Portfolio Optimization under Minimax Risk Measure with Investment Bounds

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Abstract

Minimax measure in portfolio selection problem refers to an optimization problem that maximizes the minimum portfolio return or minimizes the maximum portfolio risk. Cai et. al. (2000, Management Science) has proposed a portfolio selection model namely I_∞ model to minimize the maximum individual risk. Cai et. al. (2004, Journal of the Operational Research Society) has showed empirically that I_∞ model has a similar performance to the Markowitz's model. This thesis employs their risk measure to solve the case with investment limits. More specifically, we derive an explicit analytical solution and optimal investment policy for the portfolio selection problem with investment limits. Then we introduce an algorithm to find out the entire efficient frontier. Finally, numerical experiments on the efficient frontier and the performance of I_∞ model with investment limits in various scenarios are carried out.

Keywords: Portfolio selection, efficient frontier, Kuhn-Tucker condition, minimax risk measure, investment limits

論文摘要

極小化極大測量法運用於投資組合優化問題時，涉及求取最大的最小投資組合回報或最小的最高投資風險。蔡等人（2000，管理科學）提出一個投資組合選擇模型即l_∞模型，它採用最高個別資產風險作爲風險測量標準。蔡等人（2004，作業研究學界期刊）根據經驗證明運用l_∞模型跟運用馬科維茲模型能獲得相約的投資回報。此論文採用蔡等人的風險測量標準， 解決設有投資上限的投資組合優化問題。我們爲設有投資上限的投資組合優化問題推論出最佳的投資策略。根據這個最佳的投資策略，我們更推論出描繪整條效率前緣的方法。最後，我們會採用香港股票市場的資料進行一系列實驗，用以探討效率前緣和採用最高個別資產風險測量法於設有投資上限條件下的投資表現。

關鍵字：投資組合選擇，效率前緣，庫思一塔克爾條件，極小化極大風險測量法，投資上限

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Chapter 1

Introduction

Portfolio selection is to find an optimal allocation of wealth among a basket of assets. In most situations, an optimal allocation refers to maximizing the expected return and minimizing the risk as a portfolio basis. By formulating the portfolio selection problem using mean-variance approach, Markowitz (1956) has provided a fundamental basis for modern portfolio selection. His pioneering work has simulated and led to a proliferation of research in this area. In portfolio theory, the concept of expected return is definite. It is defined to be the total return of all assets. Nonetheless, the concept of risk falls into various schools of thinking. For example: Markowitz [19] has used the variance of returns out of portfolio as a risk measure and formulated the portfolio selection problem as a mean variance optimization problem. Konno [14], and Konno and Yamazaki [16] have proposed an l_1 function, a mean-absolute deviation function, as another risk measure. Cai, Teo, Yang and Zhou [3, 4] have proposed the l_∞ risk function, a minimax function, which regards the maximum individual risk among all assets as a risk measure.

Evidently, there exists a gap between the academic literature and the real market. Some market constraints which financial institutions encounter have not been addressed in academic study. An important subject is the consideration of investment limits. In the market, many investment institutions, like pension funds and banks, consider assets as belonging to different groups and impose investment limits on different groups of assets. For example, total fraction of the portfolio allocated to all international assets must not exceed 40% of the portfolio. Huge investment in a sector of assets would lead to volatility effect on the market, therefore in usual practice of the market, investment bounds on sectors of assets are imposed in order to restrict investors on the amount of investment made in a particular sector. Furthermore, some banks would hire financial intermediates for investments with an imposition of investment limits on different groups of assets for risk diversification. On the side of fund investment, the value of funded pension depends critically on the investment performance of the funds. In order to protect people's savings, governments often regulate pension funds strictly, particularly when contributions are mandatory. One of the regulations on pension fund is limitation on investment allowed. Quantitative restriction on the shares of particular types of assets held by the fund limits the dispersion of outcomes, particularly for defined contribution schemes. For example, asset allocation restriction for Denmark, Germany, Japan and Switzerland on domestic equities is typically 30 or 40 percent of total assets. The second common restriction on pension fund mangers is on the amount they can invest abroad. Take the example of Mandatory Pension Fund (MPF) in Hong Kong, a MPF scheme restricts its foreign currency exposure to not more than 70% of its total assets. In addition to the external constraints mentioned above, investment limits can be generated internally by investors to address their special concerns. For example, investors who concern liquidity constraints would limit their investment on less marketable securities like fixed income securities. Moreover, some investors having preference on assets of particular sectors would desire a larger proportion of these groups of assets in their portfolio.

This thesis employs the minimax risk measure of Cai et. al. [3] to solve the portfolio selection problem with investment limits. Cai et. al. has formulated the portfolio optimization problem as a bi-criteria problem with the criteria of maximizing the portfolio expected return and minimizing the I_∞ risk function. This bi-criteria problem is converted into an equivalent parameterized problem with a single criterion. With the assumption of no short selling, an analytical solution and optimal investment strategy are derived for the efficient frontier of the portfolio optimization problem without having to solve any optimization problem. Moreover, Cai et. al. [4] have showed empirically that I_∞ model has a similar performance to the Markowitz's model. In this thesis, we derive an explicit analytical solution for the portfolio selection problem with investment limits. Optimality of this solution is ensured by the KKT conditions as the problem is convex programming. Then we introduce an algorithm to find out the efficient frontier entirely, which makes the derived investment strategy an easy implementation task.

The organization of this thesis is as follows. In chapter 2, we introduce the studies on portfolio selection problem, particularly in literature on different risk measures, efficient frontiers and investment limits. In chapter 3, we make a concise review on I_∞ model which provides a referential framework of our proposed work. In chapter 4, we formulate the portfolio selection problem with group investment limits and derive the analytical solutions. In chapter 5, we study the properties of the efficient frontier and derive an algorithm to find out the efficient frontier entirely. A discussion on the time complexity of the algorithm is also provided. In chapter 6, we find the investor's optimal portfolio from the efficient frontier. In chapter 7, numerical experiments using data from Hong Kong Stock Market are reported. We conclude the thesis in Chapter 8.

Chapter 2

Literature Review

Portfolio theory was built to solve portfolio selection problem. Modern portfolio theory takes its origin from the work of Markowitz [19] in 1950s. Adopting the mean-variance criteria, Markowitz has used the portfolio variance, which corresponds to an l_2 function, as a risk measure and formulated the portfolio selection problem as a parametric quadratic programming problem, known as a mean variance optimization problem which considers the correlation among assets explicitly. He has proved the fundamental mean-variance methodology in finance, namely to maximize expected return for a given level of variance, and to minimize variance for a given level of expected return. It has led to the formulation of an efficient frontier where investors can choose their desired portfolio with their risk-return preferences. But arguments have been raised that the mean-variance model is appropriate only if the investor's utility is quadratic or the joint distribution of return is normal. Nevertheless, these arguments are rarely satisfied in practice.

Since the pioneering work of Markowitz, research on portfolio theory has been proliferated, alternative portfolio selection models have been proposed in literature. Sharpe [28] has proposed a method to allow a portfolio analysis problem to be treated as a linear programming problem. Following Sharpe, many attempts have been made to linearize the portfolio selection problem [30, 22]. Konno [14], Konno and Yamazaki [16] have proposed a mean absolute deviation risk function, which corresponds to an I_1 function, and suggested that a piecewise linear function can be used to approximate this I_1 risk function. They have demonstrated that the I_1 risk function can ease the computational difficulty associated with solving a large-scale quadratic programming problem with a dense covariance matrix and solve large-scale optimization problem on a real time basis numerically. Young [35] has introduced another linear program model using minimum return as a measure of risk. His model amounts to maximizing the minimum return over time periods with the average return on the portfolio exceeding some minimum level. Recently, Cai et. al. [3] have introduced a minimax risk function, which corresponds to an I_∞ function, in which the maximum risk of individual assets is regarded as the risk criterion. The special structure of the I_∞ risk function enables a simple analytical solution scheme for the efficient frontier of the portfolio optimization problem without having to solve any optimization problem. The investment is obtained by a simple rule of ranking the assets according to their rates of return. The assets with higher rates of return are selected according to an investor's risk aversion parameter, then the investment amount in each asset is determined based on its risk level. Cai et. al. [4] have showed empirically that I_∞ model has a similar performance to the Markowitz's model and the I_∞ model is not sensitive to data. Moreover, Teo and Yang [31] have introduced an alternative minimax risk function in portfolio optimization. This risk function is defined as the average of maximum individual risks over a number of past time periods. The practical meaning of this risk function is to satisfy the objective of an investor to minimize the average of the maximum individual risks among assets to be invested. The corresponding portfolio optimization problem is formulated as a bi-criteria piecewise linear programming problem. More recently, Deng, Li and Wang [9] have proposed a minimax model on portfolio selection with uncertainty of randomness and estimation in inputs. Their minimax model is to maximize the worst possible expected rates of returns on portfolio. By using linear programming technique, an optimal portfolio has been derived analytically. While the work of Deng, Li and Wang applies to market without frictions, Wang, Yamamoto, Yu [32] and Chen, Li, Wu [6] have solved portfolio selection problem in frictional markets. Wang, Yamamoto, Yu [32] have based on the minimax principal proposed by Deng, Li and Wang [9] and solved the portfolio selection problem with tax and dividends associated with transactions, while Chen, Li, Wu [6] have studied portfolio selection problem with transaction costs under I_∞ risk measure. Other minimax portfolio selection models include solving immunization problems for bond portfolios [12] and deriving efficient decisions in portfolio models using game theory [26].

In addition to mean-variance and minimax type models, alternative portfolio selection models with different measures of risk have been proposed in the past fifty years [20, 18, 29, 8, 15, 17, 8, 17, 2, 5]. They include mean semivariance model [20, 18], mean absolute deviation model [29], mean variance skewness model [8, 15] and mean absolute deviation skewness model

[17]. With an argument that semivariance is more conceivable than variance as a measure of risk since only adverse deviations are concerned, Markowitz [20] has considered the maximum negative deviation from the mean as the portfolio risk. Chunhachinda et. al. [8] have incorporated skewness in mean variance portfolio selection problem and showed empirically that investors trade expected portfolio return for skewness. This makes incorporation of skewness in portfolio selection lead to a significant change in the construction of the optimal portfolio. Konno, Shirakawa and Yamazaki [17] have proposed the mean-absolute deviation skewness portfolio selection model. They have formulated the portfolio selection problem with utility of investors involving the third moment, namely the skewness, as a linear programming problem.

In addition to the studies of portfolio risk measures, efficient frontier analysis is another important area of concern in portfolio selection. Under the assumption that the covariance matrix is positive definite, Merton [21] has derived the efficient frontiers of the mean variance portfolio selection model analytically with the use of Lagrange multipliers for the case that borrowing and short selling of all securities are allowed. A few years later, Elton et. al. [11] have demonstrated a method to find the efficient frontier in both cases where short selling is allowed and disallowed by assuming the correlation coefficient between all assets is identical. Recently, Goh and Yang [13] have presented analytical methods to compute the exact efficient frontier with multi-criteria convex quadratic programming problem subject to linear constraints. The efficient frontier is found under the assumption that the covariance matrix is positive definite and short selling is not allowed. A more general approach is presented by Perold [25]. He has proposed an algorithm to locate the efficient frontier for large-scale mean-variance portfolio selection problem with a positive semi-definite covariance matrix. One of the drawbacks of using mean-variance models is that it involves a large-scale quadratic programming problem with a dense covariance matrix which is usually computationally intensive to handle. Other than mean-variance models, Konno and Yamazaki [16] have proposed a mean absolute optimization problem and suggested to convert their portfolio selection problem with the I_1 risk function into a scalar linear programming problem. The I_1 risk model allows linear program instead of quadratic program to be used, nevertheless, in order to obtain the efficient frontier of the problem, a large number of linear programming problems have to be solved. Cai et. al. [3] have derived an explicit analytical solution scheme to obtain the efficient frontier of the I_∞ optimization problem. With simple equations involved in the solution scheme, much time is saved for tracing out the efficient frontier.

Based on the basic framework of portfolio selection problem in early literature, some markets constraints have been considered and incorporated into portfolio selection models, one of these studies is the consideration of investment bounds. Sharpe [27] has followed Markowitz's approach to deal with mutual fund portfolio selection problem. The problem is formulated subject to investment constraints in the form of upper bounds on the proportion of the fund to be invested in a single security. Elton et.al. [10] have proposed a method to select portfolios when upper bound constraints are imposed on individual stocks under the case that the variance-covariance matrix follows a particular structure. They have solved the problem by presenting a linear programming approximation to the usual quadratic programming problem.

Other than modeling, Pang [23] has developed an algorithm for portfolio selection problems with upper and lower bounds on investment. With a special structure of the covariance matrix, his algorithm can be applied to the index models. Around the same time, Pang [24] proposed a parametric approach to solve a similar problem. Recently, Womersley and Lau [33] have studied the portfolio selection problem with upper and lower bounds on asset allocations with semi-variance and skewness models. A skewness model is nonlinear and non-convex, making it more difficult to solve and solutions are local rather than global optimal. Other than incorporating the upper and lower bound constraints on investment together, Best and Hlouskova [1] have considered the mean variance portfolio selection problem with upper and lower bounds on asset holdings separately. A closed form solution has been developed under a technical assumption. While many literatures consider investment bounds on individual asset, Chiodi [7] has studied the lower and upper bounds of the capital invested in a group of assets regarded as a fund. He has formulated the portfolio selection problem on mutual funds in a single investment period as a mixed integer linear programming model. Since the solution of large mixed integer linear programming problems require huge computational times, some heuristics have been proposed.

Chapter 3

Review of minimax portfolio selection model

In this chapter, we will review the definition of the minimax risk measure namely the I_∞ risk function and the minimax model proposed by Cai et. al. (2000, Management Science). The I_∞ risk function is defined as the maximum individual risk among all assets. The portfolio optimization problem with I_∞ risk function is formulated as a bi-criteria problem, then it is transformed into an equivalent bi-criteria linear programming problem and further transformed into a parametric optimization problem.

3.1 The I_∞ model

Assume an investor has initial wealth M_0, which is to be invested in n possible assets $S_j, j = 1, \ldots, n$. Let R_j be a random variable representing the return rate of asset S_j, and let $x_j \geq 0$ be the allocation from M_0 to S_j. By assuming

$x_j \geq 0$, we are concerned with the situation that each asset is not allowed to short sell. The feasible region for the portfolio optimization problem is

$$\mathfrak{F} = \left\{ x = (x_1, \ldots, x_n) : \sum_{j=1}^{n} x_j = M_0, x_j \geq 0, j = 1, \ldots, n \right\}$$

Let $E(R)$ denote the mathematical expectation of a random variable R. Define

$$r_j = E(R_j) \qquad \text{and} \qquad q_j = E(|R_j - r_j|)$$

The expected return of a portfolio $x = (x_1, \ldots x_n)$ is given by

$$r(x_1, \ldots x_n) = E\left[\sum_{j=1}^{n} R_j x_j\right] = \sum_{j=1}^{n} E(R_j) x_j = \sum_{j=1}^{n} r_j x_j$$

Definition 3.1.1 *The I_∞ risk function is defined as:*

$$w_\infty(\boldsymbol{x}) = \max_{1 \leq j \leq n} E(|R_j x_j - r_j x_j|) = \max_{1 \leq j \leq n} q_j x_j.$$

With an assumption that an investor wants to maximize the expected return and on the other hand minimize the risk level, this optimization problem is aimed at two criteria in conflict, namely, a higher return is always accompanied by higher risk level. For this reason, the portfolio optimization problem can be formulated as a bi-criteria piecewise linear program as follows, which is denoted as $\mathbf{POL}_\infty$ (the **P**ortfolio **O**ptimization problem with the I_∞ risk measure).

Definition 3.1.2 *The bi-criteria portfolio optimization problem $\boldsymbol{POL}_\infty$ under the I_∞ risk measure is defined as:*

$$\begin{array}{ll} \textit{Minimize} & \left(\max_{1 \leq j \leq n} q_j x_j, -\sum_{j=1}^{n} r_j x_j \right) \\ \textit{subject to} & \boldsymbol{x} \in \mathfrak{F} \end{array}$$

We can transform $\mathbf{POL}_\infty$ to an equivalent Bi-criteria Linear Programming (**BLP**) problem

$$\begin{array}{ll}\text{Minimize} & \left(y, -\sum_{j=1}^{n} r_j x_j\right) \\ \text{subject to} & q_j x_j \leq y \qquad j = 1, \ldots, n, \\ & \mathbf{x} \in \mathfrak{F}. \end{array}$$

Now we convert the bi-criteria linear programming problem **BLP** into a parametric optimization problem with a single criterion. For a fixed λ, where $0 < \lambda < 1$, the **P**arametric **O**ptimization problem of **BLP**, denoted as **PO**(λ), is as follows:

$$\begin{array}{ll}\text{Minimize} & F_\lambda(\mathbf{x}, y) = \lambda y + (1-\lambda)\left(-\sum_{j=1}^{n} r_j x_j\right) \\ \text{subject to} & q_j x_j \leq y, \qquad j = 1, \ldots, n, \\ & \mathbf{x} \in \mathfrak{F} \end{array}$$

The equivalence relation between **BLP** and **PO**(λ) is given below (cf. Yu [34] for proof).

Proposition 3.1.1 *Consider the problems **BLP** and **PO(λ)**. The pair $(\boldsymbol{x}, y)$ is an efficient solution of **BLP** if and only if there exists a $\lambda \in (0,1)$ such that $(\boldsymbol{x}, y)$ is an optimal solution of **PO(λ)**.*

Assume

$$r_1 \leq r_2 \leq \ldots \leq r_n$$

$$q_j > 0, \quad j = 1, \ldots, n$$

Theorem 3.1.1 *For any $\lambda \in (0,1)$, an optimal solution to $\boldsymbol{PO(\lambda)}$ is given by*

$$
\begin{aligned}
x_j^* &= \begin{cases} \frac{M_0}{q_j}\left(\sum_{l\in\mathfrak{I}^*(\lambda)}\frac{1}{q_l}\right)^{-1}, & j \in \mathfrak{I}^*(\lambda), \\ 0, & j \notin \mathfrak{I}^*(\lambda); \end{cases} \\
y^* &= M_0\left(\sum_{l\in\mathfrak{I}^*(\lambda)}\frac{1}{q_l}\right)^{-1},
\end{aligned}
$$

where $\mathfrak{I}^*(\lambda)$ is the set of assets to be invested, which is determined by the following rule:

(a) If there exists an integer $k \in [0,\ n-2]$ such that

$$
\begin{aligned}
&\frac{r_n - r_{n-1}}{q_n} < \frac{\lambda}{1-\lambda}, \\
&\frac{r_n - r_{n-2}}{q_n} + \frac{r_{n-1} - r_{n-2}}{q_{n-1}} < \frac{\lambda}{1-\lambda}, \\
&\cdots \\
&\frac{r_n - r_{n-k}}{q_n} + \frac{r_{n-1} - r_{n-k}}{q_{n-1}} + \cdots + \frac{r_{n-k+1} - r_{n-k}}{q_{n-k+1}} < \frac{\lambda}{1-\lambda},
\end{aligned}
$$

and

$$
\frac{r_n - r_{n-k-1}}{q_n} + \frac{r_{n-1} - r_{n-k-1}}{q_{n-1}} + \cdots + \frac{r_{n-k+1} - r_{n-k-1}}{q_{n-k+1}} + \frac{r_{n-k} - r_{n-k-1}}{q_{n-k}} \geq \frac{\lambda}{1-\lambda},
$$

then

$$
\mathfrak{I}^*(\lambda) = \{n, n-1, \ldots, n-k\}.
$$

(b) Otherwise, if the condition above is not satisfied by any integer $k \in [0,\ n-2]$, *then*

$$\mathfrak{I}^*(\lambda) = \{n, n-1, \ldots, 1\}.$$

Chapter 4

Portfolio optimization with group investment limits

In the previous chapter, we have reviewed the portfolio optimization problem with the I_∞ risk function where investment is made by consideration of each asset individually. In this chapter, we consider the case where assets are exclusively classified into groups and an investment limit is imposed on each group.

