Quanto Options under Double Exponential Jump Diffusion

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Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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ABSTRACT

A foreign equity option (or quanto option) is a derivative security whose value depends on an exchange rate and a foreign equity. In this paper, we study the valuation of quanto options when the foreign equity price and the exchange rate follow double exponential jump diffusions. Traditionally, it is assumed that the diffusion parts of the two state variables are correlated but the Poisson processes and the jump sizes are independent across state variables. Here, we allow the two state variables to have common jumps and dependent jump sizes. The jump sizes are modelled by a multivariate exponential distribution. Analytical pricing formulas are obtained for various types of quanto options. We also study the analytical tractability of path-dependent quanto options under a joint double exponential jump diffusion. Our approach can be applied to options on two assets.

摘要

外匯股票期權（或quanto期權）為一種衍生證券，其價值同時取決於匯率和外匯股票。這文章中，我們研究quanto期權的估價，當中外匯股票的價格和匯率均遵循雙重指數跳躍擴散過程。傳統上，兩個資產的擴散過程部分會假設為相關的，但對應的泊松（Poisson）過程和跳躍幅度卻假設為互相獨立。然而，我們容許這兩個相關資產具有共同的跳躍及相關依跳躍幅度。而跳躍幅度則由一個多元指數分佈所模型。各樣的quanto期權的可解析定價公式亦能夠求出。我們還探討了聯合雙指數跳躍擴散（joint double exponential jump diffusion）過程下，路徑依賴型quanto期權的解析可處理性。我們這套方法可適用於兩種資產的期權。

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Chapter 1

Introduction

With the growth in investments in recent years, the currency translated foreign equity options (quanto options) have gained considerable popularity. Quanto options are contingent claims whose payoff is determined by a financial price or index in one currency while the actual payoff is settled in a different currency. The payoffs of these quanto options can be structured in a variety of combinations of foreign asset prices and exchange rates, thereby generating a lot of investment and hedging opportunities. Besides using fixed or floating exchange rates, their payoff structures can be made more exotic by introducing a barrier or lookback feature on either the underlying asset price or the exchange rate or both. The wider classes of payoff structures allow investors to hedge a specific risk or pursue a particular speculation of international equity investment. Expositions on hedging properties of vanilla type quanto options can be found in Reiner (1992) and Toft and Reiner (1997). Wong and Chan (2007) apply the concept of quanto lookback option for dynamic fund protection in insurance.

This thesis derives the pricing formulas and examines the pricing behaviors of quanto options with path-dependent payoff structures using the double exponential jump diffusion (DEJD) model. Under the Black-Scholes model, path-dependent quanto option pricing has been studied in Kwok and Wong (2000) and

Dai et al. (2004). However, it is well known that both the equity and exchange rate exhibit jumps in their values and the Black-Scholes model is inadequate to capture this stylized feature.

It has been a challenging task to extend jump diffusion models to quanto options because these options represent a class of options on multiple assets, or more specifically on two state variables, the exchange rate and the foreign equity. We propose a multivariate jump diffusion model to describe the joint dynamics of the exchange rate and the foreign asset. The proposed model is analytically tractable for path-dependent quanto options and allows common jumps and dependent jump sizes.

Huang and Hung (2005) investigate quanto option pricing with Lévy processes. Our approach is significantly different from theirs. Although Huang and Hung consider general Lévy processes, their method is only applicable to European quanto options that the payoff can be reduced into a one dimensional problem via a change of probability measure. Their approach cannot be used to price path-dependent quanto options or simple joint quanto options. It is well known that Lévy processes cannot be used to price path-dependent options analytically. However, using a specific Lévy process, the DEJD, such that we can obtain an analytical tractable solution for path-dependent options which is able to explain market phenomena.

Jump diffusion model was introduced by Merton (1976) to the financial industry. Kou (2002) first proposed the DEJD and gave an analytical solution to plain vanilla options. He showed that DEJD provides a psychological meaning for asset movement beyond the diffusion process. Kou and Wang (2004) derived analytical solutions for barrier and lookback option with DEJD. Kou et al. (2006) improved the computational efficiency by using double Laplace transform. Leib (2000) found evidence that the DEJD model of Kou better fits the asset return distributions.

It is unclear, however, what is the appropriate multivariate process for financial assets when both assets are marginally following DEJD. The diffusion component can be modeled by a multivariate Wiener process, but the modeling of the jump component remains unclear. We assume the financial assets to have common jumps from a common Poisson process. When a common jump arrives, we use the Marshall and Olkin (1967) multivariate exponential distribution to model the joint jump size. We show that this model not only is consistent with the marginal DEJD process, but is also analytically tractable and parsimonious.

We would also like to highlight some mathematical findings of this thesis. In the valuation of path-dependent quanto options, we often need to calculate the domestic equivalent asset price which is the product of the exchange rate and the foreign asset price. If the exchange rate and the foreign asset marginally follow DEJD process, then the domestic equivalent asset should evolve as a jump diffusion process such that the jump size follows a mixture of exponential distribution. We call this latter process the mixture exponential jump diffusion (MEJD). The distribution of the first passage time of MEJD have not been considered in the literature before. We derive the moment generating function of the first passage time when the domestic equivalent asset reaches its maximum or minimum. We also discuss the characteristics of the complex roots of a polynomial equation derived from the Laplace transform of the distribution. These characteristics are crucial for inverting the Laplace transform numerically.

In the financial perspective, we document that the common jump and dependent jump sizes have important impact on quanto option pricing. If we want to price path-dependent quanto options, the Marshall and Olkin (1967) multivariate exponential distribution is very useful to produce an analytically tractable model.

The remaining part of the thesis is organized as follows. Chapter 2 sets the ground for the thesis by introducing the notions of risk-neutral pricing and jump diffusion models. Chapter 3 summarizes results in option pricing with DEJD. We stress that the result of turbo warrant is new although it is not the major objective of this thesis. Chapter 4 presents the valuation framework for European quanto options under the DEJD process. The use of Marshall and Olkin (1967) multivariate exponential distribution in modeling common jump sizes are introduced therein. Chapter 5 contains the valuation of path-dependent quanto options. Several interesting mathematical results concerning the first passage time of the MEJD process are obtained. Chapter 6 concludes the thesis.

Chapter 2

Background

One of the most fundamental elements of derivatives (option) pricing is the notion of risk-neutral pricing theory. Consider a derivative on an asset S_t. For a European style option, the contract holder is entitled to receive a cash income determined by the payoff function on the maturity of the contract. We employ the usual notation of denoting T as the maturity date and $\Phi(S_T)$ as the payoff function. Derivative pricing means to establish a systematic framework for determining the derivative security price at any time prior to the maturity. Risk-neutral valuation asserts that the present value of a European option, $V(t, S)$, is related to its payoff function by

$$V(t, S) = e^{-r(T-t)}\mathrm{E}^{Q}[\Phi(S_T)|\mathcal{F}_t],$$

where r is the constant risk-free interest rate, $\mathcal{F}_t$ is the filtration up to time t and $\mathbf{Q}$ is the risk-neutral measure. A path-dependent option has a payoff depending on the sample path of the underlying asset instead of the terminal value alone.

2.1 Jump Diffusion Models

To better understand risk-neutral valuation , we need an appropriate probabilistic set-up. The dynamic of the asset price $\{S_t\}_{t\geq 0}$ is modeled in a complete probability space $\Pi = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, where Ω is the set of all feasible sample paths of $\{S_t\}_{t\geq 0}$, $\mathbf{P}$ is a probability measure, $(\mathcal{F}_t)$ is the filtration equipped with the σ-algebra $\mathcal{F} = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$, which represents the information accumulated up to time $t \geq 0$.

Risk-neutral valuation asserts that the present value of a European option is the discounted expectation of the option's payoff, where the expectation is taken under a risk-neutral measure. Usually, the risk-neutral measure is obtained by calibration. The risk-neutral measure $\mathbf{Q}$ should satisfy several conditions. First, the risk-neutral measure should be equivalent to the physical probability measure $\mathbf{P}$, i.e., $\mathbf{P}(\mathrm{A}) = 0 \iff \mathbf{Q}(\mathrm{A}) = 0$ for all $A \in \mathcal{F}_t$. Second, all non-dividend paying stocks in the market should be a martingale with respect to the money market account under the measure $\mathbf{Q}$, i.e., $\mathrm{E}^{\mathbf{Q}}\left[e^{-rT}S_T \middle| \mathcal{F}_t\right] = S_t$. As quanto options are considered in this thesis, there are two risk-neutral measures for the two currency worlds. Denote $\mathbf{Q}^d$ as the risk-neutral measure adopted by the domestic currency world and $\mathbf{Q}^f$ as that adopted by the foreign currency world.

To allow for general discussions, we introduce the jump diffusion model under $\mathbf{Q}$ without specifying the currency. A jump diffusion model consists of two parts: a Wiener process and a compound Poisson process. The Wiener process $W(t)$ is a fundamental continuous-time stochastic process that satisfies following conditions,

$$
\begin{aligned}
W(s) - W(t) &\sim \mathrm{N}(0, s-t), && \forall\, s \geq t \geq 0, \\
W(s) - W(t) &\perp W(u) - W(v), && \forall\, s \geq t \geq u \geq v \geq 0, \\
P(W(0) = 0) &= 1.
\end{aligned}
$$

The (homogeneous) Poisson process $N(t)$ is a right-continuous stochastic process such that

$$\begin{aligned} N(s) - N(t) &\sim \text{Poi}(\lambda(s-t)), \qquad \forall\, s \geq t \geq 0, \\ N(s) - N(t) &\perp N(u) - N(v), \;\; \forall\, s \geq t \geq u \geq v \geq 0, \\ P(N(0) = 0) &= 1, \end{aligned}$$

where λ is the intensity of the Poisson process. Obviously, $N(t)$ is a non-decreasing process.

Jump diffusion models assume that the asset price S_t evolves according to the stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t + Y_t\, dN_t, \tag{2.1}$$

or, equivalently,

$$dX_t = d\log S_t = \left(\mu - \sigma^2/2\right)\, dt + \sigma\, dW_t + y_t\, dN_t, \tag{2.2}$$

where S_t is the asset price at time t, W_t is the Wiener process, μ is the drift, σ is the volatility of the diffusion component, N_t is the Poisson process with intensity λ, and $y_t = \log(Y_t + 1)$ is the jump size. The Wiener process, W_t, Poisson process, N_t, and the jump sizes, $\{Y_t\}$, are independent random variables for all t.

2.2 Double Exponential Jump Diffusion Model

Merton (1976) first introduced a jump diffusion model in the financial literature. He considered the jump size to follow a normal distribution,

$$f_y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right], \tag{2.3}$$

where $f_y(y)$ denotes the probability density function (pdf) for the jump size y, μ_y is the mean jump size, and σ_y is the jump volatility. Using this assumption, Merton (1976) derives a closed form solution for European call and put options. Unfortunately, it is well known that the Merton model cannot be extended to price path-dependent options analytically.

Kou (2002) proposes a double exponential distribution (DED) for the jump size. The probability density function (pdf) of DED is given by

$$f_y(y) = p \cdot \eta_1 e^{-\eta_1 y} \mathbf{1}_{\{y \geq 0\}} + (1-p) \cdot \eta_2 e^{\eta_2 y} \mathbf{1}_{\{y<0\}}, \quad p \in [0,1], \eta_1 > 1, \eta_2 > 0. \tag{2.4}$$

We use the notation $y \sim \text{DED}(p, \eta_1, \eta_2)$ to indicate that y is an DED random variable with parameters p, η_1 and η_2.

Double exponential distribution has several advantages for financial modeling. By the nature of the DED, it is possible to generate heavy-tailed distribution and captures the leptokurtic feature of empirical distributions of asset returns. Kou (2002) provides a psychological interpretation for the empirical distribution using properties of DED.

The DEJD model has analytical tractability not only for plain vanilla options but also for path-dependent options. Kou (2002) and Kou and Wang (2004) derive analytical formulas for vanilla options and path-dependent options under the DEJD using Laplace transform. The computational time is typically quick as it only requires less than 2 seconds to obtain an option price.

As far as jump diffusion models are concerned, the market is incomplete so that there are infinitely many possible risk-neutral measures. Although the analysis in this paper is general enough for all risk-neutral measures and the determination of the correct measure is beyond the scope of the thesis, we would like to select one so that the discussion can be conducted smoothly. Kou (2002) characterizes the set of all risk-neutral measures using an equilibrium approach with the HARA utility functions. For simplicity, we adopt the risk-neutral measure consistent with the choice of Merton (1976), see Kou (2002) for further information. Specifically, choose

$$\mu = r - q - \lambda(\mathrm{E}[Y_t] - 1) = r - q - \lambda\left(\frac{p\eta_1}{\eta_1 - 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1\right),$$

where r is the risk free interest rate and other parameters are the same as the physical process.

Given the risk-neutral DEJD process, the moment generating function (mgf) of X_t can be obtained as

$$\begin{aligned} \mathrm{E}[e^{\theta X_t}|\mathcal{F}_0] &= e^{G(\theta)t}, \quad \theta \in (-\infty, \eta_1), \\ G(\theta) &= \left(r - q - \frac{\sigma^2}{2} - \lambda(\mathrm{E}[Y_t] - 1)\right)\theta + \frac{\sigma^2\theta^2}{2} + \\ & \quad \lambda\left(\frac{p\eta_1}{\eta_1 - \theta} + \frac{(1-p)\eta_2}{\eta_2 + \theta} - 1\right). \end{aligned} \tag{2.5}$$

It will be seen shortly that the mgf plays a key role in deriving various pricing formulae in this thesis.

Chapter 3

Option Pricing with DEJD

It has been indicated in the last chapter that option pricing requires the calculation of the expected terminal payoff. Under DEJD, direct computations may not be possible. Usually, closed-forms can be obtained only for the Black-Scholes (1973) model. Kou and Wang (2004) propose to use the Laplace transform to evaluate the expectations. They manage to derive closed-form solutions for the Laplace transforms of option prices. Their result includes the valuation of European, barrier and lookback options. This chapter aims at presenting their results. It also contains a new result for turbo warrants of Wong and Lau (2007).

3.1 Laplace Transform

Laplace transform is a useful integral transform in probability, statistics and mathematical finance because it is closely related to the moment generating function of a random variable. Usually, the Laplace transform is operating over a real domain. To convert Laplace solutions back into numerical solutions, it is necessary to perform a Laplace inversion. For simple problems, the inversion can be done with the help of a Laplace transform table. In most of the situations, we have to do the inversion numerically. The Laplace inversion generally operates over a complex domain.

Given these facts, we extend some parts of option pricing framework to the complex plane. We connect a function $f(t)$ with its Laplace transform $F(s)$ as follows.

$$
\begin{aligned}
F(s) &= \mathcal{L}_{t,s} f(t) = \int_{-\infty}^{\infty} f(t) e^{-st}\, dt, & (3.1)\\
f(t) &= \mathcal{L}_{t,s}^{-1} F(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st}\, ds, & (3.2)
\end{aligned}
$$

where $i = \sqrt{-1}$, $s > 0$ in (3.1), while s is a complex number ($s \in \mathcal{C}$) in (3.2). The integral transform in (3.1) defines the Laplace transform and the Bromwich integral in (3.2) defines the Laplace inversion.

Usually, $f(t) : \Re \to \Re$ is a real-valued function and hence $F(s)$ is real. However, the inverse transform (3.2) is preformed over the complex plane so that it is necessary to extend the function $F(s)$ to the complex plane analytically. To equip this, we recall an important expression of an elementary function on the complex plane,

$$e^{x+iy} = e^{x}(\cos y + i \sin y), \ \ x, y \in \Re.$$

The path integral in (3.2) is the Bromwich contour on the complex plane, c is a sufficiently large real number that all singularities of $F(s)$ are located in the left of the vertical line $\Re(z) = c$, where $\Re(z)$ denotes the real part of the complex number z. In addition, a singularity z^* of a function $F(s)$ means a complex number such that $F(z^*)$ is divergent or undefined.

Though the closed-form of inverse Laplace transform (3.2) is generally unavailable, numerical Laplace inversion provides efficient and accurate numerical results. We introduce a numerical Laplace inversion of Petrella (2004) here. Applying the trapezoidal rule to (3.2), with $\Delta s = ih$, we have

$$f(t) = \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} F(c + ihk)e^{t(c+ihk)}(ih) = \frac{he^{tc}}{2\pi} \sum_{k=-\infty}^{\infty} F(c + ihk)e^{thki}.$$

For a real-valued function $f(t)$, $F(s)$ is a conjugate function defined on the complex plane, i.e., $F(\overline{s}) = \overline{F(s)}$. Substituting $h = \pi/t$ into

$$e^{thki} = e^{k\pi i} = \cos(k\pi i) + i\sin(k\pi i) = (-1)^k,$$

the series becomes

$$f(t) = \frac{e^{tc}}{t} \left[\frac{F(c)}{2} + \sum_{k=1}^{\infty} \Re \left(F(c + \frac{k\pi}{t} i) \right) (-1)^k \right].$$

It can be shown that the discretization error is of $\mathcal{O}(e^{-tc})$ so that the infinite series gives an accurate result for a sufficiently large real number c.

As the infinite series happens to be an alternating series, we can implement the Euler summation to increase the speed of convergence. Consider the average of partial sums,

$$f(t) = \frac{e^{tc}}{t} \left[\frac{F(c)}{2} + \sum_{k=1}^{\infty} W_k \Re \left(F(c + \frac{k\pi}{t} i) \right) \right],$$

where

$$W_k = \begin{cases} (-1)^k, & 0 \leq k \leq n, \\ (-1)^k \sum_{r=0}^{n+m-k} C_r^m \frac{1}{2^m}, & n + 1 \leq k \leq n + m, \end{cases} \tag{3.3}$$

n and m are sufficiently large integers and $C_r^m \frac{1}{2^m}$ are coefficients of the binomial series.

In a higher dimensional case, we repeat the inversions and the general summation formula is given by

$$
\begin{aligned}
& f(t_1, \ldots, t_n) \\
= \; & \mathcal{L}^{-1}_{t_1,s_1} \cdots \mathcal{L}^{-1}_{t_n,s_n} F(s_1, \ldots, s_n) \\
= \; & \frac{1}{(2\pi i)^n} \int_{c_n - i\infty}^{c_n + i\infty} \cdots \int_{c_1 - i\infty}^{c_1 + i\infty} F(s_1, \ldots, s_n) e^{s_1 t_1 + \ldots + s_n t_n} \, ds_1 \ldots ds_n \\
= \; & \frac{\Pi_{j=1}^{k} h_j e^{\sum_{j=1}^{k} t_j c_j}}{(2\pi)^n} \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} F(c_1 + i h_1 k_1, \ldots, c_n + i h_n k_n) e^{\sum_{j=1}^{k} t_j h_j k_k i} \\
= \; & \frac{e^{\sum_{j=1}^{n} t_j c_j}}{2^n \Pi_{j=1}^{n} t_j} \left[\sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} F\left(c_1 + \frac{k_j \pi}{t_j} i, \ldots, c_n + \frac{k_j \pi}{t_j} i \right) (-1)^{\sum_{j=1}^{n} k_j} \right],
\end{aligned}
$$

where $F(\cdot)$ has no singularity on the domain $\{\zeta : \mathrm{Re}(\zeta) > c_1, \ldots, \mathrm{Re}(\zeta) > c_n\}$. The last equality follows by letting $h_j = \pi/t_j$, for $j = 1, 2, \ldots, n$. Certainly, we can implement the Euler summation by replacing $(-1)^{\sum_{j=1}^{n} k_j}$ by $\Pi_{j=1}^{n} W_{k_j}^{(j)}$ defined in (3.3) with $W_{-k_j}^{(j)} = W_{k_j}^{(j)}$.

3.2 European Option Pricing

We now demonstrate how Laplace transform is used in option pricing. The pricing representation of a European call is given by:

$$
e^{-r(T-t)} \mathrm{E}^Q[\max(S_T - K, 0) | \mathcal{F}_t].
$$

The closed-form of the expectation is not available under DEJD model. Thus, Kou (2002) considers the Laplace transform of the price. Let $k = -\log K$.

Consider the Laplace transform with respect to k. Therefore,

$$\begin{aligned} C(k) &= \mathrm{E}[\max(S_T - e^{-k}, 0)|\mathcal{F}_t], \\ \mathcal{L}_{k,\zeta}[C(k)] &= \int_{-\infty}^{\infty} e^{-\zeta k}\mathrm{E}[(S_T - e^{-k})1_{\{k>-\log S_T\}}|\mathcal{F}_t]\, dk, \\ &= \mathrm{E}\left[\int_{-\log S_T}^{\infty} e^{-\zeta k}(S_T - e^{-k})\, dk \middle| \mathcal{F}_t\right], \\ &= \frac{\mathrm{E}[S_T^{\zeta+1}|\mathcal{F}_t]}{\zeta(\zeta+1)} = \frac{S_t^{\zeta+1}e^{G(\zeta+1)(T-t)}}{\zeta(\zeta+1)}, \quad \Re(\zeta) > 0. \end{aligned} \tag{3.4}$$

The second equality follows from the Fubini's theorem. By the mgf derived in (2.5), we obtain the analytical solution. Finally, we take inverse Laplace transform in (3.4) at $\zeta = -\log K$ to obtain the option price.

