Non-linear Functional Analysis and Vector Optimization

by

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簡介

本論文主要討論向量最優化理論和介紹一些有關的研究結果。題目包括：真有效點的密度理論，向量變分不等式以及它的變化問題和集值開映像定理。泛函分析和集值分析的理論是這些研究的基礎工具。

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ABSTRACT

This thesis devotes to the subject of vector optimization. Topics covered are: density theorem of admissible points (proper efficient points), vector variational inequalities, generalized quasi-variational inequalities and set-valued open mapping theorem. The mathematical technique evolved are essentially functional analysis and set-valued analysis.

PREFACE

Chapter 1 and chapter 2 are about the theorems of the admissible points (also called the efficient points) , with respect to some order, of a convex subset in a topological vector space. In the study of vector optimization, we often encounter the situation of identifying the efficient points of a given set. Unfortunately, the behavior of the efficient points are anomalous in many cases. Therefore, the concept of positive proper efficiency (as well as other proper efficiency) was introduced. A positive proper efficient point can be regarded as the solution of a linear scalar optimization problem which we can handle with much more confidence.

Arrow, Barankin and Blackwell [1] have shown that (see Theorem 1.2.1) the efficient points and the positive proper efficient points of a finitely generated convex subset are the same in the finite dimensional spaces. This theorem has important implications in vector optimization and game theory. An application to the game theory is included at the end of chapter one.

Going to the case of infinite dimensional spaces, such as the l^p spaces, people found that the positive proper efficient points are dense in the efficient points. This is good enough in many applications. Chapter 2 introduces a recent and very general version of the density theorem. It was given by Xi Yin Zheng [26] under the inspiration of the concept of D-cone introduced by Gallagher and Saleh.

Chapter 3, chapter 4, and chapter 5 are talking about variational inequalities. In chapter 3, the fundamental concepts of (scalar and vector) variational inequalities are introduced, and also their relation with the (scalar and vector) optimization problem is explained.

Some recent existence results dealing with the vector variational inequalities can be found in chapter 4. They are given by Gue Myung Lee and Sangho Kum [14] based on the advance in the set-valued fixed-point theorem in [16].

A generalized quasi vector variational inequalities problem is a variation of the vector variational inequalities problem. We may come across this problem when solving certain kinds of PDEs. An existence theorem for the generalized quasi-variational inequalities by Nguyen Dong Yen [24] is introduced in chapter 5. The last chapter of the thesis is a set-valued open mapping theorem by Kung-Fu Ng [15], which states that a nearly open relation with its graph being closed, convex and positively homogeneous is in fact open.

Contents

Chapter 1

Admissible Points of Convex Sets

1.1 Introduction and Notations

In this chapter, we introduce the result in [1]. Throughout this chapter, S is a closed convex subset of $\mathbb{R}^k$. $s \in S$ is said to be admissible if for all $t \in S$ other than s, $t_i > s_i$ for some i, where $t \equiv (t_1, \ldots, t_k)$ and $s \equiv (s_1, \ldots, s_k)$. The set of all admissible points in S is denoted by A.

For a fixed $p \in \mathbb{R}^k$, $\langle p, t \rangle$ can be regarded as a function of t. Let $P =: \{p \in \mathbb{R}^k \mid p_i > 0 \;\; \forall i$ and $\sum\limits_{1 \le i \le k} p_i = 1\}$. Thus each $p \in P$ is a strictly positive linear functional on $\mathbb{R}^k$:

$$\langle p, t \rangle > 0, \quad \forall t > 0. \tag{1.1}$$

We define $B(p) = \{s \in S \mid \langle p, s \rangle = \inf\limits_{t \in S} \langle p, t \rangle\}$ which is closed by the continuity of dot product again. Also define $B \equiv \bigcup\limits_{p \in P} B(p)$.

1.2 The Main Result

Theorem 1.2.1 $B \subseteq A \subseteq \overline{B}$. *Also, if S is determined by (convex combination of) a finite number of vertices, then there is a finite subset $\{p_1, \ldots, p_N\}$ of P such that $B = \bigcup_{j=1}^{N} B(p_j)$, and hence $B = A = \overline{B}$.*

1.2.1 The Proof of Theorem1.2.1

The last statement $B = A = \overline{B}$ follows from the observation that $B(p)$ is closed for any fixed p. So, we are going to prove the rest of the theorem.

Suppose $s \in S$ and $s \notin A$, i.e. not admissible in S. There is a $t \in S \setminus \{s\}$ with $t_i \leq s_i$ for all i (hence at least one of the inequalities is strict). For any $p \in P$, we have $p_i t_i \leq p_i s_i$ for all i and at least one of the inequalities is strict. Thus, we get $\langle p, t \rangle < \langle p, s \rangle$ and which means $s \notin B$. Therefore, $B \subseteq A$.

To show $A \subseteq \overline{B}$, we suppose first that S is bounded. Let $a \in A$. Apply translation if necessary, we assume without lost of generality that $a = (0, \ldots, 0)$. For any $\frac{1}{n}$ with $n \in \mathbb{N}$ and $n \geq k$, let $P(\frac{1}{n})$ be the set of all $p \in P$ with $p_i \geq \frac{1}{n}$ for $i = 1, \ldots, k$. Note that $P(\frac{1}{n})$ is non-empty, compact and convex in $\mathbb{R}^k$, $P = \bigcup\limits_n P(\frac{1}{n})$. Also, $P(\frac{1}{n}) \subseteq P(\frac{1}{m})$ if $m > n$.

Recall that $\langle x, \cdot \rangle$ is in fact a linear continuous function. By the Von Neumman's Minimax principle, cf. [13], we know that for any two bounded, closed and convex subsets U and V in $\mathbb{R}^k$, there exist vectors $u^* \in U$, $v^* \in V$ with $\langle u^*, v \rangle \geq \langle u^*, v^* \rangle \geq \langle u, v^* \rangle$ for all $v \in V$, $u \in U$.

Let $t \in P$ be arbitrary. Then there is an m such that $t \in P(\frac{1}{n})$ for all $n \geq m$. We apply the Von Neumman's Minimax Principle to each pair of the closed, bounded and convex subsets $P(\frac{1}{n})$ and S. That is, $\exists p_n \in P(\frac{1}{n})$ and $\exists s_n \in S$ such that

$$\langle p_n, s \rangle \geq \langle p_n, s_n \rangle \geq \langle p, s_n \rangle \qquad (*)$$

for all $p \in P(\frac{1}{n})$ and all $s \in S$. Letting $s = a = 0$, we have

$$\langle p, s_n \rangle \leq 0$$

for all $p \in P(\frac{1}{n})$. In particular, since $t \in P(\frac{1}{n})$,

$$\langle t, s_n \rangle \leq 0 \qquad (**)$$

for each $n \geq m$. Since S is assumed to be compact, we can further assume with loss of generality that $\{s_n\}$ converges, say to $s^* \in S$. By continuninty, it follows from (**) that $\langle t, s^* \rangle \leq 0$ for all $t \in P$. Consequently, $s_i^* \leq 0$ for all i. Since $a = (0, \ldots, 0)$ which is admissible, we must have $s^* = a$. Finally, the first inequality in (*) implies that $s_n \in B(p_n) \subseteq B$, so the cluster point $a \in \overline{B}$.

Now we deal with the case of unbounded S. We need a lemma.

Lemma 1.2.1 *Let C be a convex set, $s_0 \in C$ and N be a neighborhood of s_0. Suppose that s_0 minimizes $\langle p, s \rangle$ for $s \in C \bigcap N$. Then s_0 minimizes $\langle p, s \rangle$ for $s \in C$.*

Poof.
Suppose, for some $s_1 \in C$, $\langle p, s_1 \rangle < \langle p, s_0 \rangle$. Let $s_2 = (1-\alpha)s_0 + \alpha s_1$; for $0 < \alpha < 1$, $s_2 \in C$, $\langle p, s_2 \rangle < \langle p, s_0 \rangle$. But for α sufficiently small, $s_2 \in N$, which contradicts the hypothesis.

So we come back to the theorem. Let S be a closed convex set, a an admissible point of S. Let N be a closed neighborhood of a. Then a is admissible in $S \bigcap N$. By (the bounded case of) Theorem 1.2.1, $a = \lim_{n \to \infty} s_n$, where s_n minimizes $\langle p_n, s \rangle$ for $s \in S \bigcap N$ for some p_n in P. Then, by Lemma 1.1, s_n minimizes $\langle p_n, s \rangle$ for $s \in S$ for some p_n. Hence $s_n \in B$ for each n and so $a \in \overline{B}$.

Now, we have shown that $B \subseteq A \subseteq \overline{B}$. Further suppose that S is determined by a finite set $\{s_1, \ldots, s_m\}$. That is, any element in S can be represented by a convex combination of s_1 to s_m. In particular, S is bounded and hence compact. We want to show that $B = \bigcup_{j=1}^{N} B(p_j)$ for some $\{p_1, \ldots, p_N\} \subset P$.

But before that, we need another lemma. A nonempty subset U of $\{s_1, \ldots, s_m\}$ is said to be usable if there is a $p \in P$ for which $U \subseteq B(p)$.

Lemma 1.2.2 *There exists a nonempty subset of $\{s_1, \ldots, s_m\}$ which is usable.*

Proof.
Suppose there exists a $t_0 \in S$ such that for all s_i and all $p \in P$, $\langle p, s_i \rangle >$

$\langle p, t_0 \rangle$. Since t_0 is equal to the convex combination $\sum_{i=1}^m \lambda_i s_i$, we have $\langle p, t_0 \rangle = \langle p, \sum_{i=1}^m \lambda_i s_i \rangle > \sum_{i=1}^m \lambda_i \langle p, t_0 \rangle = \langle p, t_o \rangle$. This is impossible.

To continue the proof of the theorem, let $\{U_1, \ldots, U_N\}$ be the family of all usable subsets of $\{s_1, \ldots, s_m\}$, and let $p_1, \ldots, p_N$ be the corresponding p's for which $U_j \subseteq B(p_j)$. Since $B \supseteq \bigcup_{j=1}^N B(p_j)$, we only need to show that $B \subseteq \bigcup_{j=1}^N B(p_j)$.

Let $s \in B$, say $s \in B(p)$, and $s = \sum_{i=1}^m \lambda_i s_i$. Let also $U \equiv \{s_i \mid \lambda_i \neq 0\}$. Then $U \subseteq B(p)$. Suppose not: there exists $s_i \in U$ with $\lambda_i \neq 0$ and $s_i \notin B(p)$. Then $\langle p, s_i \rangle > \inf_{t \in S} \langle p, t \rangle$. Hnece

$$\langle p, s \rangle = \sum_{i=1}^m \lambda_i \langle p, s_i \rangle > \sum_{i=1}^m \lambda_i \inf_{t \in S} \langle p, t \rangle = \inf_{t \in S} \langle p, t \rangle.$$

This contradicts to the fact that $s \in B(p)$.

Therefore, U is indeed usable and we must have $U = U_j \subseteq B(p_j)$ for some j. Now we know that $s_i \in U_j$ for all i, thus $s \in B(p_j)$ since $B(p_j)$ is convex. Q.E.D.

1.3 An Application

In 2-space, it can be shown that the set of admissible points A of a closed convex set is always closed. For an example of A which is not closed in 3-space, please refer to the original paper. Here we introduce an application of Theorem 1.2.1.

Theorem 1.3.1 *(Bohnenblust, Karlin, and Shapley) Let $A = [a_{ij}]$ be an $m \times n$ real matrix, considered as the payoff of a zero-sum two-person game, and let D be the set of all i for which*

$$A(i, q) \equiv \sum_{j=1}^n a_{ij} q_j = v$$

where q is a minimax strategy (mixed) for the person II and v is the value of the game. Then there is a minimax strategy p for player I with $p_i > 0$ for $i \in D$.

Proof.

If $i \notin D$, then $A(i,q) < v$. Therefore, it is no point to add the i^{th} row to the mixed strategy of player I. That is: if we delete from A the rows for which $i \notin D$, the resulting new game still has value v, and every minimax strategy for player I in the new game is minimax in the original game. Moreover,$A(i,q) = v$ for every $i \in D$ and every minimax strategy q for player II in the new game. The reason is: suppose $A(i_0, q_0) < v$ for some $i_0 \in D$ and some minimax strategy q_0 for player II in the new game. Let $\epsilon > 0$ and q_1 a minimax strategy for player II in the original game. Define

$$q^* \equiv \epsilon q_0 + (1-\epsilon) q_1.$$

We have

$$A(i_0, q^*) < v. \tag{1.2}$$

Also, by consider the convex combination of minimax strategies if needed, we can assume without loss of generality that $A(i, q_1) < v$ for all $i \notin D$. So, for small ϵ, we have

$$A(i, q^*) < v$$

for all $i \notin D$. It then follows from the definition of q^* that q^* will be a minimax strategy in the original game for sufficiently small ϵ. Now, from (1.2), we see that $i_0 \notin D$, a contradiction. Therefore, from now on, we assume that $i \in D$ for $i = 1, \ldots, m$.

Let S be the convex hull of the columns of A. S is a subset of m-space. By the definition of a mixed strategy in Game Theory, we see that $s_0 \equiv (v, \ldots, v) \in S$. Since v is the value of the game, s_0 is admissible. By the main theorem of this chapter, there is a $p \in P$ with $s_0 \in B(p)$. That is

$$\langle p, s_0 \rangle = v \leq \langle p, s \rangle$$

for all $s \in S$. Thus p is a minimax strategy for player I. Q.E.D.

Chapter 2

A Generalization on The Theorems of Admissible Points

2.1 Introduction and Notations

The main reference of this chapter is [26], all theorems can be found there. Let (X, τ) be a Hausdorff topological vector space (and all topological vector spaces discuss in this chapter is real), X^* its dual space. In order to make use of the Separation Theorem under the weak topology, we assume throughout this chapter that X^* separates the points of X. A subset C of X is said to be a convex cone if $C + C \subseteq C$ and $tC \subseteq C$ for all $t \geq 0$. For any subset A of X, $cone(A)$ was defined as in p. 14 of [17]. If, in addition, $C \bigcap (-C) = \{0\}$, where 0 denotes the zero vector in X, then C is said to be pointed. A subset B of C is said to be a base of C if B is convex, $C = \{tb \mid t \geq 0, b \in B\}$ and $0 \notin cl(B)$, where $cl(B)$ denotes the τ-closure of B. In addition, B is bounded (or weakly compact), then B is said to be a bounded (or weakly compact) base of C. The dual cone of C is denoted by C^+ which defined as

$$C^+ = \{f \in X^* \mid f(x) \geq 0 \ \ \forall x \in C\}.$$

All strictly positive linear functionals in C^+ is denoted by C^{+I}, that is

$$C^{+I} = \{f \in X^* \mid f(x) > 0 \, \forall x \in C \, with \, x \neq 0\}.$$

Although clearly enough, it is worthwhile to point out that $C^{+I} + C^+ \subseteq C^{+I}$.

Lemma 2.1.1 *$C^{+I} \neq \emptyset$ if and only if there is a base B of C such that $0 \notin cl_w(B)$, the weak closure of B.*

Proof. Let $f \in C^{+I}$ and $B \equiv C \bigcap f^{-1}(1)$. Then B is convex and $0 \notin cl_w(B)$ because $0 \notin f^{-1}(1)$ which is a weak-closed set containing B. Therefore the necessary part is clear. Conversely, suppose that B is a base of C with $0 \notin cl_w(B)$. Since X^* separates points of X, X is Hausdorff and locally convex under the weak topology with respect to which, $\{0\}$ is weak-compact and $cl_w(B)$ is convex and weak-closed; by the Separation Theorem in functional analysis, $\exists \, f \in X^*$ and $r_1, r_2 \in \mathbb{R}$ such that

$$0 = f(0) < r_1 < r_2 < f(b)$$

for all $b \in cl_w(B)$. Then $f \in C^{+I}$ since B is a base of C. Thus C^{+I} is nonempty. Q.E.D.

Let A be a subset of X. The set of all efficient points of A with respect to a convex cone C is denoted by $E(A, C)$, which means that $x \in E(A, C)$ if and only if $(A - x) \bigcap C = \{0\}$ and $x \in A$. That is, x is a maximal element of A with respect to the pre-order defined by C. Also, we say $\bar{x} \in Pos(A, C)$, the positive proper efficient points of A with respect to C if $\bar{x} \in A$ and $\exists \, f \in C^{+I}$ such that $f(\bar{x}) = \sup_{x \in A} f(x)$. Note that, $Pos(A, C) \subseteq E(A, C)$. In fact, $\forall a - \bar{x} \in (A - \bar{x}) \bigcap C$ and $f \in C^{+I}$, $f(a - \bar{x}) = f(a) - f(\bar{x}) \geq 0$ if and only if $a = \bar{x}$. Hence $a - \bar{x} = 0$ and $\bar{x} \in E(A, C)$. By Letting $X = \mathbb{R}^n$ and $C = \mathbb{R}^{n+}$, we find that the efficient points of A are exactly those admissible points of $-S$ we defined in chapter 1, and the positive proper efficient points of A are denoted as B of $-A$ in chapter 1.

2.2 Fundamental Lemmas

$\mathcal{N}(0)$ denotes the family of all neighborhoods of 0 in X. For each $V \in \mathcal{N}(0)$, V^0 denotes the absolute polar of V, that is

$$V^0 = \{f \in X^* \mid |f(x)| \leq 1, \forall x \in V\}.$$

Set $\mathcal{D} = \{(n, V) \mid n \in \mathbb{N},\ V \in \mathcal{N}(0)\}$. We define a partial ordering " $\leq$ " in $\mathcal{D}$ as follows: $(n_1, V_1) \leq (n_2, V_2)$ if $n_1 \leq n_2$ and $V_1 \supseteq V_2$.

