Some Topics on Nonlinear Conservation Laws

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It is well-known for the Quasi-One-Dimensional nozzle flow that the standing shock is stable in the diverging duct and is unstable in the converging duct for the idea inviscid flow. In this thesis, we will investigate viscous flows in Quasi-One-Dimensional nozzle. After a brief introduction of viscous conservation law and viscous shock profile, we study the stability of viscous shock waves. Besides initial value problem, we also study the propagation of stationary shock waves in bounded domain. Moreover, some new results about propagation of stationary shock wave for viscous transonic flow through a nozzle are obtained.

摘要

衆所周知，對於擬一維管道中的理想無粘流體而言，佇立激波在發散的管道中是穩定的，在壓縮的管道中是不穩定的。在本文中，我們將研究一維管道中的粘性流體。在很簡短的介紹粘性守恒律和粘性波段之後，我們將研究粘性激波的穩定性。對於初值問題下的激波的傳播，我們做了一定的研究，此外，還研究了有界區域上相應的問題。這裏，我們得到一些粘性跨音速流體穿過管道時關於激波傳播的結果。

Contents

Introduction

For large time behavior of convex scalar hyperbolic conservation law

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = u_0 \end{cases} \tag{0.0.1}$$

we have learn a great deal. In Lax's paper [29]1957, asymptotic behavior of solutions was discussed as t tends to infinity, it was completely described and depends on the initial data in the far field. Moreover if a system is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate, large time behavior is also determined. The deep result for 2×2 systems was presented by Glimm and Lax [15]1970, which was extended to general systems by DiPerna[7] and Liu[30].

In contrast, for viscous conservation law

$$\begin{cases} u_t + f(u)_x = u_{xx}, \\ u(x,0) = u_0. \end{cases} \tag{0.0.2}$$

it is much more difficult and to obtain asymptotic stability is not easy. Il´in and Oleinik[23]1958 using maximal principle proved that viscous shock profile in the case of a convex nonlinear term was indeed orbitally stable, and later, Peletier[41] used energy method and gave another proof. On the other hand, in [23], Il´in and Oleinik also showed that if the initial value exponentially decayed to the end-states of the profile, then the perturbation of waves decayed at a corresponding exponentially rate, which also can be obtained from energy method[58]. In the article, for convex flux function, when both initial perturbation in H^1 is small enough and has certain decay in the far field, the perturbed solution will converge to shock profile when time tends to infinity. If one consider the linearized stability of viscous shock wave, we will find that the corresponding linearized operator probably has eigenvalue 0 due to translation invariance of the equations. For this sake, we can not deduce nonlinear stability from linear stability directly [50].

Up to 1976, Sattinger [46] proved that a traveling wave is asymptotically stable with respect to small perturbations. His idea amounts to define the linearized operator in a weighted space, therefore, the corresponding eigenfunction space will become smaller. Subsequently, the eigenvalues of this linearized operator will be restricted in a smaller region in complex plane such that except for isolated simple eigenvalue 0, real parts of all other eigenvalues have a negative upper bound. Under this condition, nonlinear stability for general traveling waves can be obtained in the weighted space. Moreover, nonlinear stability of viscous shock waves becomes an example of Sattinger's theory, without restriction on convexity of flux functions. And, his result also showed that perturbation decays exponentially in time whe the initial perturbation decays exponentially to the shock profile at the far field for non-convex conservation law.

Kawashima and Matsumura [25]1985 investigated the asymptotic stability of traveling wave solutions with shock profile for several systems of gas dynamics. Particularly, for a scalar conservation law with viscosity, the solution approaches the traveling wave solution at rate $t^{-\gamma}$(for some $\gamma > 0$) as $t \to \infty$, provided that the initial disturbance is small and of integral order, and decays at an algebraic rate for $|x| \to \infty$. By energy method, the decay obtained is in a polynomial weighted L^2 space. Later, Jones, Gardner and Kapitula [24]1993 generized the algebraic decay in [25] to weighted L^∞ space, by a new technique of resolvent analysis. And the nonlinear term convexity hypothesis is not assumed. Moreover, such technique was successfully applied to system of conservation laws, even for multidimensional system of conservation laws [13, 18].

While, the aforementioned methods can only get asymptotic stability for small perturbation of travelling waves. Moreover, they can only get the convergence in L^2, L^∞, or in certain weighted space. On the other hand, we know, for one-dimensional conservation law, L^1 is indeed a suitable space, where Cauchy problem is well-posed, and L^1 space has physical significance for conservation law. Hence, L^1 stability is much more important than any other stability. This direction was originated by Osher and Ralston [42]1982, they obtained asymptotic stability for viscous shock wave when the initial data is between two shifted viscous shock waves. Along this direction, Serre and Freistuhler, have made comprehensive studies and finally obtained a complete result for L^1 stability of viscous shock profile, they showed that any L^1 perturbation will merge into the viscous shock wave, it was included in a series of work, [47], [11]. A good survey for L^1-stability of nonlinear waves in scalar conservation law is [49], where Serre also studied stability of relaxation shock, radiative shock, discreet shock and boundary layers, and so forth. The basic tools for establishing L^1 stability are some important properties for scalar viscous conservation law, L^1 contraction principle, comparison principle[27], and dispersion property for viscous conservation law[1].

For system of viscous conservation laws, the initial data without excess mass, asymptotic stability of viscous shock wave was first proved by Goodman[16], Mastumura and Nishihara [39], by energy method independently, in certain sense, it can be regarded as a generalization of Peletier's idea for scalar equation. However, the method and analysis by Goodman are more fundamental and useful in many other situations. When initial perturbation has excess mass, L^2 stability for viscous shock wave for a special class of perturbations was obtained by Liu in [30]1977 where he introduced very important diffusion waves. By introducing coupled linear diffusion waves and combining the energy estimate with pointwise estimate, Szepessy and Xin [53]1993 got rid of the restriction in [30] and obtained stability of viscous shockwave for general initial perturbation. L^1 stability for Lax shock was finally established by Liu[31] by an elaborate study of approximate Green's function and detailed pointwise estimates.

When we consider the stability of viscous shock wave in scalar conservation law for initial value problem, there is a viscous shock wave in the whole space. Although there are some small disturbances, they will merge into the shock wave, therefore, viscous shock wave will propagate in the whole space freely, so the stationary shock keeps static. For bounded domain and half space, if the shock wave is not stationary, Rankine-Hugoniot condition says that the shock wave will either be absorbed into boundary or generate a strong boundary layer. While, the propagation of stationary viscous shock wave is very subtle when the domain has boundary. In general, boundary layer will occur because usually the shock profile does not match the boundary condition exactly; moreover, since the speeds of the boundary layer and shock layer are comparable, therefore, the resonance of these two types of layer will occur. These induce fruitful phenomena for the propagation of stationary viscous shock wave in bounded domain and half space. When the viscous coefficient is small enough, viscous shock wave in bounded domain will be drifted by two boundary layers, to balance these boundary layers, the motion of shock layer will be exponentially slow in exponentially long time, this is so called metastable phenomenon. This phenomenon was first observed for Burgers equation by Kreiss and Kreiss[26] in numerics, and then studied for general equation by Laforgue and O'Malley[28], Reyna and Ward[43] independently. In [28], the authors generalized matched asymptotic analysis method. Reyna and Ward analyzed linearized problem around the shock wave, with the help of studying certain spectrum problem, and obtained the propagation of viscous shock wave in bounded domain.

As far as the half space is concerned, Ward and Reyna [55]1995 first studied the propagation of a shock by the method they developed in [43]. Since boundary layer and shock layer will be resonant, therefore, the shock layer will be drifted away from the boundary, thus the influence of boundary layer will be weaker and weaker on the shock layer, so the acceleration of shock layer away form the boundary will be smaller. Asymptotic analysis shows that shock layer will propagate with speed of order $\log t$ with respect to time t. Later on, Liu and Yu [38]1997 gave a justification for the asymptotical analysis result in [55] by detailed pointwise estimates, because they almost can write down the solution explicitly by Green's function.

In practice, balance law equation is as important as conservation law equation. Several physical situations can be modelled as hyperbolic equation with a source, for instance, the geometric effect of a nozzle on the gas flow can be expressed as source. The quasi-one-dimensional model of gas flow through a nozzle [57] is

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = -\frac{A'(x)}{A(x)}\rho, \\ \frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = -\frac{A'(x)}{A(x)}\rho u^2, \\ \frac{\partial(\rho E)}{\partial t} + \frac{\partial}{\partial x}(\rho E u + p u) = -\frac{A'(x)}{A(x)}(\rho E u + p u), \end{cases} \tag{0.0.3}$$

where ρ, u, p, E are the density, velocity, pressure and the total energy of the gas, and $A(x)$ is the area of cross section of the nozzle. For uniform nozzle $A'(x) = 0$, the system becomes one dimensional compressible Euler equation. Liu and his collaborators made comprehensive studies for the system (0.0.3), see [31, 32, 33, 14] and references therein. The main results they obtained are that the shape of the nozzle has stabilizing and distabilizing effect, and that there are a finite number of asymptotic shapes that can be constructed explicitly. Almost at the same time, Ebid, Goodman and Majda [9] studied steady states of isentropic flow through a nozzle. To analyze stability of standing transonic shock as what was done by Liu, they proposed a much simpler scalar model

$$u_t + (\frac{u^2}{2})_x = a(x)u. \tag{0.0.4}$$

analogous to isentropic flow through a nozzle.

Actually, Liu in [35]1987 also proposed a scalar model similar to (0.0.4) as

$$u_t + f(u)_x = a(x)h(u), \tag{0.0.5}$$

where he imposed conditions for strong coupling of source, $h(u) \neq 0$ and $h'(u) \neq 0$. Utilizing his modification of Glimm scheme and wave interaction estimate, he obtained a transparent and revealing qualitative understanding of wave behavior of (0.0.5), including such as existence, nonlinear stability, instability, and changing types of waves. Besides inviscid model, Liu and Hsu [22] also studied existence and nonlinear stability of steady states for viscous equation

$$u_t + f(u)_x = \epsilon u_{xx} + a(x)h(u) \tag{0.0.6}$$

by a new type of a priori estimate and spectrum analysis.

In fact, besides steady states, stability of viscous transonic shock wave is of great interests. If the life span of shock wave is very long, we will observe it easily in experiment. Therefore, we are interested with propagation of viscous shock in a nozzle as in the case of conservation law, [43]1999, where the flow is passing through a uniform nozzle. Sun and Ward [51] studied the propagation of viscous shock waves with constraint that $a(x)$ is exponentially small for the model

$$u_t + f(u)_x = \epsilon u_{xx} + a(x)u, \tag{0.0.7}$$

where the leading order approximation by matched asymptotic analysis is as same as that in [43] for viscous conservation law, applying projection method in [43] with a little bit generalization, they obtained metastability of viscous shock wave in this case again. To relax the artificial constraints in [51], we note that it is different from viscous conservation law that the shape of nozzle will help determine the location of shock wave for flow in nozzle. Motivated by the study for inviscid flow through a nozzle, we may take the leading order ansatz of location of shock wave to be static for a divergent nozzle. Then we can solve the next and higher order outer solutions, ansatz of location of shock wave and inner solutions simultaneously. It shows that the change of the ansatz of location of shock wave is very small, therefore, metastability of viscous transonic shock wave in a divergent nozzle is obtained.

We conclude this introduction by outlining the rest of this thesis. In chapter 1, we shall study nonlinear stability of shock profile by energy method and contraction principle. In chapter 2, we will use projection method to study the propagation of shock wave in bounded domain and half space, then verify the asymptotic result in half space by pointwise estimate. In chapter 3, we will study asymptotic stability by spectrum analysis then analyze leading order and higher order approximations of transonic flow through a nozzle by matched asymptotic analysis. In last chapter, chapter 4, we study the interaction between shock layer and boundary layer.

Chapter 1

Stability of Shock Waves in Viscous Conservation Laws

In this chapter, we first recall some basic properties of solutions to Cauchy problems for viscous scalar conservation laws, then define viscous shock profiles and give some basic properties of viscous shock profiles. Based on these basic knowledge, we will study the stability of shock profiles by energy method, L^1 stability of viscous shock wave by contraction principle and comparison principle respectively.

1.1 Cauchy Problem for Scalar Viscous Conservation Laws and Viscous Shock Profiles

Consider the following Cauchy problem

$$u_t + f(u)_x = u_{xx}, \tag{1.1.1}$$

$$u(x, 0) = u_0(x). \tag{1.1.2}$$

From the seminal paper of Kruzkov[27], we have

Theorem 1.1.1 *For any $u_0 \in L^\infty(\mathbb{R}^1)$, the problem* (1.1.1)-(1.1.2) *has a unique solution in $C(0,\infty; L^\infty(\mathbb{R}^1))$ and satisfies the following four properties:*

(i) : *$u \in C^\infty(\mathbb{R}^1 \times \mathbb{R}^1_+)$ when $f \in C^\infty$;*

(ii) *(Comparison principle): Assume two initial data u_0 and v_0 satisfy $u_0 \leq v_0$, then the corresponding solutions satisfy $u(x,t) \leq v(x,t)$;*

(iii) *(Conservation of mass): Let u, v be two solutions to the Cauchy problem* (1.1.1)-(1.1.2) *corresponding to the initial data u_0, v_0, if $u_0 - v_0 \in L^1(\mathbb{R})$, then $u(t) - v(t) \in L^1(\mathbb{R})$ and*

$$\int_{-\infty}^{\infty} (u(x,t) - v(x,t))dx = \int_{-\infty}^{\infty} (u_0 - v_0)dx; \tag{1.1.3}$$

(iv) *(Contraction principle): Suppose $\|u_0 - v_0\|_{L^1} < \infty$ and u, v are two solutions to the Cauchy problem* (1.1.1)-(1.1.2) *associated with initial data u_0, v_0, then*

$$\|u(t) - v(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}. \tag{1.1.4}$$

Theorem 1.1.1 is very classical, and its proof can be found in [27, 47].

Theorem 1.1.1 allows us to construct an operator $S(t)$ which with a given initial data u_0 associates at the instant $t > 0$ the solution $u(t)$ to (1.1.1)-(1.1.2). It is easy to show the family $(S(t))_{t \geq 0}$ is a semigroup.

One of the key elements in understanding the theory of viscous conservation laws is the inviscid theory. It is well-known that the shock wave

$$u(x,t) = \begin{cases} u_- & x < st, \\ u_+ & x > st, \end{cases} \tag{1.1.5}$$

is very important for hyperbolic conservation law

$$u_t + f(u)_x = 0. \tag{1.1.6}$$

If (1.1.5) is a weak solution to (1.1.6), then Rankine-Hugoniot condition implies

$$f(u_+) - f(u_-) = s(u_+ - u_-). \tag{1.1.7}$$

We will denote the shock wave (1.1.5) which satisfies Rankine-Hugoniot condition (1.1.7) by (u_-, u_+, s). If we interchange u_- with u_+ in (1.1.5), (u_+, u_-, s) also satisfies Rankine-Hugoniot condition (1.1.7). To get physical solution, we need some admissible condition. For general scalar conservation laws, Oleinik condition

$$\frac{f(u) - f(u_-)}{u - u_-} > s \quad \text{for all} \quad u \quad \text{between} \quad u_+ \quad \text{and} \quad u_-, \tag{1.1.8}$$

is an necessary condition for admissibility of shock wave (1.1.5) for hyperbolic conservation law (1.1.6), see [47]. If the flux function is convex, $f'(u_-) \neq s$ and $f'(u_+) \neq s$, then Oleinik condition becomes famous Lax geometric entropy condition, $u_- > u_+$. A natural physical entropy condition is the following viscous criteria:

Vanishing Viscosity Criteria: A weak solution u of (1.1.6) is admissible if there exists a sequence of smooth solution u^ϵ of

$$u_t + f(u)_x = \epsilon u_{xx} \tag{1.1.9}$$

which converges to u in L^1_{loc} as $\epsilon \to 0+$.

