{ri}(z_i, y_i, v, w) \\
y_i &= x_{1i} \\
e_i &= y_i - q(v, w)
\end{aligned} \tag{7.1}$$

where $r \geq 2$ is an integer, for $i = 1, \cdots, N$, $(z_i, x_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^r$ with $x_i = \mathrm{col}(x_{1i}, \cdots, x_{ri}) \in \mathbb{R}^r$ is the state, $u_i \in \mathbb{R}$ is the input, $y_i \in \mathbb{R}$ is the output, $e_i \in \mathbb{R}$ is the regulated error, $w \in \mathbb{R}^{n_w}$ is an uncertain parameter vector, and $v(t) \in \mathbb{R}^{n_v}$ is an exogenous signal generated by a linear system of the following form

$$\dot{v} = Sv \tag{7.2}$$

All functions in (7.1) are supposed to be globally defined, sufficiently smooth, and satisfy $b_i(w) > 0$, $q(0, w) = 0$, $f_i(0, 0, 0, w) = 0$, $g_{si}(0, 0, 0, w) = 0$, $i = 1, \cdots, N$, $s = 1, \cdots, r$ for all $w \in \mathbb{R}^{n_w}$.

The system composed of (7.1) and (7.2) can be viewed as a multi-agent system of $(N+1)$ agents with (7.2) as the leader and the N subsystems of (7.1) as N followers. With respect to the system composed of (7.1) and (7.2), we can define a digraph[1] $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ where $\bar{\mathcal{V}} = \{0, 1, ..., N\}$ with 0 associated with the leader system and $i = 1, \cdots, N$, associated with the N followers, respectively, and $(j, i) \in \bar{\mathcal{E}}$, $j = 0, 1, \cdots, N$ and $i = 1, \cdots, N$, if and only if the control u_i can make use of $y_i - y_j$ for feedback control. Thus our control law is of the following form:

$$\begin{aligned} u_i &= h_i(\bar{\zeta}_i, y_i - y_j, j \in \bar{\mathcal{N}_i}), \quad i = 1, \cdots, N \\ \dot{\bar{\zeta}}_i &= g_i(\bar{\zeta}_i, y_i - y_j, j \in \bar{\mathcal{N}_i}) \end{aligned} \tag{7.3}$$

where $y_0 = q(v, w)$ is the output of the leader system, $\bar{\mathcal{N}_i}$ is the neighbor set of the node i, h_i and g_i are sufficiently smooth functions vanishing at the origin, and $\bar{\zeta}_i \in \mathbb{R}^{n_{\bar{\zeta}_i}}$ with $n_{\bar{\zeta}_i}$ to be defined later. A control law of the form (7.3) is called a distributed dynamic output feedback control law because the control of each subsystem can only access the output of itself and its neighbors.

We call the composition of (7.1) and (7.3) as the closed-loop system which can be put in the following form

$$\begin{aligned} \dot{x}_c &= f_c(x_c, v, w) \\ e &= \text{col}(e_1, \cdots, e_N) \end{aligned} \tag{7.4}$$

where $x_c = \text{col}(z_1, x_1, \bar{\zeta}_1, \cdots, z_N, x_N, \bar{\zeta}_N) \in \mathbb{R}^{n_c}$ for some integer n_c, and f_c is sufficiently smooth satisfying $f_c(0, 0, w) = 0$ for all $w \in \mathbb{W}$. Then we can describe our problem as follows:

Definition 7.1 *Given the plant (7.1), the exosystem (7.2), the corresponding digraph $\bar{\mathcal{G}}$, and any compact subsets $\mathbb{V} \in \mathbb{R}^{n_v}$ and $\mathbb{W} \in \mathbb{R}^{n_w}$ containing the origins of the respective Euclidian spaces, find a control law of the form (7.3) such that, for any $v(0) \in \mathbb{V}$, $w \in \mathbb{W}$, the trajectory of the closed-loop system (7.4) starting from any initial state $x_c(0)$ exists and is bounded for all $t \geq 0$, and $\lim_{t\to\infty} e(t) = 0$.*

For each $i = 1, \cdots, N$, (7.1) is in the familiar output feedback form with the relative degree r. The control of such systems has been well studied in the literature. In particular, the global stabilization problem of such systems was studied in [36], and the global robust output regulation problem of such systems was studied in [14, 93, 94]. If, for each $i = 1, \cdots, N$, $0 \in \bar{\mathcal{N}_i}$, that is, the output of the leader system can be used by the control u_i of each subsystem of (7.1), then the control law (7.3) reduces to the following special form

$$\begin{aligned} u_i &= h_i(\bar{\zeta}_i, e_i), \quad i = 1, \cdots, N \\ \dot{\bar{\zeta}}_i &= g_i(\bar{\zeta}_i, e_i) \end{aligned} \tag{7.5}$$

[1]See [74] for a summary of digraph.

We call (7.5) a purely decentralized control law. It can be seen that, for such a special case, applying the approach of [94] to each subsystem of (7.1) will solve the problem by the purely decentralized control law (7.5). However, what makes our problem interesting is that we can solve the problem by only requiring the digraph $\bar{\mathcal{G}}$ satisfies certain connectivity condition (see Assumption 3.2 for the precise statement). This much relaxed condition on the digraph $\bar{\mathcal{G}}$ entails the employment of a distributed observer. As a result, our problem is more technically challenging than the problem in [94] in that we need to globally stabilize an extended augmented system which is a coupled multi-input nonlinear uncertain system. It should be noted that the cooperative output regulation problem of the plant (7.1) for the special case where $r = 1$ was handled recently by us in [18]. However, for this special case, there is no need to employ an observer. We only need to globally stabilize an augmented system which is a decoupled multi-input system.

The rest of this chapter is organized as follows. In Section 7.2, we will present the preliminaries for our problem. In Section 7.3, we will present our main result. In Section 7.4, we will apply our approach to solve a leader-following synchronization problem for a group of Lorenz multi-agent systems. Finally, we will close this chapter in Section 7.5 with some concluding remarks.

7.2 Preliminaries

By the general framework for handling the output regulation problem for nonlinear systems described in [30], the first step of our approach is to find an appropriate internal model for (7.1) to form an augmented system. For this purpose, we need to make some standard assumptions as follows:

Assumption 7.1 *The exosystem is neutrally stable, i.e., all the eigenvalues of S are semi-simple with zero real parts.*

Under Assumption 7.1, the exosystem can generate a combination of a step function of arbitrary amplitude and finitely many sinusoidal functions of arbitrary amplitudes and initial phases.

Assumption 7.2 *There exist globally defined smooth functions $\boldsymbol{z}_i(v, w) : \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \mapsto \mathbb{R}^{n_i}$ with $\boldsymbol{z}_i(0, w) = 0$, $i = 1, \cdots, N$, such that, for all $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$,*

$$\frac{\partial \boldsymbol{z}_i(v, w)}{\partial v} Sv = f_i(\boldsymbol{z}_i(v, w), q(v, w), v, w) \tag{7.6}$$

The above assumption is used to guarantee the solvability of the regulator equations which is a necessary condition for that of the output regulation problem. Under Assumption

7.2, let

$$\mathbf{x}_i(v,w) = \mathrm{col}(\mathbf{x}_{1i}(v,w), \cdots, \mathbf{x}_{ri}(v,w)) \tag{7.7}$$

with $\mathbf{x}_{1i}(v,w) = q(v,w)$ and for $s = 2, \cdots, r$, $i = 1, \cdots, N$,

$$\begin{aligned} \mathbf{x}_{si}(v,w) &= L_{Sv}\mathbf{x}_{(s-1)i}(v,w) - g_{(s-1)i}(\mathbf{z}_i(v,w), q(v,w), v, w) \\ \mathbf{u}_i(v,w) &= b_i^{-1}[L_{Sv}\mathbf{x}_{ri}(v,w) - g_{ri}(\mathbf{z}_i(v,w), q(v,w), v, w)] \end{aligned} \tag{7.8}$$

where $L_{Sv}\mathbf{x}_{ji}(v,w) = \frac{\partial \mathbf{x}_{ji}(v,w)}{\partial v} Sv$, $j = 1, \cdots, r$, $i = 1, \cdots, N$. Then the solution of the regulator equations associated with system (7.1) and exosystem (7.2) is given by $\mathbf{z}_i(v,w)$, $\mathbf{x}_i(v,w)$ and $\mathbf{u}_i(v,w)$, $i = 1, \cdots, N$ [31].

Assumption 7.3 *The functions $\boldsymbol{u}_i(v,w)$, $i = 1, \cdots, N$, are polynomials in v with coefficients depending on w.*

Remark 7.1 *Under Assumption 7.3, there exist integers s_i, $i = 1, \cdots, N$, such that $\boldsymbol{u}_i(v,w)$ satisfy, for all trajectories $v(t)$ of the exosystem and all $w \in \mathbb{R}^{n_w}$*

$$\frac{d^{s_i}\boldsymbol{u}_i}{dt^{s_i}} = a_{1i}\boldsymbol{u}_i + a_{2i}\frac{d\boldsymbol{u}_i}{dt} + \cdots + a_{s_i i}\frac{d^{(s_i-1)}\boldsymbol{u}_i}{dt^{(s_i-1)}} \tag{7.9}$$

where $a_{1i}, a_{2i}, \cdots, a_{s_i i}$ are real scalars such that all the roots of the polynomial $P_i(\lambda) = \lambda^{s_i} - a_{1i} - a_{2i}\lambda - \cdots - a_{s_i i}\lambda^{s_i-1}$ are distinct with zero real parts [28]. Moreover, for any controllable pair (M_i, N_i), where $M_i \in \mathbb{R}^{s_i \times s_i}$ is a Hurwitz matrix and $N_i \in \mathbb{R}^{s_i \times 1}$ is a column vector, there is a row vector $\Psi_i \in \mathbb{R}^{1 \times s_i}$ such that the following dynamic compensator

$$\dot{\eta}_i = M_i\eta_i + N_i u_i, \quad u_i = \Psi_i\eta_i, \quad i = 1, \ \cdots, N \tag{7.10}$$

is the internal model of (7.1) and (7.2) [30, 50]. It is noted that Assumption 7.3 can be relaxed if one resorts to nonlinear internal models as given in [28, 30].

The composition of the internal model (7.10) and plant (7.1) is called augmented system. Let $\tau_i(v,w) = \mathrm{col}(\mathbf{u}_i, \dot{\mathbf{u}}_i, \cdots, \mathbf{u}_i^{(s_i-1)})$, $i = 1, \cdots, N$. Then, there exists a nonsingular matrix $T_i \in \mathbb{R}^{s_i \times s_i}$ such that, under the following coordinate and input transformation,

$$\begin{aligned} \bar{z}_i &= z_i - \mathbf{z}_i(v,w), \quad \bar{x}_i = x_i - \mathbf{x}_i(v,w) \\ \bar{\eta}_i &= \eta_i - T_i\tau_i(v,w), \quad \bar{u}_i = u_i - \Psi_i\eta_i, \quad i = 1, \cdots, N \end{aligned} \tag{7.11}$$

the augmented system takes the following form [94]:

$$\begin{aligned}
\dot{\bar{z}}_i &= \dot{z}_i - \dot{\mathbf{z}}_i(v,w) = f_i(\bar{z}_i + \mathbf{z}_i, \bar{x}_{1i} + q, v, w) - f_i(\mathbf{z}_i, q, v, w)\\
&= \bar{f}_i(\bar{z}_i, \bar{x}_{1i}, v, w)\\
\dot{\bar{\eta}}_i &= \dot{\eta}_i - T_i\dot{\tau}_i(v,w) = M_i\eta_i + N_iu_i - (M_i + N_i\Psi_i)T_i\tau_i(v,w)\\
&= M_i\bar{\eta}_i + N_iu_i - N_i\Psi_iT_i\tau_i(v,w)\\
&= M_i\bar{\eta}_i + N_i\bar{u}_i + N_i\Psi_i\eta_i - N_i\Psi_iT_i\tau_i(v,w)\\
&= (M_i + N_i\Psi_i)\bar{\eta}_i + N_i\bar{u}_i \qquad (7.12)\\
\dot{\bar{x}}_{si} &= \dot{x}_{si} - \dot{\mathbf{x}}_{si}(v,w) = x_{(s+1)i} + g_{si}(z_i, y_i, v, w) - \mathbf{x}_{(s+1)i}(v,w) - g_{si}(\mathbf{z}_i, q, v, w)\\
&= \bar{x}_{(s+1)i} + \bar{g}_{si}(\bar{z}_i, \bar{x}_{1i}, v, w), \quad s = 1, \cdots, r-1\\
\dot{\bar{x}}_{ri} &= \dot{x}_{ri} - \dot{\mathbf{x}}_{ri}(v,w) = b_iu_i + g_{ri}(z_i, y_i, v, w) - b_i\mathbf{u}_i(v,w) - g_{ri}(\mathbf{z}_i, q, v, w)\\
&= b_i\bar{u}_i + b_i\Psi_i\eta_i - b_i\Psi_iT_i\tau_i(v,w) + \bar{g}_{ri}(\bar{z}_i, \bar{x}_{1i}, v, w)\\
&= b_i\bar{u}_i + b_i\Psi_i\bar{\eta}_i + \bar{g}_{ri}(\bar{z}_i, \bar{x}_{1i}, v, w)
\end{aligned}$$

