The Strong Conical Hull Intersection Property for Systems of Closed Convex Sets

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摘要

自從　F. Deutsch 及其合作者在 1997 年提出強 CHIP（the Strong Conical Hull Intersection Property, abbrev. the Strong CHIP）性質以來，此概念在優化的很多領域裡佔着重要的位置。在賦範綫性空間中給定一族閉凸集 $\{C_i : i \in I\}$；如果它們的法錐（Normal Cone）滿足以下關係：

$$N_{\bigcap_{i\in I} C_i}(x) = \sum_{i\in I} N_{C_i}(x), \quad \forall x \in \bigcap_{i\in I} C_i,$$

則稱此閉凸集族滿足強 CHIP 性質。由法錐性質可知上式等價於：

$$N_{\bigcap_{i\in I} C_i}(x) \subseteq \sum_{i\in I} N_{C_i}(x), \quad \forall x \in \bigcap_{i\in I} C_i.$$

本文將有系統地介紹強 CHIP 性質並研究它與其他在優化中十分重要的性質的相互關係，例如基本規範條件（the Basic Constraint Qualifications）、綫性正則性（the Linear Regularity）等。我們亦會建立法院強 CHIP 性質的一些充分條件，當中包括涉及到上圖和（Epi-sum）的最新發展。這些將構成論文的前三章。論文的最後一章則是簡介本人與浙江大學的　李沖教授以及我的論文導師　吳恭孚教授的一些工作，主要是關於上圖和研究的延續，詳情見香港中文大學數學研究報告２００６－０２（Mathematics Research Report Series 2006-02, the Chinese　University of Hong Kong）。

Abstract

The strong conical hull intersection property (the strong CHIP) of a system of closed convex sets in a normed linear space plays an important role in various aspects of optimization theory since it was first defined by F. Deutsch et al. in 1997. A system of closed convex sets $\{C_i : i \in I\}$ in a normed linear space X with nonempty intersection is said to have the strong CHIP if their normal cones satisfy the following property

$$N_{\bigcap_{i\in I} C_i}(x) = \sum_{i\in I} N_{C_i}(x), \ \forall x \in \bigcap_{i\in I} C_i. \tag{$*$}$$

By definition of normal cones, $(*)$ is satisfied if and only if

$$N_{\bigcap_{i\in I} C_i}(x) \subseteq \sum_{i\in I} N_{C_i}(x), \ \forall x \in \bigcap_{i\in I} C_i.$$

In this thesis, we intend to give a systematic survey on recent results regarding the relationship of the strong CHIP with other important properties in optimization like the Basic Constraint Qualifications, the linear regularity, etc. We shall also give an overview of various existing sufficient conditions for the strong CHIP. Recent development involving the study of epi-sum is also discussed. The last chapter serves as a summary of the results a joint work with Professor Chong Li from Zhejiang University and my supervisor Professor Kung Fu Ng, mainly on the extension involving epi-sum. For details, we refer the readers to Mathematics Research Report Series 2006-02, the Chinese University of Hong Kong.

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Contents

Chapter 1

Introduction

The Strong Conical Hull Intersection Property (the strong CHIP), since its introduction in a 1997 paper, has received attention from many researchers. The property originates from considerations concerning the projection algorithm. In this thesis, we aim at surveying the main development concerning this property.

In Chapter 2, as preparation for later discussions, we collect some basic facts concerning convex analysis, such as some properties of normal cones, the separation theorem and some properties of the Minkowski functional, etc. We also include some results concerning the epigraphs of convex lower semicontinuous functions in the last section of Chapter 2.

The definition of the strong CHIP is to be given in Chapter 3. In this chapter, we shall first discuss the relationship between the strong CHIP and projections onto closed convex sets, which was the first relationship between the strong CHIP and other concepts in optimization being studied historically. Digressions into cases involving systems of closed nonconvex sets are also addressed. The relationship between the strong CHIP and the Basic Constraint Qualification (the BCQ) is to be discussed in section 3.3. In the last section, we study a special pair of closed convex sets. We shall prove the result first obtained by F. Deutsch et al., which states that by suitably shrinking one of the two given sets without altering the set of intersection, we shall arrive at a pair of closed convex sets having the strong CHIP.

Sufficient conditions for the strong CHIP have been extensively studied in the literature. This will be the topic of Chapter 4. The first part concerns the sufficient condition for the strong CHIP of a system of finitely many closed convex sets. While most of them concerns interior point conditions, recent development concerning the epi-sum is also addressed. The boundedly linear regularity, an important concept originated from the projection algorithm, is also shown to be a sufficient condition for the strong CHIP. The second part addresses the results in the case when infinitely many closed convex sets are involved, which was mainly the work of Li and Ng (see [28]).

The last chapter serves as a summary of the results in a joint work with Professor Chong Li from Zhejiang University and my supervisor Professor Kung Fu Ng, mainly on the extension involving epi-sum. For the full text we refer the readers to the Mathematics Research Report Series 2006-02, the Chinese University of Hong Kong.

Chapter 2

Preliminary

2.1 Introduction

In this chapter, necessary tools for our subsequent discussion are given. They are mainly concerned with convex analysis and set-valued analysis.

2.2 Notations

The meaning of X varies from sections to sections. In some sections, it will denote a real normed linear space, while in others it may denote a Banach space or even a Hilbert space. We shall specify what X denotes at the beginning of each section and most of the time in the statement of the theorems. By X^*, we mean the dual space of X. We shall use x to denote vectors in X, and x^* to denote vectors in X^*. For $x \in X$ and $x^* \in X^*$, we shall write $\langle x^*, x\rangle$ for the value $x^*(x)$.

An extended real-valued function is said to be proper if it is not equal to $-\infty$ (the negative infinity) anywhere and there exist some point at which its value is finite. The set on which a proper function $f : X \to (-\infty, +\infty]$ is finite is called its domain, that is

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

Let f be a proper lower semicontinuous extended real-valued function on X. Then the subdifferential of f at $x \in X$, denoted by $\partial f(x)$, is defined by

$$\partial f(x) := \{z^* \in X^* : \ f(x) + \langle z^*, y - x\rangle \le f(y) \quad \text{for all } y \in X\},$$

(thus $\partial f(x) = \emptyset$ if $x \in X \backslash \operatorname{dom} f$). Let f, g be proper functions respectively defined on X and X^*. Let f^*, g^* denote their conjugate functions (with respect to the duality (X, X^*)), that is

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : \ x \in X\}, \quad \text{for each } x^* \in X^*,$$
$$g^*(x) := \sup\{\langle x^*, x\rangle - g(x^*) : \ x^* \in X^*\}, \quad \text{for each } x \in X.$$

Recall that the w^*−topology in X^* is, by definition, the weakest topology making each linear functional $x^* \mapsto \langle x^*, x\rangle$ continuous on X^*, where $x \in X$. We note that if f is a proper convex lower semicontinuous function on X, then its conjugate function f^* is a proper convex w^*−lower semicontinuous function on X^*, and

$$f^{**} = f \tag{2.2.1}$$

(see [35, Corollary 2.3.2 and Theorem 2.3.3]). The epigraph of a function f on X is denoted by epi f and defined by

$$\text{epi } f := \{(x, r) \in X \times \mathbb{R} : \ f(x) \le r\}.$$

Recall further that for proper lower semicontinuous extended real-valued convex functions f_1 and f_2 on X, the following equivalences hold:

$$f_1 \le f_2 \Longleftrightarrow f_1^* \ge f_2^* \Longleftrightarrow \text{epi } f_1^* \subseteq \text{epi } f_1^*, \tag{2.2.2}$$

where the forward direction of the first arrow and the second equivalence are easy to verify, while the backward direction of the first arrow is standard (cf. [35, Theorem 2.3.3]).

Two special types of convex lower semicontinuous functions will be used extensively. Given a closed convex set A in X, its indicator function δ_A and support function σ_A are defined by:

$$\begin{aligned} \delta_A(x) \quad &:= \begin{cases} 0, & x \in A \\ \infty, & \text{otherwise} \end{cases} . \\ \sigma_A(x^*) \quad &:= \sup\nolimits_{x \in A} \langle x^*, x\rangle, \quad \text{for all } x^* \in X^*. \end{aligned}$$

We use $B(x,\epsilon)$ to denote the closed ball with center x and radius ϵ and $\mathbf{B}$ to denote the closed unit ball in X. For a set A in X (or in $\mathbb{R}^n$), the interior (*resp.* closure, convex hull, convex cone hull, linear hull, affine hull, boundary) of A is denoted by $\operatorname{int} A$ (*resp.* $\overline{A}$, $\operatorname{co} A$, $\operatorname{cone} A$, $\operatorname{span} A$, $\operatorname{aff} A$, $\operatorname{bdry} A$). Some other basic notations and definitions will be given in the remaining parts of this chapter.

2.3 On properties of Normal Cones

The following definition of normal cones is well known, see for example [7, Section 2.1], [16, III Definition 5.2.3].

Definition 2.3.1. *Let C be a closed convex set in a normed linear space X. The normal cone $N_C(x)$ of C at a point $x \in C$ is defined by:*

$$N_C(x) := \{x^* \in X^* \ : \langle x^*, y - x\rangle \leq 0,\ \forall y \in C\}. \tag{2.3.1}$$

The next proposition collects some useful properties of normal cones. Parts **(i)** to **(v)** can be found in standard references for convex analysis, see for example [7], [11] and [35]. Since the references are scattered, we give proofs for these statements for completeness. Part **(vi)** is proved here for later use. To proceed, we first recall two definitions from the literature.

Definition 2.3.2 (see [35, Page 227]). *A Banach space X is said to be smooth if the unit ball has a unique supporting hyperplane at every point of its boundary, that is, for any boundary point x of the unit ball $B(0,1)$ in X, the set*

$$\{x^* \in X^* :\ \|x^*\| = 1 = \langle x^*, x\rangle\}$$

is a singleton.

Definition 2.3.3 (see [35, Page 230]). *Let X be a Banach space. The duality map $J : X \to X^*$ is defined by*

$$J(x) := \{x^* \in X^* :\ \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},\quad \textit{for each } x \in X.$$

Proposition 2.3.1. *Let C, D be closed convex sets in a normed linear space X. Then the following statements are true:*

(i) $N_C(x) = \partial\delta_C(x)$ *for all* $x \in C$.

(ii) *If* $C \cap D \neq \emptyset$, *then* $N_C(x) + N_D(x) \subseteq N_{C\cap D}(x)$ *for all* $x \in C \cap D$.

(iii) *If* $C \subseteq D$, *then* $N_D(x) \subseteq N_C(x)$ *for all* $x \in C$.

(iv) $N_C(x) = \bigcup_{\lambda \geq 0} \lambda \partial d(x, C)$ *for any* $x \in C$.

(v) *If X is a Banach space, then the following equivalence is valid for any $x_0 \in X$ and $u \in C$:*

$$J(x_0 - u) \cap N_C(u) \neq \emptyset \Leftrightarrow u \in P_C(x_0), \tag{2.3.2}$$

where P_C is the projection on C and J is the duality map. In particular, if X is smooth, then we have

$$J(x_0 - u) \in N_C(u) \Leftrightarrow u \in P_C(x_0). \tag{2.3.3}$$

If assume further that X is Hilbert, then we have

$$x_0 - u \in N_C(u) \Leftrightarrow u = P_C(x_0). \tag{2.3.4}$$

(vi) *If X is a smooth and reflexive Banach space, and $C = x + K$ for some vector $x \in X$ and closed convex cone K, then for any $y \in X$ and $u \in C$, the following implication holds:*

$$u \in P_C(y) \Rightarrow x \in P_C(y - u + x).$$

In particular, when X is a Hilbert space, we have

$$u = P_C(y) \Rightarrow x = P_C(y - u + x).$$

Proof. The verification of **(iii)** is straightforward. Now we start to prove **(i)**. By the definitions, the following equivalences hold:

$$\begin{aligned}
& \langle x^*, y - x\rangle \leq \delta_C(y) - \delta_C(x), \quad \forall y \in X, \\
\Leftrightarrow\quad & \langle x^*, y - x\rangle \leq \delta_C(y) - \delta_C(x), \quad \forall y \in C, \text{ (since } \delta_C(\cdot) = +\infty \text{ on } X\backslash C), \\
\Leftrightarrow\quad & \langle x^*, y - x\rangle \leq 0, \quad \forall y \in C, \text{ (since } \delta_C(\cdot) = 0 \text{ on } C).
\end{aligned}$$

Thus **(i)** is seen to hold.

To prove part **(ii)**, we take $c^* \in N_C(x)$ and $d^* \in N_D(x)$. By definition of normal cones, we have in particular that

$$\langle c^*, y - x\rangle \leq 0, \quad \forall y \in C \cap D \quad \text{and} \quad \langle d^*, y - x\rangle \leq 0, \quad \forall y \in C \cap D.$$

Adding the two inequalities, we obtain

$$\langle c^* + d^*, y - x\rangle \leq 0, \forall y \in C \cap D,$$

which means $c^* + d^* \in N_{C\cap D}(x)$. This proves part **(ii)**.

To prove **(iv)**, let $x \in C$; and thus $d(x, C) = 0$ and $\delta_C(x) = 0$. Take $y^* \in \partial d(x, C)$ and $\lambda \geq 0$. Then for any $y \in C$, it follows from the obvious inequality $\lambda d(\cdot, C) \leq \delta_C(\cdot)$ that,

$$\langle \lambda y^*, y - x\rangle \leq \lambda[d(y, C) - d(x, C)] = \lambda d(y, C) \leq \delta_C(y) = \delta_C(y) - \delta_C(x),$$

which implies $\lambda y^* \in \partial\delta_C(x)$ and hence that $\lambda y^* \in N_C(x)$, by **(i)**. This implies that $\bigcup_{\lambda\geq 0} \lambda\partial d(x, C) \subseteq N_C(x)$ for any $x \in C$. Conversely, suppose $y^* \in N_C(x)$. Then

$$\langle y^*, x - y\rangle \geq 0, \ \forall y \in C,$$

that is, x is a minimizer of $f(\cdot) := \langle y^*, x - \cdot\rangle$ over C. Since f is a Lipschitz function of rank $\|y^*\|$, it follows from [11, Proposition 2.4.3] that x is a global minimizer of the function $\langle y^*, x - y\rangle + \|y^*\|d(y, C)$ on X, i.e.,

$$\langle y^*, x - y\rangle + \|y^*\|d(y, C) \geq 0, \ \forall y \in X.$$

This gives $y^* \in \|y^*\|\partial d(x, C)$, and so $y^* \in \bigcup_{\lambda\geq 0} \lambda\partial d(x, C)$. This proves part **(iv)**.

Now, we turn to part **(v)**. By [35, Theorem 2.5.7], the subdifferential sum rule ([35, Theorem 2.8.7]) and **(i)**, we have the following equivalences:

$$\begin{aligned}
& u \in P_C(x) \\
\Leftrightarrow \quad & u \text{ minimizes } \tfrac{1}{2}\|x_0 - \cdot\|^2 + \delta_C(\cdot) \\
\Leftrightarrow \quad & 0 \in \partial\left(\tfrac{1}{2}\|x_0 - \cdot\|^2 + \delta_C(\cdot)\right)(u) \\
\Leftrightarrow \quad & 0 \in \partial\tfrac{1}{2}\|x_0 - \cdot\|^2(u) + N_C(u) \\
\Leftrightarrow \quad & 0 \in -J(x_0 - u) + N_C(u).
\end{aligned}$$

This proves (2.3.2). Moreover, if X is assumed to be smooth, then J is single valued (cf. [35, Page 230]) and hence (2.3.3) follows immediately from (2.3.2) for the case when X is smooth. Finally, assume that X is Hilbert (which is smooth in particular). Then, by the natural identification between X and X^*, $J = I$ (the identity map) and the projection onto any nonempty closed convex sets is single valued (cf. [35, Proposition 3.8.6]). Thus (2.3.4) is a re-statement of (2.3.3) in the case when X is a Hilbert space. This proves part **(v)**.

Finally, we prove part **(vi)**. Let $y \in X$ and $u \in C$, we first claim that for this specific $C = x + K$,

$$u \in P_C(y) \Rightarrow \langle J(y-u), k\rangle \leq 0, \ \forall k \in K. \tag{2.3.5}$$

To see this, let $u \in P_C(y)$. Since X is smooth, by part **(v)**, we have $J(y-u) \in N_C(u)$. This gives

$$\langle J(y-u), x+k-u\rangle \leq 0, \ \forall k \in K.$$

Since K is a cone, this gives $\langle J(y-u), k\rangle \leq 0$ for all $k \in K$. Now we start to prove the implication stated in part **(vi)**. Note that $u \in P_C(y)$ implies

$$\langle J(y-u), k\rangle \leq 0, \ \forall k \in K. \tag{2.3.6}$$

This implies that

$$\langle J((y-u+x)-x), x+k-x\rangle \leq 0, \ \forall k \in K,$$

and hence that $J((y-u+x)-x) \in N_C(x)$ and so $x \in P_C(y-u+x)$ by (2.3.3). The case when X is a Hilbert space follows readily from (2.3.4), since the projection onto any nonempty closed convex sets is single valued (cf. [35, Proposition 3.8.6]). The proof is completed. □

We shall need to use the following simple observation of normal cones, which is an easy consequence of the Hahn-Banach extension theorem. See [28, Corollary 2.1] for the case when C is a polyhedron. To proceed, we introduce the following notation.

Definition 2.3.4. *Let C be a closed subset of a normed linear space X, Z be a subspace of X such that $C \subseteq Z$. The normal cone of C as a subset of Z at the point $x \in X$ is* *defined by*

$$N_C^Z(x) := \{y^* \in Z^* : \langle y^*, y - x\rangle \leq 0, \ \forall y \in C\},$$

Proposition 2.3.2. *Let C be a closed subset of a normed linear space X, Z be a subspace of X such that $C \subseteq Z$. Then, for any $x \in C$, the normal cone of C as a subset of Z at the point x, $N_C^Z(x)$, is equal to*

$$N_C(x)|_Z := \{y^* \in Z^* : \exists\, x^* \in X^*, x^*|_Z = y^*, \langle x^*, y - x\rangle \leq 0, \ \forall y \in C\}.$$

Proof. It is obvious that $N_C(x)|_Z \subseteq N_C^Z(x)$. To show the converse inclusion, we take $y^* \in N_C^Z(x)$. By Hahn-Banach extension theorem, there exists $x^* \in X^*$ such that $x^*|_Z = y^*$. Then since $x \in C$ and $C \subseteq Z$, we have

$$\langle x^*, y - x\rangle = \langle y^*, y - x\rangle \leq 0, \ \forall y \in C,$$

where the inequality follows from the definition of $N_C^Z(x)$. Thus $y^* \in N_C(x)|_Z$. This finishes the proof. □

2.4 Polar Calculus

We recall the following separation theorem.

Theorem 2.4.1 (cf. [35, Theorem 1.1.5], [33, Theorem 2.2.28]). **(i)** *Let C be a closed convex set in a normed linear space X, and $y \notin C$. Then there exist $x^* \in X^* \backslash \{0\}$ and $\gamma \in \mathbb{R}$ such that*

$$\langle x^*, y\rangle > \gamma \geq \langle x^*, u\rangle, \forall u \in C.$$

(ii) *Let C be a w^*-closed convex set in the dual normed linear space X^*, and $y^* \notin C$. Then there exist $x \in X \backslash \{0\}$ and $\gamma \in \mathbb{R}$ such that*

$$\langle y^*, x\rangle > \gamma \geq \langle u^*, x\rangle, \forall u^* \in C.$$

Definition 2.4.1 (cf. [16, III Definition 3.2.1], [35, Page 7]). *Let C be a closed convex set in a normed linear space X. The negative polar of C is defined by*

$$C^{\ominus} := \{x^* \in X^* \ : \ \langle x^*, x\rangle \leq 0, \forall x \in C\}.$$

For a set C in X^*, we define analogously the negative polar of C as

$$C^{\ominus} := \{x \in X \;:\; \langle x^*, x\rangle \le 0, \forall x^* \in C\}.$$

Remark 2.4.1. *It follows easily from definition that for $C \subseteq X$, $C^{\ominus}$ is a w^*-closed cone in X^*, while for $C \subseteq X^*$, $C^{\ominus}$ is a closed cone in X. (cf. [17, Page 67])*

Proposition 2.4.1 ([18, 1.1.6]). *Let D be a convex set in a normed linear space X. Then $\operatorname{cone} D$ is a convex cone.*

Proof. It is easy to verify that $\operatorname{cone} D$ is homogeneous in the sense that $\lambda D \subseteq D$ for all $\lambda \ge 0$. To prove that $\operatorname{cone} D$ is convex, let $x, y \in \operatorname{cone} D$, and $t \in (0,1)$. We have to show that $tx + (1-t)y \in \operatorname{cone} D$. Since this is clearly true if $x = 0$ or $y = 0$, we suppose henceforth that $x \ne 0$ and $y \ne 0$. Then there exists λ_1, $\lambda_2 > 0$, and $d_1, d_2 \in D$ such that $x = \lambda_1 d_1$ and $y = \lambda_2 d_2$. Let $\Delta = t\lambda_1 + (1-t)\lambda_2 \ne 0$. Then $\Delta > 0$ and thus

$$tx + (1-t)y = t\lambda_1 d_1 + (1-t)\lambda_2 d_2 = \Delta(\frac{t\lambda_1}{\Delta}d_1 + \frac{(1-t)\lambda_2}{\Delta}d_2) \in \operatorname{cone} D,$$

since $\frac{t\lambda_1}{\Delta}d_1 + \frac{(1-t)\lambda_2}{\Delta}d_2 \in D$ by the convexity of D. □

Definition 2.4.2 (cf. [25], [28]). *Let $\{C_i : i \in I\}$ be a family of closed convex sets in a normed linear space X. Then*

$$\sum_{i\in I} C_i := \begin{cases} \{\sum_{j\in J} c_j : c_j \in C_j, J \subseteq I, |J| < +\infty\}, & \text{if } I \ne \emptyset \\ \{0\}, & \text{otherwise} \end{cases}$$

Some properties regarding negative polars are collected in the next proposition.

Proposition 2.4.2 (cf. [35, Theorem 1.1.9], [17, Page 113, Exercise 2.2.8]). *Let $\{A_i : i \in I\}$ be a collection of closed convex cones in a normed linear space X where I is an index set. Let C, D be closed convex sets in X. Then the following statements are true.*

(i) $(D^{\ominus})^{\ominus} = \overline{\operatorname{cone} D}$;

(ii) *If $C \subseteq D$, then $D^{\ominus} \subseteq C^{\ominus}$.*

(iii) $(\bigcap_{i\in I} A_i)^{\ominus} = \overline{\sum_{i\in I} A_i^{\ominus}}^{w^*}$,

where $\overline{Y}^{w^*}$ denotes the w^*-closure of Y for any subset Y of X^*.

Proof. The proof of **(ii)** is straightforward. We only prove **(i)** and **(iii)**.