Since shock wave (1.1.5) is dilation invariant, therefore, we expect that (1.1.9) possesses a travelling wave solution $\phi(\frac{x-st}{\epsilon})$ which converges to u in (1.1.5) as $\epsilon \to 0+$. On the other hand, if $\phi(\frac{x-st}{\epsilon})$ converges to (1.1.5) as $\epsilon \to 0+$, then $\phi(\xi) \to u_\pm$ as $\xi \to \pm\infty$. Therefore, we have

Definition 1.1.2 *$\phi(x - st)$ is called a viscous shock profile for the shock wave (u_-, u_+, s), if*

$$\begin{cases} -s\phi' + f(\phi)' = \phi'', \\ \phi(\xi) \to u_\pm \qquad as \qquad \xi \to \pm\infty, \end{cases} \tag{1.1.10}$$

where $' = \frac{d}{d\xi}$, $\xi = x - st$.

Then we have

Lemma 1.1.3 *(1) ϕ exists if and only if Oleinik condition* (1.1.8) *holds, and is unique up to phase shift;*

(2) If f is convex, then $\frac{df'(\phi)}{d\xi} < 0$;

(3) If ϕ is a shock profile to the inviscid shock (u_-, u_+, s), then there exists x_1 such that $f'(\phi(x_1)) = s$;

(4) Suppose f is convex and ϕ is a shock profile to the inviscid shock (u_-, u_+, s), then

$$|\phi'(x)| \le O(1)|u_- - u_+|^2 e^{-C|u_- - u_+|\cdot|x|}, \tag{1.1.11}$$

where $C = \min_{u\in[u_+,u_-]} f''(u)$.

Proof: (1),(2),(3) are obvious.

Since f is convex, therefore, $\phi' < 0$. From (1.1.10), we know

$$(\ln(|\phi'|))' = f'(\phi) - s. \tag{1.1.12}$$

It follows from (3) that there exists x_1 such that $f'(\phi(x_1)) = s$. Therefore, if $x > y > x_1$, integrating the equation (1.1.12) from x to y gives

$$\frac{|\phi'(x)|}{|\phi'(y)|} = e^{\int_y^x (f'(\phi(z))-s)dz}.$$

Therefore

$$|\phi'(x)| \le |\phi'(y)| e^{-C(x-y)|u_- - u_+|},$$

that is to say,

$$|\phi'(x)| e^{-C|u_- - u_+|y} = |\phi'(y)| e^{-C|u_- - u_+|x}. \tag{1.1.13}$$

Integrating both sides of (1.1.13) from x_1 to x with respect to y yields (1.1.11). Similarly, we can get (1.1.11) when $x < x_1$. □

The question is whether $\phi(x - st)$ is a global attractor for the problem

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \\ u(x,0) = u_0(x), \\ \lim_{x\to\pm\infty} u_0(x) = u_\pm. \end{cases} \tag{1.1.14}$$

The answer in general is not true, if $\phi(x-st)$ is a solution, then $\phi(x-st+\delta)$ is also a solution for any δ.

If we linearize the problem at ϕ:

$$\mathcal{L}v = \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + f'(\phi)\frac{\partial v}{\partial x} + f''(\phi)\phi' v = 0.$$

Obviously, $\mathcal{L}\phi' = 0$, therefore, 0 is an eigenvalue of $\mathcal{L}$. Hence we can not to deduce nonlinear stability for viscous shock wave from linearized stability by a standard procedure in [50].

For convex flux functions, we can deal with nonlinear stability of viscous shock wave by energy method because of $\frac{df'(\phi)}{d\xi} < 0$ by lemma 1.1.3.

We digress for a moment and consider that if $\int_{\mathbb{R}}(u_0(x)-\phi(x))dx = m \neq 0$, then

$$\int_{\mathbb{R}^1}(u(x,t)-\phi(x-st))dx = \int_{\mathbb{R}^1}(u_0(x)-\phi(x))dx = m \neq 0,$$

therefore, we do not hope that $\lim_{t\to+\infty}\int_{\mathbb{R}^1}|u(x,t)-\phi(x-st)|dx = 0$. On the other hand, for any $\delta \in \mathbb{R}^1$, we have

$$\int_{-\infty}^{\infty}(\phi(x+\delta)-\phi(x))dx = \delta(u_+ - u_-), \tag{1.1.15}$$

therefore, if we set $\delta = \frac{m}{u_+ - u_-}$, then

$$\begin{aligned}\int_{\mathbb{R}^1}(u_0(x)-\phi(x+\delta))dx &= \int_{\mathbb{R}^1}(u_0(x)-\phi(x)) + \int_{\mathbb{R}^1}(\phi(x)-\phi(x+\delta)).\\ &= 0\end{aligned}$$

Therefore, even when initial data has excess mass, for one dimensional viscous conservation law, we still expect that we can get asymptotic stability of viscous shock waves after a shift.

Remark 1.1.4 *The* (1.1.15) *only holds for one dimensional case. Therefore, high expectation to get asymptotic stability of shock profiles for multidimensional viscous conservation law only occurs when the initial perturbation has no excess mass [49].*

1.2 Stability of Shock Waves by Energy Method

We first state the main result on asymptotic stability of viscous shock waves by Il'in and Oleinik [23].

Theorem 1.2.1 *Let $f''(u) > 0$, $u_- > u_+$ and $\phi(x - st)$ be the shock profile for the shock wave (u_-, u_+, s). Set*

$$m_0 = \int_{-\infty}^{+\infty} (u_0(x) - \phi(x))dx, \tag{1.2.1}$$

$$\delta = \frac{m_0}{u_+ - u_-}. \tag{1.2.2}$$

If $\int_{\mathbb{R}^1} |x|^2(u_0(x) - \phi(x + \delta))^2 dx < \epsilon$ and $\|u_0 - \phi(\cdot + \delta)\|_{H^1} \leq \epsilon$ for some small ϵ, then

$$\sup_{x\in\mathbb{R}} |u(x,t) - \phi(x - st + \delta)| \to 0 \qquad as \qquad t \to \infty. \tag{1.2.3}$$

Proof: We will follow [58].

Step 1: For simplicity, let $s = 0$ and $\delta = 0$. Set $u(x,t) = \phi(x) + w(x,t)$, substituting into the equation, we deduce that

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(f'(\phi)w) + \frac{\partial}{\partial x}(f(\phi + w) - f(\phi) - f'(\phi)w) = \frac{\partial^2 w}{\partial x^2}, \\ w(x,0) = u_0(x) - \phi(x). \end{cases} \tag{1.2.4}$$

If we define $Q(\phi, w) = f(\phi + w) - f(\phi) - f'(\phi)w$, when w is bounded, then it is easy to obtain that

$$|Q(\phi, w)| \leq O(1)|w|^2.$$

Set $v(x,t) = \int_{-\infty}^{x} w(y,t)dy$, $v_0(x) = \int_{-\infty}^{x}(u_0(x) - \phi(x))dx$, then

$$\begin{cases} \frac{\partial v}{\partial t} + f'(\phi)\frac{\partial v}{\partial x} + Q(\phi, v_x) = \frac{\partial^2 v}{\partial x^2}, \\ v(x,0) = v_0(x). \end{cases} \tag{1.2.5}$$

Step 2: Basic energy estimate

Claim 1: There exists a constant $\epsilon_1 > 0$, such that if

$$\sup_{0\leq t\leq T} \|v(x,t)\|_{H^2(\mathbb{R}^1)} \leq \epsilon_1,$$

then the estimate

$$\|v(\cdot,t)\|_{L^2}^2+\int_0^t\|v_x(\cdot,\tau)\|_{L^2}^2d\tau+\int_0^t\int_{\mathbb{R}^1}|\frac{\partial}{\partial x}f'(\phi)|v^2dxd\tau\le C_1\|v_0\|_{L^2}^2 \quad (1.2.6)$$

holds for $0\le t\le T$.

Proof of the claim 1: We multiply both sides of (1.2.5) by v and integrate over $\mathbb{R}^1$ to get

$$\frac{1}{2}\frac{d}{dt}\parallel v(\cdot,t)\parallel_{L^2}^2+\int_{\mathbb{R}^1}(-\frac{1}{2}\frac{\partial}{\partial x}f'(\phi))v^2dx+\int_{\mathbb{R}^1}|v_x|^2dx\le-\int_{\mathbb{R}^1}vQ(\phi,v_x)dx.$$

Applying Sobolev imbedding theorem, we deduce that

$$\begin{aligned}|\int_{\mathbb{R}^1}vQ(\phi,v_x)dx| &\le \max_{\mathbb{R}^1}|v(x,t)|\cdot O(1)\int_{\mathbb{R}^1}|v_x|^2dx\\ &\le \|v\|_{H^2}\cdot O(1)\int_{\mathbb{R}^1}|v_x|^2dx.\end{aligned}$$

Combining above estimate with assumption and $\frac{\partial}{\partial x}f'(\phi)<0$ by Lemma 1.1.3, yields estimate (1.2.6).

Step 3: Higher order estimate

Claim 2: There exists a constant $\epsilon_2>0$, such that if

$$\sup_{0\le t\le T}\|v(\cdot,t)\|_{H^2(\mathbb{R}^1)}\le\epsilon_2,$$

then the estimate

$$\|v(\cdot,t)\|_{H^2}^2+\int_0^t\|v_x(\cdot,\tau)\|_{H^2}^2d\tau\le C_2\|v_0\|_{H^2}^2 \quad (1.2.7)$$

holds for $0\le t\le T$.

Proof of the Claim 2: we multiply both sides of (1.2.4) by w and integrate over $\mathbb{R}^1$ to get

$$\frac{1}{2}\frac{d}{dt}\|w(\cdot,t)\|_{L^2}^2-\int_{\mathbb{R}^1}f'(\phi)ww_xdx-\int_{\mathbb{R}^1}Q(\phi,w)w_xdx=-\int_{\mathbb{R}^1}|w_x|^2dx,$$

that is to say,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^1}|w|^2dx+\int_{\mathbb{R}^1}w_x^2dx=\int_{\mathbb{R}^1}f'(\phi)ww_xdx+\int_{\mathbb{R}^1}Q(\phi,w)w_xdx.$$

Based on Sobolev imbedding theorem and Cauchy inequality, the right hand side of above inequality can be estimated by

$$|\int_{\mathbb{R}^1} f'(\phi)ww_x dx + \int_{\mathbb{R}^1} Q(\phi,w)w_x dx| \leq \frac{1}{4}\|w_x\|_{L^2}^2 + C\|w\|_{L^2}^2 + C\|v\|_{H^2(\mathbb{R}^1)}^2\|w_x\|_{L^2}^2.$$

Then there exists $\epsilon_2' \leq \epsilon_1$ such that if

$$\sup_{[0,T]} \|v(\cdot,t)\|_{H^2(\mathbb{R}^1)}^2 \leq \epsilon_2',$$

we have the estimate

$$\|v(\cdot,t)\|_{H^1(\mathbb{R}^1)}^2 + \int_0^t \|v(\cdot,\tau)\|_{H^1(\mathbb{R}^1)}^2 d\tau \leq C\|v_0\|_{H^1(\mathbb{R}^1)}^2,$$

here we have used the estimate (1.2.6).

Similarly, we can obtain the second order derivative estimate of v by choosing a suitable $\epsilon_2 \leq \epsilon_2'$. Then the proof of the claim 2 is completed.

Step 4: Standard Continuity argument

Claim 3: There exists a constant $\epsilon'' > 0$, as long as

$$\|v_0\|_{H^2(\mathbb{R}^1)} \leq \epsilon'',$$

then

$$\sup_{0\leq t<\infty} \|v(\cdot,t)\|_{H^2(\mathbb{R}^1)}^2 + \int_0^\infty \|v_x(\cdot,\tau)\|_{H^2(\mathbb{R}^1)}^2 d\tau \leq C, \tag{1.2.8}$$

which implies

$$\lim_{t\to\infty} \|w(\cdot,t)\|_{L^\infty} = 0. \tag{1.2.9}$$

Proof of Claim 3: By fixed point theorem, we can show there exists local solution to (1.2.5) in $L^\infty(0,T_1;H^2(\mathbb{R}^1))$, for some time $T_1 < \infty$, if $v_0 \in H^2(\mathbb{R}^1)$; moreover, if $\|v_0\|_{H^2(\mathbb{R})} \leq \epsilon'$, then

$$\sup_{[0,T_1]} \|v(\cdot,t)\|_{H^2(\mathbb{R})} \leq \epsilon_2 \quad \text{and} \quad \sup_{[0,T_1]} \|t^{1/2}v(\cdot,t)\|_{H^3(\mathbb{R})} < \infty.$$

This result and local existence for more general parabolic systems can be found in [47]. Hence all the calculations above make sense. Take $\epsilon'' = \epsilon'/C_2$. If $\|v_0\|_{H^2(R)} \leq \epsilon''$, then (1.2.7) implies that

$$\|v(\cdot,T_1)\|_{H^2(\mathbb{R})} \leq C_2\|v_0\|_{H^2(\mathbb{R})} \leq \epsilon'.$$

By the local existence result, there exists solution on $[T_1, 2T_1]$ satisfying

$$\sup_{[T_1,2T_1]} \|v(\cdot,t)\|_{H^2(\mathbb{R}^1)} \le \epsilon_2.$$

Thus Claim 2 again shows

$$\sup_{[0,2T_1]} \|v(\cdot,t)\|_{H^2(\mathbb{R}^1)} \le \epsilon',$$

and so,

$$\|v(\cdot,2T_1)\|_{H^2(\mathbb{R}^1)} \le \epsilon'.$$

Continuing this procedure, ones shows that as long as

$$\|v_0\|_{H^2(\mathbb{R}^1)} \le \epsilon'',$$

then

$$\sup_{0\le t<\infty} \|v(\cdot,t)\|^2_{H^2(\mathbb{R}^1)} + \int_0^\infty \|v_x(\cdot,\tau)\|^2_{H^2(\mathbb{R}^1)} d\tau \le C.$$

Thus, we finish proving (1.2.8). In the following, C will denote a generic constant, which depends only on C in (1.2.8). Multiplying w on both sides of the equation (1.2.4), then we get

$$\begin{aligned}\int ww_t &= \int w(\frac{\partial^2 w}{\partial x^2} - \frac{\partial}{\partial x}(f'(\phi)w) - \frac{\partial}{\partial x}Q(\phi,w))dx \\ &\le C\|w\|_{H^1} + C\|w\|^2_{L^\infty}\|w\|_{H^1} \\ &\le C\|v\|_{H^2}.\end{aligned}$$

Therefore, we have

$$\int_0^\infty \|w\|^2 dx < C \qquad \text{and} \qquad \frac{d}{dt}\|w(\cdot,t)\|^2_{L^2} \le C,$$

so $\|w(\cdot,t)\|^2_{L^2} \to 0$ as $t \to +\infty$. By Sobolev imbedding theorem, we know

$$\|w\|^2_{L^\infty} = C\|w\|_{L^2}\|w\|_{H^1} \le C\|w\|_{L^2}\|v\|_{H^2},$$

hence $\lim_{t\to\infty} \|w(\cdot,t)\|_{L^\infty} = 0$ as $t \to \infty$.

