# CHARACTERISTICS OF A DETAIL PRESERVING NONLINEAR FILTER 

BY
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#### Abstract

Nonlinear image filters, especially robust estimator based filters, are renowned for their excellent performance in impulsive noise suppression while preserving edges. However, most nonlinear filters have poor detail preserving properties. This is undesirable because details as well as edges carry important information for segmentation and analysis of an image. Currently, all detail preserving image filters such as the multistage median filter and the FIRmedian hybrid filter, are median based. These filters take into account the geometric structures of a signal by utilizing subfilters in different orientations.

Researchers have been developing nonlinear image filters which are purely driven by the geometry of a signal. Mathematical morphology, which is a set-theoretical methodology deals with the geometrical features of a signal, seems to be a solution. However, morphological filters which are based on the opening and closing transformations are hardly detail preserving.

In this thesis, a novel detail preserving filter called the Multi-structuring Element Erosion Filter which is based on the erosion operator of mathematical morphology is proposed. Impulsive noises are usually spatially uncorrelated with their surroundings, however, details and edges maintain high dependencies among neighbouring pixels. In our filter, a collection of structuring elements which are two-dimensional binary signal pattern with predetermined size and shape are designed. Details and edges are represented as unions of different combinations of these structuring elements. A detail will be preserved if at least a combination of structuring elements can be found. Therefore, impulses can be distinguished from useful pictorial information and can be removed, Moreover, as the filter relies heavily on the geometrical correlation of a signal, pixels filtered by the multi-structuring element erosion filter are also highly correlated to their neighbourhoods. This is different from some median based filters which produces visually uncorrelated outputs.


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## Chapter 1 Introduction

### 1.1 Background - The Need for Nonlinear Filtering

Images are two-dimensional representations of objects. Nowadays numerous applications using images are found around the world. Engineers and scientists analyze images and produce useful information in areas like weather forecasting, geographical mapping and space probing. Moreover, digital image processing also plays an important role in our daily life. Television broadcasting, transmission of facsimile images and teleconferencing provide us a world wide communication. Medical images help specialists in analyzing and detecting disease in patients.

Images are captured by devices such as charged coupled device (CCD) camera and photoelectron tubes like vidicon camera [Lim90]. An image is usually stored and transmitted before recognition and analysis is taken place. Therefore, it is susceptible to corruption by various kinds of noise. Thermal noise and photoelectron noise may plague an image during image acquisition. The former is usually a white Gaussian noise with zero mean which is produced by the various component of electronic circuits. Photoelectron noise is created by random fluctuation of the number of photons on the light sensitive surface of the detector [Pitas90] which is approximated by a Poisson distributed noise. During the transmission and storage of digital images, another type of noise known as the salt-and-pepper noise may be generated [Pitas92]. This kind of noise appears as black and/or white impulses on the image. Film-grain noise [Pitas90] [Jain89] and speckle noise [Jain89] [Safa89] [Pitas90] are signal dependent noise. Film-grain noise occurs in the development of photographic film due to silver precipitation. The noise has a Poisson distribution and becomes a Gaussian distribution in a limiting case. Speckle noise exists in the coherent imaging of objects and is multiplicative in nature. Although a variety of noise can be observed on an image, different visual effects are perceived. The human visual system is very complicated and is still not fully understood, research indicates that the human eyes are sensitive to high frequency information. Therefore, abrupt changes in contrast are stimulating. As a result, the presence of salt-and-pepper noise is perceptually more uncomfortable than the presence of Gaussian and Poisson noise.

Image coding, segmentation, recognition and analysis are some typical operations on digital images. Noise should be removed before these tasks can be proceeded. One of the objectives of image filtering is therefore the suppression of noise in images. Linear filters, such as the moving average filters are applied successfully in many aspects. However, their performance in long tailed noise, such as Poisson and salt-and-pepper noise are not satisfactory. Furthermore, linear filters are notorious for blurring edges and destroying fine details such as texture and thin lines which are important information on an image. Consequently, researchers have been developing new filtering techniques for removing noise in an image while preserving edges and details. As linear methods are not adequate, nonlinear techniques are used.

### 1.2 Nonlinear Filtering

A system is nonlinear if the superposition principle does not hold [Oppen83]. A linear system can be decomposed into several subsystems. The output of the overall system is the sum of those of its subsystems. However, an equality between the overall system and the sum of the subsystems may not exist for nonlinear systems.

Several families of nonlinear filters are developed. Robust estimation theory [Huber81] has been extensively applied in nonlinear filtering. The principle of these filters is the correlation of neighbouring image data. The values of the neighbourhoods of a point are of similar magnitudes. As salt and pepper noise takes the maximum and minimum gray levels respectively so corruption by salt-and-pepper noise can be distinguished owing to the difference in gray values. A robust estimator rejects these outliers. The median filter is the best known filter of this family. One-dimensional median filter is found to be edge preserving, but the two-dimensional median filter is not. A lot of efforts have been put to improve the edge and detail preserving properties of median filters. Modifications of median filters, such as the weighted median filter [Huang81] [Harja91], the separable median filter [Nar81], the $\mathrm{max} / \mathrm{median}$ filter [Arce87], the multistage median filter [Neuvo87] [Arce89] and the finite impulse response median hybrid filter [Neuvo87] [Neuvo87a] [Neuvo90] are developed. Apart from the median, rank order statistics are also applied [Nodes82] [Lee87] [Palm89] [Bovik89] [Bovik88]. Other robust estimators [David81], such as the maximum likelihood estimator [Lee85] and the rank test estimator [Crin85] have been used.

One approach to achieve filtering is to utilize the geometric structure of a signal. Mathematical morphology was developed to describe the geometrical features of a binary image. A digital binary image can be represented as a set in $\boldsymbol{Z}^{2}$. A simpler and smaller set, which is called a structuring element, is a tool to quantify objects in an image. Mathematical morphology, which is based on several set operations, has been applied in image processing and analysis [Hara97] [Gour91]. The methodology was originally developed for image analysis, for instance, in counting the number of pores in a binary image of soil. Filters that are based on the morphological operators are called morphological filters. Applications in nonlinear image filters are found in [Serra82] [Arce87] [Chu89] [Safa89] [Mara90] [Kos91] [Kuos91]. One of the most important and interesting application of morphological filters is in the representation of filters in different domains. This brings about a unifying approach to describe both linear and nonlinear filters. Linear shift invariant filters, rank order filters and stack filters [Wendt86] can be represented by mathematical morphology [Mara87] [Mara87a] [Mara89].

Homomorphic filters are one of the oldest nonlinear filtering techniques, which are used for image enhancement as early as 1958. Image enhancement includes contrast enhancement and dynamic gray level modification [Lim89]. Homomorphic filters have found applications in multiplicative noise and signal dependent noise removal [Pitas90] [Jain89]. The basic idea behind this type of filters is to transform nonlinearly related signals (such as multiplicative noise on a signal) to additive signals and then to process by a linear filter. In multiplicative noise environment, the logarithm transforms multiplication into addition. If a signal is corrupted by signal dependent noise, nonlinear transformation is utilized to transform the signal dependent noise into additive noise. The additive noise can be removed conveniently by a linear filter. The image can be restored by the inverse transformation.

Another nonlinear family is the polynomial filter or known as the Volterra filter. In polynomial filters, the input and output relation is expressed in the form of a discrete Volterra series [Sicu92]. The Volterra series can be interpreted as a Taylor series with memory [Pitas92]. Owing to the complexity in calculating the high order Volterra kernels, the use of Volterra filter is quite limited.

A detail discussion on robust estimators based filters and morphological filter will be
given in Chapter 2. As the applications of the homomorphic filter and the volterra filter are limited, so further considerations of these filters will not be attempted.

### 1.3 Goal of the Work

Details and edges are most important in recognition and analysis of objects in an image. An image filter is suitable for digital image filtering if the filter has good edge and detail preserving properties. Many nonlinear filters are developed for suppressing noise in images. Pitas and Venetsanopolous have carried out a comprehensive study and comparison on robust estimator based nonlinear filters and morphological filters. They revealed that currently there are two median based filters which possess good edge and detail preserving properties, namely the multistage median filter and the FIR-median hybrid filter [Pitas92]. These filters require multistage operations. The principles of these filters are quite similar [Neuvo87a] [Arce87]. A multistage median filter is composed of several subfilters. Each subfilter is designed to preserve a detail in one direction. The subfilter may be a onedimensional line segment which are oriented horizontally, vertically and diagonally.. A subfilter outputs the median or the mean of the data on an images masked by itself. In a multistage median filter, the subfilters are median filters. In a two-dimensional FIR-median hybrid filter, the subfilters are moving average filters. The overall output of a multistage median filter (FIR-median hybrid filter) is then the median of the outputs of their subfilters. As the median is a point estimator which does not account for the geometry of a signal, so we can conclude that the geometry of a signal must be considered if details are going to be preserved.

Mathematical morphology is a methodology which is developed to deal with the geometrical structures of a signal. Morphological operators are set operators that modify the geometrical features of a binary signal. There are two primary morphological operations, namely, the dilation and erosion. Two secondary operators, known as the opening and closing, are constructed from the primary ones. Hence, nonlinear filters can be developed solely by treating the geometrical structures of a signal. The smoothing of a morphological filter is determined by the shape and size of the structuring element and the morphological operator used. Morphological filters which are based on the opening and closing operations are developed [Stev87] [Chu89] [Safa89]. However, none of these filters is detail preserving in
spite of having very fast convergence. This is apparently due to the fact that the opening and closing transformations remove details.

Two questions are raised. First of all, how can a structuring element(s) be designed so that details can be preserved? Secondly, can a filter based on dilations and erosions be detail preserving? In our works, we attempted to solve the problems. We first developed a new nonlinear filter which is based on the erosion operator of mathematical morphology. The design and implementation of the filter is discussed. Theoretical analysis of the filter is carried out. Since the filter is nonlinear, the analysis of the filter cannot adopt the method used in analyzing linear systems. Researchers characterize nonlinear filters by deterministic properties and statistical properties. Deterministic analysis reveals the signal structure which is unchanged by the particular filter. Statistical properties show the noise suppression power of the filter under various noise distributions. We will follow their approaches in analyzing our filter. We also test the filter with real images. Finally, evaluation will be made to verify the performance of the filter.

### 1.4 Organization of the Thesis

Our goal of work is to develop and to analyze a new nonlinear filter which is based on mathematical morphology. Moreover, the performance of the filter will compare against the multistage median filter. An introductory on the theories related to nonlinear filters which are based on robust estimators and mathematical morphology will be presented in Chapter 2. The Chapter will be divided into two sections. The first section gives the definition of median filter and its deviations. The theory of mathematical morphology as well as its applications in nonlinear filtering will be described briefly in the second section.

In Chapter 3, the design of the new nonlinear filter, the multi-structuring element erosion filter will be discussed. The filter is first defined for binary signals. The criteria for the structuring element used in the new binary filter will be explained. Afterwards, the filter is generalized for gray level signals. Examples will be given to improve the legibility of this chapter.

The analysis, both deterministic and statistical, will be given in Chapter 4.

Deterministic analysis aims at two objectives. Firstly, the signal that is invariant to the filter can disclose us information about the minimal details which the filter can preserve. Secondly, the number of filter passes required to bring any input signal to its invariant signal will be derived. In addition, the noise suppression of the filter under different noise distributions will be evaluated. The noise suppression is referred to the statistical properties of the filter.

The performance of the filter under the mean square error, the mean absolute error and the subjective test criteria will be given in Chapter 5. Comparison of performance between the multi-structuring element erosion filter and the multistage median filter will be carried out. All overall summary will be presented in Chapter 6.

# Chapter 2 An Overview of Robust Estimators Based Filters and Morphological Filters 

### 2.1 Introduction

Noise removal is one of the important tasks in signal processing. In image processing, noise removal is difficult. Firstly, images are non-stationary 2-D signals which contain discontinuities such as edges and corners. Edges and details represent important information in an image. Perceptually our vision relies heavily on edges information. Furthermore, images are always corrupted by additive, impulse, and signal dependent noise. Therefore, removal of noise should incorporate with the preservation of edges and details. Although linear filtering finds success in many applications and possesses very simple structures for realization [Oppen83], linear filters cannot cope with the discontinuities of an image. They blur edges, destroy thin lines and fine image details and cannot effectively remove non-Gaussian types of noises. A linear filter averages an impulse and distribute the magnitude of the impulse among the neighbourhood of the pixel. Development of nonlinear filters is initiated by these deficiencies of linear filter. Most nonlinear filters can be categorized as [Pitas92]:

1. Robust Estimator Based filters
2. Morphological filters
3. Homomorphic filters
4. Polynomial filters or Volterra filters
5. Adaptive filters

The fundamental difference between a linear filter and a nonlinear filter is that a nonlinear filter does not commute with superposition. For any linear filter, $\psi(x(\boldsymbol{n}))$, the following relation holds

$$
\begin{equation*}
\psi \cdot\left(a_{1} \cdot x_{1}(\boldsymbol{n})+a_{2} \cdot x_{2}(\boldsymbol{n})\right)=a_{1} \cdot \psi\left(x_{1}(\boldsymbol{n})\right)+a_{2} \cdot \psi\left(x_{2}(\boldsymbol{n})\right), \quad \forall a_{1}, a_{2} \in \boldsymbol{R} \tag{2.1}
\end{equation*}
$$

which may not hold for a nonlinear filter.
The most popular family of nonlinear filters are those based on robust estimators. The median, ranked order, $\alpha$-trimmed mean, multi-stage median, median hybrid, M-filters and the

Wilcoxon filters fall in this category. Morphological filters are derived from the basic operators of mathematical morphology. Mathematical morphology was developed by Matheron and Serra [Serra82] for the analysis and descriptions of objects from their geometrical features. Homomorphic filters were used as early as 1958 for image enhancement and removal of multiplicative and signal dependent noise [Lim89]. Analysis of homomorphic filters can be found in [Pitas86]. In polynomial filters, the relationship of the input and output is expressed in the form of a discrete Volterra series. The Volterra series can be interpreted as a Taylor series with memory [Pitas92] [Sicu92]. Lastly, as a nonlinear filter is usually optimal for only a specific type of noise. This limits the applicability of the filter to other signals. Adaptive nonlinear filters are therefore derived for robust nonlinear filtering for nonstationary signals. Wendt et al. introduced the stack filter in 1986 which can be regarded as a collection of nonlinear filters that commute with thresholding [Wendt86a]. Stack filters provide a means for the hardware realization of nonlinear filters. In the following, discussion on nonlinear filters that based on robust estimators, and morphological filters will be presented. This is because robust estimator based filters have found extensive applications in digital image filtering. Most detail preserving filters are median based. On the other hand, morphological filters have attracted attention of many researchers, and become a hot topic in image processing.

### 2.2 Signal Representation by Sets

Let $\boldsymbol{R}$ and $\boldsymbol{Z}$ be the set of real and integer numbers, respectively, and let $\boldsymbol{D}$ be the domain on which the signal is defined. $\boldsymbol{D}$ is either the $n$-dimensional continuous space $\boldsymbol{R}^{\mathrm{n}}$ or the discrete space $Z^{\mathrm{n}}$. An $n$-dimensional ( $n$-D) signal can be represented as a mapping $f$ from domain $\boldsymbol{D}$ to a range $\boldsymbol{R}$ or $\boldsymbol{Z}$. A binary signal is a mapping whose range takes on only two values,i.e. $f(x) \in\{0,1\}, \forall x \in \boldsymbol{D}$ [Mara87]. In particular, a binary image can be represented by the foreground $S=\{x: f(x)=1\}$ and the background $S^{c}=\{x: f(x)=0\}$, which is the complement of $S$.

Binary signals can be obtained by thresholding a gray level signal [Fitch84]. Let $G(x)$, $\boldsymbol{x} \in \boldsymbol{D}$ be a $k$-level signal in the range $[0, k-1]$, and let $S_{\mathrm{j}}(G(\boldsymbol{x}))$ be the threshold set of $G(\boldsymbol{x})$ at level $j$.

The threshold set at level $j$ is defined as :

$$
S_{j}(G(x))= \begin{cases}1, & \text { if } G(x) \geq j  \tag{2.2}\\ 0, & \text { if } G(x)<j\end{cases}
$$

The gray level signal $G(x)$ can be represented by using $k$-1 threshold sets $S_{\mathrm{j}}(G(x))$, $1 \leq j \leq k-1$. The threshold sets have two important properties.

Property 2.1 The threshold sets are linearly ordered. If $a<b$, then $S_{\mathrm{a}}(G(x)) \supseteq S_{\mathrm{b}}(G(x))$.

Property 2.2 The $k$-level signal $G(x)$ can be reconstructed uniquely from the threshold sets $S_{\mathrm{j}}(G(\boldsymbol{x})), 1 \leq j \leq k-1$ as :

$$
\begin{equation*}
G(x)=\max \left\{j: S_{j}(G(x))=1\right\}, \quad \forall x \in D . \tag{2.3}
\end{equation*}
$$

Prop.2.1 reveals that the thresholded set at gray level must be a subset of the thresholded set at a lower level. Prop. 2.2 is also known as the stacking operation which is the reverse to thresholding. An example of threshold representation of a 1-D signal is given in Table 2.1. The signal $G(x)$ has four levels as shown in the second row of the table. $G(x)$ is therefore represented by three threshold binary sets $S_{\mathrm{j}}, j=1,2,3$. Reconstruction of $G(\boldsymbol{x})$ from the threshold sets is achieved by Prop.2.2.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $G(x)$ | 3 | 2 | 1 | 0 | 0 | 1 | 2 | 3 | 3 | 3 |
| $S_{3}(G(x))$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| $S_{2}(G(x))$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $S_{1}(G(x))$ | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |

Table 2.1 Threshold Decomposition of a 1-D Signal

### 2.3 Robust Estimator Based Filters

The objective of robust estimation theory is to construct estimators that perform well for a variety of distributions and in the presence of contamination owing to wrong observation or measurement [David70]. There are three basic types of estimates, namely, the L, M and R estimates. The L-estimates are linear combinations of order statistics. The M-estimates are the maximum likelihood estimates, and lastly the R-estimates are known as the rank test. These estimators are found to have extensive applications in digital signal and image processing. For example, the M-filter proposed by Lee et al. [Lee85] and the Wilcoxon filter [Crin85] are based on the M-estimators and the R-estimators respectively. Median filters and order statistics based filters are based on the L-estimators, which are the most popular type of nonlinear filters [Pitas92].

### 2.3.1 Filters based on the L-estimators

Order statistics based filters are famous for excellent robustness under the presence of impulses. The median filter, which is the most common type of order statistic filters, is used by Tukey [Huang81] for time series analysis. In order statistics filtering, a window $W_{\mathrm{N}}$ is sliding over the points of a signal where $N$ is the index that determines the size of $W_{\mathrm{N}}$. For example, the size of $W_{\mathrm{N}}$ is $2 N+1$ for a 1-D median filter. The operation of median filters is simple. In 1-D data sequences, let $S_{2 N+1}$ be the set of $2 N+1$ elements of real numbers or integers which is masked by the window $W_{N}$, i.e. $S_{2 \mathrm{~N}+1}=\{x(n-N), \ldots, x(n), \ldots, x(n+N)\}$. Suppose that the elements in $S_{2 \mathrm{~N}+1}$ are sorted in descending order. Denote $x_{(i)}(n)$ as the $i^{\text {th }}$ ranked element. The $i^{\text {th }}$ order statistic is the $i^{\text {th }}$ largest element in $S_{2 \mathrm{~N}+1}$. In particular, the median of $S_{2 \mathrm{~N}+1}$ is the $(N+1)^{\text {th }}$ largest and the $(N+1)^{\text {th }}$ smallest ranked element. The maximum, denoted as max, is the largest element, while the minimum (min) is the smallest element in $S_{2 \mathrm{~N}+1}$. Hence, the order statistics are related :

$$
\begin{equation*}
\min =x_{(2 N+1)}(n) \leq x_{2 N}(n) \leq \ldots \leq x_{2}(n) \leq x_{1}(n)=\max \tag{2.4}
\end{equation*}
$$

After median filtering, the value at $x(n)$ is replaced by the median of $S_{2 \mathrm{~N}+1}$, i.e. $x_{(\mathrm{N}+1)}(n)$.

### 2.3.1.1 The Median Filter and its Derivations

Let $\operatorname{med}\left(X ; W_{\mathrm{N}}\right)$ be the output of a 1-D median filter on signal $X$ and with a moving window $W_{\mathrm{N}}$ of size $2 N+1$. $X$ can be a binary signal or a multilevel signal. The window length is $2 N+1$ which is always odd to facilitate a unique median of the data sequence. For even window length, the median can be defined as the average of $x_{(\mathbb{N})}(n)$ and $x_{(\mathbb{N}+1)}(n)$. The 2-D median filter is defined similarly which the shape of the window can be line segments, squares, circles, etc.. As median filters are nonlinear, so analysis cannot be done by the principle of superposition. A nonlinear filter can be characterized by two types of properties, namely deterministic and statistical properties [Huang81]. Deterministic properties show the effects of a filter on the geometrical structures of a signal, and the number of passes required to bring any non-invariant signal to its invariant. The invariant signal to the filter is also known as the root signal which is unchanged by the filter. Statistical properties indicate the efficiency and effectiveness of a filter in the removal of different type of noise. Although 1-D median filters can preserve edge and suppress impulse, the performance of 2-D median filters is not that satisfactory. Edges are blurred and fine details are removed. Examples of 2-D median filters can be found in [Pitas90]. Moreover, the computation for sorting boosts in 2-D median filtering. Modifications are made to improve the performance in edge and detail preservations as well as in the speed of computation. Attempts have been made to improve the speed of the 2-D median operation. The separable median filter was proposed by Narenda [Nar81]. In separable median filters, two passes of median filtering are used. A 2-D signal is first filtered row by row and then column by column. Deterministic analysis of separable median filter has been examined by Nodes and Gallagher [Nodes83]. In their work, the root structures are derived as well as the rate of convergence of separable median filter. Statistical analysis was performed by Liao et al. [Liao85]. Recursive filtering speeds up the 2-D median operation. In non-recursive median filtering, several passes are usually required for the convergence of a signal. Recursive median filter is defined as :

$$
\begin{equation*}
y(n)=\operatorname{med}(y(n-N), \ldots, y(n-1), x(n), . ., x(n+N)) \tag{2.5}
\end{equation*}
$$

The previous outputs $y(i), i \leq n-1$, are used for the calculation of the current output. One of the most important properties of a recursive median filter is that only one pass is needed to bring any nonroot signal to its root. A recursive median filter has a higher noise suppression and introduces more signal distortions than the non-recursive one. Separable recursive median
filter can be defined similar to the case of median filter. Analysis of recursive median filter is difficult. Arce et al. applied the principle of threshold decomposition to derive the expression for the statistical properties [Arce86] [Arce88].

Median filters commute with thresholding [Fitch84]. Thus, a multilevel (grayscale) median filter can be implemented by stacking the outputs of binary median filters. The analysis of the standard median filter is quite well established. Deterministic properties, including the root structures and rate of convergence, of 1-D median filters can be found in [Huang81] [Gal181] [Wendt86] [Astola87] [Gan91] [Fitch85] [Arce82] [Pitas92]. Statistical analysis can be found in [Huang81] [Arce86] [Pitas92]. [Huang81] provides an extensive study of statistical properties of standard median filters using different types of noise. Comparison of the relative efficiency of noise removal between a moving average filter and a median filter of the same window length can be found. It is shown that median filters are optimal for Laplacian noise. Neuvo et al. has implemented a signal processor based on a median filter in 1989 [Vainio89]. Another VLSI implementation of a median filter was performed by Richards [Rich90].

Moreover, signal distortions caused by median filter are analyzed. The effect of edge shift produced by median filters are examined by Davies [Davies89] [Davies92]. A median filter not only tries to smooth the noise in homogeneous region, but also tends to produce regions of constant or nearly constant intensity. Therefore, contours which are not present in the original image may result. The contouring phenomenon is known as streaking. Streaking is analyzed by Bovik [Bovik87]. An image, however, does not solely consist of edges and constant regions. Although noise suppression is achieved by median filtering, too much signal distortion is resulted as thin lines, sharp corners are removed or get blurred. In binary filtering, a point of 0 value will be preserved if the number of points of 0 values is greater than that of 1 . This is not likely so for points representing fine details. Images filtered by 2-D median filters can be found in [Jain89], in which edges are blurred and fine details are lost. Modifications of median filters are towards the goal of preserving fine details. In the following, several median based detail preserving filters will be discussed.

## Max/Median Filter

Arce and McLoughlin introduced a new class of median based filter, known as the max/median filter [Arce87]. The max/median filter is defined as :

$$
\begin{equation*}
y(i, j)=\max \left(z_{1}, z_{2}, z_{3}, z_{4}\right) \tag{2.6}
\end{equation*}
$$

where

$$
\begin{align*}
& z_{1}=\operatorname{med}(x(i, j-N), \ldots, x(i, j), \ldots, x(i, j+N)) \\
& z_{2}=\operatorname{med}(x(i-N, j), \ldots, x(i, j), \ldots, x(i+N, j))  \tag{2.7}\\
& z_{3}=\operatorname{med}(x(i+N, j-N), \ldots, x(i, j), \ldots, x(i-N, j+N)) \\
& z_{4}=\operatorname{med}(x(i-N, j+N), \ldots, x(i, j), \ldots, x(i+N, j-N)) .
\end{align*}
$$

The output of the max/median filter of window length $2 N+1$ is the maximum of the median subfilters $z_{\mathrm{i}}, i=1,2,3,4$, which run horizontally, vertically and diagonally through $x(i, j)$. Analysis of max/median filter [Arce87] [Neuvo88] shows that the filter is biased towards regions of higher intensity. This is caused by the max operator. Therefore, a detail will be preserved by max/median filter if it is brighter than its background. Wang [Wang90] therefore proposed the max/min median filter which is defined as :

$$
y(i, j)= \begin{cases}T_{1}(i, j), & \text { if }\left|T_{1}(i, j)-T_{0}(i, j)\right| \geq\left|T_{2}(i, j)-T_{0}(i, j)\right|  \tag{2.8}\\ T_{2}(i, j), & \text { if }\left|T_{1}(i, j)-T_{0}(i, j)\right|<\left|T_{2}(i, j)-T_{0}(i, j)\right|\end{cases}
$$

where

$$
\begin{align*}
& T_{0}(i, j)=\operatorname{med}(\text { all } x(i, j) \text { in the }(2 N+1) \text { by }(2 N+1) \text { window }) \\
& T_{1}(i, j)=\max \left(z_{1}, z_{2}, z_{3}, z_{4}\right)  \tag{2.9}\\
& T_{2}(i, j)=\min \left(z_{1}, z_{2}, z_{3}, z_{4}\right)
\end{align*}
$$

where $z_{\mathrm{i}}, i=1,2,3,4$ are the same as those in (2.9). The performance of max $/ \mathrm{min}$ filter in detail preservation is better than that of max/median. However, the max/min median filter is computational more complex than the max/median filter as $T_{0}$ requires sorting of all pixels within the moving window.

## Multi-stage Median Filter

Multi-stage median filter was proposed by Neuvo [Neuvo87]. The unidirectional multistage median filter is defined as :

$$
\begin{equation*}
\operatorname{umed}_{N}(i, j)=\operatorname{med}\left(\operatorname{med}\left(z_{1}, z_{2}, x(i, j)\right), \operatorname{med}\left(z_{3}, z_{4}, x(i, j)\right), x(i, j)\right) \tag{2.10}
\end{equation*}
$$

where $z_{\mathrm{i}}, i=1,2,3,4$ are defined by (2.9). By taking the union of orthogonal subfilters of length $2 N+1$, the bidirectional median filter can be defined. The subfilters $z_{\mathrm{b} 1}$ and $z_{\mathrm{b} 2}$ are defined as:

$$
\begin{align*}
z_{b 1}= & \operatorname{med}(x(i-N, j), x(i-N+1, j), \ldots, x(i-1, j), x(i, j), x(i+1, j), \ldots, \\
& x(i+N-1, j), x(i+N, j), x(i, j-N), x(i, j-N+1), \ldots, x(i, j-1), x(i, j+1), \ldots, \\
& x(i, j+N-1), x(i, j+N))  \tag{2.11}\\
z_{b 2}= & \operatorname{med}(x(i-N, j-N), x(i-N+1, j-N+1), \ldots, x(i-1, j-1), x(i, j), x(i+1, j+1), \ldots, \\
& x(i+N-1, j+N-1), x(i+N, j+N), x(i-N, j+N), x(i-N+1, j+N-1), \ldots, \\
& x(i-1, j+1), x(i+1, j-1), \ldots, x(i+N-1, j-N+1), x(i+N, j-N))
\end{align*}
$$

Multi-stage median filters are proved to be detail and edge preserving. Moreover, they are computational efficient. For median based filters, most of the computation is spent on the sorting of data. Obviously, in a multi-stage median filter, the number of sorting is greatly reduced as the $(2 N+1)^{2}$ pixels in the moving window is grouped into several subfilters. It has been reported that the computation for the bidirectional multi-stage filter of subfilter length 5 requires less that one fourth of computation time of the standard median filter [Neuvo87].

Owing to the asymmetric statistical properties of the max/median filter, Arce and Foster has derived another multi-stage median filter, known as the multi-stage max/median filter, which is defined as:

$$
\begin{equation*}
y(i, j)=\operatorname{med}\left(T_{1}(i, j), T_{2}(i, j), x(i, j)\right) \tag{2.12}
\end{equation*}
$$

where $T_{1}(i, j)$ and $T_{2}(i, j)$ are defined in (2.9). Analysis shows that the outputs of the multi-stage $\mathrm{max} /$ median filter is identical to the unidirectional multi-stage median filter [Arce89]. The multi-stage median filter and the multi-stage max/median filter are unbiased. In other words, the filters preserve details of higher and lower intensities. In addition, the multi-stage median filter, in its ultimate form, will be composed of numerous subfilters which span all the possible directions.

Heinonen and Neuvo introduced a new class of median based filter in 1987 [Neuvo87a]. The filter contains FIR (Finite Impulse Response) substructures and is known as the FIR-median hybrid (FMH) filter. The FMH filter with $M$ FIR filters those transfer functions are $H_{\mathrm{i}}(z), i=1,2, . ., M$, is defined as :

$$
\begin{equation*}
y(n)=\operatorname{med}\left(y_{1}(n), y_{2}(n), \ldots, y_{M}(n)\right) \tag{2.13}
\end{equation*}
$$

where $y_{\mathrm{i}}(n)$ is the output of the FIR filter $H_{\mathrm{i}}(z), i=1,2, \ldots, M$., and $M$ is odd. The hybrid median filter can be extended to 2-D by using four subfilters. Two sets of subfilters can be selected. One of them is described by :

$$
\begin{align*}
& y_{1}(i, j)=\frac{1}{N}\{x(i+1, j)+\ldots+x(i+N, j)\} \\
& y_{2}(i, j)=\frac{1}{N}\{x(i, j-N)+\ldots+x(i, j+1)\} \\
& y_{3}(i, j)=\frac{1}{N}\{x(i-n, j)+\ldots+x(i, j)\}  \tag{2.14}\\
& y_{4}(i, j)=\frac{1}{N}\{x(i, j-1)+\ldots+x(i, j+N)\}
\end{align*}
$$

The second subfilter set is resulted by rotating those subfilters in (2.14) by $45^{\circ}$. The performance of FMH filters is analyzed in [Neuvo87a] [Neuvo88] and [Neuvo89]. It has been shown that owing to the inclusion of FIR filters, the FMH filters do not commute with thresholding. However, implementation of the FMH filter is simple. 2-D median hybrid filters preserve fine details fairly well.