We first prove part **(i)**. Let $y \in \operatorname{cone} D$. Then by definition, there exist $\lambda \geq 0$ and $d \in D$ such that $y = \lambda d$. Thus, for any $u^* \in D^\ominus$, we have

$$\langle u^*, y\rangle = \lambda\langle u^*, d\rangle \leq 0,$$

which implies that $y \in (D^\ominus)^\ominus$. Since $(D^\ominus)^\ominus$ is closed (as is easily verified), it follows that $\overline{\operatorname{cone} D} \subseteq (D^\ominus)^\ominus$. To prove that the above inclusion is in fact an equality, we suppose that

$$\overline{\operatorname{cone} D} \subsetneqq (D^\ominus)^\ominus \tag{2.4.1}$$

Then there exists $x \in (D^\ominus)^\ominus \backslash \overline{\operatorname{cone} D}$. Recalling from Proposition 2.4.1 that $\overline{\operatorname{cone} D}$ is a closed convex cone. It follows from Theorem 2.4.1 that there exists $y^* \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle y^*, x\rangle > \alpha \geq \langle y^*, d\rangle, \ \forall d \in \operatorname{cone} D. \tag{2.4.2}$$

Since $0 \in \operatorname{cone} D$, (2.4.2) implies that $\alpha \geq 0$. On the other hand, since $d \in \operatorname{cone} D$ implies $td \in \operatorname{cone} D$ for any $t > 0$ by definition, we see that for any fixed $d \in \operatorname{cone} D$,

$$\alpha \geq t\langle y^*, d\rangle, \ \forall t > 0,$$

which implies $\langle y^*, d\rangle \leq 0$ for all $d \in \operatorname{cone} D$. Thus $\alpha = 0$ satisfies (2.4.2), that is

$$\langle y^*, x\rangle > 0 \geq \langle y^*, d\rangle, \ \forall d \in \operatorname{cone} D.$$

The second inequality implies $y^* \in D^\ominus$ and it follows from the first inequality that $x \notin (D^\ominus)^\ominus$. This contradicts the choice of x and **(i)** is proved.

To prove part **(iii)**, for each fixed $k \in I$, since $\bigcap_{i\in I} A_i \subseteq A_k$, it follows from part **(ii)** that $A_k^\ominus \subseteq (\bigcap_{i\in I} A_i)^\ominus$. This implies $\overline{\sum_{i\in I} A_i^\ominus}^{w^*} \subseteq (\bigcap_{i\in I} A_i)^\ominus$, since $(\bigcap_{i\in I} A_i)^\ominus$ is a w^*-closed convex cone. To prove the equality, we suppose on the contrary that

$$\overline{\sum_{i\in I} A_i^\ominus}^{w^*} \subsetneqq (\bigcap_{i\in I} A_i)^\ominus.$$

Thus there is a $y^* \in (\bigcap_{i\in I} A_i)^\ominus$ but $y^* \notin \overline{\sum_{i\in I} A_i^\ominus}^{w^*}$. By Theorem 2.4.1, there is an $x \in X$ and a constant $\gamma \in \mathbb{R}$ such that

$$\langle y^*, x\rangle > \gamma \geq \langle u^*, x\rangle, \forall u^* \in A_i^\ominus, \forall i \in I. \tag{2.4.3}$$

As in part **(i)**, (2.4.3) is true with $\gamma = 0$ and so the second inequality implies that $x \in (A_i^{\ominus})^{\ominus}$ and so $x \in A_i$ by **(i)**. Since this is true for each $i \in I$ we have $x \in \bigcap_{i \in I} A_i$ and it follows from the first inequality in (2.4.3) that $y^* \notin (\bigcap_{i \in I} A_i)^{\ominus}$. This contradicts the choice of y^*. Thus $(\bigcap_{i \in I} A_i)^{\ominus} = \overline{\sum_{i \in I} A_i^{\ominus}}^{w^*}$. □

Next we introduce a definition.

Definition 2.4.3 ([1, Definition 4.1.1 and Page 122]). *Let D be a closed convex set in a normed linear space, $x \in D$. Then the (contingent) tangent cone $T_D(x)$ to D at x is defined by*

$$T_D(x) := \{h : \exists h_n \to h, t_n \downarrow 0, \text{ such that } x + t_n h_n \in D, \text{ foreach } n \in \mathbb{N}\}.$$

We have the following proposition.

Proposition 2.4.3 ([1, Page 139]). *Let D be a closed convex set in a normed linear space, $x \in D$. Then the following statements are true.*

(i) $N_D(x) = (D - x)^{\ominus}$.

(ii) $N_D(x) = (T_D(x))^{\ominus}$.

Proof. Part **(i)** is immediate from the definition of negative polar and normal cones. We now prove **(ii)**. Let $y^* \in N_D(x)$. Take any $h \in T_D(x)$. Then there exist by definition $h_n \to h$ and $t_n \downarrow 0$ such that $x + t_n h_n \in D$ for each n. Since $y^* \in N_D(x)$, $y^* \in (D - x)^{\ominus}$ by part **(i)** and thus

$$t_n \langle y^*, h_n \rangle \leq 0 \tag{2.4.4}$$

Dividing both sides of (2.4.4) by t_n and letting n converge to infinity, we obtain $\langle y^*, h \rangle \leq 0$. Since h is an arbitrary element in $T_D(x)$, we have proved $N_D(x) \subseteq (T_D(x))^{\ominus}$. We now turn to the converse inclusion. Let $y^* \in (T_D(x))^{\ominus}$. Let $h \in D - x$. Then the line segment $[x, h] := \{x + th : 0 \leq t \leq 1\}$ is contained in D because D is a convex set containing x. Thus $x + \frac{1}{n} h \in D$ for all $n \in \mathbb{N}$ and so $h \in T_D(x)$. Consequently, $\langle y^*, h \rangle \leq 0$. Since $h \in D - x$ is arbitrary, we have $y^* \in (D - x)^{\ominus}$. By part **(i)**, $y^* \in N_D(x)$ and this completes the proof. □

2.5 Notions of Relative Interior

The first notion to be introduced here is the notion of relative interior.

Definition 2.5.1 ([6, Definition 2.1]). *Let C, D be closed convex sets in a normed linear space X. A point x is said to be in the relative interior of C, denoted by $\mathrm{ri}\, C$, if there exists $\delta > 0$ such that $B(x,\delta) \cap \mathrm{aff}\, C \subseteq C$, where $\mathrm{aff}\, C$ is the affine hull of C. A point x is said to be in the D-interior of C, denoted by $\mathrm{rint}_D C$, if there exists $\delta > 0$ such that $B(x,\delta) \cap \mathrm{aff}\, D \subseteq C$.*

Next, consider a family of closed convex sets $\{D, C_i : i \in I\}$ in X with nonempty intersection, where I is an index set. We call such a family a closed convex set system with base set D, abbreviated CCS-system with base-set D. Let $|J|$ denote the cardinality of a set J.

Definition 2.5.2 ([28, Definition 3.1]). *Let $\{D, C_i : i \in I\}$ be a CCS-system with base-set D. The CCS-system $\{D, C_i : i \in I\}$ is said to satisfy:*

i) the D-interior-point condition if

$$D \bigcap \left(\bigcap_{i \in I} \mathrm{rint}_D C_i \right) \neq \emptyset; \tag{2.5.1}$$

ii) the strong D-interior-point condition if

$$D \bigcap \left(\mathrm{rint}_D \bigcap_{i \in I} C_i \right) \neq \emptyset; \tag{2.5.2}$$

iii) the weak-strong D-interior-point condition with the pair (I_1, I_2) if there exist two disjoint finite subsets I_1 and I_2 of I such that each C_i ($i \in I_2$) is a polyhedron and

$$\mathrm{ri}\, D \bigcap \left(\mathrm{rint}_D \bigcap_{i \in I \setminus (I_1 \cup I_2)} C_i \right) \bigcap \left(\bigcap_{i \in I_1} \mathrm{ri}\, C_i \right) \bigcap \bigcap_{i \in I_2} C_i \neq \emptyset. \tag{2.5.3}$$

Any point $\bar{x}$ belonging to the set on the left-hand side of (2.5.1) (resp. (2.5.2), (2.5.3)) is called a D-interior point (*resp.* a strong D-interior point, a weak-strong D-interior point with the pair (I_1, I_2)) of the CCS-system $\{D, C_i : i \in I\}$. Similarly, the notion of an interior point (*resp.* a strong interior point, a weak-strong interior point with the pair (I_1, I_2)) of the CCS-system $\{D, C_i : i \in I\}$ is defined.

2.6 Properties of Minkowski functional

Definition 2.6.1 (cf. [35, Page 4]). *Let C be an absorbing convex subset in a linear space X, i.e. for all $x \in X$, there exists $\lambda > 0$ such that $\lambda^{-1}x \in C$. The Minkowski functional of C is defined by:*

$$p_C(x) := \inf\{\lambda > 0 : \lambda^{-1}x \in C\}, \text{ for all } x \in X.$$

Remark 2.6.1. *By [17, 3C Lemma], Minkowski functional of an absorbing convex set is a sublinear functional.*

Remark 2.6.2. *If C is a set in a normed linear space such that $0 \in \operatorname{int} C$, then C is absorbing.*

Proposition 2.6.1 (cf. [35, Proposition 1.1.1]). *Let C be an absorbing convex subset with $0 \in \operatorname{int} C$ in a normed linear space X. Then the following statements are true:*

(i) $\operatorname{int} C = \{x : p_C(x) < 1\}$;

(ii) $\overline{C} = \{x : p_C(x) \leq 1\}$.

Proof. We first show that under the assumption $0 \in \operatorname{int} C$, the Minkowski functional is a continuous sublinear functional. By Remark 2.6.1, we need only to prove the continuity. First of all, by the assumption $0 \in \operatorname{int} C$, there exists $\alpha > 0$ such that $\alpha B \subseteq C$, where B denotes the unit ball. Thus for all $x \in X \backslash \{0\}$, $\frac{\alpha x}{\|x\|} \in C$. This implies for all $x \in X$,

$$p_C(x) \leq \frac{1}{\alpha}\|x\|, \tag{2.6.1}$$

(The inequality holds trivially if $x = 0$). Thus p_C is a continuous sublinear functional.

We now begin to prove **(i)**. It follows directly from the definition of Minkowski functionals and $0 \in C$ that

$$p_C(x) < 1 \Rightarrow x \in C. \tag{2.6.2}$$

On the other hand, by continuity of p_C, $\{x : p_C(x) < 1\}$ is open. Combining this with (2.6.2), we see that $\{x : p_C(x) < 1\} \subseteq \operatorname{int} C$. Conversely, let $x \in \operatorname{int} C \backslash \{0\}$. there exists a ball $B(x, \gamma) \subseteq C$ for some $\gamma > 0$. Note that $\|\frac{\|x\|+\gamma}{\|x\|}x - x\| = \gamma$. This implies

$\frac{\|x\|+\gamma}{\|x\|}x \in B(x,\gamma) \subseteq C$. Thus $p_C(x) \le \frac{1}{\frac{\|x\|+\gamma}{\|x\|}} < 1$. On the other hand, it is obvious that $p_C(0) = 0 < 1$. Combining these we get the converse inclusion $\operatorname{int} C \subseteq \{x : p_C(x) < 1\}$. These give **(i)**.

We now turn to prove **(ii)**. Let $p_C(x) \le 1$. Then by definition, for all $\epsilon > 0$, $\frac{x}{1+\epsilon} \in C$. This implies $x \in \overline{C}$. Conversely, let $x \in \overline{C}$. Then since $0 \in \operatorname{int} C$, $tx \in \operatorname{int} C$ for all $t \in (0,1)$ by convexity of C. By **(i)**, this means that for all $t \in (0,1)$, $p_C(tx) < 1$, which implies by sublinearity that $p_C(x) < \frac{1}{t}$. Letting $t \uparrow 1$, we see that $p_C(x) \le 1$. This completes the proof. □

2.7 Properties of Epigraphs

We collect some properties of epigraphs in this section. The following lemma is easy and the proof is standard.

Lemma 2.7.1. *Let f be a proper convex function defined on X^*. Then* $\overline{\operatorname{epi} f}^{w^*} = \overline{\operatorname{epi}_S f}^{w^*}$, *where* $\operatorname{epi}_S f := \{(x^*,\alpha) \in X^* \times \mathbb{R} : \alpha > f(x^*)\}$.

Proof. It follows readily from definition that $\overline{\operatorname{epi} f}^{w^*} \supseteq \overline{\operatorname{epi}_S f}^{w^*}$. To prove the converse inclusion, take $(x^*,\alpha) \in \overline{\operatorname{epi} f}^{w^*}$. Then there exists a net $(x_V^*, \alpha_V) \in \operatorname{epi} f$ with w^*-limit (x^*,α). Consider a net $\beta_V > 0$ with limit 0. Then $(x_V^*, \alpha_V + \beta_V) \in \operatorname{epi}_S f$ and $\lim(x_V^*, \alpha_V + \beta_V) = (x^*,\alpha)$. This shows that $(x^*,\alpha) \in \overline{\operatorname{epi}_S f}^{w^*}$, which completes the proof. □

We shall need the following definition in the next lemma.

Definition 2.7.1 (cf. [35, Theorem 2.1.3 (ix)]). *Let $\{f_i : 1 \le i \le n\}$ be proper convex lower semicontinuous functions on X. The infimum convolution is defined by*

$$(f_1 \Box f_2 \cdots \Box f_n)(x) := \inf\{\sum_{i=1}^{n} f_i(x_i) : \sum_{i=1}^{n} x_i = x\}$$

The formula (2.7.1) in the next lemma was mentioned in [10, Remark 2.1] for the case when $n = 2$ and in [8, Corollary 2.3] for the general case. In both papers, the functions are defined on a Banach space.

Lemma 2.7.2. *Let $\{f_i : 1 \le i \le n\}$ be proper convex lower semicontinuous functions on X with $\sum_{i=1}^n f_i$ being proper. Then*

$$\operatorname{epi}(\sum_{i=1}^n f_i)^* = \overline{\sum_{i=1}^n \operatorname{epi} f_i^*}^{w^*}. \tag{2.7.1}$$

Proof. Since f_i are proper for each i, by [35, Theorem 2.3.3], f_i^* are proper for all i. Noting that $g^{**} = g$ for convex lower semicontinuous functions g (see [35, Theorem 2.3.3]), we have for each $x^* \in X^*$ that

$$\begin{aligned}(f_1^*\Box f_2^* \cdots \Box f_n^*)^*(x) &= \sup_{x^*\in X^*} \{\langle x^*, x\rangle - (f_1^*\Box f_2^* \cdots \Box f_n^*)(x^*)\} \\ &= \sup_{x^*\in X^*} \sup\{\langle x^*, x\rangle - \sum_{i=1}^n f_i^*(x_i^*) : \sum_{i=1}^n x_i^* = x^*\} \\ &= \sup\{\sum_{i=1}^n [\langle x_i^*, x\rangle - f_i^*(x_i^*)] : \ x_i^* \in X^*, 1 \le i \le n\} \\ &= \sum_{i=1}^n f_i^{**}(x) \\ &= \sum_{i=1}^n f_i(x).\end{aligned}$$

Thus we have

$$(f_1^*\Box f_2^* \cdots \Box f_n^*)^* = \sum_{i=1}^n f_i.$$

Taking conjugations on both sides, we see that,

$$(f_1^*\Box f_2^* \cdots \Box f_n^*)^{**} = (\sum_{i=1}^n f_i)^*. \tag{2.7.2}$$

By properness of $\sum_{i=1}^n f_i$, there exists $\alpha \in \mathbb{R}$ and $x_0 \in X$ such that $\sum_{i=1}^n f_i(x_0) \le \alpha$. By definition of conjugations, we see that for all $x^* \in X^*$,

$$(f_1^*\Box f_2^* \cdots \Box f_n^*)(x^*) \ge \langle x^*, x_0\rangle - \sum_{i=1}^n f_i(x_0) \ge \langle x^*, x_0\rangle - \alpha.$$

It then follows from the definition that $\overline{\operatorname{co}(f_1^*\Box f_2^* \cdots \Box f_n^*)}^{w^*}(x^*) \ge \langle x^*, x_0\rangle - \alpha$. Combining this with the properness of f_i^* (which follows from the properness of f_i and [35, Theorem 2.3.3]), we see that $\overline{\operatorname{co}(f_1^*\Box f_2^* \cdots \Box f_n^*)}^{w^*}$, which is equal to $\overline{(f_1^*\Box f_2^* \cdots \Box f_n^*)}^{w^*}$

by convexity of infimum convolution (cf. [35, Theorem 2.1.3 (ix)]), is proper. By (2.7.2) and [35, Theorem 2.3.4 (i)], we obtain

$$\overline{\mathrm{epi}\,(f_1^*\Box f_2^* \cdots \Box f_n^*)}^{w^*} = \mathrm{epi}\,(\sum_{i=1}^n f_i)^*.$$

An application of [35, Theorem 2.1.3 (ix)] and Lemma 2.7.1 gives the desired result. □

The next proposition was proved in [9, Lemma 3.1] for two closed convex sets in a Banach space and in [21, Lemma 3.4] for finitely many closed convex sets in a Banach space.

Proposition 2.7.1. *Let $\{C_i : 1 \le i \le n\}$ be a collection of closed convex sets in X with $C := \bigcap_{i=1}^n C_i \neq \emptyset$. Then*

$$\mathrm{epi}\ \sigma_C = \overline{\sum_{i=1}^n \mathrm{epi}\ \sigma_{C_i}}^{w^*}. \tag{2.7.3}$$

Proof. Note that $\sum_{i=1}^n \delta_{C_i} = \delta_C$ and $\delta_C^* = \sigma_C$, and it follows from (2.7.1) that

$$\begin{aligned}\mathrm{epi}\,\sigma_C = \mathrm{epi}\,\delta_C^* &= \mathrm{epi}\,(\sum_{i=1}^n \delta_{C_i})^* \\ &= \overline{\sum_{i=1}^n \mathrm{epi}\,\delta_{C_i}^*}^{w^*}.\end{aligned}$$

□

The following reveals a relationship between the epigraph of support functions of a closed convex set and the normal cone of the closed convex set.

Proposition 2.7.2. *Let A be a closed convex set in a normed linear space X and let $x \in A$, $x^* \in X^*$. Then*

$$x^* \in N_A(x) \Longleftrightarrow \sigma_A(x^*) \le \langle x^*, x\rangle \Longleftrightarrow (x^*, \langle x^*, x\rangle) \in \mathrm{epi}\sigma_A. \tag{2.7.4}$$

Proof. The following equivalences hold:

$$\begin{aligned} & x^* \in N_A(x) \\ \Longleftrightarrow\quad & \langle x^*, a - x\rangle \le 0,\ \forall a \in A, \\ \Longleftrightarrow\quad & \sigma_A(x^*) \le \langle x^*, x\rangle \\ \Longleftrightarrow\quad & (x^*, \langle x^*, x\rangle) \in \mathrm{epi}\,\sigma_A. \end{aligned}$$

□

Chapter 3

The Strong Conical Hull Intersection Property (Strong CHIP): Definition and Some Properties

3.1 Introduction

Roughly speaking, the strong conical hull intersection property (strong CHIP) is a property that concerns the decomposition of the normal vectors of the set of intersection into the sum of normal vectors of the constituent sets. The concept of the strong CHIP naturally arises in different branches of optimization. Here are some examples.

Example 3.1.1. Consider the problem of minimizing a continuous convex function g defined on $\mathbb{R}^n$ subject to an abstract closed convex constraint set C. Suppose that x_0 is a minimizer to the problem. Then by the sum rule (cf. [35, Theorem 2.8.7 iii)]) and [35, Theorem 2.5.7], we have

$$0 \in \partial g(x_0) + N_C(x_0),$$

where $N_C(x_0)$ is the normal cone of C at x_0. If it is also known that $C = \bigcap_{i \in I} C_i$ for some system of closed convex sets $\{C_i : i \in I\}$, and that the system has the strong CHIP at x_0, then we have

$$0 \in \partial g(x_0) + \sum_{i \in I} N_{C_i}(x_0),$$

or equivalently, there exists $y^* \in \partial g(x_0)$, $x_i^* \in N_{C_i}(x_0)$, $i \in I$, with only finitely many of them being nonzero, such that

$$y^* + \sum_{i \in I} x_i^* = 0.$$

The next example illustrates the original motivation for proposing the strong CHIP.

Example 3.1.2. Suppose we are given two closed convex sets A and B in a Hilbert space X. Suppose that the projection onto A is much more easy to compute than the projection onto B or $A \cap B$, then it is natural to try to express $P_{A\cap B}(x)$ in terms of $P_A(x)$, where $P_{A\cap B}(x)$ and $P_A(x)$ are projections of the point x onto $A \cap B$ and A respectively. It turns out that if $\{A, B\}$ has the strong CHIP, then for each $x \in X$, there exists a point $b \in N_B(P_{A\cap B}(x))$ such that,

$$P_{A\cap B}(x) = P_A(x - b).$$

See Theorem 3.3.1 below.

The strong CHIP was originally proposed in connection with the projection property illustrated in example 2. See for example [14], [13], [12], [24], [25]. Later on some authors identified the importance of the strong CHIP in the characterization of the point of best approximation and the minimizers in many other optimization problems as illustrated in example 1. See for example [27]. Some authors related the study of the strong CHIP with other concepts in optimization like the basic contraint qualifications (BCQ) (see for example [24], [25]).

In this chapter, we shall first define the strong CHIP. Then we shall study how different authors relate the strong CHIP with other useful properties. Finally, we shall follow [13] (see also [14] and [12]) and try to produce a pair of sets with the strong CHIP from a pair without such property. Since properties of projections onto closed convex sets would be simpler in the Hilbert space setting, we shall mainly study the strong CHIP of sets in Hilbert space in this chapter. However, the definition of the strong CHIP is given for systems of closed convex sets in general normed linear spaces for later use. Readers who are interested in the relationship between the strong CHIP and projections onto closed convex sets when the ambient space is a general Banach space are referred to [25], [26] and [28].

3.2 Definition of the strong CHIP

The following definition is due to [13, Definition 3] when the index set is finite, and to [28, Definition 2.1] when the index set is arbitrary. The definition of infinite sum used here is given in Definition 2.4.2.

Definition 3.2.1. *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in a normed linear space X with nonempty intersection, I be an index set. The system is said to have the strong conical hull intersection property (the strong CHIP) at a point $x \in C := \bigcap_{i\in I} C_i$, if*

$$N_C(x) = \sum_{i\in I} N_{C_i}(x).$$

The system is said to have the strong CHIP if it has the strong CHIP at every point in the intersection.

This concept was first introduced in [13, Definition 2.3] under a Hilbert space setting. Later on, Li and Ng studied this concept when X is a normed linear space (see for example [25] and [28]).

The next proposition, originally stated for sets in Hilbert spaces (see [13, Lemma 2.4]), gives some equivalent conditions for the strong CHIP.

Theorem 3.2.1. *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in a normed linear space X with nonempty intersection, I be an index set. Write $C := \bigcap_{i\in I} C_i$. Then the following statements are equivalent:*

(i) *$\{C_i : i \in I\}$ has the strong CHIP.*

(ii) *$N_C(x) \subseteq \sum_{i\in I} N_{C_i}(x)$, for all $x \in C$.*

(iii) $(C-x)^{\ominus} = \sum_{i\in I}(C_i - x)^{\ominus}$ *for all* $x \in C$.

(iv) $\overline{\operatorname{cone}(C-x)} = \bigcap_{i\in I}\overline{\operatorname{cone}(C_i - x)}$ *for all* $x \in C$, *and* $\sum_{i\in I}(C_i-x)^{\ominus}$ *is* w^**−closed for all* $x \in C$.

Proof. The equivalence between **(i)** and **(ii)** is immediate from Proposition 2.3.1 **(ii)**, while the equivalence between **(i)** and **(iii)** follows from Proposition 2.4.3 **(i)**.