Lemma 2.2.1 *$(\mathcal{D}, \leq)$ is a directed set.*

Proof. For any (n_1, V_1) , $(n_2, V_2) \in \mathcal{D}$, take $n = max\{n_1, n_2\}$, and $V = V_1 \bigcap V_2$. Then $(n, V) \in \mathcal{D}$ and $(n_i, V_i) \leq (n, V)$ for $i = 1, 2$. Q.E.D.

Lemma 2.2.2 *Suppose that X is a topological vector space, C is a weakly closed convex cone in X and that C^{+I} is nonempty. Fix $\bar{p} \in C^{+I}$. For each $(n, V) \in \mathcal{D}$, set $B(n, V) = \frac{1}{n}\bar{p} + V^0 \bigcap C^+$ and $A(n, V) = \bigcup\limits_{t>0} tB(n, V)$. Then*
(i) $A(n_1, V_1) \subseteq A(n_2, V_2)$ whenever $(n_1, V_1) \leq (n_2, V_2)$;
(ii) $x \in C$ if and only if $f(x) \geq 0$ for all $f \in \bigcup\limits_{(n,V) \in \mathcal{D}} A(n, V)$.

Proof. Suppose $n_1 \leq n_2$ and $V_1 \supseteq V_2$. Let $f \in A(n_1, V_1)$: $f \in tB(n_1, V_1)$ for some $t > 0$. This means $f \in \frac{tn_2}{n_1}\left(\frac{1}{n_2}\bar{p} + \frac{n_1}{n_2}V_1^0 \bigcap C^+\right)$. Since $V_1^0 \bigcap C^+$ is a convex set containing constant function zero and $\frac{n_1}{n_2} \leq 1$, we have $\frac{n_1}{n_2}(V_1^0 \bigcap C^+) \subseteq V_1^0 \bigcap C^+ \subseteq V_2^0 \bigcap C^+$. Thus $f \in \frac{n_1}{n_2}t\left(\frac{1}{n_2}\bar{p} + V_2^0 \bigcap C^+\right)$ and $A(n_1, V_1) \subseteq A(n_2, V_2)$.

To prove (ii), we first show that

$$x \in C \quad if \ and \ only \ if \ \ for \ all \quad f \in C^+, \quad f(x) \geq 0. \qquad (*)$$

The 'if' part is just the definition of C^+. Suppose conversely that $x \notin C$. Then, since C is weakly closed convex, $\exists f \in X^*$ and $r_1, r_2 \in \mathbb{R}$ such that

$$f(x) < r_1 < r_2 < f(y)$$

for all $y \in C$. Since C is positively homogenous and f is linear, it follows that $f(y) \geq 0$ for all y in C. This means that f is in C^+. Letting $y = 0$, we get $f(x) < 0$. Therefore (*) is true.

Further, we will show that $C^+ = cl_w \left(\bigcup_{(n,V) \in \mathcal{D}} A(n,V) \right)$ (cl_w means the closure w.r.t. the weak* topology). Since $\langle f, x \rangle$ is weak*-continuous with respect to f, this will finish the proof of (ii).

By the definition, we have $C^+ \supseteq \bigcup_{(n,V) \in \mathcal{D}} A(n,V)$. As C^+ is weak*-closed, $C^+ \supseteq cl_w \left(\bigcup_{(n,V) \in \mathcal{D}} A(n,V) \right)$. On the other hand, suppose $f_0 \notin cl_w \left(\bigcup_{(n,V) \in \mathcal{D}} A(n,V) \right)$. $\{f_0\}$ is weak*-compact in X^* and the set $cl_w \left(\bigcup_{(n,V) \in \mathcal{D}} A(n,V) \right)$ is weak*-closed and convex. So by the Separation Theorem, $\exists \bar{x} \in X$ and $r_1, r_2 \in \mathbb{R}$ such that

$$f_0(\bar{x}) < r_1 < r_2 < f(\bar{x}) \qquad (**)$$

for all $f \in cl_w \left(\bigcup_{(n,V) \in \mathcal{D}} A(n,V) \right)$, so particularly, for all $f \in \bigcup_{(n,V) \in \mathcal{D}} A(n,V)$. Applying similar argument as above, we have $f(\bar{x}) \geq 0$ for all $f \in \bigcup_{(n,V) \in \mathcal{D}} A(n,V)$. Now, for any $p \in C^+$, $p \in V^0 \bigcap C^+$ for some V. So, $\frac{1}{n}\bar{p} + p \in B(n,V)$ for all n, which implies that $\left(\frac{1}{n}\bar{p} + p\right)(\bar{x}) \geq 0\ \forall n$. Hence $p(\bar{x}) \geq 0$. By (*), we have $\bar{x} \in C$. Since the zero functional is clearly in $cl_w \left(\bigcup_{(n,V) \in \mathcal{D}} A(n,V) \right)$, (**) implies that $f_0(\bar{x}) < 0$ and $f_0 \notin C^+$. Q.E.D.

Lemma 2.2.3 *Let X be a topological vector space, C a convex cone in X and let C^{+I} be nonempty. Suppose that A is a closed convex subset of X and $x \in A$. Then $x \in Pos(A, C)$ if and only if there is an open convex subset U of X such that*

$$C \subseteq cone(U) \qquad \textit{and} \qquad (2.1)$$

$$cone(A - x) \bigcap U = \emptyset. \qquad (2.2)$$

Proof. Suppose there is an open convex subset U of X satisfying (2.1) (2.2). The sets $cone(A-x)$ and U satisfy the hypothesis of the Separation Theorem, so there exists $r \in \mathbb{R}$ and $g \in X^*$ such that

$$g(z) > \sup\{g(y) \mid y \in cone(A-x)\} = r \quad \text{for all} \quad z \in U. \tag{2.3}$$

Since $0 \in cone(A-x)$ and $cone(A-x)$ is positively homogeneous, it follows that $r = 0$. Consequently, (2.3) and (2.1) imply that $g \in C^{+I}$ and $g(a-x) \leq r = 0$ for all $a \in A$. Therefore $x \in Pos(A, C)$.

On the other hand, let $x \in Pos(A, C)$: $x \in A$ and $\exists f \in C^{+I}$ such that $f(x) = \sup f(A)$. Set $U = \{x \in X \mid f(x) > 0\}$. It is easy to see that (2.1) is satisfied. To show that (2.2) is also satisfied, let $u \in cone(A-x) \bigcap U$ and suppose that

$$u = t(a-x)$$

where $t > 0$ and $a \in A$. Then $0 < f(u) = t(f(a) - f(x))$ and hence $f(a) > f(x)$. This contradicts to the fact that $f(x) = \sup f(A)$. Q.E.D.

2.3 The Main Result

Theorem 2.3.1 *Let X be a topological vector space, $C \subseteq X$ be a weakly closed convex cone and let C^{+I} be nonempty. Suppose that A is a compact convex subset of X, then $E(A, C) \subseteq cl(Pos(A, C))$.*

Proof. Let $\bar{a} \in E(A, C)$. Without loss of generality, assume $\bar{a} = 0$. (If necessary, we do translation on A). For each $(n, V) \in \mathcal{D}$, by the Alaoglu-Banach theorem [18], V^0 is a weak*-compact convex subset of X^*. Hence $B(n, V) = \frac{1}{n}\bar{p} + V^0 \bigcap C^+$ is also a weak*-compact convex subset of X^*, for C^+ is weak*-closed convex and $V^0 \bigcap C^+ \subseteq V^0$.

Define a function $\Phi : B(n,V) \times A \longrightarrow \mathbb{R}$ by $\Phi(f,a) = f(a)$ for $(f,a) \in B(n,V) \times A$. The definition of weak*-topology implies that Φ satisfies the hypothesis of the minimax principle on $B(n,V) \times A$. Hence, there are $a_{(n,V)} \in A$ and $f^0_{(n,V)} \in B(n,V)$ such that for all $a \in A$ and all $f \in B(n,V)$,

$$f^0_{(n,V)}(a) \leq f^0_{(n,V)}(a_{(n,V)}) \leq f(a_{(n,V)}) \tag{2.4}$$

The first inequality implies that $a_{(n,V)} \in Pos(A,C)$. Letting $a = 0$ in (2.4),

$$f(a_{(n,V)}) \geq 0 \tag{2.5}$$

for all $f \in B(n,V)$ (in fact, for all $f \in A(n,V)$). By Lemma 2.2.1, $\{a_{(n,V)}\}_{(n,V)\in\mathcal{D}}$ is a net in $Pos(A,C)$. Since A is compact, there is a subnet of $\{a_{(n,V)}\}_{(n,V)\in\mathcal{D}}$ which converges to an $a_0 \in A$ thus in particular, $a_0 \in cl(Pos(A,C))$.

It remains to show that $a_0 = 0$. To simplify notations, we assume without loss of generality that $a_{(n,V)} \longrightarrow a_0$. From Lemma 2.2.2, we know that if $f \in A(n_0, V_0)$, then $f \in A(n,V)$ for all $(n,V) \geq (n_0,V_0)$. Therefore, by passing to the limit in (2.5), we get $f(a_0) \geq 0$ for all $f \in A(n,V)$. Since (n_0,V_0) is arbitrary, it follows from (ii) of Lemma 2.2.2, $a_0 \in C$. Now, $a_0 \geq 0 = \bar{a}$, and since $\bar{a} \in E(A,C)$, it follows from the definition of $E(A,C)$ that $a_0 = 0$. Q.E.D.

Corollary 2.3.1 *Let X be a topological vector space, $C \subseteq X$ be a weakly closed convex cone and $C^{+I} \neq \emptyset$. Also let A be a closed convex subset of X and $\bar{a} \in E(A,C)$. Suppose there is a convex neighborhood V of 0 in X such that $(\bar{a}+V)\bigcap A$ is compact. Then $\bar{a} \in cl(Pos(A,C))$.*

Proof. For each neighborhood U of 0 in X, there is a neighborhood W of 0 such that $W \subseteq U \bigcap (\frac{1}{2}V)$. We have to prove that $(\bar{a}+W)\bigcap Pos(A,C) \neq \emptyset$. By Theorem 2.3.1, $\bar{a} \in cl(Pos(A\bigcap(\bar{a}+V), C))$. Therefore, there is a point a_0 in $Pos(A\bigcap(\bar{a}+V), C)$ such that $a_0 \in \bar{a}+W$. It is sufficient to show that a_0 is in

$Pos(A,C)$. By a_0 is in $Pos(A\bigcap(\bar{a}+V),C)$, there is $f\in C^+$ such that for each x in $A\bigcap(\bar{a}+W)$,

$$f(x)\leq f(a_0). \tag{2.6}$$

Since $\frac{1}{2}V$ is a neighborhood of 0, for each $a\in A$, there is $0<t<1$ such that $t(a-a_0)\in\frac{1}{2}V$. By the convexity of A, $a_0+t(a-a_0)=(1-t)a_0+ta\in A$. Also, $\bar{a}+V\supseteq\bar{a}+\frac{1}{2}V+\frac{1}{2}V\supseteq\bar{a}+W+\frac{1}{2}V\ni a_0+t(a-a_0)$. So, $a_0+t(a-a_0)\in(\bar{a}+V)\bigcap A$. Since (2.5), $f(a_0+t(a-a_0))\leq f(a_0)$. This implies $f(a)\leq f(a_0)$ as f is linear. Since a is freely chosen from A, it follows that $a_0\in Pos(A,C)$. Q.E.D.

Theorem 2.3.2 *Let X be a locally convex topological vector space. Let $C\subseteq X$ be a convex cone with $C^{+I}\neq\emptyset$ and A a closed convex subset of X. Let $a_0\in E(A,C)$ and V be a convex neighborhood of 0 in X such that $(a_0+V)\bigcap A$ is relatively weakly compact. Suppose that C has a closed base B such that $B\subseteq V$, then*

$$(a_0+V)\bigcap Pos(A,C)\neq\emptyset.$$

Proof. Without loss of generality, we assume that $a_0=0$. We have to show $V\bigcap Pos(A,C)\neq\emptyset$. Set $V_0=cl(\frac{1}{2}V)$. Since V is convex, we have $V_0\subseteq\frac{1}{2}V+\frac{1}{2}V=V$. Set also $B_0=\frac{1}{4}B$. Then B_0 is a closed base of C and $B_0\subseteq\frac{1}{2}V_0$. Since $0\in E(A,C)$, we have $A\bigcap B_0=\emptyset$. Let $A_0=A\bigcap V_0$. A_0 is a weakly compact convex subset of X and $A_0\bigcap B_0=\emptyset$. By the Separation Theorem [18] p.74, there exists $f\in X^*$ and $\bar{a}\in A_0$ such that

$$f(\bar{a})=\sup\{f(a)\mid a\in A_0\}<\inf\{f(b)\mid b\in B_0\}=r$$

Since $\{x\mid |f(x)|<r-f(\bar{a})\}$ is a convex neighborhood of 0, $W\equiv(\frac{1}{2}V_0)\bigcap\{x\mid |f(x)|<r-f(\bar{a})\}$ is also a convex neighborhood of 0. By checking the definitions of both sides, we have,

$$B_0+W\subseteq\{x\mid f(x)>f(\bar{a})\}\bigcap V_0. \tag{2.7}$$

Since $\bar{a} \in V$, if we can show that $\bar{a} \in Pos(A, C)$, we prove the theorem. Because $C = cone(B_0) \subseteq cone(B_0 + W)$ and Lemma 2.2.3, it suffices to prove that

$$cone(A - \bar{a}) \bigcap (B_0 + W) = \emptyset \tag{2.8}$$

Suppose there exists $b \in B_0 + W$, $\hat{a} \in A$ and $t > 0$ such that $b = t(\hat{a} - \bar{a})$ i.e. $\hat{a} = \bar{a} + \frac{1}{t}b$. Since A is convex and $0 \in A$, $\frac{t}{1+t}\hat{a} \in A$. As $b \in V_0$ and $\bar{a} \in V_0$, $\frac{t}{1+t}\hat{a} = \frac{t}{1+t}\bar{a} + \frac{1}{1+t}b \in V_0$. Thus $\frac{t}{1+t}\hat{a} \in A \bigcap V_0 = A_0$ and

$$f(\frac{t}{1+t}\hat{a}) \leq f(\bar{a}). \tag{2.9}$$

On the other hand, $f(\frac{t}{1+t}\hat{a}) = \frac{t}{1+t}f(\bar{a}) + \frac{1}{1+t}f(b)$. By 2.7, we have $f(\frac{t}{1+t}\hat{a}) > f(\bar{a})$. This contradicts to 2.9. Q.E.D.

Corollary 2.3.2 *Let X be a locally convex topological vector space, C be a convex cone. Suppose $C^{+I} \neq \emptyset$ and C has a closed and bounded base in X, and let A be a closed convex subset of X and $\bar{a} \in E(A, C)$. If there is a neighborhood V of 0 in X such that $(\bar{a} + V) \bigcap A$ is relatively weak-compact, then $\bar{a} \in cl(Pos(A, C))$.*

Proof. For each neighborhood U of 0 in X, there is a convex neighborhood W of 0 such that $W \subseteq U \bigcap V$. Hence $(\bar{a}+W) \bigcap A$ is relatively weak-compact. Let B be a closed and bounded base of C, then there is a $t > 0$ such that $tB \subseteq W$. It is clear that tB is still a closed base of C. By Theorem 2.3.2, $(\bar{a} + W) \bigcap Pos(A, C) \neq \emptyset$, therefore $(\bar{a} + U) \bigcap Pos(A, C) \neq \emptyset$. This means that $\bar{a} \in cl(Pos(A, C))$. Q.E.D.

Corollary 2.3.3 *Let X be a locally convex topological vector space, $C \subseteq X$ a convex cone with $C^{+I} \neq \emptyset$ and $A \subseteq X$ a closed convex set. Let also $a_0 \in E(A, C)$ and V a convex neighborhood of 0 such that $(a_0 + V) \bigcap A$ is relatively weak-compact. Suppose C has a closed base B with $B \subseteq V$, then $a_0 \in \mu_V -$ $cl(Pos(A, C))$. (which means: there is a sequence $\{a_n\}$ in $Pos(A, C)$ such that $\mu_V(a_n - a_0) \longrightarrow 0$, where μ_V is the Minkowski functional of V.)*

Proof. For each natural number n, let $U_n = \{x \in X \mid \mu_V(x) \leq \frac{1}{n}\}$. Then U_n is a convex neighborhood of 0 in X. We know that

$$\{x \in X \mid \mu_V(x) < 1\} \subseteq V \subseteq \{x \in X \mid \mu_V(x) \leq 1\} = nU_n$$

This implies that $\frac{1}{n}V \subseteq U_n \subseteq V$. Hence $(a_0 + U_n) \bigcap A$ is relatively weak-compact and $\frac{1}{n}B \subseteq U_n$. By Theorem 2.3.2, $(a_0 + U_n) \bigcap Pos(A, C) \neq \emptyset$. Therefore, there is a_n in $Pos(A, C)$ such that $a_n - a_0 \in U_n$, that is $\mu_V(a_n - a_0) \leq \frac{1}{n}$. So $\lim \mu_V(a_n - a_0) = 0$. Q.E.D.

Corollary 2.3.4 *Let X be a locally convex topological vector space, C a convex cone in X and A a weakly compact convex subset of X. Suppose $C^{+I} \neq \emptyset$. If there are a convex neighborhood V of 0 in X and a closed base B of C such that $B \subseteq V$, then $\mu_V - cl(Pos(A, C)) \supseteq E(A, C)$, particularly $Pos(A, C) - V \supseteq E(A, C)$.*

Proof. Applying Corollary 2.3.3 .