### 2.3.1.2 Rank Order Filters and Derivations

The median filter is a particular case of rank order filters. Rank order filters were first introduced by Nodes et al. in 1982 [Nodes82]. An $i^{\text {th }}$ rank order filter replaces the original signal $x(n)$ with the $i^{\text {th }}$ order statistics of the signal data within the moving window. The root signals of rank order filters are quite different from those of median filters. Only constant signals are invariant to rank order filters. Hence, rank order filtering cannot be applied repeatedly. One of the applications of rank order filters is as a peak detector in AM detection while eliminating impulses. Lee proposed an edge gradient enhancing filter based on adaptive order statistic in 1987 [Lee87]. Two rank order filters, the maximum filter (max filter) and
the minimum filter (min filter) are of particular interest, since these filters are equivalent to the multilevel morphological dilation and erosion which will be discussed later. Similar to median filtering, rank order filtering commutes with thresholding. Coyle has derived the relation between rank order filtering and the mean absolute error [Coyle88]. Some of the derivations of the ranked order filtering will be discussed.

## Linear Combinations of Order Statistics (L-Filters)

Bovik et al. introduced a new family of nonlinear filters which are linear combinations of order statistics [Bovik83]. The new family of nonlinear filter is known as the $L$-filter [Huber81]. The 1-D $L$-filter is defined as :

$$
\begin{equation*}
y(n)=\sum_{i=1}^{2 N+1} \alpha_{i} \cdot x_{(i)}(n) \tag{2.15}
\end{equation*}
$$

where $x_{(i)}(n)$ represents the $i^{\text {th }}$ rank order statistics among the window $\{x(n-$ $N), \ldots, x(n), \ldots, x(n+N)\}$. By setting $\alpha_{N+1}=1$ and $\alpha_{i}=0, i=1,2, \ldots, N, N+2, \ldots, 2 N+1$, a $L$-filter is reduced to a median filter. Also, if all $\alpha_{\mathrm{i}}=\alpha_{\mathrm{j}}, i, j=1,2, \ldots, 2 N+1$, the filter can be regarded as a moving average filter. The 2-D $L$-filter is defined similarly. One of the advantages of $L$-filters over median filters is that no streaking effect is produced, provided that the coefficients are not similar to that of the median filter. From (2.15), it is quite obvious that the performance of a $L$-filter is somewhere in between a median filter and a moving average filter under Laplacian and Gaussian noise. The main disadvantage of $L$-filters is the high computation requirement. Moreover, a $L$-filter has poorer robustness properties if all its coefficients are nonzero. Also, time ordering between adjacent signal points is destroyed by $L$-filtering owing to the sorting operation. The time ordering is irrelevant in stationary signal such as a constant signal embedded in i.i.d. noise. However, this time ordering distortion causes problems in non-stationary signal. Deterministic analysis of $L$-filters and their relation to linear filters are discussed in [Bovik89]. Generalization of $L$-filter was done by Palmieri [Palm89] and is known as the $L l$-filter. A $L l$-filter is an FIR filter with each coefficient dependent on the rank of the specific element within the window.

## Alpha-Trimmed Mean Filter

Bednar and Batt in 1984 proposed the $\alpha$-trimmed mean filter [Bednar84]. In trimmed mean calculation, an average is taken over the trimmed data set in which some data are removed. This is different from the normal mean by which all data are averaged. The number of data points that are removed is controlled by a trimming parameter $\alpha, 0 \leq \alpha \leq 0.5$. The definition of the $\alpha$-trimmed mean filter using a moving window of $N$ points and a trimming parameter $\alpha$ is :

$$
\begin{equation*}
y_{\alpha}(n)=\frac{1}{N-2[\alpha N]} \sum_{i=[\alpha N]+1}^{N-[\alpha N]} x_{(i)}(n) \tag{2.16}
\end{equation*}
$$

where [ [] is the integral part function. An obvious difference of an $\alpha$-trimmed mean filter to that of a median filter is the restriction of filter length. If $N$ is odd and $\alpha$ is chosen to be 0.5 , (2.16) is reduced to a median filter. If $\alpha$ is chosen to be $0,(2.16)$ becomes a moving average filter. The complementary filter to the $\alpha$-trimmed mean filter is known as the $\alpha$-trimmed complementary filter, which is defined as :

$$
\begin{equation*}
\left[y_{\alpha}(n)\right]^{c}=\frac{1}{2 \alpha N}\left[\sum_{i=1}^{[\alpha N]} x_{(i)}(n)+\sum_{i=N-[\alpha N]+1}^{N} x_{(i)}(n)\right] . \tag{2.17}
\end{equation*}
$$

From the definition, edge smearing is unavoidable since averaging is required. Modifications of the $\alpha$-trimmed mean filter bring about the modified trimmed mean (MTM) filter and the double window modified trimmed mean (DWMTM) filter. Discussions of these filters can be found in [Lee85]. Lee also performed the statistical analysis of the standard $\alpha$ trimmed mean in 1988 [Lee88]. Adaptive trimmed mean filters are applied in image restoration by Bovik [Bovik88].

### 2.3.2 Filters Based on the M-estimators (M-Filters)

Let $\rho$ be an arbitrary function. The M -estimation is to find $y$ s.t. the problem [Huber81]:

$$
\begin{equation*}
\sum \rho(x, y) \tag{2.18}
\end{equation*}
$$

is minimized. The M-estimate is the solution to the implicit equation :

$$
\sum \psi(x, y)=0
$$

where $\psi(x, y)=\frac{\partial \rho(x, y)}{\partial y}$.

In nonlinear filtering, the solution $y(n)$ to the equation is of interest :

$$
\begin{equation*}
\sum_{i=n-N}^{n+N} \psi(x(i)-y(n))=0 \tag{2.20}
\end{equation*}
$$

where $\psi$ is some odd, continuous, and sign-preserving function so that $\psi(x)$ is positive (negative) whenever $x$ is positive (negative). Lee has shown that a unique solution of $y(n)$ exists if $\psi$ is strictly increasing [Lee85]. Depending on the function $\psi(x)$, different $M$-filters can be defined. The median and the arithmetic mean are special cases of $M$-filters. If $\psi=a x$, (2.20) represents a moving average filter. If $\psi$ is a hard limiter, which is defined as :

$$
\psi(x)=\left\{\begin{align*}
1 & \text { if } x \geq 0  \tag{2.21}\\
-1 & \text { if } x<0
\end{align*}\right.
$$

(2.20) becomes a median filter. If the limiter type M filter (LTM filter) is defined by:

$$
\psi(x)=\left\{\begin{align*}
g(p), & x>p  \tag{2.22}\\
g(x), & |x| \leq p \\
-g(p), & x<-p
\end{align*}\right.
$$

where $g(x)$ is a strictly increasing, odd, continuous function, and $p$ is some positive constant, then the filter is known as a LTM filter. Detail analysis of LTM filters can be found in [Lee85]. Although M-filters are proved to be robust, it is difficult to calculate the implicit solution from (2.20). Hence, the M-filter is less popular than the median based filters.

### 2.3.3 Filter Based on the R-estimators (R-Filters)[Crin85]

The most important R-filter is the Wilcoxon filter which is defined as :

$$
\begin{equation*}
y(n)=\operatorname{med}\left(\frac{x_{(i)}(n)+x_{(j)}(n)}{2}, 1 \leq i \leq j \leq 2 N+1\right) \tag{2.23}
\end{equation*}
$$

The output of the Wilcoxon filter is the median of all the averages formed pairwisely among all the data within the filter window. The filter is the locally most powerful filter for logisticdistributed noise, and it performs efficiently in estimating noisy signals in non-stationary symmetric noise environments where there are deviations from the assumed noise model. For Gaussian noise and Laplacian noise, the performance of Wilcoxon filter lies between that of the moving average and the median. However, the filter does not preserve edges well and is computational intensive.

Pitas and Venetsanopoulos in their review paper have examined many robust estimator based nonlinear filters [Pitas92]. They have compared the performance of the above mentioned median based filters. Although most filters are edge preserving, only the multistage median filter and the FIR-median hybrid filter preserve details. In addition, the multistage median filter is computationally not complicated. Both filters are unbiased. Table 2.2 is duplicated from [Pitas92], compares the performance of the multi-stage median filter and the FIR-median filter.

|  | Multi-stage Median | FIR-median Hybrid |
| :--- | :---: | :---: |
| 1. Detail Preserving | $\checkmark$ | $\checkmark$ |
| 2. Edge Preserving | $\checkmark$ | $\checkmark$ |
| 3. Bias | $\checkmark$ | $\checkmark$ |
| 4. Computation complexity |  | $\checkmark$ |
| 5. Salt-and-Pepper Noise | $\checkmark$ |  |
| 6. Gaussian Noise | $\checkmark$ |  |
| 7. Long-tailed Noise | $\checkmark$ |  |
| 8. Positive Impulses | $\checkmark$ |  |
| 9. Negative Impulses | $\checkmark$ |  |

[^0]A check mark means that the performance is the best among all filters compared by Pitas. The multi-stage median filter performs better in impulse suppression (positive, negative and salt-and-pepper) and long tailed additive white noise than the latter. We can conclude that the multi-stage median filter is the best detail preserving so far developed.

### 2.4 Filters Based on Mathematical Morphology

Mathematical morphology is a set-theoretical methodology for image analysis. Morphology can quantify many aspects, size and shape, of the geometrical structure of a signal. The method was mainly developed by Matheron and Serra [Serra82]. Applications of mathematical morphology are found in multidimensional signal processing, especially in image processing. In image processing, applications of mathematical morphology can be found in the areas of morphological edge detection [Lee87a], multiscale image analysis [Hara87], image segmentation [Serra82], morphological sampling [Hara87], image coding [Mara86], signal decomposition and representation [Mara89a], and nonlinear filtering [Mara87] [Mara87a] [Mara89] [Serra92]. Although mathematical morphology is widely used in different areas of signal processing, the following discussion will concentrate on its applications in nonlinear filtering.

### 2.4.1 Basic Morphological Operator

Morphological operators are set operators that modify the geometrical features of a binary signal. Let $X \subseteq D$ be the set representation of a binary signal, and let $S E \subseteq D$ be a compact set [Gaal64] of smaller size and simpler shape. The set $S E$ is known as a structuring element. Denote $X_{\mathrm{b}}=\{\boldsymbol{x}+\boldsymbol{b}: \boldsymbol{x} \in X\}$ the vector translate of $X$ by $\boldsymbol{b} \in \boldsymbol{D}$. The basic morphological operators for sets are dilation $\oplus$ and erosion $\Theta$ which are defined as follows. ${ }^{1}$

Let $X$ and $S E$ be subsets of $D$. The dilation of $X$ by $S E$ is defined as:

$$
\begin{equation*}
X \oplus S E=\bigcup_{b \in S E} X_{b}=\{x+b: x \in X \text { and } b \in S E\} . \tag{2.24}
\end{equation*}
$$

The erosion of $X$ by $S E$ is defined as :

[^1]\[

$$
\begin{equation*}
X \ominus S E=\bigcap_{b \in S E} X_{-b}=\left\{z: S E_{z} \subseteq X\right\} . \tag{2.25}
\end{equation*}
$$

\]

The output of the dilation operator of $X$ by $S E$ is the set of translation points $\boldsymbol{b}$ s.t. the translate of $X$ by $\boldsymbol{b}, \boldsymbol{b} \in S E$ has nonempty intersections with $X$. Therefore, $X \oplus S E=\left\{z: S E_{\mathbf{z}} \cap X \neq \varnothing\right\}$. The output of the erosion operator of $X$ by $S E$ is the set of translation points s.t. the translate of $X$ by $\boldsymbol{b}$ is a subset of $X$. Complementing the dilated set of $X^{c}$ by $S E$ produces an output which are quite similar to that of the erosion operator.

Theorem 2.1 Let $S \hat{E}$ be the reflected set of $S E$, i.e. $S \hat{E}=\{-b: b \in S E\}$, the morphological dual of erosion is :

$$
\begin{equation*}
(X \ominus S E)^{c}=X^{c} \oplus S \hat{E} \tag{2.26}
\end{equation*}
$$

where $X^{c}$ is the complement of $X$.

The proof of this theorem can be found in [Hara87]. If the structuring element is symmetrical about its origin, (2.26) can be written as :

$$
\begin{equation*}
(X \ominus S E)^{c}=X^{c} \oplus S E \tag{2.27}
\end{equation*}
$$

In other words, dilating the foreground of a binary signal using a structuring element which is symmetrical about its origin is equivalent to erode the background by the same structuring element. Unless otherwise specified, all structuring elements used in the following discussion will be symmetric.

In addition, two morphological operators, the opening $\circ$ and closing $\bullet$ transformations can be defined. The opening of $X$ by $S E$ is defined as :

$$
\begin{equation*}
X \circ S E=(X \bigoplus S E) \oplus S E \tag{2.28}
\end{equation*}
$$

The closing of $X$ by $S E$ is defined by :

$$
\begin{equation*}
X \bullet S E=(X \oplus S E) \ominus S E \tag{2.29}
\end{equation*}
$$

The operations of morphological operators are illustrated in Fig.2.1. The original binary signal $X$ and the structuring element $S E$ is shown in Fig.2.1(a). $X$ looks like a dumbbell
with projections at the square blocks. $S E$ is a symmetric horizontal structuring element of 3 points. Figs.2.1(b) to 2.1 (e) show the geometrical modifications by these operators. Dilation fills the hole in $X$ as well as expands $X$. Erosion shrinks $X$, bisects $X$ and enlarges the hole in the way opposite to that dilation does. The opening removes the vertical projection and bisects $X$ by cutting through the bar in between. The closing widens the bar and fills the holes. Obviously, the smoothing effect is determined by the shape and size of the structuring element. If the dumbbell depicted in Fig.2.1(a) is eroded by a vertically oriented structuring element, all horizontal projections will be removed.


Figure 2.1 Morphological Operations (a)X and SE (b)Dilation of $X$ by $S E$ (b)Erosion of $X$ by SE (c)Opening of $X$ by $S E$ (d)Closing of $X$ by $S E$

The binary operators can be extended to multilevel signals. A multilevel signal $G(x)$ can be operated morphologically. A typical way is to make use of the threshold sets of $G(x)$ [Serra82] [Heij89] [Mara90]. Dilating the threshold sets of $G(x)$ by $S E$ is defined as:

$$
\begin{equation*}
(G \oplus S E)(\boldsymbol{x})=\max _{\boldsymbol{b} \in S E}\{G(\boldsymbol{x}-\boldsymbol{b})\} . \tag{2.30}
\end{equation*}
$$

The erosion of $G$ by $S E$ is defined similarly.

$$
\begin{equation*}
(G \ominus S E)(\boldsymbol{x})=\min _{\boldsymbol{b} \in S E}\{G(\boldsymbol{x}+\boldsymbol{b})\} . \tag{2.31}
\end{equation*}
$$

In addition, the opening $\circ$ and closing $\bullet$ of $G$ by $S E$ are defined as $G \circ S E=(G \ominus S E) \oplus S E$ and $G \bullet S E=(G \oplus S E) \ominus S E$.

Sternberg [Stern86] have attempted to extend the binary operators to multilevel by the umbra representation. The umbra representation $U(G)$ of a $n$-D function $G(x)$ is defined as:

$$
\begin{equation*}
U(G)=\{(\boldsymbol{x}, a): a \leq G(\boldsymbol{x})\} . \tag{2.32}
\end{equation*}
$$



Figure 2.2 Umbra of a Multilevel Function $G(x)$

The umbra is the collection of points below the surface represented by $G(x)$. The shaded region in Fig.2.2 represents the umbra of a single variable function $G(x)$. The umbra of $G(x)$ is the area covered by $G(x)$. Thus, the umbra is a $(n+1)$-D function. The function $G(x)$ an be reconstructed from the umbra by for all $\boldsymbol{x} \in \boldsymbol{D}$ :

$$
\begin{equation*}
G(x)=\max \{a:(x, a) \in U(G)\} \tag{2.33}
\end{equation*}
$$

In the previous discussion, a structuring element is a subset in $\boldsymbol{D}$. A function $g(\boldsymbol{x})$ can be a structuring element, provided that $g$ is defined on a compact region of support. If $\boldsymbol{x}$ lies within the region of support of $g$, then $g(x) \neq-\infty$. Dilation and erosion of the umbra of $G$ by the umbra of $g$ are defined as:

$$
\begin{align*}
& (G \oplus g)(x)=\max _{y}\{G(y)+g(x-y)\}  \tag{2.34}\\
& (G \ominus g)(x)=\min _{y}\{G(y)-g(y-x)\} \tag{2.35}
\end{align*}
$$

where $G$ and $g$ are ranged over the intersection of the regions of support of $G$ and $g$. The opening and closing of $G$ by $g$ is $G \circ g=(G \ominus g) \oplus g$ and $G \bullet g=(G \oplus g) \ominus g$ respectively.

Property 2.3 Morphological opening and closing are idempotent, i.e.

$$
\begin{equation*}
(G \circ g) \circ g=G \circ g \text { and }(G \bullet g) \bullet g=G \bullet g \tag{2.36}
\end{equation*}
$$

In other words, repeated opening and closing give the same results as the signal is
opened and closed once. Prop. 2.3 holds for binary as well as multilevel signals. This is a very appealing property in nonlinear filtering, since one pass is required for obtaining the root signal. Therefore, the opening and closing operators are the most frequently used morphological operators in digital signal processing.

### 2.4.2 Morphological Filters



Figure 2.3 Block Diagram of a Filter Algorithm Based on morphological Closing and Opening

## Opening and Closing Filters

Examples of applications of morphological filters can be found in [Safa89] [Chu89]. These filters are combinations of the closing and opening transformations [Safa89] [Chu89]. This is due to the fact that the opening and closing have very fast convergence rates. A single filter pass can bring any nonroot signal to its root.

EKG signals are frequently plagued by impulsive noise due to muscle activities and power line interference. Moreover, background normalization is needed to correct the baseline drift of the signal caused by the respiration and motion of the subject. Chu and Delp [Chu89] have developed a filtering algorithm to remove impulses and normalize background drift. The block diagram of the filtering algorithm is depicted in Fig.2.3. Two blocks of morphological filters are in cascade. Each block consists of two morphological filters, a closing-opening and a opening closing filter. An EKG signal is processed simultaneously along two filter paths in parallel. As an EKG signal is a 1-D signal, so 1-D structuring elements are used. Along one path, the signal is first opened and then closed. On the other path, the data is closed and then opened.

Safa applied closing and openings to remove speckle noise in radar images [Safa89]. Speckle noise is a kind of noise that appears in coherent imaging. It is multiplicative and occurs whenever the roughness of the object being imaged is of the order of the wavelength of the incident radiation [Jain89]. Traditionally, statistical methods based on the minimization
of mean quadratic error between the noiseless image and its estimate, and order statistics are used to handle this problem. Safa have proposed the multidirectional filter and the comparative filter. These filters are combinations of morphological closing and opening. Results revealed that these filters are able to preserve edges.

Obviously, the closing and opening operators are the basic components of morphological filters since these transformations are idempotent. 1-D and 2-D closing-opening and opening-closing filters have been examined by Stevenson et al.[Stev87]. A $N^{\text {th }}$ order 1-D closing-opening filter ( 1 DCO ) is a two-pass filter. A signal is first closed and then opened with a structuring element $B_{\mathrm{N}}$ which is a 1-D set with $(N+1)$ points whose origin is at the leftmost point. Similarly, a 1-D opening-closing filter is an opening followed by a closing. In 2-D closing-opening (opening-closing) filtering, four structuring elements are used. These structuring elements are oriented at $0^{\circ}, 45^{\circ}, 90^{\circ}$ and $135^{\circ}$ with the horizon. Comparison shows that these morphological filters are not detail preserving [Pitas92], in spite of the onepass convergence.

## Soft Morphological Filters

Koskinen et al. proposed new morphological operations, called soft morphological operations [Kos91]. Erosion and dilation are equivalent to, respectively, local minimum and local maximum of the data masked by a structuring element. In soft morphological operations, soft erosion and soft dilation replace local minimum and local maximum by more general weighted order statistics. The soft dilation is denoted as $G \oplus[B, A, k]$ and soft erosion is denoted as $G \ominus[B, A, k] . B$ is the structuring element and $A$ is the centre of $B$. Denote $\Delta$ as the repetition operation, i.e. $k \diamond X$ implies that $X$ is repeated itself for $k$ times. The definitions of soft dilation and soft erosion are given as :

$$
\begin{align*}
G \ominus[B, A, k](\boldsymbol{x})= & k^{\text {th }} \text { smallest data in }\left\{k \diamond G(\boldsymbol{a}): \boldsymbol{a} \in A_{x}\right\}  \tag{2.37}\\
& \bigcup\left\{G(\boldsymbol{b}): \boldsymbol{b} \in(B-A)_{x}\right\}
\end{align*}
$$

and

$$
\begin{align*}
G \oplus[B, A, k](\boldsymbol{x})= & k^{\text {th }} \text { largest data in }\left\{k \diamond G(\boldsymbol{a}): \boldsymbol{a} \in A_{x}\right\} \\
& \bigcup\left\{G(\boldsymbol{b}): \boldsymbol{b} \in(B-A)_{x}\right\}
\end{align*}
$$

Hence, soft dilation and soft erosion can be regarded as ranked order filters. The definitions of soft opening and soft closing are analogous to those of standard morphological opening and closing respectively. Analysis of soft morphological filters can be found in
[Kos91][Kuos92].

### 2.5 Chapter Summary

An overview of nonlinear filters which are based on statistical robust estimators and mathematical morphology is presented. A nonlinear filter is characterized by its suppression of different kind of noise, preservation of edges and details and the computation requirement.

Robust estimators, for instance, M-estimators, L-estimators and R-estimators are applied in image processing. The M-filter are the solutions to the implicit equation which minimizes some cost functions (Maximum likelihood). The Wilcoxon filters are originated from the rank test (R-estimators). These filters are not frequently applied than those Lestimators based ones. The median, rank order filter, and their deviations belong to the filter class of L-estimators. Median based filters, such as the multi-stage median filter has excellent edge and detail preserving properties. A median based detail preserving filter usually consists of several subfilters and requires multi-stage operation. The subfilters are median filters or moving average filters which are oriented in different directions. The output of the filter is determined by those of its subfilters.

Mathematical morphology provides an alternative for nonlinear filter design. Basic principles of mathematical morphology has been discussed. There are four basic operators in mathematical morphology, namely dilation, erosion, opening and closing. The closing and opening operators are renowned for their excellent convergent rate. Morphological filters operate on the geometry of a signal. The smoothing caused by a morphological filter is determined by the interaction of the input signal with a structuring element. A structuring element is a smaller and simpler set. Owing to the idempotence of the opening and closing transformations, a single filter pass can always convert a nooroot signal to its root. This is very attractive in nonlinear filtering. Therefore, morphological filters based on these two operations are developed. Various morphological filters such as the closing-opening filter and opening-closing filters, however, are not detail preserving.

## Chapter 3 Multi-Structuring Element Erosion Filter

### 3.1 Introduction

One of the basic principles of nonlinear filters, such as order statistics based filters, alpha trimmed mean filter, etc, is to make use of the relation among the values of the picture elements within the moving window. An point is an isolated pointed if it is visually different from its neighbourhood. Nonlinear filters never treat an isolated point which has no correlation with its neighbourhood as useful detail. It seems that order-statistics based filters are not the only solution although these filters are proved to be efficacious. Mathematical morphology is the study of form and structure, has broken forth another approach in nonlinear filtering [Serra82]. In this chapter, a new detail preserving filter, Multi-structuring Element Erosion Filter, will be discussed. This filter is based on one of the morphological operator morphological erosion.

Firstly, the design criteria of Multi-Structuring Element Erosion Filter will be discussed and then the definition and construction of various structuring elements used will be given. The binary filter is defined first. The multilevel filter is defined using the selective threshold decomposition. The selective threshold decomposition is a modification of the threshold decomposition which was introduced by Fitch et al. [Fitch84]. This modified threshold decomposition can only handle set processing filters. Examples are given to demonstrate the basic operations of these filters.

### 3.2 Problem Formulation

Edges and details are important carriers of information in an image. Edges characterize object boundaries and are useful for segmentation, registration, and identification of objects. Edge points are identified by pixel locations having abrupt gray level changes. Details include small objects, sharp corners, thin lines and textures. One of the most important classes of details is texture. Textures characterize the structural patterns of surface of objects. Most image filters are used in the preprocess of an image to remove noise. The preprocessed image will then further be analyzed. Therefore, edges and details preservations are important properties of an image filter.

It is fully understood that moving average filters blur edges and remove details. Moreover, moving average filters are poor in suppressing impulses and long tailed distributed noise such as Laplacian noise. Nonlinear techniques are therefore developed. 2-D median filters have been used in image processing. A moving window of size $(2 N+1)$ by $(2 N+1)$ is slided on the entire image, the value at the centre of the window is replaced by the median of the pixels masked by the square window. Although noise suppression is achieved by 2-D median filtering, image information such as thin lines, edges and fine details are distorted or removed. Several new detail preserving nonlinear filters have been introduced [Arce89] [Neuvo87]. Most of these filters are order statistics based. Multistage max/median filters and multistage median filters consider the geometrical structure of a signal by utilizing some directional subfilters. The output of these filters are the median of their subfilters. The operation of multistage filters has been discussed in Chapter 2. Consideration of geometrical structures is key to a filter in preserving image details.

Researchers have been asked for a methodology which is able to deal with the geometrical structures as well as the magnitudes of the points of a signal. Matheron and Serra have developed the theory of mathematical morphology [Serra82]. Morphology is the study of form and structure, which can quantify many aspects of the geometrical structure of signals in a way that agrees with human perception. Four basic operators are found in mathematical morphology, which are erosion, dilation, opening and closing. Erosion and dilation correspond to the minimum and maximum operators respectively. These two operators are seldomly used in nonlinear filter design, as repeated passes by these filters reduce the input signal to a constant. A closing is a dilation followed by an erosion, and an opening is an erosion followed by a dilation. The closing and opening operators have the idempotent property, i.e. a closed (opened) signal is invariant to further closing (opening). This is a very appealing property to nonlinear filter designers, since those filters based on opening and closing operations have very fast convergent rates. However, morphological filter operations, such as erosion, dilation, closing and opening, closing-opening and opening-closing are not detail preserving [Pitas92].

In Chapter 2, Fig.2.1 has already shown some examples of the closing and opening of a dumbbell shaped binary signal by a 3-point horizontal structuring element. Fine details, such as the thin line connecting the rectangular blocks are removed. Also, the hole in the
upper rectangular block is filled. A single structuring element is not sufficient to preserve details in different orientations. Structuring elements with different directions, or more appropriately of different shapes, must be taken into account. Stevenson and Arce [Arce87] have already implemented a family of 2-D morphological filters in which several structuring elements are used. Their filters are known as the closing-opening and opening-closing filters. The former is a closing followed by an opening while the latter is an opening followed by a closing. Four linear structuring elements, which are oriented at angles $0^{\circ}, 45^{\circ}, 90^{\circ}$ and $135^{\circ}$ with the horizon are used. However, these filters are still not detail preserving [Pitas92]. Apparently, this is due to the fact that opening and closing operations remove details.

In this chapter, a new filter based on erosion is proposed. The filter is called multistructuring element erosion filter. As its name implies, the filter uses several structuring elements. The output of the filter is the union of the eroded outputs by all structuring elements.

It is reminded that one of the objectives of image filtering is to suppress noise. Many types of noise, Gaussian noise, Laplacian noise, and impulse noise are always observed in an image. As human eyes are sensitive to abrupt changes, the uncomfortable effect under the presence of salt and pepper noise is more pronounced. Therefore, our prime objective is to suppress salt and pepper noise. Under high noise environment, clustering of noise may be resulted after several filter passes. The clustered noise patches are difficult to remove thus the filter must be able to prevent noise clustering.

In the design of the Multi-Structuring Element Erosion Filter, the following considerations are taken into account. They, including fine details preservation, salt and pepper noise suppression, prevention of noise clustering under high noise environment and absence of priori knowledge, contribute to the definition and the basic operation of the filter.

### 3.3 Description of Multi-Structuring Element Erosion Filter

Let $S E_{\mathrm{i}}, i=1,2, \ldots$ be the structuring elements used in a multi-structuring element erosion filter with an index $N$. $N$ governs the size of the structuring element. If the filter index is $N$,
then the length of all structuring elements is $2 N+1$. Moreover, we adopt the notation used in Chapter 2, i.e. a gray level signal is represented as $G(x), x \in \boldsymbol{D}$.

### 3.3.1 Definition of Structuring Element for Multi-structuring Element Erosion Filter

A structuring element is a subset of the signal space $\boldsymbol{D}$ with a specific size and shape, and an origin. The origin in a structuring element is the location where the erosion outputs. The size is the area or the number of pixels composing the structuring element. Although structuring elements of any shapes can be used, we intend to impose some restrictions. Structuring elements of 1-D and 2-D filters will be discussed.

Two approaches can be used to design the structuring elements used in the multistructuring element erosion filter. One can exhausively search the details which are required to be preserved. For example, if a star shaped object needs to be preserved, then star shaped structuring elements will be used. It should be noted that a structuring element is characterized by its shape as well as the location of its origin. Fig.3.1 shows an example of this approach where five structuring elements are required. If the detail to be preserved is composed of $n$ pixels, then generally $n$ structuring elements are needed. Li has utilized this approach to remove unwanted features in an image [ Li 90 ]. The filter is known as the median based feature selective filter. To remove unwanted features, priori knowledge of the input image is required.

Another approach employs structuring elements which trace the edge of an object. Geometrically, an object can be interpreted as a stack of constant regions at different gray levels. For example, the star shaped detail in Fig. 3.1 can be decomposed into simpler objects. It can be regarded as the union of two straight lines, one orients horizontally and the other stands vertically. As a result, the problem of defining a structuring element is reduced to define a set of lines, or smaller objects, by which larger objects can be built upon. The second approach is chosen since the set of lines or smaller objects can be defined without priori knowledge on an input image.


Figure 3.1 Example of Object Based Structuring Elements

Definition 3.1 $S E$ is a structuring element used in the multi-structuring element erosion filter with index $N$ if and only if it satisfies all following conditions:
T. 1 Its size is $(2 N+1)$.
T. 2 Adjacent points must be connected.
T. 3 Any 3 connected points must not form a right angle.
T. 4 All points must be resided in the rectangle those opposite corners are the end points of the structuring element SE.
T. 1 restricts the size of all structuring elements. T. 2 imposes the requirement on the connectivity of the structuring elements. T. 3 and T. 4 define the shape of the structuring elements. In particular, T. 3 rejects those combinations of pixel which contain right angle(s). A right angle is a detail, but right angled patterns are rejected because they contain redundancy which can be accounted for. An example illustrates how the redundancy can be deal with will be given later. By T.4, a structuring element must be bounded by the rectangle formed by the end points of the structuring element. Therefore, closed signal patterns as well as oscillating structures are forbidden. In summary, the structuring elements used in a Multi-
structuring Element Erosion Filter must be simple, single pixel wide, connected, and of ( $2 N+1$ ) pixels.

Example 3.1 In these examples, combinations of pixels are given. The selection of structuring elements from these combinations are determined in accordance with Def.3.1. Fig.3.2 gives several examples which fail T.3. The right angled pixel patterns in Figs.3.2(a) and 3.2(b) are not structuring elements. The pattern shown in Fig.3.2(a) contains redundant information as there are more that one path between $p_{1}$ and $p_{2}$. In Fig.3.2(a), the pattern can be accounted for by the union of the structuring elements in Figs.3.2(c) and 3.2(d). Fig.3.2(b) is rejected since $p_{3}, p_{4}$ and $p_{5}$ form a right angle.