We now prove the equivalence between **(iii)** and **(iv)**. Suppose first that **(iii)** is true. Taking polar on both sides of the equation in **(iii)** and invoking Proposition 2.4.2 **(iii)**, we see that for each $x \in C$,

$$((C-x)^{\ominus})^{\ominus} = \bigcap_{i\in I}((C_i - x)^{\ominus})^{\ominus},$$

which gives the first half of **(iv)** by Proposition 2.4.2 **(i)**. Now observe that $\sum_{i\in I}(C_i - x)^{\ominus}$ is w^*−closed for all $x \in C$ since $(C-x)^{\ominus}$ is w^*−closed for all $x \in C$. This gives the second half of **(iv)**. Thus the implication **(iii)**⇒**(iv)** is proved. Now we turn to prove **(iv)**⇒**(iii)**. Fix any $x \in C$. Taking polar on both sides of $\overline{\operatorname{cone}(C-x)} = \bigcap_{i\in I}\overline{\operatorname{cone}(C_i - x)}$ and applying part **(iii)** of Proposition 2.4.2, we get

$$(C-x)^{\ominus} = \overline{\sum_{i\in I}(C_i - x)^{\ominus}}^{w^*}.$$

Since $\sum_{i\in I}(C_i - x)^{\ominus}$ is w^*−closed by assumption, part **(iii)** follows. This completes the proof. □

The following proposition, though simple, will be used several times in the next section.

Proposition 3.2.1. *Let* $\{C_i : i \in I\}$ *be a family of closed convex sets in a normed linear space* X, I *be an index set. Suppose there exists an affine space* Z *such that* $C_i \subseteq Z$ *for all* $i \in I$. *If* $\{C_i : i \in I\}$ *has the strong CHIP as subsets in* Z, *then* $\{C_i : i \in I\}$ *has the strong CHIP.*

Proof. Write $C := \bigcap_{i\in I} C_i$. Fix $x_0 \in C$ and let $y^* \in N_C(x_0)$. By translation, we may assume without loss of generality that $x_0 = 0$. Then Z is a subspace containing C_i for all $i \in I$. Then $y^*|_Z \in N_C(0)|_Z$. By Proposition 2.3.2 and the strong CHIP assumption, there exists $u_i^* \in N_{C_i}(0)|_Z$ for each $i \in I$ such that

$$y^*|_Z = \sum_{i \in I} u_i^*. \tag{3.2.1}$$

By the Hahn-Banach Theorem, there exists, for each $i \in I$, an extension $\tilde{u}_i^*$ of u_i^* to X^* with norm $\|u_i^*\|$. Since $C_i \subseteq Z$ for all $i \in I$, it follows directly from definition of normal cones (see (2.3.1)) that $\tilde{u}_i^* \in N_{C_i}(0)$ for $i \in I$. It now follows from (3.2.1) that

$$y^* \in \sum_{i \in I} \tilde{u}_i^* + Z^{\perp} \subseteq \sum_{i \in I} N_{C_i}(0) + Z^{\perp}. \tag{3.2.2}$$

Since $C_i \subseteq Z$ for all $i \in I$, we have $Z^{\perp} \subseteq N_{C_i}(0)$ for all i and thus (3.2.2) gives

$$y^* \in \sum_{i \in I} N_{C_i}(0),$$

which completes the proof. □

3.3 Relationship between the strong CHIP and projections onto sets

We start by proving the following theorem, from which we shall obtain several related results in the literature as corollaries.

Theorem 3.3.1. *Let $\{E, S_i : i \in I\}$ be a collection of closed convex sets in a Hilbert space X with nonempty intersection, I be an index set. Then the following statements are equivalent:*

(i) *$\{E, S_i : i \in I\}$ has the strong CHIP.*

(ii) *For all x and x_0 in X, $P_{E \cap \bigcap_{i \in I} S_i}(x) = x_0$ if and only if there exist a finite set $I_0 \subseteq I$, $x_i \in N_{S_i}(x_0)$ for each $i \in I_0$ such that $P_E(x - \sum_{i \in I_0} x_i) = x_0$.*

Proof. Let $S := E \cap \bigcap_{i \in I} S_i$. Then, by the given assumptions, S is a nonempty closed convex set.

(i)⇒ (ii) Let $x \in X$ and $x_0 \in S$ be such that $P_S(x) = x_0$. By **(i)**, we have

$$N_S(x_0) = N_E(x_0) + \sum_{i \in I} N_{S_i}(x_0). \tag{3.3.1}$$

By Proposition 2.3.1 **(v)**, it follows that the following equivalences hold:

$$x_0 = P_S(x)$$

$$\Leftrightarrow \quad x - x_0 \in N_S(x_0)$$

$\Leftrightarrow$ $\exists x_i \in N_S(x_0) \forall i \in I$ with finitely many nonzero such that

$$x - x_0 - \sum_{i=I} x_i \in N_E(x_0), \tag{3.3.2}$$

where (3.3.2) can be equivalently rewritten as

$$x_0 = P_E(x - \sum_{i \in I} x_i). \tag{3.3.3}$$

(ii)⇒(i) Let $x_0 \in S$. By Theorem 3.2.1 **(ii)**, we need only to show that

$$N_S(x_0) \subseteq N_E(x_0) + \sum_{i \in I} N_{S_i}(x_0).$$

Let $z \in N_S(x_0)$. Then, by Proposition 2.3.1 **(v)**, $P_S(z + x_0) = x_0$. By assumption in **(ii)**, there exist a finite set $I_0 \subseteq I$, $x_i \in N_{S_i}(x_0)$ for each $i \in I_0$ such that $P_E(x_0 + z - \sum_{i \in I_0} x_i) = x_0$. By Proposition 2.3.1 **(v)** again, the last equation is equivalent to $x_0 + z - \sum_{i \in I_0} x_i - x_0 \in N_E(x_0)$. This implies that

$$z \in N_E(x_0) + \sum_{i \in I_0} x_i \subseteq N_E(x_0) + \sum_{i \in I} N_{S_i}(x_0),$$

which completes the proof. □

The first corollary is the implication $1 \Leftrightarrow 2$ of [14, Theorem 3.2].

Corollary 3.3.1. *Let $\{H_i : 1 \leq i \leq n\}$ be a collection of translations of closed half-spaces in a Hilbert space X with nonempty intersection, i.e. $H_i := \{x \in X : \langle h_i, x \rangle \leq b_i\}$ for some $h_i \in X$, $b_i \in \mathbb{R}$, $1 \leq i \leq n$. Suppose C is a closed convex subset such that $C \cap \bigcap_{i=1}^{n} H_i \neq \emptyset$. Then the following statements are equivalent:*

(i) *$\{C, H_i : 1 \leq i \leq n\}$ has the strong CHIP.*

(ii) *For all x and x_0 in X, $P_{C\cap\bigcap_{i=1}^n H_i}(x) = x_0$ if and only if there exist $\lambda_i \geq 0$ for each $1 \leq i \leq n$ such that $P_C(x - \sum_{i=1}^n \lambda_i h_i) = x_0$, with $\lambda_i = 0$ for all $1 \leq i \leq n$ such that $\langle h_i, x_0\rangle = b_i$.*

Proof. In view of Theorem 3.3.1, it suffices to show that for each $1 \leq i \leq n$,

$$N_{H_i}(x_0) = \begin{cases} \operatorname{cone} h_i, & \text{if } x_0 \in \operatorname{bdry} H_i \\ \{0\}, & \text{otherwise} \end{cases}.$$

If $x_0 \in \operatorname{int} H_i$, since $H_i - x_0$ contains a neighborhood of zero, it follows that $N_{H_i}(x_0) = (H_i - x_0)^\ominus = \{0\}$. For $x_0 \in \operatorname{bdry} H_i$, it follows that $\langle h_i, x_0\rangle = b_i$. Thus $N_{H_i}(x_0) = (H_i - x_0)^\ominus = \{h \in X : \langle h, x\rangle \leq 0, \forall x \text{ such that } \langle h_i, x\rangle \leq 0\} = \operatorname{cone} h_i$. □

The next corollary is [25, Theorem 4.2] for the case when the scalar field is the field of real numbers.

Corollary 3.3.2. *Let $\{S_i : i \in I\}$ be a collection of sets in a Hilbert space X with nonempty intersection in the form $S_i := \{x \in X : \langle h_i, x\rangle \in \Omega_i\}$ for some $h_i \in X$, $\Omega_i \subseteq \mathbb{R}$ with Ω_i being closed and convex, $i \in I$. Suppose C is a closed convex subset such that $C \cap \bigcap_{i\in I} S_i \neq \emptyset$. Then the following statements are equivalent:*

(i) *$\{C, S_i : i \in I\}$ has the strong CHIP.*

(ii) *For all x and x_0 in X, $P_{C\cap\bigcap_{i\in I} S_i}(x) = x_0$ if and only if there exist $\lambda_i \in N_{\Omega_i}(\langle h_i, x_0\rangle)$ for each $i \in I$ such that $P_C(x - \sum_{i\in I} \lambda_i h_i) = x_0$.*

Proof. In view of Theorem 3.3.1 again, it suffices to show that for each $i \in I$,

$$N_{S_i}(x_0) = \{\lambda_i h_i : \lambda_i \in N_{\Omega_i}(\langle h_i, x_0\rangle)\}. \quad (3.3.4)$$

Note that $y \in N_{S_i}(x_0)$ is equivalent to

$$\langle y, x - x_0\rangle \leq 0, \ \forall x \in S_i.$$

Thus $y \in N_{S_i}(x_0)$ if and only if x_0 is a minimizer of the following optimization problem:

$$\text{Minimize}_{x\in X} \quad -\langle y, x\rangle + \delta_{\Omega_i}(\langle h_i, x\rangle)$$

By [35, Theorem 2.5.7] and [35, Theorem 2.8.3], this is further equivalent to:

$$\begin{aligned} & 0 \ \in \partial(-\langle y, \cdot\rangle)(x_0) + \partial\, \delta_{\Omega_i}(\langle h_i, \cdot\rangle)(x_0) \\ \Leftrightarrow\ & 0 \ \in -y + N_{\Omega_i}(\langle h_i, x_0\rangle)h_i, \end{aligned}$$

from which (3.3.4) follows. This completes the proof. □

The next corollary is [13, Theorem 3.2]. Before proving it, we need to prove a simple lemma on computation of normal cones. Given a continuous linear map $A : X \to Y$ from a Hilbert space X to another Hilbert space Y, we write $A^* : Y^* \to X^*$ as its conjugate map. We also write $\ker A := \{x \in X : Ax = 0\}$ and $R(A^*) := \{x^* \in X^* : \exists y^* \in Y^*, A^*y^* = x^*\}$.

Lemma 3.3.1. *Let $A : X \to Y$ be a continuous linear map from a Hilbert space X to another Hilbert space Y and $b \in Y$. Suppose that A^* has closed range. Then*

$$N_{A^{-1}b}(x) = R(A^*),\ \forall x \in A^{-1}b.$$

Proof. Let $x \in A^{-1}b$. Note that $A^{-1}b - x = \ker A$. Thus

$$\begin{aligned} N_{A^{-1}b}(x) &= (\ker A)^{\ominus} \\ &= (\ker A)^{\perp} \\ &= \overline{R(A^*)} \\ &= R(A^*), \end{aligned}$$

where the first equality follows from Proposition 2.4.2 **(i)**, the second equality is true since $\ker A$ is a subspace, while the third equality follows from (cf. [12, Lemma 8.33]) and the last one from the assumption. This completes the proof. □

It is easy to see that for any continuous linear operator $A : X \to Y$ between two Hilbert spaces, $R(A^*)$ is closed if Y is finite dimensional. This fact is to be used in the next corollary.

Corollary 3.3.3 ([13, Theorem 3.2]). *Let C be a closed convex set in a Hilbert space X, $A : X \to Y$ a continuous linear map from the Hilbert space X to a finite dimensional Hilbert space Y and $b \in Y$. Then the following statements are equivalent:*

(i) $\{C, A^{-1}b\}$ *has the strong CHIP;*

(ii) *For every* $x \in X$*, there exists* $y \in Y$ *such that*

$$A[P_C(x + A^*y)] = b; \tag{3.3.5}$$

(iii) *For every* $x \in X$*, there exists* $y \in Y$ *such that*

$$P_{C \cap A^{-1}b}(x) = P_C(x + A^*y). \tag{3.3.6}$$

Moreover, for the same $x \in X$*, the corresponding* y *in part* **(ii)** *and* **(iii)** *can be taken to be the same.*

Proof. We start by proving the equivalence of **(i)** and **(iii)**. Suppose that **(i)** is true. In view of the implication **(i)⇒(ii)** of Theorem 3.3.1, there exists $v \in N_{A^{-1}b}(P_{C \cap A^{-1}b}(x))$ such that $P_C(x - v) = P_{C \cap A^{-1}b}(x)$. On the other hand, by Lemma 3.3.1, there exists $y \in X$ such that $v = -A^*y$. Thus **(iii)** follows. We now prove the converse implication. Suppose **(iii)** holds. Since by Lemma 3.3.1, $N_{A^{-1}b}(P_{C \cap A^{-1}b}(x)) = R(A^*) = \{A^*y : y \in X\}$, condition **(iii)** implies that for any $x, x_0 \in X$,

$$P_{C \cap A^{-1}b}(x) = x_0 \iff \text{there exist } u \in N_{A^{-1}b}(P_{C \cap A^{-1}b}(x)) \text{ such that } P_C(x - u) = x_0.$$

it follows from the implication **(ii)⇒(i)** of Theorem 3.3.1 that **(i)** holds. This proves **(i)⇔(iii)**.

We now turn to the equivalence of **(iii)** and **(ii)**. Fix $x \in X$. Suppose first that **(iii)** is true for some y. Then we have,

$$P_C(x + A^*y) \in A^{-1}b \cap C. \tag{3.3.7}$$

This implies in particular that $P_C(x + A^*y) \in A^{-1}b$ and thus $A[P_C(x + A^*y)] = b$, proving **(iii)⇒(ii)**. Now suppose conversely that **(ii)** is true for some y. By **(v)** of Proposition 2.3.1, we have

$$x + A^*y - P_C(x + A^*y) \in N_C(P_C(x + A^*y)).$$

In view of Lemma 3.3.1 and $A[P_C(x + A^*y)] = b$, we see that $-A^*y \in R(A^*) = N_{A^{-1}b}(P_C(x + A^*y))$. This gives

$$x - P_C(x + A^*y) \in N_C(P_C(x + A^*y)) - A^*y \subseteq N_C(P_C(x + A^*y)) + N_{A^{-1}b}(P_C(x + A^*y)).$$

Combining this with Proposition 2.3.1 **(ii)**, we obtain

$$x - P_C(x + A^*y) \in N_{A^{-1}b\cap C}(P_C(x + A^*y)).$$

By Proposition 2.3.1 **(v)** again, we obtain $P_{A^{-1}b\cap C}(x) = P_C(x + A^*y)$. This completes the proof. □

Remark 3.3.1. *Actually, in [13], the authors devoted most parts of the paper studying the pair of closed convex sets* $\{C, A^{-1}b\}$ *rather than the family of closed convex sets* $\{C_1, \cdots, C_n\}$.

Besides projections onto closed convex sets, relationship between the strong CHIP and projections onto some kinds of nonconvex sets was also addressed in the literature, see for example [24] for the case of Hilbert space. As before, we concentrate mainly on the case when the ambient space is a Hilbert space. Let D be a closed convex set in a Hilbert space X, A_i be Fréchet differentiable functions on X, $b_i \in \mathbb{R}$, $i \in I$, $I_1 \cup I_2 = I$ and $I_1 \cap I_2 = \emptyset$. Write $K := D \cap \bigcap_{i\in I_1}\{x : A_i(x) = b_i\} \cap \bigcap_{i\in I_2}\{x : A_i(x) \le b_i\}$.

Definition 3.3.1 ([24, Definition 3.2, Definition 3.3]). *Let* $x \in K$. *A vector* d *is called a linearized feasible direction of* K *at* x *if*

$$\langle d, \nabla A_i(x)\rangle = 0, \ \forall i \in I_1,$$
$$\langle d, \nabla A_i(x)\rangle \le 0, \ \forall i \in I(x),$$

where $\nabla A_i(x)$ *is the Fréchet derivative of* A_i *at* x, $I(x)$ *is the active index set for* x, *i.e.* $I(x) := \{i \in I_2 : A_i(x) = b_i\}$. *The set of all linearized feasible directions of* K *at* x *is denoted by* $\mathrm{LFD}(x)$.

A vector d *is called a sequentially feasible direction of* K *at* x *if there exist a sequence* $\{d_k\} \subseteq X$ *and a sequence* $\{\delta_k\}$ *of real positive numbers such that*

$$d_k \to d, \quad \delta_k \to 0, \quad x + \delta_k d_k \in K, \ \forall k \in \mathbb{N}.$$

The set of all sequentially feasible directions of K *at* x *is denoted by* $\mathrm{SFD}(x)$.

Definition 3.3.2 (See the remark below [24, Proposition 3.1]). *For* $x_0 \in K$, *define*

$$K_S(x_0) = \overline{\mathrm{co}(x_0 + \mathrm{SFD}(x_0))} \cap D,$$
$$K_L(x_0) = (x_0 + \mathrm{LFD}(x_0)) \cap D.$$

Lemma 3.3.2 (See the remark below [24, Proposition 3.1]). *For $x_0 \in K$, $K_S(x_0) \subseteq K_L(x_0)$.*

Proof. Fix an $x_0 \in K$. We first show that $\mathrm{SFD}(x_0) \subseteq \mathrm{LFD}(x_0)$. Let $d \in \mathrm{SFD}(x_0)$. Then there exist $t_n \downarrow 0$ and $d_n \to d$ such that $x_0 + t_n d_n \in K$. For $i \in I_1$, $A_i(x_0+t_n d_n) - A_i(x_0) = 0 - 0 = 0$. This gives $\langle d, \nabla A_i(x_0)\rangle = 0$. For $i \in I(x_0)$, we have $A_i(x_0) = 0$. Thus $A_i(x_0 + t_n d_n) = A_i(x_0 + t_n d_n) - A_i(x_0) \leq 0$, which gives $\langle d, \nabla A_i(x_0)\rangle \leq 0$. Combining the last two sentences we have $d \in \mathrm{LFD}(x_0)$. Thus we have shown that $d \in \mathrm{LFD}(x_0)$ and hence that $\mathrm{SFD}(x_0) \subseteq \mathrm{LFD}(x_0)$. This implies that

$$\overline{\mathrm{co}(x_0 + \mathrm{SFD}(x_0))} \subseteq \overline{\mathrm{co}(x_0 + \mathrm{LFD}(x_0))}. \tag{3.3.8}$$

However, as is easily checked, $\mathrm{LFD}(x_0)$ is a closed convex set. Thus (3.3.8) gives

$$\overline{\mathrm{co}(x_0 + \mathrm{SFD}(x_0))} \subseteq x_0 + \mathrm{LFD}(x_0),$$

which gives the desired result. □

Proposition 3.3.1 ([24, Corollary 3.1]). *Let $x_0 \in K$. Consider the following statements:*

(i) *$K \subseteq K_L(x_0)$ and $K_S(x_0) = K_L(x_0)$;*

(ii) *For any $x \in X$, $x_0 \in P_K(x) \Leftrightarrow P_{K_L}(x) = x_0$;*

(iii) *For any $x \in X$, $x_0 \in P_K(x) \Rightarrow P_{K_L}(x) = x_0$.*

Then **(i)**$\Rightarrow$**(ii)**$\Rightarrow$**(iii)**. *If assume in addition that $K \subseteq K_S(x_0)$, then the three statements are equivalent.*

Proof. We first prove **(i)**$\Rightarrow$**(ii)**. Let $x_0 \in P_K(x)$ and $y \in x_0 + \mathrm{SFD}(x_0)$. We wish to show that $P_{K_L(x_0)}(x) = x_0$. By definition of sequential feasible directions, there exist a sequence $t_n \downarrow 0$ and $d_n \to y - x_0$ such that $x_0 + t_n d_n \in K$ for all n. By definition of projections, we obtain

$$\|x - x_0\| \leq \|x - (x_0 + t_n d_n)\|, \ \forall n \in \mathbb{N}.$$

Squaring both sides and expanding, we get

$$2t_n\langle d_n, x_0 - x\rangle + t_n^2\|d_n\|^2 \geq 0, \ \forall n.$$

Divide both sides by t_n and let n go to infinity, we see that for each $y \in x_0 + \text{SFD}(x_0)$, we have

$$\langle x - x_0, y - x_0 \rangle \leq 0. \tag{3.3.9}$$

It now follows readily from the bilinearity and continuity of inner product that (3.3.9) is true for all $y \in \overline{\text{co}(x_0 + \text{SFD}(x_0))}$. Invoking Proposition 2.3.1 and thanks to $K_S(x_0) \subseteq \overline{\text{co}(x_0 + \text{SFD}(x_0))}$ (see Definition 3.3.2), we obtain $x_0 = P_{K_S(x_0)}(x)$. Since by assumption $K_S(x_0) = K_L(x_0)$ given in **(i)**, we have shown that $x_0 = P_{K_L(x_0)}(x)$. Conversely, let $x_0 = P_{K_L(x_0)}(x)$. Since by assumption given in **(i)**, $K \subseteq K_L(x_0)$, we obtain $x_0 \in P_K(x)$ as desired. This finishes the proof for **(i)⇒(ii)**.

Since **(ii)⇒(iii)** is obvious, what remains is to show that the three statements are equivalent under the additional assumption $K \subseteq K_S(x_0)$.