4.1 The model

Adopting the notations used in the previous chapter, assume an investor has initial wealth M_0 to be invested in n possible assets $S_j, j = 1, \ldots, n$. Let R_j be a random variable representing the return rate of asset S_j, $r_j = E(R_j)$, $q_j = E(|R_j - r_j|)$ and let $x_j \geq 0$ be the allocation from M_0 to S_j. In order to disallow short selling, we restrict $x_j \geq 0$. Moreover, assume there are T groups $G_i, i = 1, \ldots, T$, where $G_i \cap G_j = \phi$ when $i \neq j$. Let b_i be the investment limit of the group G_i. The portfolio selection problem with group investment bounds can be formulated as:

$$\begin{aligned} \text{Minimize} \quad & \left(\max_{1 \leq j \leq n} q_j x_j, -\sum_{j=1}^{n} r_j x_j \right) \\ \text{subject to} \quad & q_j x_j \leq y, \qquad j = 1, \ldots, n \\ & \sum_{j \in G_i} x_j \leq b_i, \qquad i = 1, \ldots, T \\ & \sum_{j=1}^{n} x_j = M_0 \\ & x_j \geq 0, \qquad j = 1, \ldots, n \end{aligned}$$

With the investment risk tolerance parameter λ, using similar argument for transformation from $\mathbf{POL}_\infty$ to $\mathbf{PO}(\lambda)$, the portfolio optimization problem with group investment limits, denoted as $\mathbf{PO_B}(\lambda)$, is as follows:

$$\begin{aligned} \text{Minimize} \quad & \mathrm{F}_\lambda(\mathbf{x}, y) = \lambda y + (1-\lambda)(-\sum_{j=1}^{n} r_j x_j) \\ \text{subject to} \quad & q_j x_j \leq y, \qquad j = 1, \ldots, n \\ & \sum_{j \in G_i} x_j \leq b_i, \qquad i = 1, \ldots, T \\ & \sum_{j=1}^{n} x_j = M_0 \\ & x_j \geq 0, \qquad j = 1, \ldots, n \end{aligned}$$

4.2 The optimal investment strategy

Analogous to chapter 3, consider the problem $\mathbf{PO_B}(\lambda)$ with a given $\lambda \in (0, 1)$. Note that the parameters $r_j = E(R_j)$ and $q_j = E(|R_j - r_j|), j = 1, 2, \ldots, n$, are constants, the value of which can be computed using historical data. Without loss of generality, we assume that

$$r_1 \leq r_2 \leq \ldots \leq r_n$$

Furthermore, we assume there do not exist two assets S_i and $S_j, i \neq j$, such that $r_i = r_j$ and $q_i = q_j$. Any two assets with the same r and q are regarded as a single aggregate asset.

4.2.1 All assets are risky

In this subsection, we consider the case where only risky assets are available. This implies that all assets S_j have $q_j > 0$.

Denote $B^*(\lambda)$ as the set of groups to be invested at the investment limit (i.e. $\sum_{j\in G_i} x_j = b_i$ for all $i \in B^*(\lambda)$), $\mathcal{K}^*(\lambda)$ as the set of assets with their groups in $B^*(\lambda)$, $\mathcal{L}^*(\lambda)$ as the set of assets to be invested such that $x_j q_j = y$, for $j \in \mathcal{L}^*(\lambda)$, $\mathcal{S}^*(\lambda)$ as the set of assets to be invested such that $x_j q_j < y$ and $\sum_{j\in G_i} x_j < b_i$, for $j \in \mathcal{S}^*(\lambda)$, $\mathcal{V}^*(\lambda)$ as the set of assets where $x_j = 0$, for $j \in \mathcal{V}^*(\lambda)$ and $\mathcal{Z}^*(\lambda)$ as the set of assets belonging to $\mathcal{K}^*(\lambda)$ but not $\mathcal{L}^*(\lambda)$ nor $\mathcal{V}^*(\lambda)$.

Theorem 4.2.1 *For a given $\lambda \in (0,1)$, if the assets can be divided into the sets $\mathcal{L}^*(\lambda), \mathcal{K}^*(\lambda), \mathcal{Z}^*(\lambda), \mathcal{S}^*(\lambda)$ and $\mathcal{V}^*(\lambda)$, and the following conditions are satisfied*

(i) For $i, j \in \mathcal{S}^*(\lambda)$,

$$r_i = r_j \tag{4.1}$$

(ii) For $i \in \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda) \setminus \mathcal{V}^*(\lambda), s \in \mathcal{S}^*(\lambda)$ *and* $j \in \mathcal{V}^*(\lambda) \setminus \mathcal{K}^*(\lambda)$,

$$r_i > r_s > r_j \tag{4.2}$$

(iii) For $s \in \mathcal{S}^*(\lambda), zi \in \mathcal{Z}^*(\lambda) \cap G_i$,

$$\sum_{l \in \mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)} \frac{r_l - r_s}{q_l} - \sum_{i \in \mathcal{B}} \left(\sum_{l \in \mathcal{L}^*(\lambda)\cap G_i} \frac{r_l - r_{zi}}{q_l} \right) = \frac{\lambda}{1-\lambda} \tag{4.3}$$

(iv) For $t, l \in \mathcal{Z}^*(\lambda)$ *and* $t, l \in G_i$,

$$r_t = r_l \tag{4.4}$$

(v) For $i, z, j \in G_i, i \in \mathcal{L}^*(\lambda), z \in \mathcal{Z}^*(\lambda), j \in \mathcal{V}^*(\lambda)$,

$$r_i > r_z > r_j \tag{4.5}$$

Then an optimal solution is given by

$$x_j^* = \begin{cases} \frac{y^*}{q_j}, & j \in \mathcal{L}^*(\lambda), \\ b_i - \left(\sum_{l \in G_i \cap \mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) y^*, & j \in \mathcal{Z}^*(\lambda), \\ 0, & j \in \mathcal{V}^*(\lambda), \end{cases} \tag{4.6}$$

$$\sum_{s \in \mathcal{S}*(\lambda)} x_s = M_0 - \sum_{i \in \mathcal{B}^*(\lambda)} b_i - \sum_{j \in \mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)} \frac{y^*}{q_j} \tag{4.7}$$

$$y^* = \left(M_0 - \sum_{i \in \mathcal{B}^*(\lambda)} b_i - \sum_{s \in \mathcal{S}^*(\lambda)} x_s \right) \left(\sum_{l \in \mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)} \frac{1}{q_l} \right)^{-1}, \tag{4.8}$$

where $\mathcal{V}^*(\lambda) = \{j \notin \mathcal{L}^*(\lambda) \cup Z^*(\lambda) \cup \mathcal{S}^*(\lambda)\}$

and $\mathcal{Z}^*(\lambda) = \{j \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda) \setminus \mathcal{V}^*(\lambda)\}$.

Proof. We apply the Kuhn-Tucker conditions to $\mathbf{PO_B}(\lambda)$. First, let us introduce the Lagrangian of $\mathbf{PO_B}(\lambda)$:

$$L(x,y,\mu,\lambda_0,\gamma,\phi) = \lambda y + (1-\lambda)\left(-\sum_{j=1}^{n} r_j x_j\right) + \sum_{j=1}^{n} \mu_j (q_j x_j - y)$$
$$+\lambda_0\left(\sum_{j=1}^{n} x_j - M_0\right) + \sum_{i=1}^{T}\gamma_i\left(\sum_{j\in G_i} x_j - b_i\right) - \sum_{j=1}^{n}\phi_j x_j \quad (4.9)$$

Then the K-T conditions that an optimal solution $(\mathbf{x}, y)$ must satisfy can be written as follows:

$$\frac{\partial L}{\partial y} = \lambda - \sum_{j=1}^{n}\mu_j = 0 \quad (4.10)$$

$$\frac{\partial L}{\partial x_j} = -(1-\lambda)r_j + \mu_j q_j + \lambda_0 + \gamma_i - \phi_j = 0, \qquad j = 1,\ldots,n \quad (4.11)$$

$$\sum_{j=1}^{n} x_j = M_0 \quad (4.12)$$

$$\mu_j(q_j x_j - y) = 0, \qquad j = 1,\ldots,n \quad (4.13)$$

$$\gamma_i\left(\sum_{j\in G_i} x_j - b_i\right) = 0, \qquad i = 1,\ldots,T \quad (4.14)$$

$$\phi_j x_j = 0, \qquad j = 1,\ldots,n \quad (4.15)$$

$$\mu_j \geq 0, \qquad j = 1,\ldots,n \quad (4.16)$$

$$\gamma_i \geq 0, \qquad i = 1,\ldots,T \quad (4.17)$$

$$\phi_j \geq 0, \qquad j = 1,\ldots,n \quad (4.18)$$

Define $\mathcal{L}^*(\lambda) = \{j : \mu_j > 0\}, \mathcal{B}^*(\lambda) = \{i : \gamma_i > 0\}, \mathcal{K}^*(\lambda) = \{j : j \in G_i, i \in \mathcal{B}^*(\lambda)\}$ and $\mathcal{Z}^*(\lambda) = \{j : \mu_j = 0, \gamma_i > 0\}$. Let $\mathcal{V}^*(\lambda) = \{j \notin \mathcal{L}^*(\lambda) \cup Z^*(\lambda) \cup \mathcal{S}^*(\lambda)\}$. It follows from (4.13) and (4.14) that

$$q_j x_j = y \qquad \text{if } \ j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$$

$$0 < x_j < \frac{y}{q_j} \qquad \text{if } \ j \in \mathcal{S}^*(\lambda) \cup \mathcal{Z}^*(\lambda)$$

$$x_j = 0 \qquad \text{if } \ j \in \mathcal{V}^*(\lambda)$$

(These are conjectures, but we shall show in the following that it is in fact correct in terms of satisfying the K-T conditions.) From (4.13) and (4.14), we have

$$x_j = \frac{y}{q_j}, \qquad \text{if } \ j \in \mathcal{L}^*(\lambda)$$

$$\sum_{j \in G_i} x_j = b_i, \qquad \text{if } \ j \in \mathcal{K}^*(\lambda)$$

$$x_j = b_i - \sum_{l \in \mathcal{L}^*(\lambda) \cap G_i} x_l, \qquad \text{if } \ j \in \mathcal{Z}^*(\lambda)$$

From (4.12), we have

$$\sum_{s \in \mathcal{S}*(\lambda)} x_s \;=\; M_0 - \sum_{i \in \mathcal{B}^*(\lambda)} b_i - \sum_{j \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)} \frac{y}{q_j}$$

Thus,

$$x_j^* \;=\; \begin{cases} \frac{y^*}{q_j}, & j \in \mathcal{L}^*(\lambda), \\ b_i - \left(\sum_{l \in G_i \cap \mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) y^*, & j \in \mathcal{Z}^*(\lambda), \\ 0, & j \notin \mathcal{L}^*(\lambda) \cup \mathcal{Z}^*(\lambda) \cup \mathcal{S}^*(\lambda), \end{cases}$$

From (4.13), it follows that if $x_j q_j \neq y$, then $\mu_j = 0$. Thus $\mu_j = 0,\ \forall j \notin \mathcal{L}^*(\lambda)$. From (4.14), it follows that if $\sum_{j \in G_i} x_j \neq b_i$, then $\gamma_i = 0$. Thus $\gamma_i = 0,\ \forall i \notin \mathcal{B}^*(\lambda)$. From (4.15), it follows that if $x_j > 0$, then $\phi_j = 0$. Thus $\phi_j = 0,\ \forall j \in \mathcal{L}^*(\lambda) \cup \mathcal{Z}^*(\lambda) \cup \mathcal{S}^*(\lambda)$. Let $s \in \mathcal{S}^*(\lambda)$, from (4.11), we have

$$\lambda_0 = (1-\lambda) r_s \tag{4.19}$$

$$\mu_j = \frac{1}{q_j}[(1-\lambda) r_j - \lambda_0] = \frac{1}{q_j}(1-\lambda)(r_j - r_s), \qquad \text{for } j \in L^*(\lambda) \setminus K^*(\lambda) \tag{4.20}$$

$$\mu_j = \frac{1}{q_j}[(1-\lambda) r_j - \lambda_0 - \gamma_i] = \frac{1}{q_j}[(1-\lambda)(r_j - r_s) - \gamma_i], \quad \text{for } j \in L^*(\lambda) \cap \mathcal{K}^*(\lambda) \tag{4.21}$$

$$\gamma_i = (1-\lambda) r_j - \lambda_0 = (1-\lambda)(r_j - r_s), \qquad \text{for } j \in \mathcal{Z}^*(\lambda) \tag{4.22}$$

$$\phi_j = \gamma_i + \lambda_0 - (1-\lambda) r_j = \gamma_i + (1-\lambda)(r_s - r_j), \qquad \text{for } j \in \mathcal{K}^*(\lambda) \cap \mathcal{V}^*(\lambda) \tag{4.23}$$

$$\phi_j = \lambda_0 - (1-\lambda) r_j = (1-\lambda)(r_s - r_j), \qquad \text{for } j \in \mathcal{V}^*(\lambda) \setminus \mathcal{K}^*(\lambda) \tag{4.24}$$

From (4.19), it is clear that condition (4.1) must hold. Further, by (4.2), μ_j, γ_i and ϕ_j are all non-negative except for $j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$ and $j \in \mathcal{K}^*(\lambda) \cap \mathcal{V}^*(\lambda)$. Consider γ_i, from (4.21), (4.22) and (4.23),

$$\gamma_i = (1-\lambda)(r_j - r_s) - \mu_j q_j, \quad \text{for } j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$$

$$\gamma_i = (1-\lambda)(r_j - r_s), \quad \text{for } j \in \mathcal{Z}^*(\lambda)$$

$$\gamma_i = (1-\lambda)(r_j - r_s) + \phi_j, \quad \text{for } j \in \mathcal{K}^*(\lambda) \cap \mathcal{V}^*(\lambda)$$

It is clear that γ_i is unique for each group G_i. Let $zi \in G_i \cap \mathcal{Z}$. Thus

$$\mu_j = \frac{1}{q_j}[(1-\lambda)(r_j - r_{zi})], \quad \text{for } j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$$

$$\phi_j = (1-\lambda)(r_{zi} - r_j), \quad \text{for } j \in \mathcal{K}^*(\lambda) \cap \mathcal{V}^*(\lambda)$$

By condition (4.5), μ_j and ϕ_j are non-negative for $j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$ and $j \in \mathcal{K}^*(\lambda) \cap \mathcal{V}^*(\lambda)$. From (4.20) and (4.21), and $\mu_j = 0$ for $j \notin \mathcal{L}^*(\lambda)$, we have

$$\begin{aligned}
\sum_{j=1}^{n} \mu_j &= \sum_{j \in \mathcal{L}^*(\lambda)} \mu_j \\
&= \sum_{j \in \mathcal{L}^*(\lambda)} \frac{(1-\lambda)(r_j - r_s)}{q_j} - \sum_{j \in \mathcal{L}^*(\lambda) \cap \mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j} \\
&= \sum_{j \in \mathcal{L}^*(\lambda)} \frac{(1-\lambda)(r_j - r_s)}{q_j} - \sum_{i \in \mathcal{B}^*(\lambda)} \gamma_i \left(\sum_{j \in G_i \cap \mathcal{L}^*(\lambda)} \frac{1}{q_j} \right) \\
&= (1-\lambda) \left[\sum_{j \in \mathcal{L}^*(\lambda)} \frac{r_j - r_s}{q_j} - \sum_{i \in \mathcal{B}^*(\lambda)} (r_{zi} - r_s) \left(\sum_{j \in G_i \cap \mathcal{L}^*(\lambda)} \frac{1}{q_j} \right) \right] \\
&= \lambda \qquad \text{by condition (4.3)}
\end{aligned}$$

Thus y and x_j given by (4.6), (4.7) and (4.8) satisfy all the K-T conditions (4.10)-(4.18). Because $\mathbf{PO_B}(\lambda)$ is a convex programming problem, the K-T conditions are necessary and sufficient for optimality. Therefore, the solution given by (4.6), (4.7) and (4.8) is optimal. This completes the proof. □

Denote $B^*(\lambda)$ as the set of groups to be invested at the investment limit (i.e. $\sum_{j \in G_i} x_j = b_i$ for all $i \in B^*(\lambda)$), $\mathcal{K}^*(\lambda)$ as the set of assets with their groups in $B^*(\lambda)$, $\mathcal{L}^*(\lambda)$ as the set of assets to be invested such that $x_j q_j = y$, for $j \in \mathcal{L}^*(\lambda)$, $\mathcal{S}^*(\lambda)$ as the set of assets to be invested such that $x_j q_j < y$ and $\sum_{j \in G_i} x_j < b_i$, for $j \in \mathcal{S}^*(\lambda)$.

Theorem 4.2.2 *For a given $\lambda \in (0,1)$, if the assets can be divided into the sets $\mathcal{L}^*(\lambda), \mathcal{K}^*(\lambda)$ and $\mathcal{S}^*(\lambda)$, and the following conditions are satisfied* *(i) For $i, j \in \mathcal{S}^*(\lambda)$,*

$$r_i = r_j \tag{4.25}$$

(ii) For $i \in \mathcal{L}^*(\lambda), s \in \mathcal{S}^*(\lambda)$ and $j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda)$,

$$r_i > r_s > r_j \tag{4.26}$$

(iii) For $s \in \mathcal{S}^*(\lambda)$,

$$\left(\sum_{l \in \mathcal{L}^*(\lambda)} \frac{r_l}{q_l}\right) - r_s \left(\sum_{l \in \mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) > \frac{\lambda}{1-\lambda} \tag{4.27}$$

$$\left(\sum_{l \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)} \frac{r_l}{q_l}\right) - r_s \left(\sum_{l \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)} \frac{1}{q_l}\right) \le \frac{\lambda}{1-\lambda} \tag{4.28}$$

(iv) For $i, j \in G_i, i \in \mathcal{L}^*(\lambda), j \notin \mathcal{L}^*(\lambda)$,

$$r_i > r_j \tag{4.29}$$

Then an optimal solution is given by

$$x_j^* = \begin{cases} \frac{y^*}{q_j}, & j \in \mathcal{L}^*(\lambda), \\ 0, & j \notin \mathcal{L}^*(\lambda) \cup \mathcal{S}^*(\lambda), \end{cases} \tag{4.30}$$

$$\sum_{s \in \mathcal{S}*(\lambda)} x_s = M_0 - \sum_{i \in \mathcal{B}^*(\lambda)} b_i - \sum_{j \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)} \frac{y^*}{q_j} \tag{4.31}$$

$$y^* = \left(M_0 - \sum_{i \in \mathcal{B}^*(\lambda)} b_i - \sum_{s \in \mathcal{S}^*(\lambda)} x_s\right) \left(\sum_{l \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \tag{4.32}$$

Proof. The Kuhn-Tucker conditions that an optimal solution $(\mathbf{x}, y)$ must satisfy are given by (4.10)-(4.18). Define $\mathcal{L}^*(\lambda) = \{j : \mu_j > 0\}, \mathcal{B}^*(\lambda) = \{i :$

$\gamma_i > 0\}$ and $\mathcal{K}^*(\lambda) = \{j : j \in G_i, i \in \mathcal{B}^*(\lambda)\}$. It follows from (4.13) and (4.14) that

$$q_j x_j = y \qquad \text{if } j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$$

$$0 < x_j < \frac{y}{q_j} \qquad \text{if } j \in \mathcal{S}^*(\lambda)$$

$$x_j = 0 \qquad \text{if } j \notin \mathcal{L}^*(\lambda) \cup \mathcal{S}^*(\lambda)$$

(These are conjectures, but we shall show in the following that it is in fact correct in terms of satisfying the K-T conditions.) From (4.13) and (4.14), we have

$$x_j = \frac{y}{q_j}, \qquad \text{if } j \in \mathcal{L}^*(\lambda)$$

$$\sum_{j \in G_i} x_j = b_i, \qquad \text{if } j \in \mathcal{K}^*(\lambda)$$

From (4.12), we have

$$\sum_{s \in \mathcal{S}*(\lambda)} x_s = M_0 - \sum_{i \in \mathcal{B}^*(\lambda)} b_i - \sum_{j \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)} \frac{y}{q_j}$$

Thus,

$$x_j^* = \begin{cases} \frac{y^*}{q_j}, & j \in \mathcal{L}^*(\lambda), \\ 0, & j \notin \mathcal{L}^*(\lambda) \cup \mathcal{S}^*(\lambda), \end{cases}$$

From (4.13), it follows that if $x_j q_j \neq y$, then $\mu_j = 0$. Thus $\mu_j = 0$, $\forall j \notin \mathcal{L}^*(\lambda)$. From (4.14), it follows that if $\sum_{j \in G_i} x_j \neq b_i$, then $\gamma_i = 0$. Thus