Fig.3.3 shows some rejected pixel combinations owing to T.4. Those patterns in Figs.3.3(a) and 3.3(d) are invalid since $p_{1}$ and $p_{7}$ are not resided in the interior of the rectangles formed by the end points. Fig.3.3(b) is a pattern which tends to close itself. Fig.3.3(c) shows an oscillating combination of pixels.



Figure 3.3 Examples of Pixel Pattern Fail T. 4 (a) $(c)(d)$ oscillating (b)close itself

Structuring elements are further grouped into subgroups which are classified by their locations of the origin. As all structuring elements are of $(2 N+1)$ points long, so there are $(2 N+1)$ subgroups of structuring elements.

Theorem 3.1 The number of structuring elements in each subgroup is $7.2^{2 \mathrm{~N}}-2^{\mathrm{N}+3}$.

The proof of Theorem 3.1 is in Appendix I. In particular, three out of the ( $2 N+1$ ) subgroups are chosen as the structuring elements of a multi-structuring element erosion filter. Let $T_{1 \mathrm{~N}}$ be the subgroup of structuring elements whose origins are at the centre of the elements, and $T_{2 \mathrm{~N}}$ be the subgroups of structuring elements whose origins are at the end points. Let $T_{\mathrm{N}}$ be the collection of all structuring elements used in a multi-structuring element erosion filter with index $N$. Therefore,

$$
\begin{equation*}
T_{N}=T_{1 N} U T_{2 N} \tag{3.1}
\end{equation*}
$$

The reasons why only three subgroups of structuring elements are used are as follows. Firstly, as the filter index $N$ increases, the length of the structuring element increases accordingly. If all the $(2 N+1)$ subgroups of structuring elements are used, the total number of structuring elements becomes $(2 N+1)\left(7.2^{2 \mathrm{~N}}-2^{\mathrm{N}+3}\right)$. Moreover, the length of minimal preservable detail equals to $(2 N+1)$. The minimal length is $3 N$ if only the three subgroups are used. Table 3.1 compares the length of minimal preservable line with $N$ ranging from 1 to 4 . The first column is the filter index $N$. The second and the third columns indicate the length of preservable details and the total number of structuring elements required respectively if all subgroups are used. Similarly, the fourth and the last column show those if the above mentioned three subgroups are used.

| $N$ | All Subgroups |  | Three |  |
| :---: | :---: | :---: | :---: | :---: |
|  | Subgroups |  |  |  |
|  | Min. Length | Number of SEs | Min. Length | Number of SEs |
| 1 | 3 | 36 | 3 | 36 |
| 2 | 5 | 400 | 6 | 240 |
| 3 | 7 | 2688 | 9 | 1152 |
| 4 | 9 | 14976 | 12 | 4492 |

Table 3.1 Comparisons of Minimal Preservable Details and Number of Structuring Elements Used when (i) All Subgroups and (ii) three Subgroups are Used (SE : Structuring Element)

Referring to Table 3.1, there is no difference when $N=1$ for both the number of structuring elements used and the length of minimal preservable details. When $N=2$, the length differs by 1 point, but the number of structuring elements nearly twice than the other when
all subgroups are used. The ratio of the numbers of structuring elements increases as $N$ increases, but the difference in length is so small that the increase in the number of structuring elements is not justified. Hence, the three subgroups are used.

Corollary The total number of structuring elements used in a multi-structuring element erosion filter with index $N$ is $3\left(7.2^{2 \mathrm{~N}}-2^{\mathrm{N}+3}\right)$.

Example 3.2 Fig.3.4 illustrates some $T_{1 \mathrm{~N}}$ structuring elements for different values of $N$. Similarly, some $T_{2 \mathrm{~N}}$ structuring elements are shown in Fig.3.5. For any structuring element in Fig.3.4, the origin of the structuring element is at its mid-point. All structuring elements are of odd number of pixels, simple, connected and non-oscillating. Moreover, they are bound by the rectangles formed by their end points.


O: Origin of Structuring Element - : Component of Structuring Element

Figure 3.4 Examples of $T_{I N}$ Structuring Elements for $N=1,2,3$.

$\mathrm{N}=1$

$\mathrm{N}=3$

0 : Origin of Structuring Element

- : Component of Structuring Element

Figure 3.5 Examples of $T_{2 N}$ Structuring Elements for $N=1,2,3$.

Table 3.2 summaries the number of structuring elements, $T_{1 \mathrm{~N}}, T_{2 \mathrm{~N}}$ and $T_{\mathrm{N}}$ from $N=1$ to 4 . For $N \geqq 3$, more than 1000 structuring elements are required, which cannot be realized easily. The $T_{1 \mathrm{~N}}$ structuring elements for $N=1$ and $N=2$ are shown in Appendix II.

| $N$ | $2 N+1$ | $\left\|T_{\text {IN }}\right\|$ | $\left\|T_{2 \mathrm{~N}}\right\|$ | $\left\|T_{N}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 3 | 12 | 24 | 36 |
| 2 | 5 | 80 | 160 | 240 |
| 3 | 7 | 384 | 768 | 1152 |
| 4 | 9 | 1664 | 3328 | 4492 |

Table 3.2 Number of Structuring Elements for $N=1$ to 4

### 3.3.2 Binary Multi-structuring Element Erosion Filter

Definition 3.2 Let $X$ be a discrete binary signal defined on $\boldsymbol{Z}^{2}$. The binary multi-structuring element erosion filter with index $N$ operates on $X$ is denoted by $B M E F_{\mathrm{N}}(X)$ and is defined as:

$$
\begin{equation*}
B M E F_{N}(X)=\bigcup_{i=1}^{\left|T_{N}\right|} X \Theta S E_{i}, \quad S E_{i} \in T_{N} \tag{3.2}
\end{equation*}
$$

where $T_{\mathrm{N}}$ is the collection of structuring elements and $\left|T_{\mathrm{N}}\right|$ is the cardinal number of $T_{\mathrm{N}}$.

In mathematical morphology, operators exist in dual pair. In Section 2.4.1, dilation is the dual to erosion provided that the structuring element is symmetrical about its origin. Although the $T_{1 \mathrm{~N}}$ structuring elements are all symmetrical, the $T_{2 \mathrm{~N}}$ structuring elements are not. However, the $T_{2 \mathrm{~N}}$ structuring elements exist in mirror pairs. If $S E$ is a $T_{2 \mathrm{~N}}$ structuring elements, then its mirror images w.r.t. the horizontal axis and the vertical axis are also contained in $T_{2 \mathrm{~N}}$. It is reminded that the dual to erosion is the dilation of the complement of $X$ by the mirrored structuring element. Hence, the dual filter to $B M E F_{\mathrm{N}}$ can be defined.

Definition 3.3 The dual filter to $B M E F_{\mathrm{N}}$, denoted by $B M E F_{\mathrm{N}}{ }^{\mathrm{d}}$, is defined as:

$$
\begin{equation*}
B M E F_{N}{ }^{d}(X)=\left\{\bigcup_{i=1}^{\left|T_{N}\right|} X^{c} \Theta S E_{i}\right\}^{c}, \quad S E_{i} \in T_{N} \tag{3.3}
\end{equation*}
$$

The 2-D binary multi-structuring element erosion filter with index $N$ can be interpreted as a filter of a moving window size $(4 N+1)^{2}$. The region of the signal masked by this window is eroded by the structuring elements in $T_{\mathrm{N}}$. The window size is $(4 N+1)^{2}$ in spite that all structuring elements are of $(2 N+1)$ points. This is due to the use of $T_{2 \mathrm{~N}}$ structuring elements.

If some structuring elements are found to be subsets of the signal in the window, the output is set to 1 .

The 1-D binary multi-structuring element erosion filter can be defined similarly. In the 1-D case, only three horizontal structuring elements are defined. The operation is the same as the 2-D one, instead of the reduction in the number of structuring elements. To show the operation of the binary filters, a 1-D filtering example using a filter with index $N=2$ is given.


Figure 3.6 Example of One-dimensional $B M E F_{2}$

Example 3.3 Three structuring elements are used, and are denoted as $S E_{\mathrm{i}}$, where $i=1,2,3$. Edge conditioning is handled by end point attachment scheme [Gabb92]. The signal on the edges of the signal are repeated $2 N$ times to represent the undefined pixel beyond the edges. It should be noted that end point attachment is not the only solution to cope with the undefined pixels at the edge. Schemes such as constant value end point attachment, adaptive end point attachment are used. Pixels of constant values, either 0 or 1 , are added to the both the ends of the input signal. In morphological filters, pixels of 0 are usually added. This is done by assuming that the undefined end points belong to the background set of the binary signal. Adaptive point attachment scheme may be based on the statistics of the signal. For instance, in the 1-D case, the number of 1 as well as that of 0 are counted. If the number of 1 is greater than that of 0 , all the attachment points will be 1 or vice versa. We adopt the duplication of end points as the attachment scheme since we want to maintain the correlation of pixels at these regions. Figure 3.6 shows a detailed operation of $B M E F_{2}$. Data sequence
is first eroded by the structuring elements. The final output is then the union of the eroded results.

### 3.3.3 Selective Threshold Decomposition

From now on, the threshold decomposition will be termed as the classical threshold decomposition. Lemma 3.1 and the definition of set processing filter are stated [Mara87] by which a modification from the classical threshold decomposition is introduced.

## Set Processing Filter

A filter $\psi$ is set processing if and only if the output of the filter is chosen among the input. Examples of set processing filters are the median filter and the stack filter. In a 1-D median filter of window size $2 N+1$, the output at a point $x(n)$ is chosen from the data which is masked by the window.

Lemma 3.1 Let $G$ be a $k$-level signal defined on $D$, and $\psi$ and $\Psi$ be a multilevel set processing and a binary filter with a moving window $W$. If $\psi$ and $\Psi$ are related as follows:

$$
\begin{equation*}
\psi(G(x))=\max \left\{j: \Psi\left(S_{j}(G(x))\right)=1\right\}, \quad \forall \boldsymbol{x} \in \boldsymbol{D} \tag{3.4}
\end{equation*}
$$

where $S_{\mathrm{j}}(G), j=0,1,2, \ldots, k-1$ are the threshold sets of $G$ at level $j$, then

$$
\begin{equation*}
\psi(G(\boldsymbol{x}))=\max \left\{j \in W_{\boldsymbol{x}}: \Psi\left(S_{j}(G(\boldsymbol{x}))\right)=1\right\}, \quad \forall \boldsymbol{x} \in \boldsymbol{D} \tag{3.5}
\end{equation*}
$$

where $W_{\mathrm{x}}$ denotes the translate of $W$ by $\boldsymbol{x}$.

## Proof:

Since $\psi$ is set processing, implying that the output of the filter is chosen among the data masked by $W_{\mathrm{x}}$ which is centred at $\boldsymbol{x}$. Therefore, those input data which are not included by $W_{\mathrm{x}}$ has no contribution to the output. This is equivalent to perform filtering at the gray levels in the window. Hence, Lemma 3.1 is proved.

Lemma 3.1 has significant contributions in both the hardware and software implementation of set processing (stack) filters. In the classical threshold decomposition, all binary signals of a $k$-level signal are used. This is not feasible in the implementation of the
filter. Lemma 3.1 implies that the output by the full set of binary signals is equivalent to that of the simplified operation using the binary signals thresholded by the data in the window only.

The speed of filter operation as well as the storage requirements are the primary concerns in the hardware implementation of a set processing filter. Usually a high degree of parallel binary filtering is demanded. Therefore, an implementation of ( $k-1$ ) binary filters and a storage of all thresholded signals are required. If $k$ equals to 256 , then 255 binary signals are resulted. Let $M$ and $N$ be the width and height of the image respectively. The memory required to store these 255 binary signals become $255 M N$ bits, which is approximately equal to $32 M N$ bytes. 255 , or ( $k-1$ ) binary filters will be implemented. For a square 512 by 512 pixels image, about 8 M bytes memory is required. A window of size of $(2 N+1)$ by $(2 N+1)$ contains $(2 N+1)^{2}$ threshold levels. Each thresholded set again required $(2 N+1)^{2}$ bits for storage. As there are $M N$ pixels on an images, so the total memory required in bits is $(2 N+1)^{4} M N$. The number of binary filters to be implemented is $(2 N+1)^{2}$. Under certain circumstances, a reduction in memory is achieved provided that the number of gray levels is more than that of the number of pixels masked by the moving window. In addition, stacking requires comparisons of output bits on the threshold binary images, the number of comparisons is reduced from 255 (or $k-1$ ) to $(2 N+1)^{2}$. On the whole, a simpler hardware architecture can be resulted in which less number of binary filters are implemented.

In software implementation, suppose that the filter is implemented on a sequential machine. The effect of the usage of Lemma 3.1 is more significant in the reduction of computation time. The storage requirement and the number of comparisons are the primary considerations. If Lemma 3.1 is not applied, the output at a pixel is chosen by searching the stack of $(k-1)$ binary output at that pixel. Only $(2 N+1)^{2}$ binary outputs will be searched in the contrary.

In Chapter 2, the definition of the classical threshold decomposition is given in (2.2). The selectively threshold decomposition is derived from Lemma 3.1. which is designed to reduce the computational requirement, especially in the software implementation, of set processing filters. By selectively thresholding a multilevel signal, repeated binary filtering on binary sets which are thresholded at the same level is avoided. Furthermore, by varying the
forcing level, filters of different noise rejection property and detail preserving property are resulted. The operation of selective threshold decomposition is as follows. The threshold levels which slice a multilevel signal into binary signals is taken from a data sequence $Q$, where $Q$ contains the sorted data masked by the moving window $W$ of the filter. Let $l_{(\mathrm{j})}, j \geq 1$ be the $j^{\text {th }}$ rank order data in $Q$. If $l_{(\mathrm{j})}=l_{(\mathrm{j}-1)}$, thresholding of the multilevel signal $G$ at threshold level $l_{(\mathrm{j})}$ is skipped. A forcing threshold level, $f$, is introduced to provide a stopping threshold value for the comparison of sorted data. Whenever $f$ equals $l$, thresholding is carried out disregarding whether $l_{(\mathrm{f})}$ is equal to $l_{(\mathrm{ff}-1)}$ or not.

Definition 3.4 Let $G(\boldsymbol{x}), \boldsymbol{x} \in \boldsymbol{D}$ be a $k$-level signal taking values in $\left[0, k\right.$-1], and let $S_{f j}(G(\boldsymbol{x}))$ be the selective threshold set with a forcing level $f$ and a ranked threshold value sequence $Q$, the selective threshold set is defined as :

$$
S_{f, j}(G(\boldsymbol{x}))= \begin{cases}\varnothing, & \text { if } l_{(j-1)}=l_{(j)}  \tag{3.6}\\ S_{l_{(j)}}(G(\boldsymbol{x})), & \text { if } l_{(j)} \neq l_{(j-1)} \\ S_{l_{(j)}}(G(\boldsymbol{x})), & \text { if } j=f\end{cases}
$$

where $l_{\mathrm{j})}$ is the $j^{\text {th }}$ largest data in $Q$.

If a forcing level is set to 0 , then the selective threshold decomposition is said to be without a forcing level. In this case, the selective threshold decomposition is equivalent to the classical one. All threshold sets will be sliced. By the principle of selective threshold decomposition using a ranked threshold levels sequence Q of $k_{Q}$ levels, and a forcing level $f$. A multilevel level signal $G$ of $k$-levels can be decomposed into maximally $k_{Q}$ or less than ( $k_{Q}-f$ ) binary signals. The addition of forcing level $f$, also increase the noise rejection power. In fact, if a forcing level $f$ is used, at the forcing level, at least $f$ pixels which are equal to 1 must be found. The effect of forcing level on multilevel signal filter will be discussed in later chapters. The pixels in the window are sorted. If binary filtering is performed from the largest threshold level, stacking operation by taking the maximum level at which the output of binary filter is 1 can be eliminated. In other words, binary filtering at all threshold levels are not necessary. Hence, selective threshold decomposition has already merged the stacking operation while thresholding a signal. Example 3.4 illustrates the effect of level skipping with
and without a forcing level.

| 9 | 2 | 2 | 2 | 9 |
| :--- | :--- | :--- | :--- | :--- |
| 2 | 9 | 2 | 9 | 2 |
| 2 | 2 | 9 | 2 | 2 |
| 2 | 9 | 2 | 9 | 2 |
| 9 | 2 | 2 | 2 | 9 |

(a)

| 9 | 2 | 2 | 2 | 9 |
| :--- | :--- | :--- | :--- | :--- |
| 2 | 9 | 2 | 9 | 2 |
| 2 | 2 | 2 | 2 | 2 |
| 2 | 9 | 2 | 9 | 2 |
| 9 | 2 | 2 | 2 | 9 |

(c)

| Sorted Data |
| :--- |
| 9 <br> 9 <br> 9 <br> first <br> second <br> 9 <br> 9 <br> Third <br> 9 <br> 2 <br> 2 |


| 9 | 2 | 2 | 2 | 9 |
| :--- | :--- | :--- | :--- | :--- |
| 2 | 9 | 2 | 9 | 2 |
| 2 | 2 | 9 | 2 | 2 |
| 2 | 9 | 2 | 9 | 2 |
| 9 | 2 | 2 | 2 | 9 |

(d)

Figure 3.7 Filtering Using Selective Threshold Decomposition (a)Original (b)Sorted Array (c)O/pf=0 (d)O/p $f=2$

Example 3.4 A 2-D multilevel signal is given in Fig.3.7(a). The signal is thresholded selectively using the image data masked by a 3 by 3 window. The binary signals are filtered by $B M E F_{2}$. Fig.3.7(c) shows the multilevel filtered output with a forcing level equal to 0 . The pixel at the centre of the window become 2 after filtering. Binary filtering is performed only when the current threshold level is different from the previous one. If a forcing level is included, which is set to 2 in this example, the output at the centre is preserved as shown in Fig.3.7(d).

Property 3.1 The selectively thresholded sets are linearly ordered. If $a<b$ and $l_{(\mathrm{a})} \neq l_{(\mathrm{b})}$, then $S_{\mathrm{f}, \mathrm{a}}(G(\boldsymbol{x})) \supseteq S_{\mathrm{f}, \mathrm{b}}(G(\boldsymbol{x}))$.

The property is similar to Prop.2.1. A partial ordering between the thresholded sets at different threshold levels exists.

### 3.3.4 Multilevel Multi-Structuring Element Erosion Filter

A straightforward extension of a binary filter to a multilevel filter is by the principle of classical threshold decomposition. An $k$-level signal $G$ is sliced into ( $k-1$ ) binary signals by the classical threshold decomposition [Fitch84]. Binary filtering is performed for each thresholded signals. The multilevel output is then produced by the stacking operation, which is the reverse to the classical threshold decomposition. The merit of the classical threshold decomposition is that multilevel signal can be decomposed into binary signals. Usually, binary operations are simpler that the multilevel ones. Also, properties of a multilevel filter can be predicted from those of its binary filter. However, the binary filter must be increasing; otherwise, the stacking principle will not hold [Fitch84]. As erosion is an increasing operation [Serra82], we can simply propose a multilevel filter $y_{\mathrm{N}}(G(\boldsymbol{x}))$ which is defined as :

$$
\begin{equation*}
y_{N}(G(\boldsymbol{x}))=\max \left\{j: B M E F_{N}\left(S_{j}(G(\boldsymbol{x}))\right)=1\right\}, \quad \forall \boldsymbol{x} \in \boldsymbol{D} \tag{3.7}
\end{equation*}
$$

where

$$
S_{j}(G(\boldsymbol{x}))=\left\{\begin{array}{ll}
1 & \text { if } G(\boldsymbol{x}) \geq j \\
0 & \text { if } G(\boldsymbol{x})<j
\end{array} \quad \forall j \in[0, k-1]\right.
$$

and $B M E F_{\mathrm{N}}$ is the binary multi-structuring element erosion filter with index $N$. In the following, we will show that the multilevel erosion filter defined in (3.7) is not suitable for suppressing noise.

By Lemma 3.1, (3.7) can be reduced to:

$$
\begin{equation*}
y_{N}(G(\boldsymbol{x}))=\max \left\{l_{(j)}: B M E F_{N}\left(S_{N+1, j}(G(\boldsymbol{x}))\right)=1\right\}, \quad \forall \boldsymbol{x} \in \boldsymbol{D} \tag{3.8}
\end{equation*}
$$

where $l_{(\mathrm{j})}$ is the $j^{\text {th }}$ largest element within the $(4 N+1)^{2}$ window centred at $\boldsymbol{x}$.

However, the multilevel filter defined by (3.8) performs well only under low noise environment. Also, the filter is computational intensive. According to the interpretation of $B M E F_{N}$, a moving window of size $(4 N+1)^{2}$ slides on a signal. However, the following analysis show that the performance in computation speed as well as noise suppression are better if threshold levels are taken from the $(2 N+1)^{2}$ window than the $(4 N+1)^{2}$ one. Figure 3.8 shows the arrangement of the windows. The inner window of $(2 N+1)^{2}$ pixels is denoted as $W_{\mathrm{i}}$, the

outer one with $(4 N+1)^{2}$ pixels is $W_{0}$. The shade region is the complement of the inner window in the outer one, denoted as $W_{0} \backslash W_{\mathrm{i}}$. A structuring element is said to match the signal if the erosion of the signal by the structuring element gives a non-empty result. The pixel in $W_{\mathrm{o}} W_{\mathrm{i}}$ do not contribute to a $T_{\text {IN }}$ matching. Similarly, if the signal is matched by a $T_{2 N}$ structuring elements, at least ( $N+1$ ) 1 's are found in $W_{\mathrm{i}}$. This reveals that most of the required information is in $W_{i}$.

Figure 3.8 Inner and Outer Windows

## Computational Efficiency

It is obvious that the number of comparisons depends on the number of data to be sorted. Moreover, as the number of data increases, the number of binary filter increases as well. The computation time for sorting $(2 N+1)^{2}$ data is approximately one fourth of $(4 N+1)^{2}$.

## Noise Rejection

As different forcing levels vary the minimum number of pixels equal to 1 on a binary image. Fig. 3.9 show the outputs of a multilevel signal which is corrupted by impulse of magnitude 99 by multilevel filter with $(4 N+1)^{2}$ and $(2 N+1)^{2}$ windows. All filters are with index $N=1$. Fig.3.9(a) is the original signal, which is a constant signal of magnitude 9 corrupted by impulses of magnitude 99. Fig.3.9(b) is the selective threshold set at level 99 with a forcing level of 3 . Hence, at least 3 1's are found in Fig.3.9(b). At pixel $p_{1}$, the impulse is preserved by the filter using $(4 N+1)^{2}$ data. However, the impulse at $p_{I}$ is removed by the second filter whose threshold levels are taken in the $(2 N+1)^{2}$ window. Let $p$ be the probability of occurrence of an impulse. The expectation of number of impulses in a $(4 N+1)^{2}$ window is $(4 N+1)^{2} p$, while that of $(2 N+1)^{2}$ window is $(2 N+1)^{2} p$. The ratio of expectation of impulses occurring in these window are approximately equal to four. Hence, with the same forcing level, the effect of impulses on a filter with a $(4 N+1)^{2}$ window is greater. Therefore, the noise suppression is worse.

| 9 | 9 | 9 | 9 | 9 |
| :---: | :---: | :---: | :---: | :---: |
| 9 | 9 | 9 | 9 | 9 |
| 99 | 99 | 99 | 9 | 9 |
| 9 | 9 | 9 | 99 | 9 |
| 9 | 9 | 9 | 9 | 9 |$\quad p_{1}$

(a)Original

| 9 | 9 | 9 | 9 | 9 |
| :---: | :---: | :---: | :---: | :---: |
| 9 | 9 | 9 | 9 | 9 |
| 99 | 99 | 99 | 9 | 9 |
| 9 | 9 | 9 | 99 | 9 |
| 9 | 9 | 9 | 9 | 9 |

(c) $\mathrm{O} / \mathrm{p}$ : $(4 \mathrm{~N}+1)$ sq. window

| 0 | 0 | 0 | 0 | 0 |
| :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 |

(b)Threshold at level 99

| 9 | 9 | 9 | 9 | 9 |
| :---: | :---: | :---: | :---: | :---: |
| 9 | 9 | 9 | 9 | 9 |
| 99 | 99 | 99 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 |

(d) $\mathrm{O} / \mathrm{p}$ : $(2 \mathrm{~N}+1)$ sq. window
Impulse mag. : 99 Signal : 9

Figure 3.9 Comparison of Multilevel Multi-Structuring Element Erosion Filter Using Threshold Levels in a $(4 N+1)^{2}$ and $a(2 N+1)^{2}$ window (a)Original (b)Thresholded Set at level 99 (c)Output of (3.8) using a $(4 N+1)^{2}$ window (d)Output using the Threshold Values in a $(2 N+1)^{2}$ window only

According to the previous discussion, a structuring element will match a binary signal if there are at least $(N+1) 1^{\prime}$ in $W_{\mathrm{i}}$. The minimum forcing level $f_{\text {MIN }}$ is therefore set to $N+1$. If $f$ is less than $N+1$, the output of the binary filter must be zero. From the objective of the filter, the maximum forcing level $f_{\text {MAX }}$ will be set to $2 N+1$, so that fine details can be preserved.

Definition 3.5 Let $G(x), \boldsymbol{x} \in \boldsymbol{D}$ be an $k$-level signal. The multilevel multi-structuring element erosion filter operates on $G$ with index $N$ and forcing level $f$ is defined as:

$$
\begin{equation*}
G M E F_{f, N}(G(x))=\max _{N+1 \leq j \leq\{2 N+1)^{2}}\left\{l_{V)} \in\left(W_{i)}\right)_{x}: B M E F_{N}\left(S_{f j}(G(x))\right)=1\right\} \tag{3.9}
\end{equation*}
$$

where $\left(W_{\mathrm{i}}\right)_{\mathrm{x}}$ is a $(2 N+1)$ by $(2 N+1)$ window centred at $\boldsymbol{x}$, and $l_{(\mathrm{j})}$ is the $j^{\text {th }}$ ranked data in the $\left(W_{\mathrm{i}}\right)_{\mathrm{x}}$ window and $f$ is the forcing level $f \in\{N+1, \ldots, 2 N+1\}$.

It should be noted that sorting is performed in descending order, i.e. the largest element in $\left(W_{\mathrm{i}}\right)_{x}$ is denoted by $l_{(1)}$. This is different from the notation used by [David81] in
which $l_{(1)}$ denoted the smallest number. Similarly to the dual of binary filter, the dual of multilevel filter operates on the complement multilevel signal.

Definition 3.6 Let $k$ be the number of levels in a multilevel signal $G$, the complement of $G$, denoted as $G^{c}$ is defined as :

$$
G^{c}(x)=(k-1)-G(x), \quad \forall x \in Z^{n}
$$

The complement of a multilevel signal is so set in order to comply with the definition of binary complementation. If the number of gray levels k is set to 2 , then the signal is essentially a binary signal and Def.3.6 becomes the binary complementation. By Def.3.6, the dual filter to a multilevel erosion filter can be defined.

Definition 3.7 Let $G M E F_{\mathrm{f}, \mathrm{N}}{ }^{\text {d }}$ be the dual to $G M E F_{\mathrm{f}, \mathrm{N}} . G M E F_{\mathrm{f}, \mathrm{N}}{ }^{d}$ is defined as :

$$
\begin{equation*}
G M E F_{f, N}{ }^{d}(G)=\left\{G M E F_{f, N}\left(G^{c}\right)\right\}^{c} \tag{3.11}
\end{equation*}
$$

A flow chart of the operation of multilevel multi-structuring element erosion filter is shown in Fig.3.10. The data in the inner window of size $(2 N+1)$ by $(2 N+1)$ is first sorted. Thresholding of the output window of size $(4 N+1)$ by $(4 N+1)$ begins at the $(N+1)^{\text {th }}$ ranked threshold level since at least $(N+1)$ l's are needed in the inner window for a structuring element matching. If the current level, say $l_{(\mathrm{j})}$ equals to the previous level $l_{(\mathrm{j}-1 \mathrm{l}}$, then thresholding as well as binary filtering at this level is skipped provided that $j$ is not equal to $f$, the forcing level. The output is found by the stacking operation.

T.D. : Threshold Desomposition

Figure 3.10 Flowchart of $G M E F_{f, N}$

Fig.3.11 shows the operation of $G M E F_{f, 1}$ on a 2-D multilevel signal, where $f=\{2,3\}$. Fig.3.11(a) is the original signal, which is a plateau. An impulse of magnitude 87 is added to the signal. Originally, the values at the impulse is 3. Fig.3.11(b) and 3.11(c) show the output of the filter $G M E F_{2,1}$ and $G M E F_{3,1}$. In both cases,the impulse is removed, but the values at the corrupted pixel are different.

| 3 | 3 | 3 | 3 | 3 | 3 | 2 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 4 | 5 | 4 | 4 | 3 | 2 |
| 3 | 90 | 5 | 5 | 4 | 3 | 2 |
| 3 | 4 | 5 | 5 | 4 | 3 | 2 |
| 3 | 4 | 4 | 90 | 90 | 3 | 2 |
| 3 | 3 | 3 | 3 | 3 | 90 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 2 |

(a)

| 3 | 3 | 3 | 3 | 3 | 3 | 2 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 4 | 5 | 4 | 4 | 3 | 2 |
| 3 | 5 | 5 | 5 | 4 | 3 | 2 |
| 3 | 4 | 5 | 5 | 4 | 3 | 2 |
| 3 | 4 | 4 | 90 | 90 | 3 | 2 |
| 3 | 3 | 3 | 3 | 3 | 90 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 2 |

(b)

| 3 | 3 | 3 | 3 | 3 | 3 | 2 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 4 | 5 | 4 | 4 | 3 | 2 |
| 3 | 4 | 5 | 5 | 4 | 3 | 2 |
| 3 | 4 | 5 | 5 | 4 | 3 | 2 |
| 3 | 4 | 4 | 4 | 4 | 3 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 2 |

(c)

Figure 3.11 Examples of Multilevel Erosion Filtering using GMEF ${ }_{2,1}$ and $G M E F_{3, I}$ (a)Original Signal (b)Filtered by GMEF $_{2, I}$ (c)Filtered by GMEF $_{3, I}$

### 3.3.5 A Combination of Multilevel Multi-Structuring Element Erosion Filter and its dual

It can been seen that the binary filter is highly biased, as erosion is biased to preserve negative features. As a result, the multilevel filter is also biased. To alleviate this bias, sequential applications of $G M E F_{\mathrm{f}, \mathrm{N}}$ and its dual is proposed. Properties of the filtered signal will be dependent on the order of application. Denote the resultant filter that $G M E F_{\mathrm{f}, \mathrm{N}}$ is first used followed by its dual as $C G_{f, \mathrm{~N}}$.

Definition 3.8 Let $G(\boldsymbol{x}), \boldsymbol{x} \in \boldsymbol{Z}^{2}$ be a multilevel signal. The filter $C G_{\mathrm{f}, \mathrm{N}}$ with an index $N$ and a forcing level $f$ is defined as:

$$
\begin{equation*}
C G_{f, N}(G(\boldsymbol{x}))=G M E F_{f, N}^{d}\left(G M E F_{f, N}(G(\boldsymbol{x}))\right. \tag{3.12}
\end{equation*}
$$

Numerous combinations of the multilevel erosion filter and its dual can be made. For example, one can apply the multilevel erosion filter until the root signal is found, and then continue in this way for the dual filter. Moreover, the application sequence of the erosion filter and its dual can be changed. An image can be filtered by the dual filter first followed by the erosion filter. But only the properties of $C G_{\mathrm{f}, \mathrm{N}}$ will be discussed. Properties, both deterministic and statistical, of $C G_{\mathrm{f}, 1}$ will be discussed in the next Chapter.