Suppose that **(iii)** holds. Then for all $x \in X$,

$$P_{K_S(x_0)}(x) = x_0 \Rightarrow x_0 \in P_K(x) \Rightarrow P_{K_L(x_0)}(x) = x_0. \tag{3.3.10}$$

We claim that $K_L(x_0) \subseteq K_S(x_0)$. Suppose on the contrary that this is not true. Then there exists a $y \in K_L(x_0) \backslash K_S(x_0)$. Since by Definition 3.3.2, we have $K_L(x_0) \subseteq D$, it follows that $y \notin G$, here we use $G := \overline{\text{co}(x_0 + \text{SFD}(x_0))} = x_0 + \overline{\text{co}(\text{SFD}(x_0))}$. Thus G is a translate of a closed convex cone. Let $u = P_G(y)$. By Proposition 2.3.1 **(vi)**, we see that $x_0 = P_G(y + x_0 - u)$. Combining this with Proposition 2.3.1 **(v)**, we see that

$$y - u \in N_G(x_0) \tag{3.3.11}$$

It now follows from (3.3.11) and [35, Corollary 3.8.5] that $x_0 = P_G(x_t)$, where $x_t := x_0 + t(y - u)$ for $t > 1$. This implies $x_0 = P_{K_S(x_0)}(x_t)$ since $x_0 \in K_S(x_0) \subseteq G$ (see Definition 3.3.2). To finish the proof, we shall show however that, $x_0 \neq P_{K_L(x_0)}(x_t)$ for large t and this will contradict (3.3.10). To see this, in view of $u = P_G(y)$ and $x_0 \in G$, we have by definition of normal cones that,

$$\langle x_0 - u, y - u \rangle \leq 0.$$

Thus we obtain

$$\begin{aligned}\|x_t - y\|^2 &= \|x_0 + t(y-u) - y\|^2 \\ &= \|x_0 - u + (t-1)(y-u)\|^2 \\ &= (t-1)^2\|y-u\|^2 + 2(t-1)\langle x_0 - u, y-u\rangle + \|x_0 - u\|^2 \\ &\le (t-1)^2\|y-u\|^2 + \|x_0-u\|^2 \\ &< t^2\|y-u\|^2 = \|x_t - x_0\|^2,\end{aligned}$$

for large t. This implies $x_0 \notin P_{K_L(x_0)}(x_t)$ for large t and leads to the desired contradiction. Thus $K_L(x_0) \subseteq K_S(x_0)$. Combining with Lemma 3.3.2, we get $K_L(x_0) = K_S(x_0)$. Since it is assumed that $K \subseteq K_S(x_0)$, we obtain $K \subseteq K_L(x_0)$. **(iii)⇒(i)** is proved. □

To state the next theorem, we introduce the following notations, as was done in [24]. Based on the notations used in Definition 3.3.1, given $x_0 \in K$, we write $H_i := \{h : \langle h, \nabla A_i(x_0)\rangle \le b_i - A_i(x_0) + \langle x_0, \nabla A_i(x_0)\rangle\}$ for $i \in I_2$, $P_i := \{h : \langle h, \nabla A_i(x_0)\rangle = \langle x_0, \nabla A_i(x_0)\rangle\}$ for $i \in I_1$. Note that in this case

$$\begin{aligned}x_0 + \mathrm{LFD}(x_0) &= \{x_0 + d : \langle x_0 + d, \nabla A_i(x_0)\rangle \le \langle x_0, \nabla A_i(x_0)\rangle, i \in I(x_0), \\ &\qquad \langle x_0 + d, \nabla A_i(x_0)\rangle = \langle x_0, \nabla A_i(x_0)\rangle, i \in I_1\} \\ &= \{h : \langle h, \nabla A_i(x_0)\rangle \le \langle x_0, \nabla A_i(x_0)\rangle, i \in I(x_0), \\ &\qquad \langle h, \nabla A_i(x_0)\rangle = \langle x_0, \nabla A_i(x_0)\rangle, i \in I_1\} \\ &= \bigcap_{i\in I(x_0)} H_i \cap \bigcap_{j\in I_1} P_j. \qquad (3.3.12)\end{aligned}$$

Now we are ready for the next theorem.

Theorem 3.3.2 (cf. [24, Theorem 4.1]). *Let $x_0 \in K$. Suppose that $K \subseteq K_L(x_0)$ and $K_S(x_0) = K_L(x_0)$. Then the following statements are equivalent:*

(i) *$\{D, P_i, H_j : i \in I_1, j \in I(x_0)\}$ has the strong CHIP at x_0;*

(ii) *$\{D, P_i, H_j : i \in I_1, j \in I_2\}$ has the strong CHIP at x_0;*

(iii) *For any $x, x_0 \in X$, $x_0 \in P_K(x)$ is equivalent to the following statement:*

$$\exists \lambda_i \ge 0,\ i \in I_2, \lambda_i = 0,\ i \in I_1 \text{ such that } x_0 = P_D(x - \sum_{i\in I}\lambda_i \nabla A_i(x_0)).$$

Proof. Note that for $x_0 \in H_j$ with $j \notin I(x_0)$, $x_0 \in \text{int} H_j$ since $b_j > A_j(x_0)$ and $\nabla A_j(x_0)$ is a continuous linear map. Thus the equivalence of **(i)** and **(ii)** follows from Theorem 4.2.3 in the next chapter. We now show that **(i)** is equivalent to **(iii)**. By Proposition 3.3.1, statement **(iii)** is equivalent to

(iii*) For any $x, x_0 \in X$, $x_0 \in P_{K_L}(x)$ is equivalent to the following statement:

$$\exists \lambda_i \geq 0,\ i \in I_2, \lambda_i = 0,\ i \in I_1 \text{ such that } x_0 = P_D(x - \sum_{i \in I} \lambda_i \nabla A_i(x_0)),$$

where $K_L(x_0) = D \cap \bigcap_{i \in I(x_0)} H_i \cap \bigcap_{j \in I_1} P_j$ according to (3.3.12). Now, the equivalence between **(i)** and **(iii*)** will follow from Theorem 3.3.1 once we establish the following relationship:

(I) $N_{P_j}(x_0) = \{\lambda \nabla A_j(x_0) : \lambda \in \mathbb{R}\}$, $j \in I_1$;

(II) $N_{H_i}(x_0) = \begin{cases} \text{cone}\, \nabla A_i(x_0), & i \in I(x_0) \\ \{0\}, & i \notin I(x_0) \end{cases}$

We first prove **(I)**. Note that $N_{P_i}(x_0) = (P_i - x_0)^{\ominus} = P_i^{\perp}$ since P_i are subspaces, **(I)** follows. We now prove **(II)**. If $i \notin I(x_0)$, then $x_0 \in \text{int}\, H_i$, since $H_i - x_0$ contains a neighborhood of zero, it follows that $N_{H_i}(x_0) = (H_i - x_0)^{\ominus} = \{0\}$. Otherwise, $x_0 \in \text{bdry}\, H_i$. It follows that $\langle \nabla A_i(x_0), x_0 \rangle = b_i$. Thus $N_{H_i}(x_0) = (H_i - x_0)^{\ominus} = \{h \in X : \langle h, x \rangle \leq 0, \forall x \text{ such that } \langle \nabla A_i(x_0), x \rangle \leq 0\} = \text{cone}\, \nabla A_i(x_0)$, **(II)** follows. This completes the proof. □

3.4 Relationship between the strong CHIP and the Basic Constraint Qualifications (BCQ)

In subsequent years, the concept of the strong CHIP has been studied by some authors together with the concept of the basic constraint qualifications (BCQ) (c.f [21], [25]). We first give the definition of the BCQ. The original definition was given for continuous convex functions in $\mathbb{R}^n$ (see [30, Definition 2.1 b)]). We restate it for continuous convex functions in normed linear spaces.

Definition 3.4.1. *Let $\{g_i : i \in I\}$ be a family of continuous convex functions in a normed linear space X and $G(x) := \sup_{i\in I} g_i(x) < +\infty$ for all $x \in X$. Write $C_i := \{x : g_i(x) \le 0\}$ for each $i \in I$, $C := \bigcap_{i\in I} C_i = \{x : G(x) \le 0\}$. The BCQ is said to hold at a point $x_0 \in C$ if*

$$N_C(x_0) = \sum_{i\in I(x_0)} \text{cone}\,(\partial g_i(x_0)), \tag{3.4.1}$$

where $I(x_0) := \{i \in I : g_i(x_0) = G(x_0) = 0\}$. Here we adopt the convention that summing over a null index set equals $\{0\}$.

The main clue that leads people to relate the strong CHIP and the BCQ lies in the fact that if G satisfies some good conditions, like the Slater condition, i.e.

$$\exists\, \overline{x} \text{ such that } G(\overline{x}) < 0,$$

then for each $i \in I$ and each $x_0 \in \{x : G(x) \le 0\}$, we have (c.f [11, Theorem 2.4.7, Corollary 1])

$$N_{C_i}(x_0) = \text{cone}\,(\partial g_i(x_0)).$$

Under these assumptions, the equation (3.4.1) is just a restatement of the strong CHIP of $\{C_i : i \in I\}$, where $C_i := \{x \in X : g_i(x) \le 0\}$. One natural question is that, given a family of closed convex sets $\{C_i : i \in I\}$ having the strong CHIP, is it possible to construct continuous convex functions g_i with corresponding lower level sets C_i, $i \in I$, so that $\{g_i : i \in I\}$ satisfies the BCQ? This turns out to be true when the system of closed convex sets has suitably good properties. In [28], the authors exploited this to study the strong CHIP for a system of infinitely many closed convex sets. This is to be discussed in the second part of Chapter 3.

In [25], the authors established a relationship between the strong CHIP for a system of closed convex sets and the BCQ for a system of continuous convex functions in a Banach space X. These results have been applied to obtain new characterizations for the minimizers of a best approximation problem in the Banach space of all continuous functions defined on a compact set. See [25, Section 5] for details.

To state the results, we need the following definition. For the remainder of this subsection, X will denote a Banach space. Let $\{g_i : i \in I\}$ be a family of continuous convex functions in a Banach space X and $G(x) := \sup_{i\in I} g_i(x) < +\infty$ for all $x \in X$, D be a closed convex set with $D \cap \{x : G(x) \leq 0\} \neq \emptyset$. Write $C_i := \{x : g_i(x) \leq 0\}$ for each $i \in I$, $C := \bigcap_{i\in I} C_i = \{x : G(x) \leq 0\}$.

Definition 3.4.2 ([25, Definition 2.1]). *The system $\{g_i : i \in I\}$ is said to satisfy the BCQ relative to D at a point x_0 if*

$$N_C(x_0) = N_D(x_0) + \sum_{i\in I(x_0)} \text{cone}\,(\partial g_i(x_0)), \tag{3.4.2}$$

where $I(x_0) := \{i \in I : g_i(x_0) = G(x_0) = 0\}$. Here again we adopt the convention that summing over a null index set equals $\{0\}$.

Definition 3.4.3 ([25, Definition 2.4]). *Let $x \in D \cap C$. An element $d \in X$ is called*

(i) *a linearized feasible direction of the system $\{g_i : i \in I\}$ at x if*

$$\langle x^*, x\rangle \leq 0,\ \forall x^* \in \bigcup_{i\in I(x)} \partial\, g_i(x_0).$$

(ii) *a sequentially feasible direction of $D \cap C$ at x if there exists a sequence $d_k \to d$ and a sequence of positive real numbers $t_k \to 0$ such that $x + t_k d_k \in D \cap C$ for all k.*

Definition 3.4.4 ([25, Definition 2.5]). *For $x \in D \cap C$, define*

$$K_S(x) = \overline{\text{co}(x + \text{SFD}(x))} \cap D,$$
$$K_L(x) = (x + \text{LFD}(x)) \cap D.$$

Lemma 3.4.1 ([25, Proposition 2.1]). *For $x \in K := D \cap C$, $K \subseteq K_S(x) \subseteq K_L(x)$.*

Proof. Fix an $x \in K$. We first show that $\text{SFD}(x) \subseteq \text{LFD}(x)$. Let $d \in \text{SFD}(x)$. Then there exist $t_n \downarrow 0$ and $d_n \to d$ such that $x + t_n d_n \in K$. For $i \in I(x)$, we have $g_i(x) = 0$. Thus $g_i(x + t_n d_n) - g_i(x) = g_i(x + t_n d_n) \leq 0$, which gives $\langle z^*, d_n\rangle \leq 0$ for all $z^* \in \bigcup_{i\in I(x)} \partial\, g_i(x)$. Letting $n \to \infty$, we have $d \in \text{LFD}(x)$. Thus we have shown that $\text{SFD}(x) \subseteq \text{LFD}(x)$. This implies that

$$\overline{\text{co}(x + \text{SFD}(x))} \subseteq \overline{\text{co}(x + \text{LFD}(x))}. \tag{3.4.3}$$

However, as is easily checked, $\mathrm{LFD}(x)$ is a closed convex set. Thus (3.4.3) gives

$$\overline{\mathrm{co}(x+\mathrm{SFD}(x))} \subseteq x + \mathrm{LFD}(x),$$

which gives $K_S(x) \subseteq K_L(x)$. It now remains to show that $K \subseteq K_S(x)$. Let $d \in K$. Then since $x \in K$, we have $x + t(d-x) \in K$ for all $t \in (0,1)$ by convexity. This implies by definition that $d - x \in \mathrm{SFD}(x)$, from which we obtain $d \in x + \mathrm{SFD}(x)$. Combining this with $d \in K \subseteq D$, we see that $d \in K_S(x)$ as desired. □

Lemma 3.4.2 ([25, Lemma 3.2]). *Suppose X is reflexive and smooth, T_1, T_2 are two closed convex cones. If $x \in P_{D\cap(x+T_2)}(y) \Rightarrow x \in P_{D\cap(x+T_1)}(y)$ for all $y \in X$, then $D \cap (x+T_1) \subseteq D \cap (x+T_2)$.*

Proof. Suppose on the contrary that this is not true. Then there exists a $y \in D \cap (x + T_1) \backslash D \cap (x+T_2)$. Since $D \cap (x+T_1) \subseteq D$, we must have $y \notin G$, where $G := x + T_2$, a translate of a closed convex cone. Let $u \in P_G(y)$. Then by Proposition 2.3.1 **(vi)**, we see that $x \in P_G(y + x - u)$. Combining this with Proposition 2.3.1 **(v)**, we see that

$$J(y-u) \in N_G(x), \tag{3.4.4}$$

where J is the duality map. It now follows from (3.4.4) and [35, Corollary 3.8.5] that $x \in P_G(x_t)$, where $x_t := x + t(y-u)$ for $t > 1$. Since $x \in D \cap (x+T_2) \subseteq G$, we actually obtained $x \in P_{D\cap(x+T_2)}(x_t)$. We shall show that however, $x \notin P_{D\cap(x+T_1)}(x_t)$ for suitably large t and this will contradict the assumption. To see this, in view of $u \in P_G(y)$, $x \in G$ and the definition of normal cones, we see that,

$$\langle J(y-u), x-u \rangle \leq 0. \tag{3.4.5}$$

Thus we obtain for each $t > 1$ that,

$$\begin{aligned}
\|x_t - y\|^2 &= \langle J(x_t - y), x_t - y\rangle \\
&= t\langle J(x_t - y), y - u\rangle + \langle J(x_t - y) - J(t(y-u)), x - y\rangle \\
&\quad + \langle J(t(y-u)), x - u\rangle - \langle J(t(y-u)), y - u\rangle, \\
&\leq t\|x_t - y\|\|y-u\| - \langle J(x_t - y) - J(t(y-u)), x-y\rangle - t\|y-u\|^2, \quad (3.4.6)
\end{aligned}$$

where the last inequality holds due to (3.4.5) and the relationship $\langle J(t(y-u)), y-u\rangle = t\|y-u\|^2 > 0$, which holds by definition of duality maps (see [35, Page 230]). Note that

$$\lim_{t\to\infty}\frac{1}{t}(\langle J(x_t-y)-J(t(y-u)), x-y\rangle) = \lim_{t\to\infty}(\langle J(y-u+\frac{x-y}{t})-J((y-u)), x-y\rangle) = 0,$$

since J is norm-w^* continuous ([35, Page 230]). It now follows from (3.4.6) that for large $t > 1$

$$\|x_t - y\| < t\|y-u\| = \|x_t - x\|.$$

This implies $x \notin P_{D\cap(x+T_1)}(x_t)$ for large t and leads to the desired contradiction. Thus $D\cap(x+T_1) \subseteq D\cap(x+T_2)$. □

The following theorem stating a relationship between the strong CHIP of $\{C_i : i \in I\}$ and the BCQ of the system $\{g_i : i \in I\}$ was proved in [25, Theorem 3.1]. We write $\hat{S}_{z^*}(x_0) := \{x : \langle z^*, x - x_0\rangle \le 0\}$ for each $z^* \in \bigcup_{i\in I(x_0)} \partial g_i(x_0)$. It is direct from definition that

$$K_L(x_0) = D \cap \hat{S}(x_0), \tag{3.4.7}$$

where $\hat{S}(x_0) := \bigcap\{\hat{S}_{z^*}(x_0) : z^* \in \bigcup_{i\in I(x_0)} \partial g_i(x_0)\}$. We shall need the following lemma.

Lemma 3.4.3 ([25, Lemma 3.1]). *Let $x_0 \in K := D\cap C$. Then for any $x \in X$, we have*

$$x_0 \in P_K(x) \Leftrightarrow x_0 \in P_{K_S(x_0)}(x).$$

Proof. Since by Lemma 3.4.1, $K \subseteq K_S(x_0)$, we see that $x_0 \in P_{K_S(x_0)}(x) \Rightarrow x_0 \in P_K(x)$. It remains to show the converse implication. Let $x_0 \in P_K(x)$. By Proposition 2.3.1 **(v)**, there exists $x^* \in J(x - x_0)$ such that

$$\langle x^*, y - x_0\rangle \le 0, \tag{3.4.8}$$

for all $y \in K$. For any $d \in \mathrm{SFD}(x_0)$, there exist $t_n \downarrow 0$ and $d_n \to d$ such that $x_0 + t_n d_n \in K$. Thus (3.4.8) implies that $\langle x^*, d\rangle \le 0$ for all $d \in \mathrm{SFD}(x_0)$. Since inner product is bilinear and continuous, it then follows that $\langle x^*, y-x_0\rangle \le 0$ for all $y \in \overline{\mathrm{co}(x_0 + \mathrm{SFD}(x_0))}$ and hence for all $y \in K_S(x_0)$. Thus $x^* \in J(x - x_0) \cap N_{K_S(x_0)}(x_0) \ne \emptyset$. It follows from Proposition 2.3.1 **(v)** that $x_0 \in P_{K_S(x_0)}(x)$. The proof is completed. □

Theorem 3.4.1 ([25, Theorem 3.1]). *Let $x_0 \in K := D \cap C$. Consider the following statements:*

(a) *the system $\{g_i : i \in I\}$ satisfies the BCQ relative to D at x_0;*

(b) *$K_S(x_0) = K_L(x_0)$, and the family $\{D, \hat{S}_{z^*}(x_0) : z^* \in \bigcup_{i\in I(x_0)} \partial g_i(x_0)\}$ has the strong CHIP at x_0;*

(b*) *$K_S(x_0) = K_L(x_0)$, and the family of functions $\{\langle z^*, \cdot - x_0\rangle : z^* \in \bigcup_{i\in I(x_0)} \partial g_i(x_0)\}$ satisfies the BCQ relative to D at x_0;*

(c) *for each $x \in K$, $x_0 \in P_K(x)$ if and only if*

$$J(x - x_0) \cap \left(N_D(x_0) + \sum_{i\in I(x_0)} \operatorname{cone} \partial g_i(x_0) \right) \neq \emptyset. \tag{3.4.9}$$

Then the following implications hold:

(i) **(a)⇒(c)**; **(b)⇔(b*)⇒(c)**;

(ii) **(a)⇔(b)⇒(c)** *if X is reflexive;*

(iii) **(a)⇔(b)⇔(c)** *if X is reflexive and smooth.*

Proof. The results are trivial when $x_0 \in D \cap \operatorname{int} C$, since all the statements hold automatically. Hence we assume that $x_0 \in D \cap \operatorname{bdry} C$. Thus $G(x_0) = 0$ and that

$$N_D(x_0) + \sum_{i\in I(x_0)} \operatorname{cone} \partial g_i(x_0) \subseteq N_K(x_0). \tag{3.4.10}$$

(i) Assume **(a)**. By Proposition 2.3.1 **(v)**, $x_0 \in P_K(x)$ is equivalent to $J(x - x_0) \cap N_K(x_0) \neq \emptyset$. **(c)** follows from this and the definition of the BCQ relative to D. This proves **(a)⇒(c)**. On the other hand, the equivalence between **(b)** and **(b*)** follows directly from definition. We turn to show **(b*)⇒(c)**. We first show that $x_0 \in P_K(x)$ implies (3.4.9). Note that the BCQ assumption in **(b*)** and (3.4.7) imply that

$$N_{K_L(x_0)}(x_0) = N_D(x_0) + \sum_{i\in I(x_0)} \operatorname{cone} \partial g_i(x_0). \tag{3.4.11}$$

On the other hand, since $x_0 \in P_K(x)$ is equivalent to $x_0 \in P_{K_S(x_0)}(x)$ by Lemma 3.4.3, thus in particular

$$x_0 \in P_K(x) \Rightarrow J(x - x_0) \cap N_{K_S}(x_0) \neq \emptyset. \tag{3.4.12}$$

In view of the assumption $K_L(x_0) = K_S(x_0)$ in **(b*)** and (3.4.11), (3.4.12) implies (3.4.9). As to the converse implication, note that (3.4.10) and (3.4.9) implies that $J(x - x_0) \cap N_K(x_0) \neq \emptyset$, which gives $x_0 \in P_K(x)$ in view of Proposition 2.3.1 **(v)**. This proves **(b*)**$\Rightarrow$**(c)**.

(ii) Suppose that **(iii)** is valid, and X is reflexive. There exists an equivalent norm on X such that X is smooth (cf. [15, Page 186]). Then it follows from **(iii)** that **(a)** and **(b)** are equivalent. Combining this with the equivalences proved in **(i)**, part **(ii)** is proved.

(iii) In view of **(i)**, it remains to show that **(c)** implies **(a)** and **(b*)**. We first show that **(c)** implies **(a)**. In view of (3.4.10), we need only to prove the converse inclusion. Let $y^* \in N_K(x_0)$. By reflexivity and [35, Corollary 3.8.5], we see that $x_0 \in P_K(x_0 + u)$ for all $u \in J^{-1}(y^*)$. Then **(c)** implies that there exists $x^* \in J(x_0 + u - x_0) = J(u)$ such that

$$x^* \in N_D(x_0) + \sum_{i \in I(x_0)} \operatorname{cone} \partial\, g_i(x_0). \tag{3.4.13}$$

Since X is smooth, J is single-valued (see [35, Page 230]), $x^* = J(u) = y^*$. Thus (3.4.13) is equivalent to $y^* \in N_D(x_0) + \sum_{i \in I(x_0)} \operatorname{cone} \partial\, g_i(x_0)$. This proves **(c)**$\Rightarrow$**(a)**.

We now turn to the implication that **(c)**$\Rightarrow$**(b*)**. By Lemma 3.4.1, we see that $K \subseteq K_S(x_0) \subseteq K_L(x_0)$. This implies that for each $x \in X$, we have

$$x_0 \in P_{K_L(x_0)}(x) \Rightarrow x_0 \in P_{K_S(x_0)}(x) \Rightarrow x_0 \in P_K(x) \tag{3.4.14}$$

Conversely, if $x_0 \in P_K(x)$, by **(c)**, this is equivalent to

$$J(x - x_0) \in N_D(x_0) + \sum_{i \in I(x_0)} \operatorname{cone} \partial\, g_i(x_0). \tag{3.4.15}$$

We first check that

$$\sum_{i \in I(x_0)} \operatorname{cone} \partial\, g_i(x_0) \subseteq N_{K_L(x_0)}(x_0). \tag{3.4.16}$$

To see this, let $i \in I(x_0)$ and $z^* \in \partial g_i(x_0)$. Take $y \in K_L(x_0)$. Then by (3.4.7), we see that,

$$\langle z^*, y - x_0 \rangle \leq 0. \tag{3.4.17}$$

Since this is true for all $y \in K_L(x_0)$, we have shown that for each $i \in I(x_0)$, $\partial g_i(x_0) \subseteq N_{K_L(x_0)}(x_0)$. Since $N_{K_L(x_0)}(x_0)$ is a cone, it follows that $\sum_{i \in I(x_0)} \operatorname{cone} \partial g_i(x_0) \subseteq N_{K_L(x_0)}(x_0)$.

On the other hand, since $K_L(x_0) \subseteq D$, we have $N_D(x_0) \subseteq N_{K_L(x_0)}(x_0)$. Combining this with (3.4.15) and (3.4.16), we see that $J(x - x_0) \in N_{K_L(x_0)}(x_0)$, which in turn gives $x_0 \in P_{K_L(x_0)}(x)$, thanks to Proposition 2.3.1 **(v)**. Thus we have

$$x_0 \in P_{K_L(x_0)}(x) \Leftrightarrow x_0 \in P_{K_S(x_0)}(x) \Leftrightarrow x_0 \in P_K(x). \tag{3.4.18}$$

It then follows from Lemma 3.4.2 that $K_S(x_0) = K_L(x_0)$.

