Equations of Structured Population Dynamics

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June 1990

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A Thesis
Submitted to the Faculty of the Graduate School of
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In Partial Fulfillment of the Requirements
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Acknowledgments

I express my sincere and deepest gratitude to my supervisor Dr.W.L.Chan for his constant guidance, enthusiastic encouragement and generous support during the course of my Ph.D research work at the Chinese University of Hong Kong. This work could never have been completed without his insight, observations and many beneficial suggestions.

I am grateful to Professor G.T.Zhu of the Institute of Systems Sciences, Academia Sinica, who supervised my master thesis and introduced me to the field of control theory of distributed parameter systems.

I am deeply indebted to Professor J.Y.Yu of Beijing Institute of Information and Control for his constant advice and personal care to my studies and research.

My gratitude goes also to Dr.Paul.C.K.Kwong for providing me partial financial support for completing of this work under a grant from the International Development Research Center, Ottawa, Canada.

Finally, I would like to thank all the staff of the Department of Mathematics, the Chinese University of Hong Kong for their personal care as well as academic support to my studies in this school.

Contents

Chapter 7. Optimal Birth Control Policies II

Chapter 8. Perato Optimal Birth Control Policies

Chapter 9. Overtaking Optimal Control Problems with Infinite Horizon

Chapter 10. Viable Control in Logistic Populatiuon Model

Abstract

The study of population and its analysis using mathematical models have received increased interest in the mathematical community in recent years. It was not until the twentieth century, however, that the study of population, predominantly human population, achieved its mathematical character.

In this thesis, we are concerned with the analysis of the solution to the partial differential equations in age-dependent population dynamics, their asymptotic behaviour, numerical approximations, birth control strategies and relationship between continuous models and discrete models.

In the first part (Chapter 1-5), we study the linear McKendrick type equations of population dynamics with instantaneous time delay in the birth rate and a population model with age-size dependence and spatial diffusion in the semigroup framework. The infinitesimal generators are identified and the growth indices and the asymptotic expression of the solutions are determined explicitly. We also investigate the large time behaviour of the nonlinear population equation with a general logistic term. It is proved that the solution must have a limit when time becomes infinite and in general no oscillation is possible for the total number of population. This is in sharp contrast to the linear case. Furthermore, we study a semi-discrete model and a finite difference model obtained from the

corresponding continuous one. Properties of their solutions are investigated and the relationship between the finite difference model and the continuous model is established.

The second part (Chapter 6-10) is devoted to the control problems of population dynamics, particularly to the optimal birth control policies of equations of McKendrick types. These are distributed parameter systems involving first order partial differential equations with nonlocal bilinear boundary control. The functional analytic approach of Duboviskii and Milyutin is adopted in the investigation. Maximum principles for problems with a free end condition and fixed final horizon, time optimal control, the problem with target sets and infinite planing horizon are developed. Results in problems with free final time, phase constraints and mini-max costs are presented. Furthermore, we also deal with the unbounded time control problem and obtain the minimum principle in the overtaking sense. The Pareto optimal problem and the problem with nonsmooth criteria are also investigated. Finally, we develop viable controls for the logistic population equations.

Introduction

The interest in the study of human population or demography, in terms of its growth and decay, fertility and mortality and its relative mobility, can be traced back to ancient times. The first published table of mortality was attributed to the Roman Macer while a truly substantial work [1] in demography was published by Graunt in 1662, a study in the city of London in 1658. His work is regarded as a masterpiece in the field of demography. In 1760, it was Lenoard Euler [2] who introduced, in a virtually unknown article, the concept of a stable age structure in which proportions in all age categories would remain fixed if the population experienced no abrupt changes in migration, and if mortality were constant and births increased exponentially over time. This paper, anticipating important parts of modern stable population theory for a one-sex population closed to migration, is a cornerstone in mathematical demography. In 1798 Thomas Malthus [3] published his famous work which hypothesized that

- food is necessary for existence with only a finite amount of land on which to grow it;
- the rate of human reproduction remains constant.

From this Malthus established a (originally discrete) growth model of the human race implying really that the rate of population growth is proportional to the size of the population, which could be described by the ordinary differential equation

$$\frac{dN(t)}{dt} = \lambda N(t), \quad t \geq 0, \tag{1}$$

where $N(t)$ represents the total population size at time t, λ is the *Malthus parameter* of the given population, $\lambda=\beta-\mu$, β is the birth rate and μ is the death rate.

Perhaps the most sensational message of Malthus' model was that under normal circumstance (e.g. no famine, plagues, wars *etc.*) the size of the human population would increase geometrically (exponentially) whereas food supply would increase at best only arithmetically (linearlly).

Neglecting the difference among individuals and the *negative feedback-effects* existing in most biological world, Malthus' model was so simple that it resulted in, on one hand, both praise and censure of himself in the scientific world, and on the other hand, the further investigations on the population research. A singular achievement which links it with the stable population theory put forward by Euler was due to Alfred Lotka [4] and four years later to Sharp and Lotka [5]. From their realization, the population could be represented as a renewal process displaying some stability. They considered the age distribution of population and developed the following population model

$$p(r,t)=c(r)N(t)=B(t-r)q(r),$$
$$B(t)=\int_0^M B(t-r)q(r)dr, \qquad (2)$$

where $q(r)$ is the probability at birth that a male shall reach the age r, $B(t)$ is the male birth rate, M is the age at which a male reproductive ends, β is the rate of male birth, $p(r,t)$ is the age distribution of male, $N(t)$ is the total number. From 1922 to 1939, they studied the stability problem of stable age distribution. But the rigorous mathematical proof was not completed until Feller's paper [6] appeared in 1941 in which the following *renewal equation* was studied

$$\phi(t)=L(t)+\int_0^t p(r)m(r)\phi(t-r)dr \tag{3}$$

where $\phi(t)$ are births at time t, and are composed of births to the population alive at time zero [L(t)] and births to those born since: $\phi(t-r)$ is the number of persons born t-r years ago and, subject to their survival probability $p(r)$, currently at the age r; m(r) being their chance of giving birth in the interval x to r+dr. From which, it was deduced, under reasonable assumptions, that the age density distribution p(r,t) has the asymptotic property

$$p(r,t) \sim C_0\phi^*(r)e^{\lambda t}, \quad \text{as } t\longrightarrow \infty,$$

where C_0 is a constant, $\phi^*(r)$ is the equilibrium state of age distribution, λ is the *intrinsic growth* constant. All these efforts gave an important new impulse to the population research which was actually ushering in the quantitative mathematical research on population problem.

Based on the works of Sharp and Lotka, McKendrick [7] developed a partial differential equation model for the age-distribution p(r,t) which Foerstor [8] presented independently later. It represents the change of age structure with time, and all the other factors affecting p(r,t), e.g. system of society, natural environment, living standard, war, famine, etc., were contributed to the death process described by $\mu(r,t)$, the probability per unit of time that an individual with age r dies, and birth process is described by $\beta(r)$, the expected number of offspring per unit of time of an individual with age r. He deduced a partial differential equation with nonlocal boundary conditions:

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t}+\dfrac{\partial p(r,t)}{\partial r}=-\mu(r,t)p(r,t), \quad t>0,\ r>0, \\ p(r,0)=p_0(r), \quad r>0, \\ p(0,t)=\displaystyle\int_0^{\infty}\beta(r)p(r,t)dr, \quad t>0, \end{cases} \tag{4}$$

where $p_0(r)$ is the initial density distribution of the population.

This model is essentially equivalent to the model of Sharp-Lotka only by letting $\phi(t)=p(0,t)$ and integrating along the characteristic.

Beside, Leslie [9] formulated a fairly complete discrete age-dependent population model, which became the foundation of demography.

$$\begin{aligned} \sum_{x=0}^{m} F_x n_{x_0} &= n_{01}, \\ p_0 n_{00} &= n_{11}, \\ p_1 n_{10} &= n_{21}, \\ &\vdots \\ p_{m-1} n_{m-1,0} &= n_{m1}, \end{aligned} \tag{5}$$

where n_{11} is the number of females alive in the age group x to x+1 at time t, p_x is the probability that a female aged x to x+1 at time t will be alive in the age group x+1 to x+2 at time t+1, F_x is the number of daughters born in the interval t to t+1 per female alive aged x to x+1 at time t, who will be alive in the age group 0-1 at time t+1.

The heavy social and economical pressure in China brought about by a huge population over the past few years has urged a team of researchers under J. Song, to develop a fairly accurate model for the Chinese population. In Song and Yu [10], improved McKendrick and Leslie models were developed. In these models, the birth process was considered to be affected by four factors, which include:

- The age distribution of the population whose age lie in the interval of fecundity.
- The ratio of females aged in the fecundity periods.
- The birth number of females.
- The fertility pattern of females.

They are described by $(p(r,t),\ k(r,t),\ \beta(t),\ h(r,t))$, $r\in[r_1,r_2]$, the fecundity interval of females, respectively. The model is as follows

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), \quad t>0,\ 0<r<r_m, \\ p(r,0)=p_0(r), \quad 0\le r\le r_m, \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2} k(r,t)h(r,t)p(r,t)dr, \quad t\ge 0. \end{cases} \tag{6}$$

They found the critical specific fertility for the stable distribution of the system, and proved the existence and uniqueness of solutions by semigroup theory and spectral theory of linear operators.

It can be seen that all these models mentioned above assumed that both the birth process and the mortality process are linear functions of the population densities. Consequently, the equations of these models are linear. In spite of the introduction of the age structure, they predict the population tends to infinity when the intrinsic parameter is greater than zero. i.e. no *negative feedback-effects* are considered. The earlier efforts of overcoming this difficulty was made by the Belgian scientist Pierre-Francois Verhulst [13]. He formulated a mathematical model of a growing population with an upper limit, which allows the *Malthus parameter* to depend upon the size of the total population itself, and therefore led to a nonlinear ordinary differential equation.

$$\frac{dN(t)}{dt}=\lambda N(t)-\phi(N(t)), \quad t\geq 0. \tag{9}$$

The constant appeared in the above equation is known as the *intrinsic growth constant*. A special case is that $\phi(N)=\frac{\lambda}{K}N^2$. In this case the solution is called the *logistic growth curve*, which can be obtained explicitly

$$N(t)=\frac{\lambda N_0 e^{\lambda t}}{KN_0 e^{\lambda t}+\lambda-KN_0},$$

where $N_0=N(0)$ and K is the *environmental carrying capacity*. Hence

$$N(t) \longrightarrow \frac{\lambda}{K},$$

the *upper limit of the population*, as $t\longrightarrow \infty$.

In 1974, M.Gurtin and R.C.MacCamy [14], and F.Hoppenstoadt introduced the first models of nonlinear continuous age-dependent population dynamics by assuming that the functions β and μ appeared in the McKendrick model (3) depend on the total population size $N(t)$. They proved the existence and the uniqueness of solutions and obtained local stability results for equilibrium distributions. Their pioneering work caused an outburst of publications in several variants of their models. We refer to Webb [15] for the general nonlinear models along this direction.

One of the main applications of population dynamics is demography. But the same ideas apply to biological populations other than mankind, for instance insects, plants and micro-organisms. For such populations, age often does not give a satisfactory description of an individual. These ideas must have been in the air around 1967, because at that time there appeared more or less independently a number of publications concerning population models, in which it was argued that variables different from age such as size or maturity (sometimes in combination with age) should play a role in the considerations. In the Lecture Notes by Metz and Diekmann [16] a lot of such interesting examples can be found.

For more detailed accounts on the mathematical treatments of population dynamics one may consult Smith and Keyfitz [17] or Hallam and Levin [18].

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Chapter 1

Semigroups for Age-Dependent Population Equations with Time Delay

1.1 Introduction

In recent years there has been a rapid growth of interest in the application of the theory of semigroup to population dynamics. Very nice examples can be found in Song et al. [1, 2], Webb [3], Metz and Diekmann [4], Heijmans [5] and Greiner and Nagel [6], to name just a few. On the other hand, it is by now a firmly established tenet in population dynamics that time delays plays a very important role in the qualitative behavior of the population. One such model, possibly earliest, was due to Hutchisin [7]. Other studies include those of Gopalsamy [8], Cushing [9]. The purpose of this chapter is to study the McKendrick type models of age-dependent population dynamic with instantaneous time delay in the birth rate. These models involve first order partial differential equation with nonlocal and delayed boundary condition. This chapter consists of the following parts. In Sect. 1.2 we define the problem and show how a semigroup can be associated to it. Moreover, we identify the infinitesimal generator of this semigroup. In Sect. 1.3, the spectral properties are analyzed. An interesting result is that in the general linear case, with delays or not, if the total population converges to an equilibrium distribution it will converge to it in an oscillatory fashion, a similar phenomenon noted by Feller [10] earlier for some special cases. The last section is devoted to a logistic age-dependent model with delay and its associated nonlinear semigroup.

1.2 Problem Statement and Linear Theory

We are interested in the following model of population dynamics:

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t}+\dfrac{\partial p(r,t)}{\partial r}=-\mu(r)p(r,t), & 0<r<r_m,\ t>0, \\ p(r,\theta)=p_0(r,\theta), \quad 0\le r\le r_m,\ -\tau\le\theta\le 0, & \\ p(0,t)=\beta\displaystyle\int_{r_1}^{r_2}k(r)h(r)p(r,t-\tau)dr, & t\ge 0, \end{cases} \tag{1}$$

where $p(r,t)$ denotes the age density distribution at time t and age r, $\mu(r)$ is the relative mortality of the population, r_m is the highest age even attained by individuals of the population, $k(r)$ is the female sex ratio at age r, $h(r)$ is the fertility pattern, $[r_1,r_2]$ is the fecundity period of females, β is the specific fertility rate of females and τ is the time delay.

Let $k(r)h(r)$ be a continuous function in $[0,r_m]$ with $k(r)h(r)>0$, for every $r\in[r_1,r_2]$, $\mu(r)$ is continuous on any interval $[0,r_c]$, $r_c<r_m$. We consider the state space $\mathbf{X}$ defined by:

$$\mathbf{X}=\Big\{\ \phi(r,\theta)\ \Big|\ \phi(r,\theta)\in\mathbf{C}([0,r_m]\times[-\tau,0]),$$
$$\phi(0,0)=\beta\int_{r_1}^{r_2}k(r)h(r)\phi(r,-\tau)dr\Big\}.$$

It becomes a Banach space with the usual norm of $\mathbf{C}([0,r_m]\times[-\tau,0])$.

Let the family of operators $\{T(t), t\geq 0\}$ be defined as follows:

$T(t)\colon \mathbf{X}\longrightarrow \mathbf{X}$

$$[T(t)\phi](r,\Theta)=\begin{cases}\phi(r,t+\Theta), & \Theta+t\leq 0,\\ \phi(r-t-\Theta,0)e^{-\int_{r-t-\Theta}^{r}\mu(\rho)d\rho}, & r\geq\Theta+t>0,\\ \beta\int_{r_1}^{r_2}k(s)h(s)\phi(s,t+\Theta-r-\tau)ds\,e^{-\int_0^r\mu(\rho)d\rho}, & r<\Theta+t,\end{cases}$$

for $0\leq t\leq\tau$,

$$T(t)=[T(\tau)]^{[\frac{t}{\tau}]}T(t-[\tfrac{t}{\tau}]\tau), \qquad \text{for } t>\tau, \tag{2}$$

where $[\alpha]$ denotes the integral part of the real number α.

Theorem 1. The family of operator $\{T(t), t\geq 0\}$ defined in (2) is a one parameter strongly continuous (C_0) semigroup of bounded operators in the Banach space $\mathbf{X}$.

Proof. Let $t_1+t_2\leq\tau$, $t_i\geq 0$, $i=1, 2$ and take $\phi\in\mathbf{X}$ then

$$[T(t_2)\,T(t_1)\phi](r,\Theta)=T(t_2)\begin{cases}\phi(r,t_1+\Theta), & \Theta+t_1\leq 0,\\ \phi(r-t_1-\Theta,0)e^{-\int_{r-t_1-\Theta}^{r}\mu(\rho)d\rho}, & r\geq\Theta+t_1>0,\\ \beta\int_{r_1}^{r_2}k(s)h(s)\phi(s,t_1+\Theta-r-\tau)ds\,e^{-\int_0^r\mu(\rho)d\rho}, & r<t_1+\Theta,\end{cases}$$

$$=T(t_2)\psi(r,\Theta)$$

$$=\begin{cases}\psi(r,t_2+\Theta), & \Theta+t_2\leq 0,\\ \psi(r-t_2-\Theta,0)e^{\int_{r-t_2-\Theta}^{r}\mu(\rho)d\rho}, & r\geq\Theta+t_2>0,\\ \beta\int_{r_1}^{r_2}k(s)h(s)\psi(s,t_2+\Theta-r-\tau)de^{-\int_0^r\mu(\rho)d\rho}, & r<t_2+\Theta\end{cases}$$

$$=\begin{cases}\phi(r,t_1+t_2+\ominus), & \ominus+t_1+t_2\le 0,\\ \phi(r-t_1-t_2-\ominus,0)e^{-\int_{r-t_1-t_2+\ominus}^{r}\mu(\rho)d\rho}, & r\ge\ominus+t_1+t_2>0,\\ \beta\int_{r_1}^{r_2}k(s)h(s)\phi(s,t_1+t_2+\ominus-r-\tau)dse^{-\int_0^r\mu(\rho)d\rho}, & r<t_1+t_2+\ominus,\end{cases}$$

$$=T(t_2+t_1)\phi(r,\ominus),$$

here $\psi(r,\ominus)=T(t_1)\phi(r,\ominus)$ so that $T(t_1+t_2)=T(t_1)T(t_2)$. Furthermore, it can readily be shown that if $\psi(r,\ominus)=T(t)\phi(r,\ominus)$, $0\le t\le\tau$, then

$$\psi(0,0)=\beta\int_{r_1}^{r_2}k(r)h(r)\psi(r,-\tau)dr. \tag{2'}$$

As for the case $t_1+t_2>\tau$, $t_i>0$, $i=1,2$; similar calculations lead to

$$T(t_1+t_2)=T(t_2)T(t_1)$$

and a counterpart of (2) can also be deduced from definition (2) and (2').

To complete the proof, it suffices to prove that for every $\phi\in X$,

$$\lim_{t\to 0^+}\|T(t)\phi(r,\ominus)-\phi(r,\ominus)\|=0.$$

To this end let $t<\tau$, and we have

$$\begin{aligned}&\|T(t)\phi(r,\ominus)-\phi(r,\ominus)\|\\ &\quad\le \max_{\ominus+t\le 0}\|\phi(r,t+\ominus)-\phi(r,\ominus)\|\\ &\quad+\max_{r\ge t+\ominus\ge 0}\|\phi(r-t-\ominus,0)e^{-\int_{r-t-\ominus}^{r}\mu(\rho)d\rho}-\phi(r,\ominus)\|\\ &\quad+\max_{r\le t+\ominus}\|\beta\int_{r_1}^{r_2}k(s)h(s)\phi(s,t+\ominus-r-\tau)dse^{-\int_0^r\mu(\rho)d\rho}-\phi(0,0)\|\\ &\quad+\max_{r\le t+\ominus}\|\phi(r,\ominus)-\phi(0,0)\|\end{aligned}$$

From this and the uniform continuity of $\phi(r,\ominus)$,

$$\lim_{t\to 0^+}\|T(t)\phi(r,\ominus)-\phi(r,\ominus)\|=0.$$

follows. This completes the proof.

Let **A** be the infinitesimal generator of the C_0 semigroup $T(t)$, then we have

Theorem 2.

$$D(\mathbf{A})=\left\{\phi(r,\theta) \mid \phi\in\mathbf{X},\ \frac{\partial\phi(r,\theta)}{\partial\theta},\ \frac{\partial\phi(r,0)}{\partial r}+\mu(r)\phi(r,0) \text{ continuous}\right.$$

$$\text{and } \frac{\partial\phi(r,0)}{\partial\theta}=-\frac{\partial\phi(r,0)}{\partial r}-\mu(r)\phi(r,0),$$

$$\left.\frac{\partial\phi(0,0)}{\partial\theta}=\beta\int_{r_1}^{r_2}k(s)h(s)\frac{\partial\phi(s,-\tau)}{\partial\theta}ds\right\}$$

$$\mathbf{A}\phi(r,\theta)=\begin{cases}\dfrac{\partial\phi(r,\theta)}{\partial\theta}, & \theta\in[-\tau,0),\ r\in[0,r_m],\\ -\dfrac{\partial\phi(r,0)}{\partial r}-\mu(r)\phi(r,0), & \theta=0,\ r\in(0,r_m],\\ \beta\displaystyle\int_{r_1}^{r_2}k(s)h(s)\frac{\partial\phi(s,-\tau)}{\partial\theta}ds, & \theta=r=0,\end{cases}$$

$$\forall\ \phi\in D(\mathbf{A}). \qquad (3)$$

Proof. From the definition of **A** and the norm $\|\ \|$, the pointwise limit $\lim_{t\to0^+}\frac{T(t)-I}{t}\phi(r,\theta)$ exists. Simple calculation leads to

$$\lim_{t\to0^+}\frac{T(t)\phi-\phi}{t}\ \mathbf{A}\phi=\begin{cases}\dfrac{\partial\phi(r,\theta^+)}{\partial\theta}, & \theta\in[-\tau,0),\ r\in[0,r_m],\\ -\dfrac{\partial\phi(r,0^+)}{\partial r}-\mu(r)\phi(r,0^+), & \theta=0,\ r\in(0,r_m],\\ \beta\displaystyle\int_{r_1}^{r_2}k(s)h(s)\frac{\partial\phi(s,-\tau)}{\partial\theta}ds, & \theta=r=0.\end{cases}$$

We denote by Ω the set on the right hand side of $D(\mathbf{A})$ in the theorem. Then, $\mathbf{A}\phi\in\mathbf{X}$ implies the right and left derivatives are equal, so that $\phi\in\Omega$.

Conversely, if $\phi\in\Omega$ and let

$$\psi(r,\Theta)=\begin{cases} \dfrac{\partial\phi(r,\Theta)}{\partial\Theta}, & \Theta\in[-\tau,0),\ r\in[0,r_m], \\ -\dfrac{\partial\phi(r,0)}{\partial r}-\mu(r)\phi(r,0), & \Theta=0,\ r\in(0,r_m], \\ \beta\displaystyle\int_{r_1}^{r_2}k(s)h(s)\frac{\partial\phi(s,-\tau)}{\partial\Theta}ds, & \Theta=r=0. \end{cases}$$

It follows that

$$\left\|\frac{T(t)-I}{t}\phi-\psi\right\|$$

$$\le \max_{\Theta+t\le 0}\left\|\frac{\phi(r,t+\Theta)-\phi(r,\Theta)}{t}-\frac{\partial\phi(r,\Theta)}{\partial\Theta}\right\|$$

$$+\max_{\ge\Theta+t>0}\left\|\frac{\phi(r-\Theta-t,0)e^{-\int_{r-t-\Theta}^{r}\mu(\rho)d\rho}-\phi(r,\Theta)}{t}-\frac{\partial\phi(r,\Theta)}{\partial\Theta}\right\|$$

$$+\max_{\le\Theta+t}\left\|\frac{\beta\int_{r_1}^{r_1}k(s)h(s)\phi(s,t+\Theta-r-\tau)dse^{-\int_0^r\mu(\rho)d\rho}-\phi(r,0)}{t}-\frac{\partial\phi(r,\Theta)}{\partial\Theta}\right\|.$$

Let us examine the last term

$$\frac{\phi(r,t+\Theta)-\phi(r,\Theta)}{t}\frac{\partial\phi(r,\Theta)}{\partial\Theta}=\frac{1}{t}\int_{r_1}^{r_2}\frac{\partial\phi(r,\omega)}{\partial\Theta}d\omega-\frac{\partial\phi(r,\Theta)}{\partial\Theta},$$

the continuity of $\dfrac{\partial\phi(r,\Theta)}{\partial\Theta}$ implies the first term tends to zero $(t\rightarrow 0^+)$. Similar argument gives

$$\lim_{t\to 0^+}\left\|\frac{T(t)-I}{t}\phi-\psi\right\|=0,$$

so that $\phi\in D(\mathbf{A})$ and $\mathbf{A}\phi=\psi$. This completes the proof.

With the operator **A** at hand, equation (1) can be written as an abstract evolution equation in the Banach space **X**:

$$\begin{cases} \dfrac{\partial p(r,t+\Theta)}{\partial t}=\mathbf{A}p(r,t+\Theta), & t>0,\ 0<r<r_m, \\ p(r,\Theta)=p_0(r,\Theta), & 0\le r\le r_m,\ -\tau\le\Theta\le 0. \end{cases} \qquad (4)$$

Following [11], we have

Theorem 3. The solution, say $p(r,t+\Theta)$, of equation (4) exists and is unique. Furthermore,

<1> $p(r,t+\Theta) = T(t)p_0(r,\Theta)$,

<2> $T(t)p_0 \in \mathbf{C}([0,\infty);\mathbf{X}), \quad \forall \quad p_0 \in \mathbf{X}$,

<3>. $T(t)p_0 \in \mathbf{C}^1([0,\infty);\mathbf{X}), \quad \forall \quad p_0 \in D(\mathbf{A})$, (5)

Now, if $p(r,\Theta) \in D(\mathbf{A})$ let

$$g(h)=p(r_0+h),t_0+h), \quad r_0,t_0 \geq 0, \quad h \geq 0.$$

Hence, $g(h)$ satisfies

$$\begin{cases} \dfrac{dg(h)}{dh} = -\mu(r_0+h)g(h), \\ g(0)=p(r_0,t_0). \end{cases}$$

Solving the above gives

$$g(h)=p(r_0,t_0)e^{-\int_0^h \mu(r_0+\rho)d\rho}$$

and substituting $(r_0,h,t_0)=(r-t,t,0)$ and $(0,r,t-r)$ in turn, we have

$$p(r,t)=\begin{cases} p_0(r-t,0)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}, & r \geq t \geq 0, \\ \beta\displaystyle\int_{r_1}^{r_2} k(s)h(s)p(s,t-r-\tau)ds\,e^{-\int_0^r \mu(\rho)d\rho}, & r<t, \end{cases}$$

so that

$$p(r,t+\Theta)=\begin{cases} p_0(r,t+\Theta), & t+\Theta \leq 0, \\ p_0(r-t-\Theta,0)e^{-\int_{r-t-\Theta}^{r}\mu(\rho)d\rho}, & r \geq t+\Theta \geq 0, \\ \beta\displaystyle\int_{r_1}^{r_2} k(s)h(s)p(s,t+\Theta-r-\tau)ds\,e^{-\int_0^r \mu(\rho)d\rho}, & r<t+\Theta. \end{cases}$$

Simple calculation yields

$$T(t)p_0(r,\Theta)=\begin{cases} p_0(r,t+\Theta), & t+\Theta\le 0,\\ p_0(r-t-\Theta,o)e^{-\int_{r-t-\Theta}^{r}\mu(\rho)d\rho}, & r\ge t+\Theta\ge 0,\\ \sum_{k=0}^{n}\phi_k(t+\Theta-r-\tau)e^{-\int_0^r\mu(\rho)d\rho}+\psi_k(t+\Theta-r-\tau)e^{-\int_0^r\mu(\rho)d\rho}, & \\ \qquad 0\le t+\Theta-r\le(n+1)\tau, & \end{cases}$$

where

$$\begin{cases}\phi(t)=\beta\int_{r_1}^{r_2}k(s)h(s)p_0(s-t,0)e^{-\int_{s-t}^{s}\mu(\rho)d\rho}ds,\\ \phi_k(t)=\beta\int_{r_1}^{r_2}k(s)h(s)\phi_{k-1}(t-s-\tau)e^{-\int_0^s\mu(\rho)d\rho}ds,\quad k=1,2\ldots\end{cases}$$

$$\begin{cases}\psi_0(t)=\beta\int_{r_1}^{r_2}k(s)h(s)p_0(s,t)e^{-\int_{s-t}^{s}\mu(\rho)d\rho}ds,\\ \psi_k(t)=\beta\int_{r_1}^{r_2}k(s)h(s)\psi_{k-1}(t-s-\tau)e^{-\int_0^s\mu(\rho)d\rho}ds,\quad k=1,2\ldots,\end{cases}\qquad(6)$$

since $D(\mathbf{A})$ is dense in $\mathbf{X}$ so that (6) is true for all $p(r,\Theta)\in\mathbf{X}$. For $0\le t\le\tau$, (6) is the same as (2) which shows that (6) is an explicit representation of the C_0 semigroup $T(t)$. From the continuity of parameters and the recent results in [12], we have

Theorem 4. $T(t)$ is compact in $\mathbf{X}$ for $t\ge r_m+\tau$ but it is not for $t<r_m+\tau$ and hence $T(t)$ does not have an analytic extension.

3. Spectral Properties of the Infinitesimal Generator

In this section we develop some spectral properties of $\mathbf{A}$. We consider the following equation for all $\psi(r,\Theta)\in\mathbf{X}$ and $\lambda\in\mathbf{C}$:

$$(\lambda-\mathbf{A})\phi=\psi$$

or

$$\begin{cases} \dfrac{\partial\phi(r,\Theta)}{\partial\Theta} = \lambda\phi(r,\Theta)-\psi(r,\Theta), & \Theta\in[-\tau,0),\ r\in[0,r_m], \\ -\dfrac{\partial\phi(r,0)}{\partial r} - \mu(r)\phi(r,0)=\lambda\phi(r,0)-\psi(r,0), & \Theta=0,\ r\in(0,r_m], \\ \beta\displaystyle\int_{r_1}^{r_2} k(s)h(s)\frac{\partial\phi(s,-\tau)}{\partial\Theta}ds=\lambda\phi(0,0)-\psi(0,0), & \Theta=r=0. \end{cases}$$

Let

$$F(\lambda)=1-\beta\int_{r_1}^{r_2} k(r)h(r)e^{-\lambda(r+\tau)-\int_0^r\mu(\rho)d\rho}dr, \tag{7}$$

$$g_\psi(\lambda)=e^{-\lambda\tau}\beta\int_{r_1}^{r_2}k(r)h(r)\left[\int_0^r e^{-\lambda(r-s)-\int_s^r\mu(\rho)d\rho}\psi(s,0)+\int_{-\tau}^0 e^{-\lambda s}\psi(r,s)ds\right]dr.$$

Then when $F(\lambda)\neq 0$ so that $\lambda\in\rho(\mathbf{A})$ and

$$R(\lambda,\mathbf{A})\psi=\frac{g_\psi(\lambda)}{F(\lambda)}=e^{-\lambda(r-\Theta)-\int_0^r\mu(\rho)d\rho}+\int_0^r e^{-\lambda(r-\Theta-s)-\int_s^r\mu(\rho)d\rho}\psi(s,0)ds$$
$$-\int_0^\Theta e^{-\lambda(\Theta-s)}\psi(r,s)ds. \tag{9}$$

When $F(\lambda)=0$, $(\lambda-\mathbf{A})\phi = 0$ has the unique solution

$$\phi(r,\Theta) = e^{-\lambda r-\int_0^r\mu(\rho)d\rho}e^{\lambda\Theta}. \tag{10}$$

Applying Theorem 4, the spectral mapping theorem and in the spirit of [2] we have

Theorem 5. (1). The point spectrum of **A** consists of distinct eigenvalues of geometric multiplicity 1. They consist of the zeros of the entire function $F(\lambda)$.

(2). **A** has only one real eigenvalue, its algebraic multiplicity is 1.

(3). There is only a finite number of eigenvalues of **A** in any finite strip parallel to the imaginary axis.

Let λ_0 be the real eigenvalue of $\mathbf{A}$ and $\mathbf{p}_{\lambda_0}$ its associated projection on the eigensubspace of λ_0, then for any $\phi \in \mathbf{X}$,

$$\mathbf{p}_{\lambda 0}\phi = \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)R(\lambda,\mathbf{A})\phi = \frac{g_\psi(\lambda)}{F'(\lambda_0)}\, e^{-\lambda_0(r-\ominus)-\int_0^r\mu(\rho)d\rho}. \tag{11}$$

In view of the asymptotic formula for compact semigroup [12], we have

Theorem 6. Let $p_0 \in D(\mathbf{A})$, $\varepsilon \geq 0$ be a positive number such that

$$\sigma(\mathbf{A}) \cap \{ \lambda \mid \mathrm{Re}\lambda < \lambda_0,\ \mathrm{Re}\lambda \geq \lambda_0-\varepsilon \} = \emptyset,$$

then the solution of (4) can be written as

$$p(r,t+\ominus) = \frac{g_{p_0}(\lambda_0)}{F'(\lambda_0)}\, e^{-\lambda_0(r-\ominus)-\int_0^r\mu(\rho)d\rho} e^{\lambda_0 t} + o(e^{(\lambda_0-\varepsilon)t}),$$

$$t \longrightarrow +\infty. \tag{12}$$

In particular, the solution of equation (1) has an asymptotic form

$$p(r,t) = \frac{g_{p_0}(\lambda_0)}{F'(\lambda_0)}\, e^{-\lambda_0 r-\int_0^r\mu(\rho)d\rho} e^{\lambda_0 t} + o(e^{(\lambda_0-\varepsilon)t}),$$

$$t \longrightarrow +\infty. \tag{13}$$

Denoting $\mathbf{C}_{\lambda 0} = \frac{g_{p_0}(\lambda_0)}{F'(\lambda_0)}$, then the total population $N(t)$ at t has its asymptotic form

$$N(t) = \mathbf{C}_{\lambda 0}\| e^{-\lambda_0 r-\int_0^r\mu(\rho)d\rho} \|_{L(0,r_m)} e^{\lambda_0 t} + o(e^{(\lambda_0-\varepsilon)t}),$$

$$t \longrightarrow +\infty. \tag{14}$$

Corollary 1. Let β_{cr} be the critical fertility rate defined by

$$\beta_{cr} = \left[\int_{r_1}^{r_2} k(r)h(r)e^{-\int_0^r \mu(\rho)d\rho} dr\right]^{-1}$$

then

<1>. when $\beta > \beta_{cr}$, $N(t) \longrightarrow +\infty$ $(t \longrightarrow \infty)$;

<2>. when $\beta = \beta_{cr}$, $N(t) \longrightarrow C_0 \| e^{-\int_0^r \mu(\rho)d\rho} \|_{L(0,r_m)}$ $(t \longrightarrow \infty)$;

<3>. when $\beta < \beta_{cr}$, $N(t) \longrightarrow 0$ $(t \longrightarrow \infty)$.

Furthermore, if we let $p^*(r) = C_0 e^{-\int_0^r \mu(\rho)d\rho}$ and for $\beta=\beta_{cr}$, then in view of (13),

$$\lim_{t \to \infty} p(r,t) = p^*(r), \qquad \forall\ r \in [0, r_m] \text{ uniformly.}$$

In the squeal, we are interested in the case $\beta = \beta_{cr}$. Let λ_i, $\bar{\lambda}_i$, $\mathrm{Re}\lambda_i = \alpha < 0$, i=1, 2...n, be eigenvalues of $\mathbf{A}$ such that the set $\{\lambda \mid \alpha \le \mathrm{Re}\lambda < 0\}$ contains no other eigenvalues except λ_i, $\bar{\lambda}_i$. Denote $\mathbf{P}_{\lambda_i}$ the projection on the eigensubspace of λ_i. It can readily be shown that (cf. [13])

$$T(t)p_0 = T(t)\mathbf{p}_0 p_0 + T(t)\sum_{i=1}^{n} (\mathbf{p}_{\lambda_i} + \mathbf{p}_{\bar{\lambda}_i}) p_0 + o(e^{-(\alpha-\varepsilon)t})$$

when $\varepsilon > 0$ is such that $\sigma(\mathbf{A}) \cap \{\lambda \mid \alpha - \varepsilon \le \mathrm{Re}\lambda < \alpha\} = \emptyset$.

Let the algebraic multiplicity of λ_i be n_i, and g_k^i, $k = 1, 2 \cdots n_i$ be a basis of $\mathbf{p}_{\lambda_i} \mathbf{X}$, i.e.

$$(\mathbf{A} - \lambda_i) g_k^i = g_{k-1}^i, \quad k = 1, 2, \ldots, n_i,$$

$$(\mathbf{A} - \lambda_i) g_1^i = 0.$$

If $\mathbf{p}_{\lambda_i} P_0 = \sum_{k=1}^{n_i} \alpha_{ki} g_k$, then

$$T(t)[\mathbf{p}_{\lambda_i}+\bar{\mathbf{p}}_{\lambda_i}]P_0 = T(t)e^{-\lambda_i t}e^{\lambda_i t}\mathbf{p}_{\lambda_i}P_0+T(t)e^{-\bar{\lambda}_i t}e^{\bar{\lambda}_i t}\bar{\mathbf{p}}_{\lambda_i}P_0$$

$$= e^{\lambda_i t}\sum_{k=1}^{n_i}\sum_{m=1}^{n_i-1}\frac{\alpha_{ki}}{m!}t^m g^i_{\lambda k-m}$$

$$+e^{\bar{\lambda}_i t}\sum_{k=1}^{n_i}\sum_{m=1}^{n_i-1}\frac{\bar{\alpha}_{ki}}{m!}t^m g^i_{\lambda k-m}$$

$$=2e^{\alpha t}\cos\beta_i te^{\lambda_i t}\sum_{k=1}^{n_i}\sum_{m=1}^{n_i-1}\frac{\mathrm{Re}\alpha_{ki}}{m!}t^m g^i_{\lambda k-m}\ .$$

Notice that here we set $\lambda_k=\alpha+i\beta_k$, so that

$$p(r,t+\Theta)=p^*(r)+\sum_{i=1}^{n}2e^{\alpha t}\cos\beta_i t\sum_{k=1}^{n_i}\sum_{m=1}^{n_i-1}\frac{\mathrm{Re}\alpha_{ki}}{m!}t^m g^i_{\lambda k-m}+o(e^{(\alpha-\varepsilon)t})$$

which can also be written as

$$p(r,t+\Theta)=p^*(r)+e^{\alpha t}\sum_{i=1}^{n}\cos\beta_i t\sum_{k=1}^{M}Q_{ik}(t)g^i_k + o(e^{(\alpha-\varepsilon)t})$$

where $Q_{ik}(t)$ are polynomials of t, $g^i_k \in \mathbf{X}$ and $M<\infty$. Thus

$$p(r,t+\Theta)=p^*(r)+e^{\alpha t}\sum_{i=1}^{n}\cos\beta_i t\sum_{k=1}^{M}Q_{ik}(t)g^i_k + o(e^{(\alpha-\varepsilon)t})$$

and

$$N(t) = N^*+ e^{\alpha t}\sum_{i=1}^{n}\cos\beta_i t\ Q_i(t) + o(e^{(\alpha-\varepsilon)t})$$

where $Q_i(t)$, i=1, 2,...n are polynomials of finite degree. Now, it is not difficult to see that there exist nonnegative integers k and m distinct $\beta_1\cdots\beta_m$, among the β_i, i=1, 2,...n, with $\beta_i\neq0$, i=1, 2,...m such that

$$N(t) = N^*+t^k e^{\alpha t}\left[\sum_{i=1}^{m}\alpha_i\cos\beta_i t\right] + e^{\alpha t}\sum_{i=1}^{n}h_i\cos\beta_i t\cdot t^k$$

$$+e^{\alpha t}t^k o(t^{-2}) + o(e^{(\alpha-\varepsilon)t}) \quad (t \longrightarrow \infty). \tag{15}$$

Here we have assumed $T(t)\sum_{i=1}^{n}[p_{\lambda_i}+\bar{p}_{\lambda_i}]p_0 \neq 0$, otherwise we have to look into the projection on the third eigenvalue and so on.

Next, if there exists $t_0>0$, such that for $t\geq t_0$, $N(t)-N^*$ does not change sign, then $F(t)=\int_{t_0}^{t}\frac{N(\tau)-N^*}{\tau^k e^{\alpha\tau}}d\tau$ is monotonic in $[t_0,\infty)$ so that $\lim_{t\to\infty} F(t)$ exists. In view of (15), this is equivalent to the existence of $\lim_{t\to\infty}\sum_{i=1}^{m}\frac{\alpha_i}{\beta_i}\sin\beta_i t = B$. Moreover, if $\hat{t}_0>0$,

$$\int_{\hat{t}_0}^{\infty}\frac{1}{t}\sum_{i=1}^{m}\frac{\alpha_i}{\beta_i}\sin\beta_i t\,dt = \sum_{i=1}^{m}\frac{\alpha_i}{\hat{t}_0\beta_i^2}\cos\beta_i\hat{t}_0+\int_{\hat{t}_0}^{\infty}-t^{-2}\sum_{i=1}^{M}\frac{\alpha_i}{\beta_i^2}\cos\beta_i t\,dt$$

is a convergent integral, but $\int_{\hat{t}_0}^{\infty}\frac{1}{t}\,dt$ is not. Thus $B = 0$ or

$$\lim_{t\to\infty}\sum_{i=1}^{m}\frac{\alpha_i}{\beta_i}\sin\beta_i t = 0. \tag{16}$$

Multiplying (16) by $2\sin\beta_i t$ on both sides gives

$$\frac{\alpha_1}{\beta_1}-\frac{\alpha_1}{\beta_1}\cos 2\beta_1 t+\sum_{i=1}^{m}\left[\frac{\alpha_i}{\beta_i}\cos(\beta_1-\beta_i)t-\frac{\alpha_i}{\beta_i}\cos(\beta+\beta_i)t\right]$$

$$\longrightarrow 0, \qquad \text{as } t\longrightarrow\infty,$$

since for any $1\leq i\leq m$, $\beta_1-\beta_i\neq 0$ and $\beta_1+\beta_i\neq 0$ and it follows from the above that $\alpha_1/\beta_1 = 0$ or $\alpha_1 = 0$. But this contradicts the assumption that α_1 is nonzero. This concludes the oscillatory convergence of the total population $N(t)$ to N^*.

Theorem 7. For $\beta = \beta_{cr}$ the total population $N(t)$ tends to N^*, the equilibrium total population, in a oscillatory fashion, i.e., for any $\alpha>0$, $N(t)-N^*$ has at least one zero in $[\alpha,+\infty)$ and $\lim N(t) = N^*$.

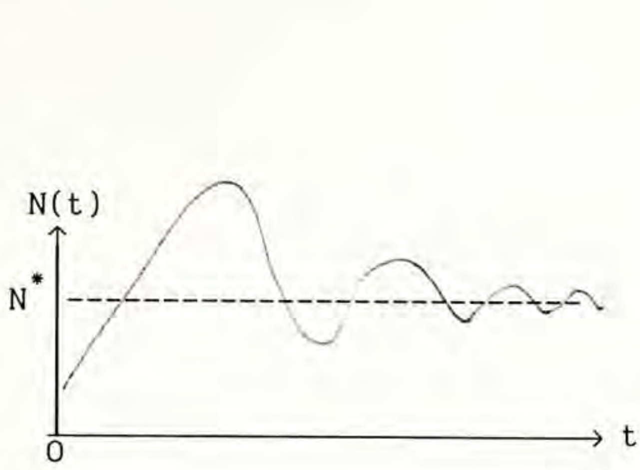

Figure. N(t) oscillates about N^*

It is interesting to note that such type of oscillatory behaviour was uncovered by Feller earlier [10] and is still of much interest [3].

1.4 A Nonlinear Semigroup of the Logistivc Age-Dependent Model with Time Delay

In this section the following nonlinear logistic age-dependent population model with delay will be considered:

$$\frac{\partial p(r,t)}{\partial t}+\frac{\partial p(r,t)}{\partial r}=-\mu(r)p(r,t)-Kf(N(t))p(r,t), \quad 0<r<r_m, \quad t>0,$$

$$p(r,\Theta)=p_0(r,\Theta), \quad 0\le r\le r_m, \quad -\tau\le\Theta\le 0,$$

$$p(0,t)=\beta\int_{r_1}^{r_2}k(r)h(r)p(r,t-\tau)dr, \qquad t\ge 0, \tag{17}$$

where K>0 is an environmental parameter reflecting a depression of growth as the population becomes large, $f(\xi)$ is a nonnegative continuous function, differentiable for $\xi>0$ such that

$$f(0)=0, \quad f(\xi)>0, \quad \forall\ \xi>0. \tag{18}$$

Other parameters are just the same as those described in section 1. **Y** is a subset of X defined by:

$$\mathbf{Y}=\Big\{\phi(r,\Theta)\,\Big|\,\phi,\frac{\partial\phi}{\partial\Theta}\in X,\ \frac{\partial\phi(r,0)}{\partial\Theta}=-\frac{\partial\phi(r,0)}{\partial r}-\mu(r)\phi(r,0)-Kf(N_0)\phi(r,0),$$

$$\phi(r,\Theta)\geq 0\}, \tag{19}$$

where $N_0=\int_0^{r_m}\phi(r,0)dr$.

We have the following existence results

Theorem 8. For any given $p_0\in Y$, the solution $p(r,t)$ to equation (17) exists and is unique with $p(r,t+\Theta)\in \mathbf{Y}\ \forall\ t\geq 0$.

Proof. Consider the nonlinear equation

$$p(r,t+\Theta)=\begin{cases} p_0(r,t+\Theta), & t+\Theta\leq 0,\\ p_0(r-t-\Theta,0)e^{-\int_{r-t-\Theta}^{r}\mu(\rho)d\rho}e^{-\int_0^{t+\Theta}Kf(N(\rho))d\rho}, & r\geq t+\Theta\geq 0,\\ \phi(t+\Theta-r)e^{-\int_0^r\mu(\rho)d\rho}e^{-\int_{t+\Theta-r}^{t+\Theta}Kf(N(\rho)d\rho}, & r<t+\Theta,\\ \phi(t)=\beta\int_{r_1}^{r_2}k(r)h(r)p(r,t-\tau)dr. \end{cases} \tag{20}$$

We see that if $N(t)\geq 0$ is continuous for $t\geq 0$ then the solution of (20), $p(r,t+\Theta)$ belongs to $\mathbf{X}$ for $t\geq 0$, $p(r,t+\Theta)\geq 0$. Moreover, $\phi(t)$ is continuously differentiable for $t\geq 0$. In fact, for $0\leq t\leq\tau$,

$$\phi(t)=\beta\int_{r_1}^{r_2}k(r)h(r)p(r,t-\tau)dr.$$

Thus $\quad \phi'(t)=\beta\int_{r_1}^{r_2}k(r)h(r)\frac{\partial p(r,t-\tau)}{\partial\Theta}dr,\quad \phi'_-(\tau)=\int_{r_1}^{r_2}k(r)h(r)\frac{\partial p_0(r,0)}{\partial\Theta}dr,$

when $t\geq\tau$,

$$\phi(t)=\beta\int_{r_1}^{r_2}k(r)h(r)\phi(t-r-\tau)e^{-\int_0^r\mu(\rho)d\rho}e^{-\int_{t-r-\tau}^{t-\tau}Kf(N(\rho))d\rho}dr\ +$$

$$\beta\int_{r_1}^{r_2}k(r)h(r)p_0(r-t+\tau,0)e^{-\int_{r-t+\tau}^{r}\mu(\rho)d\rho}e^{-\int_0^{t-\tau}Kf(N(\rho))d\rho}dr. \tag{21}$$

Here we assume the functions take on zero outside their domain of definition. For $\tau\le t\le r_1+\tau$,

$$\phi(t)=\beta\int_{r_1}^{r_2}k(r)h(r)p_0(r-t+\tau,0)e^{-\int_{r-t+\tau}^{r}\mu(\rho)d\rho}e^{-\int_0^{t-r}Kf(N(\rho))d\rho}dr$$

$$\phi'(t)=\beta\int_{r_1}^{r_2}k(r)h(r)\left[\frac{\partial p_0(r-t+\tau,0)}{\partial\Theta}+[kf(N_0)-Kf(N(t-\tau))]p_0(r-t+\tau,0)\cdot\right.$$
$$\cdot e^{-\int_{r-t+\tau}^{r}\mu(\rho)d\rho}e^{-\int_0^{t-\tau}Kf(N(\rho))d\rho}dr,$$

$$\phi'_+(\tau)=\beta\int_{r_1}^{r_2}k(r)h(r)\frac{\partial p_0(r,0)}{\partial\Theta}dr.$$

Hence $\phi(t)$ is continuous for $0\le t\le r_1+\tau$. If $r_1+\tau\le t\le 2(r_1+\tau)$ and $r\in[r_1,r_2]$, then $t-r-\tau\le r_1+\tau$,

$$\phi'(t)=\beta\int_{r_1}^{r_2}k(r)h(r)\Big[\phi'(t-r-\tau)-Kf(N(t-\tau))\phi(t-r-\tau)+Kf(N(t-r-\tau))\cdot$$
$$\phi(t-r-\tau)\Big]e^{-\int_0^{r}\mu(\rho)d\rho}e^{-\int_{t-r-\tau}^{t-\tau}Kf(N(\rho))d\rho}dr$$

$$+\beta\int_{r_1}^{r_2}k(r)h(r)\left[\frac{\partial p_0(r-t+\tau,0)}{\partial\Theta}+[Kf(N_0)-Kf(N(t-\tau))]p_0(r-t+\tau,0)\right]\cdot$$
$$\cdot e^{-\int_{r-t+\tau}^{r}\mu(\rho)d\rho}e^{-\int_0^{t-\tau}Kf(N(\rho))d\rho}dr,$$

so that $\phi'(t)$ is continuous for $0\le t\le 2(r_1+\tau)$. Inductively, it is obvious that $\phi'(t)$ is continuous in $[0,\infty)$. From (20) and for $t\ge0$,

$$\frac{\partial p(r,t+\Theta)}{\partial\Theta}=\begin{cases}\dfrac{\partial p_0(r,t+\Theta)}{\partial\Theta}, & t+\Theta\le0,\\[2mm] \left[\dfrac{\partial p_0(r-t-\Theta,0)}{\partial\Theta}+[Kf(N_0)-Kf(N(t+\Theta))]p_0(r-t+\Theta,0)\right]\cdot & \\ \quad\cdot e^{-\int_{r-t-\Theta}^{r}\mu(\rho)d\rho}e^{-\int_0^{t+\Theta}Kf(N(\rho))d\rho}dr, & r\ge t+\Theta\ge0,\\[2mm] [\phi'(t+\Theta-r)-Kf(N(t+\Theta))+Kf(N(t+\Theta-r))]\cdot & \\ \quad\cdot e^{-\int_0^{r}\mu(\rho)d\rho}e^{-\int_{t+\Theta-r}^{t+\Theta}Kf(N(\rho)d\rho}, & r<t+\Theta.\end{cases}$$

Thus $\frac{\partial p(r,t+\Theta)}{\partial\Theta} \in \mathbf{X}$, and

$$\frac{\partial p(r,t)}{\partial t} + \frac{\partial p(r,t)}{\partial\Theta} = \begin{cases} \left[-\frac{\partial p_0(r-t,0)}{\partial r} - \mu(r-t)p_0(r-t,0) - Kf(N(t))p_0(r-t,0)\right] \\ \qquad \cdot e^{-\int_{r-t}^{r}\mu(\rho)d\rho} e^{-\int_0^t Kf(N(\rho))d\rho}, & r \geq t, \\ [\phi'(t-r) - Kf(N(t)) + Kf(N(t-r))] \cdot \\ \qquad \cdot e^{-\int_0^r \mu(\rho)d\rho} e^{-\int_{t-r}^{t} Kf(N(\rho))d\rho}, & r<t, \end{cases}$$

$$= -\frac{\partial p(r,t)}{\partial r} - \mu(r)p(r,t) - Kf(N(t))p(r,t).$$

Therefore assuming $N(t)\geq 0$ to be continuous we have proved that a solution of (20) is also a solution of (17). The converse can also be readily shown to be true. Thus we have

Proposition 1. If $p_0(r,\Theta) \in \mathbf{Y}$ and $N(t)$ is continuous for $t\geq 0$ then equation (17) and equation (20) are equivalent.

