Robust Stabilization and Regulation of Nonlinear Systems in Feed Forward Form

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Contents

Abstract

This thesis contains three parts. The first part studies the small gain theorem for inter-connected nonlinear systems. The second part addresses the robust stabilization problem for nonlinear systems in feedforwad form. The third part further solves the output regulation problem for nonlinear systems in feedforwad form.

In the nonlinear control literature, a great deal of efforts have been put into the problem of finding the appropriate conditions to verify the stability of inter-connected systems. The small gain theorem is one of the effective tools, which was first introduced in the *input-to-state stability* (ISS, for short) framework by Jiang et al [26]. In [61], Teel introduced the concept of ISS with restrictions on the initial states and inputs and established a small gain theorem with restrictions for time invariant nonlinear systems. The first part of this thesis focuses on the small gain theorem for time-varying nonlinear systems. The major results are summarized as follows.

(1) The existing small gain theorem with restrictions is only available for the time invariant nonlinear system. We establish the following four types of small gain theorem with restrictions for uncertain time-varying nonlinear systems, thus closing the gap between the small gain theorem with restrictions for time-varying systems and that for time invariant systems:

(i) ISS small gain theorem with restrictions for uncertain nonlinear time-varying systems;

(ii) Semi-uniform ISS small gain theorem with restrictions for uncertain nonlinear time-varying systems;

(iii) Asymptotic small gain theorem with restrictions for uncertain nonlinear time-varying systems;

(iv) ISS small gain theorem with restrictions for uncertain time-varying systems of functional differential equations.

(2) In the past ten years, the proof of the ISS small gain theorem is based on two methods: the input-to-output formulation [8, 9, 22, 25, 61] and *Lyapunov* function argument [24, 26, 59]. The first part also explores the relation between these two methods and further gives a remark on various small gain conditions.

Since the mid-1990s, various control problems of feedforward systems have been studied by a number of people [1, 2, 3, 29, 33, 44, 45, 61]. Nevertheless, the case where the nonlinear system in feedforward form is subject to both static time-varying uncertainty and dynamic uncertainty has not been investigated. Relying upon the small gain theorem with restrictions for uncertain time-varying nonlinear systems established in the first part, the second part addresses robust semi-global and global stabilization problems for feedforward systems subject to both static and dynamic uncertainties. The stabilization solution of feedforward systems shed light on the solution of the robust output regulation problem of nonlinear systems in feedforward form.

Global robust output regulation is another challenge for feedforward systems. So far, the input disturbance suppression problem for a class of nonlinear system in feedforward form has been handled in [44]. The third part addresses the global robust output regulation problem for the same class of nonlinear systems in feedforward form as that in [44]. Our result contains the result in [44] as a special case.

摘要

本文研究關於聯結非線性系統的小增益定理以及非線性前饋系統的鎮定問題和輸出調節問題。

在非線性控制領域，大量研究集中在如何驗證聯結系統穩定性的問題上。輸入到狀態穩定框架下的小增益定理最早是 Jiang 等人提出的，並且成爲有效的工具之一。Teel 提出了帶約束的輸入到狀態穩定性的概念以及非線性時不變系統的帶約束的小增益定理。本文第一部分致力於研究非線性時變系統的小增益定理，其主要結果概述如下。

(1) 現有的帶約束的小增益定理只適用于非線性時不變系統。我們提出了如下四種不確定非線性時變系統的帶約束的小增益定理，統一了時不變系統和時變系統的帶約束的小增益定理。

(i) 不確定非線性時變系統的帶約束的輸入到狀態穩定小增益定理；

(ii) 不確定非線性時變系統的帶約束的半一致輸入到狀態穩定小增益定理；

(iii) 不確定非線性時變系統的帶約束的漸近小增益定理；

(iv) 不確定時滯系統的帶約束的輸入到狀態穩定小增益定理。

(2) 在過去的十年中，輸入到狀態穩定的小增益定理的證明主要是利用輸入到輸出框架和李亞普諾夫函數。本文第一部分探索了這兩種方法之間的關係，並且給出了一個關於多個小增益條件的注解。

從九十年代中期，許多學者開始研究各種前饋系統的控制問題。然而，至今還沒有學者研究過同時含有靜態、動態不確定性的前饋系統。利用不確定非線性時變系統的帶約束的輸入到狀態穩定小增益定理，本文第二部分解決了這類前饋系統的魯棒半全局和全局鎮定問題。前饋系統的鎮定問題的解決爲前饋系統的魯棒輸出調節問題的解決提供了前提。

全局魯棒輸出調節是前饋系統的另外一個挑戰。至今爲止，只有一類前饋系統的輸入干擾抑制問題在文[43]中得到了解決。本文第三部分解決了同一類前饋系統的全局魯棒輸出調節問題。輸入干擾抑制的結果是我們的全局魯棒輸出調節的結果特例。

Chapter 1

Introduction

1.1 Small Gain Theorem

In the nonlinear control literature, a great deal of efforts have been put into the problem of finding the appropriate conditions to verify the stability of inter-connected systems. The small gain theorem is one of the effective tools, which was originally introduced in monotone stability formulation by Hill [13]. The first small gain theorem in the ISS framework was established by Jiang *et al* [26]. Relying upon the input-to-output formulation, Jiang *et al* established a generalized small gain theorem for the following time-varying systems whose small gain condition involved two somewhat complicated inequalities [25, 26],

$$\dot{x} = f(x, u, t), \quad t \geq t_0 \geq 0 \tag{1.1}$$

viewing $x \in \mathbb{R}^n$ as the plant state, $u \in \mathbb{R}^m$ as the input, t_0 as the initial time, the function $f(x, u, t) : \mathbb{R}^n \times \mathbb{R}^m \times [t_0, \infty) \mapsto \mathbb{R}^n$ is piecewise continuous in t and locally *Lipschitz* in $\text{col}(x, u)$. In [8], Chen and Huang introduced the concept of *robust input to state stability* (RISS, for short) with respect to the external disturbance and/or the internal uncertainty and further extended the small gain theorem to uncertain system in the following form

$$\dot{x} = f(x, u, d, t), \quad t \geq t_0 \geq 0 \tag{1.2}$$

where $d(t) : [t_0, \infty) \mapsto \mathbb{R}^{n_d}$ is a family of piecewise continuous function of t, representing the external disturbance and/or the internal uncertainty. Moreover, the small gain condition was simplified into one contract mapping, leading to a more clear-cut version of the small gain theorem.

In [61], Teel introduced the concept of ISS with restrictions on the initial states and inputs and established a small gain theorem with restrictions for time invariant systems. In

Appendix B of [23], relying upon the separation principle for ISS with restrictions, Isidori *et al* established a more general small gain theorem with restrictions for time invariant systems which will be rephrased in Theorem 3.1. Nevertheless, Isidori's proof cannot be carried over to the case of time-varying systems, because the separation principle for ISS with restrictions does not hold for time-varying systems [8].

In Chapter 3, we first establish the four types of small gain theorem with restrictions for uncertain time-varying nonlinear systems, closing the gap between the small gain theorem with restrictions for time-varying systems and that for time invariant systems.

The proof of the ISS small gain theorem is usually based on two methods: the input-to-output formulation and *Lyapunov* function argument. These studies in [8, 9, 22, 25, 28, 61] are based on the concept of the gain function and the input-to-output formulation. On the other hand, it is well-known that *Lyapunov* functions play an important role in the nonlinear system and control, so it is natural to derive the small gain theorem using *Lyapunov* functions. The $ISS - Lyapunov$ function (dissipation) characterization of the small gain theorem was given in [24, 26, 60], whose small gain condition was based on the contract mapping of $ISS - Lyapunov$ functions. These functions have been applied in ISS analysis of open-loop systems and cascade inter-connected systems [22, 53, 55].

It is interesting to find out the relation between these two versions of the small gain theorems. Chapter 4 will show that the contract mapping of gain functions and that of $ISS - Lyapunov$ functions does not imply each other, i.e., if there exists two $ISS - Lyapunov$ functions for two subsystems respectively, we cannot guarantee the existence of two gain functions for two subsystems respectively which satisfy the contract mapping; on the converse, if two subsystems are both ISS and their gain functions satisfy the contract mapping, we also cannot guarantee the existence of the $ISS - Lyapunov$ functions for two subsystems which satisfy the contract mapping.

1.2 Stabilization for Feedforward Systems

In the literature of recursive control designs for nonlinear systems, two basic classes of systems are the most easily recognizable : the systems in the lower-triangular form (alternatively, strictly-feedback form) and the systems in the upper triangular form (alternatively, feedforward form). The lower-triangular systems, which occupied the attention of the nonlinear control community in the first half of the 1990s, are controlled using *backstepping*, a method that employs dominated controls necessary to suppress finite escape instabilities inherent (in open loop) to lower-triangular systems.

Since the mid-1990s, various control problems of feedforward systems have been studied by a number of people [1, 2, 3, 12, 29, 33, 44, 45, 46, 61]. Since feedforward systems do not admit any feedback path, the limitation of *backstepping* which is suitable for pure feedback systems (such as lower-triangular systems) stimulated the development of new recursive approaches for feedforward systems, such as *nested saturation* procedure or the Lyapunov *forwarding* procedure. At the absence of the dynamic uncertainty, Teel studied the disturbance attenuation with stability and gave a constructive solution for the problem based on a recursive design that utilizes the saturation function [61]. Along the same line, Arcak *et al* further considered the same problem allowing the system to contain the input unmodeled dynamics [2]. The stabilization of feedforward systems is also studied in [12, 29, 33, 44, 46]. Perhaps, a more far-reaching contribution of Teel in [61] is the tool he developed for analyzing systems containing or utilizing saturation functions. In particular, in the context of ISS with restrictions on the initial states and inputs, Teel established an asymptotic small gain theorem with restrictions for time invariant systems. This theorem is the foundation of a recursive approach that yields a closed-loop system whose state satisfies the asymptotic bound property without restriction on the initial state. This property together with the Hurwitzness of the Jacobian matrix of the closed-loop system at the origin guarantees the global asymptotic stability of the closed-loop system.

In Chapter 5 and Chapter 6, we focus on the robust stabilization problem for the feedforward systems (1.3) where, for $i = 1, .., n$, $x_i \in \mathbb{R}$, $d \in \mathbb{R}^{n_d}$, $\xi_i \in \mathbb{R}^{p_i}$, $u \in \mathbb{R}$, f_i and g_i are globally defined C^1 functions satisfying $f_i(0, \cdots, 0, d) = 0$ and $g_i(0, \cdots, 0, d) = 0$ for $d \in \mathbb{R}^{n_d}$, $c_1, \cdots, c_{n-1}$, A, B, C, D are (unknown) constants or matrices, and $d : [t_0, \infty) \to \Gamma$ is a piecewise continuous function with its range Γ a compact subset of $\mathbb{R}^{n_d}$.

$$
\begin{aligned}
\dot{x}_n &= f_n(\xi_1, x_1, ..., \xi_{n-1}, x_{n-1}, \xi_n, u, d(t)) \\
\dot{\xi}_n &= g_n(\xi_1, x_1, ..., \xi_{n-1}, x_{n-1}, \xi_n, d(t)) \\
&\vdots \\
\dot{x}_i &= f_i(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, \xi_i, u, d(t)) \\
\dot{\xi}_i &= g_i(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, \xi_i, d(t)) \\
&\vdots \\
\dot{x}_2 &= f_2(\xi_1, x_1, \xi_2, u, d(t)) \\
\dot{\xi}_2 &= g_2(\xi_1, x_1, \xi_2, d(t)) \\
\dot{x}_1 &= C\xi_1 + Du + f_1(\xi_1, u, d(t)) \\
\dot{\xi}_1 &= A\xi_1 + Bu + g_1(\xi_1, u, d(t)) \qquad (1.3)
\end{aligned}
$$

In Chapter 5, relying upon semi-uniform ISS small gain theorem with restrictions for uncertain time-varying nonlinear systems established in Chapter 3, we first study the problem of semi-global robust stabilization for (1.3) under the assumption that the linearization of each dynamic uncertainty is critically stable. In Chapter 6, appealing to asymptotic small gain theorem with restrictions for uncertain time-varying nonlinear systems proposed in Chapter 3, we further consider the problem of global robust stabilization for (1.3) under the assumption that the linearization of each dynamic uncertainty is Hurwitz.

1.3 Output Regulation for Feedforward Systems

The output regulation problem, or alternatively, servomechanism problem aims to solve the problem of designing a feedback controller to achieve asymptotic tacking for a class of reference input and disturbance rejection for a class of disturbances in an uncertain system while maintaining closed-loop stability. And the reference inputs and disturbances do not have to be known exactly so long as they are generated by a known, autonomous differential equation called exosystem.

The output regulation problem for feedforward systems has been studied in [3] and [45]. A special case of output regulation problem, input disturbance suppression problem, namely asymptotically rejecting bounded disturbances affecting the input channel, of a feedforward uncertain nonlinear system was considered in [45]. However, the paper can only handle the case when the steady-state is equal to zero. Recently, the problem of approximate and restricted tracking for a class of feedforward systems was addressed in [3]. "The term restricted refers to the fact that the disturbances to be rejected and/or the references to be tracked have to be sufficiently small. The term approximate refers to the fact that the regulated output will not vanish asymptotically, but only certain harmonic components will be canceled." [3] The output regulation problem under consideration in this thesis is more general and complicated than those studied in [3, 45] and includes the result in [45] as a special case.

A general framework for tackling the robust output regulation problem was proposed by J. Huang and Z. Chen in [18]. Under this framework, the robust output regulation for a given plant can be systematically converted into a robust stabilization problem for an appropriately defined augmented system. This general framework has been successfully applied to solve the global robust output regulation problem for lower-triangular nonlinear systems [18] and the semi-global robust output regulation problem for a class of nonlinear affine systems in normal form [35]. In Chapter 7, we will further utilize this framework to study the global robust output regulation problem for a class of feedforward systems. As in [18], our approach consists of two steps. First, the global robust output regulation problem of the given plant is converted into a global robust stabilization problem for an appropriately defined augmented system. Second, the global robust stabilization problem for the augmented system is solved on the basis of the combination of the asymptotic small gain theorem with restrictions and nested saturation technique.

1.4 Organization and Contributions

The reminder of this thesis is organized as follows.

Chapter 2: For the purpose of self-containment, the thesis starts from an introduction of various concepts concerning ISS and input-to-output stability.

Chapter 3: We establish four versions of small gain theorem with restrictions on the inputs and the initial states for uncertain time-varying nonlinear systems.

Chapter 4: A remark on various small gain conditions is given in this chapter.

Chapter 5: Relying upon semi-uniform ISS small gain theorem with restrictions established in Chapter 3, we solve semi-global robust stabilization for a class of feedforward systems subject to both static uncertainties and dynamic uncertainties.

Chapter 6: Relying upon asymptotic small gain theorem with restrictions established in Chapter 3, we address global robust stabilization for a class of feedforward systems in the presence of both static uncertainties and dynamic uncertainties.

Chapter 7: By appealing to the general framework for tackling the robust output regulation problem proposed in [18], we investigate global robust output regulation problem for a class of feedforward systems.

Chapter 8: Some concluding remarks and future prospects are given in this chapter.

The thesis was typeset using LaTeX. All numerical simulations were done using MATLAB.

Chapter 2

Input-to-State Stability for Nonlinear Systems

The notion of *input-to-state stability* (ISS, for short), first introduced in [53], provides a theoretical framework in which to formulate questions of robustness with respect to inputs acting on a system. Roughly speaking, an ISS system is one which has a finite nonlinear gain with respect to inputs and whose transient behavior can be bounded in terms of the size of the initial state and inputs. The theory of ISS systems now forms an integral part of several texts (see e.g. [23, 31]) as well as expository and research articles (see e.g. [2, 25, 44, 61]). In this chapter, we introduce some definitions and properties of ISS which will be referred to in the subsequent chapters.

Throughout the thesis, let L_∞^m be the set of all piecewise continuous bounded functions $u : [t_0, \infty) \mapsto \mathbb{R}^m$ with a finite supremum norm $\|u_{[t_0,\infty)}\| = \sup_{t \geq t_0} \|u(t)\|$. Denote the supremum norm of the truncation of $u(t)$ in $[t_1, t_2]$ by $\|u_{[t_1,t_2]}\| = \sup_{t_1 \leq t \leq t_2} \|u(t)\|$. And denote $\|u\|_a = \limsup_{t\to\infty} \|u\|$. The inequalities will involve the following functions: a function $\gamma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is of class K if it is continuous, strictly increasing and $\gamma(0) = 0$; and a function $\beta(s,t) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is of class KL if it is continuous, for each fixed $t \geq 0$, the function $\beta(s,t)$ belongs to class K with respect to s and, for each fixed s, the function $\beta(s,t)$ is decreasing with respect to t and $\beta(s,t) \to 0$ as $t \to \infty$. X is a compact set containing the origin.

Consider the following time invariant nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x,u) \quad t \geq 0 \end{aligned} \tag{2.1}$$

viewing $x \in \mathbb{R}^n$ as the plant state, $u \in \mathbb{R}^m$ as the input, $y \in \mathbb{R}^p$ as the output, the functions $f(x,u) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $h(x,u) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^p$ are locally *Lipschitz* in $\mathrm{col}(x,u)$.

Definition 2.1 [23] System (2.1) is said to be ISS with restrictions X and Δ on the initial state $x(0)$ and the input u respectively if there exist class KL function β and class K function γ such that, for any initial state $x(0) \in X$ and any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[0,\infty)}\| \leq \Delta$, the solution exists and satisfies, for all $t \geq 0$,

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u_{[0,\infty)}\|)\}. \tag{2.2}$$

∎

From a practical point of view, it turns out that the property of ISS with restrictions can be checked in terms of the existence of an local $ISS - Lyapunov$ function.

Definition 2.2 [23] A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called a local $ISS - Lyapunov$ function for system (2.1) if there exists class K_∞ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha(\cdot)$, a class K function $\chi(\cdot)$ and positive numbers δ_x and δ_u such that, for all x and u such that

$$\alpha_1(x) \leq V(x) \leq \alpha_2(x)$$
$$\chi(\|u\|) < \|x\| < \delta_x, \|u\| < \delta_u \Rightarrow \frac{\partial V}{\partial x} f(x,u) \leq -\alpha(\|x\|).$$

∎

Remark 2.1 In Lemma 3.3 [61], it was shown that if system (2.1) admits a local $ISS - Lyapunov$ function, then system (2.1) is ISS with restrictions, i.e., there exist class KL function β, class K function γ, compact set X and positive real number Δ, such that, for any initial state $x(0) \in X$ and any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[0,\infty)}\| \leq \Delta$, the solution of (2.1) exists and satisfies, for all $t \geq 0$,

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u_{[0,\infty)}\|)\},$$

where,
$\gamma(s) = \alpha_1^{-1} \circ \alpha_2 \circ \chi(s)$ and,
$X = \{x : \alpha_1^{-1} \circ \alpha_2(x) \leq \delta_x\}$,
$\Delta \leq \delta_u$,
$s \in [0, \Delta) \Longrightarrow \alpha_1^{-1} \circ \alpha_2 \circ \chi(s) \leq \delta_x$. ∎

The notion of *input-output stability* (IOS, for short) formalizes the idea that outputs depend in an "asymptotically stable" manner on inputs, while internal signals remain bounded. When the output equals the complete state, one recovers the property of ISS. When there are no inputs, one has a generalization of the classical concept of partial stability. We source the notion of IOS in [26] to the following.

Definition 2.3 System (2.1) is said to be IOS with restrictions X and Δ on the initial state $x(0)$ and the input u respectively if there exist class KL function β and class K function γ such that, for any initial state $x(0) \in X$ and any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[0,\infty)}\| \leq \Delta$, the output $y(t)$ exists and satisfies, for all $t \geq 0$,

$$\|y(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u_{[0,\infty)}\|)\}. \tag{2.3}$$

∎

Remark 2.2 By causality, the same definitions can be obtained if one would replace $\|u_{[0,\infty)}\|$ by $\|u_{[0,t]}\|$ in (2.2) and (2.3). ∎

Next, consider the following time-varying uncertain nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u,d,t), \\ y &= h(x,u,d,t) \quad t \geq t_0 \geq 0 \end{aligned} \tag{2.4}$$

viewing $x \in \mathbb{R}^n$ as the plant state, $u \in \mathbb{R}^m$ as the input, t_0 as the initial time, the function $f(x,u,d,t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_d} \times [t_0, \infty) \mapsto \mathbb{R}^n$ is piecewise continuous in $\text{col}(d,t)$ and locally *Lipschitz* in $\text{col}(x,u)$. And $d(t) : [t_0, \infty) \mapsto \mathbb{R}^{n_d}$ is a family of piecewise continuous function of t, representing the external disturbance and/or the internal uncertainty.

Definition 2.4 System (2.4) is said to be *robust input-state stable* (RISS, for short) with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exist class KL function β and class K function γ, independent of $d(t)$, such that, for any initial state $x(t_0) \in X$ and any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| \leq \Delta$, the solution of (2.4) exists and satisfies, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}.$$

∎

Remark 2.3 The concept of RISS with restrictions is generalized from the RISS concept introduced in [8]. The concept of RISS was originally introduced in [38] for a special case of system (2.4) where $d(t)$ is an unknown constant μ. In [38], it was proposed that the stability of a family of parameterized systems could be studied by verifying the stability of the following auxiliary system

$$\dot{x} = f(x, \mu), \quad \dot{\mu} = 0$$

in which both x and μ were treated as states. If $d(t) \equiv 0$ in (2.4), the stability of (2.4) can be studied through the stability of the following auxiliary system

$$\dot{x} = f(x, \lambda, u), \quad \dot{\lambda} = 1$$

in which both x and λ are treated as states. ∎

Definition 2.5 System (2.4) is said to be *robust input-output stable* (RIOS, for short) with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exist class KL function β and class K function γ, independent of $d(t)$, such that, for any initial state $x(t_0) \in X$, any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| \leq \Delta$, the output of (2.4) exists and satisfies, for all $t \geq t_0$,

$$\|y(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}.$$

∎

Definition 2.6 System (2.4) is said to have the *robust unbounded observability* (RUO, for short) property with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exist class K functions α_x, α_u and α_y, independent of $d(t)$, such that for any initial state $x(t_0) \in X$ and any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| \leq \Delta$, the solution of (2.4) exists and satisfies, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\alpha_x(\|x(t_0)\|), \alpha_u(\|u_{[t_0,t]}\|), \alpha_y(\|y_{[t_0,t]}\|)\}.$$

∎

Remark 2.4 The concepts of RIOS with restrictions and RUO with restrictions follow straightforwardly from the RIOS and RUO concepts given in [8]. ∎

The following definitions of *robust asymptotic gain* (RAG, for short) property with restrictions and *robust uniform stability* (RUS, for short) with restrictions generalize the concept of *asymptotic gain* (AG, for short) property and *uniform stability* (US, for short) in [57] respectively.

Definition 2.7 System (2.4) is said to satisfy RAG property with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exists class K function γ^u, independent of d, such that for any initial state $x(t_0) \in X$ and input $u \in L_\infty^m$ satisfying $\|u\|_a \leq \Delta$, the solution exists and satisfies, for all $t \geq t_0$,

$$\|x\|_a \leq \gamma^u(\|u\|_a). \tag{2.5}$$

∎

Definition 2.8 System (2.4) is said to be RUS with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exist class K functions γ^0 and γ^u, independent of d, such that for any initial state $x(t_0) \in X$ and input $u \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| < \Delta$, the solution exists and satisfies, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\gamma^0(\|x(t_0)\|), \gamma^u(\|u_{[t_0,\infty)}\|)\}. \tag{2.6}$$

∎

Remark 2.5 Time invariant systems (2.1) can be viewed as a special case of system (2.4) where the functions f and h are independent of the time t and the time-varying disturbance d. It is known that, system (2.1) is ISS if and only if it is US and has AG property [57]. This property is called the separation principle of ISS systems, and it greatly facilitates the establishment of many results such as the small gain theorem for time invariant systems to be described in Chapter 3. Unfortunately, a time-varying nonlinear system does not possess the separation principle [8]. Nevertheless, recently, a concept of semi-uniform ISS was introduced for time-varying nonlinear systems in [40] and it was shown that a time-varying system is semi-uniformly ISS if and only if it is US and has AG property [40].

∎

We extend the concepts of semi-uniform ISS in [40] and output asymptotic bound, asymptotic L_∞ stability in [61] to the following ones.

Definition 2.9 System (2.4) is said to be robust semi-uniformly ISS with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exist class KL function β and class K function γ and ρ, independent of $d(t)$, such that, for any initial state $x(t_0) \in X$ and any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| \leq \Delta$, the solution of (2.4) exists and satisfies, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, \frac{t-t_0}{1+\rho(t_0)}), \gamma(\|u_{[t_0,t]}\|)\}.$$

∎

Definition 2.10 System(2.4) is said to satisfy output robust asymptotic bound (o-RAG, for short) with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exist class K function γ^u, independent of d, such that for any initial state $x(t_0) \in X$ and input $u \in L_\infty^m$ satisfying $\|u\|_a \leq \Delta$, the solution exists and satisfies, for all $t \geq t_0$,

$$\|y\|_a \leq \gamma^u(\|u\|_a). \tag{2.7}$$

∎

Definition 2.11 The output function of (2.4) is said to satisfy robust asymptotic L_∞ stability (RALS, for short) with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exists class K functions γ_0 and γ, independent of $d(t)$, such that for any initial state $x(t_0) \in X$ and input $u \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| \leq \Delta$, the output exists and satisfies, for all $t \geq t_0$,

$$\begin{aligned} \|y(t)\| &\leq \max\{\gamma_0(\|x(t_0)\|), \gamma(\|u_{[t_0,\infty)}\|)\} \\ \|y\|_a &\leq \gamma(\|u\|_a). \end{aligned} \tag{2.8}$$

∎

Remark 2.6 It follows from Theorem 2 in [39] that system (2.4) is robust semi-uniformly ISS with restriction if and only if it is RUS with restrictions and RAG with restrictions. Such separation principle will play an important role in the proof of Theorem 1. ∎

Chapter 3

Small Gain Theorem with Restrictions for Uncertain Time-varying Nonlinear Systems

This chapter is to establish the following four types of small gain theorem with restrictions for uncertain time-varying nonlinear systems, thus filling the gap between the small gain theorem with restrictions for time-varying systems and that for time invariant systems:

(i) ISS small gain theorem with restrictions for uncertain nonlinear time-varying systems;

(ii) Semi-uniform ISS small gain theorem with restrictions for uncertain nonlinear time-varying systems;

(iii) Asymptotic small gain theorem with restrictions for uncertain nonlinear time-varying systems;

(iv) ISS small gain theorem with restrictions for time-varying systems of functional differential equations.

These small gain theorems will be further applied in handling the semi-global/global robust stabilization/output regulation problems in the subsequent chapters.

3.1 Input-to-State Stability Small Gain Theorem with Restrictions for Uncertain Nonlinear Time-varying Systems

3.1.1 Nonlinear Time Invariant Systems Case

Consider the feedback interconnection as depicted in Figure 3.1,

$$\dot{x}_1 = f_1(x_1, v_1, u_1), \quad y_1 = h_1(x_1, v_1, u_1) \tag{3.1}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2), \quad y_2 = h_2(x_2, v_2, u_2) \tag{3.2}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1 \tag{3.3}$$

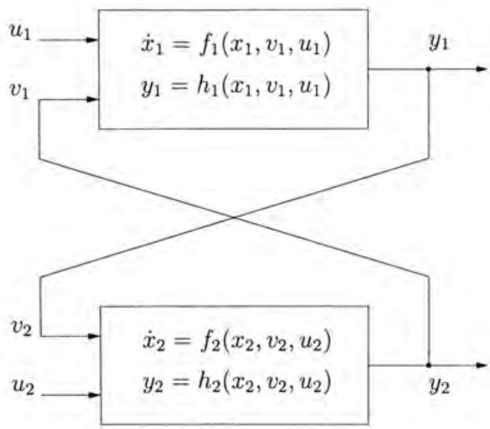

Figure 3.1: Inter-connection of (3.1) and (3.2)

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{p_i}$, $v_i \in \mathbb{R}^{q_i}$ with $p_1 = q_2$, $p_2 = q_1$, the function $f_i(x_i, v_i, u_i)$ is locally *Lipschitz* in $\text{col}(x_i, v_i, u_i)$, and $f_i(0,0,0) = 0$, $h_i(0,0,0) = 0$.

And suppose the following assumption holds.

A 3.1 There exists a C^1 function h such that

$$\text{col } (y_1, y_2) = h(x_1, x_2, u_1, u_2)$$

is the unique solution of the equations

$$y_1 = h_1(x_1, y_2, u_1), \quad y_2 = h_2(x_2, y_1, u_2).$$

The following small gain theorem with restrictions for time invariant systems was established in [23].

Theorem 3.1 Assume that subsystem (3.1) is ISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(0)$, v_1 and u_1 respectively and subsystem (3.2) is ISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(0)$, v_2 and u_2 respectively, i.e., there exist class KL functions β_1 and β_2, class K functions γ_1, γ_2, γ_1^u, γ_2^u such that, for any $x_1(0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[0,\infty)}\| \leq \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[0,\infty)}\| \leq \Delta_1^u$, the solution of (3.1) exists and satisfies, for all $t \geq 0$,

$$\|x_1(t)\| \leq \max\{\beta_1(\|x_1(0)\|, t), \gamma_1(\|v_{1[0,\infty)}\|), \gamma_1^u(\|u_{1[0,\infty)}\|)\}$$

and for any $x_2(0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[0,\infty)}\| \leq \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[0,\infty)}\| \leq \Delta_2^u$, the solution of (3.2) exists and satisfies, for all $t \geq 0$,

$$\|x_2(t)\| \leq \max\{\beta_2(\|x_2(0)\|, t), \gamma_2(\|v_{2[0,\infty)}\|), \gamma_2^u(\|u_{2[0,\infty)}\|)\}.$$

Suppose the following estimates hold for the outputs y_1 and y_2

$$\begin{aligned}
\|y_{1[0,\infty)}\| &\leq \max\{\bar{\gamma}_1^0(\|x_1(0)\|), \overline{\gamma}_1(\|v_{1[0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[0,\infty)}\|)\} \\
\limsup_{t\to\infty} \|y_1(t)\| &\leq \max\{\overline{\gamma}_1(\limsup_{t\to\infty} \|v_1(t)\|), \overline{\gamma}_1^u(\limsup_{t\to\infty} \|u_1(t)\|)\} \\
\|y_{2[0,\infty)}\| &\leq \max\{\bar{\gamma}_2^0(\|x_2(0)\|), \overline{\gamma}_2(\|v_{2[0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[0,\infty)}\|)\} \\
\limsup_{t\to\infty} \|y_2(t)\| &\leq \max\{\overline{\gamma}_2(\limsup_{t\to\infty} \|v_2(t)\|), \overline{\gamma}_2^u(\limsup_{t\to\infty} \|u_2(t)\|)\}
\end{aligned}$$

for some class K functions $\bar{\gamma}_1^0$, $\bar{\gamma}_2^0$, $\overline{\gamma}_1$, $\overline{\gamma}_2$, $\overline{\gamma}_1^u$ and $\overline{\gamma}_2^u$.

Then if

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad \forall r > 0$$

the system composed of (3.1) and (3.2) is ISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\bar{\Delta}_1$ and $\bar{\Delta}_2$ on $x(0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state and $u = \mathrm{col}(u_1, u_2)$ as input, i.e., there exist class KL function β and class K function γ, such that, for any initial state $x(0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[0,\infty)}\| \leq \bar{\Delta}_1$ and

$u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[0,\infty)}\| \leq \tilde{\Delta}_2$, the solutions of (3.1) and (3.2) exist and satisfy, for all $t \geq 0$,

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u_{[0,\infty)}\|)\} \tag{3.4}$$

where,

$\tilde{X}_1 = \{x_1 \in X_1 : \bar{\gamma}_1^0(\|x_1\|) < \Delta_2, \overline{\gamma}_2 \circ \bar{\gamma}_1^0(\|x_1\|) < \Delta_1\}$

and

$\tilde{X}_2 = \{x_2 \in X_2 : \bar{\gamma}_2^0(\|x_2\|) < \Delta_1, \overline{\gamma}_1 \circ \bar{\gamma}_2^0(\|x_2\|) < \Delta_2\}$.

$\tilde{\Delta}_1 \leq \Delta_1^u$, $\tilde{\Delta}_2 \leq \Delta_2^u$

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \Delta_1, \overline{\gamma}_1^u(s) < \Delta_2$

and

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \Delta_2, \overline{\gamma}_2^u(s) < \Delta_1$. ∎

Remark 3.1 Theorem 3.1 is slightly different from Theorem B.3.1 [23] where for $i = 1, 2$, $\gamma_i(s) = \overline{\gamma}_i(s)$ and $\gamma_i^u(s) = \overline{\gamma}_i^u(s)$. ∎

3.1.2 Uncertain Time-varying Nonlinear Systems Case

Let us introduce a technical lemma which was established in [8].

Lemma 3.1 Let β be a class KL function, γ a class K function such that $\gamma(r) < r$ ($\forall r > 0$), and $\mu \in (0, 1]$ a real number. For any nonnegative real numbers s and M, and any nonnegative real function $z(t) \in L_\infty^1$ satisfying

$$z(t) \leq \max\{\beta(s, t), \gamma(\|z_{[\mu t, t]}\|), M\}, \quad \forall t \geq 0,$$

there exists a class K_∞ function $\hat{\beta}$ such that

$$z(t) \leq \max\{\hat{\beta}(s, t), M\}, \quad \forall t \geq 0.$$

Proof: The proof is given in the Appendix.

∎

Consider the interconnection of the following two systems as depicted in Figure 3.2,

$$\dot{x}_1 = f_1(x_1, v_1, u_1, d, t), \quad y_1 = h_1(x_1, v_1, u_1, d, t) \tag{3.5}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = h_2(x_2, v_2, u_2, d, t) \tag{3.6}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1 \tag{3.7}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{p_i}$, $v_i \in \mathbb{R}^{q_i}$ with $p_1 = q_2$, $p_2 = q_1$, the functions $f_1(x_1, v_1, u_1, d, t)$ and $f_2(x_2, v_2, u_2, d, t)$ are piecewise continuous in col(d, t) and locally *Lipschitz* in col(x_1, v_1, u_1) and col(x_2, v_2, u_2) respectively, and $d : [t_0, \infty) \mapsto \mathbb{R}^{n_d}$ is piecewise continuous.

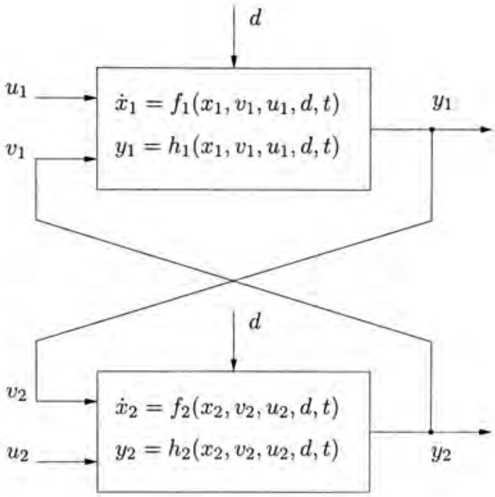

Figure 3.2: Inter-connection of (3.5) and (3.6)

The system composed of (3.5) and (3.6) is interpreted as feedback interconnection of two subsystems, the upper one with state x_1, input col(v_1, u_1) and output y_1 and the lower one with state x_2, input col(v_2, u_2) and output y_2. And suppose the following assumption holds.

A 3.2 There exists a C^1 function h such that

$$\text{col}\ (y_1, y_2) = h(x_1, x_2, u_1, u_2, d, t)$$

is the unique solution of the equations

$$\begin{aligned} y_1 &= h_1(x_1, y_2, u_1, d, t) \\ y_2 &= h_2(x_2, y_1, u_2, d, t). \end{aligned}$$

Theorem 3.2 Assume that subsystem (3.5) is RISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(t_0)$, v_1 and u_1 respectively and subsystem (3.6) is RISS with restrictions X_2, Δ_2 and

Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class KL functions β_1 and β_2, class K functions γ_1, γ_1^u, γ_2, γ_2^u, independent of $d(t)$, such that, for any $x_1(t_0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[t_0,\infty)}\| \leq \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| \leq \Delta_1^u$, the solution of (3.5) exists and satisfies, for all $t \geq t_0$,

$$\|x_1(t)\| \leq \max\{\beta_1(\|x_1(t_0)\|, t-t_0), \gamma_1(\|v_{1[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\} \tag{3.8}$$

and for any $x_2(t_0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| \leq \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| \leq \Delta_2^u$, the solution of (3.6) exists and satisfies, for all $t \geq t_0$,

$$\|x_2(t)\| \leq \max\{\beta_2(\|x_2(t_0)\|, t-t_0), \gamma_2(\|v_{2[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}. \tag{3.9}$$

Further assume that subsystem (3.5) is RIOS with restrictions $\bar{X}_1$, $\overline{\Delta}_1$ and $\overline{\Delta}_1^u$ on $x_1(t_0)$, v_1 and u_1 respectively and subsystem (3.6) is RIOS with restrictions $\bar{X}_2$, $\overline{\Delta}_2$ and $\overline{\Delta}_2^u$ on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class KL functions $\overline{\beta}_1$ and $\overline{\beta}_2$, class K functions $\overline{\gamma}_1$, $\overline{\gamma}_1^u$, $\overline{\gamma}_2$, $\overline{\gamma}_2^u$, independent of $d(t)$, such that, for any $x_1(t_0) \in \bar{X}_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[t_0,\infty)}\| \leq \overline{\Delta}_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| \leq \overline{\Delta}_1^u$, the output of (3.5) exists and satisfies, for all $t \geq t_0$,

$$\|y_1(t)\| \leq \max\{\overline{\beta}_1(\|x_1(t_0)\|, t-t_0), \overline{\gamma}_1(\|v_{1[t_0,t]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,t]}\|)\} \tag{3.10}$$

and for any $x_2(t_0) \in \bar{X}_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| \leq \overline{\Delta}_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| \leq \overline{\Delta}_2^u$, the output of (3.6) exists and satisfies, for all $t \geq t_0$,

$$\|y_2(t)\| \leq \max\{\overline{\beta}_2(\|x_2(t_0)\|, t-t_0), \overline{\gamma}_2(\|v_{2[t_0,t]}\|), \overline{\gamma}_2^u(\|u_{2[t_0,t]}\|)\}. \tag{3.11}$$

Suppose that the small gain condition

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad r > 0 \tag{3.12}$$

holds, then the system composed of (3.5) and (3.6) is RISS and RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input, i.e., there exist class KL functions β and $\overline{\beta}$, class K functions γ and $\overline{\gamma}$, independent of $d(t)$, such that, for any initial state $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| \leq \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| \leq \tilde{\Delta}_2$, the solution and output of (3.5) and (3.6) exist and satisfy, for all $t \geq t_0$,

$$\begin{aligned} \|x(t)\| &\leq \max\{\beta(\|x(t_0)\|, t-t_0), \gamma(\|u_{[t_0,t]}\|)\} \\ \|y(t)\| &\leq \max\{\overline{\beta}(\|x(t_0)\|, t-t_0), \overline{\gamma}(\|u_{[t_0,t]}\|)\} \end{aligned}$$

where,

$\gamma(s) = \max\{4\gamma_1 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 4\gamma_1 \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 4\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 4\gamma_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2^u(s), 2\gamma_2^u(s)\}$,

$\overline{\gamma}(s) = \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\}$

and,

(i) If Δ_1, Δ_2, $\overline{\Delta}_1$, $\overline{\Delta}_2$ are finite,

$\tilde{X}_1 = \{x_1 \in X_1 \cap \bar{X}_1 : \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2 \circ \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\}\}$

and

$\tilde{X}_2 = \{x_2 \in X_2 \cap \bar{X}_2 : \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\}, \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}\}$.