We now continue on the proof of **(c)**$\Rightarrow$**(b*)**. We obtain from **(c)** and (3.4.18) that $x_0 \in P_{K_L(x_0)}(x)$ if and only if $J(x - x_0) \in N_D(x_0) + \sum_{i \in I(x_0)} \operatorname{cone} \partial g_i(x_0)$, according to (3.4.7). Applying the implication **(c)**$\Rightarrow$**(a)** to the system $\{\langle z^*, \cdot - x_0 \rangle : z^* \in \bigcup_{i \in I(x_0)} \partial g_i(x_0)\}$ in place of $\{g_i : i \in I(x_0)\}$, we see that $\{\langle z^*, \cdot - x_0 \rangle : z^* \in \bigcup_{i \in I(x_0)} \partial g_i(x_0)\}$ has the BCQ relative to D at x_0. This completes the proof. □

Remark 3.4.1. *[24, Theorem 5.1] follows from part* **(c)** *of the previous theorem by taking X as a Hilbert space.*

3.5 The strong CHIP of extremal subsets

In [13], the authors considered the system $\{C, A^{-1}b\}$, where C is a closed convex set in a Hilbert space X, $A : X \to Y$ a continuous linear map from a Hilbert space X to a finite dimensional Hilbert space Y and $b \in Y$. By Corollary 3.3.3, the strong CHIP of the system of sets $\{C, A^{-1}b\}$ is equivalent to (3.3.5), from which we can find the projection onto $C \cap A^{-1}b$ by an algorithm proposed in [13]. However, it is not always true that $\{C, A^{-1}b\}$ has the strong CHIP. In [12, Chapter 10], [13] and [14], the authors considered an extremal subset C_b of C. They proved that by replacing the set C by a subset C_b, while keeping its intersection with $A^{-1}b$ unchanged, the system $\{C_b, A^{-1}b\}$ always has the strong CHIP.

We begin with the definition of extremal sets.

Definition 3.5.1 ([17, Section 8 A]). *Let X be a normed linear space, A be a convex set in X. $E \subseteq A$ is called an extremal subset of A if $x, y \in A$, and there exists some $t \in (0,1)$ such that $tx + (1-t)y \in E$, then $x, y \in E$.*

Proposition 3.5.1. *Let X be a Hilbert space, Y be a finite dimensional Hilbert space. Suppose C is a closed convex set in X, A is a continuous linear map from X to Y and $b \in R(A)$. Then the following statements are true:*

(i) *There exists a minimal extremal subset C_b of C such that*

$$C \cap A^{-1}b = C_b \cap A^{-1}b.$$

(ii) *There exists a minimal extremal subset F_b of $A(C)$ such that*

$$b \in F_b.$$

Proof. To begin with, note that by definition of extremal subsets, intersection of extremal subsets is also an extremal subset.

We now prove **(i)**. It is easy to see that C itself is an extremal subset of itself with the prescribed property. On the other hand, suppose E and B are extremal subsets of C such that

$$C \cap A^{-1}b = E \cap A^{-1}b = B \cap A^{-1}b.$$

Then $(B \cap E) \cap A^{-1}b = C \cap A^{-1}b$. Since $B \cap E$ is also extremal by our earlier note, there exists a minimal extremal subset C_b of C such that

$$C \cap A^{-1}b = C_b \cap A^{-1}b,$$

namely,

$$C_b = \bigcap \{E : E \subseteq C, E \text{ is closed convex extremal in } C, E \cap A^{-1}b = C \cap A^{-1}b\}.$$

This proves **(i)**.

Part **(ii)** follows from a similar argument, using the fact that $b \in A(C)$ and that if $b \in B$ and $b \in E$, then $b \in B \cap E$. $\square$

Write $C_{F_b} := C \cap A^{-1}(F_b)$. We have the following theorem.

Theorem 3.5.1 ([13, Proposition 4.3]). *Let X be a Hilbert space, Y be a finite dimensional Hilbert space. Suppose C is a closed convex set in X, A is a continuous linear map from X to Y and $b \in R(A)$. Then the following statements are true:*

(i) $C_b = C_{F_b}$;

(ii) $b \in \mathrm{ri}A(C_b)$;

(iii) $A(C_b) = F_b$.

Proof. The proof proceeds as follows: we shall first prove that $C_b \subseteq C_{F_b}$. Then we show that **(ii)** and **(iii)** are satisfied for C_{F_b} in place of C_b. Finally, we show that $C_{F_b} \subseteq C_b$.

We start by proving $C_b \subseteq C_{F_b}$. We shall show that C_{F_b} is an extremal set such that

$$C_{F_b} \cap A^{-1}b = C \cap A^{-1}(b). \tag{3.5.1}$$

This together with the definition of C_b proves our claim. To see this, let $x, y \in C$ and $\lambda \in (0,1)$ be such that $\lambda x + (1-\lambda)y \in C_{F_b}$. Then $\lambda Ax + (1-\lambda)Ay \in F_b$. Since F_b is extremal in $A(C)$, this implies $Ax, Ay \in F_b$. Thus $x, y \in C \cap A^{-1}(F_b) = C_{F_b}$ and hence C_{F_b} is extremal. On the other hand, (3.5.1) is true since

$$\begin{aligned} C_{F_b} \cap A^{-1}b &= C \cap A^{-1}F_b \cap A^{-1}b \\ &= C \cap A^{-1}b, \end{aligned}$$

thanks to the fact that $b \in F_b$. This shows that $C_b \subseteq C_{F_b}$ by definition of C_b, proving our first claim.

Next we show that $A(C_{F_b}) = F_b$. First of all, Since $F_b \subseteq A(C)$ and $C_{F_b} = C \cap A^{-1}(F_b)$, it is trivial that $F_b \subseteq A(C_{F_b})$. To prove the converse inclusion, let $y \in A(C_{F_b})$. There exists $x \in C_{F_b}$ such that $y = Ax$. But $C_{F_b} = C \cap A^{-1}(F_b) \subseteq A^{-1}(F_b)$, thus $y = Ax \in F_b$. This proves our second claim.

Now we show that $b \in \mathrm{ri}A(C_{F_b}) = \mathrm{ri}F_b$. By considering $\{b\}$ and F_b as subsets of $\mathrm{span}\,(F_b - b) =: Y_0$ (Note that Y_0 is closed since it is finite dimensional), the assertion becomes $b \in \mathrm{int}\, F_b$. Suppose this is not true. By Theorem 2.4.1, there exists $y \in Y_0^* \backslash \{0\}$ such that

$$\langle y, b\rangle \geq \langle y, x\rangle, \ \forall x \in F_b. \tag{3.5.2}$$

By the Hahn Banach theorem, there exists an extension $\tilde{y}$ of y to X^* preserving the norm. We claim that there exists some $f \in F_b$ such that

$$\langle \tilde{y}, b\rangle > \langle \tilde{y}, f\rangle. \tag{3.5.3}$$

Suppose this is not true, then $\tilde{y} \in (F_b - b)^{\perp}$, which means that $\tilde{y}|_{Y_0} = y = 0$, contradicting the choice of y. Thus (3.5.3) holds. This implies that $F_b \cap H$ is a proper subset of F_b, where $H := \{x \in F_b : \langle \tilde{y}, x\rangle = \langle \tilde{y}, b\rangle\}$. To see that it is extremal in $A(C)$, let $x, u \in A(C)$ and $\lambda \in (0,1)$ be such that $\lambda x + (1-\lambda)u \in F_b \cap H$. By extremality of F_b, we obtain $x, u \in F_b$. But then (3.5.2) and $\lambda x + (1-\lambda)u \in H$ implies that

$$\langle \tilde{y}, b\rangle \geq \langle \tilde{y}, \lambda x + (1-\lambda)u\rangle = \langle \tilde{y}, b\rangle,$$

which implies that $\langle \tilde{y}, x\rangle = \langle \tilde{y}, u\rangle = \langle \tilde{y}, b\rangle$, and thus $x, u \in H$. This shows that $F_b \cap H$ is a proper subset of F_b containing b which is also extremal in $A(C)$, a contradiction. Thus we have also proved $b \in \mathrm{ri} A(C_{F_b}) = \mathrm{ri} F_b$.

Finally, we prove that $C_{F_b} \subseteq C_b$. Let $x \in C_{F_b}$. Then $y := Ax \in A(C_{F_b}) = F_b$. Since $b \in \mathrm{ri} F_b$, there exists $u \in F_b$ and $\lambda \in (0,1)$ such that $\lambda y + (1-\lambda)u = b$. Take $x_0 \in A^{-1}u \cap C$. Then $\lambda x + (1-\lambda)x_0 \in A^{-1}b \cap C \subseteq C_b$. Since C_b is extremal, we obtain $x \in C_b$, which is the desired result. □

Note that $b \in \mathrm{ri} A(C)$ implies the strong CHIP of the system $\{C, A^{-1}b\}$ (see Theorem 4.2.6). In other words, by shrinking C to C_b (without changing the intersection), we obtain a system $\{C_b, A^{-1}b\}$ which has the strong CHIP.

Chapter 4

Sufficient Conditions for the Strong CHIP

4.1 Introduction

Since the strong CHIP was first proposed by F. Deutsch, W. Li and J. Ward, many researchers have given sufficient conditions for the strong CHIP. In [13], the authors have given interior point type conditions (see section 3.2 below) sufficient for the strong CHIP. In a later paper [5] by Borwein et al., the authors studied a relationship between the strong CHIP and the linear and bounded linear regularity for systems of closed convex sets in Euclidean spaces. The task of extending the results in [5] concerning the strong CHIP from a Hilbert space setting to a Banach space setting was taken up in [32]. In recent years, Jeyakumar et al. gave new sufficient conditions in terms of epigraphs of the support functions of the sets involved (cf. [9], [21]).

Concerning results in more general settings, Ng and Song (see [31, Theorem 4.2, Theorem 4.3, Corollary 4.1]) gave some sufficient conditions for the strong CHIP of systems of closed convex sets in locally convex spaces. Since we focus mainly on sets in normed linear spaces, and especially those in Banach spaces, we shall not pursue in this direction. Interested readers are referred to [31].

The case when the system consists of infinitely many closed convex sets was also addressed. In 2005, Li and Ng studied in [28] the strong CHIP for possibly infinitely many closed convex sets in normed linear spaces and gave new interior point type sufficient conditions (see section 3.2 below). These results have been applied to obtain new characterizations for the minimizers of a best approximation problem in the Banach space of all continuous functions defined on a compact set. See [27, Theorem 3.2 to 3.5] for details.

In this chapter, we give a brief overview of these sufficient conditions for the strong CHIP. To end this introductory section, we give two examples to show that the strong CHIP does not hold automatically for systems of closed convex sets.

Example 4.1.1. Consider two sets in $\mathbb{R}^2$, defined respectively by

$$A := \{(x, y) \in \mathbb{R}^2 : y \geq x^2\}$$
$$B := \{(x, y) \in \mathbb{R}^2 : y = 0\}$$

Then $A \cap B = \{(0,0)\}$. Thus $N_{A\cap B}((0,0)) = \mathbb{R}^2$. Yet $N_A((0,0)) = \{\lambda(0,-1) : \lambda \geq 0\}$ since A has smooth boundary at the origin, and $N_B((0,0)) = \{\lambda(0,1) : \lambda \in \mathbb{R}\}$. Thus $N_A((0,0)) + N_B((0,0)) = \{\lambda(0,1) : \lambda \in \mathbb{R}\}$, showing that the strong CHIP cannot hold at the origin. Thus the system of closed convex sets does not have the strong CHIP.

Example 4.1.2. We shall consider an infinite index set in this example. Consider the system of closed convex sets $\{\mathbf{B}(0, \frac{1}{n}) \subseteq X : n \in \mathbb{N}\}$ in a Banach space X. Then $\bigcap_{n\in\mathbb{N}} \mathbf{B}(0, \frac{1}{n}) = \{0\}$. Since $0 \in \mathrm{int}\mathbf{B}(0, \frac{1}{n})$ for all $n \in \mathbb{N}$, $N_{\mathbf{B}(0,\frac{1}{n})}(0) = \{0\}$. On the other hand, $N_{\{0\}}(0) = X$, showing that the system does not have the strong CHIP.

4.2 $|I|$ is finite

4.2.1 Interior point conditions

In this section, X will be a normed linear space unless otherwise specified. By Proposition 2.3.1 **(i)**, we have for each closed convex set C, $\partial\delta_C(x) = N_C(x)$ for all $x \in C$. Moreover, for two closed convex sets C and D with nonempty intersection, $\delta_C(x) + \delta_D(x) = \delta_{C\cap D}(x)$ for all $x \in X$. In this sense, the strong CHIP is a special case of the subdifferential sum rule for convex lower semicontinuous functions. To derive sufficient conditions for the strong CHIP, we first recall some sufficient conditions for the subdifferential sum rule for convex lower semicontinuous functions.

We begin our discussion with the following definitions. The definition was stated originally for $X = \mathbb{R}^n$.

Definition 4.2.1 ([7, Section 5.1]). *A set C in X is called a polyhedral set (or polyhedron) if there exist $a_i^* \in X^*$, $b_i \in \mathbb{R}$, $1 \leq i \leq n$, such that*

$$C = \{x \in X : \langle a_i^*, x \rangle \leq b_i, 1 \leq i \leq n\}.$$

A function $f : X \to (-\infty, \infty]$ is called a polyhedral function if $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : r \geq f(x)\}$ *is a polyhedral set.*

It follows immediately from the definition that polyhedral sets are closed convex sets. Thus polyhedral functions are convex lower semicontinuous functions.

Theorem 4.2.1 (c.f [35, Section 2.8], [7, Corollary 5.1.9]). *Let f, g be convex lower semicontinuous functions defined on a Banach space X. Then the sum rule holds, i.e. $\partial(f + g)(x) = \partial f(x) + \partial g(x)$ for all $x \in X$ if at least one of the following conditions holds:*

(i) $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$;

(ii) $\operatorname{dom} f \cap \operatorname{int}(\operatorname{dom} g) \neq \emptyset$;

(iii) $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$;

(iv) $\operatorname{cone}(\operatorname{dom} f - \operatorname{dom} g) = X$;

(v) *X is $\mathbb{R}^n$, f, g are polyhedral functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$.

The following theorem concerning sufficient conditions for the strong CHIP follows immediately from the above theorem.

Theorem 4.2.2 (cf. [13], [14], [32], [31], [9]). *Let C, D be closed convex sets in a Banach space X with nonempty intersection. Then the strong CHIP holds if at least one of the following conditions holds:*

(i) $0 \in \text{int}(C - D)$;

(ii) $C \cap \text{int}D \neq \emptyset$;

(iii) $0 \in \text{core}(C - D)$;

(iv) $\text{cone}(C - D) = X$;

(v) *X is $\mathbb{R}^n$, C, D are polyhedral sets.*

Proof. Note that $\text{dom}\,\delta_C = C$ and $\text{dom}\,\delta_D = D$. The first four statements follow immediately from an application of Theorem 4.2.1 with $f = \delta_C$, $g = \delta_D$. The last statement follows from Theorem 4.2.1 and the fact that δ_C and δ_D are polyhedral functions if C, D are polyhedral sets. □

The following theorem, termed DLW in [28], summarizes some sufficient conditions for the strong CHIP. It was first proposed in [14] in the Hilbert space setting.

Theorem 4.2.3 (cf. [28, Theorem 2.2]). *Let $\{D, C_1, \cdots C_n\}$ be closed convex sets in a normed linear space X. Let $D \cap \bigcap_{i=1}^n C_i \neq \emptyset$. Then the family has the strong CHIP if one of the following conditions is satisfied:*

(i) $D \cap \text{int} \bigcap_{i=1}^n C_i \neq \emptyset$;

(ii) $\text{ri}\, D \cap \bigcap_{i=1}^n C_i \neq \emptyset$ *and each C_i is a polyhedral set;*

(iii) *There exist a subset I_0 of $I := \{1, 2, \cdots, n\}$ such that C_i is a polyhedral set for each $i \in I \backslash I_0$ and*

$$\text{ri}D \cap (\text{int} \bigcap_{i \in I_0} C_i) \cap (\bigcap_{i \in I \backslash I_0} C_i) \neq \emptyset.$$

Proof. It is easy to see that part **(iii)** is true granting **(i)** and **(ii)**. To see this, fix any $x \in D \cap \bigcap_{i=1}^n C_i$. Applying **(i)** to $D \cap \bigcap_{i \in I \backslash I_0} C_i$ in place of D and to $\{C_i : i \in I_0\}$ in place of $\{C_i : 1 \leq i \leq n\}$, we obtain that

$$N_{D \cap \bigcap_{i=1}^n C_i}(x) \subseteq N_{D \cap \bigcap_{i \in I \backslash I_0} C_i}(x) + \sum_{i \in I_0} N_{C_i}(x).$$

Finally, applying **(ii)** to $\{C_i : i \in I\backslash I_0\}$ in place of $\{C_i : 1 \leq i \leq n\}$, we obtain further that

$$N_{D\cap\bigcap_{i=1}^n C_i}(x) \subseteq N_{D\cap\bigcap_{i\in I\backslash I_0} C_i}(x) + \sum_{i\in I_0} N_{C_i}(x) \subseteq N_D(x) + \sum_{i\in I} N_{C_i}(x),$$

which completes the proof for **(iii)**.

To see that part **(i)** is true, we recall that for a family $\{f_i : 0 \leq i \leq n\}$ of proper convex functions, the sum rule

$$\partial \sum_{i=0}^{n} f_i(x_0) = \sum_{i=0}^{n} \partial f_i(x_0)$$

holds if $\operatorname{dom} f_0 \cap \operatorname{int} \bigcap_{i=1}^n \operatorname{dom} f_i \neq \emptyset$ (cf. [35, Theorem 2.8.3]). Part **(i)** follows by applying this result with $f_0 := \delta_D$ and $f_i := \delta_{C_i}$ for $1 \leq i \leq n$ and noting the fact that $\operatorname{dom} \delta_E = E$ for a set E.

Now we turn to prove **(ii)**. To begin with, we claim that for polyhedral sets $P := \{x \in X : \langle a_i^*, x\rangle \leq b_i, 1 \leq i \leq n\}$ for some $a_i^* \in X^*$ and $b_i \in \mathbb{R}$, the corresponding normal cone at a point $p \in P$ is

$$N_P(p) = \operatorname{cone}\{a_i^* : i \in I(p)\}, \tag{4.2.1}$$

where $I(p) := \{i : 1 \leq i \leq n, \langle a_i^*, p\rangle = b_i\}$. To see this, we first show that (see Definition 2.4.4 for the definition of tangent cones)

$$T_P(p) = \{h : \langle a_i^*, h\rangle \leq 0, \ \forall i \in I(p)\}. \tag{4.2.2}$$

Denote the right hand side of (4.2.2) by $C(p)$. Let $h \in C(p)$. We wish to show that there exists $\delta > 0$ so that $p+th \in P$ for all $0 \leq t \leq \delta$. Note that $p+th \in P$ is equivalent to

$$\langle a_i^*, p + th\rangle \leq b_i, \ \forall i \in I.$$

If $i \in I(p)$, then $\langle a_i^*, p+th\rangle \leq b_i$ for all $t \geq 0$. Now fix any $i \notin I(p)$. If $\langle a_i^*, h\rangle = 0$, then $\langle a_i^*, p+th\rangle \leq b_i$ is again true for all $t \geq 0$. On the other hand, if $\langle a_i^*, h\rangle > 0$, then t has to be less than or equal to $\frac{b_i - \langle a_i^*, p\rangle}{\langle a^*, h\rangle}$ so that $\langle a_i^*, p + th\rangle \leq b_i$. Summarizing the above discussion, we set $c := \max\{0, \langle a_i^*, h\rangle : i \notin I(p)\} \geq 0$ and define

$$\delta := \begin{cases} \frac{\min_{i\notin I(p)}\{b_i - \langle a_i^*, p\rangle\}}{c}, & \text{if } c > 0 \\ 1, & \text{if } c = 0 \end{cases}$$

Then it follows from the above discussion that for all $0 \le t \le \delta$,

$$\langle a_i^*, p + th\rangle \le b_i, \ \forall i \in I.$$

This shows that $h \in T_P(p)$. To prove the converse inclusion, we let $h \in T_P(p)$. By definition of tangent cones, there exist $p_n \in P$, $t_n > 0$ such that $h_n := \frac{p_n - p}{t_n} \to h$ and $p + t_n h_n \in P$. This implies that for any $i \in I(p)$, $\langle a_i^*, p + t_n h_n\rangle = b_i + \langle a_i^*, t_n h_n\rangle \le b_i$. This gives $\langle a_i^*, h\rangle \le 0$ for all $i \in I(p)$. Thus $h \in C(p)$. This proves (4.2.2). From this and the fact that $N_P(p) = (T_P(p))^\ominus$ (see Proposition 2.4.3 **(i)**), we immediately see that (4.2.1) holds. We now prove that if $\{C_i : i = 1, \cdots, n\}$ is a collection of finitely many polyhedrons, then it has the strong CHIP. To see this, we note that there exist $a_{ij}^* \in X^*$, $b_{ij} \in \mathbb{R}$ with $1 \le j \le n_i$ for some integer n_i, $1 \le i \le n$, such that

$$C_i := \{x : \langle a_{ij}^*, x\rangle \le b_{ij}, 1 \le j \le n_i\}.$$

Thus for any points $c \in \bigcap_{i=1}^n C_i$, writing $I_i(c)$ as the active index of c in C_i, and noting that $I(c) = \bigcup_{i=1}^n I_i(c)$, we have, by (4.2.1),

$$\begin{aligned} N_{\bigcap_{i=1}^n C_i}(c) &= \operatorname{cone}\{a_{ij}^* : j \in I_i(c), 1 \le i \le n\} \\ &= \sum_{i=1}^n \operatorname{cone}\{a_{ij}^* : j \in I_i(c)\} \\ &= \sum_{i=1}^n N_{C_i}(c). \end{aligned}$$

Thus the strong CHIP holds for a system of finitely many polyhedrons $\{C_i : i = 1, \cdots, n\}$. In view of this and the fact that finite intersection of polyhedrons is still a polyhedron, in order to prove **(ii)**, it remains to show that if $\operatorname{ri} D \cap C \ne \emptyset$, where D is a closed convex set and C is a polyhedron, then the system $\{D, C\}$ has the strong CHIP. Without loss of generality, we may assume that $0 \in \operatorname{ri} D \cap C$. Write for simplicity $Z := \operatorname{span} D$. Fix any $x_0 \in C \cap D$. Since $(C \cap Z) \cap \operatorname{int} D \ne \emptyset$ as a subset of Z, by part **(i)**, $\{C \cap Z, D\}$ has the strong CHIP as subsets in Z. Thus, by Proposition 3.2.1, $\{C \cap Z, D\}$ has the strong CHIP, and thus in particular,

$$N_{C\cap D}(x_0) \subseteq N_{C\cap Z}(x_0) + N_D(x_0). \tag{4.2.3}$$

To complete the proof of the theorem, we claim that,

$$N_{C\cap Z}(x_0) \subseteq N_C(x_0) + N_Z(x_0). \tag{4.2.4}$$

Since C is a polyhedron, we may assume without loss of generality that there exist functionals $a_i^* \in X^*$ and $b_i \in \mathbb{R}$, $1 \leq i \leq n$ such that

$$C = \{x : \langle a_i^*, x \rangle \leq b_i, 1 \leq i \leq n\}.$$

Thus

$$C \cap Z = \{x \in Z : \langle a_i^*|_Z, x \rangle \leq b_i, 1 \leq i \leq n\}.$$

By (4.2.1),

$$N_{C \cap Z}(x_0)|_Z = \text{cone}\,\{a_i^*|_Z : i \in I(x_0)\} = N_C(x_0)|_Z$$

Thus, for any $y^* \in N_{C \cap Z}(x_0)$, there exists $u^* \in N_C(x_0)$ such that $u^* - y^* \in Z^\perp = N_Z(x_0)$. This proves (4.2.4). Combining (4.2.4) with (4.2.3) and the fact that $N_Z(x_0) \subseteq N_D(x_0)$ (which follows from Proposition 2.3.1 and the obvious inclusion $D \subseteq Z$), we complete the proof of part **(ii)**. □

4.2.2 Boundedly linear regularity

Another concept, the boundedly linear regularity (see [4], [3], [5], [32]), which is closely related to the projection algorithm, was found to imply the strong CHIP.