$\gamma_i = 0,\ \forall i \notin \mathcal{B}^*(\lambda)$. From (4.15), it follows that if $x_j > 0$, then $\phi_j = 0$. Thus $\phi_j = 0,\ \forall j \in \mathcal{L}^*(\lambda) \cup \mathcal{S}^*(\lambda)$. Let $s \in \mathcal{S}^*(\lambda)$, from (4.11), we have

$$\lambda_0 = (1-\lambda) r_s \tag{4.33}$$

$$\mu_j = \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0] = \frac{1}{q_j}(1-\lambda)(r_j - r_s), \qquad \text{for } j \in L^*(\lambda) \setminus K^*(\lambda) \tag{4.34}$$

$$\mu_j = \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0 - \gamma_i] = \frac{1}{q_j}[(1-\lambda)(r_j - r_s) - \gamma_i], \quad \text{for } j \in L^*(\lambda) \cap \mathcal{K}^*(\lambda) \tag{4.35}$$

$$\phi_j = \gamma_i + \lambda_0 - (1-\lambda)r_j = \gamma_i + (1-\lambda)(r_s - r_j), \qquad \text{for } j \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda) \tag{4.36}$$

$$\phi_j = \lambda_0 - (1-\lambda)r_j = (1-\lambda)(r_s - r_j), \qquad \text{for } j \notin \mathcal{L}^*(\lambda) \cup \mathcal{S}^*(\lambda) \tag{4.37}$$

From (4.33), it is clear that condition (4.25) must hold. Further, by (4.26), μ_j, γ_i and ϕ_j are all non-negative except for $j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$ and $j \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda)$. Consider γ_i, from (4.35) and (4.36),

$$\gamma_i = (1-\lambda)(r_j - r_s) - \mu_j q_j, \quad \text{for } j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$$

$$\gamma_i = (1-\lambda)(r_j - r_s) + \phi_j, \quad \text{for } j \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda)$$

From (4.10), (4.34) and (4.35),

$$\begin{aligned}
\lambda &= \sum_{j \in \mathcal{L}^*(\lambda)} \frac{(1-\lambda)(r_j - r_s)}{q_j} - \sum_{j \in L^*(\lambda) \cap \mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j} \\
&= \sum_{j \in \mathcal{L}^*(\lambda)} \frac{(1-\lambda)(r_j - r_s)}{q_j} - \sum_{i \in \mathcal{B}^*(\lambda)} \gamma_i \left(\sum_{j \in G_i \cap \mathcal{L}^*(\lambda)} \frac{1}{q_j} \right)
\end{aligned}$$

$$\sum_{i \in \mathcal{B}^*(\lambda)} \gamma_i \left(\sum_{j \in G_i \cap \mathcal{L}^*(\lambda)} \frac{1}{q_j} \right) = \sum_{j \in \mathcal{L}^*(\lambda)} \frac{(1-\lambda)(r_j - r_s)}{q_j} - \lambda$$

$$\sum_{i\in\mathcal{B}^*(\lambda)}\gamma_i\left(\sum_{j\in G_i\cap\mathcal{L}^*(\lambda)}\frac{1}{q_j}\right)=(1-\lambda)\left[\left(\sum_{l\in\mathcal{L}^*(\lambda)}\frac{r_l}{q_l}\right)-r_s\left(\sum_{l\in\mathcal{L}^*(\lambda)}\frac{1}{q_l}\right)\right]-\lambda$$

Let

$$\gamma_i=(1-\lambda)(r_j-r_s)+e_jq_j,\quad \text{for}\quad j\in\mathcal{K}^*(\lambda)\cap\mathcal{L}^*(\lambda) \tag{4.38}$$

For $l,t\in G_i$ and $l,t\in\mathcal{K}^*(\lambda)\cap\mathcal{L}^*(\lambda)$,

$$(1-\lambda)(r_l-r_s)+q_le_l=(1-\lambda)(r_t-r_s)+q_te_t$$

$$q_le_l-q_te_t=(1-\lambda)(r_t-r_l)$$

$$\sum_{j\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)}\frac{\gamma_i}{q_j}=(1-\lambda)\sum_{j\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)}\frac{(r_j-r_s)}{q_j}+\sum_{j\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)}e_j$$

$$\sum_{j\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)}e_j=(1-\lambda)\left(\sum_{j\in\mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)}\frac{r_j-r_s}{q_j}-\lambda\right)$$

From (4.38), if $r_t>r_l$, then $q_te_t<q_le_l$. Let pi be the least-return asset in G_i where $pi\in\mathcal{L}^*(\lambda)$ and $i\in\mathcal{B}^*(\lambda)$. Let $e_{pi}=-\epsilon$ where ϵ is a very small number and $\epsilon>0$. By (4.27), we can assign a negative value to all e_j, i.e.

$$e_j=-C_j\qquad j\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda) \tag{4.39}$$

where $C_j>0$. From (4.35), (4.38) and (4.39), for $j\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)$,

$$\begin{aligned}\mu_j&=\frac{1}{q_j}\left[(1-\lambda)(r_j-r_s)-(1-\lambda)(r_j-r_s)-q_je_q\right]\\&=-e_j>0\end{aligned}$$

By (4.27) and (4.28), we can assign a positive value for all γ_i where $i\in\mathcal{B}^*(\lambda)$, i.e.

$$\gamma_i>0\qquad\text{for}\quad i\in\mathcal{B}^*(\lambda)$$

Let

$$\gamma_i = (1-\lambda)(r_l - r_s) + q_j d_j, \quad \text{for} \quad l \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda)$$

Recall

$$\gamma_i = (1-\lambda)(r_j - r_s) + q_j e_j, \quad \text{for} \quad j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$$

For all $j, l \in G_i$, by (4.26)

$$\begin{aligned} q_j e_j - q_l d_l &= (1-\lambda)(r_l - r_j) < 0 \\ q_j e_j &< q_l d_l \end{aligned}$$

Recall $e_j < 0$, we can assign a non-negative value to all d_l, i.e.

$$d_l \geq 0 \qquad \text{for} \quad l \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda)$$

From (4.40), for $j \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda)$,

$$\begin{aligned} \phi_j &= (1-\lambda)(r_j - r_s) + q_j d_j - (1-\lambda)(r_j - r_s) \\ &= q_j d_j \geq 0 \end{aligned}$$

Therefore, $\mu_j \geq 0, \phi_j \geq 0$, $\forall j$ and $\gamma_i \geq 0$, $\forall i$. Thus y and x_j given by (4.30), (4.31) and (4.32) satisfy all the K-T conditions (4.10)-(4.18). Therefore, the solution given by (4.30), (4.31) and (4.32) is optimal. This completes the proof. □

Denote $B^*(\lambda)$ as the set of groups to be invested at the investment limit (i.e. $\sum_{j \in G_i} x_j = b_i$ for all $i \in B^*(\lambda)$), $\mathcal{K}^*(\lambda)$ as the set of assets with their groups in $B^*(\lambda)$, $\mathcal{L}^*(\lambda)$ as the set of assets to be invested such that $x_j q_j = y$, for $j \in \mathcal{L}^*(\lambda)$ and $\mathcal{Z}^*(\lambda)$ as the set of assets belong to $\mathcal{K}^*(\lambda)$ with $0 < x_j < \frac{y}{q_j}$.

Theorem 4.2.3 *For a given $\lambda \in (0,1)$, if the assets can be divided into the sets $\mathcal{L}^*(\lambda), \mathcal{K}^*(\lambda)$ and $\mathcal{Z}^*(\lambda)$, and the following conditions are satisfied*

(i) For $i \in \mathcal{L}^*(\lambda) \cup \mathcal{Z}^*(\lambda)$ and $j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda)$,

$$r_i > r_j \tag{4.40}$$

(ii) For $z \in \mathcal{Z}^*(\lambda)$,

$$\left(\sum_{l \in \mathcal{L}^*(\lambda)} \frac{r_l}{q_l}\right) - r_z \left(\sum_{l \in \mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) > \frac{\lambda}{1-\lambda} \tag{4.41}$$

(iii) For $j, l \in \mathcal{Z}^*(\lambda)$ and $j, l \in G_i$,

$$r_j = r_l \tag{4.42}$$

(iv) For $i, z, j \in G_i, i \in \mathcal{L}^*(\lambda), z \in \mathcal{Z}^*(\lambda), j \notin \mathcal{L}^*(\lambda) \cup \mathcal{Z}^*(\lambda)$,

$$r_i > r_z > r_j \tag{4.43}$$

Then an optimal solution is given by

$$x_j^* = \begin{cases} \frac{y^*}{q_j}, & j \in \mathcal{L}^*(\lambda), \\ b_i - \left(\sum_{l \in G_i \cap \mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) y^*, & j \in \mathcal{Z}^*(\lambda), \\ 0, & j \notin \mathcal{L}^*(\lambda) \cup \mathcal{Z}^*(\lambda), \end{cases} \tag{4.44}$$

$$y^* = \left(M_0 - \sum_{i \in \mathcal{B}^*(\lambda)} b_i\right) \left(\sum_{l \in \mathcal{L}^*(\lambda) \backslash \mathcal{K}^*(\lambda)} \frac{1}{q_l}\right)^{-1}, \tag{4.45}$$

Proof. The Kuhn-Tucker conditions that an optimal solution $(\mathbf{x}, y)$ must satisfy are given by (4.9)-(4.18). Define $\mathcal{L}^*(\lambda) = \{j : \mu_j > 0\}, \mathcal{B}^*(\lambda) = \{i :$

$\gamma_i > 0\}, \mathcal{K}^*(\lambda) = \{j : j \in G_i, i \in \mathcal{B}^*(\lambda)\}$ and $\mathcal{Z}^*(\lambda) = \{j : \mu_j = 0, \gamma_i > 0\}$. It follows from (4.13) and (4.14) that

$$q_j x_j = y \qquad \text{if } \ j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$$

$$0 < x_j < \frac{y}{q_j} \qquad \text{if } \ j \in \mathcal{Z}^*(\lambda)$$

$$x_j = 0 \qquad \text{if } \ j \notin \mathcal{L}^*(\lambda) \cup \mathcal{Z}^*(\lambda)$$

(These are conjectures, but we shall show in the following that it is in fact correct in terms of satisfying the K-T conditions.) From (4.13) and (4.14), we have

$$x_j = \frac{y}{q_j}, \qquad \text{if } \ j \in \mathcal{L}^*(\lambda)$$

$$\sum_{j \in G_i} x_j = b_i, \qquad \text{if } \ j \in \mathcal{K}^*(\lambda)$$

$$x_j = b_i - \sum_{l \in \mathcal{L} \cap G_i} x_l, \qquad \text{if } \ j \in \mathcal{Z}^*(\lambda)$$

Thus,

$$x_j^* = \begin{cases} \frac{y^*}{q_j}, & j \in \mathcal{L}^*(\lambda), \\ b_i - \left(\sum_{l \in G_i \cap \mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) y^*, & j \in \mathcal{Z}^*(\lambda), \\ 0, & j \notin \mathcal{L}^*(\lambda) \cup \mathcal{Z}^*(\lambda), \end{cases}$$

From (4.13), it follows that if $x_j q_j \neq y$, then $\mu_j = 0$. Thus $\mu_j = 0$, $\forall j \notin \mathcal{L}^*(\lambda)$. From (4.14), it follows that if $\sum_{j \in G_i} x_j \neq b_i$, then $\gamma_i = 0$. Thus

$\gamma_i = 0,\ \forall i \notin \mathcal{B}^*(\lambda)$. From (4.15), it follows that if $x_j > 0$, then $\phi_j = 0$. Thus $\phi_j = 0,\ \forall j \in \mathcal{L}^*(\lambda) \cup \mathcal{Z}^*(\lambda)$. From (4.11), we have

$$\mu_j = \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0], \qquad \text{for } j \in L^*(\lambda) \setminus K^*(\lambda) \tag{4.46}$$

$$\mu_j = \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0 - \gamma_i], \quad \text{for } j \in L^*(\lambda) \cap \mathcal{K}^*(\lambda) \tag{4.47}$$

$$\gamma_i = (1-\lambda)r_j - \lambda_0, \qquad \text{for } j \in \mathcal{Z}^*(\lambda) \tag{4.48}$$

$$\phi_j = \gamma_i + \lambda_0 - (1-\lambda)r_j, \qquad \text{for } j \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda) \setminus \mathcal{Z}^*(\lambda) \tag{4.49}$$

$$\phi_j = \lambda_0 - (1-\lambda)r_j, \qquad \text{for } j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda) \tag{4.50}$$

From (4.10), (4.46) and (4.47),

$$\lambda = \sum_{j \in \mathcal{L}^*(\lambda)} \frac{(1-\lambda)r_j - \lambda_0}{q_j} - \sum_{j \in L^*(\lambda) \cap \mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}$$

$$\sum_{j \in L^*(\lambda) \cap \mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j} = \sum_{j \in \mathcal{L}^*(\lambda)} \frac{(1-\lambda)r_j - \lambda_0}{q_j} - \lambda \tag{4.51}$$

$$\lambda_0 = \left(\sum_{l \in \mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left[(1-\lambda) \sum_{l \in \mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - \lambda - \sum_{j \in L^*(\lambda) \cap \mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right] \tag{4.52}$$

Let

$$(1-\lambda)r_{l0} \le \lambda_0 \le (1-\lambda)r_{z0} \tag{4.53}$$

where $r_{l0} = \max_{l \notin \mathcal{L}^*(\lambda) \cap \mathcal{K}^*(\lambda)}\{r_i\}$ and $r_{z0} = \min_{l \in \mathcal{Z}^*(\lambda)}\{r_l\}$. It is clear that $r_{z0} > r_{lo}$ by (4.40). From (4.46) and (4.53), for $j \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)$,

$$\begin{aligned} \mu_j &= \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0] \\ &\ge \frac{1}{q_j}[(1-\lambda)(r_j - r_{z0})] > 0 \end{aligned}$$

by (4.40). From (4.48) and (4.53), for $j \in \mathcal{Z}^*(\lambda)$,

$$\begin{aligned} \gamma_i &= (1-\lambda)r_j - \lambda_0 \\ &\geq (1-\lambda)(r_j - r_{z0}) > 0 \end{aligned}$$

by (4.40). Moreover, $\gamma_i \leq (1-\lambda)(r_j - r_{l0})$. Hence, we have

$$(1-\lambda)(r_j - r_{z0}) \leq \gamma_i \leq (1-\lambda)(r_j - r_{l0}) \tag{4.54}$$

From (4.47) and (4.53), for $j \in \mathcal{L}^*(\lambda) \cap \mathcal{K}^*(\lambda)$,

$$\begin{aligned} \mu_j &= \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0 - \gamma_i] \\ &= \frac{1}{q_j}[(1-\lambda)(r_j - r_{zi})] > 0 \end{aligned}$$

by (4.43), where $zi \in \mathcal{Z}^*(\lambda) \cap G_i$. From (4.49) and 4.53), for $j \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda) \setminus \mathcal{Z}^*(\lambda)$,

$$\begin{aligned} \phi_j &= \gamma_i - (1-\lambda)r_j + \lambda_0 \\ &= (1-\lambda)(r_{zi} - r_j) > 0 \end{aligned}$$

by (4.43). From (4.50) and (4.53), for $j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda)$,

$$\begin{aligned} \phi_j &= \lambda_0 - (1-\lambda)r_j \\ &\geq (1-\lambda)(r_j - r_{l0}) \geq 0 \end{aligned}$$

by (4.40). Therefore, if (4.54), then $\mu_j \geq 0$, $\forall j$ and $\gamma_i \geq 0$, $\forall i$. From (4.51),

$$\sum_{j \in \mathcal{L}^*(\lambda)} \frac{(1-\lambda)r_j - \lambda_0}{q_j} - \lambda > 0$$

$$\lambda_0 < \left(\sum_{l \in \mathcal{L}^*(\lambda)} \frac{1}{q_l} \right)^{-1} \left[(1-\lambda) \sum_{l \in \mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - \lambda \right] \tag{4.55}$$

By (4.41), we have

$$(1-\lambda)r_{z0} < \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left[(1-\lambda)\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - \lambda\right]$$

Thus, (4.55) is satisfied. Therefore, $\mu_j \geq 0, \phi_j \geq 0, \ \forall j$ and $\gamma_i \geq 0, \ \forall i$. Thus y and x_j given by (4.44) and (4.45) satisfy all the K-T conditions (4.10)-(4.18). Therefore, the solution given by (4.44) and (4.45) is optimal. This completes the proof. □

Denote $B^*(\lambda)$ as the set of groups to be invested at the investment limit (i.e. $\sum_{j\in G_i} x_j = b_i$ for all $i \in B^*(\lambda)$), $\mathcal{K}^*(\lambda)$ as the set of assets with their groups in $B^*(\lambda)$, $\mathcal{L}^*(\lambda)$ as the set of assets to be invested such that $x_j q_j = y$, for $j \in \mathcal{L}^*(\lambda)$.

Theorem 4.2.4 *For a given $\lambda \in (0,1)$, if the assets can be divided into the sets $\mathcal{L}^*(\lambda)$ and $\mathcal{K}^*(\lambda)$, and the following conditions are satisfied*

(i) For $i \in \mathcal{L}^*(\lambda)$ and $j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda)$,

$$r_i > r_j \tag{4.56}$$

(ii) For $j \in \mathcal{L}^*(\lambda)$ and $t \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda)$,

$$\sum_{l\in\mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)} \frac{r_l + r_{pi} - r_{lo}}{q_l} + \sum_{l\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{r_l - r_{pi} + r_{lo}}{q_l} - r_j \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) < \frac{\lambda}{1-\lambda} \tag{4.57}$$

$$\text{and} \quad \sum_{l\in\mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)} \frac{r_l}{q_l} + \sum_{l\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{r_l - r_{pi} + r_{lo}}{q_l} - r_t \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) \geq \frac{\lambda}{1-\lambda} \tag{4.58}$$

where lo is the greatest-return asset in G_i for $j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda)$ and pi is the greatest-return asset in Gi with $x_p = 0, p \in G_i$

(iii) For $i, z, j \in G_i, i \in \mathcal{L}^*(\lambda), j \notin \mathcal{L}^*(\lambda)$,

$$r_i > r_j \tag{4.59}$$

Then an optimal solution is given by

$$x_j^* = \begin{cases} \frac{y^*}{q_j}, & j \in \mathcal{L}^*(\lambda), \\ 0, & j \notin \mathcal{L}^*(\lambda), \end{cases} \tag{4.60}$$

$$y^* = M_0 \left(\sum_{l \in \mathcal{L}^*(\lambda)} \frac{1}{q_l} \right)^{-1}, \tag{4.61}$$

Proof. The Kuhn-Tucker conditions that an optimal solution $(\mathbf{x}, y)$ must satisfy are given by (4.9)-(4.18). Define $\mathcal{L}^*(\lambda) = \{j : \mu_j > 0\}$. It follows from (4.13) that

$$q_j x_j = y \qquad \text{if } j \in \mathcal{L}^*(\lambda)$$

Let $\sum_{j \in G_i} x_j = b_i$ for $i \in B^*(\lambda)$ and $j \in K^*(\lambda)$ and let $x_j = 0$ for $j \notin L^*(\lambda)$. (These are conjectures, but we shall show in the following that it is in fact correct in terms of satisfying the K-T conditions.) From (4.13), we have

$$x_j = \frac{y}{q_j}, \qquad \text{if } j \in \mathcal{L}^*(\lambda)$$

Thus,

$$x_j^* = \begin{cases} \frac{y^*}{q_j}, & j \in \mathcal{L}^*(\lambda), \\ 0, & j \notin \mathcal{L}^*(\lambda), \end{cases}$$

From (4.13), it follows that if $x_j q_j \neq y$, then $\mu_j = 0$. Thus $\mu_j = 0$, $\forall j \notin \mathcal{L}^*(\lambda)$. From (4.14), it follows that if $\sum_{j\in G_i} x_j \neq b_i$, then $\gamma_i = 0$. Thus $\gamma_i = 0$, $\forall i \notin \mathcal{B}^*(\lambda)$. From (4.15), it follows that if $x_j > 0$, then $\phi_j = 0$. Thus $\phi_j = 0$, $\forall j \in \mathcal{L}^*(\lambda)$. From (4.11), we have

$$\mu_j = \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0], \qquad \text{for} \quad j \in L^*(\lambda) \setminus K^*(\lambda) \tag{4.62}$$

$$\mu_j = \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0 - \gamma_i], \quad \text{for} \quad j \in L^*(\lambda) \cap \mathcal{K}^*(\lambda) \tag{4.63}$$

$$\gamma_i = (1-\lambda)r_j - \lambda_0 + \phi_j, \qquad \text{for} \quad j \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda) \tag{4.64}$$

$$\phi_j = \lambda_0 - (1-\lambda)r_j, \qquad \text{for} \quad j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda) \tag{4.65}$$