### 3.4 Chapter Summary

A new family of binary nonlinear filters based on a multi-structuring element approach is introduced. The filter is the union $3\left(7.2^{2 \mathrm{~N}}-2^{\mathrm{N}+3}\right)$ morphological erosion filters. The essence of this filter is to try to find a match between those desirable details such as edges, sharp corners, constant regions in an input signal with the set of structuring elements while removing impulse noise. All structuring elements used must satisfy the T. 1 to T. 4 constraints. These constraints govern the shape and size of the structuring element used in a multistructuring element erosion filter. The filter is characterized by an index $N$. If a filter of index $N$ is used, then all the structuring elements must be $2 N+1$ long. Constraints also restrict that the structuring element must be connected and non-oscillating. Lastly, the structuring element must not tend to close itself.

The definition of binary filter is given as the union of erosions of a binary signal by all structuring elements. Multilevel filters are derived from the binary filter using a modification of the classical threshold decomposition, known as the selective threshold decomposition. Threshold values are chosen from the inner window $W_{\mathrm{i}}$, which is a square window of size $(2 N+1)$ by $(2 N+1)$. The window is selected owing to the computation complexity and noise suppression power considerations. The objective of introducing selective threshold decomposition is to reduce the computation complexity of the filter by reducing the number of binary filtering as well as repeated filtering is avoided. Two basic gray scale filters which are defined on selective threshold decomposition and binary multi-structuring element erosion filters are proposed. In section 3.3.5, a filter which is a sequential application of the multilevel erosion filter and its dual is designed.

# Chapter 4 Properties of Multi-Structuring Element Erosion Filter 

### 4.1 Introduction

Analysis of the multi-structuring element erosion filters, both binary and multilevel, will be presented. As the multilevel element erosion filter is a nonlinear filter, the methodology of superposition is not applicable. Deterministic analysis and statistical analysis have been served to portray the properties of nonlinear filters by pioneers in this area. The former is concerning, firstly the geometrical structures of signals which are invariant to the filter, and secondly the rate of convergence of the filter. The rate of convergence shows how fast, in number of filter passes, an input signal is transformed to the corresponding invariant signal. It is reminded that the one of the purposes of image filtering is to eliminate or suppress noise. The efficiency and effectiveness of noise suppression is described by the statistical properties.

Section 4.2 will discuss the deterministic properties of the multi-structuring element erosion filter. The analysis commences with the binary erosion filters. Root signal structures of the 1-D binary erosion filter, which is a particular case of the 2-D filter, will be derived first and then those of the 2-D binary erosion filter. The root structure analysis of the multilevel erosion filters follow. The rate of convergence of the 1-D binary erosion filter will be considered, and is followed by the 2-D filters and the multilevel erosion filters.

Section 4.3 deals with the statistical analysis. The probability measure function of the 1-D and 2-D multilevel erosion filters with index $N=1$ will be derived using the statistical threshold decomposition [Arce86]. Simulations using Gaussian distribution and uniform distribution will be performed. Breakdown probabilities will be calculated under different probabilities of occurrence of impulse. We will compare the noise suppression performance with the unidirectional and bidirectional multistage median filters.

### 4.2 Deterministic Analysis

By the principle of superposition, a linear system can be analyzed by sinusoidal signals at different frequencies [Oppen83]. The passband, which is defined as the band of frequencies passed by the system, can be identified. Signals whose frequency components are within the passband are unchanged by the system. However, such analysis is not applicable to nonlinear filters because superposition does not hold. Harmonic distortion as well as phase and amplitude changes are resulted. Fig.4.1 illustrates the clipping effect of a multi-structuring element erosion filter on a sinusoidal input. A sinusoidal wave is input to the multi-structuring element filter(MEF) in Fig.4.1. The filter should be a $C G_{f, \mathrm{~N}}$. Both the peaks, maximum and minimum of the input signal are cut which brings about a clipped output.


Figure 4.1 Clipping Effect of Multi-structuring Element Erosion Filter (MEF). Sinusoidal input is fed to the erosion filter, the output is clipped

To a certain extent, the deterministic analysis is analogous to the frequency band analysis in linear systems. Deterministic properties disclose the invariant signal and the rate of convergence of a filter. Invariant signals, or known as roots, are signals which are unaltered by the filter. Let $\psi$ represents a nonlinear filter. If an input signal $G(\boldsymbol{x}), \boldsymbol{x} \in \boldsymbol{D}$, is invariant to $\psi$, then

$$
\begin{equation*}
\psi(G(x))=G(x), \quad \forall x \in D . \tag{4.1}
\end{equation*}
$$

Root signal analysis is carried out in the spatial domain. The geometrical features, such as sizes and shapes, of root signals determine a filter's applications. Finally, the number of filter passes required to convert any nonroot signal to its root, determines whether a filter can be used in real time applications. The number of filter passes is referred to the rate of
convergence. In some literature [Gallag81] [Arce82] [Arce86], the number and the construction of root signals are determined. However, the theoretical development is limited to some 1-D median based filters. Applications are found in block truncation coding (BTC) of speech [Arce83]. We will not attempt to count the number of 2-D root signals as the search space is too large.

### 4.2.1 Shape of Invariant Signal

We shall begin by defining some terminologies in deterministic analysis [Nodes83] [Fitch85] for 1-D root signal analysis. A constant neighbourhood is a region of consecutive points with identical values which are invariant to a filter. The shape and size of a constant neighbourhood are filter dependent. For a 1-D median filter, the number of consecutive points is at least $N+1$ for a filter window of $2 N+1$ points. An edge is a monotonically increasing or decreasing region which is surrounded by constant neighbourhoods. Usually, the root or invariant signals are combinations of constant neighbourhoods and edges. Nonroot structures can be described by impulses and vibration points. An impulse is a set of points whose values are different from the surrounding regions and whose surrounding regions are identically valued constant neighbourhoods. For a 1-D median filter with a moving window of $2 N+1$, an impulse is a set of $N$ or less points. Vibration points may exist for those nonlinear filters which are composed of several steps, for example, the separable median filter. A separable median filter is a two- pass filtering operation. A 2-D signal is first filtered by a horizontally oriented 1-D median filter and then, the resulting signal is filtered by a vertically oriented median filter. A vibrating point is toggled by the first pass and toggled back to its original state upon the second pass.

In 2-D root signal analysis, as geometrical structures of the signal are used, it is natural to use terms such as open curve, closed curve, etc. to describe the root structure. Descriptions of geometrical structures of a 2-D signal are given.

Two points $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$ are said to be connected if $\boldsymbol{x}_{2}$ is in the 8 -neighbourhood of $\boldsymbol{x}_{1}$. A line is said to be a single point wide line if for all points $\boldsymbol{x}$ on a single point wide line, at least one and at most three 1 's which are not fully connected are found in the 8neighbourhood of $\boldsymbol{x}$. A single point wide line, or a single point line is the simplest 2-D signal
structure. Fig.4.2 indicates some examples and counter examples of single point wide lines. In fact, the 1-D invariant signal structure is a single wide line which is oriented horizontally. Figs.4.2(a) and (b) are examples of single wide point lines. A single point wide line may contain right angle as shown in Fig.4.2(a). The patterns in Figs.4.2(c) and (d) are not single point wide since there are points on the line that more than three 1 's are located in their 8 neighbourhood.


Figure 4.2 (a)(b)Examples of single point wide lines (c)(d) Counter examples of single point wide lines

Figure 4.3 Examples of lines with $m$ segments (a) $m=1$ (b) $m=1$ (c) $m=3$ (d) $m=7$

A segment is a line which is oriented either in the horizontal or the vertical directions only. A line $l$ is said to be consisted of $m$ segments if the line can be divided into a minimal number of $m$ segments. Figs.4.3(a) and (b) are segments which are composed of 7 points. The one in Fig.4.3(a) is a horizontal segment, while the other in Fig.4.3(b) runs vertically. Fig.4.3.(c) is a curve of three segments. Two of them are composed of three points. The centre one is a single point segment. Fig.4.3(d) shows a line running diagonally. Each point on the line is a segment.

An open curve is a linkage of points in a 2-D space whose end points are not coincident. On the contrary, a closed curve is a connected linkage of points those end points are coincident. A structuring element is an open curve of $2 N+1$ points. The boundary of an object is an example of closed curve. A rectangular region is defined as a constant region in which points are arranged as a rectangle. A rectangular region can be characterized by its height and its width. An irregular region is a region which is not rectangular in shape. A curve of any arbitrary shape is an example of an irregular region.

### 4.2.1.1

 Binary Multi-Structuring Element Erosion FilterThe binary multi-structuring element erosion filter will be abbreviated as the binary erosion filter. The term binary erosion filter and $B M E F_{\mathrm{N}}$ will be used interchangeably with binary multi-structuring element erosion filter. In 1-D root signal analysis, let $[g]_{L}$ be a region of $L$ consecutive points of value $g$. A constant region is said to be (relatively) positive if the value of the constant region is greater than its surroundings. A constant region is (relatively) negative if it is of lower value than its surroundings.

## One-dimensional Binary Multi-structuring Element Erosion Filter

The 1-D binary multi-structuring element erosion filter, $B M E F_{\mathrm{N}}$, is a particular case of the 2-D one. Three collinear horizontal structuring elements are chosen among the 2-D structuring elements. A point $x(n)$ of value 1 on a 1-D binary sequence survives after one pass of $B M E F_{\mathrm{N}}$ if one of the following cases occur. A region $[1]_{2 \mathrm{~N}}$ lies either on the left or on the right of $x(n)$. Or, regions of $[1]_{\mathrm{N}}$ are found on both sides of $x(n)$. If $[1]_{2 \mathrm{~N}+1}$ is filtered, the end points and the middle point are unchanged. Thus, $[1]_{2 \mathrm{~N}+1}$ becomes $[1][0]_{\mathrm{N}-1}[1][0]_{\mathrm{N}-1}[1]$ after the filter is applied once.

Property 4.1 (1-D Binary Positive Region) Let [1] $]_{\mathrm{L}}$ be a constant region of $L$ consecutive 1 's in a 1-D binary sequence. The constant region $[1]_{\mathrm{L}}$ is a constant neighbourhood to a binary multi-structuring element erosion filter with index $N$ if and only if $L \geq 3 N$.

## Proof:

It is obvious that $L$ must be greater than $2 N+1$. Suppose $L=(2 N+1)+e$, where $e$ is the minimal number of consecutive 1 's $[1]_{e}$ appended to $[1]_{2 N+1}$. For $[1]_{2 N+1}$, only the end points and the middle point are preserved. The second to the $N^{\text {th }}$ and the $(N+2)^{\text {th }}$ to the $2 N^{\text {th }}$ are set to regions of $[0]_{\mathrm{N}-1}$. If $[1]_{1}$ is appended to the right of $[1]_{2 \mathrm{~N}+1}$ and the positive region becomes $[1]_{2 \mathrm{~N}+2}$, the second and the $2 N^{\text {th }}$ points will be preserved. The minimal number of $[1]_{1}$ appended, $e$, is $(2 N+1-3) / 2=N-1$. The minimal length is $3 N$.

Property 4.2 (1-D Binary Negative Region) Let [0] $]_{\mathrm{L}}$ be a constant region of $L$ consecutive 0's in a 1-D binary sequence. The region $[0]_{\mathrm{L}}$ is a constant neighbourhood to a binary multistructuring element erosion filter of index $N$ if and only if $L \geq 1$.

This follows from the definition of $B M E F_{\mathrm{N}}$, as erosion of $[0]_{\mathrm{L}}$ is also $[0]_{\mathrm{L}}$ for $L \geq 1$. Apparently, the binary erosion filter is biased towards regions of lower intensity. It preserves all negative details but converts regions of $[1]_{\mathrm{L}}$ into $[0]_{\mathrm{L}}$ if $L<3 N$. The bias of the filter will be verified when its statistical property is elaborated in Section 4.3. The dual filter of $B M E F_{\mathrm{N}}$, which is denoted as $B M E F_{\mathrm{N}}{ }^{\text {d }}$ in Chapter 3, on the other hand, preserves regions of higher intensities. The 1-D root signal to $B M E F_{\mathrm{N}}{ }^{\text {d }}$ is the complement to $B M E F_{\mathrm{N}}$. Constant regions of $[1]_{L}$ remain unchanged for any positive integer $L$, and regions of $[0]_{L}$ are preserved if $L \geq 3 N$.

## Two-Dimensional Binary Multi-structuring Element Erosion Filter

The root signal structures of 2-D multi-structuring element erosion filters are more complicated. It can be deducted from Property 4.1 that the length of connected 1's in a 2-D signal must be at least $3 N$ points. However, the shape of a signal also contributes to the invariance of the signal. Let $\left(r_{\mathrm{x}, \mathrm{i}}, r_{\mathrm{y}, \mathrm{i}}\right)$ be the relative coordinates of the $(i-1)^{\text {th }}$ point to the $i^{\text {th }}$ points, and $m_{s}$ be the number of segments of a structuring element. If the $i^{\text {th }}$ point lies on the same segment as $(i-1)^{\text {th }}$ point, then either (4.2) or (4.3) is true.

$$
\begin{align*}
& \left\{\begin{array}{l}
\left|r_{x, i}\right|=1 \\
\left|r_{y, i}\right|=0
\end{array}\right.  \tag{4.2}\\
& \left\{\begin{array}{l}
\left|r_{x, i}\right|=0 \\
\left|r_{y, i}\right|=1
\end{array}\right. \tag{4.3}
\end{align*}
$$

If the $(i-1)^{\text {th }}$ and the $i^{\text {th }}$ points are on different segments, then

$$
\left\{\begin{array}{l}
\left|r_{x, i}\right|=1  \tag{4.5}\\
\left|r_{y, i}\right|=1
\end{array}\right.
$$

which implies

$$
\begin{equation*}
\left|r_{x, i}\right|+\left|r_{y, i}\right|=2 . \tag{4.6}
\end{equation*}
$$



Figure 4.4 Relative coordinates between the $(i-1)^{\text {th }}$ and the $i^{\text {th }}$ points on a curve (a)both points are on the same segment (b)the points are on different segments

Fig.4.4 illustrates the determination of relative coordinates. A line of two segments are shown in Figs.4.4(a) and (b). The $(i-1)^{\text {th }}$ and the $i^{\text {th }}$ points in Fig.4.4(a) are on the same segment. The sum of the relative coordinates between the $(i-1)^{\text {th }}$ and the $i^{\text {th }}$ points is 1 . On the other hand, the $(i-1)^{\text {th }}$ and the $i^{\text {th }}$ points in Fig.4.4(b) are on different segments. The sum of relative coordinates is 2 .

Lemma 4.1 (T.4) If a single point wide line $l_{\mathrm{L}}$ of $L$ points satisfies T. 4 of Def.3.1, then the sequences

$$
\begin{equation*}
\sum_{i=2}^{l_{i}} r_{x, i} \text { and } \sum_{i=2}^{l_{i}} r_{y, i}, \quad \text { where } l_{i}=2, \ldots, L \tag{4.7}
\end{equation*}
$$

must be monotonic.

## Proof

T. 4 does not allow line which is oscillating, or have the trend to close itself. If a line is oscillating, then there exists two points, say $i_{1}$ and $i_{2}$ s.t.

$$
\begin{equation*}
\left|r_{x, i_{1}}+r_{y, i_{1}}+r_{x, i_{2}}+r_{y, i_{2}}\right|=2 \tag{4.8}
\end{equation*}
$$

since either $r_{x, i_{1}}=-r_{x, i_{2}}$ or $r_{y, i_{1}}=-r_{y, i_{2}}$. This implies that the sum of the relative coordinates will
not be monotonic. If the line is $l_{\mathrm{L}}$ is not oscillating, then (4.8) will not result, implying that the sum of the relative coordinates must be monotonic. If the line tends to close itself, (4.8) will be true for some $i_{1}$ and $i_{2}$. The above argument is applied again.

In the following property, a structuring element used by a multi-structuring element erosion filter is regarded as a single point wide line. A structuring element used by a multistructuring element erosion filter will have the property described below.

Property 4.3 (Relative Coordinates of a Structuring Element) Let $\left(r_{x}, r_{y}\right)$ be the relative coordinates between the end points of a structuring element of a multi-structuring element erosion filter with index $N$, i,e.

$$
\begin{equation*}
\left|r_{x}\right|=\left|\sum_{i=2}^{2 N+1} r_{x, i}\right|, \quad\left|r_{y}\right|=\left|\sum_{i=2}^{2 N+1} r_{y, i}\right| \tag{4.9}
\end{equation*}
$$

where $\left(r_{\mathrm{x}, \mathrm{i}}, r_{\mathrm{y}, \mathrm{i}}\right)$ is the relative coordinate of the $i^{\text {th }}$ point with respect to the $(i-1)^{\text {th }}$ point in the structuring element. If the structuring element has $m_{s}$ segments, then

$$
\begin{equation*}
\left|r_{x}\right|+\left|r_{y}\right|-m_{s}=2 N-1 \tag{4.10}
\end{equation*}
$$

## Proof:

All structuring elements must satisfy T.4. By Lemma 4.1, the sequences,

$$
\sum_{i=2}^{t_{i}} r_{x, i} \text { and } \sum_{i=2}^{l_{i}} r_{y, i}, \quad l_{i}=2, \ldots, 2 N+1
$$

are monotonic. If the structuring element has only one segment, i.e $m_{\mathrm{s}}=1$, then the end point is at a distance of $2 N$ away from the starting point. If the structuring element is of $m_{\mathrm{s}}$ segments, then

$$
\begin{align*}
\left|\sum_{i=2}^{2 N+1} r_{x, i}\right|+\left|\sum_{i=2}^{2 N+1} r_{y, i}\right| & =(2 N+1-2)+m_{s}  \tag{4.11}\\
& =2 N-1+m_{s} .
\end{align*}
$$

At every transition from one segment to another, (4.6) holds. i.e. $\left|r_{\mathrm{x}, \mathrm{i}}\right|+\left|r_{\mathrm{y}, \mathrm{i}}\right|=2$. There are totally $m_{s}$ transitions and the ' 2 ' in (4.11) sets the starting point as the new origin. Hence Property 4.3 is proved.

The following property uses the representation of relative coordinates to find a sufficient condition for invariant single point wide open lines. Only the positive lines, curves,
rectangular regions and irregular regions will be considered because negative regions of all sizes are preserved by $B M E F_{\mathrm{N}}$ (Prop.4.2).

Property 4.4 (Positive Binary Single Point Open Curve) Let $l_{\mathrm{L}}$ be a single point wide open curve and $L=3 N$. Let $\left(r_{x}, r_{y}\right)$ be the relative coordinates between the end points of $l_{\mathrm{L}}$ defined as in (4.9), and $m$ be the number of straight line segments on $l_{\mathrm{L}}$. The line $l_{\mathrm{L}}(L=3 N)$ is invariant to a binary multi-structuring element erosion filter with index $N$ if and only if

$$
\begin{equation*}
\left|r_{x}\right|+\left|r_{y}\right|-m=3 N-2 \tag{4.12}
\end{equation*}
$$

## Proof

i. $l_{\mathrm{L}}$ is invariant. Divide the line into 2 lines $l_{2 \mathrm{~N}+1}$ and $l_{\mathrm{N}+1} . l_{2 \mathrm{~N}+1}$ is a line of $(2 N+1)$ points, and $l_{\mathrm{N}+1}$ is of $N+1$ points. The last point on $l_{2 \mathrm{~N}+1}$ coincides with the first point on $l_{\mathrm{N}+1} \cdot l_{2 \mathrm{~N}+1}$ and $l_{\mathrm{N}+1}$ has $m_{1}$ and $m_{2}$ segments respectively, we must have

$$
\left|r_{x, l_{2 N, 1}}\right|+\left|r_{y, l_{2 N, 1}}\right|-m_{1}=2 N-1, \quad\left|r_{x, l_{N, 1}}\right|+\left|r_{y, l_{N, 1}}\right|-m_{2}=N-2
$$

provided that both lines satisfy T.4. For the whole line $l_{\mathrm{L}}$, the relative coordinates between the end points is

$$
\begin{equation*}
\left|r_{x}\right|+\left|r_{y}\right|=\left|r_{x, l_{2 v, 1}}\right|+\left|r_{y, l_{2 x, 1}}\right|+\left|r_{x, l_{x, 1}}\right|+\left|r_{y, l_{x, 1}}\right| \tag{4.13}
\end{equation*}
$$

The last points in $l_{2 \mathrm{~N}+1}$ coincides with the first point in $l_{\mathrm{N}+1}$, the number of segments, $m$, of $l_{\mathrm{L}}$ is $m=m_{1}+m_{2}-1$, implying

$$
\left|r_{x}\right|+\left|r_{y}\right|=2 N-1+m_{1}+N-2+m_{2}
$$

ii. $\left|r_{\mathrm{x}}\right|+\left|r_{\mathrm{y}}\right|-m=3 N-2$. This implies that $l_{\mathrm{L}}$ satisfies T. 4 and by Prop.4.3, $l_{\mathrm{L}}$ can be regarded as unions of some structuring elements. Therefore, every point on $l_{\mathrm{L}}$ can be preserved by at least one structuring element.

Property 4.4 can be extended to lines of any length $L \geq 3 N$.

Property 4.5 (Positive Binary Single Point Open Curve 2) A single point wide open curve $l_{\mathrm{L}}$, $L \geq 3 N$ is invariant to $B M E F_{\mathrm{N}}$ if $\left|r_{\mathrm{x}}\right|+\left|r_{\mathrm{y}}\right|-m \geq 3 N-2$.

A line of any length $L \geq 3 N$ can be visualized as unions of lines of length $3 N$ which are invariant to $B M E F_{\mathrm{N}}$. However, this property only gives a sufficient condition for a line to be
invariant to $B M E F_{\mathrm{N}}$. If this property does not hold, it does not imply that a line must not be invariant to $B M E F_{\mathrm{N}}$. For example, Prop.4.5 does not hold for closed curves which may be invariant to $B M E F_{\mathrm{N}}$. Fig. 4.5 gives some examples of invariance test using Props.4.4 and 4.5. Fig.4.5(a) is invariant to $B M E F_{2}$ which shows the sufficiency of Prop.4.5. Figs.4.5(b) and (c) are not root signals to $B M E F_{2}$ which do not satisfy Prop.4.4. Fig.4.5(d) is a root to $B M E F_{2}$. It satisfies Prop.4.4 but not for Prop.4.5. Since Prop.4.4 holds, the signal structure is also invariant.


Figure 4.5 Test of Invariance of Signals for BMEF 2 (a)Invariant by Prop. 4.5 (b)Nonroot Signal fails Prop.4.4 (c)Nonroot Signal Fails Props.4.4 and 4.5 (d)Invariant by Prop.4.5

As binary signal structures can be constructed by set operations, such as intersection, union, translate and complements, of other binary signals. Study of the invariance resulted by these operations are worth since this provides us more information for the root signal structure.

Property 4.6 (Binary Set Operations) Let $A$ and $B$ be invariant signals to $B M E F_{\mathrm{N}}$.

1. The intersection of $A$ and $B$ is invariant if and only if
i. $A \cap B=\varnothing$, or
ii. $A \subseteq B$ or $B \subseteq A$.
2. The union of $A$ and $B$ is invariant.

$$
B M E F_{N}(A)=A, B M E F_{N}(B)=B \Rightarrow B M E F_{N}(A \cup B)=A \cup B .
$$

The set union can be extended to the union of any number of invariant signal, either countable or uncountable.

$$
B M E F_{N}\left(\bigcup_{i=1}^{n} A_{i}\right)=\bigcup_{i=1}^{n} A_{i}, \quad n=1,2, \ldots
$$

3. Let $A_{(\mathrm{ij})}$ be the translated $A$ by $(i, j)$. If $B M E F_{\mathrm{N}}(A)=A$, then $B M E F_{\mathrm{N}}\left(A_{(\mathrm{ij}, \mathrm{j}}\right)=A_{(\mathrm{ij}, \mathrm{j}}$.
4. The complement of $A$ is invariant $B M E F_{\mathrm{N}}$ if $A$ is also invariant to $B M E F_{\mathrm{N}}{ }^{\text {d }}$.

Property 4.6 shows that the union of invariant signals is also invariant. The following properties concerning closed curves and rectangular details can be derived.

Property 4.7 (Positive single Point Closed Curve) Let $l_{\mathrm{cL}}$ be a closed curve of $L$ points. The curve $l_{\mathrm{CL}}$ is invariant to $B M E F_{\mathrm{N}}, N>1$ if $l_{\mathrm{CL}}$ is composed of at least $6 N-2$ points and $l_{\mathrm{CL}}$ can be decomposed into at least 2 invariant open curves.

## Proof:

If a closed curve can be decomposed into 2 invariant single point open curve, then the closed curve in invariant due to Property 4.6. The minimal length of an invariant signal is 3 N , that of a closed curve is $2 * 3 N-2=6 \mathrm{~N}-2$ if the end points of the open curves are overlapped.

Moreover, if a particular closed curve cannot be decomposed to two invariant open curves, a binary erosion filter will convert the closed curve to the longest open curve(s) after repeated filtering. Single point wide invariant signal structures have been discussed, details of regular shape will be treated. It is impossible to consider all 2-D structures which are invariant to the binary erosion filter. Solid rectangular structures are analyzed for their simplicity. For hollow structures, the analysis is the same as that of a closed curve.

Property 4.8 (Positive Rectangular Region) Let $d_{\mathrm{R}}$ be a solid rectangular detail with height $h$ and width $w$ on a binary signal. The solid rectangular shape detail $d_{\mathrm{R}}$ is invariant to $B M E F_{\mathrm{N}}$ if and only if the width $w$ must not be less than $w_{\text {min }}$,

$$
w_{\min }= \begin{cases}3 N & , \text { if } h \leq 3  \tag{4.14}\\ 3 N-\frac{h}{2} & , \text { if } 2 N \geq h>3 \text { and } h \text { is even } \\ 3 N-\frac{h-1}{2} & \text {, if } 2 N \geq h>3 \text { and } h \text { is odd }\end{cases}
$$

## Proof:

1. $d_{\mathrm{R}}$ is invariant. There are three cases to be considered.

Case i. $h \leq 3$. Property 4.1 has already considered the case when $h=1$. When $h=3$, if the row in the middle of the rectangle is invariant to $B M E F_{\mathrm{N}}$, then there exists a $3 N$ points invariant line. There are two possible construction of this invariant line, the $3 N$ points lie horizontally, or $3 \mathrm{~N}-1$ points lie horizontally while the last point is either below or above this line. Both lines result in a total width of $3 N$. When $h=2$, the same situation is encountered.

(a) $h$ is even

(b) h is odd

$$
B_{r}: \text { Shaded region }
$$

Figure 4.6 Definition of $B_{r}$

Case ii. $2 N \geq h \geq 3$ and $h$ is even. The rectangle of height $h$ can be divided into two identical rectangles of height $h^{\prime}$. Let $B_{\mathrm{r}}$ be the boundary of the rectangle of height $h^{\prime}$ and width $w$ as shown in Fig.4.6(a). Let $B_{\mathrm{r}}$ be the boundary of the rectangle with height $h^{\prime}+1$ and width $w$. If only the horizontal rows of $B_{\mathrm{r}}$ is invariant then $3 N \geq h^{\prime}+w \geq 2 N+1$. Consider the second point (counted from the left) on a horizontal row of $B_{\mathrm{r}}$, if it is preserved after the filter is applied once, then $h^{\prime}+w-1 \geq 2 N+1$. Continue in this way, at the $N^{\text {th }}$ point on the row, $h^{\prime}+w-(N-1) \geq 2 N+1$.

Hence, we have $h^{\prime}+w_{\min }-(N-1)=2 N+1$. This gives $h^{\prime}+w_{\min }=3 N$. Therefore, $w_{\min }=3 N-h^{\prime}=3 N-h / 2$. If the horizontal rows of $B_{\mathrm{r}}$ is preserved, the union

$$
\begin{equation*}
\bigcup_{j=1}^{h^{\prime}}\left(B_{r}\right)_{(0, j)} \tag{4.15}
\end{equation*}
$$

gives a rectangle of height $h$ and width $w_{\text {min }}$.
Case iii. $2 N \geq h \geq 3$ and $h$ is odd. The result follows if $h^{\prime}=(h-1) / 2$.
2. $w \geq w_{\min }$. The proof is similar to Prop.4.4.

Figs.4.6(a) and 4.6(b) show the shape of the boundary $B_{\mathrm{r}}$ when $h$ is even and odd respectively. The sizes of minimal rectangular details which can be preserved by the binary erosion filter with index $N, N=1,2,3$, are summarized in Table 4.1.

| Index, $N$ | Minimal 1-D Line 3 N | Minimal 2-D Rectangle |  |
| :---: | :---: | :---: | :---: |
|  |  | Height, $h$ | Width, $w$ |
| 1 | 3 | 1 | 3 |
|  |  | 2 | 3 |
|  |  | 3 | 3 |
| 2 | 6 | 1 | 6 |
|  |  | 2 | 6 |
|  |  | 3 | 5 |
|  |  | 4 | 4 |
| 3 | 9 | 1 | 9 |
|  |  | 2 | 9 |
|  |  | 3 | 9 |
|  |  | 4 | 7 |
|  |  | 5 | 7 |
|  |  | 6 | 6 |

Table 4.1 Summaries of Minimal Preservable Rectangular Details of $B M E F_{N}, N=1$ to $N=3$

A binary signal is not solely composed of single points curves, either open or closed, and rectangular regions, most of the signal patterns are irregular in shape. Denote $d_{\mathrm{IR}}$ as an
irregular region. An irregular region can be interpreted as the union of a set of invariant signals with a set of nonroot signals, i.e. $d_{\mathbb{I R}}=\{$ Invariant structures $\} \cup\{$ Nonroot structures $\}$. Any element in the set of invariant components of $d_{\mathrm{IR}}$ must have empty intersections with all the elements in the set of nonroot components. Denote $\boldsymbol{I}_{\mathrm{c}}$ and $\boldsymbol{N}_{\mathrm{c}}$ as the set of invariant components and the set of nonroot components of $d_{\mathrm{IR}}$ respectively. Therefore, we have

$$
\begin{equation*}
i_{c} \cap n_{c}=\varnothing, \quad \forall n_{c} \in N_{c}, i_{c} \in \boldsymbol{I}_{c} \tag{4.16}
\end{equation*}
$$

(4.16) implies that the two sets are disjoint. In addition, a point which is included in an element of $\boldsymbol{I}_{\mathrm{c}}$ must not appear in any element of $\boldsymbol{N}_{\mathrm{c}}$. The nonroot structures are those signals which fail to comply with Property 4.3 to 4.8 . Moreover, as single point wide open curves can build other signals by set union operations. Hence, in the discussion of Property 4.9, all signal structures are assumed to be single point wide open curves. It should noted that the set of invariant structures or the set of nonroot components composing $d_{\mathrm{IR}}$ can be empty.

Property 4.9 (Positive Irregular Region) Let $d_{\mathrm{IR}}$ be an irregular detail. $d_{\mathrm{IR}}$ can be represented as the union of a non-empty set of invariant signals $\boldsymbol{I}_{\mathrm{c}}$ and a set of nonroot signals $\boldsymbol{N}_{\mathrm{c}}$ which satisfies (4.16). The irregular detail $d_{\mathrm{IR}}$ is preserved by $B M E F_{\mathrm{N}}$ if all nonroot components of $d_{\mathrm{IR}}$ satisfy the following conditions:
i. There exists at least $N+1$ connected points on some invariant signals s.t. these points are connected to both ends of the nonroot signal.
ii. Denote $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$ as the end points of the concatenated single point open curve formed by the nonroot signal and the connected regions on some invariant signals. Suppose that the length of the concatenated curve is $L$ and of $m$ segments, and $\left(r_{x}, r_{y}\right)$ the coordinates of $\boldsymbol{x}_{2}$ relative to $\boldsymbol{x}_{1}$. The concatenated signal is invariant to $B M E F_{\mathrm{N}}$ if $\left|r_{\mathrm{x}}\right|+\left|r_{\mathrm{y}}\right|-m=L$ $2 \geq 2 N$.