Chapter 3

Introduction to Variational Inequalities

3.1 Variational Inequalities in Finite Dimensional Space

In this chapter, we introduce the concept of variational inequalities(VI). Our main references are [7] and [11].We will discuss VI from $\mathbb{R}^N$ to a general Hilbert Space. Also, we will see some problems which lead to VI. After we get familiar with VI, we will discuss some recent results in the next two chapters.

So, what is Variational Inequality or what is a VI problem (in $\mathbb{R}^n$)? Given E a subset in $\mathbb{R}^n$ and $F : E \longrightarrow \mathbb{R}^n$ is continuous. To find an $x \in E$ such that

$$\langle F(x), y - x \rangle \geq 0 \quad \forall y \in E \tag{3.1}$$

is what we call a VI problem, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. If we rewrite the above inequality as $\langle F(x), y \rangle \geq \langle F(x), x \rangle$, we realize that a VI problem is on one hand more complicate than an optimization problem in nature as the function $\langle F(x), \cdot \rangle - \langle F(x), x \rangle$ is varying for different x. On the other hand, we can say it is much simpler as $\langle F(x), \cdot \rangle - \langle F(x), x \rangle$ is an affine function.

Now, let's look at some results of VI in $\mathbb{R}^n$. For the sake of simplifying notations, we make no difference for an element in $(\mathbb{R}^n)^*$ and its identification in $\mathbb{R}^n$.

Theorem 3.1.1 *Let $K \subseteq \mathbb{R}^n$ be compact and convex; and let $F : K \longrightarrow \mathbb{R}^n$ be continuous. Then there is an $x \in K$ such that*

$$\langle F(x), y - x \rangle \geq 0 \quad \textit{for all} \quad y \in K. \tag{3.2}$$

Proof.
It is equivalent to showing that there exists $x \in K$ such that

$$\langle x, y - x \rangle \geq \langle x - F(x), y - x \rangle \quad \forall y \in K.$$

The mapping $Pr_K \circ (I - F) : K \longrightarrow K$ (where I is the identity mapping and Pr_K is the projection over the convex set K) is continuous. Moreover, in our hypothesis, K is compact and convex, we can then make use of the Brouwer's fixed point theorem [8] which promise us a fixed point $x \in K$, namely, $x = Pr_K \circ (I - F)(x)$. Consequently, by the geometric characterization of the projection, $\langle x, y - x \rangle \geq \langle x - F(x), y - x \rangle$ for all $y \in K$. Q.E.D.

Corollary 3.1.1 *Let x be a solution to (3.1) and suppose that $x \in intK$. Then* $F(x) = 0$.

Proof.
If $x \in intK$, then $K - x$ contains an neighborhood of the origin. So, for any $\zeta \in \mathbb{R}^n$, there are $\lambda \geq 0$ and $y \in K$ such that $\zeta = \lambda(y - x)$. Consequently,

$$\langle F(x), \zeta \rangle = \lambda \langle F(x), y - x \rangle \geq 0 \quad \forall \zeta \in \mathbb{R}^n,$$

from which it follows that $F(x) = 0$. Q.E.D.

Sometimes, we have to weaken the requirements of K stated in the last theorem. Suppose that K is closed instead of compact and we still assume that K is convex. Name this new VI problem (VIN). In this case, we do not always have a solution. For example, if $K = \mathbb{R}$, a closed subspace of the finite dimensional $\mathbb{R}^N$, $f(x)(y - x) \geq 0$ for all $y \in K$ has no solution for $f(x) = e^x$. Naturally, we want to find a necessary and sufficient condition for the existence of solutions of problem (VIN).

Given a convex set K, we set

$$K_R = K \bigcap \Sigma_R$$

where Σ_R is the closed ball with radius R and center 0.

Theorem 3.1.2 *Let $K \subseteq \mathbb{R}^N$ be closed and convex ;and let $F : K \longrightarrow \mathbb{R}^N$ be continuous. A necessary and sufficient condition that there exist a solution to Problem (VIN) is that there exists an $R > 0$ such that a solution $x_R \in K_R$ of*

$$\langle F(x_R), y - x_R \rangle \geq 0 \qquad \textit{for all} \quad y \in K_R \tag{3.3}$$

satisfies

$$|x_R| < R. \tag{3.4}$$

Proof.

First of all, we notice that a solution x_R of (3.2) always exists as a consequence of Theorem (3.1)(if K_R is non-empty). If there exists a solution x to Problem(VIN), then take R be large enough to make K_R nonempty and $R > |x|$.

Suppose conversely that there exists $R > 0$ and $x_R \in K_R$ which is a solution of (3.2)and satisfies (3.3). Then since $|x_R| < R$, given $y \in K$,

$$w = x_R + \epsilon(y - x_R) \in K_R$$

for all sufficiently small $\epsilon > 0$. Consequently,

$$0 \leq \langle F(x_R), w - x_R \rangle = \epsilon \langle F(x_R), y - x_R \rangle$$

and hence

$$0 \leq \langle F(x_R), y - x_R \rangle.$$

This means that x_R is a solution to Problem(VIN). Q.E.D.

Now, to prepare our discussion of variational inequalities over general Hilbert spaces, we introduce the concept of coerciveness.

Corollary 3.1.2 *Let $F : K \longrightarrow \mathbb{R}^N$ satisfy*

$$\frac{\langle F(x) - F(x_0), x - x_0 \rangle}{|x - x_0|} \longrightarrow +\infty \qquad as \quad |x| \longrightarrow +\infty, \qquad x \in K \tag{3.5}$$

for some $x_0 \in K$. Then there exists a solution to the problem(VIN).

Proof.
By hypothesis, we can choose $H > |F(x_0)|$ and $R > |x_0|$ such that $\langle F(x) - F(x_0), x - x_0 \rangle \geq H|x - x_0|$ for $|x| \geq R$, $x \in K$. Then

$$\begin{aligned} \langle F(x), x - x_0 \rangle &\geq H|x - x_0| + \langle F(x_0), x - x_0 \rangle \\ &\geq H|x - x_0| - |F(x_0)||x - x_0| \\ &\geq (H - |F(x_0)|)(|x| - |x_0|) \\ &> 0 \qquad whenever \qquad |x| \geq R. \end{aligned} \tag{3.6}$$

Now let $x_R \in K_R$ be a solution of (3.2). Note that $x_0 \in K_R$ and $\langle F(x_R), x_R - x_0 \rangle = -\langle F(x_R), x_0 - x_R \rangle \leq 0$. So, by (3.5), we must have $|x_R| < R$. Q.E.D.

Condition (3.4) is what we call the coerciveness condition.

The solution to the variational inequality is generally not unique. Suppose that $x, x' \in K$ are two distinct solutions to the Problem(VIN): $\langle F(x), y-x\rangle \geq 0$ and $\langle F(x'), y-x'\rangle \geq 0$ for all $y \in K$. Setting $y = x'$ in the first inequality, $y = x$ in the second, and adding the two we obtain $\langle F(x) - F(x'), x - x'\rangle \leq 0$. Hence a necessary and sufficient condition for uniqueness is that

$$\langle F(x) - F(x'), x - x'\rangle > 0 \qquad whenever \qquad x \neq x'. \tag{3.7}$$

Definition 3.1.1 *We call a mapping* $F : K \longrightarrow \mathbb{R}^N$ *monotone if* $\langle F(x) - F(x'), x - x'\rangle \geq 0$ *for all* $x, x' \in K$*. It is called strictly monotone if equality holds only when* $x = x'$*, that is , when condition (3.7) is valid.*

Proposition 3.1.1 *Let* $F : K_1 \longrightarrow \mathbb{R}^N$ *be a continuous strictly monotone mapping on a closed convex set* $K_1 \subseteq \mathbb{R}^N$*. Let* $K_2 \subseteq K_1$ *be closed and convex. Suppose that there exist solutions of the problems*

$$x_j \in K_j : \quad \langle F(x_j), y - x_j\rangle \geq 0 \quad for \quad all \quad y \in K_j, \quad j = 1, 2 \tag{3.8}$$

(i) If $F(x_2) = 0$*, then* $x_1 = x_2$*.*
(ii) If $F(x_2) \neq 0$ *and* $x_1 \neq x_2$*, then the hyperplane* $\langle F(x_2), y - x_2\rangle = 0$ *separates* x_1 *from* K_2

Proof.
(i) By (3.7) and the monotonicity, $\langle F(x_1) - F(x_2), x_1 - x_2\rangle = 0$. By the strict monotonicity, $x_1 = x_2$.
(ii) Since $x_1 \neq x_2$, by (3.7) and the strict monotonicity, $0 \leq \langle F(x_1), x_2 - x_1\rangle < \langle F(x_2), x_2 - x_1\rangle$. By (3.8), we also have $\langle F(x_2), y - x_2\rangle \geq 0$ for all $y \in K_2$. Q.E.D.

3.2 Problems Which Relate to Variational Inequalities

Before going on to infinite dimensional VI, we introduce some problems that are associated to variational inequalities.

Remember we have said that a VI problem is more complicate than an optimization problem. Yet in some special situations they are the same. Here is a well known example.

Let $f \in C^1(K)$, $K \subseteq \mathbb{R}^N$ be a closed convex set, and set $F(x) = \nabla f(x)$, we have the following propositions:

Proposition 3.2.1 *Suppose there exists an $x \in K$ such that $f(x) = \min_{y \in K} f(y)$. Then x is a solution of the variational inequality*

$$\langle F(x), y - x \rangle \geq 0 \qquad for\ all \qquad y \in K.$$

Proof.

If $y \in K$, then $z = x + t(y - x) \in K$ for $0 \leq t \leq 1$; therefore the function $\psi(t) = f(x + t(y - x))$, $0 \leq t \leq 1$, attains its minimum when $t = 0$. Consequently, $0 \leq \psi'(0) = \langle \nabla f(x), y - x \rangle$. Q.E.D.

Proposition 3.2.2 *Suppose f is convex and x satisfies $\langle F(x), y - x \rangle \geq 0$ for all $y \in K$. Then*

$$f(x) = \min_{y \in K} f(y).$$

Proof.

If f is convex, it is elementary nowadays with the idea of subgradient (e.g. theorem 25.1 in [17]) that

$$f(y) \geq f(x) + \langle F(x), y - x \rangle \qquad for\ any \qquad y \in K.$$

But $\langle F(x), y - x \rangle \geq 0$ for all $y \in K$, so $f(y) \geq f(x)$ for all $y \in K$. Q.E.D.

Proposition 3.2.3 *Let $f : E \longrightarrow \mathbb{R}$, be a continuously differentiable convex (strictly convex) function. Then $F(x) = \nabla f(x)$ is monotone (strictly monotone).*

Proof.

Given $x, x' \in E$. By convexity of f,

$$f(x) \geq f(x') + \langle F(x'), x - x' \rangle \qquad and$$

$$f(x') \geq f(x) + \langle F(x), x' - x \rangle.$$

Adding these,

$$\langle F(x') - F(x), x' - x \rangle \geq 0.$$

Hence F is monotone. Q.E.D.

Another example is about complementarity.

Problem 3.2.1 *Let* $\mathbb{R}^N_+ = \{x = (x_1, \dots x_N) \in \mathbb{R}^N \mid x_i \geq 0$ *for all* $i = 1, 2, \dots, N\}$ *and let* $F : \mathbb{R}^N_+ \longrightarrow \mathbb{R}^N$*. Find* $x_0 \in \mathbb{R}^N_+$ *such that* $F(x_0) \in \mathbb{R}^N_+$ *and* $\langle F(x_0), x_0 \rangle = 0$.

Theorem 3.2.1 *The point* $x_0 \in \mathbb{R}^N_+$ *is a solution to problem 3.2.1 if and only if*

$$\langle F(x_0), y - x_0 \rangle \geq 0$$

for all $y \in \mathbb{R}^N_+$.

Proof.

If x_0 is a solution to problem 3.2.1, $\langle F(x_0), y \rangle \geq 0$ for any $y \in \mathbb{R}^N_+$, so

$$\langle F(x_0), y - x_0 \rangle = \langle F(x_0), y \rangle - \langle F(x_0), x_0 \rangle = \langle F(x_0), y \rangle \geq 0.$$

On the other hand suppose that $x_0 \in \mathbb{R}^N_+$ is a solution to the variational inequality. Then, since $y = x_0 + e_i \in \mathbb{R}^N_+$ for $0 \leq i \leq N$,

$$0 \leq \langle F(x_0), x_0 + e_i - x_0 \rangle = \langle F(x_0), e_i \rangle \equiv F_i(x_0).$$

That is $F(x_0) \in \mathbb{R}^N_+$. Moreover, since $y = 0 \in \mathbb{R}^N_+$,

$$\langle F(x_0), x_0 \rangle \leq 0.$$

Finally, x_0, $F(x_0) \in \mathbb{R}^N_+$ implies that $\langle F(x_0), x_0 \rangle \geq 0$. Q.E.D.

3.3 Some Variations on Variational Inequality

In the VI problem mentioned in section (3.1), if we allow the set E to vary depending on x, we have the so call Quasi Variational Inequality(QVI) problem. That is: for $F : E \longrightarrow \mathbb{R}^n$ and $C : E \longrightarrow 2^{\mathbb{R}^n}$, we want to find a $x^* \in E$ such that

$$x^* \in C(x^*)$$

and

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in E.$$

Note that a necessary condition for a QVI problem to have nonempty solution set is that F has to have fixed points.

Another variation of VI problem is to allow the function F be set-valued. That is: for the set-valued function $F : E \longrightarrow 2^{\mathbb{R}^n}$ we want to find $x^* \in E$ and $y \in F(x^*)$ such that

$$\langle y, x - x^* \rangle \geq 0 \quad \forall x \in E.$$

The combination GVI and QVI is GQVI. We will have a chapter for this kind of problem.

Now, look at propositions 3.2.1 and 3.2.2 again. If the function to be optimized is vector-valued and we are dealing with the corresponding vector optimization problem, we still want to establish the relation between the vector optimization problem and the VI problem. In this case, ∇F is a matrix i.e. a linear operator. Therefore, we introduce the Vector Variational Inequality(VVI) problem. Let $T : E \longrightarrow L(E, \mathbb{R}^p)$, where $L(E, \mathbb{R}^p)$ is the set of linear operator from E to $\mathbb{R}^p$. The VVI problem is to find a $x^* \in E$ such that

$$\langle T(x^*), x - x^* \rangle \notin -int\mathbb{R}^p_+ \quad \forall x \in E.$$

Note that when $p = 1$, a VVI becomes a VI.

3.4 The Vector Variational Inequality Problem and Its Relation with The Vector Optimization Problem

We will generalize some result in Section (3.1). Let $C \subseteq \mathbb{R}^n$ be nonempty, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ a vector-valued function. If a vector $x^* \in C$ satisfy

$$f(x) - f(x^*) \notin -int\mathbb{R}^p_+ \quad \forall x \in C$$

then x^* is call the solution of the vector minimization problem (f, C), or simply (f, C) problem. Now set S be the set of solutions of the (f, C) problem and $W \equiv \mathbb{R}^p \setminus (-int\mathbb{R}^p_+)$, we can have the following theorem.

Theorem 3.4.1 *In the (f, C) problem, if C is convex and each component f_i of f is convex and f is differentiable, then $x^* \in S$ if and only if x^* satisfies the VVI with $E = C$ and T be defined as*

$$T(x) = \nabla f(x).$$

Proof.

The proof is similar to that of proposition 3.2.2.

Theorem 3.4.2 *Let A be a subset of a Hausdorff topological vector space Z. For each $x \in A$, let a closed set $G(x) \subseteq Z$ be given such that $G(x)$ is compact for at least one $x \in A$. If the convex hull of every finite subset $\{x_1, x_2, \ldots, x_m\} \subseteq A$ is contained in the corresponding union $\bigcup_{i=1}^m G(x_i)$ then $\bigcap_{x \in A} G(x) \neq \emptyset$.*

Proof.

Please refer to [3].

Theorem 3.4.3 *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set; let $f : C \longrightarrow \mathbb{R}_p$ be continuously differentiable, with each component f_i convex; let the set $\{x \in C \mid \nabla f(x)(q - x) \in W\}$ be compact for some $q \in C$. Then, the (f, C) problem has a solution x^*.*

Proof.
By Theorem 3.4.1 it suffices to prove that there is a vector x^* such that the vector variational inequality $\nabla f(x^*)(x - x^*) \in W$ is satisfied for all $x \in C$.

For any $y \in C$, define

$$G(y) \equiv \{x \in C \mid \nabla f(x)(y - x) \in W\}.$$

Let also

$$\{x_1, x_2, \ldots, x_m\} \subseteq C,$$

and

$$\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0, \quad \sum \alpha_i = 1.$$

Now, suppose

$$x \equiv \sum \alpha_i x_i \notin \bigcup_{i=1}^{m} G(x_i),$$

then (by the definition of $G(x_i)$) $\nabla f(x_i - x) \notin W$ for all i. Thus,

$$\nabla f(x)x = \sum \alpha_i \nabla f(x) x_i \in -int\mathbb{R}^p_+ + \nabla f(x)x,$$

which is a contradiction in itself. Therefore, we have

$$co\{x_1, x_2, \ldots, x_m\} \subseteq \bigcup_{i=1}^{m} G(x_i),$$

where $co(K)$ is the intersection of all convex sets containing K for any set K. Now let a sequence $\{x_k\} \subseteq G(y)$ such that $\|x_k - x\| \longrightarrow 0$. Since $\nabla f(\cdot)$ is continuous on C, we have the sequence of functions $\{\nabla f(x_k)\} \longrightarrow \nabla f(x)$ uniformly on C. Then,

$$\begin{aligned}\|\nabla f(x_k)(y-x_k) - \nabla f(x)(y-x)\| &\leq \|\nabla f(x_k)(y-x_k) - \nabla f(x)(y-x_k)\| \\ &\quad + \|\nabla f(x)(y-x_k) - \nabla f(x)(y-x)\} \\ &\leq \|\nabla f(x_k) - \nabla f(x)\| \left(\|y-x\| + \|x-x_k\|\right) \\ &\quad + \|\nabla f(x)\|\|x-x_k\|\end{aligned}$$

which tends to zero when k tends to infinity. Since W is closed and $\{\nabla f(x_k)(y-x_k)\} \subseteq W$, we have

$$\nabla f(x)(y-x) \in W.$$

Thus, $G(y)$ is closed. By Theorem 3.4.2, it follows that there exists a point $x^* \in \bigcap_{y \in C} G(y)$. That is

$$\nabla f(x^*)(y-x^*) \in W$$

for all $y \in C$. Q.E.D.