where, for $i = 1, \cdots, N$, $s = 1, \cdots, r$,

$$\begin{aligned}
\bar{f}_i(\bar{z}_i, \bar{x}_{1i}, v, w) &= f_i(\bar{z}_i + \mathbf{z}_i, \bar{x}_{1i} + q, v, w) - f_i(\mathbf{z}_i, q, v, w)\\
\bar{g}_{si}(\bar{z}_i, \bar{x}_{1i}, v, w) &= g_{si}(\bar{z}_i + \mathbf{z}_i, \bar{x}_{1i} + q, v, w) - g_{si}(\mathbf{z}_i, q, v, w)
\end{aligned} \qquad (7.13)$$

with the following property, for any $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$,

$$\bar{f}_i(0, 0, v, w) = 0, \quad \bar{g}_{si}(0, 0, v, w) = 0 \qquad (7.14)$$

Since the augmented system (7.12) is not in the output feedback form as displayed in (7.1), like in [94], we perform on (7.12) the following coordinate transformation

$$\tilde{\eta}_i = \bar{\eta}_i - c_{ri}\bar{x}_{ri} - \cdots - c_{1i}\bar{x}_{1i}, \quad i = 1, \cdots, N \qquad (7.15)$$

where $c_{ri} = b_i^{-1}N_i$, $c_{(j-1)i} = M_ic_{ji}$ for $j = 2, \cdots, r$, $i = 1, \cdots, N$. As a result, we have

$$\begin{aligned}
\dot{\tilde{\eta}}_i &= \dot{\bar{\eta}}_i - c_{ri}\dot{\bar{x}}_{ri} - \cdots - c_{1i}\dot{\bar{x}}_{1i}\\
&= (M_i + N_i\Psi_i)\bar{\eta}_i + N_i\bar{u}_i - c_{ri}(b_i\bar{u}_i + b_i\Psi_i\bar{\eta}_i\\
&+ \bar{g}_{ri}(\bar{z}_i, \bar{x}_{1i}, v, w)) - \cdots - c_{1i}(\bar{x}_{2i} + \bar{g}_{1i}(\bar{z}_i, \bar{x}_{1i}, v, w))\\
&= M_i\tilde{\eta}_i + M_ic_{ri}\bar{x}_{ri} + \cdots + M_ic_{1i}\bar{x}_{1i} + N_i\Psi_i\bar{\eta}_i + N_i\bar{u}_i\\
&- c_{ri}b_i\bar{u}_i - \sum_{j=1}^{r} c_{ji}\bar{g}_{ji}(\bar{z}_i, \bar{x}_{1i}, v, w) - c_{ri}b_i\Psi_i\bar{\eta}_i - c_{(r-1)i}\bar{x}_{ri} - \cdots - c_{1i}\bar{x}_{2i}\\
&= M_i\tilde{\eta}_i + M_ic_{1i}\bar{x}_{1i} - \sum_{j=1}^{r} c_{ji}\bar{g}_{ji}(\bar{z}_i, \bar{x}_{1i}, v, w)
\end{aligned}$$

$$
\begin{aligned}
&+ M_i c_{ri}\bar{x}_{ri} + \cdots + M_i c_{1i}\bar{x}_{1i} - M_i c_{ri}\bar{x}_{ri} - \cdots - M_i c_{1i}\bar{x}_{1i} \\
&= M_i\tilde{\eta}_i + M_i c_{1i}\bar{x}_{1i} - \sum_{j=1}^{r} c_{ji}\bar{g}_{ji}(\bar{z}_i, \bar{x}_{1i}, v, w) \\
\dot{\bar{x}}_{si} &= \bar{x}_{(s+1)i} + \bar{g}_{si}(\bar{z}_i, \bar{x}_{1i}, v, w), \quad s = 1, \cdots, r-1 \\
\dot{\bar{x}}_{ri} &= b_i\bar{u}_i + \bar{g}_{ri}(\bar{z}_i, \bar{x}_{1i}, v, w) + b_i\Psi_i\tilde{\eta}_i + b_i\Psi_i(c_{ri}\bar{x}_{ri} + \cdots + c_{1i}\bar{x}_{1i})
\end{aligned}
\tag{7.16}
$$

Then

$$
\begin{aligned}
\dot{\tilde{\eta}}_i &= M_i\tilde{\eta}_i + M_i c_{1i}\bar{x}_{1i} - \sum_{j=1}^{r} c_{ji}\bar{g}_{ji}(\bar{z}_i, \bar{x}_{1i}, v, w) \\
\dot{\bar{x}}_i &= A_i\bar{x}_i + b_i B\Psi_i\tilde{\eta}_i + \bar{g}_i(\bar{z}_i, \bar{x}_{1i}, v, w) + b_i B\bar{u}_i
\end{aligned}
\tag{7.17}
$$

where $\bar{g}_i(\bar{z}_i, \bar{x}_{1i}, v, w) = \mathrm{col}(\bar{g}_{1i}(\bar{z}_i, \bar{x}_{1i}, v, w), \cdots, \bar{g}_{ri}(\bar{z}_i, \bar{x}_{1i}, v, w))$,

$$
A_i = \begin{bmatrix} 0 & I_{r-1} \\ d_{ri} & d_{(r-1)i}, \cdots, d_{1i} \end{bmatrix}, \qquad B = \mathrm{col}(\underbrace{0, \cdots, 0}_{r-1}, 1)
$$

and $d_{ji} = b_i\Psi_i c_{(r+1-j)i}$, for $j = 1, \cdots, r$, $i = 1, \cdots, N$. Finally, like in [94], performing another coordinate transformation on $\bar{x}_i-$subsystem: $\xi_i = b_i^{-1}U_{si}\bar{x}_i$,

where $U_{si} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -d_{1i} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -d_{(r-2)i} & -d_{(r-3)i} & \cdots & 1 & 0 \\ -d_{(r-1)i} & -d_{(r-2)i} & \cdots & -d_{1i} & 1 \end{bmatrix}$ gives

$$
\begin{aligned}
\dot{\bar{z}}_i &= \bar{f}_i(\bar{z}_i, \bar{x}_{1i}, v, w), \quad i = 1, \cdots, N \\
\dot{\tilde{\eta}}_i &= M_i\tilde{\eta}_i + M_i c_{1i}\bar{x}_{1i} - \sum_{j=1}^{r} c_{ji}\bar{g}_{ji}(\bar{z}_i, \bar{x}_{1i}, v, w) \\
\dot{\xi}_i &= b_i^{-1}U_{si}(A_i\bar{x}_i + b_i B\Psi_i\tilde{\eta}_i + \bar{g}_i(\bar{z}_i, \bar{x}_{1i}, v, w) + b_i B\bar{u}_i) \\
&= U_{si}A_iU_{si}^{-1}\xi_i + U_{si}B\Psi_i\tilde{\eta}_i + b_i^{-1}U_{si}\bar{g}_i + U_{si}B\bar{u}_i \\
&= \begin{bmatrix} d_{[(r-1)i]} & I_{r-1} \\ d_{ri} & 0 \end{bmatrix}\xi_i + B\Psi_i\tilde{\eta}_i + B\bar{u}_i \\
&+ \begin{bmatrix} b_i^{-1}\bar{g}_{1i} \\ b_i^{-1}(-d_{1i}\bar{g}_{1i} + \bar{g}_{2i}) \\ \vdots \\ b_i^{-1}(-d_{(r-2)i}\bar{g}_{1i} - \cdots - d_{1i}\bar{g}_{(r-2)i} + \bar{g}_{(r-1)i}) \\ b_i^{-1}(-d_{(r-1)i}\bar{g}_{1i} - \cdots - d_{1i}\bar{g}_{(r-1)i} + \bar{g}_{ri}) \end{bmatrix} \\
&= A_c\xi_i + B\Psi_i\tilde{\eta}_i + G_i(\bar{z}_i, \bar{x}_{1i}, v, w) + B\bar{u}_i
\end{aligned}
\tag{7.18}
$$

where $A_c = \begin{bmatrix} 0 & I_{r-1} \\ 0 & 0 \end{bmatrix}$ and for $j = 1, \cdots, r$, $i = 1, \cdots, N$,

$$\begin{aligned} G_i(\bar{z}_i, \bar{x}_{1i}, v, w) &= \text{col}(G_{1i}(\bar{z}_i, \bar{x}_{1i}, v, w), \cdots, G_{ri}(\bar{z}_i, \bar{x}_{1i}, v, w)) \\ G_{1i}(\bar{z}_i, \bar{x}_{1i}, v, w) &= b_i^{-1}(d_{1i}\bar{x}_{1i} + \bar{g}_{1i}(\bar{z}_i, \bar{x}_{1i}, v, w)) \\ G_{ji}(\bar{z}_i, \bar{x}_{1i}, v, w) &= b_i^{-1}(d_{ji}\bar{x}_{1i} - \sum_{s=1}^{j-1} d_{(j-s)i}\bar{g}_{si}(\bar{z}_i, \bar{x}_{1i}, v, w) + \bar{g}_{ji}(\bar{z}_i, \bar{x}_{1i}, v, w)) \end{aligned} \tag{7.19}$$

It is noted that $U_{si}B = B$ and $U_{si}A_iU_{si}^{-1} = \begin{bmatrix} d_{[(r-1)i]} & I_{r-1} \\ d_{ri} & 0 \end{bmatrix}$ where $d_{[(r-1)i]} = \text{col}(d_{1i}, \cdots, d_{(r-1)i})$.

As shown in [94], under Assumption 7.4 as given in the next section, for each $i = 1, \cdots, N$, for any compact subsets $\mathbb{V} \subseteq \mathbb{R}^v$ and $\mathbb{W} \subseteq \mathbb{R}^{n_w}$ with $0 \in \mathbb{V}$ and $0 \in \mathbb{W}$, there is an output feedback control law of the form

$$\bar{u}_i = h_{Ki}(\hat{\xi}_i, e_i), \quad \dot{\hat{\xi}}_i = m_{Ki}(\hat{\xi}_i, e_i), \quad i = 1, \cdots, N \tag{7.20}$$

vanishing at the origin such that, for all $v(t) \in \mathbb{V} \subseteq \mathbb{R}^v$ and all $w \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$, the equilibrium point of the closed-loop system composed of (7.18) and (7.20) is globally asymptotically stable. As a result, the following control law

$$\begin{aligned} u_i &= h_{Ki}(\hat{\xi}_i, e_i) + \Psi_i\eta_i, \quad i = 1, \cdots, N \\ \dot{\hat{\xi}}_i &= m_{Ki}(\hat{\xi}_i, \eta_i, e_i) \\ \dot{\eta}_i &= M_i\eta_i + N_iu_i \end{aligned} \tag{7.21}$$

solves the global output regulation problem for system (7.1) and (7.2).

Nevertheless, as mentioned in the introduction, due to the communication constraints, the control law (7.20) is not admissible. Thus we can only use a distributed control law of the form (7.3). To find such a control law, let $\bar{\mathcal{A}} = [m_{ij}]_{i,j=0}^N$ be any weighted adjacency matrix of $\bar{\mathcal{G}}$. For $i = 1, \cdots, N$, let

$$e_{vi} = \sum_{j=0}^{N} m_{ij}(y_i - y_j) \tag{7.22}$$

Then we will consider a class of output feedback controllers as follows

$$\bar{u}_i = \bar{h}_i(\hat{\xi}_i, e_{vi}), \quad \dot{\hat{\xi}}_i = \bar{m}_i(\hat{\xi}_i, e_{vi}), \quad i = 1, \cdots, N \tag{7.23}$$

where $\bar{h}_i$ and $\bar{m}_i$ are globally defined sufficiently smooth functions that vanish at the origin. If the augmented system (7.18) can be globally stabilized by a control law of the form (7.23), then the global robust output regulation problem for system (7.1) and (7.2) is solved by the following distributed output feedback control law:

$$\begin{aligned} u_i &= \bar{h}_i(\hat{\xi}_i, e_{vi}) + \Psi_i \eta_i, \quad i = 1, \cdots, N \\ \dot{\hat{\xi}}_i &= \bar{m}_i(\hat{\xi}_i, \eta_i, e_{vi}) \\ \dot{\eta}_i &= M_i \eta_i + N_i u_i \end{aligned} \tag{7.24}$$