Step 5: Since $v_0(x) = \int_{\infty}^{x}(u_0(y)-\phi(y))dy$, by weighted Poincare inequality[19, 53], we deduce that

$$\|v_0\|_{L^2} \leq \left(\int_{\mathbb{R}} |x(u_0(x)-\phi(x))|^2 dx\right)^{1/2}.$$

Thus if $\int_{\mathbb{R}^1} |x|^2(u_0(x)-\phi(x))^2dx < \epsilon$ and $\|u_0-\phi\|_{H^1} \leq \epsilon$ for some ϵ small enough, we have

$$\|v_0\|_{H^2(\mathbb{R})} \leq \epsilon'',$$

so we complete the proof of the theorem. □

Remark 1.2.2 *If we assume* $|u_+ - u_-| \ll 1$, $\int_{\mathbb{R}}(1+x^2)(u_0(x)-\phi(x+\delta))^2dx \ll 1$ *and* $u_0(\cdot) - \phi(\cdot+\delta) \in H^1$ *instead of the assumptions in Theorem* 1.2.1, *we can also obtain asymptotic stability for shock profile, see[52]. Under these conditions, if each characteristic field of system of conservation laws is either genuinely nonlinear or linearly degenerate, the stability of shock profile for a Lax shock was obtained by Liu[34] for special initial data, and in general by Szepessy and Xin[53] by energy method, certainly, there are some important ingredients as we mentioned in Introduction to deal with systems.*

Remark 1.2.3 *The energy method can succeed in establishing the asymptotic stability of shock profile is due to the following two reasons. First, special form of equation for conservation law, more precisely, we can integrate the equation* (1.2.4) *once to get a Hamilton-Jacobi equation* (1.2.5). *Applying the basic energy estimate for this Hamilton-Jacobi equation, we can estimate* $v(x,t) = \int_{-\infty}^{x} w(y,t)dy$. *Then standard higher order estimate help to get the estimate for* w. *If we handle equation* (1.2.4) *directly, it is hard to deal with nonlinear term. Second,* $\frac{\partial}{\partial x}f'(\phi) < 0$. *Since two similar properties holds for a Lax shock for system of conservation laws, therefore, we can handle viscous shock wave in system of conservation laws by energy method, see [16, 34, 17, 53].*

1.3 L^1 Stability of Shock Waves in Scalar Viscous Conservation Laws

In this section, we will show the stability of shock wave with general L^1 perturbations. This is motivated by the important physical significance of the L^1 norm for conservation law and the fact that it is the norm for which the semigroup $S(t)$ is non-expansive. We first state the main result in this section.

Theorem 1.3.1 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a shock profile for the inviscid shock (u_-, u_+, s) with $u_- \neq u_+$. If $u_0 - \phi \in L^1(\mathbb{R})$, then the solution $u(t) = S(t)u_0$ to* (1.1.1)-(1.1.2) *with initial data u_0 satisfies*

$$\lim_{t\to\infty} \|u(t,\cdot) - \phi(\cdot - st + \delta)\|_1 = 0 \quad \textit{with} \quad \delta = \frac{\int_{-\infty}^{\infty}(u_0(x) - \phi(x))dx}{u_+ - u_-}, \tag{1.3.1}$$

where $S(t)$ is the semigroup defined in section 1.1.

This theorem is a consequence of long time endeavor of many mathematicians, and L^1-stability as presented in theorem 1.3.1 was first obtained by Freistuhler and Serre [11]. Here we will combine some results appeared in [42, 47, 48, 11, 49] and give a complete proof. The proof depends on several important lemmas.

Lemma 1.3.2 *If there exist α, $\beta \in \mathbb{R}$ such that $\phi(x + \alpha) \leq u_0(x) \leq \phi(x + \beta)$ almost everywhere. Then the solution $u(t) = S(t)u_0$ of the Cauchy problem satisfies* (1.3.1).

Proof: We will mainly follow [47].

Using a moving coordinate $\xi = x - st$, we can assume that the shock is stationary: $s = 0$. At the same time, if we take translation to ϕ, the assumption in lemma 1.3.2 will hold with a translation, thus we assume $\delta = 0$ for simplicity.

With the help of the comparison principle and assumption of the initial data, we have, $\phi(x + \alpha) \leq u(t, x) \leq \phi(x + \beta)$. Let us write $v(t) = u(t) - \phi$, then

$$|v(t)| \leq \phi(x + \beta) - \phi(x + \alpha) \in L^1(\mathbb{R}) \quad \text{for} \quad \forall t \geq 0.$$

In addition, contraction principle (1.1.4) yields

$$\begin{aligned} &\|v(t,\cdot+h)-v(t)\|_1 = \|u(t,\cdot+h)-u(t)\|_1 \\ \leq\ &\|u_0(\cdot+h)-u_0\|_1 = \|v(0,\cdot+h)-v(0)\|_1. \end{aligned}$$

Since $v(0) \in L^1(\mathbb{R})$, therefore

$$\|v(0,\cdot+h)-v(0)\|_1 \to 0 \quad \text{as} \quad h \to 0.$$

Thus the hypotheses in Kolmogrov compactness theorem [20, 60] are all satisfied, so the family $\{v(t)\}_{t\geq 0}$ is relatively compact in $L^1(\mathbb{R})$. the $\omega-$limit set $B = \phi + \bigcap_{s\geq 0} B_s$ where B_s is the closure in $L^1(\mathbb{R})$ of $\{v(t); t \geq s\}$. Furthermore, B is non-empty since $B_s \subset B_t$ as $s > t$ and B_s are all non-empty compact sects for all $s \geq 0$. The set B is that of all cluster points for the distance $d(z,w) = \|z-w\|_1$ of subsequences $\{u(t_n)\}_{n\in\mathbb{N}}$ where $t_n \to \infty$.

The $\omega-$limit set B is invariant under the semigroup $S(t)$ since if $b \in B$ with $b = \lim_{n\to\infty} u(t_n)$, then $S(t)(b) = \lim_{n\to\infty} u(t+t_n)$. For the same reason, $S(t): B \to B$ is onto as we also have $b = S(t)c$ where c is a cluster point of the sequence $\{u(t_n - t)\}_{n\in\mathbb{N}}$. Therefore, $b \in C^\infty$ for $\forall b \in B$ by Theorem 1.1.1.

Now, let $k \in \mathbb{R}$, the decreasing function $t \mapsto \|u(t) - \phi(\cdot - k)\|_1$ admits a limit denoted by $c(k)$ when $t \to \infty$. If $b \in B$, we deduce that $\|b - \phi(\cdot - k)\|_1 = c(k)$. However, $S(t)b \in B$, so it follows that the function $t \mapsto \|S(t)b - \phi(\cdot - k)\|_1$ is constant. Let us write $w(t) = S(t)b$ and $z(t) = w(t) - \phi(\cdot - k)$, then

$$0 = \frac{d}{dt}\|z(t)\|_1 = \int_{\mathbb{R}} z_t \cdot \mathrm{sgn} z dx. \tag{1.3.2}$$

From the equation (1.1.1) and the definition of shock profile we know

$$z_t + (f(w) - f(\phi(\cdot - k)))_x = z_{xx}.$$

Multiplying this equation both sides by sgn$z$, we deduce that

$$|z|_t + ((f(w) - f(\phi(\cdot - k)))\mathrm{sgn} z)_x = z_{xx} \cdot \mathrm{sgn} z. \tag{1.3.3}$$

Integrating over $\mathbb{R}$ gives

$$\frac{d}{dt}\int_{\mathbb{R}}|z|dx = \int_{\mathbb{R}} z_{xx}\cdot \text{sgn} z dx.$$

Thus

$$0 = \int_{\mathbb{R}} z_{xx}\cdot \text{sgn} z dx. \tag{1.3.4}$$

However, since the initial data b is the sum of a BV function ϕ and a L^1 function $b-\phi$, a priori estimate shows that w_{xx} is integrable over $\mathbb{R}$ [47] and hence also is z_{xx}. Therefore, using dominated convergence theorem, we have

$$0 = \lim_{\epsilon\to 0}\int_{\mathbb{R}} z_{xx} j_\epsilon'(z)dx,$$

where $j_\epsilon(\tau) = \sqrt{\epsilon^2+\tau^2}$. Integrating by parts, we have

$$0 = \lim_{\epsilon\to 0}\int_{\mathbb{R}} z_x^2 j_\epsilon''(z)dx. \tag{1.3.5}$$

Let x_0 be a point where z vanishes. Suppose $|z_x(x_0)| = \gamma > 0$, then there exists $\delta > 0$ such that

$$\frac{\gamma}{2} < |z_x(y)| < 2\gamma \qquad \forall y \in (x_0-\delta, x_0+\delta).$$

Choosing $\epsilon > 0$ sufficiently small such that $\frac{\epsilon}{2\gamma} < \delta$, we have $|z| < \epsilon$ on $(x_0 - \frac{\epsilon}{2\gamma}, x_0 + \frac{\epsilon}{2\gamma})$ by mean value theorem. On the other hand, $j_\epsilon''(\tau) = \epsilon^{-1}J(\tau/\epsilon)$ with $J(\tau) = (1+\tau^2)^{-3/2}$. Thus

$$\int_{\mathbb{R}} z_x^2 j_\epsilon''(z)dx \geq \frac{1}{\epsilon}\int_{x_0-\frac{\epsilon}{2\gamma}}^{x_0+\frac{\epsilon}{2\gamma}} J(1)z_x^2 dx \geq \frac{1}{\epsilon}\frac{\epsilon}{\gamma}J(1)(\frac{\gamma}{2})^2 = \frac{J(1)\gamma}{4} > 0.$$

This contradicts with (1.3.5), therefore, $z_x(x_0) = 0$. Finally, we have proved that

$$\forall b \in B, \forall k \in \mathbb{R} \qquad S(t)b(x) = \phi(x-k) \Rightarrow (S(t)b(x))_x = \phi'(x-k).$$

Since $S(t)$ is from B onto B, therefore,

$$\forall b \in B, \forall k \in \mathbb{R} \qquad b(x) = \phi(x-k) \Rightarrow b'(x) = \phi'(x-k). \tag{1.3.6}$$

To complete the proof of lemma, we note first of all that b lies between $\phi(x+\alpha)$ and $\phi(x+\beta)$ as limit of such functions, hence b takes its values strictly between u_- and u_+. Thus the function $x \mapsto k(x) = x - \phi^{-1}(b(x))$ is well defined and smooth. By construction $b(x) = \phi(x - k(x))$, the differentiation gives $b'(x) = \phi'(x - k(x))(1 - k'(x))$. Using (1.3.6) we find that

$$\phi'(x - k(x))k'(x) = 0$$

and hence that $k'(x) = 0$. Finally, k is a constant and $b = \phi(\cdot - k)$. Thank for the property of conservation of mass (1.1.3), we have

$$\int_{\mathbb{R}} (b - u_0)dx = 0.$$

Thus, we have $k = 0$ because of our assumption at the beginning of the proof.

Hence, we have proved that the ω−limit set is reduced to a single element ϕ. Since the family $\{v(t)\}_{t\geq 0}$ is relatively compact in $L^1(\mathbb{R})$ and as it has only a single limiting value when $t \to \infty$, it is convergent, that is

$$\lim_{t\to\infty} \|u(t,\cdot) - \phi(\cdot - st + \delta)\|_1 = 0.$$

□

To prove the theorem, we first extend the initial data in Lemma 1.3.2 to a larger class. Define

$$\begin{aligned}
\mathcal{U}_1 &= \{u_0 | \text{there exist} \quad \alpha, \beta \quad \text{such that} \quad \phi(x+\alpha) < u_0(x) < \phi(x+\beta)\}, \\
\mathcal{U}_2 &= \{u_0 | u_0(x) \in [\inf \phi, \sup \phi], \text{for all } x \in \mathbb{R}, \quad u_0 - \phi \in L^1\}.
\end{aligned}$$

Corollary 1.3.3 (1.3.1) *holds for* $u(t) = S(t)u_0$ *with* $u_0 \in \mathcal{U}_2$.

Proof: It is clear that $\mathcal{U}_1$ is a dense subset of $\mathcal{U}_2$ with the distance $d(z, w) = \|z - w\|_1$. Therefore, $\forall u_0 \in \mathcal{U}_2$, there exists $\{u_n\} \subset \mathcal{U}_2$ such that $\|u_n - u_0\|_1 \to 0$

as $n \to \infty$. $\forall \epsilon > 0$, there exist $N \in \mathbb{N}$ such that $\forall n > N$, $\|u_n - u_0\|_1 < \epsilon/3$, then

$$\begin{aligned} &\|S(t)u_0 - \phi(x - st + \delta)\|_1 \leq \|S(t)u_0 - S(t)u_n\|_1 \\ &+\|S(t)u_n - \phi(x - st + \delta_n)\|_1 + \|\phi(x - st + \delta_n) - \phi(x - st + \delta)\|_1 \\ \leq\ & \|u_n - u_0\|_1 + \|S(t)u_n - \phi(x - st + \delta_n)\|_1 + |\delta_n - \delta| \cdot |u_+ - u_-|, \end{aligned}$$

where $\delta_n = \frac{\int (u_n(x) - \phi(x))}{u_+ - u_-}$, hence $|\delta_n - \delta| \leq \|u_n - u_0\|_1 / |u_+ - u_-|$.

Taking $t \to \infty$ and applying Lemma 1.3.2, we get $\|S(t)u_0 - \phi(x - st + \delta)\|_1 \leq C\epsilon$ for a fixed constant C. Since ϵ is arbitrary, we finish the proof of corollary. □

In fact, we will reduce the proof of Theorem 1.3.1 to the following L^1 stability of constant states.

Lemma 1.3.4 *For every $c \in \mathbb{R}$ and any function $u_0 \in c + L^1(\mathbb{R})$ with*

$$\int_{-\infty}^{\infty} (u_0(x) - c)dx = 0, \tag{1.3.7}$$

the solution $u = S(t)u_0$ to (1.1.1)-(1.1.2) with initial data u_0 satisfies

$$\lim_{t\to\infty} \|u(t) - c\|_1 = 0. \tag{1.3.8}$$

Proof: The original proof of this lemma was due to Freistuhler and Serre [11]. Here we will follow the framework of [49].

Define $l_0(v) = \lim_{t\to\infty} \|S(t)v - c\|_1$, by contraction principle (1.1.4), l_0 is continuous on $c + L^1$. Up to the choice of a moving frame, we may always assume that $f(c) = f'(c) = 0$, after a translation, we can also assume $c = 0$. Denote $L_0^1 = \{v \in L^1(\mathbb{R}) | \int_{-\infty}^{\infty} v(x)dx = 0\}$, then the set $\mathcal{U}_3 = \{v' | v \in W^{1,1}(\mathbb{R})\}$ is dense in L_0^1. Due to continuity of l_0 on L^1, we only need to prove the lemma for $u_0 \in \mathcal{U}_3 \bigcap L^\infty$. Given $u_0 \in \mathcal{U}_3 \cap L^\infty$, $u_0 = w'$ and $w \in W^{1,1}(\mathbb{R})$, then $\|u\|_\infty = \|u_0\|_\infty$ and

$$|f(u)| \leq N_f(\|u_0\|_\infty)|u|^2, \tag{1.3.9}$$

where $N_f(\|u_0\|_\infty) = \frac{1}{2} \sup_{[-\|u_0\|_\infty, \|u_0\|_\infty]} |f''(u)|$.