For a European put option, the pricing representation is given by:

$$e^{-r(T-t)}\mathrm{E}^Q[\max(K - S_T, 0)|\mathcal{F}_t].$$

Let $k = \log(K)$ and $P(k) = \mathrm{E}[\max(e^{-k} - S_T, 0)|\mathcal{F}_t]$. Consider the Laplace transform of the option with respect to k,

$$\mathcal{L}_{k,\zeta}[P(k)] = \frac{\mathrm{E}[S_T^{-\zeta+1}|\mathcal{F}_t]}{\zeta(\zeta-1)} = \frac{S_t^{-\zeta+1}e^{G(-\zeta+1)(T-t)}}{\zeta(\zeta-1)}, \quad \Re(\zeta) > 1.$$

Again, the option price can be computed by taking inverse transform in (3.4). In the general case, this method can be applied to European options under Lévy processes (see Cont and Tankov, 2004a).

3.3 Barrier Option Pricing

Unlike a European option, the payoff of a path dependent option, such as lookback and barrier options, depends on the sample path of the underlying asset until maturity.

Kou and Wang (2004) solve lookback and barrier option pricing problems under DEJD. They consider the first passage time that the underlying asset value breaches a barrier level. The first passage time(s) are defined in the following way,

$$\tau_B := \begin{cases} \inf\{t|S_t \leq B = S_0 e^b\} & \text{for } S_0 > B, \\ \inf\{t|S_t \geq B = S_0 e^b\} & \text{for } S_0 < B. \end{cases}$$

We use an up-and-in call (UIC) option to illustrate the pricing framework although other barrier options can be valued using the same approach. The UIC option holder will knock-in a call option if the underlying asset price passes through a specified upward barrier level. The payoff function is given as follows

$$(S_T - K)^+ 1_{\{\tau_B < T\}},$$

where $1_{\{A\}}$ is the indicator function for the event A. Hence, the present value of the UIC option is,

$$\mathrm{UIC}(k, T) = e^{-rT}\mathrm{E}^Q[(S_T - e^{-k})^+ 1_{\{\tau_B < T\}}].$$

Kou, Petrella and Wang (2004) apply a double Laplace transform to the option price with respect to T and k:

$$\begin{aligned} & \mathcal{L}_{k,\zeta}\left[\mathcal{L}_{T,\alpha}\mathrm{UIC}(k,T)\right] \\ = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\zeta k} e^{-\alpha T} e^{-rT}\mathrm{E}^Q[(S_T - e^{-k})^+ 1_{\{\tau_B < T\}}]\, dk\, dT. \\ = & \frac{B^{\zeta+1}}{\zeta(\zeta+1)} \frac{1}{r + \alpha - G(\zeta+1)} \left(A(r+\alpha; b)\frac{\eta_1}{\eta_1 - \zeta - 1} + B(r+\alpha; b) \right), \end{aligned}$$

where

$$\begin{aligned} \hat{A}(z; b) &:= \mathrm{E}[e^{-z\tau} I_{\{S_\tau > B\}}] = \mathrm{E}[e^{-z\tau} I_{\{X(\tau_b) > b\}}], \\ \hat{B}(z; b) &:= \mathrm{E}[e^{-z\tau} I_{\{S_\tau = B\}}] = \mathrm{E}[e^{-z\tau} I_{\{X(\tau_b) = b\}}], \end{aligned}$$

$X_t = \log(S_t/S_0)$ as defiend in (2.2) and $b = \log(B/S_0)$.

Kou and Wang (2003) derive the analytical solutions for $\hat{A}(z;b)$ and $\hat{B}(z;b)$ for $b > 0$ under DEJD,

$$\begin{aligned} \hat{A}(z;b) &= \tfrac{(\eta_2+\beta_{1,z})(\beta_{2,z}+\eta_2)}{\eta_2(\beta_{2,z}-\beta_{1,z})}\left(e^{-b\beta_{1,z}} - e^{-b\beta_{2,z}}\right), & (3.5)\\ \hat{B}(z;b) &= -\tfrac{\eta_2+\beta_{1,z}}{\beta_{2,z}-\beta_{1,z}}e^{-b\beta_{1,z}} + \tfrac{\eta_2+\beta_{2,z}}{\beta_{2,z}-\beta_{1,z}}e^{-b\beta_{2,z}}, & (3.6)\end{aligned}$$

where $\beta_{1,z}$ and $\beta_{2,z}$ are positive real roots of the equation $G(\beta) = z$ and $G(\beta)$ is the mgf defined in (2.5). Kou and Wang (2003) show that, for $z > 0$, the equation $G(\beta) = z$ has exactly four real roots, namely $\beta_{1,z}$, $\beta_{2,z}$, $\beta_{3,z}$ and $\beta_{4,z}$, where

$$-\infty < \beta_{4,z} < -\eta_2 < \beta_{3,z} < 0 < \beta_{2,z} < \eta_1 < \beta_{1,z} < \infty. \tag{3.7}$$

As the equation $G(\beta) = z$ can be converted into a polynomial equation of degree 4, there is a closed-form solution for the roots. It is therefore possible to extend the closed-form solution to a complex domain for inverting the Laplace transform.

3.4 Lookback Options

The price of a floating strike lookback put (LP) option is given by

$$\mathrm{LP} = e^{-rT}\mathrm{E}\left[\max\left(M, \max_{t\in[0,T]} S_t\right) - S_T\right],$$

where M is the realized maximum asset value, T is the option's maturity and r is the constant risk-free interest rate.

Consider the Laplace transform of LP with respect to T,

$$\mathcal{L}_{T,\alpha}[\mathrm{LP}] = \frac{S_0^{\beta_{1,\alpha+r}}A_\alpha}{M^{\beta_{1,\alpha+r}-1}C_\alpha} + \frac{S_0^{\beta_{2,\alpha+r}}B_\alpha}{M^{\beta_{2,\alpha+r}-1}C_\alpha} + \frac{M}{\alpha+r} - \frac{S_0}{\alpha},$$

where

$$A_\alpha = \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1},\ B_\alpha = \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1},\ C_\alpha = (\alpha+r)\eta_1(\beta_{2,\alpha+r}-\beta_{1,\alpha+r}).$$

Therefore, the lookback option price can be computed via an efficient Laplace inversion method. The proof can be found in Kou, Petrella and Wang (2005).

3.5 Turbo Warrant

Results in the previous two subsections are well known. In this subsection, we present a new result taken from Wong and Lau (2007), in which we derive an analytical solution for turbo warrant using Laplace transform. The pricing formula requires us to carry out a triple Laplace inversion. Using properties of turbo warrants, we manage to reduce the computational burden of the triple Laplace inversion and calculate turbo warrant prices within a second.

Turbo warrants first appeared in Germany in late 2001. A more interesting situation appears when the barrier is set to be strictly in the money and a rebate is paid if the asset price passes the barrier. The rebate is usually calculated according to an exotic option payoff. At the end of February 2005, Societe Generale (SG) listed the first 40 turbo warrants on the Nordic Growth Market (NGM) and Nordic Derivatives Exchange. During February 2005, the turbo warrant trading revenue was 31 million kronor, 50% (31/55) of total NGM trading revenue of 55 million kronor. In June 2006, the Hong Kong Exchange and Clearing Limited (HKEx) introduced callable bull/bear contracts (CBBC), which are actually turbo warrants. There are two types of CBBC: N and R. The N-CBBC pays no rebate when the barrier is crossed whereas the R-CBBC pays an exotic rebate, see HKEx (2006).

Turbo warrants are attractive because it is believed that their prices are lower than their vanilla counterparts and are much less sensitive to the implied volatility. Thus, investors can simply bet on upward or downward movement of an asset with a lower cost and minimal volatility risk. This view is however based on the Black-Scholes asset price dynamics, the volatility of which is constant. Eriksson (2005) derives explicit solutions to turbo call and put warrants that are listed by SG using the Black-Scholes model. Wong and Lau (2007b) consider currency turbo warrants with mean reversion. The rebate of the turbo call (put) warrant is the difference between the lowest (highest) recorded stock price during a pre-specified period after the barrier is hit and the strike price. Therefore, the rebate can be viewed as a non-standard lookback option. These turbo call and put options are essentially the R-CBBC in Hong Kong. Eriksson (2005) points out that a turbo warrant is a lot less sensitive to the change in volatility than that of its vanilla counterpart.

When stochastic volatility is taken into account, Wong and Chan (2007b) find that turbo warrants can be very sensitive to the change in the shape of the volatility smile, contrasting the result of Ericsson (2005) using the Black-Scholes model. In fact, Wong and Chan (2007b) find that the sensitivity is model-dependent. Under the CEV model, a turbo warrant is more sensitive to the parallel shift of the volatility smile than its vanilla counterpart but the sensitivity to the change in skewness is similar. For the fast mean-reverting model stochastic volatility model, a turbo warrant is less sensitive to the change in volatility smile than its vanilla counterpart. For a two-scale volatility model, a turbo warrant can be more sensitive to the change in the volatility surface than a vanilla option.

Besides the stochastic volatility effect, it has been well documented that jump risk in asset price is an important factor for explaining the volatility smile, especially for short-term contracts. As there are no results on pricing turbo warrants under jump-diffusion, we investigate the turbo warrant pricing under the DEJD.

A turbo call warrant pays the option holder $(S_T - K)^+$ at maturity T if a specified barrier $B \geq K$ has not been passed by S_t at any time prior to the maturity. Denote τ_b as the first time that the asset price crosses the barrier B, i.e., $\tau_b = \inf\{t |\ S_t \leq B = S_0 e^b\}$. If $\tau_b \leq T$, then the contract is void and a new contract starts. The new contract is a call option on $m^h_{\tau_b} = \min_{\tau_b \leq t \leq \tau_b + h} S_t$, with the strike price K, and the time to maturity h.

More precisely, the turbo call option can be expressed as

$$\mathrm{TC}(t,S) = e^{-rT}\mathbf{E}_t\left[(S_T-K)^+\mathbf{1}_{\{\tau_b>T\}}\right] + \mathbf{E}_t\left[e^{-r(\tau_b+h)}(m^h_{\tau_b}-K)^+\mathbf{1}_{\{\tau_b\leq T\}}\right]. \quad (3.8)$$

It can be recognized from (3.8) that a turbo call warrant price can be decomposed into two parts. The first part resembles a down-and-out call (DOC) option, with a zero rebate, and the second part is a down-and-in lookback (DIL) option. We now define:

$$\mathrm{DOC}(t,S) = e^{-rT}\mathbf{E}_t\left[(S_T-K)^+\mathbf{1}_{\{\tau_H>T\}}\right], \quad (3.9)$$

$$\mathrm{DIL}(t,S,h) = \mathbf{E}_t\left[e^{-r(\tau_b+h)}(m^h_{\tau_b}-K)^+\mathbf{1}_{\{\tau_b\leq T\}}\right], \quad (3.10)$$

such that TC = DOC + DIL. It is important to notice that Kou and Wang (2003, 2004) and Kou et al. (2005) have developed a general pricing framework for barrier options. The pricing of DOC option can then be obtained through their approach. However, the pricing of the DIL option embedded in turbo warrants appears to be new in the literature and the derivation is not an obvious extension of any other path-dependent options.

The following lemma will be useful for a later computation. Let $b = \log(B/S_0)$ which is negative as the barrier level B is smaller than the current asset value S_0.

Lemma 3.1

$$A(z;b) := \mathrm{E}[e^{-z\tau_b} I_{\{X(\tau_b)<b\}}] = \frac{(\eta_2 + \beta_{1,z})(\beta_{2,z} + \eta_2)}{\eta_2(\beta_{2,z} - \beta_{1,z})} \left(e^{-b\beta_{1,z}} - e^{-b\beta_{2,z}}\right),$$

$$B(z;b) := \mathrm{E}[e^{-z\tau_b} I_{\{X(\tau_b)=b\}}] = -\frac{\eta_2 + \beta_{1,z}}{\beta_{2,z} - \beta_{1,z}} e^{-b\beta_{1,z}} + \frac{\eta_2 + \beta_{2,z}}{\beta_{2,z} - \beta_{1,z}} e^{-b\beta_{2,z}},$$

where η_2 is a parameter of the jump distribution in (2.4) and $\beta_{1,z}$ and $\beta_{2,z}$ are the roots defined in (3.7).

Proof The proof is similar to the one in Kou and Wang (2003) but we consider $b < 0$ in the present case.

Define

$$u(x) = \begin{cases} \gamma_1 e^{-\beta_1(b-x)} + \gamma_2 e^{-\beta_2(b-x)}, & x \geq b. \\ 1, & x < b, \end{cases},$$

with both β_1 and β_2 being negative, and constants γ_1 and γ_2 to be determined later. Denote infinitesimal generator of a jump-diffusion process X_t as

$$\mathcal{A}u(x) := \mu u'(x) + \frac{1}{2}\sigma^2 u''(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(y)] f_Y(y) d(y),$$

where $f_Y(y)$ is the pdf of the jump size for a general jump distribution. Specifically, we substitute the double exponential distribution (2.4) in the expression. Simple algebra shows that

$$\mathcal{A}u(x) = \sum_{k=1}^{2} \gamma_k G(\beta_k) e^{-\beta_k(b-x)} + \lambda(1-p) e^{\eta_2(b-x)} \left(1 - \sum_{k=1}^{2} \gamma_k \frac{\eta_2}{\eta_2 + \beta_k}\right).$$

Applying the Ito lemma to the process $\{e^{-zt} u(X_t)\}$, we find that the process

$$M(t) = e^{-z(t\wedge\tau_b)} u(X_{t\wedge\tau_b}) - \int_0^{t\wedge\tau_b} e^{-zs}(\mathcal{A}u(X_s) - zu(X_s))ds,$$

is a local martingale with $M(0) = u(0)$. If $\mathcal{A}u(X_s) - zu(X_s) = 0$, then we have

$$u(0) = \mathrm{E}M(t) = \mathrm{E}[e^{-z(t\wedge\tau_b)}u(X_{t\wedge\tau_b})] \to \mathrm{E}[e^{-z(\tau_b)}I_{\{\tau_b<\infty\}}], \qquad \text{as} \quad t \to \infty.$$

The limit is closely related to the function $A(z;b)$. Thus, it is our goal to determine the solution of the equation:

$$\mathcal{A}u(X_s) - zu(X_s) = 0.$$

Solving the equation gives

$$\begin{aligned} G(\beta_k) - z &= 0, \quad \beta_k < 0 \\ \gamma_1 \frac{\eta_2}{\eta_2 + \beta_1} + \gamma_2 \frac{\eta_2}{\eta_2 + \beta_2} &= 1, \\ \gamma_1 + \gamma_2 &= 1, \quad \text{by continuity of } u(x). \end{aligned}$$

This allows us to solve for the values of γ_1 and γ_2, and to obtain

$$\mathrm{E}[e^{-z(\tau_b)}I_{\{\tau_b<\infty\}}] = \frac{\beta_{2,z}(\eta_2 + \beta_{1,z})}{\eta_2(\beta_{2,z} - \beta_{1,z})} e^{-b\beta_{1,z}} - \frac{\beta_{1,z}(\eta_2 + \beta_{2,z})}{\eta_2(\beta_{2,z} - \beta_{1,z})} e^{-b\beta_{2,z}}. \tag{3.11}$$

Using a similar idea, we define

$$v(x) = \begin{cases} \gamma_1^* e^{-\beta_1(b-x)} + \gamma_2^* e^{-\beta_2(b-x)}, & x \geq a, \\ 0, & a - y \leq x < a, \\ 1, & x < a - y, \end{cases}$$

where $y > 0$. The infinitesimal generator applying to $v(x)$ gives

$$\mathcal{A}v(x) = \sum_{k=1}^{2} \gamma_k^* G(\beta_k) e^{-\beta_k(b-x)} + \lambda(1-p)e^{\eta_2(b-x)} \left(e^{-\eta_2 y} - \sum_{k=1}^{2} \gamma_k^* \frac{\eta_2}{\eta_2 + \beta_k} \right).$$

We construct a local martingale through the process $\{e^{-zt}v(X_t)\}$:

$$\widehat{M}(t) = e^{-z(t\wedge\tau_b)}v(X_{t\wedge\tau_b}) - \int_0^{t\wedge\tau_b} e^{-zs}(\mathcal{A}v(X_s) - zv(X_s))ds,$$

where it can be easily shown that $\mathrm{E}(e^{-\alpha\tau_b}I_{\{a-X_{\tau_a}>y\}}) = \lim_{t\to\infty}\mathrm{E}\widehat{M}(t) = \widehat{M}(0) = v(0)$ if $\mathcal{A}v(X_s) - zv(X_s) = 0$. The limit here is exactly the function $A(z;b)$ if we set $a = b$. Solving the equation, we arrive at

$$\begin{aligned} G(\beta_k) - z &= 0, \quad \beta_k < 0 \\ \gamma_1^*\frac{\eta_2}{\eta_2+\beta_1} + \gamma_2^*\frac{\eta_2}{\eta_1+\beta_2} &= e^{-\eta_2 y}, \\ \gamma_1^* + \gamma_2^* &= 0, \quad \text{by continuity of } v(x) \text{ at } x = b. \end{aligned}$$

Thus, the values of γ_1^* and γ_2^* are then obtained from the above equations.

By taking limit $y \to 0^+$ and setting $a = b$, we have

$$v(0) = \mathrm{E}(e^{-\alpha\tau_b}I_{\{b-X_{\tau_b}>0\}}) = A(z;b).$$

The expression of $A(z;b)$ is given in the Lemma 3.1. For the function $B(z;b)$, it is clear that

$$\begin{aligned} B(z;b) &= \mathrm{E}[e^{-z\tau_b}I_{\{X(\tau_b)=b\}}] \\ &= \mathrm{E}[e^{-z(\tau_b)}I_{\{\tau_b<\infty\}}] - \mathrm{E}[e^{-z(\tau_b)}I_{\{X(\tau_b)<b\}}] \\ &= \mathrm{E}[e^{-z(\tau_b)}I_{\{\tau_b<\infty\}}] - A(z;b). \end{aligned}$$

Using the expression of $A(z;b)$ and (3.11), we obtain the solution of $B(z;b)$ as in Lemma 3.1. □

With the help of Lemma 3.1, it is possible to derive the pricing formula for a turbo warrant.

Theorem 3.1

$$\mathcal{L}_{k,\zeta}\{e^{rh}DIL(T,S,K,h)\} = \frac{\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}S_{\tau_b}^{\zeta+1}\right]}{\zeta(\zeta+1)}\mathrm{E}\left[\min_{0\le t\le h}e^{X_t(\zeta+1)}\right],$$

provided that $X_0=0$ and the two expectations can be computed by using another two Laplace transforms:

$$\begin{aligned}
\mathcal{L}_{T,\alpha}\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}S_{\tau_b}^{\zeta+1}\right] &= \frac{B^{\zeta+1}}{\alpha}\left[A(r+\alpha;b)\frac{\eta_2}{\eta_2+\zeta+1}+B(r+\alpha;b)\right],\\
\mathcal{L}_{h,\beta}\mathrm{E}\left[\min_{0\le t\le h}e^{X_t(\zeta+1)}\right] &= \frac{1}{\beta}-\frac{(\zeta+1)\beta_{2,\beta}(\eta_2+\beta_{1,\beta})}{\beta\eta_2(\beta_{2,\beta}-\beta_{1,\beta})(\zeta+1-\beta_{1,\beta})}\\
&\quad+\frac{(\zeta+1)\beta_{1,\beta}(\eta_2+\beta_{2,\beta})}{\beta\eta_2(\beta_{2,\beta}-\beta_{1,\beta})(\zeta+1-\beta_{2,\beta})}.
\end{aligned}$$

Therefore, the triple Laplace transform for the DIL option respect to k, T and h is the product of above two expressions and $\frac{1}{\zeta(\zeta+1)}$. Specifically,

$$\begin{aligned}
&\mathcal{L}_{h,\beta}\mathcal{L}_{T,\alpha}\mathcal{L}_{k,\zeta}\{e^{rh}DIL(T,S,K,h)\}\\
=\ &\frac{B^{\zeta+1}}{\zeta(\zeta+1)\alpha\beta}\left[\frac{\hat{A}(r+\alpha;b)\eta_2}{\eta_2+\zeta+1}+\hat{B}(r+\alpha;b)\right]\\
&\times\left[1-\frac{\zeta+1}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})}\left(\frac{\beta_{2,\beta}(\eta_2+\beta_{1,\beta})}{\zeta+1-\beta_{1,\beta}}-\frac{\beta_{1,\beta}(\eta_2+\beta_{2,\beta})}{\zeta+1-\beta_{2,\beta}}\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
A(z;b) &:= \mathrm{E}[e^{-z\tau}I_{\{S_\tau<B\}}]=\mathrm{E}[e^{-z\tau}I_{\{X(\tau_b)<b\}}],\\
B(z;b) &:= \mathrm{E}[e^{-z\tau}I_{\{S_\tau=B\}}]=\mathrm{E}[e^{-z\tau}I_{\{X(\tau_b)=b\}}].
\end{aligned}$$

Proof By the Fubini's theorem, we have the following calculation:

$$\begin{aligned}
\mathcal{L}_{k,\zeta}\{e^{rh}\mathrm{DIL}\} &= \int_{-\infty}^{\infty}e^{-\zeta k}\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}\left(m_{\tau_b}^h-e^{-k}\right)^+\right]dk\\
&= \mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}\int_{-\infty}^{\infty}\left(m_{\tau_b}^h-e^{-k}\right)^+e^{-\zeta k}\,dk\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} \left[\frac{m_{\tau_b}^h e^{-\zeta k}}{\zeta} - \frac{e^{-(\zeta+1)k}}{\zeta+1}\right]_{-m_{\tau_b}^h}^{\infty}\right] \\
&= \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} \frac{\left(m_{\tau_b}^h\right)^{\zeta+1}}{\zeta(\zeta+1)}\right] \\
&= \frac{1}{\zeta(\zeta+1)} \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} \mathrm{E}\left[\left(m_{\tau_b}^h\right)^{\zeta+1} \middle| \mathcal{F}_{\tau_b}\right]\right], \qquad (3.12)
\end{aligned}
$$

where $\mathcal{F}_{\tau_b}$ is the information accumulated up to time τ_b. Substituting $m_{\tau_b}^h = S_{\tau_b} \min_{0 \le t \le h} e^{(X_{\tau_b+t} - X_{\tau_b})}$ into (3.12), we have

$$
\begin{aligned}
\mathcal{L}_{k,\zeta}\{e^{rh}\mathrm{DIL}\} &= \frac{1}{\zeta(\zeta+1)} \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} \mathrm{E}\left[S_{\tau_b}^{\zeta+1} \min_{0 \le t \le h} e^{(X_{\tau_b+t} - X_{\tau_b})(\zeta+1)} | \mathcal{F}_{\tau_b}\right]\right] \\
&= \frac{1}{\zeta(\zeta+1)} \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1}\right] \mathrm{E}\left[\min_{0 \le t \le h} e^{(X_{\tau_b+t} - X_{\tau_b})(\zeta+1)} | \mathcal{F}_{\tau_b}\right] \\
&= \frac{1}{\zeta(\zeta+1)} \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1}\right] \mathrm{E}\left[\min_{0 \le t \le h} e^{X_t(\zeta+1)}\right],
\end{aligned}
$$

where we have used the locally independent property of the DEJD model in the second line and the stationary property in the last line.