From (4.10), (4.62) and (4.63),

$$\lambda = \sum_{j\in\mathcal{L}^*(\lambda)} \frac{(1-\lambda)r_j - \lambda_0}{q_j} - \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}$$

$$\sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j} = \sum_{j\in\mathcal{L}^*(\lambda)} \frac{(1-\lambda)r_j - \lambda_0}{q_j} - \lambda$$

$$\lambda_0 = \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left[(1-\lambda)\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - \lambda - \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right] \tag{4.66}$$

From (4.62) and (4.66), for $j \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)$,

$$\begin{aligned}
\mu_j \quad &= \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0] \\
&= \frac{1}{q_j}\left\{(1-\lambda)r_j - \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left[(1-\lambda)\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - \lambda - \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right]\right\} \\
&= \frac{1}{q_j}\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left\{\lambda - (1-\lambda)\left[\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - r_j\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)\right]\right. \\
&\quad \left. + \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right\}
\end{aligned} \tag{4.67}$$

From (4.63) and (4.66), for $j \in \mathcal{L}^*(\lambda) \cap \mathcal{K}^*(\lambda)$,

$$\begin{aligned}
\mu_j &= \frac{1}{q_j}[(1-\lambda)r_j - \lambda_0 - \gamma_i] \\
&= \frac{1}{q_j}\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left\{\lambda - (1-\lambda)\left[\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - r_j \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)\right]\right. \\
&\quad \left. + \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j} - \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)\gamma_i\right\} \qquad (4.68)
\end{aligned}$$

$$\begin{aligned}
\gamma_i &= (1-\lambda)r_j - \lambda_0 - \mu_j q_j \\
&= (1-\lambda)r_j - \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1}\left[(1-\lambda)\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - \lambda - \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right] - \mu_j q_j \\
&= \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1}\left\{\lambda - (1-\lambda)\left[\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - r_j\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)\right]\right. \\
&\quad \left. + \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right\} - \mu_j q_j \qquad (4.69)
\end{aligned}$$

From (4.64) and (4.66), for $j \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda)$,

$$\begin{aligned}
\gamma_i &= (1-\lambda)r_j - \lambda_0 + \phi_j \\
&= (1-\lambda)r_j - \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1}\left[(1-\lambda)\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - \lambda - \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right] + \phi_j \\
&= \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1}\left\{\lambda - (1-\lambda)\left[\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - r_j\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)\right]\right. \\
&\quad \left. + \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right\} + \phi_j \qquad (4.70)
\end{aligned}$$

From (4.65) and (4.66), for $j \notin \mathcal{K}^*(\lambda) \cup \mathcal{L}^*(\lambda)$,

$$\begin{aligned}
\phi_j &= \lambda_0 - (1-\lambda)r_j \\
&= \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left[(1-\lambda)\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - \lambda - \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right] - (1-\lambda)r_j \\
&= \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left\{(1-\lambda)\left[\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l} - r_j\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)\right] - \lambda \right. \\
&\quad \left. - \sum_{j\in L^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{\gamma_i}{q_j}\right\}
\end{aligned} \tag{4.71}$$

From (4.69) and (4.70), for $j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$ and $l \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda)$, where $j, l \in G_i$,

$$(1-\lambda)r_j - \mu_j q_j = (1-\lambda)r_l + \phi_l$$

$$\mu_j q_j + \phi_l = (1-\lambda)(r_j - r_l) > 0$$

by (4.59). Therefore we can assign $\mu_j > 0$ for $j \in \mathcal{K}^*(\lambda) \cap \mathcal{L}^*(\lambda)$ and $\phi_l > 0$ for $l \in \mathcal{K}^*(\lambda) \setminus \mathcal{L}^*(\lambda)$ for any γ_i where $i \in \mathcal{B}^*(\lambda)$. Let

$$\gamma_i = (1-\lambda)(r_{pi} - r_{lo})^+$$

where lo be the greatest-return asset in G_i for $j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda)$ and pi is the greatest-return asset in G_i with $x_p = 0, p \in G_i$. It is clear that $r_{pi} > r_{lo}$, $\forall i$ by (4.56). From (4.66),

$$\lambda_0 = \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left[(1-\lambda)\sum_{l\in\mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)} \frac{r_l}{q_l} + (1-\lambda)\sum_{l\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{r_l - r_{pi} + r_{lo}}{q_l} - \lambda\right] \tag{4.72}$$

From (4.67) and (4.72), for $j \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)$,

$$\begin{aligned}
\mu_j &= \frac{1}{q_j}\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left\{\lambda - (1-\lambda)\left[\sum_{l\in\mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)} \frac{r_l}{q_l}\right.\right. \\
&\quad \left.\left. + \sum_{l\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{r_l - r_{pi} + r_{lo}}{q_l} - r_j\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)\right]\right\}
\end{aligned}$$

From (4.68) and (4.72), for $j \in \mathcal{L}^*(\lambda) \cap \mathcal{K}^*(\lambda)$,

$$\mu_j = \frac{1}{q_j}\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left\{\lambda - (1-\lambda)\left[\sum_{l\in\mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)} \frac{r_l + r_{pi} - r_{lo}}{q_l} + \sum_{l\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{r_l - r_{pi} + r_{lo}}{q_l} - r_j\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)\right]\right\}$$

From (4.71) and (4.72), for $j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda)$,

$$\phi_j = \left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left\{(1-\lambda)\left[\sum_{l\in\mathcal{L}^*(\lambda)\setminus\mathcal{K}^*(\lambda)} \frac{r_l}{q_l} + \sum_{l\in\mathcal{L}^*(\lambda)\cap\mathcal{K}^*(\lambda)} \frac{r_l - r_{pi} + r_{lo}}{q_l} - r_j\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right)\right] - \lambda\right\}$$

By (4.57) and (4.58), it is clear that $\mu_j > 0$ for $j \in \mathcal{L}^*(\lambda)$ and $\phi_j \geq 0$ for $j \notin \mathcal{L}^*(\lambda) \cup \mathcal{K}^*(\lambda)$. Therefore, $\mu_j \geq 0, \phi_j \geq 0,\ \forall j$ and $\gamma_i \geq 0,\ \forall i$. Thus y and x_j given by (4.60) and (4.61) satisfy all the K-T conditions (4.10)-(4.18). Therefore, the solution given by (4.60) and (4.61) is optimal. This completes the proof. □

Remark 4.2.1 *If $\mathcal{K}^*(\lambda) = \phi$, $r_{pi} \leq r_{lo}\ \forall i$, therefore $\gamma_i = 0\ \forall i$. (4.57) and (4.58) change to:*

$$\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l}\right) - r_j\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) < \frac{\lambda}{1-\lambda} \quad \text{for } j \in \mathcal{L}^*(\lambda)$$

$$\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{r_l}{q_l}\right) - r_j\left(\sum_{l\in\mathcal{L}^*(\lambda)} \frac{1}{q_l}\right) \geq \frac{\lambda}{1-\lambda} \quad \text{for } j \notin \mathcal{L}^*(\lambda)$$

These conform to the investment policy found by Cai et.al.[3]

4.2.2 Some riskfree assets are involved

We now consider the case where there are riskfree assets available for selection, i.e. there exists some assets such that $q_j = 0$. Without loss of generality, we may assume that there is only one riskfree asset under consideration. (All other riskfree assets whose return rates are lower than this one will be excluded by the optimal solution and therefore will not be considered.) This riskfree asset has the lowest return among all assets for investment, namely, $i = 1$. We have $q_1 = 0$ and $q_j > 0$ for $j \neq 1$. To generalize the result in section 4.2.1, we first assume that $q_1 = \epsilon > 0$, where ϵ is a sufficiently small number. We shall obtain our result by letting $\epsilon \to 0^+$. Let us consider the following two cases.

Case 1. The asset S_1 is not selected for investment.

In this case, it is obvious that the optimal solutions for $\mathbf{PO_B}(\lambda)$ as given in Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3 or Theorem 4.2.4 are unchanged.

Case 2. The asset S_1 is selected for investment.

Without loss of generality, let $S_1 \in G_1$. We can divide this case into two subcases.

(a) If $M_0 < b_1$,

In this case, $S_1 \notin \mathcal{K}^*(\lambda)$, according to Theorem 4.2.1,

$$y^* = \left(M_0 - \sum_{i \in \mathcal{B}^*(\lambda)} b_i - \sum_{s \in \mathcal{S}^*(\lambda)} x_s \right) \left(\frac{1}{\epsilon} + \sum_{l \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda), l \neq 1} \frac{1}{q_l} \right)^{-1}$$

Let $\epsilon \to 0^+$, we obtain $y^* = 0$, $x_1^* = M_0$, $x_j^* = 0$ for all $j > 1$. The results are similar for Theorem 4.2.2, Theorem 4.2.3 and Theorem 4.2.4.

(b) If $M_0 > b_1$,

In this case, $S_1 \in \mathcal{K}^*(\lambda)$, according to Theorem 4.2.1,

$$y^* = \left(M_0 - \sum_{i \in \mathcal{B}^*(\lambda)} b_i - \sum_{s \in \mathcal{S}^*(\lambda)} x_s \right) \left(\sum_{l \in \mathcal{L}^*(\lambda) \setminus \mathcal{K}^*(\lambda)} \frac{1}{q_l} \right)^{-1}$$

which is independent of q_1. In this case, it is obvious that the optimal solution for $\mathbf{PO_B}(\lambda)$ as given in Theorem 4.2.1 is unchanged. Similarly, the optimal solutions given in Theorem 4.2.2, Theorem 4.2.3 and Theorem 4.2.4 are unchanged.

4.3 Chapter summary

In this chapter, we considered the portfolio optimization problem under minimax risk measure with group investment limits and short selling being disallowed. By applying Kuhn-Tucker optimality conditions, we have solved the problem $\mathbf{PO_B}(\lambda)$ analytically. There exits four forms of the optimal solution of $\mathbf{PO_B}(\lambda)$, which are given by Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3 and Theorem 4.2.4. Fortunately, these four forms of optimal solutions have similar properties, which enable us to derive an algorithm to solve the problem $\mathbf{PO_B}(\lambda)$ completely. Before the end of this chapter, we have discussed the situation where a riskfree asset is available for investment. Solutions of which basically conform to the four theorems we established, except the case when the initial wealth is less than the investment limit of the group to which the riskfree asset belongs.

Chapter 5

Tracing out the efficient frontier

In chapter 4, we have derived the analytic optimal solutions for the parametric optimization problem $\mathbf{PO_B}(\lambda)$. This problem involves an investor's risk tolerance parameter λ. An optimal solution is generated with a given λ. The relationship between λ and x^* can be a many-to-one relationship. Recall that in the beginning of chapter 4, we defined the original objective function of our portfolio optimization problem as to minimize $\left(\max_{1\leq j\leq n} q_j x_j, -\sum_{j=1}^{n} r_j x_j\right)$, namely, maximizing the portfolio expected return and minimizing the I_∞ risk function. By denoting y to be $\max_{1\leq j\leq n} q_j x_j$, the problem is simply to maximize the expected return $\sum_{j=1}^{n} r_j x_j$ and minimize y. From the analytic solutions we have derived, x_j^* and y^* are independent of λ. If the sets $\mathcal{K}, \mathcal{L}, \mathcal{S}$ and $\mathcal{Z}$ are determined, the exact solution of the problem will be known. Therefore we need a suitable algorithm of which the general goal is to divide the assets into correct sets throughout the efficient frontier for obtaining the solution.

The efficient frontier represents the set of portfolios that will provide the highest return at each level of risk, or alternatively, the lowest risk for each level of return. Consider the efficient frontier of $\mathbf{PO_B}(\lambda)$, portfolio with larger y is accompanied by larger $\sum_{j=1}^{n} r_j x_j$. If y is small enough, the portfolio is diversified and no group reaches its investment limit. However, when y increases, investment in high-return assets also increases and some groups will attain their investment limits. Define a turning point to be a $(\sum_{j=1}^{n} r_j x_j, y)$ pair where the elements composing any sets of $\mathcal{K}, \mathcal{L}, \mathcal{S}$ and $\mathcal{Z}$ are changed. By continuously increasing y, we can get all turning points of the efficient frontier. At each turning point, the compositions of each set are determined. Furthermore, the entire efficient frontier is found by constructing a linear line between every two turning points. Before going into the details of the algorithm, we have derived the following lemmas which help us to understand the properties of the efficient frontier of $\mathbf{PO_B}(\lambda)$.

5.1 Properties of the efficient frontier

$\mathbf{PO_B}(\lambda)$ is different from $\mathbf{PO}(\lambda)$ by the additional constraints on group investment bounds. Therefore, when all of the bound constraints are ineffective, the problem $\mathbf{PO_B}(\lambda)$ will be reduced to $\mathbf{PO}(\lambda)$. For y to be sufficiently small, we are reasonable to believe that there exits a case where all bound constraints are ineffective. In the following, Lemma 5.1.1, Lemma 5.1.2 and Lemma 5.1.3 will discuss some properties of the efficient frontier when all of the bound constraints are ineffective. In this section, denote $\mathfrak{I} = \mathcal{L} \cup \mathcal{Z} \cup \mathcal{S}$.

Lemma 5.1.1 *Denote $\mathfrak{I}$ as the set of assets chosen for investment, if all of the bound constraints are ineffective, $|\mathfrak{I}|$ decreases if and only if the portfolio return and risk increase.*

Proof. Let $\mathfrak{I}_A = \{n, n-1, \ldots, t+1\}$ and $\mathfrak{I}_B = \{n, n-1, \ldots, t+1, t\}$. Let (r_A, y_A) and (r_B, y_B) be the return and risk of the portfolios with $\mathfrak{I}_A$ and $\mathfrak{I}_B$ respectively. Since $\sum_{j=1}^{n} x_j = M_0$, we can write $x_{Aj} = x_{Bj} + \delta_j$ where $\sum_{j=t+1}^{n} \delta_j = x_{Bt}$ and $\delta > 0$. Then, we have

$$
\begin{aligned}
r_A = \sum_{j=t+1}^{n} r_j x_{Aj} &= \sum_{j=t+1}^{n} (r_j x_{Bj} + r_j \delta_j) \\
&> \sum_{j=t+1}^{n} (r_j x_{Bj} + r_t \delta_j) \\
&= \sum_{j=t+1}^{n} r_j x_{Bj} + r_t \sum_{j=t+1}^{n} \delta_j \\
&= \sum_{j=t+1}^{n} r_j x_{Bj} + r_t x_{Bt} \\
&= r_B
\end{aligned}
$$

$$
\begin{aligned}
y_A = x_{Aj} q_j &= (x_{Bj} + \delta_j) q_j \\
&= x_{Bj} q_j + \delta_j q_j \\
&> x_{Bj} q_j \\
&= y_B
\end{aligned}
$$

This completes the proof. □

Lemma 5.1.2 *For $y < \min\left\{ b_i (\sum_{j \in G_i \cap \mathfrak{I}} \frac{1}{q_j})^{-1} \right\}$ $\forall i \notin \mathcal{B}$ with $\mathfrak{I} = \{n, n-1, \ldots, t\}$ as the set of assets chosen for investment. When y increases, only x_t, where t is the least-return asset in $\mathfrak{I}$, decreases, other x_j in $\mathfrak{I}$ increases.*

Proof. From Theorem 3.1.1, $y = M_0 \left(\sum_{j\in\mathfrak{J}} \frac{1}{q_j}\right)^{-1}$. With $\mathfrak{J}_A = \{n, n-1, \ldots, t+1\}$ and $\mathfrak{J}_B = \{n, n-1, \ldots, t+1, t\}$, we have $\left(\sum_{j\in\mathfrak{J}_A} \frac{1}{q_j}\right)^{-1} > \left(\sum_{j\in\mathfrak{J}_B} \frac{1}{q_j}\right)^{-1}$. Therefore when y increases, x_j for $j = t+1, \ldots, n$, increase while x_t decreases from $\frac{M_0}{q_t}\left(\sum_{j\in\mathfrak{J}} \frac{1}{q_j}\right)^{-1}$ to 0. This completes the proof. □

In the case where all of the bound constraints are ineffective, i.e. $y < \min\left\{b_i(\sum_{j\in G_i\cap\mathfrak{J}} \frac{1}{q_j})^{-1}\right\}$ $\forall i$, Lemma 5.1.1 illustrates that when y increases, the portfolio return increases while the size of the set of investment $\mathfrak{J}$ reduces. Lemma 5.1.2 shows more specifically that only the investment in the least-return asset decreases when y increases. From these two lemmas, we can deduce that when the investment in the least-return asset reduces to zero, that asset will be excluded from $\mathfrak{J}$, making the size of $\mathfrak{J}$ to become smaller. From Theorem 3.1.1, we know $y = M_0 \left(\sum_{j\in\mathfrak{J}} \frac{1}{q_j}\right)^{-1}$. If this y is less than $\min\left\{b_i(\sum_{j\in G_i\cap\mathfrak{J}} \frac{1}{q_j})^{-1}\right\}$ among all groups i, the case where all bound constraints are ineffective happens. Lemma 5.1.3 is proving it.

Lemma 5.1.3 *For* $y < \min\left\{b_i\left(\sum_{j\in G_i\cap\mathfrak{J}} \frac{1}{q_j}\right)^{-1}\right\}$ $\forall i$, *where* $y = M_0\left(\sum_{j\in\mathfrak{J}} \frac{1}{q_j}\right)^{-1}$ *from Theorem 3.1.1 with* $\mathfrak{J} = \{n, n-1, \ldots, t\}$ *being the set of assets chosen for investment, the efficient frontier is the same as that found by Theorem 3.1.1*

Proof. By Theorem 3.1.1, for $j \in \mathfrak{J}$, $x_j q_j = y$. If $\min_i\left\{b_i\left(\sum_{j\in G_i\cap\mathfrak{J}} \frac{1}{q_j}\right)^{-1}\right\} > y$, $b_i\left(\sum_{j\in G_i\cap\mathfrak{J}} \frac{1}{q_j}\right)^{-1} > y$ $\forall i$. It implies $b_i > y\left(\sum_{j\in G_i\cap\mathfrak{J}} \frac{1}{q_j}\right) = \sum_{j\in G_i\cap\mathfrak{J}} x_j = \sum_{j\in G_i} x_j$, since $x_j = 0$ for $j \notin \mathfrak{J}$. Therefore the first group to reach bound is G_i with $\min\left\{b_i\left(\sum_{j\in G_i\cap\mathfrak{J}} \frac{1}{q_j}\right)^{-1}\right\}$ among all i, and $y = \min\left\{b_i\left(\sum_{j\in G_i\cap\mathfrak{J}} \frac{1}{q_j}\right)^{-1}\right\}$ $\forall i$

at that moment. When $y < \min\left\{b_i\left(\sum_{j\in G_i\cap\mathfrak{I}}\frac{1}{q_j}\right)^{-1}\right\}$ $\forall i$, the bound constraints $\sum_{j\in G_i} x_j \leq b_i$ $\forall i$ are ineffective, thus the efficient frontier of the part with $y < \min\left\{b_i\left(\sum_{j\in G_i\cap\mathfrak{I}}\frac{1}{q_j}\right)^{-1}\right\}$ $\forall i$ is the same as that of the no-bound case. This completes the proof. □

When y increases to a certain value, namely $y \geq \min\left\{b_i\left(\sum_{j\in G_i\cap\mathfrak{I}}\frac{1}{q_j}\right)^{-1}\right\}$ among all groups i, some groups will reach their investment limits. With the effect of the bound constraints, further increase in y will result in a different response than the discussion in Lemma 5.1.2. In this case, investment in assets of the sets $\mathcal{S}$ and $\mathcal{Z}$ will reduce while allocation to other assets which have been chosen for investment will increase. The physical meaning of the assets in $\mathcal{S}$ and $\mathcal{Z}$ are the least-return asset in $\mathfrak{I}$ and the least-return asset chosen for investment for each group $G_i\cap\mathfrak{I}$ respectively.