3.5 Variational Inequalities in Hilbert Space

Before going further, we need to make our notation a little bit clearer than previous sections. Also, we will reformulate the VI problem in terms of bilinear forms.

Let H be a real Hilbert space and let H^* denote its dual. $(\cdot, \cdot)$ is the inner product on H, and $\|\cdot\|$ is its norm. For any $f \in H^*$ and $x \in H$, $\langle f, x \rangle \in \mathbb{R}$ is the pairing between H^* and H.

Observe that one side of the VI is a linear operator which vary depending on x^*. Yet, in pervious sections, we did not limit the way of this variation. Now, in this section, the linear operator vary linearly. Let $a(u, v)$ be a bilinear form on

H, i.e., $a : H \times H \longrightarrow \mathbb{R}$ is continuous and linear in each of the variables u, v. A bilinear form $a(u, v)$ is symmetric if $a(u, v) = a(v, u)$ for all $u, v \in H$. A linear and continuous mapping $A : H \longrightarrow H^*$ determines a bilinear form via the pairing

$$a(u, v) = \langle Au, v \rangle. \tag{3.9}$$

This is the consequence of the Riesz representation theorem. In fact, for any bounded linear mapping A, $\langle Au, v \rangle$ is a bilinear form. On the other hand, for any bilinear form $a(u, v)$, Riesz representation theorem [12] says that $a(u, v) = (Su, v)$ for some unique bounded linear operator S from H to H. So, we can define A by

$$u \longmapsto Su \longmapsto (Su, \cdot).$$

It can be show that A is bounded linear.

Definition 3.5.1 *The bilinear form $a(u, v)$ is coercive on H if there exists $\alpha > 0$ such that*

$$a(v, v) \geq \alpha \|v\|^2 \qquad \textit{for all} \qquad v \in H. \tag{3.10}$$

The bilinear form $a(u, v)$ is coercive if and only if the mapping A defined by (3.9) is coercive in the sense of section 3.1. This statement is to be shown. Now we reformulate the VI problem in terms of bilinear forms.

Problem 3.5.1 *Let $K \subseteq H$ be closed and convex and $f \in H^*$. To find a $u \in K$ such that*

$$a(u, v - u) \geq \langle f, v - u \rangle \qquad \textit{for all} \qquad v \in K. \tag{3.11}$$

Theorem 3.5.1 *Let $a(u, v)$ be a coercive bilinear form on H, $K \subseteq H$ closed and convex and $f \in H^*$. Then there exists a unique solution to problem 3.5.1. In addition, the mapping $f \longrightarrow u$ is Lipschitz, that is , if u_1, u_2 are solutions to problem 3.5.1 corresponding to f_1, $f_2 \in H^*$, then*

$$\|u_1 - u_2\| \leq \frac{1}{\alpha} \|f_1 - f_2\|. \tag{3.12}$$

Also, the mapping $f \longrightarrow u$ is linear if K is a subspace of H.(The norm on the right hand side is the norm in H^*.)

Proof.

We first prove inequality (3.12). Suppose there exist $u_1,\ u_2 \in K$ solutions of the variational inequalities

$$a(u_i, v - u_i) \geq \langle f_i, v - u_i \rangle \qquad for\ all \quad v \in K, \quad i = 1, 2.$$

Setting $v = u_2$ in the variational inequality for u_1 and $v = u_1$ in that for u_2 we obtain, by adding up two inequalities,

$$a(u_1 - u_2, u_1 - u_2) \leq \langle f_1 - f_2, u_1 - u_2 \rangle.$$

Hence by the coerciveness of a,

$$\alpha \|u_1 - u_2\|^2 \leq \langle f_1 - f_2, u_1 - u_2 \rangle \leq \|f_1 - f_2\| \cdot \|u_1 - u_2\|,$$

and therefore (3.12) holds. Note that we have also proved the uniqueness of the solution by taking $f_1 = f_2$. We are going back to the existence of a solution. First suppose that $a(u, v)$ is symmetric and define the functional

$$I(u) = a(u, u) - 2\langle f, u \rangle, \qquad u \in H.$$

Let $d = \inf_K I(u)$. Since

$$\begin{aligned} I(u) &\geq \alpha \|u\|^2 - 2\|f\| \cdot \|u\| \\ &\geq \alpha \|u\|^2 - \left(\frac{1}{\sqrt{\alpha}}\|f\| - \sqrt{\alpha}\|u\|\right)^2 - 2\|f\| \cdot \|u\| \\ &\geq -\frac{1}{\alpha}\|f\|^2 \end{aligned}$$

we see that

$$d \geq -\frac{1}{\alpha}\|f\|^2 > -\infty.$$

For each $n \in \mathbb{N}$, let $u_n \in K$ such that $d \leq I(u_n) \leq d + \frac{1}{n}$. Keeping in mind that K is convex, we see that

$$\begin{aligned}
\alpha\|u_n - u_m\|^2 &\leq a(u_n - u_m, u_n - u_m) \\
&= 2a(u_n, u_n) + 2a(u_m, u_m) - 4a(\frac{1}{2}(u_n + u_m), \frac{1}{2}(u_n + u_m)) \\
&\quad - 4\langle f, u_n\rangle - 4\langle f, u_m\rangle + 8\langle f, \frac{1}{2}(u_n + u_m)\rangle \\
&= 2I(u_n) + 2I(u_m) - 4I(\frac{1}{2}(u_n + u_m)) \\
&\leq 2(\frac{1}{n} + \frac{1}{m}).
\end{aligned}$$

So, the sequence $\{u_n\}$ is Cauchy and the closed set K contains an element u such that $u_n \longrightarrow u$ and $I(u_n) \longrightarrow I(u)$. Hence $I(u) = d$.

For any $v \in K$, $u + \epsilon(v - u) \in K$, $0 \leq \epsilon \leq 1$, and $I(u + \epsilon(v - u)) \geq I(u)$. In other words,

$$2\epsilon a(u, v - u) + \epsilon^2 a(v - u, v - u) - 2\epsilon\langle f, u - u\rangle \geq 0$$

or

$$a(u, v - u) \geq \langle f, v - u\rangle - \frac{1}{2}\epsilon a(v - u, v - u) \qquad \textit{for any } \epsilon, \quad 0 < \epsilon \leq 1.$$

Taking $\epsilon \longrightarrow 0^+$ we see that u is a solution to problem 3.5.1. So, we are done for symmetric a.

Now, we withdraw the assumption that a being symmetric. Define $a_0(u, v) = \frac{1}{2}(a(u, v) + a(v, u))$; $b(u, v) = \frac{1}{2}(a(u, v) - a(v, u))$ and $a_t(u, v) = a_0(u, v) + tb(u, v)$, $\ 0 \leq t \leq 1$. a_t is coercive with the same constant α.

Lemma 3.5.1 *If Problem 3.5.1 is solvable for $a_s(u, v)$ and all $f \in H^*$, then it is solvable for $a_t(u, v)$ and all $f \in H^*$, where $s \leq t \leq s + t_0$, $t_0 < \frac{\alpha}{M}$, and*

$$M = \sup \frac{|b(u, v)|}{\|u\| \cdot \|v\|} < +\infty$$

Since a_0 is symmetric, Problem 3.5.1 may be solved for a_0. Applying the lemma finite number of times, we get that Problem 3.5.1 admits a solution for $t = 1$, and the theorem is proved. Therefore we are going to show the proof of lemma 3.5.1 in the rest of the proof.

Define the mapping $T_t : H \longrightarrow K$ by $u = T_t w$ if

$$u \in K : \quad a_s(u, v-u) \geq \langle F_{(t,w)}, v-u \rangle \quad for\ all \quad v \in K,$$

where

$$\langle F_{(t,w)}, v \rangle = \langle f, v \rangle - (t-s) b(w, v)$$

and $s \leq t \leq s + t_0$.

By hypothesis, u is unique. Hence T_t is well defined. Given $u_1 = T_t w_1$ and $u_2 = T_t w_2$, we may apply (3.12) and the definition of M such that

$$\begin{aligned} \|u_1 - u_2\| &\leq \frac{1}{\alpha} \|F_{(t,w_1)} - F_{(t,w_2)}\| \\ &= \frac{(t-s)}{\alpha} \|b(w_2, \cdot) - b(w_1, \cdot)\| \\ &\leq \frac{1}{\alpha}(t-s) M \|w_1 - w_2\| \\ &\leq \frac{1}{\alpha} t_0 M \|w_1 - w_2\| \end{aligned}$$

with $t_0 \frac{M}{\alpha} < 1$. Now we find that T_t is a contraction which defined on a Hilbert space H. By completeness of H, T_t admits a unique fixed point. For this fixed point u,

$$a_t(u, v-u) \geq \langle f, v-u \rangle$$

for all $v \in K$ and every $t : \ s \leq t \leq s + t_0$. Q.E.D.

Chapter 4

Vector Variational Inequalities

4.1 Preliminaries

We have presented the idea of variational inequalities in the last chapter and now we advance to the problems of vector variational inequalities. In this chapter, we will look at some recent results on the VVIP. The main references are [25] and [14]. Before introducing the notations and the VVIP itself, we first state a fixed point theorem (for set valued functions). A detail proof of this fixed point theorem involves very much in topology, so we omit it here. For the outline of the proof, please check Theorem 1 of [16].

Theorem 4.1.1 *Let X be a nonempty and convex subset of a Hausdorff topological vector space E, K a nonempty and compact subset of X. Let A, $B : X \longrightarrow 2^X$ (the power set of X) be two multi functions. Suppose that for all $x \in X$, (i) $A(x) \subseteq B(x)$, (ii) $B(x)$ is convex, (iii) $\forall x \in K$, $A(x) \neq \phi$, (iv) $\forall y \in X$, $A^{-1}(y)$ is open, (v) for each finite subset N of X, there exists a nonempty, compact and convex subset L_N of X containing N such that $\forall x \in L - N \setminus K$, $A(x) \bigcap L_N \neq \phi$. Then B has a fixed point.*

4.2 Notations

Let E be a (Hausdorff) topological vector space and E^* separates points on E, X a nonempty convex subset of E and F another topological vector space. $C : X \longrightarrow 2^F$ is a multi function such that for all $x \in X$, $C(x)$ is a convex cone in F with $intC(x) \neq \phi$ and $C(x) \neq F$. $G : X \times X \longrightarrow F$ is a function. Denote $L(E, F)$ the space of all continuous linear maps from E to F. Let $T : X \longrightarrow L(E, F)$ be an operator. Here are some concepts which are related to the VVIP.

G is weakly C-pseudo monotone if and only if for all $x, y \in X$,

$$G(x,y) \notin -intC(x) \Longrightarrow -G(y,x) \notin -intC(x).$$

T is weakly C-pseudo monotone if and only if for all $x, y \in X$,

$$\langle T(x), y-x\rangle \notin -intC(x) \Longrightarrow \langle T(y), y-x\rangle \notin -intC(x).$$

T is C-pseudo monotone if and only if for all $x, y \in X$,

$$\langle T(x), y-x\rangle \notin -C(x)\setminus\{0\} \Longrightarrow \langle T(y), y-x\rangle \notin -C(x)\setminus\{0\}.$$

G is v-hemicontinuous if and only if for all $x, y \in X$ and all $t \in [0,1]$, the map

$$t \longmapsto G(x + t(y-x), y)$$

is continuous at 0^+.

T is v-hemicontinuous if and only if for all $x, y \in X$ and all $t \in [0,1]$, the map

$$t \longmapsto \langle T(x + t(y-x)), y-x\rangle$$

is continuous at 0^+.

Finally, these are two examples of VVIPs :

To find an $x \in X$ such that $\langle T(x), y-x\rangle \notin -intC(x)$ for all $y \in X$.

To find an $x \in X$ such that $\langle T(x)y-x\rangle \notin -C(x)\setminus\{0\}$ for all $y \in X$.

4.3 Existence Results of Vector Variational Inequality

Lemma 4.3.1 *Let E, F be two Hausdorff topological vector spaces, X a nonempty and convex subset of E. Let $C : X \longrightarrow 2^F$ be a multi function such that $\forall x \in X$, $C(x)$ is a convex cone of F with $intC(x) \neq \phi$ and $C(x) \neq F$. $G : X \times X \longrightarrow F$ is a vector valued function. Define $P \equiv \bigcap_{x \in X} C(x)$. Consider the following problems:*
(P) Find an $x \in X$ such that $G(x, y) \notin -intC(x)$, $\forall y \in X$;
(P') Find an $x \in X$ such that $-G(y, x) \notin -intC(x)$, $\forall y \in X$.
Then:
(i) Problem (P) implies problem (P') if G is weakly C-pseudo monotone.
(ii) Problem (P') implies problem (P) if the following conditions are satisfied:
(a) G is v-hemicontinuous,
(b) $\forall x \in X$, $G(x, \cdot)$ is P-convex, that is, $\forall y, z \in X$ and $\forall \alpha \in [0, 1]$, $G(x, \alpha y + (1 - \alpha)z) \in (\alpha G(x, y) + (1 - \alpha)G(x, z)) - P$;
(c) $\forall x \in X$, $G(x, x) \in P$.

Proof.
(i) It follows from the definition of weakly C-pseudo monotone.
(ii) Let $x \in X$ be a solution of problem (P') and conditions (a) to (c) be satisfied, We have

$$-G(y, x) \notin -intC(x) \quad \forall y \in X. \tag{4.1}$$

Suppose that x is not a solution of problem (P). Then there exists $\hat{y} \in X$ such that

$$G(x, \hat{y}) \in -intC(x). \tag{4.2}$$

Since X is convex, $x_t \equiv x + t(\hat{y} - x) \in X$ for all $t \in [0, 1]$. Moreover, $G(x_t, \hat{y}) \longrightarrow G(x, \hat{y})$ as $t \searrow 0$ because of the definition of G being v-hemicontinuous. This and

(4.2) imply that $\exists \hat{t} \in (0,1]$ such that

$$G(x_t, \hat{y}) \in -intC(x), \quad \forall t \in (0, \hat{t}). \tag{4.3}$$

Fix $t \in (0, \hat{t})$ By the P-convexity of $G(x_t, \cdot)$, we have

$$G(x_t, x_t) = G(x_t, t\hat{y} + (1+t)x) \in (tG(x_t, \hat{y}) + (1-t)G(x_t, x)) - P. \tag{4.4}$$

From (4.3) and condition (c), we play cancelation in (4.4) and get

$$\begin{aligned}
-(1-t)G(x_t, x) &\in (tG(x_t, \hat{y}) - G(x_t, x_t)) - P \\
&\subseteq -intC(x) - P - P \\
&\subseteq -intC(x) - C(x) - C(x) \\
&\subseteq -intC(x).
\end{aligned}$$

Hence $-G(x_t, x) \in -intC(x)$, which contradicts (4.1). Q.E.D.

Theorem 4.3.1 *Let E, F be two Hausdorff topological vector spaces, and let E^* separate points on E. Let X be a nonempty and convex subset of E, and K a nonempty and weakly compact subset of X. Let $C : X \longrightarrow 2^F$ be a multi function such that $\forall x \in X$, $C(x)$ is a convex cone of F with $-intC(x) \neq \phi$ and $C(x) \neq F$, and $G : X \times X \longrightarrow F$ a function. Define $P \equiv \bigcap_{x \in X} C(x)$ and $W : X \longrightarrow 2^F$, $W(x) \equiv F \setminus (-intC(x))$. The graph $Gr(W)$ of W is weakly closed in $X \times F$. Assume that the following conditions are satisfied:*

(i) $\forall x \in X$, $y \longmapsto G(x,y)$ is weakly continuous and P-convex;

(ii) G is weakly C-pseudo monotone and v-hemicontinuous;

(iii) $\forall x \in X$, $G(x,x) \in P$;

(iv) for each finite subset N of X, there exists a nonempty, weakly compact and convex subset L_N of X containing N, such that $\forall x \in L_N \setminus K$, $\exists y \in L_N$ satisfying $-G(y,x) \in -intC(x)$.

Then there exists $\bar{x} \in K$ such that $G(\bar{x}, x) \notin -intC(\bar{x})$, $\forall x \in X$.

Proof. Define two multi functions $A, B : X \longrightarrow 2^X$ by

$$A(x) \equiv \{y \in X \mid -G(y,x) \in -intC(x)\},$$
$$B(x) \equiv \{y \in X \mid G(x,y) \in -intC(x)\}.$$

In order to make use of Theorem 4.1.1 we need the following statements:

(I) $A(x) \subseteq B(x)$. It is because of the weak C-pseudo monotonicity of G.