$\tilde{\Delta}_1 \leq \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \leq \min\{\Delta_2^u, \overline{\Delta}_2^u\}$

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}, \overline{\gamma}_1^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}$

and

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}$.

(ii) If Δ_1, Δ_2, $\overline{\Delta}_1$, $\overline{\Delta}_2$ are infinite,

$\tilde{X}_1 = X_1 \cap \bar{X}_1$, $\tilde{X}_2 = X_2 \cap \bar{X}_2$

and

$\tilde{\Delta}_1 \leq \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \leq \min\{\Delta_2^u, \overline{\Delta}_2^u\}$.

Proof: First it is noted that the inequality $\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r$, $(r > 0)$ and the following one,

$$\overline{\gamma}_2 \circ \overline{\gamma}_1(r) < r, \quad r > 0$$

imply each other [22].

Step1: In this step, we will show that if $x_1(t_0) \in X_1 \cap \bar{X}_1$, $x_2(t_0) \in X_2 \cap \bar{X}_2$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| \leq \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| \leq \min\{\Delta_2^u, \overline{\Delta}_2^u\}$, the solution of the inter-connected system exists and is bounded for all $t \geq t_0$. For this purpose, we will consider the following two cases.

(i) Δ_1, Δ_2, $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are finite.

Toward this end, we will first prove that the outputs y_1 and y_2 exist for all $t \geq t_0$ and are bounded in a way which is similar to the proof of Theorem 10.6.1 [22]. Suppose this is not the case, for every number $R > 0$, there exists a time $T > t_0$ such that the solutions are defined on $[0, T]$ and either $\|y_1(T)\| \geq R$ or $\|y_2(T)\| \geq R$.

Without loss of generality, we only consider the case where $\|y_1(T)\| \geq R$. Choose R

such that

$R > \max\{\overline{\beta}_1(r_1, 0), \overline{\gamma}_1 \circ \overline{\beta}_2(r_2, 0), \overline{\gamma}_1^u(\Delta_1^u), \overline{\gamma}_1^u(\overline{\Delta}_1^u), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\Delta_2^u), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\overline{\Delta}_2^u)\}$,

where, $r_1 = \{x_1 \in X_1 \cap \bar{X}_1 : \sup(\|x_1\|)\}$, $r_2 = \{x_2 \in X_2 \cap \bar{X}_2 : \sup(\|x_2\|)\}$.

It follows from (3.10) and (3.11) that

$$\|y_{1[t_0,T]}\| \leq \max\{\overline{\beta}_1(\|x_1(t_0)\|, 0), \overline{\gamma}_1(\|y_{2[t_0,T]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,T]}\|)\} \tag{3.13}$$

$$\|y_{2[t_0,T]}\| \leq \max\{\overline{\beta}_2(\|x_2(t_0)\|, 0), \overline{\gamma}_2(\|y_{1[t_0,T]}\|), \overline{\gamma}_2^u(\|u_{2[t_0,T]}\|)\}. \tag{3.14}$$

Substituting (3.14) into (3.13) gives that

$$\begin{aligned}\|y_{1[t_0,T]}\| \leq \max\{&\overline{\beta}_1(\|x_1(t_0)\|, 0), \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2(t_0)\|, 0), \\ &\overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0,T]}\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,T]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,T]}\|)\}.\end{aligned} \tag{3.15}$$

Since

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0,T]}\|) < \|y_{1[t_0,T]}\|,$$

it holds that

$$\begin{aligned}\|y_{1[t_0,T]}\| \leq \max\{&\overline{\beta}_1(\|x_1(t_0)\|, 0), \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2(t_0)\|, 0), \\ &\overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,T]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,T]}\|)\} < R\end{aligned} \tag{3.16}$$

which contradicts $\|y_1(T)\| > R$. Therefore the outputs are bounded for all $t \geq t_0$.

Since the subsystems (3.5) and (3.6) are RISS with restrictions, the solution of the inter-connected system is bounded for all $t \geq t_0$.

(*ii*) At least one of Δ_1, Δ_2, $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are finite.

Toward this end, we will first prove that the outputs y_1 and y_2 exist for all $t \geq t_0$ and are bounded in a way which is similar to the proof of Theorem 1 [61].

For any given $x(t_0) \in \bar{X}_1 \times \bar{X}_2$, let $p(x(t_0), \lambda)$ be a continuous path in $\bar{X}_1 \times \bar{X}_2$ from the origin to $x(t_0)$ with the property that $p(x(t_0), 0)$ is the origin and $p(x(t_0), 1) = x(t_0)$, and let y_1^λ and y_2^λ be the outputs starting at $x^\lambda(t_0) = p(x(t_0), \lambda)$ with inputs λu_1 and λu_2. When $\lambda = 0$, the solutions and outputs are defined on $[t_0, \infty)$ and identically zero. Note that the solutions are continuous functions of λ. Hence, for any given $T > t_0$ (arbitrarily large), $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists λ^* such that the solution exists on $[t_0, T]$ and

$$\|y_{1[t_0,T]}^\lambda\| \leq \epsilon_1, \quad \|y_{2[t_0,T]}^\lambda\| \leq \epsilon_2 \tag{3.17}$$

for all $\lambda \in [0, \lambda^*]$.

Denote that

$$\tilde{\Delta}_1 = \max\{\overline{\beta}_1(\max_{\lambda\in[0,1]}\|x_1^\lambda(t_0)\|,0), \overline{\gamma}_1\circ\overline{\beta}_2(\max_{\lambda\in[0,1]}\|x_2^\lambda(t_0)\|,0), \overline{\gamma}_1\circ\overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\},$$
$$\tilde{\Delta}_2 = \max\{\overline{\beta}_2(\max_{\lambda\in[0,1]}\|x_2^\lambda(t_0)\|,0), \overline{\gamma}_2\circ\overline{\beta}_1(\max_{\lambda\in[0,1]}\|x_1^\lambda(t_0)\|,0), \overline{\gamma}_2\circ\overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\}.$$

Since $p(x(t_0),\lambda)$ belongs to $\tilde{X}_1\times\tilde{X}_2$ and $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$, $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, it holds that $\tilde{\Delta}_1 < \min\{\Delta_2, \overline{\Delta}_2\}$ and $\tilde{\Delta}_2 < \min\{\Delta_1, \overline{\Delta}_1\}$. Let $T > t_0$ be arbitrarily large and ϵ_1, ϵ_2 satisfy $\tilde{\Delta}_1 < \epsilon_1 < \min\{\Delta_2, \overline{\Delta}_2\}$, $\tilde{\Delta}_2 < \epsilon_2 < \min\{\Delta_1, \overline{\Delta}_1\}$, and let $\lambda^* \in (0,1]$ be the largest value such that (3.86) holds for all $\lambda\in[0,\lambda^*]$. Suppose $\lambda^* < 1$. Since $\|y^\lambda_{1[t_0,T]}\| < \min\{\Delta_2,\overline{\Delta}_2\}$ and $\|y^\lambda_{2[t_0,T]}\| < \min\{\Delta_1,\overline{\Delta}_1\}$, following the same lines as (i) when Δ_1, Δ_2, $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are infinite, we have that

$$\|y^\lambda_{1[t_0,T]}\| \le \tilde{\Delta}_1 < \epsilon_1, \quad \|y^\lambda_{2[t_0,T]}\| \le \tilde{\Delta}_2 < \epsilon_2.$$

By continuity of solutions, there exists $\lambda' > \lambda^*$ such that (3.86) holds, contradicting that $\lambda^* < 1$. Hence $\lambda^* = 1$. Since T can be arbitrarily large, $\|y_{1[t_0,\infty)}\| < \min\{\Delta_2,\overline{\Delta}_2\}$ and $\|y_{2[t_0,\infty)}\| < \min\{\Delta_1,\overline{\Delta}_1\}$.

In both cases, the solution of the inter-connected system exist and is bounded for all $t\ge t_0$. Moreover, $\|y_{1[t_0,\infty)}\| < \min\{\Delta_2,\overline{\Delta}_2\}$ and $\|y_{2[t_0,\infty)}\| < \min\{\Delta_1,\overline{\Delta}_1\}$. Hence, if the initial state $x(t_0)\in\tilde{X}_1\times\tilde{X}_2$, and $u_1(t)\in L_\infty^{m_1}$ satisfies $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$ and $u_2(t)\in L_\infty^{m_2}$ satisfies $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, (3.8)–(3.11) hold for $t\ge t_0$.

Step2: We will show the system composed of (3.5) and (3.6) is RIOS with restrictions $\tilde{X}_1\times\tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1,x_2)$ as state, $y = \mathrm{col}(y_1,y_2)$ as output and $u = \mathrm{col}(u_1,u_2)$ as input.

By symmetry of y_1 and y_2, it follows from (3.16) that

$$\begin{aligned}\|y_{1[t_0,\infty)}\| &\le \max\{\overline{\beta}_1(\|x_1(t_0)\|,0), \overline{\gamma}_1\circ\overline{\beta}_2(\|x_2(t_0)\|,0), \overline{\gamma}_1\circ\overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\}\\ &\le \max\{\delta_1(\|x(t_0)\|), M_1\} \qquad (3.18)\\ \|y_{2[t_0,\infty)}\| &\le \max\{\overline{\beta}_2(\|x_2(t_0)\|,0), \overline{\gamma}_2\circ\overline{\beta}_1(\|x_1(t_0)\|,0), \overline{\gamma}_2\circ\overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\}\\ &\le \max\{\delta_2(\|x(t_0)\|), M_2\} \qquad (3.19)\end{aligned}$$

where,

$$\delta_1(s) = \max\{\overline{\beta}_1(s,0), \overline{\gamma}_1\circ\overline{\beta}_2(s,0)\}, \quad \delta_2(s) = \max\{\overline{\beta}_2(s,0), \overline{\gamma}_2\circ\overline{\beta}_1(s,0)\},$$
$$M_1 = \max\{\overline{\gamma}_1\circ\overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\},$$
$$M_2 = \max\{\overline{\gamma}_2\circ\overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\}.$$

Hence,

$$\begin{aligned} \|y(t)\| &\le \|y_{1[t_0,\infty)}\| + \|y_{2[t_0,\infty)}\| \\ &\le \max\{2\delta_1(\|x(t_0)\|), 2\delta_2(\|x(t_0)\|), 2M_1, 2M_2\} \\ &\le \max\{\delta_3(\|x(t_0)\|), M_3\} \overset{def}{=} y_\infty \end{aligned} \tag{3.20}$$

where, $\delta_3 = \max\{2\delta_1(s), 2\delta_2(s)\}$ and $M_3 = \overline{\gamma}(\|u_{[t_0,\infty)}\|)$ for any K_∞ function $\overline{\gamma}$ satisfying

$$\overline{\gamma}(s) \ge \max\{2\overline{\gamma}_1^y \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2^y \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\}.$$

Relying upon (3.18) and (3.19), the restrictions $\tilde{X}_1 \times \tilde{X}_2$ on the initial state $x(t_0)$ and $\tilde{\Delta}_1$, $\tilde{\Delta}_2$ on the inputs u_1, u_2 respectively can be computed as follows:

(i) If Δ_1, Δ_2, $\overline{\Delta}_1$, $\overline{\Delta}_2$ are finite,
$\tilde{X}_1 = \{x_1 \in X_1 \cap \bar{X}_1 : \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2 \circ \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_1^y, \overline{\Delta}_1\}\}$
and
$\tilde{X}_2 = \{x_2 \in X_2 \cap \bar{X}_2 : \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\}, \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}\}$.
$\tilde{\Delta}_1 \le \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \le \min\{\Delta_2^u, \overline{\Delta}_2^u\}$
$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}, \overline{\gamma}_1^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}$
and
$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}$.

(ii) If Δ_1, Δ_2, $\overline{\Delta}_1$, $\overline{\Delta}_2$ are infinite,
$\tilde{X}_1 = X_1 \cap \bar{X}_1$, $\tilde{X}_2 = X_2 \cap \bar{X}_2$
and
$\tilde{\Delta}_1 \le \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \le \min\{\Delta_2^u, \overline{\Delta}_2^u\}$.

From (3.8) and (3.9), we could obtain that

$$\begin{aligned} \|x(t)\| &\le \|x_1(t)\| + \|x_2(t)\| \\ &\le \max\{\alpha_x(\|x(t_0)\|), \alpha_u(\|u_{[t_0,t]}\|), \alpha_y(\|y_{[t_0,t]}\|)\} \\ &\le \max\{\alpha_x(\|x(t_0)\|), \alpha_u(\|u_{[t_0,\infty)}\|), \alpha_y(y_\infty)\} \overset{def}{=} x_\infty \end{aligned}$$

where,
$\alpha_x(s) = \max\{2\beta_1(s,0), 2\beta_2(s,0)\}, \alpha_u(s) = \max\{2\gamma_1^u(s), 2\gamma_2^u(s)\}, \alpha_y(s) = \max\{2\gamma_1(s), 2\gamma_2(s)\}$.

For any time $t_1 \geq 0$, it holds that

$$\begin{aligned}\|y_1(t_0+t_1)\| &\leq \max\{\overline{\beta}_1(\|x_1(t_0+\frac{t_1}{2})\|, \frac{t_1}{2}), \overline{\gamma}_1(\|y_{2[t_0+\frac{t_1}{2}, t_0+t_1]}\|), \overline{\gamma}_1^u(\|u_{1[t_0+\frac{t_1}{2}, t_0+t_1]}\|)\} \\ &\leq \max\{\overline{\beta}_1(x_\infty, \frac{t_1}{2}), \overline{\gamma}_1(\|y_{2[t_0+\frac{t_1}{2}, t_0+t_1]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\} \qquad (3.21)\end{aligned}$$

and for $\tau \in [\frac{t_1}{2}, t_1]$, it follows that

$$\begin{aligned}\|y_2(t_0+\tau)\| &\leq \max\{\overline{\beta}_2(\|x_2(t_0+\frac{t_1}{4})\|, \tau-\frac{t_1}{4}), \overline{\gamma}_2(\|y_{1[t_0+\frac{t_1}{4}, t_0+\tau]}\|), \overline{\gamma}_2^u(\|u_{2[t_0+\frac{t_1}{4}, t_0+\tau]}\|)\} \\ &\leq \max\{\overline{\beta}_2(x_\infty, \frac{t_1}{4}), \overline{\gamma}_2(\|y_{1[t_0+\frac{t_1}{4}, t_0+t_1]}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\}. \qquad (3.22)\end{aligned}$$

Substituting (3.22) into (3.21) gives that

$$\begin{aligned}\|y_1(t_0+t_1)\| &\leq \max\{\overline{\beta}_1(x_\infty, \frac{t_1}{2}), \overline{\gamma}_1 \circ \overline{\beta}_2(x_\infty, \frac{t_1}{4}), \overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0+\frac{t_1}{4}, t_0+t_1]}\|), \\ &\qquad \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\} \\ &\leq \max\{\tilde{\beta}_1(x_\infty, t_1), \overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0+\frac{t_1}{4}, t_0+t_1]}\|), M_1\}\end{aligned}$$

for any class KL function $\tilde{\beta}_1$ satisfying

$$\tilde{\beta}_1(s,t) \geq \max\{\overline{\beta}_1(s, \frac{t}{2}), \overline{\gamma}_1 \circ \overline{\beta}_2(s, \frac{t}{4})\}. \qquad (3.23)$$

Denoting $z_1(t_1) = \|y_1(t_0+t_1)\|$ gives that

$$z_1(t_1) \leq \max\{\tilde{\beta}_1(x_\infty, t_1), \overline{\gamma}_1 \circ \overline{\gamma}_2(\|z_{1[\frac{t_1}{4}, t_1]}\|), M_1\}. \qquad (3.24)$$

Since $\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r$ $(r > 0)$, we invoke Lemma 3.1 to conclude that there exists a class KL function $\hat{\beta}_1$ such that $z_1(t_1) \leq \max\{\hat{\beta}_1(x_\infty, t_1), M_1\}$. It follows that

$$\|y_1(t)\| \leq \max\{\hat{\beta}_1(x_\infty, t-t_0), M_1\}. \qquad (3.25)$$

By symmetry of y_1 and y_2, there exists some class KL function $\hat{\beta}_2$ such that

$$\|y_2(t)\| \leq \max\{\hat{\beta}_2(x_\infty, t-t_0), M_2\}. \qquad (3.26)$$

Toward this end, consider the following two cases of y_∞ in (3.20).

(i) $\delta_3(\|x(t_0)\|) \geq M_3$: We have $y_\infty = \delta_3(\|x(t_0)\|)$, and $\|u_{[t_0,\infty)}\| = \overline{\gamma}^{-1}(M_3) \leq \overline{\gamma}^{-1} \circ \delta_3(\|x(t_0)\|)$. Hence,

$$\begin{aligned}x_\infty &= \max\{\alpha_x(\|x(t_0)\|), \alpha_u(\|u_{[t_0,\infty)}\|), \alpha_y(y_\infty)\} \\ &\leq \max\{\alpha_x(\|x(t_0)\|), \alpha_u \circ \overline{\gamma}^{-1} \circ \delta_3(\|x(t_0)\|), \alpha_y \circ \delta_3(\|x(t_0)\|)\} \leq \delta_4(\|x(t_0)\|)\end{aligned}$$

for any class K function δ_4 satisfying
$\delta_4(s) \geq \max\{\alpha_x(s), \alpha_u \circ \overline{\gamma}^{-1} \circ \delta_3(s), \alpha_y \circ \delta_3(s)\}$.

As a result, (3.25) gives

$\|y_1(t)\| \leq \max\{\hat{\beta}_1(\delta_4(\|x(t_0)\|), t - t_0), M_1\}$.

(ii) $\delta_3(\|x(t_0)\|) < M_3$: We have $y_\infty = M_3$, then $\|y_{1[t_0,\infty)}\| \leq y_\infty = M_3$.

In both case, we have obtained the following inequality:

$$\|y_1(t)\| \leq \max\{\hat{\beta}_1(\delta_4(\|x(t_0)\|), t - t_0), M_3\}. \tag{3.27}$$

By symmetry of y_1 and y_2, we could obtain the following inequality:

$$\|y_2(t)\| \leq \max\{\hat{\beta}_2(\delta_4(\|x(t_0)\|), t - t_0), M_3\}. \tag{3.28}$$

Next, we will show that the system composed of (3.5) and (3.6) is RIOS with suitable defined restrictions and gain function $\overline{\gamma}$. Combing (3.25) and (3.26) gives that

$$\begin{aligned} \|y(t)\| &\leq \|y_1(t)\| + \|y_2(t)\| \\ &\leq \max\{2\hat{\beta}_1(x_\infty, t - t_0), 2\hat{\beta}_2(x_\infty, t - t_0), 2M_1, 2M_2\} \\ &\leq \max\{\beta_3(x_\infty, t - t_0), M_3\} \end{aligned} \tag{3.29}$$

for $\beta_3(s,t) = \max\{2\hat{\beta}_1(s,t), 2\hat{\beta}_2(s,t)\}$.

Toward this end, consider the following two cases of y_∞ in (3.20).

(i) $\delta_3(\|x(t_0)\|) \geq M_3$: We have $x_\infty \leq \delta_4(\|x(t_0)\|)$. As a result, (3.29) gives that

$$\|y(t)\| \leq \max\{\beta_3(\delta_4(\|x(t_0)\|), t - t_0), M_3\}. \tag{3.30}$$

(ii) $\delta_3(\|x(t_0)\|) < M_3$: We have $y_\infty = M_3$, then $\|y(t)\| \leq M_3$.

In both case, we have obtained the following inequality:

$$\begin{aligned} \|y(t)\| &\leq \max\{\beta_3(\delta_4(\|x(t_0)\|), t - t_0), M_3\} \\ &= \max\{\overline{\beta}(\|x(t_0)\|, t - t_0), \overline{\gamma}(\|u_{[t_0,\infty)}\|)\} \end{aligned} \tag{3.31}$$

where,

$$\overline{\beta}(s,t) = \max\{2\hat{\beta}_1(\delta_4(s), t), 2\hat{\beta}_2(\delta_4(s), t)\}.$$

Since the solution $y(t)$ depends only $u(\tau)$ on $t_0 \leq \tau \leq t$, the supremum on the right hand side of (3.31) can be taken over $[t_0, t]$, which yields

$$\|y(t)\| \leq \max\{\overline{\beta}(\|x(t_0)\|, t - t_0), \overline{\gamma}(\|u_{[t_0,t]}\|)\}.$$

Hence, the system composed of (3.5) and (3.6) is RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input.

Step3: We will show that the system composed of (3.5) and (3.6) is RISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input.

Substituting (3.16) into (3.9) gives that

$$\begin{aligned}\|x_2(t)\| \;\leq\; & \max\{\beta_2(\|x_2(t_0)\|, t-t_0), \gamma_2 \circ \overline{\beta}_1(\|x_1(t_0)\|, 0), \gamma_2 \circ \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2(t_0)\|, 0), \\ & \gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,t]}\|), \gamma_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}. \qquad (3.32)\end{aligned}$$

By symmetry of x_1 and x_2, it holds that

$$\begin{aligned}\|x_1(t)\| \;\leq\; & \max\{\beta_1(\|x_1(t_0)\|, t-t_0), \gamma_1 \circ \overline{\beta}_2(\|x_2(t_0)\|, 0), \gamma_1 \circ \overline{\gamma}_2 \circ \overline{\beta}_1(\|x_1(t_0)\|, 0), \\ & \gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,t]}\|), \gamma_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\}. \qquad (3.33)\end{aligned}$$

Combing (3.32) and (3.33) gives that

$$\|x(t)\| \leq \|x_1(t)\| + \|x_2(t)\| \leq \max\{\delta_5(\|x(t_0)\|), \tilde{\gamma}(\|u_{[t_0,\infty)}\|)\} \stackrel{def}{=} x'_\infty \qquad (3.34)$$

where,

$$\begin{aligned}\delta_5(s,t) \;=\; & \max\{2\beta_1(s,0), 2\gamma_1 \circ \overline{\beta}_2(s,0), 2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\beta}_1(s,0), \\ & 2\beta_2(s,0), 2\gamma_2 \circ \overline{\beta}_1(s,0), 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\beta}_2(s,0)\} \\ \tilde{\gamma}(s) \;=\; & \max\{2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\gamma_2 \circ \overline{\gamma}_1^u(s), 2\gamma_2^u(s)\}.\end{aligned}$$

From (3.8), for any time $t_1 \geq 0$, we could obtain

$$\begin{aligned}\|x_1(t_0+t_1)\| \;\leq\; & \max\{\beta_1(\|x_1(t_0+\frac{t_1}{2})\|, \frac{t_1}{2}), \gamma_1(\|y_{2[t_0+\frac{t_1}{2}, t_0+t_1]}\|), \gamma_1^u(\|u_{1[t_0+\frac{t_1}{2}, t_0+t_1]}\|)\} \\ \leq\; & \max\{\beta_1(x'_\infty, \frac{t_1}{2}), \gamma_1(\|y_{2[t_0+\frac{t_1}{2}, t_0+t_1]}\|), \gamma_1^u(\|u_{1[t_0,\infty)}\|)\}. \qquad (3.35)\end{aligned}$$

From (3.28), for $\tau \in [\frac{t_1}{2}, t_1]$, it holds that

$$\begin{aligned}\|y_2(t_0+\tau)\| \;\leq\; & \max\{\hat{\beta}_2(\delta_4(\|x(t_0+\frac{t_1}{4})\|), \tau - \frac{t_1}{4}), \overline{\gamma}(\|u_{[t_0+\frac{t_1}{4}, t_0+\tau]}\|)\} \\ \leq\; & \max\{\hat{\beta}_2(\delta_4(x'_\infty), \frac{t_1}{4}), \overline{\gamma}(\|u_{[t_0,\infty)}\|)\}. \qquad (3.36)\end{aligned}$$

Substituting (3.36) into (3.35) gives

$$\begin{aligned}\|x_1(t_0+t_1)\| \;\leq\; & \max\{\beta_1(x'_\infty, \frac{t_1}{2}), \gamma_1 \circ \hat{\beta}_2(\delta_4(x'_\infty), \frac{t_1}{4}), \gamma_1 \circ \overline{\gamma}(\|u_{[t_0,\infty)}\|), \gamma_1^u(\|u_{1[t_0,\infty)}\|)\} \\ \leq\; & \max\{\beta_1^*(x'_\infty, t_1), \gamma_1^*(\|u_{[t_0,\infty)}\|)\} \qquad (3.37)\end{aligned}$$

where, $\beta_1^*(s,t) = \max\{\beta_1(s,\frac{t}{2}), \gamma_1 \circ \hat{\beta}_2(\delta_4(s),\frac{t}{4})\}$, $\quad \gamma_1^*(s) = \max\{\gamma_1^y \circ \overline{\gamma}(s), \gamma_1^u(s)\}$.

By symmetry of x_1 and x_2, it holds that there exist class KL function β_2^* and class K function γ_2^* such that

$$\|x_2(t)\| \leq \max\{\beta_2^*(x'_\infty, t-t_0), \gamma_2^*(\|u_{[t_0,\infty)}\|)\} \tag{3.38}$$

where, $\beta_2^*(s,t) = \max\{\beta_2(s,\frac{t}{2}), \gamma_2 \circ \hat{\beta}_1(\delta_4(s),\frac{t}{4})\}$, $\quad \gamma_2^*(s) = \max\{\gamma_2 \circ \overline{\gamma}(s), \gamma_2^u(s)\}$.

Combing (3.37) and (3.38) gives

$$\begin{aligned} \|x(t)\| &\leq \|x_1(t)\| + \|x_2(t)\| \\ &\leq \max\{\beta^*(x'_\infty, t-t_0), \gamma^*(\|u_{[t_0,\infty)}\|)\} \end{aligned} \tag{3.39}$$

where, $\beta^*(s,t) = \max\{2\beta_1^*(s,t), 2\beta_2^*(s,t)\}$,
$\gamma^*(s) = \max\{2\gamma_1^*(s), 2\gamma_2^*(s)\} = \max\{4\gamma_1 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 4\gamma_1 \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 4\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 4\gamma_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2^u(s), 2\gamma_2^u(s)\}$.

Toward this end, consider the following two cases of x'_∞ in (3.34).

(i) $\delta_5(\|x(t_0)\|) \geq \tilde{\gamma}(\|u_{[t_0,\infty)}\|)$: We have $x'_\infty = \delta_5(\|x(t_0)\|)$.
As a result, $\|x(t)\| \leq \max\{\beta^*(\delta_5(\|x(t_0)\|), t-t_0), \gamma^*(\|u_{[t_0,\infty)}\|)\}$.

(ii)$\delta_5(\|x(t_0)\|) < \tilde{\gamma}(\|u_{[t_0,\infty)}\|)$: We have $x'_\infty = \tilde{\gamma}(\|u_{[t_0,\infty)}\|)$.
As a result, $\|x(t)\| \leq x'_\infty = \tilde{\gamma}(\|u_{[t_0,\infty)}\|)$.
Since $\tilde{\gamma}(s) < \gamma^*(s)$ for all $s > 0$, in both cases, we have obtained the following inequality

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t-t_0), \gamma(\|u_{[t_0,\infty)}\|)\} \tag{3.40}$$

where, $\beta(s,t) = \beta^*(\delta_5(s),t)$, $\quad \gamma(s) = \gamma^*(s)$.
Since the solution $x(t)$ depends only $u(\tau)$ on $t_0 \leq \tau \leq t$, the supremum on the right hand side of (3.40) can be taken over $[t_0, t]$, which yields

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t-t_0), \gamma(\|u_{[t_0,t]}\|)\}.$$

Hence, the system composed of (3.5) and (3.6) is RISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state, $y = \text{col}(y_1, y_2)$ as output and $u = \text{col}(u_1, u_2)$ as input. This completes the proof. ∎

Remark 3.2 In Appendix B of [23], it was showed that, for the class of time-invariant systems, a system is ISS with restrictions X and Δ on the initial state $x(0)$ and the input

u respectively if and only if there exist class K functions γ_0 and γ, such that, for any initial state $x(0) \in X$, any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| \le \Delta$, the solution $x(t)$ exists and satisfies,

$$\begin{aligned}\|x_{[0,\infty)}\| &\le \max\{\gamma_0(\|x(0)\|), \gamma(\|u_{[0,\infty)}\|)\} \\ \|x(t)\|_a &\le \gamma(\|u(t)\|_a).\end{aligned}$$

Such equivalence is called *separation principle*. In Theorem B.3.2 [23], using the separation principle for time invariant systems, Isidori *et al* have proven the corresponding nonlinear small gain theorem with restrictions. Nevertheless, Isidori's proof cannot be carried over to the case of time-varying systems, because the *separation principle* for ISS with restrictions cannot be generalized to the time-varying case [8]. ∎

Lemma 3.2 If system (2.4) is RIOS with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively, i.e. there exist class KL function β and class K function γ, independent of $d(t)$, such that, for any initial state $x(t_0) \in X$, any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| \le \Delta$, the output of (2.4) exists and satisfies, for all $t \ge t_0$,

$$\|y(t)\| \le \max\{\beta(\|x(t_0)\|, t-t_0), \gamma(\|u_{[t_0,t]}\|)\},$$

then there exists class K function γ_0, independent of $d(t)$, such that, for any initial state $x(t_0) \in X$, any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| \le \Delta$, the output of (2.4) exists and satisfies, for all $t \ge t_0$,

$$\|y_{[t_0,\infty)}\| \le \max\{\gamma_0(\|x(t_0)\|), \gamma(\|u_{[t_0,\infty)}\|)\}, \tag{3.41}$$

$$\|y\|_a \le \gamma(\|u\|_a). \tag{3.42}$$

Proof: The proof is conducted in a way similar to the proof of Lemma *II.* 1 [57]. In the following proof, we assume that the initial state $x(t_0)$ belongs to the compact set X and the input function $u(t) \in L_\infty^m$ satisfies $\|u_{[t_0,\infty)}\| \le \Delta$.

It is clear that (3.41) holds if we define $\gamma_0(s) = \beta(s, 0)$. For any $\varepsilon > 0$, pick $\delta > 0$ such that

$$\gamma(\|u\|_a + \delta) - \gamma(\|u\|_a) < \varepsilon.$$

Pick T_1 such that

$$\|u_{[T_1,\infty)}\| \le \|u\|_a + \delta.$$

Clearly, there exists $T_2 \geq T_1$ such that

$$\beta(\|x(T_1)\|, T_2 - T_1) \leq \gamma(\|u\|_a) + \varepsilon.$$

Hence, for any $t \geq T_2$,

$$\begin{aligned}
\|y(t)\| &\leq \max\{\beta(\|x(T_1)\|, t - T_1), \gamma(\|u_{[T_1,t]}\|)\} \\
&\leq \max\{\beta(\|x(T_1)\|, T_2 - T_1), \gamma(\|u_{[T_1,\infty)}\|)\} \\
&\leq \max\{\beta(\|x(T_1)\|, T_2 - T_1), \gamma(\|u\|_a + \delta)\} \\
&\leq \gamma(\|u\|_a) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ gives (3.42). ∎

Remark 3.3 It is natural to ask whether the separation principle for ISS with restrictions can be generalized to the case for IOS with restrictions. By now, the answer is not clear to the author. ∎

3.1.3 Remarks and Corollaries

In [8, 9, 23, 25, 26, 61], many versions of small gain theorem were proposed in order to deal with different cases relevant in control applications. In this section, we will elucidate the relations between Theorem 3.2 and previous versions of the small gain theorem.

Since the time invariant system is a special case of time varying system, Theorem 3.2 also holds for time invariant case. Combining the separation principle for ISS with restrictions, we can formalize it in the following corollary.

Corollary 3.1 Assume that subsystem (3.1) is ISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(0)$, v_1 and u_1 respectively and subsystem (3.2) is ISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(0)$, v_2 and u_2 respectively, i.e., there exist class K functions γ_1^0, γ_1, γ_1^u, γ_2^0, γ_2 and γ_2^u, such that, for any $x_1(0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[0,\infty)}\| \leq \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[0,\infty)}\| \leq \Delta_1^u$, the solution of (3.1) exists and satisfies, for all $t \geq 0$,

$$\|x_{1[0,\infty)}\| \leq \max\{\gamma_1^0(\|x_1(0)\|), \gamma_1(\|v_{1[0,\infty)}\|), \gamma_1^u(\|u_{1[0,\infty)}\|)\} \tag{3.43}$$

$$\|x_1\|_a \leq \max\{\gamma_1(\|v_1\|_a), \gamma_1^u(\|u_1\|_a)\}, \tag{3.44}$$

and for any $x_2(0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[0,\infty)}\| \leq \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[0,\infty)}\| \leq \Delta_2^u$, the solution of (3.2) exists and satisfies, for all $t \geq 0$,

$$\|x_{2[0,\infty)}\| \leq \max\{\gamma_2^0(\|x_2(0)\|), \gamma_2(\|v_{2[0,\infty)}\|), \gamma_2^u(\|u_{2[0,\infty)}\|)\} \tag{3.45}$$

$$\|x_2\|_a \leq \max\{\gamma_2(\|v_2\|_a), \gamma_2^u(\|u_2\|_a)\}. \tag{3.46}$$

Further assume that subsystem (3.1) is IOS with restrictions $\bar{X}_1$, $\overline{\Delta}_1$ and $\overline{\Delta}_1^u$ on $x_1(0)$, v_1 and u_1 respectively and subsystem (3.2) is IOS with restrictions $\bar{X}_2$, $\overline{\Delta}_2$ and $\overline{\Delta}_2^u$ on $x_2(0)$, v_2 and u_2 respectively, i.e., there exist class KL functions $\overline{\beta}_1$, $\overline{\beta}_2$ and class K functions $\overline{\gamma}_1$, $\overline{\gamma}_1^u$, $\overline{\gamma}_2$, $\overline{\gamma}_2^u$, such that, for any $x_1(0) \in \bar{X}_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[0,\infty)}\| \leq \overline{\Delta}_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[0,\infty)}\| \leq \overline{\Delta}_1^u$, the output of (3.1) exists and satisfies, for all $t \geq 0$,

$$\|y_1(t)\| \leq \max\{\overline{\beta}_1(\|x_1(0)\|, t), \overline{\gamma}_1(\|v_{1[0,\infty)}\|)\}, \overline{\gamma}_1^u(\|u_{1[0,\infty)}\|)\} \tag{3.47}$$

and for any $x_2(0) \in \bar{X}_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[0,\infty)}\| \leq \overline{\Delta}_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[0,\infty)}\| \leq \overline{\Delta}_2^u$, the output of (3.2) exists and satisfies, for all $t \geq 0$,

$$\|y_2(t)\| \leq \max\{\overline{\beta}_2(\|x_2(0)\|, t), \overline{\gamma}_2(\|v_{2[0,\infty)}\|)\}, \overline{\gamma}_2^u(\|u_{2[0,\infty)}\|)\}. \tag{3.48}$$

Suppose that the small gain condition

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad r > 0$$

holds, then the system composed of (3.1) and (3.2) is ISS and IOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input, i.e., there exist class KL functions β and $\overline{\beta}$, class K functions γ and $\overline{\gamma}$, such that, for any initial state $x(0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[0,\infty)}\| \leq \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[0,\infty)}\| \leq \tilde{\Delta}_2$, the solution and output of (3.1) and (3.2) exist and satisfy, for all $t \geq 0$,

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u_{[0,\infty)}\|)\} \tag{3.49}$$

$$\|y(t)\| \leq \max\{\overline{\beta}(\|x(0)\|, t), \overline{\gamma}(\|u_{[0,\infty)}\|)\} \tag{3.50}$$

where, the gain functions $\gamma(s)$, $\overline{\gamma}(s)$ and the restrictions $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{\Delta}_1$, $\tilde{\Delta}_2$ are the same as those in Theorem 2. ∎

Remark 3.4 Observe that the restrictions are the same as those in Theorem 3.1 if we denote $\overline{\beta}_i(s, 0) = \overline{\gamma}_i^0(s)$ for $i = 1, 2$ and set $\bar{X}_1$, $\bar{X}_2$, $\overline{\Delta}_1$, $\overline{\Delta}_2$, $\overline{\Delta}_1^u$ and $\overline{\Delta}_2^u$ be infinite. It is easy to check that Corollary 3.1 provides a more general result than Theorem 3.1 does, since the restrictions are imposed on the output channels and the explicit expressions of the gain functions are given in Corollary 3.1. ∎

It is clear that RIOS with restrictions is RISS with restrictions when the state is seen as an output. The following corollary is the local version of Theorem 2.1 in [8].

Corollary 3.2 Consider the interconnection of the following two systems

$$\dot{x}_1 = f_1(x_1, v_1, u_1, d, t), \quad y_1 = x_1 \tag{3.51}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = x_2 \tag{3.52}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1$$

where, the notations are the same as those in Theorem 3.1.

Assume that subsystem (3.51) is RISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(t_0)$, v_1 and u_1 respectively and subsystem (3.52) is RISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively. In particular, (3.8), (3.9) coincide with (3.10), (3.11) respectively.

Further suppose that

$$\gamma_1 \circ \gamma_2(r) < r, \quad r > 0 \tag{3.53}$$

then the system composed of (3.51) and (3.52) is RISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state and $u = \text{col}(u_1, u_2)$ as input, i.e., there exist class KL function β and class K function γ, independent of $d(t)$, such that, for any $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| \le \tilde{\Delta}_1$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| \le \tilde{\Delta}_2$, the solutions of (3.51) and (3.52) exist and satisfy, for all $t \ge t_0$,

$$\|x(t)\| \;\le\; \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}$$

where,

$\tilde{X}_1 = \{x_1 \in X_1 : \beta_1(\|x_1\|, 0) < \Delta_2, \gamma_2 \circ \beta_1(\|x_1\|, 0) < \Delta_1\}$

and

$\tilde{X}_2 = \{x_2 \in X_2 : \beta_2(\|x_2\|, 0) < \Delta_1, \gamma_1 \circ \beta_2(\|x_2\|, 0) < \Delta_2\}$.

$\tilde{\Delta}_1 \le \Delta_1^u$, $\tilde{\Delta}_2 \le \Delta_2^u$,

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \gamma_2 \circ \gamma_1^u(s) < \Delta_1, \gamma_1^u(s) < \Delta_2$

and

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \gamma_1 \circ \gamma_2^u(s) < \Delta_2, \gamma_2^u(s) < \Delta_1$,

and, $\gamma(s) = \max\{2\gamma_1 \circ \gamma_2^u(s), 2\gamma_1^u(s), 2\gamma_2 \circ \gamma_1^u(s), 2\gamma_2^u(s)\}$.