Linear regularity and boundedly linear regularity were first defined in [4, Definition 5.1, Definition 5.6]. Boundedly linear regularity was found to be one of the sufficient conditions to guarantee linear convergence of the sequence generated by the projection algorithms, with suitable starting point. Under the even stronger assumption of linear regularity, one can even assert the linear convergence of the sequence generated by the projection algorithms, regardless of the starting point. See [4, Theorem 5.7, Theorem 5.8] for details. The projection algorithms were designed to solve the convex feasibility problem, that is, to find a point $x \in \bigcap_{i=1}^n C_i$ given $\{C_1, \cdots, C_n\}$ with nonempty intersection. See [4] for a comprehensive survey.

Recall that the strong CHIP is also closely related to projections onto closed convex sets, as was surveyed in the last chapter. Borwein et al. proved that the boundedly linear regularity implies the strong CHIP when the ambient space is $\mathbb{R}^n$ (see [5, Theorem 3]). Ng and Yang extended the result to general Banach space (see [32, Theorem 4.2]). Actually, Ng and Yang have gone further and proved that the family of closed convex sets is linearly regular if and only if they have the strong CHIP and the corresponding normal cones at every point in the intersection has the Jameson's Property (G). This point will be further elaborated in Chapter 5.

We begin our discussion with the following definition of the linear regularity and the boundedly linear regularity.

Definition 4.2.2 ([3, Definition 4.2.1]). *A collection of closed convex sets $\{C_i\}_{i\in I}$ in a normed linear space X with some index set I, is said to be linearly regular if there exists $k > 0$ such that*

$$d(x, C) \leq k \sup_{i\in I} d(x, C_i),\ \forall x \in X.$$

The family is said to be boundedly linearly regular if for each $r > 0$, there exists $k_r > 0$ such that

$$d(x, C) \leq k_r \sup_{i\in I} d(x, C_i),\ \forall x \in r\mathbf{B}.$$

The following theorem is an extension of [5, Theorem 3] to a normed linear space setting.

Theorem 4.2.4. *Let $\{C_1, \cdots, C_n\}$ be finitely many closed convex sets in a normed linear space X. Suppose $\{C_1, \cdots, C_n\}$ is boundedly linearly regular. Then the system has the strong CHIP.*

Proof. Fix $c \in C := \bigcap_{i=1}^n C_i$. Since $\{C_1, \cdots, C_n\}$ is boundedly linearly regular, for $r = \|c\| + 1$, there exists $k_r > 0$ such that

$$d(x, C) \leq k_r \sup_{1\leq i\leq n} d(x, C_i),\ \forall x \in r\mathbf{B}, \tag{4.2.5}$$

that is

$$d(x, C) \leq k_r \sup_{1\leq i\leq n} d(x, C_i) + \delta_{r\mathbf{B}}(x),\ \forall x \in X. \tag{4.2.6}$$

Let $y^* \in N_C(c)$, that is, there exist $\lambda \geq 0$ and $u^* \in \partial d(c, C)$ such that $y^* = \lambda u^*$, thanks to **(iv)** of Proposition 2.3.1. Now since $d(c, C) = d(c, C_i) = 0$ for all $1 \leq i \leq n$ and $\delta_{r\mathbf{B}}(x) = 0$ for all x in a neighborhood of c, it follows from (4.2.6) and [35, Corollary 2.8.13] (thanks to the continuity of the distance functions) that

$$\begin{aligned} y^* = \lambda u^* &\in \lambda \partial k_r \sup_{1\leq i\leq n} d(\cdot, C_i)(c) + \lambda \partial \delta_{r\mathbf{B}}(c) \\ &= \lambda \partial k_r \sup_{1\leq i\leq n} d(\cdot, C_i)(c) \\ &\subseteq \lambda k_r \mathrm{co} \bigcup_{i=1}^{n} \partial d(\cdot, C_i)(c) \\ &\subseteq \sum_{i=1}^{n} N_{C_i}(c). \end{aligned}$$

This completes the proof. □

4.2.3 Epi-sum

In [9], [21], Jeyakumar et al. have proved the following sufficient condition for finitely many closed convex sets in a Banach space to have the strong CHIP. See [21, Theorem 3.1]. His proof is actually valid when the ambient space is a normed linear space.

Theorem 4.2.5. *Let $\{C_1, \cdots, C_n\}$ be closed convex sets in a normed linear space X with nonempty intersection. Suppose $\sum_{i=1}^{n} \mathrm{epi}\, \sigma_{C_i}$ is w^*-closed. Then $\{C_1, \cdots, C_n\}$ has the strong CHIP.*

Proof. By the assumption, it follows from Proposition 2.7.1 that

$$\mathrm{epi}\, \sigma_C = \sum_{i=1}^{n} \mathrm{epi}\, \sigma_{C_i}. \tag{4.2.7}$$

To show that $\{C_1, \cdots, C_n\}$ has the strong CHIP, fix an $x \in C$ and take $y^* \in N_C(x)$. Then by (2.7.4), $(y^*, \langle y^*, x\rangle) \in \mathrm{epi}\, \sigma_C$. Applying (4.2.7), $(y^*, \langle y^*, x\rangle)$ can be written as

$$(y^*, \langle y^*, x\rangle) = \sum_{i=1}^{n} (y_i^*, \alpha_i), \tag{4.2.8}$$

for some $(y_i^*, \alpha_i) \in \mathrm{epi}\, \sigma_{C_i}$, $1 \leq i \leq n$. This implies that $\sum_{i=1}^{n} \langle y_i^*, x\rangle = \sum_{i=1}^{n} \alpha_i$. This together with the obvious inequalities $\alpha_i \geq \sigma_{C_i}(y_i^*) \geq \langle y_i^*, x\rangle$ for each i imply that

$$\alpha_i = \sigma_{C_i}(y_i^*) = \langle y_i^*, x\rangle, \text{ for each } i.$$

Thus $(y_i^*, \langle y_i^*, x\rangle) \in \operatorname{epi}\sigma_{C_i}$ for $1 \le i \le n$. By (2.7.4) again, we obtain $y_i^* \in N_{C_i}(x)$ for $1 \le i \le n$. Combining this with (4.2.8), we see that

$$y^* = \sum_{i=1}^{n} y_i^* \in \sum_{i=1}^{n} N_{C_i}(x),$$

which completes the proof. □

We shall discuss further on this topic in chapter 5, in which we shall give new sufficient conditions for the strong CHIP of a system of infinitely many closed convex sets.

The next theorem was first proved in [13] in a Hilbert space setting (see [13, Theorem 3.2], [13, Theorem 3.12]). We provide a different proof via the use of epi-sum.

Theorem 4.2.6. *Let C be a closed convex set in a normed linear space X, A is a continuous linear map from X to a finite dimensional normed linear space Y. Suppose $b \in \operatorname{ri} A(C)$. Then $\{C, A^{-1}b\}$ has the strong CHIP.*

Proof. We may assume by translation that $b = 0$, that is $0 \in \operatorname{ri} A(C)$. Thus 0 is an interior point of $A(C)$ in the space $Y_1 := \operatorname{span} A(C) = A(\operatorname{span} C)$ To complete the proof, it then suffices to show that $\{C, \ker A\}$ has the strong CHIP. Define $A_1 : \operatorname{span} C \to Y_1$ by $A_1 x = Ax$. Then

$$0 \in \operatorname{int} A_1 C. \tag{4.2.9}$$

We wish to show that $\operatorname{epi}\sigma_C + \operatorname{epi}\sigma_{\ker A_1}$ is w^*-closed as a subset of $(\operatorname{span} C)^* \times \mathbb{R}$. Let $(x^*, \alpha) \in \operatorname{epi}\sigma_C \cap (-\operatorname{epi}\sigma_{\ker A_1})$. Then $(x^*, \alpha) \in -\operatorname{epi}\sigma_{\ker A_1}$, i.e. $-\alpha \ge \sigma_{\ker A_1}(-x^*)$. This implies, since $\ker A_1$ is a subspace, that

$$\alpha \le 0, \; x^* \in (\ker A_1)^{\perp}. \tag{4.2.10}$$

It follows from [12, Lemma 8.33] and the finite dimensionality of $R(A_1^*)$ that $x^* \in (\ker A_1)^{\perp} = R(A_1^*)$. Thus, there exists $y^* \in Y_1^*$ such that $x^* = A_1^* y^*$. On the other hand, since (x^*, α) also belongs to $\operatorname{epi}\sigma_C$, we have

$$\alpha \ge \sigma_C(x^*) = \sigma_{A_1 C}(y^*). \tag{4.2.11}$$

Combining this with (4.2.9), which asserts the existence of $\delta > 0$ such that $\delta B_1 \subseteq A_1 C$, where B_1 is the unit ball in Y_1, we obtain

$$\sigma_{A_1 C}(y^*) \geq \delta \|y^*\| \geq 0.$$

Combining this with (4.2.10) and (4.2.11), we see that $y^* = 0$ and $\alpha = 0$. From $x^* = A_1^* y^*$, we see that $x^* = 0$. This shows that $\operatorname{epi} \sigma_C \cap (-\operatorname{epi} \sigma_{\ker A_1})$ is zero. Since both epigraphs are w^*-closed convex cones, and $-\operatorname{epi} \sigma_{\ker A_1}$ is finite dimensional and thus locally compact, by the Dieudonné Theorem (cf. [35, Theorem 1.1.8]), $\operatorname{epi} \sigma_C + \operatorname{epi} \sigma_{\ker A_1}$ is w^*-closed. Since $\ker A_1 = \ker A \cap \operatorname{span} C$, by Theorem 4.2.5, $\{C, \ker A \cap \operatorname{span} C\}$ has the strong CHIP as subsets in $\operatorname{span} C$. By Proposition 3.2.1, $\{C, \ker A \cap \operatorname{span} C\}$ has the strong CHIP, i.e.

$$N_{C \cap \ker A}(x) \subseteq N_C(x) + N_{\ker A \cap \operatorname{span} C}(x), \forall x \in C \cap \ker A. \tag{4.2.12}$$

On the other hand, since $0 \in (\operatorname{ri}(\operatorname{span} C)) \cap \ker A$ and $\ker A$ is a polyhedron as Y is finite dimensional, it follows from Theorem 4.2.3 that $\{\operatorname{span} C, \ker A\}$ has the strong CHIP. Combining this with (4.2.12) and the fact that $N_{\operatorname{span} C}(x) \subseteq N_C(x)$ (which follows from Proposition 2.3.1 and the obvious inclusion $C \subseteq \operatorname{span} C$) for all $x \in C$, we see that for all $x \in C \cap \ker A$,

$$\begin{aligned} N_{C \cap \ker A}(x) &\subseteq N_C(x) + N_{\ker A \cap \operatorname{span} C}(x) \subseteq N_C(x) + N_{\operatorname{span} C}(x) + N_{\ker A}(x) \\ &= N_C(x) + N_{\ker A}(x), \end{aligned}$$

i.e., $\{C, \ker A\}$ has the strong CHIP. This completes the proof. □

4.3 $|I|$ is infinite

Li and Ng ([28]) extended the notion of the strong CHIP to the case when the index set is infinite and gave some sufficient conditions for the strong CHIP to hold. Following them, who did their analysis on general normed linear spaces, we shall let X denote a normed linear space in this section unless otherwise specified.

4.3.1 A Sum Rule

The following lemma on subdifferential is crucial in the derivations in [28]. It was contained in [25, Theorem 2.1] as an intermediate step for the proof of that theorem. We isolate the statement as follows.

Lemma 4.3.1. *Let $\{g_i : i \in I\}$ be a family of continuous convex functions in X such that $\sup_{i\in I} g_i(x) < +\infty$ for all $x \in X$. Assume that I is a compact metric space and that the function $i \mapsto g_i(x)$ is upper semicontinuous on I at each $x \in X$. Let C be a nonempty closed convex subset of X such that* $\dim \operatorname{span} C < \infty$. *Then the following subdifferential formula holds:*

$$\partial \sup_{i\in I} g_i(x) \subseteq N_C(x) + \sum_{i\in I(x)} \operatorname{cone}(\partial g_i(x)),\ \forall x \in C, \tag{4.3.1}$$

where $I(x) = \{j \in I : \sup_{i\in I} g_i(x) = g_j(x)\}$.

Proof. Write $Z = \operatorname{span} C$, $G(x) = \sup_{i\in I} g_i(x)$ for each x. Fix any $x \in C$ and let $y^* \in \partial G(x)$. Then in particular, $\langle y^*, y - x\rangle \leq G(y) - G(x)$ for any $y \in Z$. Thus $y^*|_Z \in \partial G|_Z(x)$, where $f|_Z$ denotes the restriction of the function f on Z. By [16, VI Theorem 4.4.2], we have

$$y^*|_Z \in \operatorname{co} \bigcup_{i\in I(x)} \partial g_i|_Z(x).$$

Then there exist a finite index set $J \subseteq I(x)$, $x_i^* \in \partial g_i|_Z(x)$ and $\lambda_i \geq 0$ for each $i \in J$ with $\sum_{i\in J} \lambda_i = 1$ such that

$$y^*|_Z = \sum_{i\in J} \lambda_i x_i^*. \tag{4.3.2}$$

Now for each $i \in J$, $\langle y, x_i^*\rangle \leq g_i'|_Z(x; y) = g_i'(x; y) \left(:= \lim_{t\downarrow 0} \frac{g_i(x+ty)-g_i(x)}{t}\right)$ for all $y \in Z$. By continuity and convexity of g_i, we see that $g_i'(x; \cdot)$ is a continuous sublinear functional (c.f [35, Theorem 2.4.9]). By the Hahn-Banach extension theorem (cf. [33, Theorem 1.9.5]), for each $i \in J$, there exists $\hat{x}_i^* \in X^*$ such that

$$\hat{x}_i^*|_Z = x_i^* \ \text{ and } \ \langle \hat{x}_i^*, y\rangle \leq g_i'(x; y),\ \forall y \in X.$$

This inequality and [35, Theorem 2.4.4] tell us that $\hat{x}_i^* \in \partial g_i(x)$ for all $i \in J$. Combining this with (4.3.2), we see that

$$y^* - \sum_{i\in I(x)} \lambda_i \hat{x}_i^* \in Z^{\perp} \subseteq N_C(x).$$

This implies

$$y^* \in \sum_{i\in I(x)} \lambda \hat{x}_i^* + N_C(x) \subseteq N_C(x) + \sum_{i\in I(x)} \operatorname{cone}(\partial g_i(x)),$$

which completes the proof. □

4.3.2 The C-Extended Minkowski Functional

In [28], Li and Ng studied sufficient conditions for the strong CHIP of the system $\{D, C_i\}_{i\in I}$, where C_i, D are closed convex sets with $\dim \operatorname{span} D < +\infty$, in a normed linear space. Li and Ng related a closed convex set having some kinds of interior points to a continuous function having properties similar to that of the Minkowski functional of the set. Recall that the notion of interior point and the definition of the Minkowki functional were given in chapter 1. We also recall the following definition from [28].

Definition 4.3.1 ([28, Lemma 3.1, Theorem 3.2]). *Let A and C be two closed convex subsets of a normed linear space X containing the origin. A continuous sublinear functional p_A on X is called a $C-$extended Minkowski functional of A if $p_A|_{\operatorname{span} C}$ equals the Minkowski functional of $A \cap \operatorname{span} C$ in the vector space* $\operatorname{span} C$.

We will need the following lemma on closure of sets.

Lemma 4.3.2. *Let C, D be convex sets in a normed linear space. Suppose that $0 \in D \cap \operatorname{int} C$ and that D is closed. Then $\overline{C \cap D} = \overline{C} \cap D$.*

Proof. Since the inclusion $\overline{C \cap D} \subseteq \overline{C} \cap D$ is obvious, we only check the converse inclusion. Let $x \in \overline{C} \cap D$. Since $0 \in \operatorname{int} C$, it follows from Proposition 2.6.1 that for all $t \in (0,1)$, $tx \in \operatorname{int} C \subseteq C$. On the other hand, by virtue of convexity of D and the fact that $0 \in D$, we also have $tx \in D$ for all $t \in (0,1)$. Thus we have $tx \in C \cap D$ for all $t \in (0,1)$. On taking limits, we have

$$x = \lim_{t\uparrow 1} tx \in \overline{C \cap D}.$$

This completes the proof. $\square$

The next lemma is adapted from Lemma 3.1 of [28].

Lemma 4.3.3 ([28, Lemma 3.1]). *Suppose* $0 \in \mathrm{rint}_C A$*, i.e. there exists* $\alpha > 0$ *such that*

$$0 \in B(0,\alpha) \cap \operatorname{span} C \subseteq A, \tag{4.3.3}$$

where A*,* C *are closed convex sets. Let* Z *denote the closure of* $\operatorname{span} C$ *and let* $\widetilde{A} := \overline{\mathrm{co}((A \cap Z) \cup B(0,\alpha))}$*. Then* $p_{\widetilde{A}}$*, the Minkowski functional of the set* $\widetilde{A}$*, is a* $C-$*extended Minkowski functional of* A*,, and*

$$p_{\widetilde{A}}(x) \leq \frac{1}{\alpha}\|x\|, \ \forall x \in X. \tag{4.3.4}$$

Moreover,

$$\widetilde{A} \cap \overline{\operatorname{span} C} = A \cap \overline{\operatorname{span} C}. \tag{4.3.5}$$

Proof. We first check that $q_{\widetilde{A}}$ is continuous. Let $x \in X\backslash\{0\}$. Then $\frac{\alpha x}{\|x\|} \in B(0,\alpha)$. This implies $p_{\widetilde{A}}(x) \leq \frac{1}{\alpha}\|x\|$ for all $x \in X\backslash\{0\}$. Since it is direct from definition that $p_{\widetilde{A}}(0) = 0$, we have $p_{\widetilde{A}}(x) \leq \frac{1}{\alpha}\|x\|$ for all $x \in X$, which means that $p_{\widetilde{A}}(\cdot)$ is continuous since $p_{\widetilde{A}}(\cdot)$ is sublinear.

We now show that $p_{\widetilde{A}}|_{\operatorname{span} C} = p_{A \cap \operatorname{span} C}$ on $\operatorname{span} C$. We start by checking

$$D \cap Z = A \cap Z, \tag{4.3.6}$$

where $D = \mathrm{co}((\mathrm{A} \cap \mathrm{Z}) \cup \mathrm{B}(0,\alpha))$. We need only to verify $D \cap Z \subseteq A \cap Z$ as the reversed inclusion is evident. Let $x \in D \cap Z$. We have to show that $x \in A \cap Z$. Since $x \in D \cap Z$, there exists $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ and $a \in A \cap Z$, $b \in B(0,\alpha)$ such that

$$x = \lambda_1 a + \lambda_2 b. \tag{4.3.7}$$

We may assume that $\lambda_2 \neq 0$. It follows immediately from (4.3.7) that $b \in Z$. Thus there are $b_k \in \operatorname{span} C$ such that $b_k \to b$. If $b \in \mathrm{int} B(0,\alpha)$, then for large k, $b_k \in (\operatorname{span} C) \cap B(0,\alpha)$. This implies $b \in A$ by the closedness of A and (4.3.3). On the other hand, if $b \in \mathrm{bdry}\, B(0,\alpha)$, then for large k, $\|b_k\| \neq 0$. Define $\tilde{b}_k := \frac{\alpha b_k}{\|b_k\|}$. Then $\tilde{b}_k \in (\operatorname{span} C) \cap B(0,\alpha) \subseteq A$. By the closedness of A again, $b \in A$. Combining the two cases, $b \in A \cap Z$. It then follows from the convexity of $A \cap Z$ and (4.3.7) that $x \in A \cap Z$. This proves (4.3.6).

We now prove (4.3.5), that is

$$\widetilde{A} \cap Z = A \cap Z. \tag{4.3.8}$$

To this end, we observe that

$$\widetilde{A} \cap Z = \overline{D} \cap Z = \overline{D \cap Z} = A \cap Z,$$

where the second last equality follows from Lemma 4.3.2, while the last equality follows from (4.3.6) and the closedness of $A \cap Z$. This proves (4.3.5).

Now we have from (4.3.5) that

$$A \cap \operatorname{span} C = A \cap Z \cap \operatorname{span} C = \widetilde{A} \cap Z \cap \operatorname{span} C = \widetilde{A} \cap \operatorname{span} C \tag{4.3.9}$$

Thus for all $x \in \operatorname{span} C$

$$\begin{aligned} p_{\widetilde{A}}(x) &= \inf\{\lambda > 0 : \ x \in \lambda\widetilde{A} \cap \operatorname{span} C\} \\ &= \inf\{\lambda > 0 : \ x \in \lambda A \cap \operatorname{span} C\} \\ &= p_{A \cap \operatorname{span} C}(x). \end{aligned}$$

This completes the proof. □

The next lemma shows that the existence of C−extended Minkowski functional is equivalent to the existence of a type of interior points.

Lemma 4.3.4 ([28, Theorem 3.2]). *Consider the system of closed convex sets $\{D, C_i : i \in I\}$ in a normed linear space with $\dim \operatorname{aff} D < +\infty$, where I is a compact metric space and $i \mapsto (\operatorname{aff} D) \cap C_i$ is assumed to be lower semicontinuous. Let $x \in D \cap C$. Write $\widehat{C} := \bigcap_{i \in I} C_i - x$, $\widehat{C_i} := C_i - x$, $\widehat{D} := D - x$. The the following statements are equivalent:*

(i) $0 \in \widehat{D} \cap \operatorname{rint}_{\widehat{D}} \widehat{C}$;

(ii) *For each $i \in I$, there exists $\widehat{D}$−extended Minkowski functional $p_{\widehat{C}_i}$ of the set $\widehat{C}_i$ such that the function*

$$P(x) := \sup_{i \in I} p_{\widehat{C}_i}(x), \ \forall x \in X,$$

is continuous on X and $i \mapsto p_{\widehat{C}_i}(x)$ is upper semicontinuous for each $x \in X$.