Consider the Laplace transform of the first expectation.

$$
\begin{aligned}
\mathcal{L}_{T,\alpha}\left\{\mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1}\right]\right\} &= \int_{-\infty}^{\infty} e^{-r\tau_b} \mathrm{E}\left[e^{-\alpha T} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1}\right] dT \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} e^{-r\tau_b} e^{-\alpha T} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1} \, dT\right] \\
&= \mathrm{E}\left[\int_{0}^{\infty} e^{-r\tau_b} e^{-\alpha(\tau_b+t)} S_{\tau_b}^{\zeta+1} \, dt\right] \\
&= \frac{1}{\alpha} \mathrm{E}\left[e^{-(r+\alpha)\tau_b} S_{\tau_b}^{\zeta+1}\right] \\
&= \frac{B^{\zeta+1}}{\alpha}\left[A(r+\alpha; b) \frac{\eta_2}{\eta_2+\zeta+1} + B(r+\alpha; b)\right].
\end{aligned}
$$

For the second expectation, we let $x_h = \min_{0 \le t \le h} X_t \le 0$, and do the following

$$
\begin{aligned}
&= \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} \left[\frac{m^h_{\tau_b} e^{-\zeta k}}{\zeta} - \frac{e^{-(\zeta+1)k}}{\zeta+1} \right]_{-m^h_{\tau_b}}^{\infty} \right] \\
&= \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} \frac{(m^h_{\tau_b})^{\zeta+1}}{\zeta(\zeta+1)} \right] \\
&= \frac{1}{\zeta(\zeta+1)} \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} \mathrm{E}\left[(m^h_{\tau_b})^{\zeta+1} \middle| \mathcal{F}_{\tau_b} \right] \right], \qquad (3.12)
\end{aligned}
$$

where $\mathcal{F}_{\tau_b}$ is the information accumulated up to time τ_b. Substituting $m^h_{\tau_b} = S_{\tau_b} \min_{0 \le t \le h} e^{(X_{\tau_b+t} - X_{\tau_b})}$ into (3.12), we have

$$
\begin{aligned}
\mathcal{L}_{k,\zeta}\{e^{rh}\mathrm{DIL}\} &= \frac{1}{\zeta(\zeta+1)} \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} \mathrm{E}\left[S^{\zeta+1}_{\tau_b} \min_{0 \le t \le h} e^{(X_{\tau_b+t} - X_{\tau_b})(\zeta+1)} | \mathcal{F}_{\tau_b} \right] \right] \\
&= \frac{1}{\zeta(\zeta+1)} \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} S^{\zeta+1}_{\tau_b} \right] \mathrm{E}\left[\min_{0 \le t \le h} e^{(X_{\tau_b+t} - X_{\tau_b})(\zeta+1)} | \mathcal{F}_{\tau_b} \right] \\
&= \frac{1}{\zeta(\zeta+1)} \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} S^{\zeta+1}_{\tau_b} \right] \mathrm{E}\left[\min_{0 \le t \le h} e^{X_t(\zeta+1)} \right],
\end{aligned}
$$

where we have used the locally independent property of the DEJD model in the second line and the stationary property in the last line.

Consider the Laplace transform of the first expectation.

$$
\begin{aligned}
\mathcal{L}_{T,\alpha}\left\{ \mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} S^{\zeta+1}_{\tau_b} \right] \right\} &= \int_{-\infty}^{\infty} e^{-r\tau_b} \mathrm{E}\left[e^{-\alpha T} I_{\{\tau_b \le T\}} S^{\zeta+1}_{\tau_b} \right] dT \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} e^{-r\tau_b} e^{-\alpha T} I_{\{\tau_b \le T\}} S^{\zeta+1}_{\tau_b} \, dT \right] \\
&= \mathrm{E}\left[\int_{0}^{\infty} e^{-r\tau_b} e^{-\alpha(\tau_b+t)} S^{\zeta+1}_{\tau_b} \, dt \right] \\
&= \frac{1}{\alpha} \mathrm{E}\left[e^{-(r+\alpha)\tau_b} S^{\zeta+1}_{\tau_b} \right] \\
&= \frac{B^{\zeta+1}}{\alpha} \left[A(r+\alpha; b) \frac{\eta_2}{\eta_2 + \zeta + 1} + B(r+\alpha; b) \right].
\end{aligned}
$$

For the second expectation, we let $x_h = \min_{0 \le t \le h} X_t \le 0$, and do the following calculation.

$$\begin{aligned}
&\mathrm{E}\left[\min_{0\le t\le h} e^{X_t(\zeta+1)}\right] = \mathrm{E}\left[e^{x_h(\zeta+1)}\right] = \int_{-\infty}^{0} e^{y(\zeta+1)}\, d[P(x_h \le y)] \\
&= \left[e^{y(\zeta+1)} P(x_h \le y)\right]_{-\infty}^{0} - (\zeta+1)\int_{-\infty}^{0} e^{y(\zeta+1)} P(x_h \le y)\, dy \\
&= 1 - (\zeta+1)\int_{-\infty}^{0} e^{y(\zeta+1)} P(\tau_y \le h)\, dy.
\end{aligned}$$

We then consider the Laplace transform:

$$\begin{aligned}
&\mathcal{L}_{h,\beta}\left\{\int_{-\infty}^{0} e^{y(\zeta+1)} P(\tau_y \le h)\, dy\right\} = \int_{-\infty}^{0} e^{y(\zeta+1)} \left[\int_{0}^{\infty} e^{-\beta h} P(\tau_y \le h)\, dh\right] dy \\
&= \int_{-\infty}^{0} e^{y(\zeta+1)} \left[\int_{0}^{\infty} e^{-\beta h} \int_{0}^{h} P(\tau_y \in ds)\, dh\right] dy \\
&= \int_{-\infty}^{0} e^{y(\zeta+1)} \left[\int_{0}^{\infty} e^{-\beta(s+u)} \int_{0}^{\infty} P(\tau_y \in ds)\, du\right] dy, \qquad h = s+u \\
&= \int_{-\infty}^{0} e^{y(\zeta+1)} \left[\int_{0}^{\infty} e^{-\beta s} P(\tau_y \in ds) \int_{0}^{\infty} e^{-\beta u}\, du\right] dy \\
&= \int_{-\infty}^{0} e^{y(\zeta+1)} \left[\mathrm{E}\left[e^{-\beta\tau_y} I_{\{\tau_y<\infty\}}\right]/\beta\right] dy.
\end{aligned}$$

As $\mathrm{E}[e^{-z\tau_b} I_{\{\tau_b<\infty\}}] = A(z;b) + B(z;b)$, we have

$$\begin{aligned}
&\int_{-\infty}^{0} e^{y(\zeta+1)} \mathrm{E}\left[e^{-\beta\tau_y} I_{\{\tau_y<\infty\}}\right] dy \\
&= \int_{-\infty}^{0} \left[\frac{\beta_{2,\beta}(\eta_2+\beta_{1,\beta})}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})} e^{y(\zeta+1-\beta_{1,\beta})} - \frac{\beta_{1,\beta}(\eta_2+\beta_{2,\beta})}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})} e^{y(\zeta+1-\beta_{2,\beta})}\right] dy \\
&= \frac{\beta_{2,\beta}(\eta_2+\beta_{1,\beta})}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})(\zeta+1-\beta_{1,\beta})} - \frac{\beta_{1,\beta}(\eta_2+\beta_{2,\beta})}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})(\zeta+1-\beta_{2,\beta})}.
\end{aligned}$$

After recognizing $\mathcal{L}_{h,\beta}\{1\} = \frac{1}{\beta}$, we obtain an expression for $\mathcal{L}_{h,\beta}\mathrm{E}\left[e^{x_h(\zeta+1)}\right]$ as shown in the Lemma. □

From Theorem 3.1, the DIL option can be valued via a triple Laplace inversion. Appendix A presents a numerical triple Laplace inversion using the Euler summation, where we have reduced the computational burden using properties of the expectations in Theorem 3.1. More specifically, the computation is reduced to an order similar to that of the double Laplace inversion proposed by Petrella (2004).

3.6 Numerical Examples

Numerical examples are used to show that the analytical solution implemented with the triple Laplace inversion is efficient and accurate. We compare our numerical results with the Monte Carlo (MC) simulation. The simulation for jump diffusion models is classic, see Chapter 10.4 of Chan and Wong (2006). It is found from the MC simulation that the approximated turbo price converges slowly with the time step, Δt. For instance, when we run the simulation with $r = 5\%$, $q = 0\%$, $\sigma = 0.3$, $\lambda = 2.0$, $\eta_1 = 25$, $\eta_2 = 20$, $p = 0.6$, $T = 1.0$, $S_0 = 100$, $B = 95$ and $h = 1/24$, the numerical result is converging to a stable value when the Δt is approaching to zero. However, the approximated value is not close enough to the true value with $\Delta t = 10^{-5}$ and 40,000 sample paths. Figure 3.1 plots the turbo warrant price against the time step. The points marked '$*$', at $\Delta t = 0$, are produced by the Richardson extrapolation using the four data points marked 'o'. When the extrapolated value is compared to the simulated price with $\Delta t = 10^{-5}$, the difference is around 0.03 for all strike prices. Thus, we will use the warrant price produced from the Richardson extrapolation as the benchmark to verify our analytical solution.

We check the performance of the analytical solution implemented with the triple Laplace inversion in Table 4.2. The MC option price is obtained from the Richardson extrapolation based on the MC simulations with $\Delta t = 10^{-5}, 5 \times 10^{-5}$, 25×10^{-5}, 0.001 and 0.005. It can be seen that using inverse Laplace transform (ILT) produces results consistent with the MC simulation. The difference be-

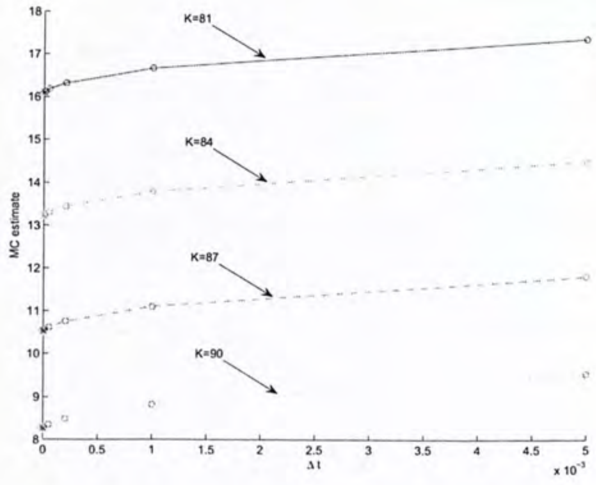

Figure 3.1: MC Estimates and Extrapolation

Strike (K)	81		84		87		90	
Time (h)	ILT	MC	ILT	MC	ILT	MC	ILT	MC
1 week	17.179	17.202	14.235	14.260	11.347	11.371	8.708	8.731
2 weeks	16.073	16.100	13.199	13.227	10.508	10.535	8.241	8.265
1 month	14.685	14.714	12.041	12.070	9.713	9.739	7.865	7.887
2 months	13.209	13.242	10.952	10.983	9.050	9.076	7.580	7.603
3 months	12.399	12.439	10.397	10.432	8.733	8.761	7.450	7.474

Table 3.1: Simulation vs. Analytical Solution

tween two approaches increases when the value of h increases. This is because the discretization error of the realized minimum asset price, after the barrier is crossed, accumulates with the value of h. The MC simulation is then biased upward once the h is getting large. We would like to stress that computing one option price only takes about 0.6 seconds using the triple Laplace transform. This computational time is remarkably efficient and useful for calibration.

As volatility smile can be partly explained by a jump diffusion model, we examine the sensitivity of turbo warrants to the volatility smile through examining the sensitivity to the jump parameters. Specifically in the DEJD model, we are interested in the change in turbo price with respect to the marginal change in

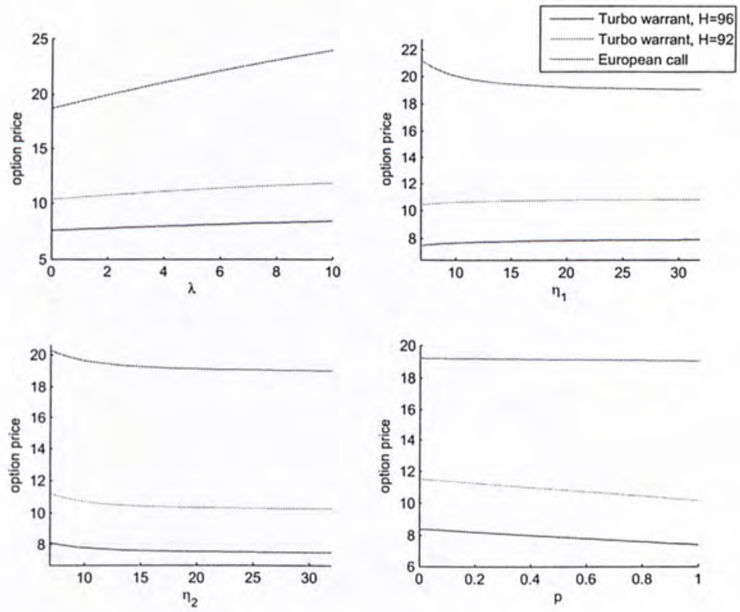

Figure 3.2: Sensitivity Analysis

λ, η_1, η_2 and p. If implied volatility is insignificant to turbo pricing, then it should be the case that a turbo warrant is much less sensitive to the jump parameters than its vanilla counterpart.

Figure 3.2 plots the turbo and vanilla call prices against jump parameters. In all cases, we fix the strike price at 90 and vary the barrier level for the turbo warrant. It can be seen that the turbo warrant is much less sensitive to the jump arrival rate λ when compared with the vanilla call option. However, the sensitivities to the mean parameters η_1 and η_2 are similar to those of the vanilla option. The turbo warrant is more sensitive to the skewness parameter p. It is reasonable as the the skewness of the asset distribution greatly affects the likelihood of falling into the knock-in region. In general, the vanilla call option is more sensitive to the jump parameters but is not very significant. Hence, the jump risk cannot be ignored in the valuation of turbo warrants.

Finally, we discuss the difference between the diffusion model and the jump

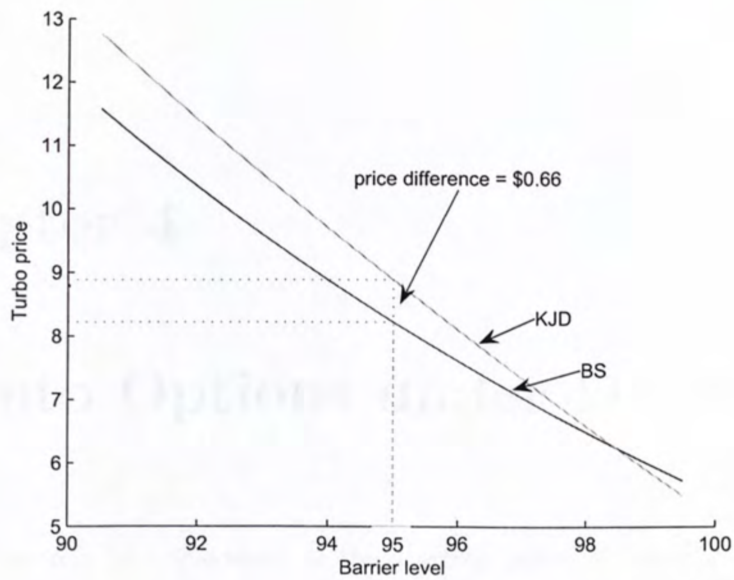

Figure 3.3: Comparing BS and DEJD Turbo Prices

diffusion model in pricing turbo warrant. Figure 3.3 plots the BS turbo price and DEJD turbo price against the barrier level using the parameters: $r = 5\%$, $q = 0\%$, $\sigma = 0.3$, $\lambda = 5.0$, $\eta_1 = 10$, $\eta_2 = 10$, $p = 0.5$, $T = 1.0$, $S_0 = 100$, $K = 90$ and $h = 1/24$. Since a turbo call warrant requires the barrier level to be larger than the strike price, we plot the graph for the barrier level being larger than 90. It can be seen that the price difference can be quite large. For instance, when the barrier is set to 95, the price difference is 0.66 while the turbo price is in between 8 and 9. Thus, the pricing error is larger than 7.3%. When the barrier level gets closer to the strike price, the difference is even larger. Given that the turbo price can be computed within a second, it is worth to include the DEJD turbo price in practice.

Chapter 4

Quanto Options under DEJD

This chapter can be considered as the starting point of the key part of this thesis. We consider option contracts where the payoff depends on two financial quantities: exchange rate, F_t, and foreign equity, S_t. According to the DEJD model, the marginal dynamics of the two quantities, exchange rate and foreign equity, should be consistent with (2.1). Specifically,

$$\begin{aligned}\frac{dS}{S} &= \mu_S\, dt + \sigma_S\, dW_S + Y_S\, dN_1,\\ \frac{dF}{F} &= \mu_F\, dt + \sigma_F\, dW_F + Y_F\, dN_2.\end{aligned}$$

To model the covariation of the two stochastic processes, we introduce the dependent structure in 3 manners: the joint Wiener process, common jump arrivals Poisson processes and correlated jumps sizes.

For the joint Wiener process (W_S, W_F), the changes of (W_S, W_F), i.e. $(\Delta W_S, \Delta W_F)$ for a time interval Δt, follows a bivariate normal distribution with mean $[0 \;\; 0]'$ and variance-covariance matrix

$$\Delta t \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

i.e., $\mathrm{E}[dW_S dW_F] = \rho\, dt$. We assume that $P(W_S(0) = 0, W_F(0) = 0) = 1$.

The dependent structure between Poisson processes can be modeled by using a common jump arrival Poisson process. Let N be the Poisson process recording the number of common jumps such that $N_1 = N_S + N$ and $N_2 = N_F + N$. We assume that the Poisson processes N_S, N_F and N are independent and have intensities λ_S, λ_F and λ, respectively.

In other words, we consider the joint asset dynamic of S_t and F_t as follows.

$$\frac{dS}{S} = \mu_S\, dt + \sigma_S\, dW_S + Y_S\, dN_S + Y_S\, dN, \tag{4.1}$$

$$\frac{dF}{F} = \mu_F\, dt + \sigma_F\, dW_F + Y_F\, dN_F + Y_F\, dN, \tag{4.2}$$

where W_S and W_F are Wiener processes with correlation coefficient ρ, μ_S and μ_F are the drifts of the foreign equity and the exchange rate to be determined later, σ_S is the volatility of S, σ_F is the volatility of F, $\ln(Y_S + 1)$ and $\ln(Y_F + 1)$ are the jump sizes, and N_S, N_F and N are independent Poisson processes with intensities λ_S, λ_F and λ respectively. We allow jump sizes Y_S and Y_F to be dependent random variables. The joint distribution of (Y_S, Y_F) will be specified in Section 4.2.