∎

Remark 3.5 If the restrictions X_1, X_2, Δ_1, Δ_1^u, Δ_2 and Δ_2^u are infinite, Corollary 3.2 reduces to Theorem 2.1 in [8]. Note that the approach of the small gain theorem with restrictions is partially different from that of the global version, since we have to keep track of the domains of attraction for the subsystems in order to utilize the inequalities characterizing the properties of RISS with restrictions and RIOS with restrictions. ∎

Corollary 3.2 can reduce to the case for time invariant systems.

Corollary 3.3 Consider the interconnection of the following two systems

$$\dot{x}_1 = f_1(x_1, v_1, u_1), \quad y_1 = x_1 \tag{3.54}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2), \quad y_2 = x_2 \tag{3.55}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1$$

where, the notations are the same as those in Theorem 3.1. Assume that subsystem (3.54) is ISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(0)$, v_1 and u_1 respectively and subsystem (3.55) is ISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(0)$, v_2 and u_2 respectively, i.e., there exist class K functions γ_1^0, γ_1, γ_1^u, γ_2^0, γ_2 and γ_2^u, such that, for any $x_1(0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[0,\infty)}\| \leq \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[0,\infty)}\| \leq \Delta_1^u$, the solution of (3.54) exists and satisfies, for all $t \geq 0$,

$$\begin{aligned} \|x_{1[0,\infty)}\| &\leq \max\{\gamma_1^0(\|x_1(0)\|), \gamma_1(\|v_{1[0,\infty)}\|), \gamma_1^u(\|u_{1[0,\infty)}\|)\} \\ \|x_1\|_a &\leq \max\{\gamma_1(\|v_1\|_a), \gamma_1^u(\|u_1\|_a)\}, \end{aligned}$$

and for any $x_2(0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[0,\infty)}\| \leq \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[0,\infty)}\| \leq \Delta_2^u$, the solution of (3.55) exists and satisfies, for all $t \geq 0$,

$$\begin{aligned} \|x_{2[0,\infty)}\| &\leq \max\{\gamma_2^0(\|x_2(0)\|), \gamma_2(\|v_{2[0,\infty)}\|), \gamma_2^u(\|u_{2[0,\infty)}\|)\} \\ \|x_2\|_a &\leq \max\{\gamma_2(\|v_2\|_a), \gamma_2^u(\|u_2\|_a)\}. \end{aligned}$$

Further suppose that

$$\gamma_1 \circ \gamma_2(r) < r, \quad r > 0 \tag{3.56}$$

then the system composed of (3.54) and (3.55) is ISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(0)$, u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state and $u = \text{col}(u_1, u_2)$ as input, i.e., there exist class KL function β and class K function γ, such that for any $x(0) \in \tilde{X}_1 \times \tilde{X}_2$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[0,\infty)}\| \leq \tilde{\Delta}_1$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[0,\infty)}\| \leq \tilde{\Delta}_2$, the solutions of (3.54) and (3.55) exist and satisfy, for all $t \geq 0$,

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u_{[0,\infty)}\|)\}$$

where, the restrictions $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ and gain function γ are the same as those in Corollary 3.2. ∎

Corollary 3.2 can be further specialized into the following three corollaries.

Corollary 3.4 Consider the interconnection of the following two systems

$$\dot{x}_1 = f_1(x_1, v_1, d, t), \quad y_1 = x_1 \tag{3.57}$$

$$\dot{x}_2 = f_2(x_2, v_2, u, d, t), \quad y_2 = x_2 \tag{3.58}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1$$

where, the notations are the same as those in Theorem 3.2.

Assume that subsystem (3.57) is RISS with restrictions X_1 and Δ_1 on $x_1(t_0)$ and v_1 respectively and subsystem (3.58) is RISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u respectively. In particular, (3.8), (3.9) coincide with (3.10), (3.11) respectively.

Further suppose that

$$\gamma_1 \circ \gamma_2(r) < r, \quad r > 0 \tag{3.59}$$

then the system composed of (3.57) and (3.58) is RISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$ and Δ on $x(t_0)$ and u respectively, viewing $x = \text{col}(x_1, x_2)$ as state and u as input, i.e., there exist class KL function β and class K function γ, independent of $d(t)$, such that, for any $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, $u(t) \in L_\infty^{m_2}$ satisfying $\|u_{[t_0,\infty)}\| \leq \Delta$, the solutions of (3.57) and (3.58) exist and satisfy, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}$$

where,

$\tilde{X}_1 = \{x_1 \in X_1 : \beta_1(\|x_1\|, 0) < \Delta_2, \gamma_2 \circ \beta_1(\|x_1\|, 0) < \Delta_1\}$

and

$\tilde{X}_2 = \{x_2 \in X_2 : \beta_2(\|x_2\|, 0) < \Delta_1, \gamma_1 \circ \beta_2(\|x_2\|, 0) < \Delta_2\}$.

$\Delta \leq \Delta_2^u$,

$s \in [0, \Delta) \Longrightarrow \gamma_1 \circ \gamma_2^u(s) < \Delta_2, \gamma_2^u(s) < \Delta_1$,

and, $\gamma(s) = \max\{2\gamma_1 \circ \gamma_2^u(s), 2\gamma_2^u(s)\}$. ∎

The following corollary is the time invariant case for Corollary 3.4.

Corollary 3.5 Consider the interconnection of the following two systems

$$\dot{x}_1 = f_1(x_1, v_1), \quad y_1 = x_1 \tag{3.60}$$

$$\dot{x}_2 = f_2(x_2, v_2, u), \quad y_2 = x_2 \tag{3.61}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1$$

where, the notations are the same as those in Theorem 3.1.

Assume that subsystem (3.60) is ISS with restrictions X_1 and Δ_1 on $x_1(0)$ and v_1 respectively and subsystem (3.61) is ISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(0)$, v_2 and u respectively, i.e., there exist class K functions γ_1^0, γ_1, γ_2^0, γ_2 and γ_2^u, such that, for any $x_1(0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[0,\infty)}\| \leq \Delta_1$, the solution of (3.60) exists and satisfies, for all $t \geq 0$,

$$\begin{aligned} \|x_{1[0,\infty)}\| &\leq \max\{\gamma_1^0(\|x_1(0)\|), \gamma_1(\|v_{1[0,\infty)}\|)\} \\ \|x_1\|_a &\leq \gamma_1(\|v_1\|_a), \end{aligned}$$

and for any $x_2(0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[0,\infty)}\| \leq \Delta_2$, $u(t) \in L_\infty^{m_2}$ satisfying $\|u_{[0,\infty)}\| \leq \Delta_2^u$, the solution of (3.61) exists and satisfies, for all $t \geq 0$,

$$\begin{aligned} \|x_{2[0,\infty)}\| &\leq \max\{\gamma_2^0(\|x_2(0)\|), \gamma_2(\|v_{2[0,\infty)}\|), \gamma_2^u(\|u_{[0,\infty)}\|)\} \\ \|x_2\|_a &\leq \max\{\gamma_2(\|v_2\|_a), \gamma_2^u(\|u\|_a)\}. \end{aligned}$$

Further suppose that

$$\gamma_1 \circ \gamma_2(r) < r, \quad r > 0 \tag{3.62}$$

then the system composed of (3.60) and (3.61) is ISS with restrictions $\bar{X}_1 \times \bar{X}_2$ and Δ on $x(0)$ and u respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state and u as input, i.e., there exist class KL function β and class K function γ, such that for any $x(0) \in \bar{X}_1 \times \bar{X}_2$, $u(t) \in L_\infty^{m_2}$ satisfying $\|u_{[0,\infty)}\| \leq \Delta$, the solutions of (3.60) and (3.61) exist and satisfy, for all $t \geq 0$,

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u_{[0,\infty)}\|)\}$$

where, the restrictions $\bar{X}_1$, $\bar{X}_2$, Δ and gain function γ are the same as those in Corollary 3.4. ∎

Remark 3.6 It is easy to check that if we set $\beta_1(s,0) = \gamma_{01}(s)$ and $\beta_2(s,0) = \gamma_{02}(s)$, Corollary 3.1 coincides with Theorem 12.2.1 [22]. ∎

Corollary 3.6 Consider the interconnection of the following two systems

$$\dot{x}_1 = f_1(x_1, u_1, d, t), \quad y_1 = x_1 \tag{3.63}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = x_2 \tag{3.64}$$

subject to the interconnection constraint:

$$v_2 = y_1$$

where, the notations are the same as those in Theorem 3.2.

Assume that subsystem (3.63) is RISS with restrictions X_1 and Δ_1^u on $x_1(t_0)$ and u_1 respectively and subsystem (3.64) is RISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively. In particular, (3.8), (3.9) coincide with (3.10), (3.11) respectively.

Then the system composed of (3.63) and (3.64) is RISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$, $\tilde{\Delta}_2$ on $x(t_0)$, u_1, u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state and $u = \text{col}(u_1, u_2)$ as input, i.e., there exist class KL function β and class K function γ, independent of $d(t)$, such that, for any $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| \leq \tilde{\Delta}_1$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| \leq \tilde{\Delta}_2$, the solutions of (3.63) and (3.64) exist and satisfy, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}$$

where,

$\tilde{X}_1 = \{x_1 \in X_1 : \beta_1(\|x_1\|, 0) < \Delta_2, \gamma_2^y \circ \beta_1(\|x_1\|, 0) < \Delta_1\}$

and

$\tilde{X}_2 = \{x_2 \in X_2 : \beta_2(\|x_2\|, 0) < \Delta_1\}$.

$\tilde{\Delta}_1 \leq \Delta_1^u$, $\tilde{\Delta}_2 \leq \Delta_2^u$,

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \gamma_2 \circ \gamma_1^u(s) < \Delta_1, \gamma_1^u(s) < \Delta_2$

and

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \gamma_2^u(s) < \Delta_1$

and, $\gamma(s) = \max\{2\gamma_1^u(s), 2\gamma_2 \circ \gamma_1^u(s), 2\gamma_2^u(s)\}$. ∎

Remark 3.7 If the restrictions are infinite, Corollary 3.4 and 3.6 reduce to Corollary 2.1 and 2.2 [8] respectively. ∎

If we only care about RIOS with restrictions of the system composed of (3.5) and (3.6), the assumption RISS with restrictions in Theorem 3.2 can be weakened to RUO with restrictions. The following corollary is the local version of Theorem 2 in [9].

Corollary 3.7 Under the assumptions of Theorem 3.2 except that class KL function β_i is replaced by class K function α_i in (3.8) and (3.9) for $i = 1, 2$, if the small gain condition (3.12) holds, the system composed of (3.5) and (3.6) is RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state, $y = \text{col}(y_1, y_2)$ as output and $u = \text{col}(u_1, u_2)$ as input, where the gain function $\overline{\gamma}$ and the restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively are the same as those in Theorem 3.2. ∎

It is worthy to mention another three special cases of Theorem 3.2, namely, the case where h_1 does not depend on y_2 explicitly, i.e.,

$$y_1 = h_1(x_1, u_1, d, t),$$

the case where h_1 does not depend on u_1 explicitly, i.e.,

$$y_1 = h_1(x_1, y_2, d, t)$$

and the case of cascade interconnected systems. The previous two cases can be specialized into the following two corollaries.

Corollary 3.8 Consider the interconnection of the following two systems

$$\dot{x}_1 = f_1(x_1, v_1, u_1, d, t), \quad y_1 = h_1(x_1, u_1, d, t) \tag{3.65}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = h_2(x_2, v_2, u_2, d, t) \tag{3.66}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1$$

where the notations are the same as those in Theorem 3.2.

Assume that subsystem (3.65) is RISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(t_0)$, v_1 and u_1 respectively and subsystem (3.66) is RISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively.

Further assume that subsystem (3.65) is RIOS with restrictions $\bar{X}_1$ and $\overline{\Delta_1^u}$ on $x_1(t_0)$ and u_1 respectively and subsystem (3.66) is RIOS with restrictions $\bar{X}_2$, $\overline{\Delta}_2$ and $\overline{\Delta_2^u}$ on $x_2(t_0)$, v_2 and u_2 respectively, in particular, (3.10) and (3.12) hold with $\overline{\gamma}_1 \equiv 0$.

Then the system composed of (3.65) and (3.66) is RISS and RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$, $\tilde{\Delta}_2$ on $x(t_0)$, u_1, u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state, $y = \text{col}(y_1, y_2)$ as output and $u = \text{col}(u_1, u_2)$ as input, where,

$\gamma(s) = \max\{4\gamma_1^y \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 4\gamma_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2^u(s), 2\gamma_2^u(s)\}$,

$\overline{\gamma}(s) = \max\{2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\}$

and,

$\tilde{X}_1 = \{x_1 \in X_1 \cap \bar{X}_1 : \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2 \circ \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\}\}$

$\tilde{X}_2 = \{x_2 \in X_2 \cap \bar{X}_2 : \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\}\}$.

$\tilde{\Delta}_1 \leq \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \leq \min\{\Delta_2^u, \overline{\Delta}_2^u\}$

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}, \overline{\gamma}_1^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}$

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_2^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}$. ∎

Corollary 3.9 Consider the interconnection of the following two systems

$$\dot{x}_1 = f_1(x_1, v_1, u_1, d, t), \quad y_1 = h_1(x_1, v_1, d, t) \tag{3.67}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = h_2(x_2, v_2, u_2, d, t) \tag{3.68}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1$$

where the notations are the same as those in Theorem 3.2.

Assume that subsystem (3.67) is RISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(t_0)$, v_1 and u_1 respectively, and subsystem (3.68) is RISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively.

And assume that subsystem (3.67) is RIOS with restrictions $\bar{X}_1$ and $\overline{\Delta}_1$ on $x_1(t_0)$ and v_1 respectively and subsystem (3.68) is RIOS with restrictions $\bar{X}_2$, $\overline{\Delta}_2$ and $\overline{\Delta}_2^u$ on $x_2(t_0)$, v_2 and u_2 respectively, in particular, (3.10) holds with $\overline{\gamma}_1^u \equiv 0$.

Further assume the small gain condition (3.12) be satisfied. Then the system composed of (3.67) and (3.68) is RISS and RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input, where,

$\gamma(s) = \max\{4\gamma_1 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 4\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 4\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 4\gamma_2 \circ \overline{\gamma}_2^u(s), 2\gamma_2^u(s)\}$.

$\overline{\gamma}(s) = \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_2^u(s)\}$

and the restriction $\tilde{X}_1 \times \tilde{X}_2$ is the same as that in Theorem 3.2 and

$\tilde{\Delta}_1 \leq \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \leq \min\{\Delta_2^u, \overline{\Delta}_2^u\}$

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}$. ∎

To conclude this section, the small gain theorem with restrictions is further specialized to the following cascade inter-connection as depicted in Figure 3.3,

$$\dot{x}_1 = f_1(x_1, u_1, d, t), \quad y_1 = h_1(x_1, u_1, d, t) \tag{3.69}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t) \tag{3.70}$$

subject to the interconnection constraint:

$$v_2 = y_1$$

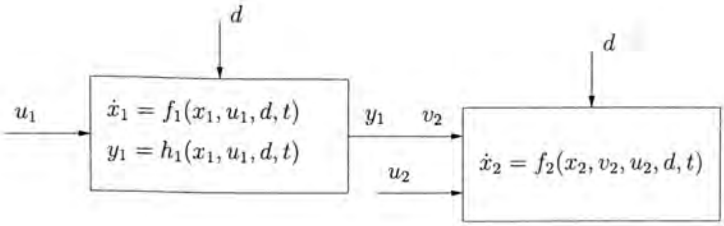

Figure 3.3: Inter-connection of (3.69) and (3.70)

It is noted that the cascade inter-connection (3.69) and (3.70) can be interpreted as feedback inter-connection (3.5) and (3.6) in which the output of lower subsystem $y_2 = h_2(x_2, y_1, u_2, d, t)$ is equal to zero. In this case, the bound estimate in Theorem 3.2 holds for $\overline{\gamma}_2(\cdot) \equiv 0$ and $\overline{\gamma}_2^u(\cdot) \equiv 0$. Hence the small gain condition is fulfilled. This implies that the cascade inter-connection is RISS with appropriate restrictions on the initial state and the input. It can be formalized in the next corollary.

Corollary 3.10 Consider the cascade inter-connection (3.69) and (3.70), where the notations are the same as those in Theorem 3.2.

Assume that subsystem (3.69) is RISS with restrictions X_1 and Δ_1^u on $x_1(t_0)$ and u_1 respectively, in particular, (3.8) holds with $\gamma_1 \equiv 0$, and subsystem (3.70) is RISS with restrictions X_2, Δ_1 and Δ_1^u on $x_2(t_0)$, v_2 and u_2 respectively.

And assume that (3.69) subsystem is RIOS with restrictions $\bar{X}_1$ and $\overline{\Delta_1^u}$ on $x_1(t_0)$ and u_1 respectively, in particular, (3.10) holds with $\overline{\gamma}_1 \equiv 0$.

Then the system composed of (3.69) and (3.70) is RISS and RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$ and $\tilde{\Delta}_1$, $\tilde{\Delta}_2$ on $x(t_0)$ and u_1, u_2, viewing $x = \mathrm{col}(x_1, x_2)$ as state, y_1 as output and $u = \text{col}(u_1, u_2)$ as input, where,

$\gamma(s) = \max\{4\gamma_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\gamma_2^u(s)\}$,

$\overline{\gamma}(s) = \max\{2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s)\}$,

and,

$\tilde{X}_1 = \{x_1 \in X_1 \cap \bar{X}_1 : \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}\}$

and

$\tilde{X}_2 = \{x_2 \in X_2 : \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\}\}$.

$\tilde{\Delta}_1 \leq \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \leq \Delta_2^u$

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_1^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}$. ∎

3.2 Semi-Uniform Input-to-State Stability Small Gain Theorem with Restrictions for Uncertain Nonlinear Time-varying Systems

Theorem 3.3 Under Assumption 3.2, assume that subsystem (3.5) is robust semi-uniformly ISS and RALS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(t_0)$, v_1 and u_1 respectively, i.e., there exist class K functions γ_1^0, γ_1, γ_1^u, $\overline{\gamma}_1^0$, $\overline{\gamma}_1$ and $\overline{\gamma}_1^u$, independent of $d(t)$, such that, for any $x_1(t_0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[t_0,\infty)}\| < \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \Delta_1^u$, the solution and output of (3.5) exist and satisfy, for all $t \geq t_0$,

$$\|x_1(t)\| \leq \max\{\gamma_1^0(\|x_1(t_0)\|), \gamma_1(\|v_{1[t_0,\infty)}\|), \gamma_1^u(\|u_{1[t_0,\infty)}\|)\} \tag{3.71}$$

$$\|x_1\|_a \leq \max\{\gamma_1(\|v_1\|_a), \gamma_1^u(\|u_1\|_a)\} \tag{3.72}$$

$$\|y_1(t)\| \leq \max\{\overline{\gamma}_1^0(\|x_1(t_0)\|), \overline{\gamma}_1(\|v_{1[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\} \tag{3.73}$$

$$\|y_1\|_a \leq \max\{\overline{\gamma}_1(\|v_1\|_a), \overline{\gamma}_1^u(\|u_1\|_a)\}. \tag{3.74}$$

And assume that subsystem (3.6) is robust semi-uniformly ISS and RALS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class K functions γ_2^0, γ_2, γ_2^u, $\overline{\gamma}_2^0$, $\overline{\gamma}_2$ and $\overline{\gamma}_2^u$, independent of $d(t)$, such that, for any $x_2(t_0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| < \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \Delta_2^u$, the solution and output of (3.6) exist and satisfy, for all $t \geq t_0$,

$$\|x_2(t)\| \leq \max\{\gamma_2^0(\|x_2(t_0)\|), \gamma_2(\|v_{2[t_0,\infty)}\|), \gamma_2^u(\|u_{2[t_0,\infty)}\|)\} \tag{3.75}$$

$$\|x_2\|_a \leq \max\{\gamma_2(\|v_1\|_a), \gamma_2^u(\|u_1\|_a)\} \tag{3.76}$$

$$\|y_2(t)\| \leq \max\{\overline{\gamma}_2^0(\|x_2(t_0)\|), \overline{\gamma}_2(\|v_{2[t_0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\} \tag{3.77}$$

$$\|y_2\|_a \leq \max\{\overline{\gamma}_2(\|v_2\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\}. \tag{3.78}$$

Suppose that the small gain condition

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad r > 0 \tag{3.79}$$

holds, then the system composed of (3.5) and (3.6) is robust semi-uniformly ISS and RALS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state, $y = \text{col}(y_1, y_2)$ as output and $u = \text{col}(u_1, u_2)$ as input, i.e., there exist class K functions γ^0, γ^u, $\overline{\gamma}^0$ and $\overline{\gamma}^u$, independent of $d(t)$, such that, for any initial state $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, the solution and output of (3.5) and (3.6) exist and satisfy, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\gamma^0(\|x(t_0)\|), \gamma^u(\|u_{[t_0,\infty)}\|)\}, \quad \|x\|_a \leq \gamma^u(\|u\|_a) \tag{3.80}$$

$$\|y(t)\| \leq \max\{\overline{\gamma}^0(\|x(t_0)\|), \overline{\gamma}^u(\|u_{[t_0,\infty)}\|)\}, \quad \|y\|_a \leq \overline{\gamma}^u(\|u\|_a). \tag{3.81}$$

where,

$$\begin{aligned}
\gamma^0(s) &= \max\{2\gamma_1^0(s), 2\gamma_1 \circ \overline{\gamma}_2^0(s), 2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^0(s), 2\gamma_2^0(s), 2\gamma_2 \circ \overline{\gamma}_1^0(s), 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^0(s)\} \\
\gamma^u(s) &= \max\{2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), \\
&\qquad 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\gamma_2 \circ \overline{\gamma}_1^u(s), 2\gamma_2^u(s)\}, \\
\overline{\gamma}^0(s) &= \max\{2\overline{\gamma}_1^0(s), 2\overline{\gamma}_1 \circ \overline{\gamma}_2^0(s), 2\overline{\gamma}_2^0(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^0(s)\} \\
\overline{\gamma}^u(s) &= \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\}
\end{aligned}$$

and,

(i) If Δ_1, Δ_2 are finite,

$\tilde{X}_1 = \{x_1 \in X_1 : \overline{\gamma}_1^0(\|x_1\|) < \Delta_2, \overline{\gamma}_2 \circ \overline{\gamma}_1^0(\|x_1\|) < \Delta_1\}$,

and

$\tilde{X}_2 = \{x_2 \in X_2 : \overline{\gamma}_2^0(\|x_2\|) < \Delta_1, \overline{\gamma}_1 \circ \overline{\gamma}_2^0(\|x_2\|) < \Delta_2\}$.

$\tilde{\Delta}_1 \leq \Delta_1^u$, $\tilde{\Delta}_2 \leq \Delta_2^u$

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \Delta_1, \overline{\gamma}_1^u(s) < \Delta_2$

and

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \Delta_2, \overline{\gamma}_2^u(s) < \Delta_1$.

(ii) If Δ_1, Δ_2 are infinite,

$\tilde{X}_1 = X_1$, $\tilde{X}_2 = X_2$

and

$\tilde{\Delta}_1 \leq \Delta_1^u$, $\tilde{\Delta}_2 \leq \Delta_2^u$

Proof: First it is noted that the inequality $\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r$, $(r > 0)$ and the following one,

$$\overline{\gamma}_2 \circ \overline{\gamma}_1(r) < r, \quad r > 0$$

imply each other [23].

Step1: In this step, we will show that if the initial state $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, and $u_1(t) \in L_\infty^{m_1}$ satisfies $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfies $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, the solution of the inter-connected system exists and is bounded for all $t \geq t_0$. For this purpose, we will consider the following two cases.

(i) Δ_1 and Δ_2 are infinite.

Toward this end, we will first prove that the outputs y_1 and y_2 exist for all $t \geq t_0$ and are bounded in a way which is similar to the proof of Theorem 10.6.1 [22]. Suppose this is not the case, for every number $R > 0$, there exists a time $T > t_0$ such that the solutions are defined on $[0, T]$ and either $\|y_1(T)\| \geq R$ or $\|y_2(T)\| \geq R$. Without loss of generality, we only consider the case where $\|y_1(T)\| \geq R$. Choose R such that
$R > \max\{\overline{\gamma}_1^0(r_1), \overline{\gamma}_1 \circ \overline{\gamma}_2^0(r_2), \overline{\gamma}_1^u(\tilde{\Delta}_1), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\tilde{\Delta}_2)\}$,
where, $r_1 = \{x_1 \in \tilde{X}_1 : \sup(\|x_1\|)\}$, $r_2 = \{x_2 \in \tilde{X}_2 : \sup(\|x_2\|)\}$.

It follows from (3.73) and (3.77) that

$$\|y_{1[t_0,T]}\| \leq \max\{\overline{\gamma}_1^0(\|x_1(t_0)\|), \overline{\gamma}_1(\|y_{2[t_0,T]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,T]}\|)\} \tag{3.82}$$

$$\|y_{2[t_0,T]}\| \leq \max\{\overline{\gamma}_2^0(\|x_2(t_0)\|), \overline{\gamma}_2(\|y_{1[t_0,T]}\|), \overline{\gamma}_2^u(\|u_{2[t_0,T]}\|)\}. \tag{3.83}$$

Substituting (3.83) into (3.82) gives that

$$\begin{aligned}\|y_{1[t_0,T]}\| \leq\ & \max\{\overline{\gamma}_1^0(\|x_1(t_0)\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^0(\|x_2(t_0)\|), \\ & \overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0,T]}\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,T]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,T]}\|)\}.\end{aligned} \tag{3.84}$$

Since

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0,T]}\|) < \|y_{1[t_0,T]}\|,$$

it holds that

$$\begin{aligned}\|y_{1[t_0,T]}\| \leq\ & \max\{\overline{\gamma}_1^0(\|x_1(t_0)\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^0(\|x_2(t_0)\|), \\ & \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,T]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,T]}\|)\} < R\end{aligned} \tag{3.85}$$

which contradicts $\|y_1(T)\| > R$. Therefore the outputs are bounded for all $t \geq t_0$.

Since the subsystems (3.5) and (3.6) are semi-uniformly ISS with restrictions, the solution of the inter-connected system is bounded for all $t \geq t_0$.

(ii) At least one of Δ_1, Δ_2 is finite.

Toward this end, we will first prove that the outputs y_1 and y_2 exist for all $t \geq t_0$ and are bounded in a way which is similar to the proof of Theorem 1 [61].

For any given $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, let $p(x(t_0), \lambda)$ be a continuous path in $\tilde{X}_1 \times \tilde{X}_2$ from the origin to $x(t_0)$ with the property that $p(x(t_0), 0)$ is the origin and $p(x(t_0), 1) = x(t_0)$, and let y_1^λ and y_2^λ be the outputs starting at $x^\lambda(t_0) = p(x(t_0), \lambda)$ with inputs λu_1 and λu_2. When $\lambda = 0$, the solutions and outputs are defined on $[t_0, \infty)$ and identically zero. Note that the solutions are continuous functions of λ. Hence, for any given $T > t_0$ (arbitrarily large), $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists λ^* such that the solution exists on $[t_0, T]$ and

$$\|y_{1[t_0,T]}^\lambda\| \leq \epsilon_1, \quad \|y_{2[t_0,T]}^\lambda\| \leq \epsilon_2 \tag{3.86}$$

for all $\lambda \in [0, \lambda^*]$.

Denote that

$$\begin{aligned}
\bar{\Delta}_1 &= \max\{\overline{\gamma}_1^0(\max_{\lambda\in[0,1]} \|x_1^\lambda(t_0)\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^0(\max_{\lambda\in[0,1]} \|x_2^\lambda(t_0)\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\}, \\
\bar{\Delta}_2 &= \max\{\overline{\gamma}_2^0(\max_{\lambda\in[0,1]} \|x_2^\lambda(t_0)\|), \overline{\gamma}_2 \circ \overline{\gamma}_1^0(\max_{\lambda\in[0,1]} \|x_1^\lambda(t_0)\|), \overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\}.
\end{aligned}$$

Since $p(x(t_0), \lambda)$ belongs to $\tilde{X}_1 \times \tilde{X}_2$ and $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$, $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, it holds that $\bar{\Delta}_1 < \Delta_2$ and $\bar{\Delta}_2 < \Delta_1$. Let $T > t_0$ be arbitrarily large and ϵ_1, ϵ_2 satisfy $\bar{\Delta}_1 < \epsilon_1 < \Delta_2$, $\bar{\Delta}_2 < \epsilon_2 < \Delta_1$, and let $\lambda^* \in (0, 1]$ be the largest value such that (3.86) holds for all $\lambda \in [0, \lambda^*]$. Suppose $\lambda^* < 1$. Since $\|y_{1[t_0,T]}^\lambda\| < \Delta_2$ and $\|y_{2[t_0,T]}^\lambda\| < \Delta_1$, following the same lines as (i) when Δ_1 and Δ_2 are infinite, we have that

$$\|y_{1[t_0,T]}^\lambda\| \leq \bar{\Delta}_1 < \epsilon_1, \quad \|y_{2[t_0,T]}^\lambda\| \leq \bar{\Delta}_2 < \epsilon_2.$$

By continuity of solutions, there exists $\lambda' > \lambda^*$ such that (3.86) holds, contradicting that $\lambda^* < 1$. Hence $\lambda^* = 1$. Since T can be arbitrarily large, $\|y_{1[t_0,\infty)}\| < \Delta_2$ and $\|y_{2[t_0,\infty)}\| < \Delta_1$.

In both cases, the solution of the inter-connected system exist and is bounded for all $t \geq t_0$. Moreover, $\|y_{1[t_0,\infty)}\| < \Delta_2$ and $\|y_{2[t_0,\infty)}\| < \Delta_1$. Hence, if the initial state $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, and $u_1(t) \in L_\infty^{m_1}$ satisfies $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfies $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, (3.71)–(3.78) hold for $t \geq t_0$.

Step 2: Substituting (3.77) into (3.73) gives that

$$\begin{aligned}\|y_1(t)\| &\le \max\{\overline{\gamma}_1^0(\|x_1(t_0)\|), \overline{\gamma}_1(\|v_{1[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\} \\ &\le \max\{\overline{\gamma}_1^0(\|x_1(t_0)\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^0(\|x_2(t_0)\|), \overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0,\infty)}\|), \\ &\quad \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\} \end{aligned} \quad (3.87)$$

Since the composite of $\overline{\gamma}_1$ and $\overline{\gamma}_2$ is a simple contraction, it follows that

$$\|y_1(t)\| \le \max\{\overline{\gamma}_1^0(\|x_1(t_0)\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^0(\|x_2(t_0)\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\}. \quad (3.88)$$

By symmetry of y_1 and y_2, it follows that

$$\|y_2(t)\| \le \max\{\overline{\gamma}_2^0(\|x_2(t_0)\|), \overline{\gamma}_2 \circ \overline{\gamma}_1^0(\|x_1(t_0)\|), \overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\}. \quad (3.89)$$

Substituting (3.89) into (3.71) gives that

$$\begin{aligned}\|x_1(t)\| &\le \max\{\gamma_1^0(\|x_1(t_0)\|), \gamma_1(\|v_{1[t_0,\infty)}\|), \gamma_1^u(\|u_{1[t_0,\infty)}\|)\} \\ &\le \max\{\gamma_1^0(\|x_1(t_0)\|), \gamma_1 \circ \overline{\gamma}_2^0(\|x_2(t_0)\|), \gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^0(\|x_1(t_0)\|), \gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|), \\ &\quad \gamma_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \gamma_1^u(\|u_{1[t_0,\infty)}\|)\}. \end{aligned} \quad (3.90)$$

By symmetry of x_1 and x_2, it follows that

$$\begin{aligned}\|x_2(t)\| &\le \max\{\gamma_2^0(\|x_2(t_0)\|), \gamma_2 \circ \overline{\gamma}_1^0(\|x_1(t_0)\|), \gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^0(\|x_2(t_0)\|), \gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \\ &\quad \gamma_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|), \gamma_2^u(\|u_{2[t_0,\infty)}\|)\}. \end{aligned} \quad (3.91)$$

Substituting (3.78) into (3.74) gives that

$$\|y_1\|_a \le \max\{\overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_1\|_a), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), \overline{\gamma}_1^u(\|u_1\|_a)\}. \quad (3.92)$$

Since the composite of $\overline{\gamma}_1$ and $\overline{\gamma}_2$ is a simple contraction, it follows that

$$\|y_1\|_a \le \max\{\overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), \overline{\gamma}_1^u(\|u_1\|_a)\}. \quad (3.93)$$

By symmetry of y_1 and y_2, it follows that

$$\|y_2\|_a \le \max\{\overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\}. \quad (3.94)$$

Substituting (3.94) into (3.72) gives that

$$\begin{aligned}\|x_1\|_a &\le \max\{\gamma_1(\|v_1\|_a), \gamma_1^u(\|u_1\|_a)\} \\ &\le \max\{\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), \gamma_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), \gamma_1^u(\|u_1\|_a)\}. \end{aligned} \quad (3.95)$$

By symmetry of x_1 and x_2, it follows that

$$\|x_2\|_a \le \max\{\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), \gamma_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), \gamma_2^u(\|u_2\|_a)\}. \tag{3.96}$$

Hence, the system composed of (3.5) and (3.6) is robust semi-uniformly ISS and RALS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state, $y = \text{col}(y_1, y_2)$ as output and $u = \text{col}(u_1, u_2)$ as input. This completes the proof. ∎

Corollary 3.11 Consider the interconnection of the following two systems

$$\dot{x}_1 = f_1(x_1, u_1, d, t), \quad y_1 = x_1 \tag{3.97}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = x_2 \tag{3.98}$$

subject to the interconnection constraint:

$$v_2 = y_1$$

where, the notations are the same as those in Theorem 3.3.

Assume that subsystem (3.97) is robust semi-uniformly ISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(t_0)$, v_1 and u_1 respectively, i.e., there exist class K functions γ_1^0, $\gamma_1 \equiv 0$ and γ_1^u, independent of $d(t)$, such that, for any $x_1(t_0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[t_0,\infty)}\| < \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \Delta_1^u$, the solution and output of (3.97) exist and satisfy, for all $t \ge t_0$, (3.71) and (3.72) holds.

And assume that subsystem (3.98) is robust semi-uniformly ISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class K functions γ_2^0, γ_2 and γ_2^u, independent of $d(t)$, such that, for any $x_2(t_0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| < \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \Delta_2^u$, the solution and output of (3.98) exist and satisfy, for all $t \ge t_0$, (3.75) and (3.76) hold.

Then the system composed of (3.97) and (3.98) is robust semi-uniformly ISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state, $y = \text{col}(y_1, y_2)$ as output and $u = \text{col}(u_1, u_2)$ as input, i.e., there exist class K functions γ^0, γ^u, $\overline{\gamma}^0$ and $\overline{\gamma}^u$, independent of $d(t)$, such that, for any initial state $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, the solution and output of (3.97) and (3.98) exist and satisfy, for all $t \ge t_0$,

$$\|x(t)\| \le \max\{\gamma^0(\|x(t_0)\|), \gamma^u(\|u_{[t_0,\infty)}\|)\}, \quad \|x\|_a \le \gamma^u(\|u\|_a)$$

where,

$$\gamma^0(s) = \max\{2\gamma_1^0(s), 2\gamma_2^0(s), 2\gamma_2 \circ \overline{\gamma}_1^0(s)\}$$
$$\gamma^u(s) = \max\{2\gamma_1^u(s), 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\gamma_2^u(s)\}$$

and,

$\bar{X}_1 = \{x_1 \in X_1 : \overline{\gamma}_1^0(\|x_1\|) < \Delta_2, \overline{\gamma}_2 \circ \overline{\gamma}_1^0(\|x_1\|) < \Delta_1\}$,

$\bar{X}_2 = \{x_2 \in X_2 : \overline{\gamma}_2^0(\|x_2\|) < \Delta_1\}$.

$\tilde{\Delta}_1 \leq \Delta_1^u$, $\tilde{\Delta}_2 \leq \Delta_2^u$

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \Delta_1, \overline{\gamma}_1^u(s) < \Delta_2$

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_2^u(s) < \Delta_1$. ∎

3.3 Asymptotic Small Gain Theorem with Restrictions for Uncertain Nonlinear Time-varying Systems

Theorem 3.4 Under Assumption 3.2, assume that both subsystems (3.5) and (3.6) are RAG and o-RAG with restrictions X_i, Δ_i and Δ_i^u on $x_i(t_0)$, v_i and u_i, $i = 1, 2$, respectively, i.e., for $i = 1, 2$, there exist class K functions γ_i, γ_i^u, $\overline{\gamma}_i$ and $\overline{\gamma}_i^u$, independent of d, such that, for any $x_i(t_0) \in X_i$, $v_i(t) \in L_\infty^{q_i}$ satisfying $\|v_i\|_a \leq \Delta_i$, $u_i(t) \in L_\infty^{m_i}$ satisfying $\|u_i\|_a \leq \Delta_i^u$, the solutions of (3.5) and (3.6) exist and satisfy, for all $t \geq t_0$,

$$\|x_i\|_a \leq \max\{\gamma_i(\|v_i\|_a), \gamma_i^u(\|u_i\|_a)\} \tag{3.99}$$
$$\|y_i\|_a \leq \max\{\overline{\gamma}_i(\|v_i\|_a), \overline{\gamma}_i^u(\|u_i\|_a)\}. \tag{3.100}$$

Suppose

A 3.3 For all initial state in $X_1 \times X_2$ and all piecewise continuous u_1, u_2, d which are bounded on $[t_0, \infty)$, the solution of (3.5) and (3.6) with connection (3.7) is defined for all $t \geq t_0$;

A 3.4 $\Delta_1 = \infty$;

A 3.5 $\overline{\gamma}_1(\infty) < \infty$ and $\overline{\gamma}_1(\infty) \leq \Delta_2$;

A 3.6 the small gain condition

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad r > 0 \tag{3.101}$$

holds.

Then, under connection (3.7), the system composed of (3.5) and (3.6) is RAG and o-RAG with restrictions $X_1 \times X_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state, $y = \text{col}(y_1, y_2)$ as output and $u = \text{col}(u_1, u_2)$ as input, i.e., there exist class K functions γ^u and $\overline{\gamma}^u$, independent of d, such that, for any initial state $x(t_0) \in X_1 \times X_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_1\|_a \leq \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_2\|_a \leq \tilde{\Delta}_2$, the solution of (3.5) and (3.6) with connection (3.7) exists and satisfies, for all $t \geq t_0$,

$$\|x\|_a \leq \gamma^u(\|u\|_a), \tag{3.102}$$

$$\|y\|_a \leq \overline{\gamma}^u(\|u\|_a). \tag{3.103}$$

where,

$$\begin{aligned} \gamma^u(s) &= \max\{2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\gamma_2 \circ \overline{\gamma}_1^u(s), 2\gamma_2^u(s)\}, \\ \overline{\gamma}^u(s) &= \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\} \end{aligned}$$

and $\tilde{\Delta}_2$ is such that $\tilde{\Delta}_2 \leq \Delta_2^u$,
and $\tilde{\Delta}_1$ is such that $\tilde{\Delta}_1 \leq \Delta_1^u$, and $\overline{\gamma}_1^u(\tilde{\Delta}_1) \leq \Delta_2$.