From the above proof, under the assumption that $N(t)\geq 0$ is continuous, we see that if (20) has a solution then $p(r,t+\Theta) \in \mathbf{X}$ and $p(r,t+\Theta)\geq 0$, for all $t \geq 0$,. We can consider, for any $T \geq 0$,

$$\phi(t) = \mathbb{B}N(t), \quad D(\mathbb{B}) = \{N(t) \mid N(t)\geq 0, \ N(t)\in \mathbf{C}[0,T]\},$$

where $\mathbb{B}$ is a nonlinear operator from $D(\mathbb{B}) \longrightarrow D(\mathbb{B})$. Also we may define from $D(\mathbb{B}) \longrightarrow D(\mathbb{B})$ the operator

$$\mathbb{A}\, N(t) = \int_0^t \mathbb{B}N(t-r) e^{-\int_0^r \mu(\rho)d\rho} e^{-\int_{t-r}^{t} Kf(N(\rho))d\rho} dr$$

$$+ \int_t^{r_m} p_0(r-t,0) e^{-\int_{r-t}^{r} \mu(\rho)d\rho} e^{-\int_0^t Kf(N(\rho))d\rho}. \qquad (22)$$

Let $\bar{p}(r,t+\Theta)$ be a solution of (20) (uniquely determined and continuous) for K=0, and the corresponding total population $\bar{N}(t)$, $t\geq 0$. Define a subset of $D(\mathbb{B})$ by

$$\Omega = \{ N(t) \mid 0 \le N(t) \le \bar{N}(t),\ 0 \le t \le T \}.$$

Then, Ω is a closed convex subset of $D(\mathbb{B})$. Further, $\mathbb{A}\Omega\subset\Omega$, f being continuous implies $\mathbb{A}$ is also a continuous mapping. It is not difficult to prove that $\mathbb{A}\Omega$ is a sequentially compact in $D(\mathbb{B})$. In fact for t, $t_0 \in [0,T]$,

$$\left| e^{-\int_0^t Kf(N(\rho))d\rho} - e^{-\int_0^{t_0} Kf(N(\rho))d\rho} \right|$$

$$\le \max_{\xi\in[0,M]} Kf(\xi)\,|t-t_0| \qquad M = \max_{t\in[0,T]} \bar{N}(t)$$

$$\left| e^{-\int_{t-r}^t Kf(N(\rho))d\rho} - e^{-\int_{t_0-r}^{t_0} Kf(N(\rho))d\rho} \right|$$

$$\le 2\max_{\xi\in[0,M]} Kf(\xi)\,|t-t_0|$$

This implies $\mathbb{B}\Omega$ is sequentially compact and so is $\mathbb{A}\Omega$. Apply Schauder's fixed point theorem, we see that $\mathbb{A}$ has at least one fixed point N(t) in Ω. This N(t), together with equation (20), gives a solution $p(r,t+\Theta)$ for equation (17). To prove the uniqueness of $p(r,t+\Theta)$ is equivalent to prove the fixed point of $\mathbb{A}$ is so. To this end, suppose N(t) and $N_0(t)$ are fixed points of $\mathbb{A}$, then

$$N(t)-N_0(t)$$

$$=\int_0^t \left[\mathbb{B}N(t-r)-\mathbb{B}N_0(t-r)\right] e^{-\int_0^r \mu(\rho)d\rho} e^{-\int_{t-r}^t Kf(N(\rho))d\rho}$$

$$+\int_0^t \mathbb{B}N_0(t-r)e^{-\int_0^r \mu(\rho)d\rho}\left[e^{-\int_{t-r}^t Kf(N(\rho))d\rho} - e^{-\int_{t-r}^t Kf(N_0(\rho))d\rho}\right]dr$$

$$+\int_t^{r_m} p_0(r-t,0)e^{-\int_{r-t}^r \mu(\rho)d\rho}\left[e^{-\int_0^t Kf(N(\rho))d\rho} - e^{-\int_0^t Kf(N_0(\rho))d\rho}\right]dr$$

$$=\int_0^t \mathbb{B}N_0(t-r)e^{-\int_0^r \mu(\rho)d\rho}\left[e^{-\int_{t-r}^t [Kf(N(\rho))-Kf(N_0(\rho))]d\rho} - 1\right]$$

$$e^{-\int_{t-r}^t Kf(N_0(\rho))d\rho} dr$$

$$+\int_t^{r^m} p_0(r-t,0)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}\left[e^{-\int_0^t[Kf(N(\rho))-Kf(N_0(\rho))]d\rho}-1\right]$$

$$\cdot e^{-\int_{t-r}^{t}Kf(N_0(\rho))d\rho}dr$$

$$\forall\ t \in [0,\tau].$$

From the fact that $|e^z-1| \le |z|e^{|z|}$, we have

$$\left|e^{-\int_0^t[Kf(N(\rho))-Kf(N_0(\rho))])d\rho}-1\right|$$

$$\le t \max_{\xi\in[0,M]} |Kf'(\xi)|\cdot C\|N-N_0\|_{C[0,\tau]},\quad C=\max_{\xi\in[0,M]} f(\xi)\ ,\forall\ t\in[0,\tau],$$

and hence there exists $\bar{K} > 0$ such that

$$|N(t)-N_0(t)| \le \bar{K}t\cdot\|N-N_0\|_{C[0,\tau]}\ ,\quad \forall\ t\in[0,\tau].$$

It follows that for small enough t, $N(t)=N_0(t)$. Repeated use of the above argument leads to the general case $N(t)=N_0(t)$, $\forall\ t\in [0,T]$. Condition on f to be continuously differentiable can be weakened to a Lipschitz type condition

$$|f(\xi_1)-f(\xi_2)| \le L(M)|\xi_1-\xi_2|,\quad \forall\ 0\le \xi_i \le M,\ i=1,\ 2,\quad M \ge 0. \qquad (23)$$

We have thus completed the proof of Theorem 9.

Since for any $p_0(r,\theta)\in$ **X** $\cap$ **P**=**Z**, **P** is the nonnegative cone of **X**, equation (20) has a unique solution $p(r,t+\theta)\in$ **Z** $\forall\ t\ge 0$ therefore we may define a nonlinear operator $\hat{T}$: $[0,\infty) \times \mathbf{Z} \longrightarrow \mathbf{Z}$ by

$$\hat{T}(t)p_0(r,\theta) = p(r,t+\theta). \qquad (24)$$

Similar to T(t), the linear case, we have

Theorem 9. $\{\hat{T}(t),\ t\ge 0\}$ is a one parameter family of nonlinear strongly continuous semigroup, its infinitesimal generator $\hat{\mathbf{A}}$ is given by

$$\hat{\mathbf{A}}\phi(r,\Theta)=\begin{cases} \dfrac{\partial\phi(r,\Theta)}{\partial\Theta}, & \Theta\in[-\tau,0),\ r\in[0,r_m], \\ -\dfrac{\partial\phi(r,0)}{\partial r}-\mu(r)\phi(r,0)-Kf(N_0)\phi(r,0), & \Theta=0,\ r\in(0,r_m], \\ \beta\displaystyle\int_{r_1}^{r_2}k(r)h(r)\frac{\partial\phi(r,-\tau)}{\partial\Theta}dr, & \Theta=r=0, \end{cases}$$

$$\forall\ \phi\in D(\hat{\mathbf{A}})=\mathbf{Y}. \qquad (25)$$

Using the operator $\hat{\mathbf{A}}$, equation (17) can be written as an abstract evolution equation

$$\begin{cases} \dfrac{\partial p(r,t+\Theta)}{\partial t}=\hat{\mathbf{A}}p(r,t+\Theta), & t>0,\ 0<r<r_m, \\ p(r,\Theta)=p_0(r,\Theta), & 0\le r\le r_m,\ -\tau\le\Theta\le 0, \end{cases} \qquad (26)$$

and Theorem 8 can be restated as

Theorem 8'. For any $p_0\in \mathbf{Z}$, equation (17) has a unique solution

$$p(r,t+\Theta)=\hat{T}(t)p_0(r,\Theta)\in \mathbf{C}([0,\infty);\mathbf{Z});$$

if $p_0\in D(\hat{\mathbf{A}})=\mathbf{Y}$ then

$$p(r,t+\Theta)=\hat{T}(t)p_0(r,\Theta)\in \mathbf{C}^1([0,\infty);\mathbf{Z}).$$

Earlier we have established that for any given $N^*>0$, the set

$$\Omega=\Big\{\phi(r,\Theta)\,\Big|\,\phi,\ \frac{\partial\phi}{\partial\Theta}\in \mathbf{X} \text{ and } \forall\ r\in[0,r_m],$$

$$\frac{\partial\phi(r,0)}{\partial\Theta}=-\frac{\partial\phi(r,0)}{\partial r}-[\mu(r)+Kf(N^*)]\phi(r,0)\Big\}$$

is dense in $\mathbf{X}$, and so is $\Omega\cap\mathbf{P}$ in $\mathbf{Z}$. Now if we take $\phi(r,\Theta)\in\Omega\cap\mathbf{P}$ and let

$$\bar{\phi}(r,\Theta)=\begin{cases} \phi(r,\Theta), & \text{if } G=\displaystyle\int_0^{r_m}\phi(r,0)dr=0 \\ \dfrac{N^*}{G}\phi(r,\Theta), & \text{if } G\neq 0, \end{cases}$$

then $\overline{\phi}(r,\theta) \in \mathbf{Y}$. Hence we have

Proposition 2. $\overline{D(\hat{A})} = \mathbf{Z}$.

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Chapter 2

Global Behaviour of Logistic Model of Age-Dependent Population Growth

2.1 Introduction

The purpose of this chapter is to study the large time behaviour of the age-dependent population model with a general logistic nonlinearity which provides a mechanism for self-limiting phenomena when the total population increases. Such a model may be described as follows:

Let $p(r,t)$ be the age density distribution of the population; r be the age; t be the time; $0\le r\le r_m$, $t\ge 0$, where r_m is the maximal age ever attained by individuals of the population. Assume that the specific fertility rate of females is a constant β; the female sex ratio $k(r)$ and fertility pattern $h(r)$ are independent of time; $h(r)$ satisfies

$$\int_{r_1}^{r_2} h(r)dr=1,$$

where$[r_1,r_2]$ denotes the fecundity period of females; the relative mortality rate $\mu(r)$ is a function that depends on age only and satisfies

$$r<r_m,\ \int_0^r \mu(\rho)d\rho <+\infty \text{ and } \lim_{r\to r_m^-} \int_0^{r_m}\mu(\rho)d\rho=+\infty.$$

Let the constant K be the environment parameter; $p_0(r)$ be the initial density, then the logistic growth model can be described by the following first order nonlinear partial differential equation

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{p(r,t)}{\partial r} = -\mu(r)p(r,t)-Kf(N(t))p(r,t), \quad 0<r<r_m, \ t>0, \\ p(r,0)=p_0(r), \quad 0\le r\le r_m, \\ p(0,t)=\beta\displaystyle\int_{r_1}^{r_2}k(r)h(r)p(r,t)dr, \quad t>0, \end{cases} \tag{1}$$

where $N(t)=\int_0^{r_m}p(r,t)dr$ is the total population, $f(N(t))$ is called the logistic term and $f(\xi)$ satisfies

$$\begin{cases} f(0)=0; \ f(\xi)>0, \ \forall \ \xi>0; \\ f(\xi) \text{ continuously differentiable.} \end{cases} \tag{2}$$

If $K=0$, (i.e. independent of habitat) then system (1) will be the well known age-dependent linear model of McKendrick. So we assume, in this chapter, that $K>0$.

The study of the nonlinear age-dependence dependent population models was initiated by Gurtin and MacCamy [4] and a recent monograph on the subject is Webb [6]. Global results in establishing the convergence of solution to equilibrium age distribution were given by Marcati [2] and Webb [7]. However, in [2] the logistic term treated was linear in the total population $N(t)$. While in [7], the nonlinear term was assumed to be increasing. Here we relax both of these assumptions and thus the method employed in proving global results are different from those of [2] and [7]. Furthermore, we emphasize on the behaviour of the total population as time goes to infinity since it is perhaps a more meaningful quantity than the density itself.

The plan of this chapter is the following. We first study the large time behaviour of the population density distribution for the general logistic model. It is proved that every solution (at least for the initial density $p_0 \geq 0$, $p_0 \in D(A)$, the domain of the population operator) must have a limit. We present conditions which guarantee the boundedness and stability of the solution. Finally we prove that no periodic solution exists for this system and no oscillation is possible for the total population about its equilibrium solution. This is in sharp contrast to the linear case where N(t) always oscillates about the equilibrium solution, as shown in chapter 1.

2.2 Global Behaviour of the Solution

Introducing the state space $X = L(0, r_m)$ and the population operator on **X**

$$A\phi = -\frac{d\phi(r)}{dr} - \mu(r)\phi(r),$$

$$D(A) = \left\{\phi \mid \phi,\ A\phi \in X, \phi(0) = \beta\int_{r_1}^{r_2} k(r)h(r)\phi(r)dr\right\}, \qquad (3)$$

we have as in [8],

Lemma 1. **A** is the infinitesimal generator of a one parameter strongly continuous semi-group of bounded operators in the Banach space **X**, and has the following asymptotic expression:

$$e^{At}\phi(r) = C_\phi e^{-\lambda_0 r - \int_0^r \mu(\rho)d\rho}[e^{\lambda_0 t} + o(e^{(\lambda_0 - \varepsilon)t})] \qquad (4)$$

where λ_0 is the real eigenvalue of **A**, which has the maximal real part in the spectrum of **A**; C_ϕ is a constant depending only on ϕ and $\varepsilon>0$ is any positive number such that $\sigma(\mathbf{A})\cap\{\lambda\,|\,\lambda_0-\varepsilon<\mathrm{Re}\lambda<\lambda_0\}=\emptyset$.

Firstly, we will prove the existence and uniqueness of solutions to equation (1).

Lemma 2. Let $p_0(r)\in D(\mathbf{A})$, $p_0\ge0$, then a necessary and sufficient condition for the equation (1) to have a classical solution is that the following nonlinear equation

$$N(t)=\|e^{\mathbf{A}t}p_0\|e^{-K\int_0^t f(N(\rho))d\rho} \tag{5}$$

has a continuous solution in $[0,\infty)$. Furthermore, the solution is unique and

$$p(r,t)=e^{\mathbf{A}t}p_0(r)e^{-K\int_0^t f(N(\rho))d\rho}. \tag{6}$$

Proof. Write equation (1) as an abstract evolution equation in the state space **X**:

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} = \mathbf{A}p(r,t) - Kf(N(t))p(r,t), \\ p(r,0) = p_0(r). \end{cases}$$

For the sufficient condition, we suppose the above equation has a classical solution $p(r,t)$, then it is obvious that $p(r,t)$ satisfies equation (6). By integrating both sides about r, we know that $N(t)$ is a continuous solution of (5). For the necessary condition, let $p(r,t)$ be the right hand side of equation (6), it can be checked directly that $p(r,t)$ satisfies equation (1). In order to prove existence and uniqueness of the solution in equation (5), let $t>0$ be an arbitrary given number, we define a set $\Omega\subset \mathbf{C}(0,T)$ by

$$\Omega = \{N(t)\,|\,0\le N(t)\le \|e^{At}p_0\|,\ N\in C(0,T)\}$$

and an operator $\mathcal{A}$ on $C(0,T)$ by

$$\mathcal{A}\,N(t)=\|e^{At}p_0\|e^{-K\int_0^t f(N(\rho))d\rho}$$

then $\mathcal{A}\Omega\subset\Omega$ and Ω is a convex closed subset of $C[0,T)$. For all $t_1, t_2\in[0,T]$

$$\mathcal{A}\,N(t_1)-\mathcal{A}\,N(t_2)=\left[\|e^{At_1}p_0\|-\|e^{At_2}p_0\|\right]e^{-K\int_0^{t_1} f(N(\rho))d\rho}$$

$$+\left[e^{-K\int_{t_2}^{t_1} f(N(\rho))d\rho}-1\right]e^{-K\int_0^{t_2} f(N(\rho))d\rho}\|e^{At_2}p_0\|$$

hence,

$$\|\mathcal{A}\,N(t_1)-\mathcal{A}\,N(t_2)\|\le\left|\,\|e^{At_1}p_0\|-\|e^{At_2}p_0\|\,\right|+\hat{M}\left[1-e^{-K\bar{M}(t_1-t_2)}\right]$$

where $\hat{M}=\max\limits_{t\in[0,T]}\|e^{At}p_0\|$, $\bar{M}=\max\limits_{\xi\in[0,M]} f(\xi)$.

Hence $\mathcal{A}\Omega$ is a relative compact set of Ω. Since the continuity of $\mathcal{A}$ is obvious by the Schauder's fixed point theorem, there exists a $N\in\Omega$ such that $\mathcal{A}\,N=N$, i.e. $N(t)$ is a solution of equation (5) in $[0,T]$. Furthermore, this solution is unique. In fact, if $N_i(t)$, $i=1,2$ are two solutions of equation (5), then

$$|N_1(t)-N_2(t)|\le L\left|1-e^{-K\int_0^t f'(\xi_t)[N_1(\xi)-N_2(\xi)]dt}\right|.$$

By $|1-e^z|\le|z|e^{|z|}$ and Growall's inequality, we have $N_1\equiv N_2$. The Lemma is proved.

Let $\beta_{cr}=\left[\int_{r_1}^{r_2}k(r)h(r)e^{-\int_0^r\mu(\rho)d\rho}\right]^{-1}$ be the critical fertility rate of females, then when $\beta\le\beta_{cr}$, $\lambda_0\le 0$ in (4), and by (5)

$$N(t)=Me^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho}+o(e^{-\varepsilon t}),\qquad t\longrightarrow\infty.$$

The first term in the above is a decreasing function of t, and so

$\lim_{t\to\infty} N(t)=c$ exists. As $Kf(\xi)>0$, $\forall\ \xi>0$, this implies $c=0$. Hence, we have

Proposition 1. For any $p_0(r)\in X$, $p_0\geq 0$, when $\beta\leq\beta_{cr}$, the solution of equation (1) $p(r,t)$ is globally asymptotic stable, i.e.

$$\lim_{t\to\infty} \|p(r,t)\|=\lim_{t\to\infty} N(t)=0.$$

In the following, we discuss mainly the case in which $\beta>\beta_{cr}$. In this case, $\lambda_0>0$ in (4).

If $Kf(\xi)<\lambda_0$, for any $\xi\geq 0$, then by the asymptotic expression

$$N(t)=[M+o(e^{-\varepsilon t})]e^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho},\quad t\longrightarrow\infty,$$

$\lim_{t\to\infty} N(t)=c$ exists. If in addition we assume

$$\text{meas}\{r\ |p_0(r)\neq 0, r\in[r_1,r_2]\}>0,$$

then $M>0$, and $c=+\infty$, i.e.

$$\lim_{t\to\infty} \|p(r,t)\|=\lim_{t\to\infty} N(t)=+\infty. \tag{7}$$

In general, there is a possibility that $\lim_{\xi\to\infty} Kf(\xi)=+\infty$. In this case, the situation like (7) can not happen. We shall discuss it in detail later. For the reason of continuity, we suppose that there exists a $\xi_0>0$ such that

$$Kf(\xi_0)=\lambda_0. \tag{8}$$

For notational simplicity, we write $q(r,t)=e^{\mathbf{A}t}p_0(r)$, $N_q(t)=\|e^{\mathbf{A}t}p_0\|$, $g(t)=\dfrac{N_q'(t)}{N_q(t)}-\lambda_0$, then

$$\begin{cases} p(r,t)=q(r,t)e^{-K\int_0^t f(N(\rho))d\rho}, \\ N(t)=N_q(t)e^{-K\int_0^t f(N(\rho))d\rho}. \end{cases} \tag{9}$$

First, we notice the fact that

$$\lim_{t\to\infty} g(t)=0. \tag{10}$$

In fact (we suppose that $p_0(r)\in D(\mathbf{A})$, and $p_0\geq 0$), by expression (4) and $\beta\int_{r_1}^{r_2}k(r)h(r)e^{-\lambda_0 r-\int_0^r\mu(\rho)d\rho}=1$, we have

$$\begin{aligned} N_q'(t)&=\int_0^{r_m}[-q_r'(r,t)-\mu(r)q(r,t)]dr \\ &=\beta\int_{r_1}^{r_2}k(r)h(r)q(r,t)dr-\int_0^{r_m}\mu(r)q(r,t)dr \\ &=C_0e^{\lambda_0 t}+\lambda_0 e^{\lambda_0 t}-C_0e^{\lambda_0 t}+o(e^{(\lambda_0-\varepsilon)t}) \\ &=\lambda_0 e^{\lambda_0 t}+o(e^{(\lambda_0-\varepsilon)t}) \end{aligned}$$

where $C_0=\| C_{p_0} e^{-\lambda_0 r-\int_0^r\mu(\rho)d\rho}\|$. Then (10) follows immediately.

Following the idea in [5] we consider the omega limit set of equation (5). Let

$$\Omega(N)=\{ N^* \mid \exists\ t_n\to\infty \text{ such that } N(t_n)\to N^*(n\to\infty)\}.$$

Taking arbitrary $N^*\in\Omega(N)$, (N^* may be infinite) then there exists $t_n\longrightarrow\infty$ such that

$$\lim_{n\to\infty} N(t_n)=N^*.$$

First, we consider the situation for which $N^*<+\infty$. If $Kf(N^*)<\lambda_0$, then there exists a small number $\varepsilon>0$ and a neighborhood of N^*, $S_\varepsilon(N^*)$, such that $kf(\xi)<\lambda_0-\varepsilon>0$ provided $\xi\in S_\varepsilon(N^*)$. Suppose that $N(t_n)\in S_\varepsilon(N^*)$ for $n>N_\varepsilon$, then there exists a $t_0>N_\varepsilon$ such that $g(t)>$

$-\varepsilon$ for $t\geq t_0$. Choose $N>N_\varepsilon$ such that $t_n\geq t_0$ for $n\geq N$ then consider the equation

$$\begin{cases} \dfrac{d\underline{N}(t)}{dt} =[\lambda_0-Kf(\underline{N}(t))]\underline{N}(t)-\varepsilon\underline{N}(t), \\ \underline{N}(t_n)=N(t_n), \qquad n>N. \end{cases}$$

By $\dfrac{dN(t)}{dt} = [\lambda_0-Kf(N(t))]N(t) + g(t)N(t)$; $\dfrac{d}{dt}[N(t)-\underline{N}(t)]\Big|_{t=t_n}>0$; $[N(t)-\underline{N}(t)]\Big|_{t=t_n}=0$, we have $N(t)>\underline{N}(t)$ in some neighborhood of t_n, along which t is increasing. We now assert that

$$N(t) > \underline{N}(t), \text{ for } t> t_n. \tag{11}$$

This is because that if there is a $\hat{t}>t_n$ such that $N(\hat{t})= \underline{N}(\hat{t})$ but $N(t) > \underline{N}(t)$ for all $t\in(t_n,\hat{t})$, then it will lead to a contradiction since $\dfrac{d}{dt}[N(t)-\underline{N}(t)]\Big|_{t=\hat{t}}>0$, so (11) holds. On theother hand, $\underline{N}(t)$ satisfies

$$\underline{N}(t)=N(t_n)e^{-\int_{t_n}^{t}[\lambda_0-\varepsilon-Kf(\underline{N}(\rho))]d\rho}, \quad t\geq t_n. \tag{12}$$

Similar to equation (5), the $\underline{N}(t)$ satisfying equation (12) is uniquely determined. As $Kf(N(t_n))< \lambda_0-\varepsilon$ in a neighborhood of t_n, along which t is increasing, $\underline{N}(t)$ is a strictly monotonic increasing function, if there exists a t^* such that $Kf(\underline{N}(t^*))=\lambda_0-\varepsilon$, and $Kf(\underline{N}(t))<\lambda_0-\varepsilon$, for all $t\in (t_n,t^*)$, then we must have $\underline{N}(t)=\underline{N}(t^*)$ for $t\geq t^*$; if for any $t\geq t_n$, $Kf(\underline{N}(t))\neq\lambda_0-\varepsilon$, then we must have $\lim_{t\to\infty} \underline{N}(t)=+\infty$. In brief, we have $\lim_{t\to\infty} \underline{N}(t)>N^*$ which leads to a contradiction. Similarly, $Kf(N^*))>\lambda_0$ is also impossible. Hence we get

$$\Omega(N)\backslash\{ \infty \} \subset \{\xi|\ Kf(\xi)=\lambda_0\}. \tag{13}$$

Let $\hat{a}=\underline{\lim}_{t\to\infty} N(t)$, $\hat{b}=\overline{\lim}_{t\to\infty} N(t)$. If $\hat{a} \neq \hat{b}$, it is apparent that for any $N^*\in (\hat{a},\hat{b})$, there exists a sequence $\{t_n\}$, $t_n\to\infty$ such that $N(t_n)=N^*$

(notice that $N(t)$ is continuous), so there must be an interval $[a,b]$, $a\neq b$, $b<+\infty$ and satisfies

$$[a,b] \subset \Omega(N)\backslash\{\infty\}. \tag{14}$$

Since $b-a>0$, we can take $\varepsilon>0$ and $\xi_0 = a+\frac{b-a}{2}$ (see Fig.1) such that

$[\xi_0-\varepsilon, \xi_0+\varepsilon]\subset(a,b)$, and then take $t_n \longrightarrow \infty$ such that $N(t_n)=\xi_0$ (this is possible). By

$$N(t)=[M+o(e^{-\varepsilon t})]^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho} = \bar{g}(t)^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho}$$

and $\lim_{t\to\infty} \bar{g}(t)=M$, we know that

$$N(t) = \xi_0 \frac{\bar{g}(t)}{\bar{g}(t)} e^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho}, \qquad \text{for } t\geq t_n.$$

If n is large enough such that for all $t\geq t_n$

$$\xi_0 \frac{\bar{g}(t)}{\bar{g}(t_n)} \in(\xi_0-\varepsilon, \xi_0+\varepsilon),$$

then $N(t)= \xi_0 \frac{\bar{g}(t)}{\bar{g}(t_n)}$, for all $t\geq t_n$, and so $\lim_{t\to\infty} N(t)=\xi_0$, this is a contradiction. Thus $b-a=0$, i.e.

$$\bar{\Omega}(N_0)=\{N^*\}.$$

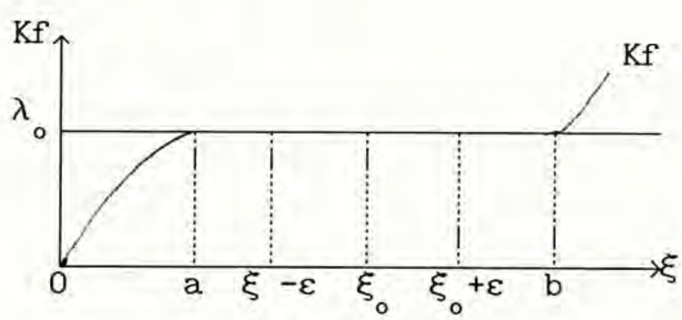

Fig. 1

Summarizing, we have

Theorem 1. For any $Kf(\xi)$ which satisfies condition (2) and $p_0(r)\in$

$D(\mathbf{A})\cap\mathbf{X}$, $p_0\geq 0$, the limit of the solution to equation (1) (i.e. solution of (6)) exists and equals to

$$\lim_{t\to\infty} p(r,t) = C\, e^{-\lambda_0 r-\int_0^r \mu(\rho)d\rho} \tag{15}$$

where $C>0$ may be infinite.

In the following we want to find necessary conditions for $N(t)$ to be bounded. Suppose that

$$\overline{\lim_{\xi\to\infty}}\, Kf(\xi) < \lambda_0, \tag{16}$$

and take $p_0(r)=Ce^{-\lambda_0 r-\int_0^r \mu(\rho)d\rho}$, $C>0$ such that $f(\xi)<\lambda_0$ for $\xi>\|p_0(r)\|$, then the solution $N(t)=\|p_0(r)\|^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho}$ corresponding to $p_0(r)$ is a strictly monotonic increasing function and $\lim\limits_{t\to\infty} N(t)=\infty$. So, in order for $N(t)$ to be bounded, it must be true that

$$\overline{\lim_{\xi\to\infty}}\, Kf(\xi) \geq \lambda_0.$$

If $\overline{\lim\limits_{\xi\to\infty}}\, Kf(\xi) >\lambda_0$ and $\lim\limits_{t\to\infty} N(t)=\infty$, then take $\varepsilon>0$ and $t_n \longrightarrow \infty$ such that $Kf(N(t_n))>\lambda_0+\varepsilon$, for this ε, there is a $t_0\geq 0$ such that the corresponding $g(t)<\varepsilon$, for all $t\geq t_0$. Suppose that $t_n\geq t_0$ for all $n>N$, and consider the limiting equations

$$\begin{cases} \dfrac{d\bar{N}(t)}{dt}=[\lambda_0-Kf(\bar{N}(t))]\bar{N}(t)+\varepsilon\bar{N}(t), \\ \bar{N}(t_n)=N(t_n), \qquad n> N. \end{cases}$$

Similar to the arguments in (11), we can get $N(t)=\bar{N}(t)$ for all $t\geq t_0$. But $\bar{N}(t)$ is a monotonic decreasing function, it follows that $N(t)\leq\bar{N}(t)\leq N(t_n)$, which is a contradiction. Hence $N(t)$ is bounded. If $\overline{\lim\limits_{\xi\to\infty}}\, Kf(\xi)=\lambda_0$ and $Kf(\xi)\geq\lambda_0$ for all sufficiently large ξ, then the boundedness of $N(t)$ can also be deduced; If $\overline{\lim\limits_{\xi\to\infty}}\, Kf(\xi)=\lambda_0$ and

$Kf(\xi)<\lambda_0$ for all sufficiently large ξ, then like (16), N(t) is also bounded.

Summarizing the above, we have

Proposition 2. A necessary condition forh the solution of (1) to be bounded is that

$$\overline{\lim_{\xi\to\infty}} Kf(\xi) \geq \lambda_0. \tag{17}$$

Furthermore, if $\overline{\lim\limits_{\xi\to\infty}} Kf(\xi)>\lambda_0$, no unbounded solution to (1) exists; if $\overline{\lim\limits_{\xi\to\infty}} Kf(\xi) =\lambda_0$ but

$$Kf(\xi) \geq\lambda_0 , \qquad \text{for sufficiently large } \xi, \tag{18}$$

$$\text{or} \quad Kf(\xi) <\lambda_0 , \qquad \text{for sufficiently large } \xi, \tag{19}$$

then no unbounded solution exists if (18) holds, and unbounded solution exist if (19) holds.

The methods used in the proof of Theorem 1 demonstrate simply the stability of non-negative equilibrium state of system (1). The conclusions are quite strong and yet no linearization is involved.

Proposition 3. Let $\xi_0>0$ be the non-negative equilibrium state of system (1) (iff $Kf(\xi_0)=\lambda_0$), then when $Kf'(\xi_0)>0$ (or $Kf(\xi)$ is strictly monotonic increasing in a neighborhood of ξ_0), the system (1) is locally asymptotic stable in D(**A**) about ξ_0; If $Kf(\xi_0)<0$, then ξ_0 is not stable.

Proof. We only give the proof for N(t), since (6) tells us that this is equivalent to the proof for p(r,t).

First, let $Kf'(\xi\ )<0$, then there is a $\Delta>0$, such that $Kf(\xi)$ is strictly monotonic decreasing in $[\xi_0-\Delta,\xi_0+\Delta]$ (see Fig.2). The state corresponds to ξ is $p_0(r)=C_0e^{-\lambda_0 r-\int_0^r\mu(\rho)d\rho}$, where $C>0$ is a constant such that $\|p_0(r)\|=\xi_0$. Take $\hat{p}(r)=Cp_0(r)$, $C<1$, such that $\|\hat{p}(r)\|\in(\xi_0-\Delta,\xi_0)$, the total number $\hat{N}(t)$ corresponding to $\hat{p}(r)$ satisfies

$$\hat{N}(t)=\|\hat{p}(r)\|e^{\int_0^t[\lambda_0-Kf(\hat{N}(\rho))]d\rho}.$$

Since $\hat{N}(t)$ is a monotonic function, and $Kf(\|\hat{p}\|)>\lambda_0$, so $\hat{N}(t)$ monotonically decreases to $\sup_{\xi\le\|\hat{p}\|}\{\xi\ |Kf(\xi)=\lambda_0\}<\xi_0-\Delta$. But $C<1$ can be chosen such that $1-C$ is small enough. Hence ξ is not stable.

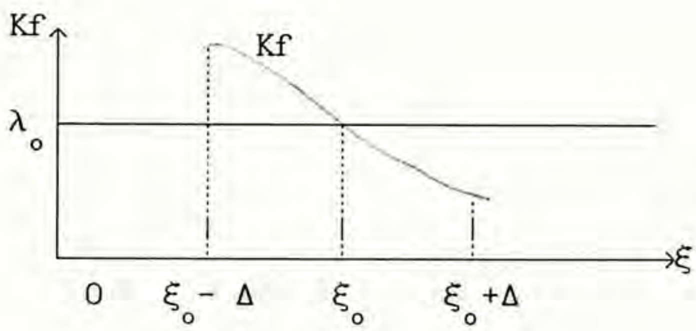

Fig.2

If $Kf'(\xi_0)>0$, then $Kf(\xi)$ is monotonic increasing in $[\xi_0-\Delta,\xi_0+\Delta]$. (see Fig.3) Choose $\varepsilon>0$ such that $Kf(\xi)=\lambda_0+\varepsilon$ has a solution in $(\xi_0,\xi_0+\Delta)$. It is not difficult to prove that there exits a neighborhood $\hat{O}_{\xi_0}$ of ξ such that $\hat{O}_{\xi_0}\subset[\xi_0-\Delta,\xi_0+\Delta]$ and for any $N_0\in\hat{O}_{\xi_0}$

$$|g(t)|<\varepsilon, \quad \forall\ t\ge t_0,$$

where $t_0>0$ is independent of N_0. By the continuous dependence of the solution with respect to initial values, we know that there is a neighborhood $O_{\xi_0}\subset\hat{O}_{\xi_0}$ such that for any $N_0\in O_{\xi_0}$, the corresponding solution $\{N(t), t \le t_0\} \subset \hat{O}_{\xi_0} \subset (\xi_0-\Delta,\ \xi_0+\Delta)$. Besides, the solution $\hat{\xi}_\varepsilon$ of $Kf(\xi)=\lambda_0+\xi$ in $(\xi_0-\Delta, \xi_0+\Delta)$ does not belongs to $\hat{O}_{\xi_0}$. For all $N_0 \in O_{\xi_0}$, consider the limiting equation

$$\begin{cases} \dfrac{d\bar{N}(t)}{dt}=[\lambda_0-Kf(\bar{N}(t))]\bar{N}(t)+\varepsilon\bar{N}(t), \\ \bar{N}(t_0)=N(t_0). \end{cases}$$

As $N(t) \le \bar{N}(t)$ for all $t \ge t_0$, and $\bar{N}(t) \le \hat{\xi}_\varepsilon$, by comparison, we have $N(t) \le \hat{\xi}_\varepsilon$ when $t \ge t_0$. Similarly, we can get ξ_ε and a neighborhood of ξ_0 such that when N_0 belongs to this neighborhood, the corresponding solution $N(t) \ge \xi_\varepsilon$. As ξ_ε and $\hat{\xi}_\varepsilon$ depend continuously on ε, $\lim\limits_{\varepsilon \to 0} \xi_\varepsilon=0$, $\lim\limits_{\varepsilon \to 0} \hat{\xi}_\varepsilon=0$, so, ξ_0 is stable. The asymptotic stability can also be deduced by Theorem 1.

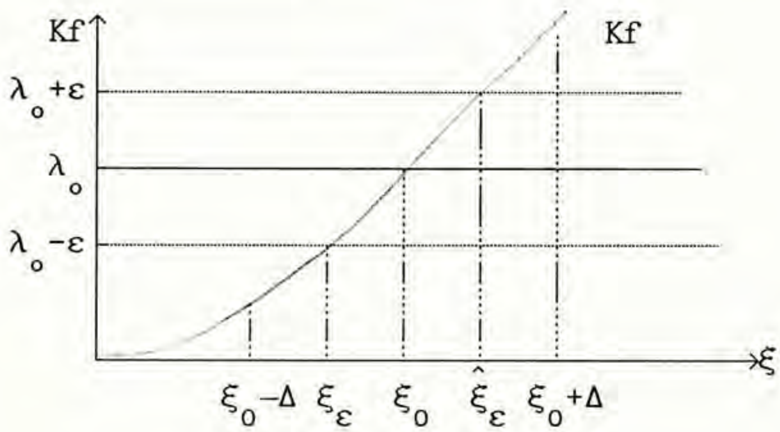

Fig.3

2.3 Oscillatory Properties

In this section we shall discuss the periodic and oscillation problem of system (1). We shall rather give a direct proof than rely on Theorem 1, although the periodic problem can be demonstrated directly by Theorem 1.

Theorem 2. System (1) does not have a non-trivial periodic solution.

Proof. Suppose there exists a $T>0$ such that $N(t)=N(t+T)$, for all $t\geq 0$, then by expression (4)

$$N(t)e^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho}=N^*+o(e^{-\varepsilon t})=\hat{g}(t).$$

It is obvious that $\hat{g}(t)=\hat{g}(t+T)$, $\forall$ $t\geq 0$. But $\lim_{t\to\infty}\hat{g}(t)=N^*$, hence $\hat{g}(t)=N^*$, and so $N(t)=N^*e^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho}$. Thus $N(t)$ is a monotonic function and hence $N(t)\equiv N^*$ and $Kf(N^*)=\lambda_0$. So for any $f(\xi)$ that satisfies (2), there does not exist non-trivial periodic solution for system (1).

Let us recall the definition in [1]

Definition. The population system is called oscillatory about its positive equilibrium position, if for any $\lim_{t\to\infty}N(t)=N^*>0$, and any interval $[\alpha,\infty)$ $\alpha>0$, there is at least one zero point of $N(t)-N^*$ in $[\alpha,\infty)$. In other words, there exists a sequence $t_n\longrightarrow\infty$ $(n\longrightarrow\infty)$ such that $N(t_n)=N^*$; if there is at least one $\lim_{t\to\infty}N(t)=N^*>0$, but $N(t)-N^*\neq 0$ for t sufficiently large, then the population system is called non-oscillatory about its positive equilibrium state.

We proved in chapter 1 that for linear system

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} = Ap(r,t), \\ \\ p(r,0) = p_0(r), \end{cases} \tag{20}$$

we have

Theorem 3. The total population of system (20) oscillates about its every positive equilibrium state (note that we must have $\beta=\beta_{cr}$, i.e. $\lambda_0=0$).

But in contrast, for nonlinear system (1) we have

Theorem 4. N(t) of the logistic population system (1) is non-oscillatory about its positive equilibrium state (it must be that $\lambda_0>0$, and $Kf(\xi)=\lambda_0$ has a solution).

Proof. Take the initial distribution $p_0(r)=Ce^{-\lambda_0 r-\int_0^r\mu(\rho)d\rho}$, $C>0$, then it is not difficult to prove that the corresponding solution $N(t)=\int_0^{r_m}p(r,t)dr$ satisfies

$$N(t)= M^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho} \tag{21}$$

where $M=C\|e^{\int_0^t[\lambda_0-Kf(N(\rho))]d\rho}\|$. We said previously that for every $f(\cdot)$ satisfying (2), the solution N(t) of equation (21) is a monotonic function on $[0,\infty)$, and hence $\lim_{t\to\infty} N(t)=N^*$ exists (it may be infinite). Choose C>0 small enough such that

$$M < N^* = \min_{\xi\geq 0}\{\xi \mid Kf(\xi)=\lambda_0\} \tag{22}$$

then the corresponding solution N(t) to (21) belongs to $[M,N^*]$ for any $t\geq 0$, and N(t) is a monotonic increasing function over $[0,\infty)$. But Since $Kf(\xi)$ is continuously differentiable, we write its Taylor expansion at $\xi=N^*$ as

$$Kf(\xi)=\lambda_0+ Kf'(N^*)(\xi-N^*) + o(\xi-N^*), \quad \forall\ \xi\in (N^*-\delta,N^*),\ \delta>0, \tag{23}$$

Hence there are constants δ_0, $c_0>0$ such that

$$\frac{Kf(\xi)-\lambda_0}{\xi-N^*}=Kf'(N^*)+o(1)\le c_0, \qquad \forall\ \xi\in(N^*-\delta_0,N^*),$$

i.e. $Kf(\xi)\ge\lambda_0+c_0(\xi-N^*)$ for all $\xi\in(N^*-\delta_0,N^*)$. Define the function

$$\hat{Kf}(\xi)=\begin{cases} N^*, & \xi\in[0,N-\hat{\delta}_0], \\ \lambda_0+c_0(\xi-N^*), & \xi\in(N^*-\hat{\delta}_0,\infty), \end{cases}$$

where $\hat{\delta}_0<\delta_0$ such that

$$\hat{Kf}(\xi)>0 \text{ for all } \xi>0,$$

$$\hat{Kf}(0)=0, \qquad \hat{Kf}\in C^1(0,\infty),$$

$$Kf(\hat{\xi})\ne\lambda_0 \text{ for } \xi\in[0,N^*].$$

Hence,

$$Kf(\xi)\ge\hat{Kf}(\xi), \qquad \text{for all } \xi\in[N^*-\hat{\delta}_0,N^*].$$

Taking $M\in(N^*-\hat{\delta}_0,N^*]$, $\hat{N}(t)$ is the solution of (1) corresponding to $\hat{Kf}(\xi)$. Then by $N(t)$, $\hat{N}(t)\in(N^*-\hat{\delta}_0,N^*)$, we have

$$N(t)\le\hat{N}(t), \qquad \text{for all } t\ge0. \tag{24}$$

But $\hat{N}(t)=Me^{\int_0^t[\lambda_0-Kf(\hat{N}(\rho))]d\rho}=Me^{\int_0^t-c_0\hat{N}(\rho)d\rho}e^{c_0N^*t}$, i.e.

$$\hat{N}(t)e^{\int_0^t c_0\hat{N}(\rho)d\rho}=Me^{c_0N^*t}.$$

Integrating on both sides, we have

$$\hat{N}(t)=\frac{N^*}{1+[N^*-M]M^{-1}e^{-c_0N^*}}<N^*, \qquad \forall\ t\ge0. \tag{25}$$

It follows from (24) that

$$\lim_{t\to\infty}N(t)=N^*, \quad N(t)<N^*, \text{ for all } t\ge0. \tag{26}$$

Thus by definition, $N(t)$ does not oscillate about N^*. The proof is complete.

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Chapter 3

Semigroups for Age-Size Dependent Population Equations with Spatial Diffusion

3.1 Introduction

In the past few years many researchers have contributed to the application of semigroup theory to structured population dynamics without spatial diffusion. Nice examples can be found in Song *et al* [1], [2], Webb [3], [4], [5], Metz and Diekmann [6], Heijmans [7], Greiner [8] and the Guo and Chan [9], among others. On the other hand the problem of spatial spread in age dependent population dynamics proposed by Gurtin [10] and Gurtin and MacCamy [11] has attracted considerable interest. Recent published works in this area include Busenberg and Iannelli [12], Kunisch *et al* [13], Langlais [14] and references therein. Our purpose here is to study diffusion models in the semigroup framework. One of the advantages of such an approach is to gain information on the behaviour of the population by an analysis of the spectrum of the infinitesimal generator of the associated semigroup. Furthermore, it is shown that the structure of the semigroup for the population with diffusion is essentially determined by those of the semigroup for the population without diffusion and the Laplacian. To our knowledge, not many results along these lines are available presently.

We are interested in the following McKendrick model of age-dependent population moving in a limited smooth domain Ω in $\mathbb{R}^n$, and shall be studying dynamics with linear, that is, random spatial diffusion processes which are applicable to many biological populations, including population of the microorganisms such as bacteria and cells.

Let $p(r,t,x)$ be the age distribution of individuals having age $r\geq 0$ at time $t\geq 0$ and spatial location x in Ω. We assume in this paper that $0\leq r\leq r_m$, r_m is the maximum life expectancy of the species. The evolution of p is governed by the following differential equation:

$$\begin{cases} \dfrac{\partial p(r,t,x)}{\partial t} + \dfrac{\partial p(r,t,x)}{\partial r} = -\mu(r)p(r,t,x)+K\Delta p(r,t,x), \ x\in\Omega, \ t,r>0, \\ p(r,0,x)=p_0(r,x), \ r\geq 0, \ x\in\bar{\Omega}, \\ p(0,t,x)=\displaystyle\int_0^{r_m}\beta(r)p(r,t,x)dr, \ t\geq 0, \ x\in\bar{\Omega}, \\ p(r,t,x)\big|_{\partial\Omega}=0, \ r, \ t\geq 0, \end{cases} \tag{1}$$

where $\mu(r)$ is the death rate function which satisfies

$$\int_0^r \mu(\rho)d\rho<\infty \text{ for } r<r_m \text{ and } \int_0^{r_m}\mu(\rho)d\rho=\infty; \tag{2}$$

$\beta(r)$ is the fertility function, bounded nonnegative measurable on $[0,r_m]$; $p_0(r,x)$ is an initial distribution, $p_0(r,x)\geq 0$; K is a positive constant; Δ denotes the Laplace operator in $\mathbb{R}^n$.

This chapter is organized as follows. In section 3.2 the infinitesimal generator $\mathbb{A}$ of the population operator with diffusion is identified. The resolvent operator $R(\lambda,\mathbb{A})$ is constructed and is shown to be compact. The dominant eigenvalue is thus determined. In section 3.3 the semigroup $T(t)$ associated with

$\mathbb{A}$ is expressed in terms of the eigenelements of the Laplacian. An asymptotic expression for the solution is obtained. In section 3.4 we consider the case when an additional internal variable in characterizing the population is introduced, i.e. the age-size dependent population model. Results in section 3.2 and 3.3 are extended to such a model. Finally in section 3.5 we study a logistic model with diffusion. A nonlinear semigroup is constructed to describe the evolution of the population. Existence and uniqueness of global solution are proved and the large time behaviour of solution is also investigated.

3.2. Properties of the Infinitesimal Generator

Introducing the state space $\mathbf{X}=L^2((0,r_m)\times\Omega)$ with the usual norm and defining the operator $\mathbb{A}:\mathbf{X}\longrightarrow\mathbf{X}$ as follows;

$$\mathbb{A}\phi(r,x)=-\frac{\partial\phi(r,x)}{\partial r}-\mu(r)\phi(r,x)+K\Delta\phi(r,x),\quad \forall\ \phi\in D(\mathbb{A}),$$

$$D(\mathbb{A})=\left\{\phi(r,x)\ \middle|\ \phi,\ \mathbb{A}\phi\in\mathbf{X},\ \phi(r,x)\big|_{\partial\Omega}=0,\ \phi(0,x)=\int_0^{r_m}\beta(r)\phi(r,x)dr\right\},\quad (3)$$

we can write equation (1) as an evolutionary equation on the state space $\mathbf{X}$:

$$\begin{cases}\dfrac{dp(r,t,x)}{dt}=\mathbb{A}p(r,t,x),\\ p(r,0,x)=p_0(r,x).\end{cases}\quad (4)$$

In the following we shall denote by $(\bar{\lambda},\phi_i)_{i\geq 0}$ the eigenvalues and eigenfunctions of the Dirichlet problem in the smooth domain Ω of $\mathbb{R}^n$, namely:

$$\begin{cases}-K\Delta\phi_i(x)=\bar{\lambda}_i\phi_i(x), & x\in\Omega,\\ \phi_i(x)\big|_{\partial\Omega}=0,\end{cases}\quad (5)$$

$$\int_{\partial\Omega}\phi_i^2(x)dx=1,\quad i\geq 0;\quad \phi_0(x)>0 \text{ in } \Omega.$$

We also assume that $0<\bar{\lambda}_0<\bar{\lambda}_1\leq\bar{\lambda}_2\leq\ldots$

Next, we denote by $\mathbf{A}$ the usual population operator without diffusion defined in $L^2(0,r_m)$ (see [2]):

$$\mathbf{A}\phi(r)=-\frac{\partial\phi(r)}{\partial r}-\mu(r)\phi(r),\qquad \forall\ \phi\in D(\mathbf{A}),$$

$$D(\mathbf{A})=\left\{\phi(r)\,\Big|\ \phi,\ \mathbf{A}\phi\in L^2(0,r_m),\ \phi(0)=\int_0^{r_m}\beta(r)\phi(r)dr\right\},\tag{6}$$

and $\{\hat{\lambda}_j\}_{j\geq 0}$ the eigenvalues of $\mathbf{A}$, i.e. the solution of following equation

$$1-\int_0^{r_m}\beta(r)e^{-\hat{\lambda}_j r-\int_0^r\mu(\rho)d\rho}dr=0,\tag{7}$$

and arrange $\hat{\lambda}_j$ in the following way:

$$\hat{\lambda}_0>\operatorname{Re}\hat{\lambda}_1\geq\operatorname{Re}\hat{\lambda}_2\geq\ldots\ .$$

We solve the following equation

$$(\lambda-\mathbb{A})\phi=\psi,\qquad \forall\ \psi\in \mathbf{X}.\tag{8}$$

If for any i, $j\geq 0$, $\lambda+\bar{\lambda}_i\neq\hat{\lambda}_j$, then define

$$\phi_\psi(r,x)=\sum_{i=0}^{\infty}R(\lambda+\bar{\lambda}_i,\mathbf{A})<\psi(r,x),\phi_i(x)>\phi_i(x),\tag{9}$$

where $<\psi(r,x),\phi_i(x)>=\int_\Omega\psi(r,x)\phi_i(x)dx$, $R(\lambda,\mathbf{A})=(\lambda-\mathbf{A})^{-1}$, the resolvent of $\mathbf{A}$. Since $\mathbf{A}$ is the infinitesimal generator of a bounded strongly continuous semigroup, so there exist constants M, $\omega>0$ such that

$$\|R(\lambda,\mathbf{A})\|\leq\frac{M}{\operatorname{Re}\lambda-\omega},\qquad \text{for } \operatorname{Re}\lambda>\omega.\tag{10}$$

Recall that $\bar{\lambda}_i\longrightarrow\infty$ as $i\longrightarrow\infty$ and suppose that $\operatorname{Re}(\lambda+\bar{\lambda}_i)>\omega$ when $i>N$, some positive integer, we can see that

$$\sum_{i=0}^{\infty}\|R(\lambda+\bar{\lambda}_i,\mathbf{A})<\psi(r,x),\phi_i(x)>\|^2$$

$$\le \sum_{i=0}^{N} \|R(\lambda+\bar{\lambda}_i, \mathbf{A})<\psi(r,x),\phi_i(x)>\|^2$$

$$+\left[\frac{M}{\mathrm{Re}(\lambda+\lambda_i)-\omega}\right]^2 \sum_{i=N+1}^{\infty} \|<\psi(r,x),\phi_i(x)>\|^2$$

$$\le \sum_{i=0}^{N} \|R(\lambda+\bar{\lambda}_i, \mathbf{A})<\psi(r,x),\phi_i(x)>\|^2+\left[\frac{M}{\mathrm{Re}(\lambda+\lambda_i)-\omega}\right]^2 \|\psi\|^2 <\infty,$$

so $\phi_\psi(r,x)\in\mathbf{X}$ is well defined. Furthermore, for any $n>0$

$$(\lambda-\mathbb{A}) \sum_{i=0}^{n} R(\lambda+\bar{\lambda}_i, \mathbf{A})<\psi(r,x),\phi_i(x)>\phi_i(x)$$

$$= \sum_{i=0}^{n} <\psi(r,x),\phi_i(x)>\phi_i(x)\longrightarrow\psi(r,x) \text{ in } \mathbf{X} \quad \text{as} \quad n\longrightarrow\infty.$$

Since $\mathbf{A}$ and Δ are both closed operators on $\mathbf{X}$, so is $\mathbb{A}$, Hence

$$(\lambda-\mathbb{A})\phi_\psi=\psi$$

i.e. $\phi_\psi(r,x)$ is a solution of (8).