The proof of Property 4.9 is contained in Appendix III. Fig. 4.7 shows two examples of irregular details. Fig.4.7(a) and (c) are two irregular details. The binary filter $B M E F_{2}$ is used. The detail in Fig.4.7(a) can be decomposed into two invariant components, which are shown in Fig.4.7(b). The components are invariant line of 6 points. No nonroot signal is found. Therefore, the detail in Fig.4.7(a) is invariant to $B M E F_{2}$. The detail in Fig.4.7(c) can be decomposed into an invariant component which is shown in Fig.4.7(d) and a nonroot signal in Fig.4.7(e). Since the nonroot signal of (c) satisfies Property 4.9. The detail in Fig.4.7(b)
is also invariant to $B M E F_{2}$.


Figure 4.7 Decomposition of Irregular Details into Invariant Components and Nonroot Signals (a)Irregular Detail 1 (b)Invariant components of (a), there is no nonroot signals (c)Irregular Detail 2 (d)Invariant Component of (c) (e)Nonroot Component of (c)

Property 4.10 A signal is invariant to $B M E F_{\mathrm{N}+1}$ is also invariant to $B M E F_{\mathrm{N}}$.

As the length of the structuring element used in $B M E F_{\mathrm{N}+1}$ is longer than that of $B M E F_{\mathrm{N}}$, hence Prop.4.10 follows. Similar to the 1-D root signals, the complement of root signal of $B M E F_{\mathrm{N}}$ is invariant to $B M E F_{\mathrm{N}}{ }^{\mathrm{d}}$. Properties of the root signals of $B M E F_{\mathrm{N}}{ }^{\text {d }}$ will not be stated here. The deduction of root signals of $B M E F_{\mathrm{N}}{ }^{\text {d }}$ can be done by replacing regions of 1 's by those of 0 's complementing those of $B M E F_{\mathrm{N}}$. The size of minimal preservable details are the same.

The geometrical structures of an invariant signal to a multilevel multi-structuring element erosion filter can be deducted from those of the binary filter with the same index as the selective threshold decomposition also possesses the stacking property(Property 3.1). Before further our discussion of invariant signals to the multilevel multi-structuring element erosion filter with an arbitrary forcing level $f$, where $f \in\{N+1, N+2, \ldots, 2 N+1$, the filter with forcing level $N+1$ is considered, which is also known as the standard multilevel erosion filter. In the following discussion, the multilevel multi-structuring element erosion filter with index $N$ and forcing level $f$ is denoted as $G M E F_{\mathrm{f}, \mathrm{N}}$. The multilevel multi-structuring element erosion filter will also be abbreviated as the multilevel erosion filter. The term multilevel erosion filter and $G M E F_{\mathrm{f}, \mathrm{N}}$ will be used interchangeably with multilevel multi-structuring element erosion filter. During the analysis, a (relatively) positive region is a region $[g]_{\mathrm{L}}$ whose level $g$ is greater than those of its surroundings. Similarly, a (relatively) negative region is defined if the level of the constant region is less than those of its surroundings.

## One-Dimensional Multilevel Multi-structuring Element Erosion Filter

The 1-D multilevel multi-structuring element erosion filter is a particular case of the 2-D one which is defined in Definition 3.5. Three horizontal structuring elements are used. The analysis is quite similar to that of the binary 1-D filters. Let $[g]_{\mathrm{L}}$ be a region of $L$ consecutive points which are of identically value $g$. It can be observed from the following properties that the root signal structures of the standard multilevel erosion filter thresholded at level $g$ resemble those of the binary erosion filter with the same index. In the meantime, only the standard 1-D multilevel erosion filter will be considered since those filters with forcing level greater than $N+1$, the invariant signals are constant signals.

Property 4.11 (1-D Multilevel Positive Region) Let $[g]_{\mathrm{L}}$ be a constant region which is surrounded by $\left[g_{1}\right]_{\mathrm{L} 1}$ and $\left[g_{2}\right]_{\mathrm{L} 2}$ on both sides, and $g \geq g_{1}, g_{2} .[g]_{\mathrm{L}}$ is a constant neighbourhood to the 1-D $G M E F_{\mathrm{N}+1, \mathrm{~N}}$ if and only if $L \geq 3 \mathrm{~N}$.

## Proof:

$\ln \left[g_{1}\right]_{\mathrm{L} 1}[g]_{\mathrm{L}}\left[g_{2}\right]_{\mathrm{L} 2}$, since $g$ is greater than $g_{1}$ and $g_{2}$, thresholding the signal at level $g$ results in a binary signal $[0]_{L}[1]_{L}[0]_{L}$. By Prop.4.1, $L \geq 3 N$.

Property 4.12 (1-D Multilevel Negative Region) Let $[g]_{\mathrm{L}}$ be a constant region which is surrounded by $\left[g_{1}\right]_{\mathrm{L} 1}$ and $\left[g_{2}\right]_{\mathrm{L} 2}$ on both sides, and $g \leq g_{1}, g_{2}$. $[g]_{\mathrm{L}}$ is invariant to the 1-D $G M E F_{\mathrm{N}+1, \mathrm{~N}}$ if and only if $L \geq 1$.

Property 4.13 (1-D Multilevel Positive Region 2) Let $[g]_{\mathrm{L}}$ be a constant region which is surrounded by $\left[g_{1}\right]_{\mathrm{L} 1}$ and $\left[g_{2}\right]_{\mathrm{L} 2}$ on both sides, and $g_{1} \geq g \geq g_{2}$. $[g]_{\mathrm{L}}$ is invariant to the 1-D $G M E F_{\mathrm{N}+1, \mathrm{~N}}$ if and only if $L+L_{1} \geq 3 \mathrm{~N}$.

Property 4.14 (Monotonic Sequence) A monotonic sequence

$$
\left[g_{1}\right]_{L_{1}}\left[g_{2}\right]_{L_{2}} \ldots\left[g_{n}\right]_{L_{n}}, \quad \text { where } g_{1} \leq g_{2} \leq \ldots \leq g_{n} \text { or } g_{1} \geq g_{2} \geq \ldots \geq g_{n} \text {. }
$$

is invariant to $G M E F_{\mathrm{N}+1, \mathrm{~N}}$ if and only if $\sum_{i=1}^{n} L_{i} \geq 3 N$, which implies $L_{n} \geq 3 N$ when $g_{1} \leq \ldots \leq g_{\mathrm{n}}$ or $L_{1} \geq 3 N$ when $g_{1} \geq \ldots \geq g_{n}$.

The proofs of Props.4.12 to 4.14 can be observed on the threshold set at level $g$. Props.4.11 and 4.12 indicate that the multilevel erosion filter is edge preserving. But the preservation of edge depends on the sequence of appearance of falling edge and rising edge. If a falling edge appears before a rising edge, a negative region is found. By Prop.4.12, a negative region of any length is invariant to the multilevel erosion filter with index $N$ and forcing level $N+1$. On the other hand, if a falling edge follows a rising edge, a positive region is enclosed by the edges. Unless the length of the positive constant region is greater than or equal to $3 N$, the region is not invariant to the standard multilevel erosion filter. If the signal is an increasing signal, i.e. the gray level increases gradually, the signal at level $g$ is invariant to the multilevel erosion filter if the number of consecutive 1 's in the threshold set at level $g$ satisfy Prop.4.1. This means that the length of the region those value is greater than or equal to $g$ must be at least $3 N$. The preservation of monotonic sequence is given by Props.4.13 and 4.14. It should be noted that at the maximum level of the monotonic sequence, a constant region of at least $3 N$ points must be found. This complies with Prop.4.11. However, at the minimum level of the sequence, any number of points will be preserved. In summary, the multilevel erosion filter is biased towards regions of lower intensities. Fig.4.8 displays some examples of 1-D invariant signals. The multilevel erosion filter used is with index 2 . Hence, for positive details, the number of consecutive points of constant value must
not be less than 6. Fig.4.8(a) shows a positive region of 7 points, which is greater than the minimum length of 6 points for $N=2$. A staircase signal, which is a monotonic increasing signal, is shown in Fig.4.8(b). The number of points at the level 10 or above is greater than 6. If the number of points at this top level is less that 6 , by Prop.4.11, the points originally of value 10 will be converted to 5 . If the number of consecutive 5 's is still less than 6 , this level will be converted to the next lower level. This operation proceeds until Prop.4.11 is satisfied. Fig.4.8(c) is a negative region. Fig.4.8(d) gives an examples of an input sequence which is increasing when $x \leq 10$ and decreasing when $x \geq 16$.


Figure 4.8 Examples of One-dimensional Invariant Signal to $G M E F_{3,2}$ (a)Positive region of length 7 (b)A discrete monotonic sequence(staircase) (c)Negative regions of length 1 and 2 (d)A combination of monotonic sequence, increasing when $x \leq 10$ and decreasing when $x \geq 16$.

The dual to the multilevel erosion filter with index $N$ and forcing level $N+1$ can be deduced by complementing the root signal structures of the multilevel erosion filter with the same index and forcing level.

## Two-Dimensional Multilevel Multi-structuring Element Erosion Filter

The standard 2-D multilevel multi-structuring element erosion filter is first considered, i.e. the forcing level of the multilevel erosion is set to its minimal, which is equal to $N+1$. The multilevel erosion filter with the minimal forcing level can be analyzed using the classical threshold decomposition. The root signal structure analysis of the 2-D multilevel erosion filter follows the same approach as in the analysis of the binary erosion filter. The sufficient condition for a single point wide open curve with gray level $g$ is first derived. A line is defined as a connection of points with constant gray level $g$. The relative coordinates between the end points of a single point wide open curve is used to characterize its invariance to a filter. A positive open curve is an open curve whose gray level is greater than its surroundings. A negative detail is a region of constant gray level which is of lower intensity than its surroundings.

Since the standard erosion filter uses the classical threshold decomposition, the root signal structures follow from the binary structures. Therefore, in spite of repeating all the binary root signal properties, we have:

Property 4.15 (Root Signal Structure of Standard Erosion Filter) A constant region of value $g$ is invariant to a standard multilevel erosion filter if the threshold set at level $g$ is invariant to the 2-D binary erosion filter with the same index.

A single point wide line of $L$ points with constant gray level $g$ and $m$ segments which is surrounded by regions of lower gray levels will be preserved by a standard erosion filter with index $N$, if the relative coordinates between the end points of the line satisfies (4.10), i.e. Props.4.4 and 4.5. By Prop.4.2, a constant region of any size and any shape which is of intensity lower than its surroundings is invariant to the 2-D $G M E F_{\mathrm{N}+1, \mathrm{~N}}$.

In binary 2-D signals, the unions of invariant signals are root signal to the binary erosion filter. In multilevel signals, the union operation is replaced by the logical OR operation. Hence, the following property is resulted.

Property 4.16 (Logical Operations) Let $G_{1}(x)$ and $G_{2}(x), x \in \boldsymbol{Z}^{2}$, be invariant multilevel signals to $G M E F_{\mathrm{N}+1, \mathrm{~N}}$.

1. The logical AND of $G_{1}(x)$ and $G_{2}(x)$, which is denoted as $G_{1}(x) \wedge G_{2}(x)$, is invariant to $G M E F_{\mathrm{N}+1, \mathrm{~N}}$ if
i. $G_{1}(x) \wedge G_{2}(x)=0$, or
ii. $G_{1}(x) \wedge G_{2}(x)=G_{1}(x)$ or $G_{1}(x) \wedge G_{2}(x)=G_{2}(x)$.
2. The logical OR of $G_{1}(x)$ and $G_{2}(x)$, which is denoted as $G_{1}(x) \vee G_{2}(x)$, is invariant to GMEF $F_{\mathrm{N}+1, \mathrm{~N}}$.
3. The complement of $G_{1}(x)$ is invariant to $G M E F_{\mathrm{N}+1, \mathrm{~N}}$ if $G_{1}(x)$ is also invariant to $G M E F_{\mathrm{N}+1, \mathrm{~N}}{ }^{\mathrm{d}}$.

For regular and irregular root signal structures, please refer to Props.4.7 to 4.9. Props.4.11 to 4.14 accounts for the signal requirements in preserving edges and monotonic regions.

The performance of a multilevel multi-structuring element erosion filter depends mainly on the binary signals decomposed by the selective threshold decomposition. The selective threshold decomposition is a modification of the classical threshold decomposition. Modification includes:

1. Threshold levels are taken from a set of sorted data. In a multilevel erosion filter, the threshold levels are selected from the pixels masked by the $(2 N+1)$ by $(2 N+1)$ window of the filter.
2. Binary filtering at the current threshold level is skipped if the current threshold level equals to the previous one. This reduces repeated binary filtering at the same threshold level as the value of the pixels within the $(2 N+1)$ by $(2 N+1)$ square window may not be all distinct.
3. A forcing level, $f$, is introduced. If the rank of the threshold level equals to $f$, then threshold decomposition is carried out even the current threshold level equals to the previous one.

In a multilevel erosion filter, the forcing level $f$ ranges from $N+1$ to $2 N+1$. The single point wide open curves are not invariant to multilevel erosion filters with forcing level greater than $N+1$. An irregular signal may be invariant to these filters if the signal is a closed curve, or a open curve those ends are clustered. Fortunately, there is no change in other invariant structures such as closed curves and rectangular details. This can be explained by the fact that a match by a $T_{1 \mathrm{~N}}$ structuring element requires at least $2 N+1$ connected points. Hence, the forcing level $f \leq 2 N+1$ has no effect on the result.

Property 4.17 (Clustering at the Ends of Open Curve) Let $l$ be an open curve of constant gray level $g . l$ is invariant to $G M E F_{\mathrm{f}, \mathrm{N}}$ where $N+1 \leq f \leq 2 N+1$ if constant regions which are composed of at least $f g$ 's are found at the ends of $l$.

Property 4.18 (Closed Curve or Rectangular Details) If the invariant structure is either a positive closed curve or a rectangular detail, variations of $f$ from $N+1$ to $2 N+1$ do not affect the invariance of the structure.

## Proof:

At any point on a positive closed curve, there must be at least $2 N+1$ pixels of value $g$ which are included by the $(2 N+1)$ by $(2 N+1)$ moving window. Hence, if $N+1 \leq f \leq 2 N+1$, then the closed curve is still invariant. Moreover, as a positive rectangular detail can be treated as union of invariant positive rectangular closed curve. Therefore, Prop.4.18 is proved.

The root signal structures to the dual filter of a multilevel erosion filter with index N and forcing level $f$ can be deducted by complementing those of the multilevel erosion filter. Fig.4.9 shows some examples of root signals of the dual filter with $N=2$. Fig.4.9(a) is a line of 6 points at gray level 9. A negative impulse of magnitude 1 is found near the upper right angle of the signal. The signal is invariant to $G M E F_{3,2}$ since each threshold set is invariant to $B M E F_{2}$. Fig.4.9(b) is a stacking of rectangular regions. Each region satisfies Prop.4.8. The signal is invariant to both $G M E F_{3,2}$ and $G M E F_{4,2}$. Fig.4.9(c) is a root signal to $G M E F_{4,2}{ }^{\text {d }}$. The signal is a stacking of rectangular constant regions with values ranging from 5 to 10 . The negative rectangular details in Fig.4.9(c) are of size greater than or equal to those given by Prop.4.8. The multilevel erosion dual filter is insensitive to positive details. In other words, all positive details will be preserved. This property is shown in Fig.4.9(d) as the positive
impulse is preserved.


Figure 4.9 Examples of Invariant Signals to Multilevel Erosion Filter (a)GMEF ${ }_{3,2}(b) G M E F_{3,2}$ and $G M E F_{4,2}$ (c) $G M E F_{3,2}{ }^{d}$ and $G M E F_{4,2}{ }^{d}{ }^{d}(d) G M E F_{3,2}{ }^{d}$

## Root Signal Analysis of $C G_{f, N}$

A $C G_{f, N}$ filter is a two-pass operation. A signal is first filtered by a multilevel erosion filter, and then by the dual of the multilevel erosion filter. The root signal structure is similar to those of the multilevel erosion filter and its dual.

Property 4.19 Let $G(x)$ be a signal which is invariant to both $G M E F_{f, N}$ and $G M E F_{\mathrm{f}, \mathrm{N}}$, i.e.

$$
G M E F_{f, N}(G(x))=G(x) \text { and } G M E F_{f, N}^{d}(G(x))=G(x)
$$

$G(x)$ is also invariant to $C G_{\mathrm{f}, \mathrm{N}}$.

If a signal satisfies Property 4.19, then the minimal number of points in the signal depends on the forcing level. If the forcing level $f$ equals to $N+1$, the minimal number of points in a constant region is $3 N . C G_{\mathrm{N}+1, \mathrm{~N}}$ is not biased towards regions of higher intensities nor lower intensities as much as $G M E F_{\mathrm{N}+1, \mathrm{~N}}$ does. Edges and monotonic regions are preserved if Property $4.11,4,13$ and 4.14 are satisfied. Rectangular details and irregular details are already discussed in the previous subsections.

In $C G_{\mathrm{f}, \mathrm{N}}$, a special type of signal structure exists. This is the vibrating point. A vibrating point is defined as a point those value is toggled after each subfilter pass. A vibrating point is resulted at a positive region which is not invariant to $G M E F_{\mathrm{f}, \mathrm{N}}$ or at a negative region which is not invariant to $G M E F_{\mathrm{f}, \mathrm{N}}{ }^{\text {d }}$. In 1-D $C G_{\mathrm{f}, \mathrm{N}}$ filtering, numerous vibrating points are found. For example, a signal sequence $\left[g_{1}\right]_{L_{1}}[g]_{\mathrm{L}}\left[g_{2}\right]_{\mathrm{L} 2}$, where $3 N-1 \geq L \geq 2 N+1$ and $g \geq g_{1}, g_{2}$, after filtered by $G M E F_{\mathrm{f}, \mathrm{N}}$ becomes

$$
\left[g_{2}\right]_{L_{1}}[g]_{L \bmod N}\left[g_{1}\right]_{\frac{1}{2}(L-3(L \bmod N)}[g]_{L \bmod N}\left[g_{2}\right]_{\frac{1}{2}(L-3(L \bmod N)}\left[g_{2}\right]_{L_{2}}
$$

since $(L-3(L \bmod N)) / 2$ is not invariant to $G M E F_{f, \mathrm{~N}}{ }^{\mathrm{d}}$. The above intermediate output signal will be converted back to $\left[g_{1}\right]_{\mathrm{L} 1}[g]_{\mathrm{L}}\left[g_{2}\right]_{\mathrm{L} 2}$ after the dual filter pass. Another example of vibrating point is given. Another signal sequence $\left[g^{\prime}\right]_{\mathrm{L}^{\prime}}\left[g^{\prime}\right]_{L^{\prime}} \cdot\left[g^{\prime}\right]_{\mathrm{L}^{\prime} 2}$, where $3 N-1 \geq L^{\prime} \geq 2 N+1, L^{\prime}{ }_{1}, L_{2}^{\prime} \geq 3 N$ and $g^{\prime} \leq g^{\prime}{ }_{1}, g^{\prime}{ }_{2}$, is input to the filter. The intermediate output is the input signal itself. In this example, positive regions of length less that $2 N+1$ are resulted. If $g^{\prime}=g^{\prime}{ }_{2}=5$ and $g^{\prime}=3, L^{\prime}=5$ and $L_{1}^{\prime}=L_{2}^{\prime}=6$, the output of $C G_{3,2}$ is $[5]_{6}[3]_{1}[5]_{1}[3]_{1}[5]_{1}[3]_{1}[5]_{6}$. The regions of $[5]_{1}$ are not desirable.

Although the intermediate output is transparent to a user, it is worth mentioning that such
vibrating structure is invariant to $C G_{f, N}$. The invariant signals produces by vibrating points does not comply with the properties described previously. Obviously, the length of a constant neighbourhood consecutive points of identical value is reduced to $2 N+1$.

The analysis of 2-D vibrating point is very complicated. Vibrating points occurs at the regions which are not invariant to both $G M E F_{\mathrm{f}, \mathrm{N}}$ and its dual. However, owing to the geometrical correlation of the neighbourhood points, the effect of vibrating point is not as serious as the 1-D case. Detailed analysis of vibrating structures are not performed. Fig.4.10 shows two vibrating structures. Figs.4.10(a) and (b) show two vibrating points which occurs at a negative region and a positive region respectively. These vibrating points are obtained by filtering a real image called Baboon with $C G_{4,2}$. It is quite interesting that not all filtered images possess such vibrating structures.

| 98 | 58 | 69 | 69 | 47 |
| :--- | :--- | :--- | :--- | :--- |
| 88 | 47 | 47 | 47 | 47 |
| 117 | 47 | 52 | 47 | 47 |
| 86 | 47 | 47 | 47 | 47 |
| 76 | 66 | 75 | 52 | 70 |

(a)

| 98 | 58 | 69 | 69 | 47 |
| :---: | :--- | :--- | :--- | :--- |
| 88 | 47 | 47 | 47 | 47 |
| 117 | 47 | 48 | 47 | 47 |
| 86 | 47 | 47 | 47 | 47 |
| 76 | 66 | 75 | 52 | 70 |

(c)

(b)

(d)
$\square$ : Vibrating Point

Figure 4.10 Examples of Vibrating Points of $C G_{4,2}(a)$ and $(b)$ are the taken from the output of $C G_{4,2}(c)$ and (d) are the outputs of $G M E F_{4,2}$ using (a) and (c) as inputs. The vibrating point is shaded.

### 4.2.2 Rate of Convergence of Multi-Structuring Element Erosion Filter

The rate of convergence of a filter is defined as the number of filter passes required to bring any nonroot signal to its root. The faster the rate of convergence, the lesser the number of passes needed. The convergent rate of the binary erosion filter will first be considered. Analysis of the 2-D binary filter follows the 1-D case. Finally, the 2-D multilevel erosion filter convergent rate will be elaborated.

### 4.2.2.1 Convergent Rate of Binary Multi-structuring Element Erosion Filter

The following properties are concerning the rate of convergence of the 1-D binary erosion filters. Although our main objective is the 2-D filters, studies of the 1-D binary filters help us to gain insight about those of the 2-D filters.

Property 4.19 A signal $[1]_{\mathrm{L}}, L<2 N+1$, is removed by a 1-D $B M E F_{\mathrm{N}}$ after one pass.

## Proof:

If the length of the consecutive 1 's is less than that of the structuring element, then erosion simply produces no 1 after the first pass. This implies that the signal is removed after one pass.

Property 4.20 For any nonroot signal $[1]_{\mathrm{L}}, 2 N+1 \leq L<3 N$, a filter pass of a 1-D $B M E F_{\mathrm{N}}$ trisects the nonroot signal into 3 equal portions of length equal to $L^{\prime}<N+1$, where $L^{\prime}=L \bmod N$.

## Proof:

As $2 N+1 \leq L<3 N$, so we have $3 N-L \leq N-1$. Perform filtering from left to right, at the ( $L$ $2 N+1)^{\text {th }}$ to the $N^{\text {th }}$ points, since both $T_{1 \mathrm{~N}}$ and $T_{2 \mathrm{~N}}$ structuring elements are not matched, so these points are cleared to $[0]_{3 \mathrm{~N}-\mathrm{L}}$ after one pass of the $1-\mathrm{D} B M E F_{\mathrm{N}}$. The number of points counted from the left side that are preserved is $N-(3 N-L)=L-2 N$, which is equal to $L \bmod N$, as $L \geq 2 N+1$. Similarly, the number of points counted from the right side that is preserved by the $B M E F_{\mathrm{N}}$ after one pass is $L \bmod N$. At the middle region of $[1]_{\mathrm{L}}$, the $(N+1)^{\text {th }}$ to the $(L-(N+1))^{\text {th }}$ points are also preserved by $T_{\text {iN }}$ structuring elements. This counts up to another $L \bmod N$ region of $[1]_{\mathrm{LmodN}}$.

By Property 4.19 and 4.20 , an important property concerning the rate of convergence of the $1-\mathrm{D} B M E F_{\mathrm{N}}$ can be considered.

Property 4.21 (Convergence of 1-D signals) In 1-D binary erosion filters, at most two passes are required to convert any nonroot signal to its invariant.

## Proof:

Case i. Binary positive regions. If $L$ is greater than $3 N$ in $[1]_{\mathrm{L}}$, then the signal is invariant and no filter pass is needed. If $L \leq 2 N$, by Property 4.19 , one pass is required to remove the region. If $2 N+1 \leq L<3 N$, by Property $4.20,[1]_{\mathrm{L}}$ is cut into three separated regions [11] $]_{\text {Lmodn }}$ after the first pass. Since $L \bmod N<2 N+1$, an additional pass is required to remove these residual signals. Thus, at most two passes are needed for the conversion of any nonroot signal to root signal.

Case ii. Binary negative regions. By Property 4.2, a region of $[0]_{\mathrm{L}}$ is invariant to the 1-D $B M E F_{\mathrm{N}}$.

The 1-D binary erosion filter has a very fast convergent rate, only two passes are needed for the convergence of all binary signals. Moreover, the convergent rate is independent of the size of the structuring element. The following properties are about the convergent rate of the 2-D $B M E F_{\mathrm{N}}$. It is expected that the convergent rate analysis of the 2-D binary filters will be more complicated, as the geometrical structures of the signal are taken into account. However, we arrive at some interesting properties. We follow the same approach as in the root signal analysis. Firstly, we consider the convergent rate of single point wide curves, and the followed by rectangular details. Finally, that of irregular details are considered.

Property 4.22 (Convergence of Single Point Wide Curves) At most two passes are required to converted a single point wide line $l$, either closed or open, to its root.

## Proof:

There are two types of curves, open and closed, to be considered.
Case i. Open Curves. The convergent rate of open curve is considered first. Two more cases are found. Case (a) $L<3 N$. The length of $l$ is less than the minimal invariant length. By some mapping of the coordinates, the curve can be transformed to a 1-D line. The
convergent rate of the transformed $l$ follows from Prop.4.21.
Case (b) $L \geq 3 N$ but the relative coordinates between the end points do not satisfy Property 4.5, i.e. $\left|r_{\mathrm{x}}\right|+\left|r_{\mathrm{y}}\right|-m<3 N-2$ for all subcollections of $3 N$ connected points on $l$. This implies that no invariant line can be selected from $l$. Thus, $l$ can be decomposed into lines of length less that $3 N$. If there exist some invariant components in $l$, then the nonroot components will be removed after at most two filter passes. Hence, the result follows.

Case ii. Closed curve. If the closed curve can be represented as the union of at least two invariant open curves, then no pass is required since the line is already invariant. However, if Prop.4.7 fails, the closed curve can be regarded as the union of an invariant line(if any) and a nonroot signal. Therefore, at most two passes are needed.

Property 4.23 Any nonroot signals which are bounded by a rectangle of size $h^{*} w$, where $h$, $w \leq N+1$, are removed by $B M E F_{\mathrm{N}}$ after one pass.

Property 4.24 (Convergence of 2-D signals) Any nonroot signal is converted to its root by $B M E F_{\mathrm{N}}$ in at most two passes.

## Proof:

A detail can be classified into regular and irregular in shape.
Case i. Rectangular details. The largest nonroot rectangular detail must be a subset of the rectangles of size given by Property 4.8. Without loss of generality, suppose that the nonroot structure is a rectangle of size $h^{*}\left(w_{\min }-1\right)$, denoted by $d_{\mathrm{R}}$. The rectangle $d_{\mathrm{R}}$ can be regarded as the union of the translates of rectangular boundary $B_{\mathrm{r}}$, which is defined in Prop.4.8. Since $B_{\mathrm{r}}$ can be converted to its invariant structure by at most two passes (Prop.4.22), so at most two passes are required for rectangular details.

Case ii. Irregular details. An irregular detail can be represented as a union of its invariant components and its nonroot components. It is proved that some nonroot components are invariant to the $B M E F_{\mathrm{N}}$ (Prop.4.9). For the remaining nonroot components, at most two passes are needed for the convergence.

As a result, at most two passes are required for the convergence of any 2-D binary signals. This result is encouraging. The convergent rates (number of filter passes) of other nonlinear filters, such as the median filter, increase as the length of the filter window as well
as the size of the input signal increases.

### 4.2.2 $\quad$ Convergent Rate of Multilevel Multi-structuring Element Erosion Filter

In a sequential computer, the filter window moves from left to right and from top to bottom on an input image. Binary filtering at each threshold set is processed sequentially. Hence, the convergent rate of the multilevel erosion filter depends on the number of binary filtering, which is proportional to the size of the $(2 N+1)$ by $(2 N+1)$ window.

Property 4.25 (Convergence of $2-D G M E F_{\mathrm{N}+1, \mathrm{~N}}$ ) The maximum number of passes required for $G M E F_{\mathrm{N}+1, \mathrm{~N}}$ to bring any nonroot signal to its root is $2\left[(2 N+1)^{2}-(N+2)\right]$.

## Proof:

There are at most $(2 N+1)^{2}$ distinct threshold levels in a $(2 N+1)$ by $(2 N+1)$ window. As thresholding starts from the $(N+1)^{\text {th }}$ ranked data and ends at the $\left((2 N+1)^{2}-1\right)^{\text {th }}$ one. Therefore, the maximum number of threshold level is $\left[(2 N+1)^{2}-(N+2)\right]$. By Prop.4.24, at most two passes are needed for the convergence of a signal. Hence, the result follows.

It is obvious that the root signal to $G M E F_{\mathrm{f}, \mathrm{N}}$ is also invariant to $G M E F_{\mathrm{f}-1, \mathrm{~N}}$. However, the reverse is not true. The property below shows the convergent rate of filtering the root signals of $G M E F_{\mathrm{f}-1, \mathrm{~N}}$ by $G M E F_{\mathrm{f}, \mathrm{N}}$. It is reminded that only the invariant open curves of $G M E F_{\mathrm{f}-1, \mathrm{~N}}$ can be changed by $G M E F_{\mathrm{f}, \mathrm{N}}$. Closed curves and rectangular details are invariant to $G M E F_{\mathrm{f}, \mathrm{N}}$ (Prop.4.18).

Property 4.26 Let $L$ be the number of points of a single point wide (except at the ends) open curve $l_{\mathrm{L}}$ of value $g$ which is invariant to $G M E F_{\mathrm{f}-1, \mathrm{~N}}, f \in\{N+2, \ldots, 2 N+1\}$. The number of passes required to convert $l_{\mathrm{L}}$ to the root of $G M E F_{\mathrm{f}, \mathrm{N}}$ is

$$
\begin{cases}\frac{1}{2}[L-(2 N+1)]+2, & \text { if } L \text { is odd }  \tag{4.17}\\ \frac{1}{2}[L-2 N]+1, & \text { if } L \text { is even } .\end{cases}
$$

## Proof:

Case i. $L$ is odd. Suppose $l_{L}$ is a single point open curve except at the ends, which is the worst case for convergence. By thresholding the multilevel signal at $g$. If the ends of $l_{\mathrm{L}}$ are clustered by $i+11$ 's, then the curve $l_{\mathrm{L}}$ is also invariant to $G M E F_{i+1, \mathrm{~N}}$. If $i 1$ 's are found at the ends of $l_{\mathrm{L}}$, then $G M E F_{\mathrm{f}, \mathrm{N}}$ removes one point are each pass on both sides. When the filtered curve is reduced to ( $2 N+1$ ) points, 2 more passes are needed (Property 4.24). The total number of passes is $[L-(2 N+1)] / 2+2$.