Lemma 5.1.4 *When y increases, only x_s, where $s\in\mathcal{S}$, and x_z, where $z\in\mathcal{Z}$, decrease, x_j for $j\in G_i\cap(\mathcal{L}\cup\mathcal{Z})\setminus\{l_i\}, i\in\mathcal{B}$, l_i is the least-return asset in $G_i\cap(\mathcal{L}\cup\mathcal{Z})$ and x_j for $j\in\mathcal{L}\setminus\mathcal{K}$ would increase.*

Proof. Let l_i be the least-return asset in $G_i\cap(\mathcal{L}\cup\mathcal{Z})$. From Theorems 4.2.1-4.2.4, the optimal solutions satisfy the following system of equations:

$$\begin{cases} y = \left(M_0 - \sum_{i\in\mathcal{B}} b_i - \sum_{s\in\mathcal{S}} x_s\right)\left(\sum_{j\in\mathcal{L}\setminus\mathcal{K}}\frac{1}{q_j}\right)^{-1} \\ y = x_j q_j, \qquad j\in G_i\cap(\mathcal{L}\cup\mathcal{Z})\setminus\{l_i\}, i\in\mathcal{B} \\ \sum_{j\in G_i} x_j = b_i, \qquad i\in\mathcal{B} \\ x_s q_s < y, \qquad s\in\mathcal{S} \\ y = x_j q_j, \qquad j\in\mathcal{L}\setminus\mathcal{K} \end{cases}$$

From the system of equations, we have:

$$\Delta y = -\left(\sum_{j\in G_i\cap(\mathcal{L}\cup\mathcal{Z})\setminus\{l_i\}} \frac{1}{q_j}\right)^{-1} \Delta x_{li}, \quad l_i \in \mathcal{Z}, i \in \mathcal{B}$$

$$\Delta y = -\left(\sum_{j\in\mathcal{L}\setminus\mathcal{K}} \frac{1}{q_j}\right)^{-1} \Delta x_s, \quad s \in \mathcal{S}$$

$$\Delta y = q_j \Delta x_j, \qquad j \in G_i \cap (\mathcal{L}\cup\mathcal{Z}) \setminus \{l_i\}, i \in \mathcal{B}$$

$$\Delta y = q_j \Delta x_j, \qquad j \in \mathcal{L}\setminus\mathcal{K} \tag{5.1}$$

Thus, $\frac{\Delta x_{li}}{\Delta y} < 0$, $\frac{\Delta x_s}{\Delta y} < 0$ and $\frac{\Delta x_j}{\Delta y} > 0$ for $j \in G_i \cap (\mathcal{L}\cup\mathcal{Z}) \setminus \{l_i\}$ and $j \in \mathcal{L}\setminus\mathcal{K}$. This completes the proof. □

In the beginning of this chapter, we have mentioned that efficient frontier is composed of many turning points and a turning point is defined as a $(\sum_{j=1}^{n} r_j x_j, y)$ pair where the composition of any sets of $\mathcal{K}, \mathcal{L}, \mathcal{S}$ and $\mathcal{Z}$ is changed. The following lemma shows that only three cases are possible to give a turning point.

Lemma 5.1.5 *For each turning point, only one of the following cases would happen:*

(a) $x_s = 0$, *where* $s \in \mathcal{S}$

(b) $x_z = 0$, *where* $z \in \mathcal{Z}$

(c) $\sum_{j\in G_i} x_j = b_i$, *where* $i \notin \mathcal{B}$

Proof. In the proof of Lemma 5.1.4, we know $\Delta z < 0$ and $\Delta s < 0$ when $\Delta y > 0$. From (5.1), y increases with x_j for $j \in \mathcal{L} \setminus \mathcal{K}$. If $y = b_p \left(\sum_{j \in G_i \cap (\mathcal{L} \cup \mathcal{Z})} \frac{1}{q_j} \right)^{-1}$, where p is the group with $\min \left\{ b_i \left(\sum_{j \in G_i \cap (\mathcal{L} \cup \mathcal{Z})} \frac{1}{q_j} \right)^{-1} \right\}$, $\forall i \notin \mathcal{B}$, then $\sum_{j \in G_p} x_j = b_p$. This completes the proof. □

Remark 5.1.1 *In Lemma 5.1.5, case (c) happens when* $\Delta y = \min_{i \notin \mathcal{B}} \left\{ b_i \left(\sum_{j \in G_i \cap (\mathcal{L} \cup \mathcal{Z})} \frac{1}{q_j} \right)^{-1} \right\} - y_t$, *where* y_t *is the* y *of current portfolio.*

With the presence of constraints, the efficient frontier is a finite curve. The highest efficient point is the terminate point of the efficient frontier. It corresponds to a portfolio with greatest y and greatest expected return. After reaching the highest efficient point, we cannot find any portfolio with higher expected return when we increase y.

Lemma 5.1.6 *For all* $j > s$, *if* $j \in G_i$ *and* $i \in \mathcal{B}$ *(i.e.* $j \in \mathcal{K}$*), then the portfolio return cannot increase further with any increase in* y.

Proof. Denote r_p as the portfolio return.

$$\begin{aligned} \Delta r_p &= -r_s \Delta x_s + \sum_{j \notin S} r_j \Delta x_j \qquad \text{where} \quad \Delta x_s = \sum_{j \notin S} \Delta x_j \\ &= -\sum_{j \notin S} r_s \Delta x_j + \sum_{j \notin S} r_j \Delta x_j \\ &= \sum_{j \notin S} (-r_s + r_j) \Delta x_j \\ &= \sum_{j > s} (-r_s + r_j) \Delta x_j + \sum_{j < s} (-r_s + r_j) \Delta x_j \end{aligned}$$

If all $j > s \in \mathcal{K}$, then $\Delta x_j = 0$ for $j > s \in \mathcal{K}$. This leads to

$$\Delta r_p = \sum_{j<s}(-r_s + r_j)\Delta x_j < 0$$

Therefore the portfolio return cannot increase further, i.e. the portfolio with highest return is reached. This completes the proof. □

Lemma 5.1.7 *The linear combination of two successive efficient points are efficient.*

Proof. Let $(\mathbf{x}^1, y^1)$ and $(\mathbf{x}^2, y^2)$ be two successive efficient points on the efficient frontier. Suppose $(\mathbf{x}^0, y^0)$ is an optimal solution for $\mathbf{PO_B}(\lambda)$ with $y^0 = \alpha y^1 + (1-\alpha)y^2$ and $x^0 \neq \alpha x^1 + (1-\alpha)x^2$.

Case 1: $x^0 = \alpha x^1 + (1-\alpha)x^2 + \Delta x$

$$x^0 = \alpha x^1 + (1-\alpha)x^2 + \Delta x = M_0 + \Delta x$$

which is infeasible.

Case 2: $x^0 = \alpha x^1 + (1-\alpha)x^2 - \Delta x$

$$\begin{aligned}
F(\mathbf{x}^0, y^0) &= \alpha[\alpha y^1 + (1-\alpha)y^2] - (1-\alpha)\sum_{j=1}^{n} r_j[\alpha x^1 + (1-\alpha)x^2 - \Delta x] \\
&= \alpha F(\mathbf{x}^1, y^1) + (1-\alpha)F(\mathbf{x}^2, y^2) + \sum_{j=1}^{n} r_j \Delta x \\
&> \alpha F(\mathbf{x}^1, y^1) + (1-\alpha)F(\mathbf{x}^2, y^2) \\
&= F(\alpha x^1 + (1-\alpha)x^2, \alpha y^1 + (1-\alpha)y^2)
\end{aligned}$$

It contradicts the fact that the solution $(\mathbf{x}^0, y^0)$ is optimal. This completes the proof. □

Lemma 5.1.8 *In an optimal portfolio, $\forall i \in \mathcal{B}$, for $p, l, t \in G_i, p \in \mathcal{L}, l \in \mathcal{Z}$ and $t \notin \mathcal{L} \cup \mathcal{Z}$,*

$$r_p \geq r_l \geq r_t$$

Proof. Suppose $(\mathbf{x}^0, y^0)$ is an optimal portfolio with a group G_a, where $a \in \mathcal{B}$. Denote r_p as the portfolio return. For $p, l, t \in G_a, p \in \mathcal{L}, l \in \mathcal{Z}$ and $t \notin \mathcal{L} \cup \mathcal{Z}$,

Case 1: If $r_l > r_p$,

$$\begin{aligned}
r_p = \sum_{j=1}^{n} r_j x_j &= \sum_{j \notin G_a} r_j x_j + \sum_{j \in G_a} r_j x_j \\
&= \sum_{j \notin G_a} r_j x_j + r_p x_p + r_l x_l + r_t x_t \\
&= \sum_{j \notin G_a} r_j x_j + r_p \left(\frac{y}{q_p} \right) + r_l \left(b_i - \frac{y}{q_p} \right) \\
&= \sum_{j \notin G_a} r_j x_j + r_l b_i + (r_p - r_l) \left(\frac{y}{q_p} \right) \\
&< \sum_{j \notin G_a} r_j x_j + r_l b_i
\end{aligned}$$

Therefore with the same y and without changing x_j for $j \notin G_a$, we can construct a portfolio with higher return by investing $\frac{y}{q_p}$ more on asset l. It contradicts the fact that the solution $(\mathbf{x}^0, y^0)$ is optimal.

Case 2: If $r_t > r_l$,

$$\begin{aligned} r_p = \sum_{j=1}^{n} r_j x_j &= \sum_{j \notin G_a} r_j x_j + \sum_{j \in G_a} r_j x_j \\ &= \sum_{j \notin G_a} r_j x_j + r_p x_p + r_l x_l \\ &< \sum_{j \notin G_a} r_j x_j + r_p x_p + r_t x_l \end{aligned}$$

Similar to case 1, with the same y and without changing x_j for $j \notin G_a$, we can construct a portfolio with higher return by investing x_l on asset t. It contradicts the fact that the solution $(\mathbf{x}^0, y^0)$ is optimal. This completes the proof. □

From the above two lemmas, Lemma 5.1.7 and Lemma 5.1.8, we can obtain some insights for developing the algorithm to find the entire efficient frontier. Lemma 5.1.7 suggests that if all turning points are determined, the entire efficient frontier is found by constructing a linear line between every two successive efficient points. Moreover, Lemma 5.1.8 suggests that we should consider the assets with higher expected return to be invested first. Recall from Lemma 5.1.3, this lemma is established with the argument that investing in all assets would lead to the case where all bound constraints are ineffective, then the initial efficient point is ensured by Theorem 3.1.1. However it is not always true. Making investment in all assets are possible to result in some groups exceeding their investment limits. In this case, those groups are invested equal to their investment limits. The following lemma will discuss about it.

Lemma 5.1.9 *If $\exists t$ such that $b_t\left(\sum_{j\in G_t}\frac{1}{q_j}\right)^{-1} > M_0\left(\sum_{j\in\mathcal{L}}\frac{1}{q_j}\right)^{-1}$ where $\mathcal{L}=\{n,n-1,\ldots,1\}$, then $\forall j\in G_t, j\in\mathcal{K}$ in an optimal portfolio.*

Proof. Obviously, the solution form follows Theorem 4.2.3, thus it is optimal. Let the solution be $(\mathbf{x}^0, y^0)$. The following shows y^0 is the smallest y such that $\sum_{j\in G_t} x_j \le b_t\ \forall t$, where $y^0 = \left(M_0 - \sum_{i\in\mathcal{B}} b_i\right)\left(\sum_{l\in\mathcal{L}\setminus\mathcal{K}}\frac{1}{q_l}\right)^{-1}$, $\mathcal{B}=\left\{t : b_t\left(\sum_{j\in G_t}\frac{1}{q_j}\right)^{-1} > M_0\left(\sum_{j\in\mathcal{L}}\frac{1}{q_j}\right)^{-1}\right\}$. Suppose $(\mathbf{x}^1, y^1)$ is an optimal solution where $y^1 = y^0 - \Delta y$. Therefore

$$x_j^1 = \frac{y^0-\Delta y}{q_j} = x_j^0 - \Delta x_j \qquad \text{for } j\in\mathcal{L}$$

$$\sum_{j\in\mathcal{L}}\Delta x_j = \sum_{j\in\mathcal{L}\setminus\mathcal{K}}\Delta x_j + \sum_{j\in\mathcal{L}\cap\mathcal{K}}\Delta x_j > \sum_{j\in\mathcal{L}\cap\mathcal{K}}\Delta x_j$$

Therefore to maintain $\sum_{j\in G_i} x_j \le b_i\ \forall i$, there will be $\sum_{j=1}^{n} x_j < M_0$ which results in an infeasible solution. It contradicts the fact that the solution $(\mathbf{x}^1, y^1)$ is optimal. This completes the proof. □

5.2 The algorithm

The above Lemmas have disclosed some properties of the efficient frontier of $\mathbf{PO_B}(\lambda)$. In this subsection, we make use of these properties to derive an algorithm to trace out the efficient frontier entirely. The following describes the procedure of the algorithm.

Algorithm 5.2.1 *Algorithm for finding the efficient frontier of $\boldsymbol{PO_B}(\lambda)$.*

Step 1: *(a) Sort the assets $S_j, j = 1, \ldots, n$, in ascending rates of return, i.e.* $r_1 < r_2 < \cdots < r_n$.

(b) Define $\mathcal{K} = \{j : \sum_{j\in G_i} x_j = b_i\}, \mathcal{B} = \{i : \sum_{j\in G_i} x_j = b_i\}$

$$x_j = \begin{cases} \frac{y}{q_j}, & j \in \mathcal{L}, \\ b_i - \left(\sum_{l\in G_i\cap\mathcal{L}} \frac{1}{q_l}\right) y, & j \in \mathcal{Z}, \\ 0, & j \in \mathcal{V}, \end{cases}$$

$$\sum_{s\in S} x_s = M_0 - \sum_{i\in\mathcal{B}} b_i - \sum_{j\in\mathcal{L}\setminus\mathcal{K}} \frac{y}{q_j}$$

Note that $\mathcal{Z} = \{j : j \in \mathcal{K} \setminus \mathcal{L} \setminus \mathcal{V}\}$ *and* $\mathcal{V} = \{j : j \notin \mathcal{L} \cup \mathcal{Z} \cup \mathcal{S}\}$.

Set $\mathcal{K} = \phi, \mathcal{L} = \phi$ *and* $\mathcal{S} = \phi$. *Therefore* $\mathcal{Z} = \phi$ *and* $\mathcal{B} = \phi$.

Step 2: *(a) Set* $\mathcal{L} = \{n, n-1, \ldots, 1\}$. *By Theorem 3.1.1, get y and x_j.*

(b) Let $c = \min\{b_i(\sum_{j\in G_i\cap\mathcal{L}} \frac{1}{q_j})^{-1}\}\forall i \notin \mathcal{B}$. *Increase y along the efficient frontier found by Theorem 3.1.1. Find two successive turning points A and B where* $\sum_{j\in\mathfrak{I}_B} x_j \leq c$ *and* $\sum_{j\in\mathfrak{I}_A} x_j > c$.

(c) If point B does not exist, go to step 3; else if point A does not exist, then the efficient frontier is the same as that found by Theorem 3.1.1, stop; else, set $\mathcal{L} = \mathfrak{I}_A, \mathcal{S} = \mathfrak{I}_B \setminus \mathfrak{I}_A, \mathcal{V} = \{j : j \notin \mathfrak{I}_A \cup \mathfrak{I}_B\}$ *and* $s = \{j : j \in \mathcal{S}\}$.

Step 3: *(a) Define a set $\mathcal{K}$", where for any integer* $j \in \mathcal{L}, j \in G_t$ *with* $b_t(\sum_{j\in G_t} \frac{1}{q_j})^{-1} < y$, *then $j \in \mathcal{K}$".*

(b) If $\mathcal{K}$" is empty, do step 8c and go to step 4, else continue.

(c) *Set* $t \in \mathcal{B}$, *put* $\mathcal{K}' = \mathcal{K} \cup \mathcal{K}", \mathcal{L}' = \mathcal{L} \setminus \mathcal{K}". \mathcal{K} = \mathcal{K}'$ *and* $\mathcal{L} = \mathcal{L}'$,

$y = \left(M_0 - \sum_{i \in \mathcal{B}} b_i\right)\left(\sum_{l \in \mathcal{L}} \frac{1}{q_l}\right)^{-1}$,

(d) *Repeat step 3.*

Step 4: *For all* $j > s$, *if* $j \in G_i$ *and* $i \in \mathcal{B}$, *then stop, the portfolio with highest return and risk is given by step 9a; else continue.*

Step 5: *Increase* y *until one of the following cases happen:*

(a) $x_s = 0$, *go to step 6,*

(b) $x_z = 0$ *where* $z \in \mathcal{Z}$, *go to step 7,*

(c) $y = \min_{i \notin \mathcal{B}}\{b_i(\sum_{j \in G_i \cap \mathcal{L}} \frac{1}{q_j})^{-1}\}$, *go to step 8.*

Step 6: (a) *Put* $\mathcal{V}' = \mathcal{V} \cup \{s\}, \mathcal{V} = \mathcal{V}'$. *Let* $m = s - 1, s = m$. *If* $x_s = 0$, *repeat step 6a, else if* $s \in \mathcal{L}, \mathcal{L}' = \mathcal{L} \setminus \{s\}, \mathcal{L} = \mathcal{L}'$.

(b) *If* $s \in G_i$ *and* $i \in \mathcal{B}$, *then* $\mathcal{B}' = \mathcal{B} \setminus \{i\}, \mathcal{B} = \mathcal{B}', \mathcal{K}' = \mathcal{K} \setminus \{j : j \in G_i\}, \mathcal{K}' = \mathcal{K}, \mathcal{Z}' = \mathcal{Z} \setminus \{j : j \in G_i\}, \mathcal{Z}' = \mathcal{Z}$.

(c) *Go to step 9.*

Step 7: (a) *Set* $\mathcal{Z}' = \mathcal{Z} \setminus \{z\}$, *find* $t = j$ *where* $j, z \in G_i$ *and* $r_j - r_z > 0$ *is min. Then* $\mathcal{Z} = \mathcal{Z}' \cup \{t\}, \mathcal{L}' = \mathcal{L} \setminus \{t\}, \mathcal{L} = \mathcal{L}'$ *and* $\mathcal{V}' = \mathcal{V} \cup \{z\}, \mathcal{V} = \mathcal{V}'$.

(b) *Go to step 9.*

Step 8: (a) *Let* $c = \min_{i \notin \mathcal{B}}\{b_i(\sum_{j \in G_i \cap \mathcal{L}} \frac{1}{q_j})^{-1}\}$, *p is the group where* $b_p(\sum_{j \in G_p \cap \mathcal{L}} \frac{1}{q_j})^{-1} = c$. *Set* $y = c, \mathcal{K}' = \mathcal{K} \cup \{j : j \in G_p\}, \mathcal{K} = \mathcal{K}', \mathcal{B}' = \mathcal{B} \cup \{p\}, \mathcal{B} = \mathcal{B}', \mathcal{L}' = \mathcal{L} \setminus \{j : j \in G_p\}, \mathcal{L} = \mathcal{L}'$.

(b) *For all* $j \in G_p$, *set* $x_j = 0, \mathcal{V} = \mathcal{V} \cup \{j\}$ *and* $\mathcal{V} = \mathcal{V}'$

(c) *Repeat* $j \in G_p$, *let* $m = j$ *where* r_j *is maximum* $\forall j \in G_p$ *and* $x_j = 0$. *If* $b_p - \sum_{j \in G_p} x_j \geq \frac{y}{q_m}$, *then* $x_m = \frac{y}{q_m}, \mathcal{L}' = \mathcal{L} \cup \{m\}, \mathcal{L} = \mathcal{L}'$; *else if* $b_p - \sum_{j \in G_p} x_j > 0$, *then* $x_m = b_p - \sum_{j \in G_p} x_j, \mathcal{Z}' = \mathcal{Z} \cup \{m\}, \mathcal{Z} = \mathcal{Z}'$; *else* $x_m = 0, \mathcal{V}' = \mathcal{V} \cup \{m\}, \mathcal{V} = \mathcal{V}'$.