Proof. Write for simplicity that $Z := \operatorname{span} \widehat{C}$. We first show that **(ii)** implies **(i)**. Note that by definition of $\widehat{D}$−extended Minkowski functionals, we have $(\operatorname{span} \widehat{D}) \cap \widehat{C}_i = \{x \in \operatorname{span} \widehat{D} : p_{\widehat{C}_i}(x) \leq 1\}$, and thus

$$\widehat{D} \cap \widehat{C} = \widehat{D} \cap \{x \in \operatorname{span} \widehat{D} : \sup_{i \in I} p_{\widehat{C}_i}(x) \leq 1\}.$$

On the other hand, by assumption in **(ii)**, $x \mapsto \sup_{i \in I} p_{\widehat{C}_i}(x)$ is continuous. Continuity of $\sup_{i \in I} p_{\widehat{C}_i}(\cdot)$ implies that $0 \in \widehat{D} \cap \operatorname{rint}_{\widehat{D}} \widehat{C}$, that is, **(i)** holds.

Now we prove the converse implication. Assume that **(i)** holds. By Lemma 4.3.3, there exist $\widehat{D}$−extended Minkowski functional for the sets $\widehat{C}_i$ for each $i \in I$. Moreover, by (4.3.4), for each $i \in I$, we have

$$p_{\widehat{C}_i}(x) \leq \frac{1}{\alpha} \|x\|, \ \forall x \in X,$$

where $\alpha > 0$ is such that $B(0, \alpha) \cap \operatorname{span} \widehat{D} \subseteq \widehat{D} \cap \widehat{C}$. By definition, we have

$$\sup_{i \in I} p_{\widehat{C}_i}(x) \leq \frac{1}{\alpha} \|x\|, \ \forall x \in X,$$

thus P is continuous on X.

To complete the proof, it remains to show that under assumption **(i)**, $i \mapsto p_{\widehat{C}_i}(x)$ is upper semicontinuous for every $x \in X$, i.e. for any $i_0 \in I$ and $x \in X$,

$$\limsup_{i \to i_0} p_{\widehat{C}_i}(x) \leq p_{\widehat{C}_{i_0}}(x). \tag{4.3.10}$$

Suppose not. Then by sublinearity of Minkowski functional, there exists $x \in X$ such that

$$\limsup_{i \to i_0} p_{\widehat{C}_i}(x) > 1 \geq p_{\widehat{C}_{i_0}}(x). \tag{4.3.11}$$

From the second inequality and property of Minkowski functional (see Proposition 2.6.1), we have $x \in \overline{\operatorname{co}((\widehat{C}_{i_0} \cap Z) \cup B(0, \alpha))}$. Then for each $n \in \mathbb{N}$, there exist $\lambda_{1n}, \lambda_{2n} \geq 0$ with $\lambda_{1n} + \lambda_{2n} = 1$, $b_n \in B(0, \alpha)$ and $z_n \in \widehat{C}_{i_0} \cap Z$ such that

$$x_n := \lambda_{1n} b_n + \lambda_{2n} z_n \to x.$$

By lower semicontinuity of $i \mapsto \widehat{C}_i \cap Z$ (which follows from the lower semicontinuity assumption of $i \mapsto (\operatorname{aff} D) \cap C_i$), for each $n \in \mathbb{N}$, there exists $c_{n_i} \in \widehat{C}_i \cap Z$ such that

$\|z_n - c_{n_i}\| \to 0$ as $i \to i_0$. Define $x_{n_i} = \lambda_{1n} b_n + \lambda_{2n} c_{n_i}$ for each i and $n \in \mathbb{N}$. Then $x_{n_i} \in \operatorname{co}((\widehat{C}_i \cap Z) \cup B(0, \alpha))$, so $p_{\widehat{C}_i}(x_{n_i}) \leq 1$, but

$$p_{\widehat{C}_i}(x) \leq p_{\widehat{C}_i}(x - x_{n_i}) + p_{\widehat{C}_i}(x_{n_i}) \leq \frac{1}{\alpha}\|x - x_{n_i}\| + 1 \leq \frac{1}{\alpha}(\|x - x_n\| + \lambda_{2n}\|z_n - c_{n_i}\|) + 1.$$

Thus $\limsup_{i \to i_0} p_{\widehat{C}_i}(x) \leq 1$, contradicting (4.3.11). Thus (4.3.10) holds for all $x \in X$ and $i_0 \in I$. □

4.3.3 Relative Interior Point Conditions

The following theorem is the first part of [28, Theorem 4.1]. The proof of it reveals the main underlying idea of that paper. The authors tried to define continuous convex functions with lower level sets equal to some given closed convex sets. If the BCQ also holds (see Definition 3.4.1), proving the strong CHIP would be transformed into a problem of proving a subdifferential rule. Then it is possible to make use of Lemma 4.3.1, which was developed in their earlier paper [25, Theorem 2.1].

In the rest of this chapter, I will always be a compact metric space. We shall write $I(x) := \{i \in I : x \text{ is a relative boundary point of } C_i \text{ in } \operatorname{aff} D\}$.

Theorem 4.3.1 ([28, Theorem 4.1]). *Let $\{D, C_i : i \in I\}$ be a family of closed convex sets, I be a compact metric space. Write $C := \bigcap_{i \in I} C_i$. Let $x_0 \in D \cap C$. Then $\{D, C_i : i \in I\}$ has the strong CHIP at x_0 if the following conditions are satisfied.*

- **(i)** *The system $\{D, C_i : i \in I\}$ has the strong $D-$relative interior point condition (See Section 1.5);*
- **(ii)** *The set valued mapping $i \mapsto (\operatorname{aff} D) \cap C_i$ is lower semicontinuous on I;*
- **(iii)** *The pair $\{\operatorname{aff} D, C_i\}$ has the strong CHIP at x_0 for each $i \in I(x_0)$;*
- **(iv)** *D is finite dimensional.*

Proof. By translation, we may assume $0 \in D \cap \operatorname{rint}_D C$. According to the relative interior point assumption and the lower-semicontinuity assumption, by Lemma 4.3.3 and Lemma 4.3.4, we see that for each i, there exists an extended Minkowski functional of C_i such that for any $x \in \operatorname{span} D$, $p_{C_i}(x) < 1$ if and only if $x \in \operatorname{rint}_D C_i$ and

$p_{C_i}(x) \le 1$ if and only if $x \in C_i$. Define $g_i(\cdot) := p_{C_i}(\cdot) - 1$ and $G(\cdot) := \sup_{i \in I} g_i(\cdot)$. Then by Lemma 4.3.4, G is continuous and $i \mapsto g_i$ is upper semicontinuous at each $x \in X$. Moreover, for each $x \in D \cap C \subseteq \operatorname{span} D$, we have $I(x) := \{i \in I : x \text{ is a relative boundary point of } C_i \text{ in } \operatorname{span} D\} = \{i \in I : p_{C_i}(x) = 1\} = \{i \in I : g_i(x) = G(x) = 0\}$

Applying Lemma 4.3.1 to $\{g_i : i \in I\}$ and the set D, we get, for each $x \in D \cap C$, that

$$N_D(x) + \partial G(x) \subseteq N_D(x) + \sum_{i \in I(x)} \operatorname{cone}(\partial g_i(x)). \tag{4.3.12}$$

Now since $g_i(0) < 0$ for all $i \in I$ and $G(0) < 0$, we have from [11, Theorem 2.4.7, Corollary 1] that for each $x \in D \cap C$,

$$N_{G^{-1}(\mathbb{R}^-)}(x) = \operatorname{cone}(\partial G(x))$$

$$N_{g_i^{-1}(\mathbb{R}^-)}(x) = \begin{cases} \operatorname{cone}(\partial g_i(x)), & \forall i \in I(x) \\ 0, & \text{otherwise} \end{cases} \tag{4.3.13}$$

On the other hand,

$$N_{D \cap C}(x) = N_{D \cap G^{-1}(\mathbb{R}^-)}(x) = N_D(x) + N_{G^{-1}(\mathbb{R}^-)}(x),$$

where the first equality is by definition of G, while the second one follows from Theorem 4.2.1 and the fact that $0 \in D \cap \operatorname{int}(G^{-1}(\mathbb{R}^-))$ (note that $G(0) = -1$). Thus we get

$$\begin{aligned} N_{D \cap C}(x) &= N_D(x) + N_{G^{-1}(\mathbb{R}^-)}(x) \\ &= N_D(x) + \partial G(x) \\ &\subseteq N_D(x) + \sum_{i \in I(x)} \operatorname{cone}(\partial g_i(x)) \\ &= N_D(x) + \sum_{i \in I(x)} N_{g_i^{-1}(\mathbb{R}^-)}(x) \\ &= N_D(x) + \sum_{i \in I(x)} N_{C_i \cap \operatorname{span} D}(x), \end{aligned}$$

where the inclusion is due to (4.3.12), the third equality is due to (4.3.13), while the last equality is due to $C_i \cap \operatorname{span} D \subseteq g_i^{-1}(\mathbb{R}^-)$ and Proposition 2.3.1 **(iii)**. Finally, invoking assumption **(iii)**, we have at the point $x = x_0$ that, $N_{C_i \cap \operatorname{span} D}(x_0) = N_{C_i}(x_0) + N_{\operatorname{span} D}(x_0)$. This gives

$$N_{D\cap C}(x_0) \subseteq N_D(x_0) + \sum_{i\in I(x_0)} (N_{C_i}(x_0) + N_{\operatorname{span} D}(x_0)) \subseteq N_D(x_0) + \sum_{i\in I(x_0)} N_{C_i}(x_0),$$

where the last inclusion follows from $D \subseteq \operatorname{span} D$ and Proposition 2.3.1 **(iii)**. This completes the proof. □

The next theorem concerns the weak-strong interior point condition (see Section 1.5). Before proving it, we need the following lemma.

Lemma 4.3.5. *Let $\{D, C_1, C_2, \cdots, C_n\}$ be a collection of closed convex sets in a normed linear space X with $\dim D < +\infty$ and C_i being subspaces for all $1 \leq i \leq n$. Suppose $0 \in \operatorname{ri} D \cap \bigcap_{i=1}^n C_i$. Then $\{D, C_1, C_2, \cdots, C_n\}$ has the strong CHIP.*

Proof. First observe that $D \cap \bigcap_{i=1}^n C_i = D \cap \bigcap_{i=1}^n (C_i \cap \operatorname{span} D)$ and that the interior point assumption is equivalent to $0 \in \operatorname{ri} D \cap \bigcap_{i=1}^n (C_i \cap \operatorname{span} D)$. Since $C_i \cap \operatorname{span} D$ are finite dimensional subspaces for all $1 \leq i \leq n$, by Theorem 4.2.3, $\{D, C_1 \cap \operatorname{span} D, C_2 \cap \operatorname{span} D, \cdots, C_n \cap \operatorname{span} D\}$, as subsets of $\operatorname{span} D$, has the strong CHIP. Hence, by Proposition 3.2.1, $\{D, C_1 \cap \operatorname{span} D, C_2 \cap \operatorname{span} D, \cdots, C_n \cap \operatorname{span} D\}$ has the strong CHIP. To finish the proof, it suffices to show that $\{C_i, \operatorname{span} D\}$ has the strong CHIP for each $1 \leq i \leq n$. For each $1 \leq i \leq n$, consider $Z_i := C_i + \operatorname{span} D$. By finite dimensionality of $\operatorname{span} D$, C_i are finite co-dimensional subspaces in Z_i, and thus are polyhedrons. Since $0 \in \operatorname{ri} D \cap C_i$ for $1 \leq i \leq n$, we have $0 \in \operatorname{ri}(\operatorname{span} D) \cap C_i$ for $1 \leq i \leq n$. By Theorem 4.2.3, $\{C_i, \operatorname{span} D\}$ has the strong CHIP as a subset in Z_i, and thus the strong CHIP by Proposition 3.2.1. This completes the proof. □

Theorem 4.3.2 ([28, Theorem 4.3]). *Let $\{D, C_i : i \in I\}$ be a family of closed convex sets, I be a compact metric space. Write $C := \bigcap_{i\in I} C_i$. Let $x_0 \in D \cap C$. Then $\{D, C_i : i \in I\}$ has the strong CHIP at x_0 if the following conditions are satisfied.*

(i) *The system $\{D, C_i : i \in I\}$ has the weak-strong $D-$relative interior point condition;*

(ii) *The set valued mapping $i \mapsto C_i \cap (\operatorname{aff} D)$ is lower semicontinuous on I;*

(iii) *The pair* $\{\text{aff}\, D, C_i\}$ *has the strong CHIP at* x_0 *for each* $i \in I\backslash(I_1 \cup I_2)$*;*

(iv) D *is finite dimensional.*

Proof. Let I^l denote the closure of the set $I\backslash(I_1 \cup I_2)$. By assumption **(i)** and translation, we may assume without loss of generality that

$$0 \in \text{ri}\, D \bigcap \left(\text{rint}_D \bigcap_{i \in I\backslash(I_1 \cup I_2)} C_i \right) \cap \left(\bigcap_{i \in I_1} \text{ri}\, C_i \right) \cap \bigcap_{i \in I_2} C_i. \tag{4.3.14}$$

For each $i \in I_1$, by Lemma 4.3.3 as applied to C_i in place of C and A, there exists a closed convex set $\tilde{C}_i$ having 0 as interior such that

$$\tilde{C}_i \cap \overline{\text{span}\, C_i} = C_i. \tag{4.3.15}$$

Write $J = I \setminus (I_1 \cup I_2)$. For $i_0 \in I^l$, define

$$\widehat{C}_{i_0} := \begin{cases} \liminf_{i \to i_0, i \in I^l} (C_i \cap \text{span}\, D), & \text{if } i \in I^l \setminus J \\ C_{i_0} \cap \text{span}\, D, & \text{otherwise.} \end{cases} \tag{4.3.16}$$

Then by definition of lower limits and assumption **(ii)**, we have

$$C_i \cap \text{span}\, D \subseteq \widehat{C}_i, \ \forall i \in I^l \tag{4.3.17}$$

To proceed, we need to establish the following claims:

i) $C \cap D = D \cap \bigcap_{i \in I^l} \widehat{C}_i \cap \bigcap_{i \in I_2} C_i \cap \bigcap_{i \in I_1} \tilde{C}_i \cap \bigcap_{i \in I_1} \overline{\text{span}\, C_i}$

ii) $0 \in \text{ri}(D \cap \bigcap_{i \in I^l} \widehat{C}_i \cap \bigcap_{i \in I_1} \overline{\text{span}\, C_i}) \cap \bigcap_{i \in I_2} C_i$;

iii) $0 \in \text{ri}(D \cap \bigcap_{i \in I^l} \widehat{C}_i) \cap \bigcap_{i \in I_1} \overline{\text{span}\, C_i}$

iv) I^l is a compact metric space, $i \mapsto \widehat{C}_i \,(= \widehat{C}_i \cap \text{span}\, D)$ is lower semicontinuous for $i \in I^l$ and $\{D, \widehat{C}_i : i \in I^l\}$ satisfies the strong $D-$relative interior condition.

i) is obvious on invoking (4.3.15) and (4.3.17). We go on to prove *ii)*. To see this, we first show that

$$0 \in \text{ri} D \cap \text{rint}_D \bigcap_{i \in I^l} \widehat{C}_i. \tag{4.3.18}$$

Note that by (4.3.14), we have

$$0 \in \mathrm{ri} D \cap \mathrm{rint}_D \bigcap_{i \in J} C_i.$$

Thus in particular, there exists $\delta > 0$ such that for all $i \in J$,

$$\delta \mathbf{B} \cap \mathrm{span}\, D \subseteq C_i \cap \mathrm{span}\, D. \tag{4.3.19}$$

Passing to lower limits and invoking assumption **(ii)**, (4.3.16) and (4.3.17), we see that $\delta \mathbf{B} \cap \mathrm{span}\, D \subseteq \widehat{C}_i$ for all $i \in I^l$. This proves (4.3.18). Combining this with (4.3.14), we obtain

$$\begin{aligned} 0 &\in \mathrm{ri} D \cap \mathrm{rint}_D \bigcap_{i \in I^l} \widehat{C}_i \cap \left(\bigcap_{i \in I_1} \mathrm{ri}\, C_i \right) \cap \bigcap_{i \in I_2} C_i \qquad (4.3.20) \\ &\subseteq \mathrm{ri}(D \cap \bigcap_{i \in I^l} \widehat{C}_i \cap \bigcap_{i \in I_1} \overline{\mathrm{span}\, C_i}) \cap \bigcap_{i \in I_2} C_i. \end{aligned}$$

This proves Claim *ii*). Claim *iii*) follows from (4.3.20) and the fact that $\mathrm{ri} D \cap \mathrm{rint}_D \bigcap_{i \in I^l} \widehat{C}_i \subseteq \mathrm{ri}\left(D \cap \bigcap_{i \in I^l} \widehat{C}_i \right)$. Finally, we turn to Claim *iv*). I^l, being a closed subset of the compact metric space I, is compact. Note that $i \mapsto (\mathrm{span}\, D) \cap C_i$ is lower semicontinuous on I. Thus, for $i_0 \in J$, we have from the definition of lower limits and (4.3.17) that,

$$\widehat{C}_{i_0} = C_{i_0} \cap \mathrm{span}\, D \subseteq \liminf_{i \to i_0} (C_i \cap \mathrm{span}\, D) \subseteq \liminf_{i \to i_0, i \in I^l} (C_i \cap \mathrm{span}\, D) \subseteq \liminf_{i \to i_0, i \in I^l} \widehat{C}_i.$$

For $i_0 \in I^l \setminus J$, we have

$$\widehat{C}_{i_0} = \liminf_{i \to i_0, i \in I^l} (C_i \cap \mathrm{span}\, D) \subseteq \liminf_{i \to i_0, i \in I^l} \widehat{C}_i.$$

Thus $I^l \ni i \mapsto \widehat{C}_i (\subseteq \mathrm{span}\, D)$ is lower semicontinuous. Finally, the fact that $\{D, \widehat{C}_i : i \in I^l\}$ satisfies the strong D-relative interior condition follows from (4.3.18). The four claims are proved.

Now we have

$$\begin{aligned} N_{C \cap D}(x_0) &= N_{D \cap \bigcap_{i \in I^l} \widehat{C}_i \cap \bigcap_{i \in I_2} C_i \cap \bigcap_{i \in I_1} \bar{C}_i \cap \bigcap_{i \in I_1} \overline{\mathrm{span}\, C_i}}(x_0) \\ &\subseteq N_{D \cap \bigcap_{i \in I^l} \widehat{C}_i \cap \bigcap_{i \in I_2} C_i \cap \bigcap_{i \in I_1} \overline{\mathrm{span}\, C_i}}(x_0) + \sum_{i \in I_1} N_{\bar{C}_i}(x_0) \\ &\subseteq N_{D \cap \bigcap_{i \in I^l} \widehat{C}_i \cap \bigcap_{i \in I_1} \overline{\mathrm{span}\, C_i}}(x_0) + \sum_{i \in I_2} N_{C_i}(x_0) + \sum_{i \in I_1} N_{\bar{C}_i}(x_0), \qquad (4.3.21) \end{aligned}$$

where the first inclusion is due to $0 \in \operatorname{int} \tilde{C}_i$ for $i \in I_1$ and Theorem 4.2.3, the second inclusion follows from Theorem 4.2.3, Claim *ii)* and the fact that C_i are polyhedrons for $i \in I_2$.

Applying Lemma 4.3.5 and Claim *iii)*, we deduce from (4.3.21) that,

$$N_{C\cap D}(x_0) \subseteq N_{D\cap\bigcap_{i\in I^l}\widehat{C}_i}(x_0) + \sum_{i\in I_1} N_{\overline{\operatorname{span} C_i}}(x_0) + \sum_{i\in I_2} N_{C_i}(x_0) + \sum_{i\in I_1} N_{\tilde{C}_i}(x_0). \tag{4.3.22}$$

Invoking (4.3.15) and Theorem 4.2.3, we have,

$$\sum_{i\in I_1} N_{C_i}(x_0) = \sum_{i\in I_1} N_{\overline{\operatorname{span} C_i}}(x_0) + \sum_{i\in I_1} N_{\tilde{C}_i}(x_0). \tag{4.3.23}$$

Combining (4.3.22), (4.3.23) and applying Theorem 4.3.1 to the system $\{D, \widehat{C}_i : i \in I^l\}$, we see that

$$N_{C\cap D}(x_0) \subseteq N_D(x_0) + \sum_{i\in I^l} N_{\widehat{C}_i\cap\operatorname{span} D}(x_0) + \sum_{i\in I_2} N_{C_i}(x_0) + \sum_{i\in I_1} N_{C_i}(x_0). \tag{4.3.24}$$

To finish the proof, it suffices to show that

$$N_{\widehat{C}_i\cap\operatorname{span} D}(x_0) \subseteq N_{C_i}(x_0) + N_D(x_0),\ \forall i \in I^l. \tag{4.3.25}$$

For $i \in J$, (4.3.25) follows from assumption **(iii)**. Suppose first that $i \in I_2 \cap J$. Then it follows from (4.3.17) that $(\operatorname{span} D) \cap C_i \subseteq \widehat{C}_i$. Thus we have

$$N_{\widehat{C}_i\cap\operatorname{span} D}(x_0) \subseteq N_{C_i\cap\operatorname{span} D}(x_0) = N_{C_i}(x_0) + N_D(x_0),$$

where the last equality follows from Theorem 4.2.3, thanks to the fact that C_i is a polyhedron and that $0 \in \operatorname{ri} D \cap C_i$.

Now we turn to show that (4.3.25) holds also for $i \in I_1 \cap J$, which in turns finishes the proof. Let $j \in I_2 \cap I^l$. We have from (4.3.15), Theorem 4.2.3 and Lemma 4.3.5 that

$$\begin{aligned}
N_{\widehat{C}_i\cap\operatorname{span} D}(x_0) &\subseteq N_{C_i\cap\operatorname{span} D}(x_0)\\
&= N_{\tilde{C}_i\cap\overline{\operatorname{span} C_i}\cap\operatorname{span} D}(x_0)\\
&\subseteq N_{\tilde{C}_i}(x_0) + N_{\overline{\operatorname{span} C_i}}(x_0) + N_{\operatorname{span} D}(x_0)\\
&= N_{\tilde{C}_i\cap\overline{\operatorname{span} C_i}}(x_0) + N_{\operatorname{span} D}(x_0)\\
&\subseteq N_D(x_0) + N_{C_i}(x_0).
\end{aligned}$$

This completes the proof. □

4.3.4 Bounded Linear Regularity

The relationship between the strong CHIP and bounded linear regularity when the index set is infinite was not sufficiently addressed in the literature. The case when X is a general normed linear space would be addressed in the next chapter. We discuss the case when X is finite dimensional and I is a compact metric space. The proof is just similar to that of Theorem 4.2.4.

Theorem 4.3.3. *Let $\{C_i : i \in I\}$ be a family of closed convex sets in a finite dimensional space X, I be a compact metric space. Suppose that $i \mapsto C_i$ is lower semicontinuous and that $\{C_i : i \in I\}$ is boundedly linearly regular. Then the system has the strong CHIP.*

Proof. First recall that from [34, Corollary 4.7], The fact that $i \mapsto C_i$ is lower semicontinuous implies that $i \mapsto d_{C_i}(x)$ is upper semicontinuous for each $x \in X$.