Proof: Theorem 3.4 is a slight extension of Theorem 2 [61] and the proof of Theorem 3.4 is similar to that of Theorem 2 [61].

Since $\Delta_1 = \infty$ and $\tilde{\Delta}_1 \leq \Delta_1^u$, the following estimates hold

$$\begin{aligned} \|y_1\|_a &\leq \max\{\overline{\gamma}_1(\|v_1\|_a), \overline{\gamma}_1^u(\|u_1\|_a)\} \\ &\leq \max\{\overline{\gamma}_1 \circ \overline{\gamma}_2(\|v_2\|_a), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), \overline{\gamma}_1^u(\|u_1\|_a)\}. \end{aligned} \tag{3.104}$$

Since the small gain condition holds,

$$\|y_1\|_a \leq \max\{\overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), \overline{\gamma}_1^u(\|u_1\|_a)\}. \tag{3.105}$$

Since $\overline{\gamma}_1^u(\tilde{\Delta}_1) \leq \Delta_2$, $\overline{\gamma}_1(\infty) \leq \Delta_2$ and

$$\begin{aligned} \|y_2\|_a &\leq \max\{\overline{\gamma}_2(\|v_2\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\} \\ &\leq \max\{\overline{\gamma}_2 \circ \overline{\gamma}_1(\|v_1\|_a), \overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\}. \end{aligned} \tag{3.106}$$

Since the small gain condition holds,

$$\|y_2\|_a \leq \max\{\overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\}. \tag{3.107}$$

Combining (3.105) and (3.107) gives that

$$\begin{aligned} \|y\|_a &\leq \|y_1\|_a + \|y_2\|_a \\ &\leq \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), 2\overline{\gamma}_1^u(\|u_1\|_a), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), 2\overline{\gamma}_2^u(\|u_2\|_a)\}. \end{aligned} \tag{3.108}$$

Therefore,

$$\begin{aligned}\|x_1\|_a &\le \max\{\gamma_1(\|v_1\|_a), \gamma_1^u(\|u_1\|_a)\} \\ &\le \max\{\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), \gamma_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), \gamma_1^u(\|u_1\|_a)\}. \end{aligned} \tag{3.109}$$

Similarly,

$$\|x_2\|_a \le \max\{\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), \gamma_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), \gamma_2^u(\|u_2\|_a)\}. \tag{3.110}$$

Therefore,

$$\begin{aligned}\|x\|_a \le\ & \max\{2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), 2\gamma_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), 2\gamma_1^u(\|u_1\|_a), \\ & 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_2\|_a), 2\gamma_2 \circ \overline{\gamma}_1^u(\|u_1\|_a), 2\gamma_2^u(\|u_2\|_a)\}. \end{aligned} \tag{3.111}$$

∎

A special version of Theorem 3.4 which will be used in next section is the case when the first subsystem does not rely on u_1 and the second subsystem does not rely on v_1. In this case, conditions Assumptions 3.4, 3.5 and 3.6 always hold. Therefore, we have the following

Corollary 3.12 Consider the interconnection $v_2 = y_1$ of the following two systems

$$\dot{x}_1 = f_1(x_1, u_1, d, t), \quad y_1 = x_1 \tag{3.112}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = x_2 \tag{3.113}$$

where, the notations are the same as those in Theorem 3.4. Suppose:

A 3.7 For all initial state in $X_1 \times X_2$ and all piecewise continuous u_1, u_2, d which are bounded on $[t_0, \infty)$, the solution of (3.112) and (3.113) with connection $v_2 = y_1$ is defined for all $t \ge t_0$;

A 3.8 Subsystem (3.112) is RAG with restrictions X_1 and Δ_1^u on $x_1(t_0)$ and u_1 respectively;

A 3.9 Subsystem (3.113) is RAG with restrictions X_2 and Δ_2^u on $x_2(t_0)$ and u_2 respectively.

Then system (3.112) and (3.113) with connection $v_2 = y_1$ is RAG with restrictions $X_1 \times X_2$, Δ_1^u, Δ_2^u on $(x_1(t_0), x_2(t_0))$ and u_1^u, u_2^u respectively, i.e., there exist class K

function γ^u, independent of d, such that, for any initial state $x(t_0) \in X_1 \times X_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_1\|_a \leq \Delta_1^u$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_2\|_a \leq \Delta_2^u$, the solution of (3.112) and (3.113) with connection $v_2 = y_1$ exists and satisfies, for all $t \geq t_0$,

$$\|x\|_a \leq \gamma^u(\|u\|_a)$$

where $\gamma^u(s) = \max\{2\gamma_2 \circ \gamma_1^u(s), 2\gamma_1^u(s), 2\gamma_2^u(s)\}$ and all the gain functions are defined the same way as those in Theorem 3.4. ∎

We introduce the following corollary which can be directly applied in the next section.

Corollary 3.13 Consider the interconnections

$$v_{21} = y_{11}, \quad v_{22} = y_{12}, \quad v_1 = y_2 \tag{3.114}$$

of the following two systems

$$\begin{aligned} \Sigma_1 \quad &: \quad \dot{x}_1 = f_1(x_1, v_1, d, t) \\ &\qquad y_{11} = h_{11}(x_1, v_1, d, t), \quad y_{12} = h_{12}(x_1, v_1, d, t) \\ \Sigma_2 \quad &: \quad \dot{x}_2 = f_2(x_2, v_{21}, v_{22}, u_2, d, t), \\ &\qquad y_2 = h_2(x_2, v_{21}, v_{22}, u_2, d, t). \end{aligned}$$

Suppose:

A 3.10 For all initial state in $X_1 \times X_2$ and all piecewise continuous u_2, d which are bounded on $[t_0, \infty)$, the solution of Σ_1 and Σ_2 with connection (3.114) is defined for all $t \geq t_0$;

A 3.11 Subsystem Σ_2 is RAG and o-RAG with restriction Δ_{22} on the input v_{22}, i.e., there exist class K functions γ_{21}, γ_{22}, γ_2^u, $\overline{\gamma}_{21}$, $\overline{\gamma}_{22}$ and $\overline{\gamma}_2^u$, independent of d, such that for any initial state $x_2(t_0) \in \mathrm{Re}^{n_2}$ and any input $v_{22}(t)$ satisfying $\|v_{22}\|_a \leq \Delta_{22}$, the solution $x_2(t)$ exists and satisfies, for all $t \geq t_0$,

$$\|x_2\|_a \leq \max\{\gamma_{21}(\|v_{21}\|_a), \gamma_{22}(\|v_{22}\|_a), \gamma_2^u(\|u_2\|_a)\} \tag{3.115}$$

$$\|y_2\|_a \leq \max\{\overline{\gamma}_{21}(\|v_{21}\|_a), \overline{\gamma}_{22}(\|v_{22}\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\}. \tag{3.116}$$

A 3.12 Subsystem Σ_1 is RAG and o-RAG without restriction, i.e., there exist class K functions γ_1, $\overline{\gamma}_{11}$, $\overline{\gamma}_{12}$, $\overline{\gamma}_{11}^u$ and $\overline{\gamma}_{12}^u$, independent of d, such that for any initial state $x_1(t_0) \in \mathrm{Re}^{n_1}$ and any input $v_1(t)$, the solution $x_2(t)$ exists and satisfies, for all $t \geq t_0$,

$$\|x_1\|_a \leq \gamma_1(\|v_1\|_a), \tag{3.117}$$

$$\|y_{11}\|_a \leq \overline{\gamma}_{11}(\|v_1\|_a), \tag{3.118}$$

$$\|y_{12}\|_a \leq \overline{\gamma}_{12}(\|v_1\|_a). \tag{3.119}$$

A 3.13 $\overline{\gamma}_{11}(\infty) < \infty$, $\overline{\gamma}_{12}(\infty) < \infty$ and $\overline{\gamma}_{12}(\infty) \leq \Delta_{22}$

A 3.14 The small gain conditions

$$\overline{\gamma}_{11} \circ \overline{\gamma}_{21}(r) < r, \quad \overline{\gamma}_{12} \circ \overline{\gamma}_{22}(r) < r, \quad r > 0$$

hold.

Then under the interconnection (3.114), systems Σ_1 and Σ_2 are RAG.

Proof: Since Assumption 3.13 holds, substituting (3.118) and (3.119) into (3.116) gives that

$$\begin{aligned} \|y_2\|_a &\leq \max\{\overline{\gamma}_{21}(\|v_{21}\|_a), \overline{\gamma}_{22}(\|v_{22}\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\} \\ &\leq \max\{\overline{\gamma}_{21} \circ \overline{\gamma}_{11}(\|v_1\|_a), \overline{\gamma}_{22} \circ \overline{\gamma}_{12}(\|v_1\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\} \end{aligned} \tag{3.120}$$

Since the small gain condition (B5) holds, it holds that

$$\|y_2\|_a \leq \overline{\gamma}_2^u(\|u_2\|_a). \tag{3.121}$$

Therefore,

$$\begin{aligned} \|x_2\|_a &\leq \max\{\gamma_{21}(\|v_{21}\|_a), \gamma_{22}(\|v_{22}\|_a), \gamma_2^u(\|u_2\|_a)\} \\ &\leq \max\{\gamma_{21} \circ \overline{\gamma}_{11}(\|v_1\|_a), \gamma_{22} \circ \overline{\gamma}_{12}(\|v_1\|_a), \gamma_2^u(\|u_2\|_a)\} \\ &\leq \max\{\gamma_{21} \circ \overline{\gamma}_{11} \circ \overline{\gamma}_2^u(\|u_2\|_a), \gamma_{22} \circ \overline{\gamma}_{12} \circ \overline{\gamma}_2^u(\|u_2\|_a), \gamma_2^u(\|u_2\|_a)\} \end{aligned} \tag{3.122}$$

Substituting (3.121) into (3.117)-(3.119) gives that

$$\|x_1\|_a \leq \gamma_1(\|v_1\|_a) \leq \gamma_1 \circ \overline{\gamma}_2^u(\|u_2\|_a) \tag{3.123}$$

$$\begin{aligned} \|y_1\|_a &\leq \|y_{11}\|_a + \|y_{12}\|_a \leq \max\{2\overline{\gamma}_{11}(\|v_1\|_a), 2\overline{\gamma}_{12}(\|v_1\|_a)\} \\ &\leq \max\{2\overline{\gamma}_{11} \circ \overline{\gamma}_2^u(\|u_2\|_a), 2\overline{\gamma}_{12} \circ \overline{\gamma}_2^u(\|u_2\|_a). \end{aligned} \tag{3.124}$$

∎

Remark 3.8 Corollary 3.13 is similar to Proposition 1 [2]. ∎

3.4 Input-to-State Stability Small Gain Theorem with Restrictions for Uncertain Time-varying Systems of Functional Differential Equations

Consider the interconnection of the following two systems as,

$$\dot{x}_1(t) = f_1(x_{1d}(t), v_{1d}(t), u_{1d}(t), d, t), \quad y_1(t) = h_1(x_{1d}(t), v_{1d}(t), u_{1d}(t), d, t) \tag{3.125}$$

$$\dot{x}_2(t) = f_2(x_{2d}(t), v_{2d}(t), u_{2d}(t), d, t), \quad y_2(t) = h_2(x_{2d}(t), v_{2d}(t), u_{2d}(t), d, t) \tag{3.126}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1 \tag{3.127}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{p_i}$, $v_i \in \mathbb{R}^{q_i}$ with $p_1 = q_2$, $p_2 = q_1$, the functions $f_1(x_1, v_1, u_1, d, t)$ and $f_2(x_2, v_2, u_2, d, t)$ are piecewise continuous in $\text{col}(d, t)$ and locally *Lipschitz* in $\text{col}(x_1, v_1, u_1)$ and $\text{col}(x_2, v_2, u_2)$ respectively, and $d : [t_0, \infty) \mapsto \mathbb{R}^{n_d}$ is piecewise continuous. The notation $\|x_{id}(t)\| = \sup_{t-t_d \le s \le t} \|x(s)\|$ will be used throughout this part, and $\|u_{id}(t)\|$, $\|v_{id}(t)\|$ are defined in the similar way.

The system composed of (3.125) and (3.126) is interpreted as feedback interconnection of two subsystems, the upper one with state x_1, input $\text{col}(v_1, u_1)$ and output y_1 and the lower one with state x_2, input $\text{col}(v_2, u_2)$ and output y_2. And suppose the following assumption holds.

A 3.15 There exists a C^1 function h such that

$$\text{col}\ (y_1, y_2) = h(x_{1d}, x_{2d}, u_{1d}, u_{2d}, d, t)$$

is the unique solution of the equations

$$\begin{aligned} y_1 &= h_1(x_{1d}, y_{2d}, u_{1d}, d, t) \\ y_2 &= h_2(x_{2d}, y_{1d}, u_{2d}, d, t). \end{aligned}$$

Theorem 3.5 Assume that subsystem (3.125) is RISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(t_0)$, v_1 and u_1 respectively and subsystem (3.126) is RISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class KL functions β_1 and β_2, class K functions γ_1, γ_1^u, γ_2, γ_2^u, independent of $d(t)$, such that, for any $x_1(t_0) \in X_1$,

$v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[t_0,\infty)}\| \le \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| \le \Delta_1^u$, the solution of (3.125) exists and satisfies, for all $t \ge t_0$,

$$\|x_1(t)\| \le \max\{\beta_1(\|x_{1d}(t_0)\|, t-t_0), \gamma_1(\|v_{1d[t_0,t]}\|), \gamma_1^u(\|u_{1d[t_0,t]}\|)\} \tag{3.128}$$

and for any $x_2(t_0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| \le \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| \le \Delta_2^u$, the solution of (3.126) exists and satisfies, for all $t \ge t_0$,

$$\|x_2(t)\| \le \max\{\beta_2(\|x_{2d}(t_0)\|, t-t_0), \gamma_2(\|v_{2d[t_0,t]}\|), \gamma_2^u(\|u_{2d[t_0,t]}\|)\}. \tag{3.129}$$

Further assume that subsystem (3.125) is RIOS with restrictions $\bar{X}_1$, $\overline{\Delta}_1$ and $\overline{\Delta}_1^u$ on $x_1(t_0)$, v_1 and u_1 respectively and subsystem (3.126) is RIOS with restrictions $\bar{X}_2$, $\overline{\Delta}_2$ and $\overline{\Delta}_2^u$ on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class KL functions $\overline{\beta}_1$ and $\overline{\beta}_2$, class K functions $\overline{\gamma}_1$, $\overline{\gamma}_1^u$, $\overline{\gamma}_2$, $\overline{\gamma}_2^u$, independent of $d(t)$, such that, for any $x_1(t_0) \in \bar{X}_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[t_0,\infty)}\| \le \overline{\Delta}_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| \le \overline{\Delta}_1^u$, the output of (3.125) exists and satisfies, for all $t \ge t_0$,

$$\|y_1(t)\| \le \max\{\overline{\beta}_1(\|x_{1d}(t_0)\|, t-t_0), \overline{\gamma}_1(\|v_{1d[t_0,t]}\|), \overline{\gamma}_1^u(\|u_{1d[t_0,t]}\|)\} \tag{3.130}$$

and for any $x_2(t_0) \in \bar{X}_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| \le \overline{\Delta}_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| \le \overline{\Delta}_2^u$, the output of (3.126) exists and satisfies, for all $t \ge t_0$,

$$\|y_2(t)\| \le \max\{\overline{\beta}_2(\|x_{2d}(t_0)\|, t-t_0), \overline{\gamma}_2(\|v_{2d[t_0,t]}\|), \overline{\gamma}_2^u(\|u_{2d[t_0,t]}\|)\}. \tag{3.131}$$

Suppose that the small gain condition

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad r > 0 \tag{3.132}$$

holds, then the system composed of (3.125) and (3.126) is RISS and RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state, $y = \text{col}(y_1, y_2)$ as output and $u = \text{col}(u_1, u_2)$ as input, i.e., there exist class KL functions β and $\overline{\beta}$, class K functions γ and $\overline{\gamma}$, independent of $d(t)$, such that, for any initial state $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| \le \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| \le \tilde{\Delta}_2$, the solution and output of (3.125) and (3.126) exist and satisfy, for all $t \ge t_0$,

$$\begin{aligned} \|x(t)\| &\le \max\{\beta(\|x(t_0)\|, t-t_0), \gamma(\|u_{[t_0,t]}\|)\} \\ \|y(t)\| &\le \max\{\overline{\beta}(\|x(t_0)\|, t-t_0), \overline{\gamma}(\|u_{[t_0,t]}\|)\} \end{aligned}$$

where,

$\gamma(s) = \max\{4\gamma_1 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 4\gamma_1 \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 4\gamma_2 \circ \overline{\gamma}_1 \circ$

$\overline{\gamma}_2^u(s), 4\gamma_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2^u(s), 2\gamma_2^u(s)\}$,

$\overline{\gamma}(s) = \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\}$

and,

(i) If Δ_1, Δ_2, $\overline{\Delta}_1$, $\overline{\Delta}_2$ are finite,

$\tilde{X}_1 = \{x_1 \in X_1 \cap \bar{X}_1 : \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2 \circ \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\}\}$

and

$\tilde{X}_2 = \{x_2 \in X_2 \cap \bar{X}_2 : \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\},\ \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}\}$.

$\tilde{\Delta}_1 \leq \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \leq \min\{\Delta_2^u, \overline{\Delta}_2^u\}$

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}, \overline{\gamma}_1^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}$

and

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}$.

(ii) If Δ_1, Δ_2, $\overline{\Delta}_1$, $\overline{\Delta}_2$ are infinite,

$\tilde{X}_1 = X_1 \cap \bar{X}_1$, $\tilde{X}_2 = X_2 \cap \bar{X}_2$

and

$\tilde{\Delta}_1 \leq \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \leq \min\{\Delta_2^u, \overline{\Delta}_2^u\}$.

Proof: The proof is similar to that of Theorem 3.2. ∎

Remark 3.9 Theorem 3.5 extends the IOS small gain theorem with restrictions for time invariant systems of functional differential equations in [47] to the time-varying case. ∎

Chapter 4

A Remark on Various Small Gain Conditions

4.1 Introduction

ISS small gain theorem is one of the powerful tools to verify the stability of the interconnected systems. The proof of the ISS small gain theorem is usually based on two methods: the input-to-output formulation [8, 9, 22, 25, 26, 61] and *Lyapunov* function argument [24, 26, 59]. Relying upon the the input-to-output formulation, Jiang *et al* established a generalized small gain theorem for time-varying systems whose small gain condition involved two somewhat complicated inequalities [25, 26]. In [8], Chen and Huang introduced the concept of RISS and further simplified the small gain condition into one contract mapping, giving a more clear-cut version of the small gain theorem. These studies are based on the concept of the gain function. On the other hand, it is well-known that *Lyapunov* functions play an important role in the nonlinear system and control, so it is natural to derive the small gain theorem using *Lyapunov* functions. The $ISS-Lyapunov$ function (dissipation) characterization of the small gain theorem was given in [24, 26, 59], whose small gain condition was based on the contract mapping of $ISS-Lyapunov$ functions. These functions have been applied in ISS analysis of open-loop systems and cascade interconnected systems [22, 53, 55].

It is interesting to find the connection between these two versions of the small gain theorems. This chapter will show that the contract mapping of gain functions and that of $ISS-Lyapunov$ functions does not imply each other, i.e., if there exists two $ISS-Lyapunov$ functions for two subsystems respectively, we cannot guarantee the existence of two gain functions for two subsystems respectively which satisfy the contract mapping; on the converse, if two subsystems are ISS respectively and their gain functions satisfy the contract mapping, we also cannot guarantee the existence of the $ISS - Lyapunov$ functions for two subsystems that satisfy the contract mapping.

4.2 Preliminary

Consider the following time-varying system:

$$\dot{x} = f(t, x, u) \qquad t \geq t_0 \geq 0 \tag{4.1}$$

viewing $x \in \mathbb{R}^n$ as the plant state, $u \in \mathbb{R}^m$ as the input, t_0 as the initial time, the function $f(t, x, u) : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is piecewise continuous in t and locally *Lipschitz* in $\mathrm{col}(x, u)$ for all $t \geq t_0$. And suppose $f(t, 0, 0) = 0$ for all $t \geq t_0$.

Definition 4.1 System (4.1) is said to be *weakly robustly stable* if there exists a smooth function φ, satisfying $\psi(\|x\|) \leq \varphi(x)$ for some K_∞ function ψ, such that the following system

$$\dot{x}(t) = f(t, x(t), d(t)\varphi(x(t))) \tag{4.2}$$

is $uniform$ *global asymptotic stability* (UGAS, for short) for any $d(t) \in M_D =$ the set of all measurable functions $\mathbb{R} \mapsto D = [-1, 1]^m$. ∎

Definition 4.2 Consider the following system

$$\dot{x} = f(x, u) \qquad y = h(x) \tag{4.3}$$

where, $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $h : \mathbb{R}^n \mapsto \mathbb{R}^p$ are both locally *Lipschitz* continuous, $f(0, 0) = 0$ and $h(0) = 0$. System (4.3) is $state - independent\ uniformly\ output\ stable$ with respect to input in M_Ω, where Ω is a compact subset of $\mathbb{R}^m$, if there exists KL function β and K function γ such that

$$\|y(t)\| \leq \max\{\beta(\|h(x(0))\|, t), \gamma(\|u_{[0,t]}\|)\} \tag{4.4}$$

for all $u \in M_\Omega$. ∎

The following small gain theorem based on gain functions was given in [8].

Theorem 4.1 Consider the inter-connection of the following two systems

$$\dot{x}_1 = f_1(t, x_1, v_1, u) \tag{4.5}$$

$$\dot{x}_2 = f_2(t, x_2, v_2, u) \tag{4.6}$$

subject to the interconnection constraints:

$$v_1 = x_2, \quad v_2 = x_1$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $v_i \in \mathbb{R}^{q_i}$, $u \in \mathbb{R}^m$ with $n_1 = q_2$ and $n_2 = q_1$, the functions $f_i(t, x_i, v_i, u)$ is piecewise continuous in t and locally *Lipschitz* in $\text{col}(x_i, v_i, u)$.

Assume that subsystem (4.5) is ISS viewing x_1 as state and $\text{col}(v_1, u)$ as input and subsystem (4.6) is ISS viewing x_2 as state and $\text{col}(v_2, u)$ as input, i.e., there exists class KL functions β_1 and β_2, class K functions γ_1^x, γ_2^x, γ_1^u, γ_2^u such that, for any $x_1(t_0) \in \mathbb{R}^{n_1}$, $\text{col}(v_1, u) \in L_\infty^{q_1+m}$, the solution of (4.5) exists and satisfies, for all $t \geq t_0 \geq 0$,

$$\|x_1(t)\| \leq \max\{\beta_1(\|x_1(t_0)\|, t - t_0), \gamma_1^x(\|v_{1[t_0,t]}\|), \gamma_1^u(\|u_{[t_0,t]}\|)\} \tag{4.7}$$

and for any $x_2(t_0) \in \mathbb{R}^{n_2}$, $\text{col}(v_2, u) \in L_\infty^{q_2+m}$, the solution of (4.6) exists and satisfies, for all $t \geq t_0 \geq 0$,

$$\|x_2(t)\| \leq \max\{\beta_2(\|x_2(t_0)\|, t - t_0), \gamma_2^x(\|v_{2[t_0,t]}\|), \gamma_2^u(\|u_{[t_0,t]}\|)\}. \tag{4.8}$$

Further assume

$$\gamma_1^x \circ \gamma_2^x(r) < r, \qquad \forall r > 0 \tag{4.9}$$

then the system composed of (4.5) and (4.6) is ISS viewing $x = \text{col}(x_1, x_2)$ as state and u as input, in particular,

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\} \qquad \forall t \geq t_0 \tag{4.10}$$

for some class KL class function β and any class K function satisfying

$$\gamma(r) \geq \max\{2\gamma_1^x \circ \gamma_2^u(r), 2\gamma_1^u(r), 2\gamma_2^x \circ \gamma_1^u(r), 2\gamma_2^u(r)\} \qquad \forall r \geq 0. \tag{4.11}$$

∎

Remark 4.1 The inequality (4.9), which is equivalent to $\gamma_2^x \circ \gamma_1^x(r) < r$ ($\forall r > 0$), is called the contract mapping of gain functions. ∎

Consider the following inter-connected system,

$$\dot{x}_1 = f_1(t, x_1, v_1, u_1) \tag{4.12}$$

$$\dot{x}_2 = f_2(t, x_2, v_2, u_2) \tag{4.13}$$

subject to the inter-connection constraints:

$$v_1 = x_2, \quad v_2 = x_1, \tag{4.14}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $v_i \in \mathbb{R}^{p_i}$ with $p_1 = n_2$ and $p_2 = n_1$, the functions $f_1(t, x_1, v_1, u_1)$ and $f_2(t, x_2, v_2, u_2)$ are piecewise continuous in t and locally *Lipschitz* in $\text{col}(x_1, v_1, u_1)$ and (x_2, v_2, u_2) respectively and $f_1(t, 0, 0, 0) = f_2(t, 0, 0, 0) = 0$ for all $t \geq t_0$.

Assume that x_1 subsystem admits an $ISS - Lyapunov$ function $V_1(t, x_1)$ such that there exists K_∞ functions $\underline{\alpha}_1$, $\overline{\alpha}_1$ such that

$$\underline{\alpha}_1(\|x_1\|) \leq V_1(t, x_1) \leq \overline{\alpha}_1(\|x_1\|) \tag{4.15}$$

for any $x_1 \in \mathbb{R}^{n_1}$ and for some K_∞ functions χ_1, γ_1 and α_1,

$$\begin{aligned} & V_1(t, x_1) \geq \max\{\chi_1(V_2(t, x_2)), \gamma_1(\|u_1\|)\} \\ & \Rightarrow \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1, x_2, u_1) \leq -\alpha_1(V_1(t, x_1)) \end{aligned} \tag{4.16}$$

for any $x_2 \in \mathbb{R}^{n_2}$ and any $u_1 \in L_\infty^{m_1}$.

And x_2 subsystem admits an $ISS - Lyapunov$ function $V_2(t, x_2)$ such that there exists K_∞ functions $\underline{\alpha}_2$, $\overline{\alpha}_2$ such that

$$\underline{\alpha}_2(\|x_2\|) \leq V_2(t, x_2) \leq \overline{\alpha}_2(\|x_2\|) \tag{4.17}$$

for any $x_2 \in \mathbb{R}^{n_2}$ and for some K_∞ functions χ_2, γ_2 and α_2,

$$\begin{aligned} & V_2(t, x_2) \geq \max\{\chi_2(V_1(t, x_1)), \gamma_2(\|u_2\|)\} \\ & \Rightarrow \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_1, x_2, u_2) \leq -\alpha_2(V_2(t, x_2)) \end{aligned} \tag{4.18}$$

for any $x_1 \in \mathbb{R}^{n_1}$ and any $u_2 \in L_\infty^{m_2}$.

With this, we have the following theorem relying upon the $ISS - Lyapunov$ functions which was introduced in [10].

Theorem 4.2 Assume that, for $i = 1, 2$, the x_i subsystem admits an $ISS - Lyapunov$ function V_i satisfying (4.15)-(4.16) and (4.17)-(4.18) respectively. If the following small gain condition holds

$$\chi_1 \circ \chi_2(r) < r, \qquad \forall r > 0$$

then the inter-connected system composed of (4.12) and (4.13) is ISS with $x = \text{col}(x_1, x_2)$ as state, $u = \text{col}(u_1, u_2)$ as input. ∎

Remark 4.2 The inequality $\chi_1 \circ \chi_2(r) < r$, which is equivalent to $\chi_2 \circ \chi_1(r) < r$ ($\forall r > 0$), is called the contract mapping of $ISS - Lyapunov$ functions. ∎

The following theorem was proven using $ISS - Lyapunov$ functions in [24].

Theorem 4.3 Suppose that there exists C^1 functions $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_i} \mapsto \mathbb{R}_{\geq 0}$, $i = 1, 2$ such that for all $x_i \in \mathbb{R}^{n_i}$ and $t \in \mathbb{R}_{\geq 0}$,

$$\underline{\alpha}_i(\|x_i\|) \leq V_i(t, x_i) \leq \overline{\alpha}_i(\|x_i\|) \tag{4.19}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1, x_2, u_1) \leq -\alpha_1(\|x_1\|) + \gamma_1^x(\|x_2\|) + \gamma_1^u(\|u_1\|) \tag{4.20}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_1, x_2, u_2) \leq -\alpha_2(\|x_2\|) + \gamma_2^x(\|x_1\|) + \gamma_2^u(\|u_2\|) \tag{4.21}$$

are satisfied with some $\underline{\alpha}_i$, $\overline{\alpha}_i$, $\alpha_i \in K_\infty$ and γ_i^x, $\gamma_i^u \in K$. If there exists $c_i > 1$ ($i = 1, 2$) such that

$$\underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \alpha_1^{-1} \circ c_1 \cdot \gamma_1^x \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ c_2 \cdot \gamma_2^x(s) < s, \quad \forall s \in (0, \infty) \tag{4.22}$$

$$(c_1 - 1)(c_2 - 1) > 1 \tag{4.23}$$

are satisfied, the interconnected system (4.12) and (4.13) is ISS with respect to the state $x = \text{col}(x_1, x_2)$ and the input $u = \text{col}(u_1, u_2)$. ∎

4.3 The Sufficient and Necessary Condition for Input-to-State Stability of Time-varying Systems

4.3.1 ISS-Lyapunov functions for Time-varying Systems

The following is a fundamental result on ISS theory for time-varying systems, that was proposed by H. Edwards *et al* [10]. However, the authors did not provide the proof in [10]. Here we give the detail of the proof, laying the fundamentals for the subsequent sections.

Theorem 4.4 Assume that the origin is the equilibrium point for zero-input system of system (4.1), then system (4.1) is ISS if and only if it admits an $ISS - Lyapunov$ function.

Proof:

$Sufficiency$: Assume that $V(t, x) : [t_0, \infty) \times \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ is an $ISS - Lyapunov$ function of system (4.1), i.e., there exists K_∞ functions $\underline{\alpha}$, $\overline{\alpha}$ such that, for all $x(t_0) \in \mathbb{R}^n$, (4.2) and (4.3) hold.

Let $M = \|u_{[t_0,\infty)}\|$, $\eta = \overline{\alpha}\circ\chi(M)$ and $\rho = \underline{\alpha}(r)$. Choose ρ and r such that $\eta < \rho < \underline{\alpha}(r)$ and $\bar{\alpha}(\|x(t_0)\|) \leq \rho$. Define the following time-dependent sets for $t \geq t_0$,

$$\begin{aligned}\Omega_{t,\eta} &= \{x \in B_r | V(t,x) \leq \eta\} \\ \Omega_{t,\rho} &= \{x \in B_r | V(t,x) \leq \rho\}.\end{aligned}$$

Then it holds that

$$\Omega_{t,\eta} \subset \Omega_{t,\rho} \subset B_r.$$

Observe that

$$\overline{\alpha}(\|x(t_0)\|) \leq \rho \Rightarrow x(t_0) \in \Omega_{t_0,\rho}. \tag{4.24}$$

Since

$$\begin{aligned}&\overline{\alpha}\circ\chi(M) = \eta < \rho = V(t,x) \leq \overline{\alpha}(\|x\|) \\ &\Rightarrow \|x\| \geq \chi(\|u\|) \\ &\Rightarrow \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x,u) < 0,\end{aligned}$$

we can conclude that if $\|x(t_0)\| \leq \overline{\alpha}^{-1}(\rho)$, then $x(t) \in \Omega_{t,\rho}$, hence $x(t) \in B_r$ for all $t \geq t_0$.

Next, we will claim the two facts below:

(i) A solution starting inside $\Omega_{t_0,\eta}$ will stay inside $\Omega_{t,\eta}$ for all $t \geq t_0$;

(ii) A solution starting inside $\{\Omega_{t_0,\rho} - \Omega_{t_0,\eta}\}$ will enter $\Omega_{t,\eta}$ in finite time $T \geq t_0$.

Claim (i) follows from the fact that on the boundary of $\Omega_{t_0,\eta}$,

$$\begin{aligned}&\overline{\alpha}\circ\chi(M) = \eta = V(t,x) \leq \overline{\alpha}(\|x\|) \\ &\Rightarrow \|x\| \geq \chi(\|u\|) \\ &\Rightarrow \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x,u) < 0.\end{aligned}$$

To prove claim (ii), assume that T be the first time when the solution enters $\Omega_{t,\eta}$. If the solution never enters $\Omega_{t,\eta}$, then $T = \infty$. For $t \in [t_0, T)$,

$$\begin{aligned}&\overline{\alpha}\circ\chi(M) = \eta < V(t,x) \leq \overline{\alpha}(\|x\|) \\ &\Rightarrow \|x\| \geq \chi(\|u\|) \\ &\Rightarrow \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x,u) \leq -\alpha(\|x\|) \leq -\alpha\circ\overline{\alpha}^{-1}(V(t,x)) = -\tilde{\alpha}(V(t,x)).\end{aligned}$$

where $\tilde{\alpha} = \alpha\circ\overline{\alpha}^{-1}$. Let $y(t)$ satisfy the autonomous first-order differential equation

$$\dot{y} = -\tilde{\alpha}(y), \quad y(t_0) = V(t_0, x(t_0)) \geq 0.$$

By Lemma 3.4 (Comparison Lemma) in [31],

$$V(t, x(t)) \leq y(t), \quad \forall t \geq t_0.$$

By Lemma 4.4 in [31], there exists a class KL function $\tilde{\beta}(r,s)$ such that

$$V(t, x(t)) \leq \tilde{\beta}(V(t_0, x(t_0)), t - t_0), \quad \forall t \geq t_0.$$

Therefore, the solution satisfies the equality

$$\begin{aligned}\|x(t)\| &\leq \underline{\alpha}^{-1}(V(t,x)) \leq \underline{\alpha}^{-1} \circ \tilde{\beta}(V(t_0, x(t_0)), t - t_0) \\ &\leq \underline{\alpha}^{-1} \circ \tilde{\beta}(\overline{\alpha}(\|x(t_0)\|), t - t_0) = \beta(\|x(t_0)\|, t - t_0) \quad \forall t \geq t_0\end{aligned}$$

where β is a class KL function, which has been proven in Lemma 4.2 [31]. According to the property of KL functions, there exists a finite time T' such that $\beta(\|x(t_0)\|, t-t_0) \leq \overline{\alpha}^{-1}(\eta)$ for $t \geq T'$, implying that $\|x(t)\| \leq \overline{\alpha}^{-1}(\eta)$. Hence, $V(t,x) \leq \eta$, implying that the solution stays inside $\Omega_{t,\eta}$ for all $t \geq T'$. Since T is the first time when the solution enters $\Omega_{t,\eta}$, we have $T \leq T'$. Since T is finite, this completes claim (ii).

Claim (ii) implies that $x(t) \in \Omega_{t,\eta}$ for all $t \geq T$. That is

$$\|x(t)\| \leq \underline{\alpha}^{-1}(V(t,x)) \leq \underline{\alpha}^{-1}(\eta) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(M) = \gamma(M), \quad \forall t \geq T. \tag{4.25}$$

And claim (i) implies that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \forall t_0 \leq t \leq T. \tag{4.26}$$

As a result,

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,\infty)}\|)\} \quad \forall t \geq t_0.$$

Since the solution $x(t)$ only depends on $u(\tau)$ for $t_0 \leq \tau \leq t$, taking the supremum $\|u_{[t_0,\infty)}\|$ over $[t_0, t]$ yields

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\} \quad \forall t \geq t_0.$$

Necessity : The proof of this part will be conducted in a similar way as that of Theorem 1 in [59]. And the proof will be based on the following implications:

(4.1) is ISS $\Rightarrow$ (4.1) is *weakly robustly stable* $\Rightarrow$ (4.1) admits an $ISS - Lyapunov$ function.

Assume that system (4.1) is ISS, i.e., there exists a KL class function β and a K function γ such that

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t-t_0), \gamma(\|u_{[t_0,t]}\|)\} \qquad \forall t \geq t_0 \tag{4.27}$$

for any initial state $x(t_0) \in \mathbb{R}^n$, and any input function $u(t) \in L_\infty^m$.

Let $\rho(s) = \beta(s, 0)$ implying that ρ be a K-function. One can always assume that $\rho(s) > s$ for all $s > 0$, ρ and γ be K_∞. It follows that ρ^{-1} is K_∞ and $\rho^{-1}(s) < s$ for all $s > 0$. Now let σ be a K_∞ function satisfying $\sigma(s) < \gamma^{-1}(\frac{1}{b}\rho^{-1}(s))$ for all $s > 0$, where $b > 1$. And one can simply set $\sigma(s) = \frac{1}{c}\gamma^{-1}(\frac{1}{b}\rho^{-1}(s))$ for all $s > 0$, where $1 < c < b$. Lemma 2.11 in [59] shows that there exists a smooth function φ and a K_∞ function ψ such that $\psi(\|x\|) \leq \varphi(x) \leq \sigma(\|x\|)$ for all $x \in \mathbb{R}^n$. For the fixed function φ, one can shows that

$$\dot{x}(t) = f(t, x(t), d(t)\varphi(x(t))) \tag{4.28}$$

is UGAS.

Toward this end, we first show that

$$\gamma(\|d(t)\varphi(x(t))\|) \leq \frac{\|x(t_0)\|}{c} \quad \forall t \geq t_0 \tag{4.29}$$

for any $x(t_0) \in \mathbb{R}^n$ and any $d(t) \in M_D$. It is sufficient to show that

$$\gamma(\varphi(x(t))) \leq \frac{\|x(t_0)\|}{c} \quad \forall t \geq t_0, \tag{4.30}$$

since γ is K_∞ and $\|d(t)\| \leq 1$. For any $x(t_0) \in \mathbb{R}^n$ and $d(t) \in M_D$, since

$$\gamma(\varphi(x(t_0))) \leq \gamma(\sigma(\|x(t_0)\|)) \leq \frac{1}{b}\rho^{-1}(\|x(t_0)\|) \leq \frac{1}{b}\|x(t_0)\|, \tag{4.31}$$

$\gamma(\varphi(x(t))) \leq \frac{1}{b}\|x(t_0)\|$ for all t small enough.

Let $t_1 = \inf\{t > t_0 : \gamma(\varphi(x(t))) > \frac{1}{c}\|x(t_0)\|\}$. Assume that $t_1 < \infty$. Then (4.29) holds for $t \in [t_0, t_1)$, implying that $\gamma(\|d(t)\varphi(x(t))\|) \leq \frac{1}{c}\rho(\|x(t_0)\|)$ for $t \in [t_0, t_1)$. (4.27) gives that $\|x(t)\| \leq \beta(\|x(t_0)\|, 0) = \rho(\|x(t_0)\|)$ for $t_0 \leq t \leq t_1$, which implies that $\gamma(\varphi(x(t_1))) \leq \gamma(\sigma(\|x(t_1)\|)) \leq \frac{1}{b}\rho^{-1}(\|x(t_1)\|) \leq \frac{1}{b}\|x(t_0)\| \leq \frac{1}{c}\|x(t_0)\|$, which contradicts the definition of t_1. Hence $t_1 = \infty$, and then (4.29) holds.