(d) *Go to step 9.*

Step 9: (a) *For a given* y,

$$x_j = \begin{cases} \frac{y}{q_j}, & j \in \mathcal{L}, \\ b_i - \left(\sum_{l \in G_i \cap \mathcal{L}} \frac{1}{q_l}\right) y, & j \in \mathcal{Z}, \\ 0, & j \in \mathcal{V}; \end{cases}$$

$$\sum_{s \in S} x_s = M_0 - \sum_{i \in \mathcal{B}} b_i - \sum_{j \in \mathcal{L} \setminus \mathcal{K}} \frac{y}{q_j}$$

(b) *Go to step 4.*

Theorem 5.2.1 *Algorithm 5.2.1 finds the entire efficient frontier.*

Proof. Prove by induction. By Lemma 5.1.1, the portfolio with smallest return and risk is given by investing in all assets. Therefore the initial point of the efficient frontier is given by including all assets in the set of investment,

$\mathfrak{I}$. The initial efficient point is proven by Theorem 3.1.1 if no group exceeds its investment limit, otherwise Lemma 5.1.9 proves it. Lemmas 5.1.2, 5.1.3, 5.1.4, 5.1.5 and 5.1.8 prove that the algorithm finds the next efficient point. By Lemma 5.1.7, the linear combination of two successive effective turning points is proved to be efficient. The stopping condition of the algorithm is proved by Theorem 5.1.6. Therefore, we conclude that the efficient frontier is found by the algorithm entirely. This completes the proof. $\square$

Algorithm 5.2.1 has outlined the optimal algorithm for finding the efficient frontier of $\mathbf{PO_B}(\lambda)$. Initially, the algorithm starts at a portfolio with the smallest return and risk. It denotes the lowest efficient point on the efficient frontier. In most cases, the lowest efficient point is a fully diversified portfolio by investing in all available assets. However, there exists a case that making investment in all assets results in some groups exceeding their investment limits. In order to remain feasible, those groups are invested up to their limits and the remaining wealth is distributed to other assets. When the portfolio risk, y, increases, the amount of investment in assets changes while the elements in the investment sets remain unchanged until a turning point is reached. A turning point represents a $(\sum_{j=1}^{n} r_j x_j, y)$ pair where the elements composing any sets of $\mathcal{K}, \mathcal{L}, \mathcal{S}$ and $\mathcal{Z}$ are changed. If all the bound constraints are ineffective, $\mathbf{PO_B}(\lambda)$ is reduced to $\mathbf{PO}(\lambda)$. Therefore the efficient frontier of $\mathbf{PO_B}(\lambda)$ and $\mathbf{PO}(\lambda)$ are the same. A turning point then refers to a situation that the elements in $\mathcal{L}$ are changed. If these exists at least one effective bound constraint, the efficient frontier of $\mathbf{PO_B}(\lambda)$ will be different from that of $\mathbf{PO}(\lambda)$. In this case, a turning point is reached if the elements in the sets $\mathcal{K}, \mathcal{S}$ or $\mathcal{Z}$ are changed. It will lead to changes in the composition of $\mathcal{L}$ as well. By continuously increasing y, all turning points can be obtained until the highest-return portfolio is reached. The highest-return portfolio represents a portfolio of which any increase in y cannot further increase the return. With the special nature of $\mathbf{PO_B}(\lambda)$, a linear combination of any successive efficient turning points are efficient. Therefore the efficient frontier is achieved by constructing a linear line between every two turning points.

5.3 Time complexity of the algorithm

We follow the assumptions in the previous sections that there are n assets and T groups. There are two situations of $\mathbf{PO_B}(\lambda)$: the bound constraints are ineffective and effective. If all of the bound constraints are ineffective, Algorithm 5.2.1 stops at step 2. The time complexity of the algorithm is $O(n)$. If the bound constraints are effective, we need to consider the complexity of each step. When finding the initial point of the efficient frontier, if investing in all assets does not violate the bound constraints, the complexity is $O(n)$, otherwise the algorithm goes through step 3. The complexity of step 3(a) is $O(nT)$ which is repeated for at most T times where $\sum_{i=1}^{T} b_i = M_0$ in this case. Therefore the complexity for finding the initial efficient point is $O(nT^2)$. According to Lemma 5.1.5, the complexity of finding the next efficient point by step 5 is $O(n)$. After that, the complexity of updating the compositions of the sets at an efficient point by step 6, step 7 and step 8 is also $O(n)$. Step 4 is for checking whether the stopping condition of the algorithm is reached, the complexity is $O(n)$. Next, the solution is calculated by step 9 with complexity $O(n)$. Step 4 to step 9 are repeated for at most $3n$ times. Therefore the complexity of the algorithm is $O(nT^2 + n^2) = O(n^2T)$, as $T <= n$.

5.4 Chapter summary

In this chapter, based on the special nature of linear programming of $\mathbf{PO_B}(\lambda)$, the portfolio optimization problem with investment bounds, some properties of the efficient frontier are revealed and an algorithm with polynomial time complexity is derived to solve $\mathbf{PO_B}(\lambda)$. The algorithm starts at the initial efficient point, which is the lowest point of the efficient frontier with smallest expected portfolio return and risk. By increasing the portfolio risk y, the entire efficient frontier is found until the highest-return portfolio is obtained. With the analytic solutions found in chapter 4, the investments made to each asset in the portfolio are known. Similar to the situation without bound constraints, the algorithm suggests that for solving $\mathbf{PO_B}(\lambda)$, assets with higher expected return should be considered first. Then the amount of investment on each asset is determined by the risk level of each asset.

Chapter 6

Finding the investor's optimal portfolio

In the previous section, we have found the efficient frontier of $\mathbf{PO_B}(\lambda)$. The efficient frontier is composed of infinite return-risk pairs, each corresponds to an optimal portfolio with a particular expected rate of return or risk. Therefore an optimal portfolio can be acquired easily if an expected return rate is given by an investor. However, recall in chapter 4, $\mathbf{PO_B}(\lambda)$ is formulated as a parametric optimization problem with λ as a parameter. In this section, we will discuss the case when the risk tolerance parameter λ is given by an investor.

6.1 Investor's portfolio with given λ

From Algorithm 5.2.1, we get the efficient frontier of the problem. The efficient frontier consists of many efficient turning points. Now denote the initial turning point as (R^1, y^1). Without loss of generality, assume there are p turning points on the efficient frontier. Therefore, (R^t, y^t) for $t = 1, 2, \ldots, p$.

Theorem 6.1.1 *With a given λ, the investor's optimal portfolio is given by the portfolio with* $\min\{\lambda y^t - (1-\lambda)R^t\}$.

Proof. Suppose $(\mathbf{x}^0, y^0)$ is an optimal portfolio with return $R^0 = \sum_{j=1}^n r_j x_j^0$, therefore $F(\mathbf{x}^0, y^0) = \lambda y^0 - (1-\lambda)R^0$.

Case1: Obviously, if $F(\mathbf{x}^0, y^0)$ is not minimum, there exists an efficient turning point $(\mathbf{x}^1, y^1)$ such that $F(\mathbf{x}^1, y^1) < F(\mathbf{x}^0, y^0)$. It contradicts the fact that $(\mathbf{x}^0, y^0)$ is optimal.

Case 2: If $(\mathbf{x}^0, y^0)$ is not a turning point on the efficient frontier. Therefore $(\mathbf{x}^0, y^0)$ is a linear combination of two efficient turning points $(\mathbf{x}^1, y^1)$ and $(\mathbf{x}^2, y^2)$. Without loss of generality, let $\mathbf{x}^1 < \mathbf{x}^2$, therefore $y^1 < y^2$. Thus, $\mathbf{x}^1 < \mathbf{x}^0 < \mathbf{x}^2$ and $y^1 < y^0 < y^2$. Let the line joining (R^1, y^1), (R^2, y^2) and (R^0, y^0) be $R = my + c$, where m and c are constants.

$$\begin{aligned}
F(\mathbf{x}^0, y^0) &= y^0 - (1-\lambda)R^0 \\
&= y^0 - (1-\lambda)(my^0 + c) \\
&= [\lambda - (1-\lambda)m]\, y^0 - (1-\lambda)c \\
&> [\lambda - (1-\lambda)m]\, y^1 - (1-\lambda)c \\
&= F(\mathbf{x}^1, y^1)
\end{aligned}$$

It contradicts the fact that $(\mathbf{x}^0, y^0)$ is optimal. This completes the proof. □

6.2 Chapter summary

In this chapter, we find the investor's optimal portfolio after the efficient frontier is obtained. With a given expected rate of return, the optimal portfolio of a particular investor can be achieved easily with reference to the efficient frontier. For investor with a given risk tolerance parameter λ, the optimal portfolio would be one of the turning points on the efficient frontier.

Chapter 7

Numerical experiments

In this chapter, two series of numerical experiments are provided to evaluate the performance of the I_∞ model with investment bounds. In the first section, an example is used to illustrate the characteristics of the efficient frontier of the problem we discussed in the previous chapter. Two comparisons of the efficient frontiers: in the case with and without bound constraints, and in the case with different bound constraints are conducted. In the second section, numerical testing is carried out to compare the performance of the I_∞ model with the classical mean variance (I_2) model in different situations.

7.1 Finding the efficient frontier numerically

To compare the entire efficient frontiers with and without bound constraints, we have implemented Algorithm 5.2.1 proposed in chapter 5 to find the efficient frontier of the case with investment bound constraints, while for the case without bound constraints, we have followed the method proposed by

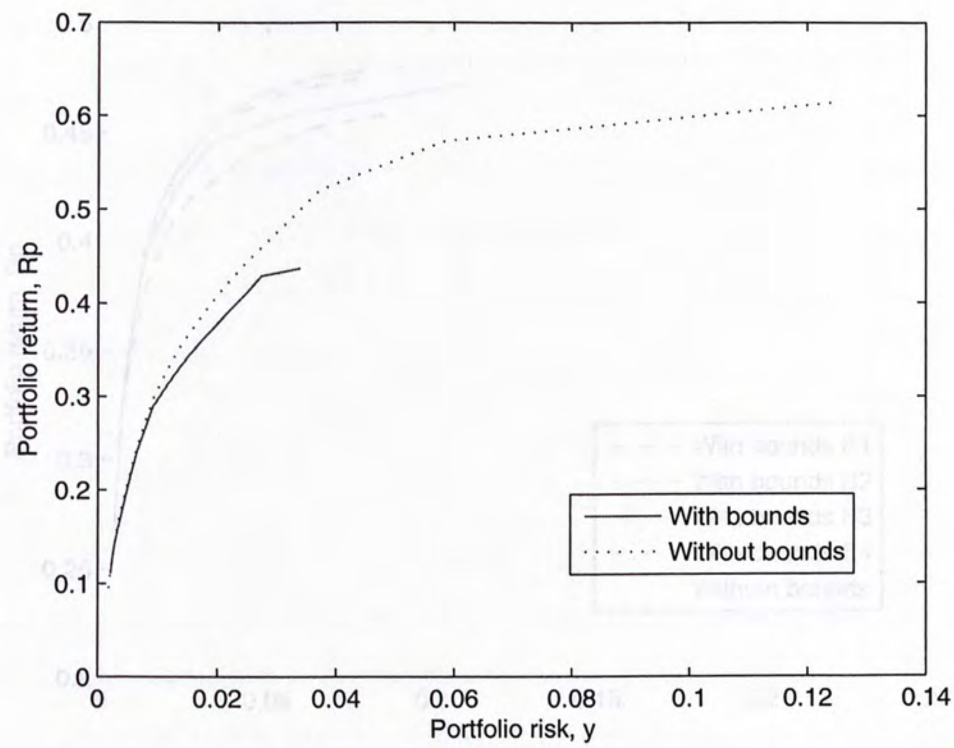

Figure 7.1: Efficient frontiers with and without bound constraints

Cai et. al. [3] to obtain the efficient frontier. In this experiment, assume $M_0 = 1$ and there are 35 stocks with 4 groups. The bounds of the groups are $0.7, 0.2, 0.25$ and 0.3 respectively. Table A.1 in Appendix A shows the expected rate of return (r_i) and the expected risk (q_i) of the assets. (The calculations of r_i and q_i will be provided in details in the next section.) In this example, there are 35 and 28 turning points for the cases without and with investment bound constraints respectively. Figure 7.1 shows the corresponding efficient frontiers.

Moreover, different values of the bound constraints are studied. To pro-

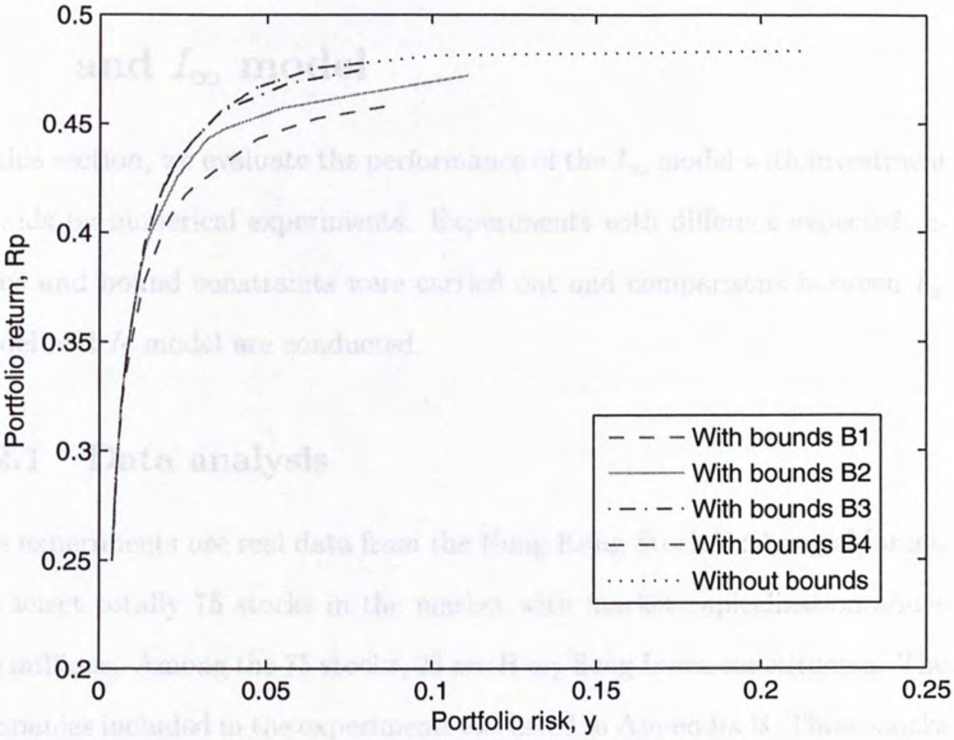

Figure 7.2: Efficient frontiers with different bound constraints

vide a better visualization of the results, 20 out of the 35 stocks are chosen for the experiment. The bounds are magnified by 1.5 times each for 4 times. For example, $B1 = [0.49; 0.7; 0.25; 0.21]$, $B2 = [0.735; 1.05; 0.375; 0.315]$, which is 1.5 times B1. In the same way, $B3 = [1.10; 1.58; 0.57; 0.48]$ and $B4 = [1.65; 2.37; 0.86; 0.72]$. Figure 7.2 shows the conforming result that when the bounds of the groups increase, the efficient frontier with bound constraints approaches that of without bound constraints.

7.2 Performance between mean-variance model and I_∞ model

In this section, we evaluate the performance of the I_∞ model with investment bounds by numerical experiments. Experiments with different expected returns and bound constraints were carried out and comparisons between I_∞ model and I_2 model are conducted.

7.2.1 Data analysis

The experiments use real data from the Hong Kong Stock Exchange Market. We select totally 75 stocks in the market with market capitalization above 900 millions. Among the 75 stocks, 23 are Hang Seng Index constituents. The companies included in the experiments are listed in Appendix B. These stocks are divided into 4 groups according to the business nature of the companies: Properties, Utilities, Commerce and Industry, and Finance. Moreover, three investment periods, namely, short term (1 week and 1 month) and intermediate term (6 months) are considered. The number of consecutive working days for one-week, one-month and five-month investment period are 5, 21 and 131 days respectively. In the experiment, historical data for the relevant stocks are used to estimate the parameters r_i and q_i. Specifically, the return rates of 100 trading days prior to the investment day (the day when investment is made) are used in the estimation. Let R_{ij} be the jth past return rate of stock i, and θ be the number of trading days in the investment period. Then

$$R_{ij} = \frac{p_{i,j} - p_{i,j+\theta}}{p_{i,j+\theta}}, \qquad j = 1, 2, \ldots, 100$$

where $p_{i,j}$ is the closing price of stock i on the jth trading day before the investment day. The expected return rate of stock i is calculated by

$$r_i = \frac{\sum_{j=1}^{100} R_{ij}}{100}$$

and the expected risk of stock i is calculated according to the formula

$$q_i = \frac{\sum_{j=1}^{100} |R_{ij} - r_{ij}|}{100}.$$

To obtain the mean variance portfolio, the covariances of the return rates of any two stocks are needed. The covariance, σ_{ij}, is calculated as follows:

$$\sigma_{ij} = \frac{1}{100}\sum_{k=1}^{100}(R_{ik} - r_i)(R_{jk} - r_j).$$

7.2.2 Experiment description and discussion

The experiments are carried out with the assumption that the amount of initial wealth $M_0 = 1$. The expected rate of return of the portfolio, p , is used as a parameter in the comparison between the I_∞ model and mean-variance (I_2) model. In any case, same expected rate of return of portfolio in I_∞ model and mean-variance model is used. The experiments are conducted with serval different values of p for different periods of investment (see Table 7.1). In

Investment period	Expected return of the portfolio, p
1 week	0.2% , 0.5%, 0.8%
1 month	1%, 3%, 5%
6 months	5%, 10%, 15%

Table 7.1: Expected return rates for different investment periods

our experiments, stocks are classified into 4 groups according to the business nature of the companies. For each group, the investment limit is chosen with reference to its market capitalization rate. For example, group-one stocks in total approximately contribute to 10% by capitalization of the total stocks we considered in our experiments, therefore 0.1 times a magnification factor is selected to be the investment limit of group 1. The investment limits for other groups are selected similarly. In our experiments, the capitalization rate for the 4 groups are 0.1, 0.1, 0.5 and 0.3 respectively. Specifically, our numerical experiments are conducted for different investment periods, different expected rates of return of the portfolio and different bound constraints. An actual return graph is constructed for each experiment to illustrate the outcome (see Appendix C). On the graph, the horizontal axis represents the day of investment, while the vertical axis represents the actual return rate of the corresponding portfolio. Moreover, same starting date is chosen for different investment periods. The graphical results are included in Appendix C. In most cases, the trends of the actual portfolio return using I_∞ model are similar regardless of the size of the bounds. Generally, the return of portfolio by the I_∞ model fluctuates more than the I_2 model. For one-week investment, it can be observed from the graphs that the actual returns of the portfolio obtained by the I_∞ model and the I_2 model are very close to each other, however, the I_∞ model results in more fluctuations. The portfolio of the I_∞ model is less sensitive to the investment bounds for smaller expected rate of return of the portfolio. Similar results are found for investment period of one month. For one-month investment, the performance of the I_∞ model and the I_2 model is comparable regardless of the size of the expected returns and the investment bounds. Both the I_∞ model and the I_2 model result in more fluctuations of the actual returns for large expected returns. For intermediate investment period of six months, the I_∞ model and the I_2 model performs similarly for small expected return regardless of the size of the investment bounds, their deviations become larger for larger expected return.

7.3 Chapter summary

In this chapter, two series of experiments using data collected from the Hong Kong Stock Exchange are conducted. In the first series of experiments, by adopting Algorithm 5.2.1 proposed in chapter 5, the efficient frontiers of the problem with and without bound constraints are found. A conforming result has revealed that the efficient frontier with bound constraints approaches that of without bound constraints when the bounds become larger. In the second series of experiments, the performance of the I_∞ model and classical mean-variance (I_2) model are studied. Generally, the trends of the I_∞ model and the I_2 model are close to each other regardless of the investment bounds. Moreover, the I_∞ model shows more fluctuations than the I_2 model, especially for short investment period.

Chapter 8

Conclusion

In this thesis, we explored and solved the portfolio optimization problem under minimax risk measure model with group investment bound constraints and short selling is disallowed. In the market, assets are classified into sectors according to the types of securities to which they belong and the nature of their companies. Investment limits on sectors of assets are imposed both externally by regulations and internally by investors. In this thesis, We employed the I_∞ risk model proposed by Cai et. al [3] in our problem. The ultimate objective of our portfolio optimization problem is to maximize the expected return of the portfolio while minimize the portfolio risk defined by the I_∞ risk function.