Fix a $c \in C := \bigcap_{i \in I} C_i$. Since $\{C_i : i \in I\}$ is boundedly linearly regular, for $r = \|c\| + 1$, there exists $k_r > 0$ such that

$$d(x, C) \leq k_r \sup_{i \in I} d(x, C_i), \ \forall x \in r\mathbf{B}, \tag{4.3.26}$$

that is

$$d(x, C) \leq k_r \sup_{i \in I} d(x, C_i) + \delta_{r\mathbf{B}}(x), \ \forall x. \tag{4.3.27}$$

Let $y^* \in N_C(c)$, that is, there exist $\lambda \geq 0$ and $u^* \in \partial d(c, C)$ such that $y^* = \lambda u^*$, thanks to **(iv)** of Proposition 2.3.1. Now since $d(c, C) = d(c, C_i) = 0$ for all $i \in I$ and $\delta_{r\mathbf{B}}(x) = 0$ for all x in a neighborhood of c, it follows from (4.3.27) and [16, VI Theorem 4.4.2] (thanks to the continuity of the distance functions) that

$$\begin{aligned} y^* = \lambda u^* &\in \lambda \partial k_r \sup_{i \in I} d(\cdot, C_i)(c) + \lambda \partial \delta_{r\mathbf{B}}(c) \\ &= \lambda \partial k_r \sup_{i \in I} d(\cdot, C_i)(c) \\ &\subseteq \lambda k_r \mathrm{co} \bigcup_{i \in I} \partial d(\cdot, C_i)(c) \\ &\subseteq \sum_{i \in I} N_{C_i}(c). \end{aligned}$$

This completes the proof. $\square$

Chapter 5

The SECQ, Linear Regularity and the Strong CHIP for Infinite System of Closed Convex Sets in Normed Linear Spaces

5.1 Introduction

This chapter states some of the main results in the recent joint work by Professor Chong Li from Zhejiang University, my thesis supervisor Professor Kung Fu Ng and me. For details of the proofs, readers are referred to [29].

One of our fundamental lemmas is the following, which is a result similar to Lemma 2.7.2. It was stated without proof in [22, P.902]. (Note that the condition that "$\sup_{i\in I} g_i$ is proper" is needed).

Lemma 5.1.1 ([29, Lemma 2.2]). *Let $\{g_i : i \in I\}$ be a system of proper convex lower semicontinuous functions on a normed linear space X with $\sup_{i\in I} g_i(x_0) < +\infty$ for some $x_0 \in X$. Then*

$$\operatorname{epi}(\sup_{i\in I} g_i)^* = \overline{\operatorname{co}\bigcup_{i\in I} \operatorname{epi} g_i^*}^{w^*}. \tag{5.1.1}$$

The following example shows that the above proposition is not true without the properness assumption.

Example 5.1.1. Consider $C_1 = [0,1] \times \mathbb{R}$ and $C_2 = [-2,-1] \times \mathbb{R}$. Then $C_1 \cap C_2 = \emptyset$, thus $\max\{\delta_{C_1}(x), \delta_{C_2}(x)\} = \delta_{C_1 \cap C_2}(x) \equiv +\infty$ is not proper. By definition of conjugation and epigraph, we obtain $\operatorname{epi} \sigma_{C_1 \cap C_2} = \mathbb{R}^3$. On the other hand, we obtain by direct computation that

$$\sigma_{C_1}(x_1, x_2) = \begin{cases} 0, & \text{if } x_2 = 0, x_1 < 0, \\ x_1, & \text{if } x_2 = 0, x_1 \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\sigma_{C_2}(x_1, x_2) = \begin{cases} -2x_1, & \text{if } x_2 = 0, x_1 < 0, \\ -x_1, & \text{if } x_2 = 0, x_1 \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus $\operatorname{epi} \sigma_{C_1} + \operatorname{epi} \sigma_{C_2} \subseteq \mathbb{R} \times \{0\} \times \mathbb{R}$. This implies

$$\overline{\operatorname{co}(\operatorname{epi} \sigma_{C_1} \cup \operatorname{epi} \sigma_{C_2})}^{w^*} = \overline{\operatorname{epi} \sigma_{C_1} + \operatorname{epi} \sigma_{C_2}}^{w^*} \neq \operatorname{epi} \sigma_{C_1 \cap C_2}$$

By putting $g_i = \delta_{C_i}$ and using the fact that $\delta^*_{C_i} = \sigma_{C_i}$, we obtain the following analogue of Proposition 2.7.1.

Proposition 5.1.1 ([29, Proposition 2.1]). *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i \in I} C_i \neq \emptyset$. Then*

$$\operatorname{epi}\ \sigma_C = \overline{\sum_{i \in I} \operatorname{epi}\ \sigma_{C_i}}^{w^*}. \tag{5.1.2}$$

Lemma 5.1.1 motivates the following definition, in which we isolate the weakly* closedness of sum of epigraphs as a property to be studied.

Definition 5.1.1 ([29, Definition 2.1 c)]). *Let $\{C_i : i \in I\}$ be a collection of convex subsets of X. The collection is said to have the SECQ if* $\operatorname{epi} \sigma_{\bigcap_{i \in I} C_i} = \sum_{i \in I} \operatorname{epi} \sigma_{C_i}$.

The following result is direct from definition and Lemma 5.1.1.

Corollary 5.1.1 ([29, Corollary 2.1]). *Let $\{C_i : i \in I\}$ be a collection of closed convex sets in X with $C := \bigcap_{i\in I} C_i \neq \emptyset$. Then the following equivalences are true:*

$$\{C_i : i \in I\} \text{ satisfies the SECQ} \iff \textstyle\sum_{i\in I} \operatorname{epi} \sigma_{C_i} \text{ is } w^*\text{-closed} \iff \operatorname{epi} \sigma_C \subseteq \sum_{i\in I} \operatorname{epi} \sigma_{C_i}. \tag{5.1.3}$$

The intention of studying the SECQ is clear. It is expected that the SECQ should be a sufficient condition for the strong CHIP of the corresponding family of closed convex sets, since this is true in the particular case when the index set is finite. See Theorem 4.2.5. That this is really the case is the content of Theorem 5.2.1.

To close this beginning section, we give some notes on the following notion of semicontinuity of set-valued maps, which are to be used in sections 4.4 and 4.5. Readers may refer to standard texts such as [1].

Definition 5.1.2. *Let Q be a compact metric space. Let X be a normed linear space and let $t_0 \in Q$. A set-valued function $F : Q \to 2^Y \setminus \{\emptyset\}$ is said to be*

(i) *lower semicontinuous at t_0, if, for any $y_0 \in F(t_0)$ and any $\epsilon > 0$, there exists a neighborhood $U(t_0)$ of t_0 such that $\mathbf{B}(y_0, \epsilon) \cap F(t) \neq \emptyset$ for each $t \in U(t_0)$;*

(ii) *lower semicontinuous on Q if it is lower semicontinuous at each $t \in Q$.*

The following characterization regarding the lower semicontinuity is a reformulation of the equivalence of **(i)** and **(ii)** in [28, Proposition 3.1]. Let $\liminf_{t\to t_0} F(t)$ denote the lower limit of the set-valued function F at $t_0 \in Q$ which is defined by

$$\liminf_{t\to t_0} F(t) := \{z \in X : \exists \{z_t\}_{t\in Q} \text{ with } z_t \in F(t) \text{ such that } z_t \to z \text{ as } t \to t_0\}.$$

5.2 The strong CHIP and the SECQ

Recall that I is an arbitrary index set and $\{C_i : i \in I\}$ is a collection of nonempty closed convex subsets of X. We denote $\bigcap_{i\in I} C_i$ by C and assume that $0 \in C$ throughout the remaining parts of the chapter. The following theorem describes a relationship between the strong CHIP and the SECQ for the system $\{C_i : i \in I\}$, which is an infinite index analogue of Theorem 4.2.5.

Theorem 5.2.1 ([29, Theorem 3.1]). *If $\{C_i : i \in I\}$ satisfies the SECQ, then it has the strong CHIP; the converse conclusion holds if* $\operatorname{dom}\sigma_C \subseteq \operatorname{Im}\partial\,\delta_C$, *that is if*

$$\operatorname{dom}\sigma_C \subseteq \bigcup_{x\in C} N_C(x). \tag{5.2.1}$$

That means the SECQ is a sufficient condition for the strong CHIP. The converse is true if the set of intersection satisfies some property, namely (5.2.1). It is then natural to think of sufficient conditions for (5.2.1) to hold. This is the context of the next proposition.

Let f be a proper extended real valued function on X and $\overline{x} \notin \operatorname{dom} f$. Recall that the continuity of f at $\overline{x}$ means that there exists a neighborhood V of $\overline{x}$ such that $f(\cdot) = +\infty$ on V.

Proposition 5.2.1 ([29, Proposition 3.1]). *Let C be a nonempty closed convex set in X. Then the condition* (5.2.1) *holds in each of the following cases.*

(i) *There exists a weakly compact convex set D and a closed convex cone K such that $C = D + K$.*

(ii) $\dim C < \infty$, $\operatorname{Im}\partial\delta_C$ *is convex and the restriction $\sigma_C|_{(\operatorname{span} C)^*}$ of σ_C to the dual of the linear hull of C is continuous.*

Remark 5.2.1. **(i)** *By [2, Theorem 2.4.1], for a closed convex set C with $\dim C < \infty$, the last condition in* **(ii)** *of Proposition 5.2.1 is satisfied if and only if there does not exist a half-line ρ such that $\rho \subseteq \operatorname{bd} C$ nor exist a half-line ρ in $(\operatorname{span} C)\backslash C$ such that $\inf\{\|x - y\| : x \in \rho, y \in C\} = 0$.*

(ii) *Since* $\operatorname{Im}\partial\,\delta_C \subseteq \operatorname{dom}\sigma_C$ *holds automatically,* (5.2.1) *is equivalent to* $\operatorname{Im}\partial\,\delta_C = \operatorname{dom}\sigma_C$. *Thus, by the convexity of* $\operatorname{dom}\sigma_C$, *the convexity assumption of* $\operatorname{Im}\partial\,\delta_C$ *in (ii) of Proposition 2.1 is necessary for* (5.2.1).

Combining Theorem 5.2.1 and Proposition 5.2.1, we immediately have the following corollary, stating the situations under which the SECQ and the strong CHIP become equivalent. Part **(i)** was known in some special cases; see [9, Proposition 4.2] for the case when I is a two point set and $D = \{0\}$, and [21] for the case when I is a finite set and $D = \{0\}$.

Corollary 5.2.1 ([29, Corollary 3.1]). *Let $\{C_i : i \in I\}$ be a family of closed convex sets in X. Then the strong CHIP and the SECQ are equivalent for $\{C_i : i \in I\}$ in each of the following cases.*

(i) *There exists a weakly compact convex set D and a closed convex cone K such that $C = D + K$.*

(ii) $\dim C < \infty$, $\operatorname{Im} \partial \delta_C$ *is convex and the restriction* $\sigma_C|_{(\operatorname{span} C)^*}$ *of* σ_C *to the dual of the linear hull of* $\operatorname{span} C$ *is continuous..*

5.3 Linear regularity and the SECQ

Let I be an arbitrary index set and let $\{C_i : i \in I\}$ be a CCS-system with $0 \in C$, where $C = \bigcap_{i \in I} C_i$ as before. Throughout this section, we shall use $\mathbf{\Sigma}^*$ to denote the set $\mathbf{B}^* \times \mathbb{R}^+$, where $\mathbf{B}^*$ is the closed unit ball of X^* while $\mathbb{R}^+$ consists of all nonnegative real numbers. This section is devoted to a study of the relationship between the linear regularity and the SECQ. For a closed convex set S in a normed linear space X, let $d_S(\cdot)$ denote the distance function of S defined by $d_S(x) = \inf\{\|x - y\| : y \in S\}$ for each $x \in X$.

The following definition is just a duplication of Definition 4.2.2. We restate here for convenience.

Definition 5.3.1. *The system $\{C_i : i \in I\}$ is said to be*

(i) *linearly regular if there exists a constant $\gamma > 0$ such that*

$$d(x, C) \leq \gamma \sup_{i \in I} d(x, C_i) \quad \text{for all } x \in X. \tag{5.3.1}$$

(ii) *boundedly linearly regular if, for each $r > 0$, there exists a constant $\gamma_r > 0$ such that*

$$d(x, C) \leq \gamma_r \sup_{i \in I} d(x, C_i) \quad \text{for all } x \in r\mathbf{B}. \tag{5.3.2}$$

We intend to use epigraphs to study the linear regularity. Similar techniques have been used by Jeyakumar et al. in their study of the Farkas Lemmas and Constraint

Qualifications concerning epi-graphs. See [9, 19, 20, 21, 23], etc. To do this, we need to take conjugation on both sides of (5.3.1) and consider the corresponding epigraphs. We thus need Lemma 5.1.1 together with the following lemma. The formula (5.3.3) is obtained by direct computation.

Lemma 5.3.1 ([29, Lemma 4.2]). *Let $\gamma > 0$ and let $f_\gamma := \frac{1}{\gamma} d_S$. If $0 \in S$, then*

$$\operatorname{epi} f_\gamma^* = \operatorname{epi} \sigma_S \cap \left(\frac{1}{\gamma}\mathbf{B}^* \times \mathbb{R}^+\right). \tag{5.3.3}$$

In the next theorem, we shall use the graph $\operatorname{gph} f$ of a function f which is defined by

$$\operatorname{gph} f := \{(x, f(x)) \in X \times \mathbb{R} : x \in \operatorname{dom} f\}.$$

Clearly, $\operatorname{gph} f \subseteq \operatorname{epi} f$ for a function f on X.

The following theorem gives new characterizations of the linear regularity for families of closed convex sets in normed linear spaces.

Theorem 5.3.1 ([29, Theorem 4.1]). *Let $\gamma > 0$. Then the following conditions are equivalent.*

(i) *For all $x \in X$, $d(x, C) \leq \gamma \sup_{i \in I} d(x, C_i)$.*

(ii) $\operatorname{epi} \sigma_C \cap \mathbf{\Sigma}^* \subseteq \overline{\operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i} \cap \gamma \mathbf{\Sigma}^*)}^{w^*}$.

(iii) $\operatorname{gph} \sigma_C \cap \mathbf{\Sigma}^* \subseteq \overline{\operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i} \cap \gamma \mathbf{\Sigma}^*)}^{w^*}$.

One immediate application of our new characterizations of the linear regularity is to give yet another proof for the important characterization of the linear regularity of finitely many closed convex sets in a Banach space, given in [32, Theorem 4.2]. For details, readers are referred to [29, Theorem 4.2].

Since linear regularity and SECQ are both sufficient conditions for the strong CHIP, it is interesting to see when linear regularity would become a sufficient condition for the SECQ. This is the context of the next theorem.

Theorem 5.3.2 ([29, Theorem 4.3]). *Suppose that*

$$\overline{\operatorname{co} \bigcup_{i \in I} (\operatorname{epi} \sigma_{C_i} \cap \mathbf{\Sigma}^*)}^{w^*} \subseteq \sum_{i \in I} \operatorname{epi} \sigma_{C_i}, \tag{5.3.4}$$

and that $\{C_i : i \in I\}$ is linearly regular. Then it satisfies the SECQ.

The following example shows that (5.3.4) in Theorem 5.3.2 cannot be dropped.

Example 5.3.1 ([29, Example 4.1]). Let $X = \mathbb{R}^2$ and $I = \mathbb{N}$. Define $C_i := \{x \in X : \|x\| \leq \frac{1}{i}\}$ for each $i \in I$. Then $C = \bigcap_{i \in I} C_i = \{0\}$ and $d(x, C_i) = \max\{0, \|x\| - \frac{1}{i}\}$ for each $x \in X$. It follows that

$$\sup_{i \in I} d(x, C_i) = \|x\| = d(x, 0) = d(x, \bigcap_{i \in I} C_i).$$

Hence the system $\{C_i : i \in I\}$ is linearly regular. On the other hand, since $C = \{0\}$ and $N_{C_i}(0) = \{0\}$ for each $i \in I$, this system does not have the strong CHIP. Consequently, it does not satisfy the SECQ.

Since (5.3.4) is a condition that cannot be removed in Theorem 5.3.2, in the next theorem, we shall study some sufficient conditions for it. The first one is a technical lemma concerning convergence of elements from epigraphs. Recall that $\{C_i : i \in I\}$ is a CCS-system with $0 \in C$. We assume in the remainder of this section that I is a compact metric space.

Lemma 5.3.2 ([29, Lemma 4.3]). *Suppose that $i \mapsto C_i$ is lower semicontinuous. Consider elements $i_0 \in I$, $(x_0^*, \alpha_0) \in X^* \times \mathbb{R}$ and nets $\{i_k\} \subseteq I$, $\{(x_k^*, \alpha_k)\} \subseteq X^* \times \mathbb{R}$ with each $(x_k^*, \alpha_k) \in \operatorname{epi}\sigma_{C_{i_k}}$. Suppose further that $i_k \to i_0$, $\alpha_k \to \alpha_0$, and that $x_k^* \to^{w^*} x_0^*$. If $\{x_k^*\}$ is bounded, then $(x_0^*, \alpha_0) \in \operatorname{epi}\sigma_{C_{i_0}}$.*

Theorem 5.3.3 ([29, Theorem 4.4]). *Suppose that $i \mapsto C_i$ is lower semicontinuous on I and that either I is finite or there exists an index $i_0 \in I$ such that $\dim C_{i_0} < +\infty$. Then (5.3.4) holds. Consequently, if $\{C_i : i \in I\}$ is, in addition, linearly regular, then it satisfies the SECQ.*

A similar relationship holds between the boundedly linear regularity and the strong CHIP. For details of the derivation, we refer the readers to our paper.

Corollary 5.3.1 ([29, Corollary 4.1]). *Suppose that $i \mapsto C_i$ is lower semicontinuous on I and that either I is finite or there exists an index $i_0 \in I$ such that $\dim C_{i_0} < +\infty$. If $\{C_i : i \in I\}$ is boundedly linearly regular, then it has the strong CHIP.*

5.4 Interior-point conditions and the SECQ

Recall that I is an index-set and $C = \bigcap_{i\in I} C_i \subseteq X$. As in [28], the family $\{D, C_i : i \in I\}$ is called a closed convex set system with base-set D (CCS-system with base-set D) if D and each C_i are closed convex subsets of X. Furthermore, throughout the remainder of this section, we always assume that I is a compact metric space and $0 \in D \cap C$. Let $|J|$ denote the cardinality of the set $J \subseteq I$. The following definition is a generalization of the interior point conditions given in section 5 of Chapter 1.

Definition 5.4.1 ([29, Definition 5.1]). *Let $\{D, C_i : i \in I\}$ be a CCS-system with base-set D. Let m be a positive integer. Then the CCS-system $\{D, C_i : i \in I\}$ is said to satisfy:*

(i) *the m-D-interior-point condition if, for any subset J of I with $|J| \leq \min\{m, |I|\}$,*

$$D \bigcap \left(\bigcap_{i\in J} \mathrm{rint}_D C_i \right) \neq \emptyset; \tag{5.4.1}$$

(ii) *the m-interior-point condition if, for any subset J of I with $|J| \leq \min\{m, |I|\}$,*

$$D \bigcap \left(\bigcap_{i\in J} \mathrm{int} C_i \right) \neq \emptyset. \tag{5.4.2}$$

Our main theorem in this section is on some sufficient conditions for the SECQ of a family of infinitely many closed convex sets. Since the SECQ implies the strong CHIP (see Theorem 5.2.1), we simultaneously obtain sufficient conditions for the strong CHIP.

Theorem 5.4.1 ([29, Theorem 5.1]). *Let $m \in \mathbb{N}$ and let $\{D, C_i : i \in I\}$ be a CCS-system with the base-set D. We consider the following conditions.*

(a) *D is of finite dimension m.*

(b) *The set-valued mapping $i \mapsto (\mathrm{span}\, D) \cap C_i$ is lower semicontinuous on I.*

(c) *The system $\{D, C_i : i \in I\}$ satisfies $(m+1)$-D-interior-point condition.*

(d) *For each $i \in I$, the pair $\{D, C_i\}$ has the property:*

$$\operatorname{epi} \sigma_{(\operatorname{span} D) \cap C_i} \subseteq \operatorname{epi} \sigma_D + \operatorname{epi} \sigma_{C_i} \tag{5.4.3}$$

(e.g. $\{D, C_i\}$ satisfies the SECQ),

(c*) *The system $\{D, C_i : i \in I\}$ satisfies m-D-interior-point condition.*

(d*) *For each finite subset J of I with $|J| = \min\{m+1, |I|\}$, the subsystem $\{D, C_j : j \in J\}$ satisfies the SECQ.*

Then the following assertions hold.

(i) *If* **(a)**, **(b)**, **(c)** *are satisfied, then $\{D, (\operatorname{span} D) \cap C_i : i \in I\}$ satisfies the SECQ.*

(ii) *If* **(a)**, **(b)**, **(c)**, **(d)** *are satisfied, then $\{D, C_i : i \in I\}$ satisfies the SECQ.*

(iii) *If D is bounded and* **(a)**, **(b)**, **(c*)**, **(d*)** *are satisfied, then $\{D, C_i : i \in I\}$ satisfies the SECQ.*

Replacing the D-interior conditions by interior point conditions, we obtain the following corollary.

Corollary 5.4.1. *Let $m \in \mathbb{N}$ and let $\{D, C_i : i \in I\}$ be a CCS-system with the base-set D satisfying the following conditions.*

(a) *D is of finite dimension m.*

(b) *The set-valued mapping $i \mapsto (\operatorname{span} D) \cap C_i$ is lower semicontinuous on I.*

(c$^+$) *The system $\{D, C_i : i \in I\}$ satisfies $(m+1)$-interior-point condition.*

Then $\{D, C_i : i \in I\}$ satisfies the SECQ.

The last two corollaries explain our original intention for completing this work: we intended to generalize the main results of [28]. The following corollary, which is a direct consequence of Theorem 5.4.1 **(i)**, is an improvement of [28, Theorem 4.1].

Corollary 5.4.2 ([29, Corollary 5.1])**.** *Let $\{D, C_i : i \in I\}$ be a CCS-system with the base-set D. Let $m \in \mathbb{N}$ and let $x_0 \in D \cap C$. Suppose that the following conditions are satisfied.*

(a) *D is of finite dimension m.*

(b) *The set-valued mapping $i \mapsto (\operatorname{span} D) \cap C_i$ is lower semicontinuous on I.*

(c) *The system $\{D, C_i : i \in I\}$ satisfies $(m+1)$-D-interior-point condition.*

($\tilde{\text{d}}$) *For each $i \in I$, the pair $\{D, C_i\}$ has the property:*

$$N_{(\operatorname{span} D)\cap C_i}(x_0) \subseteq N_D(x_0) + N_{C_i}(x_0). \tag{5.4.4}$$

Then the system $\{D, C_i : i \in I\}$ has the strong CHIP at x_0.

The following corollary is an important improvement of [28, Proposition 5.1]. Our main improvement lies in the fact that we need not require the upper semicontinuity of the set valued map $i \mapsto (\operatorname{span} D) \cap C_i$ and that **(d)** can be weakened to required only the subsystems $\{D, C_j : j \in J\}$ with $|J| = l + 1$ have the strong CHIP.

Corollary 5.4.3 ([29, Corollary 5.2]). *Let $m \in \mathbb{N}$ and let $\{D, C_i : i \in I\}$ be a CCS-system with the base-set D satisfying the following conditions.*

(a) *D is of finite dimension m.*

(b) *The set-valued mapping $i \mapsto (\operatorname{span} D) \cap C_i$ is lower semicontinuous on I.*

(c*) *The system $\{D, C_i : i \in I\}$ satisfies m-D-interior-point condition.*

(d) *For each finite subset J of I with $|J| = \min\{m+1, |I|\}$, the subsystem $\{D, C_j : j \in J\}$ has the strong CHIP.*

Then the system $\{D, C_i : i \in I\}$ has the strong CHIP.

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