(II) $\forall x \in X$, $B(x)$ is convex. For any $y, z \in B(x)$ and $\alpha \in [0,1]$, we have

$$\begin{aligned} G(x, \alpha y + (1-\alpha)z) &\in (\alpha G(x,y) + (1-\alpha)F(x,z)) - P \\ &\subseteq \alpha(-intC(x)) + (1-\alpha)(-intC(x)) - P \\ &\subseteq -intC(x) - C(x) \\ &\subseteq -intC(x). \end{aligned}$$

Hence $\alpha y + (1-\alpha)z \in B(x)$.

(III) $\forall y \in X$, $A^{-1}(y)$ is weakly open. In fact, let $\{x_\lambda\}$ be a net in $\sim (A^{-1}(y))$ (the negation of $A^{-1}(y)$) weakly convergent to $x \in X$. Then $-G(y, x_\lambda) \notin -intC(x_\lambda)$, hence $-G(y,x_\lambda) \in W(x_\lambda)$. Since $(x_\lambda, -G(y,x_\lambda)) \in Gr(W)$ and weakly converges to $(x, -G(y,x))$, by (i) and the weak closeness of $Gr(W)$, we have $-G(y,x) \in W(x)$, i.e. $-G(y,x) \notin -intC(x)$. Thus $x \in \sim (A^{-1}(y))$. Therefore $\sim (A^{-1}(y))$ is weakly close, hence $A^{-1}(y)$ is weakly open.

(IV) By the hypothesis (iv), for each finite subset N of X, there exists a nonempty, weakly compact and convex subset L_N of X containing N such that $\forall x \in L_N \setminus K$, $\exists y \in L_N$ satisfying $-G(y,x) \in -intC(x)$. Thus $A(x) \bigcap L_N \neq \phi$.

(V) B has no fixed point. If not, $\exists x \in X$ such that $G(x,x) \in -intC(x)$. By (iii), $G(x,x) \in -intC(x) \bigcap P \subseteq -intC(x) \bigcap C(x) = \emptyset$, a contradiction. (Please note that $-intC(x) \bigcap C(x) = \emptyset$ is the consequence of $C(x) \neq F$.)

Now, all hypothesis in Theorem 4.1.1 have satisfied except $A(x) \neq \emptyset \quad \forall x$. In order to avoid contradiction against Theorem 4.1.1 which promises a fixed point for B, there must be an $\bar{x} \in K$ such that $A(\bar{x}) = \emptyset$, i.e. ,

$$-G(y, \bar{x}) \notin -intC(\bar{x}) \quad \forall y \in X.$$

Finally, as the hypothesis of Lemma 4.3.1 are also fulfilled, we have

$$G(\bar{x}, y) \notin -intC(\bar{x}) \quad \forall y \in X.$$

Q.E.D.

Although with the help of [25] and Theorem 8 in [16] the achievement of Theorem 4.3.1 can be comprehended, the proof above is in itself very elegant. For example, condition (iii) and the P-convexity are used to make Lemma 4.3.1 valid, yet, at the same time, P-convexity is related to the convexity of the set $B(x)$ and condition (iii) is needed to show that B has no fixed point. It seems that the given conditions are fully used.

Now, since it is a linear operator T which directly related to VVIP, we have the following corollary.

Corollary 4.3.1 *Let $E, F, E^*, X, K, C, W,$ and P be the same as in Theorem 4.3.1. Let $T : X \longrightarrow L(E, F)$ be weakly C-pseudo monotone and v-hemicontinuous. Assume that for each finite subset N of X, there exists a nonempty, weakly compact and convex subset L_N of X containing N such that $\forall x \in L_N \setminus K$, $\exists y \in L_N$ satisfying $\langle T(x), y - x\rangle \in -intC(x)$. Then $\exists \bar{x} \in K$ such that $\langle T(\bar{x}), x - \bar{x}\rangle \notin -intC(\bar{x})$ for all $x \in X$.*

Proof.

Putting $G(x, y) = \langle T(x), y - x\rangle$ in Theorem 4.3.1, we should get the result. We have to check if $\langle T(x), y - x\rangle$ satisfies conditions (i) and (iii). This is easy and we omit it.

Another result in [14] is also an existence theorem of VVI but without the pseudo monotonicity.

Theorem 4.3.2 *Let E, F, E^*, X, K, C, W, and P be the same as in Theorem 4.3.1. Let $G : X \times X \longrightarrow F$ a function satisfying the following conditions:*
(i) $\forall x \in X$, $y \mapsto G(x,y)$ is P-convex,
(ii) $\forall x \in X$, $G(x,y)$ is weakly continuous,
(iii) $\forall x \in X$, $G(x,x) \in C(x)$.
(iv) for each finite subset N of X, there exists a nonempty, weakly compact and convex subset L_N of X containing N such that $\forall x \in L_N \setminus K$, $\exists y \in L_N$ satisfying $G(x,y) \in -intC(x)$.
Then $\exists \bar{x} \in K$ such that $G(\bar{x},x) \notin -intC(\bar{x})$ $\forall x \in X$.

Proof.
Define $A(x) \equiv \{y \in X \mid G(x,y) \in -intC(x)\}$, where $x \in X$. In order to prove the theorem, we need the following statements.
(I) With similar reasoning in the proof of Theorem 4.3.1, for all $x \in X$, $A(x)$ is convex and A has no fixed point.
(II) For all $y \in X$, $A^{-1}(y) = \{x \in X \mid G(x,y) \in -intC(x)\}$ is weakly open. The reason is this: Let $\{x_\lambda\}$ be a net in $\sim A_{-1}(y)$ weakly convergent to $x \in X$. Then $G(x_\lambda, y) \notin -intC(x_\lambda)$, hence $G(x_\lambda, y) \in W(x_\lambda)$. Since $(x_\lambda, G(x_\lambda, y)) \in Gr(W)$ and converges weakly to $(x, G(x,y))$ by (ii), it follows from the weak closeness of $Gr(W)$ that $G(x,y) \in W(x)$. So, $G(x,y) \notin -intC(x)$ and $x \in \sim (A^{-1}(y))$. Therefore $\sim (A^{-1}(y))$ is weakly closed and $A^{-1}(y)$ is weakly open.
(III) Condition (v) of Theorem 4.1.1 is satisfied. By hypothesis (iv), for each finite subset N of X, there exists a nonempty, weakly compact and convex subset L_N of X containing N such that for all $x \in L_N \setminus K$, $\exists y \in L_N$ satisfying $G(x,y) \in -intC(x)$. By the definition of A, $A(x) \bigcap L_N \neq \phi$.
Now, in order not to contradict to Theorem 4.1.1, there must be an $\bar{x} \in K$ such

that $A(\bar{x}) = \phi$, or equivalently, $G(\bar{x}, x) \notin -intC(\bar{x})$ for all x in X. Q.D.E.

By Theorem 4.3.2, we have the following corollary.

Corollary 4.3.2 *Let E, F, E^*, X, K,C, W, and P be the same as in Theorem 4.3.1. Let $T : X \longrightarrow L(E, F)$ be a map satisfying for all $y \in X$, $x \mapsto \langle T(x), y - x\rangle$ is weakly continuous. Assume that for each finite subset N of X, there exists a nonempty, weakly compact and convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, $\exists y \in L_N$ satisfying $\langle T(x), y - x\rangle \in -intC(x)$. Then $\exists \bar{x} \in K$ such that $\langle T(\bar{x}, x - \bar{x}\rangle \notin -intC(\bar{x})$ for all $x \in X$.*

Proof.
It can be obtained directly by applying Theorem 4.3.2.

Chapter 5

The Generalized Quasi-Variational Inequalities

5.1 Introduction

The main reference of this chapter is [24]. The generalized quasi-variational inequality problem or the GQVI(X, f, F) problem is this: Given a closed convex subset $X \subseteq \mathbb{R}^n$ and two multi functions, $f : X \longrightarrow 2^{\mathbb{R}^n}$ and $F : X \longrightarrow 2^X$. We are asked to find a vector $\hat{x} \in X$ and a vector $\hat{y} \in f(\hat{x})$ such that $\hat{x} \in F(\hat{x})$ and

$$\langle \hat{y}, \hat{x} - x \rangle \leq 0$$

for all $x \in F(\hat{x})$.

First, we introduce some notations and definitions. A multi function $G : Y \longrightarrow 2^Z$, where Y and Z are topological spaces, is said to be lower (upper) semi continuous on Y if and only if for any open (closed) subset $V \subseteq Z$ the set $\{a \in Y \mid G(a) \bigcap V \neq \emptyset\}$ is open (closed). A multi function is continuous if and only if it is simultaneously lower and upper semi continuous. G is upper semi continuous at a point $a \in Y$ if and only if for every open subset $V \subseteq Z$ containing $G(a)$, there is a neighborhood U of a such that $G(U) \subseteq V$.

Proposition 5.1.1 *G is upper semi continuous on Y if and only if G is upper semi continuous at every point $a \in Y$.*

Proof.

Necessity. Let V be an open set in Z such that $G(a) \subseteq V$. Define $U \equiv \{y \in Y \mid G(y) \bigcap \sim V \neq \emptyset\}$. Clearly, $\sim U$ is open and $a \in \sim U$. Also, $G(\sim U) \subseteq V$.

Sufficiency. For any closed set V in Z, we want to show that $S \equiv \{y \in Y \mid G(y) \bigcap V \neq \emptyset\}$ is closed. It is equivalent to show that $\sim S = \{y \in Y \mid G(y) \subseteq \sim V\}$ is open. Let $\hat{a} \in \overline{S}$. By hypothesis, there is a neighborhood U of $\hat{a}$ such that $G(U) \subseteq \overline{V}$. Hence $\sim S$ is open. Q.E.D.

Furthermore, define $grG \equiv \{(a,b) \in Y \times Z \mid b \in G(a)\}$. grG is called the graph of G. G is said to be with closed graph if its graph is closed in $Y \times Z$. Finally, a subset $A \subseteq Y$ is said to be contractible if there is a point $a \in A$ and a continuous map $\mu : A \times [0,1] \longrightarrow A$ such that $\mu(u,0) = u$ and $\mu(u,1) = a$ for all $u \in A$. Now, let us state a classic theorem on the GQVI in [4]. We will make use of it to establish the main theorem of this chapter.

Theorem 5.1.1 *Assume that:*

(i) X is a nonempty compact and convex set;

(ii) $f : X \longrightarrow 2^{\mathbb{R}^n}$ is an upper semi continuous multi function with nonempty contractible compact values;

(iii) $F : X \longrightarrow 2^X$ is a continuous multi function with nonempty, closed and convex values.

Then there exists a solution to GQVI(X, f, F).

In fact, we will replace the upper semi continuity of f by the following weaker requirement:

$$\text{For each } y \in X - X \text{ the set } \{x \in X \mid \inf_{z \in f(x)} \langle z, y \rangle \leq 0\} \text{ is closed.} \tag{5.1}$$

We denote by $\mathbb{F}$ the class of multi functions $f : X \longrightarrow 2^{\mathbb{R}^n}$ satisfying (5.1).

Proposition 5.1.2 *If f is upper semi continuous then $f \in \mathbb{F}$.*

Proof.

We fix a point $y \in X - X$. For every natural number n, define $V_n \equiv \{z \in \mathbb{R}^n \mid \langle z, y\rangle \leq \frac{1}{n}\}$. V_n is closed. By hypothesis, $\{x \in X \mid f(x) \bigcap V_n \neq \emptyset\}$ is also closed. Now, note that $\{x \in X \mid \inf_{z \in f(x)} \langle x, y\rangle \leq 0\} = \bigcap_{n \in \mathbb{N}} \{x \in \mid f(x) \bigcap V_n \neq \emptyset\}$, which is a closed set. Q.E.D.

5.2 Properties of The Class $\mathbb{F}_0$

$\mathbb{F}_0$ is a subclass of $\mathbb{F}$ formed by multi functions with compact and convex values. For a subset $A \subseteq \mathbb{R}^n$, define $f^{-1}(A) \equiv \{x \in X \mid f(x) \bigcap A \neq \emptyset\}$. Symbol B will stand for the closed unit ball in $\mathbb{R}^n$.

Let P be the linear subspace in $\mathbb{R}^n$ generated by $X - X$. There is $\epsilon > 0$ such that $\epsilon(B \bigcap P) \subseteq X - X$. If Q is the subspace in $\mathbb{R}^n$ orthogonal to P, then $\mathbb{R}^n = P + Q$. π is the orthogonal projection of $\mathbb{R}^n$ onto P. We know that $\pi(z) \neq 0$ if and only if $z \notin Q$.

For any finite number of nonzero vectors γ_1 to γ_k in P, it corresponds an open polyhedral cone which is defined as:

$$T \equiv \{z \in \mathbb{R}^n \mid \langle z, \gamma_i\rangle > 0 \quad \forall i\}. \tag{5.2}$$

If we define

$$T_0 \equiv \{z \in P \mid \langle z, \gamma_i\rangle > 0 \quad \forall i\} \tag{5.3}$$

then $T = T_0 + Q$ and $\pi(T) = T_0$. Let τ_1 be the weakest topology in $\mathbb{R}^n$ containing all the open polyhedral cones i.e. τ_1 consists of all possible unions of the cones of the form (5.2) and plus $\mathbb{R}^n$. Let τ denotes the norm topology on $\mathbb{R}^n$.

Theorem 5.2.1 *A multi function $f : X \longrightarrow 2^{\mathbb{R}^n}$ with nonempty compact and convex values belongs to $\mathbb{F}_0$ if and only if f is (τ, τ_1)-upper semi continuous.*

Proof.

Necessity. Assume that $f \in \mathbb{F}_0$ and is compact convex valued. By Proposition 5.1.1, for each point $\bar{x} \in X$, we have to show that for any open subset $V \in \tau_1$ with $f(\bar{x}) \subseteq V$ there is a neighborhood $U \in \tau$ of $\bar{x}$ such that $f(U) \subseteq V$.

If $V = \mathbb{R}^n$, we can set $U = X$. So, let us assume that $V \neq \mathbb{R}^n$. By the definition of τ_1, we have $V \bigcap Q = \emptyset$ and $f(\bar{x}) \bigcap Q = \emptyset$. Since $f(\bar{x}) \subseteq V$, then $\pi(f(\bar{x})) \subseteq V_0 \equiv \pi(V) = \bigcup_{\alpha \in I} V_0^{\alpha}$, where $\{V_0^{\alpha}\}$ is a collection of open polyhedral cones in P. Also, as $\pi(f(\bar{x})) \bigcap (P \setminus V_0) = \emptyset$ and $\pi(f(\bar{x}))$ being compact, there is a number ρ such that

$$\rho = \inf\{\|z - z'\| \mid z \in \pi(f(\bar{x})), \;\; z' \in P \setminus V_0\} > 0. \tag{5.4}$$

For any $\delta \in (0, \rho)$, define

$$D \equiv \pi(f(\bar{x})) + \delta(B \bigcap P)$$

and

$$C \equiv \pi(f(\bar{x})) + \frac{\delta}{2}(B \bigcap P)$$

We have $C \subseteq D \subseteq V_0$ and C, D are compact and convex. Consider the induced topology of P, riC (the relative interior of C, cf. [17], p. 44) and riD coincide with the interiors of C and D respectively. By the separation theorem, for every $z \in C_1 \equiv C \setminus riC$ there exists a nonzero vector $\gamma(z) \in P$ such that

$$C \subseteq P_z^{(1)} \equiv \{w \in P \mid \langle w - z, \gamma(z) \rangle \geq 0\}. \tag{5.5}$$

We are going to show that the family of open (relative to P) half-spaces

$$P_z^{(2)} \equiv \{w \in P \mid \langle w - z, \gamma(z) \rangle < 0\}, \quad z \in C_1$$

covers the compact set $D_1 \equiv D \setminus riD$. So, Let $a_0 \in riC$ and $b \in D_1$. Consider the segment

$$[b, a_0] = \{(1-t)b + ta_0 \mid 0 \le t \le 1\},$$

which connects $b \notin C$ and $a_0 \in riC$. Since C is compact and convex, there exists a unique value $\bar{t} \in (0,1)$ such that

$$\bar{z} \equiv (1-\bar{t})b + \bar{t}a_0 \in C_1.$$

Let $\gamma(\bar{z}) \in P$ be a nonzero vector satisfying (5.5). Since $\langle a_0 - \bar{z}, \gamma(\bar{z})\rangle \ge 0$ and $\bar{z} \in (b, a_0)$, then $\langle b - \bar{z}, \gamma(\bar{z})\rangle \le 0$. If $\langle b - \bar{z}, \gamma(\bar{z})\rangle = 0$, then $\langle a_0 - \bar{z}, \gamma(\bar{z})\rangle = 0$. This contradicts to $a_0 \in riC$. Therefore, $\langle b - \bar{z}, \gamma(\bar{z})\rangle < 0$, i.e. $b \in P_{\bar{z}}^{(2)}$. As b being arbitrary, we have $D_1 \subseteq \{P_z^{(2)}\}_{z \in C}$. In fact, since D_1 is compact, we even have

$$D_1 \subseteq \bigcup \{P_{z_i}^{(2)} \mid i = 1, \ldots, s\}. \tag{5.6}$$

We claim that

$$\Delta \equiv \bigcap \{P_{z_i}^{(1)} \mid i = 1, \ldots, s\} \tag{5.7}$$

is a polyhedral set in P satisfying the inclusions

$$\pi(f(\bar{x})) + \frac{\delta}{2}(B \bigcap P) \subseteq \Delta \subseteq D. \tag{5.8}$$

The first inclusion is a direct result of (5.5). Now suppose there is a point $w \in \Delta \setminus D$. By the convexity of Δ, the whole segment connecting w with a point $z \in \pi(f(\bar{x}))$ belongs to Δ. Clearly, this segment must contain at least one point from D_1. But this is impossible because of (5.6) and (5.7).