Case ii. $L$ is even. When the line is reduced to a length of $2 N$ points, one pass is needed to remove the remaining curve completely, hence, the total number of passes required is [ $L$ $2 N \mathrm{~J} / 2+1$. Thus, the upper bound of the number of passes to bring the root of $G M E F_{\mathrm{f}-1, \mathrm{~N}}$ to that of $G M E F_{\mathrm{f}, \mathrm{N}}$ is proved.

## Convergent Rate of Two-dimensional CG $_{f, N}$

The convergence of the filter $C G_{f, \mathrm{~N}}$ is difficult to define. Ideally a signal is invariant to $C G_{f, \mathrm{~N}}$ if the signal is a root to both the dual filter pair. However, this is not possible owing to the presence of vibrating points. A signal $G(x)$ is said to be invariant to $C G_{f, \mathrm{~N}}$ if

$$
C G_{f, N}(G(x))=G(x), \quad \forall x \in \boldsymbol{D} .
$$

Property 4.27 (Convergence of 2-D $C G_{\mathrm{N}+1, \mathrm{~N}}$ ) The maximum number of passes required for $C G_{\mathrm{N}+1, \mathrm{~N}}$ to bring any nonroot signal to its root is $2\left[(2 N+1)^{2}-(N+2)\right]$.

## Proof:

A pass in $C G_{\mathrm{N}+1, \mathrm{~N}}$ includes a $G M E F_{\mathrm{N}+1, \mathrm{~N}}$ pass followed by its dual. The multilevel signal converges from the maximum and minimum levels and moves towards the level within these bounds. After at most two binary filter passes, the multilevel erosion filter and its dual will proceed to the next level. This will stop until both filters operate on the same gray level. Thus the total number of passes required is $\left[(2 N+1)^{2}-(N+2)\right]$. Vibrating point may be resulted. Consider the worst case, if only one of the subfilter (either the multilevel erosion filter or its dual) can proceed to the next gray level, then a total number of $2\left[(2 N+1)^{2}-(N+2)\right]$ is needed.

The number of passes required to bring the root signals of $C G_{f-1, \mathrm{~N}}$ to those of $C G_{f, \mathrm{~N}}$ can also be treated by Property 4.26 .

### 4.3 Statistical Analysis

The performance of the multi-structuring element erosion filter can be described by the statistics of input images. However, accurate statistical descriptions of input images are difficult to obtain. In Section 4.2, deterministic properties have been discussed to evaluate how well the filter preserves fine details. In the following discussion, the noise attenuation resulted by the multi-structuring element erosion filter is described by a simple model. A constant signal on which noise are added is used. The corrupted signal is denoted by $G(x)$, $\boldsymbol{x} \in \boldsymbol{Z}^{2}$ which is a $k$-level signal.

In this section, the probability measure function of 1-D multi-structuring element erosion filters is described. The probability measure function of the 2-D multi-structuring element erosion filter using 3-pixel long structuring elements are also derived. Ideally, a general description of the probability measure function of 2-D multi-structuring element erosion filters using different forcing level should be given. However, it is not possible to obtain the expression since the number of structuring elements increase exponentially as $N$ increases and are geometrically related.

### 4.3.1 Output Distribution of Multi-structuring Element Erosion Filter

The statistical analysis of the multilevel element erosion filter is based on the statistical threshold decomposition [Arce86] [Arce88]. Let $\left\{\boldsymbol{G}(\boldsymbol{x}), \boldsymbol{x} \in \boldsymbol{Z}^{2}\right\}$ be an independent, identically distributed (i.i.d.), discrete random sequence, with a probability space $(\Omega, \boldsymbol{B}, \boldsymbol{P})$, where the sample sequence $\Omega=\{0,1,2, \ldots, k-1\}$. The event space $\boldsymbol{B}$ is the power set and $\boldsymbol{P}$ is the probability measure function defined on $\boldsymbol{B} \boldsymbol{P}(F)$ assigns a non-negative real number to every member $F$ of $\boldsymbol{B}$. Random variables $x_{\mathrm{i}}\left(\zeta_{1}, \zeta_{2}, \ldots, \zeta_{\mathrm{n}}\right)$ are said to be independent, identically distributed if and only if $x_{\mathrm{i}}\left(\zeta_{1}, \zeta_{2}, \ldots, \zeta_{\mathrm{n}}\right)=x\left(\zeta_{\mathrm{i}}\right), i=1,2, \ldots, n$. Hence, the distribution of the random variables $x_{\mathrm{i}}$ equal to that of $x$. The thresholded binary sequence $\left\{S_{\mathrm{j}}(\boldsymbol{x})\right\}$ is also i.i.d. with a probability space $\left\{\Omega_{\mathrm{b}}, \boldsymbol{B}_{\mathrm{b}}, \boldsymbol{P}_{\mathrm{b}}\right\}$, where the sample space $\Omega_{\mathrm{b}}=\{0,1\}, \boldsymbol{B}_{\mathrm{b}}$ is the binary power set and $\boldsymbol{P}_{\mathrm{b}}$ is the probability measure function defined by

$$
\begin{gather*}
P_{r}[G(x) \leq j-1]=P_{r}\left[S_{j}(G(x))=0\right]=F(j-1)  \tag{4.18}\\
\quad \text { and } P_{r}\left[S_{j}(G(x))=1\right]=1-F(j-1)
\end{gather*}
$$

where $S_{\mathrm{j}}(G)$ is the binary set thresholded by the classical threshold decomposition at level $j$.

### 4.3.1.1 One-Dimensional Statistical Analysis of Multilevel Multi-structuring Element Erosion Filters

A general expression of the probability measure function of the 1-D multilevel erosion filters is derived. It is assumed that the input signal is a constant signal with additive noise (Gaussian, uniformly distributed noises or impulses). The constant signal is original at zero level.

Property 4.28 If $\left\{G M E F_{\mathrm{f}, \mathrm{N}}(G(x)): x \in \boldsymbol{Z}\right\}$ and $\left\{B M E F_{\mathrm{N}}\left(S_{\mathrm{j}}(G(x))\right): x \in Z, 1 \leq j \leq k-1\right\}$ are the $k$ level, and thresholded multistage filtered signals, then the probability measure function is :

$$
\begin{align*}
& P_{r}\left[G M E F_{f, N}(G(x)) \leq j\right] \\
= & P_{r}\left[B M E F_{N}\left(S_{j+1}(G(x))\right)=0\right] \\
= & F(j)+F(j)^{2} \cdot[1-F(j)] \cdot\left[\sum_{i=0}^{2 N-1}(i+1) \cdot[1-F(j)]^{i}+\sum_{i=2 N}^{3 N-2}((i+1)-3((i+1) \bmod N)) \cdot[1-F(j)]^{i}\right] .  \tag{4.19}\\
& +2[f-(N+1)] \cdot F(j) \cdot[1-F(j)]^{2 N+1}
\end{align*}
$$

where $F(j)$ is the probability measure function of the i.i.d. input sequence defined in (4.18).

The proof of Property 4.28 is in Appendix III. It can be seen that by setting $f$ to $N+1$, the probability measure function of the standard 1-D multilevel erosion filter with index $N$ is resulted. The probability measure function of the dual filter to the 1-D multi-structuring element erosion filter is derived. The expression of the probability measure function is quite different from those of the median based nonlinear filters, as geometrical structures of the signal are taken into accounted. For example, the output of filtering a region of $[g]_{2 \mathrm{~N}+1}$, the is $[g]_{1}\left[g^{\prime}\right]_{\mathrm{N}-1}[g]_{1}\left[g^{\prime \prime}\right]_{\mathrm{N}-1}[g]_{1}$. However, if the same signal is fed to a 1-D median filter of a $2 N+1$ points moving window, the output will be $[g]_{2 N+1}$ itself. Therefore, the output of $G M E F_{\mathrm{f}, \mathrm{N}}$ depends on the current location of the moving window.

Property 4.29 If $\left\{G M E F_{\mathrm{f}, \mathrm{N}}{ }^{\mathrm{d}}(G(x)): x \in \boldsymbol{Z}\right\}$ and $\left\{B M E F_{\mathrm{N}}{ }^{\mathrm{d}}\left(S_{\mathrm{j}}(G(x))\right): x \in \boldsymbol{Z}, 1 \leq j \leq k-1\right\}$ are the $k$ level, and thresholded binary filtered signals which are dual to $\left\{G M E F_{\mathrm{f}, \mathrm{N}}(G(x))\right\}$ and $\left\{B M E F_{\mathrm{N}}\left(S_{\mathrm{j}}(G(x))\right)\right\}$ respectively, then the probability measure function is :

$$
\begin{align*}
& P_{r}\left[G M E F_{f, N}{ }^{d}(G(x)) \leq j\right] \\
= & 1-\left[1-F(j)+F(j) \cdot[1-F(j)]^{2} \cdot\left[\sum_{i=0}^{2 N-1}(i+1) F(j)^{i}\right.\right.  \tag{4.20}\\
& \left.\left.+\sum_{i=2 N}^{3 N-2}(i+1-3((i+1) \bmod N)) \cdot F(j)^{i}\right]+2[f-(N+1)] \cdot F(j)^{2 N+1}[1-F(j)]\right] .
\end{align*}
$$

The proof of Property 4.29 is in Appendix III. The probability measure function of the 1-D multilevel filter with index $N$ and forcing level $N+1$ is:

$$
\begin{align*}
& P_{r}\left[G M E F_{N+1, N}(G(x)) \leq j\right] \\
= & F(j)+F(j)^{2} \cdot[1-F(j)] \cdot\left[\sum_{i=0}^{2 N-1}(i+1) \cdot[1-F(j)]^{i}\right.  \tag{4.21}\\
& \left.+\sum_{i=2 N}^{3 N-2}(i+1-3((i+1) \bmod N)) \cdot[1-F(j)]^{i}\right] .
\end{align*}
$$

It can be seen that a partial ordering exists between $P_{r}\left[G M E F_{f, \mathrm{~N}}(G(x)) \leq j\right]$ and $P_{r}\left[G M E F_{\mathrm{f}, \mathrm{N}+1}(G(x)) \leq j\right]$ which implies that:

$$
\begin{aligned}
P_{r}\left[G M E F_{2,1}(G(x)) \leq j\right] & \leq P_{r}\left[G M E F_{3,2}(G(x)) \leq j\right] \\
& \leq \ldots \\
& \leq P_{r}\left[G M E F_{N+1, N}(G(x)) \leq j\right] \\
& \leq P_{r}\left[G M E F_{N+2, N+1}(G(x)) \leq j\right]
\end{aligned}
$$

Props.4.28 and 4.29 reveal that as the length of the structuring element increases, the noise suppression of the filter increases as well. If the forcing level is different from $N+1$, an additional term $2[f-(N+1)] F(j)[1-F(j)]^{2 N+1}$ is included in the expression. The derivation of this term can be found in the proof of Prop.4.28 in Appendix III. Therefore, the noise suppression is even better if the forcing level is increased.

### 4.3.1.2 Two-Dimensional Statistical Analysis of Multilevel Multi-structuring Element Erosion Filter

A general expression for the 2-D multilevel filter is very complicated. Not only the values of the points, but also the geometrical structure should be taken into account. The
following two properties give the probability measure function of the multilevel erosion filters with index 1 .

Property 4.30 If $\left\{G M E F_{f, 1}(G(\boldsymbol{x})): \boldsymbol{x} \in \boldsymbol{Z}^{2}\right\}$ and $\left\{B M E F_{1}\left(S_{j}(G(\boldsymbol{x}))\right): \boldsymbol{x} \in \boldsymbol{Z}^{2}, 1 \leq j \leq k-1\right\}$, where $f=\{2,3\}$, are the $k$-level, and thresholded multistage filtered signals, then the probability measure function is :

$$
\begin{align*}
& P_{r}\left[G M E F_{2,1}(G(x)) \leq j\right] \\
= & P_{r}\left[B M E F_{1}\left(S_{j+1}(G(x))\right)=0\right]  \tag{4.23}\\
= & F(j)+F(j)^{8} \cdot[1-F(j)]\left\{1+8 \cdot F(j)^{2} \cdot[1-F(j)]+8 \cdot F(j)^{3} \cdot[1-F(j)]^{2}\right. \\
+ & \left.8 \cdot F(j)^{4} \cdot[1-F(j)]^{2}+8 \cdot F(j)^{4} \cdot[1-F(j)]^{3}\right\}
\end{align*}
$$

and

$$
\begin{equation*}
P_{r}\left[G M E F_{3,1}(G(x)) \leq j\right]=P_{r}\left[G M E F_{2,1}(G(x)) \leq j\right]+8 F(j)^{7} \cdot[1-F(j)]^{2} \cdot\left[1-F(j)^{3}\right] \tag{4.24}
\end{equation*}
$$

where $F(j)$ is the probability measure function of the i.i.d. input sequence.

It can been seen that by varying the forcing level from 2 to 3 , an additional term is added to the expression as seen in (4.24). Therefore, if the input noise distribution is unchanged, the noise suppression of the filter increases if the forcing level is increased. Similarly, the probability measure function of the dual to the 2-D $G M E F_{\mathrm{f}, 1}, f=\{2,3\}$ is given by Property 4.31 .

Property 4.31 If $\left\{G M E F_{f, 1}{ }^{d}(G(\boldsymbol{x})): \boldsymbol{x} \in \boldsymbol{Z}^{2}\right\}$ and $\left\{B M E F_{1}{ }^{\mathrm{d}}\left(S_{\mathrm{j}}(G(\boldsymbol{x}))\right): \boldsymbol{x} \in \boldsymbol{Z}^{2}, 1 \leq j \leq k-1\right\}, f=\{2,3\}$, are the $k$-level and thresholded binary filtered signals which are dual to $\left\{G M E F_{\mathrm{f}, 1}(G(x))\right\}$ and $\left\{B M E F_{1}{ }^{\mathrm{d}}\left(S_{\mathrm{j}}(G(\boldsymbol{x}))\right)\right\}$ respectively, then the probability measure function is :

$$
\begin{align*}
& P_{r}\left[G M E F_{2,1}{ }^{d}(G(x)) \leq j\right] \\
= & P_{r}\left[B M E F_{1}{ }^{d}\left(S_{j+1}(G(x))\right)=0\right] \\
= & 1-\left\{1-F(j)+F(j) \cdot[1-F(j)]^{8} \cdot\left[1+8 F(j) \cdot[1-F(j)]^{2}\right.\right.  \tag{4.25}\\
+ & \left.\left.8 F(j)^{2} \cdot[1-F(j)]^{3}+8 F(j)^{2} \cdot[1-F(j)]^{4}+8 F(j)^{3} \cdot[1-F(j)]^{4}\right]\right\}
\end{align*}
$$

and

$$
\begin{align*}
& P_{r}\left[G M E F_{3,1}{ }^{d}(G(x)) \leq j\right]  \tag{4.26}\\
= & P_{r}\left[G M E F_{2,1}{ }^{d}(G(x)) \leq j\right]-8 F(j)^{2} \cdot[1-F(j)]^{7} \cdot\left[1-(1-F(j))^{3}\right]
\end{align*}
$$

where $F(j)$ is the probability measure function of the i.i.d. input sequence.

Under the corruption of impulsive noise, the probability of getting an impulse at the filtered image is also helpful to quantify the performance of a filter. Let the Breakdown Probability be the probability of an impulse which is output by a multi-structuring element erosion filter, and $p$ be the occurrence probability of an impulse at the input image. It is assumed that the impulse is of unique polarity, either positive or negative but not both. The breakdown probability in homogeneous region is described. This model of impulse is used owing to the bias of the multilevel erosion filter. For example, a $G M E F_{f, \mathrm{~N}}$ filter cannot remove negative impulses.

Property 4.32 Let $p$ be the probability of an impulse occurring at the input, then the breakdown probabilities for the 2-D multi-structuring element erosion filter $P_{r}\left[G M E F_{\mathrm{f}, 1}(G(x))\right]$ and its dual $P_{r}\left[G M E F_{\mathrm{f}, 1}{ }^{\mathrm{d}}(G(x))\right], f=\{2,3\}$, are :

$$
\begin{align*}
& P_{r}\left[G M E F_{2,1}(G(x))\right] \\
&= \begin{cases}1-\left\{p+p^{8} \cdot[1-p]\left\{1+8 \cdot p^{2} \cdot[1-p]\right.\right. & \\
\left.\left.+8 \cdot p^{3} \cdot[1-p]^{2}+8 \cdot p^{4} \cdot[1-p]^{2}+8 \cdot p^{4} \cdot[1-p]^{3}\right\}\right\}, & \\
p, & \text { for positive impulse } \\
\text { for negative impulse }\end{cases} \tag{4.27}
\end{align*}
$$

and

$$
\begin{align*}
& P_{r}\left[G M E F_{3,1}(G(x))\right] \\
= & \begin{cases}P_{r}\left[G M E F_{2,1}(G(x))\right]-8 p^{7} \cdot[1-p]^{2} \cdot\left[1-p^{3}\right], & \text { for positive impulse } \\
p, & \text { for negative impulse }\end{cases} \tag{4.28}
\end{align*}
$$

The breakdown probabilities of the dual filters are:

$$
\begin{align*}
& P_{r}\left[G M E F_{2,1}^{d}(G(x))\right] \\
&= \begin{cases}p, & \text { for positive impulse } \\
1-\left\{p+p^{8} \cdot[1-p]\left\{1+8 \cdot p^{2} \cdot[1-p]+8 \cdot p^{3} \cdot[1-p]^{2}\right.\right. & \text { for negative impulse } \\
\left.\left.+8 \cdot p^{4} \cdot[1-p]^{2}+8 \cdot p^{4} \cdot[1-p]^{3}\right\}\right\}, & \end{cases}
\end{align*}
$$

and

$$
\begin{align*}
& P_{r}\left[G M E F_{3,1}{ }^{d}(G(x))\right] \\
= & \left\{\begin{array}{cc}
p, & \text { for positive impulse } \\
P_{r}\left[G M E F_{2,1}{ }^{d}(G(x))\right]-8 p^{2} \cdot[1-p]^{7} \cdot\left[1-(1-p)^{3}\right], & \text { for negative impulse }
\end{array}\right. \tag{4.30}
\end{align*}
$$

The ordering of probability measure functions as well as the breakdown probabilities is
true even the filters are 2-D. Hence,

$$
\begin{aligned}
& P_{r}\left[G M E F_{N+1, N}[G](x) \leq j\right] \leq P_{r}\left[G M E F_{N+2, N+1}[G](x) \leq j\right] \\
& \text { and } P_{r}\left[G M E F_{f, N}(G(x)) \leq j\right] \leq P_{r}\left[G M E F_{f+1, N}(G(x)) \leq j\right] \text {. }
\end{aligned}
$$

The probability measure function of the dual to multilevel erosion filter possesses a reverse partial ordering relation.

## Two-dimensional Statistical Analysis of $C G_{f, l}, f=\{2,3\}$

The probability measure function of $C G_{\mathrm{f}, 1}$ can be derived by replacing the input probability measure $F(j)$ function of $G M E F_{\mathrm{f}, \mathrm{N}}{ }^{d}$ by the probability measure function of $G M E F_{\mathrm{f}, \mathrm{N}}$.

Property 4.33 If $\left\{C G_{\mathrm{f}, 1}(G(x))\right\}$ is output of the 2-D $C G_{\mathrm{f}, 1}$ using an $k$-level input signal, then the probability measure function of the filter with a forcing level of 2 is:

$$
\begin{align*}
& P_{r}\left[C G_{2,1}(G(x)) \leq j\right] \\
= & 1-\left\{1-F^{\prime}(j)+F^{\prime}(j) \cdot\left[1-F^{\prime}(j)\right]^{8}\left\{1+8 F^{\prime}(j) \cdot\left[1-F^{\prime}(j)\right]^{2}\right.\right.  \tag{4.31}\\
& \left.\left.+8 F^{\prime}(j)^{2} \cdot\left[1-F^{\prime}(j)\right]^{3}+8 F^{\prime}(j)^{2} \cdot\left[1-F^{\prime}(j)\right]^{4}+8 F^{\prime}(j)^{3} \cdot\left[1-F^{\prime}(j)\right]^{4}\right\}\right\}
\end{align*}
$$

where $F^{\prime}(j)=P_{r}\left[G M E F_{2,1}(G(x)) \leq j\right]$. Similarly, the probability measure function the filter with a forcing level of 2 is:

$$
\begin{align*}
& P_{r}\left[C G_{2,1}(G(x)) \leq j\right] \\
= & 1-\left\{1-F^{\prime \prime}(j)+F^{\prime \prime}(j) \cdot\left[1-F^{\prime \prime}(j)\right]^{8} \cdot\left[1+8 F^{\prime \prime}(j) \cdot\left[1-F^{\prime \prime}(j)\right]^{2}\right.\right.  \tag{4.32}\\
& \left.\left.+8 F^{\prime \prime}(j)^{2} \cdot\left[1-F^{\prime \prime}(j)\right]^{3}+8 F^{\prime \prime}(j)^{2} \cdot\left[1-F^{\prime \prime}(j)\right]^{4}+8 F^{\prime \prime}(j)^{3} \cdot\left[1-F^{\prime \prime}(j)\right]^{4}\right]\right\} \\
& -8 F^{\prime \prime}(j) \cdot\left[1-F^{\prime \prime}(j)\right]^{7} \cdot\left[1-\left(1-F^{\prime \prime}(j)\right)^{3}\right]
\end{align*}
$$

where $F^{\prime \prime}(j)=P_{r}\left[G M E F_{3,1}(G(x)) \leq j\right]$.

The breakdown probability of $C G_{f, 1}$ can also be derived.

Property 4.34 Let $p$ be the probability of an impulse occurring at the input, then the breakdown probabilities for the 2-D multi-structuring element erosion filter $P_{r}\left[C G_{2,1}[G](x)\right]$ and $P_{r}\left[C G_{3,1}[G](x)\right]$ are :

$$
\begin{align*}
& P_{r}\left[C G_{2,1}(G(x))\right] \\
= & 1-\left\{p+p^{8} \cdot[1-p]\left\{1+8 \cdot p^{2} \cdot[1-p]+8 \cdot p^{3} \cdot[1-p]^{2}+8 \cdot p^{4} \cdot[1-p]^{2}+8 \cdot p^{4} \cdot[1-p]^{3}\right\}\right\} \tag{..}
\end{align*}
$$

and

$$
\begin{equation*}
P_{r}\left[C G_{3,1}(G(x))\right]=P_{r}\left[C G_{2,1}(G(x))\right]-8 p^{7} \cdot[1-p]^{2} \cdot\left[1-p^{3}\right] \tag{4.34}
\end{equation*}
$$

It can be observed that the breakdown probability of $C G_{f, 1}, f=\{2,3\}$ is independent of the polarity of the impulse. If the impulses are positive impulses, the dual filter has no effect to further suppress positive impulses of the filtered signal or vice versa.

### 4.3.2 Discussions on Statistical Properties

The multi-structuring element erosion filter is biased towards lower intensity regions of an image. On the contrary, its dual performs in the opposite manner and tends to preserve higher intensity details. The combination of the multi-structuring elements is almost symmetric about the median of the gray level. Comparisons of probability measure functions of other detail preserving filters are made. The multi-structuring element erosion filters are compared with the unidirectional multistage median filters with subfilter length 5 and 7 points and the bidirectional multistage median filters with subfilter filter length 3 and 5 points. In our simulation, a homogeneous signal is superimposed by noise of gaussian and uniform distributions, resulting in a multilevel signal of 200 levels. Figs.4.11 and 4.12 indicate the plots of probability density functions of these filters. The probability density function is the first derivative of the probability measure function. The degree of bias of a filter can be observed easier using its density function. If the filter is median unbiased, the density function is symmetrical about the median; otherwise, the density function is skewed. The unidirectional filter with subfilters of length $2 N+1$ is denoted as umed $_{\mathrm{N}}$. The bidirectional filter with subfilters of size $4 N+1$ is denoted as bmed $_{\mathrm{N}}$.

The input Gaussian distribution is with a mean of 100 and a standard deviation of 30 . Pronounced bias for $G M E F_{2,1}$ and its dual are observed. This is explained by the definition of the multi-structuring element erosion filter. Morphological erosion operates on the regions of higher intensity only. $C G_{2,1}$ shows a very little bias towards the lower intensity region. The $b m e d_{2}$ filter has the best Gaussian noise suppression. The bmed $_{1}$ also has good performance under Gaussian noise. The $C G_{2,1}$, umed $_{2}$ and umed $_{3}$ perform poorer than the bidirectional median filters. The noise attenuation by the multi-structuring element erosion filter is not as good as those of the multistage median filters, especially the bidirectional one. The


Figure 4.11 Plots of Probability Density Functions for Various Filters under Gaussian Distributed Input


Figure 4.12 Plots of Probability Density Functions of Various Filter under Uniform Distributed Input
bidirectional multistage median filter has higher rejection on Gaussian noise since the number of pixels used equals to $4 N+1$, which in unidirectional one and in multi-structuring element erosion filter $2 N+1$ are used. However, the noise attenuation of $C G_{2,1}$ and the unidirectional multistage filters umed $_{2}$ and umed ${ }_{3}$ are quite similar. Under uniformly distributed noise, the bidirectional median filter outperforms the others. The $u m e d_{3}$ has better performance than the $C G_{2,1}$ and umed ${ }_{2}$.

Since the multistage median filter is a median unbiased estimator. It follows that the expectations of the multistage filter are approximately equal to the expectation of the input if the standard deviation is not too large. The results in Table 4.2 seem to comply with this fact. When the standard deviation of the input distribution is 10 or 30 , the expectation is approximately equal to that of the input. This can also be observed in Fig.4.11 as the curve of probability measure functions of umed $_{2}$, umed $_{3}$, bmed $_{1}$ and bmed $_{2}$ are highly symmetric about the median. Expectations of $C G_{\mathrm{f}, 1}$, which are less that the median values of the input, are also complied with the plot on Fig.4.12.

| Filter | Gaussian | Distribution | Mean $=100$ | Uniform Distribution |
| :---: | :---: | :---: | :---: | :---: |
|  | S.D. $=10$ | S.D. $=30$ | S.D. $=50$ |  |
| Input | 100.0 | 100.0 | 95.6 | 100.5 |
| GMEF $_{2,1}$ | 98.9 | 96.7 | 98.4 | 95.9 |
| GMEF ${ }_{2,1}{ }^{\text {d }}$ | 100.1 | 102.4 | 108.0 | 104.2 |
| $C G_{2,1}$ | 99.0 | 99.0 | 102.5 | 99.5 |
| umed $_{2}$ | 100.0 | 100.0 | 103.1 | 100.5 |
| umed $_{3}$ | 100.5 | 101.4 | 105.0 | 100.5 |
| bmed $_{1}$ | 100.0 | 100.0 | 103.2 | 100.5 |
| bmed $_{2}$ | 100.6 | 101.7 | 105.5 | 104.2 |

Table 4.2 Expectations of Various Filters under Gaussian and Uniform Distribution

It can be observed that the $C G_{\mathrm{f}, \mathrm{N}}$ filters are not median-unbiased. This is owing to the bias of the multilevel erosion filter and its dual. The filter $C G_{f, \mathrm{~N}}$ therefore tends to preserve details of lower intensities.

In the analysis of breakdown probabilities, impulses of unique polarity is assumed. The impulse is either salt (positive impulses) or pepper (negative impulses). Mixture of salt and pepper is not used in the simulation, since the multi-structuring element erosion filters are highly biased. Moreover, as the multistage median filters are unbiased estimators, any change in polarity of impulse causes no effect on the breakdown probability. Tables 4.3 and 4.4 summarize the breakdown probabilities of $G M E F_{2,1}, G M E F_{2,1}{ }^{\text {d }}, C G_{2,1}$, umed $_{2}$, umed ${ }_{3}$, bmed $_{1}$ and bmed $_{2}$.

| $p$ | $G M E F_{2,1}$ | $G M E F_{2,1}{ }^{\text {d }}$ | $G M E F_{3,1}$ | $G M E F_{3,1}{ }^{\text {d }}$ | $C G_{2,1}$ | $C G_{3,1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.063 | 0.063 | 0.007 | 0.063 | 0.003 | 0.007 | 0.003 |
|  | 0.007 | 0.063 | 0.003 | 0.063 |  |  |
|  | 0.125 | 0.042 | 0.125 | 0.026 | 0.042 | 0.026 |
|  | 0.042 | 0.125 | 0.026 | 0.125 |  |  |
| 0.250 | 0.250 | 0.189 | 0.250 | 0.148 | 0.189 | 0.148 |
|  | 0.189 | 0.250 | 0.148 | 0.250 |  |  |
| 0.375 | 0.375 | 0.352 | 0.375 | 0.320 | 0.352 | 0.320 |
|  | 0.352 | 0.375 | 0.320 | 0.375 |  |  |
| 0.500 | 0.500 | 0.495 | 0.500 | 0.482 | 0.495 | 0.482 |
|  | 0.495 | 0.500 | 0.482 | 0.500 |  |  |

Table 4.3 Breakdown Probabilities for some Multi-structuring Element Erosion Filters

| Probability, $p$ | umed $_{2}$ | umed $_{3}$ | bmed $_{1}$ | bmed $_{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.063 | 0.005 | 0.001 | 0.003 | 0.0001 |
| 0.125 | 0.035 | 0.014 | 0.019 | 0.003 |
| 0.250 | 0.176 | 0.131 | 0.116 | 0.054 |
| 0.375 | 0.348 | 0.328 | 0.288 | 0.227 |
| 0.500 | 0.500 | 0.500 | 0.500 | 0.500 |

Table 4.4 Breakdown Probabilities for Multistage Median Filters

Again the bidirectional multistage median filters, $b m e d_{1}$ and $b m e d_{2}$, have the Iowest values of breakdown probabilities. Those of multi-structuring element erosion filters and their duals are highly dependent on the polarity of the impulse. If positive impulse is filtered by $G M E F_{\mathrm{f}, 1}{ }^{\text {d }}$, all impulses will be preserved. The breakdown probability of $C G_{3,1}$ takes the minimal values among those of $G M E F_{3,1}$ and its dual, which lies between those of umed $_{2}$ and umed $_{3}$ when $p \leq 0.250$. Recalling that the minimal preservable details of $G M E F_{3,1}$ are of 3 or 4 pixels.

### 4.4 Chapter Summary

Both the root signal structures as well as the rate of convergence of the erosion filter, binary and multilevel, are treated. The root signals of the 1-D binary erosion filters are accounted by Props.4.1 and 4.2. The filter tends to preserve negative details of any length. Positive details are preserved unless at least $3 N$ consecutive pixels of identical values are present. The analysis of 2-D root signal structures is more difficult. We started at the root structures of single point wide open curves, followed by single point wide closed curves, and rectangular details and finally irregular details. 2-D binary root signal structures are given by Props.4.3 to 4.10. The size of the minimal preservable rectangular details is summarized in Table 4.1. By extending the binary root signal structures, those of the multilevel erosion filters are examined. It can be shown that the multilevel erosion filter is edge preserving, and can preserve monotonic region, provided that Props.4.11 to 4.14 are satisfied. In the last section of root signal analysis, the $C G_{\mathrm{f}, \mathrm{N}}$ is analyzed. As the $C G_{\mathrm{f}, \mathrm{N}}$ is a two-pass operation, vibrating structures are resulted.

Convergent rate analysis is another aspect in the deterministic properties of the filter. It is proved that the binary filter, both 1-D and 2-D, has a constant convergent rate. At most two passes are demanded for transforming any nonroot signal to its root. For multilevel erosion filter, with forcing level $f$, the upper bound for the convergent rate is given by Prop.4.25. The maximum number of passes for the convergence of the roots of a multilevel with forcing $f-1$ by the filter with forcing level $f$ is also derived.