4.1 Domestic Risk-neutral Dynamics

As quanto options are traded in domestic currency, we concentrate on the processes of S and F under the domestic risk neutral measure, Q^d. Let B_d denote the domestic bank account and B_f the foreign bank account. Therefore,

$$\frac{dB_d}{B_d} = r_d\, dt \quad \text{and} \quad \frac{dB_f}{B_f} = r_f\, dt,$$

where r_d is the domestic interest rate and r_f is the foreign interest rate. If a domestic investor deposits to the foreign bank account, her position can be described as FB_f. The dynamics of her position is given by

$$\frac{d(FB_f)}{FB_f} = (\mu_F + r_f)\,dt + \sigma_F\,dW_F + Y_F\,dN_F + Y_F\,dN.$$

Under Q^d, FB^f/B^d is a martingale. Therefore,

$$\mu_F = r_d - r_f - (\lambda_F + \lambda)\mathrm{E}(Y_F), \tag{4.3}$$

which completes the process (4.2).

Under Q^d, consider the process of S

$$\frac{dS}{S} = \mu_S\,dt + \sigma_S\,dW_S^d + Y_S\,dN_S^d + Y_S\,dN^d, \tag{4.4}$$

where μ_S is a constant, W_S^d is a Wiener process in Q^d, and N_S^d and N^d are independent Poisson processes in Q^d. If the investor purchases the foreign asset, the dynamic of her domestic position, $S^* = FS$, calculated using (4.2) and (4.4) becomes

$$\begin{aligned}\frac{dS^*}{S^*} &= \frac{dS}{S} + \frac{dF}{F} + \frac{dS}{S}\frac{dF}{F}\\ &= (\mu_S + \mu_F + \rho\sigma_S\sigma_F)\,dt + (\sigma_S\,dW_S^d + \sigma_F\,dW_F)\\ &\quad + Y_S\,(dN_S^d + dN^d) + Y_F\,(dN_F + dN) + Y_S Y_F\,dN dN^d.\end{aligned} \tag{4.5}$$

As S^* can be viewed as a domestic asset, S^*/B_d is a martingale under Q^d. Hence,

$$\begin{aligned}&\mu_S + \mu_F + \rho\sigma_S\sigma_F\\ =\ & r_d - (\lambda_S + \lambda)\mathrm{E}(Y_S) - (\lambda_F + \lambda)\mathrm{E}(Y_F) - \lambda\mathrm{E}(Y_S Y_F).\end{aligned} \tag{4.6}$$

Putting (4.6) and (4.3) together,

$$\mu_S = r_f - \rho\sigma_S\sigma_F - (\lambda_S + \lambda)\mathrm{E}(Y_S) - \lambda\mathrm{E}(Y_S Y_F). \tag{4.7}$$

Using (4.2) and (4.1), we obtain

$$\begin{aligned}\frac{dS^*}{S^*} &= (r_d - (\lambda_S + \lambda)\mathrm{E}(Y_S) + (\lambda_F + \lambda)\mathrm{E}(Y_F) + \rho\sigma_S\sigma_F)\, dt + \sigma_S\, dW_S \\ &\quad + \sigma_F\, dW_F + Y_S\,(dN_S + dN) + Y_F\,(dN_F + dN) + Y_S Y_F\, dN,\end{aligned}$$

which is consistent with (4.5) and (4.6) if the following equalities hold,

$$\begin{aligned}\rho\sigma_F\, dt + dW_S &= dW_S^d, \\ dN_S &= dN_S^d, \\ \lambda\, dt + dN &= dN^d.\end{aligned} \tag{4.8}$$

Hence, the processes of S and S^* under Q^d are respectively (4.4) and (4.5) with N^d, N_S^d, W^d defined in (4.8) and μ_S defined in (4.7).

4.2 The Exponential Copula

Let $y_S = \ln(Y_S + 1)$ and $y_F = \ln(Y_F + 1)$. According to DEJD, the marginal distributions of y_S and y_F are double exponential distributions (DED). Specifically, we assume that $y_S \sim \mathrm{DED}(p_S, \eta_S^u, \eta_S^d)$ and $y_F \sim \mathrm{DED}(p_F, \eta_F^u, \eta_F^d)$ where the pdf is given in (2.4). However, one can separately consider upward and downward jumps so that the one-sided jump sizes are exponentially distributed. In other words, we use the following decomposition:

$$\begin{aligned}(y_S, y_F) &= (b_S^1, b_F^1)1_{\{a_S \geq 0,\ a_F \geq 0\}} + (-b_S^2, b_F^2)1_{\{a_S < 0,\ a_F \geq 0\}} \\ &\quad + (-b_S^3, -b_F^3)1_{\{a_S < 0,\ a_F < 0\}} + (b_S^4, -b_F^4)1_{\{a_S \geq 0,\ a_F < 0\}},\end{aligned} \tag{4.9}$$

where the superscript i of b_S and b_F represents the i-th quadrant of the y_S-y_F plane, i.e., $i = 1, 2, 3$ and 4. In addition, we denote

$$\begin{aligned} p^1 &:= \Pr(y_S \geq 0,\ y_F \geq 0), \\ p^2 &:= \Pr(y_S < 0,\ y_F \geq 0), \\ p^3 &:= \Pr(y_S < 0,\ y_F < 0), \\ p^4 &:= \Pr(y_S \geq 0,\ y_F < 0). \end{aligned}$$

The random variables b_S^i and b_F^i follow exponential distributions for all $i = 1, 2, 3$ and 4. Let us concentrate on b_S^i and b_F^i for a fixed i. As common jumps and dependent jump sizes are allowed, the joint distribution for b_S^i and b_F^i should be specified such that their marginal distributions are univariate exponential distributions.

We employ the Marshall and Olkin (1967) copula for exponential marginals:

$$\Pr(b_S^i > u, b_F^i > v) = \exp\left(-\eta_S^i u - \eta_F^i v - \eta_i \max(u, v)\right). \tag{4.10}$$

Marshall and Olkin (1967) showed that the joint distribution of (4.10) leads b_S^i and b_F^i to follow exponential marginals with parameters $\eta_S^i + \eta_i$ and $\eta_F^i + \eta_i$ respectively. Hence y_S and y_F are marginally degenerated to double exponential distributions if the following conditions are satisfied:

$$\begin{aligned} \eta_S^1 + \eta_1 &= \eta_S^4 + \eta_4 = \eta_S^u, \\ \eta_S^2 + \eta_2 &= \eta_S^3 + \eta_3 = \eta_S^d, \\ \eta_F^1 + \eta_1 &= \eta_F^2 + \eta_2 = \eta_F^u, \\ \eta_F^3 + \eta_3 &= \eta_F^4 + \eta_4 = \eta_F^d. \end{aligned}$$

There are some nice properties of this multivariate exponential distribution. For instance, the joint density function can easily be obtained as,

$$
\begin{aligned}
f_i(u,v) &= (\eta_S^i + \eta_i)\eta_F^i e^{-(\eta_S^i+\eta_i)u-\eta_F^i v}, & u > v, \\
f_i(u,v) &= (\eta_F^i + \eta_i)\eta_S^i e^{-\eta_S^i u-(\eta_F^i+\eta_i)v}, & u < v, \\
f_i(u,v) &= \eta_i e^{-(\eta_S^i+\eta_F^i+\eta_i)u}\delta(u-v), & u = v,
\end{aligned}
$$

where $\delta(\cdot)$ is the Dirac delta function. It shows that the random variables has a mass density along the line $u = v$, which contributes the correlation structure between them. To show this, observe that the first two expressions are trivial consequences of direct differentiation. For the last expression, evaluate the following probability,

$$
\begin{aligned}
& \Pr(b_S^i \in (t, t+\Delta], b_F^i \in (t, t+\Delta]) \\
= \; & \exp\left(-\eta_S^i t - \eta_F^i t - \eta_i \max(t,t)\right) \\
& - \exp\left(-\eta_S^i (t+\Delta) - \eta_F^i t - \eta_i \max(t+\Delta, t)\right) \\
& - \exp\left(-\eta_S^i t - \eta_F^i (t+\Delta) - \eta_i \max(t, t+\Delta)\right) \\
& + \exp\left(-\eta_S^i (t+\Delta) - \eta_F^i (t+\Delta) - \eta_i \max(t+\Delta, t+\Delta)\right) \\
= \; & e^{-(\eta_S^i+\eta_F^i+\eta_i)t}\left[1 - e^{-(\eta_S^i+\eta_i)\Delta} + e^{-(\eta_F^i+\eta_i)\Delta} + e^{-(\eta_S^i+\eta_F^i+\eta_i)\Delta}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
f_i(t,t) &= \lim_{\Delta\to 0} \frac{1}{\Delta^2} \Pr(b_S^i \in (t, t+\Delta], b_F^i \in (t, t+\Delta]) \\
&= \lim_{\Delta\to 0} \frac{1}{\Delta^2} e^{-(\eta_S^i+\eta_F^i+\eta_i)t}\left[1 - e^{-(\eta_S^i+\eta_i)\Delta} - e^{-(\eta_F^i+\eta_i)\Delta} + e^{-(\eta_S^i+\eta_F^i+\eta_i)\Delta}\right] \\
&= e^{-(\eta_S^i+\eta_F^i+\eta_i)t} \lim_{\Delta\to 0} \frac{1}{2\Delta}\left[(\eta_S^i+\eta_i) + (\eta_F^i+\eta_i) - (\eta_S^i+\eta_F^i+\eta_i)\right] \\
&= e^{-(\eta_S^i+\eta_F^i+\eta_i)t}\eta_i\delta(u-v).
\end{aligned}
$$

Given the pdf, the moment generating function (mgf) is then derived as

$$
\begin{aligned}
E(e^{\alpha b_S^i+\beta b_F^i}) &= E(e^{\alpha b_S^i+\beta b_F^i}1_{\{b_S^i>b_F^i\}}) + E(e^{\alpha b_S^i+\beta b_F^i}1_{\{b_S^i<b_F^i\}}) + E(e^{\alpha b_S^i+\beta b_F^i}1_{\{b_S^i=b_F^i\}}) \\
&= \frac{(\eta_S^i+\eta_i)\eta_F^i}{(\eta_S^i+\eta_i-\alpha)(\eta_S^i+\eta_F^i+\eta_i-\alpha-\beta)} \\
&\quad +\frac{(\eta_F^i+\eta_i)\eta_S^i}{(\eta_F^i+\eta_i-\beta)(\eta_S^i+\eta_F^i+\eta_i-\alpha-\beta)} \\
&\quad +\frac{\eta_i}{\eta_S^i+\eta_F^i+\eta_i-\alpha-\beta}.
\end{aligned} \tag{4.11}
$$

4.3 The moment generating function

The mgf (4.11) of the multivariate exponential distribution of Marshall and Olkin (1967) leads to the mgf of (F_T, S_T), which is defined as $E(S_T^\alpha F_T^\beta)$. The expectation can be taken under either Q^d or Q^f depending on the problem of interest.

Consider $X_t = \ln S_t/S_0$ and $Y_t = \ln F_t/F_0$. Applying Itô's lemma on (4.1) and (4.2) yields

$$
dX_t = d\log S = \mu_X\,dt + \sigma_S\,dW_S + y_S\,dN_S + y_S\,dN, \tag{4.12}
$$

$$
dY_t = d\log F = \mu_Y\,dt + \sigma_F\,dW_F + y_F\,dN_F + y_F\,dN, \tag{4.13}
$$

or, equivalently,

$$
X_T = \mu_X T + \sigma_S W_S + \sum_{i=1}^{N_S(T)} y_{S,i} + \sum_{k=1}^{N(T)} y_S^{(k)}, \tag{4.14}
$$

$$
Y_T = \mu_Y T + \sigma_F W_F + \sum_{j=1}^{N_F(T)} y_{F,j} + \sum_{k=1}^{N(T)} y_F^{(k)}, \tag{4.15}
$$

where $y_S = \ln(Y_S+1)$, $y_F = \ln(Y_F+1)$, $E[dW_S\,dW_F] = \rho\,dt$, $E[N_S] = \lambda_S$,

$E[N_F] = \lambda_F$, $E[N] = \lambda$ and

$$\mu_X = \mu_S - \sigma_S^2/2 = r_f - \rho\sigma_S\sigma_F - (\lambda_S + \lambda)E(Y_S) - \lambda E(Y_S Y_F) - \sigma_S^2/2,$$
$$\mu_Y = \mu_F - \sigma_F^2/2 = r_d - r_f - (\lambda_F + \lambda)E(Y_F) - \sigma_F^2/2. \quad (4.16)$$

Theorem 4.1 *If the exchange rate and the foreign equity price follow DEJD, then the moment generating function is given by*

$$E_t\left[S_T^\alpha F_T^\beta\right] = S_t^\alpha F_t^\beta \exp\left(G(\alpha,\beta;\mu_X,\mu_Y)(T-t)\right),$$

where

$$G(\alpha,\beta;\mu_X,\mu_Y) := \alpha\mu_X + \beta\mu_Y + \frac{1}{2}[\alpha^2\sigma_S^2 + 2\rho\alpha\beta\sigma_S\sigma_F + \beta^2\sigma_F^2] \quad (4.17)$$
$$+\lambda_S E[e^{\alpha y_S} - 1] + \lambda_F E[e^{\beta y_F} - 1] + \lambda E[e^{\alpha y_S + \beta y_F} - 1].$$

The expected values in (4.17) can be calculated using the identity,

$$\begin{aligned} & E[e^{\alpha a_S + \beta a_F}] \\ = \; & p^1 E[e^{\alpha b_S^1 + \beta b_F^1}] + p^2 E[e^{\alpha(-b_S^2) + \beta b_F^2}] + p^3 E[e^{\alpha(-b_S^3) + \beta(-b_F^3)}] + p^4 E[e^{\alpha b_S^4 + \beta(-b_F^4)}] \\ = \; & \sum_{i=1}^{4} \frac{p^i}{\eta_1^i + \eta_2^i + \eta^i - \delta_i\alpha - \epsilon_i\beta}\left(\frac{(\eta_1^i + \eta^i)\eta_2^i}{\eta_1^i + \eta^i - \delta_i\alpha} + \frac{(\eta_2^i + \eta^i)\eta_1^i}{\eta_2^i + \eta^i - \epsilon_i\beta} + \eta^i\right), \end{aligned}$$

where $\delta_1 = \delta_4 = \epsilon_1 = \epsilon_2 = 1$ and $\delta_2 = \delta_3 = \epsilon_3 = \epsilon_4 = -1$.

Proof Consider the following calculation,

$$\begin{aligned} \alpha \log S_T + \beta \log F_T = \; & \alpha X_t + \beta Y_t + (\alpha\mu_X + \beta\mu_Y)(T-t) \\ & + \alpha\sigma_S\left[W_S(T) - W_S(t)\right] + \beta\sigma_F\left[W_F(T) - W_F(t)\right] \\ & + \alpha \sum_{i=N_S(t)}^{N_S(T)} y_{S,i} + \beta \sum_{j=N_F(t)}^{N_F(T)} y_{F,j} + \sum_{k=N(t)}^{N(T)} \left[\alpha y_S^{(k)} + \beta y_F^{(k)}\right]. \end{aligned}$$

We then directly substitute the above expression into the mgf so that

$$\begin{aligned}
& \mathrm{E}\left[e^{\alpha \log S_T+\beta \log F_T}\right] \\
= & \mathrm{E}\left[e^{\alpha X_t+\beta Y_t}\right] \mathrm{E}\left[e^{(\alpha\mu_X+\beta\mu_Y)(T-t)}\right] \mathrm{E}\left[e^{\alpha\sigma_S\,[W_S(T)-W_S(t)]+\beta\sigma_F\,[W_F(T)-W_F(t)]}\right] \\
& \times \mathrm{E}\left[e^{\alpha\sum_{i=N_S(t)}^{N_S(T)} y_{S,i}}\right] \mathrm{E}\left[e^{\beta\sum_{j=N_F(t)}^{N_F(T)} y_{F,j}}\right] \mathrm{E}\left[e^{\sum_{k=N(t)}^{N(T)}\left[\alpha y_S^{(k)}+\beta y_F^{(k)}\right]}\right] \\
= & S_t^\alpha F_t^\beta e^{(\alpha\mu_X+\beta\mu_Y)(T-t)} e^{(\alpha^2\sigma_S^2+2\rho\alpha\beta\sigma_S\sigma_F+\beta^2\sigma_F^2)\frac{T-t}{2}} \mathrm{E}\left[\mathrm{E}[e^{\alpha\sum_{i=N_S(t)}^{N_S(T)} y_{S,i}}|\Delta N_S]\right] \\
& \times \mathrm{E}\left[\mathrm{E}[e^{\beta\sum_{j=N_F(t)}^{N_F(T)} y_{F,j}}|\Delta N_F]\right] \mathrm{E}\left[\mathrm{E}[e^{\sum_{k=N(t)}^{N(T)}\left[\alpha y_S^{(k)}+\beta y_F^{(k)}\right]}|\Delta N]\right] \\
= & S_t^\alpha F_t^\beta e^{(\alpha\mu_X+\beta\mu_Y)(T-t)} e^{(\alpha^2\sigma_S^2+2\rho\alpha\beta\sigma_S\sigma_F+\beta^2\sigma_F^2)\frac{T-t}{2}} \\
& \times \mathrm{E}\left[\mathrm{E}[e^{\alpha y_S}]^{\Delta N_S}\right] \mathrm{E}\left[\mathrm{E}[e^{\beta y_F}]^{\Delta N_F}\right] \mathrm{E}\left[\mathrm{E}[e^{\alpha y_S+\beta y_F}]^{\Delta N}\right] \\
= & S_t^\alpha F_t^\beta e^{(\alpha\mu_X+\beta\mu_Y)(T-t)} e^{(\alpha^2\sigma_S^2+2\rho\alpha\beta\sigma_S\sigma_F+\beta^2\sigma_F^2)\frac{T-t}{2}} \exp\left[\lambda_S(T-t)(\mathrm{E}[e^{\alpha y_S}]-1)\right] \\
& \times \exp\left[\lambda_F(T-t)(\mathrm{E}[e^{\beta y_F}]-1)\right] \exp\left[\lambda(T-t)(\mathrm{E}[e^{\alpha y_S+\beta y_F}]-1)\right].
\end{aligned}$$

It is now easy to identify that

$$\begin{aligned}
G(\alpha,\beta) &= \alpha\mu_X+\beta\mu_Y+\frac{1}{2}(\alpha^2\sigma_S^2+2\rho\alpha\beta\sigma_S\sigma_F+\beta^2\sigma_F^2) \\
&\quad +\lambda_S\mathrm{E}[e^{\alpha y_S}-1]+\lambda_F\mathrm{E}[e^{\beta y_F}-1]+\lambda\mathrm{E}[e^{\alpha y_S+\beta y_F}-1]. \ \square
\end{aligned}$$

4.4 European Quanto Options

We now apply Laplace transform to value four types of quanto options.

4.4.1 Floating Exchange Rate Foreign Equity Call

The first one, perhaps the simplest one, is the floating exchange rate foreign equity call. The payoff is,

$$C_{fl}(T) = F_T \max(S_T - K_f, 0),$$

where K_f is specified in terms of foreign currency. The present value of this option can be calculated as

$$\begin{aligned} C_{fl}(t,S,F) &= \mathrm{E}_t^{Q^d}\left[e^{-r_d(T-t)}F_T\max(S_T-K_f,0)\right] \\ &= \mathrm{E}_t^{Q^d}\left[\frac{B_d(t)}{B_d(T)}F_T\max(S_T-K_f,0)\right]. \end{aligned}$$

As $C_{fl}(t,S,F)/(FB_f)$ is a martingale under Q^f, we have

$$C_{fl}(t,S,F) = \mathrm{E}_t^{Q^f}\left[\frac{B_f(t)F}{B_f(T)F_T}F_T\max(S_T-K_f,0)\right], \tag{4.18}$$

which implies that the Randon-Nikodym derivative is given by

$$\left.\frac{dQ^f}{dQ^d}\right|_{\mathcal{F}_t} = \frac{F_TB_f(T)/B_d(T)}{F_tB_f(t)/B_d(t)}, \tag{4.19}$$

where $\mathcal{F}_t$ is the filtration up to time t. From (4.18), we recognize that

$$C_{fl}(t,S,F) = F\times C(t,S;K_f,r_f,q=0),$$

where $C(t,S;K_f,r_f,q)$ is the vanilla call price with the strike price K_f and the foreign interest rate r_f. As we assume that the foreign asset pays no dividend, the dividend yield q is set to zero. From (4.1), we know that the process of S under Q^f follows DEJD so that the standard call option pricing formula derived by Kou and Wang (2004) can be applied. For such a floating exchange rate foreign equity option, we see that the correlation between two Wiener processes, the common jumps and dependent jump sizes have no impact on the option price. The reason is that the present value of this option is not affected by the future fluctuation of exchange rate.

Here we provide the analytical solution for vanilla call under DEJD in domestic measure.