From (4.27) and (4.29), it follows that $\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t-t_0), \frac{1}{c}\|x(t_0)\|\}$ for $x(t_0) \in \mathbb{R}^n$. Since $\beta \in KL$, for each $r > 0$, there exists $T(r) \geq 0$, independent of t_0, such that $\beta(r, t-t_0) \leq \frac{1}{c}r, \quad \forall t \geq t_0 + T(r)$. It holds that for each $r > 0$, there exists some $T(r) \geq 0$, independent of t_0, such that

$$\|x(t)\| \leq \frac{r}{c}, \quad t \geq t_0 + T(r), \quad \|x(t_0)\| \leq r. \tag{4.32}$$

Choose $r > 0$, independent of t_0. For any $\varepsilon > 0$, let k be a positive number such that $c^{-k}r < \varepsilon$. Let $r_1 = r$ and $r_i = \frac{1}{c}r_{i-1}$. Denote $\tau = T(r_1) + T(r_2) + ... + T(r_k)$, which is independent of t_0. For any $t \geq t_0 + \tau$, it follows that $\|x(t)\| \leq c^{-k}r \leq \varepsilon$. This shows the global uniform convergence of the origin.

To show that the origin is uniformly stable, (4.27) and (4.29) give that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, 0) = \rho(\|x(t_0)\|), \quad t \geq t_0, \quad x(t_0) \in \mathbb{R}^n, \quad \forall d(t) \in M_D.$$

For each $\varepsilon > 0$, there exists $\delta \leq \rho^{-1}(\varepsilon)$, independent of t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \leq \varepsilon, \quad t \geq t_0.$$

Then the system (4.28) is UGAS, implying that system (4.1) is *weakly robustly stable*.

Consider the auxiliary system:

$$\begin{aligned} \dot{x}(t) &= f(\lambda(t), x(t), d(t)\varphi(x(t))) \\ \dot{\lambda}(t) &= 1 \\ y &= h(x, \lambda) = x. \end{aligned} \tag{4.33}$$

Since system (4.28) is UGAS, system (4.33) is $state - independent\ uniformly\ output$ *stable* (as a matter of fact, the converse also holds). Applying theorem 3.2 in [59], there exists a smooth Lyapunov function V for system (4.33), that is, a smooth function V so that, for some positive definite K_∞ functions $\underline{\alpha}$, $\overline{\alpha}$ and α, it holds that

$$\underline{\alpha}(\|x\|) \leq V(\lambda, x) = V(t, x) \leq \overline{\alpha}(\|x\|)$$

and,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, d(t)\varphi(x)) \leq -\alpha(V) \tag{4.34}$$

for any initial state $x(t_0) \in \mathbb{R}^n$, $d(t) \in M_D$. Note that from (4.34), it follows that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -\alpha(V) \tag{4.35}$$

where $x(t_0) \in \mathbb{R}^n$ and $\|u\| \leq \varphi(x)$. Let $\chi(r) = \psi^{-1}(r)$, then $V(t, x)$ is an $ISS - Lyapunov$ function for system (4.1). This completes the proof. ∎

Remark 4.3 Inequalities (4.27) and (4.29) show that $x(t)$ is uniformly bound for all $t \geq t_0$ and uniformly ultimately bounded with the ultimate bound $\underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(M) = \gamma(M)$. Since γ is a class K function, as $M \to 0$, the ultimate bound approaches zero. ∎

4.3.2 A Remark on Input-to-State Stability for Time-varying Systems

First, we will introduce a property on ISS.

Definition 4.3 A locally *Lipschitz* K_∞ function ρ is called a $K_\infty - stability\ margin$ for the system (4.1) if the system

$$\dot{x}(t) = f(t, x(t), d(t)\rho(\|x(t)\|)) \tag{4.36}$$

is UGAS for all $d(t) \in M_D$, that is, for some $\beta \in KL$, it holds that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad t \geq t_0$$

for all trajectories of system (4.36). ∎

Remark 4.4 The concept of $K_\infty - stability\ margin$ for time-varying systems with respect to a closed subset $A \in \mathbb{R}^n$ was introduced in [10]. ∎

Next, we will extend two lemmas on ISS properties for time invariant systems introduced in [59] to the case of time-varying systems.

Lemma 4.1 If system (4.1) admits a $K_\infty - stability\ margin$, then it is *weakly robustly stable.*

Proof: Arguing as Theorem 4.4, for any K_∞ function σ, there exists a smooth function $\varphi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and a K_∞ function ψ such that $\psi(\|x\|) \leq \varphi(x) \leq \sigma(\|x\|)$ $(\forall x \in \mathbb{R}^n)$. Under the particular feedback control law $d(t)\varphi(x)$ which is bounded by σ, the following system

$$\dot{x}(t) = f(t, x(t), d(t)\varphi(x(t)))$$

is UGAS. Arguing as the necessity part of Theorem 4.4, there exists an $ISS - Lyapunov$ function for system (4.1), then (4.1) is *weakly robustly stable.* ∎

Lemma 4.2 If system (4.1) admits an $ISS - Lyapunov$ function, the it also admits a $K_\infty - stability\ margin.$

Proof: Assume that $V(t, x)$ be an $ISS - Lyapunov$ function. Without loss of generality, we assume that $\chi \in K_\infty$. Let $\rho(r) = \chi^{-1}(r)$, then $\rho \in K_\infty$, and

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)f(t, x, u) \leq -\alpha(\|x\|) \leq -\alpha \circ \overline{\alpha}^{-1}(V(t, x))$$

when $\|u\| \leq \rho(\|x\|)$.

It follows that

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x,k(t,x)) \le -\tilde{\alpha}(V(t,x))$$

for any state feedback control law $k(t,x)$ bounded by ρ, where $\tilde{\alpha} = \alpha \circ \overline{\alpha}^{-1}$. Let $y(t)$ satisfy the autonomous first-order differential equation

$$\dot{y} = -\tilde{\alpha}(y), \quad y(t_0) = V(t_0, x(t_0)) \ge 0.$$

By Lemma 3.4 (Comparison Lemma) in [31],

$$V(t, x(t)) \le y(t), \quad \forall t \ge t_0$$

By Lemma 4.4 in [31], there exists a class KL function $\tilde{\beta}(r,s)$ such that

$$V(t,x(t)) \le \tilde{\beta}(V(t_0, x(t_0)), t - t_0), \quad \forall t \ge t_0$$

for every solution $x(t)$ and feedback control law $k(t,x)$ bounded by ρ. Therefore, the solution satisfies the equality

$$\begin{aligned} \|x(t)\| &\le \underline{\alpha}^{-1}(V(t,x)) \le \underline{\alpha}^{-1}(\tilde{\beta}(V(t_0,x(t_0)), t - t_0)) \\ &\le \underline{\alpha}^{-1}(\tilde{\beta}(\overline{\alpha}(\|x(t_0)\|), t - t_0)) = \beta(\|x(t_0)\|, t - t_0) \quad \forall t \ge t_0 \end{aligned}$$

where β is a class KL function, which has been proven in Lemma 4.2 [31].

Thus, system (4.1) admits a $K_\infty - \textit{stability margin}\ \rho$. ∎

According to Lemma 4.1, Lemma 4.2 and Theorem 4.2, the following properties are equivalent for any time-varying system:

1. It is ISS.
2. It admits an $ISS - Lyapunov$ function.
3. It admits a $K_\infty - \textit{stability margin}$.
4. It is *weakly robustly stable*.

Proof:

1.⇒4. See Necessity part of Theorem 4.2.

4.⇒2. See Necessity part of Theorem 4.2.

2.⇒1. See Sufficient part of Theorem 4.2.

2.⇒3. See Lemma 4.2.

3.⇒4. See Lemma 4.1.

4.4 Comparison of Various Small Gain Theorems

4.4.1 Comparison of Theorem 4.1 and Theorem 4.2

In this part, we will show that if the conditions of Theorem 4.2 hold, it is not guaranteed that there exist two gain functions that satisfy the contract mapping of gain functions, $\gamma_1 \circ \gamma_2(r) < r$ $(\forall r > 0)$.

Assume that in systems (4.12) and (4.13), for $i = 1, 2$, the x_i subsystem admits an $ISS - Lyapunov$ function $V_i(t, x_i)$ satisfying (4.15)-(4.16) and (4.17)-(4.18) respectively.

From the estimate of $V_i(t, x_i)$, we obtain that

$$\|x_1\| \geq \underline{\alpha}_1^{-1} \circ \chi_1 \circ \overline{\alpha}_2(\|x_2\|) \;\Rightarrow\; \underline{\alpha}_1(\|x_1\|) \geq \chi_1 \circ \overline{\alpha}_2(\|x_2\|) \Rightarrow V_1(t, x_1) \geq \chi_1(V_2(t, x_2))$$
$$\|x_1\| \geq \underline{\alpha}_1^{-1} \circ \gamma_1(\|u_1\|) \;\Rightarrow\; \underline{\alpha}_1(\|x_1\|) \geq \gamma_1(\|u_1\|) \Rightarrow V_1(t, x_1) \geq \gamma_1(\|u_1\|)$$

According to the sufficiency part of Theorem 4.4, it follows that

$$\|x_1(t)\| \leq \max\{\beta_1(\|x_1(t_0)\|, t - t_0), \gamma_1^x(\|x_{2[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\}$$

for all $t \geq t_0$, where,

$$\gamma_1^x = \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \{\underline{\alpha}_1^{-1} \circ \chi_1 \circ \overline{\alpha}_2\}, \qquad \gamma_1^u = \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \{(\underline{\alpha}_1)^{-1} \circ \gamma_1\}.$$

Similarly, we could obtain that

$$\|x_2\| \geq \underline{\alpha}_2^{-1} \circ \chi_2 \circ \overline{\alpha}_1(\|x_1\|) \;\Rightarrow\; \underline{\alpha}_2(\|x_2\|) \geq \chi_2 \circ \overline{\alpha}_1(\|x_1\|) \Rightarrow V_2(t, x_2) \geq \chi_2(V_1(t, x_1))$$
$$\|x_2\| \geq \underline{\alpha}_2^{-1} \circ \gamma_2(\|u_2\|) \;\Rightarrow\; \underline{\alpha}_2(\|x_2\|) \geq \gamma_2(\|u_2\|) \Rightarrow V_2(t, x_2) \geq \gamma_2(\|u_2\|)$$

According to the sufficiency part of Theorem 4.4, it follows that

$$\|x_2(t)\| \leq \max\{\beta_2(\|x_2(t_0)\|, t - t_0), \gamma_2^x(\|x_{1[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}$$

for all $t \geq t_0$, where,

$$\gamma_2^x = \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \{\underline{\alpha}_2^{-1} \circ \chi_2 \circ \overline{\alpha}_1\}, \qquad \gamma_2^u = \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \{\underline{\alpha}_2^{-1} \circ \gamma_2\}.$$

Then

$$\gamma_1^x \circ \gamma_2^x(r) = \{\underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \underline{\alpha}_1^{-1} \circ \chi_1 \circ \overline{\alpha}_2\} \circ \{\underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \underline{\alpha}_2^{-1} \circ \chi_2 \circ \overline{\alpha}_1\}(r) \qquad (4.37)$$

Under the assumption $\chi_1 \circ \chi_2(r) < r$ $(\forall r > 0)$, we cannot guarantee that $\gamma_1^x \circ \gamma_2^x(r) < r$ $(\forall r > 0)$.

□

In this part, we will show that if the conditions of Theorem 4.1 hold, it is not guaranteed that there exists two $ISS-Lyapunov$ functions that satisfy the contract mapping of $ISS-Lyapunov$ functions $\chi_1 \circ \chi_2(r) < r$ $(\forall r > 0)$.

Assume the conditions of Theorem 4.1 hold. Let $\rho_i(r) = \beta_i(r, 0)$. Arguing as the necessity part of Theorem 4.4, one can always assume that $\rho_i(r) > r$ for all $r > 0$ and ρ_i, γ_i^x be K_∞. It follows that ρ_i^{-1} is K_∞ and $\rho_i^{-1}(r) < r$ for all $r > 0$. Now let $\sigma_1^{(i)}(r)$ and $\sigma_2^{(i)}(r)$ be K_∞ functions such that

$$\sigma_1^{(i)}(r) < (\gamma_i^x)^{-1}(\frac{1}{b_i}\rho_i^{-1}(r)) \qquad \sigma_2^{(i)}(r) < (\gamma_i^u)^{-1}(\frac{1}{\bar{b}_i}\rho_i^{-1}(r))$$

for all $r > 0$, where $b_i > 1$ and $\bar{b}_i > 1$.

There exists smooth functions $\varphi_1^{(i)}(\cdot)$, $\varphi_2^{(i)}(\cdot)$ and K_∞ functions $\psi_1^{(i)}(\cdot)$, $\psi_2^{(i)}(\cdot)$ such that

$$\begin{aligned} \psi_1^{(1)}(\|r\|) \le \varphi_1^{(1)}(r) \le \sigma_1^{(1)}(\|r\|) \qquad \psi_2^{(1)}(\|r\|) \le \varphi_2^{(1)}(r) \le \sigma_2^{(1)}(\|r\|) \\ \psi_1^{(2)}(\|r\|) \le \varphi_1^{(2)}(r) \le \sigma_1^{(2)}(\|r\|) \qquad \psi_2^{(2)}(\|r\|) \le \varphi_2^{(2)}(r) \le \sigma_2^{(2)}(\|r\|). \end{aligned}$$

Then there exists a smooth $Lyapunov$ function $V_i(t, x_i)$ for x_i subsystem, that is, a smooth function $V_i(t, x_i)$ so that, for some positive definite K_∞ functions $\underline{\alpha}_i$, $\overline{\alpha}_i$ and α_i, it holds that $\underline{\alpha}_i(\|x_i\|) \le V_i(t, x_i) \le \overline{\alpha}_i(\|x_i\|)$ and

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1, d(t)\varphi_1^{(1)}(x_1), d(t)\varphi_2^{(1)}(x_1)) \le -\alpha_1(V_1) \tag{4.38}$$
$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_2, d(t)\varphi_1^{(2)}(x_2), d(t)\varphi_2^{(2)}(x_2)) \le -\alpha_2(V_2) \tag{4.39}$$

for $\forall x_i \in \mathbb{R}^{n_i}$, $d(t) \in M_D$. Note that from (4.38) and (4.39), it follows that

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1, x_2, u_1) \le -\alpha_1(V_1) \tag{4.40}$$
$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_1, x_2, u_2) \le -\alpha_2(V_2) \tag{4.41}$$

where $\|x_2\| \le \varphi_1^{(1)}(x_1)$, $\|u_1\| \le \varphi_2^{(1)}(x_1)$ and $\|x_1\| \le \varphi_1^{(2)}(x_2)$, $\|u_2\| \le \varphi_2^{(2)}(x_2)$.

Since,

$$\begin{aligned} & (\underline{\alpha}_2 \circ \psi_1^{(1)} \circ \overline{\alpha}_1^{-1})^{-1}(V_2(t, x_2)) \le V_1(t, x_1) \\ \Rightarrow\ & V_2(t, x_2) \le \underline{\alpha}_2 \circ \psi_1^{(1)} \circ \overline{\alpha}_1^{-1}(V_1(t, x_1)) \\ \Rightarrow\ & \underline{\alpha}_2(\|x_2\|) \le \underline{\alpha}_2 \circ \psi_1^{(1)} \circ \overline{\alpha}_1^{-1}(V_1(t, x_1)) \\ \Rightarrow\ & \|x_2\| \le \psi_1^{(1)} \circ \overline{\alpha}_1^{-1}(V_1(t, x_1)) \le \psi_1^{(1)}(\|x_1\|) \le \varphi_1^{(1)}(x_1), \end{aligned}$$

$$(\psi_2^{(1)} \circ \overline{\alpha}_1^{-1})^{-1}(\|u_1\|) \leq V_1(t, x_1) \Rightarrow \|u_1\| \leq \psi_2^{(1)} \circ \overline{\alpha}_1^{-1}(V_1(t, x_1)) \leq \psi_2^{(1)}(\|x_1\|)$$

and

$$\begin{aligned}
& (\underline{\alpha}_1 \circ \psi_1^{(2)} \circ \overline{\alpha}_2^{-1})^{-1}(V_1(t, x_1)) \leq V_2(t, x_2) \\
\Rightarrow \quad & V_1(t, x_1) \leq \underline{\alpha}_1 \circ \psi_1^{(2)} \circ \overline{\alpha}_2^{-1}(V_2(t, x_2)) \\
\Rightarrow \quad & \underline{\alpha}_1(\|x_1\|) \leq \underline{\alpha}_1 \circ \psi_1^{(2)} \circ \overline{\alpha}_2^{-1}(V_2(t, x_2)) \\
\Rightarrow \quad & \|x_1\| \leq \psi_1^{(2)} \circ \overline{\alpha}_2^{-1}(V_2(t, x_2)) \leq \psi_1^{(2)}(\|x_2\|) \leq \varphi_1^{(2)}(x_2),
\end{aligned}$$

$$(\psi_2^{(2)} \circ \overline{\alpha}_2^{-1})^{-1}(\|u_2\|) \leq V_2(t, x_2) \Rightarrow \|u_2\| \leq \psi_2^{(2)} \circ \overline{\alpha}_2^{-1}(V_2(t, x_2)) \leq \psi_2^{(2)}(\|x_2\|)$$

let, $\chi_1(r) = \overline{\alpha}_1 \circ (\psi_1^{(1)})^{-1} \circ \underline{\alpha}_2^{-1}(r) \qquad \chi_2(r) = \overline{\alpha}_2 \circ (\psi_1^{(2)})^{-1} \circ \underline{\alpha}_1^{-1}(r)$
and $\gamma_1 = \overline{\alpha}_1 \circ (\psi_2^{(1)})^{-1} \qquad \gamma_2 = \overline{\alpha}_2 \circ (\psi_2^{(2)})^{-1}$

Hence, if $V_1(t, x_1) \geq \max\{\chi_1(V_2(t, x_2)), \gamma_1(\|u_1\|)\}$, (4.40) holds, implying that $V_1(t, x_1)$ is an $ISS - Lyapunov$ function for x_1 subsystem.

Similarly, if $V_2(t, x_2) \geq \max\{\chi_2(V_1(t, x_1)), \gamma_2(\|u_2\|)\}$, (4.41) holds, implying that $V_2(t, x_2)$ is an $ISS - Lyapunov$ function for x_2 subsystem.

If $\gamma_1^x \circ \gamma_2^x(r) < r \ (\forall r > 0)$, then

$$\chi_1 \circ \chi_2(r) = \{\overline{\alpha}_1 \circ (\psi_1^{(1)})^{-1} \circ \underline{\alpha}_2^{-1}\} \circ \{\overline{\alpha}_2 \circ (\psi_1^{(2)})^{-1} \circ \underline{\alpha}_1^{-1}\}(r) \tag{4.42}$$

Under the assumption $\gamma_1^x \circ \gamma_2^x(r) < r \ (\forall r > 0)$, we cannot guarantee that $\chi_1 \circ \chi_2(r) < r$ $(\forall r > 0)$.

□

Example 4.1 Consider the following system

$$\dot{x}_1 = -x_1 + g(t)x_2 + u \tag{4.43}$$

$$\dot{x}_2 = x_1 - x_2 + u \tag{4.44}$$

where $g(t)$ is continuously differentiable and satisfies $0 \leq g(t) \leq k$ and $\dot{g}(t) \leq g(t)$, $\forall t \geq 0$.

Taking $V_1(t, x_1) = \frac{1}{2}x_1^2$ as an $ISS - Lyapunov$ function candidate for x_1 subsystem, it can be seen that

$$\begin{aligned}
\|x_1\| \geq \max\{\frac{3}{2}k\|x_2\|, 4\|u\|\} \Rightarrow \dot{V}_1 &= x_1\dot{x}_1 \\
&= -x_1^2 + g(t)x_2x_1 + ux_1 \\
&\leq -x_1^2 + kx_2x_1 + ux_1 \\
&\leq -x_1^2 + \frac{2}{3}x_1^2 + \frac{1}{4}x_1^2 \\
&= -\frac{1}{12}x_1^2
\end{aligned}$$

$$\begin{aligned} V_1(t,x_1) &\geq \max\{\frac{9}{8}k^2V_2(t,x_2), 8u^2\} = \max\{\chi_1(V_2(t,x_2)), \gamma_1(\|u\|)\} \\ &\Rightarrow \|x_1\| \geq \max\{\frac{3}{2}k\|x_2\|, 4\|u\|\} \Rightarrow \dot{V}_1 \leq -\frac{1}{12}x_1^2 = -\alpha_1(\|x_1\|) \end{aligned}$$

Taking $x_2^2 \leq V_2(t,x_2) = [1+g(t)]x_2^2 \leq (1+k)x_2^2$ as an $ISS - Lyapunov$ function candidate for x_2 subsystem, it can be seen that

$$\begin{aligned} \|x_2\| \geq \max\{(2+k)\|x_1\|, 2\|u\|\} \Rightarrow \dot{V}_2 &= 2[1+g(t)]x_2\dot{x}_2 + \dot{g}(t)x_2^2 \\ &= 2[1+g(t)]x_2(x_1 - x_2 + u) + \dot{g}(t)x_2^2 \\ &= -[-\dot{g}(t) + g(t) + 2]x_2^2 + [2+g(t)]x_1x_2 + ux_2 \\ &\leq -2x_2^2 + (2+k)x_1x_2 + ux_2 \\ &\leq -2x_2^2 + (2+k)\frac{1}{2+k}x_2^2 + \frac{1}{2}x_2^2 \\ &= -\frac{1}{2}x_2^2 \end{aligned}$$

$$\begin{aligned} V_2(t,x_2) &\geq \max\{2(2+k)^2(1+k)V_1(t,x_1), 4(1+k)u^2\} = \max\{\chi_2(V_2(t,x_2)), \gamma_2(\|u\|)\} \\ &\Rightarrow \|x_2\| \geq \max\{(2+k)\|x_1\|, 2\|u\|\} \Rightarrow \dot{V}_2 \leq -\frac{1}{2}x_2^2 = -\alpha_2(\|x_2\|) \end{aligned}$$

Then, if $k = 0.26$,

$$\begin{aligned} \chi_1 \circ \chi_2(r) &= \frac{9}{8}k^2 \cdot 2(2+k)^2(1+k) \cdot r = 0.9789r < r \\ \gamma_1^x \circ \gamma_2^x(r) &= \{(\underline{\alpha}_1)^{-1} \circ \overline{\alpha}_1 \circ \underline{\alpha}_1^{-1} \circ \chi_1 \circ \overline{\alpha}_2\} \circ \{(\underline{\alpha}_2)^{-1} \circ \overline{\alpha}_2 \circ \underline{\alpha}_2^{-1} \circ \chi_2 \circ \overline{\alpha}_1\}(r) \\ &= 1.0987r > r \end{aligned} \quad (4.45)$$

However, in the case when $k = 0.25$, $\chi_1 \circ \chi_2(r) = 0.8899r < r$ and $\gamma_1^x \circ \gamma_2^x(r) = 0.9831r < r$. ∎

Example 4.2 Consider the following inter-connected system,

$$\dot{x}_1 = -x_1 + \frac{1}{3}\sin t \cdot x_2 + \frac{1}{3}\sin t \cdot u \quad (4.46)$$

$$\dot{x}_2 = \frac{1}{3}\sin t \cdot x_1 - x_2 + \frac{1}{3}\sin t \cdot u \quad (4.47)$$

The solution of subsystem (4.46) can be obtained as follow,

$$\begin{aligned} x_1(t) &= \exp^{(-(t-t_0))} x_1(t_0) + \int_{t_0}^{t} \frac{1}{3}\exp^{(-(t-\tau))}(\sin\tau \cdot x_2(\tau) + u(\tau))d\tau \\ &\leq \exp^{(-(t-t_0))} x_1(t_0) + \int_{t_0}^{t} \frac{1}{3}\exp^{(-(t-\tau))}(\|x_{2[t_0,\infty)}\| + \|u_{[t_0,\infty]}\|)d\tau \\ &= \exp^{(-(t-t_0))} x_1(t_0) + \frac{1}{3}(\|x_{2[t_0,\infty)}\| + \|u_{[t_0,\infty]}\|)\int_{t_0}^{t}\exp^{(-(t-\tau))}\sin\tau \cdot d\tau \\ &= \exp^{(-(t-t_0))} x_1(t_0) + \frac{1}{3}(\|x_{2[t_0,\infty)}\| + \|u_{[t_0,\infty]}\|)(1 - \exp^{t_0 - t}) \\ &\leq \exp^{(-(t-t_0))} x_1(t_0) + \frac{1}{3}\|x_{2[t_0,\infty)}\| + \frac{1}{3}\|u_{[t_0,\infty)}\| \end{aligned} \quad (4.48)$$

Similarly,

$$x_2(t) \leq \exp^{(-(t-t_0))} x_2(t_0) + \frac{1}{3}\|x_{1[t_0,\infty)}\| + \frac{1}{3}\|u_{[t_0,\infty)}\| \tag{4.49}$$

(4.48) and (4.49) give that (4.46) and (4.47) are ISS respectively, i.e.,

$$\|x_1(t)\| \leq \max\{\beta_1(\|x_1(t_0)\|, t-t_0), \gamma_1^x(\|x_{2[t_0,\infty)}\|), \gamma_1^u(\|u_{[t_0,\infty)}\|)\} \tag{4.50}$$

$$\|x_2(t)\| \leq \max\{\beta_2(\|x_2(t_0)\|, t-t_0), \gamma_2^x(\|x_{1[t_0,\infty)}\|), \gamma_2^u(\|u_{[t_0,\infty)}\|)\} \tag{4.51}$$

where, $\beta_1(s,r) = \beta_2(s,r) = 2 \cdot e^{-r} \cdot s$ and $\gamma_1^x(r) = \gamma_2^x(r) = \gamma_1^u(r) = \gamma_2^u(r) = \frac{2}{3}r$.

Let $\rho_1(s) = \max\{2\beta_1(s,0), 2s\} = \max\{2e^{-s}, 2s\}$ implying that ρ_1 is K_∞ and $\rho_1(s) > s$ for all $s > 0$. It follows that $\rho_1^{-1}(s) = \min\{0.5e^s, 0.5s\}$ is K_∞ and $\rho_1(s) < s$ for all $s > 0$. Let $\sigma_1(s) = 0.5(\gamma_1^x)^{-1} 0.25\rho_1^{-1}(s) = 0.1875 \min\{0.5e^s, 0.5s\}$. Hence there exists a smooth function $\varphi_1(s) = 0.1\min\{0.5e^{\|s\|}, 0.5\|s\|\}$ and a K_∞ function $\psi_1(s) = 0.1\min\{0.5e^s, 0.5s\}$ such that

$$\psi_1(\|x_1\|) \leq \varphi_1(x_1) \leq \sigma_1(\|x_1\|) \tag{4.52}$$

for all $x_1 \in \mathbb{R}$.

By appealing to Theorem 4.4, there exists a smooth Lyapunov function $V_1(t, x_1)$ for system (4.46) such that, for some positive definite K_∞ functions $\underline{\alpha}_1$, $\overline{\alpha}_1$ and α_1, it holds that

$$\underline{\alpha}_1(\|x_1\|) \leq V_1(t, x_1) \leq \overline{\alpha}_1(\|x_1\|)$$

and,

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1, d(t)\varphi_1(x_1)) \leq -\alpha_1(V_1) \tag{4.53}$$

for any initial state $x_1(t_0) \in \mathbb{R}$, $d(t) \in M_D$.

Note that from (4.53), it follows that

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1, u) \leq -\alpha_1(V_1)$$

where $x_1(t_0) \in \mathbb{R}$ and $\|u\| \leq \varphi_1(x_1)$. Let $\chi_1(s) = \psi_1^{-1}(s) = 10\max\{2e^{-s}, 2s\}$, then $V_1(t, x_1)$ is an $ISS - Lyapunov$ function for system (4.46).

By symmetry of subsystems x_1 and x_2, it follows that there exists there exists a smooth Lyapunov function $V_2(t, x_2)$ for system (4.47) such that, for some positive definite K_∞ functions $\underline{\alpha}_2$, $\overline{\alpha}_2$ and α_2, it holds that

$$\underline{\alpha}_2(\|x_2\|) \leq V_2(t, x_2) \leq \overline{\alpha}_2(\|x_2\|)$$

and

$$\|x_2\| \geq \chi_2(\|u\|) \Rightarrow \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_2, u) \leq -\alpha_2(V_2)$$

where $\chi_2(s) = 10\max\{2e^{-s}, 2s\}$.

Observe that although $\gamma_1^x \circ \gamma_2^x(s) = 4s/9 < s$ for all $s > 0$, $\chi_1 \circ \chi_2(s) > s$ for all $s > 0$.

∎

4.4.2 Comparison of Theorem 4.1 and Theorem 4.3, Theorem 4.2 and Theorem 4.3

Consider the following interconnected system,

$$\dot{x}_1 = f_1(t, x_1, v_1, u_1) \quad (4.54)$$

$$\dot{x}_2 = f_2(t, x_2, v_2, u_2) \quad (4.55)$$

subject to the inter-connection constraints:

$$v_1 = x_2, \quad v_2 = x_1. \quad (4.56)$$

Suppose that there exists C^1 functions $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_i} \mapsto \mathbb{R}_{\geq 0}$, $i = 1, 2$ such that for all $x_i \in \mathbb{R}^{n_i}$ and $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}
\underline{\alpha}_i(\|x_i\|) &\leq V_i(t, x_i) \leq \overline{\alpha}_i(\|x_i\|) \\
\frac{\partial V_1}{\partial x_1} f_1(x_1, x_2, u_1) &\leq -\alpha_1(\|x_1\|) + \gamma_1^x(\|x_2\|) + \gamma_1^u(\|u_1\|) \quad (4.57) \\
\frac{\partial V_2}{\partial x_2} f_2(x_1, x_2, u_2) &\leq -\alpha_2(\|x_2\|) + \gamma_2^x(\|x_1\|) + \gamma_2^u(\|u_2\|) \quad (4.58)
\end{aligned}$$

are satisfied with some $\underline{\alpha}_i$, $\overline{\alpha}_i$, α_i , γ_i^x and $\gamma_i^u \in K_\infty$. Following the similar steps as Lemma 10.4.2 [22], it holds that

$$\begin{aligned}
&\|x_1\| \geq \max\{\chi_1(\|x_2\|), \gamma_1(\|u_1\|)\} \\
&\Rightarrow \frac{\partial V_1}{\partial x_1} f_1(t, x_1, x_2, u_1) \leq -(1 - \frac{1}{\tau_1^x} - \frac{1}{\tau_1^u})\alpha_1(\|x_1\|) \quad (4.59) \\
&\|x_2\| \geq \max\{\chi_2(\|x_1\|), \gamma_2(\|u_2\|)\} \\
&\Rightarrow \frac{\partial V_2}{\partial x_2} f_2(t, x_1, x_2, u_2) \leq -(1 - \frac{1}{\tau_2^x} - \frac{1}{\tau_2^u})\alpha_2(\|x_2\|) \quad (4.60)
\end{aligned}$$

where,

$$\chi_1(r) = \alpha_1^{-1}(\tau_1^x \cdot \gamma_1^x(r)), \qquad \gamma_1(r) = \alpha_1^{-1}(\tau_1^u \cdot \gamma_1^u(r))$$

$$\chi_2(r) = \alpha_2^{-1}(\tau_2^x \cdot \gamma_2^x(r)), \qquad \gamma_2(r) = \alpha_2^{-1}(\tau_2^u \cdot \gamma_2^u(r))$$

with $\tau_i^x > 1$ and $\tau_i^u > 1$. Then (4.7) and (4.8) are satisfied with

$$\tilde{\gamma}_1^x(r) = \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \chi_1(r) \qquad \tilde{\gamma}_2^x(r) = \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \chi_2(r). \tag{4.61}$$

If the small gain condition (4.22) holds, i.e., there exists $\tau_i^x > 1$, $i = 1, 2$ such that

$$\underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_1^x \cdot \gamma_1^x \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ \tau_2^x \cdot \gamma_2^x(s) \;<\; s, \qquad \forall s \in (0, \infty) \tag{4.62}$$

$$(\tau_1^x - 1)(\tau_2^x - 1) \;>\; 1, \tag{4.63}$$

the contract mapping of gain functions $\tilde{\gamma}_1^x \circ \tilde{\gamma}_2^x(r) < r$ also holds, which means that Theorem 4.3 implies Theorem 4.1. However, the converse is not always true.

□

Consider the systems (4.54) and (4.55). Suppose that there exists C^1 functions $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_i} \mapsto \mathbb{R}_{\geq 0}$, $i = 1, 2$ such that (4.57) and (4.58) hold with some $\underline{\alpha}_i$, $\overline{\alpha}_i$, α_i , γ_i^x and $\gamma_i^u \in K_\infty$. Denote that

$$\chi_1(r) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_1^x \circ \gamma_1^x \circ \underline{\alpha}_2^{-1}(r), \qquad \gamma_1(r) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_1^u \circ \gamma_1^u(r)$$

$$\chi_2(r) = \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \tau_2^x \circ \gamma_2^x \circ \underline{\alpha}_1^{-1}(r), \qquad \gamma_2(r) = \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \tau_2^u \circ \gamma_2^u(r)$$

with $\tau_i^x > 1$ and $\tau_i^u > 1$ for $i = 1, 2$.

It follows that

$$\begin{aligned}
& V_1(x_1) \geq \max\{\chi_1(V_2(x_2)), \gamma_1(\|u_1\|)\} \\
& \Rightarrow \bar{\alpha}_1(\|x_1\|) \geq \max\{\chi_1 \circ \underline{\alpha}_2(\|x_2\|), \gamma_1(\|u_1\|)\} \\
& \Rightarrow \|x_1\| \geq \max\{\alpha_1^{-1} \circ \tau_1^x \circ \gamma_1^x(\|x_2\|), \alpha_1^{-1} \circ \tau_1^u \circ \gamma_1^u(\|u_1\|)\} \\
& \Rightarrow \frac{\partial V_1}{\partial x_1} f_1(x_1, x_2, u_1) \leq -(1 - \frac{1}{\tau_1^x} - \frac{1}{\tau_1^u})\alpha_1(\|x_1\|), \\
& V_2(x_2) \geq \max\{\chi_2(V_1(x_1)), \gamma_2(\|u_2\|)\} \\
& \Rightarrow \bar{\alpha}_2(\|x_2\|) \geq \max\{\chi_2 \circ \underline{\alpha}_1(\|x_1\|), \gamma_2(\|u_2\|)\} \\
& \Rightarrow \|x_2\| \geq \max\{\alpha_2^{-1} \circ \tau_2^x \circ \gamma_2^x(\|x_1\|), \alpha_2^{-1} \circ \tau_2^u \circ \gamma_2^u(\|u_2\|)\} \\
& \Rightarrow \frac{\partial V_2}{\partial x_2} f_2(x_2, x_1, u_2) \leq -(1 - \frac{1}{\tau_2^x} - \frac{1}{\tau_2^u})\alpha_2(\|x_2\|).
\end{aligned}$$

If the small gain condition (4.22) holds, i.e., there exists $\tau_i^x > 1$, $i = 1, 2$ such that

$$\underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_1^x \cdot \gamma_1^x \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ \tau_2^x \cdot \gamma_2^x(s) \;<\; s, \qquad \forall s \in (0, \infty) \tag{4.64}$$

$$(\tau_1^x - 1)(\tau_2^x - 1) \;>\; 1 \tag{4.65}$$

the contract mapping of *ISS-Lyapunov* functions $\chi_1 \circ \chi_2(r) < r$ also holds, which means that Theorem 4.3 implies Theorem 4.2. However, the converse is not always true.

□

4.5 Two Small Gain Theorems for Time-varying Systems

In [59], the definitions of IOS, OL-IOS and SIIOS for time invariant systems were proposed. And IOS and OL-IOS small gain theorem for time invariant systems were given respectively in [28] and [40]. Following the same lines as Theorem 2.1 [8], we can obtain the following small gain theorem related to SIIOS for time-varying systems.

First, we extend the concept of SIIOS to *robust state − independent input − to − output stability* (RSIIOS).

Definition 4.4 Consider the following system

$$\dot{x} = f(t, x, u, d) \qquad y = h(t, x, u, d). \tag{4.66}$$

System (4.66) is RSIIOS, if there exists KL function β and K function γ such that

$$\|y(t)\| \leq \max\{\beta(\|y(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}. \tag{4.67}$$

∎

Consider the inter-connection of the following two systems

$$\dot{x}_1 = f_1(t, x_1, v_1, u_1, d), \quad y_1 = h_1(t, x_1, v_1, u_1, d) \tag{4.68}$$

$$\dot{x}_2 = f_2(t, x_2, v_2, u_2, d), \quad y_2 = h_2(t, x_2, v_2, u_2, d) \tag{4.69}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $v_i \in \mathbb{R}^{q_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{p_i}$ with $p_1 = q_2$ and $p_2 = q_1$.

Theorem 4.5 Assume that subsystem (4.68) is RSIIOS viewing x_1 as state and $\text{col}(v_1, u)$ as input and subsystem (4.69) is RSIIOS viewing x_2 as state and $\text{col}(v_2, u)$ as input, i.e., there exists class KL functions β_1 and β_2, class K functions γ_1^x, γ_2^x, γ_1^u, γ_2^u such that, for any $x_1(t_0) \in \mathbb{R}^{n_1}$, $\text{col}(v_1, u) \in L_\infty^{q_1+m}$, the output of (4.68) exists and satisfies, for all $t \geq t_0 \geq 0$,

$$\|y_1(t)\| \leq \max\{\beta_1(\|y_1(t_0)\|, t - t_0), \gamma_1^y(\|v_{1[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\} \tag{4.70}$$

and for any $x_2(t_0) \in \mathbb{R}^{n_2}$, $\text{col}(v_2, u) \in L_\infty^{q_2+m}$, the output of (4.69) exists and satisfies, for all $t \geq t_0 \geq 0$,

$$\|y_2(t)\| \leq \max\{\beta_2(\|y_2(t_0)\|, t - t_0), \gamma_2^y(\|v_{2[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}. \tag{4.71}$$

Further assume

$$\gamma_1^y \circ \gamma_2^y(r) < r, \qquad \forall r > 0 \tag{4.72}$$

then the system composed of (4.68) and (4.69) is RSIIOS viewing $x = \text{col}(x_1, x_2)$ as state, $u = \text{col}(u_1, u_2)$ as input and $y = \text{col}(y_1, y_2)$ as output, in particular,

$$\|y(t)\| \leq \max\{\beta(\|y(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\} \qquad \forall t \geq 0 \tag{4.73}$$

for some class KL class function β and any class K function satisfying

$$\gamma(r) \geq \max\{2\gamma_1^y \circ \gamma_2^u(r), 2\gamma_1^u(r), 2\gamma_2^y \circ \gamma_1^u(r), 2\gamma_2^u(r)\} \qquad \forall r \geq 0. \tag{4.74}$$

∎

Remark 4.5 Since SIIOS is independent of the initial state $x(t_0)$, it is redundant to require that subsystems (4.68) and (4.69) are UO respectively, which is essential to IOS small gain theorem [9, 60]. ∎

Before giving Theorem 4.6, we extend the concept of OL-IOS to *robust output Lagrange input − to − output stability* (ROL-IOS).