7.3 Solvability of Problem

In this section, we will focus on globally stabilizing the augmented system (7.18) by a control law of the form (7.23). For this purpose, let us first propose a distributed observer for the variables ξ_i, $i = 1, \cdots, N$, as follows:

$$\dot{\hat{\xi}}_i = A_c \hat{\xi}_i + \lambda(e_{vi} - \hat{\xi}_{1i}) + B\bar{u}_i, \quad i = 1, \cdots, N \tag{7.25}$$

where $\lambda = \mathrm{col}(\lambda_1, \cdots, \lambda_r)$ is chosen such that the matrix $A_0 = \begin{bmatrix} -\lambda_{[r-1]} & I_{r-1} \\ -\lambda_r & 0 \end{bmatrix}$ is Hurwitz. System (7.25) is called a distributed observer because it depends on e_{vi} instead of e_i. It can be verified that the observation error $\tilde{\xi}_i = \xi_i - \hat{\xi}_i$, $i = 1, \cdots, N$, satisfies

$$\begin{aligned} \dot{\tilde{\xi}}_i &= \dot{\xi}_i - \dot{\hat{\xi}}_i \\ &= A_c \xi_i + B\Psi_i \tilde{\eta}_i + G_i(\bar{z}_i, \bar{x}_{1i}, v, w) + B\bar{u}_i - A_c \hat{\xi}_i - \lambda(e_{vi} - \hat{\xi}_{1i}) - B\bar{u}_i \\ &= A_c \tilde{\xi}_i + B\Psi_i \tilde{\eta}_i + G_i(\bar{z}_i, \bar{x}_{1i}, v, w) - \lambda(e_{vi} - \xi_{1i} + \tilde{\xi}_{1i}) \\ &= (A_c \tilde{\xi}_i - \lambda \tilde{\xi}_{1i}) + B\Psi_i \tilde{\eta}_i + G_i(\bar{z}_i, \bar{x}_{1i}, v, w) + \lambda(b_i^{-1} \bar{x}_{1i} - e_{vi}) \\ &= A_0 \tilde{\xi}_i + \lambda(b_i^{-1} e_i - e_{vi}) + B\Psi_i \tilde{\eta}_i + G_i(\bar{z}_i, \bar{x}_{1i}, v, w) \end{aligned} \tag{7.26}$$

Attaching (7.26) to (7.18) and replacing the state variable vector ξ_i by $(e_i, \hat{\xi}_{2i}, \cdots, \hat{\xi}_{ri})$ gives the following system

$$\begin{aligned} \dot{\bar{z}}_i &= \bar{f}_i(\bar{z}_i, e_i, v, w) \\ \dot{\tilde{\eta}}_i &= M_i \tilde{\eta}_i + M_i c_{1i} e_i - \sum_{j=1}^{r} c_{ji} \bar{g}_{ji}(\bar{z}_i, e_i, v, w) \\ \dot{\tilde{\xi}}_i &= A_0 \tilde{\xi}_i + \lambda(b_i^{-1} e_i - e_{vi}) + B\Psi_i \tilde{\eta}_i + G_i(\bar{z}_i, e_i, v, w) \\ \dot{e}_i &= b_i \dot{\xi}_{1i} = b_i(\xi_{2i} + G_{1i}(\bar{z}_i, e_i, v, w)) \\ &= b_i \tilde{\xi}_{2i} + b_i \hat{\xi}_{2i} + b_i G_{1i}(\bar{z}_i, e_i, v, w) \\ \dot{\hat{\xi}}_{si} &= \hat{\xi}_{(s+1)i} + \lambda_s(e_{vi} - \hat{\xi}_{1i}), \quad s = 2, \cdots, r-1 \\ \dot{\hat{\xi}}_{ri} &= \bar{u}_i + \lambda_r(e_{vi} - \hat{\xi}_{1i}), \quad i = 1, \cdots, N \end{aligned} \tag{7.27}$$

Due to the employment of the distributed observer (7.25), system (7.27) is a coupled multi-input system. The problem of the global stabilization of a system of the form (7.27) has never been encountered before. Nevertheless, it is still possible to treat system (7.27) as a block lower triangular system. For this purpose, let $\bar{z} = \mathrm{col}(\bar{z}_1, \cdots, \bar{z}_N)$, $\tilde{\eta} = \mathrm{col}(\tilde{\eta}_1, \cdots, \tilde{\eta}_N)$, $\tilde{\xi} = \mathrm{col}(\tilde{\xi}_1, \cdots, \tilde{\xi}_N)$, $\hat{\xi}_s = \mathrm{col}(\hat{\xi}_{s1}, \cdots, \hat{\xi}_{sN})$, $s = 2, \cdots, r$, $M = \mathrm{diag}(M_1, \cdots, M_N)$, $\bar{A}_0 = \mathrm{diag}(A_0, \cdots, A_0)$, $\bar{\Psi} = \mathrm{diag}(B\Psi_1, \cdots, B\Psi_N)$, $b = \mathrm{diag}(b_1, \cdots, b_N)$, $\bar{\xi}_1 = \mathrm{col}(\tilde{\xi}_{11}, \cdots, \tilde{\xi}_{1N})$, $e_v = \mathrm{col}(e_{v1}, \cdots, e_{vN})$, and $\bar{u} = \mathrm{col}(\bar{u}_1, \cdots, \bar{u}_N)$. Then (7.27) can be put into the following compact from:

$$
\begin{aligned}
\dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\
\dot{\tilde{\eta}} &= M\tilde{\eta} + f_\eta(\bar{z}, e, v, w) \\
\dot{\tilde{\xi}} &= \bar{A}_0\tilde{\xi} + \bar{\Psi}\tilde{\eta} + f_\xi(\bar{z}, e, v, w) \\
\dot{e} &= b\hat{\xi}_2 + J_1(\bar{z}, \tilde{\xi}, e, v, w) \\
\dot{\hat{\xi}}_s &= \hat{\xi}_{s+1} + J_s(\tilde{\xi}, e, \hat{\xi}_2, \cdots, \hat{\xi}_s, v, w), \quad s = 2, \cdots, r-1 \\
\dot{\hat{\xi}}_r &= \bar{u} + J_r(\tilde{\xi}, e, \hat{\xi}_2, \cdots, \hat{\xi}_r, v, w)
\end{aligned}
\tag{7.28}
$$

where, for $i = 1, \cdots, N$, $s = 2, \cdots, r$,

$$
\begin{aligned}
\bar{f}(\bar{z}, e, v, w) &= \mathrm{col}(\bar{f}_1(\bar{z}_1, e_1, v, w), \cdots, \bar{f}_N(\bar{z}_N, e_N, v, w)) \\
f_\eta(\bar{z}, e, v, w) &= \begin{bmatrix} M_1 c_{11} e_1 - \sum_{j=1}^{r} c_{j1}\bar{g}_{j1}(\bar{z}_1, e_1, v, w) \\ \vdots \\ M_N c_{1N} e_N - \sum_{j=1}^{r} c_{jN}\bar{g}_{jN}(\bar{z}_N, e_N, v, w) \end{bmatrix} \\
f_\xi(\bar{z}, e, v, w) &= \begin{bmatrix} \lambda(b_1^{-1}e_1 - e_{v1}) + G_1(\bar{z}_1, e_1, v, w) \\ \vdots \\ \lambda(b_N^{-1}e_N - e_{vN}) + G_N(\bar{z}_N, e_N, v, w) \end{bmatrix} \\
J_1(\bar{z}, \tilde{\xi}, e, v, w) &= \mathrm{col}(J_{11}, \cdots, J_{1N}) \\
J_{1i}(\bar{z}, \tilde{\xi}, e, v, w) &= b_i\tilde{\xi}_{2i} + b_i G_{1i}(\bar{z}_i, e_i, v, w) \\
J_s(\tilde{\xi}, e, \hat{\xi}_2, \cdots, \hat{\xi}_s, v, w) &= \lambda_s(e_v - b^{-1}e + \bar{\xi}_1)
\end{aligned}
\tag{7.29}
$$

It can be seen that if (7.27) can be globally stabilized by a control law of the form (7.23), then the output regulation problem of the original system (7.1) will be solved by the distributed control law (7.24). For this purpose, we need some assumptions and lemmas.

Assumption 7.4 *For any compact subset $\Omega \subset \mathbb{R}^{n_v} \times \mathbb{W}$, there exists a C^1 function $V_{\bar{z}_i}$ satisfying $\underline{\iota}_i(||\bar{z}_i||) \leq V_{\bar{z}_i}(\bar{z}_i) \leq \bar{\iota}_i(||\bar{z}_i||)$, for some class $\mathcal{K}_\infty$ functions $\underline{\iota}_i(\cdot)$ and $\bar{\iota}_i(\cdot)$ such that, for any $(v, w) \in \Omega$ along the trajectory of the subsystem $\dot{\bar{z}}_i = \bar{f}_i(\bar{z}_i, e_i, v, w)$,*

$\dot{V}_{\bar{z}_i} \leq -\alpha_i(||\bar{z}_i||) + \gamma_i(e_i)$, *where* $\alpha_i(\cdot)$ *is some known class* $\mathcal{K}_\infty$ *function satisfying* $\lim_{s\to 0^+}\sup(\alpha_i^{-1}(s^2)/s) < \infty$, *and* $\gamma_i(\cdot)$ *is some known smooth positive definite function.*

Remark 7.2 *The* $(\bar{z}, \tilde{\eta}, \tilde{\xi})-$*subsystem in (7.28) can be put as follows:*

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\ \begin{bmatrix} \dot{\tilde{\eta}} \\ \dot{\tilde{\xi}} \end{bmatrix} &= \begin{bmatrix} M & 0 \\ \bar{\Psi} & \bar{A}_0 \end{bmatrix} \begin{bmatrix} \tilde{\eta} \\ \tilde{\xi} \end{bmatrix} + \begin{bmatrix} f_\eta(\bar{z}, e, v, w) \\ f_\xi(\bar{z}, e, v, w) \end{bmatrix} \end{aligned} \tag{7.30}$$

Let $V_{\bar{z}} = \sum_{i=1}^{N} V_{\bar{z}_i}$. *Under Assumption 7.4,* $V_{\bar{z}}$ *satisfies* $\underline{\iota}(||\bar{z}||) \leq V_{\bar{z}}(\bar{z}) \leq \bar{\iota}(||\bar{z}||)$ *for some class* $\mathcal{K}_\infty$ *functions* $\underline{\iota}(\cdot)$ *and* $\bar{\iota}(\cdot)$, *and*

$$\dot{V}_{\bar{z}} \leq \sum_{i=1}^{N}(-\alpha_i(||\bar{z}_i||) + \gamma_i(e_i)) \leq -\alpha(||\bar{z}||) + \gamma(e)$$

for some known class $\mathcal{K}_\infty$ *function* $\alpha(\cdot)$ *satisfying* $\lim_{s\to 0^+}\sup(\alpha^{-1}(s^2)/s) < \infty$, *and some known smooth positive definite function* $\gamma(\cdot)$.

By changing supply rate technique[69], given any smooth function $\Delta_1(\bar{z})$, *there exists a* C^1 *function* $\bar{V}_{\bar{z}}$ *satisfying* $\underline{\varpi}(||\bar{z}||) \leq \bar{V}_{\bar{z}}(\bar{z}) \leq \overline{\varpi}(||\bar{z}_i||)$, *for some class* $\mathcal{K}_\infty$ *functions* $\underline{\varpi}(\cdot)$ *and* $\overline{\varpi}(\cdot)$ *such that, for any* $(v, w) \in \Omega$ *along the trajectory of the subsystem* $\dot{\bar{z}} = \bar{f}(\bar{z}, e, v, w)$, $\dot{\bar{V}}_{\bar{z}} \leq -\Delta_1(\bar{z})||\bar{z}||^2 + \Delta_2(e)||e||^2$, *for some known smooth function* $\Delta_2(e) \geq 1$. $\begin{bmatrix} M & 0 \\ \bar{\Psi} & \bar{A}_0 \end{bmatrix}$ *is Hurwitz since* M *and* $\bar{A}_0$ *are both Hurwitz, thus there exists* $\bar{P}$ *such that* $\begin{bmatrix} M & 0 \\ \bar{\Psi} & \bar{A}_0 \end{bmatrix}^T \bar{P} + \bar{P}\begin{bmatrix} M & 0 \\ \bar{\Psi} & \bar{A}_0 \end{bmatrix} \leq -2I$. *Let* $\tilde{z} = \begin{bmatrix} \tilde{\eta} \\ \tilde{\xi} \end{bmatrix}$ *and* $V_Z(\bar{z}, \tilde{\eta}, \tilde{\xi}) = \bar{V}_{\bar{z}} + \tilde{z}^T\bar{P}\tilde{z}$. *Then*

$$\begin{aligned} \dot{V}_Z(\bar{z}, \tilde{z}) &= \dot{\bar{V}}_{\bar{z}} + \dot{\tilde{z}}^T\bar{P}\tilde{z} + \tilde{z}^T\bar{P}\dot{\tilde{z}} \\ &= \dot{\bar{V}}_{\bar{z}} + \tilde{z}^T(\begin{bmatrix} M & 0 \\ \bar{\Psi} & \bar{A}_0 \end{bmatrix}^T \bar{P} + \bar{P}\begin{bmatrix} M & 0 \\ \bar{\Psi} & \bar{A}_0 \end{bmatrix})\tilde{z} + 2\tilde{z}^T\bar{P}\begin{bmatrix} f_\eta(\bar{z}, e, v, w) \\ f_\xi(\bar{z}, e, v, w) \end{bmatrix} \\ &\leq -\Delta_1(\bar{z})||\bar{z}||^2 + \Delta_2(e)||e||^2 - 2||\tilde{z}||^2 + ||\tilde{z}||^2 + ||\bar{P}\begin{bmatrix} f_\eta(\bar{z}, e, v, w) \\ f_\xi(\bar{z}, e, v, w) \end{bmatrix}||^2 \end{aligned} \tag{7.31}$$

Since $f_\eta(\bar{z}, e, v, w)$ *and* $f_\xi(\bar{z}, e, v, w)$ *are both sufficiently smooth with* $f_\eta(0, 0, v, w) = 0$ *and* $f_\xi(0, 0, v, w) = 0$, *by Lemma 7.8 of [28], there exist some known smooth functions* $\delta_1(\bar{z}) \geq 1$ *and* $\delta_2(e) \geq 1$, *such that for all* $\bar{z} \in \mathbb{R}^{n_1+\cdots+n_N}$, $e \in \mathbb{R}^N$ *and* $(v, w) \in \Omega$,

$$||\begin{bmatrix} f_\eta(\bar{z}, e, v, w) \\ f_\xi(\bar{z}, e, v, w) \end{bmatrix}||^2 \leq \delta_1(\bar{z})||\bar{z}||^2 + \delta_2(e)||e||^2$$

Then,

$$\begin{aligned}\dot{V}_Z(\bar{z},\tilde{z}) &\leq -\Delta_1(\bar{z})||\bar{z}||^2 + \Delta_2(e)||e||^2 - ||\tilde{z}||^2 + ||\bar{P}||^2\delta_1(\bar{z})||\bar{z}||^2 + ||\bar{P}||^2\delta_2(e)||e||^2 \\ &= -(\Delta_1(\bar{z}) - ||\bar{P}||^2\delta_1(\bar{z}))||\bar{z}||^2 - ||\tilde{z}||^2 + (\Delta_2(e) + ||\bar{P}||^2\delta_2(e))||e||^2\end{aligned} \tag{7.32}$$

Letting $\Delta_1(\bar{z}) \geq ||\bar{P}||^2\delta_1(\bar{z}) + 1$ *and* $\pi(e) = \Delta_2(e) + ||\bar{P}||^2\delta_2(e)$ *gives*

$$\dot{V}_Z(\bar{z},\tilde{\eta},\tilde{\xi}) \leq -||\bar{z}||^2 - ||\tilde{\eta}||^2 - ||\tilde{\xi}||^2 + \pi(e) \tag{7.33}$$

To introduce the next assumption, define a subgraph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ of $\bar{\mathcal{G}}$ where $\mathcal{V} = \{1,\cdots,N\}$, and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is obtained from $\bar{\mathcal{E}}$ by removing all edges between the node 0 and the nodes in $\mathcal{V}$.