Theorem 4.2 *If the foreign equity and the exchange rate follow (4.2) and (4.1) respectively, then the floating exchange rate foreign equity call price is given by*

$$C_{fl}(t,S,F) = e^{-r_d(T-t)}\mathcal{L}^{-1}_{k,\zeta}\left[\frac{S^{\zeta+1}F}{\zeta(\zeta+1)}\exp\left(G(\zeta+1,1;\mu_X,\mu_Y)(T-t)\right)\right], \quad (4.20)$$

where $G(\alpha,\beta;\mu_X,\mu_Y)$, μ_X and μ_Y are defined in (4.17) and (4.16) respectively.

Proof Consider the Laplace transform on (4.18):

$$\begin{aligned}\mathcal{L}_{k,\zeta}\left[e^{r_d(T-t)}C_{fl}(t,S,F)\right] &= \int_{-\infty}^{\infty} e^{-\zeta k}\mathrm{E}^{Q^d}[F_T(S_T-e^{-k})^+]dk \\ &= \mathrm{E}^{Q^d}\left[F_T\int_{-\ln S}^{\infty}(S_Te^{-\zeta k}-e^{-(\zeta+1)k})dk\right] \\ &= \mathrm{E}^{Q^d}\left[S_T^{\zeta+1}F_T\right]\frac{1}{\zeta(\zeta+1)}.\end{aligned}$$

Under Q^d, the processes of S and F are defined in (4.4) and (4.2), respectively. Applying Itô's lemma on $\ln S$ and $\ln F$ with respect to (4.4) and (4.2), the drifts are easily obtained. □

4.4.2 Fixed Exchange Rate Foreign Equity Call

The fixed exchange rate foreign equity call option is also known as standard quanto call option, which gives the holder the right to purchase a foreign asset with a fixed exchange rate on the expiration date of the contract. The payoff function takes the form:

$$C_{fix}(T) = F_0\max(S_T-K_f,0).$$

The present value of this option has the representation:

$$C_{fix}(t,S,F) = F_0e^{-r_d(T-t)}\mathrm{E}_t^{Q^d}\left[\max(S_T-K_f,0)\right],$$

which implies that

$$\begin{aligned} C_{fix}(t,S,F) &= F_0 \times C(t,S;K_f,r_d,\widehat{q}), \qquad (4.21) \\ \widehat{q} &= r_f - r_d - \rho\sigma_S\sigma_F - \lambda \mathrm{E}(A_S A_F), \end{aligned}$$

where $C(t,S;K_f,r_d,\widehat{q})$ is the vanilla call option price with strike K_f using the domestic interest rate r_d and the pseudo dividend yield $\widehat{q}$. As the process of S under Q^d follows DEJD as shown in (4.4), the pricing formula for $C(t,S;K_f,r_d,\widehat{q})$ can be obtained using the result of Kou and Wang (2004). To understand the pseudo dividend yield $\widehat{q}$, we consider the drift, μ_S, of S under Q^d defined in (4.7). The drift can be written as

$$\mu_S = r_d - (\lambda_S + \lambda)\mathrm{E}(A_S) + [\mu_S - r_d + (\lambda_S + \lambda)\mathrm{E}(A_S)] .$$

Under Q^d, all non-dividend paying assets have drift $r_d - (\lambda_S + \lambda)\mathrm{E}(A_S)$ so that the remaining term can be regarded as the pseudo dividend yield induced by the foreign interest rate r_f and fluctuation of the exchange rate F. The fluctuation comes from the volatility of the diffusion component and the jump size when common jumps occur. Here, we see that common jumps and dependent jump sizes have a significant impact on standard quanto options.

Theorem 4.3 *If the foreign equity and the exchange rate follow (4.2) and (4.1) respectively, then the floating exchange rate foreign equity call price is given by*

$$C_{fix}(t,S,F) = e^{-r_d(T-t)}\mathcal{L}^{-1}_{k,\zeta}\left[\frac{S^{\zeta+1}F_0}{\zeta(\zeta+1)}\exp\left(G(\zeta+1,0;\mu_X,\mu_Y)(T-t)\right)\right], \qquad (4.22)$$

where $G(\alpha,\beta;\mu_X,\mu_Y)$, μ_X and μ_Y are defined in (4.17) and (4.16) respectively.

Proof Similarly, consider the Laplace transform on (4.22):

$$\begin{aligned}\mathcal{L}_{k,\zeta}\left[e^{r_d(T-t)}C_{fix}(t,S,F)\right] &= \int_{-\infty}^{\infty} e^{-\zeta k}\mathrm{E}^{Q^d}[F_0(S_T-e^{-k})^+]dk \\ &= \mathrm{E}^{Q^d}\left[F_0\int_{-\ln S}^{\infty}(S_Te^{-\zeta k}-e^{-(\zeta+1)k})dk\right] \\ &= F_0\mathrm{E}^{Q^d}\left[S_T^{\zeta+1}\right]\frac{1}{\zeta(\zeta+1)}. \ \square\end{aligned}$$

4.4.3 Domestic Foreign Equity Call

The third common quanto option is the call option on foreign equity denominated in domestic currency. The payoff is,

$$C_d(T)=\max(F_TS_T-K_d,0),$$

where the strike price K_d is set in terms of domestic currency. We write the pricing representation as

$$C_d(t,S,F) = e^{-r_d(T-t)}\mathrm{E}_t^{Q^d}\left[\max(S_T^*-K_d,0)\right], \tag{4.23}$$

where the process of $S^*=FS$ is obtained in (4.5) in which S^* does not follow DEJD. There is no existing result for the pricing this product which needs to be derived.

Let $k=-\ln K_d$ and $\mathcal{L}_{k,\zeta}$ be the Laplace transform operator with respect to k. The following theorem gives the pricing formula for the domestic foreign equity call in terms of Laplace inversion.

Theorem 4.4 *If the foreign equity and the exchange rate follow (4.2) and (4.1) respectively, then the domestic foreign equity call price is given by*

$$C_d(t,S,F)=e^{-r_d(T-t)}\mathcal{L}_{k,\zeta}^{-1}\left[\frac{S^{\zeta+1}F^{\zeta+1}}{\zeta(\zeta+1)}\exp\left(G(\zeta+1,\zeta+1;\mu_X,\mu_Y)(T-t)\right)\right] \tag{4.24}$$

where $G(\alpha, \beta; \mu_X, \mu_Y)$, μ_X and μ_Y are defined in (4.17) and 4.16 respectively.

Proof Consider the Laplace transform on (4.23):

$$
\begin{aligned}
\mathcal{L}_{k,\zeta}\left[e^{r_d(T-t)}C_d(t,S,F)\right] &= e^{-r_d(T-t)}\int_{-\infty}^{\infty} e^{-\zeta k}\mathrm{E}^{Q^d}[(S_T^* - e^{-k})^+]dk \\
&= \mathrm{E}^{Q^d}\left[\int_{-\ln S^*}^{\infty}(S_T^* e^{-\zeta k} - e^{-(\zeta+1)k})dk\right] \\
&= \mathrm{E}^{Q^d}\left[(S_T^*)^{\zeta+1}\right]\frac{1}{\zeta(\zeta+1)}.
\end{aligned}
$$

The expectation can be computed as

$$
\mathrm{E}^{Q^d}\left[(S_T^*)^{\zeta+1}\right] = \mathrm{E}^{Q^d}\left[S_T^{\zeta+1}F_T^{\zeta+1}\right] = (S_tF_t)^{\zeta+1}\exp\left(G(\zeta+1, \zeta+1; \mu_X, \mu_Y)(T-t)\right),
$$

which is the mgf defined in Theorem 4.1. We remind that μ_X is the drift of $\ln S$ and μ_Y that of $\ln F$. Under Q^d, the processes of S and F are defined in (4.4) and (4.2) respectively. Applying Itô's lemma on $\ln S$ and $\ln F$ with respect to (4.4) and (4.2), the drifts are easily obtained. □

4.4.4 Joint Quanto Call

The joint quanto call option gives the holder the right to purchase a foreign equity with exchange rate no less than a predetermined rate. Specifically, the payoff function is,

$$
C_J(T) = \max(F_T, F_0)\max(S_T - K_f, 0).
$$

Even when jumps disappear in the model, the pricing formula involves the bivariate normal distribution function, see Kwok and Wong (2000). Unlike the preceding quanto options, it is impossible to reduce the valuation of the joint quanto option to a one dimensional problem. We now obtain an analytical formula for the joint quanto call under DEJD, which is new.

Theorem 4.5 *If S and F follow DEJD under Q^d, then the joint quanto call price is*

$$\begin{aligned} C_J(t,S,F) &= \mathcal{L}^{-1}_{f,\delta}\mathcal{L}^{-1}_{k,\zeta}\left[\frac{S^{\zeta+1}F^{\delta+1}}{\zeta(\zeta+1)\delta(\delta+1)}e^{G(\zeta+1,\delta+1;\mu_S,\mu_F)(T-t)}\right] \\ &\quad \times e^{-r_d(T-t)} + C_{fix}(t,S,F), \end{aligned} \tag{4.25}$$

where $f=-\ln F_0$, $k=-\ln K_f$, $C_{fix}(t,S,F)$ is the standard quanto call obtained in (4.21) and the function G can be found in (4.17).

Proof The joint quanto call can be fully replicated by a portfolio of a double call and a standard quanto call. To see this, we write the payoff as

$$C_J(T) = \max(F_T - F_0, 0)\max(S_T - K_f, 0) + F_0 \max(S_T - K_f, 0).$$

The standard quanto call has pricing formula C_{fix} obtained in (4.21). It remains to determine the double call.

Consider a double Laplace transform on the future value of double call as

$$\begin{aligned} &\mathcal{L}_{f,\delta}\mathcal{L}_{k,\zeta}\left\{\mathrm{E}^{Q^d}\left[\max(F_T-e^{-f},0)\max(S_T-e^{-k},0)\right]\right\} \\ = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\zeta k}e^{-\delta f}\mathrm{E}^{Q^d}[(F_T-e^{-f})^+(S_T-e^{-k})^+]\,dk\,df \\ = &E^{Q^d}\left[\int_{-\infty}^{\infty} e^{-\zeta k}(S_T-e^{-k})^+dk\int_{-\infty}^{\infty} e^{-\delta f}(F_T-e^{-f})^+df\right] \\ = &\mathrm{E}^{Q^d}\left[S_T^{\zeta+1}F_T^{\delta+1}\right]\frac{1}{\zeta(\zeta+1)}\frac{1}{\delta(\delta+1)}, \end{aligned}$$

where the mgf $\mathrm{E}^{Q^d}[S_T^{\zeta+1}F_T^{\delta+1}]$ has been established in Theorem 4.1. □

The joint quanto call is numerically more sophisticated than standard quanto products as it requires the computation of the double Laplace inversion. Fortunately, the numerical double Laplace inversion is very efficient. The time for obtaining an option price is less one second. An algorithm for the double Laplace inversion is shown in Section 3.1.

As a summary, we list the results in Table 4.1. It can be seen that European quanto options can be priced using Laplace transform and the mgf between S_t and F_t.

Option Type	Payoff	Pricing Formula
Floating FX Quanto	$F_T(S_T - K)^+$	$e^{-r_d(T-t)}\mathcal{L}_{k,\zeta}^{-1}\left[\frac{S^{\zeta+1}F}{\zeta(\zeta+1)}e^{G(\zeta+1,1)(T-t)}\right]$
Fixed FX Quanto	$F_0(S_T - K)^+$	$e^{-r_d(T-t)}F_0\mathcal{L}_{k,\zeta}^{-1}\left[\frac{S^{\zeta+1}}{\zeta(\zeta+1)}e^{G(\zeta+1,0)(T-t)}\right]$
Domestic Quanto	$(F_T S_T - K)^+$	$e^{-r_d(T-t)}\mathcal{L}_{k,\zeta}^{-1}\left[\frac{S^{\zeta+1}F^{\zeta+1}}{\zeta(\zeta+1)}e^{G(\zeta+1,\zeta+1)(T-t)}\right]$
Joint Quanto	$\max(F_T, F_0)(S_T - K)^+$	$\mathcal{L}_{f,\delta}^{-1}\mathcal{L}_{k,\zeta}^{-1}\left[\frac{S^{\zeta+1}F^{\delta+1}}{\zeta(\zeta+1)\delta(\delta+1)}e^{G(\zeta+1,\delta+1)(T-t)}\right]$ $\times e^{-r_d(T-t)} + C_{fix}(t,S,F)$

Table 4.1: List of Pricing Formulae of European Style Options

4.5 Numerical Examples

We demonstrate the implementation of European quanto option pricing under DEJD model. To simplify matters, we only consider an upward-jump model, i.e., $p^1 = 1$, $p^2 = p^3 = p^4 = 0$. Set: $S = 100$, $F = 1$, $r_d = 5\%$, $r_f = 3\%$, $\sigma_S = 0.3$, $\sigma_F = 0.1$, $\rho = 0.8$, $\lambda_S = 5$, $\lambda_F = 2$, $\lambda = 5$, $\eta_S^1 = 10$, $\eta_F^1 = 15$, $\eta^1 = 25$, $\eta_S = \eta_S^1 + \eta^1 = 35$ and $\eta_F = \eta_F^1 + \eta^1 = 40$.

The results of using Laplace inversion are compared with Monte Carlo simulations. We use strike prices (K): 85, 90, 95, 100 and 105, and the time to maturity T equals to 1/4 and 1/2. Table 4.2 shows the result for the floating quanto and Table 4.3 shows that of the domestic quanto. The third and the sixth columns report the simulation time for 1,000,000 scenarios. The valuation time takes about 0.001 second on average for Laplace inversion, and takes about 5 seconds for Monte Carlo simulation. Therefore, the analytical solution

Strike(K)	3 months				6 months			
	ILT	MC	SD	time	ILT	MC	SD	time
90	12.8729	12.8612	0.0137	5.89	15.4692	15.4801	0.0200	7.67
95	9.5627	9.5623	0.0116	5.77	12.4663	12.4778	0.0177	7.67
100	6.8698	6.8648	0.0106	5.95	9.9059	9.9178	0.0151	8.02
105	4.7807	4.7807	0.0083	5.83	7.7695	7.7836	0.0139	7.82

Table 4.2: Simulation vs. Analytical Solution: Floating FX Quanto

Strike(K)	3 months				6 months			
	ILT	MC	SD	time	ILT	MC	SD	time
90	14.7604	14.7484	0.0170	5.89	18.4338	18.4478	0.0255	7.67
95	11.7077	11.7092	0.0161	5.77	15.6550	15.6666	0.0223	7.67
100	9.1352	9.1311	0.0141	5.95	13.2140	13.2197	0.0196	8.02
105	7.0218	7.0249	0.0123	5.83	11.0928	11.1066	0.0188	7.82

Table 4.3: Simulation vs. Analytical Solution: Domestic Quanto

is computationally more efficient and real time calibration becomes possible.

In terms of accuracy, Table 4.2 and Table 4.3 both show that the Monte Carlo price and the analytical price are close to each other, verifying numerically that our solution is correct and contains no error from calculation and computation.

As seen from the pricing formulas, the fixed FX quanto and domestic quanto option are sensitive to the common jump and the dependent jump size. We would like to examine the impact of the common jump intensity λ and the common size distribution to the prices of these two options. Figure 4.1 plots the option price against jump parameters. The top two graphs of Figure 4.1 correspond to the fixed FX quanto with $F_0 = 1$, by varying the common jump intensity λ and the dependence rate η (fixing $\eta_S = \eta_S^1 + \eta^1$ and $\eta_F = \eta_F^1 + \eta^1$), respectively. The bottom two correspond to the domestic quanto with varying λ and η, respectively. It is interesting to notice that the parameter η indicates the "correlation" of the jump sizes when a common jump occurs.

It is seen from Figure 4.1 that both options are increasing with the λ.

Therefore, the options are more expensive if the investor expects common jumps for the FX and equity markets. When the jump direction is more likely to the same for the foreign asset and the exchange rate (or η is small), the option price will be higher. It is intuitively reasonable because we only consider upward jumps. When both assets are likely to jump up at time same time, it is expected that the terminal values are likely to be higher and hence increases the option price.

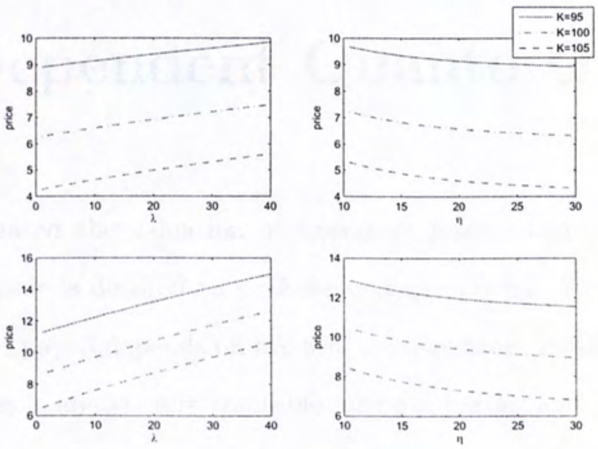

Figure 4.1: Sensitivity Analysis of Fixed FX Quanto and Domestic Quanto

Chapter 5

Path-Dependent Quanto Options

We have investigated the valuation of European foreign equity options under DEJD. This chapter is devoted to path-dependent options. We concentrate on contracts that the payoff depends on the first passage time. It will be shown that the DEJD process is analytically tractable for both barrier and lookback foreign equity options if the joint exponential distribution of Marshall and Olkin (1967) is used to model the joint jump sizes when common jumps appear.

5.1 The Domestic Equivalent Asset S_t^*

Before deriving the pricing formula for path-dependent quanto options, it is helpful to look closely at the dynamic of S_t^* first. Form the relation shown in (4.5)

$$\frac{dS_t^*}{S_t^*} = \frac{dS_tF_t}{S_tF_t} = \frac{dS_t}{S_t} + \frac{dF_t}{F_t} + \frac{dS_t}{S_t}\frac{dF_t}{F_t},$$

we can easily conclude that the dynamic of S_t^* is governed by a jump-diffusion process. Therefore, we follow Kou's methodology on domestic foreign equity and we first obtain the jump sizes distribution with lemma 5.1.

Lemma 5.1 *Suppose y_S and y_F follow the joint double exponential distribution* *using the Marshall and Olkin (1967) copula, then the probability density function (pdf) of* $\hat{Y} = y_S + y_F$ *takes the form*

$$f_{\hat{Y}}(y) = P_1 \sum_{i=1}^{5} q_i^u \gamma_i^u e^{-\gamma_i^u y} 1_{\{y>0\}} + P_2 \sum_{j=1}^{5} q_j^d \gamma_j^d e^{\gamma_j^d y} 1_{\{y<0\}} + P_3 \delta(y),$$

where $\delta(\cdot)$ *is the Dirac delta function. The proof and the values of* P_1*,* P_2*,* P_3*,* q_j^u γ_j^u q_j^d *and* γ_d^u *please refer appendix E.*

Lemma 5.1 enables us to derive the probability density function for the jump size of S^*. As the jump component of $d \ln S^*$ is $y_S \, dN_S + y_F \, dN_F + (y_S + y_F) \, dN$, we can write it in an alternative way as $Y \, d\hat{N}$. Conventionally, we must take out the events with jump size 0 from the Poisson process. Thus, the resulting Poisson process $\hat{N}$ has the intensity $\hat{\lambda} = \lambda_S + \lambda_F + \lambda(P_1 + P_2)$ and the jump size Y follows a mixture of exponential distributions. The pdf of Y takes the form:

$$\begin{aligned} f_Y(y) &= \frac{\lambda_S}{\hat{\lambda}} f_S(y) + \frac{\lambda_F}{\hat{\lambda}} f_F(y) + \frac{\lambda(P_1 + P_2)}{\hat{\lambda}} f_{\hat{Y}}(y | y \neq 0) \\ &= \left[\frac{\lambda_S}{\hat{\lambda}} p_S \eta_S^u e^{-\eta_S^u y} + \frac{\lambda_F}{\hat{\lambda}} p_F \eta_F^u e^{-\eta_F^u y} + \frac{\lambda}{\hat{\lambda}} P_1 \sum_{j=1}^{5} q_j^u \gamma_j^u e^{-\gamma_j^u y} \right] 1_{\{y>0\}} \\ &\quad + \left[\frac{\lambda_S}{\hat{\lambda}} (1 - p_S) \eta_S^d e^{\eta_S^d y} + \frac{\lambda_F}{\hat{\lambda}} (1 - p_F) \eta_F^d e^{\eta_F^d y} + \frac{\lambda}{\hat{\lambda}} P_2 \sum_{j=1}^{5} q_j^d \gamma_j^u e^{\gamma_j^d y} \right] 1_{\{y<0\}}. \end{aligned} \tag{5.1}$$

To allow a general discussion, we consider $dX_t = d \ln S^* = \mu \, dt + \sigma \, dW + Y \, dN$, where the intensity of the Poisson process N_t is λ and the pdf of the jump size, $f_Y(y)$, is a mixture of exponential distributions,

$$f_Y(y) = P_u \sum_{i=1}^{n} p_i \eta_i e^{-\eta_i y} 1_{\{y \geq 0\}} + P_d \sum_{j=1}^{m} q_j \kappa_j e^{\kappa_j y} 1_{\{y<0\}} \geq 0, \tag{5.2}$$

where n and m are the numbers of upward and downward jumps components respectively, $\sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j = 1$, $P_u + P_d = 1$, $P_u, P_d \in [0, 1]$ and $\eta_i, \kappa_j > 0$. We call this new model the mixture exponential jump diffusion (MEJD) model.