Now, since D is bounded and Δ is closed, we know that Δ is compact. Also, since $0 \in P \setminus V_0$, we get $0 \notin D$ and thus $0 \notin \Delta$. This follows that the set $cone\Delta \equiv \{tw \mid t \le 0 \quad and \quad w \in \Delta\}$ is closed. Applying Theorem 19.7 in [17], we find that $cone\Delta$ is a polyhedral cone in P, i.e., there exist nonzero vectors $\gamma_1, \ldots, \gamma_k \in P$ such that $cone\Delta = \{w \in P \mid \langle w, \gamma_i \rangle \geq 0 \ \forall i\}$. Define $W \equiv \{w \in \mathbb{R}^n \mid \langle w, \gamma_i \rangle > 0 \ \forall i\}$. Since V_0 is a union of a collection of polyhedral cones in P by (5.8), we have

$$\pi(f(\bar{x})) + \frac{\delta}{2}(B \bigcap P) \subseteq cone\Delta \subseteq V_0$$

thus

$$f(\bar{x}) \subseteq W \subseteq V.$$

It can be shown that

$$\{x \in X \mid f(x) \subseteq W\} = \bigcap_i \{x \in X \mid \inf_{z \in f(x)} \langle z, \gamma_i \rangle > 0\}.$$

Since $f \in \mathbb{F}_0$, the set $\{x \in X \mid f(x) \subseteq W\}$ is open in τ and this is the neighborhood of $\bar{x}$ we want to find.

Sufficiency. Assume that f is (τ, τ_1)-upper semi continuous. Since we have also assumed that f being compact and convex valued, it is sufficient to show that f satisfies (5.1). For any $y \in X - X$, the case that $y = 0$ is easy. So, suppose $y \neq 0$. We have

$$\{x \in X \mid \inf_{z \in f(x)} \langle z, y \rangle \leq 0\} = \{x \in X \mid f(x) \bigcap T \neq \emptyset\},$$

where $T \equiv \{z \in \mathbb{R}^n \mid \langle z, y \rangle \leq 0\}$ is a τ_1-closed set. Since f is (τ, τ_1)-upper semi continuous, we find that $\{x \in X \mid f(x) \bigcap T \neq \emptyset\}$ is closed, i.e. f satisfies condition (5.1). Q.E.D.

Now, another definition. for any vector $z \in \mathbb{R}^n$

$$p(z) \equiv \begin{cases} \frac{\pi(z)}{\|\pi(z)\|} & \pi(z) \neq 0 \\ 0 & \pi(z) = 0. \end{cases}$$

Theorem 5.2.2 *A multi function $f : X \longrightarrow 2^{\mathbb{R}^n}$ with nonempty compact and convex values belongs to $\mathbb{F}_0$ if and only if the following two conditions are satisfied:*

(i) $Z(f) \equiv \{x \in X \mid f(x) \bigcap Q \neq \emptyset\}$ *is closed;*

(ii) the factorization ψ *of* f, *defined by*

$$\psi(x) = \begin{cases} \{p(z) \mid z \in f(x)\} & x \notin Z(f) \\ \{0\} & x \in Z(f) \end{cases}$$

is upper semi continuous on $X \setminus Z(f)$ *in the usual sense (i.e. w.r.t. the norm).*

Proof.

Necessity. Recall that Q is the orthogonal subspace of P. It can be shown that Q is a τ_1-closed set. Since f is (τ, τ_1)-upper semi continuous, $Z(f)$ is closed. To show (ii), let $\bar{x} \in X \setminus Z(f)$ be given and W be any open set (w.r.t. the norm topology) in $\mathbb{R}^n$ containing $\psi(\bar{x})$. Since $\bar{x} \notin Z(f)$, $\psi(\bar{x})$ is a subset of the unit sphere B_P of the subspace P. Clearly,

$$\psi(\bar{x}) \subseteq V_0 \equiv \bigcup_{t>0} \{t(W \bigcap B_P)\}. \tag{5.9}$$

V_0 is an open cone in P, and $\pi(f(\bar{x})) \subseteq V_0$. Since $0 \notin V_0$, we claim that

$$V_0 = \bigcup_{\alpha \in I} V_0^{\alpha},$$

where

$$V_0^{\alpha} = \{z \in P \mid \langle z, \gamma_i^{\alpha} \rangle > 0 \quad i = 1, \dots, k_{\alpha}\}.$$

To show this, assume that the dimension of P is k and fix a point $z \in V_0$. Since $z \neq 0$, $P_1 \equiv \{w \in P \mid \langle w - z, z \rangle = 0\}$ is an affine manifold of dimension k-1 in P which contains z but does not contain 0. Since $P_1 \bigcap V_0$ is open w.r.t. P_1 and $z \in P_1 \bigcap V_0$, then there exists a simplex Δ with vertexes $z_1, z_2, \dots, z_k \in P_1$ such that $\Delta \subseteq P_1 \bigcap V_0$, $z \in ri\Delta$ and $ri\Delta$ is an open set w.r.t. P_1. Denote by γ_j the unique unit vector in P satisfying $\langle z_i, \gamma_j \rangle = 0$ for all $i \in \{1, \dots, k\} \setminus \{j\}$ and $\langle z, \gamma_j \rangle > 0$. It can be shown that

$$V_0(z) \equiv \{w \in P \mid \langle w, \gamma_i \rangle > 0 \quad \forall j = 1, \dots, k\} \subseteq V_0.$$

The reason is this: we note that the vectors $z_1, z_2, \dots, z_k$ are linearly independent. Suppose the contrary, one of these vectors, say z_k, can be expressed as a linear combination of the others

$$z_k = \sum_{i=1}^{k-1} \alpha_i z_i, \qquad \alpha_i \in \mathbb{R}. \tag{5.10}$$

Let L be the linear subspace generated by $z_1, \dots, z_{k-1}$. For any vector $v \in P_1$, if $v \in \Delta$, then v is a convex combination of $z_1, \dots, z_k$. Hence, by (5.10), $v \in L$. If $v \notin \Delta$, the interval (z, v) has a nonempty intersection with $ri\Delta$ because $z \in ri\Delta$ and $ri\Delta$ is open w.r.t. P_1. So, v can be expressed as a linear combination of z and a point from $ri\Delta$. As $\Delta \subseteq L$, this implies that $v \in L$. Thus we conclude that $P_1 \subseteq L$. Since P_1 is an affine manifold of dimension $k-1$ and the dimension of the linear subspace L is not greater than $k-1$, we have $P_1 = L$. This implies that $0 \in P_1$ which contradicts to the nature of P_1.

Furthermore, since $z \in ri\Delta$, we have $z = \sum_{i=1}^{k} \lambda_i z_i$, where $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i > 0$ for all i. By the definition of γ_i, we also have

$$0 < \langle z, \gamma_j \rangle = \sum_{i=1}^{k} \langle \lambda_i z_i, \gamma_j \rangle = \lambda_j \langle z_j, \gamma_j \rangle.$$

Therefore

$$\langle z_j, \gamma_j \rangle > 0 \qquad \forall j = 1, \dots, k. \tag{5.11}$$

Now. let w be any vector from $V_0(z)$. Since $z_1, \dots, z_k$ are linearly independent and P has dimension k, there exist $\mu_1, \dots, \mu_k \in \mathbb{R}$ such that $w = \sum_{i=1}^{k} \mu_i z_i$. For each j,

$$0 < \langle w, \gamma_j \rangle = \sum_{i=1}^{k} \langle \mu_i z_i, \gamma_j \rangle = \mu_j \langle z_j, \gamma_j \rangle.$$

By (5.11), we have $\mu_j > 0$ for every j. Therefore, $\frac{1}{\sum_{i=1}^{k} \mu_i} w \in ri\Delta$. Since $ri\Delta \subseteq V_0$ and $tV_0 \subseteq V_0$ for every $t > 0$, $w \in V_0$. So, we find that V_0 is a polyhedral cone of

P as V_0^α and $z \in v_0(z)$, thus our claim follows.

If we set $V = V_0 + Q$, then $V = \bigcup_{\alpha \in I} V^\alpha$, where

$$V^\alpha = \{z \in \mathbb{R}^n \mid \langle z, \gamma_i^\alpha \rangle > 0 \quad \forall i = 1, \dots, k_\alpha\}.$$

Therefore $V \in \tau_1$. The inclusion $\pi(f(\bar{x})) \subseteq v_0$ yields $f(\bar{x}) \in V$. Since $f \in \mathbb{F}_0$, Theorem 5.2.1 shows that there exists a neighborhood $U \in \tau$ of $\bar{x}$ satisfying $f(U) \subseteq V$. This implies

$$\psi(U) \subseteq V_0 \bigcap S_P.$$

This and (5.9) give $\psi(U) \subseteq W \bigcap S_P \subseteq W$. That is, ψ is upper semi continuous on $X \setminus Z(f)$.

Sufficiency. Assume that $f : X \longrightarrow 2^{\mathbb{R}^n}$ is a multi function with nonempty compact convex values which satisfies (i) and (ii). Given any $\bar{x} \in X$ and any τ_1-neighborhood V of $f(\bar{x})$, by Theorem 5.2.1 we want to find an open neighborhood $U \in \tau$ of $\bar{x}$ such that $f(U) \subseteq V$. If $\bar{x} \in Z(f)$, V contains a point in Q. This means that V cannot be the union of the form (5.2), thus V is $\mathbb{R}^n$. In this case, any $U \in \tau$ will do. If $\bar{x} \in X \setminus Z(f)$, then (i) and (ii) imply that there is a τ-neighborhood U of $\bar{x}$ which is contained in $X \setminus Z(f)$ such that

$$\psi(U) = \{\frac{\pi(z)}{\|\pi(z)\|} \mid z \in f(x), \quad x \in U\} \subseteq V.$$

Since V is positive homogeneous, $\pi(f(U)) \subseteq V$. Moreover, V is the union of a family of open polyhedral cones of the form (5.2), we conclude that $f(U) \subseteq V$. Q.E.D.

Proposition 5.2.1 *If $f(x)$ is compact convex, then the set $\psi(x)$ defined in the statement of Theorem 5.2.2 is compact and contractible.*

Proof.

If $f(\bar{x}) \bigcap Q \neq \emptyset$, we have $\psi(\bar{x}) = \{0\}$ which is compact and contractible.

Now, suppose $f(\bar{x})\bigcap Q = \emptyset$, then $\psi(\bar{x}) = p(f(\bar{x})) \subseteq S_P$. Since $f(\bar{x})$ is compact and p is continuous on $\mathbb{R}^n \setminus Q$, $\psi(\bar{x})$ is compact. To show that $\psi(\bar{x})$ is contractible, we fix a point $a \in \psi(\bar{x})$ and set

$$\mu(z,t) \equiv p((1-t)z + ta)$$

for $z \in \psi(\bar{x})$ and $t \in [0,1]$. Clearly, $\mu(z,0) = z$ and $\mu(z,1) = a$ for all $z \in \psi(\bar{x})$. Now, since $f(\bar{x})\bigcap Q = \emptyset$ and $f(\bar{x})$ is convex, we must have $(1-t)z+ta \in P\setminus\{0\}$ for all $z \in \psi(\bar{x})$ and $t \in [0,1]$. But p is continuous on $P \setminus \{0\}$, we see that μ is continuous.

Finally, we will show that the image of μ is inside $\psi(\bar{x})$. As $z,\ \ a \in \psi(\bar{x})$, there exist $z',\ \ a' \in \pi(f(\bar{x}))$ such that $p(z') = z$ and $p(a') = a$. This means that there are two real numbers α and β such that $z = \alpha z'$ and $a = \beta a'$. Since

$$\begin{aligned}(1-t)z + ta &= \alpha(1-t)z' + \beta t a' \\ &= (\alpha(1-t)+\beta t)\left(\frac{\alpha(1-t)}{\alpha(1-t)+\beta t}z' + \frac{\beta t}{\alpha(1-t)+\beta t}a'\right)\end{aligned}$$

and

$$\alpha(1-t) + \beta t > 0$$

therefore,

$$\begin{aligned}p((1-t)z+ta) &= p\left(\frac{\alpha(1-t)}{\alpha(1-t)+\beta t}z' + \frac{\beta t}{\alpha(1-t)+\beta t}a'\right) \\ &\in p(\pi(f(\bar{x}))) \\ &= \psi(\bar{x}).\end{aligned}$$

Q.E.D.

5.3 Main Theorem

First, we state a lemma and a fixed-point theorem.

Lemma 5.3.1 *Assume that the multi function $F : X \longrightarrow 2^X$ is lower semi continuous on X and is with convex values and closed graph. If $B_\rho \subseteq \mathbb{R}^n$ is a closed ball with the radius $\rho > 0$ and the center 0, such that $F(x) \bigcap B_\rho$ is nonempty for every $x \in X$, then the multi function $F_\rho(x) \equiv F(x) \bigcap B_\rho$ is continuous on X.*

For the proof of this Lemma, please refer to Lemma (3.1) of [23].

Theorem 5.3.1 *(Kakutani)*
Suppose that
(i) K is a nonempty, compact, convex set in a Hausdorff locally convex space X;
(ii) the multivalued map $T : K \longrightarrow 2^K$ is upper semi-continuous;
(iii) the set $T(x)$ is nonempty, closed, and convex for all $x \in K$.
Then T has a fixed point.

Now we introduce the main theorem of this chapter.

Theorem 5.3.2 *Assume that X is closed and convex, $f \in \mathbb{F}$, and there is a compact subset $K \subseteq X$ such that the following conditions hold:*
(i) $f(x)$ is nonempty and compact for every $x \in X$, convex for all $x \in K$ satisfying $x \in F(x)$;
(ii) The multi function F is lower semi continuous on X, with closed graph; for all $x \in X$ the set $F(x)$ is convex and $F(x) \bigcap K \neq \emptyset$;
(iii) for every $x \in X \setminus K$, $x \in F(x)$, one has

$$\sup_{y \in F(x) \bigcap K} \inf_{z \in f(x)} \langle z, x - y \rangle > 0. \tag{5.12}$$

Then GQVI(X, f, F) problem has a solution $(\hat{x}, \hat{y}) \in K \times \mathbb{R}^n$.

As we mentioned before, we replace the upper semi continuity of f by the weaker condition (5.1). Yet Theorem 5.3.2 is not a complete generalization of Theorem 5.1.1. It is because in the main theorem, we ask $f(x)$ to be convex if $x \in K$ and $x \in F(x)$. This is not necessary in Theorem 5.1.1.

Now we show the proof of Theorem 5.3.2:

Denote $\tilde{f}(x)$ the convex hull of $f(x)$. By hypothesis,

$$\tilde{f}(x) = f(x) \tag{5.13}$$

for those $x \in K$ and $x \in F(x)$.

Since K is compact, we can define B_i as the closed ball in $\mathbb{R}^n$ centered at 0 with the radius i, where i is an integer number such that $K \subseteq B_i$. $\tilde{\psi}$ is the factorization of $\tilde{f}$ (see Theorem 5.2.2 for definition). For every i described as above, there are two possible cases:

(1) For all $x \in X \bigcap B_i$, $\tilde{f}(x) \bigcap Q = \emptyset$.

(2) There exists at least one point $x \in X \bigcap B_i$ such that $\tilde{f}(x) \bigcap Q \neq \emptyset$.

If (1) is the case, by Theorem 5.2.2, $\tilde{\psi}$ is upper semi continuous on $X \bigcap B_i$. Let

$$F_i(x) = F(x) \bigcap B_i. \tag{5.14}$$

Now, apply Proposition 5.2.1 to $\tilde{f}$ and Lemma 5.3.1 to F_i, we can find that GQVI$(X \bigcap B_i, \tilde{\psi}, F_i)$ fulfill all hypothesis of Theorem 5.1.1. Hence there are $a_i \in X \bigcap B_i$ and $b_i \in \tilde{\psi}(a_i)$ such that $a_i \in F_i(a_i)$ and

$$\langle b_i, a_i - y \rangle \leq 0 \quad \forall y \in F_i(a_i). \tag{5.15}$$

If (2) is the case, we further consider the sets

$$C \equiv \{x \in X \bigcap B_i \mid \tilde{f}(x) \bigcap Q \neq \emptyset\}$$

and

$$D \equiv \{x \in X \bigcap B_i \mid x \in F_i(x)\}.$$

If $C \bigcap D \neq \emptyset$, then taking any $a_i \in C \bigcap D$ and $b_i = 0 \in \tilde{\psi}(a_i)$, we see that (5.14) holds.

Suppose $C \bigcap D = \emptyset$. By assumption, C is nonempty and closed. Applying Theorem 5.3.1 to $X \bigcap B_i$ and F_i, we find that D is also nonempty. Since the graph of F is closed, D is also closed. Therefore, we can apply the Urysohn lemma and get a continuous function $\alpha : X \bigcap B_i \longrightarrow [0, 1]$ such that $\alpha(x) = 0$ for $x \in C$ and $\alpha(x) = 1$ for $x \in D$. Now we claim that

$$\tilde{\xi}(x) \equiv \{\alpha(x)y \mid y \in \tilde{\psi}(x)\}$$

is upper semi continuous on $X \bigcap B_i$.

The reason is this:

Since α is continuous, and by Theorem 5.2.2 $\tilde{\psi}$ is upper semi continuous and compact-valued on $(X \bigcap B_i) \backslash C$, it can be verified that $\tilde{\xi}$ is upper semi continuous on $(X \bigcap B_i) \setminus C$. Now let $\bar{x} \in C$ and ϵB be a small ball containing the set $\tilde{\xi}(x) = \{0\}$. By the continuity of α at $\bar{x}$, we have

$$\begin{aligned} \sup\{\|z\| \mid z \in \tilde{\xi}(x)\} &= \sup\{|\alpha(x)|\|y\| \mid y \in \tilde{\psi}(x)\} \\ &\leq |\alpha(x)| \\ &< \epsilon \end{aligned}$$

if x is sufficiently close to $\bar{x}$. So, $\tilde{\xi}$ is also upper semi continuous on C.