Assumption 7.5 *Every node* $i = 1,\cdots,N$ *is reachable from the node 0 in the digraph* $\bar{\mathcal{G}}$*, and* $\mathcal{G}$ *is an undirected graph.*

Remark 7.3 *Let* $H = [h_{ij}]_{i,j=1}^N$ *with* $h_{ii} = \sum_{i=0}^N m_{ij}$ *and* $h_{ij} = -m_{ij}$*. Then it can be seen that* $e_v = He$*. Moreover, by Lemma 4 in [27], all the eigenvalues of* H *have positive real parts if and only if Assumption 7.5 is satisfied. Since* $\mathcal{G}$ *is an undirected graph,* H *is symmetric positive definite.*

Before introducing our main result, we need to establish one more lemma.

Lemma 7.1 *Consider the following system*

$$\begin{aligned}\dot{\zeta} &= \bar{F}(\zeta,\varphi,\mu(t)) \\ \dot{\varphi} &= \bar{J}(\zeta,\varphi,\mu(t)) + b(\mu(t))u\end{aligned} \tag{7.34}$$

where $\zeta \in \mathbb{R}^{n_0}$*,* $\varphi = col(\varphi_1,\cdots,\varphi_N) \in \mathbb{R}^N$*,* $u = col(u_1,\cdots,u_N) \in \mathbb{R}^N$*,* $\mu(t) \in \Omega \subseteq \mathbb{R}^{n_\mu}$ *with* Ω *being some compact subset,* $\bar{F}(\zeta,\varphi,\mu(t))$ *and* $\bar{J}(\zeta,\varphi,\mu(t))$ *are sufficiently smooth with* $\bar{F}(0,0,\mu(t)) = 0$ *and* $\bar{J}(0,0,\mu(t)) = 0$ *for all* $\mu(t) \in \Omega$*.* $b(\mu(t)) \in \mathbb{R}^{N\times N}$ *is a diagonal matrix with the ith diagonal element* $b_i(\cdot)$ *a continuous function of* μ *satisfying* $b_i(\mu(t)) > 0$ *for all* $\mu(t) \in \Omega$*. Assume that there exists a* C^1 *function* $\bar{U}_0(\zeta)$ *satisfying* $\underline{\alpha}_0(||\zeta||) \leq \bar{U}_0(\zeta) \leq \overline{\alpha}_0(||\zeta||)$ *for some* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}_0(\cdot)$ *and* $\overline{\alpha}_0(\cdot)$*, such that, for all* $\mu(t) \in \Omega$*, along the trajectory* $\dot{\zeta} = \bar{F}(\zeta,\varphi,\mu(t))$*,*

$$\dot{\bar{U}}_0(\zeta) \leq -\alpha_0(||\zeta||) + \gamma_0(\varphi) \tag{7.35}$$

where $\alpha_0(\cdot)$ *is some class* $\mathcal{K}_\infty$ *function with* $\lim_{s\to 0^+}\sup(\alpha_0^{-1}(s^2)/s) < +\infty$*, and* $\gamma_0(\cdot)$ *is some known smooth positive definition function.*

Let $\varphi_v = P\varphi$ *where* $\varphi_v = col(\varphi_{v1}, \cdots, \varphi_{vN}) \in \mathbb{R}^N$ *and* $P \in \mathbb{R}^{N\times N}$ *is a positive definite and symmetric matrix. Then there exists a controller of the form*

$$u_i = -\rho_{1i}(\varphi_{vi})\varphi_{vi} + \nu_i, \quad i = 1, \cdots, N \tag{7.36}$$

where $\rho_{1i}(\cdot)$, $1 = 1, \cdots, N$ *are positive smooth functions,* $\nu_i \in \mathbb{R}$*, and a* C^1 *function* $\bar{U}_1(\zeta, \varphi)$ *satisfying*

$$\underline{\beta}_0(||(\zeta,\varphi)||) \leq \bar{U}_1(\zeta,\varphi) \leq \overline{\beta}_0(||(\zeta,\varphi)||) \tag{7.37}$$

for some class $\mathcal{K}_\infty$ *functions* $\underline{\beta}_0(\cdot)$ *and* $\overline{\beta}_0(\cdot)$*, such that, for all* $\mu(t) \in \Omega$*, along the trajectory of (7.34) and (7.36),*

$$\dot{\bar{U}}_1(\zeta,\varphi) \leq -||\zeta||^2 - a||\varphi||^2 + \sum_{i=1}^{N} \nu_i^2, \quad \text{for some } a > 0 \tag{7.38}$$

Proof: By the changing supply rate technique [69], given any smooth function $\vartheta(\zeta) > 0$, there exists a C^1 function $V_\zeta(\zeta)$ satisfying $\underline{\alpha}_4(||\zeta||) \leq V_\zeta(\zeta) \leq \overline{\alpha}_4(||\zeta||)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_4(\cdot)$ and $\overline{\alpha}_4(\cdot)$, such that, for all $\mu(t) \in \Omega$, along the trajectory of $\zeta-$subsystem in (7.34),

$$\dot{V}_\zeta(\zeta) \leq -\vartheta(\zeta)||\zeta||^2 + \varrho(\varphi)||\varphi||^2 \tag{7.39}$$

where $\varrho(\cdot)$ is some known smooth positive definite function.

Since $\varphi = P^{-1}\varphi_v$, then

$$\begin{aligned} \dot{V}_\zeta(\zeta) &\leq -\vartheta(\zeta)||\zeta||^2 + \varrho(P^{-1}\varphi_v)||P^{-1}\varphi_v||^2 \\ &\leq -\vartheta(\zeta)||\zeta||^2 + \tilde{\varrho}(\varphi_v)||\varphi_v||^2 \end{aligned} \tag{7.40}$$

where $\tilde{\varrho}(\cdot)$ is some known smooth positive definite function.

By Lemma 7.8 in [28], there exist known smooth definite positive functions $\bar{\varrho}_i(\varphi_{vi})$, $i = 1, \cdots, N$, such that $\tilde{\varrho}(\varphi_v)||\varphi_v||^2 \leq \sum_{i=1}^{N} \bar{\varrho}_i(\varphi_{vi})|\varphi_{vi}|^2$. Then

$$\dot{V}_\zeta(\zeta) \leq -\vartheta(\zeta)||\zeta||^2 + \sum_{i=1}^{N} \bar{\varrho}_i(\varphi_{vi})|\varphi_{vi}|^2 \tag{7.41}$$

Since $b_i(\mu(t)) > 0$ for all $\mu(t) \in \Omega$, there exist $b_{\min}$ and $b_{\max}$ such that $0 < b_{\min} \leq b_i(\mu(t)) \leq b_{\max} < +\infty$ for all $\mu(t) \in \Omega$.

Let $\bar{J}(\zeta,\varphi,\mu(t)) = \begin{bmatrix} \bar{J}_1(\zeta,\varphi,\mu(t)) \\ \vdots \\ \bar{J}_N(\zeta,\varphi,\mu(t)) \end{bmatrix}$. Since $\bar{J}_i(\zeta,\varphi,\mu(t))$, $i = 1, \cdots, N$, are all smooth and satisfy $\bar{J}_i(0,0,\mu(t)) = 0$ for all $\mu(t) \in \Omega$, by Lemma 7.8 in [28], there exist some smooth functions $\varpi_i(\zeta) \geq 1$ and $\sigma_{ji}(\varphi_{vj}) \geq 1$, $j = 1, \cdots, N$, such that for any $\zeta \in \mathbb{R}^{n_0}$, $\varphi_{vi} \in \mathbb{R}$ and $\mu(t) \in \Omega$,

$$\begin{aligned}|\bar{J}_i(\zeta, \varphi, \mu(t))| &= |\bar{J}_i(\zeta, P^{-1}\varphi_v, \mu(t))| \\ &\leq \varpi_i(\zeta)||\zeta|| + \sum_{j=1}^{N} \sigma_{ji}(\varphi_{vj})|\varphi_{vj}|\end{aligned}$$

Thus,

$$\begin{aligned}|\varphi_{vi}||\bar{J}_i(\zeta, \varphi, \mu(t))| &\leq |\varphi_{vi}|(\varpi_i(\zeta)||\zeta|| + \sum_{j=1}^{N} \sigma_{ji}(\varphi_{vj})|\varphi_{vj}|) \\ &\leq (\frac{1}{4}|\varphi_{vi}|^2 + \varpi_i^2(\zeta)||\zeta||^2) + \sum_{j=1}^{N}(\frac{1}{4}|\varphi_{vi}|^2 + \sigma_{ji}^2(\varphi_{vj})|\varphi_{vj}|^2) \\ &= \frac{1+N}{4}|\varphi_{vi}|^2 + \varpi_i^2(\zeta)||\zeta||^2 + \sum_{j=1}^{N} \sigma_{ji}^2(\varphi_{vj})|\varphi_{vj}|^2\end{aligned} \tag{7.42}$$

Let $V_\varphi = \frac{1}{2}\varphi_v^T P^{-1}\varphi_v$. Then the derivative of V_φ along the subsystem $\dot{\varphi}_i = \bar{J}_i(\zeta, \varphi, \mu(t)) + b_i(\nu_i - \rho_{1i}(\varphi_{vi})\varphi_{vi})$ satisfies

$$\begin{aligned}\dot{V}_\varphi &= \varphi_v^T P^{-1}\dot{\varphi}_v = \varphi_v^T\dot{\varphi} = \sum_{i=1}^{N} \varphi_{vi}\dot{\varphi}_i \\ &= \sum_{i=1}^{N} -b_i\rho_{1i}(\varphi_{vi})\varphi_{vi}^2 + \sum_{i=1}^{N} b_i\varphi_{vi}\nu_i + \sum_{i=1}^{N} \varphi_{vi}\bar{J}_i(\zeta, \varphi, \mu(t)) \\ &\leq \sum_{i=1}^{N} -(b_{\min}\rho_{1i}(\varphi_{vi}) - \frac{b_{max}^2}{4})|\varphi_{vi}|^2 + \sum_{i=1}^{N} |\nu_i|^2 + \sum_{i=1}^{N} |\varphi_{vi}||\bar{J}_i(\zeta, \varphi, \mu(t))|\end{aligned} \tag{7.43}$$

Thus

$$\begin{aligned}\dot{V}_\varphi = &\sum_{i=1}^{N} -(b_{\min}\rho_{1i}(\varphi_{vi}) - \frac{b_{max}^2}{4} - \frac{1+N}{4} - \sum_{j=1}^{N} \sigma_{ij}^2(\varphi_{vi}))|\varphi_{vi}|^2 \\ &+ \sum_{i=1}^{N} \varpi_i^2(\zeta)||\zeta||^2 + \sum_{i=1}^{N} |\nu_i|^2\end{aligned} \tag{7.44}$$

Let $\bar{U}_1 = V_\zeta + V_\varphi$. Then the derivative of the trajectory of $(\zeta, \varphi)-$system satisfies

$$\begin{aligned}\dot{\bar{U}}_1 \leq &\sum_{i=1}^{N} -(b_{\min}\rho_{1i}(\varphi_{vi}) - \frac{b_{max}^2}{4} - \frac{1+N}{4} - \sum_{j=1}^{N} \sigma_{ij}^2(\varphi_{vi}) \\ &- \bar{\varrho}_i(\varphi_{vi}))|\varphi_{vi}|^2 - (\vartheta(\zeta) - \sum_{i=1}^{N} \varpi_i^2(\zeta))||\zeta||^2 + \sum_{i=1}^{N} |\nu_i|^2\end{aligned} \tag{7.45}$$

Let $\vartheta(\zeta) \geq \sum_{i=1}^{N} \varpi_i^2(\zeta) + 1$ and $\rho_{1i}(\varphi_{vi}) \geq b_{\min}^{-1}(\frac{b_{max}^2}{4} + \frac{5+N}{4} + \sum_{j=1}^{N} \sigma_{ij}^2(\varphi_{vi}) + \bar{\varrho}_i(\varphi_{vi}))$. Then

$$\begin{aligned}\dot{U}_1 &\leq -||\zeta||^2 - ||\varphi_v||^2 + \sum_{i=1}^{N} |\nu_i|^2 \\ &\leq -||\zeta||^2 - ||P\varphi||^2 + \sum_{i=1}^{N} |\nu_i|^2 \\ &\leq -||\zeta||^2 - \lambda_m^2(P)||\varphi||^2 + \sum_{i=1}^{N} |\nu_i|^2\end{aligned} \tag{7.46}$$

with $\lambda_m(P) > 0$ denoting the smallest eigenvalue of P.