5.1.1 Mathematical Results on the First Passage Time of the Mixture Exponential Jump Diffusion Model

Lemma 5.2 *For any sufficiently large real number α, the equation $G(\beta) = \alpha$ has exactly $n+1$ positive real roots and $m+1$ negative real roots.*

Proof As $G(\beta)$ is a rational function, it is analytic, on the complex plane except for the singularities at $\{\eta_i, -\kappa_j\}$ for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. Denote $G_i(\beta) := G(\beta) - \lambda P_u \frac{p_i \eta_i}{\beta - \eta_i}$, for $i = 1, 2, \ldots, n$. Then $\exists\ \delta_i > 0$ s.t. $G_i(\beta)$ is continuous and bounded on the interval $[\eta_i - \delta_i, \eta_i + \delta_i]$. In other words, $\exists\ |l_i|, |u_i| < \infty$, such that

$$l_i \leq G_i(\beta) \leq u_i,\ \forall\ \beta \in [\eta_i - \delta_i, \eta_i + \delta_i].$$

Consider the first case in which $\lambda P_u p_i > 0$. We have

$$\lim_{\beta \to \eta_i^+} G(\beta) = \lim_{\beta \to \eta_i^+} \left[G_i(\beta) + \lambda P_u p_i \frac{\eta_i}{\beta - \eta_i} \right] = G_i(\eta_i) + \lambda P_u p_i \lim_{x \to 0^+} \frac{\eta_i}{x} \to +\infty.$$

and

$$G(\eta_i + \delta_i) = G_i(\eta_i + \delta_i) + \lambda P_u p_i \frac{\eta_i}{(\eta_i + \delta_i) - \eta_i} < u_i + \lambda P_u p_i \frac{\eta_i}{\delta_i} := c_i.$$

By the Intermediate Value Theorem, $\forall\ \alpha \in [c_i, \infty)$, $\exists \beta \in (\eta_i, \eta_i + \delta_i]$ such that $G(\beta) = \alpha$. For the second case in which $\lambda P_u p_i < 0$, we have

$$\lim_{\beta \to \eta_i^-} G(\beta) = \lim_{\beta \to \eta_i^-} \left[G_i(\beta) + \lambda P_u p_i \frac{\eta_i}{\beta - \eta_i} \right] = G_i(\eta_i) + \lambda P_u p_i \lim_{x \to 0^-} \frac{\eta_i}{x} \to +\infty,$$

and

$$G(\eta_i - \delta_i) = G_i(\eta_i - \delta_i) + \lambda P_u p_i \frac{\eta_i}{(\eta_i - \delta_i) - \eta_i} < u_i + |\lambda P_u p_i| \frac{\eta_i}{\delta_i} := c_i.$$

Hence, $\forall\ \alpha \in [c_i, \infty)$, $\exists \beta \in (\eta_i, \eta_i - \delta_i]$ such that $G(\beta) = \alpha$. Combining two cases, we know that, $\forall\ \alpha \in [c_i, \infty)$, $\exists \beta \in [\eta_i - \delta_i, \eta_i + \delta_i]$ such that $G(\beta) = \alpha$.

Using similar arguments, there are ϵ_j and d_j such that $\forall\ \alpha \in [d_j, \infty)$, $\exists \beta \in [-\kappa_j - \epsilon_j, -\kappa_j + \epsilon_j]$ such that $G(\beta) = \alpha$.

It is easy to see that

$$\lim_{\beta \to \pm\infty} G(\beta) \to +\infty.$$

For $\eta > \max\{\eta_i\}$, $G(\beta)$ is continuous on the interval $[\eta, \infty)$. Hence $\forall\ \alpha > G(\eta)$, $G(\beta) = \alpha$ has at least one real root on the interval $[\eta, \infty)$. Similarly, $\forall\ \kappa < \min\{\kappa_j\}$ and $\alpha > G(-\kappa)$, $G(\beta) = \alpha$ has at least one real root on the interval $(-\infty, -\kappa]$.

Let $c = \max\{c_i, d_j, G(\eta), G(-\kappa)\}$. The previous analysis shows that the equation $G(\beta) = \alpha > c$ has at least one real root on each of the intervals $[\eta_i - \delta_i, \eta_i + \delta_i]$, $[-\kappa_j - \epsilon_j, -\kappa_j + \epsilon_j]$, $[\eta, \infty)$ and $(-\infty, -\kappa]$. Thus, there are at least $m + n + 2$ real roots. However, the equation $G(\beta) = \alpha$ can be transformed into a polynomial equation of degree $m + n + 2$. By the Fundamental Theorem of Algebra, it must have exactly $m + n + 2$ roots so that there must be exactly one real root lying on each of the intervals. Hence, there are $n + 1$ positive roots and $m + 1$ negative roots. □

Lemma 5.3 *Let $G_k^{-1}(\alpha) = \beta_k$ be the inverse (k^{th}-root finding) function of the equation $G(\beta) = \alpha$, $\mathcal{C}^+ := \{z \in \mathcal{C} | \Re(z) > 0\}$ and $\mathcal{C}^- := \{z \in \mathcal{C} | \Re(z) < 0\}$. Then, either $G_k^{-1}(\mathcal{C}^+) \subset \mathcal{C}^+$ or $G_k^{-1}(\mathcal{C}^+) \subset \mathcal{C}^-$ holds.*

Proof We start by showing that $G^{-1}(\mathcal{C}^+) \subset \mathcal{C}^+ \cup \mathcal{C}^-$. Consider $G(\beta) = \alpha$ and $\Re(\beta) = 0$. Then,

$$e^{\Re(\alpha)} = e^{\Re(G(\beta))} = \left|e^{G(\beta)}\right| = \left|\mathrm{E}[e^{\beta X}]\right| \leq \mathrm{E}\left[\left|e^{\beta X}\right|\right] = \mathrm{E}\left[e^{\Re(\beta)X}\right] = \mathrm{E}\left[e^{0}\right] = e^{0}.$$

We conclude that $\Re(\beta) = 0 \Rightarrow \Re(\alpha) \leq 0$. The equivalent statement is that $\Re(\alpha) > 0 \Rightarrow \Re(\beta) \neq 0$. Hence, $G^{-1}(C^+) \subset \mathcal{C}^+ \cup \mathcal{C}^-$.

As $G(\beta)$ is a rational function, the equation $G(\beta) = \alpha$ can be transformed into a polynomial equation of β with coefficients being continuous functions of α. Harris and Martin (1987) prove that the roots of a polynomial over complex plane vary continuously as a function of the coefficients and the inverse mapping is a homeomorphism with the quotient topology. Therefore, $G_k^{-1}(\alpha)$ are continuous functions for all $k = 1, 2, \ldots, n+m+2$ and should map a connected domain onto a connected set.

As $G^{-1}(\mathcal{C}^+) \subset \mathcal{C}^+ \cup \mathcal{C}^-$, we must have $G_k^{-1}(\mathcal{C}^+) \subset \mathcal{C}^+ \cup \mathcal{C}^-$. We highlight that $\mathcal{C}^+$ and $\mathcal{C}^-$ are two connected subsets of $\mathcal{C}$, but $\mathcal{C}^+$ and $\mathcal{C}^-$ are separated. Because G_k^{-1} is a continuous function, it should be that either $G_k^{-1}(\mathcal{C}^+) \subset \mathcal{C}^+$ or $G_k^{-1}(\mathcal{C}^+) \subset \mathcal{C}^-$ holds. The proof is completed. □

Theorem 5.1 *There is a positive real number c such that the equation $G(\beta) = \alpha$ has exactly $n+m+2$ complex roots for all $\Re(\alpha) > c, \alpha \in \mathcal{C}$. In addition, there are $n+1$ complex roots with positive real parts and $m+1$ complex roots with negative real parts.*

Proof By Lemma 5.2, $G(\beta) = \alpha$ has exactly $n+1$ positive real roots and $m+1$ negative real roots for all real number $\alpha > c$. Consider $G_k^{-1}(\alpha) = \beta_k$ such that $\beta_k > 0$ for $k = 1, 2, \ldots, n+1$ and $\beta_k < 0$ for $k = n+2, n+3, \ldots, n+m+2$. By Lemma 5.3, we can extend the domain of $G^{-1}(\alpha)$ to $\mathcal{C}^+$ such that

$$G_k^{-1} : \mathcal{C}^+ \to \mathcal{C}^+ \quad \text{for} \quad k = 1, 2, \ldots, n+1,$$
$$G_k^{-1} : \mathcal{C}^+ \to \mathcal{C}^- \quad \text{for} \quad k = n+2, n+3, \ldots, n+m+2.$$

i.e. $G(\beta) = \alpha \in \mathcal{C}^+$ has exactly $n+m+2$ complex roots β_k, where $\beta_k \in \mathcal{C}^+$ for $k = 1, 2, \ldots, n+1$ and $\beta_k \in \mathcal{C}^-$ for $k = n+2, n+3, \ldots, n+m+2$. The proof is then completed. □

Lemma 5.2 and Theorem 5.1 constitute the building blocks for deriving the moment generating function of the first passage time(s).

Lemma 5.4 *Let $\tau_b = \inf\{t | X_t \geq b\}$ for $b > X_0$, where X_t follows the mixture exponential jump diffusion process. Suppose the equation $G(\beta) = \alpha$ has no multiple roots, then for any $\alpha \in \mathcal{C}$ with a large positive real part,*

$$\mathrm{E}\left[e^{-\alpha\tau_b} 1_{\{\tau<\infty\}}\right] = \begin{cases} \sum_{k=1}^{n+1} \gamma_k e^{-\beta_k(b-x)}, & x \leq b. \\ 1, & x > b. \end{cases}$$

$$\mathrm{E}\left[e^{-\alpha\tau_b} 1_{\{X_\tau \neq b\}}\right] = \begin{cases} \sum_{k=1}^{n+1} \hat{\gamma}_k e^{-\beta_k(b-x)}, & x < b. \\ 0, & x = b, \\ 1, & x > b. \end{cases}$$

where β_i's are the $n+1$ roots of the equation $G(\beta) = \alpha$ with $\Re(\beta_k) > 0$, and γ_k and $\hat{\gamma}_k$ are respectively the solutions of the systems of linear equations

$$\begin{cases} \sum_{k=1}^{n+1} \gamma_k \frac{\eta_i}{\eta_i - \beta_k} & = 1, \quad i = 1, 2, \ldots n, \\ \sum_{k=1}^{n+1} \gamma_k & = 1, \end{cases}$$

and

$$\begin{cases} \sum_{k=1}^{n+1} \hat{\gamma}_k \frac{\eta_i}{\eta_i - \beta_k} & = 1, \quad i = 1, 2, \ldots n, \\ \sum_{k=1}^{n+1} \hat{\gamma}_k & = 0. \end{cases}$$

Proof Please refer to Appendix D. □

Lemma 5.5 *Let $\tau_a = \inf\{t | X_t \leq a\}$ for $a < X_0$, where X_t follows the mixture exponential jump diffusion process. Suppose the equation $G(\beta) = \alpha$ has no multiple* *roots, then for any $\alpha \in C$ with a large positive real part,*

$$\mathrm{E}\left[e^{-\alpha\tau_a}1_{\{\tau<\infty\}}\right] = \begin{cases} \sum_{k=1}^{m+1}\gamma_k e^{-\beta_k(a-x)}, & x \geq a. \\ 1, & x < a, \end{cases}$$

$$\mathrm{E}\left[e^{-\alpha\tau_a}1_{\{X_\tau\neq a\}}\right] = \begin{cases} \sum_{k=1}^{m+1}\hat{\gamma}_k e^{-\beta_k(a-x)}, & x > a, \\ 0, & x = a, \\ 1, & x < a, \end{cases}$$

where β_k are the $m+1$ roots of the equation $G(\beta) = \alpha$ with $\Re(\beta_k) < 0$, and γ_k, $\hat{\gamma}_k$ are respectively the solutions of the systems of linear equations

$$\begin{cases} \sum_{k=1}^{m+1}\gamma_k \frac{\kappa_j}{\kappa_j+\beta_k} & = 1 \quad i = 1, 2, \ldots m, \\ \sum_{k=1}^{m+1}\gamma_k & = 1. \end{cases}$$

and

$$\begin{cases} \sum_{k=1}^{m+1}\hat{\gamma}_k \frac{\kappa_j}{\kappa_j+\beta_k} & = 1, \quad i = 1, 2, \ldots m, \\ \sum_{k=1}^{n+1}\hat{\gamma}_k & = 0. \end{cases}$$

Proof The proof is similar to that of Lemma 5.4 and hence is omitted here. □

5.2 Quanto Lookback Option

We are now ready to derive the analytical pricing formula for the lookback option. Lookback option payoff contains an extreme value of the underlying asset over a period of time. Under the Black-Scholes model, the pricing of quanto lookback options has been discussed by Dai et al. (2003). There are many different payoff structures for quanto lookbacks. For instance, the floating exchange rate foreign equity lookback put option has the payoff: $F_T(M_0^T - S_T)$, where $M_0^t = \sup\{S_\tau | 0 \leq \tau \leq t\}$ is the realized maximum asset value from time 0 to t. The fixed exchange rate foreign equity lookback put has the payoff: $F_0(M_0^T - S_T)$. These two quanto lookback options are relatively simple to price because the foreign equity price S_t follows DEJD under both Q^f and Q^d as shown in Chapter 3. Thus, the change of measure technique developed in Chapter 4 can be applied to transform the problem into a standard lookback option pricing problem under DEJD. The result of Kou and Wang (2004) can then be used directly.

Unfortunately, in pricing of domestic foreign equity lookback put, the payoff is $M_0^{*T} - S_t^*$ where $M_0^{*T} = \sup\{S_\tau^* | t \le \tau \le T\}$, it is a mathematically challenging task and has not been considered before. The difficulty comes from the fact that the jump component of the domestic equivalent asset (FS) is no longer double exponentially distributed, it follows a mixture of exponential distributions. To the best of our knowledge, this is the first study to consider path-dependent option pricing under mixture exponential jump diffusion.

According to Wong and Kwok (2003), a model-free representation for the domestic foreign equity lookback put can be written as

$$\mathrm{LP}_d(S_t^*, M_0^{*t}) = e^{-r_d(T-t)} \left[\int_{M_0^{*t}}^{\infty} \mathrm{P}(M_t^{*T} > y)dy + M_0^{*t} - \mathrm{E}[S_T^* | \mathcal{F}_t] \right]. \tag{5.3}$$

It is clear that the key to valuing the lookback option is to determine the cumulative distribution function (cdf) for the future maximum value of the domestic equivalent asset. To derive the cdf, we need the following results.

Theorem 5.2 *Suppose S^* follows a mixture exponential jump diffusion process. Then the price of the lookback put option on S^* is given by*

$$\mathrm{LP}_d(S_t^*, M_0^{*t}) = e^{-r(T-t)} \left[\mathcal{L}_{T,\alpha}^{-1} \left\{ \frac{1}{\alpha} \sum_{k=1}^{n+1} \frac{\gamma_k}{\beta_k - 1} \frac{(S_t^*)^{\beta_k}}{(M_0^{*t})^{\beta_k - 1}} \right\} + M_0^{*t} - \mathrm{E}[S_T^* | \mathcal{F}_t] \right],$$

where β_k are the $n+1$ roots of the equation $G(\beta) = \alpha$ with $\Re(\beta_k) > 0$ for all $\alpha \in \mathcal{C}^+$ and γ_k are defined in Lemma 5.4.

Proof Let $\hat{M}_0^t = \ln M_0^{*t}$, $X_t = \ln S_t^*$ and $\hat{y} = \log y$. Consider the Laplace transform of the integral $\int_{M_0^{*t}}^{\infty} \mathrm{P}(M_t^{*T} > y)dy$ with respect to the maturity T,

$$
\begin{aligned}
& \mathcal{L}_{T,\alpha}\left\{\int_{M_0^{*t}}^{\infty} \mathrm{P}(M_t^{*T} > y)dy\right\} \\
= & \int_{\hat{M}_0^t}^{\infty} \mathcal{L}_{T,\alpha}[\mathrm{P}(\hat{M}_t^T > \hat{y})]e^{\hat{y}}d\hat{y} = \int_{\hat{M}_0^t}^{\infty}\left[\int_0^{\infty} e^{-\alpha T}\mathrm{E}[1_{\{\tau_{\hat{y}} \leq T\}}]dT\right]e^{\hat{y}}d\hat{y} \\
= & \int_{\hat{M}_0^t}^{\infty} \frac{\mathrm{E}[e^{-\alpha\tau_{\hat{y}}}]e^{\hat{y}}}{\alpha}d\hat{y} = \frac{1}{\alpha}\int_{\hat{M}_0^t}^{\infty}\sum_{k=1}^{n+1}\gamma_k e^{-\beta_k(\hat{y}-X_t)}e^{\hat{y}}d\hat{y} \\
= & \frac{1}{\alpha}\sum_{k=1}^{n+1}\frac{\gamma_k}{\beta_k - 1}e^{-\beta_k(\hat{M}_0^t - X_t)}e^{\hat{M}_0^t} = \frac{1}{\alpha}\sum_{k=1}^{n+1}\frac{\gamma_k}{\beta_k - 1}\frac{(S_t^*)^{\beta_k}}{(M_0^{*t})^{\beta_k - 1}}.
\end{aligned}
$$

By Theorem 5.1, $\beta_k \in \mathcal{C}^+$ are the roots of the equation $G(\beta) = \alpha$ and γ_k are solved according to Lemma 5.4. Using (5.3), the proof is completed. □

Theorem 5.2 gives an analytical solution to lookback put option under the mixture exponential jump diffusion process. To implement the pricing formula, the complex roots of the polynomial equation $G(\beta) = \alpha$ are solved numerically, roots with positive real parts are identified and a numerical Laplace inversion algorithm is carried out to obtain the option price. There is an efficient built-in function in MATLAB to solve complex roots for a polynomial equation. We employ the Laplace inversion algorithm proposed by Petrella (2004) to compute the option price.

Remarks:

1. When the Laplace transform is applied to a function, it is usually assumed that the variable α is a positive real number. In such a situation, Theorem 5.2 still holds as β_k becomes a positive real roots of the equation $G(\beta) = \alpha$ for $k = 1, 2, \cdots, n+1$ and sufficiently large α. Lemma 5.2 ensures that there are exactly $n+1$ positive real roots. We consider $\alpha \in \mathcal{C}^+$ because

an efficient Laplace inversion algorithm works with the complex domain by extending α to a complex number with a positive real part. In fact, our consideration is more general.

2. If S^* follows DEJD, i.e. $n = m = 1$, the equation $G(\beta) = \alpha$ has exactly 4 real roots for a sufficiently large positive α, see Lemma 5.2. In such a situation, there is a closed-form solution for the polynomial equation of degree 4 and Theorem 5.2 reduces to the pricing formula of Kou and Wang (2004). However, our result is more general. The approach of Kou and Wang (2004) requires the analytical form of quadratic polynomial roots and then extend them to the complex domain. Our approach only needs to identify roots with positive real parts. This enables us to generalize the pricing formula to the case of $n+m+2 > 4$, where is no closed-form solution for the corresponding polynomial equation.

3. For the domestic foreign equity lookback put option, we have $S^* = SF$ which does not follow DEJD in general. Lemma 5.1 asserts that S^* follows a mixture exponential jump diffusion process. Theorem 5.2 can infer the corresponding pricing formula.

5.3 Quanto Barrier Option

Barrier options are no longer regarded as exotic options as their trading volume is higher than their vanilla counterparts in certain option markets. The barrier is often provided on the foreign asset (S) for floating and fixed exchange rate foreign equity options or on the domestic equivalent asset (FS) for the domestic foreign equity options. When there is a barrier provision on the foreign asset, the valuation is simple because it only requires a change of the probability measure, as being done in Section 3, and then applies the barrier option pricing mechanism for the single asset case. We stress that S_t follows DEJD under both Q^f and Q^d. Using DEJD process, the barrier option pricing formula for single asset options has been derived by Kou and Wang (2004). The challenge arises in the valuation of domestic foreign equity options with a barrier provision on the domestic equivalent asset, because the domestic equivalent asset does not follow the DEJD process but a mixture exponential jump diffusion model, see (4.5).

To illustrate the ideas, we consider the up-and-in domestic foreign equity call, whose payoff is given by

$$\mathrm{UIC_d}(T) = \max(S_T F_T - K, 0) 1_{\{\tau_H < T\}},$$

where the strike price K is in domestic currency and

$$\tau_H = \inf\{t | S_t F_t \leq H\}.$$

The pricing representation is given by

$$\mathrm{UIC_d}(0, S, F) \quad = \quad e^{-r_d T} \mathrm{E}^{\mathrm{Q^d}} \left[(S_T^* - e^{-k})^+ 1_{\{\tau_H < T\}} \right], \tag{5.4}$$

where the process of $S^* = FS$ is obtained in (4.5). The following theorem summarizes the pricing formula of the barrier option.