On the other hand, it can be shown that ϕ is the unique solution of (8), and thus $\lambda\in\rho(\mathbb{A})$, the resolvent set of $\mathbb{A}$ and

$$R(\lambda,\mathbb{A})\psi= \sum_{i=0}^{\infty} R(\lambda+\bar{\lambda}_i, \mathbf{A})<\psi(r,x),\phi_i(x)>\phi_i(x). \tag{11}$$

If there are some i, j such that $\lambda+\bar{\lambda}_i=\hat{\lambda}_j$, then

$$\phi_i(r,x)=e^{-(\lambda+\bar{\lambda}_i)r-\int_0^r\mu(\rho)d\rho}\phi_i(x) \tag{12}$$

satisfies $(\lambda-\mathbb{A})\phi_i=0$, i.e. $\lambda\in \sigma(\mathbb{A})=\sigma_p(\mathbb{A})$, the point spectrum of $\mathbb{A}$. Furthermore, if $(\lambda-\mathbb{A})\phi=0$, then expanding the known initial function $\phi(0,x)$ as

$$\phi(0,x)= \sum_{i=0}^{\infty} \alpha_i\phi_i(x), \qquad \text{in } L^2(\Omega)$$

then

$$\phi(r,x)= \sum_{i=0}^{\infty} \alpha_i e^{-(\lambda+\bar{\lambda}_i)r-\int_0^r\mu(\rho)d\rho}\phi_i(x),$$

and from the condition

$$\phi(0,x)=\int_0^{r_m}\beta(r)\phi(r,x)dr$$

we have for each i

$$\text{either } \alpha_i=0 \text{ or } \int_0^{r_m}\beta(r)e^{-(\lambda+\bar{\lambda}_i)r-\int_0^r\mu(\rho)d\rho}dr=0.$$

In particular, for $\lambda_0=\hat{\lambda}_0-\bar{\lambda}_0$, which is the dominant eigenvalue of $\mathbb{A}$, $(\lambda-\mathbb{A})\phi=0$ has only one independent linear solution, which is

$$\phi_{\lambda_0}(r,x)=e^{-\hat{\lambda}_0 r-\int_0^r\mu(\rho)d\rho}\phi_0(x). \tag{13}$$

so λ_0 is of geometric multiplicity one.

For every n>0, define the operator

$$R_n\psi(r,x)=\sum_{i=0}^{n}R(\lambda+\bar{\lambda}_i,\mathbf{A})\langle\psi(r,x),\phi_i(x)\rangle\phi_i(x),\quad \forall\ \psi\in \mathbf{X}.$$

If $\mathrm{Re}(\lambda+\bar{\lambda}_i)>\omega$ when i>n then

$$\|R(\lambda,\mathbb{A})\psi-R_n\psi\|$$

$$\le\sum_{i=n+1}^{\infty}\left[\frac{M}{\mathrm{Re}(\lambda+\lambda_i)-\omega}\right]^2\|\langle\psi(r,x),\phi_i(x)\rangle\|^2$$

$$\le\left[\frac{M}{\mathrm{Re}(\lambda+\lambda_n)-\omega}\right]^2\|\psi\|^2,$$

so $\lim_{n\to\infty}\|R(\lambda,\mathbb{A})-R_n\|=0$. Since for every i, $R(\lambda+\lambda_i,\mathbf{A})$ is compact on $L^2(0,r_m)$, so is R_n on $\mathbf{X}$. Hence it follows that $R(\lambda,\mathbb{A})$ is compact on $\mathbf{X}$ for every $\lambda\in\rho(\mathbb{A})$.

Take $\phi\in D(\mathbb{A})$, one has

$\langle\mathbb{A}\phi(r,x),\phi(r,x)\rangle$

$$=\int_{(0,r_m)\times\Omega} \frac{\partial\phi(r,x)}{\partial r}\,\bar{\phi}(r,x)drdx - \int_{(0,r_m)\times\Omega} \mu(r)|\phi(r,x)|^2 drdx$$

$$+K\int_{(0,r_m)\times\Omega} \Delta\phi(r,x)\bar{\phi}(r,x)drdx$$

$$\le\int_\Omega \frac{1}{2}|\phi(0,x)|^2 dx$$

$$=\frac{1}{2}\int_\Omega\left[\int_0^{r_m}\beta(r)\phi(r,x)dr\right]^2 dx$$

$$\le\frac{1}{2}\int_\Omega\left[\int_0^{r_m}\beta(r)^2 dr\right]\left[\int_0^{r_m}\phi(r,x)^2 dr\right] dx$$

$$=\frac{1}{2}\,\|\beta(r)\|^2\|\phi(r,x)\|^2,$$

i.e. $\mathbb{A}$ is a m-dissipative operator on $\mathbf{X}$. Since $\mathbf{X}$ is a Hilbert space, so $\overline{D(\mathbb{A})}=\mathbf{X}$. Thus from Pazy [15] we have

Theorem 1.

(1). The operator $\mathbb{A}$ defined in (3) generates a strongly continuous semigroup on $\mathbf{X}$;

(2). For every $\lambda\in\rho(\mathbb{A})$, $R(\lambda,\mathbb{A})$ given by (9) is a compact operator;

(3). $\sigma(\mathbb{A}) = \sigma_p(\mathbb{A}) = \{\bar{\lambda}_i-\hat{\lambda}_j\}_{i,j=0}^{\infty}$;

(4). $\mathbb{A}$ has a real dominant eigenvalue λ_0, i.e. λ_0 is greater than any real parts of the eigenvalues of $\mathbb{A}$;

(5). λ_0 is a simple eigenvalue of $\mathbb{A}$.

Proof. Parts (1)-(4) have been proved in the above. For part (4), it suffices to note that

$$R(\lambda,\mathbb{A}) = R(\lambda+\bar{\lambda}_0,\mathbf{A})<\cdot,\phi_0(x)>\phi_0(x)+\sum_{i=1}^{\infty} R(\lambda+\bar{\lambda}_i,\mathbf{A})<\cdot,\phi_i(x)>\phi_i(x)$$

and

$$R(\lambda,\mathbf{A})\phi = \frac{g_\phi(\lambda)}{F(\lambda)}\, e^{-\lambda r-\int_0^r \mu(\rho)d\rho} + \int_0^r e^{-\lambda(r-s)-\int_s^r \mu(\rho)d\rho}\phi(s)ds$$

$$\forall\ \phi\in L^2(0,r_m), \tag{14}$$

where

$$F(\lambda)=1-\int_0^{r_m}\beta(r)e^{-\lambda r-\int_0^r \mu(\rho)d\rho}dr,$$

$$g_\phi(\lambda)=\int_0^{r_m}\beta(r)\left[\int_0^r e^{-\lambda(r-s)-\int_s^r \mu(\rho)d\rho}\phi(s)ds\right]dr,$$

so $\lambda=\lambda_0$ is a pole of order one of $R(\lambda,\mathbb{A})$. The conclusion (5) thus follows from the closeness of $\mathbb{A}$.

3.3 Properties of the Semigroup

In this section, we shall discuss the C_0 semigroup $T(t)$ generated by the operator $\mathbb{A}$. For every $\phi\in \mathbf{X}$, define the operators $\{\hat{T}(t), t\ge 0\}$ as follows:

$$\hat{T}(t)\phi(r,x)=\sum_{i=0}^{\infty} e^{\mathbf{A}t}e^{-\bar{\lambda}_i t}<\phi(r,x),\phi_i(x)>\phi_i(x) \tag{15}$$

where $e^{\mathbf{A}t}$ is the semigroup generated by $\mathbf{A}$.

It is obvious that $\hat{T}(t)$ is a well defined bounded linear operator on $\mathbf{X}$ for every $t\ge 0$ and for all $\phi_{n,q}(r,x)=\sum_{j=0}^{n} q_j(r)\phi_j(x)$, $q_j(r)\in L^2(0,r_m)$, $j=0,1,\ldots n$, $n>0$ we can directly verify that

$$\hat{T}(t+s)\phi_{n,q}(r,x)=\hat{T}(t)\hat{T}(s)\ \phi_{n,q}(r,x), \qquad \forall\ t,s\ge 0.$$

Since $\{\phi_{n,q}(r,x)\}$ is dense in $\mathbf{X}$, so $\hat{T}(t+s)=\hat{T}(t)\hat{T}(s)$ for all $t,s\ge 0$. Moreover,

$$\lim_{t\to 0}\hat{T}(t)\phi_{n,q} = \phi_{n,q}$$

and since $\|\hat{T}(t)\| \le Me^{(\omega-\hat{\lambda}_0)t}$, so

$$\lim_{t\to 0} \hat{T}(t)\phi = \phi \qquad \text{for all } \phi\in \mathbf{X}.$$

This shows that $\hat{T}(t)$ is also a C_0 semigroup on $\mathbf{X}$. A simple calculation shows that

$$\lim_{t\to 0} \frac{\hat{T}(t)-I}{t}\,\phi_{n,q} = \mathbb{A}\phi_{n,q}$$

for all $\phi_{n,q}$, hence $\hat{T}(t)=T(t)$ for all $t\ge 0$.

Take $\phi(r,x)=q(r)\phi_0(x)$, $q(r)\in L^2(0,r_m)$, then

$$\hat{T}(t)\phi(r,x)=e^{-\bar{\lambda}_0 t}[e^{At}q(r)]\phi\,(x)$$

$T(t)$ is not compact when $t<r_m$, since e^{At} is not so when $t<r_m$. On the other hand, when $t\ge r_m$, e^{At} is a compact operator on $L^2(0,r_m)$.

Let

$$T_n(t)(r,x)=\sum_{i=0}^{n} e^{At}e^{-\bar{\lambda}_i t}\langle\phi(r,x),\phi_i(x)\rangle\phi_i(x)$$

$$\forall\ \phi\in \mathbf{X},\ n\ge 0,\ t\ge r_m,$$

then $T_n(t)$ is a compact operator on $\mathbf{X}$ for every $n\ge 0$, $t\ge r_m$ and

$$\|[\hat{T}(t)-T_n(t)]\phi\|^2 = \sum_{i=n+1}^{\infty} \|e^{At}e^{-\bar{\lambda}_i t}\langle\phi,\phi_i\rangle\|^2$$

$$\le [e^{-\bar{\lambda}_n t}Me^{\omega t}]^2\|\phi\|^2, \qquad \forall\ \phi\in \mathbf{X}$$

so $\|\hat{T}(t)-T_n(t)\| \le Me^{(\omega-\bar{\lambda}_n)t}$, for all $n\ge 0$, $t\ge r_m$ and

$$\lim_{n\to\infty} \|\hat{T}(t)-T(t)\| = 0.$$

We therefore have proved the following

Theorem 2. The C_0 semigroup $T(t)$ on $\mathbf{X}$ generated by operator $\mathbb{A}$ is compact when $t\ge r_m$, but not for $t<r_m$, and $T(t)=\hat{T}(t)$ given by (15).

In the spirit of Yu. *et al* [16], we have the following

Theorem 3.

(1). For every $p_0 \in \mathbf{X}$, there exists a unique solution $p(r,t,x)$ to equation (4), which is given by

$$p(r,t,x)=T(t)p_0(r,x)\in C(0,\infty;\mathbf{X}); \tag{17}$$

(2). If $p_0 \in D(\mathbb{A})$, then $p(r,t,x)=T(t)p_0(r,x)\in C^1(0,\infty;\mathbf{X})$;

(3). $p(r,t,x)$ has the following asymptotic expression

$$p(r,t,x)=C_{p_0}(x)e^{-\hat{\lambda}_0 r-\int_0^r \mu(\rho)d\rho}e^{\lambda_0 t}\phi_0(x) + o(e^{(\lambda_0-\varepsilon)t}) \tag{18}$$

where $\lambda_0=\hat{\lambda}_0-\hat{\lambda}_0$ is the dominant eigenvalue of $\mathbb{A}$, ε is a small positive number such that $\sigma(\mathbb{A})\cap\{\lambda \mid \lambda_0-\varepsilon\le \mathrm{Re}\lambda<\lambda_0\}=\emptyset$, and

$$C_{p_0} = \frac{\displaystyle\int_0^{r_m} \beta(r)\left[\int_0^r e^{-\hat{\lambda}_0(r-s)-\int_s^r \mu(\rho)d\rho}\langle p_0(s,x),\phi_0(s)\rangle ds\right]dr}{\displaystyle\int_0^{r_m} r\beta(r)e^{-\hat{\lambda}_0 r-\int_0^r \mu(\rho)d\rho}dr} \tag{19}$$

(4). $T(t)$ is a positive semigroup.

Proof. We only need to prove (4), since the other parts can be proved along the lines in [16]. Notice that $T(t)=e^{\mathbf{A}t}e^{K\Delta t}$, here $e^{K\Delta t}$ is the positive semigroup generated by $K\Delta$ in $L^2(\Omega)$. So the positiveness of $T(t)$ follows by that of $e^{\mathbf{A}t}$. The proof is thus completed.

3.4. Dynamics with Age-Size Structures

It has been recognized that age structure alone is not adequate to explain the population dynamics of some species (see [5], [6], [17], [18]). The size of individuals could also be used to distinguish cohorts. In principle there are many ways to differentiate individuals in addition to age, such as body size and dietary requirements or some other physiological variables and behavioural parameters. For the sake of simplicity and the reason of similarity of mathematical treatment we assume here that only one internal variable is involved. Meanwhile, we consider the velocity of internal variable to be constant. Note that this assumption is not restrictive since in [5] it was pointed out that the problem in which the growth of an internal variable does not increase at the same rate as age can be converted to the constant velocity case. Thus we consider the following population model

$$\begin{cases} \frac{\partial p(r,t,x,g)}{\partial t} + \frac{\partial p(r,t,x,g)}{\partial r} + \frac{\partial p(r,t,x,g)}{\partial g} = -\mu(r)p(r,t,x,g) \\ \qquad\qquad +K\Delta p(r,t,x,g), \\ p(r,0,x,g)=p_0(r,x,g), \\ p(0,t,x,g)=\int_0^{r_m}\beta(r)p(r,t,x,g)dr, \\ p(r,t,x,0)=\int_0^{b}h(g)p(r,t,x,g)dg, \\ p|_{\partial\Omega}=0 \end{cases} \tag{20}$$

where g is the internal variable (e.g. size), $0 \le g \le b$. The other parameters are just like those of equation (1). h(g), a nonnegative bounded measurable function, is the reproduction rate.

Note. Here we assume that the death rate function $\mu_d(r,g)=\mu(r)$ is independent of the internal variable g. For the case of $\mu_d(r,g)=\mu(r)+\mu_s(g)$, if the corresponding state is denoted by $\bar{p}(r,t,x,g)$, then by making the following transformation

$$p(r,t,x,g)=e^{\int_0^g\mu_s(\rho)d\rho}\bar{p}(r,t,x,g),$$

it can be verified directly that $p(r,t,x,g)$ satisfies (20) with $h(g)$ changed to $h(g)e^{-\int_0^g \mu_s(\rho)d\rho}$ and $p_0(r,x,g)$ to $p_0(r,x,g)e^{-\int_0^g \mu_s(\rho)d\rho}$.

We consider the equation (20) in the state space $\tilde{X} = L^2((0,r_m)\times\Omega\times(0,b))$, and define the operator $\mathcal{A}$: $\tilde{X}\longrightarrow\tilde{X}$ as follows:

$$\mathcal{A}\phi(r,x,g)=-\frac{\partial\phi(r,x,g)}{\partial r}-\frac{\partial\phi(r,x,g)}{\partial g}-\mu(r)\phi(r,x,g)+K\Delta\phi(r,x,g), \quad \forall\ \phi \in D(\mathcal{A})$$

$$D(\mathcal{A})=\Big\{\phi(r,x,g)\ \Big|\ \phi,\ \mathcal{A}\phi\in \tilde{X},\ \phi(0,x,g)=\int_0^{r_m}\beta(r)\phi(r,x,g)dr$$

$$\phi(r,x,0)=\int_0^b h(g)\phi(r,x,g)dg,\ \phi|_{\partial\Omega}=0\Big\}. \tag{21}$$

With the operator $\mathcal{A}$, (20) can be written as an evolutionary equation on $\tilde{X}$

$$\begin{cases} \dfrac{dp(r,t,x,g)}{dt}=\mathcal{A}p(r,t,x,g), \\ p(r,0,x,g)=p_0(r,x,g). \end{cases} \tag{22}$$

Consider the operators $\mathbb{A}$ given by (3) and $\mathbf{B}$, mapping $L^2(0,B)$ into itself as follows

$$\mathbf{B}\phi(g)=-\frac{\partial\phi(g)}{\partial g}, \quad \forall\ \phi\in D(\mathbf{B})$$

$$D(\mathbf{B})=\Big\{\phi(g)\ \Big|\ \phi,\mathbf{B}\phi\in L^2(0,b),\phi(0)=\int_0^b h(g)\phi(g)dg\Big\}, \tag{23}$$

then define the operators $T_g(t)$: $\tilde{X}\longrightarrow\tilde{X}$ for all $t\geq 0$

$$T_g(t)\phi(r,x,g)=e^{\mathbf{B}t}e^{\mathbb{A}t}\phi(r,x,g)=e^{\mathbb{A}t}e^{\mathbf{B}t}\phi(r,x,g), \quad \forall\ \phi\in \tilde{X}. \tag{24}$$

It is readily seen that $T_g(t)$ is a strongly continuous semigroup of linear bounded operators on $\tilde{X}$ and

$$\lim_{t\to 0}\frac{T_g(t)-I}{t}\phi(r,x,g)=\mathcal{A}\phi(r,x,g),\qquad \forall\ \phi\in D(\mathcal{A}). \tag{25}$$

If $\mathcal{A}$ is a closed operator, then (25) implies that the operator $\mathcal{A}$ is the infinitesimal generator of $T_g(t)$. In order to demonstrate the closeness of $\mathcal{A}$ define the operator $\mathbf{C}$ on $L^2((0,r_m)\times(0,b))$ as follows

$$\mathbf{C}\phi(r,g)=-\frac{\partial\phi(r,g)}{\partial r}-\frac{\partial\phi(r,g)}{\partial g}-\mu(r)\phi(r,g),\qquad \forall\ \phi\in D(\mathbf{C})$$

$$D(\mathbf{C})=\Big\{\phi(r,g)\ \Big|\ \phi,\mathbf{C}\phi\in L^2((0,r_m)\times(0,b)),\phi(0,g)=\int_0^{r_m}\beta(r)\phi(r,g)dr,$$

$$\phi(r,0)=\int_0^b h(g)\phi(r,g)dg\Big\}. \tag{26}$$

Solving the equation

$$(\lambda-\mathbf{C})\phi(r,g)=\psi(r,g),\qquad \forall\ \psi\in L^2((0,r_m)\times(0,b)),\ \lambda\in\mathbb{C}, \tag{27}$$

i.e.

$$\begin{cases}\dfrac{\partial\phi(r,g)}{\partial g}+\dfrac{\partial\phi(r,g)}{\partial r}=-\lambda\phi(r,g)-\mu(r)\phi(r,g)+\psi(r,g),\\[2mm] \phi(r,0)=\displaystyle\int_0^b h(g)\phi(r,g)dg,\\[2mm] \phi(0,g)=\displaystyle\int_0^{r_m}\beta(r)\phi(r,g)dr,\end{cases} \tag{28}$$

so that

$$\phi(r,g)=e^{-\lambda g}e^{\mathbf{A}g}\phi(r,0)+\int_0^g e^{-\lambda(g-s)}e^{\mathbf{A}(g-s)}\psi(r,s)ds. \tag{29}$$

A necessary and sufficient condition for the equation (29) to have solution for all $\psi\in L^2((0,r_m)\times(0,b))$ is that

$$\left[I-\int_0^b h(g)e^{-\lambda g}e^{\mathbf{A}g}dg\right]^{-1} \tag{30}$$

exists in $L^2(0,r_m)$. In this case $\lambda\in\rho(\mathbf{C})$ and

$$R(\lambda,\mathbf{C})\psi(r,g)= e^{-\lambda g}e^{\mathbf{A}g}\left[I-\int_0^b h(g)e^{-\lambda g}e^{\mathbf{A}g}dg\right]^{-1}\cdot$$

$$\cdot\int_0^b h(g)dg\int_0^g e^{-\lambda(g-s)}e^{\mathbf{A}(g-s)}\psi(r,s)ds$$

$$+\int_0^g e^{-\lambda(g-s)}e^{\mathbf{A}(g-s)}\psi(r,s)ds \tag{31}$$

It is obvious that when $\mathrm{Re}\lambda$ is large enough then $\lambda\in\rho(\mathbf{C})$, so $\mathbf{C}$ is a closed operator on $L^2((0,r_m)\times(0,b))$. A simple calculation shows that $\mathbf{C}$ is the infinitesimal generator of C_0 semigroup $e^{\mathbf{A}t}e^{\mathbf{B}t}$ on $L^2((0,r_m)\times(0,b))$. Furthermore, since $e^{\mathbf{A}t}$, $e^{\mathbf{B}t}$ are compact for $t\geq\max(r_m, b)$ on $L^2(0,r_m)$ and $L^2(0,b)$, respectively, so $e^{\mathbf{A}t}e^{\mathbf{B}t}$ is compact for $t\geq\max(r_m, b)$ on $L^2((0,r_m)\times(0,b))$. Hence $R(\lambda,\mathbf{C})$ is compact for all $\lambda\in\rho(\mathbf{C})$. Denoting by μ_i, $i=0,1,\ldots$, the eigenvalues of $\mathbf{C}$ then similar arguments to those of previous parts show that if $\lambda\in\mathbb{C}$ such that $\lambda+\bar{\lambda}_i\neq\mu_j$ for all $i,j\geq0$ then $\lambda\in\rho(\mathcal{A})$. This shows that $\mathcal{A}$ is a closed operator on $\tilde{X}$.

It is readily seen that if $\lambda+\hat{\lambda}_i=\hat{\mu}_j$ for some $i,j\geq0$, here $\{\hat{\mu}_j, j\geq0\}$ are the eigenvalues of $\mathbf{B}$, i.e. the solution of following

$$1-\int_0^b h(g)e^{-\mu_j g}dg=0, \qquad j=0,1,\ldots \tag{32}$$

then $\lambda\in\rho(\mathbf{C})$ and the corresponding eigenfunction is

$$\phi_{ij}(r,g)=e^{-\hat{\mu}_j g}e^{-\hat{\lambda}_i r-\int_0^r\mu(\rho)d\rho}. \tag{33}$$

We have the following results on large time behaviour of system (20)

Theorem 5. For any $p_0(r,x,g)\in\tilde{X}$, $p_0\geq0$, the solution $p(r,t,x,g)$ of equation (20) has the following asymptotic expression

$$p(r,t,x,g)=e^{\mathcal{A}t}p_0(r,x,g)$$

$$=C_0(x)e^{-\hat{\mu}_0 g}e^{-\hat{\lambda}_0 r-\int_0^r\mu(\rho)d\rho}e^{(\lambda_0+\hat{\mu}_0)t}\phi_0(x) + o(e^{(\lambda_0+\hat{\mu}_0-\varepsilon)t})$$

$$\text{as } t\longrightarrow \infty, \qquad (34)$$

where

$$C_0(x)=\frac{\int_0^b h(g)\left[\int_0^g e^{-\hat{\mu}_0(g-s)}D(x,s)ds\right]dg}{-\int_0^b gh(g)e^{-\hat{\mu}_0 g}dg} \qquad (35)$$

$$D(x,g)=\frac{\int_0^{r_m}\beta(r)\left[\int_0^r e^{-\hat{\lambda}_0(r-s)-\int_s^r\mu(\rho)d\rho}<p_0(s,x,g),\phi_0(s)>ds\right]dr}{-\int_0^{r_m} r\beta(r)e^{-\hat{\lambda}_0 r-\int_0^r\mu(\rho)d\rho}dr}$$

(36)

Proof. By (18) and the asymptotic expression of semigroup e^{Bt} on $L^2(0,r_m)$ the proof can readily be constructed.

3.5. Logistic Model with Diffusion

In the following, we shall consider a nonlinear model with a logistic term, namely,

$$\begin{cases} \dfrac{\partial p(r,t,x)}{\partial t}+\dfrac{\partial p(r,t,x)}{\partial r}=-\mu(r)p(r,t,x)+K\Delta p(r,t,x)-f(N(t))p(r,t,x), \\ p(r,0,x)=p_0(r,x), \\ p(r,t,x)=\int_0^{r_m}\beta(r)p(r,t,x)dr, \\ p(r,t,x)|_{\partial\Omega}=0,\ r,\ t\geq 0, \end{cases} \qquad (37)$$

where $N(t)=\int_\Omega\int_0^{r_m}p(r,t,x)drdx$ is the total population per unit volume at time t and location x in Ω. The logistic term $f(\xi)$ is a nonnegative function satisfying

$$f(0)=0,\quad f(\xi)>0 \text{ for all } \xi>0,$$

$$f(\xi) \text{ is continuously differentiable.} \tag{38}$$

Let $\mathbf{A}$ be the operator defined by (6), $\mathbb{P}_0$ be the eigenprojection of $\mathbf{A}$ corresponding to $\hat{\lambda}_0$, then we have (see [14])

$$e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbb{P}_0 + o(e^{(\hat{\lambda}_0-\varepsilon_0)t}),\qquad t\longrightarrow\infty, \tag{39}$$

in the operator norm of $L^2(0,r_m)$, where ε_0 is any number such that $\sigma(\mathbf{A})\cap\{\lambda|\ \hat{\lambda}_0-\varepsilon_0\le\mathrm{Re}\lambda<\hat{\lambda}_0\}=\emptyset$.

Now, take $p_0(r,x)\in D(\mathbb{A})$, $p_0\ge 0$, then $\|p_0(r,x)\|_{L^2(0,r_m)}$ is a continuous function of x. By (39)

$$e^{\mathbf{A}t}p_0(r,x)=C_{\hat{\lambda}_0}(x)e^{-\hat{\lambda}_0 r-\int_0^r\mu(\rho)d\rho}e^{\hat{\lambda}_0 t} + o(e^{(\lambda_0-\varepsilon)t}) \tag{40}$$

holds uniformly for all $x\in\Omega$ in the sense of $L^2(0,r_m)$ norm, where

$$C_{p_0} = \frac{\int_0^{r_m}\beta(r)\left[\int_0^r e^{-\hat{\lambda}_0(r-s)-\int_s^r\mu(\rho)d\rho}p_0(s,x)ds\right]dr}{\int_0^{r_m}r\beta(r)e^{-\hat{\lambda}_0 r-\int_0^r\mu(\rho)d\rho}dr} \tag{41}$$

Integrating with respect to r from 0 to r_m on both sides of (40), we have

$$\int_0^{r_m}e^{\mathbf{A}t}p_0(r,x)dr =\hat{C}_0C_{\hat{\lambda}_0}(x)e^{\hat{\lambda}_0 t} + o(e^{(\hat{\lambda}_0-\varepsilon)t}), \tag{42}$$

uniformly for $x\in\Omega$, and where $C_0=\|e^{-\hat{\lambda}_0 r-\int_0^r\mu(\rho)d\rho}\|_{L(0,r_m)}$.

From [13], it is known that $K\Delta$ generates an analytic semigroup in the space

$$\mathbf{Y}=\{\phi(x)\,|\,\phi(x)\in C(\bar{\Omega}),\ \phi|_{\partial\Omega}=0\} \tag{43}$$

Since $p_0 \in D(\mathbb{A})$, so $\int_0^{r_m} e^{\mathbf{A}t} p_0(r,x)dr \in \mathbf{Y}$ for all $t \geq 0$, and hence we have

$$N(t,x)=C_0(x)e^{\lambda_0 t} + o(e^{(\lambda_0-\varepsilon)t}) \tag{44}$$

where ε, λ_0 are similarly defined in (18) and

$$C_0(x)=\hat{C}_0 C_{p_0}(x)\phi_0(x) \tag{45}$$

For any fixed $x\in\Omega$, $\hat{N}(t,x)=\int_0^{r_m} e^{\mathbf{A}t} p_0(r,x)dr$ is a continuous function of t and $\hat{N}(t,x)\in\mathbf{Y}$ for any fixed t, considering $e^{K\Delta t}$ as an analytic semigroup in the space $\mathbf{Y}$, it is clear that $N(t,x)$ is a continuous function of t.

Lemma 1. For any $p_0(r,x)\in D(\mathbb{A})$, $p_0 \geq 0$, $p(r,t,x)$ is the solution of t for any fixed x in Ω of the following nonlinear equation

$$N(t,x)=\|e^{\mathbb{A}t}p_0(r,x)\|_{L(0,r_m)} e^{-\int_0^t f(N(\rho))d\rho} \tag{46}$$

and in this case

$$p(r,t,x)=e^{\mathbb{A}t}p_0(r,x)e^{-\int_0^t f(N(\rho))d\rho}. \tag{47}$$

Proof. This can be verified directly.

The following propositions can be proved using an approach similar to that of Chapter 2.

Proposition 1. For any fixed $x\in\Omega$, there exists a unique nonnegative continuous solution to (46).

By proposition 1 and lemma 1, we can associate a nonlinear semigroup $\bar{T}(t)$ on $\mathbf{p}=\{\phi(r,x)\,|\,\phi\in\mathbf{X},\ \phi\geq 0\}$ as follows:

$$\bar{T}(t)p_0(r,x) = e^{\mathbb{A}t}p_0(r,x)e^{-\int_0^t f(N(\rho))d\rho} \tag{48}$$

for all $p_0 \in \mathbf{P}$, $t \geq 0$, where $N(t,x)$ is determined by (46).

Theorem 4. Under the hypothesis of (38), for any fixed $x \in \Omega$, we have

$$\lim_{t\to\infty} N(t,x) = N^*(x), \tag{49}$$

$$\lim_{t\to\infty} p(r,t,x) = C_0(x)e^{-\hat{\lambda}_0 r - \int_0^r \mu(\rho)d\rho}, \tag{50}$$

where $N^*(x) \in \{0,\ \infty,\ \omega(\lambda_0)\}$, $\omega(\lambda_0) = \{\xi \mid f(\xi) = \lambda_0\}$.

Proposition 2. For every $x \in \Omega$, a necessary condition for the solution of (37) to be bounded is that

$$\overline{\lim_{\xi\to\infty}} f(\xi) \geq \lambda_0. \tag{51}$$

Furthermore, if $\overline{\lim\limits_{\xi\to\infty}} f(\xi) > \lambda_0$ then no bounded solution to (37) exists; if $\overline{\lim\limits_{\xi\to\infty}} f(\xi) = \lambda_0$ but

$$f(\xi) \geq \lambda_0, \quad \text{for sufficiently large } \xi, \tag{52}$$

$$\text{or} \quad f(\xi) < \lambda_0, \quad \text{for sufficiently large } \xi, \tag{53}$$

then no unbounded solution exists, if (52) holds; and unbounded solution exists if (53) holds.

Corollary 1. If $f(\xi)$ is increasing and $\lim\limits_{\xi\to\infty} f(\xi) = \infty$, then for every $x \in \Omega$

$$\lim_{t\to\infty} N(t,x) = \xi_0, \tag{54}$$

$$\lim_{t\to\infty} p(r,t,x) = C_0 e^{-\hat{\lambda}_0 r - \int_0^r \mu(\rho)d\rho}, \tag{55}$$

where ξ_0 is the unique solution of $f(\xi) = \lambda_0$.

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Chapter 4

Semi-Discrete Population Equations with Time Delay

4.1 Introduction

In this chapter, we are concerned with the evolution of the population of a relative stationary society. The population density p(r,t) is governed by the following partial differential equation with a nonlinear logistic term and time delay

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t}+\dfrac{\partial p(r,t)}{\partial r}=-\mu(r)p(r,t)-Kf(N(t))p(r,t), & r,t>0, \\ p(r,\theta)=p_0(r,\theta),\ 0\le r\le r_m,\ -\tau\le\theta\le 0, \\ p(r,0)=\beta\displaystyle\int_{r_1}^{r_2}k(r)h(r)p(r,t-\tau)dr,\ t\ge 0, \end{cases} \tag{1}$$

where r is the age; t is the time; τ is the time delay; $0\le r\le m+1$, $t\ge 0$, where m+1 is the maximum age ever attained by individuals of the population. Assume that the specific fertility rate of females is a constant β; the female sex ratio k(r) and fertility pattern are independent of time; h(r) satisfies

$$\int_{r_1}^{r_2} h(r)dr=1,$$

where $[r_1,r_2]$ denotes the fecundity period of females; the relative mortality rate $\mu(r)$ is a function that depends on age only and satisfies

$$\int_0^{r}\mu(\rho)d\rho <+\infty, \qquad \lim_{r\to r_m^-}\int_0^{r_m}\mu(\rho)d\rho=+\infty;$$

the constant K is the environment parameter; $p_0(r,\theta)$ is the initial density; $N(t)=\int_0^{r_m}p(r,t)dr$ is the total population, $f(N(t))$ is called the logistic term and $f(\xi)$ satisfies

$$\begin{cases} f(0)=0;\ f(\xi)>0,\ \forall\ \xi>0; \\ f(\xi)\ \text{continuously differentiable.} \end{cases} \tag{2}$$

If K=0, (i.e.independent of the habitat) and $\tau=0$ (neglecting the effect of time delay) then equation (1) is the well-known age-dependent linear model of McKendrick discussed in [2].

Let $x_i(t)=\int_i^{i+1}p(r,t)dr$, $i=0,1,2,\ldots m$ be the number of individuals whose age is of i full years but not exceed i+1, so that $x_0(t)$ is the number of infants whose age is less than one full year.

Discretizing equation (1) with respect to age r but keeping time t unchanged, we have

$$\dot{x}_i(t)=-(1+\eta_i)x_i(t)+x_{i-1}(t)-Kf(N(t))x_i(t), \quad i=1,2,\ldots m, \tag{3}$$

where $\int_i^{i+1}\mu(r)p(r,t)dr=\eta_i x_i(t)$. Similarly we discretize the boundary condition of (1) yielding

$$x_0(t)=(1-\mu_{00})\beta\sum_{i=r_1}^{r_2}k_i h_i x_i(t-\tau), \quad t\geq 0, \tag{4}$$

where μ_{00} is the infanr death rate. For notationalsimplicity we set $h_i=0$ for $i\notin[r_1,r_2]$.

Let $\mathbf{x}(t)=(x_1(t),\ldots x_m(t))^T$, $N(t)=\sum_{i=1}^{m}x_i(t)$, $b_i=(1-\mu_{00})k_i h_i$,

$i=0,1,\ldots m,$

$$A=\begin{bmatrix} -(1+\eta_1) & 0 & \ldots & 0 & 0 \\ 1 & -(1+\eta_2) & & & \\ \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & -(1+\eta_{m-1}) & \\ 0 & 0 & \cdot & 1 & 1+\eta_m) \end{bmatrix}, \quad B=\begin{bmatrix} b_1 & b_2 & \ldots & b_m \\ 0 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \tag{5}$$

then equation (3) can be written as

$$\begin{cases} \mathbf{x}(t)=A\mathbf{x}(t)+\beta B\mathbf{x}(t-\tau)-Kf(N(t))\mathbf{x}(t), & t\ge 0, \\ \mathbf{x}(\theta)=\mathbf{x}_0(\theta), & -\tau\le\theta\le 0. \end{cases} \tag{6}$$

4.2 Linear Semi-Discrete Population Equation with Time Delay

Letting K=0 in equation (6) we get the linear semi-discrete population equation with time delay

$$\begin{cases} \mathbf{x}(t)=A\mathbf{x}(t)+\beta B\mathbf{x}(t-\tau), & t\ge 0, \\ \mathbf{x}(0)=\mathbf{x}_0(\theta), & -\tau\le\theta\le 0. \end{cases} \tag{7}$$

We introduce the state space $\mathbf{Y}=C([-\tau,0];\mathbb{R}^m)$. It is well-known that (cf. [6]) there exists a unique solution to equation (7) for every $\mathbf{x}_0(\theta)\in \mathbf{Y}$ such that $\mathbf{x}(t+\theta)\in \mathbf{Y}$ for all $t\ge 0$. The solution can be obtained by the following iterative process:

$$\begin{cases} \mathbf{x}(t)=e^{At}\mathbf{x}_0(0) + \int_0^t e^{A(t-s)}\beta B\mathbf{x}_0(s-\tau)ds, & 0\le t\le\tau, \\ \mathbf{x}(t)=e^{At}\mathbf{x}_0(n\tau) + \int_{n\tau}^t e^{A(t-s)}\beta B\mathbf{x}(s-\tau)ds, & n\tau\le t\le(n+1)\tau, \\ & n=1,2,\ldots. \end{cases} \tag{8}$$

It can easily be shown that e^{At} is a nonnegative matrix for all $t\ge 0$, so $\mathbf{x}(t)\ge 0$ provided $\mathbf{x}_0(\theta)\ge 0$. Let

$$T(t)=\begin{cases} \mathbf{x}_0(t+\theta), & \text{for } t+\theta\le 0, \\ \mathbf{x}(t+\theta), & \text{for } t+\theta)> 0. \end{cases} \tag{9}$$

where $\mathbf{x}(t)$, $t\geq 0$ are defined by (8). The following theorem is obvious

Theorem 1. $T(t)$ is a strongly continuous semigroup on $\mathbf{Y}$ and is compact for all $t\geq\tau$ but not for $t<\tau$. The infinitesimal generator $\mathbb{A}$ of $T(t)$ is given by

$$\mathbb{A}\mathbf{x}(\theta)=\begin{cases} \mathbf{x}'(\theta), & \text{for } -\tau\leq\theta\leq 0, \\ A\mathbf{x}(0)+\beta B\mathbf{x}(-\tau), & \text{for } \theta=0, \end{cases}$$

$$D(\mathbb{A})=\{\mathbf{x}(\theta) \mid \mathbf{x},\ A\mathbf{x}\in \mathbf{Y}\}. \tag{10}$$

With $T(t)$ and $\mathbb{A}$, we can write equation (7) as an evolutionary equation on $\mathbf{Y}$

$$\begin{cases} \dot{\mathbf{x}}(t+\theta)=\mathbb{A}\mathbf{x}(t+\theta), & t\geq 0, \\ \mathbf{x}(\theta)=\mathbf{x}_o(\theta), & -\tau\leq\theta\leq 0. \end{cases} \tag{11}$$

Corollary 1. (1). For any $\mathbf{x}_o(\theta)\in \mathbf{Y}$, $\mathbf{x}(t+\theta)=T(t)\mathbf{x}_o(\theta)\in C([0,\infty);\mathbf{Y})$; (2). when $\mathbf{x}_o(\theta)\in D(\mathbb{A})$ then $\mathbf{x}(t+\theta)=T(t)\mathbf{x}_o(\theta)\in C^1([0,\infty);\mathbf{Y})$.

We now consider the spectrum of the operator $\mathbb{A}$. First, if

$$(\lambda-\mathbb{A})\mathbf{x}(\theta)=0, \qquad \mathbf{x}(\theta)\in\mathbf{Y},\ \mathbf{x}(\theta)\neq 0,\ \lambda\in\mathbb{C},$$

then

$$\begin{cases} \mathbf{x}(\theta)=\mathbf{x}(0)e^{\lambda\theta}, \\ (A+\beta Be^{-\lambda\tau})\mathbf{x}(0)=\lambda\mathbf{x}(0). \end{cases} \tag{12}$$

So

$$\Delta(\lambda)=\det(\lambda-A-\beta Be^{-\lambda\tau})=0$$

and hence

$$\Delta(\lambda)=\prod_{k=1}^{m}(\lambda+1+\eta_k)-\left[\sum_{i=1}^{m} b_i\prod_{j=i+1}^{m}(\lambda+1+\eta_j)\right]\beta e^{-\lambda\tau}. \tag{13}$$

Let

$$\beta_{cr}=\frac{(1+\eta_1)(1+\eta_2)\dots(1+\eta_m)}{b_m+b_{m-1}(1+\eta_m)+\dots+b_1(1+\eta_2)\dots(1+\eta_m)}. \tag{14}$$

Lemma 1. (1). When $\beta=\beta_{cr}$, 0 is a simple root of $\Delta(\lambda)=0$;

(2). when $\beta>\beta_{cr}$, $\Delta(\lambda)$ has only one positive root λ_0 which is simple, and for all other root λ, $\mathrm{Re}\lambda<\lambda_0$;

(3). when $0<\beta<\beta_{cr}$, every root of $\Delta(\lambda)$ has a negative real part;

(4). when $\beta=\beta_{cr}$, all nonzero roots of $\Delta(\lambda)$ have negative real parts.

Proof. (1). Since $\beta=\beta_{cr}$, it is obvious that 0 is a root of $\Delta(\lambda)$ by [5], and is a simple root of $\Delta_0(\lambda)$

$$\Delta_0(\lambda)=\prod_{k=1}^{m}(\lambda+1+\eta_k)-\left[\sum_{i=1}^{m}b_i\prod_{j=i+1}^{m}(\lambda+1+\eta_j)\right].$$

Notice that $\Delta'(0)=\Delta_0'(0)\neq0$, so 0 is a simple root of $\Delta(\lambda)=0$.

(2). Let $0<\beta<\beta_{cr}$, $\lambda=\mu+\sigma i$ be a root of $\Delta(\lambda)$ then

$$\beta e^{-\lambda\tau}=\frac{(\lambda+\nu_1)(\lambda+\nu_2)\dots(\lambda+\nu_m)}{b_m+b_{m-1}(\lambda+\nu_m)+\dots+b_1(\lambda+\nu_2)\dots(\lambda+\nu_m)}$$

$$\nu_k=1+\eta_k,\quad k=1,2,\dots m. \tag{15}$$

Let $\lambda+\nu_k=\rho_k e^{i\phi_k}$, $\rho_k=\sqrt{(\nu_k+\mu)^2+\sigma^2}$, $0\le\phi_k\le\frac{\pi}{2}$. We can assume without lost of generality that $\sigma\ge0$ (since if $\Delta(\lambda)=0$ then $\Delta(\bar{\lambda})=0$), then

$$\beta e^{-\lambda\tau}=\frac{1}{b_m\rho_1^{-1}\rho_2^{-1}\dots\rho_m^{-1}e^{-i(\phi_1+\phi_2+\dots\phi_m)}+\dots+b_1\rho_1^{-1}e^{-i\phi_1}},$$

and taking the real part, one has

$$\beta e^{-\mu\tau}\cos\sigma\tau=\frac{1}{b_m\rho_1^{-1}\rho_2^{-1}\dots\rho_m^{-1}\cos(\phi_1+\phi_2+\dots\phi_m)+\dots+b_1\rho_1^{-1}\cos\phi_1}.$$

If $\mu>0$ or $\mu=0$, $\sigma>0$ ($\mu=\sigma=0$ is impossible), then it follows that

$$\beta e^{-\mu\tau}|\cos\sigma\tau| > \frac{1}{|b_m\rho_1^{-1}\rho_2^{-1}\ldots\rho_m^{-1}+\ldots+b_1\rho_1^{-1}|} = \beta_{cr}$$

which leads to a contradiction. Hence $\mu<0$, i.e. $\mathrm{Re}\lambda<0$.

Furthermore, when $\beta=\beta_{cr}$, if $\Delta(\lambda)=0$ has a pure imaginary root, say $i\sigma$ ($\sigma>0$), then

$$\beta|\cos\sigma\tau| > \frac{1}{|b_m\rho_1^{-1}\rho_2^{-1}\ldots\rho_m^{-1}+\ldots+b_1\rho_1^{-1}|} > \beta_{cr},\quad \rho_k=\sqrt{\nu_k^2+\sigma^2}$$

which also leads to a contradiction. So $\sigma=0$, and conclusion (4) follows.

(3). When $\beta>\beta_{cr}$, since $\Delta(0)=\nu_1\nu_2\ldots\nu_m(1-\beta\beta_{cr}^{-1})<0$ and $\Delta(\lambda)\longrightarrow\infty$ as $\lambda\longrightarrow\infty$ along the real axis, so $\Delta(\lambda)=0$ must have a positive root λ_0. Moreover, $\Delta(\lambda)=(\lambda+\nu_1)(\lambda+\nu_2)\ldots(\lambda+\nu_m)[1-\beta e^{-\lambda\tau}(b_1((\lambda+\nu_1)^{-1}+\ldots+b_m(\lambda+\nu_1)^{-1}(\lambda+\nu_2)^{-1}\ldots(\lambda+\nu_m)^{-1})]$ is a strictly monotonic function for $\lambda>0$, so λ is unique. Now, $\Delta'(\lambda_0)=\Delta_0'(\lambda_0)e^{-\lambda_0\tau}+g'(\lambda_0)(1-e^{-\lambda_0\tau})\cdot b(\lambda_0)e^{-\lambda_0\tau}$, where

$$g(\lambda)=\prod_{k=1}^{m}(\lambda+\nu_k),\quad b(\lambda)=\beta[b_m+b_{m-1}(\lambda+\nu_m)+\ldots+b_1(\lambda+\nu_2)\ldots(\lambda+\nu_m)]$$

so that $\Delta'(\lambda_0)>0$ and hence λ_0 is a simple root of $\Delta(\lambda)=0$. Let $\lambda=\mu+i\sigma$, $\lambda\neq\lambda_0$, $\mu>\lambda_0$ be another root of $\Delta(\lambda)=0$, let

$$\lambda+\nu_k=\rho_k e^{i\phi_k},\quad \rho_k=\sqrt{(\nu_k+\mu)^2+\sigma^2},\quad 0\le\phi_k\le\frac{\pi}{2}$$

then

$$\beta e^{-\mu\tau}|\cos\sigma\tau| = \left|\frac{1}{b_m\rho_1^{-1}\rho_2^{-1}\ldots\rho_m^{-1}\cos(\phi_1+\phi_2+\ldots\phi_m)+\ldots+b_1\rho_1^{-1}\cos\phi_1}\right|.$$
$$>\beta e^{-\lambda_0\tau}$$

This is a contradiction. So $\mu<\lambda_0$, i.e. $\mathrm{Re}\lambda<\lambda_0$. The proof is complete.

Take ϕ, $\psi\in$ **Y**, we solve the equation

$$(\lambda-\mathbb{A})\phi=\psi.$$

One has

$$\phi(\Theta)=\phi(0)e^{\lambda\Theta}-\int_0^{\Theta}e^{\lambda(\Theta-s)}\psi(s)ds,$$

$$(\lambda-A-\beta e^{-\lambda\tau}B)\phi(0)=\psi(0)+\beta e^{-\lambda\tau}B\int_s^0 e^{-\lambda s}\psi(s)ds=g_{\psi}\in \mathbb{R}^m,$$

so if $\Delta(\lambda)\neq 0$ then $\lambda\in\rho(\mathbb{A})$ and

$$R(\lambda,\mathbb{A})\psi(\Theta)=(\lambda-A-\beta e^{-\lambda\tau}B)^{-1}g_{\psi}e^{\lambda\Theta}-\int_0^{\Theta}e^{\lambda(\Theta-s)}\psi(s)ds. \tag{16}$$

By (14), we see that when $\beta\geq\beta_{cr}$, λ_0 is a pole of order one of $R(\lambda.\mathbb{A})$, and hence λ_0 is a simple eigenvalue of the operator $\mathbf{A}$. Since $T(t)$ is compact for $t>\tau$, the spectral mapping principle can be used to yield

Proposition 1. (1) when $\beta\geq\beta_{cr}$, the operator $\mathbb{A}$ has only one dominant nonnegative eigenvalue λ_0, which is algebraic simple;

(2). there is only a finite number of eigenvalues of $\mathbb{A}$ in every finite strip parallel to the imaginary axis;

(3). $\sigma(\mathbb{A})=\sigma_p(\mathbb{A})$ consists of all the eigenvalues of $\mathbb{A}$.

Let $\beta\geq\beta_{cr}$, $\mathbb{P}_{\lambda_0}$ be the eigenprojection corresponding to λ_0, then for any $\mathbf{x}(\Theta)\in\mathbf{Y}$

$$\mathbb{P}_{\lambda_0}\mathbf{x}(\Theta)=\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)R(\lambda,\mathbb{A})\mathbf{x}(\Theta)$$
$$=C_x e^{\lambda_0\Theta}, \tag{17}$$

where $C_x=\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)(\lambda-A-\beta e^{-\lambda\tau}B)^{-1}g_x$.