By Lemma 5.3.1 and Proposition 5.2.1, we can apply Theorem 5.1.1 to the GQVI($X \bigcap b_i, \tilde{\xi}, F_i$) problem. Let $a_i \in X \bigcap B_i$ and $b_i \in \tilde{\xi}(a_i)$ be a solution to this GQVI. As $a_i \in D$, $\tilde{\xi}(a_i) = \alpha(a_i)\tilde{\psi}(a_i) = \tilde{\psi}(a_i)$. Therefore, $b_i \in \tilde{\psi}(a_i)$ and

$$\langle b_i, a_i - y \rangle \leq 0 \quad \forall y \in F_i(a_i).$$

We now find that there always exists $a_i \in X \bigcap B_i$ and $b_i \in \tilde{\psi}(a_i)$ such that (5.15) holds.

We further claim that a_i belongs to K.

Suppose the contrary: $a_i \in (X \bigcap B_i) \setminus K$. Since $a_i \in F(a_i)$ and $\tilde{f}(a_i)$ is the convex hull of $f(a_i)$, by assumption (iii), we get

$$\sup_{y \in F(a_i) \bigcap K} \inf_{z \in \tilde{f}(a_i)} \langle z, a_i - y \rangle > 0.$$

Let $\bar{y}$ be the point such that

$$\inf_{z \in \tilde{f}(a_i)} \langle z, a_i - \bar{y} \rangle > 0. \tag{5.16}$$

Since $a_i - \bar{y} \in X - X \subseteq P$, then $\tilde{f}(a_i) \bigcap Q = \emptyset$, hence we even have

$$\inf_{z \in \tilde{\psi}(a_i)} \langle z, a_i - \bar{y} \rangle > 0.$$

This is a contradiction against (5.15).

Due to the compactness of K, we may assume that $a_i \longrightarrow \hat{x} \in K$. Since the graph of F is closed, $\hat{x} \in F(\hat{x})$. Also, as the b_i's belong to the unit closed ball, we can assume that $b_i \longrightarrow \hat{z}$.

Now, if $\tilde{f}(\hat{x}) \bigcap Q \neq \emptyset$, then for any $\hat{y} \in \tilde{f}(\hat{x}) \bigcap Q$ one has

$$\langle \hat{y}, \hat{x} - y \rangle \leq 0 \quad \forall y \in F(\hat{x}). \tag{5.17}$$

According to (5.13), $\tilde{f}(\hat{x}) = f(\hat{x})$ because $\hat{x} \in K \bigcap F(\hat{x})$. Hence $(\hat{x}, \hat{y})$ is a solution to GQVI(X, f, F).

If $\tilde{f}(\hat{x}) \bigcap Q = \emptyset$, then (by Theorem 5.2.2) $\hat{x} \in X \setminus Z(f)$ and $\tilde{\psi}$ is upper semi continuous in a neighborhood of $\hat{x}$ in τ. Let us assume that all those a_i belong to the neighborhood. This and the fact that $\tilde{\psi}$ is compact-valued ([17] Theorem 17.2) make $\hat{z} \in \tilde{\psi}(\hat{x})$. since $\hat{x} \in K \bigcap F(\hat{x})$, we have $\tilde{f}(\hat{x}) = f(\hat{x})$ and $\tilde{\psi}(\hat{x}) = \{\frac{\pi(z)}{\|\pi(z)\|} \mid z \in f(\hat{x})\}$. Therefore,

$$\hat{z} = \frac{\pi(\hat{y})}{\|\pi(\hat{y})\|} \tag{5.18}$$

for a point $\hat{y} \in f(\hat{x})$. Given any point $y \in F(\hat{x})$, the lower semi continuity of F shows that there is a sequence $\{y_i\}$ such that $y_i \in F(a_i)$ for all i and $y_i \longrightarrow y$. As $\{y_i\}$ is convergent, $y_i \in B_i$ for large i, thus $y_i \in F(a_i) \bigcap B_i = F_i(a_i)$ for these large i. By (5.15), $\langle b_i, a_i - y_i \rangle \leq 0$ for all large i. Passing to the limit, we get

$$\langle \hat{z}, \hat{x} - y \rangle \leq 0. \tag{5.19}$$

With the help of (5.18) and the fact that $\hat{x} - y \in X - X \subseteq P$, we can deduce from (5.19) to get

$$\langle \hat{y}, \hat{x} - y \rangle \leq 0.$$

Therefore, $(\hat{x}, \hat{y})$ is a solution to the GQVI(X, f, F) problem. Q.E.D.

5.4 Remarks

In this section, we will show the proof of the original version of Theorem 5.1.1. A function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasi-concave if for each $\lambda \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n \mid \varphi(x) \geq \lambda\}$ is convex.

Theorem 5.4.1 *Let φ be a continuous function mapping $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$. Let f and K be multi functions from $\mathbb{R}^n$ to itself. Suppose that C is a nonempty convex compact set in $\mathbb{R}^n$ such that the following conditions hold:*
(i) for each fixed (u, w), $\varphi(v, u.w)$ is a quasi-concave function in $v \in C$;
(ii) f is a nonempty contractible compact valued upper semi continuous mapping on C;
(iii) $V(x) \equiv K(x) \bigcap C$ is a nonempty continuous convex valued mapping on C.
Then there exist vectors $u^* \in V(u^*)$ and $w^* \in f(u^*)$ such that

$$\varphi(v, u^*, w^*) \leq \varphi(u^*, u^*, w^*)$$

for all $v \in V(u^*)$.

Proof.
We observe that the set $f(C) = \bigcup_{x \in C} f(x)$ is compact. In fact, fix a point $x \in C$ and an open cover $\{D_i\}$ of $f(C)$. Since $f(x)$ is compact, there are finite number of D_is which cover $f(x)$. Let the subcover of $f(x)$ be B_x. Since f is upper semi-continuous, there is a neighborhood N_x of x such that $f(N_x) \subseteq B_x$. C is

compact, finite number of N_xs can cover C. Therefore, finite number of B_xs can cover $f(C)$.

The convex hull E of $f(C)$ is also a compact set. Define the multi function $\pi : C \times E \longrightarrow C$ as below:

$$\pi(u,w) = \{c \in V(u) \mid \varphi(c,u,w) = \max_{v \in V(u)} \varphi(v,u,w)\}.$$

By the Maximum Theorem of [9], π is a nonempty, compact-valued and upper semi-continuous mapping. Also, since φ is quasi-concave in v, $\pi(u,w)$ is a convex and hence contractible set. Consequently, the multi function $F : C \times E \longrightarrow C \times E$ defined by

$$F(u,w) = (\pi(u,w), f(u))$$

is nonempty, contractible compact valued and upper semi-continuous. Apply the Eilenberg-Montgomery fixed point theorem [5], F has a fixed point, i.e., there exists $(u^*, w^*) \in F(u^*, w^*)$. Thus $w* \in f(u^*)$ and $u^* \in \pi(u^*, w^*)$. This means that $u^* \in V(u^*)$ and for every $v \in V(U^*)$, we have

$$\varphi(v, u^*, w^*) \leq \varphi(u^*, u^*, w^*).$$

Q.E.D.

Now, by letting $\varphi(v, u.w) = -\langle w, (v-u)\rangle$, we have the following corollary which is the original version of Theorem 5.1.1.

Corollary 5.4.1 *Let f and K be multi functions from $\mathbb{R}^n$ into itself. Suppose that there exists a nonempty compact convex set C such that*
(i) $K(C) \subseteq C$;
(ii) f is a nonempty contractible compact valued upper semi continuous mapping on C;

(iii) K is a nonempty continuous convex valued mapping on C.

Then there exists a solution to the GQVI(C,f,K).

Chapter 6

A set-valued open mapping theorem and related results

6.1 Introduction and Notations

Let X, Y be two topological vector spaces (different from pervious chapters, a topological vector space needs not be Hausdorff here), g a mapping between them. We say that g is an open mapping at a point $x \in X$ if for any neighborhood V of x, $g(V)$ contains a neighborhood of $g(x)$. g is say to be open if it is open at every point of X. g is open if and only if $g(U)$ is open in Y whenever U is an open subset of X.

Let E be a space. A pseudo-metric for E is a function d on the Cartesian product space $X \times X$ to the non-negative real numbers such that for all points $x, y, z \in X$

(a) $d(x, y) = d(y, x)$;

(b) $d(x, y) = 0$ if $x = y$;

(c) $d(x, y) + d(y, z) \geq d(x, z)$.

Sometimes, pseudo-metric is called semi-metric. In addition, if d satisfies:

(d) $d(x, y) > 0$ for all $x \neq y$,

then d is call a metric.

Now, we construct a topology on E by using d. For the each point x_0 of E, define the open sphere around x_0 of radius $r > 0$ as

$$\{y \mid d(x_0, y) < r\}.$$

Let the union of spheres of all radii of all points be the subbase of the topology τ on E. τ is obtained by d. For any topological space $(E, \hat{\tau})$,
(1) if $\hat{\tau}$ can be obtained by a semi-metric, then $(E, \hat{\tau})$ is called semi-metrizable.
(2) if $\hat{\tau}$ can be obtained by a metric, then $(E, \hat{\tau})$ is called metrizable.

Now suppose E is a linear space. Let p be a function from E into $\mathbb{R}$ such that:
(i) $|\lambda| \leq 1$ implies $p(\lambda x) \leq p(x)$ for all $x \in E$;
(ii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in E$;
(iii) $p(x) = 0$ if and only if $x = 0$.
p is called a pseudo-norm on E. Define d as

$$d(x, y) \equiv p(x - y).$$

It can be shown that d is a semi-metric. We should note that the topology induced by d does not always make E a topological vector space. If it does lead to a Hausdorff topological vector space, then d is a metric.

6.2 An Open Mapping Theorem

Let E and F be two metrizable topological vector space and d be the metric (which generated the topology) on E. For each positive real number r, set

$$V_r = \{x \in E \mid d(x, 0) \leq r\}.$$

Now, suppose f is a continuous linear function from E into F. Banach's open mapping theorem states that:

Theorem 6.2.1 *For E being complete, if $\overline{f(V_r)}$ of each $f(V_r)$ is a neighborhood of 0 in F, then $f(V_\beta) \supseteq \overline{f(V_\alpha)}$ whenever $\beta > \alpha > 0$. In particular, each $f(V_r)$ is a neighborhood of 0.*

Proof. We omit it.

Since E and F are topological vector spaces, we in fact have f to be open.

We observe that since F is Hausdorff (because F is metrizable), f is linear and continuous if and only if the graph of f is a closed linear subspace in $E \times F$. In [15], the Banach open mapping theorem was set up for the relations that their graphs are closed cones in $E \times F$. We will introduce the result in the rest of this chapter.

6.3 Main Result

A relation S between E and F is a subset of $E \times F$. The inverse relation of S is the set S^{-1} defined by

$$S^{-1} \equiv \{(y,x) \in F \times E \mid (x,y) \in S\}.$$

If $A \subseteq E$, we denote

$$S(A) = \{y \in F \mid (a,y) \in S \quad for \ \ some \quad a \in A\}.$$

If B is a subset of F, we denote

$$S^{-1}(B) = \{x \in E \mid (b,x) \in S^{-1} \quad for \ \ some \quad b \in B\}$$

Theorem 6.3.1 *Let (E,τ) be a topological vector space, p a pseudo-norm on it, and d be the semi-metric induced by p such that τ is induced by d. Let V_r be as defined in the last section. Let F be a topological vector space. Let S be a closed cone in $E \times F$, and suppose that $\overline{S(V_r)}$ is a neighborhood of 0 in F, for each $r > 0$. Then whenever $\beta > \alpha > 0$, we have that $S(V_\beta) \supseteq \overline{S(V_\alpha)}$; consequently each $S(V_r)$ is a neighborhood of 0 in F.*

Proof.

We first claim that: if $y \in \overline{S(A)}$ for some $A \subseteq E$, then there exists a set B of arbitrary small diameter such that $A \bigcap B \neq \emptyset$ and $y \in \overline{S(B)}$.

The reason is this: let $\epsilon > 0$ be arbitrary. By hypothesis, $\overline{S(V_\epsilon)}$ is a neighborhood of 0 in F, hence contains a symmetric neighborhood W of 0 ([18], p.10). Since $y \in \overline{S(A)}$, we must have the neighborhood $y + W$ of y intersects with $S(A)$. So, let $y' \in (y + W) \bigcap S(A)$ and $a \in A$ such that $(a, y') \in S$. Set

$$B \equiv a + V_\epsilon.$$

The diameter of B is less than or equal to 2ϵ and $A \bigcap B \neq \emptyset$. To see that $y \in \overline{S(B)}$, we note that since W is symmetric, $y \in y' + W$ implies there is a $w \in W \subseteq \overline{S(V_\epsilon)}$ such that

$$y = y' + w.$$

Since $w \in \overline{S(V_\epsilon)}$, there is a net $\{w_\alpha\} \subseteq S(V_\epsilon)$ such that

$$w_\alpha \longrightarrow w.$$

For each of these w_α there exists $v_\alpha \in V_\epsilon$ such that

$$(v_\alpha, w_\alpha) \in S.$$

Since $(a, y') \in S$ and S is a cone, we have

$$(a + v_\alpha, w_\alpha + y') \in S$$

therefore,

$$w_\alpha + y' \in S(a + V_\alpha) \subseteq S(B).$$

Passing to the limit, we get

$$w + y' = y \in \overline{S(B)}.$$

So this proved our claim.

Next, let $\beta > \alpha > 0$ and set $\epsilon = \beta - \alpha$. Let also $y_0 \in \overline{S(V_\alpha)}$. We want to show that $y_0 \in S(V_\beta)$. Write A_0 for V_α. By the construction of τ, there is a neighborhood of A_1 of 0 such that:

(1) $|A_1| < \frac{1}{2}\epsilon$;

(2) $A_0 \bigcap A_1 \neq \emptyset$;

(3) $y_0 \in \overline{S(A_1)}$;

where $|A_0|$ is the diameter of the set A_0. Further using A_1 to construct A_2 and so on. Now we have a sequence of sets $\{A_n\}$ such that:

(1') $|A_n| < \frac{\epsilon}{2^n}$;

(2') $A_n \bigcap A_{n+1} \neq \emptyset$;

(3') $y_0 \in \overline{S(A_n)}$,

for each $n = 0, 1, 2, \ldots$. Also, any neighborhood of 0 will contains A_n for n large enough. Take $a_n \in A_n \bigcap A_{n+1}$. $\{a_n\}$ is a Cauchy sequence in E by (2'). Since E is complete, we suppose $a_n \longrightarrow a$. By (2'), we know that $a \in V_{\alpha+\epsilon} = V_\beta$.

We claim that $(a, y_0) \in S$. To see this, let D be a arbitrary neighborhood of (a, y_0). Let $G \subseteq E$ and $H \subseteq F$ be two neighborhoods such that $a \in G$, $y_0 \in H$, and $G \times H \subseteq D$ being a neighborhood of (a, y_0) in the space $E \times F$. For large n, $A_n \subseteq G$. Therefore $y_0 \in \overline{S(A_n)} \subseteq \overline{S(G)}$. Hence H intersects $S(G)$. That is, there exists $(g, h) \in S$ where $g \in G$ and $h \in H$. Since S is closed, this proved that $(a, y_0) \in S$.

Finally, $(a, y_0) \in S$ implies $y_0 \in S(a) \subseteq S(V_\beta)$. Q.E.D.

By considering the inverse relation S^{-1} of S, we have the following variant of Theorem 6.3.1:

Theorem 6.3.2 *Let E, V_r and F be as in Theorem 6.3.1. Let T be a closed cone in $F \times E$, and suppose that $\overline{T^{-1}(V_r)}$ is a neighborhood of 0 in F for each $r > 0$.*

Then, whenever $\beta > \alpha > 0$, we have

$$T^{-1}(V_\beta) \supseteq T^{-1}(V_\alpha);$$

consequently each $T^{-1}(V_r)$ is a neighborhood of 0 in F.

We will see the theorem's application to ordered normed spaces later.

Corollary 6.3.1 *Let E, V_r and F be as in Theorem 6.3.1, and suppose further that F is Hausdorff. Let f be a continuous linear function from E into F. If $\overline{f(V_r)}$ is a neighborhood of 0 in F for each $r > 0$, then $\overline{f(V_\beta)} \supseteq \overline{f(V_\alpha)}$ whenever $\beta > \alpha > 0$. Consequently, f is an open mapping.*

Proof. Since f is linear and F is Hausdorff, the graph of f is thus closed. Also, since f is linear, its graph is a linear subspace, that is a closed cone, in $E \times F$.

Please compare the corollary with Theorem 6.2.1. Requirements of E and F are weaken.

6.4 An Application on Ordered Normed Spaces

Let F and G be two linear spaces over the field $\mathbb{R}$. If there exists a bilinear functional $\langle \cdot, \cdot \rangle$ on $F \times G$ such that

(1) if $0 \neq f \in F$, then $\exists g \in G$ such that $\langle f, g \rangle \neq 0$;

(2) if $0 \neq g \in G$, then $\exists f \in F$ such that $\langle f, g \rangle \neq 0$;

then (F, G) is called a duality.

Let (F, G) be a duality. For any subset M of F, the polar M^o of M is defined by

$$M^o \equiv \{g \in G \mid \langle f, g \rangle \leq 1 \ \ \forall f \in M\}.$$

Lemma 6.4.1 *Let (F, G) be a duality. For any subset $M \subseteq F$, the bipolar M^{oo} is the $\sigma(F, G)$-closed, convex hull of $M \bigcup \{0\}$.*

Proof. Please refer to [19], p.126.