The standard energy estimate for equation (1.1.1), and the fact that $uf'(u) = g'(u)$ yield

$$\frac{d}{dt}\|u\|_2^2 + 2\|u_x\|_2^2 = 0.$$

Using one dimensional Nash inequality[40]

$$\|u\|_2^3 \leq C\|u_x\|_2\|u\|_1^2, \tag{1.3.10}$$

and decreasing property of $t \to \|u(t)\|_1$, we obtain

$$\|u_0\|_1^4 \frac{d}{dt}\|u\|_2^2 + C\|u\|_2^6 \leq 0.$$

This differential inequality obviously implies the following dispersion relation[1]

$$\|u\|_2 \leq C\|u_0\|_1 t^{-1/4}. \tag{1.3.11}$$

Write u as mild solution

$$u(t) = K(t) * u_0 - \int_0^t (\partial_x K(t-s) * f(u(s)))ds,$$

then by (1.3.9) and (1.3.11),

$$\begin{aligned} \|u(t)\|_1 &\leq \|K(t) * u_0\|_1 + \int_0^t \|\partial_x K(t-s)\|_1 \|f(u(s))\|_1 ds \\ &\leq \|K(t) * u_0\|_1 + CN_f(\|u_0\|_\infty)\|u_0\|_1^2 \int_0^t \frac{ds}{\sqrt{s(t-s)}}. \end{aligned}$$

Since

$$\begin{aligned} \|K(t) * u_0\|_1 &= \|\partial_x K(t) * w\|_1 \leq \|\partial_x K(t)\|_1 \|w\|_1 \\ &\leq \frac{C}{\sqrt{t}}\|w\|_1 \end{aligned}$$

and

$$\int_0^t \frac{ds}{\sqrt{s(t-s)}} = \pi,$$

therefore

$$l_0(u_0) \leq c_3 N_f(\|u_0\|_\infty)\|u_0\|_1^2. \tag{1.3.12}$$

On the other hand, $l_0(u_0) = l_0(u(t))$, we may apply (1.3.12) to $u(t)$, instead to get

$$l_0(u_0) \leq c_3 N_f(\|u_0\|_\infty)\|u(t)\|_1^2.$$

Taking $t \to \infty$, we get

$$l_0(u_0) \leq c_3 N_f(\|u_0\|_\infty)(l_0(u_0))^2. \tag{1.3.13}$$

Fix a real number $R > 0$ and consider the ball B_R defined by $\|u_0\|_\infty < R$ in $L_0^1 \cap L^\infty$. In the connected set B_R, (1.3.13) tells either $l_0(u_0) = 0$ or $l_0 > 1/(c_3 N_f(R))$. Since l_0 is continuous and take the value zero for $u_0 = 0$, this implies $l_0 \equiv 0$ on B_R, hence on the union $L_0^1 \cap L^\infty$ of these balls. This ends the proof of the lemma. □

After these plenty of preparations, we can prove the theorem easily.

Proof of theorem 1.3.1: We will follow [11].

Denote $\sup \phi$ and $\inf \phi$ by c_+ and c_-, respectively. Set

$$m_+ = \int_{-\infty}^{\infty} (u_0(x) - c_+)_+ dx \qquad \text{and} \qquad m_- = \int_{-\infty}^{\infty} (c_- - u_0(x))_+ dx.$$

These are well-defined since $0 \leq m_\pm \leq \|u_0 - \phi\|_1$. As

$$\phi(\pm\infty) = u_\pm \qquad \text{and} \qquad u_0 - \phi \in L^1(\mathbb{R}),$$

therefore the Lebesgue measures of two sets $Z_+ = \{x | u_0(x) \leq \frac{c_- + c_+}{2}\}$ and $Z_- = \{x | u_0(x) \geq \frac{c_- + c_+}{2}\}$ are infinite. Thus there exist sets $M_+ \subset Z_+$ and $M_- \subset Z_-$ of Lebesgue measure $2m_+/(c_+ - c_-)$ and $2m_-/(c_+ - c_-)$. Set

$$a_1(x) = \begin{cases} \max\{u_0(x), c_+\} & x \in \mathbb{R} \setminus M_+, \\ (c_- + c_+)/2 & x \in M_+, \end{cases}$$

$$a_2(x) = \begin{cases} \min\{u_0(x), c_-\} & x \in \mathbb{R} \setminus M_-, \\ (c_- + c_+)/2 & x \in M_-, \end{cases}$$

then

$$a_2(x) \le u_0(x) \le a_1(x) \qquad \forall x \in \mathbb{R}.$$

Clearly

$$a_1 - c_+ \in L^1(\mathbb{R}) \qquad \text{and} \qquad \int_{-\infty}^{\infty} (a_1(x) - c_+)dx = 0.$$

Set $u_1(t) = S(t)a_1$, $u_2(t) = S(t)a_2$, then Lemma 1.3.4 implies

$$\lim_{t\to\infty} \|(u_1(t,\cdot) - c_+)_+\|_1 = 0.$$

Similarly, we can prove that

$$\lim_{t\to\infty} \|(u_2(t,\cdot) - c_-)_-\|_1 = 0. \tag{1.3.14}$$

By comparison principle, the solution $u(t) = S(t)u_0$ satisfies

$$u_2(t,x) \le u(t,x) \le u_1(t,x) \qquad \forall x \in \mathbb{R}.$$

Fix an arbitrary $\epsilon > 0$, then there exists a $t_\epsilon > 0$ such that

$$\|(u(t_\epsilon,\cdot) - c_+)_+\|_1 < \epsilon/2, \qquad \|(u(t_\epsilon,\cdot) - c_-)_-\|_1 < \epsilon/2.$$

Let $\bar{u}_\epsilon(t) = S(t - t_\epsilon)\bar{u}_0$ for $t \ge t_\epsilon$ with

$$\bar{u}_0 = \begin{cases} c_- & \text{if} \quad u(t_\epsilon,x) < c_-, \\ u(t_\epsilon,x) & \text{if} \quad c_- \le u(t_\epsilon,x) \le c_+, \\ c_+ & \text{if} \quad c_+ < u(t_\epsilon,x). \end{cases} \tag{1.3.15}$$

While

$$\|\bar{u}_0 - u(t_\epsilon,\cdot)\|_1 \le \epsilon$$

and

$$\|\bar{u}_0 - \phi(\cdot - st_\epsilon)\|_1 \le \|\bar{u}_0 - u(t_\epsilon,\cdot)\| + \|u(t_\epsilon,\cdot) - \phi(\cdot - st_\epsilon)\|_1 < \infty,$$

so Corollary 1.3.3 implies that

$$\lim_{t\to\infty} \|\bar{u}_\epsilon(t) - \phi(\cdot - st_\epsilon - s(t - t_\epsilon) + \delta_\epsilon)\|_1 = 0$$

with appropriate δ_ϵ. By contraction property

$$\limsup_{t\to\infty} \|u(t,\cdot) - \phi(\cdot - st + \delta_\epsilon)\|_1 \le \epsilon.$$

As $\int_{-\infty}^{\infty}(\phi(x+\delta_1) - \phi(x+\delta_2))dx = (\delta_1 - \delta_2)(u_+ - u_-)$, therefore

$$\begin{aligned}|\delta_{\epsilon_1} - \delta_{\epsilon_2}| \cdot |u_+ - u_-| \le \quad & \limsup_{t\to\infty} \|u(t,\cdot) - \phi(\cdot - st + \delta_{\epsilon_1})\|_1 \\ & + \limsup_{t\to\infty} \|u(t,\cdot) - \phi(\cdot - st + \delta_{\epsilon_2})\|_1 \\ \le \; & \epsilon_1 + \epsilon_2.\end{aligned}$$

Thus δ_ϵ converges, as $\epsilon \downarrow 0$, to some limit δ and $\lim_{t\to\infty} \|u(t,\cdot) - \phi(\cdot - st + \delta)\|_1 = 0$. Therefore $\delta = \frac{\int_{-\infty}^{\infty}(u_0(x) - \phi(x))dx}{u_+ - u_-}$. □

Chapter 2

Slow Motion of a Viscous Shock

After studying the stability of viscous shock wave for initial value problem, we consider propagation of viscous stationary shock waves in bounded domain and half space. For bounded domain case, we will use two asymptotic analysis methods and projection method to study the location of shock layer and study the effect of boundary conditions for the propagation of shock layer. In the case of propagation of stationary shock wave in half space, we first study the problem by asymptotic analysis, then verify this asymptotic analysis results by careful pointwise estimate.

2.1 Propagation of a Viscous Shock in Bounded Domain

In this section we will study the internal layer behavior associated with the following viscous shock problem in the limit $\epsilon \to 0$

$$u_t + (f(u))_x = \epsilon u_{xx}, \quad 0 < x < L, \quad t > 0, \qquad u \in \mathbb{R}, \tag{2.1.1}$$

$$u(x,0) = u_0(x), \qquad u(0,t) = \alpha_-, \quad u(L,t) = \alpha_+, \tag{2.1.2}$$

where $\alpha_- > 0$, $\alpha_+ < 0$, and the smooth nonlinearity $f(u)$ has the following properties:

$$f(0) = f'(0) = 0, \quad f(\alpha_-) = f(\alpha_+) = f(\alpha), \quad f''(u) > 0. \tag{2.1.3}$$

Two important examples for the flux function are: $f(u) = \frac{u^2}{2}$, this is well-known Burgers equation; $f(u) = u - 1 + \frac{1}{u+1}$ which arises the study of one dimensional transonic gas in a straight channel[21].

To get some insights of the problem, we first focus on the steady problem.

2.1.1 Steady Problem

For problem (2.1.1)-(2.1.2), the corresponding steady problem is

$$(f(u))_x = \epsilon u_{xx}, \qquad 0 < x < L, \tag{2.1.4}$$

$$u(0) = \alpha_-, \quad u(L) = \alpha_+, \tag{2.1.5}$$

where $\alpha_- > 0$, $\alpha_+ < 0$, and (2.1.3) hold, the problem (2.1.4)-(2.1.5) has a unique solution with a shock type internal layer.

For $\epsilon \to 0$, the leading order matched asymptotic expansion solution for (2.1.4)-(2.1.5) is given by $u \sim \phi((x - x_0)/\epsilon)$, [10], where $\phi(z)$ is the shock profile satisfying

$$\phi'(z) = f(\phi(z)) - f(\alpha), \quad -\infty < z < \infty, \qquad \phi(0) = 0, \tag{2.1.6}$$

$$\phi(z) \sim \alpha_- - a_- e^{\nu_- z}, \quad z \to -\infty, \tag{2.1.7}$$

$$\phi(z) \sim \alpha_+ + a_+ e^{-\nu_+ z}, \quad z \to \infty, \tag{2.1.8}$$

where

$$\nu_\pm = \mp f'(\alpha\pm), \tag{2.1.9}$$

$$\log(\mp\frac{a_\pm}{\alpha_\pm}) = \pm\nu_\pm \int_0^{\alpha_\pm} (\frac{1}{f(\eta) - f(\alpha_\pm)} \pm \frac{1}{\nu_\pm(\eta - \alpha_\pm)}) d\eta. \tag{2.1.10}$$

Since $f(u)$ is convex, direct computation shows $a\pm > \mp\alpha_\pm$. Notice that for any $x_0 \in (0,1)$, with $O(\epsilon) \ll x_0 \ll 1 - O(\epsilon)$, the matched asymptotic expansion solution satisfies the equation exactly and it satisfies (2.1.5) to with exponentially small terms. Therefore the location x_0 of the shock layer can not be determined only by matched asymptotic expansions.

The deviation $\tilde{w} = u - \phi(z)$ between the steady state and internal layer should satisfy a nonlinear differential equation

$$f(\tilde{w} + \phi(z))_x - f(\phi(z))_x = \epsilon \tilde{w}_{xx}.$$

Since we expect that the deviation is small enough, therefore, at least we need that the solution to the corresponding linearized problem is small enough. Thus we will consider the following linearized problem.

$$\epsilon w_{xx} - (f'(\phi)w)_x = 0, \qquad 0 < x < L, \tag{2.1.11}$$

$$w(0) = \alpha_- - \phi(-x_0/\epsilon) \sim a_- e^{-\nu_- x_0/\epsilon}, \tag{2.1.12}$$

$$w(L) = \alpha_+ - \phi((L - x_0)/\epsilon) \sim -a_+ e^{-\nu_+(L-x_0)/\epsilon}. \tag{2.1.13}$$

To solve the problem (2.1.11)-(2.1.13), we first transform the differential equation (2.1.11) into a self-adjoint form. Introducing a new variable

$$\hat{w} = w(x)\exp(-g(z)), \qquad g(z) = \frac{1}{2}\log(\frac{\phi'(z)}{\phi'(0)}), \quad \text{where} \quad z = \frac{x - x_0}{\epsilon}. \tag{2.1.14}$$

After a simple calculation, we find that $\hat{w}$ satisfies

$$\epsilon^2 \hat{w}_{xx} - V(\frac{x - x_0}{\epsilon})\hat{w} = 0, \tag{2.1.15}$$

$$\hat{w}(0) \sim (f(\alpha)a_-)^{1/2}\nu_-^{-1/2} e^{-\nu_- x_0/(2\epsilon)}, \tag{2.1.16}$$

$$\hat{w}(L) \sim -(f(\alpha)a_+)^{1/2}\nu_+^{-1/2} e^{-\nu_+(L-x_0)/(2\epsilon)}, \tag{2.1.17}$$

where the potential $V(z)$ is defined by

$$V(z) = \frac{1}{4}(f'(\phi(z)))^2 + \frac{1}{2}f''(\phi(z))\phi'(z). \tag{2.1.18}$$

Define

$$\mathcal{L}_\epsilon \psi = \epsilon^2 \psi_{xx} - V(\frac{x - x_0}{\epsilon})\psi. \tag{2.1.19}$$

Therefore we can represent $\hat{w}$ as linear combination of eigenfunctions of self-adjoint linear operator $\mathcal{L}_\epsilon$ and a correction term which is induced by inhomogeneous boundary conditions (2.1.16)-(2.1.17). Hence we consider associated eigenvalue problem

$$\begin{cases} \mathcal{L}_\epsilon \psi = \lambda \psi, & 0 < x < L, \\ \psi(0) = 0, \quad \psi(L) = 0, & (\psi, \psi) = 1. \end{cases} \tag{2.1.20}$$

It is obvious that λ is real. Suppose λ is an eigenvalue, then

$$\begin{aligned} \lambda &= -\epsilon^2 \int \psi_x^2 dx - \int V(z)\psi^2 dx \\ &= -\epsilon^2 \int \psi_x^2 dx - \int \frac{1}{4}(f'(\phi(z)))^2 \psi^2 dx - \frac{1}{2}\int f''(\phi(z))\phi'(z)\psi^2 dx. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{2}\left|\int f''(\phi'(z))\phi(z)\psi^2 dx\right| &= \frac{1}{2}\left|\int \epsilon(f'(\phi(z)))_x \psi^2 dx\right| \\ &= \left|\int \epsilon f'(\phi(z))\psi\psi_x dx\right| \\ &\leq \int \epsilon^2 \psi_x^2 dx + \frac{1}{4}\int (f'(\phi(z)))^2 \psi^2 dx, \end{aligned}$$

thus $\lambda \leq 0$.