By [7], we have

Theorem 2. When $\beta \geq \beta_{cr}$ the solution to equation (7) has the following asymptotic form

$$\mathbf{x}(t+\theta)=T(t)\mathbf{x}_0(\theta)=C_x\, e^{\lambda_0 \theta} e^{\lambda_0 t} + o(e^{(\lambda_0-\varepsilon)t}), \quad t \longrightarrow \infty, \tag{18}$$

where $\varepsilon>0$ is any number such that $\sigma(\mathbb{A}) \cap \{\lambda \,|\, \lambda_0-\varepsilon \leq \mathrm{Re}\lambda < \lambda_0\} = \emptyset$, λ_0 is the dominant eigenvalue of $\mathbb{A}$; when $0<\beta<\beta_{cr}$,

$$\|\mathbf{x}(t+\theta)\| \leq Me^{-\alpha t}, \quad \text{for all } t \geq 0 \tag{19}$$

where $\alpha>0$ is any number such that $\sup\{\mathrm{Re}\lambda \,|\, \lambda \in \sigma(\mathbb{A})\} < -\alpha$.

Note. For $0<\beta<\beta_{cr}$, there still exists a maximal real root, also denoted by λ_0, such that any other root λ of $\Delta(\lambda)=0$ satisfies $\mathrm{Re}\lambda<\lambda_0$. Indeed this follows from the fact that $\Delta(0)=\nu_1\nu_2\ldots\nu_m(1-\beta\beta_{cr})>0$ and $\Delta(-\nu_k)\leq 0$, $\nu_k=\min_{1\leq i\leq m}\nu_i$, so that the maximal real root λ_0 of $\Delta(\lambda)=0$ must satisfy $\lambda_0>-\nu_k$. If $\lambda=\mu+i\sigma$, $\lambda\neq\lambda_0$, $\mu>\lambda_0$ is another root of $\Delta(\lambda)=0$, then

$$\beta e^{-\mu\tau}|\cos\sigma\tau| = \left| \frac{1}{b_m\rho_1^{-1}\rho_2^{-1}\ldots\rho_m^{-1}\cos(\phi_1+\phi_2+\ldots\phi_m)+\ldots+b_1\rho_1^{-1}\cos\phi_1} \right|$$

$$>\beta e^{-\lambda_0\tau}$$

where $\lambda+\nu_k=\rho_k e^{i\phi_k}$, $\rho_k=\sqrt{(\nu_k+\mu)^2+\sigma^2}$, $0\leq\phi_k\leq\frac{\pi}{2}$. This leads to a contradiction. So $\mu<\lambda_0$, i.e. $\mathrm{Re}\lambda<\lambda_0$. Generally λ_0 is not simple. Instead of (19), we have

$$\|\mathbf{x}(t+\theta)\| \leq Me^{(\lambda_0+\varepsilon)t}, \quad \text{for all } t\geq 0 \tag{20}$$

where $\varepsilon>0$ is a small number.

$\beta=\beta_{cr}$ defined by (15), is called the critical fertility rate of females (cf. [5]). When $\beta=\beta_{cr}$, the population density stabilizes to an ideal state as time goes to infinity. In this case,

$$C_{x_0} = \lim_{\lambda\to 0} \lambda(\lambda-A-\beta e^{-\lambda\tau}B)^{-1}g_{x_0}$$

$$= \lim_{\lambda\to 0} \lambda(\lambda-A-\beta_{cr}B)^{-1}g_{x_0}$$

$$=\frac{\langle\tilde{\mathbf{x}}_0,\tilde{\mathbf{x}}\rangle}{\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle}\tilde{\mathbf{x}},$$

where

$$\tilde{\mathbf{x}} =(\nu_1\nu_2\ldots\nu_m, \ \ldots\nu_m, \ 1)^T,$$

$$\tilde{\mathbf{y}}=(1, \quad b_2\nu_2^{-1}+\ldots+b_m\nu_2^{-1}\nu_3^{-1}\ldots\nu_m^{-1}, \ b_m\nu_m^{-1})^T,$$

$$\tilde{\mathbf{x}}_0=\mathbf{x}_0(0)+\beta_{cr}B\int_{-\tau}^{0}\mathbf{x}_0(s)ds. \tag{21}$$

We then have

Theorem 3. When $\beta=\beta_{cr}$, the solution of linear system (7) has the following asymptotic expression

$$\mathbf{x}(t)= \frac{\langle\tilde{\mathbf{x}}_0,\tilde{\mathbf{x}}\rangle}{\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle}\tilde{\mathbf{x}} + o(e^{-\varepsilon t}) \tag{22}$$

and the total population N(t) is given by

$$N(t)= \frac{\langle\tilde{\mathbf{x}}_0,\tilde{\mathbf{x}}\rangle}{\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle}\sum_{k=1}^{m}(1+\beta_{cr}b_k)\tilde{x}_k+ o(e^{-\varepsilon t}),$$

$$\tilde{\mathbf{x}}=(\tilde{x}_1,\tilde{x}_2,\ldots\tilde{x}_m)^T. \tag{23}$$

Finally, we want to show that $\sigma(\mathbb{A})$ is a infinite set. In fact, we have demonstrated that $\lambda\varepsilon\sigma(\mathbb{A})$ if and only if $\Delta(\lambda)=0$. By (13), since $\Delta(\lambda)$ has a pole of at most order one at $\lambda=0$. We can assume that $\Delta(0)\neq 0$ (otherwise we consider $\Delta(\lambda)/\lambda$ instead of $\Delta(\lambda)$). If $\Delta(\lambda)=0$ has only a finite number of zeros, then by [8]

$$\Delta(\lambda)=e^{a\lambda+b}p(\lambda)$$

where $p(\lambda)$ is a polynomial of finite order, a, b are complex constants.

It is obvious that $a\neq 0$, and let $a=u+iv$, then

$$\Delta(\lambda)=e^{\lambda u}e^{iv\lambda+b}p(\lambda);$$

(1). if $u>0$, then letting $\lambda\longrightarrow-\infty$ along the real axis leads to a contradiction;

(2). if $u<0$, then letting $\lambda\longrightarrow\infty$ along the real axis also leads to a contradiction;

(3). if $u=0$, then $v\neq 0$, we can assume without lost of generality that $v>0$. Let $\lambda=i\sigma$, $\sigma>0$, then letting $\sigma\longrightarrow\infty$ again leads to a contradiction.

Summarizing, we have

Proposition 2. $\sigma(\mathbb{A})$ is a infinite set.

In the following, we shall study further the asymptotic properties of the eigenvalues of $\mathbb{A}$ (or the zeros of $\Delta(\lambda)$). By [11], the zeros of $\Delta(\lambda)$ asymptotically the same as that of $g(\lambda)=\lambda^m e^{\lambda\tau}-b_{r_1}\beta\lambda^{m-r_1}$, or $e^{\lambda\tau}\lambda^{r_1}-b_{r_1}\beta$, i.e.

$$\begin{cases} \mathrm{Re}\lambda_n=\frac{r_1}{\tau}[\lg|\omega|-\lg|2n\pi|\frac{r_1}{\tau} + \frac{r_1}{\tau}\arg\omega+\frac{\pi}{2}\frac{r_1}{\tau}\varepsilon|] + o(1) \\ \mathrm{Im}\lambda_n=\frac{r_1}{\tau}[2n\pi+\arg\omega+\frac{\pi}{2}\varepsilon] + o(1), \qquad n=\pm 1, \pm 2, \ldots \end{cases} \tag{25}$$

where

$$\varepsilon=\begin{cases} -1, & \text{if } n>0; \\ 1, & \text{if } n<0. \end{cases}$$

Furthermore, it is also known from [11] that all the zeros, except for a finite number of $\Delta(\lambda)=0$, are simple. This is equivalent to saying that all the eigenvalues of $\mathbb{A}$ except a finite number of them are algebraic simple. Notice that $\sigma(\mathbb{A})=\sigma_p(\mathbb{A})$ locates inside a fan-shaped sector as show in Fig. 1, where $0<\theta<\frac{\pi}{2}$. Thus T(t) can not be extended to an analytic semigroup on $C(-\tau,0)$, but T(t) is differentiable for $t>\tau$ by the previous discussion.

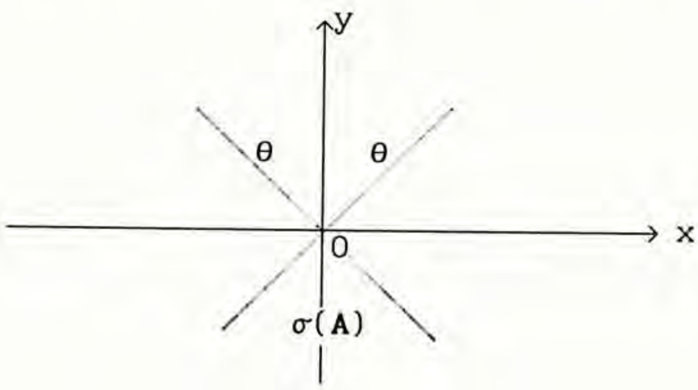

FIGURE 1

In order to study the eigenprojection of $\mathbb{A}$ and some control problems, we need a larger space than $C(-\tau,0)$. Thus we consider the following initial value problem:

$$\begin{cases} \mathbf{x}(t)=A\mathbf{x}(t)+\beta\mathbf{x}(t-\tau), & t\geq 0, \\ \mathbf{x}(0)=\mathring{\mathbf{x}}, \quad \mathbf{x}(\theta)=\mathbf{x}_0(\theta), & -\tau\leq\theta\leq 0. \end{cases} \tag{26}$$

Define the new state space $\hat{\mathbf{Y}}=\mathbb{R}^m\times L^p(-\tau,0;\mathbb{R}^m)$ and the operators on $\hat{\mathbf{Y}}$ as follows:

$$\hat{T}(t)(\mathring{\mathbf{x}},\mathbf{x}_0(\theta))=(\mathbf{x}(t),\mathbf{x}(t+\theta)), \quad \forall\ (\mathring{\mathbf{x}},\mathbf{x}_0(\theta))\in\hat{\mathbf{Y}}, \quad t\geq 0 \tag{27}$$

$\hat{\mathbf{Y}}$ as follows:

$$\hat{T}(t)(\mathring{\mathbf{x}},\mathbf{x}_0(\theta))=(\mathbf{x}(t),\mathbf{x}(t+\theta)),\quad \forall\ (\mathring{\mathbf{x}},\mathbf{x}_0(\theta))\in\hat{\mathbf{Y}},\quad t\geq 0 \tag{27}$$

where $\mathbf{x}(t)$, $\mathbf{x}(t+\theta)$ is defined iteratively

$$\begin{cases} \mathbf{x}(t)=e^{At}\mathring{\mathbf{x}}+\beta\int_0^t e^{A(t-s)}B\mathbf{x}_0(s-\tau)ds, & 0\leq t\leq\tau, \\ \mathbf{x}(t)=e^{At}\mathbf{x}(n\tau)+\beta\int_{n\tau}^t e^{A(t-s)}B\mathbf{x}(s-\tau)ds, & n\tau\leq t\leq(n+1)\tau, \end{cases}$$

$$n=1,2,\ldots. \tag{28}$$

It is obvious that $\mathbf{x}(t)$ defined by (28) belongs to $C(0,\infty;\mathbb{R}^m)$, and satisfies (26) for $t>\tau$. When $0\leq t\leq\tau$, $\mathbf{x}(t)$ is absolutely continuous and satisfies (24) almost everywhere. Following the definition in [12], $\mathbf{x}(t)$ is the solution of (26) in $\hat{\mathbf{Y}}$ (the uniqueness of solution is apparent). Hence $\hat{T}(t)$ determines a strongly continuous semigroup in $\hat{\mathbf{Y}}$, and its infinitesimal generator is given by

$$D(\hat{\mathbb{A}})=\{(\mathbf{x}(0),\mathbf{x}(\theta))\,|\,\mathbf{x}(\theta)\in H^1(-\tau,0)\},$$

$$\hat{\mathbb{A}}(\mathbf{x}(0),\mathbf{x}(\theta))=(A\mathbf{x}(0)+\beta B\mathbf{x}(-\tau),\mathbf{x}'(\theta)). \tag{29}$$

$\sigma(\hat{\mathbb{A}})$ is identical to $\sigma(\mathbb{A})$ and consists of zeros of $\Delta(\lambda)$. The resolvent of $\hat{\mathbb{A}}$ is as follows

$$R(\lambda,\hat{\mathbb{A}})(\mathring{\mathbf{y}},\mathbf{y}(\theta))$$

$$=((\lambda-A-\beta e^{-\lambda\tau}B)^{-1}[\mathbf{y}_0+\int_{-\tau}^0 e^{-\lambda(\tau+s)}\mathbf{y}(s)ds],\mathring{\mathbf{x}}e^{\lambda\theta}+\int_\theta^0 e^{-\lambda(s-\theta)}\mathbf{y}(s)ds), \tag{30}$$

where $\mathring{\mathbf{x}}=((\lambda-A-\beta e^{-\lambda\tau}B)^{-1}[\mathring{\mathbf{y}}+\int_{-\tau}^0 e^{-\lambda(\tau+s)}\mathbf{y}(s)ds]$.

$\hat{T}(t)$ is compact and differentiable for $t\geq\tau$ but can not be extended to an analytic semigroup.

Let λ_0 be the dominant eigenvalue of $\hat{\mathbb{A}}$, then when $\beta\geq\beta_{cr}$, the

$$\hat{\mathbb{P}}_{\lambda_0}(\overset{\circ}{\mathbf{x}},\mathbf{x}(\theta))=\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)R(\lambda,\hat{\mathbb{A}})(\overset{\circ}{\mathbf{x}},\mathbf{x}(\theta))$$

$$=(E_{\lambda_0}[\overset{\circ}{\mathbf{x}}+\int_{-\tau}^{0}e^{-\lambda_0(\tau+s)}\mathbf{x}(s)ds],\ E_{\lambda_0}[\overset{\circ}{\mathbf{x}}+\int_{-\tau}^{0}e^{-\lambda_0(\tau+s)}\mathbf{x}(s)ds]e^{\lambda_0\theta}],\quad (31)$$

where $E_{\lambda_0}=\lim_{\lambda\to\lambda_0}(\lambda-A-\beta e^{-\lambda\tau}B)^{-1}$.

Proposition 3. When $\beta<\beta_{cr}$, $\hat{T}(t)(\overset{\circ}{\mathbf{x}},\mathbf{x}_0(\theta))$ converges to zero exponentially as $t\to\infty$; when $\beta\ge\beta_{cr}$,

$$\hat{T}(t)(\overset{\circ}{\mathbf{x}},\mathbf{x}_0(\theta))=\hat{\mathbb{P}}_{\lambda_0}(\overset{\circ}{\mathbf{x}},\mathbf{x}(\theta))e^{\lambda_0 t}+o(e^{(\lambda_0-\varepsilon)t}).\quad (32)$$

Define the linear operator $\mathbb{B}$: $\hat{\mathbf{Y}}^*=\mathbb{R}^m\times L^q(-\tau,0;\mathbb{R}^m)\longrightarrow\hat{\mathbf{Y}}^*$ as follows:

$$\mathbb{B}(\overset{\circ}{\mathbf{y}},\mathbf{y}(\theta))=(A^*\overset{\circ}{\mathbf{y}}+\mathbf{y}(0),\ -\dot{\mathbf{y}}(\theta)),\quad \forall\ (\overset{\circ}{\mathbf{y}},\mathbf{y}(\theta))\in D(\mathbb{B}),$$

$$D(\mathbb{B})=\{(\overset{\circ}{\mathbf{y}},\mathbf{y}(\theta))\,|\,\mathbf{y}(\theta)\in H^1(-\tau,0),\ \beta B^*\overset{\circ}{\mathbf{y}}=\mathbf{y}(-\tau)\}.\quad (33)$$

Then

$$\langle\hat{\mathbb{A}}(\mathbf{x}(0),\mathbf{x}(\theta)),(\overset{\circ}{\mathbf{y}},\mathbf{y}(\theta))\rangle_{L^p\times L^q}$$

$$=\langle(\mathbf{x}(0),\mathbf{x}(\theta)),\mathbb{B}(\overset{\circ}{\mathbf{y}},\mathbf{y}(\theta))\rangle,\ \forall\ (\mathbf{x}(0),\mathbf{x}(\theta))\times(\overset{\circ}{\mathbf{y}},\mathbf{y}(\theta))\in D(\hat{\mathbb{A}})\times D(\mathbb{B}),$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\langle\cdot,\cdot\rangle$ denotes the inner product between L^p and L^q.

Taking $(\overset{\circ}{\mathbf{z}},\mathbf{z}(\theta))\in\hat{\mathbf{Y}}^*$ and solve the equation

$$(\lambda-\mathbb{B})(\overset{\circ}{\mathbf{y}},\mathbf{y}(\theta))=(\overset{\circ}{\mathbf{z}},\mathbf{z}(\theta)).$$

If $\det(\lambda-A^*-\beta e^{-\lambda\tau}B^*)^{-1}=\Delta(\lambda)\neq0$, then $\lambda\in\rho(\mathbb{B})$ and

$$\dot{\mathbf{y}}=(\lambda-A^*-\beta e^{-\lambda\tau}B^*)^{-1}[\overset{\circ}{\mathbf{z}}+\int_{-\tau}^{0}e^{\lambda s}\mathbf{z}(s)ds],$$

$$y(\theta)=[\beta e^{-\lambda\tau}B^*\mathring{y}+\int_{-\tau}^{0}e^{\lambda s}z(s)ds\,e]^{-\lambda\theta}+\int_{0}^{\theta}e^{\lambda(s-\theta)}z(s)ds. \tag{34}$$

If $\Delta(\lambda)=0$, then $\lambda\in\sigma(\mathbb{B})=\sigma_p(\mathbb{B})$ and the corresponding eigenfunction

$$(\mathring{y},\ \beta e^{-\lambda\tau}e^{-\lambda\theta}\ B^*\mathring{y}),\qquad (\lambda-A^*-\beta e^{-\lambda\tau}B^*)\mathring{y}=0. \tag{35}$$

Since $D(\mathbb{B})\subset D(\hat{\mathbb{A}}^*)$, and for $(\mathring{y},y(\theta)\in D(\mathbb{B})$

$$\mathbb{B}(\mathring{y},\ y(\theta))=\hat{\mathbb{A}}^*(\mathring{y},\ y(\theta))$$

On the other hand, $\hat{\mathbb{A}}$ is a dense operator, so $\lambda\in\rho(\hat{\mathbb{A}}^*)$ provided $\lambda\in\rho(\hat{\mathbb{A}})$. Take $\lambda\in\rho(\mathbb{B})\cap\rho(\hat{\mathbb{A}}^*)$, then

$$(\lambda-\hat{\mathbb{A}}^*)D(\mathbb{B})=(\lambda-\mathbb{B})D(\mathbb{B})=\hat{Y}^*,$$

so $D(\mathbb{B})=(\lambda-\hat{\mathbb{A}}^*)\hat{Y}^*=D(\hat{\mathbb{A}}^*)$, i.e. $B=\hat{\mathbb{A}}^*$. Hence $\rho(\hat{\mathbb{A}})=\rho(\hat{\mathbb{A}}^*)$ and for $\lambda\in\rho(\hat{\mathbb{A}})$

$$R(\lambda,\hat{\mathbb{A}})^*=R(\lambda,\hat{\mathbb{A}}^*).$$

Let λ be an eigenvalue of $\hat{\mathbb{A}}$, denote by Γ the circle with centre at λ and has no other eigenvalues of $\hat{\mathbb{A}}$ inside except λ, then the eigenprojection corresponding to λ is

$$\mathbb{E}(\lambda)=\frac{1}{2\pi i}\int_{\Gamma}(\xi-\hat{\mathbb{A}})d\xi=\frac{1}{2\pi i}\int_{\Gamma}R(z,\hat{\mathbb{A}})dz,$$

so $\mathbb{E}^*(\lambda)$ is the eigenprojection of $\hat{\mathbb{A}}^*$ corresponding to λ.

For a linear operator L with compact resolvent in a Banach space **E**, define

$$\sigma_\infty(L)=\{\ x\mid x\in\mathbf{E},\ \mathbb{E}(\lambda)x=0,\ \text{for all } \lambda\in\sigma(L)\}$$

$$sp(L)=\{\ y\mid y\in\mathbf{E},\ \text{and there exist a complex } \lambda \text{ and an integer n such that } (\lambda-L)^n=0\}$$

then one has [13]

Lemma. (1). σ_∞ is a invariant subspace of $R(\lambda,L)$;

(2). $\sigma_\infty(L)=\{\ x\mid x\in\mathbf{E}$, and $R(\lambda,L)x$ is a entire function of $\lambda\}$;

(3). $sp(L)=\sigma_\infty(L^*)^\perp$.

Now let us consider the structure of $\sigma_\infty(\hat{\mathbb{A}}^*)$. Take any $(\mathring{z},z(\theta))\in\sigma_\infty(\hat{\mathbb{A}}^*)$, assume without lost of generality that $z(\theta)$ is continuous, since otherwise we can consider $R(\lambda,\hat{\mathbb{A}}^*)(\mathring{z},z(\theta))$ instead of $z(\theta)$ in the following. So $R(\lambda,\hat{\mathbb{A}}^*)(\mathring{z},z(s))$ is an entire function of λ in the space $\hat{Y}^*$. By (34)

$$\alpha(\lambda)=(\lambda-A^*-\beta e^{-\lambda\tau}B^*)^{-1}[\mathring{z}+\int_{-\tau}^{0}e^{\lambda s}z(s)ds] \tag{36}$$

is a vector-valued entire function of λ. A simple calculation shows that if

$$(\lambda-A^*-\beta e^{-\lambda\tau}B^*)\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

then

$$\begin{cases} y_1=\dfrac{x_1(\lambda+\nu_2)\dots(\lambda+\nu_m)+\dots+x_m}{\Delta(\lambda)}, \\ y_{i+1}=(\lambda+\nu_i)y_i-x_i-\beta b_i e^{-\lambda\tau}y_1 \qquad i=1,2,\dots m-1. \end{cases} \tag{37}$$

Let $\mathring{z}+\int_{-\tau}^{0}e^{\lambda s}z(s)ds=(x_1(\lambda),x_2(\lambda),\dots x_m(\lambda))^T$, and solve the equation (37), with solution denoted by $y_i(\lambda)$. In order that y_i to be an entire function of λ we must have

$$y_1=\frac{x_1(\lambda)(\lambda+\nu_2)\dots(\lambda+\nu_m)+\dots+x_m(\lambda)}{\Delta(\lambda)}. \tag{38}$$

Notice that $\mathring{z}+\int_{-\tau}^{0}e^{\lambda s}z(s)ds$ is at most an entire function of order one in λ, and so is $y_1(\lambda)$. When λ goes to infinity along the imaginary axis, $\int_{-\tau}^{0}e^{\lambda s}z(s)ds$ goes to zero, so that

$$|y_1(\lambda)|\le M=\text{const.} \qquad \text{for } \lambda=|\lambda|e^{\pm\frac{\pi}{2}i}.$$

and

$$|y_1(\lambda)|=o(e^{|\lambda|^{\rho+\varepsilon}}), \quad |\lambda|\to\infty.$$

Applying the Phragmén-Lindelöf theorem (see [8]) in the angular sectors $\{\lambda|\lambda=|\lambda|e^{i\Theta}, -\frac{\pi}{2}\le\Theta\le\frac{\pi}{2}\}$ and $\{\lambda|\lambda=|\lambda|e^{i\Theta}, \frac{\pi}{2}\le\Theta\le\frac{3\pi}{2}\}$, respectively. We know that $|y(\lambda)|\le M$ on the whole complex plane, and hence

$$y_1(\lambda)=0.$$

So $x_i(\lambda)=0$, for $i=1,2,\ldots m$. i.e. $\mathring{\mathbf{z}}=\mathbf{z}(\Theta)=0$.

If $\rho=1$, define the Phragmén-Lindelöf function

$$h(\Theta)=\overline{\lim_{|\lambda|\to\infty}}\frac{\lg|y_1(|\lambda|e)^{i\Theta}}{|\lambda|}$$

and it is obvious that

$$h(\pi)=h(\frac{\pi}{2})=0.$$

Let $H_\varepsilon(\Theta)=\varepsilon(\sin\Theta-\cos\Theta)$, $\varepsilon>0$, then $H_\varepsilon(\frac{\pi}{2})=\varepsilon$. Further let

$$F(\lambda)=y_1(\lambda)e^{-\varepsilon(1+i)\lambda}$$

then $|F(\lambda)|=|y_1(\lambda)|e^{H_\varepsilon(\Theta)|\lambda|}$. When λ goes to infinity along $\Theta=\frac{\pi}{2}$

$$|F(\lambda)|=o(e^{-\varepsilon|\lambda|})o(e^{\varepsilon|\lambda|})=o(1);$$

when λ goes to infinity along $\Theta=\pi$

$$|F(\lambda)|=o(1).$$

Applying again the Phragmén-Lindelöf theorem in the angular sector $\{\lambda|\lambda=|\lambda|e^{i\Theta},\frac{\pi}{2}\le\Theta\le\pi\}$ it is clear that $|F(\lambda)|$ is bounded, and so

$$|y_1(\lambda)|=o(e^{H_\varepsilon(\Theta)|\lambda|}).$$

But $H_{2\varepsilon}\le 2\varepsilon$, hence

$$y_1(\lambda)=o(e^{2\varepsilon|\lambda|}), \quad \lambda=|\lambda|e^{i\Theta}, \ \frac{\pi}{2}\le\Theta\le\pi \tag{39}$$

Similarly, we can also show that

$$y_1(\lambda)=o(e^{\varepsilon_o|\lambda|}), \quad \lambda=|\lambda|e^{i\theta}, \ \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}, \tag{40}$$

where $\varepsilon_0>\varepsilon$ is any number.

Applying the generalized Phragmén-Lindelöf theorem, it is known that $y_1(\lambda)$ is bounded uniformly on the left half plane. The boundedness of $y_1(\lambda)$ on the right half plane is apparent, so $y_1(\lambda)$ is a bounded entire function, and hence is a constant by Liouvill's theorem. Notice that $y_1(\lambda)$ goes to zero along the imaginary axis, so $y_1(\lambda)=0$, or

$$\mathring{z}=z(\theta)=0.$$

Theorem 4. $\sigma_\infty(\hat{\mathbb{A}}^*)=0$, and hence the root subspace of $\hat{\mathbb{A}}$ is complete in $\mathbb{R}^m\times L^p(-\tau,0;\mathbb{R}^m)$.

Let $\{\lambda_n\}$ be the eigenvalues of $\hat{\mathbb{A}}$, and λ_n be algebraic simple for $n>N$, $\{\phi_n(0),\phi_n(0)e^{\lambda_n\theta}\}$ be the corresponding eigenfunctions, then the solution $\mathbf{x}(t)$ of equation (26) can be written as

$$\mathbf{x}(t)=\mathbb{P}_1\hat{T}(t)[\sum_{i=0}^{N}\mathbb{P}_i(\mathring{\mathbf{x}},\mathbf{x}(\theta))] + \sum_{n=N+1}^{\infty}C_n e^{\lambda_n t}, \tag{41}$$

where $\mathbb{P}_1$ is the projection operator from $\mathbb{R}^m\times L^q(-\tau,0;\mathbb{R}^m)$ to $\mathbb{R}^m$; $\mathbb{P}_i$ is the eigenprojection corresponding to λ_i; C_n is a constant depending on the initial value $(\mathring{\mathbf{x}},\mathbf{x}(\theta))$.

4.3 Nonlinear Semi-Discrete Population Equation with Time Delay

In this section, we are concerned with the nonlinear equation (6) where $f(\xi)$ satisfies the condition (2).

Suppose that $\mathbf{x}_0(\Theta)\in \mathbf{Y}\cap\mathbf{P}$, here $\mathbf{P}$ is the nonnegative cone of $\mathbf{Y}$, then $N(t)=\langle\mathbf{x}(t) + \beta B\mathbf{x}(t-\tau),e\rangle$, $e=(1,1,\dots 1)^T$, $\langle\cdot,\cdot\rangle$ is the Euclidean inner product of $\mathbb{R}^m$. Let $\bar{\mathbf{x}}(t)$, $\bar{N}(t)$ be the solution to the linear equation (i.e. K=0 in (6)) with the same initial condition. Define the operator $\mathbb{S}$: $C[0,\tau)\to C[0,\tau)$ as follows

$$\mathbb{S}N(t)=e^{-\int_0^t Kf(N(\rho))d\rho}\langle e^{At}\mathbf{x}_0,e\rangle$$

$$+\int_0^t e^{-\int_s^t Kf(N(\rho))d\rho}\langle e^{A(t-s)}\beta B\mathbf{x}_0(s-\tau),e\rangle ds+\langle\beta B\mathbf{x}_0(t-\tau),e\rangle \quad (42)$$

It is obvious that $\mathbb{S}\Omega\subset\Omega$, here $\Omega=\{ N(t)\,|\, N(t)\in C[0,\tau],\ 0\le N(t)\le\bar{N}(t)$, for all $t\in[0,\tau]\}$ is a bounded closed convex set of $C[0,\tau]$. Take any t_1, $t_2\in [0,\tau]$, $t_1\le t_2$, one has

$$|\mathbb{S}N(t_1)-\mathbb{S}N(t_2)|$$

$$\le|\langle(e^{At_1} -e^{At_2})\mathbf{x}_0,e\rangle|+|\langle e^{At}\mathbf{x}_0,e\rangle|G|t_1-t_2|e^{G|t_1-t_2|}$$

$$+\int_{t_2}^{t_1}|\langle e^{A(t_2-s)}\beta B\mathbf{x}_0(s-\tau),e\rangle|ds$$

$$+\int_0^{t_1}|\langle e^{A(t_1-s)}\beta B\mathbf{x}_0(s-\tau),e\rangle-\langle e^{A(t_2-s)}\beta B\mathbf{x}_0(s-\tau),e\rangle|ds+$$

$$+|\langle\beta B\mathbf{x}_0(t_1-\tau),e\rangle-\beta B\mathbf{x}_0(t_2-\tau),e\rangle|+$$

$$+\int_0^{t_1}G|t_1-t_2|e^{G|t_1-t_2|}|\langle e^{A(t_2-s)}\beta B\mathbf{x}_0(s-\tau),e\rangle|ds \quad (43)$$

where $G=\max_{\xi\in[0,\bar{M}]} Kf(\xi)$, $\bar{M}=\max_{t\in[0,\tau]}|\bar{N}(t)|$. Notice that we have used here the fact that $|e^z-1|\le|z|e^{|z|}$. So $\mathbb{S}$ is a compact operator and hence $\mathbb{S}$ has a fixed point $N(t)$ in Ω by Schauder's fixed point theorem with

$$N(t)=e^{-\int_0^t Kf(N(\rho))d\rho}\langle e^{At}\mathbf{x}_0,e\rangle$$

$$+\int_0^t e^{-\int_s^t Kf(N(\rho))d\rho} \langle e^{A(t-s)}\beta B\mathbf{x}_0(s-\tau), e\rangle ds + \langle \beta B\mathbf{x}_0(t-\tau), e\rangle$$

(44)

From this N(t) we construct

$$\mathbf{x}(t) = e^{-\int_0^t Kf(N(\rho))d\rho} e^{At}\mathbf{x}_0$$

$$+\int_0^t e^{-\int_s^t Kf(N(\rho))d\rho} e^{A(t-s)}\beta B\mathbf{x}_0(s-\tau)ds. \tag{45}$$

Then $\mathbf{x}(t)$ is a solution of (42) in $[0,\tau]$, and this solution is unique. In fact if N_1, N_2 are two fixed points then

$$N_1(t)-N_2(t) = [e^{-\int_0^t Kf(N_1(\rho))d\rho} - e^{-\int_0^t Kf(N_2(\rho))d\rho}]\langle e^{At}\mathbf{x}_0, e\rangle$$

$$+\int_0^t [e^{-\int_s^t Kf(N_1(\rho))d\rho} - e^{-\int_s^t Kf(N_2(\rho))d\rho}]\langle e^{A(t-s)}\beta B\mathbf{x}_0(s-\tau), e\rangle ds$$

$$= [e^{\int_0^t Kf(\xi_\rho)[N_2(\rho)-N_1(\rho)]d\rho} - 1]e^{-\int_s^t Kf(N_2(\rho))d\rho}\langle e^{At}\mathbf{x}_0, e\rangle$$

$$+\int_0^t [e^{-\int_s^t Kf(\xi_\rho)[N_2(\rho)-N_1(\rho)]d\rho} - 1]e^{-\int_s^t Kf(N_2(\rho))d\rho}.$$

$$\cdot\langle e^{A(t-s)}\beta B\mathbf{x}_0(s-\tau), e\rangle ds$$

where ξ_ρ is between $N_1(\rho)$ and $N_2(\rho)$. By $|e^z-1| \le |z|e^{|z|}$ and the Gronwall's inequality it follows immediately that $N_1(t)=N_2(t)$. So $\mathcal{S}$ has a unique fixed point, i.e. the solution of (25) is unique. Considering $[n\tau,(n+1)\tau]$ as $[(n-1)\tau,n\tau]$ iteratively for $n=1,2,\ldots,$ we have

Proposition 4. For any $\mathbf{x}_0(\theta) \in \mathbf{Y}\cap\mathbf{P}$, there exists a unique solution to (42), which is given by (44) and (45).

Take any $\mathbf{x}_0(\theta)\in Y\cap P$, define

$$\tilde{T}(t)\mathbf{x}_0(\theta)=\mathbf{x}(t+\theta), \quad \text{for } t\geq 0, \tag{46}$$

where $\mathbf{x}(t)$ is the solution to (42) with the initial value $\mathbf{x}_0(\theta)$. It can be verified easily that $\tilde{T}(t)$ is a nonlinear strongly continuous semigroup in **P** and its infinitesimal generator $\tilde{A}$ is given by

$$\tilde{A}\mathbf{x}(\theta)=\begin{cases} \mathbf{x}'(\theta), & \text{for } -\tau\leq\theta\leq 0, \\ A\mathbf{x}(0)+\beta B\mathbf{x}(-\tau)-Kf(N_0)\mathbf{x}_0, & \text{for } \theta=0, \end{cases}$$

$$D(\tilde{A})=\{\mathbf{x}(\theta)\mid \mathbf{x}\in \mathbf{P},\ A\mathbf{x}\in \mathbf{Y}\}. \tag{47}$$

Since $\overline{D(\tilde{A})}=\mathbf{Y}$, so for any fixed constant $N^*>0$

$$\Omega=\{\mathbf{x}(\theta)\mid \mathbf{x},\mathbf{x}'\in \mathbf{Y}, \mathbf{x}'(0)=A\mathbf{x}(0)+\beta B\mathbf{x}(t-\tau)-Kf(N^*)\mathbf{x}(0)\}$$

is dense in **Y**. For any $\mathbf{x}(\theta)\in\mathbf{P}$, $\mathbf{x}(\theta)>0$, there is a sequence $\mathbf{x}_n(\theta)\in\Omega$ such that $\|\mathbf{x}_n-\mathbf{x}\|\to 0$ as $n\to\infty$, so we may suppose that $\mathbf{x}_n(\theta)\in\mathbf{P}$, i.e.

$$\Omega^+=\{\mathbf{x}(\theta)\mid \mathbf{x}\in\mathbf{P}, \mathbf{x}'\in \mathbf{Y} \text{ and } \mathbf{x}'(0)=A\mathbf{x}(0)+\beta B\mathbf{x}(t-\tau)-Kf(N^*)\mathbf{x}(0)\}$$

is dense in **P**. Let $\mathbf{x}(\theta)\in\Omega^+$, define

$$\bar{\mathbf{x}}(\theta)=\begin{cases} \mathbf{x}(\theta), & \text{when } G_0=\langle\mathbf{x}(0)+\beta B\mathbf{x}(-\tau),e\rangle=0, \\ \dfrac{N^*}{G_0}, & \text{when } G_0\neq 0. \end{cases}$$

then it is apparent that $\bar{\mathbf{x}}(\theta)\in D(\tilde{A})$. So we have

Proposition 5. Writing equation (42) as an abstract nonlinear evolutionary equation in **Y**

$$\begin{cases} \dot{\mathbf{x}}(t+\theta)=\tilde{A}\mathbf{x}(t+\theta), & t\geq 0, \\ \mathbf{x}(0)=\mathbf{x}_0(\theta), & -\tau\leq\theta\leq 0, \end{cases} \tag{48}$$

then for any $\mathbf{x}_0(\theta)\in Y\cap P$, the solution to (48) exists and is unique with

$$\mathbf{x}(t+\theta)\in C([0,\infty);\ \mathbf{P});$$

if $\mathbf{x}_0 \in D(\tilde{\mathbb{A}})$ then

$$\mathbf{x}(t+\theta) \in C^1([0,\infty);\ \mathbf{P}).$$

Furthermore, $D(\tilde{\mathbb{A}})$ is dense in $\mathbf{P}$, i.e. $\overline{D(\tilde{\mathbb{A}})}=\mathbf{P}$.

Next, we turn to the stability problem of the nonlinear population system with time delay. Since for $\mathbf{x}_0(\theta) \in \mathbf{Y} \cap \mathbf{P}$

$$\mathbf{x}(t) = e^{-\int_0^t Kf(N(\rho))d\rho} e^{At} \mathbf{x}_0(0)$$

$$+\int_0^t e^{-\int_s^t Kf(N(\rho))d\rho} e^{A(t-s)} \beta B\mathbf{x}(s-\tau)ds. \qquad (49)$$

Let $\bar{\mathbf{x}}(t)$ be the solution of the linear equation (7) with the same initial condition, then $\mathbf{x}(t+\theta) \le \bar{\mathbf{x}}(t+\theta)$ for all $t \ge 0$. So we have

Proposition 6. If $\beta \le \beta_{cr}$, then equation (42) is globally stable in $\mathbf{P} \cap \mathbf{Y}$; when $\beta < \beta_{cr}$, the system is asymptotically stable (or exponentially stable).

If there exists a nonnegative equilibrium state for (31) (i.e. the state which is independent of time) $\mathbf{x}^*(\theta)$, then $\tilde{\mathbb{A}}\mathbf{x}^*(\theta)=0$, or $\frac{d}{d\theta}\mathbf{x}^*(\theta)=0$ for $\theta \ne 0$, so $\mathbf{x}^*(\theta)=\mathbf{x}^*$ is a constant vector. By $A\mathbf{x}^*+\beta B\mathbf{x}^*-Kf(N^*)\mathbf{x}^*=0$, one has

$$(A+\beta B)\mathbf{x}^* = Kf(N^*)\mathbf{x}^*$$

where $N^* = \langle A\mathbf{x}^*+\beta B\mathbf{x}^*, e\rangle > 0$. Hence $\beta > \beta_{cr}$, $Kf(N^*)=\hat{\lambda}_0$ the dominant eigenvalue of $A+\beta B$ [5]. On the other hand, if $\beta > \beta_{cr}$, then the eigenvector $\mathbf{x}^*$ which satisfies $Kf(N^*)=\hat{\lambda}_0$ is an equilibrium state of equation (48).

Proposition 7. A necessary and sufficient condition for system

(48) to have a nonnegative equilibrium state is that there is a constant $N^*>0$ such that $Kf(N^*)=\hat{\lambda}_o$. In such a case, we must have $\beta>\beta_{cr}$, and the nonnegative state $\mathbf{x}^*$ satisfies

$$N^*=\langle A\mathbf{x}^*+\beta B\mathbf{x}^*,e\rangle.$$

Let $\mathbf{x}(t)$ be the solution of (42), $\mathbf{x}^*$ be its nonnegative equilibrium state, $z(t)=\mathbf{x}(t)-\mathbf{x}^*$, then

$$\begin{aligned}\dot{z}(t)&=A\mathbf{x}(t)-\beta B\mathbf{x}(t-\tau)-Kf(N(t))\mathbf{x}(t)\\&\quad -A\mathbf{x}^*(t)-\beta B\mathbf{x}^*(t-\tau)-Kf(N^*(t))\mathbf{x}^*(t)\\&=Az(t)-\beta Bz(t-\tau)-Kf'(N^*(t))X^*z(t)-Kf(N^*(t))z(t)\\&\quad -Kf'(N^*(t))X^*\beta Bz(t-\tau)+\circ(z(t),z(t-\tau))\\&=[A-Kf(N^*(t))]z(t)-Kf'(N^*(t))X^*z(t)+\\&\quad +\beta Bz(t-\tau)-Kf'(N^*(t))X^*\beta Bz(t-\tau)+\circ(z(t),z(t-\tau)),\end{aligned}\tag{51}$$

where $X^*=[\mathbf{x}^*,\mathbf{x}^*\ldots\mathbf{x}^*]$, $\circ(u,v)$ satisfies

$$\lim_{\substack{u\to 0\\ v\to 0}}\frac{|\circ(u,v)|}{\|u\|+\|v\|}=0.$$

The first order approximation of (51) is

$$\begin{aligned}\dot{z}(t)&=[A-Kf(N^*(t))-Kf'(N^*(t))X^*]z(t)\\&\quad +[I-Kf'(N^*(t))X^*]\beta Bz(t-\tau).\end{aligned}\tag{53}$$

For any real number $\lambda\ge 0$, define the matrix

$$E(\lambda)=A-Kf(N^*(t))-Kf'(N^*(t))X^*+\beta Be^{-\lambda\tau}-Kf'(N^*(t))X^*\beta Be^{-\lambda\tau}\tag{54}$$

When $Kf'(N^*(t))<0$, $\tilde{E}(\lambda)=E(\lambda)+Kf(N^*)+\max_{1\le i\le m}\nu_i$ is a positive matrix, so that $\tilde{E}(\lambda)$ has a maximal real eigenvalue (cf. [9]), say μ_λ. Define the function

$$g(\lambda)=-Kf(N^*)+\max_{1\le i\le m}\nu_i+\mu_\lambda,\quad \forall\ \lambda\ge 0.\tag{55}$$

Then $g(\lambda)$ is the maximal real eigenvalue of $E(\lambda)$ (the dominant eigenvalue). Let $\tilde{g}(\lambda)=g(\lambda)-\lambda$, so that $\tilde{g}(0)=g(0)$. For any eigenvalue λ of $E(\lambda)$, by definition, there exists a $\mathbf{y}\neq0$ such that

$$[A-Kf(N^*(t))-Kf'(N^*(t))X^*+\beta B -Kf'(N^*(t))X^*\beta B]\mathbf{y}=\lambda\mathbf{y}$$

$\lambda\neq0$, since otherwise, taking inner product with $\mathbf{y}^*$ on both sides above, one has $<\mathbf{y}+\beta B\mathbf{y},e>=0$. Thus $(A+\beta B)\mathbf{y}=Kf(N^*)\mathbf{y}$. This is a contradiction. Let $\mathbf{z} = \frac{<\mathbf{y},\mathbf{y}^*>}{<\mathbf{x}^*,\mathbf{y}^*>}\mathbf{x}^*$, we have

$$(A+\beta B-Kf(N^*))\mathbf{z} =\lambda\mathbf{z}.$$

If $\mathbf{z}\neq0$, then $\mathrm{Re}\lambda<0$; if $\mathbf{z}=0$ then $\lambda=-Kf'(N^*)N^*>0$. It can be verified directly that $\lambda=-Kf'(N^*)N^*$ is an eigenvalue of $E(0)$, so $g(0)=-Kf'(N^*)N^*>0$, and hence $\tilde{g}(0)>0$. From [9]

$$g(\lambda)= \max_{\substack{x\geq0\\ \|x\|=1}} \min_{1\leq i\leq m} \frac{\sum_{k=1}^{m} e_i^T E(\lambda)e_k x_k}{x_i}, \qquad \mathbf{x}=(x_1,x_2,\ldots x_m)^T$$

and hence $g(\lambda)$ is continuous for $\lambda\geq0$. But $\sup_{\lambda\geq0}\|E(\lambda)\|<+\infty$, so $\sup_{\lambda\geq0}|g(\lambda)|<+\infty$. From the fact that $\lim_{\lambda\to+\infty}\tilde{g}(\lambda)=-\infty$, we know that there exists a $\lambda_0>0$ such that $\tilde{g}(\lambda_0)=0$, i.e.

$$g(\lambda_0)=\lambda_0,$$

or $\det(\lambda_0-E(\lambda_0))=0$.

When $Kf'(N^*)>0$, if $\mathbf{x}$ is an eigenvector of $E(\lambda)$, then

$$[A-Kf(N^*)-Kf'(N^*)X^*+\beta Be^{-\lambda\tau}-Kf'(N^*)X^*\beta Be^{-\lambda\tau})\mathbf{x}=\lambda\mathbf{x}$$

$$\begin{aligned}(\lambda-A+Kf(N^*)-\beta Be^{-\lambda\tau})\mathbf{x}&=-f'(N^*)<\mathbf{x},e>\mathbf{x}^*-Kf'(N^*)<B\mathbf{x},e>\beta e^{-\lambda\tau}\mathbf{x}^*\\ &=-f'(N^*)[<\mathbf{x},e>+<B\mathbf{x},e>\beta e^{-\lambda\tau}]\mathbf{x}^*\\ &\hat{=}\ \mathbf{y} = (y_1,y_2,\ldots y_m)^T.\end{aligned}$$

If $(\lambda-A+Kf(N^*)-\beta Be^{-\lambda\tau})^{-1}$ exists, then a straightforward computation shows that

$$\begin{cases} (\lambda+\nu_1)x_1-[\hat{b}_1x_1+b_2x_2+\ldots+b_mx_m]\beta e^{-\lambda\tau}=y_1 \\ -x_{i-1}+(\lambda+\hat{\nu}_i)x_i=y_i, \quad i=2,3,\ldots m, \quad \hat{\nu}_i=\nu_i+Kf(N^*). \end{cases}$$

Hence

$$x_{m-1}=(\lambda+\hat{\nu}_m)x_m-y_m$$

$$x_{m-2}=(\lambda+\hat{\nu}_{m-1})(\lambda+\hat{\nu}_m)x_m-(\lambda+\hat{\nu}_{m-1})y_m-y_{m-1}$$

$$\ldots \quad \ldots \quad \ldots.$$

$$x_1=(\lambda+\hat{\nu}_2)\ldots(\lambda+\hat{\nu}_m)x_m-(\lambda+\hat{\nu}_2)\ldots(\lambda+\hat{\nu}_{m-1})y_m$$

$$-(\lambda+\hat{\nu}_2)\ldots(\lambda+\hat{\nu}_{m-2})y_{m-1}-\ldots-(\lambda+\hat{\nu}_2)y_3-y_2$$

$$\Delta(\lambda)=y_1+(\lambda+\hat{\nu}_2)y_2+\ldots+(\lambda+\hat{\nu}_1)\ldots(\lambda+\hat{\nu}_{m-1})y_m$$

$$-\tilde{b}_1y_2-[\tilde{b}_1(\lambda+\hat{\nu}_2)+\tilde{b}_2]y_3-\ldots-$$

$$-[\tilde{b}_1(\lambda+\hat{\nu}_2)+\ldots+(\lambda+\hat{\nu}_{m-1})+\ldots+\tilde{b}_{m-1})]y_m$$

$$\tilde{b}_i=b_i\beta e^{-\lambda\tau}, \tag{55}$$

$$\langle\mathbf{x},e\rangle+\langle B\mathbf{x},e\rangle\beta e^{-\lambda\tau}=(\lambda+\tilde{\nu}_1)(\lambda+\hat{\nu}_2)\ldots(\lambda+\hat{\nu}_m)x_m-(\lambda+\hat{\nu}_1)\ldots(\lambda+\hat{\nu}_{m-1})y_m$$

$$-(\lambda+\hat{\nu}_1)y_2-y_1$$

$$+[1+(\lambda+\hat{\nu}_2)\ldots(\lambda+\hat{\nu}_m)+\ldots+(\lambda+\hat{\nu}_m)]x_m$$

$$-[1+(\lambda+\hat{\nu}_2)\ldots(\lambda+\hat{\nu}_{m-1})+\ldots+(\lambda+\hat{\nu}_{m-1})]y_m$$

$$-\ldots-[[1+(\lambda+\hat{\nu}_2)]y_3-y_2$$

When $Kf'(N^*)>0$, we have shown that (similar to the case of $Kf'(N^*)<0$) $E(0)\mathbf{x}=0$ has no nonzero solution. Since $(-A-\beta B+Kf(N^*))\mathbf{x}^*=0$, so when $\mathrm{Re}\lambda\geq 0$ and $\lambda\neq 0$, $(\lambda-A+Kf(N^*)-\beta Be^{-\lambda\tau})^{-1}$ exists. Now, we suppose that $\mathrm{Re}\lambda\geq 0$ $\lambda\neq 0$, then $\langle\mathbf{x},e\rangle+\langle B\mathbf{x},e\rangle\neq 0$, so we have

$$-\frac{1}{Kf'(N^*)}=[1+(\lambda+\hat{\nu}_1)\dots(\lambda+\hat{\nu}_m)+\dots+(\lambda+\hat{\nu}_m)]\hat{x}_m$$

$$-[1+(\lambda+\hat{\nu}_1)\dots(\lambda+\hat{\nu}_{m-1})+\dots+(\lambda+\hat{\nu}_{m-1})]x^*_m$$

$$-[1+(\lambda+\hat{\nu}_1)x^*_2-x^*_1$$

where $\Delta(\lambda)\hat{x}_m=x^*_1+(\lambda+\hat{\nu}_1)x^*_2+\dots+(\lambda+\hat{\nu}_1)\dots(\lambda+\hat{\nu}_{m-1})x^*_m$

$$-b_1\beta e^{-\lambda\tau}x^*_2-[b_1(\lambda+\hat{\nu}_2)+b_2]\beta e^{-\lambda\tau}x^*_3-\dots-$$

$$-[b_1(\lambda+\hat{\nu}_2)+\dots+(\lambda+\hat{\nu}_{m-1})+\dots+b_{m-1})]\beta e^{-\lambda\tau}x^*_m$$

and

$$\frac{1}{Kf'(N^*)}(\lambda+\hat{\nu}_1)\dots(\lambda+\hat{\nu}_m)+\dots+(\lambda+\hat{\nu}_m)$$

$$+\sum_{i=1}^{m}[1+(\lambda+\hat{\nu}_{i+1})\dots(\lambda+\hat{\nu}_m)+\dots+(\lambda+\hat{\nu}_m)](\lambda+\hat{\nu}_1\quad)\dots(\lambda+\hat{\nu}_{i+1})x^*_i$$

$$=\frac{1}{Kf'(N^*)}[b_1(\lambda+\hat{\nu}_2)+\dots+(\lambda+\hat{\nu}_m)+\dots+b_m)]$$

$$+\sum_{i=1}^{m}(\lambda+\hat{\nu}_{i+1})\dots(\lambda+\hat{\nu}_m)[x^*_1+\dots+[1+(\lambda+\hat{\nu}_1\dots(\lambda+\hat{\nu}_{i-1})+\dots+(\lambda+\hat{\nu}_{i-1})]x^*_i]$$

$$-\sum^{m}(\lambda+\hat{\nu}_{i+1})\dots(\lambda+\hat{\nu}_{j-1})[1+\dots+(\lambda+\hat{\nu}_{j+1})\dots(\lambda+\hat{\nu}_m)+\dots+(\lambda+\hat{\nu}_m)]x^*_j]\cdot$$

$$\cdot b_i\beta e^{-\lambda\tau} \qquad (56)$$

If we can show under the condition $Kf'(N^*)>0$ that equation (56) has no solution $i\sigma$ with $\sigma>0$, then all the eigenvalues of $E(0)$ have negative real parts (the case of $Kf'(N^*)<0$), and hence the results of [15] can be applied to show that $\det(\lambda-E(\lambda))=0$ has roots with only negative real parts.