Lemma 7.2 *Under Assumptions 7.1 to 7.3, 7.4 and 7.5, the global stabilization problem of system (7.28) can be solved by the distributed output feedback control law of the form*

$$\bar{u}_i = \alpha_{ri}(\check{\xi}_{ri}), \quad i = 1, \cdots, N \tag{7.47}$$

where $\alpha_{ri}(\cdot)$ *is recursively defined by*

$$\begin{aligned}&\check{\xi}_{2i} = \hat{\xi}_{2i} + \rho_{1i}(e_{vi})e_{vi}, \quad i = 1, \cdots, N \\ &\alpha_{si}(\check{\xi}_{si}) = -\rho_{si}(\check{\xi}_{si})\check{\xi}_{si}, \quad s = 2, \cdots, r \\ &\check{\xi}_{(s+1)i} = \hat{\xi}_{(s+1)i} - \alpha_{si}(\check{\xi}_{si}), \quad s = 2, \cdots, r-1\end{aligned} \tag{7.48}$$

Proof: Let $Z = \text{col}(\bar{z}, \tilde{\eta}, \tilde{\xi})$. Then the $(Z, e)-$subsystem in (7.28) can be put into the compact form

$$\begin{aligned}\dot{Z} &= F(Z, e, v, w) \\ \dot{e} &= b\hat{\xi}_2 + J_1(Z, e, v, w)\end{aligned} \tag{7.49}$$

with $F(Z, e, v, w) = \begin{bmatrix} \bar{f}(\bar{z}, e, v, w) \\ M\tilde{\eta} + f_\eta(\bar{z}, e, v, w) \\ \bar{A}_0\tilde{\xi} + \bar{\Psi}\tilde{\eta} + f_\xi(\bar{z}, e, v, w) \end{bmatrix}$. Then (7.33) implies that

$$\dot{V}_Z \leq -||Z||^2 + \pi(e) \tag{7.50}$$

for some known smooth positive definite function $\pi(e)$.

It is noted that (7.49) is in the form of (7.34) with $\zeta = Z$, $\varphi = e$, $\varphi_v = e_v$ and $u = \hat{\xi}_2$, and $Z-$subsystem satisfies the condition (7.35). Then, by Lemma 7.1, with $P = H$, there exists

$$\hat{\xi}_{2i} = -\rho_{1i}(e_{vi})e_{vi} + \check{\xi}_{2i}, \quad i = 1, \cdots, N \tag{7.51}$$

and a C^1 function $U_1(Z,e)$ satisfying $\underline{\beta}_1(||(Z,e)||) \le U_1(Z,e) \le \overline{\beta}_1(||(Z,e)||)$ for some class $\mathcal{K}_\infty$ functions $\underline{\beta}_1(\cdot)$ and $\overline{\beta}_1(\cdot)$, such that, for all $(v,w) \in \Omega$, along the trajectory of (7.49) and (7.51),

$$\dot{U}_1(Z,e) \le -||Z||^2 - a||e||^2 + \sum_{i=1}^{N} \check{\xi}_{2i}^2 \tag{7.52}$$

Let $Z_e = \mathrm{col}(Z,e)$. Then the system (7.28) can be put in the compact form as follows:

$$\begin{aligned} \dot{Z}_e &= F_e(Z_e, \hat{\xi}_2, v, w) \\ \dot{\hat{\xi}}_s &= \hat{\xi}_{s+1} + J_s(Z_e, \hat{\xi}_2, \cdots, \hat{\xi}_s, v, w), \quad s = 2, \cdots, r-1 \\ \dot{\hat{\xi}}_r &= \bar{u} + J_r(Z_e, \hat{\xi}_2, \cdots, \hat{\xi}_r, v, w) \end{aligned} \tag{7.53}$$

with $F_e(Z_e, \hat{\xi}_2, v, w) = \begin{bmatrix} F(Z,e,v,w) \\ b\hat{\xi}_2 + J_1(Z,e,v,w) \end{bmatrix}$.

Let $X_1 = Z_e$, $X_s = \mathrm{col}(X_{s-1}, \check{\xi}_s)$, $s = 2, \cdots, r$, and $b_m = \min\{1, a\} > 0$. Then (7.52) implies that

$$\dot{U}_1(Z_e) \le -b_m||Z_e||^2 + \sum_{i=1}^{N} \check{\xi}_{2i}^2 \tag{7.54}$$

It is noted that the X_s-subsystem, $s = 2, \cdots, r$, also has the form of (7.34) with $\zeta = X_{s-1}$, $\varphi = \check{\xi}_s$, $P = I_{N\times N}$ and $b(\mu(t)) = I_{N\times N}$, and $X_{s-1}-$subsystem satisfies the condition (7.35). Recursively applying Lemma 7.1 to the X_s-subsystem shows the existence of smooth functions $\rho_{si}(\check{\xi}_{si})$ and C^1 functions $U_s(X_s)$ satisfying $\underline{\beta}_s(||X_s||) \le U_s(X_s) \le \overline{\beta}_s(||X_s||)$ for some class $\mathcal{K}_\infty$ functions $\underline{\beta}_s(\cdot)$ and $\overline{\beta}_s(\cdot)$, such that, under the following virtual control law

$$\hat{\xi}_{(s+1)i} = \check{\xi}_{(s+1)i} - \rho_{si}(\check{\xi}_{si})\check{\xi}_{si}, \quad s = 2, \cdots, r-1, \qquad i = 1, \cdots, N \tag{7.55}$$

$U_s(X_s)$ satisfies, for all $(v,w) \in \Omega$,

$$\dot{U}_s \le -||X_s||^2 + ||\check{\xi}_{s+1}||^2 \tag{7.56}$$

Letting $\check{\xi}_{(r+1)i} = 0$, $i = 1, \cdots, N$, in (7.56) yields $\dot{U}_r \le -||X_r||^2$. That is, under the output feedback control law (7.47), the equilibrium of the closed-loop system is uniformly globally robustly asymptotically stable.

Thus we can get our main result as follows.

Theorem 7.1 *Under Assumptions 7.1 to 7.3, 7.4 and 7.5, the cooperative global robust output regulation problem of system (7.1) and (7.2) can be solved by the distributed output* *feedback control law of the form*

$$
\begin{aligned}
u_i &= \alpha_{ri}(\check{\xi}_{ri}) + \Psi_i \eta_i, \quad i = 1, \cdots, N \\
\dot{\hat{\xi}}_i &= A_c \hat{\xi}_i + \lambda(e_{vi} - \hat{\xi}_{1i}) + B(u_i - \Psi_i \eta_i) \\
\dot{\eta}_i &= M_i \eta_i + N_i u_i
\end{aligned}
\tag{7.57}
$$

Remark 7.4 *The main technical challenge of this section is that we need to globally stabilize a coupled multi-input system (7.27) by a distributed control law of the form (7.23). It is noted that the global stabilization problem of a system of the form (7.27) has not even been studied by a centralized control law before. By establishing Lemmas 7.1 and 7.2 here, we have managed to overcome the difficulty in globally stabilizing the coupled multi-input system (7.27) by a distributed control law (7.57).*

7.4 Application to Hyper-Chaotic Lorenz Multi-agent Systems

Consider a group of Hyper-chaotic Lorenz systems taken from [94]:

$$
\begin{aligned}
\dot{z}_{1i} &= a_{11i} z_{1i} + a_{12i} x_{1i} \\
\dot{z}_{2i} &= a_{3i} z_{2i} + z_{1i} x_{1i} \\
\dot{x}_{1i} &= x_{2i} + a_{21i} z_{1i} + a_{22i} x_{1i} - z_{1i} z_{2i} \\
\dot{x}_{2i} &= b_i u_i + a_{4i} z_{1i} \\
e_i &= x_{1i} - v_1, \quad i = 1, \cdots, 5.
\end{aligned}
\tag{7.58}
$$

The exosystem is in the form (7.2) with $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The interconnection among various subsystems is determined by Figure 7.1. To make our problem more interesting, as in

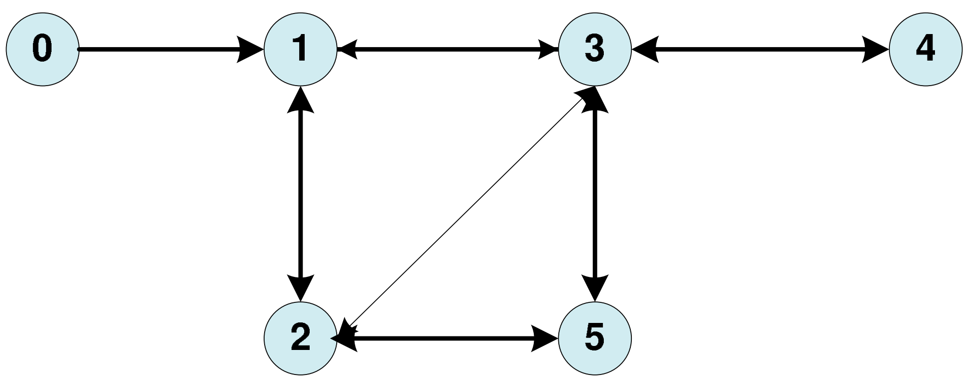

Figure 7.1: The network topology

[94], we allow the parameters $(a_{11i}, a_{12i}, a_{21i}, a_{22i}, a_{3i}, a_{4i})$ and b_i to undergo some perturbation. To be more specific, let $(a_{11i}, a_{12i}, a_{21i}, a_{22i}, a_{3i}, a_{4i}) = (\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3, \bar{a}_4) + (0.5w_{1i}, 0.5w_{2i}, 0.5w_{3i}, 0.1w_{4i}, 0.1w_{5i}, 0.1w_{6i})$ and $b_i = (0.5 + w_{7i})^2 * (0.5 + 0.1 * i)$, $i = 1, \cdots, 5$, where $(\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3, \bar{a}_4) = (-10, 10, 28, -1, -8/3, -1)$ represents the normal value of $(a_{11i}, a_{12i}, a_{21i}, a_{22i}, a_{3i}, a_{4i})$ and $w_{ij} \in [0, 1]$ for $j = 1, \cdots, 7$, $i = 1, \cdots, 5$, represents the uncertainty.

From Figure 7.1, Assumption 7.5 is satisfied. It was shown in [94] that, for each $i = 1, \cdots, 5$, system (7.58) and the exosystem satisfy Assumptions 7.1 to 7.3 and 7.4. In particular, the solution of the regulator equations of (7.58) is

$$\begin{aligned}
\mathbf{x}_{1i}(v, w) &= v_1, \quad \mathbf{z}_{1i}(v, w) = r_{11i}v_1 + r_{12i}v_2 \\
\mathbf{z}_{2i}(v, w) &= r_{21i}v_1^2 + r_{22i}v_1v_2 + r_{23i}v_2^2 \\
\mathbf{x}_{2i}(v, w) &= r_{31i}v_1 + r_{32i}v_2 + r_{33i}v_1^3 + r_{34i}v_1^2v_2 + r_{35i}v_1v_2^2 + r_{36i}v_2^3 \\
\mathbf{u}_i(v, w) &= r_{41i}v_1 + r_{42i}v_2 + r_{43i}v_1^3 + r_{44i}v_1^2v_2 + r_{45i}v_1v_2^2 + r_{46i}v_2^3
\end{aligned} \tag{7.59}$$

where

$$\begin{aligned}
r_{11i} &= -\frac{a_{11i}a_{12i}}{1 + a_{11i}^2}, \quad r_{12i} = -\frac{a_{12i}}{1 + a_{11i}^2} \\
r_{21i} &= -\frac{r_{11i} + r_{22i}}{a_{3i}}, \quad r_{22i} = -\frac{r_{12i}a_{3i} + 2r_{11i}}{4 + a_{3i}^2}, \quad r_{23i} = \frac{r_{22i}}{a_{3i}} \\
r_{31i} &= -(a_{21i}r_{11i} + a_{22i}), \quad r_{32i} = 1 - a_{21i}r_{12i} \\
r_{33i} &= r_{11i}r_{21i}, \quad r_{34i} = r_{11i}r_{22i} + r_{12i}r_{21i} \\
r_{35i} &= r_{11i}r_{23i} + r_{12i}r_{22i}, \quad r_{36i} = r_{12i}r_{23i} \\
r_{41i} &= -b_i^{-1}r_{32i}, \quad r_{42i} = b_i^{-1}r_{31i}, \quad r_{43i} = -b_i^{-1}r_{34i} \\
r_{44i} &= b_i^{-1}(3r_{33i} - 2r_{35i}), \quad r_{45i} = b_i^{-1}(2r_{34i} - 3r_{36i}), \quad r_{46i} = b_i^{-1}r_{35i}
\end{aligned}$$