Suppose $\{\lambda_j\}_{j\geq 0}$ and $\{\psi_j\}_{j\geq 0}$ are eigenvalues and corresponding eigenfunctions to $\mathcal{L}_\epsilon$. We now give an asymptotic estimate for the principal eigenvalue λ_0 and for the corresponding eigenfunction ψ_0. Define $\tilde{\psi}_0(x) = \phi'(z) \cdot \exp(-g(z))$, then $\mathcal{L}_\epsilon \tilde{\psi}_0 = 0$. Then Green's identity shows

$$\lambda_0(\tilde{\psi}_0, \psi_0) = \epsilon^2(\tilde{\psi}_0(L)\psi_{0x}(L) - \tilde{\psi}_0(0)\psi_{0x}(0)). \tag{2.1.21}$$

To estimate λ_0 from (2.1.21) we construct $\psi_0(x)$ asymptotically and then calculate $\psi_{0x}(0)$ and $\psi_{0x}(L)$. Since $\tilde{\psi}_0$ satisfies $\mathcal{L}_\epsilon \tilde{\psi}_0 = 0$, is exponentially small at $x = 0$ and $x = L$, and is of one sign, then $\psi_0 \sim N_0 \tilde{\psi}_0$, except possibly near the endpoints.

Here N_0 is a normalization constant, thus we must add a boundary layer term to $N_0\tilde{\psi}_0(x)$ near each endpoint to approximate ψ_0.

We first consider the region near $x = 0$. Since $V(z) \sim \nu_-^2/4$ for $z \to -\infty$, then for $x \approx 0$, we have

$$\psi_0(x) \sim N_0(\tilde{\psi}_0(x) + b_l e^{-\nu_- x/(2\epsilon)}). \tag{2.1.22}$$

Using $\tilde{\psi}_0(x) \sim -(a_-\nu_- f(\alpha))^{1/2} e^{\nu_-(x-x_0)/(2\epsilon)}$ for $x \approx 0$ and enforcing $\psi_0(0) = 0$, we find

$$b_l = (a_-\nu_- f(\alpha))^{1/2} e^{-\nu_- x_0/(2\epsilon)}.$$

Therefore

$$\psi_{0x}(0) \sim -\epsilon^{-1}\nu_- N_0(a_-\nu_- f(\alpha))^{1/2} e^{-\nu_- x_0/(2\epsilon)}. \tag{2.1.23}$$

A similar calculation for the region near $x = L$ gives

$$\psi_{0x}(L) \sim \epsilon^{-1}\nu_+ N_0(a_+\nu_+ f(\alpha))^{1/2} e^{-\nu_+(L-x_0)/(2\epsilon)}. \tag{2.1.24}$$

Now to evaluate the left hand side of (2.1.21), we use the estimate $(\tilde{\psi}_0, \psi_0) \sim N_0(\tilde{\psi}_0, \tilde{\psi}_0)$ where

$$(\tilde{\psi}_0, \tilde{\psi}_0) \sim \epsilon \int_{-\infty}^{\infty} (\phi'(z))^2 \exp(-2g(z))dz = 2\epsilon(\alpha_- - \alpha_+) f(\alpha). \tag{2.1.25}$$

Hence, we obtain the following estimate for $\lambda_0 = \lambda_0(x_0)$:

$$\lambda_0(x_0) \sim -\frac{1}{\alpha_- - \alpha_+}(a_+\nu_+^2 e^{-\nu_+(L-x_0)/\epsilon} + a_-\nu_-^2 e^{-\nu_- x_0/\epsilon}). \tag{2.1.26}$$

Motivated by analysis for Burgers equation in [26], we assume that $\{\lambda_j\}_{j\geq 1}$ are away from 0. Now we project $\hat{w}$ to subspace which is spanned by ψ_j, then

$$0 = (\mathcal{L}_\epsilon \hat{w}, \psi_j) = B_j + \lambda_j(\hat{w}, \psi_j), \tag{2.1.27}$$

where

$$B_j = \epsilon^2(\hat{w}(L)\psi_{jx}(L) - \hat{w}(0)\psi_{jx}(0)). \tag{2.1.28}$$

Then by previous asymptotic analysis $B_0 = O(1)\epsilon(e^{-\nu_+(L-x_0)/\epsilon} + e^{-\nu_- x_0/\epsilon})$. On the other hand, to satisfy well-posedness for the linearized problem, there will be $\|\hat{w}\|_{L^\infty} \leq O(1)\max\{|\hat{w}(0)|, |\hat{w}(L)|\}$. Using asymptotic analysis for λ_0 in (2.1.26), to balance two terms in right hand side of (2.1.27) for $j = 0$, B_0 must be 0. Thus

$$a_-\nu_- e^{-\nu_- x_0/\epsilon} = a_+\nu_+ e^{-\nu_+(L-x_0)/\epsilon}. \tag{2.1.29}$$

The solution to (2.1.29) is $x_0 = x_e$ where

$$x_e = \frac{\nu_+ L}{\nu_- + \nu_+} - \frac{\epsilon}{\nu_- + \nu_+}\log(\frac{a_+\nu_+}{a_-\nu_+}). \tag{2.1.30}$$

Summarizing, we have

Proposition 2.1.1 *The shock layer solution for* (2.1.4), (2.1.5) *is given asymptotically by* $u \sim \phi(\frac{x-x_e}{\epsilon})$, *where* x_e *is defined by* (2.1.30).

This proposition and several propositions below were obtained in [43] for a special case $\alpha_- = -\alpha_+$, $L = 1$, and some results about more general case where α_- may not equal to $-\alpha_+$ had appeared in [44], [55], [54] diversely either without derivation or by different treatment. Here we present analysis and results all for more general case where α_- may not equal to $-\alpha_+$ and L is arbitrary by a uniform treatment.

2.1.2 Time-Dependent Problem

For time dependent problem

$$u_t + (f(u))_x = \epsilon u_{xx}, \qquad 0 < x < L, \quad t > 0, \tag{2.1.31}$$

$$u(0,t) = \alpha_-, \qquad u(L,t) = \alpha_+, \tag{2.1.32}$$

we will track the propagation of the shock wave by the method developed in the previous subsection.

Starting from initial data a shock layer is formed on an $O(1)$ time scale. To describe the subsequent slow motion of the shock layer we look for a solution to (2.1.31)-(2.1.32) of the form $u(x,t) \sim \phi((x - x_0(t))/\epsilon)$, where $\phi(z)$ is the shock profile defined in (2.1.6)-(2.1.8) and $x = x_0(t)$ is the unknown location of the shock layer. Since $\phi(0) = 0$, then $x_0(t)$ is an approximation to zero of $u(x,t)$ during the slow evolutionary period. In a strict sense, labelled by x_0^0, corresponding to the location of the zero of u for the shock layer initial data of the form $u(x,0) \sim \phi(x - x_0^0)$. For more general initial data, however, we will interpret x_0^0 as the location of the shock layer at the onset of the slow evolution. Although a precise definition of x_0^0 is not needed for our purposes, one possible definition is that x_0^0 is the location of the zero of u at the time when the inviscid problem($\epsilon = 0$) first forms a shock. Since the slow evolution occurs on an exponentially long time scale, we only incur an $O(1)$ error in the total elapsed time by assuming that the slow motion begins at $t = 0$, that is to say, $x_0(0) = x_0^0$.

For $t \gg O(1)$, we look for a solution to (2.1.31)-(2.1.32) of the form $u(x,t) \sim \phi(z) + w(x,t)$, where $z = (x - x_0(t))/\epsilon$, $w \ll \phi$. Linearize the problem at $\phi(z)$, then

$$\epsilon w_{xx} - (f'(\phi(z))w)_x = -\epsilon^{-1}\dot{x}_0\phi'(z) + w_t, \tag{2.1.33}$$

$$w(0,t) \sim a_-\nu_- e^{-\nu_- x_0/\epsilon}, \tag{2.1.34}$$

$$w(L,t) \sim a_+\nu_+ e^{-\nu_-(L-x_0)/\epsilon}. \tag{2.1.35}$$

As same as before, to get an adjoint linear operator, we use the transformation

$$\hat{w}(x,t) = \exp(-g(z))w(x,t),$$

then we can convert the boundary value problem (2.1.33)-(2.1.35) to

$$\epsilon^2\hat{w}_{xx} - V(z)\hat{w} = -\dot{x}_0\phi'(z)e^{-g(z)} + \epsilon\hat{w}_t - \frac{\dot{x}_0}{2}f'(\phi(z))\hat{w}, \tag{2.1.36}$$

$$\hat{w}(0,t) \sim (a_-f(\alpha))^{1/2}\nu_-^{-1/2}e^{-\nu_- x_0/(2\epsilon)}, \tag{2.1.37}$$

$$\hat{w}(L,t) \sim (a_+f(\alpha))^{1/2}\nu_+^{-1/2}e^{-\nu_+(L-x_0)/(2\epsilon)}. \tag{2.1.38}$$

Suppose $\{\psi_j(x)\}_{j\geq 0}$ are eigenfunctions for the eigenvalue problem (2.1.20), then

$$-\dot{x}_0(\phi' e^{-g}, \psi_j) + \epsilon(\hat{w}_t, \psi_j) - \frac{\dot{x}_0}{2}(f'(\phi)\hat{w}, \psi_j) - \lambda_j(\hat{w}, \psi_j) = -B_j(t), \tag{2.1.39}$$

where $B_j(t) = \epsilon^2(\hat{w}(L,t)\psi_{jx}(L) - \hat{w}(0,t)\psi_{jx}(0))$.

Since $\psi_0(x) \sim N_0\phi'(z)\exp(-g(z))$ is exponentially small outside a narrow region of width $O(\epsilon)$ centered at $x = x_0$, thus the dominant contribution to the inner product integrals in (2.1.39) for $j = 0$ arise from the region near $x = x_0$. In this region, we assume $\hat{w}_t \ll e^{-g}\phi'$, thus we neglect the second term on the left side of (2.1.39). Moreover, since $\hat{w} \ll 1$ and $f'(\phi) \approx 0$ when x is in a small neighborhood of x_0, the third term on the left side of (2.1.39) is asymptotically smaller than the first term. Noting that $\lambda_0 \ll 0$, then letting $\epsilon \to 0$ in (2.1.39), we obtain the following approximate equation of motion for x_0:

$$\dot{x}_0(\phi' e^{-g}, \psi_0) = B_0(t). \tag{2.1.40}$$

Proposition 2.1.2 *For $\epsilon \to 0$, the exponentially slow evolution of the shock layer for* (2.1.31)-(2.1.32) *is described by $u \sim \phi(\frac{x-x_0(t)}{\epsilon})$, where $x_0(t)$ satisfies the ordinary differential equation*

$$\dot{x}_0 = \frac{1}{\alpha_- - \alpha_+}(a_-\nu_- e^{-\nu_- x_0/\epsilon} - a_+\nu_+ e^{-\nu_+(L-x_0)/\epsilon}), \tag{2.1.41}$$

here $\phi(z)$ is defined by (2.1.6) − (2.1.8)*and $\nu_\pm$, $a_\pm$ are defined in* (2.1.9), (2.1.10). *The initial position of the shock layer $x_0^0 = x_0(0)$ is determined by the transient process describing the formation of the shock layer from the initial data.*

Remark 2.1.3 *If $x_0^0 > x_e$, then $\dot{x}_0 < 0$, therefore the shock layer will move to x_e at last; Conversely, if $x_0^0 < x_e$, then $\dot{x}_0 > 0$, therefore the shock layer will move to x_e after exponentially long time. So we can see that the location x_e of shock for the steady problem* (2.1.4)-(2.1.5) *is stable.*

2.1.3 Super-Sensitivity of Boundary Conditions

Now we look at the problem we solved again. The problem can be written in an abstract form as:

$$lx = y,$$

where l is an abstract operator, in our case, it relates to the differential operator; y represents the effect of boundary conditions; x is what we want to solve, the location of shock layer. The differential operator has small eigenvalues, that is to say, the norm of the operator l is very small, on the other hand, we know that boundary condition is also quite small, therefore, y can be viewed as a small quantity. So, as a matter of fact, we are solving an ill-conditioned problem. To verify this ill-condition, we perturb y a little bit and solve some problems with a little bit different boundary conditions .

First, we study the steady problem (2.1.4) with boundary conditions

$$u(0) = \alpha_- - A_l e^{-c_l/\epsilon}, u(L) = \alpha_+ + A_r e^{-c_r/\epsilon}, \tag{2.1.42}$$

here A_l, A_r, c_l, $c_r > 0$. As same as in section 2.1.1, studying the new boundary layer terms and then we have

Proposition 2.1.4 *The shock layer solution for* (2.1.4), (2.1.42) *is given asymptotically by* $u \sim \phi(\frac{x-x_e}{\epsilon})$, *where* x_e *is solution of following equation:*

$$a_-\nu_- e^{-\nu_- x_e/\epsilon} - a_+\nu_+ e^{-\nu_+(L-x_e)/\epsilon} = A_l\nu_- e^{-c_l/\epsilon} - A_r\nu_+ e^{-c_r/\epsilon}.$$

When f *is even, then* $\nu_- = \nu_+ = \nu$, $\alpha_- = -\alpha_+ = \alpha$, $a_- = a_+ = a$, *and* x_e *can be explicitly represented by*

$$x_e = \frac{L}{2} + \frac{\epsilon}{\nu}\log(\gamma + (\gamma^2 + 1)^{1/2}), \tag{2.1.43}$$

where

$$\gamma = \frac{A_r e^{(\frac{\nu L}{2} - c_r)/\epsilon} - A_l e^{(\frac{\nu L}{2} - c_l)/\epsilon}}{2a}.$$

If we choose $c_l = c_r = \nu L/2$ and $A_l \neq A_r$, this example shows that the exponentially small changes in the boundary conditions induce on $O(\epsilon)$ changes in the location of the shock layer.

The second example is the steady problem (2.1.4) with boundary conditions

$$\epsilon u_x(0) - k_l(u(0) - \alpha_-) = 0, \qquad \epsilon u_x(L) + k_r(u(L) - \alpha_+) = 0, \tag{2.1.44}$$

here k_l, $k_r > 0$. As same as in section 2.1.1, we study certain eigenvalue problem, give an asymptotic estimate for the principal eigenvalue, and apply solvability condition for associated linearized problem, then we have

Proposition 2.1.5 *When* $(\nu_+ - k_r)(\nu_- - k_l) > 0$, *the shock layer solution for* (2.1.4), (2.1.44) *is given asymptotically by* $u \sim \phi(\frac{x-x_e}{\epsilon})$, *where* x_e *is defined by*

$$x_e = \frac{\nu_+ L}{\nu_+ + \nu_-} - \frac{\epsilon}{\nu_+ + \nu_-} \log\left(\frac{a_+\nu_+ k_l}{a_-\nu_- k_r}(\frac{\nu_+ - k_r}{\nu_- - k_l})\right). \tag{2.1.45}$$

Alternatively, when $(\nu_+ - k_r)(\nu_- - k_l) < 0$, *there is no shock layer solution for* (2.1.4), (2.1.44).

We find that if we perturb the boundary conditions (2.1.5) a little bit to (2.1.44), the shock layer may disappear. This again implies that the problem is very sensitive to its boundary conditions.