Definition 4.5 System (4.66) is ROL-IOS, if there exists KL function β and K functions γ, σ_1 and σ_2 such that

$$\|y(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}, \tag{4.75}$$

$$\|y(t)\| \leq \max\{\sigma_1(\|y(t_0)\|), \sigma_2(\|u_{[t_0,t]}\|)\}. \tag{4.76}$$

∎

Remark 4.6 System (4.66) is RIOS if (4.75) holds. System (4.66) is *robust output Lagrange stable* (ROL) if (4.76) holds. Hence, ROL-IOS is equivalent to the conjunction of RIOS and ROL. ∎

Theorem 4.6 Assume that subsystem (4.68) is ROL-IOS viewing x_1 as state and $\text{col}(v_1, u)$ as input and subsystem (4.69) is ROL-IOS viewing x_2 as state and $\text{col}(v_2, u)$ as input, i.e., there exists class KL functions β_1 and β_2, class K functions γ_1^x, γ_2^x, γ_1^u, γ_2^u such that, for any $x_1(t_0) \in \mathbb{R}^{n_1}$, $\text{col}(v_1, u) \in L_\infty^{q_1+m}$, the output of (4.68) exists and satisfies, for all $t \geq t_0 \geq 0$,

$$\|y_1(t)\| \leq \max\{\beta_1(\|x_1(t_0)\|, t - t_0), \gamma_1^y(\|v_{1[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\}, \tag{4.77}$$

$$\|y_1(t)\| \leq \max\{\sigma_1^o(\|y_1(t_0)\|), \gamma_1^y(\|v_{1[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\}, \tag{4.78}$$

and for any $x_2(t_0) \in \mathbb{R}^{n_2}$, $\text{col}(v_2, u) \in L_\infty^{q_2+m}$, the output of (4.69) exists and satisfies, for all $t \geq t_0 \geq 0$,

$$\|y_2(t)\| \leq \max\{\beta_2(\|x_2(t_0)\|, t - t_0), \gamma_2^y(\|v_{2[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}, \tag{4.79}$$
$$\|y_2(t)\| \leq \max\{\sigma_2^o(\|y_2(t_0)\|), \gamma_2^y(\|v_{2[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}. \tag{4.80}$$

Further assume that subsystems (4.68) and (4.69) have the RUO property. Then under the assumption

$$\gamma_1^y \circ \gamma_2^y(r) < r, \qquad \forall r > 0 \tag{4.81}$$

the system composed of (4.68) and (4.69) is ROL-IOS viewing $x = \text{col}(x_1, x_2)$ as state, $u = \text{col}(u_1, u_2)$ as input and $y = \text{col}(y_1, y_2)$ as output, in particular,

$$\|y(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}, \tag{4.82}$$
$$\|y(t)\| \leq \max\{\gamma_1(\|y(t_0)\|), \gamma_2(\|u_{[t_0,t]}\|)\} \qquad \forall t \geq t_0 \tag{4.83}$$

for some class KL class function β and any class K function satisfying

$$\begin{aligned}
\gamma(r) &= \max\{2\gamma_1^y \circ \gamma_2^u(r), 2\gamma_1^u(r), 2\gamma_2^y \circ \gamma_1^u(r), 2\gamma_2^u(r)\} \\
\gamma_1(r) &= \max\{2\sigma_1^o(r), 2\sigma_2^o(r), 2\gamma_1^y \circ \sigma_2^o(r), 2\gamma_2^y \circ \sigma_1^o(r), 2\gamma_1^y \circ \gamma_2^y \circ \sigma_1^o(r), 2\gamma_2^y \circ \gamma_1^y \circ \sigma_2^o(r)\} \\
\gamma_2(r) &= \max\{2\gamma_1^y \circ \gamma_2^y \circ \gamma_1^u(r), 2\gamma_2^y \circ \gamma_1^y \circ \gamma_2^u(r), 2\gamma_1^y \circ \gamma_2^u(r), 2\gamma_2^y \circ \gamma_1^u(r), 2\gamma_1^u(r), 2\gamma_2^u(r)\}.
\end{aligned}$$

Proof: The system composed of (4.68) and (4.69) is RIOS has been proven in [9]. Next we will show that ROL also holds.

(4.77) gives that

$$\|y_{1[t_0,t]}\| \leq \max\{\sigma_1^o(\|y_1(0)\|), \gamma_1^y(\|y_{2[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\}. \tag{4.84}$$

Substituting (4.84) into (4.78) gives that

$$\begin{aligned}
\|y_2(t)\| \leq \max\{&\sigma_2^o(\|y_2(t_0)\|), \gamma_2^y \circ \sigma_1^o(\|y_1(0)\|), \gamma_2^y \circ \gamma_1^y(\|y_{2[t_0,t]}\|), \\
&\gamma_2^y \circ \gamma_1^u(\|u_{1[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}.
\end{aligned} \tag{4.85}$$

Hence,

$$\begin{aligned}
\|y_{2[t_0,t]}\| \leq \max\{&\sigma_2^o(\|y_2(t_0)\|), \gamma_2^y \circ \sigma_1^o(\|y_1(t_0)\|), \gamma_2^y \circ \gamma_1^y(\|y_{2[t_0,t]}\|), \\
&\gamma_2^y \circ \gamma_1^u(\|u_{1[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}.
\end{aligned} \tag{4.86}$$

Since

$$\gamma_1^y \circ \gamma_2^y(r) < r, \qquad \forall r > 0 \tag{4.87}$$

it holds that

$$\|y_{2[t_0,t]}\| \leq \max\{\sigma_2^o(\|y_2(t_0)\|), \gamma_2^y \circ \sigma_1^o(\|y_1(t_0)\|), \gamma_2^y \circ \gamma_1^u(\|u_{1[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}. \quad (4.88)$$

Substituting (4.88) into (4.78) gives that

$$\begin{aligned} \|y_1(t)\| \leq\ & \max\{\sigma_1^o(\|y_1(t_0)\|), \gamma_1^y \circ \sigma_2^o(\|y_2(t_0)\|), \gamma_1^y \circ \gamma_2^y \circ \sigma_1^o(\|y_1(t_0)\|), \\ & \gamma_1^y \circ \gamma_2^y \circ \gamma_1^u(\|u_{1[t_0,t]}\|), \gamma_1^y \circ \gamma_2^u(\|u_{2[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\}. \end{aligned} \quad (4.89)$$

By symmetry of y_1 and y_2, it follows that

$$\begin{aligned} \|y_2(t)\| \leq\ & \max\{\sigma_2^o(\|y_2(t_0)\|), \gamma_2^y \circ \sigma_1^o(\|y_1(t_0)\|), \gamma_2^y \circ \gamma_1^y \circ \sigma_2^o(\|y_2(t_0)\|), \\ & \gamma_2^y \circ \gamma_1^y \circ \gamma_2^u(\|u_{2[t_0,t]}\|), \gamma_2^y \circ \gamma_1^u(\|u_{1[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}. \end{aligned} \quad (4.90)$$

Hence (4.83) holds for

$$\begin{aligned} \gamma_1(r) &= \max\{2\sigma_1^o(r), 2\sigma_2^o(r), 2\gamma_1^y \circ \sigma_2^o(r), 2\gamma_2^y \circ \sigma_1^o(r), 2\gamma_1^y \circ \gamma_2^y \circ \sigma_1^o(r), 2\gamma_2^y \circ \gamma_1^y \circ \sigma_2^o(r)\} \\ \gamma_2(r) &= \max\{2\gamma_1^y \circ \gamma_2^y \circ \gamma_1^u(r), 2\gamma_2^y \circ \gamma_1^y \circ \gamma_2^u(r), 2\gamma_1^y \circ \gamma_2^u(r), 2\gamma_2^y \circ \gamma_1^u(r), 2\gamma_1^u(r), 2\gamma_2^u(r)\}. \end{aligned}$$

∎

4.6 Conclusion

In this chapter, we give the complete proof of the theorem which provides sufficient and necessary conditions of ISS in terms of $ISS-Lyapunov$ functions. And the equivalence of the notions related to ISS for time-varying systems is stated. Moreover, the connection of the two methods: input-output formulation and $ISS-Lyapunov$ functions, is illustrated.

Chapter 5

Semi-global Robust Stabilization for A Class of Feedforward Systems

5.1 Introduction

In this chapter, we will study the robust stabilization of nonlinear systems of the form:

$$
\begin{aligned}
\dot{x}_n &= c_{n-1}x_{n-1} + f_n(\xi_1, x_1, ..., \xi_{n-1}, x_{n-1}, \xi_n, u, d) \\
\dot{\xi}_n &= g_n(\xi_1, x_1, ..., \xi_{n-1}, x_{n-1}, \xi_n, u, d), \\
&\vdots \\
\dot{x}_i &= c_{i-1}x_{i-1} + f_i(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, \xi_i, u, d) \\
\dot{\xi}_i &= g_i(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, \xi_i, u, d), \\
&\vdots \\
\dot{x}_2 &= c_1x_1 + f_1(\xi_1, x_1, \xi_2, u, d) \\
\dot{\xi}_2 &= g_2(\xi_1, x_1, \xi_2, u, d), \\
\dot{x}_1 &= C\xi_1 + Du + f_1(\xi_1, u, d) \\
\dot{\xi}_1 &= A\xi_1 + Bu + g_1(\xi_1, u, d), \quad t \geq t_0 \geq 0
\end{aligned}
\tag{5.1}
$$

where, for $i = 1, .., n$, $x_i \in \mathbb{R}$, $d \in \mathbb{R}^{n_d}$, $\xi_i \in \mathbb{R}^{p_i}$, $u \in \mathbb{R}$, f_i and g_i are globally defined C^1 functions satisfying $f_i(0, \cdots, 0, d) = 0$ and $g_i(0, \cdots, 0, d) = 0$ for $d \in \mathbb{R}^{n_d}$, $c_1, \cdots, c_{n-1}$, A, B, C, D are (unknown) constants or matrices, and $d : [t_0, \infty) \rightarrow \Gamma$ is a piecewise continuous function with its range Γ a compact subset of $\mathbb{R}^{n_d}$. It is noted that system (5.1) contains two classes of well known nonlinear systems as special cases. First, when $p_i = 0$ for $i = 1, \cdots, n$, system (5.1) becomes a subclass of the standard feedforward system studied in [61], and second, when $p_i = 0$ for $i = 2, \cdots, n$ and $p_1 \neq 0$, system (5.1) becomes the feedforward system subject to input unmodeled dynamics as studied in [2]. Thus we can view system (5.1) as nonlinear systems in feedforward form subject to dynamic uncertainty $\xi_1, \xi_2, \cdots, \xi_n$ and static uncertainty d. System (5.1) is interesting on its own, on one hand, because dynamic uncertainty is ubiquitously present in real systems. On the other hand, it is known that the robust output regulation problem for a nonlinear system can be converted into a robust stabilization problem of an appropriately augmented system which can be viewed as the original system subject to dynamic and static uncertainties where the dynamic uncertainty models the internal model [18] and [19]. Thus the stabilization solution of system (5.1) also shed light on the solution of the robust output regulation problem of nonlinear systems in the standard feedforward form.

The objective of this chapter is to design a static partial state feedback control law to semi-globally stabilize the equilibrium of the system. More precisely, given any compact subsets $0 \in X \subset \mathbb{R}^n$ and $0 \in \Xi \subset \mathbb{R}^p$ with $p = p_1 + p_2 \cdots + p_n$, design a control law of the form $u = k(x)$ with $x = \text{col}(x_1, x_2, \cdot, x_n)$ and $k(0) = 0$, such that, for all d, the equilibrium of the closed-loop system is asymptotically stable with $X \times \Xi$ contained in the basin of attraction.

In addition to the fact that system (5.1) contains dynamic uncertainty, there is another difference between the above problem formulation and the one studied in [2] and [61] in that our formulation implies asymptotic disturbance rejection of time-varying exogenous signal d in contrast with the disturbance attenuation in [61].

In comparison with the papers [2] and [61], a special difficulty of our problem arises from the fact that we will not assume that the Jacobian matrix of the closed-loop system at the origin can be made exponentially stable. As a result, we cannot employ the technique of the asymptotic small gain theorem. To overcome this difficulty, we will employ the technique of a small gain theorem with restrictions for uncertain time-varying nonlinear systems. This theorem can guarantee both the uniform stability and asymptotic gain properties with restrictions of two interconnected systems. An advantage of our problem formulation and technique is that we can handle a larger class of systems than those in [2]. In particular, as a bonus of our problem formulation, we can obtain some non-local stabilization result on nonlinear systems whose linearization at the origin has uncontrollable modes in the imaginary axis. Nevertheless, unlike asymptotic small gain theorem employed in [2] or [61], our small gain theorem does not guarantee that the interconnected system is ISS without restriction on the initial state even though the two subsystems are assumed to be ISS without restrictions on the initial states (but with restriction on the inputs). Therefore, our method does not have the capability of global asymptotic stabilization. The best we can do is semi-global stabilization and we have indeed achieved this goal.

5.2 Main result

Like [2] and [61], our approach will utilize saturation functions characterized as follows.

Definition 5.1 A locally *Lipschitz* function $\sigma(\cdot) : \mathbb{R} \to [-\lambda, \lambda]$ is said to be a saturation function with saturation level $\lambda > 0$ if:
1) $\sigma(x) = x$ when $\|x\| \leq \frac{\lambda}{2}$,
2) $\frac{\lambda}{2} \leq sgn(x)\sigma(x) \leq \min\{\|x\|, \lambda\}$ when $\|x\| \geq \frac{\lambda}{2}$. ∎

To state our assumptions, we rewrite, for $i = 2, \cdots, n$,

$$g_i(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, \xi_i, u, d) = \tilde{g}_i(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, \xi_i) + h_i(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, \xi_i, u, d) \quad (5.2)$$

where $\tilde{g}_i(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, \xi_i)$ is linear in $\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, \xi_i$.

A 5.1 For all $d \in \Gamma$, the following equations hold

$$\lim_{\|(\xi_1,u)\|\to 0} \frac{\|f_1(\xi_1, u, d)\|}{\|(\xi_1, u)\|} = 0, \quad \lim_{\|(\xi_1,u)\|\to 0} \frac{\|g_1(\xi_1, u, d)\|}{\|(\xi_1, u)\|} = 0,$$

and for $i = 2, .., n$,

$$\lim_{\|(\xi_1,x_1,...,\xi_i,u)\|\to 0} \frac{\|f_i(\xi_1, x_1, ..., \xi_i, u, d)\|}{\|(\xi_1, x_1, ..., \xi_i, u)\|} = 0, \quad \lim_{\|(\xi_1,x_1,...,\xi_i,u)\|\to 0} \frac{\|h_i(\xi_1, x_1, ..., \xi_i, u, d)\|}{\|(\xi_1, x_1, ..., \xi_i, u)\|} = 0.$$

A 5.2 For $i = 1, ..., n-1$, $c_i^L < c_i < c_i^H$ for some positive numbers c_i^L, c_i^H. And the *dc* gain $\vartheta^L < \vartheta = D - CA^{-1}B < \vartheta^H$ for some positive numbers ϑ^L, ϑ^H, where A is invertible.

A 5.3 ξ_1 subsystem is RUS and satisfies RAG with a linear gain function viewing ξ_1 as state and u as input. For $i = 2, ..., n$, ξ_i subsystem is RUS and satisfies RAG viewing ξ_i as state and $\text{col}(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, u)$ as input.

Proposition 5.1 Let $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^p \to \mathbb{R}$ be smooth function satisfying

$$\lim_{\|(x_1,x_2)\|\to 0} \frac{\|f(x_1, x_2, d(t))\|}{\|(x_1, x_2)\|} = 0 \tag{5.3}$$

for all $x_1 \in X_1$ with X_1 being a compact subset of $\mathbb{R}^{n_1}$, $x_2 \in X_2$ with X_2 being a compact subset of $\mathbb{R}^{n_2}$ and $d(t) \in D$ with D being a compact subset of $\mathbb{R}^p$. Then there exists a nondecreasing, continuous function $\gamma_o(s)$ such that

$$\|f(x_1, x_2, d(t))\| \leq \max\{\gamma_o(\|x_1\|), \gamma_o(\|x_2\|)\}$$

where

$$\lim_{\|s\|\to 0} \frac{\|\gamma_o(s)\|}{\|s\|} = 0.$$

Proof: By Lemma 7.8 of [5], there exist smooth functions $F_1(x_1)$ such that

$$\|f(x_1, x_2, d(t))\| \leq F_1(x_1) + F_2(x_2).$$

Moreover, due to (5.3), we can assume $\lim_{\|x_1\|\to 0} \frac{\|F_1(x_1)\|}{\|x_1\|} = 0$ and $F_2(x_2)$ with $\lim_{\|x_2\|\to 0} \frac{\|F_2(x_2)\|}{\|x_2\|} = 0$. By Taylor Theorem, there exist smooth positive definite function $M_1(x)$ and $M_2(x)$ such that

$$\|F_1(x_1)\| \leq M_1(x_1)\|x_1\|^2.$$

Similarly,

$$\|F_2(x_2)\| \leq M_2(x_2)\|x_2\|^2.$$

It completes the proof. ∎

Lemma 5.1 Consider the following system

$$\begin{aligned} \dot{x} &= C\xi + Du + G_2(\xi, u, d) \\ \dot{\xi} &= A\xi + Bu + G_1(\xi, u, d) \end{aligned} \tag{5.4}$$

where, $x \in \mathbb{R}$, $\xi \in \mathbb{R}^l$, and, for $i = 1, 2$,

$$\lim_{\|(\xi,u)\|\to 0} \frac{\|G_i(\xi, u, d)\|}{\|(\xi, u)\|} = 0. \tag{5.5}$$

Assume the *dc* gain $\vartheta^L < \vartheta = D - CA^{-1}B < \vartheta^H$ for positive numbers ϑ^L, ϑ^H. And the ξ subsystem is RUS and satisfies RAG with restrictions Ξ, Δ_u on $\xi(t_0)$, u respectively and has a linear gain function, i.e., for all d, there exist class K function γ_1^0 and positive numbers N_1^u, Δ_u such that for any $\xi(t_0) \in \Xi$ and any $u(t) \in L_\infty^1$ satisfying $\|u_{[t_0,\infty)}\| < \Delta_u$, the solution $\xi(t)$ exists and satisfies, for all $t \geq t_0$,

$$\|\xi(t)\| \leq \max\{\gamma_1^0(\|\xi(t_0)\|), N_1^u \|u_{[t_0,\infty)}\|\}, \quad \|\xi\|_a \leq N_1^u \|u\|_a. \tag{5.6}$$

Then for any compact set $X \subset \mathbb{R}$, there exists a control law $u = -\sigma(kx - \bar{u})$, where σ is a saturation function with level λ, such that the system

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} C\xi + Du + G_2(\xi, u, d) \\ A\xi + Bu + G_1(\xi, u, d) \end{bmatrix}_{u=-\sigma(kx-\bar{u})} \tag{5.7}$$

is RUS and satisfies RAG with restrictions $X \times \Xi$, $\Delta_{\bar{u}}$ on $\bar{x}(t_0)$, $\bar{u}$ respectively and a linear gain function, i.e., for all d, there exist class K function γ^0 and positive numbers N^u, $\Delta_{\bar{u}}$ such that for any $\bar{x}(t_0) \in X \times \Xi$ and any $\bar{u}(t) \in L_\infty^1$ satisfying $\|\bar{u}_{[t_0,\infty)}\| < \Delta_{\bar{u}}$, the solution $\bar{x}(t)$ exists and satisfies, for all $t \geq t_0$

$$\|\bar{x}(t)\| \leq \max\{\gamma^0(\|\bar{x}(t_0)\|), N^u \|\bar{u}_{[t_0,\infty)}\|\}, \quad \|\bar{x}\|_a \leq N^u \|\bar{u}\|_a.$$

Further, let $(\bar{A}_1, \bar{B}_1)$ be the Jacobian linearization of system (5.7), $\bar{C}_1 = [c_1 \quad 0_{(1\times l)}]$ with $c_1^L < c_1 < c_1^H$ for some positive numbers c_1^L, c_1^H, and $\bar{D}_1 = 0$. Then, $\vartheta_1^L < \bar{\vartheta}_1 = \bar{D}_1 - \bar{C}_1 \bar{A}_1^{-1} \bar{B}_1 < \vartheta_1^H$ for some positive numbers ϑ_1^L, ϑ_1^H.

Remark 5.1 Lemma 5.1 is an extension of Lemma 1 of [2] in that it concerns both US and asymptotic gain property. Also, d of Lemma 1 of [2] is treated as a disturbance to be attenuated while d here is treated as a disturbance to be rejected. ▮

Proof: The spirit of the proof is similar to that of the proof of Lemma 1 in [2]. That is, we need to employ the small gain theorem. For this purpose, introduce the same coordinate transformation $z = x - CA^{-1}\xi$ as in [2] to change system (5.7) into the following:

$$\dot{\bar{x}} = \begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \vartheta u + G(\xi, u, d) \\ A\xi + Bu + G_1(\xi, u, d) \end{bmatrix}_{u=-\sigma(kx-\bar{u})} \tag{5.8}$$

where $G(\xi, u, d) = G_2(\xi, u, d) - CA^{-1}G_1(\xi, u, d)$. Clearly,

$$\lim_{\|(\xi,u)\| \to 0} \frac{\|G(\xi, u, d)\|}{\|(\xi, u)\|} = 0. \tag{5.9}$$

Define $\tilde{\lambda} = \vartheta\lambda$ and $\tilde{k} = \vartheta k$. Then

$$\vartheta u = -\vartheta\sigma(k(x - \frac{\bar{u}}{k})) = -\tilde{\sigma}(\tilde{k}(x - \frac{\bar{u}}{k})) = -\tilde{\sigma}(\tilde{k}(z + CA^{-1}\xi - \frac{\bar{u}}{k})) \tag{5.10}$$

where $\bar{\sigma}(s) = \vartheta\sigma(s/\vartheta)$ is a saturation function with level $\tilde{\lambda}$.

With (5.10), system (5.8) can be viewed as the interconnection

$$v_1 = y_2, \quad v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = y_1 \tag{5.11}$$

of the following two subsystems

$$\Sigma_1 \;:\quad \dot{\xi} = A\xi - B\sigma(kv_1) + G_1(\xi, -\sigma(kv_1), d), \quad y_1 = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} = \begin{bmatrix} CA^{-1}\xi \\ G(\xi, -\sigma(kv_1), d)/\tilde{k} \end{bmatrix},$$

$$\Sigma_2 \;:\quad \dot{z} = -\bar{\sigma}(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})) + \tilde{k}v_{22}, \quad y_2 = z + v_{21} - \frac{\bar{u}}{k}.$$

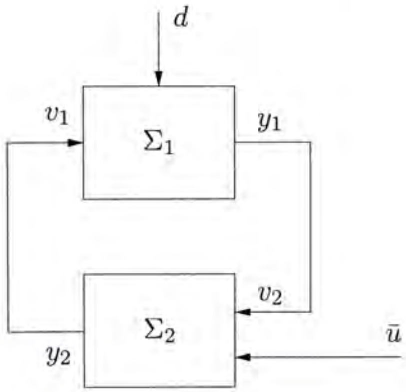

Figure 5.1: Inter-connection of Σ_1 and Σ_2

We will now apply Theorem 3.2 to show that system (5.7) is RUS and satisfies RAG with appropriate restrictions in four steps.

Step 1. Show subsystem Σ_2 is ISS and satisfies ALS with no restrictions on $z(t_0)$ and $\bar{u}$ and with restriction $\Delta_2 = \lambda/(2k)$ on v_2. Let $V(z) = z^2/2$. Then its derivative along the trajectory of Σ_2 satisfies

$$\dot{V} = -(\bar{\sigma}(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})) - \tilde{k}v_{22})z.$$

Under the restriction $\|v_{22[t_0,\infty)}\| < \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$, consider the following three cases:
(1) $\tilde{k}\|z + v_{21} - \frac{\bar{u}}{k}\| \leq \tilde{\lambda}/2$. We have

$$\dot{V} = -(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k}) - \tilde{k}v_{22})z$$

Thus,

$$\|z\| > \max\{3\|v_{21}\|, 3\|\frac{\bar{u}}{k}\|, 3\|v_{22}\|\} \geq \|v_{21}\| + \|\frac{\bar{u}}{k}\| + \|v_{22}\| \Rightarrow \dot{V} \leq -a\|z\|^2$$

for some positive numbers a.

(2) $\tilde{k}\|z + v_{21} - \frac{\bar{u}}{k}\| > \tilde{\lambda}/2$ and $z > 0$: We have
$\tilde{\lambda}/2 \leq sgn(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k}))\tilde{\sigma}(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})) \leq \min\{\|\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})\|, \tilde{\lambda}\}$.
Noting $\|v_{22}\| < \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$ gives

$$\|z\| > \max\{2\|v_{21}\|, 2\|\frac{\bar{u}}{k}\|\} \geq \|v_{21}\| + \|\frac{\bar{u}}{k}\| \Rightarrow \dot{V} \leq -(\frac{\tilde{\lambda}}{2} - \tilde{k}v_{22})z \leq -b\|z\|$$

for some positive numbers b.

(3) $\tilde{k}\|(z + v_{21} - \frac{\bar{u}}{k})\| > \tilde{\lambda}/2$ and $z < 0$: We have
$\tilde{\lambda}/2 \leq sgn(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k}))\tilde{\sigma}(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})) \leq \min\{\|\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})\|, \tilde{\lambda}\}$.
Since $\|v_{22}\| < \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$, it follows that

$$\begin{aligned}
&\|z\| > \max\{2\|v_{21}\|, 2\|\frac{\bar{u}}{k}\|\} \geq \|v_{21}\| + \|\frac{\bar{u}}{k}\| \\
&\Rightarrow -\min\{-\tilde{k}(z + v_{21} - \frac{\bar{u}}{k}), \tilde{\lambda}\} \leq \tilde{\sigma}(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})) \leq -\tilde{\lambda}/2 \\
&\Rightarrow \dot{V} \leq -(-\frac{\tilde{\lambda}}{2} - \tilde{k}v_{22})z \leq -c\|z\|
\end{aligned}$$

for some positive numbers c.

Therefore there exists a class K_∞ function $\alpha(\cdot)$ such that if $\|v_{22[t_0,\infty)}\| < \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$,

$$\|z\| > \max\{3\|v_{21}\|, 3\|\frac{\bar{u}}{k}\|, 3\|v_{22}\|\} \Rightarrow \dot{V} \leq -\alpha(\|z\|).$$

Hence, by Lemma 3.3 [61], $V(z) = z^2/2$ is a local ISS Lyapunov function for subsystem Σ_2. Therefore, for all d, there exists class K function γ_2^0 such that for any $z(t_0) \in \mathrm{Re}$, and any $v_{22}(t) \in L_\infty^1$ satisfying $\|v_{22[t_0,\infty)}\| < \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$, $v_{21}(t) \in L_\infty^1$ and $\bar{u}(t) \in L_\infty^1$, the solution $z(t)$ exists for all $t \geq t_0$ and satisfies

$$\begin{aligned}
&\|z(t)\| \leq \max\{\gamma_2^0(\|z(t_0)\|), 3\|v_{21[t_0,\infty)}\|, 3\|v_{22[t_0,\infty)}\|, \frac{3}{k}\|\bar{u}_{[t_0,\infty)}\|\}, \\
&\|z\|_a \leq \max\{3\|v_{21}\|_a, 3\|v_{22}\|_a, \frac{3}{k}\|\bar{u}\|_a\}.
\end{aligned} \tag{5.12}$$

It follows from (5.12) and $y_2 = z + v_{21} - \frac{\bar{u}}{k}$ that for any $z(t_0) \in \mathrm{Re}$, and any $v_{22}(t) \in L_\infty^1$ satisfying $\|v_{22[t_0,\infty)}\| < \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$, $v_{21}(t) \in L_\infty^1$ and $\bar{u}(t) \in L_\infty^1$, the output $y_2(t)$

exists for all $t \geq t_0$ and satisfies

$$\begin{aligned}
\|y_2(t)\| &\leq \max\{\overline{\gamma}_2^0(\|z(t_0)\|), 9\|v_{21[t_0,\infty)}\|, 9\|v_{22[t_0,\infty)}\|, \frac{9}{k}\|\bar{u}_{[t_0,\infty)}\|\} \\
&\leq \max\{\overline{\gamma}_2^0(\|z(t_0)\|), \overline{\gamma}_2(\|v_{2[t_0,\infty)}\|), \overline{\gamma}_2^u(\|\bar{u}_{[t_0,\infty)}\|)\}, \\
\|y_2\|_a &\leq \max\{9\|v_{21}\|_a, 9\|v_{22}\|_a, \frac{9}{k}\|\bar{u}\|_a\} \\
&\leq \max\{\overline{\gamma}_2(\|v_2\|_a), \overline{\gamma}_2^u(\|\bar{u}\|_a\}),
\end{aligned} \tag{5.13}$$

where $\overline{\gamma}_2^0(s) = 3\gamma_2^0(s)$ and $\overline{\gamma}_2(s) = 9s$. Obviously, Σ_2 is ISS and ALS with restriction $\Delta_2 = \lambda/(2k)$ on v_2.

Step 2. Show subsystem Σ_1 is RUS and satisfies RAG and RALS with restriction Ξ on $\xi(t_0)$ and without restriction on v_1. Choose λ such that $\lambda < \Delta_u$ to guarantee $\|u_{[t_0,\infty)}\| < \Delta_u$. Noting

$$\|\sigma(kv_1)\| \leq \min\{k\|v_1\|, \lambda\} \leq k\|v_1\| \tag{5.14}$$

and substituting (5.14) into (5.6) gives that for any $\xi(t_0) \in \Xi$ and any $v_1(t) \in L_\infty^1$, the solution $\xi(t)$ exists for all $t \geq t_0$ and satisfies

$$\begin{aligned}
\|\xi(t)\| &\leq \max\{\gamma_1^0(\|\xi(t_0)\|), N_1^u\|\sigma(kv_{1[t_0,\infty)})\|\} \leq \max\{\gamma_1^0(\|\xi(t_0)\|), N_1^u k\|v_{1[t_0,\infty)}\|\}, \\
\|\xi\|_a &\leq N_1^u k\|v_1\|_a.
\end{aligned} \tag{5.15}$$

Let l be a positive number such that $\|CA^{-1}\| < l$. Then, for any $\xi(t_0) \in \Xi$ and any $v_1(t) \in L_\infty^1$, the solution $\xi(t)$ exists for all $t \geq t_0$ and satisfies

$$\begin{aligned}
\|y_{11}(t)\| &= \|CA^{-1}\xi(t)\| \\
&\leq \max\{\overline{\gamma}_{11}^0(\|\xi(t_0)\|), lN_1^u k\|v_{1[t_0,\infty)}\|\} \\
&= \max\{\overline{\gamma}_{11}^0(\|\xi(t_0)\|), \overline{\gamma}_{11}(\|v_{1[t_0,\infty)}\|)\} \\
\|y_{11}\|_a &= \|CA^{-1}\xi\|_a \leq lN_1^u k\|v_1\|_a = \overline{\gamma}_{11}(\|v_1\|_a)
\end{aligned} \tag{5.16}$$

where $\overline{\gamma}_{11}^0(s) = l\gamma_1^0(s)$.

Next consider y_{12}. It follows from (5.9) that there exists a nondecreasing, continuous function $\gamma_o(s)$ such that

$$\|G(\xi(t), -\sigma(kv_1(t), d)\| \leq \max\{\gamma_o(\|\xi(t)\|), \gamma_o(\|\sigma(kv_1(t))\|)\}.$$

where

$$\lim_{\|s\|\to 0} \frac{\|\gamma_o(s)\|}{\|s\|} = 0. \tag{5.17}$$

Using (5.6) and (5.14) gives that for any $\xi(t_0) \in \Xi$ and any $v_1(t) \in L^1_\infty$, the following estimates hold

$$\begin{aligned}
\|y_{12}(t)\| &\le \max\{\gamma_o \circ \gamma_1^0(\|\xi(t_0)\|)/\tilde{k}, \gamma_o(N_1^u\|\sigma(kv_{1[t_0,\infty)})\|)/\tilde{k}, \gamma_o(\|\sigma(kv_{1[t_0,\infty)})\|)/\tilde{k}\} \\
&\le \max\{\overline{\gamma}_{12}^0(\|\xi(t_0)\|), \overline{\gamma}_o(\|\sigma(kv_{1[t_0,\infty)})\|)/\tilde{k}\} \\
&\le \max\{\overline{\gamma}_{12}^0(\|\xi(t_0)\|), \overline{\gamma}_{12}(\|v_{1[t_0,\infty)}\|)\}, \\
\|y_{12}\|_a &\le \max\{\gamma_o(N_1^u\|\sigma(kv_1)\|_a)/\tilde{k}, \gamma_o(\|\sigma(kv_1)\|_a)/\tilde{k}\} \\
&\le \overline{\gamma}_o(\|\sigma(kv_1)\|_a)/\tilde{k} \le \overline{\gamma}_{12}(\|v_1\|_a) \qquad (5.18)
\end{aligned}$$

where $\overline{\gamma}_{12}^0(s) = \gamma_o \circ \gamma_1^0(s)/\tilde{k}$, $\overline{\gamma}_o(s) = \max\{\gamma_o(N_1^u s), \gamma_o(s)\}$, and $\overline{\gamma}_{12}(s) = \overline{\gamma}_o(\min\{ks, \lambda\})/\tilde{k}$. The gain function $\overline{\gamma}_{12}(s)$ can be written as follows:

$$\overline{\gamma}_{12}(s) = \begin{cases} \frac{\overline{\gamma}_o(ks)}{\vartheta k}, & 0 < s \le \frac{\lambda}{k} \\ \frac{\overline{\gamma}_o(\lambda)}{\vartheta k}, & s \ge \frac{\lambda}{k} \end{cases}$$

Due to (6.13), for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\overline{\gamma}_o(s) < \epsilon s,\ 0 < s \le \delta. \qquad (5.19)$$

Thus, letting $\lambda = \delta$ gives

$$\overline{\gamma}_{12}(s) < \frac{\epsilon}{\vartheta} s,\ s > 0. \qquad (5.20)$$

Combining (5.16), (5.18) and (5.20) gives that for any $\xi(t_0) \in \Xi$ and any $v_1(t) \in L^1_\infty$, the following estimates hold

$$\|y_1(t)\| \le \max\{\overline{\gamma}_1^0(\|\xi(t_0)\|), \overline{\gamma}_1(\|v_{1[t_0,\infty)}\|)\}, \quad \|y_1\|_a \le \overline{\gamma}_1(\|v_1\|_a), \qquad (5.21)$$

where $\overline{\gamma}_1(s) = (lN_1^u k + \frac{\epsilon}{\vartheta})s$, $\overline{\gamma}_1^0(s) = \overline{\gamma}_{11}^0(s) + \overline{\gamma}_{12}^0(s)$.

Step 3. Choose k and ϵ to satisfy the small gain condition. For this purpose, note that

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(s) = 9(lN_1^u k + \frac{\epsilon}{\vartheta})s.$$

It suffices to choose k and ϵ sufficiently small such that

$$9(lN_1^u k + \frac{\epsilon}{\vartheta^L}) < 1.$$

Note that λ is determined by ϵ and is independent of k. Therefore, it is possible to choose k and λ such that the small gain condition is satisfied while $\frac{\lambda}{k}$ is sufficiently large.

By Theorem 3.2, system (5.8) is RUS and satisfies RAG with restrictions $\tilde{X} \times \tilde{\Xi}$, $\Delta_{\bar{u}}$ on $\bar{x}(t_0)$, $\bar{u}$ respectively, viewing $\bar{x} = \mathrm{col}(z, \xi)$ as state and $\bar{u}$ as input, i.e., for all d, there exist class K function $\tilde{\gamma}^0$ and positive numbers $\tilde{N}^u$ and $\Delta_{\bar{u}}$, such that, for any $\tilde{x}(t_0) \in \tilde{X} \times \tilde{\Xi}$, and $\bar{u}(t) \in L^1_\infty$ satisfying $\|\bar{u}_{[t_0,\infty)}\| < \Delta_{\bar{u}}$, the solution of Σ_1 and Σ_2 with connection (5.11) exists and satisfies, for all $t \geq t_0$,

$$\|\tilde{x}(t)\| \leq \max\{\tilde{\gamma}^0(\|\tilde{x}(t_0)\|), \tilde{N}^u \|\bar{u}_{[t_0,\infty)}\|\}, \quad \|\tilde{x}\|_a \leq \tilde{N}^u \|\bar{u}\|_a \tag{5.22}$$

where,
$\tilde{\gamma}^0(s) = \max\{2\gamma_1^0(s), 2N_1^u k \overline{\gamma}_2^0(s), 18N_1^u k \overline{\gamma}_1^0(s), 2\gamma_2^0(s), 6\overline{\gamma}_1^0(s), 6(lN_1^u k + \frac{\epsilon}{\vartheta^L})\overline{\gamma}_2^0(s)\}$,
$\tilde{N}^u = \max\{18N_1^u, 54(lN_1^u k + \frac{\epsilon}{\vartheta^L})/k, 6/k\}$, and
$\tilde{\Xi} = \{\xi \in \Xi : \overline{\gamma}_1^0(\|\xi\|) < \lambda/(2k)\}$, $\tilde{X} = \{z \in \mathrm{Re} : (lN_1^u k + \frac{\epsilon}{\vartheta^L})\overline{\gamma}_2^0(\|z\|) < \lambda/(2k)\}$, and
$\Delta_{\bar{u}} = \lambda/(18(lN_1^u k + \frac{\epsilon}{\vartheta^L}))$.

Step 4. Since $CA^{-1}\xi = x - z$, for any $\tilde{x}(t_0) \in \tilde{X} \times \tilde{\Xi}$, and $\bar{u}(t) \in L^1_\infty$ satisfying $\|\bar{u}_{[t_0,\infty)}\| < \Delta_{\bar{u}}$, the solution of system (5.7) exists and satisfies, for all $t \geq t_0$,

$$\begin{aligned}
\|\bar{x}(t)\| &\leq \max\{2\|\xi(t)\|, 2\|x(t)\|\} = \max\{2\|\xi(t)\|, 2\|(z + CA^{-1}\xi)(t)\|\} \\
&\leq 4(1+l)\max\{\|z(t)\|, \|\xi(t)\|\} \leq 4(1+l)\|\tilde{x}(t)\| \\
&\leq \max\{4(1+l)\tilde{\gamma}^0(\|\tilde{x}(t_0)\|), 4(1+l)\tilde{N}^u\|\bar{u}_{[t_0,\infty)}\|\} \\
&\leq \max\{4(1+l)\tilde{\gamma}^0(4(1+l)\|\bar{x}(t_0)\|), 4(1+l)\tilde{N}^u\|\bar{u}_{[t_0,\infty)}\|\} \\
&= \max\{\gamma^0(\|\bar{x}(t_0)\|), N^u\|\bar{u}_{[t_0,\infty)}\|\}
\end{aligned} \tag{5.23}$$

where $\gamma^0(s) = 4(1+l)\tilde{\gamma}^0(4(1+l)s)$ and $N^u = 4(1+l)\tilde{N}^u$.