Adopting Cai et. al approach, the portfolio optimization problem is formulated as a bi-criteria problem and converted into an equivalent parameterized problem with an investor's risk tolerance parameter. By applying Kuhn-Tucker optimality conditions, we have solved the problem analytically. The solution exists in four different forms with similar properties. Optimality of the solutions are ensured by the KKT conditions as the problem is convex programming.

With similar properties of the optimal solutions and the special nature of linear programming, we have revealed some properties of the efficient frontier of the portfolio optimization problem. The properties enable us to derive an algorithm with polynomial time complexity to solve the problem completely. The algorithm starts at the initial efficient point, which is a portfolio with smallest expected return and risk. Then by increasing the portfolio risk, all turning points, which are defined as return-risk pairs where the assets in the investment sets are changed, are obtained The algorithm stops and the entire efficient frontier is found when the highest-return portfolio is reached. Similar to the situation without bound constraints, for the portfolio optimization problem with investment limits, the algorithm suggests that assets with higher expected return should be considered first. Then the amount of investment on each asset is determined by the risk level of the asset. After the efficient frontier is obtained, an investor's optimal portfolio with particular expected return is achieved easily. For investor with a given risk tolerance parameter, the optimal portfolio is restricted to one of the efficient turning points on the efficient frontier.

In the last section of this thesis, two series of numerical experiments were carried out. One of the series is to demonstrate the efficient frontiers of the problem with and without investment bounds. A conforming result has illustrated that the efficient frontier of the problem with bounds approach to that of without bounds when the bounds increase. Another series of experiments was to evaluate the performance of the I_∞ model and compare it with that of the classical mean-variance I_2 model. Various scenarios have been tested with different expected returns of the portfolio and different investment bounds in short and intermediate terms of investment. In general, the trends of the actual return of the portfolio found by the I_∞ model and I_2 model are similar regardless of the investment bounds. Moreover, the I_∞ model shows more fluctuations than the I_2 model.

The portfolio optimization problem under I_∞ model with investment limits first determines which assets should be invested, then the amount to be invested is decided according to the risk level of each asset. No correlations among assets are involved in the process of finding the optimal solutions. Without directly calculating the correlations of the covariance, less time is required to trace out the efficient frontier. The I_∞ model can be further studied to apply to situations with more constraints. Possible extensions include consideration that some assets are subjects to lower investment bounds, transaction costs are taken into account and generalization of the I_∞ model to the multi-period case.

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Appendix A

Stocks for finding the efficient frontiers with and without bound constraints

Stock	Group	Expected Return r_i	Expected Risk q_i
1	1	0.0567	0.0461
2	1	0.0377	0.0556
3	1	0.0359	0.0458
4	1	0.1991	0.0969
5	1	0.1734	0.0809
6	2	0.1149	0.0478
7	2	-0.0178	0.0251
8	2	0.0255	0.0322
9	3	-0.0693	0.0623
10	3	-0.0584	0.0496
11	3	-0.0115	0.0467
12	3	0.0964	0.1087
13	3	0.0821	0.0339
14	3	0.0482	0.0789
15	3	0.0838	0.1320
16	3	0.0237	0.0811
17	3	0.1579	0.1686
18	3	0.2235	0.1298
19	3	0.1843	0.1053
20	3	0.0170	0.1988
21	3	0.2485	0.1433
22	3	0.0348	0.0732
23	3	0.2320	0.1028
24	3	0.0830	0.0689
25	3	0.1107	0.1557
26	3	0.4294	0.1019
27	3	-0.0024	0.0466
28	3	0.0283	0.0893
29	3	0.5359	0.1108
30	4	0.0752	0.0171
31	4	0.0087	0.0439
32	4	0.2882	0.0719
33	4	0.2724	0.1529
34	4	0.1377	0.0752
35	4	0.6141	0.1239

Table A.1: Expected return and expected risk of the stocks for finding the efficient frontiers of the problem with and without bound constraints

Appendix B

List of companies

	Code	Name	Sector
1	0001.HK	Cheung Kong (Holdings) Ltd.	Properties
2	0002.HK	CLP Holdings Ltd.	Utilities
3	0003.HK	Hong Kong and China Gas Co. Ltd.	Utilities
4	0004.HK	Wharf (Holdings) Ltd.	Commerce & Industry
5	0005.HK	HSBC Holdings Ltd.	Finance
6	0006.HK	Hong kong Electric Holdings Ltd.	Utilities
7	0008.HK	PCCW Ltd.	Commerce & Industry
8	0011.HK	Hang Seng Bank Ltd.	Finance
9	0012.HK	Henderson Land Development Co. Ltd.	Properties
10	0013.HK	Hutchison Whampoa Ltd.	Commerce & Industry
11	0014.HK	Hysan Development Co. Ltd.	Properties
12	0016.HK	Sun Hung Kai Properties Ltd.	Properties
13	0017.HK	New World Development Co. Ltd.	Commerce & Industry
14	0019.HK	Swire Pacific Ltd. 'A'	Commerce & Industry
15	0020.HK	Weelock and Co Ltd	Commerce & Industry
16	0023.HK	Bank of East Asia Ltd	Finance
17	0041.HK	Great Eagle Holdings Ltd.	Properties
18	0044.HK	Hong Kong Aircraft Engineering Co Ltd	Commerce & Industry
19	0049.HK	Wheelock Properties Ltd.	Properties
20	0052.HK	Fairwood Holdings Ltd.	Commerce & Industry
21	0057.HK	Chen Hsong Holdings Ltd.	Commerce & Industry
22	0066.HK	MTE Corporation Ltd.	Commerce & Industry8
23	0069.HK	Shangri-La Asia Ltd	Commerce & Industry
24	0083.HK	Sino Land Co. Ltd.	Properties
25	0086.HK	Sun Hung Kai & Co. Ltd	Properties
26	0096.HK	Wing Lung Bank Ltd	Finance

Table B.1: Lists of companies included in the numerical experiments

	Code	Name	Sector
27	0097.HK	Henderson Investment Ltd	Commerce & Industry
28	0101.HK	Hang Lung Properties Ltd.	Properties
29	0101.HK	Hang Lung Properties Ltd.	Properties
30	0116.HK	Chow Sang Sang Holdings Ynternational Ltd.	Commerce & Industry
31	0123.HK	Guangzhou Investment Co. Ltd.	Finance
32	0144.HK	China Merchants Holdings (International) Co. Lrd.	Commerce & Industry
33	0145.HK	Hong Kong Building and Loan Agency Ltd.	Finance
34	0165.HK	China Everbright Ltd	Commerce & Industry
35	0173.HK	K. Wah International Holdings Ltd.	Commerce & Industry
36	0203.HK	Denway Motors Ltd	Commerce & Industry
37	0210.HK	Prime Success International Group Ltd.	Commerce & Industry
38	0227.HK	First Shanghai Investments Ltd.	Commerce & Industry
39	0242.HK	Shun Tak Holdings Ltd	Commerce & Industry
40	0247.HK	Tsim Sha Tsui Properties Ltd.	Properties
41	0267.HK	CITIC Pacific Ltd.	Commerce & Industry
42	0291.HK	China Resources Enterprise Ltd.	Commerce & Industry
43	0293.HK	Cathay Pacific Airways Ltd	Commerce & Industry
44	0302.HK	Wing Hang Bank Ltd	Finance
45	0303.HK	Vtech Holdings Ltd.	Commerce & Industry
46	0308.HK	China Travel International Investment Hong Kong Ltd	Finance
47	0322.HK	Tingyi (Cayman Islands) Holdings Corp.	Commerce & Industry
48	0338.HK	Sinopec Shanghai Petrochemical Co. Ltd. - H Shares	Commerce & Industry
49	0341.HK	Caf de Carol Holdings Ltd.	Commerce & Industry
50	0347.HK	Angang Steel Co Ltd. - H Shares	Utilities
51	0386.HK	China Petroleum & Chemical Corporation -H Shares	Commerce & Industry
52	0388.HK	Hong Kong Exchanges and Cleatinh Ltd	Finance

Table B.2: Lists of companies included in the numerical experiments

	Code	Name	Sector
53	0440.HK	Dah Sing Financial Holdings Ltd	Finance
54	0480.HK	HKR International	Commerce & Industry
55	0511.HK	Television Broadcasts Ltd.	Commerce & Industry
56	0662.HK	Asia Financial Holdings Ltd.	Finance
57	0683.HK	Kerry Properties Ltd	Commerce & Industry
58	0754.HK	Hopson Development Holdings Ltd	Commerce & Industry
59	0762.HK	China Unicom Ltd.	Commerce & Industry
60	0857.HK	PetroChina Co. Ltd. - H Shares	Commerce & Industry
61	0903.HK	TPV Technology Ltd.	Commerce & Industry
62	0914.HK	Anhui Conch Cement Co. Ltd. -H Shares	Commerce & Industry
63	0917.HK	New World China Land Ltd.	Properties
64	0941.HK	China Mobile Ltd.	Commerce & Industry
65	0983.HK	Shui On Construction and Materials Ltd.	Commerce & Industry
66	0991.HK	Datang International Power Generation Co. Ltd. - H Shares	Utilities
67	0992.HK	Lenovo Group Ltd	Commerce & Industry
68	1038.HK	Cheung Kong Infrastructure Holdings Ltd.	Commerce & Industry
69	1044.HK	Hengan International Group Co Ltd	Commerce & Industry
70	1098.HK	Road King Infrastructure	Properties
71	1111.HK	Chong Hing Bank Ltd.	Finance
72	1114.HK	Brilliance China Automative Holdings Ltd.	Commerce & Industry
73	1136.HK	TCC International oldings Ltd.	Commerce & Industry
74	1171.HK	Tanzhou Coal Mining Co Ltd. - H Shares	Utilities
75	1199.HK	COSCO Pacific Ltd	Commerce & Industry

Table B.3: Lists of companies included in the numerical experiments

Appendix C

Graphical Results

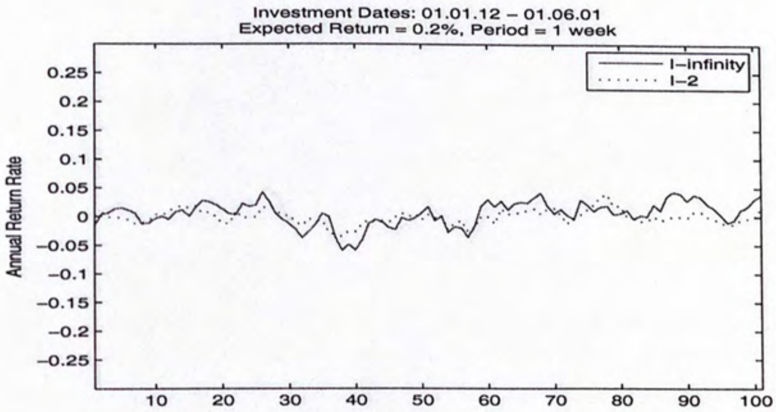

Figure C.1: Actual return graph with investment bounds equal to 1.5 times the group capitalization rates for 1-week investment with expected return equal to 0.2%

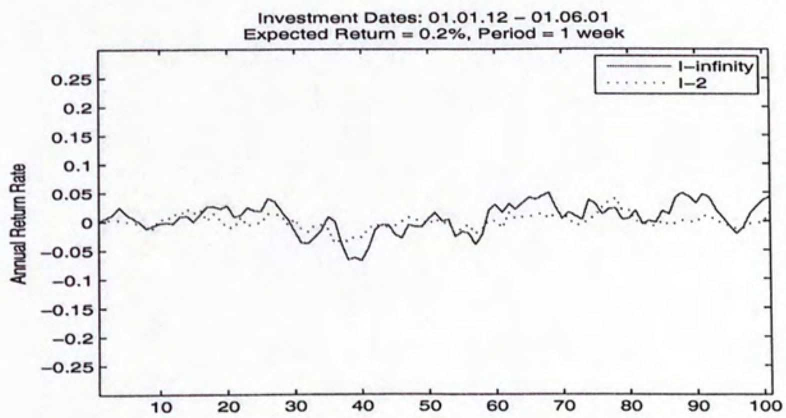

Figure C.2: Actual return graph with investment bounds equal to 3 times the group capitalization rates for 1-week investment with expected return equal to 0.2%

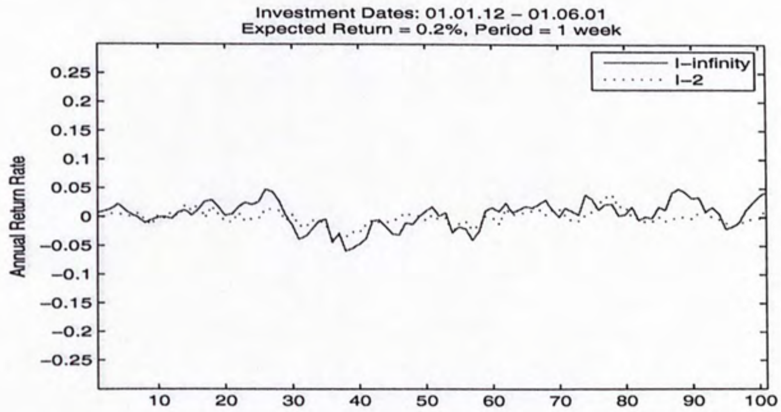

Figure C.3: Actual return graph with investment bounds equal to 5 times the group capitalization rates for 1-week investment with expected return equal to 0.2%

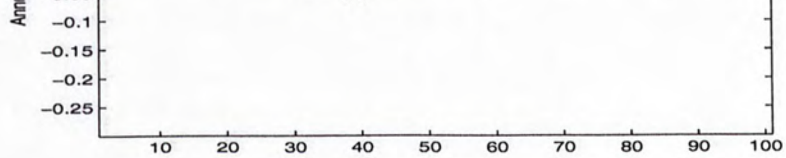

Figure C.4: Actual return graph with investment bounds equal to 1.5 times the group capitalization rates for 1-week investment with expected return equal to 0.5%

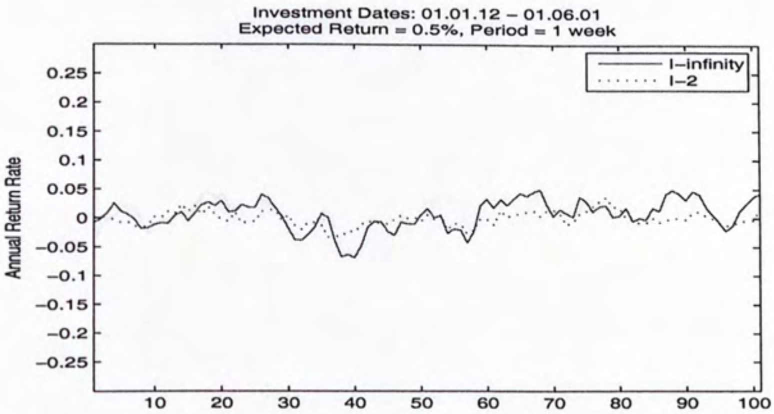

Figure C.5: Actual return graph with investment bounds equal to 3 times the group capitalization rates for 1-week investment with expected return equal to 0.5%

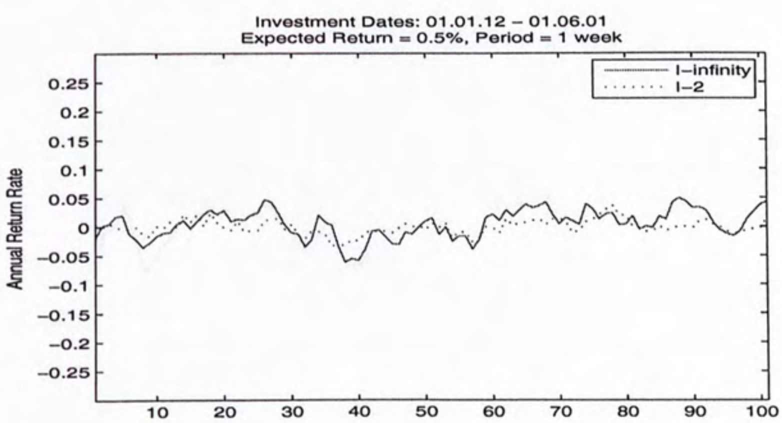

Figure C.6: Actual return graph with investment bounds equal to 5 times the group capitalization rates for 1-week investment with expected return equal to 0.5%

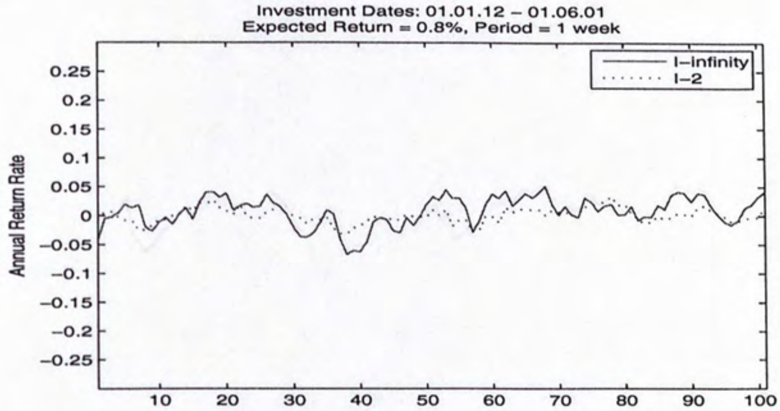

Figure C.7: Actual return graph with investment bounds equal to 1.5 times the group capitalization rates for 1-week investment with expected return equal to 0.8%

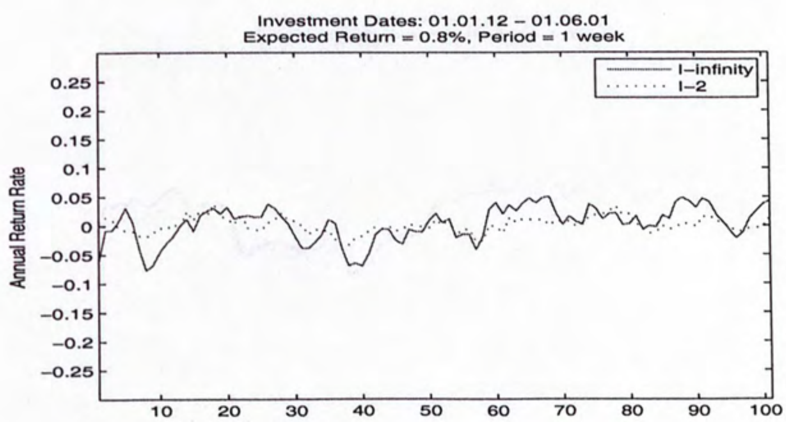

Figure C.8: Actual return graph with investment bounds equal to 3 times the group capitalization rates for 1-month investment with expected return equal to 0.8%

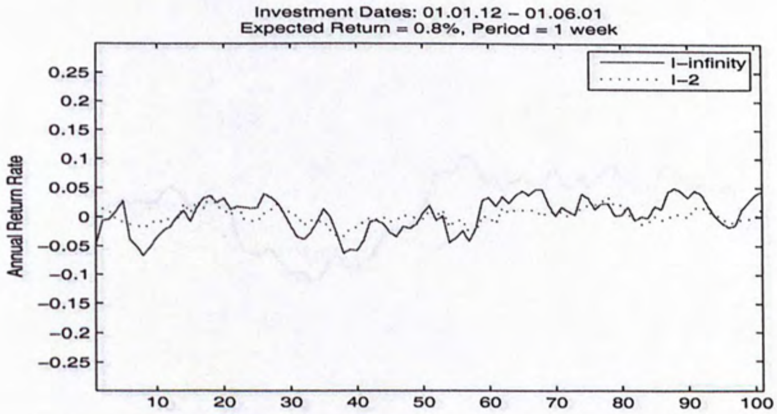

Figure C.9: Actual return graph with investment bounds equal to 5 times the group capitalization rates for 1-week investment with expected return equal to 0.8%