Proposition 8. When $Kf'(N^*)<0$, system (48) is not stable about its positive equilibrium state; when $Kf'(N^*)>0$, and (56) has no solutions $i\sigma$ with $\sigma>0$, then system (48) is locally asymptotically stable about its positive equilibrium state in $C[-\tau,0]\cap\mathbf{P}$.

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Chapter 5

A Finite Difference Scheme for the Equations of Population Dynamics

5.1 Introduction

We consider here a discrete population model which results from applying a certain finite difference scheme to the McKendrick type partial differential model considered in Song and Yu [1]. By proving some qualitative properties of this discrete system and the convergence of the sequence of solutions of the discrete problems (as the age step approaches zero), we thereby establish that the discrete model, indeed converges to the continuous model. Finite difference scheme has been presented in [2] but here our approach is different in that the resulting discrete system remains to be a population model with the important critical fertility index converging to the counterpart of the continuous model, i.e. our discretization is both mathematically and biologocally meaningful.

As before, $p(r,t)$ denotes the population density at time t and in the age interval $(r,r+dr)$, $0\le r\le r_m$, $t\ge 0$, where r_m is the highest age ever attained by the individuals of the population. Let $\mu(r)$ be the relative death-modulus, i.e. the death-rate per unit population of age r, satisfying

$\mu(r)>0$ and is continuous on any interval $[0,r_c]$, $r_c<r_m$;

$$\int_0^{r_c}\mu(r)dr<+\infty, \quad \int_0^{r_m}\mu(r)dr=+\infty. \tag{1}$$

Let $k(r)$ be the female sex ratio at age r, $h(r)$ the fertility pattern. We assume, through out this chapter, that

$$\begin{aligned}&k(r)h(r) \text{ is continuous on } [r_1,r_2],\\ &k(r)h(r)>0, \text{ for } r\in(r_1,r_2);\ k(r)h(r)=0, \text{ elsewhere;}\end{aligned} \tag{2}$$

where $[r_1,r_2]$ is the fecundity period of females. Let β be the specific fertility rate of females, $p_0(r)$ be the initial age distribution, then the age-dependent population model is given by the following first order partial differential equations

$$\begin{cases}\dfrac{\partial p(r,t)}{\partial t}+\dfrac{\partial p(r,t)}{\partial r}=-\mu(r)p(r,t), & 0<r<r_m,\ t>0\\ p(r,0)=p_0(r,t), & 0\le r\le r_m,\\ p(0,t)=\displaystyle\int_{r_1}^{r_2}k(r)h(r)p(r,t)dr, & t\ge 0.\end{cases} \tag{3}$$

Let $\mathcal{X}=L^1(0,r_m)$ be the underlying Banach space with the usual norm, then the associate population operator $\mathbb{A}$ is

$$\begin{cases}\mathbb{A}\phi(r)=-\dfrac{d\phi(r)}{\partial r}-\mu(r)\phi(r), \quad \forall\ \phi\in D(\mathbb{A}),\\ D(\mathbb{A})=\left\{\phi(r)\,\middle|\,\phi,\mathbb{A}\phi\in\mathcal{X},\ \phi(0)=\displaystyle\int_{r_1}^{r_2}k(r)h(r)\phi(r)dr\right\}.\end{cases} \tag{4}$$

The following results can be found in [1]

Theorem 1.

(i). There exists a unique solution $p(r,t)=e^{\mathbb{A}t}p_0(r)$ to Eq.(3) for any $p_0\in\mathcal{X}$, where $e^{\mathbb{A}t}$ is the C_0-semigroup generated by $\mathbb{A}$

$$p(r,t)\in C([0,\infty);\mathcal{X}),$$

$$p(r,t)\in C^1([0,\infty);D(\mathbb{A})), \text{ if } p_0\in D(\mathbb{A}).$$

(ii). $\lambda\in\sigma(\mathbb{A})$, the spectrum of operator $\mathbb{A}$, if and only if λ is the root of

$$F(\lambda)=1-\beta\int_{r_1}^{r_2}k(r)h(r)e^{-\lambda r-\int_0^r\mu(\rho)d\rho}dr=0, \tag{5}$$

(iii). The operator $\mathbb{A}$ has a real dominant eigenvalue λ_0. $\lambda_0=0$ if and only if β equals to the critical fertility β_{cr} given by

$$\beta_{cr}=\left[\int_{r_1}^{r_2}k(r)h(r)e^{-\int_0^r\mu(\rho)d\rho}dr\right]^{-1}. \tag{6}$$

When $\beta=\beta_{cr}$,

$$\lim_{t\to\infty} p(r,t)=C_0e^{-\int_0^r\mu(\rho)d\rho}, \quad \forall\ r\in[0,r_m], \tag{7}$$

where

$$C_0=\frac{\int_{r_1}^{r_2}k(r)h(r)\left[\int_0^r p_0(s)e^{\int_0^s\mu(\rho)d\rho}ds\right]e^{-\int_0^r\mu(\rho)d\rho}dr}{\int_{r_1}^{r_2}rk(r)h(r)e^{-\int_0^r\mu(\rho)d\rho}dr}. \tag{8}$$

For a given integer $M>0$, let $\delta=\frac{r_m}{M}$. We denote by I_i the interval $[\frac{(i-1)r_m}{M}, \frac{ir_m}{M}]$, then a discrete system relating to Eq.(3) is defined as follows:

$$\begin{cases} x_{i+1}(j+1)=(1-\delta\eta_i)x_i(j), \quad i=0,1,2,\ldots M-1, \quad j=0,1,2\ldots, \\ x_i(0)=x_{0i}, \quad i=1,2,\ldots M, \\ x_0(j)=\beta\sum_{i=1}^{M}\delta(kh)_ix_i(j), \quad j=0,1,2,\ldots \end{cases} \tag{9}$$

where

$$\begin{cases} \delta\eta_i=\dfrac{\delta\mu(i\delta)}{1+\delta\mu(i\delta)}, \quad i=0,1,2,\ldots M-1, \\ (kh)_i=k(i\delta)h(i\delta), \quad i=1,2,\ldots M, \\ p_{0\delta}(r)=\begin{cases} x_{0i}, & \text{if } r\in I_i, \quad i=1,2,\ldots M, \\ 0, & \text{elsewhere}, \end{cases} \end{cases} \tag{10}$$

and

$$\int_0^{r_m} |p_0(r)-p_{0\delta}(r)|dr \longrightarrow 0, \quad \text{as } \delta \longrightarrow 0. \tag{11}$$

Define

$$p_\delta(r,t)= x_i(j), \text{ for } (r,t)\in I_i\times I_j,$$
$$i=1,2,\dots M, \; j=0,1,\dots \tag{12}$$

We shall prove later that $p_\delta(r,t)$ defined by (12) is an approximation of $p(r,t)$, the solution of system (3).

5.2 The discrete System

W first study system (9) since it can be considered as a population model by itself alone.

Let

$$X(j)=\begin{bmatrix} x_1(j) \\ x_2(j) \\ \vdots \\ x_M(j) \end{bmatrix}, \qquad X_0=\begin{bmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0M} \end{bmatrix},$$

then system (9) can be written as

$$\begin{cases} X(j+1)=AX(j)+\beta BX(j), \; j=0,1,\dots \\ X(0)=X_0, \end{cases} \tag{13}$$

where

$$A=\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1-\delta\eta_1 & \dots & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & 1-\delta\eta_{M-1} & 0 \end{bmatrix}, \quad B=\begin{bmatrix} b_1 & b_2 & \dots & b_M \\ 0 & 0 & \dots & 0 \\ & & & \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

$$b_i=\delta(1-\delta\eta_0)(kh)_i, \; i=1,2,\dots M. \tag{14}$$

The characteristic polynomial of (13) is

$$F_\delta(\lambda)=\det(\lambda-A-\beta B)$$

$$=\lambda^M-\beta[b_1\lambda^{M-1}+b_2(1-\delta\eta_1)\lambda^{M-2}+\ldots+b_M\prod_{i=1}^{M-1}(1-\delta\eta_i)]. \qquad (15)$$

Lemma 1. Let the $m\times m$ matrix E be of the form

$$E=\begin{bmatrix} * & \Delta & \ldots & \Delta & * \\ * & * & \ldots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & * & * \end{bmatrix}_{m\times m},\quad *>0,\ \Delta\geq 0,$$

then there exists a positive p such that $E^p>0$.

proof. We employ mathematical induction. For m=1, $E=(*)>0$. Assume that there exists an integer $q>0$ such that for $m=k$, $E^q>0$, consider $m=k+1$, then

$$E=\begin{bmatrix} A_k & B \\ C & D \end{bmatrix},\quad A_k=\begin{bmatrix} * & \Delta & \ldots & \Delta & \Delta \\ * & * & \ldots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & * & * \end{bmatrix}_{m\times m},$$

$$B=\begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix},\quad C=(0,0,\ldots 0,*),\quad D=(*).$$

We see that $E^2\geq\begin{bmatrix} A_k+BC & BD \\ DC & D^2 \end{bmatrix}$, $\bar{A}_k=A_k^2+BC$, $\bar{B}=BD$, $\bar{C}=DC$, $\bar{D}=D^2$ have the same form as E, B, C, D, respectively. Let q be such an integer that $\bar{A}_k^q>0$, then a straightforward calculation shows that

$$E^q\geq\begin{bmatrix} \bar{A}_k^q & \bar{B}\bar{D}^{q-1} \\ \bar{D}^{q-1}\bar{C} & \bar{D}^q \end{bmatrix},$$

and hence

$$E^{2q}\geq\begin{bmatrix} \bar{A}_k^{2q} & \bar{A}^q\bar{B}\bar{D}^{q-1} \\ \bar{D}^{q-1}\bar{C}\bar{A}^q & \bar{D}^q \end{bmatrix}>0.$$

Thus by induction the lemma is true for all m. The proof is complete.

Let m be such that $b_m \neq 0$, $b_i=0$ for $i=m+1,\ldots M$, then for the matrix $C=I+A+\beta B$, A and B defined by (14), we have

$$\det(\lambda - C)=(\lambda-1)^{M-m}\det(\lambda - E)$$

where the matrix E is of the form of lemma 1. Hence from the theory of positive matrix, (Warge [3]), we have immediately

Theorem 2.

(i). The matrix $A+\beta B$ has a real dominant eigenvalue $\lambda_{0\delta} \geq 0$, all the real parts of the other eigenvalues of $A+\beta B$ are strictly less than $\lambda_{0\delta}$;

(ii). when $\lambda_{0\delta}>0$, $\lambda_{0\delta}$ is algebraic simple;

(iii). $\lambda_{0\delta}=1$ if and only if $\beta=\beta_{cr}^{(\delta)}$, the critical fertility of system (9), given by

$$\beta_{cr}^{(\delta)}=\frac{1}{b_1+b_2(1-\delta\eta_1)+\ldots+b_M\prod_{i=1}^{M-1}(1-\delta\eta_i)}$$

$$=\frac{\prod_{i=1}^{M-1}(1+\delta\mu(i\delta))}{b_M+b_{M-1}(1+\delta\mu((M-1)\delta))+\ldots+b_1\prod_{i=1}^{M-1}(1+\delta\mu(i\delta))}; \qquad (16)$$

In this case the unique positive eigenvector corresponding to $\lambda_{0\delta}=1$ is

$$X^{(\delta)}=(x_1^{(\delta)}, x_2^{(\delta)},\ldots,x_m^{(\delta)})^T \qquad (17)$$

where

$x_M^{(\delta)}=1$,

$x_i^{(\delta)}= (1+\delta\mu(i\delta))(1+\delta\mu(i+1)\delta))\ldots(1+\delta\mu((M-1)\delta))$, $i=1,2,\ldots M-1$,

and the solution of (13) satisfies

$$\lim_{j\to\infty} X(j)=C_{0\delta}X^{(\delta)}, \tag{18}$$

where $C_{0\delta}=a/b$

$$\begin{aligned}
a= &\ x_{01} \\
&+(1-\beta_{cr}b_1)(1+\delta\mu(\delta))x_{02} \\
&+[(1-\beta_{cr}b_1)(1+\delta\mu(\delta))(1+\delta\mu(2\delta))-b_2(1-\delta\mu(2\delta))]x_{03}+\cdots \\
&+[(1-\beta_{cr}b_1)(1+\delta\mu(\delta))\cdots(1+\delta\mu((M-1)\delta)) \\
&\qquad -\beta_{cr}b_2(1+\delta\mu(2\delta))\cdots(1+\delta\mu((M-1)\delta)) \\
&\qquad -\cdots-\beta_{cr}b_{M-1}(1+\delta\mu((M-1)\delta))]x_{0M},
\end{aligned} \tag{19}$$

$$b=(1+\delta\mu(\delta))(1+\delta\mu(2\delta))\cdots(1+\delta\mu((M-1)\delta))F'_\delta(1)$$

(iv). when $\beta>\beta^{(\delta)}_{cr}$, $\lambda_{0\delta}>1$;

(v). when $\beta<\beta^{(\delta)}_{cr}$, $\lambda_{0\delta}<1$.

Theorem 2 tells us that the critical fertility $\beta^{(\delta)}_{cr}$ plays a similar role as β_{cr} defined by (6) for the continuous system. It is an important index concerning the stability of the discrete system (9). The relation between the two indices is described by

Proposition 1. Let β_{cr} and $\beta^{(\delta)}_{cr}$ be defined by (9) and (16), and let $C_{0\delta}$ and $X^{(\delta)}$ be defined by (17) and (19), respectively, then

(i). $$\beta_{cr}=\beta^{(\delta)}_{cr}+o(\delta), \tag{20}$$

(ii). $$\int_0^{r_m} |p^*_\delta(r)-C_0e^{-\int_0^r\mu(\rho)d\rho}|dr\to 0, \text{ as } \delta\to 0, \tag{21}$$

where $p^*_\delta(r)=C_{0\delta}x^{(\delta)}_i$, for $r\in I_i$, $i=1,2,\ldots M$.

Proof. For $\delta<r_1$,

$$\beta_{cr}^{-1} = \int_{r_1}^{r_2} k(r)h(r)e^{-\int_0^r \mu(\rho)d\rho}dr = \int_0^{r_m} k(r)h(r)e^{-\int_0^r \mu(\rho)d\rho}dr$$

$$= \sum_{i=1}^{M} \delta(kh)_i e^{-\int_0^{i\delta} \mu(\rho)d\rho}dr + o(\delta)$$

$$= \sum_{i=1}^{M} \delta(kh)_i e^{-\delta[\mu(\delta)+\mu(2\delta)+\ldots+\mu(i\delta)]} + o(\delta).$$

$$= \sum_{i=1}^{M} \left[\frac{\delta(kh)_i}{(1+\delta\mu(\delta))(1+\delta\mu(2\delta))\ldots(1+\delta\mu(i\delta))}\cdot \right.$$
$$\left. \cdot e^{[o(\mu(\delta))+o(\mu(2\delta))+\ldots+o(\mu(i\delta))]}\right] + o(\delta)$$

$$= \sum_{i=1}^{M} \frac{\delta(kh)_i}{(1+\delta\mu(\delta))(1+\delta\mu(2\delta))\ldots(1+\delta\mu(i\delta))}$$
$$+ \sum_{i=1}^{M} \frac{\delta(kh)_i}{(1+\delta\mu(\delta))(1+\delta\mu(2\delta))\ldots(1+\delta\mu(i\delta))}\cdot$$
$$\cdot\left[e^{[o(\mu(\delta))+o(\mu(2\delta))+\ldots+o(\mu(i\delta))]}-1\right] + o(\delta)$$

$$= \left[\beta_{cr}^{(\delta)}\right]^{-1} + o(\delta).$$

Notice that we have used in the above the facts that

$$\log[(1+\delta\mu(\delta))(1+\delta\mu(2\delta))\ldots(1+\delta\mu(i\delta))]$$
$$=\delta[\mu(\delta)+\mu(2\delta)+\ldots+\mu(i\delta)]+[o(\mu(\delta))+o(\mu(2\delta))+\ldots+o(\mu(i\delta))]$$

and

$$e^{-\delta[\mu(\delta)+\mu(2\delta)+\ldots+\mu(i\delta)]} = (1+\delta\mu(\delta))(1+\delta\mu(2\delta))\ldots(1+\delta\mu(i\delta))\cdot$$
$$\cdot e^{[o(\mu(\delta))+o(\mu(2\delta))+\ldots+o(\mu(i\delta))]}$$

and $(kh)_i=0$ when $i\delta\notin[r_1,r_2]$. This can also be shown to be true for any $r_c<r_m$,

$$e^{-\int_0^r \mu(\rho)d\rho}dr = (1-\delta\eta_0)(1-\delta\eta_1)\cdots(1-\delta\eta_{i-1})+o(\delta)$$
$$\text{for all } r\in I_i\subseteq[0,r_c]. \qquad (22)$$

This completes the proof of part (i). The second part is similar but involves some tedious computations which we omit.

Lemma 2. Let $N(t)=\int_0^{r_m} p_\delta(r,t)dr$, then for all $t\in[0,r_m]$

$$N(t)\le \text{Const.}\cdot N(0). \tag{23}$$

Proof. Let $t\in I_{j+1}$, then

$$\begin{aligned}
N(t)=N(j+1)&=\delta\sum_{i=1}^{M} p_\delta(i,j+1)\\
&\le\delta\sum_{i=1}^{M} p_\delta(i-1,j)=\delta p_\delta(0,j)+\delta\sum_{i=1}^{M} p_\delta(i,j)\\
&=\delta p_\delta(0,j)+N(j)\le[\delta\max_i(kh)_i+1]N(j)\\
&\le\cdots\le[1+\delta\,\overline{\overline{kh}}]^j N(0),
\end{aligned}$$

and hence $N(j)\le N(0)e^{\overline{\overline{kh}}r_m}=\text{Const.}\cdot N(0)$, here $\overline{\overline{kh}}=\max_{r\in[r_1,r_2]}k(r)h(r)$.

5.3 The Main Result

Our main result is the following

Theorem 3. Let $p_\delta(r,t)$ be defined by (12) and let $p(r,t)$ be the solution of Eq.(3) then we have

$$\int_0^{r_m}|p_\delta(r,t)-p(r,t)|dr\longrightarrow 0,\ \text{as}\ \delta\longrightarrow 0,$$

which holds in any finite interval $t\in[0,T]$, $0<T<\infty$, as δ goes to zero.

Proof. It is sufficient to prove it for $t\in[0,r_m]$. Since if (22) holds in $[0,r_m]$ then

$$\int_0^{r_m}|p_\delta(r,r_m)-p(r,r_m)|dr\longrightarrow 0,\ \ \text{as}\ \delta\longrightarrow 0.$$

We can consider $p_\delta(r,r_m)$ and $p(r,r_m)$ as the initial conditions with the starting point $t=r_m$ of (13) and (3), respectively, and deduce that (12) holds in $[r_m, r_{2m}]$. Hence the general situation follows. Furthermore, by lemma 2, we only need to prove that for any fixed $r_c < r_m$

$$\int_0^{r_c} |p_\delta(r,r_m) - p(r,r_m)| dr \to 0, \text{ as } \delta \to 0.$$

The reason is that we want to use the estimate (22), but for convenience, we consider (22) still holds in $[0, r_m]$.

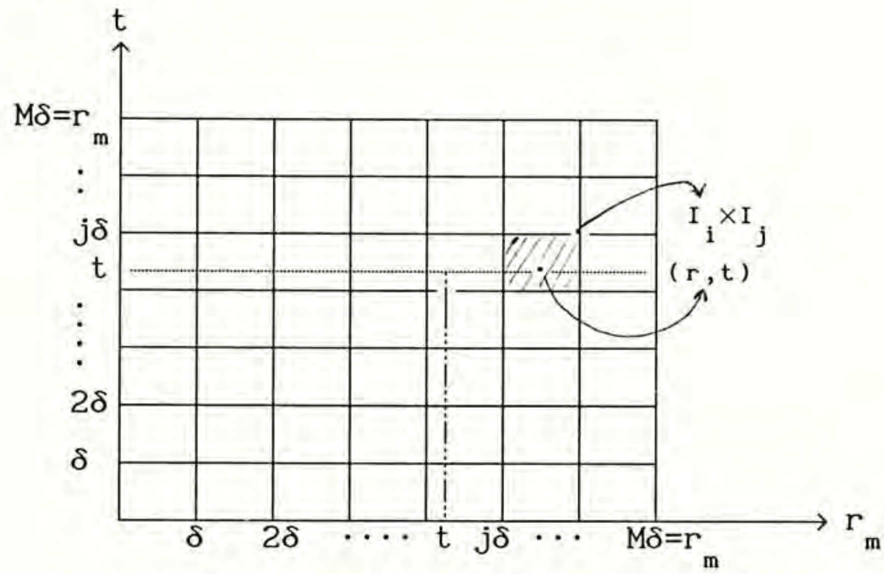


Fig1. The Finite difference scheme for the population system in $[0,r_m]\times[0,r_m]$

For $r \geq t$, $t \in I_j$, $j=1,2,\ldots M$

$$\int_t^{r_m} |p_\delta(r,t) - p(r,t)| dr$$

$$= \int_t^{r_m} \left| p_\delta(r,t) - p_0(r-t) e^{-\int_{r-t}^{r} \mu(\rho) d\rho} \right| dr$$

$$\leq \int_t^{r_m} \left| p_\delta(r,t) - p_{0\delta}(r-t) e^{-\int_{r-t}^{r} \mu(\rho) d\rho} \right| dr + \| p_0 - p_{0\delta} \|$$

$$= \int_t^{j\delta} \left| p_\delta(r,t) - p_{0\delta}(r-t) e^{-\int_{r-t}^{r} \mu(\rho) d\rho} \right| dr$$

$$+ \sum_{i=q}^{M} \int_{I_i} \left| p_\delta(r,t) - p_{0\delta}(r-t) e^{-\int_{r-t}^{r} \mu(\rho) d\rho} \right| dr + \| p_0 - p_{0\delta} \|$$

$$=\int_{t}^{j\delta}\left|p_\delta(r,t)-p_{0\delta}(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}\right|dr$$

$$+\sum_{i=j}^{M}\int_{I_i((r-t)\in I_{i-j})}\left|p_\delta(i,j)-p_\delta(i-j,0)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}\right|dr$$

$$+\sum_{i=j}^{M}\int_{I_i((r-t)\in I_{i-j-1})}\left|p_\delta(i,j)-p_\delta(i-j-1,0)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}\right|dr$$

$$+\|p_0-p_{0\delta}\| = L_1 + L_2 + L_3 + \|p_0-p_{0\delta}\|,$$

$$L_1=\int_{t}^{j\delta}\left|p_\delta(r,t)-p_{0\delta}(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}\right|dr$$

$$=\int_{t}^{j\delta}\left|p_\delta(j,j)-p_{0\delta}(1)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}\right|dr$$

$$=\int_{t}^{j\delta}\left|p_{0\delta}(1)\left[e^{-\int_{r-t}^{r}\mu(\rho)d\rho}-(1-\delta\eta_1)(1-\delta\eta_2)\cdots(1-\delta\eta_{j-1})\right]\right|dr$$

$$\le\int_{I_j}\left|x_{01}\left[e^{-\int_{r-t}^{r}\mu(\rho)d\rho}-(1-\delta\eta_1)(1-\delta\eta_2)\cdots(1-\delta\eta_{j-1})\right]\right|dr$$

$$= o(\delta)\int_{I_j} x_{01}dr \le o(\delta)\|p_{0\delta}(r)\|;$$

$$L_2=\sum_{i=j}^{M}\int_{I_i((r-t)\in I_{i-j})}\left|p_\delta(i,j)-p_\delta(i-j,0)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}\right|dr$$

$$=\sum_{i=q}^{M}\int_{(t+i-j)\delta}^{i\delta}\left|p_\delta(i,j)-p_\delta(i-j,0)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}\right|dr$$

$$=\sum_{i=q}^{M}\int_{(t+i-j)\delta}^{i\delta}\left|p_\delta(i-j)\cdot\right.$$

$$\left.\left[e^{-\int_{r-t}^{r}\mu(\rho)d\rho}-(1-\delta\eta_i)(1-\delta\eta_{i-1})\cdots(1-\delta\eta_{i-j})\right]\right|dr$$

$$=o(\delta)\sum_{i=q}^{M}\int_{(t+i-j)\delta}^{i\delta}\left|p_\delta(i-j)\right|dr\le o(\delta)\|p_\delta(r)\|;$$

$$L_3 = \sum_{i=j}^{M} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| p_\delta(i,j) - p_\delta(i-j-1,0) e^{-\int_{r-t}^{r}\mu(\rho)d\rho} \right| dr$$

$$\le \sum_{i=j}^{M} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| p_\delta(i,j) - p_\delta(i-j,0) e^{-\int_{r-t}^{r}\mu(\rho)d\rho} \right| dr$$

$$+ \sum_{i=j}^{M} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| p_\delta(i-j,0) - p_\delta(i-j-1,0) \right| dr$$

$$\le o(\delta)\| p_{0\delta}(r)\| + \sum_{i=j}^{M} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| p_\delta(i-j) - p_\delta(i-j-1) \right| dr$$

$$\le o(\delta)[\| p_{0\delta}(r)\| + 1] + 2\| p_0(r) - p_{0\delta}(r)\|;$$

and hence $\int_t^{r_m} |p_\delta(r,t) - p(r,t)| dr \le 3\|p_0 - p_{0\delta}\| + o(\delta)\|p_0\| \to 0$ for all $t\in[0,r_m]$ uniformly when δ tend to zero.

For $r<t$, $t\in I_j$, $j=1,2,\dots M$

$$\int_0^t \left| p_\delta(r,t) - \beta\int_{r_1}^{r_2} k(s)h(s)p_\delta(s,t-r)ds e^{-\int_0^r \mu(\rho)d\rho} \right| dr$$

$$= \int_{(j-1)\delta}^{t} \left| p_\delta(r,t) - \beta\int_{r_1}^{r_2} k(s)h(s)p_\delta(s,t-r)ds e^{-\int_0^r \mu(\rho)d\rho} \right| dr$$

$$+ \int_0^{(j-1)\delta} \left| p_\delta(r,t) - \beta\int_{r_1}^{r_2} k(s)h(s)p_\delta(s,t-r)ds e^{-\int_0^r \mu(\rho)d\rho} \right| dr$$

$$= \int_{(j-1)\delta}^{t} \left| p_\delta(j,j) - \beta\int_{r_1}^{r_2} k(s)h(s)p_\delta(s,t-r)ds e^{-\int_0^r \mu(\rho)d\rho} \right| dr$$

$$+ \sum_{i=1}^{j-1} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| p_\delta(i,j) - \sum_{m=1}^{M} \beta\int_{I_m} k(s)h(s)p_\delta(m,j-i+1)ds e^{-\int_0^r \mu(\rho)d\rho} \right| dr$$

$$+ \sum_{i=1}^{j-1} \int_{(t+i-j)\delta}^{i\delta} \left| p_\delta(i,j) - \sum_{m=1}^{M} \beta\int_{I_m} k(s)h(s)p_\delta(m,j-i)ds e^{-\int_0^r \mu(\rho)d\rho} \right| dr$$

$$= M_1 + M_2 + M_3,$$

$$M_1 = \int_{(j-1)\delta}^{t} \left| p_\delta(j,j) - \beta\int_{r_1}^{r_2} k(s)h(s)p_\delta(s,t-r)ds e^{-\int_0^r \mu(\rho)d\rho} \right| dr$$

$$\le \int_{(j-1)\delta}^{t} \left| p_\delta(j,j) - \sum_{i=1}^{M} \beta\delta(kh)_i p_\delta(i,1) e^{-\int_0^r \mu(\rho)d\rho} \right| dr$$

$$+ \int_{(j-1)\delta}^{t} \left| \sum_{i=1}^{M} \beta \int_{I_i} [k(s)h(s)-(kh)_i] ds p_\delta(i,1) \right| dr$$

$$\le \int_{(j-1)\delta}^{t} \left| (1-\delta\eta_0)\cdots(1-\delta\eta_{j-1}) - e^{-\int_0^r \mu(\rho)d\rho} \right| p_\delta(0,0) dr$$

$$+ \int_{(j-1)\delta}^{t} \sum_{i=1}^{M} \beta\delta(kh)_i \left[x_{0i} + x_{0i-1} \right] dr$$

$$+\delta\cdot o(\delta) \sum_{i=1}^{M} \int_{I_i} p_\delta(i,1) ds$$

$$\le \delta\cdot o(\delta)\cdot\|p_{0\delta}\| + \delta\cdot \text{Const.}\cdot\|p_{0\delta}\| + \delta\cdot o(\delta)\cdot\|p_{0\delta}\|$$

$$= o(\delta)\cdot\|p_{0\delta}\|$$

$$M_2 = \sum_{i=1}^{j-1} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| p_\delta(i,j) - \sum_{m=1}^{M} \beta \int_{I_m} k(s)h(s) p_\delta(m,j-i+1) ds e^{-\int_0^r \mu(\rho)d\rho} \right| dr$$

$$= \sum_{i=1}^{j-1} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| \left[(1-\delta\eta_0)\cdots(1-\delta\eta_{i-1}) - e^{-\int_0^r \mu(\rho)d\rho} \right] \cdot \right.$$

$$\left. \cdot \left[p_\delta(0,j-i) - \sum_{m=1}^{M} \beta \int_{I_m} k(s)h(s) p_\delta(m,j-i+1) ds \right] \right| dr$$

$$= o(\delta)\cdot \sum_{i=1}^{j-1} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| p_\delta(0,j-i) - \sum_{m=1}^{M} \beta \int_{I_m} k(s)h(s) p_\delta(m,j-i+1) ds \right| dr$$

$$= o(\delta)\cdot \left[\sum_{i=1}^{j-1} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| \sum_{m=1}^{M} \beta \int_{I_m} [k(s)h(s)-(kh)_m] p_\delta(m,j-i-1) ds \right| dr \right.$$

$$\left. + \sum_{i=1}^{j-1} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| \sum_{m=1}^{M} \beta \int_{I_m} (kh)_m [p_\delta(m,j-i-1) - p_\delta(m,j-i)] ds \right| dr \right]$$

$$\le o(\delta)\cdot \left[o(\delta)\cdot \sum_{i=1}^{j-1} \int_{(i-1)\delta}^{(t+i-j)\delta} \sum_{m=1}^{M} \int_{I_m} p_\delta(m,j-i-1) ds dr \right.$$

$$+ \sum_{i=1}^{j-1} \int_{(i-1)\delta}^{(t+i-j)\delta} \left| \sum_{m=1}^{M} \beta \int_{I_m} (kh)_m [p_\delta(m,j-i-1) - p_\delta(m-1,j-i-1)] ds \right| dr \quad ,$$

$$+\sum_{i=1}^{j-1}\int_{(i-1)\delta}^{(t+i-j)\delta}\sum_{m=1}^{M}\beta\int_{I_m}(kh)_m\delta\eta_{m-1}p_\delta(m-1,j-i-1)dsdr\Big]$$

$$\le o(\delta)\cdot\Big[o(\delta)\cdot\sum_{i=1}^{j-1}\int_{(i-1)\delta}^{(t+i-j)\delta}\sum_{m=1}^{M}\int_{I_m}p_\delta(m,j-i-1)dsdr$$

$$+\sum_{i=1}^{j-1}\int_{(i-1)\delta}^{(t+i-j)\delta}\sum_{m=1}^{M}\beta\int_{I_m}|(kh)_m-\delta(kh)_{m-1}|p_\delta(m-1,j-i-1)dsdr$$

$$+\sum_{i=1}^{j-1}\int_{(i-1)\delta}^{(t+i-j)\delta}\sum_{m=1}^{M}\beta\int_{I_m}(kh)_m\delta\eta_{m-1}p_\delta(m-1,j-i-1)dsdr\Big]$$

$$\le o(\delta)\cdot o(\delta)\cdot\sum_{i=1}^{j-1}\int_{(i-1)\delta}^{(t+i-j)\delta}\sum_{m=1}^{M}\int_{I_m}p_\delta(m,j-i-1)dsdr$$

$$\le o(\delta)\cdot o(\delta)\sum_{m=1}^{M}\delta\cdot\sum_{i=1}^{j-1}\delta p_\delta(m,j-i-1)$$

$$\le \text{Const.}\cdot o(\delta)\cdot o(\delta)=o(\delta).$$

The estimate about M_3 is similar to that of M_2. Here it is omitted. So we have proved that for $t\le r$

$$\int_t^{r_m}|p_\delta(r,t)-p(r,t)|dr\longrightarrow 0,$$

$$\int_0^t\Big|p_\delta(r,t)-\beta\int_{r_1}^{r_2}k(s)h(s)p_\delta(s,t-r)dse^{-\int_0^r\mu(\rho)d\rho}\Big|dr\longrightarrow 0,$$

$$\text{as } \delta\longrightarrow 0,$$

uniformly for $t\in[0,r_m]$. Observe that

$$\int_0^t|p_\delta(r,t)-p(r,t)|dr$$

$$\le\int_0^t\Big|p_\delta(r,t)-\beta\int_{r_1}^{r_2}k(s)h(s)p_\delta(s,t-r)dse^{-\int_0^r\mu(\rho)d\rho}\Big|dr$$

$$+\int_0^t\beta\int_{r_1}^{r_2}k(s)h(s)|p_\delta(s,t-r)-p(r,t-r)|dsdr$$

$$\le o(\delta)+\text{Const.}\|p-p_{0\delta}\|$$

$$+\int_0^t \beta\int_{r_1}^{r_2} k(s)h(s)\int_{r_1}^{r_2} k(\tau)h(\tau)\,|p_\delta(\tau,t-r-s)-p(\tau,t-r-s)|\,d\tau ds dr$$

$$\le\cdots\le o(\delta)+\text{Const.}\|p-p_{0\delta}\|$$

$$+\int_0^t \beta\int_{r_1}^{r_2} k(s)h(s)\int_{r_1}^{r_2} k(\tau)h(\tau)\cdots\int_{r_1}^{r_2} k(\tau)h(\tau)\cdot$$

$$\cdot|p_\delta(\Theta,t-r-s-\tau-\cdots)-p(\Theta,t-r-s-\tau-\cdots)|\,d\tau ds\cdots d\Theta dr$$

$$\le o(\delta)+\text{Const.}\|p-p_{0\delta}\|\longrightarrow 0,\quad (\delta\longrightarrow 0).$$

In the last step above, we stop the process until $t-r-s-\tau-\cdots\le r_1$. The proof is now complete.

Theorem 3 tells us that the discrete $p_\delta(r,t)$ can be considered as the age distribution of the population if the initial discrete $p_{0\delta}(r)$ is, or in fact $p_\delta(i,j)$, can be considered as the total number of individuals aged in $[(i-1)\delta,i\delta)$ and in time period $[(j-1)\delta,j\delta)$ if the initial condition $p_{0\delta}(i)$ is the number aged between $(i-1)\delta$ and $i\delta$ in the time period $[0,\delta)$.

5.4 A Finite Difference Scheme for the Logistic Population Model

In this section we shall design a finite difference scheme for the following logistic population model

$$\begin{cases} \dfrac{\partial p^L(r,t)}{\partial t}+\dfrac{\partial^L p(r,t)}{\partial r}=-\mu(r)p^L(r,t)-Kf(N(t))p^L(r,t), & 0<r<r_m,\ t>0, \\ p^L(r,0)=p_0(r), & 0\le r\le r_m, \\ p^L(0,t)=\beta\int_{r_1}^{r_2} k(r)h(r)p^L(r,t)dr, & t\ge 0. \end{cases} \tag{24}$$

where

$$N(t)=\int_0^{r_m} p^L(r,t)dr,$$

K is the environmental constant and the logistic function $f(\xi)$ satisfies

$$f(\xi) \text{ is continuous differentiable for } \xi\geq 0;$$

$$f(0)=0,\ f(\xi)>0 \text{ if } \xi>0. \tag{25}$$

The difference scheme is defined as follows:

$$\begin{cases} N_\delta^L(j)=\dfrac{N_\delta^0(j)}{\prod\limits_{m=0}^{j-1}(1+\delta Kf(N_\delta(m)))}, \\ \\ p_{L\delta}(i,j)=\dfrac{p_\delta(i,j)}{\prod\limits_{m=0}^{j-1}(1+\delta Kf(N_\delta(m)))}, \quad i,j=0,1,\ldots M-1, \end{cases} \tag{26}$$

where $p_\delta(r,t)$ is defined by (12) and

$$N_\delta^0(j)=\delta\sum_{i=0}^{M-1}p_\delta(i,j). \tag{27}$$

Define similarly

$$\begin{cases} p_{L\delta}(r,t)=p_{L\delta}(i,j), \quad \text{for } (r,t)\in I_i\times I_j, \\ \\ N_\delta^0(t)=N_\delta^0(j),\ N_\delta^L(t)=N_\delta^L(j), \text{ for } t\in I_j. \end{cases} \tag{28}$$

By lemma 2, for any $T>0$, there exists a constant C independent of δ such that

$$N_\delta^L(t)\leq N_\delta^0(t)\leq C, \qquad t\in[0,T]. \tag{29}$$

For $t\in I_j$, imitating the proof of proposition 1 we have

$$N_\delta^L(t)=N_\delta^0(t)e^{-\int_0^t Kf(N_\delta^L(\rho))d\rho}+o(\delta). \tag{30}$$

Furthermore, if $\varepsilon<\delta$, then

$$|N_\delta^L(t)-N_\varepsilon^L(t)|\leq|N_\delta^0(t)-N_\varepsilon^0(t)|+o(\delta)$$
$$+C|\ e^{-\int_0^t K[f(N_\delta^L(\rho))-f(N_\varepsilon^L(\rho))]d\rho}-1|.$$

$$\le |N_\delta^0(t)-N_\varepsilon^0(t)|+o(\delta)+\text{Const.}\int_0^t |N_\delta^L(\rho)-N_\varepsilon^L(\rho)|d\rho.$$

Notice that we have used the differentiability of $f(\xi)$ and the fact that $|e^z-1|\le|z|e^{|z|}$ for all complex z. We then can establish, by an application of the Gronwall's inequality, that

$$|N_\delta^L(t)-N_\varepsilon^L(t)|\le\text{Const.}\,|N_\delta^0(t)-N_\varepsilon^0(t)|+o(\delta). \tag{31}$$

By theorem 3

$$\lim_{\varepsilon,\delta\searrow 0} |N_\delta^0(t)-N_\varepsilon^0(t)|=0, \qquad \lim_{\delta\searrow 0} N_\delta^0(t)=N^0(t),$$

uniformly for $t\in[0.T]$, and hence

$$\lim_{\varepsilon,\delta\searrow 0} |N_\delta^L(t)-N_\varepsilon^L(t)|=0, \qquad \lim_{\delta\searrow 0} N_\delta^L(t)=N^L(t) \tag{32}$$

uniformly for $t\in[0.T]$. Then the Lebesgue dominant theorem shows that

$$\int_0^t f(N_\delta^L(\rho))d\rho \longrightarrow \int_0^t f(N^L(\rho))d\rho, \text{ as } \delta\longrightarrow 0, \quad \forall\ t\in[0.T].$$

Hence $N^L(t)$ satisfies

$$N^L(t)=N^0(t)e^{-\int_0^t Kf(N^L(\rho))d\rho}, \text{ for } t\in[0.T]. \tag{33}$$

This means that $N^L(t)$ is a continuous function since $N^0(t)$ is. We know that (33) has only one nonnegative continuous solution, so $N^L(t)=N(t)$ for all $t\in[0,T]$. From theorem 3 and (26), we therefore have

Theorem 4. Let (26) be the difference scheme for equation (23) then

$$\int_0^{r_m} |p_{L\delta}(r,t)-p^L(r,t)|dr \longrightarrow 0, \text{ as } \delta\longrightarrow 0 \tag{34}$$

uniformly in any bounded time interval, where $p(r,t)$ is the solution of (23).

5.5 Numerical Simulation

Take $\beta=2.2$ and the parametric functions are as follows:

$$p_0(r)=\begin{cases} 1+\sin(\frac{x}{10}\pi)+700000000, & 0\le x\le 20, \\ 2-150(\frac{x-20}{100}-0.2)^2/9+0.3\times\sin(\frac{x-20}{6}\pi)+700000000, & 20<x\le 50, \\ -\frac{x^2}{750}+\frac{2.5}{3}+700000000, & 50<x\le 100. \end{cases}$$

$$h(r)=\begin{cases} 1-\frac{400}{9}(\frac{x-35}{100})^2, & 20\le x\le 50, \\ 0, & \text{elsewhere}. \end{cases}, \quad k(r)=0.5.$$

$$\mu(r)=\begin{cases} (-15000+\frac{1}{9})x+15, & 0\le x\le 10, \\ \frac{100}{100-x}, & 10<x<100. \end{cases}$$

We obtain the corresponding approximate distribution for $\delta=1/30$, $1/60$, respectively and the approximate critical fertility ($N=10,1000$; step=100, $\delta=1/N$). $\beta_{cr} = 6.1973710$

$\beta_{cr}^{(\delta)}$	$\beta_{cr}^{(\delta)}-\beta_{cr}$
7.5000000	1.3026290
6.3313280	1.339569E-001
6.2687690	7.139778E-002
6.2460580	4.868650E-002
6.2342960	3.692532E-002
6.2270930	2.972174E-002
6.2222920	2.492094E-002
6.2187850	2.141380E-002
6.2161520	1.878119E-002
6.2141400	1.676893E-002

(1).

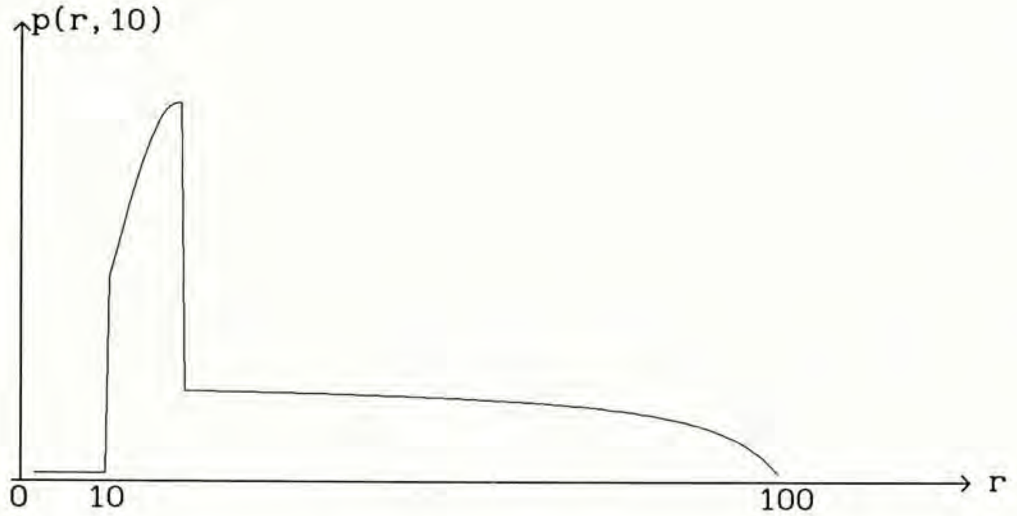
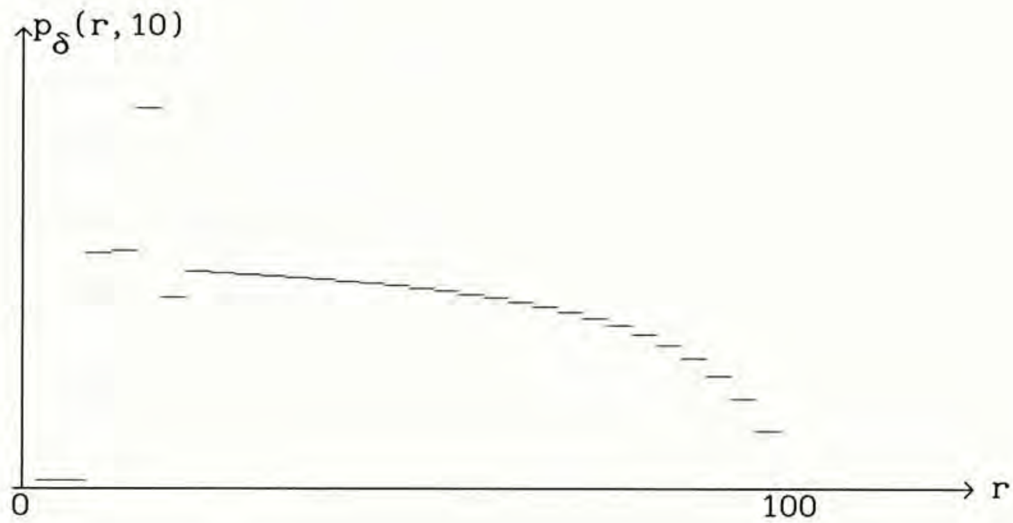
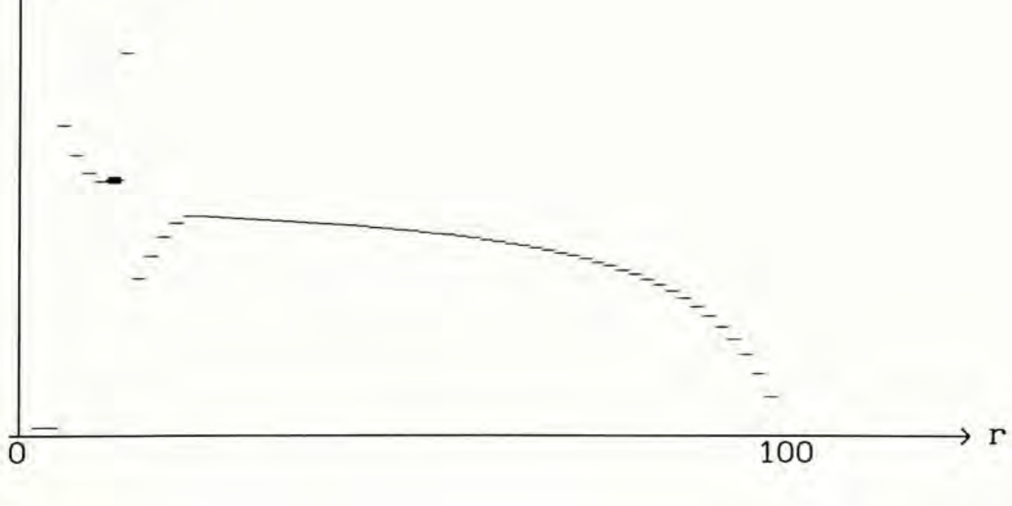


Fig. 1. p(r, 10) for r≥10

(2). δ=1/30



(3). δ=1/60



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Chapter 6

Optimal Birth Control Policies I

6.1 Intruduction

A wide variety of problems dealing with biological populations and resource management have been formulated in an optimal control setting [1,2]. Much work has been done on models described by ordinary differential equations. On the other hand, age-structured population models involving partial differential equations are becoming increasingly emphasized [3-5]. Analysis of such distributed systems in the optimal control theory framework has only recently been reported [3,6]. In this chapter we shall work in the spirit of [3] on optimal birth control policies of the human population using the McKendrick type model. We adopt Dubovitskii and Milyutin's functional analytical approach [7] in the optimization yielding more transparent results. We first study the "standard" problem with a free end condition and fixed final horizon (time). Other aspects which are not treated in [6], such as the time optimal control problem, the problem with target sets and the infinite planning horizon case are investigated. The role of controllability [9,10] is also discussed.

6.2 Fixed Horizon and Free End Point Problem

Consider the control of the following population distributed parameter system [3]

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), & 0<r<r_m, t>0, \\ p(r,0)=p_0(r), \quad 0\le r\le r_m, & \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2}(r)h(r)p(r,t)dr, & t\ge 0 \end{cases} \tag{1}$$

in which $p(r,t)$ is the population density, r denotes age, t represents time, r_m is the maximum age, $\beta(t)$ is the specific fertility rate of females at time t; $k(r)$ and $h(r)$ denote respectively, the female ratio and the fertility pattern and $[r_1,r_2]$ is the fertility interval with $\int_{r_1}^{r_2}h(r)dr=1$. The initial population density $p_0(r)$ and the mortality rate $\mu(r)$ satisfy

$$\int_0^r \mu(\rho)d\rho< +\infty \text{ for } r< r_m, \text{ and } \int_0^{r_m}\mu(\rho)d\rho = +\infty.$$

Generally speaking, the population parameters $\mu(r)$, $k(r)$ and $h(r)$ are time dependent. Here we assume that they are time independent functions in order to simplify arguments. However, under suitable smoothness assumption, the results obtained for the optimal control problems continue to hold.

For the population dynamical system there are two independent controlling variables $\beta(t)$ and $h(r)$ (may be $h(r,t)$ and can be combined into one). The latter reflects the fertility pattern of the female such as late marriage and fertility. $\beta(t)$ reflects an average birth rate. We study here under certain demands of the society, what the optimal birth policy is. This is an optimal control problem in control theory. We determine necessary conditions for the optimal control, extending Pontryagin's maximum principle to population systems with distributed parameters.

Assume that the population parameters in equation (1) are nonnegative and are measurable functions. Furthermore, let β, h, and k be bounded functions whose values outside their domain of definition are zero.

By the method of characteristics, the solution of equation (1) can be written (formally) as

$$p(r,t)=\begin{cases} p_0(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}, & r\geq t,\\ \beta(t-r)\int_{r_1}^{r_2}k(s)h(s)p(s,t-r)ds\,e^{-\int_0^r\mu(\rho)d\rho}, & r<t, \end{cases} \quad (2)$$

The classical solution of (1) is a solution of (2). Under certain smoothness conditions on the population parameters, the two are equivalent. For a detailed discussion, see [3].