Similarly, it holds that

$$\|\bar{x}\|_a \leq N^u \|\bar{u}\|_a.$$

Since $\frac{\lambda}{k}$ can be arbitrarily large so that $\tilde{\Xi} = \Xi$ and $X \subset \tilde{X}$, system (5.7) is RUS and satisfies RAG with restrictions $X \times \Xi$, $\Delta_{\bar{u}}$ on $\bar{x}(t_0)$, $\bar{u}$ and has a linear gain function $N^u s$, viewing $\bar{u}$ as input, $\bar{x} = \mathrm{col}(x, \xi)$ as state.

Finally, note that the Jacobian linearization of (5.7) at the origin is

$$\bar{A}_1 = \begin{bmatrix} -kD & C \\ -kB & A \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} D \\ B \end{bmatrix}. \tag{5.24}$$

Using $\bar{C}_1 = [c_1 \quad 0_{(1\times l)}]$, $\bar{D}_1 = 0$ gives that $\vartheta_1^L = \frac{c_1^L}{k} < \bar{\vartheta}_1 = \bar{D}_1 - \bar{C}_1\bar{A}_1^{-1}\bar{B}_1 = \frac{c_1}{k} < \frac{c_1^H}{k} = \vartheta_1^H$. ∎

Lemma 5.2 Consider the following system

$$\begin{aligned}\dot{\zeta} &= G_3(\zeta,x,\xi)+H_3(\zeta,x,\xi,u,d)\\ \dot{x} &= C\xi+Du+G_2(\xi,u,d)\\ \dot{\xi} &= A\xi+Bu+G_1(\xi,u,d)\end{aligned} \tag{5.25}$$

where, $x\in\mathbb{R}$, $\xi\in\mathbb{R}^l$, $\zeta\in\mathbb{R}^p$, G_3 is linear in ζ,x,ξ, and

$$\lim_{\|(\xi,u)\|\to 0}\frac{\|G_i(\xi,u,d)\|}{\|(\xi,u)\|}=0,\ \ i=1,2,\qquad \lim_{\|(\zeta,x,\xi,u)\|\to 0}\frac{\|H_3(\zeta,x,\xi,u,d)\|}{\|(\zeta,x,\xi,u)\|}=0. \tag{5.26}$$

Assume that (x,ξ) subsystem satisfies all assumptions in Lemma ??. Moreover, ζ subsystem is RUS and satisfies RAG with a linear gain function viewing ζ as state and $\mathrm{col}(x,\xi,u)$ as input. Then for any compact sets $Z\subset\mathbb{R}^p$ and $X\subset\mathbb{R}$, there exists a control law $u=-\sigma(kx-\bar{u})$, where σ is a saturation function with level λ such that the system

$$\dot{\bar{\zeta}}=\begin{bmatrix}\dot{\zeta}\\ \dot{x}\\ \dot{\xi}\end{bmatrix}=\begin{bmatrix}G_3(\zeta,x,\xi)+H_3(\zeta,x,\xi,u,d)\\ C\xi+Du+G_2(\xi,u,d)\\ A\xi+Bu+G_1(\xi,u,d)\end{bmatrix}_{u=-\sigma(kx-\bar{u})} \tag{5.27}$$

is RUS and satisfies RAG with restrictions $Z\times X\times\Xi$, $\Delta_{\bar{u}}$ on $\bar{\zeta}(t_0)$, $\bar{u}$ respectively and has a linear gain function. Moreover, $\vartheta_1^L<\vartheta_1=D_1-C_1A_1^{-1}B_1<\vartheta_1^H$ for positive numbers ϑ_1^L, ϑ_1^H, where (A_1,B_1) is the Jacobian linearization of system (5.27) and $C_1=[0_{(1\times p)}\quad \bar{C}_1]$, $D_1=0$.

Proof: It follows from Lemma 5.1 that for any compact set $X\subset\mathbb{R}$, there exists a control law $u=-\sigma(kx-\bar{u})$, where σ is a saturation function with level λ, such that the system

$$\dot{\bar{x}}=\begin{bmatrix}\dot{x}\\ \dot{\xi}\end{bmatrix}=\begin{bmatrix}C\xi+Du+G_2(\xi,u,d)\\ A\xi+Bu+G_1(\xi,u,d)\end{bmatrix}_{u=-\sigma(kx-\bar{u})} \tag{5.28}$$

is RUS and satisfies RAG with restrictions $X\times\Xi$, $\Delta_{\bar{u}}$ on $\bar{x}(t_0)$, $\bar{u}$ respectively and has a linear gain function. And $\bar{\vartheta}_1=\bar{D}_1-\bar{C}_1\bar{A}_1^{-1}\bar{B}_1=\frac{c_1}{k}$ with $c_1^L<c_1<c_1^H$ for some positive numbers c_1^L, c_1^H, where $(\bar{A}_1,\bar{B}_1)$ is the Jacobian linearization of system (5.28) at the origin and $\bar{C}_1=[c_1\quad 0_{(1\times l)}]$, $\bar{D}_1=0$.

Since ζ subsystem is RUS and satisfies RAG with a linear gain function viewing ζ as state and $\mathrm{col}(x,\xi,u)$ as input, i.e., for all d, there exist positive numbers N_x, N_ξ, N_u and class K function $\gamma^0(s)$ such that for any $\zeta(t_0)\in\mathrm{Re}^p$, the following estimates hold for all $t\geq t_0$,

$$\begin{aligned}\|\zeta(t)\| &\leq \max\{\gamma^0(\|\zeta(t_0)\|),N_x\|x_{[t_0,\infty)}\|,N_\xi\|\xi_{[t_0,\infty)}\|,N_u\|u_{[t_0,\infty)}\|\},\\ \|\zeta\|_a &\leq \max\{N_x\|x\|_a,N_\xi\|\xi\|_a,N_u\|u\|_a\}.\end{aligned} \tag{5.29}$$

Note that

$$\begin{aligned} \|u_{[t_0,\infty)}\| &\leq \|(kx-\bar{u})_{[t_0,\infty)}\| \leq \max\{2k\|x_{[t_0,\infty)}\|, 2\|\bar{u}_{[t_0,\infty)}\|\} \\ \|u\|_a &\leq \|kx-\bar{u}\|_a \leq \max\{2k\|x\|_a, 2\|\bar{u}\|_a\}. \end{aligned} \tag{5.30}$$

Substituting (5.30) into (5.29) gives that

$$\begin{aligned} \|\zeta(t)\| &\leq \max\{\gamma^0(\|\zeta(t_0)\|), N_x\|x_{[t_0,\infty)}\|, N_\xi\|\xi_{[t_0,\infty)}\|, 2kN_u\|x_{[t_0,\infty)}\|, 2N_u\|\bar{u}_{[t_0,\infty)}\|\} \\ \|\zeta\|_a &\leq \max\{N_x\|x\|_a, N_\xi\|\xi\|_a, 2kN_u\|x\|_a, 2N_u\|\bar{u}\|_a\}. \end{aligned} \tag{5.31}$$

Hence, ζ subsystem is RUS and satisfies RAG with a linear gain function viewing ζ as state and $\mathrm{col}(x,\xi,\bar{u})$ as input. We invoke Corollary 3.6 to conclude that for any compact set $Z \subset \mathrm{Re}^p$, $\bar{\zeta}$ system is RUS and satisfies RAG with a linear gain function and restrictions $Z \times X \times \Xi$ and $\Delta_{\bar{u}}$ on $\bar{\zeta}(t_0)$ and $\bar{u}$ respectively.

The Jacobian linearization of system (5.27) at the origin is in the form:

$$A_1 = \begin{bmatrix} * & * \\ 0 & \bar{A}_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \bar{B}_1 \end{bmatrix} \tag{5.32}$$

with $\bar{A}_1$, $\bar{B}_1$ being given by (5.24). Therefore,

$$\frac{c_1^L}{k} < \vartheta_1 = D_1 - C_1A_1^{-1}B_1 = \bar{D}_1 - \bar{C}_1\bar{A}_1^{-1}\bar{B}_1 = \frac{c_1}{k} < \frac{c_1^H}{k}.$$

∎

Using Lemmas 5.1 and 5.2, it is possible to establish the following result.

Theorem 5.1 Consider system (5.1), under Assumptions 5.1-5.3, there exists a control law of the form $u = -\sigma_1(k_1x_1 + \sigma_2(k_2x_2 + \cdots + \sigma_n(k_nx_n)))$ that solves the semi-global robust stabilization problem for system (5.1).

Proof: First note that the subsystem consisting of the last three equations of (5.1) is in the form (5.25) and, under Assumptions 5.1-5.3, this subsystem satisfies all conditions of Lemma 5.2. Therefore, for any compact sets $X_1 \subset \mathbb{R}$, $\Xi_1 \subset \mathbb{R}^{p_1}$ and $\Xi_2 \subset \mathbb{R}^{p_2}$, there exists a saturation control law $u = -\sigma_1(k_1x_1 - u_1)$ such that $\zeta_2 = [\xi_2^T, x_1, \xi_1^T]^T$ subsystem is RUS and satisfies RAG with restrictions $\Xi_2 \times X_1 \times \Xi_1$, Δ_{u_1} on $\zeta_2(t_0)$, u_1 respectively and has a linear gain function.

Now, consider the system

$$\begin{bmatrix} \dot{x}_2 \\ \dot{\xi}_2 \\ \dot{x}_1 \\ \dot{\xi}_1 \end{bmatrix} = \begin{bmatrix} c_1x_1 + f_2(\xi_1, x_1, \xi_2, u, d) \\ \bar{g}_2(\xi_1, x_1, \xi_2) + h_2(\xi_1, x_1, \xi_2, u, d) \\ C\xi_1 + Du + f_1(\xi_1, u, d) \\ A\xi_1 + Bu + g_1(\xi_1, u, d) \end{bmatrix}_{u=-\sigma_1(k_1x_1-u_1)} \tag{5.33}$$

which can be put in the form (5.4) as follows

$$\begin{bmatrix} \dot{x}_2 \\ \dot{\zeta}_2 \end{bmatrix} = \begin{bmatrix} C_1\zeta_2 + D_1 u_1 + G_2(\zeta_2, u_1, d) \\ A_1\zeta_2 + B_1 u_1 + F_2(\zeta_2, u_1, d) \end{bmatrix} \tag{5.34}$$

where $C_1 = [0_{1\times p_2} \quad c_1 \quad 0_{1\times p_1}]$, $D_1 = 0$, and A_1 and B_1 are given by (5.32) with $\bar{A}_1$, $\bar{B}_1$ being given by (5.24) with $k = k_1$.

Due to Assumption 5.1,

$$\lim_{\|(\zeta_2,u_1)\|\to 0} \frac{\|G_2(\zeta_2, u_1, d)\|}{\|(\zeta_2, u_1)\|} = 0, \quad \lim_{\|(\zeta_2,u_1)\|\to 0} \frac{\|F_2(\zeta_2, u_1, d)\|}{\|(\zeta_2, u_1)\|} = 0.$$

It can be verified, under Assumption 5.2, that

$$\frac{c_1^L}{k_1} < \vartheta_1 = D_1 - C_1 A_1^{-1} B_1 = \bar{D}_1 - \bar{C}_1 \bar{A}_1^{-1} \bar{B}_1 = \frac{c_1}{k_1} < \frac{c_1^H}{k_1}. \tag{5.35}$$

Thus, applying Lemma 5.1 to subsystem (5.34) shows that, for any compact set $X_2 \subset \mathbb{R}$, there exists a saturation control law $u_1 = -\sigma_2(k_2 x_2 - u_2)$ such that subsystem (5.34) is RUS and satisfies RAG with restrictions $X_2 \times \Xi_2 \times X_1 \times \Xi_1$, Δ_{u_2} on $(x_2(t_0), \zeta_2(t_0))$, u_2 respectively and has a linear gain function.

Repeating the above procedure $(n-1)$ times leads to the following system

$$\dot{\bar{x}}_n = \begin{bmatrix} \dot{x}_n \\ \dot{\zeta}_n \end{bmatrix} = \begin{bmatrix} C_{n-1}\zeta_n + D_{n-1} u_{n-1} + G_n(\zeta_n, u_{n-1}, d) \\ A_{n-1}\zeta_n + B_{n-1} u_{n-1} + F_n(\zeta_n, u_{n-1}, d) \end{bmatrix} \tag{5.36}$$

where, $\zeta_n = [\xi_n^T, x_{n-1}, \xi_{n-1}^T \cdots, x_1, \xi_1^T]^T$,

$$\lim_{\|(\zeta_n,u_{n-1})\|\to 0} \frac{\|G_n(\zeta_n, u_{n-1}, d)\|}{\|(\zeta_n, u_{n-1})\|} = 0, \quad \lim_{\|(\zeta_n,u_{n-1})\|\to 0} \frac{\|F_n(\zeta_n, u_{n-1}, d)\|}{\|(\zeta_n, u_{n-1})\|} = 0,$$
$$C_{n-1} = [0_{1\times p_n} \quad c_{n-1} \quad 0_{1\times(p_1+\ldots+p_{n-1}+n-2)}] = [0_{1\times p_n} \quad \bar{C}_{n-1}], \quad D_{n-1} = 0$$

where ζ_n subsystem is RUS and satisfies RAG with restrictions $\Xi_n \times X_{n-1} \times \Xi_{n-1} \cdots \times X_1 \times \Xi_1 \subset \mathbb{R}^{p_1+\ldots+p_n+n-1}$ and $\Delta_{u_{n-1}}$ on $\zeta_n(t_0)$ and u_{n-1} respectively and linear gain function, viewing ζ_n as state, u_{n-1} as input. And the *dc* gain $\vartheta_{n-1}^L = \frac{c_{n-1}^L}{k_{n-1}} < \vartheta_{n-1} = D_{n-1} - C_{n-1} A_{n-1}^{-1} B_{n-1} = \frac{c_{n-1}}{k_{n-1}} < \frac{c_{n-1}^H}{k_{n-1}} = \vartheta_{n-1}^H$. System (5.36) is in the form (5.4) and satisfies all conditions of Lemma 5.1. Thus, applying Lemma 5.1 to system (5.36) shows that, for any compact set $X_n \subset \mathrm{Re}$, there exists a saturation control law $u_{n-1} = -\sigma_n(k_n x_n - u_n)$ such that $\bar{x}_n$ subsystem is RUS and satisfies RAG with restrictions $X_n \times \Xi_n \times X_{n-1} \times \Xi_{n-1} \cdots \times X_1 \times \Xi_1$, Δ_{u_n} on $\bar{x}_n(t_0)$, u_n respectively and has a linear gain function viewing $\bar{x}_n$ as state and u_n as input. Setting $u_n \equiv 0$ gives the result of semi-global robust stabilizaiton for system (5.1). This completes the proof. ∎

Example 5.1 Consider the following system

$$\begin{aligned}\dot{x}_2 &= 0.1x_1 + (x_1^2 + \xi_1^2 + \xi_2^2) \times d \\ \dot{\xi}_2 &= -\xi_2 + x_1 \\ \dot{x}_1 &= \xi_1 \\ \dot{\xi}_1 &= -10\xi_1 + u. \end{aligned} \tag{5.37}$$

where the external disturbance $d(t) = 0.001 \times sin(t)$. It is not difficult to verify that this system satisfies Assumptions 5.1-5.3. In fact, given a compact set, $X_2 \times \Xi_2 \times X_1 \times \Xi_1 = [-0.01, 0.01] \times [-0.01, 0.01] \times [-0.01, 0.01] \times [-0.01, 0.01]$, using the procedure detailed in Theorem 5.1, it is possible to show that the following nested saturation controller $u = -\sigma_1(k_1x_1 + \sigma_2(k_2x_2))$with parameters $\lambda_1 = 50$, $k_1 = 5$, $\lambda_2 = 0.04$, $k_2 = 0.172$ robustly stabilize the equilibrium of the system at the origin with $X_2 \times \Xi_2 \times X_1 \times \Xi_1$ contained in the basin of attraction.

Step 1: Consider the lower subsystem of (5.37)

$$\begin{aligned}\dot{x}_1 &= \xi_1 \\ \dot{\xi}_1 &= -10\xi_1 + u \end{aligned} \tag{5.38}$$

where $\vartheta_1 = 0.1 > 0$.

ξ_1 subsystem is ISS with linear gain function $N_1^u s = 0.2s$.

The coordinate transformation $z_1 = x_1 - CA^{-1}\xi_1$ gives

$$\dot{z}_1 = 0.1u + G_1(\xi_1, d(t)) = 0.1u. \tag{5.39}$$

Since $G_1(\xi_1, d(t)) = 0$, $\epsilon_1 = 0$. Choose $k_1 = 5$ such that the small gain condition $9 \times N_1^u \times \|CA^{-1}\| \times k_1 < 1$ hold. And choose $\lambda_1 = 50$.

Hence, $u = -\sigma_1(5x_1 - u_1)$ with level $\lambda_1 = 50$.

Note that $X_1' = \{\xi_1 \in [-0.01, 0.01] : \|\xi_1\| \leq 100\} = [-0.01.0.01]$ and $z_1(t_0) \in \{\|z_1\| \leq 33\}$. It follows from $z_1 = x_1 - 0.1\xi_1$ that $x_1(t_0) \in [-0.01, 0.01]$ is contained in the domain of attraction.

Step 2. Putting $u = -\sigma_1(5x_1 - u_1)$ into system (5.37) gives that

$$\begin{aligned}\dot{x}_2 &= 0.1x_1 + (x_1^2 + \xi_1^2 + \xi_2^2) \times d(t) \\ \dot{\xi}_2 &= -\xi_2 + x_1 \\ \dot{x}_1 &= \xi_1 \\ \dot{\xi}_1 &= -10\xi_1 - \sigma_1(5x_1 - u_1). \end{aligned} \tag{5.40}$$

The linearization of subsystem ζ_2= col(ξ_2, x_1, ξ_1) is in the following form

$$\begin{aligned}\dot{\zeta}_2 &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -10 \end{bmatrix} \zeta_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ &= A_1\zeta_2 + B_1 u_1. \end{aligned} \tag{5.41}$$

Using C_1=[0, 0.1, 0] and $D_1 = 0$ gives $\vartheta_2 = D_1 - C_1A_1^{-1}B_1 = 0.02 > 0$, $\|C_1A_1^{-1}\| = 0.2$, $C_1A_1^{-1} = [0, -0.2, -0.02]$.

Put subsystem ζ_2 into the following structure

$$\dot{\zeta}_2 = A_1\zeta_2 + B_1u_1 + \tilde{G}_1(\zeta_2, u_1, d(t))$$

where,

$$\tilde{G}_1(\zeta_2, u_1, d(t)) = \begin{bmatrix} 0 \\ 0 \\ -\sigma_1(5x_1 - u_1) + (5x_1 - u_1) \end{bmatrix}. \tag{5.42}$$

And ζ_2 subsystem is RISS with linear gain function $N_2^u = 3.2s$.

The coordinate transformation $z_2 = x_2 - C_1A_1^{-1}\zeta_2$ gives

$$\dot{z}_2 = 0.02u_1 + \tilde{G}_2(\zeta_2, u_1, d(t)) \tag{5.43}$$

where,

$$\begin{aligned}\tilde{G}_2(\zeta_2, u_1, d(t)) &= (x_1^2 + \xi_1^2 + \xi_2^2) \times d(t) - C_1A_1^{-1}\tilde{G}_1(\zeta_2, u_1, d(t)) \\ &= (x_1^2 + \xi_1^2 + \xi_2^2)^2) \times d(t) \\ &- 0.02 \times (\sigma_1(5x_1 - u_1) - (5x_1 - u_1)) \end{aligned} \tag{5.44}$$

Clearly,

$$\lim_{(\|(\zeta_2, u_1)\|)\to 0} \frac{\|\tilde{G}_2(\zeta_2, u_1, d(t))\|}{\|(\zeta_2, u_1)\|} = 0. \tag{5.45}$$

And the following estimate holds

$$\begin{aligned}\tilde{G}_2(\zeta_2, u_1, d(t)) &\leq 0.001 \times (x_1^2 + \xi_1^2 + \xi_2^2) + 0.02 \times \frac{1}{4\lambda_1}(5x_1 + u_1)^2 \\ &\leq \max\{0.011\|\zeta_2\|^2, 0.0025\|u_1\|^2\} \\ &\leq \max\{\beta_2(\zeta_2(t_0), t - t_0), \gamma_{o2}(\|u_1\|)\} \end{aligned} \tag{5.46}$$

where, $\gamma_{o2}(s) = 0.011 \times (N_2^u)^2 s^2 = 0.11s^2$. Choose $\lambda_2 = 0.01$ and $k_2 = 0.086$. Hence, $\epsilon_2 = 0.056$. Thus, the small gain condition $9(N_2^u \times \|C_1A_1^{-1}\| \times k_2 + \epsilon_2) < 1$ holds. Note that

$\{\zeta_2(t_0) : \|\zeta_2\| \leq 0.048\}$ and $z_2(t_0) \in \{\|z_2\| \leq 0.051\}$. It follows from $z_2 = x_2 - 0.2\zeta_2$ that $x_2(t_0) \times \xi_2(t_0) \times x_1(t_0) \times \xi_1(t_0) \subset [-0.01, 0.01] \times [-0.01, 0.01] \times [-0.01, 0.01] \times [-0.01, 0.01]$ is contained in the domain of attraction.

Hence, we can obtain the following nested saturation controller

$$u = -\sigma_1(k_1 x_1 + \sigma_2(k_2 x_2)) \tag{5.47}$$

with parameters $\lambda_1 = 50$, $k_1 = 5$, $\lambda_2 = 0.01$, $k_2 = 0.086$. ∎

Example 5.2 Consider the following system,

$$\begin{aligned}
\dot{x}_2 &= x_1 + f_2(x_1, \xi, u) = x_1 + (\xi_1^2 + \xi_2^2 + \xi_3^2) + u^2 \\
\dot{x}_1 &= u + f_1(\xi, u) = u - (\xi_1^2 + \xi_2^2 + \xi_3^2) \\
\dot{\xi} &= \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} 0.25 \\ \xi_3 \\ -\xi_2 \end{bmatrix} u + \begin{bmatrix} 0 \\ -\xi_2^3 \\ -\xi_3^3 \end{bmatrix} \\
&= A\xi + Bu + g_1(\xi, u).
\end{aligned} \tag{5.48}$$

Note that the linearization of (5.48) at the origin has a pair of uncontrollable modes in the imaginary axis. Hence, the approach in [2] cannot solve the stabilization problem for (5.48). However, (5.48) satisfies the solvability conditions in Theorem ??. Thus, given any compact subsets $X_2, X_1 \subset \mathrm{Re}$ and $\Xi_1 \subset \mathrm{Re}^3$ which contain the origin, it is possible to design a controller of the following form $u = -\sigma_1(k_1 x_1 + \sigma_2(k_2 x_2))$ such that $X_2 \times X_1 \times \Xi_1$ is contained in the basin of attraction.

Choose $V(\xi_2, \xi_3) = (\xi_2^2 + \xi_3^2)/2$ as *Lyapunov* function candidate for subsystems ξ_2 and ξ_3. Its derivative along the trajectories of subsystems ξ_2 and ξ_3 satisfies

$$\dot{V} = -\xi_2^4 - \xi_3^4. \tag{5.49}$$

Thus, the system composed of ξ_2 and ξ_3 is globally asymptotically stable. It is not hard to check that subsystem ξ_1 is ISS with linear gain function $0.5s$. Hence subsystem ξ=col(ξ_1, ξ_2, ξ_3) is ISS with linear gain function $N_1^u s = s$.

Step 1:

Consider the lower subsystem of (5.48)

$$\begin{aligned}
\dot{x}_1 &= u - \xi^2 \\
\dot{\xi} &= A\xi + Bu + h_1(\xi, u).
\end{aligned} \tag{5.50}$$

It is not difficult to check that system (5.50) is in the same form as system (5.8) with $\vartheta_1 = 1$, $C = 0$. Hence, no coordinate transformation is needed.
And

$$\|G_1(\xi, u)\| = \gamma_{1o}(\|\xi\|)$$

with $\gamma_{1o}(s) = s^2$. Thus

$$\kappa_1 = \sup_{s\in[0,\lambda_1]} \frac{\gamma_{1o}(s)}{\vartheta_1 s} = \lambda_1.$$

Choose $\lambda_1 = 0.111$ such that the small gain condition $9 \times \kappa_1 < 1$ hold.
Choosing $k_1 = 1$ gives that $u = -\sigma_1(x_1 - u_1)$ with level $\lambda_1 = 0.111$.

Step 2.

Putting $u = -\sigma_1(10x_1 - u_1)$ into system (5.48) gives that

$$\begin{aligned}
\dot{x}_2 &= x_1 + (\xi_1^2 + \xi_2^2 + \xi_3^2) \\
\dot{x}_1 &= -\sigma_1(x_1 - u_1) - (\xi_1^2 + \xi_2^2 + \xi_3^2) \\
\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} -0.25\sigma_1(x_1 - u_1) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\xi_2^3 \\ -\xi_3^3 \end{bmatrix}. \quad (5.51)
\end{aligned}$$

The linearization of subsystem ζ_2= col(x_1, ξ) is in the following form

$$\begin{aligned}
\dot{\zeta}_2 &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ -0.25 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \zeta_2 + \begin{bmatrix} 1 \\ 0.25 \\ 0 \\ 0 \end{bmatrix} u_1 \\
&= A_1\zeta_2 + B_1 u_1. \quad (5.52)
\end{aligned}$$

Using C_1=[1, 0, 0, 0] and $D_1 = 0$ gives $\vartheta_2 = D_1 - C_1 A_1^{-1} B_1 = 1 > 0$, $\|C_1 A_1^{-1}\| = 1$, $C_1 A_1^{-1} = [-1, 0, 0, 0]$.

Put subsystem ζ_2 into the following structure

$$\dot{\zeta}_2 = A_1\zeta_2 + B_1 u_1 + \tilde{G}_1(\zeta_2, u_1)$$

where,

$$\tilde{G}_1(\zeta_2, u_1) = \begin{bmatrix} -\xi^2 - \sigma_1(10x_1 - u_1) + (x_1 - u_1) \\ ** \\ ** \\ * \end{bmatrix} \quad (5.53)$$

And x_1 subsystem is ISS with linear gain function $N_2^u = s$.

The coordinate transformation $z_2 = x_2 - C_1 A_1^{-1} \zeta_2$ gives

$$\dot{z}_2 = u_1 + \tilde{G}_2(\zeta_2, u_1) \tag{5.54}$$

where,

$$\begin{aligned} \tilde{G}_2(\zeta_2, u_1) &= \xi^2 - C_1 A_1^{-1} \tilde{G}_1(\zeta_2, u_1, d(t)) \\ &= -\sigma_1(x_1 - u_1) + (x_1 - u_1). \end{aligned} \tag{5.55}$$

Clearly,

$$\lim_{(\|(\zeta_2, u_1)\|) \to 0} \frac{\|\tilde{G}_2(\zeta_2, u_1)\|}{\|(\zeta_2, u_1)\|} = 0. \tag{5.56}$$

And the following estimate holds

$$\begin{aligned} \tilde{G}_2(\zeta_2, u_1) &\leq \frac{1}{4\lambda_1}(x_1 + u_1)^2 \\ &\leq \max\{2.25\|x_1\|^2, 2.25\|u_1\|^2\} \\ &\leq \max\{\beta_2(\zeta_2(t_0), t - t_0), \gamma_{o2}(\|u_1\|)\} \end{aligned} \tag{5.57}$$

where, $\gamma_{2o}(s) = 2.25 \times (N_2^u)^2 s^2 = 2.25 s^2$.

Hence,

$$\kappa_2 = \sup_{s \in [0, \lambda_2]} \frac{\gamma_{2o}(s)}{\vartheta_2 s} = 2.25 \lambda_2.$$

Choose $\lambda_2 = 0.05$ and $k_2 = 0.125$ such that the small gain conditions $9 \times \kappa_2 < 1$ and $8 \times N_2^u \times \|C_1 A_1^{-1}\| \times k_2 < 1$ hold.

Hence, we can obtain the following nested saturation controller

$$u = -\sigma_1(k_1 x_1 + \sigma_2(k_2 x_2)) \tag{5.58}$$

with parameters $\lambda_1 = 0.111$, $k_1 = 1$, $\lambda_2 = 0.05$, $k_2 = 0.125$.

∎

5.3 Conclusion

So far, we have solved semi-global robust stabilization problem for system (5.1). Compared with the results obtained in [2], our results are more general in the following aspects:

(i) System (5.1) is subject to dynamic uncertainties, thus is more complicated than the system considered in [2].

(ii) The restriction on *Hurwitzness* of the Jacobian linearization of the input unmodeled dynamics is weakened to critical stability.

(iii) Disturbance rejection instead of disturbance attenuation is achieved.

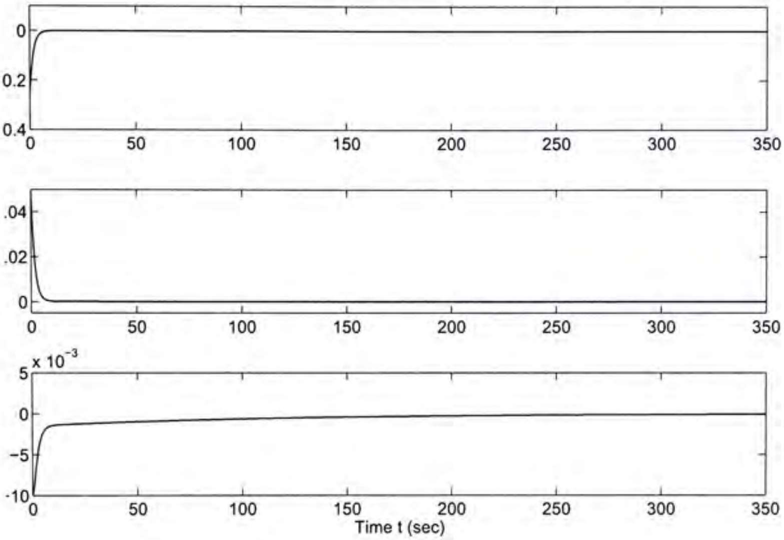

Figure 5.2: Profile of the states of the closed-loop system for Example 5.1

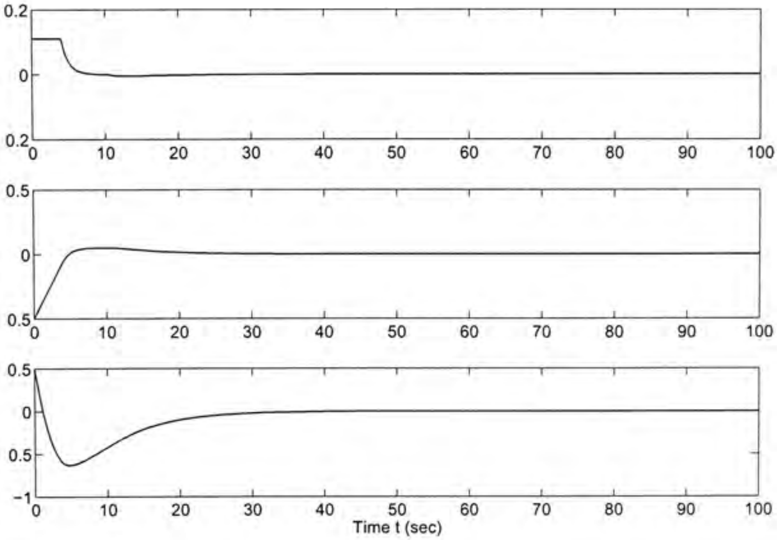

Figure 5.3: Profile of the states of the closed-loop system for Example 5.2

Chapter 6

Global Robust Stabilization for A Class of Feedforward Systems

Systems of the form (5.1) are studied in last chapter in which the semi-global robust stabilization for (5.1) is considered assuming each dynamic uncertainty in (5.1) satisfies some ISS assumption. In this chapter, we further apply Theorem 3.4 to solve the global robust stabilization for (5.1) assuming each (unforced) dynamic uncertainty in (5.1) is locally exponentially stable.

6.1 Main Result

To state our assumptions, we rewrite system (5.1), for $i = 2, \cdots, n$,

$$g_i(\xi_1, x_1, ..., \xi_i, u, d) = \tilde{g}_i(\xi_1, x_1, ..., \xi_i) + h_i(\xi_1, x_1, ..., \xi_i, u, d)$$

where $\tilde{g}_i(\xi_1, x_1, ..., \xi_i)$ is linear in $\xi_1, x_1, ..., \xi_i$.

A 6.1

$$\lim_{\|(\xi_1,u)\|\to 0} \frac{\|f_1(\xi_1, u, d)\|}{\|(\xi_1, u)\|} = 0, \quad \lim_{\|(\xi_1,u)\|\to 0} \frac{\|g_1(\xi_1, u, d)\|}{\|(\xi_1, u)\|} = 0,$$

and for $i = 2, .., n$,

$$\lim_{\|(\xi_1,x_1,...,\xi_i,u)\|\to 0} \frac{\|f_i(\xi_1, x_1, ..., \xi_i, u, d)\|}{\|(\xi_1, x_1, ..., \xi_i, u)\|} = 0, \quad \lim_{\|(\xi_1,x_1,...,\xi_i,u)\|\to 0} \frac{\|h_i(\xi_1, x_1, ..., \xi_i, u, d)\|}{\|(\xi_1, x_1, ..., \xi_i, u)\|} = 0.$$

A 6.2 For $i = 1, ..., n-1$, $c_i^L < c_i < c_i^H$ for some positive numbers c_i^L, c_i^H. And the *dc* gain $\vartheta^L < \vartheta = D - CA^{-1}B < \vartheta^H$ for some positive numbers ϑ^L, ϑ^H.

A 6.3 ξ_1 subsystem is RAG with restriction Δ_u on u and has a linear gain function viewing ξ_1 as state and u as input. Moreover, A is Hurwitz. For $i = 2, ..., n$, ξ_i subsystem is RAG viewing ξ_i as state and $\text{col}(\xi_1, x_1, ..., \xi_{i-1}, x_{i-1}, u)$ as input. Moreover, $M_i = \frac{\partial g_i(0,\cdots,0)}{\partial \xi_i}$ is Hurwitz.

Remark 6.1 The major difference between Assumption 6.3 and Assumption 5.3 is that the uniform stability assumption in Assumption 5.3 is strengthened by requiring M_i be Hurwitz. ∎

Lemma 6.1 Consider the following system

$$\begin{aligned} \dot{x} &= C\xi + Du + G_2(\xi, u, d) \\ \dot{\xi} &= A\xi + Bu + G_1(\xi, u, d) \end{aligned} \tag{6.1}$$

where, $x \in \mathbb{R}$, $\xi \in \mathbb{R}^l$, and,

$$\lim_{\|(\xi,u)\|\to 0} \frac{\|G_1(\xi, u, d)\|}{\|(\xi, u)\|} = 0, \qquad \lim_{\|(\xi,u)\|\to 0} \frac{\|G_2(\xi, u, d)\|}{\|(\xi, u)\|} = 0.$$

Assume the *dc* gain $\vartheta^L < \vartheta = D - CA^{-1}B < \vartheta^H$ for positive numbers ϑ^L, ϑ^H. And the ξ subsystem is RAG with restriction Δ_u on u and has a linear gain function, i.e., there exist positive numbers N_1^u, Δ_u such that for any initial state $\xi(t_0) \in \mathbb{R}^l$ and any input $u(t) \in L_\infty^1$ satisfying $\|u\|_a \le \Delta_u$, the solution $\xi(t)$ exists and satisfies, for all $t \ge t_0$,

$$\|\xi\|_a \le N_1^u \|u\|_a. \tag{6.2}$$

Then there exists a control law $u = -\sigma(kx - \bar{u})$, where σ is a saturation function with level λ, such that the system

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} C\xi + Du + G_2(\xi, u, d) \\ A\xi + Bu + G_1(\xi, u, d) \end{bmatrix}_{u=-\sigma(kx-\bar{u})} \tag{6.3}$$

is RAG with restriction $\Delta_{\bar{u}}$ on $\bar{u}$ and a linear gain function. i.e., there exist positive numbers N^u, $\Delta_{\bar{u}}$ such that for any initial state $\bar{x}(t_0) \in \mathbb{R} \times \mathbb{R}^l$ and any input $\bar{u}(t) \in L_\infty^1$ satisfying $\|\bar{u}\|_a \le \Delta_{\bar{u}}$, the solution $\bar{x}(t)$ exists and satisfies, for all $t \ge t_0$

$$\|\bar{x}\|_a \le N^u \|\bar{u}\|_a.$$

Further, let $(\bar{A}_1, \bar{B}_1)$ be the Jacobian linearization of system (6.3) at the origin, $\bar{C}_1 = [c_1 \quad 0_{(1\times l)}]$ with $c_1^L < c_1 < c_1^H$ for positive numbers c_1^L, c_1^H, and $\bar{D}_1 = 0$. Then, $\bar{A}_1$ is Hurwitz and $\vartheta_1^L < \bar{\vartheta}_1 = \bar{D}_1 - \bar{C}_1 \bar{A}_1^{-1} \bar{B}_1 < \vartheta_1^H$ for positive numbers ϑ_1^L, ϑ_1^H.

Remark 6.2 Lemma 6.1 is an extension of Lemma 1 of [2] in that d of Lemma 1 of [2] is treated as a disturbance to be attenuated while d here is treated as a disturbance to be rejected. ∎

Proof: The spirit of the proof is similar to that of the proof of Lemma 1 in [2]. That is, we need to employ the asymptotic small gain theorem. For this purpose, introduce the same coordinate transformation $z = x - CA^{-1}\xi$ as in [2] to change system (6.3) into the following:

$$\dot{\bar{x}} = \begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \vartheta u + G(\xi, u, d) \\ A\xi + Bu + G_1(\xi, u, d) \end{bmatrix}_{u=-\sigma(kx-\bar{u})} \tag{6.4}$$

where $\vartheta = D - CA^{-1}B$ and $G(\xi, u, d) = G_2(\xi, u, d) - CA^{-1}G_1(\xi, u, d)$. Clearly,

$$\lim_{\|(\xi,u)\|\to 0} \frac{\|G(\xi, u, d)\|}{\|(\xi, u)\|} = 0. \tag{6.5}$$

Define $\tilde{\lambda} = \vartheta\lambda$ and $\tilde{k} = \vartheta k$, then

$$\begin{aligned} \vartheta u &= -\vartheta\sigma(k(x - \frac{\bar{u}}{k})) = -\tilde{\sigma}(\tilde{k}(x - \frac{\bar{u}}{k})) \\ &= -\tilde{\sigma}(\tilde{k}(z + CA^{-1}\xi - \frac{\bar{u}}{k})) \end{aligned} \tag{6.6}$$

where $\tilde{\sigma}(s) = \vartheta\sigma(s/\vartheta)$ is a saturation function with level $\tilde{\lambda}$.

With (6.6), system (6.4) can be viewed as the interconnection

$$v_1 = y_2, \quad v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = y_1 \tag{6.7}$$

of the following two subsystems

$$\begin{aligned} \Sigma_1 \ &: \quad \dot{\xi} = A\xi - B\sigma(kv_1) + G_1(\xi, -\sigma(kv_1), d), \\ &\qquad y_1 = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} = \begin{bmatrix} CA^{-1}\xi \\ G(\xi, -\sigma(kv_1), d)/\tilde{k} \end{bmatrix}, \\ \Sigma_2 \ &: \quad \dot{z} = -\tilde{\sigma}(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})) + \tilde{k}v_{22}, \\ &\qquad y_2 = z + v_{21} - \frac{\bar{u}}{k}. \end{aligned}$$

Following the similar steps of Lemma 1 in [2], we will now apply Corollay 3.13 to show that system (6.3) is RAG with appropriate restrictions. For this purpose, we divide the rest of the proof into five steps.