A general formula for the probability measure function is obtained for the 1-D multistructuring element erosion filters, while that for the 2-D one is too complicated to be
described. In the latter case, it is still possible to have the expression for $N=1$ and different forcing levels.

Multi-structuring element erosion filter is not a median unbiased estimator as most of the median based nonlinear filters do. This is not desirable as the filter will tend to preserve details of either lower or higher intensity. The sequential application $C G_{f, 1}$ help to alleviate this problem by reduce this bias. A strictly symmetrical probability density function is still not obtainable, although simulation results show that the density function is quite symmetric. In other words, the sequential filter $C G_{\mathrm{f}, \mathrm{N}}$ is not median-unbiased.

Comparisons have been made between another detail preserving nonlinear filter family, the multistage median filters, for rating the noise attenuation of the erosion filters. Since we are only able to obtain the probability measure function of 2-D multi-structuring element erosion filter for $N=1$. The unidirectional multistage median filters with subfilters those length are 5 and 7, and the bidirectional one with length 3 and are compared. Simulations using Gaussian, uniform and uni-polarity impulse are done. It is revealed that the multi-structuring element erosion filter is of comparable performance to the unidirectional multistage median filter, but is less powerful that the bidirectional one.

## Chapter 5 Performance Evaluation

### 5.1 Introduction

Robust estimation theory has served as an excellent tool for nonlinear filtering. The best known and the most widely used order statistic filter is the median filter which is from estimation theory. But the performance of the median filter in 2-D case is not as good as the 1-D case, derivations of median filters are implemented which include weighted median filters, FIR-hybrid median filters and multistage median filters. Although analysis of these filters are available, these analyses are based on a simpler model in which the noise is added on a homogeneous signal. However, real signals do not often follow this model. Therefore, performance comparison and evaluation using real images must be done.

In some literature, performance of a new filter is compared with the moving average filter and the standard 2-D median filter. Nevertheless, such comparison gives limited information about the performance of the new filter. This is because the moving average filter and the median filter are notorious for their edge blurring and detail removal properties. For a new nonlinear image filter, it should be taken for granted that the filter must outperform these filters. Moreover, the usefulness of a filter should be compared to those of better performance. This provides a selection guide for those who want to apply a nonlinear filter for some specific applications. In fact, there is no image filter which outperforms all other filters in all aspects. For example, the L-filter ${ }^{1}$ has the highest suppression in short-tailed additive white noise, however, it performs poorly in salt-and-pepper noise. As a result, different applications require different nonlinear filters.

In this chapter, the performance of the multilevel erosion filter is compared against the multistage median filter. The multistage median filter, according to Pitas and Venetsanopoulous, has the best detail and edge preserving property. Also, the filter is computationally simple and is effective in suppressing salt-and-pepper noise.

[^2]
### 5.2 Performance Criteria

There is no universally accepted, or standard criteria for measuring the quality and grading the performance of an image processing system. Usually the performance of a filter is evaluated using the mean-square-error (mse), the mean-absolute-error (mae), a subjective visual criteria and the computation requirement. The last criterion first considers the numbers of comparison, addition and multiplication needed per output pixel. In addition, the number of passes required to bring the input images to an invariant image is accounted as well. The subjective criterion usually grades a filtered image on the distortion resulted by the filter with the original image. The mean-square-error and the mean-absolute-error are defined as:

$$
\begin{equation*}
m s e=\frac{\sum_{x_{1}=0}^{M-1} \sum_{x_{1}=0}^{N-1}[G(\boldsymbol{x})-\hat{G}(\boldsymbol{x})]^{2}}{\sum_{x_{1}=0}^{M-1} \sum_{x_{2}=0}^{N-1}[G(\boldsymbol{x})]^{2}} \tag{5.1}
\end{equation*}
$$

and

$$
\begin{equation*}
m a e=\frac{\sum_{x_{1}=0}^{M-1} \sum_{x_{1}=0}^{N-1}|G(\boldsymbol{x})-\hat{G}(\boldsymbol{x})|}{\sum_{x_{1}=0}^{M-1} \sum_{x_{2}=0}^{N-1} G(\boldsymbol{x})} \tag{5.2}
\end{equation*}
$$

where $G(x)$ and $\hat{G}(\boldsymbol{x}), \boldsymbol{x}=\left(x_{1}, x_{2}\right) \in \boldsymbol{Z}^{2}$ are the original and the filtered images.

In our simulation, grayscale images of size 512 by 512 and 256 gray levels are used, i.e. $M=N=512$. Fig. 5.1 shows the original test images used. The images are Lenna, Baboon, Peppers and Bridge. These are quite frequently used as test images.

Comparisons of the multi-structuring element erosion filter are made against the multistage median filters. Pitas and Venetsanopoulous has already compared the performance of various nonlinear filters in their work [Pitas92].

Among those filter described in [Pitas92], the multistage median filter has excellent performance on following four aspects:

1. Good noise attenuation on positive and negative impulse (salt-and-pepper noise)
2. Detail preserving
3. Edge Preserving
4. Computation Complexity.

Moreover, the minimal size of details that can be preserved by the filter chosen for comparison should be equal. For example, if $C G_{2,1}$ is to be compared, the filter used in the comparison should be able to preserve details of length at least 3 points. The performance of $C G_{2,1}, C G_{3,1}$ and $C G_{3,2}$ will be evaluated. Therefore, the unidirectional multistage median filters with subfilters of lengths 5 and 7 , denoted as $u m e d_{2}$ and $u m e d_{3}$, and the bidirectional multistage median filters with subfilters of lengths 3 and 5 , denoted as bmed $_{1}$ and bmed $_{2}$ respectively, are compared.

In the noise suppression comparison, an input image will be filtered repeatedly until the root image is produced. A visual comparison on the number of noise suppression will be given. The distortion of the root image is compared against the original noise free image. The mean-square-error and mean-absolute-error are used to quantify the efficiencies of noise attenuation as well as the image distortion. Lower the mean-square-error and mean-absoluteerror, lower the distortion resulted by the filter during noise suppression. Subjective visual test is used in concluding the goodness of preservation of details and edges.


Figure 5.1 Test Images Used (a)Lenna (b)Baboon (c)Pepper (d)Bridge

### 5.2.1 Noise Suppression

In this test, images which are corrupted by three kinds of noise will be filtered. Salt and pepper noise, impulsive noise which is of uniformly distributed magnitudes and Gaussian white noise are added to the original images. The invariant signal will be compared with the original one. In salt-and-pepper test, equal number of positive and negative impulses are added. Fig. 5.2 shows the original image-Peppers, the corrupted Peppers and the filtered images by various filters. Fig.5.2(a) shows the input image in which Peppers is corrupted by 7500 salt and pepper which corresponds to a probability of occurrence of impulses of 0.06.


(e) umed ${ }_{2}$

(e) $\mathrm{bmed}_{1}$

(e) umed $_{3}$

(e)bmed 2

Figure 5.2 (a)Peppers corrupted by 15000 Salt-and-pepper, and the Filtered Images by (b)CG $G_{2,1}$ (c)CG $G_{3,1}$ (d) CG $_{3,2}$ (e) umed $_{2}(f)$ umed $_{3}(\mathrm{~g})$ bmed $_{1}(\mathrm{~h})$ bmed $_{2}$

Six different input images corrupted by different numbers of salt-and-pepper noise are used for each tested images.The numbers of salt and pepper are $0,5000,10000,15000$, 20000, 30000, 40000 and 50000 which correspond to probabilities of occurrence of impulses of $0.02,0.04,0.06,0.08,0.11,0.15$ and 0.19 . Among all the filtered images, the $b_{m e d_{2}}$ and the $C G_{3,2}$ have the best salt-and-pepper noise suppression, since most of the impulses are removed. The results of the $C G_{3,1}$, the bmed $_{1}$ and the umed $_{3}$ have several clusters of impulses still remaining. The umed $_{2}$ performs poorly. However, $C G_{2,1}$ has the worst performance. This
complies with the breakdown probability calculations in Chapter 4. The probability of occurrence of noise is 0.06 . According to Table 4.1, we can arrange the filters in ascending order of breakdown probabilities, i.e. bmed $_{2}, C G_{3,2}$, umed $_{3}, C G_{3,1}$, bmed $_{1}$, umed ${ }_{5}$ and $C G_{2,1}$. By considering the number of impulses left after filtering, the sequence holds. The following Figs. show the filtered images of $C G_{2,1}, C G_{3,1}, C G_{3,2}$, umed $_{2}$, umed $_{3}$, bmed $_{1}$ and bmed $_{2}$. Figs.5.3 and 5.4 are the outputs of the above mentioned filters under impulsive noise those magnitudes are uniformly distributed and Gaussian noise respectively. All outputs are root to the corresponding filters. Fig.5.3(a) is a Lenna corrupted by 15000 impulsive noise. Fig.5.4 is a Lenna corrupted by Gaussian noise with an overall signal to noise ratio of 15 dB .


(e)Filtered by umed ${ }_{2}$

(e)Filtered by bmed ${ }_{I}$
(e)Filtered by umed $_{3}$

(h) Filtered by bmed ${ }_{2}$

Figure 5.3 (a)Lenna corrupted by 15000 Impulsive Noise those Magnitudes are uniformly distributed, and the Filtered Images by $(b) C G_{2, I}(c) C G_{3,1}(d) C G_{3,2}(e)$ umed $_{2}(f)$ umed $_{3}(g)$ bmed $_{I}(h)$ bmed $_{2}$

Fig.5.4 exhibits the filtered images by these filters. The input image is a Lenna added with Gaussian noise. The signal to noise ratio of the input image is 15 dB .

(a)Lenna Corrupted by Gaussian Noise

(c) Filtered by $C G_{3,1}$

(b)Filtered by $C G_{2,1}$

(d)Filtered by $\mathrm{CG}_{3,2}$

(e)Filtered by umed ${ }_{2}$

(g)Filtered by bmed ${ }_{1}$

(f)Filtered by umed $_{3}$

(h)Filtered by bmed ${ }_{2}$

Figure 5.4 (a)Lenna corrupted by Gaussian Noise ( $S / N: 15 d B$ ), and the Filtered Images by (b)CG $\boldsymbol{q}_{2,1}$ (c) $C G_{3,1}(d) C G_{3,2}$ (e) umed ${ }_{2}$ (f)umed ${ }_{3}(g)$ bmed $_{1}(h)$ bmed $_{2}$

Another type of impulses is defined by Li [ Li 90 ]. According to Li , a small isolated feature which is visually different from its surroundings can be classified as an impulsive noise. In the following discussion, such impulsive noises are modelled by impulses those magnitudes are uniformly distributed. Images which are corrupted by this type of impulsive noise are tested. Again, the numbers of impulses are $0,5000,10000,15000,20000,30000$ and 40000. Gaussian white noises are also added to Lenna, the resulting input images are of $5,10,15,20$ and 25 dB in signal-to-noise-ratios.

Although the bidirectional median filters have the best noise suppression, too much signal distortion is resulted. This can be accounted by the mean-square-error and the mean-absolute-error. Fig. 5.5 shows the mean-square-error of the filtered images under salt-andpepper noise. All test images are used. Figs.5.5(a), 5.5(b), 5.5(c) and 5.5(d) show the mean-square-error's of the filtered images by the above mentioned filters. Each graph gives the mean-square-error's of the filtered images under different noise corruptions. The x -axis gives the probabilities of occurrence of impulses. The $y$-axis is the logarithms of the mean-squareerror's. The mean-square-error can be regarded as an amplification of error by squaring the difference between the images. Under the salt and pepper noisy environment, impulses are either of the greatest and the lowest magnitudes. Therefore, the higher is the mean-squareerror, the higher is the image distortion. It can be observed that one point is missing in all mean-square-error plots. The point corresponds to a zero mean-square-error when the original image is tested. As the probability of impulses increases, the performance of $C G_{3,1}$ becomes the best. At a noisier environment, the bidirectional multistage median filter is the best. Referring to the figures, $C G_{3,1}$ has the best performance in most conditions. The mean-squareerror of $C G_{3,1}$ always ranks among the lowest two mean-square-errors. Although $C G_{2,1}$ and $u^{u m e d}{ }_{5}$ are good at lower noise environment, its performance with respect to the mean-squareerror criterion deteriorates rapidly as the number of impulses increases.

When the total number of impulses is 15000 , the $C G_{3,1}$ has the lowest mean-squareerror. This implies that the filtered images by the $C G_{3,1}$ gives the least distortion.

Fig.5.6 shows the mean-absolute-error of the filtered images. The mean-absolute-error accounts for the absolute difference between the filtered images and the original one. If a constant signal is embedded with noise, the mean-absolute-error indicates the average of
absolute gray level shift per pixel. All the data within the image are treated with unity weight. This is different from the mean-square-error, by which the weight of the difference is proportional to the difference itself. Hence, the mean-absolute-error does not amplify the difference. In particular, the mean-absolute-error will give the shift of gray level resulted by the noise. Therefore, the lower is the mean-absolute-error, the lesser is the gray level shift, and the higher is the image fidelity.


Figure 5.5 Mean-square-error for Various Nonlinear Filters (a)Lenna (b)Baboon (c)Peppers (d)Bridge


Figure 5.6 Mean-absolute-error of Filtered Images (a)Lenna (b)Baboon (c)Pepper (d)Bridge

Similar to the case of the mean-square-error, $C G_{2,1}$, performs best in reducing the mean-absolute-error when the noise probabilities is low enough. The difference between the mean-absolute-error of the $C G_{2,1}$ and the umed $_{2}$ is distinguishable. Results reveal that $C G_{2,1}$ maintains the least mean-absolute-error if the number of impulses is well below 10000. In other words, the probability of impulses occurrence $p$ is equal to or below 0.04 . As the number of impulses increases, $C G_{3,1}, C G_{3,2}$, umed $_{2}$ and bmed $_{1}$ produce error of comparable values. However, on the whole, $C G_{3.1}$ seems to have better performance among these filters. Table 5.1 summaries the performance of rejecting salt and pepper noise with different probabilities.

| Probability, $p$ | Filter |
| :---: | :---: |
| $\geq 0.04$ | $C G_{2,1}$, umed $_{2}$ |
| $0.04 \leq p \leq 0.06$ | umed $_{2}, C G_{3,1}$ |
| $0.06 \leq p \leq 0.20$ | $C G_{3,1}{ }^{*}, C G_{3,2}$, umed $_{3}$, bmed $_{1}$ |

Table 5.1 Selection Guide of Nonlinear Filters under Salt and Pepper Noise

Impulsive noise those magnitudes are uniformly distributed are added to the test image Lenna. The numbers of impulsive noise contaminations are $0,5000,10000,15000,20000$, 30000 and 40000 . These correspond to probabilities of occurrence of impulses of $0.02,0.04$, $0.06,0.08,0.11,0.15$ and 0.19 . Figs.5.7(a) and 5.7(b) show the mean-square-error and the mean-absolute-error of the root images respectively. Under the mean-square-error criterion, the $C G_{2,1}$ and the $C G_{3,1}$ outperforms other filters. The umed ${ }_{2}$ performs as good as the $C G_{2,1}$ under low noise environment. As the environment becomes more noisy ( $\geq 15000$ ), the performance of $C G_{3,1}$ becomes the best. The $C G_{2,1}$ has the lowest mean-absolute-error under all test conditions as shown in Fig.5.7(b) and is followed by the umed ${ }_{2}$. The $C G_{3,1}$ exhibits poorer performance under the mean-absolute-error criterion. Both the bidirectional median filters have the worst performance in these criteria.

Gaussian noise are added to Lenna. The resulted images are of different signal-to-noise-ratios ranging from 5 dB to 25 dB . Figs.5.8(a) and 5.8(b) show the mean-square-error and the mean-absolute-error of the filtered images. Under both criteria, the multilevel multi-
structuring element erosion filters $C G_{2,1}, C G_{3,1}$ and $C G_{3,2}$ and the unidirectional median filters, $u^{u m e d} d_{2}$, and umed ${ }_{3}$, perform poorly. However, the bidirectional median filters, bmed ${ }_{1}$ and bmed $_{2}$, have the best performance because the bidirectional median filters consider more points that other filters.

### 5.2.2 Subjective Criterion

A filter with high noise rejection power may not be good at preserving edges and details. Fig.5.9 displays enlarged pictures at the left eye of lenna. The probability of occurrence of impulses is 0.08 which is corrupted by 10000 salt and 10000 pepper approximately. The eyes are taken from the filtered images of this corrupted Lenna. Fig.5.9(a) is the enlarged left eye of the original Lenna. Figs.5.9(b), 5.9(c) and 5.9(d) are the root images to $C G_{2,1}, C G_{3,1}$ and $C G_{3,2}$ respectively. The output of the unidirectional median filters, umed $_{2}$ and umed $_{3}$ are given in Figs.5.9(e) and 5.9(d). Those of the bidirectional filters bmed $d_{1}$ and bmed $_{2}$ are shown in Figs. $5.9(\mathrm{~g})$ and $5.9(\mathrm{~h})$ respectively. Each square block in the enlarged view represents a pixel on the filtered image. Let us first describe the fine details in the eye. The eyeball of Lenna contains 3 distinct gray levels which clearly cut the eye into different regions. A darker ring separates the iris from the eyeball. Inside the iris is the pupil which is represented by a darker circle. Surrounding the eyeball is the eyelashes. Separate eyelash can be observed clearly. The eyebrow is composed of two almost homogeneous regions of gray levels. Shadows on the hat are also important details to be preserved.
$C G_{2,1}$ performs poorly since much impulses remain after repeated filtering. $C G_{3,1}$ has very good noise rejection, except that there are two patches of impulses, one is at the rib of Lenna's hat and the other is at the right edge of the enlarged view. Details in the eyeball as well as the shadow on the hat can still be observed clearly. Eyelashes are preserved although little blurring is resulted. $C G_{3,2}$ removes all the impulses. umed $d_{2}$ cannot remove all the impulses, and some impulses are converted to values which seem to be visually uncorrelated with their neighbourhoods. In other words, some salt-and-pepper impulses are converted to another type of impulses. These pixels are distinguishable and are regarded as artifacts caused by the filter. These are not desirable as they are visually uncomfortable. The result of umed ${ }_{3}$ is similar but higher noise rejection is attained. The bidirectional multistage median filters


5.17


Figure 5.7 Error of Filtered Images by Different Filters Under Impulsive noise with Uniform Distributed Magnitudes (a)mse (b)mae
have very pronounced noise rejections, however, most details are destroyed. The blurring effect is more significant in bmed $_{2}$. Therefore, among these filter, $C G_{3,1}$ has better detail preservation properties.

Another important characteristics of the nonlinear image filter is the preservation of edges. Referring to Fig. 5.9 again, the rib of Lenna's hat is an example of a sharp edge. Jittering of edges are resulted in $C G_{2,1}, C G_{3,2}$, umed $_{2}$ and umed ${ }_{3}$. Pixels of perceptually different gray levels are located along the edge. Hence, a smooth edge is changed into a zig zag shape. $C G_{3,1}$, bmed $_{1}$ and $b m e d_{2}$ have relatively little jittering effects. However, the bidirectional filters tend to average the gray levels, which result in reduction of image contrast. In conclusion, $C G_{3,1}$ outperforms other filters in the preservation of edges.

Artifacts are also resulted by the multilevel erosion filters if impulses appear at an edge or on the boundary of an object. In Fig.5.9(d), it can be seen that all spatially uncorrelated output pixels are found in those locations where a matching of structuring element is found. However, the same explanation cannot be applied to those of the unidirectional median filters. Since these filters have not accounted for the geometrical structures of a signal, isolated visually uncorrelated pixels are found. As the length of the subfilter of the unidirectional median filter increases, more artifacts are produced.

(a)

(c)

(e)

(g)

(b)

(d)

(f)

(h)

Figure 5.9 Enlarged View of Lenna Eyes for Comparisons of Edge and Detail Preservation (a)Original


### 5.2.3 Computational Requirement

## Computation Complexity

Computational complexity is defined as the numbers of additions, multiplications and comparisons required per pixels of an image. The lesser are the numbers of these operations, the more efficient is in computing the filter. The definition of the multilevel erosion filter is already given in Chapter 3. We consider the simplest multilevel erosion filter, the standard multilevel erosion filter. For each pixel, the values within the $(2 N+1)$ by $(2 N+1)$ window is sorted. Thresholding begins from the $(N+1)^{\text {th }}$ threshold level of the sorted set. The determination of the output at a point requires information contained in a $(4 N+1)$ by $(4 N+1)$ window centred at the point. This should not be confused with the $(2 N+1)$ by $(2 N+1)$ window in which the threshold levels are taken. During the thresholding of the $(4 N+1)$ by $(4 N+1)$ window, $(4 N+1)^{2}$ comparisons are required. If a quick sorting algorithm is used, the number of comparisons needed can be summarized as follows.

| Number of Comparisons Required |  |  |
| :---: | :---: | :---: |
| Operations | Average | Worst |
| Sorting Threshold Levels | $O\left((2 N+1)^{2} \log _{2}(2 N+1)^{2}\right)$ | $(2 N+1)^{4} / 2$ |
| Thresholding | $(4 N+1)^{2}$ | $(4 N+1)^{2}$ |

Table 5.2 Number of Comparisons Required in the Formation of Threshold Sets in Multilevel Erosion Filtering

In a selectively thresholding of the image data masked by a $(4 N+1)$ by $(4 N+1)$ window, suppose that each point is compared independently. A binary comparison is used to compare a bit at a point. If the output can be determined by considering the first structuring element only, then the number of binary comparison is $2 N+1$. For the worst case, if the output is determined until the last structuring element, then the number of binary comparison is $3(2 N+1)\left(7.2^{2 \mathrm{~N}}-2^{\mathrm{N}+3}\right)$. An average number of comparisons is $(2 N+1)\left(3\left(7.2^{2 \mathrm{~N}}-2^{\mathrm{N}+3}\right)-1\right) / 2$. The numbers of binary comparisons are given in Table 5.3.

| Binary Comparisons Required For Binary Erosion Filtering |  |  |
| :---: | :---: | :---: |
| Best | Average | Worst |
| $2 N+1$ | $\left.3(2 N+1)\left(\left(7.2^{2 \mathrm{~N}}-2^{\mathrm{N}+3}\right)-1\right)\right) / 2$ | $3(2 N+1)\left(7.2^{2 \mathrm{~N}}-2^{\mathrm{N}+3}\right)$ |

Table 5.3 Number of Binary Comparisons Required for Binary Erosion Filtering
Suppose that the output the multilevel erosion filter is determined at the $j^{\text {th }}$ level, then the average number of binary comparisons required is $2[j-(N+1)]\left[3\left(7.2^{2 N}-\right.\right.$ $\left.\left.2^{N+3}\right)(2 N+1)\right]+(2 N+1)\left[3\left(7.2^{2 N}-2^{N+3}\right)-1\right) / 2$. In addition, the average number of integer comparisons is $(2 N+1)^{4} / 2+2 j(4 N+1)^{2}$.

Table 5.4 shows the comparisons required by the subfilters of the unidirectional multistage median filter and the bidirectional median filter with index $N$. There are four subfilters in the unidirectional median filter, and two for the bidirectional one. The output of the multistage median filter is the median of the outputs of the subfilter. Therefore, the number of comparisons needed for finding the median of the medians is negligible.

| ${\text { Unidirectional Median Filter, } \text { umed }_{\mathrm{N}}}^{2}$ Bidirectional Median Filter, bmed $_{\mathrm{N}}$ |  |  |  |
| :---: | :---: | :---: | :---: |
| Average | Worst | Average | Worst |
| $O\left((2 N+1) \log _{2}(2 N+1)\right)$ | $2(2 N+1)^{2}$ | $O\left((4 N+1) \log _{2}(4 N+1)\right)$ | $(4 N+1)^{2}$ |

Table 5.4 Number of Integer Comparisons Required by Multistage Median Filters

It can be seen that the multilevel erosion filter is more computational intensive than the multistage median filters. The number of integer comparisons required by the multilevel erosion filter is $O\left(N^{2} \log _{2} N^{2}\right)$. However, the multistage median filter is $O\left(\log _{2} N\right)$.

Table 5.5 gives the time of computation for these filters. The computer used is a DEC5240 workstation. The time is input dependent, however, an average is listed.

|  | $C G_{2,1}$ | $C G_{3,1}$ | $C G_{3,2}$ | umed $_{2}$ | umed $_{3}$ | bmed $_{1}$ | bmed $_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Time(s) | 120 | 120 | 650 | 15 | 60 | 60 | 200 |

Table 5.5 Computation Time Required for Different Filters(One Pass)

Owing to the large number of comparisons required by the multilevel erosion filters, the time for each filter pass is much more that the multistage filters. Therefore, we can conclude that the multilevel erosion filters have the worst performance in computation complexity.

## Rate of Convergence in Real Images

The theoretical bound for the rate of convergence of the multilevel erosion filter has been analysed in Chapter 4. The rates of convergence using different input images are given in the following tables.

| Impulses | $C G_{2,1}$ | $C G_{3,1}$ | $C G_{3.2}$ | umed $_{2}$ | umed $_{3}$ | bmed $_{1}$ | bmed $_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 6 | 35 | 8 | 9 | 11 | 34 | 47 |
| 0.02 | 6 | 22 | 7 | 8 | 11 | 30 | 38 |
| 0.04 | 6 | 22 | 8 | 9 | 12 | 30 | 44 |
| 0.06 | 6 | 22 | 8 | 9 | 11 | 30 | 35 |
| 0.08 | 5 | 22 | 8 | 9 | 10 | 30 | 47 |
| 0.11 | 5 | 21 | 9 | 9 | 13 | 28 | 34 |
| 0.15 | 5 | 22 | 6 | 8 | 13 | 35 | 31 |
| 0.19 | 5 | 21 | 7 | 9 | 13 | 27 | 30 |

Table 5.6 Convergent rate of Lenna Based Test Images

| Impulses | $C G_{2.1}$ | $C G_{3.1}$ | $C G_{3.2}$ | umed $_{2}$ | umed $_{3}$ | bmed $_{1}$ | bmed $_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 7 | 32 | 9 | 8 | 12 | 50 | 28 |
| 0.02 | 7 | 34 | 9 | 8 | 14 | 50 | 29 |
| 0.04 | 6 | 34 | 9 | 8 | 11 | 35 | 30 |
| 0.06 | 6 | 34 | 9 | 8 | 13 | 36 | 28 |
| 0.08 | 6 | 34 | 9 | 7 | 15 | 39 | 28 |
| 0.11 | 6 | 38 | 9 | 10 | 15 | 25 | 30 |
| 0.15 | 6 | 30 | 8 | 9 | 12 | 25 | 28 |
| 0.19 | 7 | 33 | 10 | 11 | 11 | 24 | 26 |

Table 5.7 Convergent rate of Baboon Based Test Images

The $C G_{2,1}$ has the least number of passes for the conversion of root signal. The multilevel images usually required no more than 7 filter passes for convergence. The $C G_{3,2}$ also required very few passes for convergence. However, the bidirectional median filters and the $C G_{3,1}$ perform poorly since more than 20 passes are needed for convergence. The
unidirectional median filters, umed $_{2}$ and umed $_{3}$, have the moderate rates of convergence among the filters tested.

Although the standard multilevel erosion filter requires the least number of passes, the overall performance is still worse than other filters used in the comparison. This is because it takes too much computation time for each erosion filter pass. The unidirectional filters have the best computation performance.

### 5.3 Chapter Summary

Comparisons have been made to evaluate the performance of the multi-structuring element erosion filters against the multistage median filters. Performance criteria are based on the mean-square-error, the mean-absolute-error, a subjective visual criteria and the computation requirement. Real images such as Lenna, Baboon, Peppers and Bridges are used for testing. A table is constructed for the selection of filter which suit for some range of impulses occurrence. In fact, the mean-square-error and the mean-absolute-error among these detail preserving filter are very close to each other. For Gaussian noise, the multi-structuring element erosion filter is not as suitable as the multistage median filter, since Chapter 4 has already shows that the bidirectional filter is the best in handling Gaussian noise.

In the preservation of details as well as edges, the multi-structuring element erosion filter has better performance. This may be due to the geometric consideration of the filter. However, the median is a point estimator which does not take into account the geometric relation of the pixels.

Unfortunately, in the computation aspect, the unidirectional multistage median filter definitely outperforms our new filter with forcing level other than. $(N+1)$. the number of filter passes required to convert the input image to its roots is far less for the former filter. In conclusion, the multi-structuring element erosion filter can be regarded as a good detail preservation nonlinear filter.

## Chapter 6 Recapitulation and Suggestions for Further Work

### 6.1 Recapitulation

The objective of the work is to develop a morphological image filter which can suppress salt-and-pepper noise effectively and efficiently, while preserves details and edges. At present, most of the best detail preserving nonlinear filters are median based filter, or more precisely, robust estimator based. Examples of these detail preserving filters include the multistage median filters, the multistage max/median filters, the finite impulse response median hybrid filters and the adaptive L-filters (linear combinations of order statistic). Although the motive for the development of mathematical morphology was to describe the geometrical structure of a signal, currently all morphological filters are not detail preserving. To achieve this goal, a filter called multi-structuring element erosion filter is designed. In Chapter 1, the background for the need of nonlinear filtering is examined. Nonlinear filters are motivated by the deficiencies of linear filters. Different families of nonlinear filters are briefly discussed.

An overview of nonlinear filtering techniques which are based on robust estimation theory and mathematical morphology are presented in Chapter 2. In robust estimator based filters, most of the discussions are on the median filter and its deviations as median filtering is the most frequently used nonlinear filtering technique. Emphasis are put on the definition and performance of some median based detail preserving filters. From which we are able to conclude that geometric structures of a signal must be considered in order to preserve details. This leads us to an alternative in nonlinear filtering techniques. An introduction on mathematical morphology is given. The definitions as well as the applications in nonlinear filtering are presented. Moreover, it is discovered that the morphological filters based on the opening and closing transformations are not detail preserving. This gives rise to the idea of the multi-structuring element erosion filter.

The definition of the multi-structuring element erosion filter is given in Chapter 3. The filter is composed of many subfilters. Each subfilter is a morphological erosion of a binary signal with a specific structuring element. The overall output of the filter is the union of all those of its subfilters. The chapter first begins with the definition of the structuring elements
used in the filter. A signal pattern is a structuring element in the filter if T. 1 to T. 4 of Def.3.1 are all satisfied. The total number of structuring elements used by the filter is $3\left(7.2^{2 \mathrm{~N}}-2^{\mathrm{N}+3}\right)$, where $N$ is the index of the filter. If the index of the filter is $N$, the length of all structuring elements is $2 N+1$. Direct extension of the binary filter to multilevel signals using the principle of threshold decomposition (classical threshold decomposition) is not applicable owing to the high computation complexity and poor noise suppression. A modification of the classical threshold decomposition by Fitch et al. called the selective threshold decomposition is proposed to help alleviating the computation requirement. In selective threshold decomposition, a limited number of threshold levels are used instead of all the threshold levels as in the classical threshold decomposition. By means of the selective threshold decomposition, a family of multilevel erosion filters is defined. The multilevel filters are characterized by two parameters, the filter index $N$ and the forcing level $f$. The forcing level is a parameter introduced by the selective threshold decomposition, which takes values from $\{N+1, \ldots, 2 N+1\}$. If the forcing level is set to the minimum value, i.e. $N+1$, the filter is called the standard multilevel filter. By varying the forcing level, the properties of the filter varies.

Chapter 4 describes the deterministic and statistical properties of the multi-structuring element erosion filter. The deterministic properties disclose the signal structures which are invariant to the filter. In Section 4.2, the sizes of the minimal preservable details are analyzed. Estimations of the rate of convergence of the filter follow. The results are quite encouraging. For the binary erosion filter, at most two passes are needed to convert any nonroot signals to their roots. The convergent rates for other detail preserving filters are functions of the sizes as well as the indices of the filters.