While, for time dependent problem (2.1.31) with boundary conditions similar to (2.1.44)

$$\epsilon u_x(0,t) - k_l(u(0,t) - \alpha_-) = 0, \quad \epsilon u_x(L,t) + k_r(u(L,t) - \alpha_+) = 0, \tag{2.1.46}$$

parallel to section 2.1.2, we have

Proposition 2.1.6 *For* $\epsilon \to 0$, *the exponentially slow evolution of the shock layer for* (2.1.33), (2.1.44) *is described by* $u \sim \phi(\frac{x-x_0(t)}{\epsilon})$, *where* $x_0(t)$ *satisfies the ordinary differential equation*

$$\dot{x}_0 = \frac{1}{\alpha_- - \alpha_+}\left(a_+\nu_+(\frac{\nu_+}{k_r} - 1)e^{-\nu_+(L-x_0)/\epsilon} - a_-\nu_-(\frac{\nu_-}{k_l} - 1)e^{-\nu_- x_0/\epsilon}\right), \tag{2.1.47}$$

here $\phi(z)$ *is defined by* (2.1.6) *and* $\nu_\pm$, $a_\pm$*are defined in* (2.1.9), (2.1.10). *The initial position of the shock layer* $x_0^0 = x_0(0)$ *is determined by the transient process describing the formation of the shock layer from the initial data.*

Remark 2.1.7 *If* $\nu_- < k_l$ *and* $\nu_+ < k_r$, *for any* $x_0^0 \in (0, L)$, *the solution* $x_0(t)$ *tends to* x_e *in the equilibrium location. When* $\nu_- > k_l$ *and* $\nu_+ > k_r$, *the equilibrium location is unstable; more precisely, when* $x_0^0 > x_e (x_0^0 < x_e)$, *the shock will* *eventually hit the boundary $x = L(x = 0)$. Finally, if $(\nu_+ - k_r)(\nu_- - k_l) < 0$, the shock layer will hit the boundary at $x = L(x = 0)$ when $\nu_+ > k_r(\nu_+ < k_r)$.*

Remark 2.1.8 *If we take k_l, $k_r \to \infty$ in the boundary condition (2.1.44),(2.1.46), formally, we get the boundary condition(2.1.5), (2.1.32) respectively. Meanwhile, when k_l, $k_r \to \infty$, the location of shock layer x_e in (2.1.45) will tend to (2.1.30), similarly, the propagation of shock layer $x_0(t)$ defined by (2.1.47) will go to (2.1.41). Thus we can regard (2.1.4)- (2.1.5) as a special case of (2.1.4)- (2.1.44) and (2.1.31)- (2.1.32) as a special case of (2.1.31)- (2.1.46) for $k_l = k_r = \infty$.*

2.2 Propagation of a Stationary Shock in Half Space

In chapter 1, we know that if there is no excess mass, for Cauchy problem, the location of shock can be regarded as static. While in Section 2.1, we find that the location of shock will move slowly due to the effect of boundary layer. Since there are two boundary layers, in certain sense, as a result of balance of two boundary layers, the shock will not move a lot. In this section, we will see that when there is only one boundary, the shock still moves slowly, but they will move away from the boundary farther and farther.

2.2.1 Asymptotic Analysis

First of all, we will use projection method developed in section 2.1 to give the propagation of shocks in half space.

Consider the problem

$$u_t + f(u)_x = \epsilon u_{xx}, \quad 0 < x < \infty, \quad t > 0, \quad u(x,0) = u_0(x), \tag{2.2.1}$$

$$u(0,t) = \alpha_-, \quad u(x,t) \to \alpha_+ \quad \text{as} \quad x \to \infty. \tag{2.2.2}$$

Starting from $u_0(x)$, we assume that a shock layer is formed in an $O(1)$ time interval with the shock layer location an $O(1)$ distance away form $x = 0$.

If we take $k_l \to \infty$, $L \to \infty$ in the boundary conditions (2.1.46), then we get the boundary conditions (2.2.2); at the same time, the location of shock layer (2.1.47) will become the propagation of shock layer for the problem (2.2.1)-(2.2.2). Thus we have

Proposition 2.2.1 *[55] For $t \gg O(1)$ and $\epsilon \to 0$, the slow shock layer motion for (2.2.1), (2.2.2) is given by $u \sim \phi(\frac{x-x_0(t)}{\epsilon})$, where $x_0(t)$ satisfies*

$$x_0(t) \sim x_0^0 + \frac{\epsilon}{\nu_-}\log(1+\frac{t}{t_s}), \quad t_s \equiv \frac{\epsilon(\alpha_- - \alpha_+)}{a_-\nu_-^2}e^{\nu_- x_0^0/\epsilon}, \tag{2.2.3}$$

here ν_- and a_- are defined in (2.1.9) *and* (2.1.10).

2.2.2 Pointwise Estimate

In section 2.2.1, we only give the propagation of shock waves as (2.2.3) by asymptotic analysis, but it is not rigorous mathematical proof. In this subsection, we will justify the above asymptotic result by careful pointwise estimate.

More precisely, after a scaling, we consider the following initial boundary value problem

$$\begin{cases} u_t + uu_x = u_{xx}, \\ u(0,t) = 1, \quad u(\infty,t) = -1, \\ u(x,0) = u_0(x). \end{cases} \tag{2.2.4}$$

Since we know for Burgers equation, the inviscid shock $(1,-1,0)$ has a shock profile

$$\phi(x) = -\tanh\frac{x}{2}. \tag{2.2.5}$$

In this section, we will consider the initial value $u_0(x)$ which is a perturbation of the stationary wave solution $\phi(x-x_0)$ with a location $x_0 = \frac{1}{\epsilon}$ for $\epsilon > 0$ sufficiently small with the following two properties:

$$\int_0^\infty (u_0(x) - \phi(x - x_0))dx = 0, \tag{2.2.6}$$

$$|u_0(x) - \phi(x - x_0) - e^{-x}(1 - \phi(x - x_0))| < H(x, x_0), \tag{2.2.7}$$

where $H(x, y)$ is a function of x and y defined as

$$H(x,y) = \begin{cases} \frac{xe^{-y/3}}{\cosh\frac{x-y}{2}} & \text{for} \quad 0 \le x \le 1, \\ \frac{e^{-y/3}}{\cosh\frac{x-y}{2}} & \text{for} \quad x > 1. \end{cases} \tag{2.2.8}$$

In order to trace the asymptotic behavior of the solution $u(x, t)$, we define the wave front $X(t)$ of the solution $u(x, t)$ in terms of the stationary wave $\phi(x)$. $X(t)$ is given by the implicit relation

$$\int_0^\infty (u(x,t) - \phi(x - X(t)))dx = 0. \tag{2.2.9}$$

It is easy to see that for each $t \ge 0$, $X(t)$ is unique. we will explain later that for each $t \ge 0$, $X(t)$ exists. with the help of $X(t)$, we have

Theorem 2.2.2 *[38] Suppose the initial data $u_0(x)$ satisfies* (2.2.6)-(2.2.7), *then the solution $u(x, t)$ of the initial boundary value problem* (2.2.4) *has the properties:*

$$|u(x,t) + \tanh\frac{x - X(t)}{2}| < \frac{e^{-X(t)/3}}{\cosh\frac{x-X(t)}{2}} = \frac{(e^{x_0} + O(1)t)^{-1/3}}{\cosh\frac{x-X(t)}{2}}, \tag{2.2.10}$$

$$X(t) = x_0 + \log(1 + te^{-x_0}) + e(t), \tag{2.2.11}$$

where $e(t)$ is a function satisfying

$$\lim_{t\to\infty} e(t) = 0.$$

Remark 2.2.3 *We can see the location of wave front in* (2.2.11) *coincides with what we have got in* (2.2.3) *by asymptotic analysis.*

To prove the theorem, we need introduce some notations.

For any $a \in (\frac{1}{6}, \frac{1}{3})$ we define a sequence $\{X_n\}_{n\geq 0}$:

$$X_0 = x_0 = 1/\epsilon > 0, \tag{2.2.12}$$

$$X_n = X_{n-1} + \delta_{n-1} \quad \text{for} \quad n \geq 1, \tag{2.2.13}$$

where δ_n is any constant with $\frac{1}{2}e^{-aX_n} < \delta_n < 2e^{-aX_n}$. This induces a sequence $\{T_n\}_{n\geq 0}$ given implicitly by $X(T_n) = X_n$, we will show the existence and uniqueness of T_n later.

Lemma 2.2.4 *If the solution $u(x,t)$ to the initial boundary value problem* (2.2.4) *satisfies*

$$|u(x, T_n) - \phi(x - X_n) - (1 - \phi(-X_n))e^{-x}| < H(x, X_n) \tag{2.2.14}$$

for some $n \geq 0$, then the following initial boundary value problem

$$\begin{cases} \partial_t v + \partial_x(\phi(x - X_n)v) - \partial_{xx} v = -\frac{1}{2}\partial_x v^2, \\ v(0,t) = 1 - \phi(-X_n), \qquad v(\infty, t) = 0, \\ v(x, 0) = u(x, T_n) - \phi(x - X_n), \end{cases} \tag{2.2.15}$$

has a solution $v_n(x,t) = v(x,t)$ for $0 \leq t \leq X_n\delta_n \exp(X_n)$, furthermore, the following boundary gradient estimate

$$|\partial_x v_n(0,t)| = O(1)e^{-X_n}(X_n\delta_n + e^{X_n/6 - t/4}) \tag{2.2.16}$$

holds for any $t \in [0, X_n\delta_n \exp(X_n)]$.

This lemma is a summary of several lemmas in [38]. The proof is quite long, but the idea is very clear. Here we only sketch some basic ideas of proof for lemma 2.2.4, the details can be found in [38]. The local existence for this nonlinear differential equation is proved by fixed point theorem. We first study the iterative initial boundary value problem for the linear partial differential equation

$$\begin{cases} \partial_t v_n^k + \partial_x(\phi(x - X_n)v_n^k) - \partial_{xx} v_n^k = -\frac{1}{2}\partial_x (v_n^{k-1})^2, \\ v_n^k(0,t) = 1 - \phi(-X_n), \qquad v_n^k(\infty, t) = 0, \\ v_n^k(x, 0) = u(x, T_n) - \phi(x - X_n), \end{cases} \tag{2.2.17}$$

for $k \geq 1$ and $v_n^0 = 0$. We represent the solution for (2.2.17) by its Green's function which can be explicitly written down. Then a detailed pointwise estimate yields convergence of iterative approximate solution at least on $[0, X_n\delta_n \exp(X_n)]$. As far as the boundary gradient estimate (2.2.16) is concerned, first, we can represent the solution by Green's function, therefore, when we take the derivative to solution, the derivative will transfer to the derivative of Green's function, thus, we need only to estimate the derivative of Green's function and a sharper estimate for solution itself, essentially, a sharper estimate for the solution to the linear equation.

Now we apply Lemma 2.2.4 to prove Theorem 2.2.2.

Proof of theorem 2.2.2: Define $X_n(t) = X(t+T_n)$, then $T_{n+1}-T_n$ is the time the wave front $X(t)$ drifts from X_n to X_{n+1}. Actually, $u(x,t) = v_n(x, t-T_n) + \phi(x - X_n)$ on $[T_n, T_n + X_n\delta_n \exp(X_n)]$, therefore, we use translation $t \mapsto t + T_n$, and define $u_n(x,t) = u(x, t+T_n)$ for $0 < t \leq X_n\delta_n \exp(X_n)$. We know that $X_n(t)$ exist locally. From the definition of $X(t)$, we differentiate (2.2.6) with respect to t, then

$$\begin{aligned} 0 &= \int \left(\partial_t u_n(x,t) - \phi'(x - X_n(t))\dot{X}_n(t)\right) dx \\ &= \int \partial_t v_n(x,t) dx - \dot{X}_n(t) \int \phi'(x - X_n(t)) dx. \end{aligned}$$

Using the equation (2.2.15) and boundary gradient estimate (2.2.16), then we have for $t \leq X_n$, we first assume $X_n(t) > 0$, then

$$|\dot{X}_n(t)| \leq e^{-X_n}(1 + O(1)(e^{-t/4+X_n/6} + X_n\delta_n)),$$

thus we find $X_n + \delta_n > X_n + Ce^{-2X_n/3} > X_n(t) > X_n - Ce^{2X_n/3} > 0$ for some constant C, when $t \leq X_n$. Do above computation again for $t > X_n$, then

$$\dot{X}_n(t) > 0.$$

Therefore, $X_n(t)$ are well-defined for $t < X_n\delta_n \exp(X_n)$ and $e^{-X_n(t)} = O(1)e^{-X_n}$, moreover, we have

$$|\dot{X}_n(t)| \leq e^{-X_n}(1 + O(1)(e^{-t/4+X_n/6} + X_n\delta_n))$$

for $t \leq X_n\delta_n \exp(X_n)$. Thus

$$\begin{aligned} 0 &= \int_0^\infty (u(x,t) - \phi(x - X_n(t)))dx \\ &= \int_0^\infty (u(x,t) - \phi(x - X_n)) + (\phi(x - X_n) - \phi(x - X_n(t)))dx \\ &= \int_0^t \int_0^\infty \partial_s v_n(x,s)dxds - (X_n(t) - X_n)(2 + O(1)e^{-X_n}) \\ &= \int_0^t (-\partial_x v_n(0,s) - \phi_x(-X_n))ds - (X_n(t) - X_n)(2 + O(1)e^{-X_n}), \end{aligned}$$

where we use the equation and $\phi(x) = -\tanh\frac{x}{2}$. Then using the boundary gradient estimate and $\phi_x(-X_n) = -2e^{-X_n}(1 + O(1)e^{-X_n})$, we have

$$\begin{aligned} X_n(t) - X_n &= (\frac{1}{2} + O(1)e^{-X_n})\int_0^t (-\partial_x v_n(0,s) - \phi_x(-X_n))ds \\ &= te^{-X_n} + O(1)(e^{-t/4+X_n/6} + X_n\delta_n)te^{-X_n}. \end{aligned} \tag{2.2.18}$$

Suppose that both $t \leq X_n\delta_n e^{X_n}$ and $X_n \gg 1$. Since

$$\delta_n = X_{n+1} - X_n = (T_{n+1} - T_n)e^{-X_n} + O(1)(e^{-5X_n/6} + (X_n\delta_n)^2)$$

and $\delta_n = e^{-aX_n} \gg e^{-5X_n/6}$, we have that

$$T_{n+1} - T_n = (1 + O(1)X_n^2\delta_n)\delta_n e^{X_n}, \tag{2.2.19}$$

therefore $X_n < T_{n+1} - T_n < X_n\delta_n e^{X_n}$, hence T_{n+1} is uniquely determined.

At the same time, by a delicate pointwise estimate, we can get

$$|v_n(x, T_{n+1} - T_n) - (1 - \phi(x - X_n))| \leq H(x, X_n).$$

Since at initial time, $T_0 = 0$, (2.2.7) holds, therefore, by induction, for the solution $u(x,t)$ to (2.2.4) defined globally in time, similar to (2.2.7), we have

$$|u(x, T_n) - \phi(x - X_n) - (1 - \phi(x - X_n))| \leq H(x, X_n) \tag{2.2.20}$$

for $n \geq 0$.