Let $(X, \|\cdot\|)$ be a normed space ordered by a cone C. That is, for any $x, y \in X$, $x \le y$ means $y-x \in C$. Let U be the closed unit ball in X, and let α be a positive real number. We say that C is α-normal if

$$(U + C) \bigcap (U - C) \subseteq \alpha U.$$

We say that C is α-generating if

$$U \subseteq \alpha co(U^+ \bigcup -U^+),$$

where $U^+ \equiv U \bigcap C$.
The relation between the definitions, say α-normal, and the order is this: suppose $x \in (U+C) \bigcap (U-C)$, then there exists $y, z \in U$ such that $x \in z+C$ and $x \in y-C$. This means:

$$z \le x \le y.$$

Since C is positively homogeneous, C being α-normal means: if $z \le x \le y$, then $\|x\| \le \alpha \max\{\|y\|, \|z\|\}$. For the case of α-generating, please refer to [6].

Theorem 6.4.1 *Let $(X, \|\cdot\|)$ be a normed space and let C be a complete cone in X. Let X' be the dual space and the dual cone $C' \equiv \{f \in X' \mid f(x) \ge 0, \ \ \forall x \in C\}$ be α-normal for some $\alpha > 0$. Then C is $(\alpha + \epsilon)$-generating for each $\epsilon > 0$, and $(X, \|\cdot\|)$ is complete.*

Proof.
Let U' be the closed unit ball in X'. Since C is complete, $0 \in C$ and hence $U' \subseteq U' + C'$. We can show that

$$(U' + C')^o = (U')^o \bigcap (C')^o.$$

By the Hahn-Banach theorem, we know that for any $x \in X$,

$$\|x\| = \sup\{|f(x)| \mid f \in U'\}.$$

By this, we can show that

$$(U')^o \subseteq U.$$

Therefore

$$(U')^o \bigcap (C')^o = U \bigcap (-C).$$

But

$$U \bigcap (-C) = -U^+,$$

by definition, we have

$$(U' + C')^o = -U^+. \tag{6.1}$$

By the Banach-Alaoglu theorem, U' is weak*-compact. Therefore, $U' + C'$ and $U' - C'$ are weak*-closed. They are also convex sets. Also, since C' is α-normal,

$$(U' + C') \bigcap (U' - C') \subseteq \alpha U'.$$

By (1.3) on page 125 of [19], we get

$$\frac{1}{\alpha} U \subseteq ((U' + C') \bigcap (U' - C'))^o.$$

It follows from the Lemma 6.4.1 that

$$((U' + C') \bigcap (U' - C'))^o = \overline{co((U' + C')^o \bigcup (U' - C')^o)}.$$

By (6.1), we have

$$\overline{co((U' + C')^o \bigcup (U' - C')^o)} = \overline{co(-U^+ \bigcup U^+)},$$

which means

$$\frac{1}{\alpha}U = \overline{co(-U^+\bigcup U^+)}. \tag{6.2}$$

Now let p be the Minkowski functional with respect to $co(U^+\bigcup -U^+)$. Let $X_1 \equiv C - C$. X_1 is a subspace. Since

$$\{x \in X \mid p(x) < 1\} \subseteq co(U^+\bigcup -U^+) \subseteq \{x \in X \mid p(x) \leq 1\} \tag{6.3}$$

and C is complete, we can use Lemma 2 of p.221 of [19] and conclude that (X_1, p) is complete. Let

$$V_r \equiv \{x \in X_1 \mid p(x) \leq r\},$$

and let i be the identity map from (X_1, p) into $(X, \|\cdot\|)$. By (6.2) and (6.3), we have

$$\frac{1}{\alpha}U \subseteq \overline{i(V_1)}.$$

Hence each $\overline{i(V_1)}$ is a neighborhood of 0 in $(X, \|\cdot\|)$. Here, applying Theorem 6.3.1, we conclude that

$$i(V_\beta) \supseteq \overline{i(V_\alpha)}$$

whenever $\beta > \alpha > 0$. Write D for $co(U^+\bigcup -U^+)$, then $D \subseteq V_1$ and $V_\alpha \subseteq \beta D$ whenever $\beta > \alpha > 0$. Hence

$$\begin{aligned}\overline{co(-U^+\bigcup U^+)} &= \overline{i(D)} \\ &\subseteq \overline{i(V_1)} \\ &\subseteq \overline{i(V_{1+\epsilon})} \\ &\subseteq i((1+2\epsilon)D),\end{aligned}$$

where $\epsilon > 0$. By (6.2), we have

$$\begin{aligned}\frac{1}{\alpha(1+2\epsilon)}U &\subseteq D \\ &= co(U^+\bigcup -U^+).\end{aligned}$$

Therefore C is $\alpha(1+2\epsilon)$-generating for each $\epsilon > 0$, that is equivalently, $(\alpha+\epsilon)$-generating.

Finally, the continuous function i is open by Corollary 6.3.1. So $(X, \|\cdot\|)$ is homeomorphic to (X_1, p). (X_1, p) is complete, so is $(X, \|\cdot\|)$. Q.E.D.

Readers can compare Theorem 6.4.1 with Theorem 8 of [6].

6.5 An Application on Open Decomposition

Let (F, τ) be a topological vector space with a cone C. For each neighborhood U of 0, if $U\bigcap C - U\bigcap C$ (resp. $\overline{U\bigcap C - U\bigcap C}$) is also a neighborhood of 0, then (F, C, τ) is said to have the open decomposition property (resp. semi-open decomposition property). Since $C - C$ is a subspace, we see that open decomposition property implies $C - C = F$ and semi-open decomposition property implies $\overline{C - C} = F$.

A subset A of an ordered set (in our case, let $A \subseteq F$ and F ordered by C) is said to be order-convex if for any $a, b \in A$ with $a \le x \le b$, then $x \in A$. It can be shown that A is order-convex if and only if $(A+C)\bigcap(A-C) = A$. In fact $(A+C)\bigcap(A-C)$ is the smallest order-convex set which containing A. M. Duhoux has shown that if the topological dual E' is order-convex (w.r.t. the dual cone C') in the algebraic dual then the two decomposition properties are equivalent.

Suppose (E, τ) is semi-metrizable, E_1 is the subspace spanned by C (in fact $E_1 = C - C$) and let p be a pseudo-norm inducing τ. For $r > 0$ let

$$U_r \equiv \{x \in E \mid p(x) \le r\},$$

and

$$U_r^* \equiv U_r\bigcap C - U_r\bigcap C.$$

Since $\{U_r \mid r > 0\}$ forms a local base in (E, τ), it can be shown that U_r^* satisfies:

(a) For each $r_1 > 0$, there exists $r_2 > 0$ such that $U_{r_2}^* + U_{r_2}^* \subseteq U_{r_1}^*$;

(b) Every U_r^* is circled and absorbing (cf. [19], p. 11) in $C - C$;

(c) There exists $0 < |\lambda| < 1$ such that for all $r > 0$, $\lambda U_r^* = U_s^*$ for some $s > 0$.

Therefore, by section 1.2 of [19], U_r^* uniquely determines a semi-metrizable vector topology τ_1 in E_1. For each $x \in E_1$, we define

$$p^*(x) \equiv \inf\{r > 0 \mid x \in U_r^*\},$$

and

$$V_r \equiv \{x \in E_1 \mid p^*(x) \leq r\}.$$

Since $V_\alpha \subseteq U_\beta^* \subseteq V_\alpha$ whenever $0 < \alpha < \beta$, it follows that p^* is a pseudo-norm inducing the topology $\tau_1 \in E_1$. Now we give another proof of Theorem 3.5.1 in [10].

Theorem 6.5.1 *Let (F, τ) be a metrizable topological vector space and let C be a complete cone in F. Suppose for each $r > 0$, $\overline{U_r \bigcap C - U_r \bigcap C}$ is a neighborhood of 0. Then $U_\beta \bigcap C - U_\beta \bigcap C$ contains $\overline{U_\alpha \bigcap C - U_{alpha} \bigcap C}$ and hence is a neighborhood of 0, whenever $\beta > \alpha > 0$. In particular, if (F, C, τ) has the semi-open decomposition property then it has the open decomposition property.*

Proof.

By lemma 2 on page 221 of [19], we see that (F_1, τ_1) is a complete metrizable space. Since τ_1 cannot be coaster than τ, the identity map i from F_1 to F is continuous. An identity map is a linear map. Therefore the graph of i is a closed cone in $F_1 \times F$. By the construction of V_r, $\overline{i(V_r)}$ is a neighborhood of 0. Applying Theorem 6.3.1, we have

$$i(V_\beta) \supseteq \overline{i(V_\alpha)}$$

whenever $\beta > \alpha > 0$. Therefore, suppose $\beta > \gamma > \alpha$, we have

$$i(U_\beta^*) \supseteq i(V_\gamma) \supseteq \overline{i(V_\alpha)} \supseteq \overline{i(U_\alpha^*)},$$

that is, by the definition of U_r^*,

$$U_\beta \bigcap C - U_\beta \bigcap C \supseteq \overline{U_\alpha \bigcap C - U_\alpha \bigcap C}$$

in the space (F, τ). Q.E.D.

6.6 An Application on Continuous Mappings from Order-infrabarreled Spaces

We first introduce some definitions.

A subset in a topological vector space is call a barrel if it is convex, closed, absorbing and circled. A locally convex space E is barreled if each barrel in E is a neighborhood of 0. Therefore, we can say that a barreled space is a l.c.s. in which the family of all barrels forms a local base. If E is ordered, we have a corresponding concept called order-infra barreled which means that: each barrel in E which absorbs all order-intervals (the set of the from: $\{z \in E \mid x \leq z \leq y\}$) is a neighborhood of the origin. A barreled space with a cone must be order-infra barreled. K. F. Ng has shown that the converse is not true.

Let E, F be two ordered topological vector spaces and suppose that the positive cone (the cone which defines the order) F^+ of F is closed. Let t be a sublinear function (that means being convex and positively homogeneous) from E to F and let T be defined as

$$T \equiv \{(x, y) \in E \times F \mid t(x) \leq y\}.$$

Since t is sublinear, it can be shown that T is a cone in $E \times F$. If t is continuous, T is closed. To see this, suppose there is a net $\{(x_\alpha, y_\alpha)\}$ in T with $\{(x_\alpha, y_\alpha)\} \longrightarrow$

(x_0, y_0). Since the net is in T, we have the order

$$y_\alpha - t(x_\alpha) \geq 0$$

in F. This means

$$y_\alpha - t(x_\alpha) \in F^+.$$

Since t is continuous and F^+ is closed, passing to the limit, we get

$$y_0 - t(x_0) \in F^+,$$

which means

$$y_0 - t(x_0) \geq 0.$$

Hence $(x_0, y_0) \in T$. Conversely, the following theorem gives some sufficient conditions for t be continuous when T is closed.

t is said to be monotonic if $t(x) \geq t(y)$ whenever $x \geq y \geq 0$. A cone in E is said to be normal if the order-convex (w.r.t. the cone) and convex neighborhoods of 0 in E form a local basis.

Lemma 6.6.1 *Let t be a sublinear mapping of E into F. Suppose that F^+ is normal. If t is continuous at 0, then t is continuous.*

Proof.

Since every neighborhood of 0 contains a balanced neighborhood of 0, we can assume V be a symmetric, order-convex and convex neighborhood of 0 in F. Let U be a symmetric neighborhood of 0 in E such that $t(U) \subseteq V$. Let x be an arbitrary point in E and x_1 an arbitrary point in U. We want to show that $t(x + U) \subseteq t(x) + V$. By the convexity of t, it can be shown that

$$-t(x_1) \leq t(x) - t(x + x_1) \leq t(-x_1).$$

Since V and U are symmetric, we must have $-t(x_1), t(-x_1) \in V$. V is order-convex, so $t(x) - t(x + x_1) \in V$. $-t(x) + t(x + x_1)$ also belongs to V. Therefore, $t(x + x_1) \in t(x) + V$. Hence, we see that t is continuous at x. Q.E.D.

Theorem 6.6.1 *Let E be and order-infra barreled space with the open decomposition property, and let F be a complete, metrizable locally convex topological vector space with a normal cone. Let t be a monotonic and sublinear function from E into F, and let*

$$T \equiv \{(x, y) \in E \times F \mid t(x) \leq y\}.$$

If T is closed, then t is continuous.

Proof.
By lemma 6.6.1, it is sufficient to show that t is continuous at the origin of E. Let V be an order-convex, convex, and balanced neighborhood of 0 in F. We want to show that $t^{-1}(V)$ is a neighborhood of 0 in E. Consider the non-empty set $S \equiv t^{-1}(V) \bigcap E^{+}$. Suppose $x, y \in S$. Let

$$0 \leq \lambda, \mu \leq 1 \quad \textit{and} \quad \lambda + \mu = 1.$$

Since E^{+} is convex, $\lambda x + \mu y \in E^{+}$ and this means

$$0 \leq \lambda x + \mu y.$$

By the monotonicity and sublinearity of t, we get

$$0 \leq t(\lambda x + \mu y) \leq \lambda t(x) + \mu t(y).$$

Since $\lambda t(x) + \mu t(y) \in V$, so

$$t(\lambda x + \mu y) \in V$$

thus

$$\lambda x + \mu y \in S.$$

We have shown that S is convex. Since V is a neighborhood of 0, by the monotonicity and sublinearity of t again we know that S absorbs each positive element in E. Now define

$$U \equiv S - S.$$

By definition, U is a symmetric convex set in E. Also, $t(U) \subseteq V$. To see this, set $u \equiv x_1 - x_2 \in U$ where $x_1, x_2 \in S$. Since t is monotonic and sublinear, we have

$$-t(x_2) \leq t(x_1) - t(x_2) \leq t(x_1 - x_2) \leq t(x_2).$$

Since V is balanced (hence symmetric) and order-convex, we have $t(x_1), -t(x_2) \in V$ and

$$t(u) = t(x_1 - x_2) \in V.$$

This shows that $t(U) \subseteq V$. Since E has the open decomposition property, $E = E^+ - E^+$. We know that S absorbs E^+, therefore U absorbs E.

Now, we want to show that U absorbs all order-intervals in E. It is sufficient to show that U absorbs $I \equiv [0, x]$ for all $x \in E^+$. Note that $t(x) \geq 0$. Since 0 is an interior point of V, we can find $M > 0$ such that

$$t(x) \in MV.$$

Since t is monotonic and V is order-convex, we have

$$t(I) \subseteq [0, t(x)] \subseteq MV,$$

for all $x \in E^+$. Thus

$$I \subseteq MS \subseteq MU.$$

This shows that U absorbs I. Consequently, since $E = E^+ - E^+$, we can apply 3.10.2 on p.132 of [10] and conclude that U absorbs all order-intervals.

We have shown that $\overline{U}$ is a barrel and absorbs all intervals in E which is order-infra barreled by hypothesis. Thus $\overline{U}$ is a neighborhood of 0 in E. Since

$$T^{-1}(V) \supseteq t^{-1}(V) \supseteq U,$$

it follows that $\overline{T^{-1}(V)}$ is a neighborhood of 0 in E. V is arbitrary, applying Theorem 6.3.1, we find that $T^{-1}(V)$ is also a neighborhood of 0 in E.

Let

$$W \equiv T^{-1}(V) \bigcap E^+ - T^{-1}(V) \bigcap E^+.$$

Since E has the open decomposition property, W is a neighborhood of 0 in E. To complete the proof, we will show that

$$W \subseteq t^{-1}(V).$$

Let $w \equiv w_1 - w_2 \in W$ where $w_1, w_2 \in T^{-1}(V) \bigcap E^+$. For each w_i, there exists $v_i \in V$ such that $(w_i, v_i) \in T$, that means: $0 \leq t(w_i) \leq v_i$ for each i. Therefore, we have

$$-v_2 \leq -t(w_2) \leq t(w_1) - t(w_2) \leq t(w) = t(w_1 - w_2) \leq t(w_1) \leq v_1.$$

Since V is symmetric and order-convex, it follows that $t(w) \in V$. Therefore $W \subseteq t^{-1}(V)$. Q.E.D.

Let H be a Hausdorff linear space. We do not assume there is any relation between the topological and the algebraic structure of H. That is, H may not be a topological vector space. An FH space X is a complete metric linear space such that:

(1) X is a linear subspace of H;

(2) the topology of X is stronger than that of H.

Let s be the space of sequences of numbers and s^+ the space of sequences of positive numbers. An FK space F is a special kind of FH space where $H = s$. F is ordered by the cone

$$F^+ \equiv F \bigcap s^+.$$

Corresponding to Theorem 1. on p.203 of [22], we have the following corollary.

Corollary 6.6.1 *Let E be as in Theorem 6.6.1, F and ordered FK space. Let t be a monotonic and sublinear function from E into F. Then t is continuous if and only if it is continuous as a map into s.*

Proof.

Suppose that t is continuous from E into s. Then the set

$$T \equiv \{(x, y) \in E \times F \mid t(x) \leq y\}$$

is closed in $E \times F$. Hence t is continuous into F by Theorem 6.6.1. The other half of the proof is obvious because the definition of a FK space asks the topology of F be stronger than that of s. Q.E.D.

Corollary 6.6.2 *Let E, F be FK spaces, and suppose that the cone E^+ is generating in E and F^+ is normal in F. Then any monotonic and sublinear function from E into F is continuous.*

Proof. From Section 3.5 on [10], we know that E has the open decomposition property. Thus E satisfies the conditions in Theorem 6.6.1. Since the topology in E is always finer than that of s, t is continuous from E into s. Thus Corollary 6.6.1 can be applied. Q.E.D.

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