Statistical analysis are also presented in Chapter 4. A general expression of the probability measure function of the 1-D multilevel erosion filter is derived using the method of statistical threshold decomposition [Arce88]. The breakdown probabilities, which is defined as the probability that an impulse is output, is evaluated. In the 2-D case, only the probability measure function and the breakdown probability are derived for filter with index equal to 1 . The expression for higher index is not obtained owing to the geometrical correlation among the points which make the expression too complicated. The probability measure functions and the breakdown probabilities of the multistage median filter, both the unidirectional and the bidirectional, are compared against those of the 2-D multilevel erosion filter with index equal
to 1 . It is shown that the multilevel erosion filter outperforms the unidirectional multistage median filter and is of comparable performance in salt-and-pepper noise rejection as by the bidirectional one. Under Gaussian and uniform noise distributions, the unidirectional median filters have similar noise rejection to those of the multilevel filter.

Chapter 5 gives a performance comparison using real images between the multistage median filter and the multi-structuring element erosion filter. Standard images, such as Lenna, Baboon, Bridge and Peppers are used for evaluation. The comparison is made according to the following criteria : the mean-square-error, the mean-absolute-error, a subjective test and the computation complexity. Test images are corrupted by salt-and-pepper noise, impulsive noise and Gaussian white noise. Our filter produces images with good visual quality, in the sense that the number of clusters of impulses are always among the least while preserving details. Moreover, the new filter does not result the artifacts which appears in other median based filter. This is because the new filter always transforms an impulse into value which are visually correlated with its neighbourhood. Artifacts will be resulted by the new filter if the impulse occurs on the boundary an object. However, the new filter is computational complicated. A detailed calculation of the computational requirements is given in Chapter 5.

Conclusively, we have proposed a new nonlinear image filter and have compared the performance of this filter against the multistage median filter. The new filter possesses some properties which are different from the multistage median filter. Firstly, the convergent rate of non-recursive median based filter is dependent on the size of the input signal. A standard 1-D median filter of window size $2 N+1$ requires a number of passes of $3(L-2) / 2(N+2)$. The number of passes required increases as the size of the input signal increases. In binary multistructuring element erosion filtering, at most 2 passes are needed for the convergence which is independent of both the window size and the input signal. Secondly, a pixel is preserved by a median based filter if the value of the pixel is the median of those pixels spanned by the subfilters. Hence, a preserved pixel does not imply that it is visually correlated with its neighbourhoods. However, the new filter performs filtering by considering the geometric features of a signal. If an impulse occurs near the boundary of an object, the erosion filter always converts it to a value which are spatially correlated with its neighbourhood. On the other hand, owing to the lack of consideration of geometrical structures, median based filters may fail to produce visually correlated output and generate artifacts. Thirdly, it is observed
that the new filter, especially the standard one, also posseses good edge and detail preserving properties and good noise suppression while the non-standard filter has good noise suppression. Therefore, non-standard multi-structuring element erosion filter can be applied together with the standard one. A non-standard filter can be used to break down the clustered impulses, and a standard filter is used for better rate of convergence. This scheme not only help to improve the convergent rate, but also reduces the computation complexity.

In summary, we have developed a new nonlinear image filter which possesses good detail and edge preserving properties. In particular, the binary filter has a constant rate of convergence. These nice properties seem to outweigh the non-recursive median filters. Nevertheless, the only drawback of the filter is the computation complexity.

### 6.2 Suggestions for Further Work

There are plenty of works that can be continued. First of all, a general expression for the probability measure function is not derived. A general expression can help to reveal the noise suppression capability of the filter under different noise distributions. It has been mentioned that our filter is computational intensive. Obviously, one can proceed to design a fast algorithm for the implementation of the filter on a sequential computer. However, the numbers of binary comparison is still hinder its software implementation. Therefore, it seems better to consider the hardware implementation of our filter.

### 6.2.1 Probability Measure Function for the Two-dimensional Filter

In Chapter 4, the general expression for the probability measure function cannot be obtained for the 2-D multi-structuring element erosion filter. This is owing to the geometrical correlation between the points. A point in the $(4 N+1)$ by $(4 N+1)$ window may be utilized by several structuring elements. It is very difficult to account for such correlation. Arce have attempted to perform statistical analysis of 2-D closing-opening filters [Arce87] by the Monte Carlo method so that the output distribution of the morphological filter can be simulated. Owing to the computation complexity of the multilevel erosion filter, the Monte Carlo method is not feasible. Although a rough idea of the output distribution can be predicted from the 1-D probability measure functions and that of multilevel erosion filter with index 1 . It is ideal
to obtain such expression.

## 6.2 . 2 Hardware Implementation

The binary multi-structuring element erosion filter with index $N$ can be represented as a logic operation on the bits inside the moving window of size $(4 N+1)$ by $(4 N+1)$. In a binary signal, the value at a point is either a 0 or a 1 . If a matching between the structuring element and the signal is found, the logic AND of those relevant bits must be a 1 . The union of erosion can be represented as the logic OR of all the outputs of the erosion with different structuring elements. As a result, the


Figure 6.1 Relative Coordinates within a $5 \times 5$ window operation can be expressed in a minimal sum-of-products logic function. Fig. 6.1 shows a $5 \times 5$ window of the binary erosion filter with index 1. Denote $p_{\mathrm{a}, \mathrm{b}}$ be the point at $(a, b)$. The binary erosion filter with index 1 at the threshold level $j$ of a multilevel signal $G(x)$ which is defined on $Z^{2}$ is :

$$
\begin{align*}
& B M E F_{1}\left(S_{j}(G(x))\right) \\
= & p_{0,0} p_{0,1} p_{0,-1}+p_{0,0} p_{1,0} p_{-1,0}+p_{0,0} p_{1,1} p_{-1,-1}+p_{0,0} p_{1,-1} p_{-1,1} \\
& +p_{0,0} p_{1,0} p_{-1,1}+p_{0,0} p_{1,0} p_{-1,-1}+p_{0,0} p_{0,-1} p_{-1,1}+p_{0,0} p_{0,-1} p_{1,1} \\
& +p_{0,0} p_{-1,0} p_{1,-1}+p_{0,0} p_{01,0} p_{1,1}+p_{0,0} p_{0,1} p_{-1,-1}+p_{0,0} p_{0,1} p_{1,-1} \\
& +p_{0,0} p_{-1,1} p_{-2,1}+p_{0,0} p_{-1,1} p_{-2,2}+p_{0,0} p_{-1,1} p_{-1,2}+p_{0,0} p_{0,1} p_{-1,2}  \tag{6.1}\\
& +p_{0,0} p_{0,1} p_{0,2}+p_{0,0} p_{0,1} p_{1,2}+p_{0,0} p_{1,1} p_{1,2}+p_{0,0} p_{1,1} p_{2,2} \\
& +p_{0,0} p_{1,1} p_{2,1}+p_{0,0} p_{1,0} p_{2,1}+p_{0,0} p_{1,0} p_{2,0}+p_{0,0} p_{1,0} p_{2,-1} \\
& +p_{0,0} p_{1,-1} p_{2,-1}+p_{0,0} p_{1,-1} p_{2,-2}+p_{0,0} p_{1,-1} p_{1,-2}+p_{0,0} p_{0,-1} p_{1,-2} \\
& +p_{0,0} p_{0,-1} p_{0,-2}+p_{0,0} p_{0,-1} p_{-1,-2}+p_{0,0} p_{-1,-1} p_{-1,-2}+p_{0,0} p_{-1,-1} p_{-2,-2} \\
& +p_{0,0} p_{-1,-1} p_{-2,-1}+p_{0,0} p_{-1,0} p_{-2,-1}+p_{0,0} p_{-1,0} p_{-2,0}+p_{0,0} p_{-1,0} p_{-2,1}
\end{align*}
$$

where $S_{\mathrm{j}}(G)$ is the threshold signal at level $j$ and $\boldsymbol{x}$ is the centre of the $(4 N+1)$ by $(4 N+1)$ window.

The binary erosion filter can be implemented solely by AND and OR logic operations.
Fig.6.2. exhibits a block diagram of the hardware implementation of the multilevel filter.


Figure 6.2 Block Diagram for Hardware Implementation of Multilevel Erosion Filter

By Lemma 3.1, the output of the multilevel multi-structuring element erosion filter is restricted to those values in the $(2 N+1)$ by $(2 N+1)$ window. Therefore, the threshold level generator reads the data masked by the $(2 N+1)$ by $(2 N+1)$ window as threshold levels. If the forcing level $f$ is set to the minimal value, i.e. $f=N+1$, no sorting for the threshold value is required. If the forcing level is not $N+1$, sorting is needed to reject those threshold levels which are not utilized in the binary filtering. For example, if the forcing level is $N+2$, the first to the $N+1$ threshold levels have no contribution to the output of the filter. Hence, these levels can be discarded. The threshold device slices the region of an input signal, which is masked by the $(4 N+1)$ by $(4 N+1)$ window, into binary signals. This window contains all the information to determine the output. An array of binary multi-structuring element is set up. Each binary filter is identical and is implemented by the logic function in (6.1). The output of each binary filter is stacked by the stacking device. The stacking device is in fact a binary comparator. The comparator finds out the maximum gray level at which the binary output is 1. The output of the multilevel is therefore this maximum gray level.

The merits of hardware implementation of the filter is on the rate of convergence. In Chapter 4, it has been shown that at most two passes are required for the conversion of nonroot signal to its root. The rate of convergence of multilevel erosion filter is equal to $2\left((2 N+1)^{2}-(N+2)\right)$ if the index of the filter is $N$. Moreover, since the multilevel erosion filter commutes with the stacking property, binary filtering at the $i^{\text {th }}$ threshold level will not affect the output of that at the $(i-1)^{\text {th }}$ level. The rate of convergence of the multilevel filter is given by Property 4.25 . Owing to the assumption that a sequential computer is used, we can only proceed to the next gray level until the current level is converged. However, such sequential operation is not necessary as the outputs of each level are independent to each others. In other words, the output of the multilevel filter is the same if each threshold set is processed simultaneously. This speeds up the operations of the filter with forcing level $N+1$. Only two passes can bring all nonroot signals to their roots. Also, the number of passes is independent of the index of the filter.

Nevertheless, if the forcing level is not $N+1$, the rate of convergence is no longer two passes only. Chapter 4 shows that the rate of convergence depends on the length of the invariant signal at forcing level $f-1$. Although the number of filter passes are not bound to two only, the implementation can still reduce the total number of passes.

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## Appendix I Proof of Theorem 3.1

It can be seen that as the length of a structuring element is $2 N+1$, all structuring elements can be bounded in a $(2 N+1)$ by $(2 N+1)$ window. The structuring elements in a subgroup can further be classified into the following types:

1. [Type 1] Structuring elements that end at a corner (corners) of the window.
2. [Type 2] Structuring elements that end at the centre row/column of the window, but excluding the horizontal and the vertical structuring elements.
3. [Type 3] Structuring elements that end at opposite edges of the window.
4. [Type 4] Structuring elements that end at adjacent edges of the window.

In the following, $C_{\mathrm{i}}$ and $E_{\mathrm{i}}, i=1,2,3,4$, denote the corner and the edge indicated by the figures.
i.Type 1 structuring elements.


Figure A1.1 Type I structuring elements

For a structuring element that ends at one of the corners of the $(2 N+1)$ by $(2 N+1)$ window, the structuring element must be of the shape as shown in Fig.A1.1. The lower right triangle in the window, which is shaded in Fig.A1.1 consists of the structuring element which started at $C_{1}$ and passes through the point $(0,0)$. Generally, the pixel $(i, j)$ which lies in the shaded region has two connected pixels $(i, j+1)$ and $(i+1, j+1)$. Therefore, the number of Type 1 structuring elements in the shaded region is $2^{\mathrm{N}}$. There are 8 such triangular regions, resulting in $2^{\mathrm{N}+3}$ structuring elements.

Excessive number of diagonal structuring element is counted. In each triangular region, the corresponding diagonal is counted. There are actually only 2 diagonals. Also, Type 1 structuring element excludes the 4 structuring elements which end at $(0, N),(N, 0),(-N, 0)$ and $(0,-N)$. Thus, the number of Type 1 structuring elements is $2^{\mathrm{N}+3}-6$.
ii.Type 2 structuring elements. In Fig.A1.2, consider the structuring start from $E_{1}$ to $E_{2}$. The
number of structuring elements start from a point on the shaded region of $E_{1}$ and end at the shaded region of $E_{2}$ are $\left(2^{\mathrm{N}-1}-1\right)$ by excluding the one which ends at the corner $C_{1}$ or $C_{3}$. Hence, the number of Type 2 structuring element $8\left(2^{\mathrm{N}-1}-1\right)$ since there are 8 segments of edges.


Figure A1.2 Type 2 Structuring Elements


Figure A1.3 Type 3 Structuring Elements
iii.Type 3 Structuring elements. Without loss of generality, consider the structuring elements start from the upper left quadrant to the lower right quadrant of the window. First of all, the number of structuring elements which begin at $(-N, i)$ are considered. The number of paths from $(-N, i)$ to the origin of the window is ${ }_{N} C_{i}$. The length of the structuring element which is left to the centre is $2 N$ pixels. The offset of the end point relative to the centre is $i$. Thus the offset $i$ must be only accomplished by $i$ pixels only owing to the definition of a valid structuring element. The number of paths from this end point to the centre is therefore equal to the number of ways to select from the $2 N$ pixels a group of $i$ pixels, which is equal to ${ }_{N} \mathrm{C}_{\mathrm{i}}$, where ${ }_{N} \mathrm{C}_{\mathrm{i}}=N!/ i!(N-i)!$. In Fig.A1.3, the number of valid paths from the point $(-N, i)$ to the centre is ${ }_{N} \mathrm{C}_{\mathrm{i}}$. Therefore, the total number of Type 3 structuring elements started at the point $(-N, i)$ is ${ }_{N} \mathrm{C}_{\mathrm{i}}\left(2^{\mathrm{N}}-1\right)$. The total number of structuring elements start from the upper half of $E_{1}$ (including the centre level) to the lower half of $E_{2}$ is therefore :
For the edge pairs $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$, the number of valid structuring elements is $2\left(2^{\mathrm{N}}-1\right)^{2}-1$. A -1 is subtracted from the number since the horizontal structuring element is counted twice. The total number of Type 3 structuring elements is $2\left[2\left(2^{\mathrm{N}}-1\right)^{2}-1\right]$ when both the edges pairs $E_{1}$ and $E_{2}, E_{3}$ and $E_{4}$ are taken into account.

$$
\begin{gathered}
\sum_{i 0}^{N-1}\left(2^{N}-1\right) \cdot{ }_{N} C_{i}=\sum_{i 0}^{N}\left({ }_{N} C_{i}-1\right) \cdot\left(2^{N}-1\right) \\
=\left(2^{N}-1\right)^{2} \\
\because \sum_{i 0}^{N}\left({ }_{N} C_{i}\right)=2^{N}
\end{gathered}
$$

iv.Type 4 Structuring Elements


Figure A1.4 Type 4 Structuring Elements

There are two cases to be considered. Fig.A1.4(a) shows the first case, a structuring element start at the upper half of $E_{1}$ and end at the right half of $E_{4}$, must contain the points $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$. The number of structuring element is equal to $\left(2^{\mathrm{N}-1}-1\right)^{2}$. The second case is illustrated in Fig.A1.4(b), the structuring element must contain $x_{4}$ and at least one point at the centre level, say $x_{3}$. The number of segments which passes the point $x_{4}$ is $\left(2^{\mathrm{N}-1}-1\right)$. The number of segments which possess pixels along the centre level is $\left(2^{\mathrm{N}}-2\right)$, since the structuring which ends at the centre level is excluded. The total number of Type 4 structuring elements is $4\left[\left(2^{\mathrm{N}-1}-1\right)^{2}+\left(2^{\mathrm{N}-1}-\right.\right.$ 1) $\left.\left(2^{\mathrm{N}-1}-2\right)\right]$.

The number of structuring elements in a subgroup is the sum of the number of these four types of structuring elements.

## Appendix II : Shape of Structuring Elements for $\mathbf{N}=1$ and $\mathbf{N}=\mathbf{2}$

This appendix displays the shape of $T_{1 \mathrm{~N}}$ structuring elements used for $N=1$ and $N=2$. A structuring element belongs to $T_{\text {IN }}$ if the origin of the structuring element is at the centre. Only $T_{1 \mathrm{~N}}$ structuring element are given since $T_{2 \mathrm{~N}}$ structuring elements are of the same shape but different locations of origin. A structuring element belongs to $T_{2 \mathrm{~N}}$ if its origin is at one of its end points. Fig.A2.1 shows the shapes of all $T_{11}$ structuring elements. Those of $T_{12}$ is depicted in Fig.A2.2.


Figure A2.1 $T_{\text {II }}$ Structuring Elements


Figure A2.2 $T_{12}$ Structuring Elements

## Appendix III Proofs of Properties in Chapter 4

In this appendix, the proofs of following properties will be given.

1. Property 4.9
2. Property 4.28
3. Property 4.29
4. Property 4.30

## Proof of Property 4.9:

It is obvious that the invariant components of $d_{\mathrm{IR}}$ is unaltered by $B M E F_{\mathrm{N}}$. If there is no invariant component, then $d_{\mathrm{IR}}$ must not be root signal to $B M E F_{\mathrm{N}}$. Suppose that a nonroot component $l_{\mathrm{L}}$ of $L$ points are connected to some connected regions $l_{\mathrm{L} 1}$ and $l_{\mathrm{L} 2}$ on some invariant signals. Let $l_{\mathrm{L}}$, be the concatenated curve by $l_{\mathrm{L}}, l_{\mathrm{L} 1}$ and $l_{\mathrm{L} 2}$.
i. Existence of $N$-point connected regions. Suppose that there exists at most one such connected region. The length of the nonroot component must be less than $3 N$. After the concatenation of a region of $L^{\prime}$ points to the nonroot component, if the concatenated component becomes then a contradiction is resulted. Since if this concatenated component is invariant to the filter, then it must belong to $I_{\mathrm{C}}$. Therefore, at least two regions of connected points are found. Consider the worst case, if the nonroot component is a single point, then at least $N$ connected points on some invariant components are connected to this point. The minimal number of connected points for a structuring element matching is $N$.
ii. If there exist at least two such regions as described by $i$, a concatenated signal of at least $2 N+1$ points is resulted. The nonroot signal will be preserved if Property 4.3 is satisfied.

Proof of Property 4.28:
Using the total law of probability, $P_{\mathrm{r}}\left[G M E F_{\mathrm{f}, \mathrm{N}}(G(\boldsymbol{x})) \leq j\right]$ can be written as:

$$
\begin{align*}
& P_{r}\left[B M E F_{N}\left(s_{j+1}(G(x))\right)=0\right] \\
= & P_{r}\left[B M E F_{N}\left(S_{j+1}(G(x))\right)=0 \mid S_{j+1}(G(x))=0\right] \cdot P_{r}\left[S_{j+1}(G(x))=0\right]  \tag{A3.1}\\
& +P_{r}\left[B M E F_{N}\left(S_{j+1}(G(x))\right)=0 \mid S_{j+1}(G(x))=1\right] \cdot P_{r}\left[S_{j+1}(G(x))=1\right] .
\end{align*}
$$

According to the definition of multi-structuring element erosion filter, if the value at $x$ equals to 0 , the output must be zero.

Hence,

$$
\begin{aligned}
& P_{r}\left[B M E F_{N}\left(S_{j+1}(G(x))\right)=0 \mid S_{j+1}(G(x))=0\right]=1 \text { and } P_{r}\left[S_{j+1}(G(x))=0\right]=F(j) \\
& \quad \therefore, P_{r}\left[B M E F_{N}\left(S_{j+1}(G(x))\right)=0 \mid S_{j+1}(G(x))=0\right] \cdot P_{r}\left[S_{j+1}(G(x))=0\right]=F(j) .
\end{aligned}
$$

As the second term in (A3.1) can be written as by the total law of probability,

$$
\begin{align*}
& P_{r}\left[\text { BMEF }{ }_{N}\left(S_{j+1}(G(x))\right)=0 \mid S_{j+1}(G(x))=1\right] \\
& =P_{r}\left[B M E F_{N}\left(S_{j+1}(G(x))\right)=0 \mid S_{j+1}(G(x))=1\right. \text { and case 1] } \\
& \quad \cdot P_{r}\left[\operatorname{case} 1 \mid S_{j+1}(G(x))=1\right]  \tag{A3.2}\\
& +P_{r}\left[B M E F_{N}\left(S_{j+1}(G(x))\right)=0 \mid S_{j+1}(G(x))=1 \text { and case } 2\right] \\
& \quad \cdot P_{r}\left[\text { case } 2 \mid S_{j+1}(G(x))=1\right]
\end{align*}
$$

where case 1 and case 2 are as follows.
Case 1. $[1]_{\mathrm{L}} \subset[1]_{2 \mathrm{~N}+1}$
If $L<2 N+1$, then neither of the $T_{\text {IN }}$ nor $T_{2 \mathrm{~N}}$ structuring element matchings are possible. The centre of the window, $x$, can be at any location along $L$. The limiting value of $L$ is $2 N$, thus,

$$
\begin{equation*}
\operatorname{Pr}\left[\text { case } 1 \text { and } S_{j+1}(G(x))=1\right]=\sum_{i=1}^{2 N}\binom{i}{1} \cdot F(j)^{2} \cdot[1-F(j)]^{i} \tag{A3.3}
\end{equation*}
$$

Case 2. $[1]_{2 \mathrm{~N}+1} \subseteq[1]_{\mathrm{L}} \subset[1]_{3 \mathrm{~N}}$
Using Property 4.20 , filtering by a multi-structuring element erosion filter trisects [1] $]_{\mathrm{L}}$ into three equal portions $[1]_{\text {Lmodn }}$. Thus,

$$
\begin{equation*}
\operatorname{Pr}\left[\text { case } 2 \text { and } S_{j+1}(G(x))=1\right]=\sum_{i=2 N+1}^{3 N-1}\binom{i-3(i \bmod N)}{1} F(j)^{2} \cdot[1-F(j)]^{i} \tag{A3.4}
\end{equation*}
$$

In addition, the probability measure function of a 1-D multi-structuring element filter with a forcing level $f$, where $N+1 \leq f \leq 2 N+1$, are described. The root of $G M E F_{f+1, N}$ is also invariant to $G M E F_{\mathrm{f}, \mathrm{N}}$. On the contrary, if a signal point is not preserved by $G M E F_{f, \mathrm{~N}}$, then it is not matched by $G M E F_{f+1, \mathrm{~N}}$ as well. Moreover, if the centre of the binary filter window lies between the first to the $[f-(N+1)]^{\text {th }}$ and the $\left[L-(f-(N+1)]^{\text {th }}\right.$ to the last position on $[1]_{L}$, these [1]'s will be set to [0]'s upon filtering. Hence, an addition term:

$$
\begin{equation*}
(1-F(j)) \cdot \sum_{i=2 N+1}^{\infty} 2[f-(N+1)] \cdot F(j)^{2} \cdot[1-F(j)]^{i}=2[f-(N+1)] \cdot F(j) \cdot[1-F(j)]^{2 N+1} \tag{A3.5}
\end{equation*}
$$

is added. Hence, Property 4.28 is proved.

## Proof of Property 4.29:

From the definition of the dual filter, we have

$$
\begin{align*}
G M E F_{f, N}{ }^{d}(G(x)) \leq j & \Rightarrow\left\{G M E F_{f, N}\left(G^{c}(x)\right)\right\}^{c} \leq j \\
& \Rightarrow(k-1)-G M E F_{f, N}\left(G^{c}(x)\right) \leq j  \tag{A3.6}\\
& \Rightarrow G M E F_{f, N}\left(G^{c}(x)\right) \leq(k-1)-j
\end{align*}
$$

Thus, the probability measure function can be expressed as :

$$
\begin{aligned}
& \because P_{r}\left[G M E F_{f, N}\left(G^{c}(x)\right) \geq(k-1)-j\right]=1-P_{r}\left[G M E F_{f, N}\left(G^{c}(x)\right) \leq(k-1)-j-1\right] \\
& \text { and } P_{r}\left[G M E F_{f, N}\left(G^{c}(x)\right) \leq(k-1)-j-1\right]=P_{r}\left[B M E F_{N}\left(S_{(k-1)-j}\left(G^{c}(x)\right)\right)=0\right]
\end{aligned}
$$

Since,

$$
\begin{aligned}
S_{j}\left(G^{c}(x)\right) & = \begin{cases}1 \text { if } G^{c}(x) \geq j \\
0 & \text { if } G^{c}(x)<j\end{cases} \\
& = \begin{cases}1 & \text { if }(k-1)-G(x) \geq j \\
0 & \text { if }(k-1)-G(x)<j\end{cases} \\
& = \begin{cases}1 & \text { if } G(x) \leq(k-1)-j \\
0 & \text { if } G(x) \geq k-j\end{cases}
\end{aligned}
$$

we have

$$
\begin{aligned}
P_{r}\left[S_{j}\left(G^{c}(x)\right)=0\right] & =P_{r}\left[S_{k-j}(G(x))=1\right] \\
& =1-F(k-j-1)
\end{aligned}
$$

which implies,

$$
\begin{equation*}
P_{r}\left[S_{(k-1)-j}\left(G^{c}(x)\right)=\right]=1-F(j) \tag{A3.7}
\end{equation*}
$$

By Property 4.28,

$$
\begin{align*}
& P_{r}\left[B M E F_{N}\left(S_{(k-1)-j}\left(G^{c}(x)\right)\right)=0\right] \\
= & 1-F(j)+F(j) \cdot[1-F(j)]^{2} \cdot\left[\sum_{i=0}^{2 N-1}(i+1) \cdot F(j)^{i}\right.  \tag{A3.8}\\
+ & \left.\sum_{i=2 N}^{3 N-2}(i+1-3((i+1) \bmod N)) \cdot F(j)^{i}\right]+2[f-(N+1)] \cdot F(j)^{2 N+1} \cdot[1-F(j)]
\end{align*}
$$

(4.20) of Property 4.29 can be obtained by subtracting (A3.8) from 1.

Proof of Property 4.30:
Using the total law of probability,

$$
\begin{align*}
& P_{r}\left[B M E F_{1}\left(S_{j+1}(G(x))\right)=0\right] \\
= & P_{r}\left[B M E F_{1}\left(S_{j+1}(G(x))\right)=0 \mid S_{j+1}(G(x))=0\right] \cdot P_{r}\left[S_{j+1}(G(x))=0\right]  \tag{A3.9}\\
+ & P_{r}\left[B M E F_{1}\left(S_{j+1}(G(x))\right)=0 \mid S_{j+1}(G(x))=1\right] \cdot P_{r}\left[S_{j+1}(G(x))=1\right] .
\end{align*}
$$

By the same argument as in Property 4.28 , the first term in which $P_{r}\left[S_{j+1}[G](x)=0\right]$ is considered equals to $F(j)$.

Figure A3.1 indicates the cases needed to be considered when $S_{\mathrm{j}+1}[G](x)=1$.


Figure A3.1 Cases for the derivation of probability measure function

There are 8 pixels in the $N_{8}(\boldsymbol{x})$ neighbourhood.
Case 1. All $N_{8}(\boldsymbol{x})$ pixels are zero
If all the $N_{8}(\boldsymbol{x})$ points are less than $j$, then on the threshold signal at level $j$, all $N_{8}(\boldsymbol{x})$ points are zero.

$$
\begin{gather*}
P_{r}\left[B M E F_{1}\left(S_{j+1}(G(x))\right)=0 \mid \text { case } 1 \text { and } S_{j+1}(G(x))=1\right]=1  \tag{A3.10}\\
\text { and } P_{r}\left[\text { case } 1 \text { and } S_{j+1}(G(x))=1\right]=F(j)^{8} \cdot[1-F(j)]
\end{gather*}
$$

Case 2. One out of the $8 N_{8}(x)$ is equal to 1

If any one out of the $N_{8}(\boldsymbol{x})$ neighbourhood is equal to 1 , then $b_{1}, b_{2}$ and $b_{3}$ must not be equal to 1 ; otherwise, a $T_{2 \mathrm{~N}}$ structuring element matching will be resulted.

$$
\begin{align*}
P_{r}\left[\text { case } 2 \text { and } S_{j+1}(G(x))=1\right] & =[1-F(j)] \cdot\binom{8}{1}[1-F(j)] \cdot F(j)^{7} \cdot F(j)^{3}  \tag{A3.11}\\
& =8 F(j)^{10} \cdot[1-F(j)]^{2} .
\end{align*}
$$

Case 3. Two adjacent $N_{8}(\boldsymbol{x})$ are equal to 1

The number of combinations of 1 's which are adjacent to each other is $\binom{8}{1}$, and no structuring element matching if $c_{1}$ to $c_{5}$ are all non unity.

$$
\begin{align*}
P_{r}\left[\text { case } 3 \text { and } S_{j+1}(G(x))=1\right] & =[1-F(j)] \cdot\binom{8}{1} \cdot[1-F(j)]^{2} \cdot F(j)^{6} \cdot F(j)^{5}  \tag{A3.12}\\
& =8 F(j)^{11} \cdot[1-F(j)]^{3} .
\end{align*}
$$

Case 4. Two, but not adjacent $N_{8}(\boldsymbol{x})$ are equal to 1
Figure A3.1(d) shows the two possible cases when there are two out of the $N_{8}(x)$ are equal to 1 and these 1 's are not adjacent to each other. Pixels $d_{1}$ to $d_{6}$ or $d_{1}^{\prime}$ to $d_{6}^{\prime}$ must be non-unity in order to facilitate a condition that no structuring element is matched.

$$
\begin{equation*}
\operatorname{Pr}\left[\text { case } 4 \text { and } S_{j+1}(G(x))=1\right]=8 \cdot F(j)^{12} \cdot[1-F(j)]^{2} \tag{A3.13}
\end{equation*}
$$

Case 5. Three out of $N_{8}(x)$ are equal to 1
Figure A3.1(e) and A3.1(f) indicate the two combinations that $T_{1 \mathrm{~N}}$ structuring elements are not matched. No $T_{2 \mathrm{~N}}$ structuring element is matched if in each situation $e_{1}$ to $e_{7}$ and $f_{1}$ to $f_{7}$ are non unity. Hence, the probability of achieving neither $T_{1 \mathrm{~N}}$ and $T_{2 \mathrm{~N}}$ are matched is:

$$
\begin{equation*}
\operatorname{Pr}\left[\text { case } 5 \text { and } S_{j+1}(G(x))=1\right]=8 \cdot F(j)^{12} \cdot[1-F(j)]^{4} . \tag{A3.14}
\end{equation*}
$$

Moreover, there are two forcing levels when $N=1$. The probability measure function of $G M E F_{3,1}$ is expressed. Similar to the proof of the 1-D filter, the probability of $\mathrm{T}_{2 \mathrm{n}}$ structuring element matching with the minimal number of pixels is :

$$
\begin{equation*}
8 \cdot F(j)^{7} \cdot[1-F(j)] \cdot\left[1-F(j)^{3}\right] \tag{A3.15}
\end{equation*}
$$

By summing up these probabilities, the probability measure function is resulted.


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[^0]:    Table 2.2 Comparison of Performance of the Multi-stage Median Filter and the FIR-median Hybrid Filter

[^1]:    ${ }^{1}$ The definition of morphological operators are based on those work of Haralick et al. [Hara87]. For other definitions of basic morphological operators, please refer to [Serra82][Mara90].

[^2]:    ${ }^{1}$ Refers to Chapter for the definition of the L-filter