For an arbitrary $p_0(r)\in L^2(0,r_m)$, equation (2) in $L^2(0,r_m)$ has a unique solution $p(r,t)\in C(0,\infty;L(0,r_m))$; moreover,

$$\begin{cases} p(r,t)=p_0(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}+\sum\limits_{k=0}^{\infty}\phi_k(t-r)e^{-\int_0^r\mu(\rho)d\rho},\\ \phi_0(t)=\beta(t)\int_{r_1}^{r_2}k(s)h(s)p_0(s-t)e^{-\int_{s-t}^{s}\mu(\rho)d\rho}ds,\\ \phi_k(t)=\beta(t)\int_{r_1}^{r_2}k(s)h(s)\phi_{k-1}(t-s)e^{-\int_0^s\mu(\rho)d\rho}ds,\ k=1,2... \end{cases} \quad (3)$$

and $\phi_k(t)$ does not vanish only in $[kr_1,(k+1)r_2]$.

Because of the above reasons, we call the solution of equation (2) a weak solution of equation (1). Unless otherwise stated, in what follows when we speak of a solution of equation (1) we shall mean the weak solution.

Consider now the optimal control problem: to determine (β^*,p^*), $\beta^*(\cdot)\in U_{ad}$ such that

$$J(\beta^*,p^*) = \min_{\beta(\cdot)\in U_{ad}} J(\beta,p),$$

$$J(\beta,p) = \int_0^T\int_0^{r_m} L(p(r,t),\beta(t),r,t))drdt + \frac{1}{2}\int_0^{r_m} [p(r,T)-\bar{p}(r)]^2dr \tag{4}$$

where $p(r,t)$ is the trajectory of the control $\beta(t)$, $\bar{p}(r)\in L^2(0,r_m)$ is an arbitrary fixed function and L is a functional defined on $L^2(0,r_m)\times[\beta_0,\beta_1]\times[0,r_m]\times[0,T]$ satisfing the following conditions:

(1). $\dfrac{\partial L(p(r),\beta,r,t)}{\partial p}, \dfrac{\partial L(p(r),\beta,r,t)}{\partial \beta}$ exist for every $(p(r),\beta,r,t)\in L^2(0,r_m)\times[\beta_0,\beta_1]\times[0,r_m]\times[0,T]$ and L is continuous about its variables.

(2). $\int_0^{r_m}\left|\dfrac{\partial L(P(r),\beta,r,t)}{\partial p}\right|dr, \int_0^{r_m}\left|\dfrac{\partial L(p(r),\beta,r,t)}{\partial \beta}\right|dr$ are bounded for $t\in[0,T]$ and any bounded subset of $L^2(0,r_m)\times[\beta_0,\beta_1]\times[0,r_m]\times[0,T]$.

$$U_{ad} = \{ \beta(t) \mid 0\le\beta_0\le\beta(t)\le\beta_1,\ t\in[0,T] \text{ a.e}, \ \beta(t) \text{ is measurable on } [0,T]\}. \tag{5}$$

Let (β^*,p^*) be an optimal solution of problem (4) and define the adjoint equation of equation (1) to be

$$\begin{cases}\dfrac{\partial q(r,t)}{\partial r}+\dfrac{\partial q(r,t)}{\partial t}=\mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\dfrac{\partial L(p^*,\beta^*,r,t)}{\partial p},\\ q(r,T)=\bar{p}(r)-p^*(r,T),\\ q(0,t)=q(t).\end{cases} \tag{6}$$

As with equation (1), we call solutions (weak solutions) of equation (6) to be the solutions of the following equation:

$$q(t)=e^{-\int_0^{T-t}\mu(\rho)d\rho}q(T-t,T)+\int_t^T e^{-\int_0^{s-t}\mu(\rho)d\rho}\beta^*(s)k(s-t)h(s-t)q(s)ds$$
$$-\int_t^T e^{-\int_0^{s-t}\mu(\rho)d\rho}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(s-t,s)}ds,$$

$$q(r,t)=e^{-\int_0^{r+T-t}\mu(\rho)d\rho}q(r+T-t,T)$$
$$+\int_t^T e^{-\int_r^{r+s-t}\mu(\rho)d\rho}\beta^*(s)k(r+s-t)h(r+s-t)q(s)ds$$
$$-\int_t^T e^{-\int_r^{r+s-t}\mu(\rho)d\rho}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(r+s-t,s)}ds$$
$$0\le t\le T,\ 0\le r\le r_m, \tag{7}$$

In $L^2(0,r_m)$, equation (6) has a unique solution.

Firstly, we have the following

Lemma 1. The solutions of equation (1) and its adjoint equation (6) satisfy the following relation:

$$\int_0^{r_m}q(r,T)[p(r,T)-p^*(r,T)]dr$$

$$=\int_0^T\int_{r_1}^{r_2} q(t)k(r)h(r)p(r,t)[\beta(t)-\beta^*(t)]drdt$$

$$+\int_0^T\int_0^{r_m} \frac{\partial L(p^*,\beta^*,r,t)}{\partial p}[p(r,t)-p^*(r,t)]drdt \qquad (8)$$

Proof. $\int_0^{r_m} q(r,T)p(r,t)dr \quad (T>r)$

$$=\int_0^{r_m} q(r,T)\beta(T-r)\int_{r_1}^{r_2} k(\tau)h(\tau)p(\tau,T-r)d\tau e^{-\int_0^r \mu(\rho)d\rho}$$

$$=\int_{T-r}^{T} q(T-r,T)\beta(t)\int_{r_1}^{r_2} k(\tau)h(\tau)p(\tau,t)d\tau e^{-\int_0^{T-t} \mu(\rho)d\rho}dt$$

$$=\int_0^T q(T-r,T)\beta(t)\int_{r_1}^{r_2} k(\tau)h(\tau)p(\tau,t)d\tau e^{-\int_0^{T-t}\mu(\rho)d\rho}dt$$

$$=\int_0^T [q(t)-\int_t^T e^{-\int_0^{s-t} \mu(\rho)d\rho}\beta^*(s)k(s-t)h(s-t)q(s)ds$$

$$+\int_t^T e^{-\int_0^{s-t}\mu(\rho)d\rho} \frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\Big|_{(s-t,s)}ds]$$

$$\cdot\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr$$

$$=\int_0^T q(t)\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr$$

$$+\int_0^T\int_t^T e^{-\int_0^{s-t}\mu(\rho)d\rho} \frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\Big|_{(s-t,s)}ds$$

$$\cdot\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr$$

$$-\int_0^T\int_t^T e^{-\int_0^{s-t} \mu(\rho)d\rho}\beta^*(s)k(s-t)h(s-t)q(s)ds$$

$$\cdot\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)drdt$$

$$=\int_0^T q(t)\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr$$

$$\int_0^T \beta^*(s)q(s)ds\int_{r_1}^{r_2} k(r)h(r)dr$$

$$\cdot\int_0^s e^{-\int_0^{s-t}\mu(\rho)d\rho}\beta(t)k(s-t)h(s-t)p(r,t)dt$$

$$+\int_0^T ds\int_{r_1}^{r_2} k(r)h(r)dr$$

$$\cdot\int_0^s e^{-\int_0^{s-t}\mu(\rho)d\rho}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(s-t,s)}\beta(t)k(s-t)h(s-t)p(r,t)dt$$

$$=\int_0^T q(t)\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)drdt$$

$$-\int_0^T \beta^*(s)q(s)ds\int_0^s e^{-\int_0^t\mu(\rho)d\rho}k(t)h(t)dt$$

$$\cdot\beta(s-t)\int_{r_1}^{r_2} k(r)h(r)p(r,s-t)dr$$

$$+\int_0^T ds\int_0^s e^{-\int_0^t\mu(\rho)d\rho}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(t,s)}\beta(s-t)\int_{r_1}^{r_2} k(r)h(r)p(s-t)dt$$

$$=\int_0^T q(t)\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)drdt$$

$$-\int_0^T q(s)\beta^*(s)\int_0^s k(r)h(r)p(r,s)dr$$

$$+\int_0^T ds\int_0^s \left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(t,s)}p(r,s)dr$$

so $\int_0^{r_m} q(r,T)[p(r,T)-p^*(r,T)]dr$

$$=\int_0^T q(t)\beta(t)\int_{r_1}^{r_2}k(r)h(r)p(r,t)drdt -\int_0^T q(s)\beta^*(s)\int_r^s k(r)h(r)p(r,s)dr$$

$$-\int_0^T q(t)\beta^*(t)\int_{r_1}^{r_2}k(r)h(r)p(r,t)drdt$$

$$+\int_0^T ds\int_0^s \frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\Big|_{(r,s)}p(r,s)dr$$

$$+\int_0^T q(s)\beta^*(s)\int_{r_1}^s k(r)h(r)p^*(r,s)dr$$

$$-\int_0^T ds\int_0^s \frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\Big|_{(r,s)}p^*(r,s)dr$$

$$=\int_0^T q(t)\beta(t)\int_{r_1}^{r_2}k(r)h(r)p(r,t)drdt-\int_0^T q(t)\beta^*(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)drdt$$

$$-\int_0^T q(t)\beta^*(t)\int_{r_1}^{r_2}k(r)h(r)[p(r,t)-p^*(r,t)drdt$$

$$+\int_0^T\int_0^{r_m} \frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\Big|_{(r,t)}[p(r,t)-p^*(r,t)]dr$$

$$=\int_0^T\int_{r_1}^{r_2}q(t)k(r)h(r)p(r,t)[\beta(t)-\beta^*(t)]drdt$$

$$+\int_0^T\int_0^{r_m} \frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\Big|_{(r,t)}[p(r,t)-p^*(r,t)]dr$$

This is Lemma 1.

It can be easily deduced from Lemma 1 that

$J(\beta,p)-J(\beta^*,P^*)$

$$=\int_0^T\left[\int_0^{r_m} \frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial\beta}\Big|_{(r,t)}dr\right][\beta(t)-\beta^*(t)]dt$$

$$-\int_0^T q(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr[\beta(t)-\beta^*(t)]dt$$

$$-\int_0^T q(t)\int_{r_1}^{r_2}k(r)h(r)[p(r,t)-p^*(r,t)]dr[\beta(t)-\beta^*(t)]dt$$

$$+\frac{1}{2}\int_0^{r_m} [p(r,T)-p^*(r,T)]^2dr$$

$$+\int_0^T\int_0^{r_m}[o(p(r,t)-p^*(r,t))+ o(\beta(t)-\beta^*(t))]drdt. \qquad (9)$$

From equation (3), we can show that for $T>0$ there exists $M_1>0$ such that

$$\int_0^{r_m} [p(r,t)-p^*(r,t)]^2dr \le M_1\int_0^T [\beta(t)-\beta^*(t)]^2dt$$

$$\forall\ (\beta,p)\in U_{ad},\quad t\in [0,T]. \qquad (10)$$

In (9), substitute $\theta\beta(t)+(1-\theta)\beta^*(t)$, $\theta \in(0,1)$, for $\beta(t)$; paying attention to (10), we obtain immediately that (note that the integrand is bounded and measurable)

$$\left[q(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr-\int_0^{r_m} \frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\Big|_{(r,t)}dr\right]$$

$$\cdot[\beta-\beta^*(t)]\le 0,\qquad \forall\ \beta\in [\beta_0,\beta_1],t\in [0,T]\ \text{a.e} \qquad (11)$$

Theorem 1. The solution of the problem (4) satisfies the maximum principle:

$$\beta^*(t)H(t)= \max_{\beta_0\le\beta\le\beta_1} \beta H(t),\qquad \forall\ t\in [0,T]\ \text{a.e.}$$

$$H(t)=q(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr-\int_0^{r_m} \frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial\beta}\Big|_{(r,t)}dr, \qquad (12)$$

from which we have

$$\beta(t)=\begin{cases} \beta_0, & H(t)<0, \\ \beta_1, & H(t)>0, \\ \text{indeterminate}, & H(t)=0, \end{cases}$$

$H(t)$ is the switching function.

6.3 Time Optimal Control problem

We consider the time optimal control problem for system (1); that is, determine $T^*>0$ and $(\beta^*, p^*)\in \bar{U}_{ad}$ such that

$$\begin{aligned} &T^*=\min\{T\mid p(r,T)\cap V\neq\emptyset,\ \forall\ (\beta,p)\in \bar{U}_{ad}\}, \\ &p^*(r,T^*)\cap V\neq\emptyset, \\ &V=\{\phi(r)\mid \|\phi(r)-\bar{p}(r)\|\leq M,\ \phi,\bar{p}\in L^2(0,r_m)\}, \\ &U_{ad}=\{(\beta,p)\mid 0\leq \beta_0\leq \beta(t)\leq\beta_1,\ (\beta,p) \text{ satisfies } (1)\}. \end{aligned} \qquad (13)$$

If (β^*,p^*,T^*) is the solution of the time optimal control problem (13), then in [9] it is shown that

$$\int_0^{r_m} [p^*(r,T^*)-\bar{p}(r)][p^*(r,T^*)-p(r,T^*)]dr\geq 0$$

$$\forall\ (\beta,p)\in U_{ad}. \qquad (14)$$

Define the adjoint equation

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r}+\dfrac{\partial q(r,t)}{\partial t}=\mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t), \\ q(r,T)=p_\varepsilon(r,T)-\bar{p}(r), \\ q(0,t)=q(t), \end{cases} \qquad (15)$$

its solution is understood to be as in (7). Combining (14) and Lemma 1, we obtain

Theorem 2. (Maximum principle for Time optimal control) The time optimal control satisfies the following maximum principle

$$\beta^*(t)H(t) = \max_{\beta_0 \le \beta \le \beta_1} \beta H(t), \qquad \forall\ t \in [0,T] \text{ a.e.}$$

$$H(t) = q(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr. \tag{16}$$

6.4 Infinite Horizon Problem

We consider further the optimal control problem on an infinite time interval

$$\min_{\beta(\cdot)\in U_{ad}} J(\beta,p) = \min_{\beta(\cdot)\in U_{ad}} \int_0^\infty\int_0^{r_m} L(p(r,t),\beta(t),r,t)drdt, \tag{17}$$

with other conditions similar to (4). We will assume that L is continuously differentiable with respect to its arguments. Moreover, for each admissible (β,p), the integral in (17) is convergent.

Lemma 2. Let (β^*,p^*) be the solution of the optimal control problem (17). Then for each arbitrary $T>0$, (β^*,p^*) is a solution of the following optimal control problem:

$$\begin{cases} J_T(\beta,p) = \min_{\beta(\cdot)\in U_{ad}} \int_0^T\int_0^{r_m} L(p(r,t),\beta(t),r,t)drdt, \\ \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), \qquad 0<r<r_m,\ t>0, \\ p(r,0)=p_0(r), \qquad 0\le r\le r_m, \\ p(r,T)=p^*(r,T), \\ p(0,t)=\beta(t)\int_{r_1}^{r_2}(r)h(r)p(r,t)dr, \qquad t\ge 0, \end{cases}$$

$$U_{ad}=\{\beta(t) \mid 0\le \beta_0\le \beta(t)\le\beta_1,\ t\in[0,T] \text{ a.e.}\}. \tag{18}$$

Proof. If not, let $(\hat{\beta},\hat{p})$ satisfy equation (18), and

$$\int_0^T\int_0^{r_m} L(\hat{p}(r,t),\hat{\beta}(t),r,t)drdt < \int_0^T\int_0^{r_m} L(p^*(r,t),\beta^*(t),r,t)drdt,$$

then define

$$\beta^*(t)=\begin{cases}\hat{\beta}(t), & 0\le t\le T,\\ \beta^*(t), & t>T,\end{cases}$$

$$\hat{P}^*(r,t)=\begin{cases}\hat{p}(r,t), & 0\le t\le T,\\ p^*(r,t), & t>T.\end{cases}$$

$(\hat{\beta}^*(t),\hat{p}^*(r,t))$ is admissible and

$$J(\hat{\beta}^*(\cdot),\hat{p}^*(\cdot,\cdot))< J(\beta^*(\cdot),p^*(\cdot,\cdot)).$$

This is a contradiction. So, Lemma 2 holds.

Let $\mathbf{X}=\mathbf{C}(0,T;L^2(0,r_m))\times L^\infty(0,T)$. We consider the necessary conditions that must be satisfied for the optimal control problem (18). From the definition of solution (2), each admissible control $(p,\beta)\in \mathbf{X}$. Define

$$\Omega_1=\{(p(r,t),\beta(t))\in X \mid \beta_0\le\beta(t)\le\beta_1,\ t\in [0,T] \text{ a.e.}\},$$

$$\Omega_2=\{(p(r,t),\beta(t))\in X \mid p_t+p_r=-\mu p,\ p(0,t)=\beta(t)\int_{r_1}^{r_2}k(r)h(r)p(r,t)dr,$$

$$p(r,0)=p_0(r),\ p(r,T)=p^*(r,T)\}. \tag{19}$$

Then problem (18) is equivalent to finding $(p^*,\beta^*)\in \Omega_1\cap\Omega_2$ such that

$$J_T(\beta^*,p^*)= \min_{(p,\beta)\in \Omega_1\cap \Omega_2} J_T(\beta,p). \tag{20}$$

This is a minimum problem formed by the inequality constraint Ω_1

and the equality constraint Ω_2. We can use the general theory of Dubovitskii and Milyutin for extremum problems.

Theorem 3. [7]. Let the functional $J_T(\beta,p)$ assume a local minimum at the point (p^*,β^*) in $\Omega_1\cap\Omega_2$. Assume that $J_T(\beta,p)$ is regularly decreasing at (p^*,β^*) with directions of decrease cone K_0; assume that the inequality constraint is regular at (p^*,β^*) with feasible directions cone K_1, and that the equality constraint is also regular at (p^*,β^*) with tangent directions cone K_2. Then there exist continuous linear functionals f_0, f_1, f_2, not all identically zero, such that $f_i\in K_i$, i=0, 1, 2, and satisfy the condition

$$f_0+f_1+f_2=0. \tag{21}$$

We will now determine systematically the corresponding cones in problem (20). Under the assumptions for $J(\beta,p)$, the functional $J_T(\beta,p)$ is differentiable at any point (β_0,p_0) and

$$J_T'(\beta_0,p_0)(p,\beta)=\int_0^T\int_0^{r_m}\left[\frac{\partial L(p_0(r,t),\beta_0(t),r,t)}{\partial p}\,p(r,t)\right.$$
$$\left.+\int_0^T\int_0^{r_m}\frac{\partial L(p_0(r,t),\beta_0(t),r,t)}{\partial \beta}\,\beta(t)\right]drdt.$$

Since $J_T(p,\beta)$ is regularly decreasing at (p^*,β^*), its directions of decrease cone is

$$K_0=\{(p,\beta)\in X \mid J_T'(\beta^*,p^*)(p,\beta)<0\}. \tag{22}$$

If $K_0\neq\varnothing$, then for arbitrary $f_0\in K_0^*$, there exists $\lambda_0\geq 0$ such that

$$f_0(p,\beta)=-\lambda_0\int_0^T\int_0^{r_m}\left[\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial p}\,p(r,t)\right.$$

$$\left.+\int_0^T\int_0^{r_m}\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial \beta}\,\beta(t)\right]drdt. \qquad (23)$$

Note that $\Omega_1=C(0,T;L^2(0,r_m))\times\hat{\Omega}_1$, $\hat{\Omega}_1=\{\beta(t)\in L^\infty(0,T)\,|\,\beta_0\le\beta(t)\le\beta_1\}$ is a closed convex subset of $L^\infty(0,T)$. Thus, $\mathring{\Omega}_1=C\times\mathring{\Omega}_1\neq\emptyset$, and for Ω, at the point (p^*,β^*) the feasible directions cone is

$$K_1=\{\lambda(\mathring{\Omega}_1-(p^*,\beta^*))\,|\,\lambda>0\}.$$

For an arbitrary $f_1\in K^*$, if there exists $a(t)\in L(0,T)$ such that [7]

$$f_1(p,\beta)=\int_0^T a(t)\beta(t)dt \qquad (24)$$

then

$$a(t)[\beta-\beta^*(t)]\ge 0,\forall\ \beta\in[\beta_0,\beta_1],t\in[0,T]\text{ a.e.} \qquad (25)$$

In order to determine the tangent directions cone, we define the operator

$$G:\mathbf{X}\longrightarrow C(0,T;L^2(0,r_m)\times L^2(0,r_m)),$$

by

$$G(p,\beta)=\left[p(r,t)-\begin{cases}p_0(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}, & r\ge t,\\ \beta(t-r)\int_{r_1}^{r_2}k(s)h(s)p(s,t-r)ds\,e^{-\int_0^r\mu(\rho)d\rho}, & r<t,\end{cases}\right.$$

$$\left.p(r,T)-p^*(r,T)\right],$$

By $G(p,\beta)$, we can write

$$\Omega_2=\{(p,\beta)\in \mathbf{X}\,|\,G(p,\beta)=0\} \qquad (26)$$

$$G(p_0+\hat{p},\beta_0+\hat{\beta})-G(p_0,\beta_0)$$

$$=\left[\hat{p}(r,t)-\begin{cases} 0, & r\geq t, \\ \beta_0(t-r)\int_{r_1}^{r_2}k(s)h(s)\hat{p}(s,t-r)ds\,e^{-\int_0^r\mu(\rho)d\rho}+ & \\ \quad\hat{\beta}(t-r)\int_{r_1}^{r_2}k(s)h(s)p_0(s,t-r)ds\,e^{-\int_0^r\mu(\rho)d\rho}, & r<t, \end{cases}\quad \hat{p}\ (r,T)\right],$$

$$+\left[-\begin{cases} 0, & r\geq t, \\ \hat{\beta}(t-r)\int_{r_1}^{r_2}k(s)h(s)\hat{p}(s,t-r)ds\,e^{-\int_0^r\mu(\rho)d\rho}, & r<t, \end{cases}\ 0\right],$$

from which $G'(p_0,\beta_0)$ exists and

$$G'(p_0,\beta_0)(\hat{p},\hat{\beta})$$

$$=\left[\hat{p}(r,t)-\begin{cases} 0, & r\geq t, \\ \beta_0(t-r)\int_{r_1}^{r_2}k(s)h(s)\hat{p}(s,t-r)ds\,e^{-\int_0^r\mu(\rho)d\rho}+ & \\ \quad\hat{\beta}(t-r)\int_{r_1}^{r_2}k(s)h(s)p_0\ (s,t-r)ds\,e^{-\int_0^r\mu(\rho)d\rho}, & r<t, \end{cases}\quad \hat{p}\ (r,T)\right],$$

Let (p^*,β^*) be the solution of (20). Then $G(p^*,\beta^*)=0$. Choose arbitrary $(q(r,t),g(r))\in \mathbf{C}(0,T);L^2(0,r_m)\times L^2(0,r_m)$, and solve the equation

$$G'(p^*,\beta^*)(\hat{p},\hat{\beta})=(q,g).$$

Then

$$\begin{cases} \hat{p}(r,t)=q(r,t), \quad r\ge t, \\ \hat{p}(r,t)-\beta^*(t-r)\int_{r_1}^{r_2} k(s)h(s)\hat{p}(s,t-r)ds\, e^{-\int_0^r \mu(\rho)d\rho} \\ \qquad =q(r,t), \quad r<t, \\ \hat{p}(r,t)=g(r). \end{cases} \tag{27}$$

Assume that the linearized system

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), \quad 0<r<r_m, t>0, \\ p(r,0)=0, \quad 0\le r\le r_m, \\ p(0,t)=\beta^*(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr+\beta(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr, \end{cases} \tag{28}$$

is controllable. Then choose $\beta(t)=\hat{\beta}(t)\in L^\infty$ such that $p(r,t)=g(r)-f(r,T)$, and let $\hat{p}(r,t)=p(r,t)+f(r,t)$, here $f(r,t)=q(r,t)$ for $r\ge t$, and $\beta^*(t-r)\int_{r_1}^{r_2} k(r)h(r)f(r,t-r)dr$ for $r< t$, $(\hat{p},\hat{\beta})$ satisfies equation (27). Now the tangent directions cone K_2 is formed by the kernel of $G'(p,\beta)$. In other words (p,β), satisfying the following equation, belongs to X

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), \quad 0<r<r_m, \ t>0, \\ p(r,0)=0, \quad 0\le r\le r_m, \\ p(0,t)=\beta^*(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr+\beta(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr, \end{cases} \tag{29}$$

$$p(r,T)=0. \tag{30}$$

Define

$$K_{11}=\{(p(r,t),\beta(t))\in X \ |(p,\beta) \text{ satisfies equation (29)}\}$$

$$K_{12}=\{(p(r,t),\beta(t))\in X \ |(p,\beta) \text{ satisfies equation (30)}\}.$$

Then the tangent directions cone

$$K_2=K_{11}+K_{12} \tag{31}$$

and K_{11}, K_{12} are linear subspaces of **X**. Because

$$K_2^*=K_{11}^*+K_{12}^*,$$

for arbitrary $f_2\in K_2^*, f_2=f_{11}+f_{12}$, $f_{1i}\in K_{12}^*$, $i=1,2$, $f_{12}=(f'_{12},0)$, $f'_{12}(p(r,t))=0$ and for all $p(r,t)\in C(0,T;L^2(0,r_m))$ satisfying $p(r,t)=0$, there exists $\alpha(r)\in L^2(0,r_m)$ such that

$$f_{12}(p,\beta)=\int_0^{r_m}p(r,T)\alpha(r)dr. \tag{32}$$

From Theorem 3, there exists in **X** not identically zero linear functional f_0, f_1, f_{11}, f_{12} such that

$$f_0+f_1+f_{11}+f_{12}=0.$$

In particular, for $\beta(t)\in L^\infty(0,T)$, select p such that (p,β) satisfies (29). Then $(p,\beta)\in K_{11}$ and $f_{11}(p,\beta)=0$, from which

$$\begin{aligned} f_1(p,\beta)&=-f_0(p,\beta)-f_{12}(p,\beta)\\ &=\lambda_0\int_0^T\int_0^{r_m}\left[\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial p}\,p(r,t)\right.\\ &\quad\left.+\int_0^T\int_0^{r_m}\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial \beta}\,\beta(t)\right]drdt\\ &\quad-\int_0^{r_m}p(r,T)\alpha(r)dr. \end{aligned} \tag{33}$$

Define the adjoint system

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r}+\dfrac{\partial q(r,t)}{\partial t}=\mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\lambda_o\dfrac{\partial L(p^*,\beta^*,r,t)}{\partial p},\\ q(r,T)=\alpha(t),\\ q(0,t)=q(t). \end{cases} \tag{34}$$

As in Lemma 1, we can prove

Lemma 3. The following relation holds between the solution (p,β)

of equation (29) and the solution of (34):

$$\lambda_0\int_0^T\int_0^{r_m}\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial p}\,p(r,t)drdt-\int_0^{r_m}p(r,t)\alpha(r)dr$$

$$= -\int_0^T q(t)\beta(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)drdt. \qquad (35)$$

From Lemma 3, (33) can be written as

$$f_1(p,\beta)=\int_0^T\left[\lambda_0\int_0^{r_m}\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial \beta}dr\right.$$

$$\left.-q(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)\right]\beta(t)dt. \qquad (36)$$

Then inequality (25) states that

$$\int_0^{r_m}\left[\lambda_0\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial \beta}-q(t)k(t)h(t)p^*(r,t)\right]dr[\beta-\beta^*(t)]\geq 0$$

$$\forall\ t\in[0,T] \text{ a.e.} \qquad (37)$$

We claim that there can not exist situations in which λ_0, $\alpha(r)$ are simultaneously zero. For then $f_1\equiv 0$, $q(r,t)\equiv 0$, $f_{12}=0$, $f_0=0$, from which $f_{11}=0$. This contradicts the fact that f_0, f_1, f_{11}, f_{12} are not all identically zero.

On the other hand, if $K_0=\emptyset$ then for arbitrary $(p,\beta)\in \mathbf{X}$.

$$\lambda_0\int_0^T\int_0^{r_m}\left[\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial p}p(r,t)\right.$$

$$\left.+\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial \beta}\beta(t)\right]drdt=0.$$

In particular choose $\lambda_0=1$, $\alpha(r)=0$; then (35) shows that

$$\lambda_0\int_0^T\int_0^{r_m}\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial p}\,p(r,t)drdt$$

$$= -\int_0^T q(t)\beta(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)drdt.$$

Thus,

$$\int_0^T\Bigg[\int_0^{r_m}\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial \beta}$$

$$-q(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr\Bigg]\beta(t)dt=0, \qquad \forall\ \beta(t)\in L^\infty(0,T),$$

from which

$$\int_0^{r_m}\Bigg[\frac{\partial L(p^*(r,t),\beta^*(t),r,t)}{\partial \beta}$$

$$-q(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)\Bigg]dr=0, \qquad \forall\ t\in [0,T]\ \text{a.e.}$$

Therefore, (37) still holds.

Finally, if the adjoint system

$$\begin{cases} \dfrac{\partial \hat{q}(r,t)}{\partial t}+\dfrac{\partial \hat{q}(r,t)}{\partial r}=\mu(r)\hat{q}(r,t)-\beta^*(t)k(r)h(r)\hat{q}(t), \\ \hat{q}(0,t)=\hat{q}(t) \end{cases} \tag{38}$$

has a nonzero solution $\hat{q}(r,t)$ (in which case $\hat{q}(r,T)\not\equiv 0$) such that

$$\hat{q}(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr=0, \qquad \forall\ t\in [0,T]\ \text{a.e.}$$

then choose $\lambda_0=0$, $\alpha(r)=\hat{q}(r,T)$; we know (37) is still valid. Otherwise, if for an arbitrary nonzero solution $\hat{q}(r,t)$ of equation (38) we always have

$$\hat{q}(r,t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr\not\equiv 0, \tag{39}$$

in which case we say the situation is nondegenerate, then the linearized system (28) is controllable. For if not there exist $\alpha(t)\in L^2(0,r_m)$ such that

$$\int_0^{r_m}\alpha(r)p(r,T)dr=0, \qquad \alpha(r)\not\equiv 0.$$

Then selecting $\lambda_0=0$ and the solution $\hat{q}(r,t)$ of equation (34) corresponding to $\alpha(r)$ (note that it is also a solution of (35)), we have from Lemma 3

$$\int_0^T \hat{q}(r,t)\beta(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr= 0, \quad \forall \quad \beta(t)\in L^\infty(0,T),$$

from which

$$\hat{q}(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr= 0, \quad \forall \quad t\in [0,T] \text{ a.e.}$$

This is a contradiction. So under assumption (39)the linearized system is controllable.

Under all circumstances, we obtain

Theorem 4. Assume that (p^*,β^*) is the solution of the optimal control problem. Then there exist $\lambda_0\geq 0$, $\alpha(r)\in L^2(0,r_m)$, not identically zero, such that the following maximum principle holds:

$$\beta^*(t)H_\beta(p^*,\beta^*)= \max_{\beta_0\leq\beta\leq\beta_1} \beta H_\beta(p^*,\beta^*),$$

$$H(p,\beta)=q(t)\beta(t)\int_{r_1}^{r_2}k(r)h(r)p(r,t)dr-\lambda_0 L(p,\beta,r,t),$$

$$H_\beta(p^*,\beta^*)= \frac{\partial H(p^*,\beta^*)}{\partial\beta}. \tag{40}$$

Note. In reference [9], it is shown that for an arbitrarily given ideal situation $p^*(r)$ and $\varepsilon > 0$, if the initial function $p_0(r)$ satisfies suitable conditions and provided

$$\beta_1 > \beta_{cr} = \left[\int_{r_1}^{r_2} k(r)h(r)e^{-\int_0^r \mu(\rho)d\rho} dr\right]^{-1},$$ there exist a control $\beta(t) \in U_{ad}$ and a time $T > 0$ such that $\|p(r,T) - p^*(r)\| \le \varepsilon$. This suggests us to pose the following optimal control problem.

Determine the optimal control $(p^*, \beta^*) \in X$ such that

$$\begin{cases} J(p^*, \beta^*) = \displaystyle\min_{\beta(\cdot)\in U_{ad}} \int_0^T \int_0^{r_m} L(p(r,t), \beta(t), r, t)\,dr\,dt, \\ \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), \qquad 0<r<r_m, t>0 \\ p(r,0) = p_0(r), \qquad 0 \le r \le r_m, \\ p(r,T) \in V = \{p(r) \,|\, \|p(r) - p^o(r)\| \le M\} \\ p(0,t) = \beta(t) \displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)\,dr, \qquad t \ge 0, \\ U_{ad} = \{\beta(t) \,|\, 0 \le \beta_0 \le \beta(t) \le \beta_1,\ t \in [0,T] \text{ a.e.}\}. \end{cases} \tag{41}$$

The assumptions on L are the same as before. Let

$$\Omega_1 = \{p(r,t), \beta(t) \in X \,|\, \beta_0 \le \beta(t) \le \beta_1,\ t \in [0,T] \text{ a.e.}\},$$
$$\Omega_2 = \{(p(r,t), \beta(t)) \in X \,|\, p(r,T) \in V\},$$
$$\Omega_3 = \{(p(r,t), \beta(t)) \in X \,|\, p_t + p_r = -\mu p,\ p(r,t) = p_0(r),$$
$$p(0,t) = \beta(t) \int_{r_1}^{r_2} k(r)h(r)p(r,t)\,dr\}. \tag{42}$$

Now, the directions of decrease cone and its adjoint are as in (22) and (23). Corresponding to Ω_1 the feasible directions cone and its adjoint are as in (24) and (25). Because Ω_2 is a closed convex set and $\mathring{\Omega}_2 \neq \emptyset$, the dual f_2 corresponding to its feasible directions is a supporting functional, that is,

$$f_2(p,\beta) \geq f_2(p^*,\beta^*), \quad \forall\ p(r,T) \in V.$$

Thus, there exists $\alpha(r) \in L^2(0,r_m)$ such that

$$f_2(p,\beta) = \int_0^{r_m} \alpha(r)p(r,T)dr \tag{43}$$

Therefore,

$$\alpha(r) = \hat{\lambda}_0[p^o(r) - p^*(r,T)], \qquad \hat{\lambda}_0 \geq 0.$$

For Ω_3, define the operator $G: \mathbf{X} \longrightarrow C(0,T;L^2(0,r_m))$ by

$$G(p,\beta) = p(r,t) - \begin{cases} p_0(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}, & r \geq t, \\ \beta(t-r)\int_{r_1}^{r_2} k(s)h(s)p(s,t-r)ds e^{-\int_0^r \mu(\rho)d\rho}, & r < t, \end{cases} \tag{44}$$

then $\Omega_3 = \{(p,\beta) \,|\, G(p,\beta) = 0\}$. As before,

$$G'(p^*, \beta^*)(\hat{p},\hat{\beta})$$

$$= \hat{p}(r,t) - \begin{cases} 0, \quad r \geq t, \\ \beta^*(t-r)\int_{r_1}^{r_2} k(s)h(s)\hat{p}(s,t-r)ds e^{-\int_0^r \mu(\rho)d\rho} + \\ \quad \hat{\beta}(t-r)\int_{r_1}^{r_2} k(s)h(s)p^*(s,t-r)ds e^{-\int_0^r \mu(\rho)d\rho}, \quad r < t, \end{cases}$$

For an arbitrary $g(r,t) \in \mathbf{C}(0,T;L^2(0,r_m))$, the equation

$$G'(p^*,\beta^*)(\hat{p},0) = g(r,t)$$

has a unique solution. Thus, $G'(p^*,\beta^*)\mathbf{X} = \mathbf{C}(0,T;L^2(0,r_m))$. Therefore, for Ω_3 the tangent directions cone $K_3 = \{(p,\beta) \,|\, G'(p^*,\beta^*)(p,\beta) = 0\}$, from which we have

Theorem 5. Assume that (p^*,β^*) is the solution of the optimal control problem (41). Then there exist $\lambda_0\geq 0$, $\hat{\lambda}_0\geq 0$, not identically zero, such that

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r}+\dfrac{\partial q(r,t)}{\partial t}=\mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\lambda_0\dfrac{\partial L(p^*,\beta^*,r,t)}{\partial p}, \\ q(r,T)=\hat{\lambda}_0[p^0(r)-p^*(r,T)], \\ q(0,t)=q(t). \end{cases} \tag{45}$$

(p^*,β^*) satisfies the maximum principle

$$\beta^*(t)H_\beta(p^*,\beta^*)=\max_{\beta_0\leq\beta\leq\beta_1}\beta H_\beta(p^*,\beta^*). \tag{46}$$

Here, H_β is as shown in (40).

Return to the infinite time problem (17). Assume that

$$\lambda_{OT}+\|q_T(t)\|_{L^2(0,T)}=1. \tag{47}$$

The subscripts show the relationship to T. Take $T_N\longrightarrow\infty$ such that $\lambda_{OT_N}\longrightarrow\lambda_\infty$. From (6) and for fixed t and T_N sufficiently large t,

$$q_{T_N}=\int_t^{T_N}e^{-\int_0^{s-t}\mu(\rho)d\rho}\beta^*(s)k(s-t)h(s-t)q_{T_N}(s)ds$$

$$-\lambda_{TN}\int_t^{T}e^{-\int_0^{s-t}\mu(\rho)d\rho}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(s-t,s)}ds.$$

It is easily proven that

$$q_{T_N}\longrightarrow q(t),\quad \text{as } N\longrightarrow\infty,$$

and q(t) satisfies

$$q(t)=\int_t^{\infty}e^{-\int_0^{s-t}\mu(\rho)d\rho}\beta^*(s)k(s-t)h(s-t)q(s)ds$$

$$-\lambda_\infty\int_t^{T}e^{-\int_0^{s-t}\mu(\rho)d\rho}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(s-t,s)}ds.$$

Theorem 6. For the optimal control problem on an infinite time interval, there exist $\lambda_\infty \geq 0$ and $q(t)$, both not identically zero, such that

$$\beta^*(t)H_\beta(p^*,\beta^*)= \max_{\beta_0\leq\beta\leq\beta_1} \beta H_\beta(p^*,\beta^*), \quad \forall\ t\in [0,T] \text{ a.e.},$$

$$H(p,\beta)=q(t)\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr-\lambda_\infty L(p,\beta,r,t),$$

$$H_\beta(p^*,\beta^*)= \frac{\partial H(p^*,\beta^*)}{\partial\beta} .$$

The function $q(r,t)$ satisfies

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r} + \dfrac{\partial q(r,t)}{\partial t}=\mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\lambda_\infty \dfrac{\partial L(p^*,\beta^*,r,t)}{\partial p}, \\ q(r,\infty)=0, \\ q(0,t)= q(t). \end{cases} \tag{48}$$

6.5 Results of the Nolinear System with Logistic Term

In this section we will discuss the control problem of the following logistic population distributed parameter system

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t)-Kf(N(t))p(r,t), \quad 0<r<r_m, t>0, \\ p(r,0)=p_0(r), \quad 0\leq r\leq r_m, \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr, \quad t\geq 0, \end{cases} \tag{49}$$

where $N(t)=\int_0^{r_m} p(r,t)dr$ is the total population at t, K is an environment constant and the logistic function $f(\xi)$ satisfies

$$f(0)=0, \ f(\xi)>0 \text{ for } \xi>0;$$

$f(\xi)$ is continuously differentiable. (50)

Since the methods are very similar, we omit the proof and only list the results

It was proved in chapter 2 that for an arbitrary $p_0(r)\in L^2(0,r_m)$, equation (49) in $L^2(0,r_m)$ has a unique solution (weak solution exactly) $p(r,t)\in C([0,\infty);L^2(0,r_m))$; moreover,

$$p(r,t)=\begin{cases} p_0(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}e^{-\int_0^t Kf(N(\rho))d\rho}, & r\geq t,\\ \beta(t-r)\int_{r_1}^{r_2}k(s)h(s)p(s,t-r)ds\, e^{-\int_0^r\mu(\rho)d\rho}e^{-\int_0^t Kf(N(\rho))d\rho}, & r<t, \end{cases}$$

or

$$\begin{cases} p(r,t)=\left[p_0(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}+\sum_{k=0}^{\infty}\phi_k(t-r)e^{-\int_0^r\mu(\rho)d\rho}\right]e^{-\int_0^t Kf(N(\rho))d\rho}\\ \phi_0(t)=\beta(t)\int_{r_1}^{r_2}k(s)h(s)p_0(s-t)e^{-\int_{s-t}^{s}\mu(\rho)d\rho}ds,\\ \phi_k(t)=\beta(t)\int_{r_1}^{r_2}k(s)h(s)\phi_{k-1}(t-s)e^{-\int_0^s\mu(\rho)d\rho}ds, \quad k=1,2\ldots \end{cases} \tag{51}$$

and $\phi_k(t)$ does not vanish only in $[kr_1,(k+1)r_2]$.

The optimal control problem is to determine (β^*,p^*), $\beta^*(\cdot)\in U_{ad}$, such that

$$J(\beta^*,p^*) = \min_{\beta(\cdot)\in U_{ad}} J(\beta,p),$$

$$J(\beta,p) = \int_0^T\int_0^{r_m} L(p(r,t),\beta(t),r,t))drdt + \frac{1}{2}\int_0^{r_m} [p(r,T)-\bar{p}(r)]^2dr. \tag{52}$$

As before $p(r,t)$ is the trajectory of the control $\beta(t)$, $\bar{p}(r)\in$

$L^2(0,r_m)$ an arbitrary fixed function, L is a functional defined on $L^2(0,r_m)\times[\beta_0,\beta_1]\times[0,r_m]\times[0,T]$ satisfying the same condition as previous sections.

$$U_{ad} = \{\ \beta(t)\ |\ 0\le\beta_0\le\beta(t)\le\beta_1,\ t\in[0,T] \text{ a.e},$$
$$\beta(t) \text{ is measurable on } [0,T]\}. \tag{53}$$

The adjoint equation of equation (49) is

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r} + \dfrac{\partial q(r,t)}{\partial t} = \mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\dfrac{\partial L(p^*,\beta^*,r,t)}{\partial p} \\ \qquad +Kf(N^*(t))q(r,t)+Kf'(N^*(t))\displaystyle\int_0^{r_m} p^*(r,t)q(r,t)dr, \\ q(r,T)=\bar{p}(r)-p^*(r,T), \\ q(0,t)=q(t). \end{cases} \tag{54}$$

As with equation (49), we take solutions (weak solutions) of equation (53) to be the solutions of the following equation

$$\bar{q}(t)=e^{-\int_0^{T-t}\mu(\rho)d\rho}\bar{q}(T-t,T)+\int_t^T e^{-\int_0^{s-t}\mu(\rho)d\rho}\beta^*(s)k(s-t)h(s-t)\bar{q}(s)ds$$
$$-\int_t^T e^{-\int_0^{s-t}\mu(\rho)d\rho}\left[\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\bigg|_{(s-t,s)}\right.$$
$$\left.+Kf'(N^*(t))\int_0^{r_m} p^*(r,t)\bar{q}(r,t)dr\right]ds,$$

$$\bar{q}(r,t)=e^{-\int_0^{r+T-t}\mu(\rho)d\rho}\bar{q}(r+T-t,T)$$
$$+\int_t^T e^{-\int_r^{r+s-t}\mu(\rho)d\rho}\beta^*(s)k(r+s-t)h(r+s-t)\bar{q}(s)ds$$
$$-\int_t^T e^{-\int_r^{r+s-t}\mu(\rho)d\rho}\left[\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\bigg|_{(r+s-t,s)}\right.$$

$$+f'(N^*(t))\int_0^{r_m} p^*(r,t)\bar{q}(r,t)dr\Big]ds, \quad 0\le t\le T,\ 0\le r\le r_m, \qquad (55)$$

$$q(r,t)=\bar{q}(r,t)e^{\int_0^t Kf(N(\rho))d\rho}, \quad q(t)=\bar{q}(t)e^{\int_0^t Kf(N(\rho))d\rho}.$$

In $L^2(0,r_m)$, equation (54) has a unique solution.

Theorem 7. The solution of the problem (52) satisfies the maximum principle:

$$\beta^*(t)H(t)=\max_{\beta_0\le\beta\le\beta_1} \beta H(t), \qquad \forall\ t\in[0,T] \text{ a.e.}$$

$$H(t)=q(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr-\int_0^{r_m} \frac{\partial L(p,\beta,\cdot,\cdot)}{\partial\beta}\Big|_{(r,t)}dr, \qquad (56)$$

from which we have

$$\beta(t)=\begin{cases}\beta_0, & H(t)<0,\\ \beta_1, & H(t)>0,\\ \text{indeterminate}, & H(t)=0.\end{cases}$$

$H(t)$ is the switching function.

Lastly we discuss the fixed horizon and target set problems. It was proved in chapter 2 that if $\beta(t)=\beta$ then there exists a constant $c_0\ge 0$ such that every solution $p(r,t)$ of (49) converges to $c_0 e^{-\int_0^r \mu(\rho)d\rho}$ as time goes to infinity

$$\lim_{t\to\infty} p(r,t)=c_0 e^{-\int_0^r \mu(\rho)d\rho}$$

and when $\lim_{\xi\to\infty} f(\xi)=\infty$, $c_0<\infty$. This suggests us to pose the following optimal control problem:

$$
\begin{cases}
J_T(p^*,\beta^*) = \displaystyle\min_{\beta(\cdot)\in U_{ad}} \int_0^T\int_0^{r_m} L(p(r,t),\beta(t),r,t)\,dr\,dt, \\
\dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t) - Kf(N(t))p(r,t), \\
p(r,0) = p_0(r), \\
p(r,T) \in V = \{p(r) \mid \|p(r)-p^{\circ}(r)\| \le M\}, \\
p(0,t) = \beta(t)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)\,dr, \\
U_{ad} = \{\beta(t) \mid 0 \le \beta_0 \le \beta(t) \le \beta_1,\ t\in[0,T] \text{ a.e.}\}.
\end{cases} \tag{57}
$$

Define the adjoint system

$$
\begin{cases}
\dfrac{\partial q(r,t)}{\partial t} + \dfrac{\partial q(r,t)}{\partial r} = \mu(r)q(r,t) + Kf(N^*)q(r,t) + Kf'(N^*)\displaystyle\int_0^{r_m} p^* q(r,t)\,dr \\
\qquad + Kf'(N^*)\displaystyle\int_t^T q(s)p^*(0,t) \ -\beta^*(t)k(r)h(r)q(t) + \lambda_0 \dfrac{\partial L(p^*,\beta^*,r,t)}{\partial p}, \\
q(r,T) = \hat{\lambda}_0[p^{\circ}(r) - p^*(r,T)], \\
q(0,t) = q(t).
\end{cases} \tag{58}
$$

Theorem 8. Assume that (p^*,β^*) is the solution of the optimal control problem. Then there exist $\lambda_0 \ge 0$, $\hat{\lambda}_0 \ge 0$ not identically zero, such that the following maximum principle holds:

$$
\beta^*(t)H_\beta(p^*,\beta^*) = \max_{\beta_0\le\beta\le\beta_1} \beta H_\beta(p^*,\beta^*),
$$

$$
H(p,\beta) = q(t)\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)\,dr - \lambda_0 L(p,\beta,r,t),
$$

$$
H_\beta(p^*,\beta^*) = \frac{\partial H(p^*,\beta^*)}{\partial \beta}, \tag{59}
$$

where q(t) is the solution of adjoint equation (58).

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Chapter 7

Optimal Birth Control of Policies II

In chapter 6 we discussed optimal birth control of population systems of McKendrick type. The present chapter presents further results of current interests. These include problems with free final time, of which the minimum time problem is a special case (but relaxing many convexity assumptions). Systems with phase constraints are also studied. Finally, mini-max control for population regulation is characterized.

7.1 Free Final Time Problem

Consider the free final time optimal control problem of the population control system

Problem (P): Minimize $J(\beta,p)=\int_0^{t_1}\int_0^{r_m}L(p(r,t),\beta(t))drdt$

subject to

$$\begin{cases}\dfrac{\partial p(r,t)}{\partial t}+\dfrac{\partial p(r,t)}{\partial r}=-\mu(r)p(r,t), & 0<r<r_m,\ t>0,\\ p(r,0)=p_0(r), & 0\le r\le r_m,\\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2}k(r)h(r)p(r,t)dr, & t\ge 0,\\ p(r,t_1)=p^0(r),\quad t_1>0\ ,\ \beta(t)\in M\subset \mathbb{R}^+ & \end{cases} \tag{1}$$

where L is a function defined on $L^2(0,r_m)\times\mathbb{R}^+$ satisfying

(1). $L(p(r),\beta)$ is continuous in β,

(2). $\left|\frac{\partial L(p(r),\beta)}{\partial p}\right|$ is bounded for every bounded subset of $L^2(0,r_m)\times\mathbb{R}^+$.

For any measurable function $v(s)\geq 0$, define the time transformation

$$t(\tau)=\int_0^\tau v(s)ds, \quad t(1)=t_1, \tag{2}$$

and let $p(r,\tau)=p(r,t(\tau))$,

$$\beta(\tau)=\begin{cases}\beta(t(\tau)), & \tau\in S_1,\\ \text{arbitrary}, & \tau\in S_2\end{cases} \tag{3}$$

then $(p(r,\tau),\beta(\tau))$ satisfies the following equation

$$\begin{cases}\dfrac{\partial p(r,\tau)}{\partial\tau}+v(\tau)\dfrac{\partial p(r,\tau)}{\partial r}=-\mu(r)v(\tau)p(r,\tau), & 0<r<r_m,\ 0<\tau<1,\\ p(r,0)=p_0(r), & 0\leq r\leq r_m,\\ v(\tau)p(0,\tau)=v(\tau)\beta(\tau)\displaystyle\int_{r_1}^{r_2}k(r)h(r)p(r,\tau)dr, & 0\leq\tau\leq 1,\\ p(r,1)=p^0(r), & \end{cases} \tag{4}$$

where
$$S_1=\{\tau\mid \tau\in[0,1],\ v(\tau)>0\},$$
$$S_2=\{\tau\mid \tau\in[0,1],\ v(\tau)=0\}. \tag{5}$$

Conversely, if $(p(r,\tau),\beta(\tau))$ solves equation (4), define $p(r,t)=p(r,\tau(t))$, $\beta(t)=\beta(\tau(t))$,

$$\tau(t)=\inf\{\tau\mid t(\tau)=t\} \tag{6}$$

then $(p(r,t),\beta(t))$ satisfies equation (1) for $t=t(\tau)$, $v(\tau)>0$, but for the monotone function $t(\tau)=\int_0^\tau v(s)ds$, $\text{mes}\{\, t=t(\tau)\mid v(\tau)>0\}=t_1$

$=\int_0^1 v(s)ds$, so $(p(r,t),\beta(t))$ satisfies equation (1) for $t\in[0,t_1]$ a.e.