It can be further verified that, for $i = 1, \cdots, 5$, $\frac{d^4\mathbf{u}_i(v,w)}{dt^4} + 9\mathbf{u}_i(v, w) + 10\frac{d^2\mathbf{u}_i(v,w)}{dt^2} = 0$. Thus, $s_i = 4$ for $i = 1, \cdots, 5$.
Let

$$M_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -m_1 & -m_2 & -m_3 & -m_4 \end{bmatrix}, \quad N_i = \text{col}(0, 0, 0, 1)$$

where the parameters (m_1, m_2, m_3, m_4) are chosen such that M_i is Hurwitz. Also, it was shown that with $(m_1, m_2, m_3, m_4) = (4, 12, 13, 6)$, we have $\Psi_i = [-5, 12, 3, 6]$. Thus, we can obtain a distributed internal model of the form (7.10). Performing the coordinate and input transformation (7.11) gives the augmented system as follows:

$$\begin{aligned} \dot{\bar{z}}_{1i} &= a_{11i}\bar{z}_{1i} + a_{12i}\bar{x}_{1i} \\ \dot{\bar{z}}_{2i} &= a_{3i}\bar{z}_{2i} + (\bar{z}_{1i} + \mathbf{z}_{1i})(\bar{x}_{1i} + v_1) - \mathbf{z}_{1i}v_1 \\ \dot{\bar{\eta}}_i &= (M_i + N_i\Psi_i)\bar{\eta}_i + N_i\bar{u}_i \\ \dot{\bar{x}}_{1i} &= \bar{x}_{2i} + \bar{g}_{1i}(\bar{z}_i, \bar{x}_{1i}, v, w) \\ \dot{\bar{x}}_{2i} &= b_i\bar{u}_i + b_i\Psi_i\bar{\eta}_i + a_{4i}\bar{z}_{1i} \end{aligned} \tag{7.60}$$

with $\bar{g}_{1i}(\bar{z}_{1i}, \bar{z}_{2i}, \bar{x}_{1i}, v, w) = a_{21i}\bar{z}_{1i} + a_{22i}\bar{x}_{1i} - \bar{z}_{1i}\bar{z}_{2i} - \mathbf{z}_{1i}\bar{z}_{2i} - \bar{z}_{1i}\mathbf{z}_{2i}$.

Let $\tilde{\eta}_i = \bar{\eta}_i - c_{2i}\bar{x}_{ri} - c_{1i}\bar{x}_{1i}$ with $c_{2i} = b_i^{-1}N_i$, $c_{1i} = b_i^{-1}M_iN_i$ for $i = 1, \cdots, 5$, and $\xi_i = b_i^{-1}U_{si}\bar{x}_i$ with $U_{si} = \begin{bmatrix} 1 & 0 \\ -d_{1i} & 1 \end{bmatrix}$. Then,

$$\begin{aligned} \dot{\bar{z}}_{1i} &= a_{11i}\bar{z}_{1i} + a_{12i}\bar{x}_{1i} \\ \dot{\bar{z}}_{2i} &= a_{3i}\bar{z}_{2i} + (\bar{z}_{1i} + \mathbf{z}_{1i})(\bar{x}_{1i} + v_1) - \mathbf{z}_{1i}v_1 \\ \dot{\tilde{\eta}}_i &= M_i\tilde{\eta}_i + M_ic_{1i}\bar{x}_{1i} - c_{1i}\bar{g}_{1i}(\bar{z}_{1i}, \bar{z}_{2i}, \bar{x}_{1i}, v, w) - c_{2i}a_{4i}\bar{z}_{1i} \\ \dot{\xi}_i &= A_c\xi_i + B\Psi_i\tilde{\eta}_i + G_i(\bar{z}_i, \bar{x}_{1i}, v, w) + B\bar{u}_i \end{aligned} \tag{7.61}$$

where $A_c = \begin{bmatrix} 0 & I_{r-1} \\ 0 & 0 \end{bmatrix}$ and for $i = 1, \cdots, 5$,

$$G_i(\bar{z}_i, \bar{x}_{1i}, v, w) = \begin{bmatrix} b_i^{-1}(d_{1i}\bar{x}_{1i} + \bar{g}_{1i}) \\ b_i^{-1}(d_{2i}\bar{x}_{1i} - d_{1i}\bar{g}_{1i} + a_{4i}\bar{z}_{1i}) \end{bmatrix} \tag{7.62}$$

The distributed observer for ξ_i is

$$\dot{\hat{\xi}}_i = A_c\hat{\xi}_i + \lambda(e_{vi} - \hat{\xi}_{1i}) + B\bar{u}_i, \quad i = 1, \cdots, 5 \tag{7.63}$$

Then

$$\begin{aligned} \dot{\bar{z}}_{1i} &= a_{11i}\bar{z}_{1i} + a_{12i}\bar{x}_{1i} \\ \dot{\bar{z}}_{2i} &= a_{3i}\bar{z}_{2i} + (\bar{z}_{1i} + \mathbf{z}_{1i})(\bar{x}_{1i} + v_1) - \mathbf{z}_{1i}v_1 \\ \dot{\tilde{\eta}}_i &= M_i\tilde{\eta}_i + M_ic_{1i}\bar{x}_{1i} - c_{1i}\bar{g}_{1i}(\bar{z}_{1i}, \bar{z}_{2i}, \bar{x}_{1i}, v, w) - c_{2i}a_{4i}\bar{z}_{1i} \\ \dot{\tilde{\xi}}_i &= A_0\tilde{\xi}_i + \lambda(b_i^{-1}e_i - e_{vi}) + B\Psi_i\tilde{\eta}_i + G_i(\bar{z}_i, e_i, v, w) \\ \dot{e}_i &= b_i\tilde{\xi}_{2i} + b_i\hat{\xi}_{2i} + d_{1i}\bar{x}_{1i} + \bar{g}_{1i} \\ \dot{\hat{\xi}}_{2i} &= \bar{u}_i + \lambda_2(e_{vi} - \hat{\xi}_{1i}) \end{aligned} \tag{7.64}$$

where $A_0 = \begin{bmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 0 \end{bmatrix}$, with $\lambda = [\lambda_1, \lambda_2] = [2, 3]$, is Hurwitz.

For the $(\bar{z}_{1i}, \bar{z}_{2i})-$subsystem in (7.64), let

$$V_{\bar{z}_i} = \frac{h_1}{2}\bar{z}_{1i}^2 + \frac{h_1}{4}\bar{z}_{1i}^4 + \frac{h_1}{8}\bar{z}_{1i}^8 + \frac{h_2}{2}\bar{z}_{2i}^2 + \frac{h_2}{4}\bar{z}_{2i}^4 \tag{7.65}$$

The time derivative of $(\bar{z}_{1i}, \bar{z}_{2i})-$subsystem is given by

$$\begin{aligned}
\dot{V}_{\bar{z}_i} &= h_1\bar{z}_{1i}\dot{\bar{z}}_{1i} + h_1\bar{z}_{1i}^3\dot{\bar{z}}_{1i} + h_1\bar{z}_{1i}^7\dot{\bar{z}}_{1i} + h_2\bar{z}_{2i}\dot{\bar{z}}_{2i} + h_2\bar{z}_{2i}^3\dot{\bar{z}}_{2i} \\
&= h_1a_{11i}\bar{z}_{1i}^2 + h_1a_{12i}\bar{z}_{1i}e_i + h_1a_{11i}\bar{z}_{1i}^4 + h_1a_{12i}\bar{z}_{1i}^3e_i \\
&+ h_1a_{11i}\bar{z}_{1i}^8 + h_1a_{12i}\bar{z}_{1i}^7e_i + a_{3i}h_2\bar{z}_{2i}^2 + h_2\bar{z}_{1i}\bar{z}_{2i}e_i \\
&+ h_2\bar{z}_{2i}\mathbf{z}_{1i}e_i + h_2\bar{z}_{2i}\bar{z}_{1i}v_1 + a_{3i}h_2\bar{z}_{2i}^4 + h_2\bar{z}_{1i}\bar{z}_{2i}^3e_i + h_2\bar{z}_{2i}^3\mathbf{z}_{1i}e_i + h_2\bar{z}_{2i}^3\bar{z}_{1i}v_1
\end{aligned} \tag{7.66}$$

Using the following inequalities in (7.66)

$$\begin{aligned}
h_1a_{12i}\bar{z}_{1i}e_i &\le \frac{0.01}{2}\bar{z}_{1i}^2 + \frac{h_1^2a_{12i}^2}{0.02}e_i^2 \\
h_1a_{12i}\bar{z}_{1i}^3e_i &\le \frac{3}{4}\bar{z}_{1i}^4 + \frac{h_1^4a_{12i}^4}{4}e_i^4 \\
h_1a_{12i}\bar{z}_{1i}^7e_i &\le \frac{7}{8}\bar{z}_{1i}^8 + \frac{h_1^8a_{12i}^8}{8}e_i^8 \\
h_2\bar{z}_{1i}\bar{z}_{2i}e_i &\le \frac{1}{2}\bar{z}_{2i}^2 + \frac{h_2^2}{2}\bar{z}_{1i}^2e_i^2 \le \frac{1}{2}\bar{z}_{2i}^2 + \frac{1}{4}\bar{z}_{1i}^4 + \frac{h_2^4}{4}e_i^4 \\
h_2\bar{z}_{2i}\mathbf{z}_{1i}e_i &\le \frac{1}{2}\bar{z}_{2i}^2 + \frac{h_2^2\mathbf{z}_{1i}^2}{2}e_i^2 \\
h_2\bar{z}_{2i}\bar{z}_{1i}v_1 &\le \frac{v_1^2h_2^2}{2}\bar{z}_{1i}^2 + \frac{1}{2}\bar{z}_{2i}^2 \\
h_2\bar{z}_{1i}\bar{z}_{2i}^3e_i &\le \frac{3}{4}\bar{z}_{2i}^4 + \frac{h_2^4}{4}\bar{z}_{1i}^4e_i^4 \le \frac{3}{4}\bar{z}_{2i}^4 + \frac{1}{8}\bar{z}_{1i}^8 + \frac{h_2^8}{8}e_i^8 \\
h_2\bar{z}_{2i}^3\mathbf{z}_{1i}e_i &\le \frac{3}{4}\bar{z}_{2i}^4 + \frac{h_2^4\mathbf{z}_{1i}^4}{4}e_i^4 \\
h_2\bar{z}_{2i}^3\bar{z}_{1i}v_1 &\le \frac{3}{4}\bar{z}_{2i}^4 + \frac{h_2^4v_1^4}{4}\bar{z}_{1i}^4
\end{aligned} \tag{7.67}$$

gives

$$\begin{aligned}
\dot{V}_{\bar{z}_i} &\le -(-h_1a_{11i} - \frac{0.01}{2} - \frac{v_1^2h_2^2}{2})\bar{z}_{1i}^2 - (-h_1a_{11i} - 1)\bar{z}_{1i}^8 \\
&- (-h_1a_{11i} - 1 - \frac{h_2^4v_1^4}{4})\bar{z}_{1i}^4 + (a_{3i}h_2 + \frac{3}{2})\bar{z}_{2i}^2 + (h_2a_{3i} + \frac{9}{4})\bar{z}_{2i}^4 + (\frac{h_1^2a_{12i}^2}{0.02} + \frac{h_2^2\mathbf{z}_{1i}^2}{2})e_i^2 \\
&+ (\frac{h_1^4a_{12i}^4}{4} + \frac{h_2^4}{4} + \frac{h_2^4\mathbf{z}_{1i}^4}{4})e_i^4 + (\frac{h_1^8a_{12i}^8}{8} + \frac{h_2^8}{8})e_i^8 \\
&\le -l_{1i}\bar{z}_{1i}^2 - l_{2i}\bar{z}_{1i}^4 - l_{3i}\bar{z}_{1i}^8 - l_{4i}\bar{z}_{2i}^2 - l_{5i}\bar{z}_{2i}^4 + l_{6i}e_i^2 + l_{7i}e_i^4 + l_{8i}e_i^8
\end{aligned} \tag{7.68}$$

where

$$\begin{aligned}
l_{1i} &= -h_1 a_{11i} - \frac{0.01}{2} - \frac{v_1^2 h_2^2}{2}, \quad l_{2i} = -h_1 a_{11i} - 1 - \frac{h_2^4 v_1^4}{4} \\
l_{3i} &= -h_1 a_{11i} - 1, \quad l_{4i} = -a_{3i} h_2 - \frac{3}{2} \\
l_{5i} &= -h_2 a_{3i} - \frac{9}{4}, \quad l_{6i} = \frac{h_1^2 a_{12i}^2}{0.02} + \frac{h_2^2 \mathbf{z}_{1i}^2}{2} \\
l_{7i} &= \frac{h_1^4 a_{12i}^4}{4} + \frac{h_2^4}{4} + \frac{h_2^4 \mathbf{z}_{1i}^4}{4}, \quad l_{8i} = \frac{h_1^8 a_{12i}^8}{8} + \frac{h_2^8}{8}
\end{aligned} \tag{7.69}$$