From the estimate (2.2.19), it follows that

$$\frac{X_{n+1}-X_n}{T_{n+1}-T_n} = \frac{\delta_n}{(1+O(1)X_n^2\delta_n)\delta_n e^{X_n}} = e^{-X_n}(1+O(1)X_n^2 e^{-aX_n}). \tag{2.2.21}$$

This is discretization of the ordinary differential equation

$$\frac{dX}{dt} = e^{-X(t)}$$

with initial value $X(0) = X_0$, we have

$$X(t) = \log(e^{x_0}+t), \tag{2.2.22}$$

then there exists a constant $C > 0$ such that

$$|T_{n+1}-T_n-(e^{X_n+1}-e^{X_n})| \leq Ce^{-\tilde{a}X_n}(e^{X_{n+1}}-e^{X_n}) \tag{2.2.23}$$

holds with $a-\tilde{a} > 0$ sufficiently small. When X_0 is sufficiently large, we have

$$\frac{1}{2}(T_{n+1}-T_n) < e^{X_n}-e^{X_0} < 2(T_{n+1}-T_0) \quad \text{for all} \quad n \geq 1. \tag{2.2.24}$$

According to (2.2.23) and (2.2.24), we have

$$\frac{T_{n+1}-T_n}{1+C(\frac{1}{2}T_{n+1}+e^{X_0})^{-\tilde{a}}} \leq e^{X_{n+1}}-e^{X_n} \leq \frac{T_{n+1}-T_n}{1-C(\frac{1}{2}T_{n+1}+e^{X_0})^{-\tilde{a}}} \tag{2.2.25}$$

and

$$T_n - \frac{4C}{1-\tilde{a}}(\frac{1}{2}T_n+e^{X_0})^{1-\tilde{a}} \leq e^{X_n}-e^{X_0} \leq T_n + \frac{4C}{1-\tilde{a}}(\frac{1}{2}T_n+e^{X_0})^{1-\tilde{a}}, \tag{2.2.26}$$

then

$$X_n = \log(e^{X_0}+T_n(1+O(1)(T_n+e^{X_0})^{-\tilde{a}})). \tag{2.2.27}$$

Thus

$$X_n = \log(T_n+1)+E_n, \tag{2.2.28}$$

Here E_n satisfies

$$E_n = \log(1+O(1)((e^{X_0}+T_n)^{-\tilde{a}}+\frac{e^{X_0}}{T_{n+1}})). \tag{2.2.29}$$

From (2.2.20), we conclude that

$$|u(x,T_n) - \phi(x - X_n)| \leq \frac{(e^{X_0} + O(1)T_n)^{-1/3}}{\cosh\frac{x-X_n}{2}}. \tag{2.2.30}$$

From (2.2.19), we know $T_n \to \infty$ as $n \to \infty$, then by (2.2.28), $X_n \to \infty$ as $n \to \infty$. Moreover, from the definition of $\{X_n\}_n$ in (2.2.12) and (2.2.13), it follows that for any $y \geq X_0 + e^{-aX_0}$ we can construct a sequence $\{X_n\}_n$ such that $y = X_m$ for a $X_m \in \{X_n\}$. Thus for any $t > e^{(1-a)X_0}$, there is a sequence $\{X_n\}_n$ satisfying such that $X(t) = X_m \in \{X_n\}$. Therefore, we have that when $t \geq e^{(1-a)X_0}$,

$$|u(x,t) + \tanh\frac{x - X(t)}{2}| < \frac{e^{-X(t)/3}}{\cosh\frac{x-X(t)}{2}} = \frac{(e^{x_0} + O(1)t)^{-1/3}}{\cosh\frac{x-X(t)}{2}}. \tag{2.2.31}$$

So we complete the proof of the theorem. □

Chapter 3

Viscous Transonic Flow Through a Nozzle

In this chapter, the model we consider is

$$u_t + (f(u))_x = \varepsilon u_{xx} + a(x)h(u) \tag{3.0.1}$$

The flux function $f(u)$ is assumed to be convex. This is motivated by gas dynamics, where the sound speed depends monotonically on the density. By composition with a simple translation, we may assume that:

$$\begin{cases} f''(u) > 0 & \text{for all } u \text{ under consideration} \\ f(0) = f'(0) = 0 \end{cases} \tag{3.0.2}$$

The function $h(u)$ represents the coupling of the source due to the geometry and the gas flow. The following strong nonlinear coupling assumption is dictated by physical consideration:

$$h(u) \neq 0, \qquad h'(u) \neq 0 \qquad \text{for all } u \text{ under consideration} \tag{3.0.3}$$

The function $a(x)$ denotes the strength of the source and may change sign.

Firstly, we recall some observation of propagation of a shock wave in nozzle. Then, we shall study the nonlinear stability and instability of stationary of (3.0.1). At the end, we will take our focus on the movement of a viscous shock wave by matched asymptotic analysis. Furthermore, some problems which are still unsolved are left in next chapter.

3.1 Nonlinear Stability and Instability of Shock Waves

Before studying the viscous cases, we first consider the propagation of a shock wave through stationary waves. The stationary wave to the right(left) of the shock is denoted by $u_r(x)(u_l(x))$and the location of the shock wave is $x = x(t)$.

The speed of the shock wave is governed by the Rankine-Hugoniot condition

$$\begin{aligned} x'(t) &= \tfrac{f(u_+(t))-f(u_-(t))}{u_+(t)-u_-(t)} \equiv \sigma(u_-(t), u_+(t)) \\ u_+(t) &\equiv u_r(x(t)); \quad u_-(t) \equiv u_l(x(t)) \end{aligned} \tag{3.1.1}$$

Since $u_-(t)$ and $u_+(t)$ are the values of the stationary waves $u_l(t)$ and $u_r(t)$ at $x = x(t)$, we have

$$\begin{aligned} \tfrac{du_-(t)}{dt} &= \tfrac{du_l(x(t))}{dx} x'(t) = f'(u_-(t))^{-1} a(x(t)) h(u_-(t)) x'(t) \\ \tfrac{du_+(t)}{dt} &= \tfrac{du_r(x(t))}{dx} x'(t) = f'(u_+(t))^{-1} a(x(t)) h(u_+(t)) x'(t) \end{aligned} \tag{3.1.2}$$

Differentiate (3.1.1) and use (3.1.2) to obtain

$$\frac{x''(t)}{x'(t)} = \frac{a(x(t))}{u_+(t) - u_-(t)} (h(u_+(t) - h(u_-(t)) - x'(t)(\frac{h(u_+(t))}{f'(u_+(t)} - \frac{h(u_-(t))}{f'(u_-(t))})) \tag{3.1.3}$$

The system of ordinary differential equations (3.1.1) and (3.1.2) determines the location $x = x(t)$ and the states $u(t)$ of the shock wave. It also shows that when a shock wave propagates through a stationary wave it leaves behind another stationary wave, which is the extension of the original stationary wave behind the shock wave.

Base on (3.1.3), we may investigate the stability of shock waves. A supersonic shock wave $u_- > u_+ > 0$, accelerates to the right and moves away from the sonic state; a subsonic shock wave $0 > u_- > u_+$ accelerates to the left and also moves away from the sonic state. For transonic shock waves $u_- > 0 > u_+$, from (3.1.3), for nearly stationary shock waves

$$\frac{x''(t)}{x'(t)} \sim \frac{a(x(t))(h(u_+(t) - h(u_-(t))}{u_+(t) - u_-(t)}, \quad \text{when} \quad x'(t) \sim 0 \tag{3.1.4}$$

Thus a nearly stationary transonic shock wave decelerates and is nonlinearly stable if

$$a(x)h'(u) < 0 \quad \text{(stability)} \tag{3.1.5}$$

and accelerates and nonlinearly unstable if

$$a(x)h'(u) > 0 \quad \text{(instability)} \tag{3.1.6}$$

3.2 Asymptotic Stability and Instability

In this part, we study the nonlinear stability and instability of stationary solution of

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} + a(x)h(u), \\ u(x,0) = \tilde{u}(x), \\ u(0,t) = u_l, \qquad u(1,t) = u_r, \end{cases} \tag{3.2.1}$$

Let $U(x) = U(x;\varepsilon)$ be a stationary solution of

$$\begin{cases} (f(U))_x = \varepsilon U_{xx} + a(x)h(U) \\ U(0) = u_l, \qquad U(1) = u_r, \end{cases} \tag{3.2.2}$$

Theorem 3.2.1 *Suppose $u(x,t)$ is a solution of (3.2.1) and $U(x)$ is a solution of corresponding stationary equation (3.2.2), then we will have following things:*

(i) (Divergent duct):If $a(x)h'(u) < 0$, then

$$\frac{\partial u}{\partial x}(0,t) > \frac{dU}{dx}(0) \Longrightarrow u(x,t) \geq U(x)$$

and

$$\frac{\partial u}{\partial x}(0,t) < \frac{dU}{dx}(0) \Longrightarrow u(x,t) \leq U(x)$$

(ii) (Convergent duct):If $a(x)h'(u) < 0$,

Moreover, if $U(x)$ contains no interior layer, then it is asymptotically stable; If $U(x)$ contains interior layer, then it is asymptotically unstable.

(iii) (Special Model): For the model

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2} - \frac{A'(x)}{A(x)}u \tag{3.2.3}$$

If $-\frac{A'(x)}{A(x)} - \frac{\partial u}{\partial x}(x) < 0$, then $U(x)$ is stable.

Proof:

Define

$$w(x,t) = u(x,t) - U(x)$$

From (3.2.1) and (3.2.2), we have

$$\begin{cases} w_t + f'(w+U)(w_x+U_x) = \epsilon w_{xx} + f'(U)U_x - a(x)h(U) + a(x)h(w+U) \\ w(0,t) = 0, \qquad w(1,t) = 0, \end{cases} \tag{3.2.4}$$

The linearized equation is

$$\begin{cases} w_t + f'(U)w_x + f'(U)U_x = \epsilon w_{xx} + a(x)h'(U)w \\ w(0,t) = 0, \qquad w(1,t) = 0, \end{cases} \tag{3.2.5}$$

Set $w(x,t) = e^{\lambda t}\eta(x)$ and obtain

$$\begin{cases} \epsilon\eta'' - (f'(U)\eta)' + a(x)h'(U)\eta = \lambda\eta \\ \eta(0) = 0, \qquad \eta(1) = 0, \end{cases} \tag{3.2.6}$$

(1)

Firstly we may assume that $u_x(0,t) > U_x(0)$ for all t.

Step 1:

Claim: $\lambda < 0$

Proof: We prove by contradiction. Assume $\lambda \geq 0$,

From (3.2.6) we have

$$\epsilon\eta'' - (f'(U)\eta)' + (a(x)h'(U) - \lambda)\eta = 0 \tag{3.2.7}$$

Integrate (3.2.7) from x' to x with $0 \leq x' < x \leq 1$,

$$\epsilon\eta'(x) = \epsilon\eta'(x') + f'(U(x))\eta(x) - f'(U(x'))\eta(x') - \int_{x'}^{x}(ah' - \lambda)\eta(s)ds \tag{3.2.8}$$

Integrate again from x' to x,

$$\begin{aligned} \eta(x) = {} & \eta(x')e^{\int_{x'}^{x}\frac{f'(U)(\xi)}{\epsilon}d\xi} \\ & +(\eta'(x') - \tfrac{f'(U(x'))}{\epsilon}\eta(x'))\int_{x'}^{x} e^{-\int_{x}^{y}\frac{f'(y(\xi))}{\epsilon}d\xi}dy \\ & -\tfrac{1}{\epsilon}\int_{x'}^{x} dy \int_{x'}^{y}(a(s)h'(U(s)) - \lambda)\eta(s)e^{-\int_{x}^{y}\frac{f'(\xi)}{\epsilon}d\xi}ds \end{aligned} \tag{3.2.9}$$

Setting $x' = 0$, previous hypothesis $ch' < 0$ and $\lambda \geq 0$ implies that $\eta(x) > 0$ as long as $\eta(y) > 0$ for $0 < y < x$. By the initial assumption $u_x(0,t) > U_x(0)$, it is clear that $\eta(y) > 0$ for y close to 0.

Integrate (3.2.6), that

$$\epsilon(\eta'(1) - \eta'(0)) + \int_0^1 a(x)h'(U(x))\eta(x)dx = \lambda\int_0^1 \eta(x)dx \tag{3.2.10}$$

Here $\eta(x) \geq 0$ implies $\lambda < 0$, which is a contradiction.

Step 2:

By define

$$\begin{cases} p = e^{\int_0^x -\frac{f'(U(\xi))}{\epsilon} d\xi} \\ q = p(x)(a(x)h'(U) - f''(U)U_x - \lambda) \end{cases} \tag{3.2.11}$$

(3.2.6) becomes

$$\begin{cases} (p\eta')' + q\eta = 0 \\ \eta(0) = 0, \qquad \eta(1) = 0, \end{cases} \tag{3.2.12}$$

Use Comparison Theorem in [6], we may have the desired result.

(2) In this case $a(x)h'(u) < 0$,

We followed by [37],

For the case $U(x)$ contains no interior layer, and $\lambda = \lambda(\varepsilon)$ is bounded uniformly in ε, we may rewrite (3.2.6) as

$$\begin{cases} \epsilon\eta'' - (f'(U)\eta)' + a(x)h'(U)\eta = \lambda\eta \\ \eta(0) = 0, \qquad \eta'(0) = 1, \end{cases} \tag{3.2.13}$$

Thus it follows form Lemma 4.4 in [37] that $\eta(1) > 0$, which is a contradicts (3.2.6), q(1)=0. If $\lambda = \lambda(\varepsilon)$ becomes large $\lambda \gg 1$, then it follows from integrating (3.2.6) that

$$(\eta(x)E(x))' = E(x) - \frac{1}{\varepsilon}\int_0^x (a(x)h'(U) - \lambda)\eta(\tau)d\tau,$$

where

$$E(x) \equiv exp\{-\frac{1}{\varepsilon}\int_0^x f'(U)(\tau)d\tau\}$$

Therefor, we have

$$\eta(x) > \frac{1}{E(x)}\int_0^x E(\tau)d\tau.$$

In particular, $q(1) > 0$, again a contradiction. Thus, for $U(x)$ contains no interior layer, $\lambda < 0$ and $U(x)$ is stable.

On the other hand, we consider the case $U(x)$ contains an interior layer. We want to prove by contradiction that $\lambda > 0$. If not, then

$$a(x)h'(U) - \lambda > 0$$

and so from adapting Lemma 4.5 [37] to (3.2.13), we have $\eta(1) < 0$ which contradicts (3.2.6), $\eta(1) = 0$. Thus $\lambda > 0$ and $U(x)$ is unstable.

(3)

By define

$$\begin{cases} p = e^{\int_0^x -\frac{f'(U(\xi))}{\epsilon} d\xi} \\ q = p(x)(-\frac{A'(x)}{A(x)} - U_x - \lambda) \end{cases} \tag{3.2.14}$$

(3.2.3) becomes

$$\begin{cases} (p\eta')' + q\eta = 0 \\ \eta(0) = 0, \qquad \eta(1) = 0, \end{cases} \tag{3.2.15}$$

The initial assumption $-\frac{A'(x)}{A(x)} - \frac{\partial u}{\partial x}(x) < 0$ guarantees that we can use Strong Minimum Principle in [56], which implies that $\eta \geq 0$. By a similar argument of (1)step 1, we could deduce $\lambda < 0$ and so $U(x)$ is stable.

□

3.3 Matched Asymptotic Analysis

The model we consider is a simplified scalar model related to the model proposed in [9]. More precisely, consider the following initial boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} + a(x)u, \\ u(x,0) = \tilde{u}(x), \\ u(0,t) = u_l, \qquad u(1,t) = u_r, \end{cases} \tag{3.3.1}$$

where

$$f''(u) > 0, \qquad f'(0) = f(0) = 0. \tag{3.3.2}$$

Motivated by the study for inviscid flow, we first study the divergent nozzle case, that is $a(x) < 0$, where standing shock in the inviscid flow is stable.