Based on the above arguments, we consider the optimal (fixed final time) problem

Problem (Q): Minimize $J(\beta,p)=\int_0^{t_1}\int_0^{r_m} L(p(r,t),\beta(t))drdt$

subject to equation (4)

If (p^*,β^*,t_1) solves problem (P), then for any $v^*(\tau)\ge 0$ satisfying $\int_0^1 v^*(s)ds=t_1$, $\beta^*(\tau)$ defined similar to (3), $(p^*(r,\tau),\beta^*(\tau),v^*)$ solves problem (Q). By this $\beta(\tau)$, we put forward another problem as follows

Problem (L): Minimize $J(\beta^*,p,v)=\int_0^1\int_0^{r_m} v(\tau)L(p(r,\tau),\beta^*(\tau))drd\tau$

subject to

$$\begin{cases} \dfrac{\partial p(r,\tau)}{\partial \tau}+v(\tau)\dfrac{\partial p(r,\tau)}{\partial r}=-\mu(r)v(\tau)p(r,\tau), & 0<r<r_m,\ 0<\tau<1, \\ p(r,0)=p_0(r), & 0\le r\le r_m, \\ v(\tau)p(0,\tau)=v(\tau)\beta^*(\tau)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,\tau)dr, & 0\le\tau\le 1, \\ p(r,1)=p^0(r), \end{cases} \tag{7}$$

and (p^*,v^*) solves problem (L). Consider the solution of (7) as that of the integral equation

$$\begin{cases} \displaystyle\int_0^r p(s,\tau)ds-\int_0^r p_0(s)ds+\int_0^\tau v(\xi)[p(r,\xi)-\beta^*(\xi)\int_{r_1}^{r_2} k(r)h(r)p(r,\xi)dr]d\xi \\ \qquad\qquad +\displaystyle\int_0^r\int_0^\tau v(\xi)\mu(s)p(s,\xi)dsd\xi, & (8) \\ p(r,1)=p^0(r). & (9) \end{cases}$$

Similarly, we consider that the solution of the differential equation is equivalent to that of the corresponding integral equation.

Simple arguments can be found that the equation (8) has a unique solution on $C(0,1;L^2(0,r_m))$ and so we take $X=C(0,1;L^2(0,r_m))\times L^\infty(0,1)$ as the state space. Define the inequality constraint

$$\Omega_1=\{(p(r,\tau),v(\tau))\in X \mid v(\tau)\geq 0, \text{ for } \tau\in[0,1] \text{ a.e.}\} \tag{10}$$

and the equality constraint

$$\Omega_2=\{(p(r,\tau),v(\tau))\in X \mid (p,v) \text{ satisfies (8) and (9)}\} \tag{11}$$

Under these notations, we can write problem (L) as

$$\begin{cases} \text{Minimize } J(\beta^*,p,v)=\int_0^1\int_0^{r_m} v(\tau)L(p(r,\tau),\beta^*(\tau))drd\tau \\ \text{subject to } (p(r,\tau),v(\tau))\in \Omega_1\cap\Omega_2\subset X. \end{cases} \tag{12}$$

$J(\beta^*,p,v)$ is Frechét differentiable at any point $(\hat{p},\hat{v})$ and

$$J'(\beta^*,p_0,v_0)(p,\beta)=\int_0^1\int_0^{r_m}\Big[v_0(\tau)\frac{\partial L(p_0(r,\tau),\beta^*(\tau))}{\partial p}p(r,\tau) +v(\tau)L(p_0,\beta^*)\Big]drd\tau \tag{13}$$

and so the decreasing direction cone of $J'(\beta^*,p_0,v_0)$ at (p^*,v^*) is

$$K_0=\{(p,v) \mid J'(\beta^*,p^*,v^*)(p,v)< 0\} \tag{14}$$

If $K_0\neq\emptyset$, then for any $f_0\in K_0^*$, there exists a constant $\lambda_0\geq 0$, such that

$$f_0(p,v)=-\lambda_0\int_0^1\int_0^{r_m} v^*(\tau)\Big[\frac{\partial L(p^*(r,\tau),\beta^*(\tau))}{\partial p}p(r,\tau) +L(p^*,\beta^*)v(\tau)\Big]drd\tau. \tag{15}$$

Notice that $\Omega_1=C(0,1;L^2(0,r_m))\times\hat{\Omega}_1$, $\hat{\Omega}_1=\{v(\tau)\in L^\infty(0,1)\mid v(\tau)\geq 0\}$ is a closed convex subset of $L^\infty(0,1)$, $\mathring{\Omega}_1=C\times\mathring{\hat{\Omega}}_1\neq\emptyset$, and so the feasible direction cone of Ω_1 at (p^*,v^*) is

$$K_1=\{\lambda(\mathring{\Omega}_1-(p^*,v^*))\mid \lambda>0\}. \tag{16}$$

For any $f_1\in K_1^*$, if $c(t)\in L(0,1)$ such that

$$f_1(p,v)=\int_0^1 c(\tau)v(\tau)d\tau \tag{17}$$

then [1]

$$c(\tau)[v-v^*(\tau)]\geq 0, \qquad \forall\ v\in(0,\infty),\ \tau\in[0,1]\ \text{a.e.} \tag{18}$$

In order to determine the tangent direction cone of Ω_2 at (p^*,v^*), we define the operator as follows $G\colon X\longrightarrow X$

$G(p,v)=$

$$\Big[\int_0^r p(s,\tau)ds-\int_0^r p_0(s)ds+\int_0^\tau v(\xi)[p(r,\xi)-\beta^*(\xi)\int_{r_1}^{r_2}k(r)h(r)p(r,\xi)dr]d\xi$$

$$+\int_0^r\int_0^\tau v(\xi)\mu(s)p(s,\xi)dsd\xi,\ p(r,1)-p^0(r)\Big] \tag{19}$$

then $\Omega_2=\{(p,v)\mid G(p,v)=0\}$. (20)

Now $G'(p^*,v^*)(p,v)=$

$$\Big[\int_0^r p(s,\tau)ds+\int_0^\tau\Big[[v(\xi)p^*(r,\xi)+v^*(\xi)p(r,\xi)]$$

$$-\beta^*(\xi)\int_{r_1}^{r_2}k(r)h(r)[v(\xi)p^*(r,\xi)+v^*(\xi)p(r,\xi)]dr\Big]d\xi$$

$$+\int_0^r\int_0^\tau \mu(s)[v(\xi)p^*(s,\xi)+v^*(\xi)p(s,\xi)]dsd\xi,\quad p(r,1)\Big] \tag{21}$$

and we solve the equation

$$G'(p^*,v^*)(p,v)=(q,g)\in X$$

i.e.

$$\begin{cases} \int_0^r p(s,\tau)ds+\int_0^\tau\Big[[v(\xi)p^*(r,\xi)+v^*(\xi)p(r,\xi)] \\ -\beta^*(\xi)\int_{r_1}^{r_2}k(r)h(r)[v(\xi)p^*(r,\xi)+v^*(\xi)p(r,\xi)]dr\Big]d\xi \\ +\int_0^r\int_0^\tau \mu(s)[v(\xi)p^*(s,\xi)+v^*(\xi)p(s,\xi)]dsd\xi=q(r,\tau), \\ p(r,1)=g(r) \end{cases} \tag{22}$$

If the linearized system

$$\begin{cases} \frac{\partial p(r,\tau)}{\partial \tau} + v^*(\tau)\frac{\partial p(r,\tau)}{\partial r} = -\mu(r)[v(\tau)p^*(r,\tau)+v^*(\tau)p(r,\tau)] \\ \qquad -v(\tau)\frac{\partial p^*(r,\tau)}{\partial r}, \\ p(r,0)=0, \\ v(\tau)p^*(0,\tau)+v^*(\tau)p(0,\tau)=v(\tau)\beta^*(\tau)\int_{r_1}^{r_2}k(r)h(r)p^*(r,\tau)dr \\ \qquad +v^*(\tau)\beta^*(\tau)\int_{r_1}^{r_2}k(r)h(r)p(r,\tau)dr \end{cases} \tag{23}$$

is controllable, then let $\hat{p}(r,\tau)=p(r,\tau)+d(r,\tau)$, $d(r,\tau)$ is determined by

$$\int_0^r d(s,\tau)ds+\int_0^\tau v^*(\xi)[d(r,\xi)-\beta^*(\xi)\int_{r_1}^{r_2}k(r)h(r)d(r,\xi)dr]d\xi$$

$$+\int_0^r\int_0^\tau v^*(\xi)\mu(s)d(s,\xi)dsd\xi=q(r,\tau)$$

(p,β), $\beta=\hat{\beta}$ solves equation (23) and $p(r,1)=g(r)-d(r,1)$, so $(\hat{p},\hat{\beta})$

solves equation (22). In this case, the tangent direction cone of Ω_2 at (p^*,v^*) is determined by

$$K_2=\{(p,v)\,|\,G'(p^*,v^*)(p,v)=0\}$$

i.e.

$$\begin{cases} \dfrac{\partial p(r,\tau)}{\partial \tau} + v^*(\tau)\dfrac{\partial p(r,\tau)}{\partial r} = -\mu(r)[v(\tau)p^*(r,\tau)+v^*(\tau)p(r,\tau)] \\ \qquad\qquad -v(\tau)\dfrac{\partial p^*(r,\tau)}{\partial r}, \\ p(r,0)=0, \\ v(\tau)p^*(0,\tau)+v^*(\tau)p(0,\tau)=v(\tau)\beta^*(\tau)\displaystyle\int_{r_1}^{r_2}k(r)h(r)p^*(r,\tau)dr \\ \qquad\qquad +v^*(\tau)\beta^*(\tau)\displaystyle\int_{r_1}^{r_2}k(r)h(r)p(r,\tau)dr \\ p(r,1)=0. \end{cases} \tag{24}$$

$K_2=K_{11}\cap K_{12}$, $K_{12}=\{(p,v)\,|\,p(r,1)=0\}$, K_{11} consists of such $(p,v)\in X$ such that

$$\begin{cases} \dfrac{\partial p(r,\tau)}{\partial \tau} + v^*(\tau)\dfrac{\partial p(r,\tau)}{\partial r} = -\mu(r)[v(\tau)p^*(r,\tau)+v^*(\tau)p(r,\tau)] \\ \qquad\qquad -v(\tau)\dfrac{\partial p^*(r,\tau)}{\partial r}, \\ p(r,0)=0, \\ v(\tau)p^*(0,\tau)+v^*(\tau)p(0,\tau)=v(\tau)\beta^*(\tau)\displaystyle\int_{r_1}^{r_2}k(r)h(r)p^*(r,\tau)dr \\ \qquad\qquad +v^*(\tau)\beta^*(\tau)\displaystyle\int_{r_1}^{r_2}k(r)h(r)p(r,\tau)dr \end{cases} \tag{25}$$

For any $f\in K_2^*$, $f=f_{11}+f_{12}$, $f_{1i}\in K_{1i}^*$, $i=1,2$,

$$f_{12}(p,v)=\int_0^{r_m}\alpha(r)p(r,1)dr, \quad \alpha(r)\in L^2(0,1). \tag{26}$$

By the Dubovitskii-Milyutin Theorem, there exist functionals $f_i\in K_i^*$, $i=0, 1, 2$, not all identically zero such that

$$f_0+f_1+f_{11}+f_{12}=0. \tag{27}$$

In particular for any (p,v) satisfying (25), $f_{11}(p,v)=0$, and so

$$f_1(p,v)=-f_0(p,v)-f_{12}(p,v)$$

$$=\lambda_0\int_0^1\int_0^{r_m}\left[\frac{\partial L(p^*(r,\tau),\ \beta^*(\tau))}{\partial p}v^*(\tau)p(r,\tau)\right.$$

$$\left.+L(p^*,\beta^*)v(\tau)\right]drd\tau-\int_0^{r_m}\alpha(r)p(r,1)dr \tag{28}$$

where the solution of (25) is considered as that of the integral equation

$$\int_0^r p(s,\tau)ds+\int_0^\tau\left[[v(\xi)p^*(r,\xi)+v^*(\xi)p(r,\xi)]\right.$$

$$\left.-\beta^*(\xi)\int_{r_1}^{r_2}k(r)h(r)[v(\xi)p^*(r,\xi)+v^*(\xi)p(r,\xi)]dr\right]d\xi$$

$$+\int_0^r\int_0^\tau\mu(s)[v(\xi)p^*(s,\xi)+v^*(\xi)p(s,\xi)]dsd\xi=0, \tag{29}$$

Define the adjoint equation

$$\begin{cases}\dfrac{\partial q(r,\tau)}{\partial r}+v^*(\tau)\ \dfrac{\partial q(r,\tau)}{\partial \tau}=v^*(\tau)\mu(r)q(r,\tau)-\beta^*(\tau)k(r)h(r)q(\tau)\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\lambda_0\ \dfrac{\partial L(p^*,\ \beta^*)}{\partial p},\\ q(r,1)=\alpha(r),\\ v^*(\tau)q(0,\tau)=v^*(\tau)q(\tau),\end{cases} \tag{30}$$

and

$$\int_r^{r_m}\hat{q}(s,\tau)ds=q(r,\tau). \tag{31}$$

As lemma 1 of chapter 6, we have

Lemma 1. The solution of equation (25) and the solution of equations (30), (31) have the following relation

$$\lambda_0\int_0^1\int_0^{r_m}\left[\frac{\partial L(p^*(r,\tau),\ \beta^*(\tau))}{\partial p}v^*(\tau)p(r,\tau)\right]drdt$$

$$-\int_0^{r_m}\alpha(r)p(r,1)dr$$

$$=\int_0^1\left[\int_0^{r_m}p^*(r,\tau)\hat{q}(r,\tau)+\beta^*(\tau)\int_{r_1}^{r_2}k(r)h(r)p^*(r,\tau)q(\tau)\right.$$

$$\left.+\int_0^{r_m}\mu(r)p^*(r,\tau)q(r,\tau)dr\right]v(\tau)d\tau. \tag{32}$$

Lemma 1 together with (28)and (18) imply that

$$\left[\int_0^{r_m}p^*(r,\tau)\hat{q}(r,\tau)drd\tau+\beta^*(\tau)\int_{r_1}^{r_2}k(r)h(r)p^*(r,\tau)q(\tau)drd\tau\right.$$

$$\left.+\int_0^{r_m}\mu(r)p^*(r,\tau)q(r,\tau)dr-\lambda_0\int_0^{r_m}L(p^*,\beta^*)dr\right][v-v^*(\tau)]\geq 0$$

$$\text{for all } v\geq 0. \tag{33}$$

It follows from (33) that

$$\int_0^{r_m}p^*(r,\tau)\hat{q}(r,\tau)+\beta^*(\tau)\int_r^{r_2}k(r)h(r)p^*(r,\tau)q(\tau)dr$$

$$+\int_0^{r_m}\mu(r)p^*(r,\tau)q(r,\tau)dr-\lambda_0\int_0^{r_m}L(p^*,\beta^*)dr=0,\ \forall\ \tau\in S_1, \tag{34}$$

$$\int_0^{r_m}p^*(r,\tau)\hat{q}(r,\tau)+\beta^*(\tau)\int_r^{r_2}k(r)h(r)p^*(r,\tau)q(\tau)dr$$

$$+\int_0^{r_m}\mu(r)p^*(r,\tau)q(r,\tau)dr-\lambda_0\int_0^{r_m}L(p^*,\beta^*)dr\geq 0,\ \forall\ \tau\in S_2. \tag{35}$$

We say that λ_0 and $\alpha(r)$ can not be both zero, since otherwise, $f_0=0$, $q(s,\tau)=0$, $f_{12}=0$, $f_1=0$ and hence $f_{11}=0$. This contradicts the Dubovitskii-Milyutin Theorem. Furthermore, if $K_0=\emptyset$, take $\lambda_0=1$, $\alpha(r)=0$, then (32) implies (33) and hence (34) and (35) are valid. Finally, if equation (30) has a nonzero solution $q(r,\tau)$ such that

$$\int_0^{r_m} p^*(r,\tau)\hat{q}(r,\tau)+\beta^*(\tau)\int_{r_1}^{r_2} k(r)h(r)p^*(r,\tau)q(\tau)dr$$

$$+\int_0^{r_m}\mu(r)p^*(r,\tau)q(r,\tau)dr=0, \qquad (36)$$

then take $\lambda_0=0$, and (33) is also valid. On the other hand, for any nonzero solution of (30)

$$\int_0^{r_m} p^*(r,\tau)\hat{q}(r,\tau)+\beta^*(\tau)\int_{r_1}^{r_2} k(r)h(r)p^*(r,\tau)q(\tau)dr$$

$$+\int_0^{r_m}\mu(r)p^*(r,\tau)q(r,\tau)dr\neq 0, \qquad (37)$$

we call this situation the nondegenerate case, since here the linearized system must be controllable. This is because otherwise there exists $\alpha(r)\in L^2(0,r_m)$ such that $\int_0^{r_m}\alpha(r)p(r,1)dr=0$, $\alpha(r)\neq 0$, and taking $\lambda_0=0$, we have a contradiction to (36). Hence, no matter what happened, (33) and hence (34) and (35) are always valid.

Define $q(r,t)=q(r,\tau(t))$, $\hat{q}(r,t)=\hat{q}(r,\tau(t))$, $q(t)=q(0,\tau(t))$, then (34) can be written as

$$\int_0^{r_m} p^*(r,t)\hat{q}(r,t)+\beta^*(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)q(t)dr$$

$$+\int_0^{r_m}\mu(r)p^*(r,t)q(r,t)dr-\lambda_0\int_0^{r_m} L(p^*,\beta^*)dr=0,$$

$$\text{for all } t\in[0,t_1] \text{ a.e.} \qquad (38)$$

Choose S_1 to be a perfect nowhere dense subset of [0,1] (see [1]) and define

$$v^*(\tau)=\begin{cases}\dfrac{t_1}{\mu(s_1)}, & \tau\in S_1,\\ 0, & \tau\in S_2=[0,1]\setminus S_2.\end{cases} \tag{39}$$

Now, analyzing the condition (35) as in [1], we can define $\beta^*(\tau)$ on S_2 and get (with the same notation as before)

$$\int_0^{r_m} p^*(r,t)\hat{q}(r,t)+\beta\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)q(t)dr$$

$$+\int_0^{r_m}\mu(r)p^*(r,t)q(r,t)dr-\lambda_0\int_0^{r_m}L(p^*,\beta\ )dr\geq 0,\qquad \forall\ \beta\in M, \tag{40}$$

for all $t\in[0,t_1]$. We have thus proved the following

Theorem 1. (Maximum Principle) Under the conditions on L mentioned in the beginning of this chapter, and let (β^*,p^*,t_1) be a solution of problem (P), then there exist $q(r,t)$, $\lambda_0\geq 0$, not both zero, such that

$$\int_0^{r_m} p^*(r,t)\hat{q}(r,t)+\beta^*(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)q(t)dr$$

$$+\int_0^{r_m}\mu(r)p^*(r,t)q(r,t)dr-\lambda_0\int_0^{r_m}L(p^*,\beta^*)dr=0,\quad \forall\ t\in[0,t_1]\ \text{a.e.}$$

$$[\beta-\beta^*(t)]\left[-\lambda_0\int_0^{r_m}\left[\frac{L(p^*,\beta^*)}{\partial\beta}+k(r)h(r)p^*(r,t)q(t)\right]dr\right]\geq 0,$$

$$\forall\ \beta\in M, t\in[0,t_1]\ \text{a.e.}$$

where

$$\begin{cases}\dfrac{\partial q(r,t)}{\partial r}+\dfrac{\partial q(r,t)}{\partial t}=\mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\lambda_0\dfrac{\partial L(p^*,\beta^*)}{\partial p},\\ q(r,t_1)=\alpha(r),\\ q(0,t)=q(t),\end{cases} \tag{41}$$

$$q(r,t)=\int_r^{r_m}\hat{q}(s,t)ds.$$

Note. If the end point condition $p(r,t_1)= p^0(r)$ is imposed instead of

$$p(r,t_1)\in \{p(r) \mid \|p(r)-p^0(r)\|\le \varepsilon\} \tag{42}$$

then $\alpha(r)$ should be taken as $\alpha(r)=p^*(r,t_1)-p^0(r)$ and λ_0 can be set to 1.

Corollary 1. If L=1, then problem (P) is the time optimal control problem considered in chapter 6 and the time optimal control satisfies the maximum principle

$$\beta^*(t)H(t)= \max_{\beta\in M} \beta H(t), \qquad \forall\ t\in [0,t_1] \text{ a.e.},$$

$$H(t)=\lambda_0\int_0^{r_m}\frac{\partial L(p^*,\beta^*)}{\partial\beta}-q(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr, \tag{43}$$

where t_1 is the minimum time. $q(t)$ is the solution of adjoint equation (41).

The result is the same as that of chapter 6 but here the convexity assumption on M is not assumed.

7.2 System with Phase Constraints

In this part, we consider the optimal control problem of a population system with phase constraints

$$\text{Problem (Q): Minimize } \hat{J}(\beta,p)=\int_0^T\int_0^{r_m}Q(p(r,t),\beta(t),t)drdt \tag{44}$$

under the constraints:

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), & 0<r<r_m, \ t>0, \\ p(r,0)=p_0(r), & 0\le r\le r_m, \\ p(r,T)=p^0(r), & 0\le r\le r_m, \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr, & t\ge 0, \\ \beta(t)\in [\beta_0,\beta_1] \text{ for } t\in [0,T] \text{ a.e.} \\ \displaystyle\int_0^{r_m} G(p(r,t),t)dr\le 0, & t\ge 0. \end{cases} \tag{45}$$

in the class of

$$(p(r,t),\beta(t))\in \mathbf{X}=C(0,T;L^2(0,r_m))\times L^\infty(0,T) \tag{46}$$

The time T is fixed.

Define

$$Q_1=\{ (p(r,t),\beta(t))\in \mathbf{X} \mid \beta(t)\in [\beta_0,\beta_1],\ t\in [0,T] \text{ a.e.} \} \tag{47}$$

$$Q_2=\{(p(r,t),\beta(t))\in \mathbf{X} | p_t+p_r=-\mu p,\ p(0,t)=\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr,$$

$$p(r,0)=p_0(r), p(r,T)=p^0(r) \}. \tag{48}$$

$$Q_3=\{(p(r,t),\beta(t))\in \mathbf{X}| \int_0^{r_m} G(p(r,t),t)dr\le 0\} \tag{49}$$

Then problem (Q) is equivalent to finding $(p^*,\beta^*)\in Q_1\cap Q_2\cap Q_3$ such that

$$\hat{J}(\beta^*,p^*)= \min_{(p,\beta)\in Q_1\cap Q_2\cap Q_3} \hat{J}(\beta,p). \tag{50}$$

This is a minimum problem formed by the inequality constraints Q_1, Q_3 and the equality Ω_2. We can use again the general theory of Dubovitskii-Milyutin for the extremum problem.

We had already investigated the corresponding cones of Q_1 and

Q_2 of the Dubovitskii-Milyutin theorem. Now we need only to consider constraint Q_3. Notice that Q_3 can be written as

$$Q_3=\{(p(r,t),\beta(t))\in \mathbf{X} \mid F(p)\le 0\} \tag{51}$$

where $F(p)= \max_{0\le t\le 1}\int_0^{r_m} G(p(r,t),t)dr$ and we assume

(1). $\int_0^{r_m} G(p(r),t)dr$ is a continuous functional on $L^2(0,r_m)\times [0,\infty]$;

(2). $\int_0^{r_m} G(p_0(r),0)dr< 0$, $\int_0^{r_m} G(p^0(r),T)dr< 0$;

(3). $\int_0^{r_m} G'_p(p(r),t)dr$ is also continuous on $L^2(0,r_m)\times [0.\infty)$ and $\int_0^{r_m} G'_p(p(r),t)dr\ne 0$ if $\int_0^{r_m} G(p(r),t)dr= 0$.

Let (β^*,p^*) solve problem (Q), then we consider $F(p^*)=0$. Since otherwise, the feasible direction cone K_3 of Q_3 at (β^*,p^*) is the whole space, i.e. $K_3=\mathbf{X}$. So $Q_3=\{(p(r,t),\ \beta(t))\in \mathbf{X} \mid F(p)\le F(p^*)\}$. Applying arguments as in [1] we can prove that

Lemma 2. F(p) is differentiable at any point in any direction and

$$F'(\hat{p},p)= \max_{t\in S}\int_0^{r_m} G'_p(\hat{p}(r,t),t)p(r,t)dr \tag{52}$$

where $S=\{ t\in [0,T] \mid \int_0^{r_m} G(\hat{p}(r,t),t)dr=F(\hat{p})\}$.

Furthermore, F(p) satisfies a Lipschitz condition in any ball.

Notice that $F'(p^*,G'_p(p^*,t))< 0$, we know that

$$K_3=\{(p,\beta)\in X \mid F'(p^*,p)< 0\}. \tag{53}$$

Define the linear operator $\mathbf{A}: \mathbf{x}\longrightarrow C[0,T]$ by

$$\mathbf{A}p(r,t)=-\int_0^{r_m} G'_p(p^*(r,t),t)p(r,t)dr \tag{54}$$

and

$$K=\{y(t)\in C[0,T] \mid y(t)\geq 0,\ \forall\ t\in S\}$$

then $K_3=\{p(r,t)\in X \mid \mathbf{A}p\in K\}$. Since $\mathbf{A}(-G'_p(p^*(r,t),t))\in \mathring{K}$, so $K_3^*=\mathbf{A}^*K^*$, i.e. for any $f\in K_3^*$, there exists a measure $dm(t)$, nonnegative and with support on S, such that

$$f(p(r,t))=\int_0^T \mathbf{A}p(r,t)dm(t)=\int_S \mathbf{A}p(r,t)dm(t)$$

$$=-\int_S\int_0^{r_m} G'_p(p^*(r,t),t)p(r,t)drdm(t) \tag{55}$$

Based on the previous results, there exist $\lambda_0\geq 0$, $\alpha(r)\in L^2(0,r_m)$ such that

$$f_1(p,\beta)=\lambda_0\int_0^T\int_0^{r_m}\left[\frac{\partial Q(p^*,\beta^*,t)}{\partial p}p(r,t)+\frac{\partial Q(p^*,\beta^*,t)}{\partial \beta}\beta(t)\right]drdt$$

$$-\int_0^{r_m} p(r,T)\alpha(r)dr+\int_0^T\int_0^{r_m} G'_p(p^*(r,t),t)p(r,t)drdm(t) \tag{56}$$

where (p,β) satisfies

$$\begin{cases}\dfrac{\partial p(r,t)}{\partial t}+\dfrac{\partial p(r,t)}{\partial r}=-\mu(r)p(r,t),\\ p(r,0)=0,\\ p(0,t)=\beta^*(t)\displaystyle\int_{r_1}^{r_2}k(r)h(r)p(r,t)dr+\beta(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr,\end{cases} \tag{57}$$

with the assumption that the decreasing direction cone of J at (p^*,β^*) is not empty and system (57) is controllable.

Define the adjoint system

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r} + \dfrac{\partial q(r,t)}{\partial t} = \mu(r)q(r,t) - \beta^*(t)k(r)h(r)q(t) + \lambda_0 \dfrac{\partial Q(p^*,\beta^*,t)}{\partial p} \\ \qquad\qquad + G'_p(p^*(r,t),t)\dfrac{dm(t)}{dt}, \\ q(r,T) = \alpha(r), \\ q(0,t) = q(t). \end{cases} \tag{58}$$

The solution of equation (58) should be considered as that of the following integral equation

$$-\int_0^r q(s,t)ds = -\int_0^r \alpha(s)ds - \int_0^r\int_t^T [q(s,\tau)-q(\tau)]dsd\tau + \int_0^r\int_t^T \mu(s)q(s,\tau)dsd\tau$$

$$-\int_0^r k(s)h(s)ds\int_t^T \beta^*(\tau)q(\tau)d\tau + \lambda_0\int_0^r\int_t^T \frac{\partial Q}{\partial p}\, dsd\tau$$

$$+\int_0^{r_m} \int_t^T G'_p(p^*(s,\tau),\tau)dm(\tau)dr. \tag{59}$$

As before, we have

Lemma 3. The solution of equation (57) and the adjoint equation has the relation

$$\lambda_0\int_0^T\int_0^{r_m} \frac{\partial Q(p^*,\beta^*,t)}{\partial p} p(r,t)drdt - \int_0^{r_m} p(r,T)\alpha(r)dr$$

$$+\int_0^T\int_0^{r_m} G'_p(p^*(r,t),t)p(r,t)drdm(t)$$

$$= \int_0^T\left[\int_0^{r_m}\left[\lambda_0 \frac{\partial Q(p^*,\beta^*,t)}{\partial \beta} - q(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr\right]\beta(t)dt. \tag{60}$$

Same reason as before, whether or not the decreasing direction cone of J at (p^*,β^*) is empty and the system (57) is controllable, we always have

Theorem 2. (Maximum principle) Let (p^*,β^*) solves the problem

(Q), then there exist $\lambda_0 \geq 0$, $q(t)$ not both zero, such that

$$\int_0^{r_m}\left[\lambda_0 \frac{\partial Q(p^*,\beta^*,t)}{\partial\beta} - q(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr\right][\beta-\beta^*(t)] \geq 0,$$

$$\forall\ t\in[0,T] \text{ a.e.} \tag{61}$$

We can also consider the free final time problem with phase constraints

Problem (W): Minimize $\bar{J}(\beta,p)=\int_0^{t_1}\int_0^{r_m} W(p(r,t),\beta(t),t)drdt$

under the constraints:

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), & 0<r<r_m,\ t>0, \\ p(r,0)=p_0(r), & 0\leq r\leq r_m, \\ p(r,t_1)=p^0(r), & 0\leq r\leq r_m, \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr, & t\geq 0, \\ \beta(t)\in M, \quad \text{for } t\in [0,t_1] \text{ a.e.} & \\ \displaystyle\int_0^{r_m} G(p(r,t),t)dr\leq 0, & t\geq 0. \end{cases} \tag{62}$$

in the class of

$$(p(r,t),\beta(t))\in X=C(0,t_1;L^2(0,r_m))\times L^\infty(0,t_1) \tag{63}$$

The time t_1 is free.

Following the same lines of reasoning of section 5, we can prove

Theorem 3. Let (p^*,β^*,t_1) solve the problem (W), then there exist λ_0, $\alpha(r)\in L^2(0,r_m)$ with support on $S=\left\{ t\in [0,T] \,\middle|\, \int_0^{r_m} G(\hat{p}(r,t),t)dr= F(\hat{p})\right\}$ and a nonnegative measure $dm(t)$ such that

$$\int_0^{r_m} p^*(r,t)\hat{q}(r,t)+\beta^*(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)q(t)dr$$

$$+\int_0^{r_m}\mu(r)p^*(r,t)q(r,t)dr-\lambda_0\int_0^{r_m} W(p^*,\beta^*)dr=0, \quad \forall\ t\in [0,t_1]\ \text{a.e.} \qquad (64)$$

$$[\beta-\beta^*(t)]\left[-\lambda_0\int_0^{r_m}\left[\frac{\partial W(p^*,\beta^*)}{\partial\beta}+k(r)h(r)p^*(r,t)q(t)\right]dr\right]\geq 0,$$

$$\text{for all } \beta\in[\beta_0,\beta_1], \text{ and } t\in [0,t\ ]\ \text{a.e.} \qquad (65)$$

where

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r}+\dfrac{\partial q(r,t)}{\partial t}=\mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\lambda_0\dfrac{\partial W(p^*,\beta^*)}{\partial p}, \\ \qquad\qquad\qquad\qquad +G'_p(p^*(r,t),t)\dfrac{dm(t)}{dt}, \\ q(r,t_1)=\alpha(r), \\ q(0,t)=q(t) \end{cases} \qquad (66)$$

$$q(r,t)=\int_r^{r_m}\hat{q}(s,t)ds.$$

7.3 Mini-Max Problems

The min-max control problem of population control system can be stated as

$$\text{Problem (Y): Minimize } F(p)=\max_{0\leq t\leq t_1}\int_0^{r_m} G(p(r,t),t)drdt \qquad (67)$$

with respect to $(p(r,t),\beta(t))\in \mathbf{X}$ and t_1 under the constraints:

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t}+\dfrac{\partial p(r,t)}{\partial r}=-\mu(r)p(r,t), & 0<r<r_m, t>0, \\ p(r,0)=p_0(r), & 0\leq r\leq r_m, \\ p(r,t_1)=p^0(r), & 0\leq r\leq r_m, \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr, & t\geq 0, \\ \beta(t)\in M, \quad \text{for } t\in [0,t_1]\ \text{a.e.} \end{cases} \qquad (68)$$

We only state the results since the proof is similar.

Theorem 4. Let $\int_0^{r_m} G(p(r),t)dr$ be continuously differentiable with respect to $p(r)$, $\int_0^{r_m} G'_p(p(r),t)\neq 0$ when $G(p(r),t)\neq 0$. Let (p^*,β^*,t_1) solve problem (Y), then there exist $q(r,t)$, $\alpha(r)\in L^2(0,r_m)$ and nonnegative measure $dm(t)$ with support on the set

$$S=\left\{ t\in [0,t_1] \,\middle|\, \int_0^{r_m} G(p^*(r,t),t)dr= \max_{0\le t\le t_1} \int_0^{r_m} G(p^*(r,t),t)drdt \right.$$

such that

$$\int_0^{r_m} p^*(r,t)\hat{q}(r,t)+\beta^*(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)q(t)dr$$

$$+\int_0^{r_m} \mu(r)p^*(r,t)q(r,t)dr=0, \quad \forall\ t\in [0,t_1] \text{ a.e.} \tag{69}$$

$$[\beta-\beta^*(t)]\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)q(t)dr\ge 0,$$

$$\forall\ \beta\in M,\ t\in [0,t_1] \text{ a.e.} \tag{70}$$

where $q(r,t)$ is the solution of the adjoint equation

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r} + \dfrac{\partial q(r,t)}{\partial t} = \mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+ G'_p(p^*)\dfrac{dm(t)}{dt}, \\ q(r,t_1)=\alpha(r), \\ q(0,t)=q(t), \end{cases} \tag{71}$$

$$q(r,t)=\int_r^{r_m} \hat{q}(s,t)ds.$$

Reference

[1] I. V. Girsanov, Lectures on Mathematical Theory of Extrem Problem, Lecture Notes in Eco.Math.Sys., **67**, Springer-Verlag, 1972.

Chapter 8

Pareto Optimal Birth Control Policies

8.1 Introduction

It is well known that multicriteria optimization provides the mathematical framework to accommodate the demands of decision making when conflicting criteria are involved. In this chapter we shall study multicriteria optimization of birth control policies for age-structured population system of McKendrick type. It involves a distributed parameter system described by a first order partial differential equation with nonlocal bilinear boundary control. Single objective optimization i.e. optimal control problems concerning age-dependent population system have been discussed in chapter 6 and chapter 7 (see also [1]). Here we incorporate the multicriteria aspect in the optimization.

The multicriteria optimization problem of population control is to determine a weak Pareto minimum of the following criteria, i.e.

$$\text{W-Pareto minimize} \quad J_i(\beta,p)=\int_0^T\int_0^{r_m} L_i(p(r,t),\beta(t),r,t)\,drdt$$

$$i=1,2\ldots m, \tag{1}$$

for all (β,p) satisfying

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r.t)}{\partial r} = -\mu(r)p(r,t), \quad 0\le r\le r_m, \ t>0, \\ p(r,0)=p_0(r), \quad 0\le r\le r_m, \\ p(r,T)\in \Omega_T, \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr, \qquad t\ge 0, \\ \beta(t)\in \Omega = \{\beta(t) \mid 0\le\beta_0\le\beta(t)\le\beta_1,\ t\in[0,T] \text{ a.e}, \ \beta(t) \text{ measurable}\}, \end{cases} \tag{2}$$

where $p(r,t)$ denotes the age density distribution at time t and age r, $\mu(r)$ is the relative mortality of the population, r_m is the highest age even attained by individuals of the population, $k(r)$ is the female sex ratio at age r, $h(r)$ is the fertility pattern, $[r_1, r_2]$ is the fecundity period of females with

$$\int_{r_1}^{r_2} h(r)dr=1,$$

β is the specific fertility rate of females. The initial population density $p_0(r)$ is a nonnegative measurable function and the mortality $\mu(r)$ satisfies

$$\int_0^r \mu(\rho)d\rho<+\infty \text{ for } r<r_m \text{ and } \int_0^{r_m}\mu(\rho)d\rho=+\infty;$$

Ω_T is a convex subset of $L^2(0,r_m)$. Taking $\mathbf{X}=\mathbf{C}(0,T;L(0,r_m))\text{x}$ $L^\infty(0,T)$ as the state space, and for a given $\beta(t)\in \Omega$ the solution of (2) is considered to be the solution of the integral equation

$$p(r,t)=\begin{cases} p_0(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}, & r\ge t, \\ \beta(t-r)\displaystyle\int_{r_1}^{r_2} k(s)h(s)p(s,t-r)ds\, e^{-\int_0^r\mu(\rho)d\rho}, & r<t, \end{cases} \tag{3}$$

then $p(r,t)\in \mathbf{X}$ as long as $p_0(r)\in L^2(0,r_m)$, $\beta(t)\in L^\infty(0,T)$. We also assume that

(1) $\int_0^{r_m} L_i(p(r),\beta,r,t)dr$ are continuous on $(p(r),\beta)\in L^2(0,r_m)\times \mathbb{R}$,

(2) $\int_0^{r_m}\left|\frac{\partial L_i(p(r),\beta,r,t)}{\partial p}\right|dr$, $\int_0^{r_m}\left|\frac{\partial L_i(p(r),\beta,r,t)}{\partial \beta}\right|dr$ are bounded for all bounded $(p(r),\beta)\in L^2(0,r_m)\times\mathbb{R}$. In section 8.2 the analytic approach of Dubovitskii and Milyutin is adapted for the multicriteria optimization. In section 8.3 weak Pareto minimum principle for problem with target set and fixed finite horizon is developed. New results on problem with nonsmooth criteria are presented in section 8.4.

8.2 Dubovitskii-Milyutin Theorem [2]

Here, we first generalized the Dubovitskii-Milyutin theorem to handle vector minimization problem.

Theorem 1. Let the vector functional $F(x)=(f_1(x),f_2(x),\dots f_m(x))$ assume a weak Pareto minimum on $Q=\bigcap_{j=1}^{n+1}Q_j$ at a point $x_0\in Q$ (i.e. there is no $x\in Q$ such that $f_i(x)< f_i(x_0)$ for every $i=1,2,\dots m$). Assume that for any $1\le i\le m$, $f_i(x)$ is regularly decreasing at x_0, with directions of decrease K_i^0; the inequality constraints Q_j, $j=1, 2,\dots n$, are regular at x_0, with feasible directions K_j; the equality constraint Q_{n+1} is also regular at x_0, with tangent direction K_{n+1}. Then there exist continuous linear functionals g_i^0, $i=1,2,\dots m$, g_j, $j=1, 2,\dots n+1$, not all identically zero, such that $g_i^0\in K_i^{0*}$, $g_j\in K_j^*$, which satisfy the Euler-Langrange equation

$$g_1^0 + g_2^0 + \cdots + g_m^0 + g_1 + g_2 + \ldots + g_{n+1} = 0. \tag{4}$$

Proof. We shall first prove that a necessary condition for the vector functional to have a weak Pareto minimum at x_0 is $(\bigcap_{j=1}^{n+1} K_j) \cap (\bigcap_{i=1}^{m} K_i^0) = \emptyset$. Suppose that this is false, so that there exists $h \in K_i^0$, $i=1, 2, \ldots m$, $h \in K_j$, $j=1, 2, \ldots n+1$. By the definition of K_i^0, K_j, there exists a neighborhood U of the vector h such that, whenever $0<\varepsilon<\varepsilon_0$, any vector $x_0+ \varepsilon\bar{h}$, $\bar{h} \in U$, lies in $\bigcap_{j=1}^{n} Q_j$ and satisfies inequalities $f_i(x_0+\varepsilon\bar{h}) \le f_i(x_0)+\varepsilon\alpha$, here $\alpha_i=\alpha_i(f_i, x_0, h)$, $\alpha_i<0$. Now consider the vector $x(\varepsilon) = x_0+\varepsilon h+r(\varepsilon) \in Q_{n+1}$ as in the definition of tangent directions,and let ε_1 be such that $\frac{1}{\varepsilon}r(\varepsilon) \in U-h$, or $\bar{h}(\varepsilon)=h+\frac{1}{\varepsilon}r(\varepsilon) \in U$ for $0<\varepsilon<\varepsilon_1$.The vectors $x(\varepsilon)=x_0+\varepsilon\bar{h}$ are, on the one hand, in $\bigcap_{j=1}^{n} Q_j$, and on the other hand, in Q_{n+1}. In other words, these vectors satisfy all the constraints. But they also satisfy inequalities:

$$f_i(x_0+\varepsilon\bar{h}(\varepsilon)) \le f_i(x_0)+\varepsilon\alpha_i < f_i(x_0), \qquad i=1,2,\ldots m,$$

which contradicts the assumption that x_0 is a weak Pareto minimum point. Thus $(\bigcap_{j=1}^{n+1} K_j) \cap (\bigcap_{i=1}^{m} K_i^0) = \emptyset$. Now, by definition, K_i^0, K_j are convex open cones with vertex at 0, and K_{n+1} is a convex cone. Dubovitskii-Milyutin theorem is therefore applicable, and this implies the required result.

8.3 Weak Pareto Minimum Principle

Coming back to the problem (1), we consider the situation where

$$\Omega_T = \{ p_T(r) \}, \qquad p_T(r) \in L^2(0, r_m). \tag{5}$$

Define

$$Q_1=\{ (p(r,t),\beta(t))\in X \mid \beta(t)\in [\beta_0,\beta_1],\ t\in [0,T] \text{ a.e.} \},$$

$$Q_2=\{(p(r,t),\beta(t))\in X \mid p_t+p_r=-\mu p,\ p(0,t)=\beta(t)\int_{r_1}^{r_2}k(r)h(r)p(r,t)dr,$$

$$p(r,0)=p_0(r),\ p(r,T)=p_T(r)\}. \qquad (6)$$

$J(\beta,p)=(J_1(\beta,p),\dots J_m(\beta,p))$, then problem (1) can be state as

$$\text{W-Pareto minimize } J(\beta,p),$$
$$\text{subject to } (p,\beta)\in Q_1\cap Q_2. \qquad (7)$$

Let (β^*,p^*) solves problem (7), then we have

Theorem 2. There exist $\lambda_i^0\geq 0$, $i=1,2,\dots m$, $q(r,t)$, which satisfy the adjoint equation

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r}+\dfrac{\partial q(r,t)}{\partial t}=\mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\displaystyle\sum_{i=1}^m\lambda_i^0\frac{\partial L_i}{\partial p}, \\ q(r,T)=\alpha(r), \\ q(0,t)=q(t), \end{cases} \qquad (8)$$

and $\sum_{i=1}^m \lambda_i^0+\|\alpha(r)\|_{L^2(0,r_m)}\neq 0$ such that

$$\beta^*(t)H_\beta(\beta^*,p^*)=\max_{\beta_0\leq\beta\leq\beta_1}\beta H_\beta(\beta^*,p^*), \qquad \forall\ t\in[0,T] \text{ a.e.},$$

$$H(\beta,p)=q(t)\beta(t)\int_{r_1}^{r_2}k(r)h(r)p(r,t)dr-\beta(t)\sum_{i=1}^m\lambda_i^0\int_0^{r_m}L_i(p,\beta,r,t)dr,$$

$$H_\beta(\beta^*,p^*)=\frac{\partial H(\beta^*,p^*)}{\partial\beta}. \qquad (9)$$

Proof. We only notice the fact that under the assumptions on $J_i(\beta,p)$, $i=1,2,\dots m$, $J_i(\beta,p)$ is regularly decreasing at (p^*,β^*). Let K_i^0 be the directions of decreasing cone of J_i, then if $K_i^0 \neq \emptyset$ then for arbitrary $f_i^0 \in K_i^0$, there exists a $\lambda_i^0 \geq 0$ such that

$$f_i^0(p,\beta) = -\lambda_i^0 \int_0^T \int_0^{r_m} \left[\frac{\partial L_i(p^*,\beta^*,r,t)}{\partial p} p(r,t) + \frac{\partial L_i(p^*,\beta^*,r,t)}{\partial \beta} \beta(t) \right] drdt$$

$$i=1,2,\ldots m.$$

The determination of feasible direction cone of Q_1 and the tangent direction cone of Q_2 are the just the same as that of chapter 6. Proceeding the same lines as in chapter 6, we know that there exists a $f_1 \in K_1^*$, the dual of the feasible direction cone of Q_1 at (p^*,β^*), and $\alpha(r) \in L^2(0,r_m)$, $\sum_{i=1}^m \lambda_i^0 + \|\alpha(r)\|_{L^2(0,r_m)} \neq 0$ such that

$$f_1(p,\beta) = \sum_{i=1}^m \lambda_i^0 \int_0^T \int_0^{r_m} \left[\frac{\partial L_i(p^*,\beta^*,r,t)}{\partial p} p(r,t) + \frac{\partial L_i(p^*,\beta^*,r,t)}{\partial \beta} \beta(t) \right] drdt$$

$$-\int_0^{r_m} \alpha(r) p(r,T)$$

for all (β,p) satisfying the following controllable system

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r.t)}{\partial r} = -\mu(r)p(r,t), \quad 0 \leq r \leq r_m, \ t>0, \\ p(r,0)=0, \quad 0 \leq r \leq r_m, \\ p(0,t) = \beta^*(t)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr + \beta(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr, \quad t \geq 0. \end{cases}$$

Let $q(r,t)$ be the solution of (8), then (see chapter 6 Lemma 3) the following relation holds

$$\sum_{i=1}^m \lambda_i^0 \int_0^T \int_0^{r_m} \frac{\partial L_i(p^*,\beta^*,r,t)}{\partial p} p(r,t)drdt - \int_0^{r_m} \alpha(r)p(r,T)dr$$

$$= -\int_0^T q(t)\beta(t) \int_{r_1}^{r_2} k(r)h(r)p^*(r,t)drdt.$$

Hence we have

$$f_1(p,\beta)=\int_0^T\Big[\sum_{i=1}^m \lambda_i^0\int_0^{r_m}\frac{\partial L_i(p^*,\beta^*,r,t)}{\partial\beta}dr$$

$$-q(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr\Big]\beta(t)dt.$$

Since $f_1\in K_1^*$, it follows from [2] that

$$\Big[\sum_{i=1}^m \lambda_i^0\int_0^{r_m}\frac{\partial L_i(p^*,\beta^*,r,t)}{\partial\beta}dr -q(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr\Big]\cdot$$

$$\cdot[\beta-\beta^*(t)\geq 0,\ \forall\ t\in [0.T]\text{ a.e.}$$

This leads to (9). Following the arguments in chapter 6, we can prove that the minimum principle (9) always hold whether or not $K_i^0\neq\varnothing$ and the linearized system is controllable. The proof of the theorem is thus completed.

Next, for the situation where

$$\Omega_T \text{ is a convex set with } \mathring{\Omega}_T \neq \varnothing, \tag{10}$$

similar to the optimal control case, we can shown that the minimum principle (9) still holds, and the $\alpha(r)$ in the adjoint equation (8), is a supporting functional of convex set Ω_T at $p^*(r,T)$, i.e.

$$\int_0^{r_m}\alpha(r)p(r)dr \geq \int_0^{r_m}\alpha(r)p^*(r,T)dr, \quad \forall\ p(r)\in \Omega_T, \tag{11}$$

and at the same time, we can choose $\lambda_i^0\geq 0$ such that

$$\sum_{i=1}^m \lambda_i^0 = 1. \tag{12}$$

8.4 Problems with nonsmooth criteria

In general, the assumptions (1) and (2) for $L_i(p(\cdot),\beta)$, $i=1,2,\ldots m$, may not hold. We are interested in the case that J_i satisfies the Lipschitz condition

$$|J_i(\beta,p)-J_i(\hat{\beta},\hat{p})| \le M\ [\|p-\hat{p}\|+\|\beta-\hat{\beta}\|].$$

Generally, a real-valued function F on a Banach space **X** is said to be locally Lipschitz if any point in **X** admits a neighborhood U such that, for some constant K, for all y and z in U, we have

$$|f(z)-f(y)| \le K\|z-y\|.$$

For a locally Lipschitz function F and for x_0, x in **X**, $F^0(x_0;x)$ defined as follows is said to be the generalized directional derivative in the direction x at x_0:

$$F^0(x_0;x) = \overline{\lim_{\substack{t\to 0\\ h\searrow 0}}}\ \frac{F(x_0+h+tx)-F(x_0)}{t},$$

we denote by $\partial^* F(x_0)$ the subdifferential of the convex continuous function $F^0(x_0;\cdot)$ at 0, that is

$$\partial^* F(x_0)=\{\phi \in \mathbf{X}^* \,|\, F^0(x_0;x)\ge \phi(x),\ \forall\ x\in \mathbf{X}\}.$$

We call it the Clarke [3] gradient of F at x_0.

Lemma 3. [4] Let F: $\mathbf{X}\longrightarrow \mathbb{R}^1$ be a locally Lipschitz function and $x_0\in \mathbf{X}$. If $F^0(x_0;\bar{x})<0$, then $\bar{x}$ lie in the direction cone of decrease of F at x_0.

Lemma 4. [4] Let F: $\mathbf{X}\longrightarrow \mathbb{R}^1$ be a locally Lipschitz function and $x_0\in \mathbf{X}$. Define $C=\{x\in \mathbf{X}\,|\,F^0(x_0;x)<0\}$ and assume $0\notin \partial^* F(x_0)$, then $C^*=\{\lambda y\,|\,\lambda\ge 0,\ y\in \partial^* F(x_0)\}$.

By Lemma 3 and Lemma 4, we have

Proposition 1. Let the vector functional $F(x)=(f_1(x),\ldots f_m(x))$ assume a weak Pareto minimum on $Q = \bigcap_{j=1}^{n+1} Q_j$ at a point $x_0 \in Q$. Assume that for any $1 \le i \le m$, $f_i(x)$ are locally Lipschitz function $0 \notin \partial^* f_i(x_i)$. Let $C_i^* = \{\lambda g_i^0 \mid \lambda \ge 0,\ g_i^0 \in \partial^* f_i(x_0)\}$, then there exist $\lambda_i^0 \ge 0$, $i=1,2,\ldots m$, $g_i^0 \in \partial^* f_i(x_0)$, $g_j \in K_j^*$, the cone defined on theorem, such that

$$\lambda_1 g_1^0 + \lambda_2 g_2^0 + \ldots + \lambda_m g_m^0 + g_1 + g_2 + \ldots + g_{n+1} = 0, \tag{13}$$

where $\{\lambda_i g_i^0\}$, $\{g_j\}$, $i=1,2,\ldots m$, $j=1,2, \ldots n+1$ are not identically zero.