Let $V_Z = V(\bar{z}, \tilde{\eta}, \tilde{\xi}) + \tilde{z}^T \bar{P} \tilde{z}$. And it can be verified that

$$\begin{aligned}
&|| \begin{bmatrix} f_\eta(\bar{z}, e, v, w) \\ f_\xi(\bar{z}, e, v, w) \end{bmatrix} ||^2 \le \sum_{i=1}^{5} ||M_i c_{1i} \bar{x}_{1i} - c_{1i} \bar{g}_{1i} - c_{2i} a_{4i} \bar{z}_{1i}||^2 + \sum_{i=1}^{5} (\lambda_1 (b_i^{-1} e_i - e_{vi}) \\
&+ b_i^{-1} (d_{1i} e_i + \bar{g}_{1i}))^2 + \sum_{i=1}^{5} (\lambda_2 (b_i^{-1} e_i - e_{vi}) + b_i^{-1} (d_{2i} e_i - d_{1i} \bar{g}_{1i} + a_{4i} \bar{z}_{1i}))^2
\end{aligned}$$

And

$$\begin{aligned}
&||M_i c_{1i} \bar{x}_{1i} - c_{1i} \bar{g}_{1i} - c_{2i} a_{4i} \bar{z}_{1i}||^2 \\
&\le 3||M_i c_{1i} e_i||^2 + 3||c_{2i} a_{4i} \bar{z}_{1i}||^2 + 3||c_{1i} \bar{g}_{1i}||^2 \\
&\le 3||M_i c_{1i} e_i||^2 + 3||c_{2i} a_{4i} \bar{z}_{1i}||^2 + 3||c_{1i}||^2 (a_{21i} \bar{z}_{1i} + a_{22i} e_i - \bar{z}_{1i} \bar{z}_{2i} - \mathbf{z}_{1i} \bar{z}_{2i} - \bar{z}_{1i} \mathbf{z}_{2i})^2 \\
&\le 3||M_i c_{1i} e_i||^2 + 3||c_{2i} a_{4i} \bar{z}_{1i}||^2 + 15||c_{1i}||^2 (a_{21i}^2 \bar{z}_{1i}^2 + a_{22i}^2 e_i^2 + \frac{1}{2} \bar{z}_{1i}^4 + \frac{1}{2} \bar{z}_{2i}^4 + \mathbf{z}_{1i}^2 \bar{z}_{2i}^2 + \bar{z}_{1i}^2 \mathbf{z}_{2i}^2) \\
&= (3||M_i c_{1i}||^2 + 15||c_{1i}||^2 a_{22i}^2) e_i^2 + 15||c_{1i}||^2 \mathbf{z}_{1i}^2 \bar{z}_{2i}^2 \\
&+ (3||c_{2i} a_{4i}||^2 + 15||c_{1i}||^2 (a_{21i}^2 + \mathbf{z}_{2i}^2)) \bar{z}_{1i}^2 + \frac{15}{2} ||c_{1i}||^2 \bar{z}_{1i}^4 + \frac{15}{2} ||c_{1i}||^2 \bar{z}_{2i}^4
\end{aligned} \tag{7.70}$$

$$\begin{aligned}
&\sum_{i=1}^{5} (\lambda_1 (b_i^{-1} e_i - e_{vi}) + b_i^{-1} (d_{1i} e_i + \bar{g}_{1i}))^2 \\
&\le \sum_{i=1}^{5} 4|\lambda_1 b_i^{-1}|^2 e_i^2 + 4\lambda_1^2 e_{vi}^2 + 4|b_i^{-1} d_{1i}|^2 e_i^2 + 4|b_i^{-1}|^2 |\bar{g}_{1i}|^2 \\
&\le \sum_{i=1}^{5} (4|\lambda_1 b_i^{-1}|^2 + 4|b_i^{-1} d_{1i}|^2) e_i + 4\lambda_1^2 ||e_v||^2 \\
&+ \sum_{i=1}^{5} 4|b_i^{-1}|^2 (a_{21i} \bar{z}_{1i} + a_{22i} e_i - \bar{z}_{1i} \bar{z}_{2i} - \mathbf{z}_{1i} \bar{z}_{2i} - \bar{z}_{1i} \mathbf{z}_{2i})^2 \\
&\le \sum_{i=1}^{5} (4|\lambda_1 b_i^{-1}|^2 + 4|b_i^{-1} d_{1i}|^2) e_i + 4\lambda_1^2 ||H||^2 \sum_{i=1}^{5} e_i^2 \\
&+ \sum_{i=1}^{5} 20|b_i^{-1}|^2 (a_{21i}^2 \bar{z}_{1i}^2 + a_{22i}^2 e_i^2 + \frac{1}{2} \bar{z}_{1i}^4 + \frac{1}{2} \bar{z}_{2i}^4 + \mathbf{z}_{1i}^2 \bar{z}_{2i}^2 + \bar{z}_{1i}^2 \mathbf{z}_{2i}^2)
\end{aligned}$$

$$= \sum_{i=1}^{5}(4|\lambda_1 b_i^{-1}|^2 + 4|b_i^{-1}d_{1i}|^2 + 4\lambda_1^2||H||^2 + 20|b_i^{-1}|^2 a_{22i}^2)e_i^2$$
$$+ \sum_{i=1}^{5} 20|b_i^{-1}|^2((a_{21i}^2 + \mathbf{z}_{2i}^2)\bar{z}_{1i}^2 + \frac{1}{2}\bar{z}_{1i}^4 + \frac{1}{2}\bar{z}_{2i}^4 + \mathbf{z}_{1i}^2\bar{z}_{2i}^2) \tag{7.71}$$

$$\begin{aligned}
&\sum_{i=1}^{5}(\lambda_2(b_i^{-1}e_i - e_{vi}) + b_i^{-1}(d_{2i}e_i - d_{1i}\bar{g}_{1i} + a_{4i}\bar{z}_{1i}))^2 \\
&\leq \sum_{i=1}^{5} 5|\lambda_2 b_i^{-1}|^2 e_i^2 + 5\lambda_2^2 e_{vi}^2 + 5|b_i^{-1}d_{2i}|^2 e_i^2 + 5|b_i^{-1}d_{1i}|^2|\bar{g}_{1i}|^2 + 5|b_i^{-1}a_{4i}|^2\bar{z}_{1i}^2 \\
&\leq \sum_{i=1}^{5}(5|\lambda_2 b_i^{-1}|^2 + 5|b_i^{-1}d_{2i}|^2)e_i + 5\lambda_2^2||H||^2 e^2 \\
&+ \sum_{i=1}^{5} 25|b_i^{-1}d_{1i}|^2(a_{21i}^2\bar{z}_{1i}^2 + a_{22i}^2 e_i^2 + \frac{1}{2}\bar{z}_{1i}^4 + \frac{1}{2}\bar{z}_{2i}^4 + \mathbf{z}_{1i}^2\bar{z}_{2i}^2 + \bar{z}_{1i}^2\mathbf{z}_{2i}^2) + 5|b_i^{-1}a_{4i}|^2\bar{z}_{1i}^2 \\
&\leq \sum_{i=1}^{5}(5|\lambda_2 b_i^{-1}|^2 + 5|b_i^{-1}d_{2i}|^2 + 5\lambda_2^2||H||^2 + 25|b_i^{-1}d_{1i}|^2 a_{22i}^2)e_i^2 \\
&+ \sum_{i=1}^{5}(25|b_i^{-1}d_{1i}|^2(a_{21i}^2 + \mathbf{z}_{2i}^2) + 5|b_i^{-1}a_{4i}|^2)\bar{z}_{1i}^2 + \sum_{i=1}^{5} 25|b_i^{-1}d_{1i}|^2(\frac{1}{2}\bar{z}_{1i}^4 + \frac{1}{2}\bar{z}_{2i}^4 + \mathbf{z}_{1i}^2\bar{z}_{2i}^2)
\end{aligned} \tag{7.72}$$

Then

$$\begin{aligned}
&||\begin{bmatrix} f_\eta(\bar{z}, e, v, w) \\ f_\xi(\bar{z}, e, v, w) \end{bmatrix}||^2 \leq \sum_{i=1}^{5}(3||M_i c_{1i}||^2 + 15||c_{1i}||^2 a_{22i}^2 + 4|\lambda_1 b_i^{-1}|^2 + 4|b_i^{-1}d_{1i}|^2 + 4\lambda_1^2||H||^2 \\
&+ 20|b_i^{-1}|^2 a_{22i}^2 + 5|\lambda_2 b_i^{-1}|^2 + 5|b_i^{-1}d_{2i}|^2 + 5\lambda_2^2||H||^2 + 25|b_i^{-1}d_{1i}|^2 a_{22i}^2)e_i^2 \\
&+ \sum_{i=1}^{5}(3||c_{2i}a_{4i}||^2 + 15||c_{1i}||^2(a_{21i}^2 + \mathbf{z}_{2i}^2) + 20|b_i^{-1}|^2(a_{21i}^2 + \mathbf{z}_{2i}^2) \\
&+ 25|b_i^{-1}d_{1i}|^2(a_{21i}^2 + \mathbf{z}_{2i}^2) + 5|b_i^{-1}a_{4i}|^2)\bar{z}_{1i}^2 + \sum_{i=1}^{5}(\frac{15}{2}||c_{1i}||^2 + 10|b_i^{-1}|^2 + \frac{25}{2}|b_i^{-1}d_{1i}|^2)\bar{z}_{1i}^4 \\
&+ \sum_{i=1}^{5}(15||c_{1i}||^2\mathbf{z}_{1i}^2 + 20|b_i^{-1}|^2\mathbf{z}_{1i}^2 + 25|b_i^{-1}d_{1i}|^2\mathbf{z}_{1i}^2)\bar{z}_{2i}^2 \\
&+ \sum_{i=1}^{5}(\frac{15}{2}||c_{1i}||^2 + 10|b_i^{-1}|^2 + \frac{25}{2}|b_i^{-1}d_{1i}|^2)\bar{z}_{2i}^4 \\
&= \sum_{i=1}^{5} \delta_{3i}e_i^2 + \delta_{4i}\bar{z}_{1i}^2 + \delta_{5i}\bar{z}_{1i}^4 + \delta_{6i}\bar{z}_{2i}^2 + \delta_{7i}\bar{z}_{2i}^4
\end{aligned}$$

with

$$\delta_{3i} = 3||M_i c_{1i}||^2 + 15||c_{1i}||^2 a_{22i}^2 + 4|\lambda_1 b_i^{-1}|^2 + 4|b_i^{-1} d_{1i}|^2$$
$$+ 4\lambda_1^2||H||^2 + 20|b_i^{-1}|^2 a_{22i}^2 + 5|\lambda_2 b_i^{-1}|^2 + 5|b_i^{-1} d_{2i}|^2 + 5\lambda_2^2||H||^2 + 25|b_i^{-1} d_{1i}|^2 a_{22i}^2$$
$$\delta_{4i} = 3||c_{2i} a_{4i}||^2 + 15||c_{1i}||^2 (a_{21i}^2 + \mathbf{z}_{2i}^2) + 20|b_i^{-1}|^2 (a_{21i}^2 + \mathbf{z}_{2i}^2) + 25|b_i^{-1} d_{1i}|^2 (a_{21i}^2 + \mathbf{z}_{2i}^2)$$
$$+ 5|b_i^{-1} a_{4i}|^2$$
$$\delta_{5i} = \frac{15}{2}||c_{1i}||^2 + 10|b_i^{-1}|^2 + \frac{25}{2}|b_i^{-1} d_{1i}|^2$$
$$\delta_{6i} = 15||c_{1i}||^2 \mathbf{z}_{1i}^2 + 20|b_i^{-1}|^2 \mathbf{z}_{1i}^2 + 25|b_i^{-1} d_{1i}|^2 \mathbf{z}_{1i}^2$$
$$\delta_{7i} = \frac{15}{2}||c_{1i}||^2 + 10|b_i^{-1}|^2 + \frac{25}{2}|b_i^{-1} d_{1i}|^2$$

Then

$$\begin{aligned} \dot{V}_Z \le & -(l_{1i} - ||\bar{P}||^2 \delta_{4i}) \bar{z}_{1i}^2 - (l_{2i} - ||\bar{P}||^2 \delta_{5i}) \bar{z}_{1i}^4 - l_{3i} \bar{z}_{1i}^8 - (l_{4i} - ||\bar{P}||^2 \delta_{6i}) \bar{z}_{2i}^2 \\ & - (l_{5i} - ||\bar{P}||^2 \delta_{7i}) \bar{z}_{2i}^4 - ||\tilde{z}||^2 + (l_{6i} + ||\bar{P}||^2 \delta_{3i}) e_i^2 + l_{7i} e_i^4 + l_{8i} e_i^8 \end{aligned} \tag{7.73}$$

Finally, by the procedure of Section III, we can design a control law as follows:

$$\begin{aligned} \check{\xi}_{2i} &= \hat{\xi}_{2i} + \rho_{1i}(e_{vi}) e_{vi}, \quad i = 1, \cdots, 5 \\ \alpha_{2i}(\check{\xi}_{2i}) &= -500(\check{\xi}_{2i}^6 + 1)\check{\xi}_{2i} \\ u_i &= \alpha_{2i}(\check{\xi}_{2i}) + \Psi_i \eta_i \\ \dot{\hat{\xi}}_i &= A_c \hat{\xi}_i + \lambda(e_{vi} - \hat{\xi}_{1i}) + B(u_i - \Psi_i \eta_i) \\ \dot{\eta}_i &= M_i \eta_i + N_i u_i \end{aligned} \tag{7.74}$$

where $\rho_{1i}(e_{vi}) = 500(e_{vi}^6 + 1)$.