Theorem 5.3 *If S and F follow DEJD under Q^d, then the up-and-in domestic foreign equity call is*

$$\mathrm{UIC_d}(0, S^*, H) \quad = \quad \mathcal{L}_{T,\theta}^{-1} \mathcal{L}_{k,\zeta}^{-1} \left[\frac{1}{\zeta(\zeta+1)} \frac{\mathrm{E}^{\mathrm{Q^d}} \left[e^{-(\theta + r_d)\tau_H} S_{\tau_H}^{*\zeta+1} \right]}{\theta + r_d - G(\zeta+1, \zeta+1)} \right], \tag{5.5}$$

where $k = -\ln K$.

Proof We take a double Laplace transform to the representation (5.4),

$$\begin{aligned}
& \mathcal{L}_{T,\theta}\mathcal{L}_{k,\zeta}\left[\mathrm{UIC_d}(0,S,F,H)\right] \\
= & \int_0^\infty \int_{-\infty}^\infty e^{-\zeta k} e^{-\theta T} e^{-r_d T} \mathrm{E}^{\mathrm{Q^d}}[(S_T^* - e^{-k})^+] 1_{\{\tau_H < T\}} dk\, dT \\
= & \mathrm{E}^{\mathrm{Q^d}}\left[\int_0^\infty e^{-(\theta+r_d)T}\left(\int_{-\infty}^\infty e^{-\zeta k}(S_T^* - e^{-k})^+\, dk\right) 1_{\{\tau_H<T\}} dT\right] \\
= & \mathrm{E}^{\mathrm{Q^d}}\left[\int_0^\infty e^{-(\theta+r_d)T} \frac{S_T^{*\zeta+1}}{\zeta(\zeta+1)} 1_{\{\tau_H<T\}} dT\right] \\
= & \frac{1}{\zeta(\zeta+1)}\mathrm{E}^{\mathrm{Q^d}}\left[\int_0^\infty e^{-(\theta+r_d)(T'+\tau_H)} S_{T'+\tau_H}^{*\zeta+1} dT'\right] \\
= & \frac{1}{\zeta(\zeta+1)}\mathrm{E}^{\mathrm{Q^d}}\left[\int_0^\infty e^{-(\theta+r_d)(T'+\tau_H)} \mathrm{E}^{\mathrm{Q^d}}[S_{T'+\tau_H}^{*\zeta+1}|\mathcal{F}_{\tau_H}] dT'\right] \\
= & \frac{1}{\zeta(\zeta+1)}\mathrm{E}^{\mathrm{Q^d}}\left[\int_0^\infty e^{-(\theta+r_d)(T'+\tau_H)} S_{\tau_H}^{*\zeta+1} \mathrm{E}^{\mathrm{Q^d}}[(S_{T'}/S_0)^{*\zeta+1}|\mathcal{F}_0] dT'\right] \\
= & \frac{1}{\zeta(\zeta+1)}\mathrm{E}^{\mathrm{Q^d}}\left[e^{-(\theta+r_D)\tau_H} S_{\tau_H}^{*\zeta+1} \int_0^\infty e^{-(\theta+r_d)T'} e^{G(\zeta+1,\zeta+1)T'} dT'\right] \\
= & \frac{1}{\zeta(\zeta+1)} \frac{1}{\theta + r_d - G(\zeta+1,\zeta+1)} \mathrm{E}^{\mathrm{Q^d}}\left[e^{-(\theta+r_d)\tau_H} S_{\tau_H}^{*\zeta+1}\right]. \quad \square
\end{aligned}$$

To implement the pricing formula given in Theorem 5.3, we have to determine the expectation $\mathrm{E}^{\mathrm{Q^d}}\left[e^{-(\theta+r_d)\tau_H} S_{\tau_H}^{*\zeta+1}\right]$, which can be decomposed into two parts,

$$H^\zeta \left\{\mathrm{E}\left[e^{-(\theta+r_d)\tau_H} 1_{\{S_\tau^*=H\}}\right] + \mathrm{E}\left[e^{-(\theta+r_d)\tau_H} e^{\zeta\chi} 1_{\{S_\tau^* \neq H\}}\right]\right\},$$

where χ is a random variable representing the overshot, see Kou and Wang (2004). The first expectation can be computed using Lemma 5.4, but the computation of the second expectation requires the random variable χ to have the memoryless property. It can be seen from (4.5) that the common jump component in $d\ln S^*$ is $(a_S + a_F)\,dN$, which, according to Lemma 5.1, follows a mixture of exponential distributions. The memoryless property is not always guaranteed for a mixture of exponential distributions. The following theorem gives a useful case when the quanto barrier options admit an analytical solution using the double Laplace transform.

Theorem 5.4 *When the distribution of the aggregate jump size, $\hat{f}(y)$, takes the following form,*

$$\hat{f}(y) = \begin{cases} P_u \left[\sum_{i=1}^{k} p_i \gamma_i e^{-\gamma_i y}\right] 1_{\{y>0\}} + P_d \gamma e^{\gamma y} 1_{\{y<0\}}, & \text{for lower barrier,} \\ P_u \gamma e^{-\gamma y} 1_{\{y>0\}} + P_d \left[\sum_{j=1}^{k} q_j \gamma_i e^{\gamma_j y}\right] 1_{\{y<0\}}, & \text{for upper barrier,} \end{cases}$$

where $\sum_{i=1}^{k} p_i = \sum_{j=1}^{k} q_j = 1$, $P_u + P_d = 1$, $P_u, P_d \in [0,1]$ and $\gamma, \gamma_i > 0$, we have

$$\mathrm{E}\left[e^{-\alpha\tau} S_\tau^{*\zeta}\right] = H^\zeta \left\{ \mathrm{E}\left[e^{-\alpha\tau} 1_{\{S_\tau^*=H\}}\right] + \frac{\eta}{\eta \pm (-\zeta)} \mathrm{E}\left[e^{-\alpha\tau} 1_{\{S_\tau^* \neq H\}}\right] \right\},$$

where H and τ are the barrier level and the stopping time respectively, the positive sign '+' is used for a positive overshot and '−' for a negative overshot.

Proof When the condition is satisfied, the overshot (χ) at stopping time (τ) is exponentially (γ) distributed. Thus, the expectation $\mathrm{E}\left[e^{-\alpha\tau_H} S_{\tau_H}^{*\zeta}\right]$ can be calculated using the memoryless property of the exponential distribution. Following Kou and Wang (2004), the formula (5.6) is easily established. □

Under the condition of Theorem 5.4, the UIC option price on S^* can be obtained through the use of Laplace transform. The Laplace transforms of the expectations $\mathrm{E}\left[e^{-\alpha\tau} 1_{\{S_\tau^*=H\}}\right]$ and $\mathrm{E}\left[e^{-\alpha\tau} 1_{\{S_\tau^* \neq H\}}\right]$ can be evaluated by Lemmas 5.4 and 5.5. By substituting (5.6) into Theorem 5.3, the quanto barrier option can be valued using a numerical Laplace inversion.

There are situations in which the condition of Theorem 5.4 holds. For instance, when either the exchange rate or the foreign asset has no upward jumps, the aggregated upward jump is exponentially distributed. In this situation, an up-barrier option can be valued through Theorem 5.3. If we assume the upward jumps of the exchange rate and the foreign asset follow the same exponential distribution, then the condition also holds. For other cases in which both upward and downward jumps are a mixture of exponential distributions, the memoryless property is lost however and Theorem 5.3 may not be useful.

Apparently quanto barrier option pricing is more difficult than the quanto lookback option pricing. The major reason is that barrier option pricing requires the joint distribution of the first passage time and the underlying asset value; whereas, lookback option pricing only requires the distribution of the first passage time.

To value discrete quanto barrier and lookback options, Borovkov and Novikov (2002) and Petrella and Kou (2004) find that the Laplace transforms of discrete barrier and lookback options can be calculated recursively using Spitzer's formula. This method can be applied to a general class of Lévy processes if the following two conditions are satisfied. First, the characteristic (or moment generating) function of the process is available. Second, the analytical solution of the corresponding European option price is available. Generally, the Laplace transform can be applied to compute the European option price under general Lévy processes. For the joint DEJD with the Marshall and Olkin (1967) copula, the moment generating function and formulas for European quanto options are obtained in Section 4.

5.4 Numerical results

We demonstrate the implementation of the analytical solution for a lookback option under MEJD. Similar to Chapter 4, we use the following parameters: $S = 100$, $F = 1$, $r_d = 5\%$, $r_f = 3\%$, $\sigma_S = 0.3$, $\sigma_F = 0.15$, $\rho = 0.6$, $\lambda_S = 5$, $\lambda_F = 2$, $\lambda = 5$ $\eta_S^1 = 20$, $\eta_F^1 = 15$, $\eta^1 = 25$, $\eta_S = \eta_S^1 + \eta^1 = 45$ and $\eta_F = \eta_F^1 + \eta^1 = 40$.

Realized Maximun (M)	ILT	Extrapolated values	MC			
Δt			0.0005	0.002	0.01	0.05
105	19.599	19.2803	19.114	18.619	17.472	15.326
110	21.004	20.7518	20.617	20.215	19.292	17.630
115	23.163	22.9706	22.863	22.541	21.811	20.527
120	25.950	25.8078	25.723	25.469	24.897	23.920
125	29.248	29.1426	29.077	28.880	28.440	27.706
130	32.953	32.8768	32.826	32.675	32.341	31.796

Table 5.1: Simulation vs. Analytical Solution: Lookback Option under MEJD

After transforming the moment generating function of $S^* = SF$, we have

$$G(\beta) = -0.4442\beta + 0.4080^2\frac{\beta^2}{2} + 12\left(1.0417\frac{30}{30-\beta} + 0.2083\frac{45}{45-\beta} - 0.2500\frac{40}{40-\beta}\right).$$

Therefore, the equation $G(\beta) = \alpha$ is equivalent to a polynomial equation of degree five. The roots cannot be solved analytically and the method of Kou and Wang (2004) fails. Using our approach, the roots are obtained numerically through the subroutine "roots" provided by MATLAB. Then, we identify positive roots and use Lemma 5.4 to calculate the required parameters for Theorem 5.2.

Table 5.1 shows the numerical result for floating strike lookback put, with the realized maximum ranging from 105 to 130. The analytical solution is compared with Monte Carlo simulations. We stress that the Monte Carlo price is biased downward because the simulation records maximum values in discrete time points but the option is assumed to be monitored continuously. To reduce the bias, we use the Richardson extrapolation to estimate the continuous lookback price.

From Table 5.1, we see that the inverse Laplace transform (ILT) provides a high quality of estimate for the lookback option price. All the ILT prices are close to the MC prices The ILT is typically efficient as it only takes 0.02 seconds to obtain an option price while Monte Carlo simulation requires more than 20 minutes.

Chapter 6

Conclusion

In summary, we propose a [illegible]

We derive [illegible]

Chapter 6

Conclusion

In summary, we propose a multivariate jump diffusion model for the joint movement of the exchange rate and a foreign equity. In the diffusion component, the multivariate Wiener process is used. For the jump component, we consider a common Poisson process for the two assets. The jump sizes of the two assets are jointly modeled by the Marshall and Olkin (1967) multivariate exponential distribution. We show that this model maintains the analytical tractability for European quanto options and path-dependent quanto options.

We derive analytical solutions to several European quanto options, quanto lookback options and quanto barrier options using Laplace transform. As the domestic lookback equity call option price depends on the moment generating function of the first passage time that the domestic equivalent asset reaches its maximum, the process for the domestic equivalent asset is shown to be mixture exponential jump diffusion and we obtain an analytical expression for the moment generating function using the Laplace transform. To perform inversion of the Laplace transform, we have to identify complex roots of a polynomial which is derived from the moment generating function. We show that we only have to identify roots with a positive (negative) real part for computing the first passage time of reaching the maximum (minimum). We prove that these roots exist for all the Laplace parameter α with positive real part.

Appendix A

Numerical Laplace Inversion for Turbo Warrants

First, we write the triple Laplace transform for DIL option as

$$\mathcal{L}_{h,\beta}\mathcal{L}_{T,\alpha}\mathcal{L}_{k,\zeta}e^{rh}\mathrm{DIL}(T,S,K,h) = F_\zeta(\zeta)F_\alpha(\alpha,\zeta)F_\beta(\beta,\zeta),$$

where

$$\begin{aligned} F_\zeta(\zeta) &= \frac{H^{\zeta+1}}{\zeta(\zeta+1)}, \\ F_\alpha(\alpha,\zeta) &= \frac{1}{\alpha}\left[\frac{A(r+\alpha;b)\eta_2}{\eta_2+\zeta+1} + B(r+\alpha;b)\right], \\ F_\beta(\beta,\zeta) &= \frac{1}{\beta}\left[1 - \frac{\zeta+1}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})}\left(\frac{\beta_{2,\beta}(\eta_2+\beta_{1,\beta})}{\zeta+1-\beta_{1,\beta}} - \frac{\beta_{1,\beta}(\eta_2+\beta_{2,\beta})}{\zeta+1-\beta_{2,\beta}}\right)\right]. \end{aligned}$$

The analytical solution of expression of DIL is given by

$$\mathrm{DIL}(x,y,z) = \int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_3-i\infty}^{c_3+i\infty}\frac{e^{x\zeta+y\alpha+z\beta}}{(2\pi i)^3}F_\zeta(\zeta)F_\alpha(\alpha,\zeta)F_\beta(\beta,\zeta)\,d\alpha\,d\beta\,d\zeta,$$

where c_1, c_2, c_3 are arbitrary positive real numbers, such that there is no singularity on the domain $\{\mathrm{Re}(\zeta) > c_1, \mathrm{Re}(\alpha) > c_2, \mathrm{Re}(\beta) > c_3\}$.

Specifically for the present case, we have the alternating series,

$$A \sum_{j_1=-n_1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}),$$

where $A = \frac{e^{xc_1+yc_2+zc_3}}{8xyz}$.

The order of number of terms is $O(n_1 n_2 n_3)$, here we can reduce the order by factorization. Since Laplace transform has a nice property that $F(\bar{z}) = \overline{F(z)}$, we can simplify the expression. Specifically,

$$\begin{aligned}
& A \sum_{j_1=-n_1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) \\
= \; & A \sum_{j_1=1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) \\
& + A \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_2+j_3} F_\zeta(\zeta_0) F_\alpha(\alpha_{j_2}, \zeta_0) F_\beta(\beta_{j_3}, \zeta_0) \\
& + A \sum_{j_1=-n_1}^{-1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) \\
= \; & A \sum_{j_1=1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) \\
& + A F_\zeta(\zeta_0) \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_2+j_3} F_\alpha(\alpha_{j_2}, \zeta_0) F_\beta(\beta_{j_3}, \zeta_0) \\
& + A \sum_{j_1=1}^{n_1} \sum_{j_2=n_2}^{-n_2} \sum_{j_3=n_3}^{-n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\bar{\zeta}_{j_1}) F_\alpha(\bar{\alpha}_{j_2}, \bar{\zeta}_{j_1}) F_\beta(\bar{\beta}_{j_3}, \bar{\zeta}_{j_1}) \\
= \; & 2\mathrm{Re}\left\{ A \sum_{j_1=1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) \right\} \\
& + A F_\zeta(\zeta_0) \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_2+j_3} F_\alpha(\alpha_{j_2}, \zeta_0) F_\beta(\beta_{j_3}, \zeta_0) \\
= \; & 2A \sum_{j_1=1}^{n_1} (-1)^{j_1} \mathrm{Re}\left\{ F_\zeta(\zeta_{j_1}) \left[\sum_{j_2=-n_2}^{n_2} (-1)^{j_2} F_\alpha(\alpha_{j_2}, \zeta_{j_1}) \right] \left[\sum_{j_3=-n_3}^{n_3} (-1)^{j_3} F_\beta(\beta_{j_3}, \zeta_{j_1}) \right] \right\} \\
& + A F_\zeta(\zeta_0) \sum_{j_2=-n_2}^{n_2} (-1)^{j_2} F_\alpha(\alpha_{j_2}, \zeta_0) \sum_{j_3=-n_3}^{n_3} (-1)^{j_3} F_\beta(\beta_{j_3}, \zeta_0). \qquad \text{(A.1)}
\end{aligned}$$

Notice that $F_\alpha(\alpha_{j_2}, \zeta_{j_1})$ and $F_\beta(\beta_{j_3}, \zeta_{j_1})$ are no longer conjugate functions with respect to α and β respectively, unless ζ_{j_1} is real. Thus, we cannot simplify the two summations in the first line of (A.1). For the last term, we can fortunately simplify it as follows,

$$\begin{aligned}\sum_{j_2=-n_2}^{n_2}(-1)^{j_2}F_\alpha(\alpha_{j_2},\zeta_0) &= F_\alpha(\alpha_0,\zeta_0)+2\sum_{j_2=1}^{n_2}(-1)^{j_2}\mathrm{Re}[F_\alpha(\alpha_{j_2},\zeta_0)],\\ \sum_{j_3=-n_3}^{n_3}(-1)^{j_3}F_\beta(\beta_{j_3},\zeta_0) &= F_\beta(\beta_0,\zeta_0)+2\sum_{j_3=1}^{n_3}(-1)^{j_3}\mathrm{Re}[F_\beta(\beta_{j_3},\zeta_0)].\end{aligned}$$

To implement the Euler summation, we consider partial sums of the series, and aggregate the partial sums by binomial coefficient weights. The final result is

$$A\sum_{j_1=-n_1-m_1}^{n_1+m_1}\sum_{j_2=-n_2-m_2}^{n_2+m_2}\sum_{j_3=-n_3-m_3}^{n_3+m_3}W_{j_1}W_{j_2}W_{j_3}F_\zeta(\zeta_{j_1})F_\alpha(\alpha_{j_2},\zeta_{j_1})F_\beta(\beta_{j_3},\zeta_{j_1})h_1h_2h_3,$$

where W_j is defined in (3.3) with $W_{-k_j}^{(j)}=W_{k_j}^{(j)}$.

After similar operation, we have the following summation formula,

$$\begin{aligned}&2A\sum_{j_1=1}^{n_1+m_1}W_{j_1}\mathrm{Re}\left\{F_\zeta(\zeta_{j_1})\left[\sum_{j_2=-n_2-m_2}^{n_2+m_2}W_{j_2}F_\alpha(\alpha_{j_2},\zeta_{j_1})\right]\left[\sum_{j_3=-n_3-m_3}^{n_3+m_3}W_{j_3}F_\beta(\beta_{j_3},\zeta_{j_1})\right]\right\}+\\ &AF_\zeta(\zeta_0)\left[F_\alpha(\alpha_0,\zeta_0)+2\sum_{j_2=1}^{n_2+m_2}W_{j_2}\mathrm{Re}[F_\alpha(\alpha_{j_2},\zeta_0)]\right]\left[F_\beta(\beta_0,\zeta_0)+2\sum_{j_3=1}^{n_2+m_3}W_{j_3}\mathrm{Re}[F_\beta(\beta_{j_3},\zeta_0)]\right].\end{aligned}$$

Here the order of number of terms becomes $O(n_1(n_2+n_3))$. In our computation, $n_1=500$, $m_1=250$, $n_2=m_2=25$ and $n_3=m_3=25$.

Appendix B

The Relation Among Barrier Options

By no-arbitrage argument, we have

$$\begin{aligned} \mathrm{UIC} + \mathrm{UOC} = \quad & \mathrm{C} \quad = \mathrm{DIC} + \mathrm{DOC}, \\ \mathrm{UIP} + \mathrm{UOP} = \quad & \mathrm{P} \quad = \mathrm{DIP} + \mathrm{DOP}, \end{aligned}$$

where UIC stands for up-and-in call, UOC for up-and-out call, DIC for down-and-in call, DOC for down-and-out call, UIP for up-and-in put, UOP for up-and-out put, DIP for down-and-in put and DOP for down-and-out put.

For DIC option, the payoff at maturity is given by

$$\mathrm{DIC}(k,T) = e^{-rT}\mathrm{E}^{Q}[(S_T - e^{-k})^{+}1_{\{\tau_B<T\}}],\ B < S_0.$$

Similarly, we have

$$\begin{aligned} & \mathcal{L}_{k,\zeta}\left[\mathcal{L}_{T,\alpha}DIC(k,T)\right] \\ = \quad & \frac{B^{\zeta+1}}{\zeta(\zeta+1)}\frac{1}{r+\alpha-G(\zeta+1)}\left(\mathrm{E}[e^{-(r+\alpha)\tau}I_{\{S_\tau<B\}}]\frac{\eta_2}{\eta_2+\zeta+1} + \mathrm{E}[e^{-(r+\alpha)\tau}I_{\{S_\tau=B\}}]\right). \end{aligned}$$

For UIP/DIP option, let $k = log(K)$, the payoff at maturity is given by

$$\text{UIP}(k,T) = e^{-rT}\text{E}^Q[(e^k - S_T)^+ 1_{\{\tau_B < T\}}],\ B > S_0,$$

$$\text{DIP}(k,T) = e^{-rT}\text{E}^Q[(e^k - S_T)^+ 1_{\{\tau_B < T\}}],\ B < S_0.$$

Laplace transforms are given by

$$\frac{B^{-\zeta+1}}{\zeta(1-\zeta)}\frac{1}{r+\alpha-G(-\zeta+1)} \times \left(\text{E}[e^{-(r+\alpha)\tau}I_{\{S_\tau > B\}}]\frac{\eta_1}{\eta_1+\zeta-1} + \text{E}[e^{-(r+\alpha)\tau}I_{\{S_\tau = B\}}]\right),$$
$$\frac{B^{-\zeta+1}}{\zeta(1-\zeta)}\frac{1}{r+\alpha-G(-\zeta+1)} \times \left(\text{E}[e^{-(r+\alpha)\tau}I_{\{S_\tau < B\}}]\frac{\eta_2}{\eta_2-\zeta+1} + \text{E}[e^{-(r+\alpha)\tau}I_{\{S_\tau = B\}}]\right).$$

The general form of the expectations can be found in Section 3.