As application of these results to the population control system we consider the following problem:

$$\begin{aligned} &\text{W-Pareto minimize } J_i(\beta,p)=G_i(p(r,T)),\ i=1,2,\ldots m, \\ &\text{subject to (2)}, \end{aligned} \tag{14}$$

where G_i is a functional on $L^2(0,r_m)$, which satisfies the locally Lipschitz condition:

$$|G_i(p(r))-G_i(\hat{p}(r))| \le K_i \| p(r)-\hat{p}(r) \|, \tag{15}$$

$$\forall\ p^0(r) \in L^2(0,r_m),\ p,\ \hat{p} \in U_i,\ \text{some neighborhood of } p^0.$$

By (15), J_i is also locally Lipschitz, and by virtue of the definition

$$J_i^0(\beta^*,p^*;\beta,p)=G_i^0(p^*(r,T);p(r,T)), \quad \text{for all } (p,\beta) \in \mathbf{X}.$$

For any $\xi^* \in \partial^* J_i^0(\beta^*,p^*)$, then for all $(p,\beta) \in \mathbf{X}$,

$$\xi^*(p,\beta) \le J_i^0(\beta^*,p^*;\beta,p)=G_i^0(p^*(r,T);p(r,T)).$$

In particular taking $p(r,t)$ with $p(r,T)=0$, then $\xi^*(p,\beta)\le 0$, and hence $\xi^*(p,\beta)\le 0$. Since $p(r,t) = p(r,T) + (p(r,t) - p(r,T))$, so $\xi^*(p,\beta)=\xi^*(p(r,T),\beta)$. This says that

$$\partial J_i^0(\beta^*,p^*)=\partial G_i^0(p^*(r,T)). \tag{16}$$

For any $g^*\in \partial G_i^0(p^*(r,T))$, there exists $\alpha(r)\in L^2(0,r_m)$ such that

$$g^*(p)=\int_0^{r_m}\alpha(r)p(r)dr, \qquad \text{for all } p(r)\in L^2(0,r_m).$$

At the same time we see that $\alpha(r)\in \partial G_i^0(p^*(r,T))$. Proceeding same as in problem (1), we can deduce the necessary conditions for (β^*,p^*) to be a weak Pareto optimal solution of problem (14) as follows

Theorem 4. Let (β^*,p^*) be the solution of problem (14), suppose $0\notin \partial G_i^0(p^*(r,T))$, $i=1,2,...m$, then there exist $\lambda_i^0\ge 0, \alpha(r)\in N_{\Omega_T}$, N_{Ω_T} is the normal cone of Ω_T at $p^*(r,T)$ and $\alpha_i(r)\in L^2(0,r_m)$, $i=1,2,...m$, such that

$$\beta^*(t)H(t)= \max_{\beta_0\le\beta\le\beta_1} \beta H(t), \qquad \forall\ t\in [0,T] \text{ a.e.}$$

where

$$H(t)=q(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr,$$

$q(t)$ satisfies adjoint equation

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r}+\dfrac{\partial q(r,t)}{\partial t}= \mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t), \\ q(r,T)= -\sum\limits_{i=1}^{m}\lambda_i^0\alpha_i(r)+\alpha(r), \\ q(0,t)=q(t), \end{cases} \tag{17}$$

$\lambda_i^0\alpha_i(r)$, $\alpha(r)$, $q(t)$ can not be identically zero.

Corollary. If m=1, and p(r,T) is free, then (17) becomes

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r} + \dfrac{\partial q(r,t)}{\partial t} = \mu(r)q(r,t) - \beta^*(t)k(r)h(r)q(t), \\ q(r,T) \in -\partial G(p^*(r,T)), \\ q(0,t) = q(t). \end{cases}$$

Next, we come back to the problem (1), but now we assume that for all $1 \le i \le m$, there exist functions $M_i(r,t) \in \mathbf{X}$ such that

$$|L_i(p,\beta,r,t) - L_i(\hat{p},\hat{\beta},r,t)| \le M_i(r,t)[\,|p-\hat{p}| + |\beta-\hat{\beta}|\,]$$

$$\text{for all } p,\beta,\hat{p},\hat{\beta} \in \mathbb{R}^1,\ (r,t) \in [0,r_m]\times[0,T], \tag{18}$$

Under these assumptions $J_i(\beta,p)$, $i=1,2,\dots m$ satisfy the global Lipschitz condition

$$|J_i(p,\beta) - J_i(\hat{p},\hat{\beta})| \le K_i[\,|p-\hat{p}| + |\beta-\hat{\beta}|\,]. \tag{19}$$

By definition

$$J_i^0(\beta^*,p^*;\beta,p) = \overline{\lim_{\substack{(\hat{p},\hat{\beta})\to(p^*,\beta^*)\\ \lambda\searrow 0}}} \frac{J_i(\hat{\beta}+\lambda\beta,\hat{p}+\lambda p) - J_i(\hat{\beta},\hat{p})}{\lambda}$$

$$= \overline{\lim_{\substack{(\hat{p},\hat{\beta})\to(p^*,\beta^*)\\ \lambda\searrow 0}}} \int_0^T\!\!\int_0^{r_m} \frac{L_i(\hat{p}+\lambda p,\hat{\beta}+\lambda\beta,r,t) - L_i(\hat{p},\hat{\beta})}{\lambda}\,drdt.$$

It is obvious that

$$J_i^0(\beta^*,p^*;\beta,p) \le$$

$$\int_0^T\!\!\int_0^{r_m}\Big[L_{ip}^0(p^*(r,t);p(r,t),\beta^*(t)) + L_{i\beta}(p^*(r,t),\beta^*(t);\beta(t))\Big]drdt.$$

Following arguments in [3], we can show that

$$\partial J_i(p^*,\beta^*) \subset \int_0^T\int_0^{r_m}\left[\partial^* L_{ip}(p^*,\beta^*)+ \partial L^*_{i\beta}(p^*,\beta^*)\right]drdt, \qquad (20)$$

and hence we have the following

Theorem 4. Suppose the condition (18) is satisfied, then if (p^*,β^*) solves problem (1), then there exist $\lambda_i \geq 0$, $i=1,2,\dots m$, $\Theta_{ip}(r,t)\in \partial L_{ip}(p^*(r,t),\beta^*(t))$, $\Theta_{i\beta}\in \partial L_{i\beta}(p^*(r,t),\beta^*(t))$, $\alpha(r)\in L^2(0,r_m)$ such that

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r} + \dfrac{\partial q(r,t)}{\partial t} = \mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\sum\limits_{i=1}^{m}\lambda_i^0\Theta_{ip}(r,t), \\ q(r,T)=\alpha(r), \\ q(0,t)=q(t), \end{cases}$$

and

$$\beta^*(t)H_\beta(\beta^*,p^*)= \max_{\beta_0\le\beta\le\beta_1} \beta H_\beta(\beta^*,p^*), \qquad \forall\ t\in [0,T] \text{ a.e.}$$

$$H_\beta(\beta^*,p^*)=q(t)\int_{r_1}^{r_2}k(r)h(r)p^*(r,t)dr - \sum_{i=1}^{m}\lambda_i^0\int_0^{r_m}\Theta_{i\beta}(r,t)dr. \qquad (21)$$

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Chapter 9

Overtaking Optimal Control Problems with Infinite Horizon

9.1 Introduction

The problem of controlling and managing age-dependent biological population has been studied in an optimal control setting in chapter 6 and chapter 7 with a finite or infinite time horizon and various terminal conditions (see also [1], [2]). The aim of this chapter is to study conditions under which the optimal birth control over an infinite time horizon of the McKendrick model has a stabilizing effect. As opposed to chapter 6, here, we do not a priori assume that the cost functional, an improper integral, converges. This leads us to consider a weaker type of optimality, known as the overtaking optimality. Such a concept has a long history in the economic and operation research literature. It is hoped that our study will lead to a proper understanding of the open-endedness of the future in age-dependent population management.

Recently in [3], the overtaking optimal control of an infinite dimensional linear control system with unbounded time interval has been considered. However, the results there cannot be applied directly to our situation since the McKendrick model involves a bilinear (nonlinear) boundary birth control of a distributed system described by a first order partial differential equation. We are, in fact, extending some of the results of [2] to a nonlinear case.

The chapter is organized as follows. In section 9.2 the optimal birth control problem is formulated. In section 9.3 the minimum principle which must be satisfied by the overtaking optimal control is established via an associated finite horizon optimal control problem. Section 9.4 deals with the large time behaviour of the overtaking optimal trajectory, i.e., the turnpike property. Generally speaking, this property says that an optimal trajectory on any finite horizon will stay most of the time in the vicinity of an extremal steady state and will ultimately converge to it if the time interval becomes unbounded. Finally in section 9.5, some existence results for overtaking optimal control are presented.

9.2 Problem Statement

we consider the population evolution system described by the following first order partial differential equation with boundary control

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), & 0<r<r_m, t>0, \\ p(r,0)=p_0(r), & 0\le r\le r_m, \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr, & t\ge 0 \end{cases} \tag{1}$$

in which p(r,t) is the population density, r denotes age, t represents time; r_m is the maximum age; $\beta(t)$, the control variable, is the specific fertility rate of females at time t; k(r) and h(r) denote, respectively, the female ratio and the fertility pattern; $[r_1, r_2]$ is the fertility interval with $\int_{r_1}^{r_2} h(r)dr=1$. The initial population density $p_0(r)$ is a nonnegative function and the mortality rate $\mu(r)$ satisfies

$$\int_0^r \mu(\rho)d\rho < +\infty \text{ for } r< r_m \text{ and } \int_0^{r_m} \mu(\rho)d\rho = +\infty.$$

Assume that the population parameters in equation (1) are nonnegative and measurable functions. Furthermore, let β, h, and k be bounded functions whose values outside their domain of definitions are zero.

As in chapter 6, we consider the solution of equation (1) to be

$$p(r,t)=\begin{cases} p_0(r-t)e^{-\int_{r-t}^{r}\mu(\rho)d\rho}, & r\geq t, \\ \beta(t-r)\int_{r_1}^{r_2} k(s)h(s)p(s,t-r)ds\,e^{-\int_0^r \mu(\rho)d\rho}, & r<t. \end{cases} \tag{2}$$

Then for an arbitrary $p_0(r)\in L^2(0,r_m)$, equation (2) in $L^2(0,r_m)$ has a unique solution $p(r,t)\in C([0,\infty);L^2(0,r_m))$. Unless otherwise stated, in what follows when we speak of solution of equation (1) we shall mean the weak solution (2).

Consider now the optimal control problem. The performance of the system on any interval [0,t] is evaluated by the cost functional

$$J(\beta,p,t) = \int_0^t\int_0^{r_m} L(p(r,t),\beta(t))drdt \tag{3}$$

where L: $L^2(0,r_m)\times[0,\infty)\longrightarrow L^2(0,r_m)$ is a continuously differential function. We call $\mathcal{A}(p_0)$ the set of pairs (β,p) which satisfy

(1) $\beta(\cdot)\in U_{ad}=\{\beta(t)\mid 0\le\beta_0\le\beta(t)\le\beta_1,\ t\in[0,\infty)$ a.e., $\beta(t)$ is measurable on $[0,\infty)\}$.

(2) $p(\cdot,\cdot)$ is given by (2).

Then $\beta(\cdot)$ is called an admissible control at p_0, and $p(\cdot,\cdot)$ is the associated trajectory.

In this chapter, we consider our problem on an infinite horizon, and we do not a priori assume the convergence of (3) as $t\longrightarrow\infty$. Hence we need to consider the following weaker notions of optimality.

Definition 1. $(\beta^*,p^*)\in\mathcal{A}(p_0)$ is overtaking optimal at p_0 if for any other pair $(\beta,p)\in\mathcal{A}(p_0)$

$$\varliminf_{t\to\infty}[J(\beta,p,t)-J(\beta^*,p^*,t)]\ge 0. \tag{4}$$

In other words, for every $(p,\beta)\in\mathcal{A}(p_0)$, any fixed $T>0$, and every $\varepsilon>0$, there exists t with $t\ge T$ such that

$$J(\beta^*,p^*,t)\le J(\beta,p,t)+\varepsilon. \tag{5}$$

For any fixed T and an overtaking optimal control pair (β^*,p^*), define the finite horizon optimal control problem:

Minimize $J(\beta,p,T)$,

subject to

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), \qquad 0<r<r_m, t>0, \\ p(r,0)=p_0(r), \qquad 0\le r\le r_m, \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr, \qquad t\ge 0, \\ p(r,T)=p^*(r,T), \beta(\cdot)\in U_{ad}. \end{cases} \tag{6}$$

For notational convenience, we denote the infinite horizon problem the IHP problem, and the associate finite horizon problem the FHP problem. First, we have the following apparent result:

Proposition 1. If (β^*, p^*) is optimal for IHP, then it is optimal for FHP.

Proof. If $(\beta^*, p^*)$is not FHP optimal for IHP, then for some $(\hat{\beta}, \hat{p})$ satisfying (6), $\hat{\beta}(\cdot)\in U_{ad}$, and some $\varepsilon > 0$ we have

$$\int_0^T\int_0^{r_m} L(\hat{p}(r,t),\hat{\beta}(t))drdt < \int_0^T\int_0^{r_m} L(p^*(r,t),\beta^*(t))drdt-\varepsilon.$$

Let (β, p) be defined by

$$\begin{aligned}(\beta(t),p(r,t)) &= (\beta^*(t),p^*(r,t)) \quad \text{for all } t\in (T,\infty), \\ &= (\hat{\beta}(t),\hat{p}(r,t)) \quad \text{for all } t\in [0,T,],.\end{aligned}$$

We then have $(\beta, p)\in \mathcal{A}(p_0)$ and

$$\int_0^t\int_0^{r_m} L(\hat{p}(r,t),\hat{\beta}(t))drdt < \int_0^t\int_0^{r_m} L(p^*(r,t),\beta^*(t))drdt-\varepsilon$$

for all $t\ge T$. This last statement contradicts the optimality of (β^*, p^*). This concludes the proof of the proposition.

We proved minimum principle for FHP problem in chapter 6

Theorem 1. Let (β^*, p^*) be the solution of FHP, then there exist $\lambda_{0T} \geq 0$, $\alpha_T(r) \in L^2(0, r_m)$, not both zero, such that the following minimum principle holds

$$\beta^*(t) H_\beta(\beta^*, p^*) = \max_{\beta_0 \leq \beta \leq \beta_1} \beta H_\beta(\beta, p^*), \qquad \forall\, t \in [0, T] \text{ a.e.}, \tag{7}$$

where

$$H(\beta, p) = q_T(t)\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr - \lambda_{0T} L(p, \beta),$$

$$H_\beta(\beta^*, p^*) = \frac{\partial H(\beta^*, p^*)}{\partial \beta},$$

$q_T(t)$ is the solution of the adjoined equation

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r} + \dfrac{\partial q(r,t)}{\partial t} = \mu(r)q(r,t) - \beta^*(t)k(r)h(r)q(t) + \lambda_{0T}\dfrac{\partial L(p^*, \beta^*)}{\partial p}, \\ q(r,T) = \alpha_T(r), \\ q(0,t) = q_T(t). \end{cases} \tag{8}$$

As with equation (1), we call solutions (weak solutions) of equation (8) to be the solutions of

$$q_T(t) = e^{-\int_0^{T-t}\mu(\rho)d\rho}\alpha_T(T-t) + \int_t^T e^{-\int_0^{s-t}\mu(\rho)d\rho}\beta^*(s)k(s-t)h(s-t)q_T(s)ds$$
$$-\lambda_{0T}\int_t^T e^{-\int_0^{s-t}\mu(\rho)d\rho}\left.\frac{\partial L(p^*, \beta^*)}{\partial p}\right|_{(s-t,s)} ds,$$

$$q(r,t) = e^{-\int_r^{r+T-t}\mu(\rho)d\rho}\alpha_T(r+T-t)$$
$$+\int_t^T e^{-\int_r^{r+s-t}\mu(\rho)d\rho}\beta^*(s)k(r+s-t)h(r+s-t)q_T(s)ds$$
$$-\lambda_{0T}\int_t^T e^{-\int_r^{r+s-t}\mu(\rho)d\rho}\left.\frac{\partial L(p^*, \beta^*)}{\partial p}\right|_{(r+s-t,\ s)} ds,$$

$$0 \le t \le T,\ 0 \le r \le r_m. \qquad (9)$$

Proposition 1 tells us that if (β^*, p^*) is optimal for IHP, then it must satisfy the minimum principle (7) on [0,T]. (7) is equivalent to

$$\left[q_T(t)\int_{r_1}^{r_2} k(r)h(r)p^*(r,t)dr - \int_0^{r_m} \frac{\partial L(p^*,\beta^*)}{\partial p}\bigg|_{(r,t)} dr\right]$$

$$\cdot[\beta-\beta^*(t)] \le 0, \quad \forall\ \beta \in [\beta_0, \beta_1],\ t \in [0,T] \text{ a.e.} \qquad (10)$$

Since λ_{OT}, $\alpha_T(r)$ can not vanish simultaneously, so we may assume that $\|(\lambda_{OT}, p_{T_i}(r,0))\|$, as $T_i \longrightarrow \infty$ to be a monotone increasing series, such that $\lambda_{OT_i} \longrightarrow \lambda_\infty$ and $p_{T_i}(r) \longrightarrow \alpha(r)$ (in the weak sense). By (9), it can be shown easily that

$$q_{T_i} \longrightarrow q(t),$$

$$q(t) = \int_0^{t+r_m} e^{-\int_0^{s-t} \mu(\rho)d\rho}\, \beta^*(s)k(s-t)h(s-t)q(s)ds$$

$$-\lambda_\infty \int_t^{t+r_m} e^{-\int_0^{s-t} \mu(\rho)d\rho}\, \frac{\partial L(p^*,\beta^*)}{\partial p}\bigg|_{(s-t,s)} ds$$

$$q(r,t) = \int_t^{t+r_m-r} e^{-\int_r^{r+s-t} \mu(\rho)d\rho}\, \beta^*(s)k(r+s-t)h(r+s-t)q(s)ds$$

$$-\lambda_\infty \int_t^{t+r_m-r} e^{-\int_r^{r+s-t} \mu(\rho)d\rho}\, \frac{\partial L(p^*,\beta^*)}{\partial p}\bigg|_{(r+s-t,\ s)} ds$$

$$(11)$$

Under the assumption that

Assumption 1.

$$\int_0^{r_m} e^{-\int_0^r \mu(\rho)d\rho} \left|\frac{\partial L(p^*,\beta^*)}{\partial p}\right|_{(r,t)} dr < \infty, \qquad \forall\ t\in [0,\infty) \text{ a.e.}, \qquad (12)$$

equation (11) has a unique solution and $q(r,t)$ is the mild solution of adjoint system

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r} + \dfrac{\partial q(r,t)}{\partial t} = \mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t) + \lambda_\infty \dfrac{\partial L(p^*,\beta^*)}{\partial p} \\ q(0,t)=q(t). \end{cases} \qquad (13)$$

Furthermore, if we assume

Assumption 2.

$$\lim_{t\to\infty} \int_0^{r_m} e^{-\int_0^r \mu(\rho)d\rho} \left|\frac{\partial L(p^*,\beta^*)}{\partial p}\right|_{(r,t)} dr = 0. \qquad (14)$$

then there is a transversality condition

$$q(r,\infty)= 0. \qquad (15)$$

Theorem 2. (minimum principle) Under assumptions 1 and 2, the overtaking optimal control (β^*,p^*) satisfies

$$\beta^*(t)H_\beta(\beta^*,p^*)= \max_{\beta_0\le\beta\le\beta_1} \beta H_\beta(\beta^*, p^*), \qquad \forall\ t\in [0,\infty] \text{ a.e.},$$

where

$$H(\beta,p)=q(t)\beta(t)\int_{r_1}^{r_2} k(r)h(r)p(r,t)dr-\lambda_\infty L(p,\beta),$$

$$H_\beta(\beta^*,p^*)= \frac{\partial H(\beta^*,p^*)}{\partial \beta},$$

and $q(t)$ is the solution of the adjoint equation

$$\begin{cases} \dfrac{\partial q(r,t)}{\partial r} + \dfrac{\partial q(r,t)}{\partial t} = \mu(r)q(r,t)-\beta^*(t)k(r)h(r)q(t)+\lambda_\infty \dfrac{\partial L(p^*,\beta^*)}{\partial p}, \\ q(r,\infty)=0, \\ q(0,t)= q(t), \end{cases}$$

where $\lambda_\infty \geq 0$, $q(t)$ are not both zero.

9.3 The Turnpike Property

In this section we investigate the asymptotic convergence properties of overtaking optimal trajectories. In the literature these are the so-called turnpike properties. We assume the following:

Assumption 3. $L(p(\cdot),\beta)$ satisfies the following growth condition: there exist $K_1>0$ and $K>0$ such that

$$\|p(r)\|^2+\beta^2 > K_1 \Rightarrow \int_0^{r_m} L(p(r),\beta)dr \geq K(\|p(r)\|^2+\beta^2) \tag{16}$$

and $L(p(\cdot),\beta)$ is convex on $L^2(0,r_m)\times[\beta_0,\beta_1]$.

Assumption 4. There is an unique constant $\bar{c}\geq 0$, $\beta_0\leq\bar{\beta}\leq\beta_1$ such that

$$\int_0^{r_m} L(\bar{c}e^{-\int_0^r \mu(\rho)d\rho},\bar{\beta})dr = \min_{\substack{c\geq 0 \\ \beta_0\leq\beta\leq\beta_1}} \int_0^{r_m} L(ce^{-\int_0^r \mu(\rho)d\rho},\beta)dr. \tag{17}$$

We can now establish the weak turnpike theorem

Theorem 3. Under Assumptions 3 and 4 if $(\tilde{p}(r,t),\tilde{\beta}(t))\in\mathcal{A}(p_0)$ is such that

$$\overline{\lim_{T\to\infty}}\int_0^T\int_0^{r_m}[L(\tilde p(r,t),\tilde\beta(t))-L(\hat p(r),\bar\beta)]drdt=\alpha<\infty, \tag{18}$$

then necessarily

$$\frac{1}{T}\int_0^{r_m}\tilde p(r,t)dr\longrightarrow \hat p(r),\quad \frac{1}{T}\int_0^T\tilde\beta(t)dt\longrightarrow\bar\beta \tag{19}$$

where $\hat p(r)=e^{-\int_0^r\mu(\rho)d\rho}$.

Proof. First we show that there exists a constant $\bar M>0$ such that

$$\int_0^{r_m}\tilde p(r,t)dr\le\bar M,\quad \forall\ t\ge 0. \tag{20}$$

In fact, by (2) for $T>r_m$

$$\int_0^{r_m}\tilde p(r,T)dr=\int_0^{r_m}\tilde\beta(T-r)\int_{r_1}^{r_2}k(s)h(s)\tilde p(s,T-r)dse^{-\int_0^r\mu(\rho)d\rho}dr$$

$$=\int_{t-r_m}^T\tilde\beta(t)\int_{r_1}^{r_2}k(r)h(r)\tilde p(r,t)dre^{-\int_0^{T-t}\mu(\rho)d\rho}dt$$

$$=\int_{T-r_m}^T\tilde\beta(t)e^{-\int_0^{T-t}\mu(\rho)d\rho}\int_0^{r_m}k(r)h(r)\tilde p(r,t)drdt$$

$$\le M\int_{T-r_m}^T\int_0^{r_m}\tilde p(r,t)drdt$$

where M is a constant. If $T_k\longrightarrow\infty$ such that $\int_0^{r_m}\tilde p(r,T_k)dr\longrightarrow\infty$, then the above expression says that

$$\int_{T_k-r_m}^{T_k}\int_0^{r_m}\tilde p(r,t)drdt\longrightarrow+\infty\quad\text{as } k\longrightarrow\infty. \tag{21}$$

Using Jensen's inequality on L

$$\frac{1}{r_m}\int_{T_k-r_m}^{T_k}\int_0^{r_m}L(\tilde p(r,t),\tilde\beta(t))drdt$$

$$\geq\int_0^{r_m} L\Big(\frac{1}{r_m}\int_{T_k-r_m}^{T_k}\tilde{p}(r,t)dt,\ \frac{1}{r_m}\int_{T_k-r_m}^{T_k}\tilde{\beta}(t)dt\Big)dr$$

$$\geq K\Big[\ \Big\|\frac{1}{r_m}\int_{T_k-r_m}^{T_k}\tilde{p}(r,t)dt\Big\|^2\Big] \longrightarrow \infty,\quad \text{as } k\longrightarrow\infty,$$

i.e., $\lim\limits_{k\to\infty}\int_{T_k-r_m}^{T_k}\int_0^{r_m} L(\tilde{p}(r,t),\tilde{\beta}(t))drdt=+\infty$. This contradicts (18), and hence (20) holds.

Secondly, we show that there exists a constant $\hat{M}$ such that

$$\Big\|\frac{1}{T}\int_0^T\tilde{p}(r,t)dt\Big\| \leq \hat{M}\ ,\quad \forall\ T>0. \tag{22}$$

Suppose the contrary, that there exists a sequence $\{T_k\}$, $T_k\longrightarrow\infty$ such that

$$\Big\|\frac{1}{T}\int_0^{T_k}\tilde{p}(r,t)dt\Big\| \longrightarrow +\infty,\quad \text{as } t\longrightarrow\infty.$$

Using Jensen's inequality again on L

$$\frac{1}{T_k}\int_0^{T_k}\int_0^{r_m} L(\tilde{p}(r,t),\tilde{\beta}(t))drdt$$

$$\geq\int_0^{r_m} L\Big(\frac{1}{T_k}\int_0^{T_k}\tilde{p}(r,t)dt,\ \frac{1}{T_k}\int_0^{T_k}\tilde{\beta}(t)dt\Big)dr$$

$$\geq K\Big[\ \Big\|\frac{1}{T_k}\int_0^{T_k}\tilde{p}(r,t)dt\Big\|^2\Big],$$

we obtain

$$K\Big[\Big\|\frac{1}{T_k}\int_0^{T_k}\tilde{p}(r,t)dt\Big\|^2\Big]-\int_0^{r_m} L(\hat{p}(r),\beta_{cr})dr$$

$$\leq\frac{1}{T_k}\int_0^{T_k}\Big[\int_0^{r_m} L(\tilde{p}(r,t),\tilde{\beta}(t))-\int_0^{r_m} L(\hat{p}(r),\beta_{cr})\Big]drdt.$$

This contradicts (18) and so (22) holds.

Finally, for every $z(r)\in C^1(0,r_m)$, $z(r)=0$ on (r_c,r_m), for some

$r_c<r_m$, it can be shown that

$$<\tilde{p}(r,t)-p_0(r),z(r)>$$

$$=\int_0^t\tilde{p}(0,\tau)d\tau z(0)-\int_0^t<\mu(r)\tilde{p}(r,\tau),z(r)>d\tau+\int_0^t<\tilde{p}(r,\tau),z'(r)>d\tau \qquad (23)$$

so

$$\frac{1}{T}<\tilde{p}(r,t)-p_0(r),z(r)>$$

$$=\frac{1}{T}\int_0^T\tilde{\beta}(t)\int_{r_1}^{r_2}k(r)h(r)\tilde{p}(r,t)drdt\cdot z(0)$$

$$-\frac{1}{T}\int_0^T<\mu(r)\tilde{p}(r,\tau),z(r)>d\tau + \frac{1}{T}\int_0^T<\tilde{p}(r,\tau),z'(r)>d\tau \qquad (24)$$

Suppose $(p^*(r),\beta^*)$ is a weak cluster point of the set

$$\left\{ (\frac{1}{T}\int_0^T\tilde{p}(r,t)dt,\frac{1}{T}\int_0^T\tilde{\beta}(t)dt)\right\}.$$

When T goes to infinite in (24), we have

$$<\mu(r)p^*(r),z(r)> +<p^*(r),z'(r)>=0$$

for all $z(r)\in C_0^1(0,r)$. So $p^*(r)=ce^{-\int_0^r\mu(\rho)d\rho}$, $c\geq0$. By (18); Jensen's inequality and the continuity of $\int_0^{r_m}L(p(r),\beta)dr$, we see that

$$\int_0^{r_m}L(p^*(r),\beta^*)dr\leq\int_0^{r_m}L(\bar{p}(r),\bar{\beta})dr.$$

Therefore, by the uniqueness of $(\bar{p},\bar{\beta})$, we have

$$p^*(r)=\bar{p}(r),\quad \beta^*=\bar{\beta},$$

and this completes the proof.

Define the operator $\mathbf{A}:L^2(0,r_m)\longrightarrow L^2(0,r_m)$ by

$$\mathbf{A}\phi(r)=\phi'(r)+\mu(r)\phi(r),$$

$$D(\mathbf{A})=\{\phi(r)\,|\,\phi(r),\mathbf{A}\phi(r)\in L^2(0,r_m)\}, \qquad (25)$$

then it follows that

$$\mathbf{A}^*\psi(r)=-\psi'(r)+\mu(r)\psi(r),$$

$$D(\mathbf{A}^*)=\{\psi(r)\,|\,\psi(r),\mathbf{A}^*\psi(r)\in L^2(0,r_m)\}. \qquad (26)$$

By the assumptions already made on L, we know that there exists a $\psi(r)\in D(\mathbf{A}^*)$ such that

$$\int_0^{r_m} L(\bar{p}(r),\bar{\beta})dr\le \int_0^{r_m} L(p(r),\beta)dr-\langle p(r),\mathbf{A}^*\psi(r)\rangle,$$

$$\text{for all } p(r)\ge 0,\ \beta\in[\beta_0,\beta_1]. \qquad (27)$$

Let $L_0(p(\cdot),\beta):L^2(0,r_m)\times\mathbb{R}^1\longrightarrow[0,\infty)$ be defined by

$$L_0(p(\cdot),\beta)=\begin{cases}\displaystyle\int_0^{r_m} L(p(r),\beta)dr-\int_0^{r_m} L(\bar{p}(r),\bar{\beta})dr-\langle p(r),\mathbf{A}^*\psi(r)\rangle, \\ \qquad\qquad\qquad \text{for all } p(r)\ge 0,\ \beta\in[\beta_0,\beta_1], \\ +\infty, \qquad \text{otherwise.}\end{cases} \qquad (28)$$

Then $L_0(\bar{p}(\cdot),\bar{\beta})=0$. Furthermore, L_0 also satisfies the growth condition:

$$\|p(r)\|^2+\beta^2> K_2 \Rightarrow L_0(p(\cdot),\beta) \ge K(\|p(r)\|^2+\beta^2). \qquad (29)$$

Lemma 1. If an admissible pair $(\tilde{p}(\cdot,\cdot),\tilde{\beta}(\cdot))\in\mathcal{A}(p_0)$ satisfies

$$\int_0^\infty L_0(\tilde{p}(\cdot,t),\tilde{\beta}(t))dt <\infty, \qquad (30)$$

then necessarily $\|\tilde{p}(\cdot,t)\|$ is bounded for $t\ge 0$.

Proof. As in [3], we define

$$\Omega_T=\{t\ge T\,|\,\|\tilde{p}(\cdot,t)\|^2\ge K_2\}$$

for each $T>0$ and similar arguments show that

$$\lim_{T\to\infty}\text{mes}(\Omega_T)=0.$$

Choose $t>1$ sufficiently large so that

$$\text{mes}(\Omega_T)<1.$$

Then for each $t\in\Omega_T$, there exists $h\in[0,1]$ so that $t-h\notin\Omega_T$. Let

$\sigma=t-h$, then

$$\tilde{p}(r,t)=\tilde{p}(r,\sigma+h)=S(h)\tilde{p}(r,\sigma)$$

$$=\begin{cases}\tilde{p}(r-h,\sigma)e^{-\int_{r-h}^{r}\mu(\rho)d\rho}, & r\ge h,\\ \beta(h-r)\int_{r_1}^{r_2}k(s)h(s)[S(h-r)\tilde{p}(s,\sigma)]dse^{-\int_0^r\mu(\rho)d\rho}, & r<h.\end{cases}$$

By this we can show easily that

$$\|\tilde{p}(\cdot,t)\| \le M\|\tilde{p}(\cdot,\sigma)\|, \quad M=\text{const}.$$

This is the desired result.

Remark. It can be shown that under the condition (18) and the assumption of Theorem 2 it follows that

$$\int_0^\infty L_0(\tilde{p}(\cdot,t),\tilde{\beta}(t))dt < \infty$$

and therefore $\|\tilde{p}(\cdot,t)\|$ is bounded for $t\ge 0$.

We introduce the set

$$G=\{p(r)\in L^2(0,r_m)\,|\,\ni \beta\in [\beta_0,\beta_1] \text{ s.t. } L_0(p(\cdot),\beta)=0\} \tag{31}$$

and the following

Definition 2. Let $\mathcal{F}$ be the family of all trajectories $p(r,\cdot)\ge 0$ such that

$$p(\cdot,t)\in G \text{ a.e. on } [0,\infty).$$

We say that G has property $\mathcal{G}$ (for convergence) if $p(\cdot,t)\xrightarrow{W}\bar{p}(\cdot)$ as $t\to\infty$ uniformly in $\mathcal{F}$.

The following results are true.

Theorem 4. Under Assumption 4, if G has the property $\mathcal{G}$ and if a feasible pair $(\tilde{p},\tilde{\beta})$ is such that

$$\int_0^\infty L_0(\tilde{p}(\cdot,t),\tilde{\beta}(t))dt<\infty, \tag{32}$$

then, necessarily, $\tilde{p}(\cdot,t)$ converges weakly to $\bar{p}(\cdot)$ as $t\longrightarrow\infty$.

Corollary. In addition to the hypotheses given in Theorem 3, let us suppose that there exists a pair $(\tilde{p},\tilde{\beta})\in\mathcal{A}(p_0)$ such that (32) holds, then if in the class of all bounded trajectories there exists an overtaking optimal solution, say $(\hat{p},\hat{\beta})$, it follows that

$$\lim_{t\to\infty}\hat{p}(\cdot,t)=\bar{p}(\cdot) \text{ in the weak sense.}$$

Remark. If the system (1) is controllable, i.e. there exist $\beta(t)\in U$, and $T>0$, such that the corresponding trajectory $p(r,t)$ satisfy

$$p(r,T)=\bar{p}(r)$$

and define

$$(\tilde{p}(r,t),\tilde{\beta}(t))=\begin{cases}(p(r,t),\beta(t)), & 0\le t\le T,\\ (\bar{p}(r),\bar{\beta}), & t\ge T,\end{cases}$$

then condition (32) is satisfied.

9.4 Existence of Overtaking Optimal Solutions

Assumption 5. There exists $(\tilde{p}(r,t),\tilde{\beta}(t)\in\mathcal{A}(p_0)$ such that

$$\int_0^\infty\int_0^{r_m}L(\tilde{p}(\cdot,t),\tilde{\beta}(t))dt<\infty. \tag{33}$$

Theorem 5. Under Assumption 5 there exists an overtaking optimal solution $(\hat{p},\hat{\beta})$.

Proof. Let

$$\Phi = \inf \left\{ \int_0^{\infty}\int_0^{r_m} L(p(r,t),\beta(t))drdt, (p,\beta)\in \mathcal{A}(p_0) \right\}.$$

By assumption, Φ is finite. Let $(p_n,\beta_n)\in \mathcal{A}(p_0)$ be a minimizing sequence. For any fixed $T>0$, since $\beta_n(t)\in [\beta_0,\beta_1]$ for $t\geq 0$, we may extract, if necessary, a subsequence $\hat{\beta}(t)$ such that

$$\beta_n(t) \longrightarrow \hat{\beta}(t) \text{ weakly in } L^2(0,T).$$

Since $\{\beta(t) \,|\, \beta(t)\in [\beta_0,\beta_1] \text{for } t\in [0,T] \text{ a.e.}\}$ is a closed convex subset of $L^2(0,T)$, it is weakly closed, and hence $\hat{\beta}(t)\in [\beta_0,\beta_1]$ for $t\in [0,T]$ a.e. By (2) $p_n(\cdot,t) \longrightarrow \hat{p}(\cdot,t)$ weakly in $L^2(0,T;L^2(0,r_m)$ and $(\hat{p},\hat{\beta})\in \mathcal{A}(p_0)$. By convexity

$$\int_0^T\int_0^{r_m} L(p(r,t),\beta(t))drdt$$

is weak l.s.c. over $L^2(0,T;L^2(0,r_m))\times L^2(0,T)$. This shows that

$$\int_0^{\infty}\int_0^{r_m} L(\hat{p}(r,t),\hat{\beta}(t))drdt \leq \Phi$$

and $(\hat{p},\hat{\beta})$ is an overtaking optimal solution.

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Chapter 10

Viable Control Policy in Logistic Population Model

10.1 Introduction

Denoting by r the age of some population, $0 \le r < \infty$, t the time, $p(r,t)$ the age distribution, $\beta(r)$ the fertility function, $\mu(r)$ the age -dependent mortality function, $N(t)$ the total population, $f(N(t))$ the contribution from population pressure and $c(t)$ the age-indiscriminate culling control, which is associated with the amount of culling or harvesting in the population system. We are concerned with the evolution of the following logistic population system with culling control

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -[\mu(r)+f(N(t))]p(r,t)-c(t)p(r,t), \\ p(r,0)=p_0(r), \\ p(0,t)=\displaystyle\int_0^\infty \beta(r)p(r,t)dr, \\ N(t)=\displaystyle\int_0^\infty p(r,t)dr. \end{cases} \tag{1}$$

Firstly, we consider the problem which has been investigated in [1], i.e. to determine the conditions under which the population system may be steered to a specified population level and held there. However the study in [1] appears to be incomplete. Specifically, apart from the assumption that $\mu(r)=\alpha_0$ is a steady death rate it is asserted that the fertility function $\beta(r)$ can be approximated by a function of form $g(r)e^{-\alpha r}$ in the sense that $|g(r)e^{-\alpha r}-\beta(r)|<\varepsilon$ for all $r\geq 0$ and the given positive number $\varepsilon>0$. Here $g(r)$ is a suitable nth degree polynomial and $\alpha>0$. We know that this is possible only for $\beta(r)$ decaying to zero exponentially as r goes to infinity. In fact it was also stated that $\beta(r)$ and all its derivatives rapidly tend to zero beyond a certain value of r. Furthermore, it was claimed that $\tilde{p}(t)\geq p(t)$ and $p_b(t)\leq p(t)$ for all $t\in[0,t_1]$ as obvious facts in the proof of (16) in page 304, lines 13 and 18. But these are just facts required to be proved since they are only equivalent to (16).

10.2. Viable Control

In view of the above shortcomings, we will approach the problem via another route, namely, methods developed in chapter 2. It is assumed here that the fertility function $\beta(r)$ is a bounded measurable function in $[0,\infty)$ and $\mu(r)$ is locally integrable and

$$\int_0^\infty \mu(r)dr=\infty. \tag{2}$$

Let $\mathbf{X}=L(0,\infty)$ be the state space with the usual norm, the population operator $\mathbb{A}$ is defined by

$$\begin{cases} \mathbb{A}\phi(r)=-\phi'(r)-\mu(r)\phi(r), & \forall\ \phi\in D(\mathbb{A}), \\ D(\mathbb{A})=\{\phi(r)\,|\,\phi,\ \mathbb{A}\phi\in\mathbf{X},\ \phi(0)=\int_0^\infty\beta(r)\phi(r)dr\}, \end{cases} \tag{3}$$

then the following proposition was proved in Yu et al [2]

Proposition 1.

(i). The operator $\mathbb{A}$ is the infinitesimal generator of a semigroup of bounded linear operators on **X**.

(ii). The spectrum of $\mathbb{A}$ consists of eigenvalues of $\mathbb{A}$, i.e. $\sigma(\mathbb{A})=\sigma_p(\mathbb{A})$, which are the zeros of following equation

$$1-\int_0^\infty \beta(r)e^{-\lambda r-\int_0^r\mu(\rho)d\rho}dr=0. \tag{4}$$

(iii). $\mathbb{A}$ has only one real eigenvalue λ_0 great than any real part of the other eigenvalues of $\mathbb{A}$.

(iv). The following asymptotic expression holds

$$e^{\mathbb{A}t}\phi(r)=C_0e^{-\lambda_0 r-\int_0^r\mu(\rho)d\rho}e^{\lambda_0 t}+o(e^{(\lambda_0-\varepsilon)t}),$$
$$\forall\ \phi\in\mathbf{X}, \tag{5}$$

where $C_0\geq 0$ depends only on the initial condition $\phi(r)$; $C_0>0$ provided that $\text{meas}\{r\,|\,p_0(r)\neq 0,\ r\in[0,\infty)\}>0$, and $\varepsilon>0$ is a positive number such that $\{\lambda_0-\varepsilon\leq\text{Re}\lambda<\lambda_0\}\cap\sigma(\mathbb{A})=\emptyset$.

For the bounded nonnegative measurable control function $c(t)$ and differentiable function $f(\xi)$, using the same arguments as in [2], we can prove

Proposition 2. For any nonnegative initial $p_0\in\mathbf{X}$, there exists a unique solution to equation (1)

$$p(r,t)=e^{\mathbb{A}t}p_0(r)e^{-\int_0^t f(N(\rho))d\rho}e^{-\int_0^t c(\rho)d\rho}, \tag{6}$$

$p(r,t)\in C(0,\infty;\mathbf{X})$ if $p_0\in\mathbf{X}$; $p(r,t)\in C^1(0,\infty;\mathbf{X})$ if $p_0\in D(\mathbb{A})$, where $N(t)$ is the unique continuous solution of

$$N(t)=\|e^{\mathbb{A}t}p_0(r)\|e^{-\int_0^t f(N(\rho))d\rho}e^{-\int_0^t c(\rho)d\rho}. \tag{7}$$

Proposition 3. If

$$\lim_{t\to\infty} c(t)=c \tag{8}$$

and $f(\xi)$ is continuously differentiable

$$f(0)=0,\ f(\xi)>0 \text{ for } \xi>0, \tag{9}$$

then for all initial value $N_0>0$

$$\lim_{t\to\infty} N(t)=0, \quad \text{if } \lambda_0-c\le 0; \tag{10}$$

$$\lim_{t\to\infty} N(t)=\xi_0, \quad \text{if } \lambda_0-c\ge 0; \tag{11}$$

where $\xi_0\in \{\xi \mid f(\xi)=\lambda_0-c\}\cup\{\infty\}$.

For $p_0\in D(\mathbb{A})$, (7) implies that

$$\frac{dN(t)}{dt}=[g(t)-f(N(t))-c(t)]N(t), \tag{12}$$

where $g(t)=N_0'(t)/N_0(t), N_0(t)=\|e^{\mathbb{A}t}p_0(r)\|$ and we know from [2] that

$$\lim_{t\to\infty} g(t)=\lambda_0 \tag{13}$$

in a oscillatory fashion.

Equation (9) is equivalent to (1) through (6) and (7), this is the same as that of [1] but it is simple and some strong assumptions are avoided.

In the following, physical requirements lead us to assume that the control $c(t)$ is bounded, i.e.

$$0\le c(t)\le b \quad \forall\ t\ge 0, \tag{14}$$

and $f(\xi)$ is an increasing function of ξ.

In order to choose the control which bring the population from $N_0>0$ to a specified value $N^*>0$ in a finite time t^* and hold it steady at that value, we define similarly as in [1]

$$c(t)=\begin{cases} c_1(t) & \text{for } t\in[0,t_1], \\ c_2(t) & \text{for } t>t_1, \end{cases} \tag{15}$$

where t_1 and c_1, c_2 are as yet unspecified.

It follows directly from (12) that if $N(t)=N^*$ for all $t\ge t_1$, then the control $c_2(t)$ is determined uniquely by

$$c_2(t)=g(t)-f(N^*). \tag{16}$$

This is a viable control iff

$$0\le g(t)-f(N^*)\le b \tag{17}$$

Noticing (13), we see that

$$0\le \lambda_0-f(N^*)\le b$$

or

$$\lambda_0-b\le f(N^*) \le\lambda_0. \tag{18}$$

This can be considered as a restriction on N^*.

Since usually, $g(t)$ tend to λ_0 in a oscillatory fashion, so we only consider the situation

$$\lambda_0-b< f(N^*) <\lambda_0. \tag{19}$$

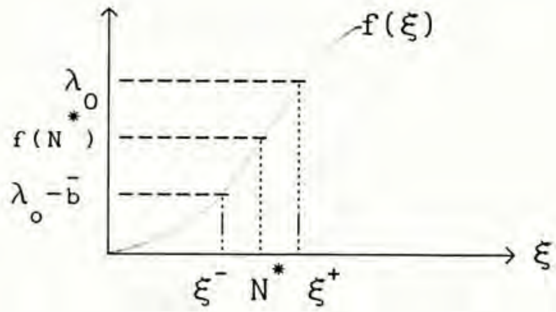
If (19) is satisfied, then the possible minimum time in which the system can be brought to the level N^* and held there can be chosen to be the largest solution of

$$g(t)-f(N^*)=0, \qquad g(t)-f(N^*)-b=0. \tag{20}$$

Having determined $c_2(t)$ and t_1 we now turn to the control which will bring the population from N_0 to N^* in a time t_1 (or longer perhaps).

We assert that it is always possible to choose $t^*\ge t_1$ and a admissible control $c_1(t)$ bring the state to N^* in time t^*. In fact, since $\lambda_0-b< f(N^*) <\lambda_0$ and $f(\xi)$ is an increasing function of ξ, there exists a $\bar{b}>0$ such that

$$0< \lambda_0-\bar{b}< f(N^*) <\lambda_0.$$



Firstly, take $c_1(t)=0$, if the corresponding state $N(t_1)<N^*$ then by proposition 3, $N(t)$ tends to ξ^+ as t goes to infinity, here $f(\xi^+)=\lambda_0$ (see figure), $\xi^+\geq N^*$. By the continuity of $N(t)$, there exists $t^*\geq t_1$ such that

$$N(t^*)=N^*.$$

Therefore, the viable control $c(t)$ can be chosen as

$$c(t)=\begin{cases} 0, & \text{for } t\in[0,t^*], \\ g(t)-f(N^*), & \text{for } t>t^*. \end{cases} \tag{21}$$

Secondly, if for $c_1(t)=0$, the corresponding state $N(t_1)>N^*$, then take $\lambda_0-\bar{b}$ to be the control in $t\geq t_1$, by proposition 3 the corresponding solution tends to $\xi^-\leq N^*$. Hence there exists a t^* such that $N(t^*)=N^*$. In this case the viable control is taken to be

$$c(t)=\begin{cases} 0, & 0\leq t\leq t_1, \\ \lambda_0-\bar{b}, & t_1<t\leq t^*, \\ g(t)-f(N^*), & t>t^*. \end{cases} \tag{22}$$

Summarizing, we have proved

Theorem 1. Let $N^*>0$ be a specified value satisfying (19), $f(\xi)$ a continuously differentiable increasing function of ξ, $f(0)=0$, then for any initial value $p_0\in D(\mathbb{A})$, $N_0>0$, there exist an admissible culling control $c(t)$, i.e. $c(t)$ is measurable and $0 \le c(t) \le b$ for all $t \ge 0$, and a time t^* such that the corresponding total population

$$N(t)=N^*, \text{ for all } t \ge t^*. \tag{23}$$

10.3. Minimum Time Problem

Now, we are interested in the minimum time t^* in which the population system (12) can be brought to the level N^* and held there under the assumptions of theorem 1. The existence of t^* have been proved in theorem 1 and we know from previous discussion that t^* is not less than the largest solution t_1 of

$$g(t)-f(N^*)=0, \qquad g(t)-f(N^*)-b=0. \tag{24}$$

This problem can be written as a standard time optimal control problem described by ordinary differential equation:

$$\text{minimize } J(N(T),T)$$

$$J(N(T),T)=T \tag{25}$$

subject to

$$\begin{cases} \dfrac{dN(t)}{dt}=[g(t)-f(N(t))-c(t)]N(t), \quad N(0)=N_0, \\ g_1(N(T),T)=N(T)-N^*=0; \quad g_2(N(T),T)=T-t_1 \ge 0, \\ c(t) \in [0,b], \ \forall \ t \in [0,T]. \end{cases} \tag{26}$$

The Hamiltonian function of (26) is

$$H(N,\Psi,t,c)=\Psi \cdot [g(t)-f(N)-c]N. \tag{27}$$

In order to apply the Pontryagin's maximum principle [3], we assume in the following that the initial value $p_0 \in D(\mathbb{A}^2)$, so that $g(t)$ is continuously differentiable. So for the optimal solution $(T^*,c^*(t),N^*(t))$ of (25)-(26), we deduce from the Pontryagin maximum principle a nonzero function $\Psi(t)$, $t \in [0,T^*]$, and constants μ_0, μ_1, ν, $\mu_0 \ge 0$, $\nu \ge 0$ not all identically zero, satisfying (i). $\begin{cases} \dfrac{d\Psi(t)}{dt} = -[g(t)-f(N^*(t))-c^*(t)-f'(N^*(t))N^*(t)]\Psi(t), \\ \Psi(T^*)=-\mu_1; \end{cases}$

(ii). $\mu_1 c^*(t) = \max\limits_{c\in[0,b]} \mu_1 c;$

(iii) $-\mu_1[g(T^*)-f(N^*)-c^*(T^*)]N^*=\mu_0+\nu;$

(iv). $\nu(T^*-t_1)=0;$

(v). $\Psi(t)[g(t)-f(N^*(t))-c^*(t)]N^*(t)=\mu_0+\nu+\int_{T^*}^{t}\Psi(s)N^*(s)g'(s)ds.$

Since $\Psi(t)\not\equiv 0$, so $\mu_1\neq 0$. A computation shows that (v) is a equivalent to $\frac{d}{dt}c^*(t)=0$ and hence is a consequence of (ii). By (ii), the optimal control

$$c^*(t)\equiv 0, \quad \text{for all } t\in[0,T^*]$$

$$\text{or} \quad c^*(t)\equiv b \quad \text{for all } t\in[0,T^*], \tag{28}$$

according to $\mu_1<0$ or $\mu_1>0$.

If $\nu\neq 0$, then the optimal time $T^*=t_1$, we may, through computing the values of solutions corresponding to the controls $c(t)=0$ and $c(t)=b$ at t_1 respectively, determining $c^*(t)\equiv 0$ or $c^*(t)\equiv b$ to be the optimal control.

Considering the situation of $\nu=0$, if $\mu_0=0$ then $T^*=t_1$ since in this case (iii) becomes

$$g(T^*)-f(N^*)-c^*=0, \qquad c^*=0 \text{ or } b.$$

Reasons same as above lead us to the optimal control. Otherwise, we can assume $\zeta N^*=-\mu_0/\mu_1\neq 0$ and (iii) becomes

$$g(T^*)-f(N^*)-c^*=\zeta, \; c^*=0 \text{ or } b. \tag{29}$$

This is a necessary condition for T^* to be the optimal time, but unfortunately, we can not determine T^* definitely from (29), the reason is that (v) is a consequence of (ii). The only information we have is that the optimal culling control is 0 or b.

Theorem 2. Under the assumptions of theorem 1, if the initial value $p_0 \in D(\mathbb{A}^2)$, then the optimal culling control which bring the total population from N_0 to N^* in minimum time is either 0 or b.

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