The simulation is conducted with the following initial values of various variables:

$$\begin{aligned} & v(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \\ & [z_1(0), x_1(0)] = \begin{bmatrix} -0.1 & 2 & 1 & 0 \end{bmatrix}^T, \quad \hat{\xi}_1(0) = \begin{bmatrix} 7 & 1 \end{bmatrix}^T, \\ & [z_2(0), x_2(0)] = \begin{bmatrix} 1 & 3 & 1 & 0 \end{bmatrix}^T, \quad \hat{\xi}_2(0) = \begin{bmatrix} 7 & -1 \end{bmatrix}^T, \\ & [z_3(0), x_3(0)] = \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}^T, \quad \hat{\xi}_3(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \\ & [z_4(0), x_4(0)] = \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}^T, \quad \hat{\xi}_4(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \\ & [z_5(0), x_5(0)] = \begin{bmatrix} -1 & 2 & 1 & 0 \end{bmatrix}^T, \quad \hat{\xi}_5(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \\ & \eta_1(0) = \begin{bmatrix} 0 & 0 & 0 & 8 \end{bmatrix}^T, \quad \eta_2(0) = \begin{bmatrix} -1 & 0 & 2 & 2 \end{bmatrix}^T, \\ & \eta_3(0) = \begin{bmatrix} 0 & 0 & 2 & 2 \end{bmatrix}^T, \quad \eta_4(0) = \begin{bmatrix} -1 & 4 & -2 & -2 \end{bmatrix}^T, \\ & \eta_5(0) = \begin{bmatrix} -1 & 3 & -2 & 0 \end{bmatrix}^T. \end{aligned}$$

The simulation results are shown in Figures 7.2-7.5 and satisfactory tracking performance can be observed.

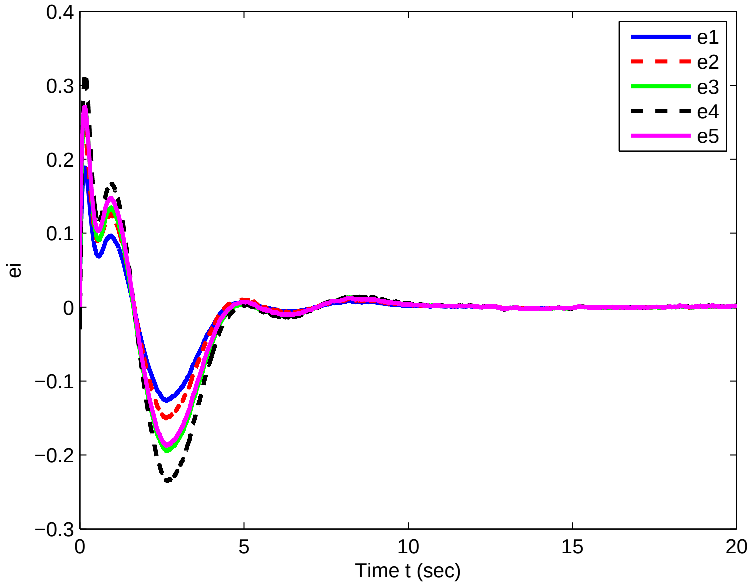

Figure 7.2: The profiles of the tracking errors e_i

7.5 Concluding Remarks

In this chapter we have considered the cooperative global robust output regulation problem for a class of nonlinear multi-agent systems in output feedback form with relative degree greater than unity. We have introduced a distributed internal model and a distributed observer to convert our problem into the global stabilization problem of a coupled multi-input multi-output system. By solving this stabilization problem by a distributed control law, we have successfully solved our original problem.

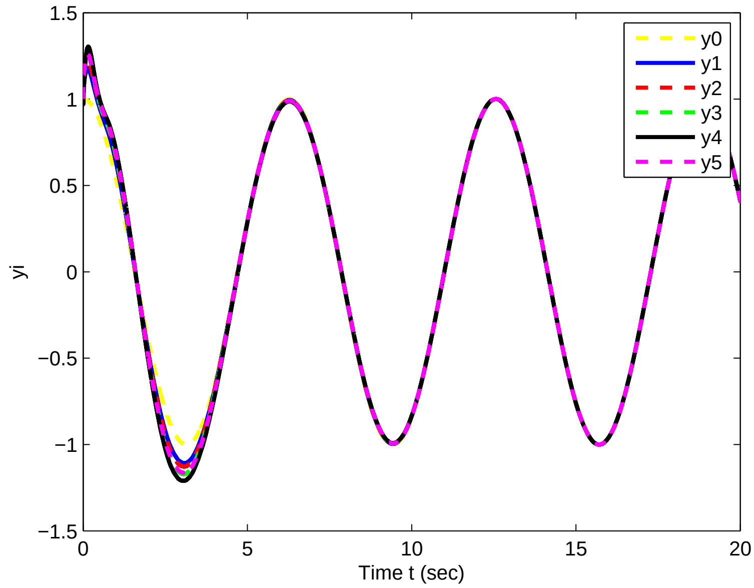

Figure 7.3: The output y_i of all agents

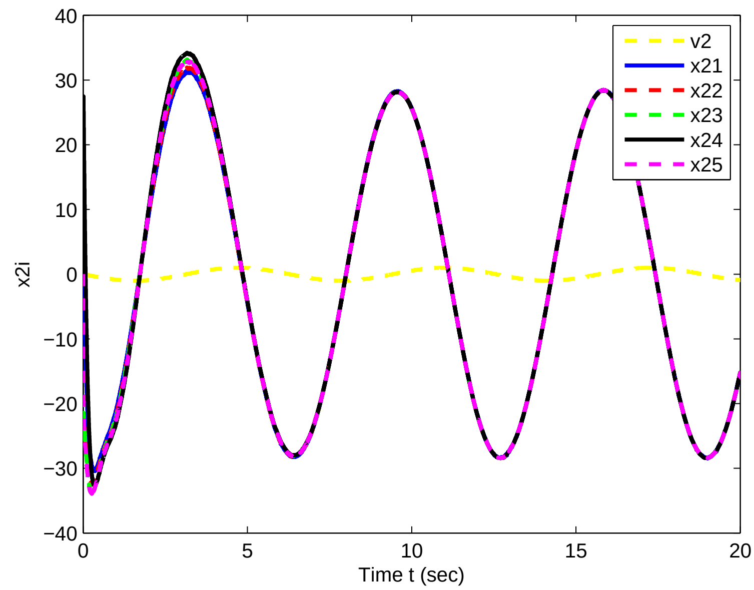

Figure 7.4: The state x_{2i} of all agents

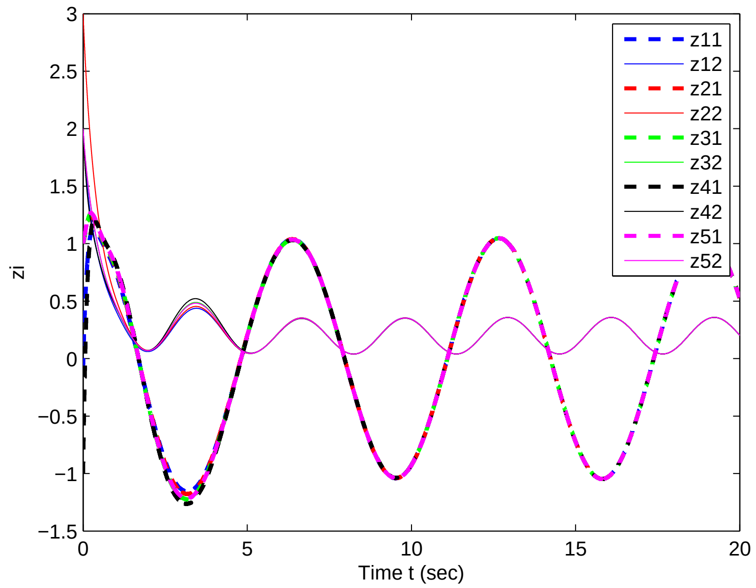

Figure 7.5: The state z_i of all agents

Chapter 8

Conclusions and Future Work

8.1 Conclusions

In this chapter, we will summarize the main results in this thesis, which can be divided into two parts.

The first part is about the leader-following rendezvous with connectivity preservation problem.

1. We have considered the leader-following problem of rendezvous with connectivity preservation for a multiple single-integrator system where the leader system can be any linear autonomous system. We have proposed a very simple continuous distributed state feedback control protocol, independent of the information of the upper bound of the leader's signal. Our control law is able to maintain the connectivity of the system, and, at the same time, achieve asymptotic tracking of all followers to the output of the leader system.

2. we have also designed both dynamic state feedback control law and position feedback control law to solve the leader-following rendezvous problem of a set of double integrator systems subject to a class of external disturbances. The leader system can have different dynamics from the followers', which can not only generate a ramp signal, but also sinusoidal signals with arbitrary amplitudes and initial phases. Furthermore, we allow all the followers to be subject to external disturbances. In particular, in chapter 5, we even allow the external disturbances to various followers to be different. Both of these two protocols can maintain the connectivity of the network graph as well as achieve asymptotic tracking for a class of leader system.

The second part is about the cooperative output regulation problem of nonlinear multi-agent systems.

1. The global robust output regulation problem for a class of nonlinear multi-agent systems in output feedback form with unity relative degree has been studied. First, a distributed internal model is applied to convert the output regulation problem into the global stabilization problem of an augmented system. Then the augmented system is further globally stabilized via a distributed output feedback control law, which leads to the solution of the global robust output regulation problem of the original system.

2. The global robust output regulation problem for a class of nonlinear multi-agent systems in output feedback form with relative degree greater than unity has been further studied. Similar to the cooperative output regulation problem of nonlinear multi-agent systems in output feedback form with unity relative degree, we first design a distributed internal model to convert the problem into the global stabilization problem of an augmented system. However, instead of globally stabilizing a decoupled multi-input augmented system, we have to further design a distributed observer and develop techniques to globally stabilize a coupled multi-input nonlinear uncertain system. It is interesting to note that our main result can be summarized as follows: under the assumption that the communication graph is connected, then the cooperative output regulation problem of a multi-agent system can be solved by a distributed output feedback control law if the global robust output regulation problem for each subsystem of the multi-agent system can be solved by an output feedback control law.

8.2 Future Work

In the near future, we will further consider the connectivity preservation problem for nonlinear multi-agent systems, for example, Lagrange multi-agent systems and nonlinear multi-agent systems in the form of (7.1).

We will apply the methods for solving connectivity preservation problem of linear multi-agent systems to other coordinated control problems, such as flocking, swarming, formation and so on.

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Biography

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The main research result summarized in this thesis leads to some journal papers and conference papers.

Journal Papers

1. **Y. Dong** and J. Huang, "A Leader-following Rendezvous Problem of Double Integrator Multi-agent Systems," *Automatica*, vol. 49, no. 5, pp. 1386-1391, 2013.

2. **Y. Dong** and J. Huang, "Cooperative Global Robust Output Regulation for Nonlinear Multi-agent Systems in Output Feedback Form," *Journal of Dynamic Systems Measurement and Control-Transactions of ASME*, conditionally accepted.

3. **Y. Dong** and J. Huang, "Cooperative Global Output Regulation for a Class of Nonlinear Multi-agent Systems," *IEEE Transactions on Automatic Control*, revised and resubmitted.

Conference Papers

1. **Y. Dong** and J. Huang, "Leader-following Connectivity Preservation Rendezvous of linear Multi-agent Systems Based Only Position Measurements," in *the 52nd IEEE Conference on Decision and Control*, Palazzo dei Congressi, Florence, Italy, December 10 – 13, 2013, submitted.

2. **Y. Dong** and J. Huang, "Cooperative Global Output Regulation for a Class of Nonlinear Multi-agent Systems," in *the third annual IEEE International Conference*

on CYBER Technology in Automation, Control, and Intelligent Systems, Nanjing, China, May 26 – 29, 2013, pp. 211–216.

3. **Y. Dong** and J. Huang, "Cooperative Global Robust Output Regulation for Nonlinear Multi-agent Systems in Output Feedback Form," in *the Third IASTED Asian Conference on Modelling, Identification and Control*, Phuket, Thailand, April 10 – 12, 2013, pp. 138–143.

4. **Y. Dong** and J. Huang, "Leader-following Rendezvous with Connectivity Preservation of Single-integrator Multi-agent Systems," *in Proceedings 12th International Conference on Control, Automation, Robotics and Vision (ICARCV)*, Guangzhou, China, Dec. 5 – 7, 2012, pp. 1688–1690.

5. **Y. Dong** and J. Huang, "Leader-following rendezvous with connectivity preservation of a class of multi-agent systems," in *Proceedings of the 31st Chinese Control Conference*, Hefei, China, July 25 – 27, 2012, pp. 6477-6482.