First of all, we generalize the study of Ebid, Goodman and Majda [9]. Suppose u_l and u_r satisfy that there exist $\bar{x}$, u_-, u_+ such that

$$Q(u_-) - Q(u_l) = A(\bar{x}), \quad Q(u_r) - Q(u_+) = A_1 - A(\bar{x}) \tag{3.3.3}$$

and

$$f(u_+) = f(u_-), \qquad f'(u_+) < f'(u_-), \tag{3.3.4}$$

where $Q(u) = \int_0^u \frac{f'(y)}{y} dy$, $Q(0) = 0$, and $A(x) = \int_0^x a(s)ds$, $A_1 = \int_0^1 a(s)ds$, then there is a standing transonic shock (u_-, u_+) at $\bar{x}$ in the steady flow

$$\frac{df(u)}{dx} = a(x)u \tag{3.3.5}$$

with boundary condition $u(0) = u_l$ and $u(1) = u_r$. When $f(u) = \frac{u^2}{2}$, (3.3.3) and (3.3.4) will reduce to the results in [9]

$$\frac{u_l + u_r - A_1}{2} + A(\bar{x}) = 0 \quad \text{for some} \quad \bar{x} \in [0, 1] \quad \text{and} \quad u_l > u_r - A_1. \tag{3.3.6}$$

Suppose u_l, u_r in (3.3.1) satisfy (3.3.3) and (3.3.4), then a transonic shock layer will be generated in bounded domain $[0, 1]$ when ϵ is sufficiently small. As same as stability or instability of standing shock for inviscid flow, the stability and instability of stationary viscous shock wave are of great interest and importance. To reach this goal, we first study the propagation of viscous transonic shock wave in a bounded nozzle.

First of all, we use matched asymptotic analysis to study the internal shock layer and the solution in outer region. Although this process is known in principle [10], we would like to carry it out in detail here so that we can explain the problem easily later.

We start with the outer expansion. In the region away from the shock layer, the solution may be approximated by truncation of the formal series

$$u(x,t) \sim u_0(x,t) + \epsilon u_1(x,t) + \epsilon^2 u_2(x,t) + \cdots . \tag{3.3.7}$$

Substituting this into (3.3.1) and equating coefficients of powers of ϵ, we get

$$O(1): \qquad u_{0t} + f(u_0)_x - a(x)u_0 = 0, \tag{3.3.8}$$

$$O(\epsilon): \qquad u_{1t} + (f'(u_0)u_1)_x - a(x)u_1 = u_{0xx}, \tag{3.3.9}$$

$$O(\epsilon^2): \qquad u_{2t} + (f'(u_0)u_2)_x - a(x)u_2 = u_{1xx} - \frac{1}{2}(f''(u_0)u_1^2)_x. \tag{3.3.10}$$

In the shock layer region, u should be represented by an inner expansion:

$$u(x,t) \sim U_0(\xi,t) + \epsilon U_1(\xi,t) + \epsilon^2 U_2(\xi,t) + \cdots , \tag{3.3.11}$$

where ξ is the stretched variable given by

$$\xi = \frac{x - x_0(t)}{\epsilon} + \delta_0(t) + \epsilon\delta_1(t) + \epsilon^2\delta_2(t) + \cdots . \tag{3.3.12}$$

This time we substitute (3.3.11) into (3.3.1) and obtain

$$O(\frac{1}{\epsilon}): \qquad U_{0\xi\xi} + \dot{x}_0 U_{0\xi} - f(U_0)_\xi = 0, \tag{3.3.13}$$

$$O(1): \qquad U_{1\xi\xi} + \dot{x}_0 U_{1\xi} - (f'(U_0)U_1)_\xi = \dot{\delta}_0(t)U_{0\xi} + U_{0t} - a(x_0(t))U_0, \tag{3.3.14}$$

$$O(\epsilon): \qquad U_{2\xi\xi} + \dot{x}_0 U_{2\xi} - (f'(U_0)U_2)_\xi = \dot{\delta}_1(t)U_{0\xi} + \dot{\delta}_0(t)U_{1\xi} + U_{1t} + \frac{1}{2}(f''(U_0)U_1^2)_\xi - a(x_0(t))U_1 - a'(x_0(t))(\xi - \delta_0(t))U_0. \tag{3.3.15}$$

On the other hand, in a zone somewhat farther from the shock layer, the matching zone, for example, $\epsilon^\nu < x - x_0(t) + \epsilon\delta_0(t) + \cdots \leq \epsilon^\mu$, for some $0 < \mu < \nu < 1$, we expect both the inner and the outer expansions to be valid. Therefore, the two expansions must agree there. As explained in [10], we can express the outer solutions in terms of ξ and use Taylor series to find the following matching conditions as $\xi \to \infty$:

$$\begin{aligned}
U_0(\xi,t) &= u_0(x_0(t) \pm 0, t) + o(1), && (3.3.16)\\
U_1(\xi,t) &= u_1(x_0(t) \pm 0, t) + (\xi - \delta_0)\partial_x u_0(x_0(t) \pm 0, t) + o(1), && (3.3.17)\\
U_2(\xi,t) &= u_2(x_0(t) \pm 0, t) + (\xi - \delta_0)\partial_x u_1(x_0(t) \pm 0, t)\\
&\quad -\delta_1 \partial_x u_0(x_0(t) \pm 0, t) + \frac{1}{2}(\xi - \delta_0)^2 \partial_x^2 u_0(x_0(t) \pm 0, t)\\
&\quad +o(1). && (3.3.18)
\end{aligned}$$

Now let us look at the leading order outer solution, it is described by (3.3.8), which is a quasi-linear hyperbolic differential equation and can be viewed as equation for inviscid flow through a divergent nozzle. Since for inviscid flow, the shape of divergent nozzle has stabilizing effect, therefore, the leading order term in the ansatz for the location of shock wave will not move a lot. Suppose it is generated at some time $t = \tilde{t}$, $x_0 = \tilde{x}$, and the change of location of shock layer is a quantity $O(\epsilon)$, thus we may assume $\dot{x}_0 = 0$. Since our interest is the propagation of this shock layer after its generation, and usually the time of generating a shock layer is quite short, so we can assume $\tilde{t} = 0$ without loss of generality. Thus the leading order term U_0 for the inner solution satisfies

$$U_{0\xi\xi} - f(U_0)_\xi = 0, \qquad (3.3.19)$$

and the equation for leading order expansion u_0 for the outer solution reads

$$u_{0t} + f(u_0)_x = a(x)u_0. \qquad (3.3.20)$$

Since up to the leading order, the speed and location of shock wave does not change as time goes on, therefore, combining with our assumption (3.3.3) and (3.3.4), we deduce that u_0 will be the steady state of (3.3.20), that is u_0 satisfies

$$\frac{df(u_0)}{dx} = a(x)u_0 \qquad (3.3.21)$$

on $[0, \bar{x}]$ and $[\bar{x}, 1]$ respectively, and has a jump (u_-, u_+) at $\bar{x}$. Using the matching condition (3.3.16), U_0 will be the shock profile ϕ for the standing shock (u_-, u_+)

at $\bar{x}$ in the steady flow (3.3.21) for all time t. In the following we choose ϕ such that $f'(\phi(0)) = 0$, for example, for $f(u) = \frac{u^2}{2}$, we have $\phi(\xi) = u_+ \tanh \frac{u-\xi}{2}$.

To get more accurate propagation of the shock layer, we must analyze the next order approximations. First, we solve the first order outer solution, u_1, from the linear hyperbolic equation (3.3.9). Since u_0, the solution of (3.3.21) satisfies the boundary condition in the initial boundary value problem (3.3.1), therefore, we impose the boundary condition $u_1(0,t) = 0$ and $u_1(1,t) = 0$ when we solve (3.3.9) in the domain $[0,\bar{x}] \times \mathbb{R}^+$ and $[\bar{x},1] \times \mathbb{R}^+$ respectively. Since $u_0(x) > 0$ when $x \in [0,\bar{x}]$ and $u_0(x) < 0$ for $x \in [\bar{x},1]$, therefore, the initial boundary value problems

$$\begin{cases} u_{1t} + (f'(u_0)u_1)_x - a(x)u_1 = u_{0xx}, & x \in [0,\bar{x}], \quad t > 0, \\ u_1(x,0) = u_1^-(x), & x \in [0,\bar{x}], \\ u_1(0,t) = 0, & t > 0, \end{cases} \tag{3.3.22}$$

and

$$\begin{cases} u_{1t} + (f'(u_0)u_1)_x - a(x)u_1 = u_{0xx}, & x \in [\bar{x},1], \quad t > 0, \\ u_1(x,0) = u_1^+(x), & x \in [\bar{x},1], \\ u_1(1,t) = 0, & t > 0, \end{cases} \tag{3.3.23}$$

are both well-posed. Moreover, since u_0 does not depend on t and $f'(u_0) > f'(u_-)$, for $x \in [0,\bar{x}]$ and $f'(u_0) < f'(u_+)$, for $x \in [\bar{x},1]$, by characteristic method it is easy to see that u_1 is independent of time when t is sufficiently large.

Now we go back to the first order approximation of inner solution, with the help of knowledge of x_0, u_0, U_0 and $\alpha = a(\bar{x})$, we can rewrite (3.3.14) as

$$U_{1\xi\xi} - (f'(\phi)U_1)_\xi = \dot{\delta}_0(t)\phi' - \alpha\phi.$$

If we define a smooth function $D_1(\xi)$ satisfies

$$D_1(\xi) = \begin{cases} \beta_+\xi & \text{for} \quad \xi > 1, \\ \beta_-\xi & \text{for} \quad \xi < -1, \end{cases} \tag{3.3.24}$$

where $\beta_\pm = u_0'(\bar{x}\pm)$, and set $V_1(\xi,t) = U_1(\xi,t) - D_1(\xi)$, then

$$V_{1\xi\xi}(\xi,t) - (f'(\phi)V_1)_\xi = -D_{1\xi\xi} + (f'(\phi)D_1)_\xi + \dot{\delta}_0(t)\phi' - \alpha\phi. \tag{3.3.25}$$

Thank for the nice property (1.1.11) of shock profile and $f'(u_\pm)\beta_\pm = \alpha u_\pm$, we know that $g(\xi) = -D_{1\xi\xi} + (f'(\phi)D_1)_\xi - \alpha\phi$ satisfies $\int_{-\infty}^{\infty} |g(\xi)|d\xi < \infty$. Therefore, if we integrate equation (3.3.25) from 0 to ξ, we have

$$V_{1\xi}(\xi,t) - f'(\phi)V_1(\xi,t) + c(t) = G(\xi) + \dot{\delta}_0(t)\phi, \tag{3.3.26}$$

where $G(\xi) = \int_0^\xi g(\xi)d\xi$ and $c(t)$ is related to $V_1(0,t)$ and $V_{1\xi}(0,t)$. Solve this ordinary differential equation, we get the one of solutions

$$V_1(\xi,t) = \int_0^\xi (\dot{\delta}_0(t)\phi(\eta) - G(\eta) - c(t)) \exp\left(\int_\eta^\xi f'(\phi(\zeta))d\zeta\right) d\eta. \tag{3.3.27}$$

After a simple analysis, we will get

$$V_1(\xi,t) \;\to\; -\frac{\dot{\delta}_0(t)u_\pm - G_\pm - c(t)}{f'(u_\pm)} \quad \text{as} \quad \xi \to \pm\infty, \tag{3.3.28}$$

where $G_\pm = \lim_{\xi\to\pm\infty} G(\xi)$.

On the other hand, for $t > \bar{t}$ sufficient large, u_1 is independent of time, then using the matching condition (3.3.17), we have

$$V_1(\xi,t) \to \gamma_\pm - \beta_\pm\delta_0(t) \quad \text{as} \quad \xi \to \pm\infty, \tag{3.3.29}$$

here $\gamma_\pm = u_1(\bar{x}\pm)$. Combing (3.3.28) with (3.3.29), we have

$$\begin{aligned}\gamma_- - \beta_-\delta_0(t) &= \lim_{\xi\to-\infty} V_1(\xi,t) = -\frac{\dot{\delta}_0(t)u_- - G_- - c(t)}{f'(u_-)} \\ &= -\frac{\dot{\delta}_0(t)u_- - G_- - ((\gamma_+ - \beta_+\delta_0(t))f'(u_+) + \dot{\delta}_0(t)u_+ - G_+)}{f'(u_-)},\end{aligned}$$

hence

$$\dot{\delta}_0(t) + \frac{\beta_+f'(u_+) - \beta_-f'(u_-)}{u_- - u_+}\delta_0(t) = \frac{G_- - G_+ + \gamma_+f'(u_+) - \gamma_-f'(u_-)}{u_- - u_+}.$$

Using $f'(u_\pm)\beta_\pm = \alpha u_\pm$, then

$$\dot{\delta}_0(t) - \alpha\delta_0(t) = h, \tag{3.3.30}$$

where

$$h = \frac{G_- - G_+ + \gamma_+ f'(u_+) - \gamma_- f'(u_-)}{u_- - u_+}. \tag{3.3.31}$$

Thus

$$\delta_0(t) = e^{\alpha(t-\bar{t})}\delta_0(\bar{t}) + h(e^{\alpha(t-\bar{t})} - 1)/\alpha. \tag{3.3.32}$$

Similarly, we can solve outer solution u_2 and derive

$$\dot{\delta}_1(t) - \alpha\delta_1(t) = \tilde{h}, \tag{3.3.33}$$

where $\tilde{h} = \frac{M_- - M_+ + u_2(\bar{x}+)f'(u_+) - u_2(\bar{x}-)f'(u_-)}{u_- - u_+}$ for some M_- and M_+. Thus

$$\delta_1(t) = e^{\alpha(t-\bar{t})}\delta_1(\bar{t}) + \tilde{h}(e^{\alpha(t-\bar{t})} - 1)/\alpha. \tag{3.3.34}$$

Using $\delta_1(t)$, we can solve inner solution U_2.

For divergent nozzle, $\alpha < 0$, therefore, we find that the location of shock wave will not be drifted far away from the original location from above asymptotic analysis. Moreover, the time that shock wave exists is very long, this is nothing but metastability of viscous shock wave.

The main difference between equation (3.3.1) and viscous conservation law is that the shape of nozzle helps determine the location of the shock wave. Therefore, the propagation of viscous shock wave in a nozzle can be determined only by matched asymptotic analysis.

For the rigorous mathematical proof for this asymptotic analysis result, we leave for the future.

Moreover, to our knowledge, the propagation and dynamic stability or instability of viscous shock wave in a convergent nozzle are all unknown.

Chapter 4

C

Consider

$$\begin{cases} UU_x = U_{xx} + a(x)U \\ U(0) = u_l, \qquad U(1) = u_r, \end{cases} \tag{4.0.1}$$

where $a(x) = -\frac{A'(x)}{A(x)}$ as before.

Lemma 4.0.1 *There exist a positive M depending on u_l and u_r such that any solution of $U(x)$ of* (4.0.1) *statisfies*

$$|U(x)| < M, \qquad 0 \leq x \leq 1.$$

Moreover,

$$|U'(x)| < C(x)M + M^2, \qquad 0 \leq x \leq 1.$$

where $C(x) \triangleq \max_{0 \leq y \leq x}\{|a(y)|\}$

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