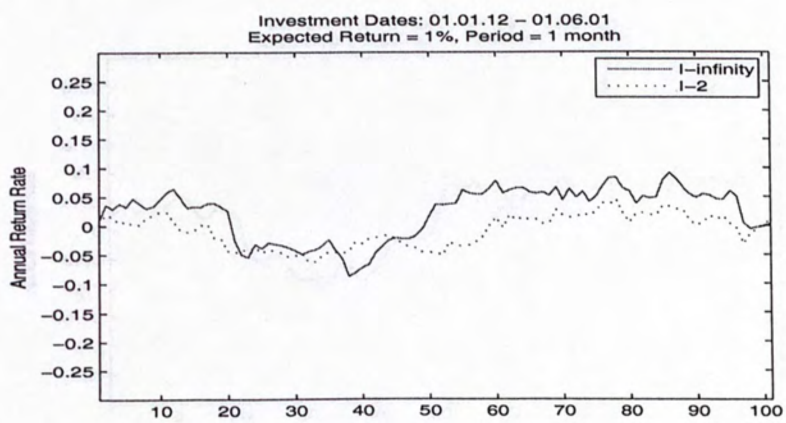

Figure C.10: Actual return graph with investment bounds equal to 1.5 times the group capitalization rates for 1-month investment with expected return equal to 1%

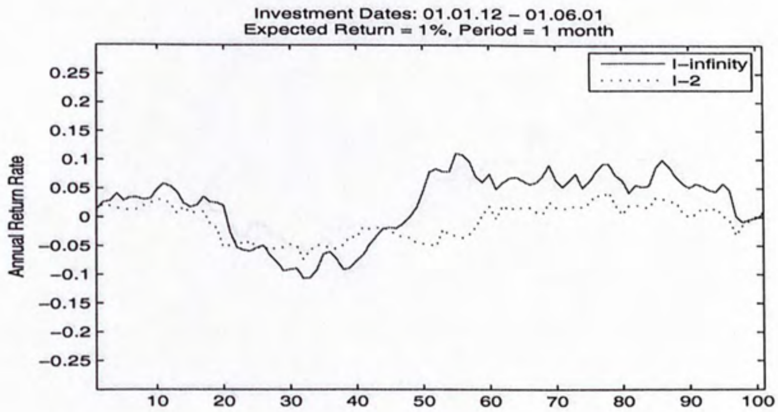

Figure C.11: Actual return graph with investment bounds equal to 3 times the group capitalization rates for 1-month investment with expected return equal to 1%

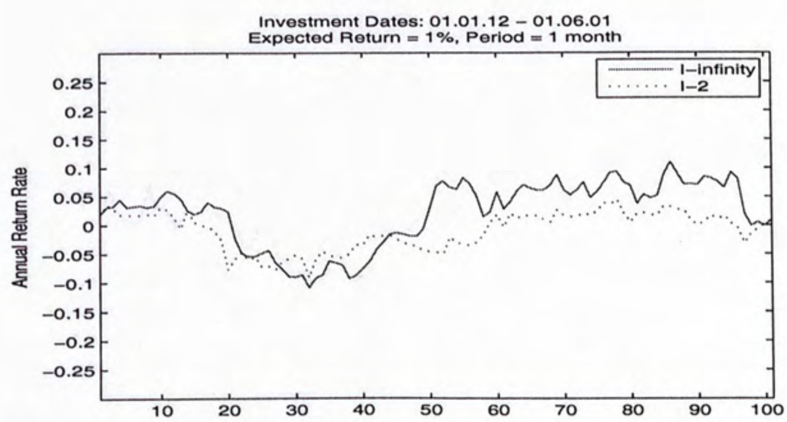

Figure C.12: Actual return graph with investment bounds equal to 5 times the group capitalization rates for 1-month investment with expected return equal to 1%

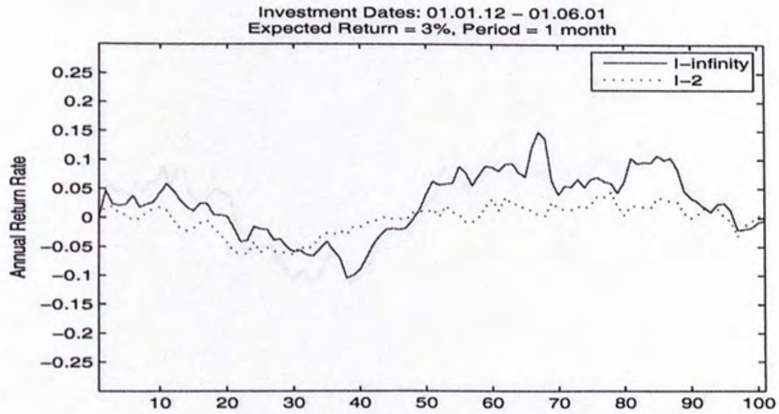

Figure C.13: Actual return graph with investment bounds equal to 1.5 times the group capitalization rates for 1-month investment with expected return equal to 3%

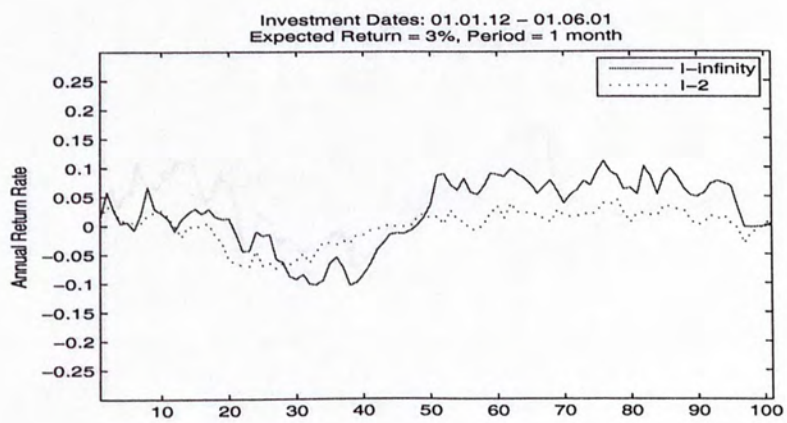

Figure C.14: Actual return graph with investment bounds equal to 3 times the group capitalization rates for 1-month investment with expected return equal to 3%

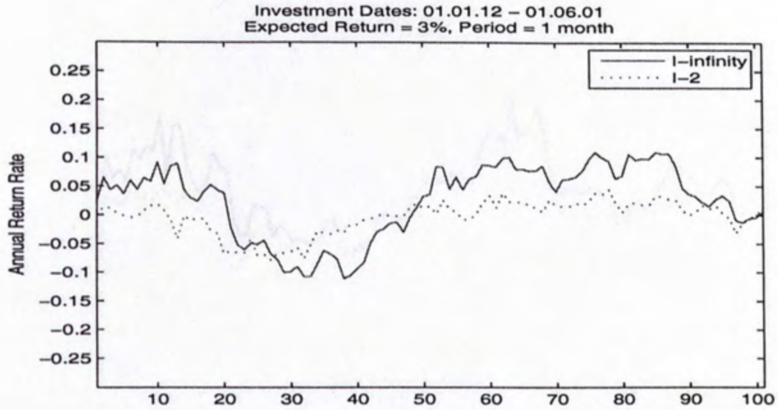

Figure C.15: Actual return graph with investment bounds equal to 5 times the group capitalization rates for 1-month investment with expected return equal to 3%

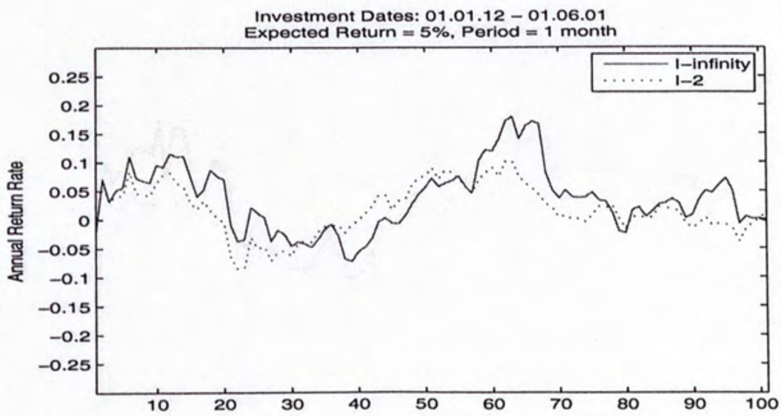

Figure C.16: Actual return graph with investment bounds equal to 1.5 times the group capitalization rates for 1-month investment with expected return equal to 5%

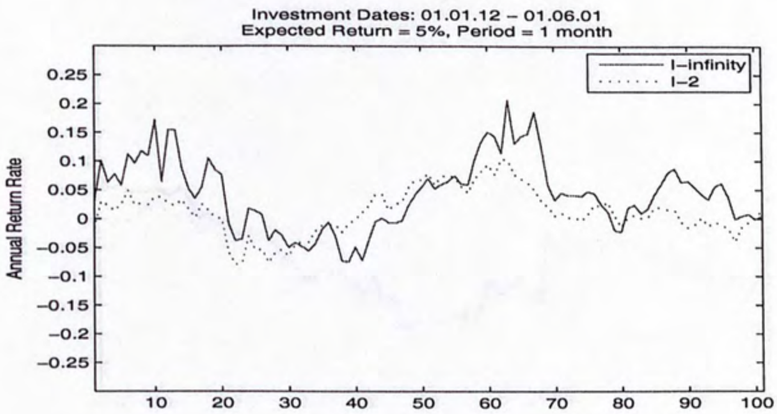

Figure C.17: Actual return graph with investment bounds equal to 3 times the group capitalization rates for 1-month investment with expected return equal to 5%

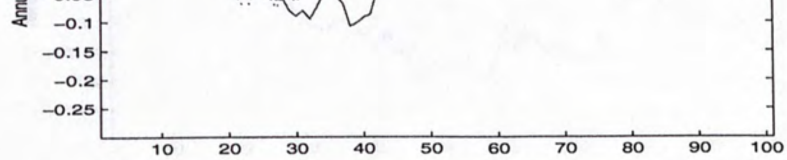

Figure C.18: Actual return graph with investment bounds equal to 5 times the group capitalization rates for 1-month investment with expected return equal to 5%

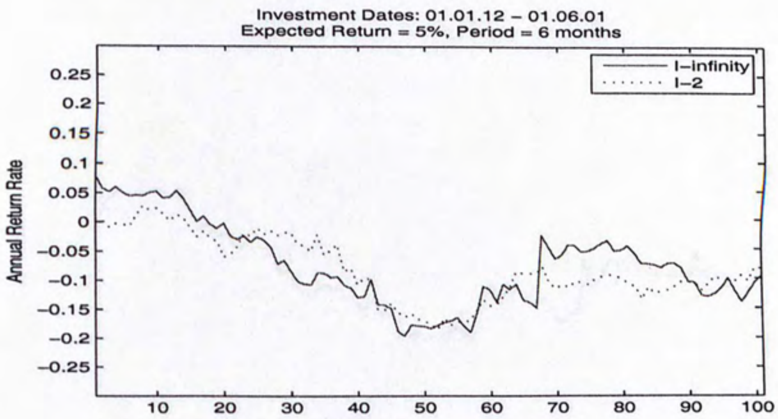

Figure C.19: Actual return graph with investment bounds equal to 1.5 times the group capitalization rates for 6-month investment with expected return equal to 5%

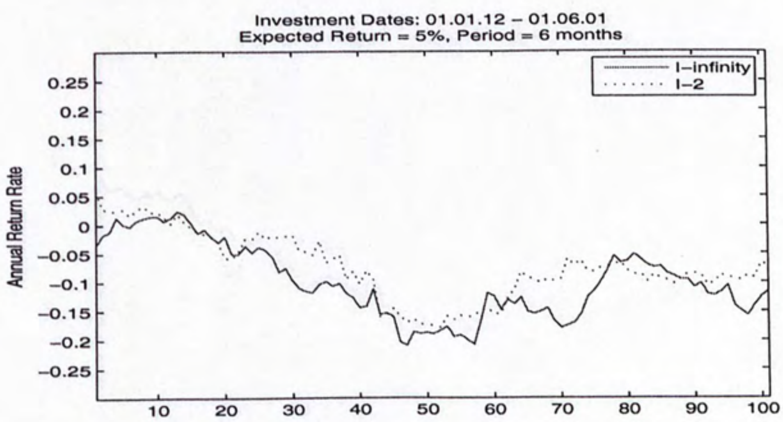

Figure C.20: Actual return graph with investment bounds equal to 3 times the group capitalization rates for 6-month investment with expected return equal to 5%

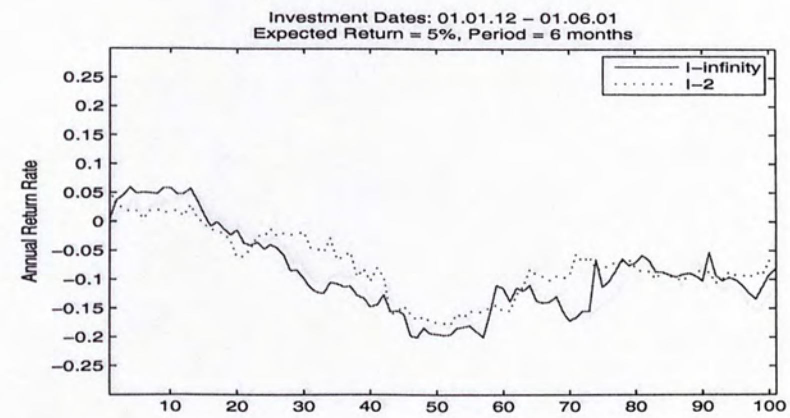

Figure C.21: Actual return graph with investment bounds equal to 5 times the group capitalization rates for 6-month investment with expected return equal to 5%

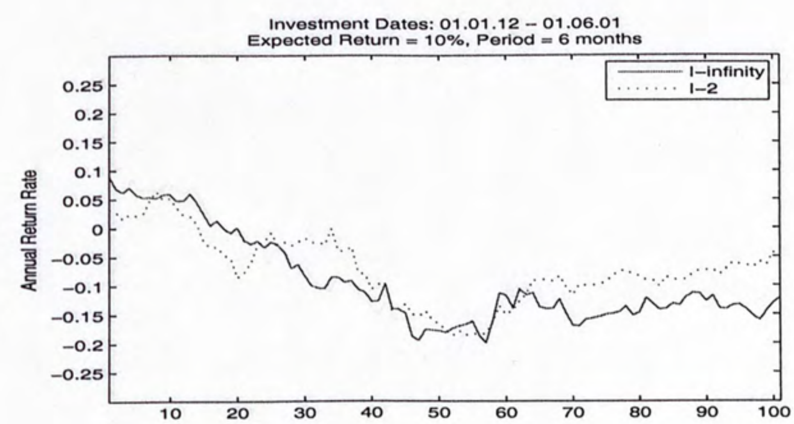

Figure C.22: Actual return graph with investment bounds equal to 1.5 times the group capitalization rates for 6-month investment with expected return equal to 10%

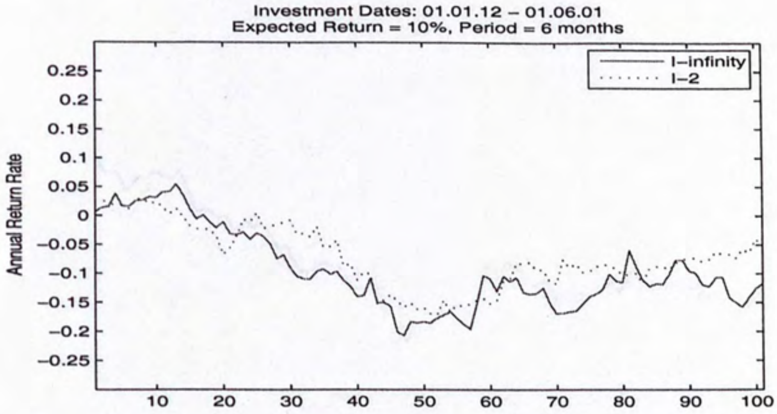

Figure C.23: Actual return graph with investment bounds equal to 3 times the group capitalization rates for 6-month investment with expected return equal to 10%

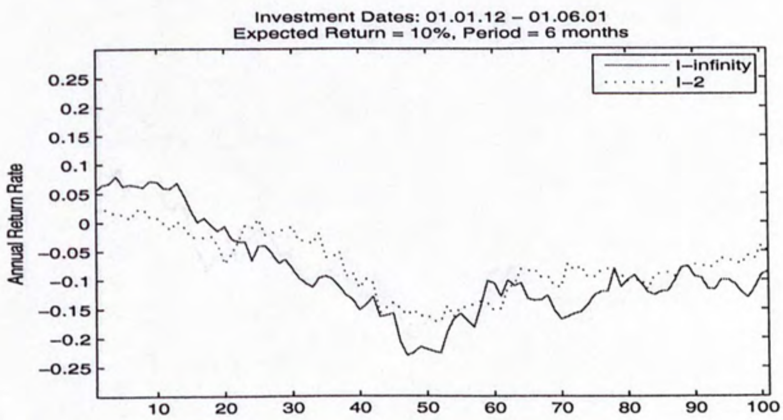

Figure C.24: Actual return graph with investment bounds equal to 5 times the group capitalization rates for 6-month investment with expected return equal to 10%

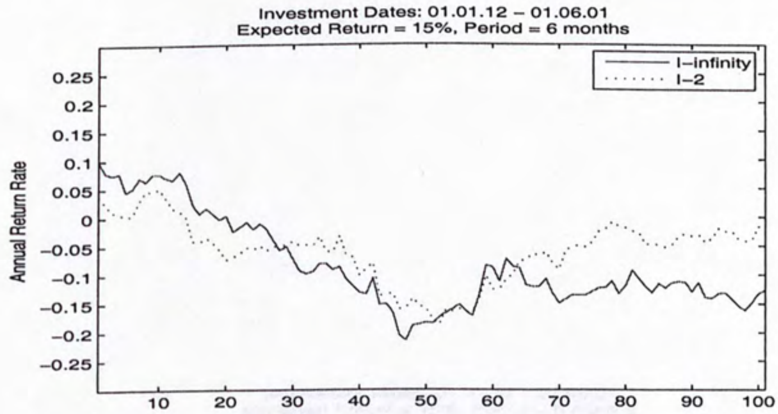

Figure C.25: Actual return graph with investment bounds equal to 1.5 times the group capitalization rates for 6-month investment with expected return equal to 15%

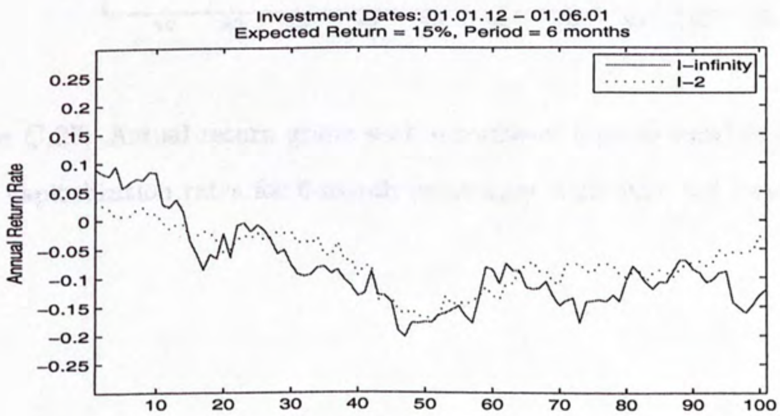

Figure C.26: Actual return graph with investment bounds equal to 3 times the group capitalization rates for 6-month investment with expected return equal to 15%

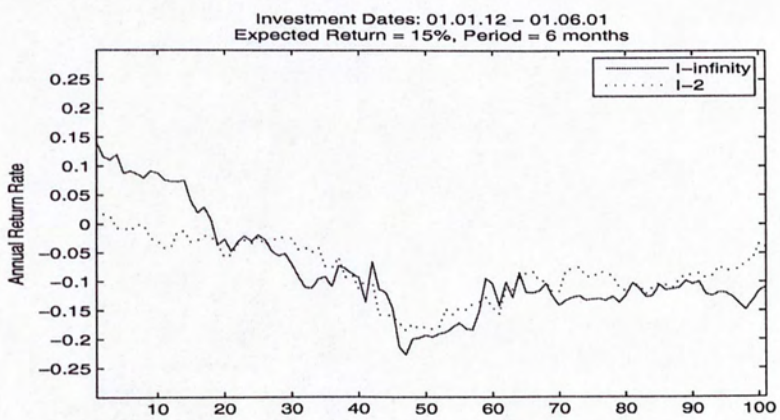

Figure C.27: Actual return graph with investment bounds equal to 5 times the group capitalization rates for 6-month investment with expected return equal to 15%