Appendix C

Proof of Lemma 5.1

Proof We derive the convolution of a_S and a_F on each quadrant

$$
\begin{aligned}
f_y(y|a_S > 0, a_F > 0) &= \int_{y/2}^{y} f_1(u, y-u)\,du + \int_0^{y/2} f_1(u, y-u)\,du + \frac{1}{2} f_1(\frac{y}{2}, \frac{y}{2}) \\
&= \left(\frac{(\eta_1^1 + \eta^1)\eta_2^1}{\eta_1^1 + \eta^1 - \eta_2^1} + \frac{(\eta_2^1 + \eta^1)\eta_1^1}{\eta_2^1 + \eta^1 - \eta_1^1} + \frac{\eta^1}{2} \right) e^{-\frac{\eta_1^1 + \eta_2^1 + \eta^1}{2} y} \\
&\quad - \frac{(\eta_1^1 + \eta^1)\eta_2^1}{\eta_1^1 + \eta^1 - \eta_2^1} e^{-(\eta_1^1 + \eta^1)y} - \frac{(\eta_2^1 + \eta^1)\eta_1^1}{\eta_2^1 + \eta^1 - \eta_1^1} e^{-(\eta_2^1 + \eta^1)y}, \quad y \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
f_y(y|a_S < 0, a_F > 0) &= \begin{cases} \int_0^\infty f_2(u, y+u)\,du, & y > 0, \\ \int_0^\infty f_2(u, u)\,du, & y = 0, \\ \int_0^\infty f_2(y+u, u)\,du, & y < 0, \end{cases} \\
&= \begin{cases} \frac{(\eta_1^2 + \eta^2)\eta_2^2}{\eta_1^2 + \eta_2^2 + \eta^2} e^{-(\eta_1^2 + \eta^2)y}, & y > 0, \\ \frac{\eta^2}{\eta_2^2 + \eta_2^2 + \eta^2} \delta(y), & y = 0, \\ \frac{(\eta_2^2 + \eta^2)\eta_1^2}{\eta_2^2 + \eta_2^2 + \eta^2} e^{(\eta_2^2 + \eta^2)y}, & y < 0. \end{cases}
\end{aligned}
$$

$$
f_y(y|a_S < 0, a_F < 0) = \int_{y/2}^{t} f_3(u, y-u)\,du + \int_0^{y/2} f_3(u, y-u)\,du + \frac{1}{2} f_3(\frac{y}{2}, \frac{y}{2})
$$

$$= \left(\frac{(\eta_1^3 + \eta^3)\eta_2^3}{\eta_1^3 + \eta^3 - \eta_2^3} + \frac{(\eta_2^3 + \eta^3)\eta_1^3}{\eta_2^3 + \eta^3 - \eta_1^3} + \frac{\eta^3}{2} \right) e^{\frac{\eta_1^3 + \eta_2^3 + \eta^1}{2} y}$$

$$- \frac{(\eta_1^3 + \eta^3)\eta_2^3}{\eta_1^3 + \eta^3 - \eta_2^3} e^{(\eta_1^3 + \eta^3)y} - \frac{(\eta_2^3 + \eta^3)\eta_1^3}{\eta_2^3 + \eta^3 - \eta_1^3} e^{(\eta_2^3 + \eta^3)y}, \quad y \le 0.$$

$$f_y(y|a_S > 0, a_F < 0) = \begin{cases} \int_0^\infty f_4(y+u, u)\, du, & y > 0, \\ \int_0^\infty f_4(u, u)\, du, & y = 0, \\ \int_0^\infty f_4(u, y+u)\, du, & y < 0, \end{cases}$$

$$= \begin{cases} \frac{(\eta_1^4 + \eta^4)\eta_2^4}{\eta_1^4 + \eta_2^4 + \eta^4} e^{-(\eta_1^4 + \eta^4)y}, & y > 0, \\ \frac{\eta^4}{\eta_2^4 + \eta_2^4 + \eta^4} \delta(y), & t = 0, \\ \frac{(\eta_2^4 + \eta^4)\eta_1^4}{\eta_2^4 + \eta_2^4 + \eta^4} e^{(\eta_2^4 + \eta^4)y}, & y < 0. \end{cases}$$

Then we combine the terms by using following identity

$$f_y(y) = p^1 f_y(y|a_S \ge 0, a_F \ge 0) + p^2 f_y(y|a_S < 0, a_F \ge 0)$$

$$+ p^3 f_y(y|a_S < 0, a_F < 0) + p^4 f_y(y|a_S \ge 0, a_F < 0),$$

$$f_y(y) = P_1 \sum_{i=1}^{5} q_i^u \gamma_i^u e^{-\gamma_i^u y} 1_{\{y>0\}} + P_2 \sum_{j=1}^{5} q_j^d \gamma_j^d e^{\gamma_j^d y} 1_{\{y<0\}} + P_3 \delta(y),$$

where $\delta(.)$ is the Dirac delta function and

$$c_1^u = p^1 \frac{2}{\eta_1^1 + \eta_2^1 + \eta^1} \left(\frac{(\eta_1^1 + \eta^1)\eta_2^1}{\eta_1^1 + \eta^1 - \eta_2^1} + \frac{(\eta_2^1 + \eta^1)\eta_1^1}{\eta_2^1 + \eta^1 - \eta_1^1} + \frac{\eta^1}{2} \right),$$

$$c_2^u = -p^1 \frac{\eta_2^1}{\eta_1^1 + \eta^1 - \eta_2^1}, \qquad c_3^u = -p^1 \frac{\eta_1^1}{\eta_2^1 + \eta^1 - \eta_1^1},$$

$$c_4^u = p^2 \frac{\eta_1^2}{\eta_2^2 + \eta_2^2 + \eta^2}, \qquad c_5^u = p^4 \frac{\eta_2^4}{\eta_1^4 + \eta_2^4 + \eta^4},$$

$$c_1^d = p^3 \frac{2}{\eta_1^3 + \eta_2^3 + \eta^1} \left(\frac{(\eta_1^3 + \eta^3)\eta_2^3}{\eta_1^3 + \eta^3 - \eta_2^3} + \frac{(\eta_2^3 + \eta^3)\eta_1^3}{\eta_2^3 + \eta^3 - \eta_1^3} + \frac{\eta^3}{2} \right),$$

$$c_2^d = -p^3 \frac{\eta_2^3}{\eta_1^3 + \eta^3 - \eta_2^3}, \qquad c_3^d = -p^3 \frac{\eta_1^3}{\eta_2^3 + \eta^3 - \eta_1^3},$$

$$c_4^d = p^2 \frac{\eta_2^2}{\eta_1^2 + \eta_2^2 + \eta^2}, \qquad c_5^d = p^4 \frac{\eta_1^4}{\eta_2^4 + \eta_2^4 + \eta^4},$$

$$
\begin{aligned}
\gamma_1^u &= \frac{\eta_1^1+\eta_2^1+\eta^1}{2}, \ \gamma_2^u=\eta_1^1+\eta^1, \quad \gamma_3^u=\eta_2^1+\eta^1, \quad \gamma_4^u=\eta_2^2+\eta^2, \quad \gamma_5^u=\eta_2^4+\eta^4,\\
\gamma_1^d &= \frac{\eta_1^3+\eta_2^3+\eta^1}{2}, \ \gamma_2^d=\eta_1^3+\eta^3, \quad \gamma_3^d=\eta_2^3+\eta^3, \quad \gamma_4^d=\eta_1^2+\eta^2, \quad \gamma_5^d=\eta_2^4+\eta^4,\\
P_1 &= \sum_{i=1}^5 \frac{c_i^u}{\gamma_i^u}, \quad P_2=\sum_{i=1}^5 \frac{c_i^d}{\gamma_i^d}, \quad P_3=p^2\frac{\eta^2}{\eta_2^2+\eta_2^2+\eta^2}+p^4\frac{\eta^4}{\eta_2^4+\eta_2^4+\eta^4},\\
q_i^u &= \frac{c_i^u}{P_1\gamma_i^u}, \quad q_i^d=\frac{c_i^d}{P_2\gamma_i^d}, \quad i=1,2,3,4,5.
\end{aligned}
$$

The jump process of $S^* = SF$ is $\{a_S\,dN_S + a_F\,dN_F + (a_S+a_F)\,dN\}$, which can be reduced to $\{\hat{y}\,d\hat{N}\}$. Therefore the jump process $d\hat{N}$ has intensity $\hat{\lambda} = \lambda_S + \lambda_F + \lambda(P_1+P_2)$ and $\hat{y}$ follows a mixture of distribution, which is given by

$$
\begin{aligned}
\hat{f}(\hat{y}) &= \frac{\lambda_S}{\hat{\lambda}} f_S(\hat{y}) + \frac{\lambda_F}{\hat{\lambda}} f_F(\hat{y}) + \frac{\lambda(P_1+P_2)}{\hat{\lambda}} f_y(\hat{y}|\hat{y}\neq 0)\\
&= \left[\frac{\lambda_S}{\hat{\lambda}} p_S\eta_S^u e^{-\eta_S^u\hat{y}} + \frac{\lambda_F}{\hat{\lambda}} p_F\eta_F^u e^{-\eta_F^u\hat{y}} + \frac{\lambda}{\hat{\lambda}} P_1 \sum_{j=1}^5 q_j^u\gamma_j^u e^{-\gamma_j^u\hat{y}}\right] 1_{\{\hat{y}>0\}}\\
&\quad + \left[\frac{\lambda_S}{\hat{\lambda}}(1-p_S)\eta_S^d e^{\eta_S^d\hat{y}} + \frac{\lambda_F}{\hat{\lambda}}(1-p_F)\eta_F^d e^{\eta_F^d\hat{y}} + \frac{\lambda}{\hat{\lambda}} P_2 \sum_{j=1}^5 q_j^d\gamma_j^u e^{\gamma_j^d\hat{y}}\right] 1_{\{\hat{y}<0\}}\\
&= P_u\sum_{i=1}^7 p_i\eta_i e^{-\eta_i y} 1_{\{y\geq 0\}} + P_d\sum_{j=1}^7 q_j\kappa_j e^{\kappa_j y} 1_{\{y<0\}}.
\end{aligned}
$$

Here,

$\eta_i = \gamma_i^u, \qquad \eta_6 = \eta_S^u, \qquad \eta_7 = \eta_F^u,$

$\kappa_i = \gamma_i^d, \qquad \kappa_6 = \eta_S^d, \qquad \kappa_7 = \eta_F^d,$

$P_u = \frac{\lambda_S}{\hat{\lambda}} p_S + \frac{\lambda_F}{\hat{\lambda}} p_F + \frac{\lambda}{\hat{\lambda}} P_1,$

$P_d = \frac{\lambda_S}{\hat{\lambda}}(1-p_S) + \frac{\lambda_F}{\hat{\lambda}}(1-p_F) + \frac{\lambda}{\hat{\lambda}} P_2,$

$p_i = \frac{\lambda P_1 q_i^u}{\hat{\lambda} P_u}, \quad p_6 = \frac{\lambda_S}{\hat{\lambda} P_u} p_S, \quad p_7 = \frac{\lambda_F}{\hat{\lambda} P_u} p_F,$

$q_i = \frac{\lambda P_2 q_i^d}{\hat{\lambda} P_d}, \quad q_6 = \frac{\lambda_S}{\hat{\lambda} P_d}(1-p_S), \quad q_7 = \frac{\lambda_F}{\hat{\lambda} P_d}(1-p_F),$

for $i = 1, 2, 3, 4$ and 5.

Appendix D

Proof of Theorem 5.4 and 5.5

The main idea is to construct a martingale on the process $\mathrm{E}[e^{-\alpha\tau}1_{\{\tau<\infty\}}|X_0 = x]$. We first define the infinitesimal generator

$$\begin{aligned}\mathcal{L}'u(X_t) &= \mu u'(X_t) + \frac{\sigma^2}{2}u''(X_t) + \mathrm{E}[u(X_t + y) - u(X_t)] \\ &= \mu u'(X_t) + \frac{\sigma^2}{2}u''(X_t) + \lambda \int_{-\infty}^{\infty} [u(X_t + y) - u(X_t)]f(y)\,dy.\end{aligned}$$

Proof of Theorem 5.4:

Proof Applying Itô's lemma to the process $\{e^{-\alpha t}u(X_t)\}$, we construct a local martingale with $M(0) = u(X_0)$,

$$M(t) = e^{-\alpha(t\wedge\tau_b)}u(X_{t\wedge\tau_b}) - \int_0^{t\wedge\tau_b} e^{-\alpha s}(-\alpha u(X_s) + \mathcal{L}'u(X_s))\,ds.$$

Suppose $-\alpha u(X_s) + \mathcal{L}'u(X_s) = 0$, we have

$$u(X_0) = \mathrm{E}M(t) = \mathrm{E}[e^{-\alpha(t\wedge\tau_b)}u(X_{t\wedge\tau_b})] = \mathrm{E}[e^{-\alpha(\tau_b)}1_{\{\tau_b<\infty\}}], \qquad \text{as } t \to \infty.$$

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The main idea is to construct a martingale on the process $\mathrm{E}[e^{-\alpha\tau}1_{\{\tau<\infty\}}|X_0 = x]$. We first define the infinitesimal generator

$$\begin{aligned}\mathcal{L}'u(X_t) &= \mu u'(X_t) + \frac{\sigma^2}{2}u''(X_t) + \mathrm{E}[u(X_t+y) - u(X_t)] \\ &= \mu u'(X_t) + \frac{\sigma^2}{2}u''(X_t) + \lambda\int_{-\infty}^{\infty}[u(X_t+y) - u(X_t)]f(y)\,dy.\end{aligned}$$

Proof of Theorem 5.4:

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The solution of $\mathcal{L}'u(x) - \alpha u(x) = 0$ is given by the form

$$u(x) = \sum_{k=1}^{n+1} \gamma_k e^{-\beta_k(b-x)}, \quad x \le b.$$

For $x < b$,

$$\begin{aligned}
&\mathcal{L}'u(x)\\
=\ & \mu \sum_{k=1}^{n+1} \gamma_k \beta_k e^{-\beta_k(b-x)} + \frac{1}{2}\sigma^2 \sum_{k=1}^{n+1} \gamma_k \beta_k^2 e^{-\beta_k(b-x)}\\
&+\lambda(\int_{-\infty}^{b-x} + \int_{b-x}^{\infty})[u(x+y) - u(x)] f_Y(y)\, dy\\
=\ & \sum_{k=1}^{n+1} \gamma_k(\mu\beta_k + \frac{1}{2}\sigma^2\beta_k^2) e^{-\beta_k(b-x)} + \lambda \left[\int_{-\infty}^{b-x} \sum_{k=1}^{n+1} (e^{\beta_k y} - 1)\gamma_k e^{-\beta_k(b-x)} f_Y(y)\, dy \right.\\
&\left. + \int_{b-x}^{\infty} \left(1 - \sum_{k=1}^{n+1} \gamma_k e^{-\beta_k(b-x)}\right) f_Y(y)\, dy \right]\\
=\ & \sum_{k=1}^{n+1} \gamma_k(\mu\beta_k + \frac{1}{2}\sigma^2\beta_k^2) e^{-\beta_k(b-x)} + \lambda \sum_{k=1}^{n+1} \gamma_k e^{-\beta_k(b-x)} \mathrm{E}[(e^{\beta_k Y} - 1)1_{\{Y<b-x\}}]\\
&+\lambda \left(1 - \sum_{k=1}^{n+1} \gamma_k e^{-\beta_k(b-x)}\right) P_u \sum_{i=1}^{n} p_i e^{-\eta_i(b-x)}\\
=\ & \sum_{k=1}^{n+1} \gamma_k \left[\mu\beta_k + \frac{1}{2}\sigma^2\beta_k^2 + \mathrm{E}[(e^{\beta_k Y} - 1)1_{\{Y<b-x\}}]\right] e^{-\beta_k(b-x)}\\
&+\lambda \left(1 - \sum_{k=1}^{n+1} \gamma_k e^{-\beta_k(b-x)}\right) P_u \sum_{i=1}^{n} p_i e^{-\eta_i(b-x)}\\
=\ & \sum_{k=1}^{n+1} \gamma_k \left(G(\beta_k) - \sum_{i=1}^{n} p_i \left(\frac{\eta_k e^{-(\eta_i-\beta_k)(b-x)}}{\eta_i - \beta_k} - e^{-\eta_i(b-x)}\right)\right) e^{-\beta_k(b-x)}\\
&+\lambda P_u \sum_{i=1}^{n} p_i e^{-\eta_i(b-x)} \left(1 - \sum_{k=1}^{n+1} \gamma_k e^{-\beta_k(b-x)}\right)\\
=\ & \sum_{k=1}^{n+1} \gamma_k G(\beta_k) e^{-\beta_k(b-x)} + \lambda P_u \sum_{i=1}^{n} p_i e^{-\eta_i(b-x)} \left(1 - \sum_{k=1}^{n+1} \gamma_k \frac{\eta_i}{\eta_i - \beta_k}\right).
\end{aligned}$$

Therefore,

$$\mathcal{L}'u(x) - \alpha u(x) = \sum_{k=1}^{n+1} \gamma_k [G(\beta_k) - \alpha] e^{-\beta_k(b-x)} + \lambda P_u \sum_{i=1}^{n} p_i e^{-\eta_i(b-x)} \left(1 - \sum_{k=1}^{n+1} \gamma_k \frac{\eta_i}{\eta_i - \beta_k}\right),$$

with boundary conditions

$$u(x) = \begin{cases} 1, & x \geq b, \\ 0, & x \to -\infty. \end{cases}$$

By Theorem 5.1, $G(\beta) = \alpha$ has exactly $n+1$ roots s.t. $\beta \in C^+$. With the continuity condition, the solution of $\mathcal{L}'u(x) - \alpha u(x) = 0$ is given by:

$$\begin{aligned} G(\beta_k) - \alpha &= 0, \quad \Re(\beta_k) > 0, \\ \sum_{k=1}^{n+1} \gamma_k \frac{\eta_i}{\eta_i - \beta_k} &= 1, \quad i = 1, 2, \dots n, \\ \sum_{k=1}^{n+1} \gamma_k &= 1. \end{aligned}$$

By a similar argument, we can solve the solution of $\mathrm{E}\left[e^{-\alpha\tau_b} 1_{\{X_\tau \neq b\}}\right]$ by changing the boundary condition as follows:

$$u(x) = \begin{cases} 1, & x > b, \\ 0, & x = b, \\ 0, & x \to -\infty. \end{cases}$$

And the value of $\hat{\gamma}_k$ is obtained by solving the following of system of linear equation

$$\begin{aligned} \sum_{k=1}^{n+1} \hat{\gamma}_k \frac{\eta_i}{\eta_i - \beta_k} &= 1, \quad i = 1, 2, \dots n, \\ \sum_{k=1}^{n+1} \hat{\gamma}_k &= 0. \quad \square \end{aligned}$$

Proof of Theorem 5.5:

Proof The proof is similar to that of Lemma 5.4, here we just outline the difference, again we construct a local martingale with $M(0) = u(X_0)$, and finally it leads the integral equation

$$\mathcal{L}'u(x) - \alpha u(x) = 0.$$

The solution is given by the form

$$u(x) = \sum_{k=1}^{m+1} \gamma_k e^{-\beta_k(a-x)}, \quad x \geq a.$$

Then for $x \geq a$,

$$\mathcal{L}'u(x) - \alpha u(x) = \sum_{k=1}^{m+1} \gamma_k [G(\beta_k) - \alpha] e^{-\beta_k(a-x)} + \lambda P_d \sum_{j=1}^{m} q_j e^{\kappa_j(a-x)} \left(1 - \sum_{k=1}^{m+1} \gamma_k \frac{\kappa_j}{\kappa_j + \beta_k}\right),$$

with boundary conditions

$$u(x) = \begin{cases} 0, & x \to +\infty, \\ 1, & x \leq a. \end{cases}$$

By Theorem 5.1, $G(\beta) = \alpha$ has $m+1$ roots s.t. $\beta \in \mathcal{C}^-$. The solution is shown as lemma 5.5.

Similarly, the solution of $u(X_0) = \mathrm{E}[e^{-\alpha(\tau_a)} 1_{\{X_\tau \neq a\}}]$ is followed by the boundary conditions

$$u(x) = \begin{cases} 0, & x \to +\infty, \\ 0, & x = a, \\ 1, & x < a. \end{cases} \qquad \square$$

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