Step 1. Show under the assumption that Σ_1 and Σ_2 are both RAG, there is no finite escape time for the inter-connected system.

Toward this end, suppose the trajectories are defined on $[t_0, T)$. Then $y_1(t)$ is bounded on $[t_0, T)$. Define the input v_2 for subsystem Σ_2: $v_2(t) = y_1(t)$ for $t \in [t_0, T)$, $v_2(t) = 0$ otherwise. Let $\tilde{z}(t)$ be the response from the initial state $z(t_0)$. Since Σ_2 is RAG, $\|v_2\|_a = 0$ and $\bar{u}$ is bounded, $\tilde{z}(t)$ is defined for all $t \geq t_0$ and hence bounded on $[t_0, T)$. By casuality, $\tilde{z}(t) = z(t)$ for all $t \in [t_0, T)$ and thus $z(t)$ is bounded on $[t_0, T)$.

Now consider subsystem Σ_1. Since $y_2(t)$ is bounded on $[t_0, T)$, an identical argument shows that $\xi(t)$ is bounded on $[t_0, T)$. Therefore, there is no finite escape time for Σ_1 and Σ_2.

Step 2. Show subsystem Σ_2 is RAG and o-RAG with no restrictions on $z(t_0)$ and $\bar{u}$ and with restriction $\Delta_{22} = \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$ on v_{22}. Choose the Lyapunov candidate $V(z) = z^2/2$ for subsystem Σ_2, then its derivative along the trajectory of Σ_2 satisfies

$$\dot{V} = -(\tilde{\sigma}(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})) - \tilde{k}v_{22})z.$$

It can be shown that under the restriction $\|v_{22[t_0,\infty)}\| < \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$, the following implication holds

$$\|z\| > \max\{3\|v_{21}\|, 3\|\frac{\bar{u}}{k}\|, 3\|v_{22}\|\} \Rightarrow \dot{V} < 0.$$

Therefore for any initial state $z(t_0) \in \mathbb{R}$, and any input $v_{22}(t) \in L^1_\infty$ satisfying $\|v_{22}\|_a < \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$, $v_{21}(t) \in L^1_\infty$ and $\bar{u}(t) \in L^1_\infty$, the solution $z(t)$ exists for all $t \geq t_0$ and satisfies

$$\|z\|_a \leq \max\{3\|v_{21}\|_a, 3\|v_{22}\|_a, \frac{3}{k}\|\bar{u}\|_a\}. \tag{6.8}$$

It follows from (6.8) and $y_2 = z + v_{21} - \frac{\bar{u}}{k}$ that for any initial state $z(t_0) \in \mathbb{R}$, and any input $v_{22}(t) \in L^1_\infty$ satisfying $\|v_{22}\|_a \leq \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$, $v_{21}(t) \in L^1_\infty$ and $\bar{u}(t) \in L^1_\infty$, the output $y_2(t)$ exists for all $t \geq t_0$ and satisfies

$$\begin{aligned} \|y_2\|_a &\leq \max\{9\|v_{21}\|_a, 9\|v_{22}\|_a, \frac{9}{k}\|\bar{u}\|_a\} \\ &\leq \max\{\overline{\gamma}_{21}(\|v_{21}\|_a), \overline{\gamma}_{22}(\|v_{22}\|_a), \overline{\gamma}_2^u(\|\bar{u}\|_a\}), \end{aligned} \tag{6.9}$$

where $\overline{\gamma}_{21}(s) = \overline{\gamma}_{22}(s) = 9s$. Obviously, Σ_2 is RAG and o-RAG with restriction $\Delta_{22} = \tilde{\lambda}/(2\tilde{k}) = \lambda/(2k)$ on v_{22}.

Step 3. Show subsystem Σ_1 is RAG and o-RAG without restriction.

Choose λ such that $\lambda \leq \Delta_u$ to guarantee $\|u\|_a \leq \Delta_u$.

Noting

$$\|\sigma(kv_1)\|_a \leq \min\{k\|v_1\|_a, \lambda\} \leq k\|v_1\|_a \tag{6.10}$$

and substituting (6.10) into (6.2) gives that for any initial state $\xi(t_0) \in \mathbb{R}^l$ and any input $v_1(t) \in L^1_\infty$, the solution $\xi(t)$ exists for all $t \geq t_0$ and satisfies

$$\|\xi\|_a \leq N_1^u \|\sigma(kv_1)\|_a \leq N_1^u \min\{k\|v_1\|_a, \lambda\} \leq N_1^u k\|v_1\|_a. \tag{6.11}$$

Let l be a positive number such that $\|CA^{-1}\| < l$. Then, for any initial state $\xi(t_0) \in \mathrm{Re}^l$ and any input $v_1(t) \in L^1_\infty$, the solution $\xi(t)$ exists for all $t \geq t_0$ and satisfies

$$\|y_{11}\|_a = \|CA^{-1}\xi\|_a \leq lN_1^u \min\{k\|v_1\|_a, \lambda\} = \overline{\gamma}_{11}(\|v_1\|_a). \tag{6.12}$$

Next consider y_{12}. It follows from (6.5) that there exists a nondecreasing, continuous function $\gamma_o(s)$ such that

$$\|G(\xi(t), -\sigma(kv_1(t), d)\| \leq \max\{\gamma_o(\|\xi(t)\|), \gamma_o(\|\sigma(kv_1(t))\|)\}.$$

where

$$\lim_{\|s\| \to 0} \frac{\|\gamma_o(s)\|}{\|s\|} = 0. \tag{6.13}$$

Using (6.2) and (6.10) gives that for any initial state $\xi(t_0) \in \mathbb{R}^l$ and any input $v_1(t) \in L^1_\infty$, the following estimates hold

$$\begin{aligned}\|y_{12}\|_a &\leq \max\{\gamma_o(N_1^u\|\sigma(kv_1)\|_a)/\tilde{k}, \gamma_o(\|\sigma(kv_1)\|_a)/\tilde{k}\} \\ &= \overline{\gamma}_o(\|\sigma(kv_1)\|_a)/\tilde{k} \\ &= \overline{\gamma}_{12}(\|v_1\|_a)\end{aligned} \tag{6.14}$$

where $\overline{\gamma}_o(s) = \max\{\gamma_o(N_1^u s), \gamma_o(s)\}$, $\overline{\gamma}_{12}(s) = \overline{\gamma}_o(\min\{ks, \lambda\})/\tilde{k}$.

The gain function $\overline{\gamma}_{12}(s)$ can be written as follows:

$$\overline{\gamma}_{12}(s) = \begin{cases} \frac{\overline{\gamma}_o(ks)}{\vartheta k}, & 0 < s \leq \frac{\lambda}{k} \\ \frac{\overline{\gamma}_o(\lambda)}{\vartheta k}, & s \geq \frac{\lambda}{k} \end{cases}$$

Due to (6.13), for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\overline{\gamma}_o(s) < \epsilon s, \ 0 < s \leq \delta. \tag{6.15}$$

Thus, letting $\lambda = \delta$ gives

$$\overline{\gamma}_{12}(s) < \frac{\epsilon}{\vartheta}s, \; s > 0. \tag{6.16}$$

Combining (6.12), (6.14) and (6.19) gives that for any initial state $\xi(t_0) \in \mathbb{R}^l$ and any input $v_1(t) \in L^1_\infty$, the following estimates hold

$$\|y_1\|_a \leq \overline{\gamma}_1(\|v_1\|_a), \tag{6.17}$$

where $\overline{\gamma}_1(s) = (lN_1^u k + \frac{\epsilon}{\vartheta})s$.

The gain function $\overline{\gamma}_{12}(s)$ can be written as follows:

$$\overline{\gamma}_{12}(s) = \begin{cases} \frac{\overline{\gamma}_o(ks)}{\vartheta k}, & 0 < s \leq \frac{\lambda}{k} \\ \frac{\overline{\gamma}_o(\lambda)}{\vartheta k}, & s \geq \frac{\lambda}{k} \end{cases}$$

For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\overline{\gamma}_o(s) < \epsilon s, \; 0 < s \leq \delta. \tag{6.18}$$

Thus, letting $\lambda = \delta$ gives

$$\overline{\gamma}_{12}(s) < \frac{\epsilon}{\vartheta}s, \; s > 0. \tag{6.19}$$

Note that λ is determined by ϵ and is independent of k. Choose k and ϵ to satisfy the small gain condition and the restriction Δ_{22}. For this purpose, note that

$$\overline{\gamma}_{11} \circ \overline{\gamma}_{21}(s) < 9lN_1^u ks, \quad \overline{\gamma}_{12} \circ \overline{\gamma}_{22}(s) < 9\frac{\epsilon}{\vartheta}s. \tag{6.20}$$

It suffices to choose k and ϵ sufficiently small such that

$$9lN_1^u k < 1, \quad 9\frac{\epsilon}{\vartheta^L} < 1.$$

Note that $\overline{\gamma}_{12}(\infty) = \frac{\overline{\gamma}_o(\lambda)}{\vartheta k} < \infty$ where

$$\lim_{\|s\| \to 0} \frac{\|\overline{\gamma}_o(s)\|}{\|s\|} = 0.$$

Thus, λ can be chosen sufficiently small such that the restriction Δ_{22} can be satisfied

$$\overline{\gamma}_{12}(\infty) = \frac{\overline{\gamma}_o(\lambda)}{\vartheta k} \leq \frac{\tilde{\lambda}}{2\tilde{k}} = \frac{\lambda}{2k} = \Delta_{22}$$

By Corollary 3.13, system (6.4) is RAG with linear gain function, viewing $\tilde{x} = \mathrm{col}(z, \xi)$ as state and $\bar{u}$ as input, i.e., there exist positive numbers $\tilde{N}^u$ and $\Delta_{\bar{u}}$, independent of

d, such that, for any $\bar{x}(t_0) \in \mathbb{R} \times \mathbb{R}^l$, and $\bar{u}(t) \in L^1_\infty$, the solution of Σ_1 and Σ_2 with connection () exists and satisfies, for all $t \geq t_0$,

$$\|\bar{x}\|_a \leq \bar{N}^u \|\bar{u}\|_a. \tag{6.21}$$

Since $CA^{-1}\xi = x - z$, for any $\bar{x}(t_0) \in \mathbb{R} \times \mathbb{R}^l$, and $\bar{u}(t) \in L^1_\infty$, the solution of system (6.3) exist and satisfy, for all $t \geq t_0$,

$$\begin{aligned} \|\bar{x}\|_a &\leq \max\{2\|\xi\|_a, 2\|x\|_a\} \\ &= \max\{2\|\xi\|_a, 2\|(z + CA^{-1}\xi)\|_a\} \\ &\leq 4(1 + \|CA^{-1}\|)\bar{N}^u \|\bar{u}\|_a \\ &= N^u \|\bar{u}\|_a \end{aligned} \tag{6.22}$$

Therefore, system (6.3) is RAG and has a linear gain function $N^u s$, viewing $\bar{u}$ as input, $\bar{x} = \mathrm{col}(x, \xi)$ as state.

Note that the Jacobian linearization of (6.3) at the origin is

$$\bar{A}_1 = \begin{bmatrix} -kD & C \\ -kB & A \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} D \\ B \end{bmatrix}. \tag{6.23}$$

Using $\bar{C}_1 = [c_1 \quad 0_{(1\times l)}]$, $\bar{D}_1 = 0$ gives that $\vartheta_1^L = \frac{c_1^L}{k} < \bar{\vartheta}_1 = \bar{D}_1 - \bar{C}_1 \bar{A}_1^{-1} \bar{B}_1 = \frac{c_1}{k} < \frac{c_1^H}{k} = \vartheta_1^H$. Like Lemma 1 in [2], the small analysis for Σ_1 and Σ_2 also holds with $G \equiv 0$, $\bar{u} \equiv 0$ and $G_1 \equiv 0$,

$$\begin{aligned} \Sigma_1 \quad : \quad & \dot{\xi} = A\xi + Bu, \\ & y_1 = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} = \begin{bmatrix} CA^{-1}\xi \\ 0 \end{bmatrix}, \\ \Sigma_2 \quad : \quad & \dot{z} = -\bar{k}(z + v_{21}) + \bar{k}v_{22}, \\ & y_2 = z + v_{21} \end{aligned} \tag{6.24}$$

under the interconnection

$$v_1 = y_2, \quad v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = y_1. \tag{6.25}$$

and input $u = -k(z + CA^{-1}\xi)$, which is $\dot{\bar{x}} = \bar{A}_1 \bar{x}$ in the $\bar{x}$ coordinate. It follows from Lemma 2 in [2] that the asymptotic gain of $\dot{\xi} = A\xi + Bu$ is the same as that of $\dot{\xi} = A\xi + Bu + G_1(\xi, u, d)$. Since the small gain condition (6.20) holds, the origin of $\dot{\bar{x}} = \bar{A}_1 \bar{x}$ is globally attractive, namely, $\bar{A}_1$ is Hurwitz. This completes the proof. ∎

The following lemma can be directly derived from Lemma 5.2.

Lemma 6.2 Consider the following system

$$\begin{aligned}\dot{\zeta} &= H_3(\zeta, x, \xi) + G_3(\zeta, x, \xi, u, d) \\ \dot{x} &= C\xi + Du + G_2(\xi, u, d) \\ \dot{\xi} &= A\xi + Bu + G_1(\xi, u, d)\end{aligned} \tag{6.26}$$

where, $x \in \mathbb{R}$, $\xi \in \mathbb{R}^l$, $\zeta \in \mathbb{R}^p$, H_3 is linear in ζ, x, ξ, and for i=1,2,

$$\lim_{\|(\xi,u)\| \to 0} \frac{\|G_i(\xi, u, d)\|}{\|(\xi, u)\|} = 0,$$
$$\lim_{\|(\zeta,x,\xi,u)\| \to 0} \frac{\|G_3(\zeta, x, \xi, u, d)\|}{\|(\zeta, x, \xi, u)\|} = 0.$$

Assume that there exists a control law $u = -\sigma(kx - \bar{u})$, where σ is a saturation function with level λ, such that the system

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} C\xi + Du + G_2(\xi, u, d) \\ A\xi + Bu + G_1(\xi, u, d) \end{bmatrix}_{u=-\sigma(kx-\bar{u})} \tag{6.27}$$

is RAG and has a linear gain function $N^u \cdot s$. And $\bar{\vartheta}_1 = \bar{D}_1 - \bar{C}_1\bar{A}_1^{-1}\bar{B}_1 = \frac{c_1}{k}$ with $c_1^L < c_1 < c_1^H$ for positive numbers c_1^L, c_1^H, where $(\bar{A}_1, \bar{B}_1)$ is the Jacobian linearization of system (6.27) and $\bar{C}_1 = [c_1 \quad 0_{(1\times l)}]$, $\bar{D}_1 = 0$ with $\bar{A}_1$ is Hurwitz. Moreover, subsystem ζ is RAG with a linear gain function and $M = \frac{\partial H_3(0,0,0)}{\partial \zeta}$ is Hurwitz. Then the system

$$\bar{\zeta} = \begin{bmatrix} \dot{\zeta} \\ \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} H_3(\zeta, x, \xi) + G_3(\zeta, x, \xi, u, d) \\ C\xi + Du + G_2(\xi, u, d) \\ A\xi + Bu + G_1(\xi, u, d) \end{bmatrix}_{u=-\sigma(kx-\bar{u})} \tag{6.28}$$

is RAG and has a linear gain function. Moreover, A_1 is Hurwitz and $\vartheta_1^L < \vartheta_1 = D_1 - C_1A_1^{-1}B_1 < \vartheta_1^H$ for positive numbers ϑ_1^L, ϑ_1^H, where (A_1, B_1) is the Jacobian linearization of system (6.28) and $C_1 = [0_{(1\times p)} \quad \bar{C}_1]$, $D_1 = 0$. ∎

Remark 6.3 Note that the Jacobian linearization of system (6.28) at the origin is in the form:

$$A_1 = \begin{bmatrix} M & * \\ 0 & \bar{A}_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \bar{B}_1 \end{bmatrix}. \tag{6.29}$$

In Lemma 5.1, M is not assumed to be Hurwitz. In the case that M is Hurwitz, we can further obtain that A_1 is also Hurwitz. ∎

Using Lemmas 6.1 and 6.2, it is possible to establish the following result.

Theorem 6.1 Consider system (5.1), under assumptions 6.1-6.3, there exists a control law of the form

$$u = -\sigma_1(k_1x_1 + \sigma_2(k_2x_2 + \cdots + \sigma_n(k_nx_n)))$$

that solves the global robust stabilization problem for system (5.1).

Proof: Similar to Theorem 5.1, the proof is constructed from repeated applications of Lemma 6.1 and Lemma 6.2. ∎

Example 6.1 Consider the following system

$$\begin{aligned} \dot{x}_2 &= c_1x_1 + (x_1^2 + \xi_1^2 + \xi_2^2) \times d \\ \dot{\xi}_2 &= -\xi_2 + x_1 \\ \dot{x}_1 &= \xi_1 + \xi_1^2 \times d \\ \dot{\xi}_1 &= -5\xi_1 + u. \end{aligned} \tag{6.30}$$

where the uncertain parameter $1 \leq c_1 \leq 2$ and the external disturbance $d(t) = 0.01 \times sin(t)$. It is not difficult to verify that this system satisfies Assumptions 6.1-6.3. Using the procedure detailed in Theorem 6.1, it is possible to show that the following nested saturation controller $u = -\sigma_1(k_1x_1 + \sigma_2(k_2x_2))$with parameters $\lambda_1 = 2.22$, $k_1 = 1.35$, $\lambda_2 = 0.002$, $k_2 = 0.086$ robustly globally stabilize the equilibrium of the system at the origin.

Step 1: Consider the lower subsystem of (6.30)

$$\begin{aligned} \dot{x}_1 &= \xi_1 + \xi_1^2 \times d \\ \dot{\xi}_1 &= -5\xi_1 + u \end{aligned} \tag{6.31}$$

where $\vartheta_1 = 0.2 > 0$.

ξ_1 subsystem is ISS with linear gain function $N_1^u s = 0.4s$.

The coordinate transformation $z_1 = x_1 - CA^{-1}\xi_1$ gives

$$\dot{z}_1 = 0.1u + G_1(\xi_1, d(t)) = 0.1u + \xi_1^2 \times d. \tag{6.32}$$

Following the same step in Lemma 6.1, we choose $k_1 = 1.35$ and $\lambda_1 = 0.022$.

Hence, $u = -\sigma_1(1.35x_1 - u_1)$ with level $\lambda_1 = 0.022$.

Step 2. Putting $u = -\sigma_1(1.35x_1 - u_1)$ into system (6.30) gives that

$$\begin{aligned} \dot{x}_2 &= c_1x_1 + (x_1^2 + \xi_1^2 + \xi_2^2) \times d \\ \dot{\xi}_2 &= -\xi_2 + x_1 \\ \dot{x}_1 &= \xi_1 + \xi_1^2 \times d \\ \dot{\xi}_1 &= -5\xi_1 - \sigma_1(1.35x_1 - u_1). \end{aligned} \tag{6.33}$$

The linearization of subsystem ζ_2= col(ξ_2, x_1, ξ_1) is in the following form

$$\begin{aligned} \dot{\zeta}_2 &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1.35 & -5 \end{bmatrix} \zeta_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ &= A_1\zeta_2 + B_1u_1. \end{aligned} \tag{6.34}$$

Using $C_1 = [0,\ c_1,\ 0]$ and $D_1 = 0$ gives $\vartheta_2 = D_1 - C_1A_1^{-1}B_1 = 0.74c_1 > 0$, $\|C_1A_1^{-1}\| = 3.77c_1$, $C_1A_1^{-1} = [0, -3.7, -0.74]c_1$.

Put subsystem ζ_2 into the following structure

$$\dot{\zeta}_2 = A_1\zeta_2 + B_1u_1 + \tilde{G}_1(\zeta_2, u_1, d(t))$$

where,

$$\tilde{G}_1(\zeta_2, u_1, d(t)) = \begin{bmatrix} 0 \\ 0 \\ -\sigma_1(1.35x_1 - u_1) + (1.35x_1 - u_1) \end{bmatrix}. \tag{6.35}$$

And ζ_2 subsystem is RISS with linear gain function $N_2^u = 3.2s$.

The coordinate transformation $z_2 = x_2 - C_1A_1^{-1}\zeta_2$ gives

$$\dot{z}_2 = 0.74c_1u_1 + \tilde{G}_2(\zeta_2, u_1, d(t)) \tag{6.36}$$

where,

$$\begin{aligned} \tilde{G}_2(\zeta_2, u_1, d(t)) &= (x_1^2 + \xi_1^2 + \xi_2^2) \times d(t) - C_1A_1^{-1}\tilde{G}_1(\zeta_2, u_1, d(t)) \\ &= (x_1^2 + \xi_1^2 + \xi_2^2)^2) \times d(t) \\ &- \ 0.74 \times (\sigma_1(1.35x_1 - u_1) - (1.35x_1 - u_1)) \end{aligned} \tag{6.37}$$

Clearly,

$$\lim_{(\|(\zeta_2,u_1)\|)\to 0} \frac{\|\tilde{G}_2(\zeta_2, u_1, d(t))\|}{\|(\zeta_2, u_1)\|} = 0. \tag{6.38}$$

And the following estimate holds

$$\begin{aligned}\tilde{G}_2(\zeta_2, u_1, d(t)) &\leq 0.001 \times (x_1^2 + \xi_1^2 + \xi_2^2) + 0.74 \times \frac{1}{4\lambda_1}(1.35x_1 + u_1)^2 \\ &\leq \max\{8.5\|\zeta_2\|^2, 8.5\|u_1\|^2\} \\ &\leq \max\{\beta_2(\zeta_2(t_0), t - t_0), \gamma_{o2}(\|u_1\|)\} \end{aligned} \tag{6.39}$$

where, $\gamma_{o2}(s) = 8.5 \times (N_2^u)^2 s^2 = 87s^2$. Choose $\lambda_2 = 0.002$ and $k_2 = 0.086$ such that the small gain conditions hold.

Following the same step in Lemma 6.1, we choose $k_2 = 0.086$ and $\lambda_2 = 0.002$

Hence, we can obtain the following nested saturation controller

$$u = -\sigma_1(k_1 x_1 + \sigma_2(k_2 x_2)) \tag{6.40}$$

with parameters $\lambda_1 = 50$, $k_1 = 5$, $\lambda_2 = 0.01$, $k_2 = 0.086$. ∎

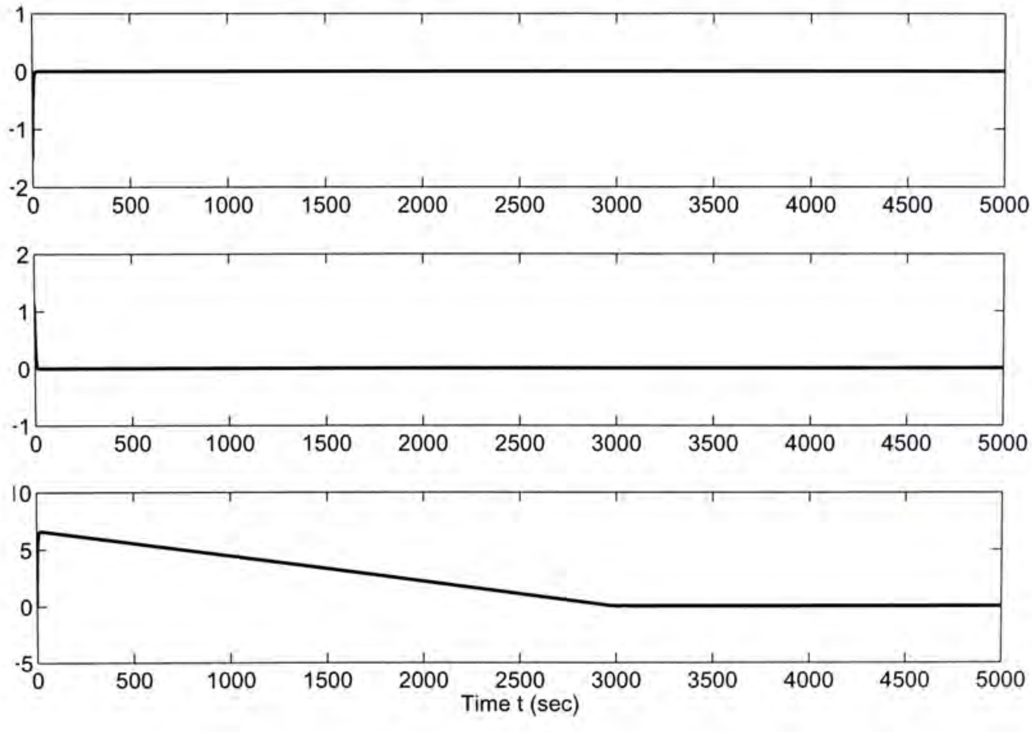

Figure 6.1: Profile of the states of the closed-loop system for Example 6.1

6.2 Conclusion

In this chapter, we have studied the global robust stabilization problem for a class of feedforward systems subject to both static time-varying disturbances and dynamic uncertainties. Compared with the results obtained in [2], our results are more general in the following aspects:

(i) System (5.1) is subject to dynamic uncertainties, thus is more complicated than the system considered in [2].

(ii) Disturbance rejection instead of disturbance attenuation is achieved. This objective is relevant to our work on the robust output regulation of feedforward systems in the subsequent chapters.

Chapter 7

Global Robust Stabilization and Output Regulation for A Class of Feedforward Systems

7.1 Introduction

In [45], the problem of global robust stabilization for the following class of feedforward systems is solved where χ subsystem can be viewed as input unmodeled dynamics unavailable for feedback control.

$$\begin{aligned}
\dot{\bar{x}}_1 &= \mu_1 \bar{x}_2 + g_1(\dot{\bar{x}}_2, ..., \dot{\bar{x}}_n, \chi, \mu) \\
&\vdots \\
\dot{\bar{x}}_i &= \mu_i \bar{x}_{i+1} + g_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \chi, \mu) \\
&\vdots \\
\dot{\bar{x}}_{n-1} &= \mu_{n-1} \bar{x}_n + g_{n-1}(\dot{\bar{x}}_n, \chi, \mu) \\
\dot{\bar{x}}_n &= \mu_n u \\
\dot{\chi} &= M\chi + Nu.
\end{aligned} \tag{7.1}$$

In this chapter, we will first address global robust stabilization problem for the following feedforward system

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \mu_1 \bar{x}_2 + g_1(\dot{\bar{x}}_2, ..., \dot{\bar{x}}_n, \dot{\chi}_1, ..., \dot{\chi}_n, \mu, v) \\
\dot{\chi}_1 &= M_1 \chi_1 + \bar{g}_1(\dot{\bar{x}}_2, ..., \dot{\bar{x}}_n, \dot{\chi}_2, ..., \dot{\chi}_n, \mu, v) \\
&\vdots \\
\dot{\bar{x}}_i &= \mu_i \bar{x}_{i+1} + g_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\chi}_i, ..., \dot{\chi}_n, \mu, v) \\
\dot{\chi}_i &= M_i \chi_i + \bar{g}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\chi}_{i+1}, ..., \dot{\chi}_n, \mu, v) \\
&\vdots \\
\dot{\bar{x}}_{n-1} &= \mu_{n-1} \bar{x}_n + g_{n-1}(\dot{\bar{x}}_n, \dot{\chi}_{n-1}, \dot{\chi}_n, \mu, v) \\
\dot{\chi}_{n-1} &= M_{n-1} \chi_{n-1} + \bar{g}_{n-1}(\dot{\bar{x}}_n, \dot{\chi}_n, \mu, v) \\
\dot{\bar{x}}_n &= \mu_n u + g_n(\chi_n) \\
\dot{\chi}_n &= M_n \chi_n
\end{aligned}
\tag{7.2}
$$

where, for $i = 1, ..., n,$, $\bar{x}_i \in \mathbb{R}$ is the state of the system (7.2), $\chi_i \in \mathbb{R}^{p_i}$ is dynamic uncertainty which is unavailable for feedback control. μ=col$(\mu_1, ..., \mu_n)$ is uncertain parameter (possibly time-varying). And the external disturbance $v : [t_0, \infty) \mapsto \Gamma$ is a family of piecewise continuous function of t with its range Γ a compact subset of $\mathbb{R}^{n_v}$. And for $i = 1, ..., n-1$, g_i are locally *Lipschitz* in col$(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\chi}_i, ..., \dot{\chi}_n)$ and piecewise continuous in (μ, v). And for $i = 1, ..., n-1$, $\bar{g}_i$ are locally *Lipschitz* in col$(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\chi}_{i+1}, ..., \dot{\chi}_n)$ and piecewise continuous in (μ, v). g_n is locally *Lipschitz* in χ_n.

It is noted that system (7.2) is more general than the system studied in [45] in that system (7.2) contains the dynamic uncertainties as apposed to the existing case where only the input unmodeled dynamics is present. We can also address the global robust stabilization problem for system (7.2) when the last subsystem in system (7.2) is in the form

$$\dot{\chi}_n = M_n \chi_n + N_n u.$$

Therefore our global robust stabilization result includes the stabilization result in [45] as a special case.

A general framework for tackling the robust output regulation problem was proposed by Huang and Chen in [18]. Under this framework, the robust output regulation for a given plant can be systematically converted into a robust stabilization problem for an appropriately defined augmented system. This general framework has been successfully applied to solve the global robust output regulation problem for lower triangular nonlinear systems [6] and the semi-global robust output regulation problem for a class of nonlinear affine systems in normal form [35]. In this paper, we will further utilize this framework to study the global robust output regulation problem for a class of feedforward systems. As in [18], our approach consists of two steps. First, the global robust output regulation problem of the given plant is converted into a global robust stabilization problem for an appropriately defined augmented system. Second, the global robust stabilization problem for the augmented system is solved on the basis of the combination of the small gain theorem with restrictions and nested saturation technique.

7.2 Preliminary

Consider the feedback interconnection,

$$\dot{x}_1 = f_1(x_1, v_1, u_1, d, t), \quad y_1 = h_1(x_1, v_1, u_1, d, t) \tag{7.3}$$

$$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = h_2(x_2, v_2, u_2, d, t) \tag{7.4}$$

subject to the interconnection constraints:

$$v_1 = y_2, \quad v_2 = y_1 \tag{7.5}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{p_i}$, $v_i \in \mathbb{R}^{q_i}$ with $p_1 = q_2$, $p_2 = q_1$, the functions $f_i(x_i, v_i, u_i, d, t)$ and $h_i(x_i, v_i, u_i, d, t)$ are locally *Lipschitz* in $\text{col}(x_i, v_i, u_i)$ and piecewise continuous in $\text{col}(d, t)$. And $f_i(0, 0, 0, d, t) = 0$, $h_i(0, 0, 0, d, t) = 0$. And suppose the assumption, *Lipschitz well posed*, holds.

A 7.1 The equations

$$\begin{aligned} y_1 &= h_1(x_1, h_2(x_2, y_1, u_2, d, t), u_1, d, t) \\ y_2 &= h_2(x_2, h_1(x_1, y_2, u_1, d, t), u_2, d, t) \end{aligned}$$

have unique solutions $y_1 \in \mathbb{R}^{p_1}$ and $y_2 \in \mathbb{R}^{p_2}$ so that (7.3) and (7.4) can be written in the following form

$$\dot{x} = f(x, u, d, t), \quad y = h(x, u, d, t)$$

where x=col(x_1, x_2), y=col(y_1, y_2), u=col(u_1, u_2), and the resulting f and h are locally *Lipschtz*.

The following is rephrased from Theorem B.3.2 of [23].

Theorem 7.1 Assume that subsystem (7.3) is RAG and RoALS with restrictions Δ_1 and Δ_1^u on v_1 and u_1 respectively, i.e., there exist class K functions γ_1, γ_1^u, $\overline{\gamma}_1$, $\overline{\gamma}_1^u$, such that, for any initial state $x_1(t_0) \in \mathbb{R}^{n_1}$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_1\|_a \leq \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_1\|_a \leq \Delta_1^u$, the solution of (7.3) exists and satisfies, for all $t \geq t_0$,

$$\|x_1\|_a \leq \max\{\gamma_1(\|v_1\|_a), \gamma_1^u(\|u_1\|_a)\}, \tag{7.6}$$

$$\|y_1\|_a \leq \max\{\overline{\gamma}_1(\|v_1\|_a), \overline{\gamma}_1^u(\|u_1\|_a)\}. \tag{7.7}$$

Assume that subsystem (7.4) is RAG and RoALS with restrictions Δ_2 and Δ_2^u on v_2 and u_2 respectively, i.e., there exist class K functions γ_2, γ_2^u, $\overline{\gamma}_2$, $\overline{\gamma}_2^u$, such that, for any initial state $x_2(t_0) \in \mathbb{R}^{n_2}$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_2\|_a \leq \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_2\|_a \leq \Delta_2^u$, the solution of (7.4) exists and satisfies, for all $t \geq t_0$,

$$\|x_2\|_a \leq \max\{\gamma_2(\|v_2\|_a), \gamma_2^u(\|u_2\|_a)\}, \tag{7.8}$$

$$\|y_2\|_a \leq \max\{\overline{\gamma}_2(\|v_2\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\}. \tag{7.9}$$

Suppose also that:

(1) for all $x_1(t_0)$ and $x_2(t_0)$ and all u_1, u_2 which are bounded on $[t_0, \infty)$, $x_1(t)$ and $x_2(t)$ are defined for all $t \geq t_0$;

(2) the small gain condition

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad r > 0$$

holds;

(3) there exists a finite time $T^* > t_0$ such that

$$\|y_{1[T^*,\infty)}\| \leq \Delta_2, \quad \|y_{2[T^*,\infty)}\| \leq \Delta_1.$$

Then the system composed of (7.3) and (7.4) is RAG with restrictions Δ_1^u and Δ_2^u on u_1 and u_2 respectively, viewing $x = \text{col}(x_1, x_2)$ as state, $y = \text{col}(y_1, y_2)$ as output and $u = \text{col}(u_1, u_2)$ as input. ∎

7.3 Global Robust Stabilization via Partial State Feedback

Like [23, 44, 45], the control applications will involve saturation functions, which satisfy the following properties:

(1) $\sigma : \mathbb{R} \to \mathbb{R}$;

(2) $\|\sigma'(s)\| = \|\frac{d\sigma(s)}{ds}\| \leq 2$;

(3) $s\sigma(s) > 0$ for all $s \neq 0$, $\sigma(0) = 0$;

(4) $\sigma(s) = sgn(s)$ for $\|s\| \geq 1$;

(5) $\|s\| < \|\sigma(s)\| < 1$ for $\|s\| < 1$.

A 7.2 For $i = 1, .., n$, M_i is Hurwitz.

A 7.3 $\mu_i^L \leq \mu_i \leq \mu_i^U$, $i = 1, ..., n$ for some positive (or, equivalently negative) numbers μ_i^L, μ_i^U.

Similar to [45], we first consider the following system

$$\begin{aligned}
\dot{\tilde{x}}_1 &= \mu_1\tilde{x}_2 + v_1 \\
\dot{\chi}_1 &= M_1\chi_1 + \bar{g}_1(\dot{\tilde{x}}_2, ..., \dot{\tilde{x}}_n, \dot{\chi}_2, ..., \dot{\chi}_n, \mu, v) \\
&\vdots \\
\dot{\tilde{x}}_i &= \mu_i\tilde{x}_{i+1} + v_i \\
\dot{\chi}_i &= M_i\chi_i + \bar{g}_i(\dot{\tilde{x}}_{i+1}, ..., \dot{\tilde{x}}_n, \dot{\chi}_{i+1}, ..., \dot{\chi}_n, \mu, v) \\
&\vdots \\
\dot{\tilde{x}}_{n-1} &= \mu_{n-1}\tilde{x}_n + v_{n-1} \\
\dot{\chi}_{n-1} &= M_{n-1}\chi_{n-1} + \bar{g}_{n-1}(\dot{\tilde{x}}_n, \dot{\chi}_n, \mu, v) \\
\dot{\tilde{x}}_n &= \mu_n u + v_n \\
\dot{\chi}_n &= M_n\chi_n
\end{aligned} \tag{7.10}$$

Under the change of coordinate introduced in [45]

$$\begin{aligned}
&\tilde{x}_1 \to z_1 = \tilde{x}_1, \\
&\tilde{x}_i \to z_i = \tilde{x}_i + \lambda_{i-1}\sigma(K_{i-1}z_{i-1}/\lambda_{i-1}), \quad i = 2, ..., n
\end{aligned} \tag{7.11}$$

and the control law

$$\begin{aligned}
u &= -\lambda_n\sigma(K_n\frac{x_n + \lambda_{n-1}\sigma(K_{n-1}z_{n-1}/\lambda_{n-1})}{\lambda_n}) \\
&= -\lambda_n\sigma(K_n\frac{z_n}{\lambda_n}),
\end{aligned} \tag{7.12}$$

system (7.10) is converted into the following form (for convenience we have left g_i and χ_i in the original coordinates)

$$\begin{aligned}
\dot{z}_1 &= -\mu_1\lambda_1\sigma(\frac{K_1 z_1}{\lambda_1}) + \mu_1 z_2 + v_1 \\
\dot{\chi}_1 &= M_1\chi_1 + \bar{g}_1(\dot{\bar{x}}_2, ..., \dot{\bar{x}}_n, \dot{\chi}_2, ..., \dot{\chi}_n, \mu, v) \\
&\vdots \\
\dot{z}_i &= -\mu_i\lambda_i\sigma(\frac{K_i z_i}{\lambda_i}) + \mu_i z_{i+1} + K_{i-1}\sigma'(\frac{K_{i-1}z_{i-1}}{\lambda_{i-1}})\dot{z}_{i-1} + v_i \\
\dot{\chi}_i &= M_i\chi_i + \bar{g}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\chi}_{i+1}, ..., \dot{\chi}_n, \mu, v) \\
&\vdots \\
\dot{z}_n &= -\mu_n\lambda_n\sigma(\frac{K_n z_n}{\lambda_n}) + K_{n-1}\sigma'(\frac{K_{n-1}z_{n-1}}{\lambda_{n-1}})\dot{z}_{n-1} + v_n \\
\dot{\chi}_n &= M_n\chi_n.
\end{aligned} \quad (7.13)$$

7.3.1 RAG with restrictions

z_i subsystem

Like Aappendix C of [45], define

$$\gamma^i = 24\mu_i^U, \quad \gamma_{v_i}^i = 24\frac{\mu_i^U}{\mu_i^L}, \quad i = 1, ..., n-1$$

$$\gamma_{v_j}^i = 48\frac{\mu_i^U}{\mu_i^L}K_{i-1}\gamma_{v_j}^{i-1}, \quad i = 2, ..., n, \ j = 1, ..., i-1.$$

Then we can give the following lemma which is established in [45] as Lemma C.2.1.

Lemma 7.1 Consider the following system

$$\begin{aligned}
\dot{z}_1 &= -\mu_1\lambda_1\sigma(\frac{K_1 z_1}{\lambda_1}) + \mu_1 z_2 + v_1 \\
&\vdots \\
\dot{z}_i &= -\mu_i\lambda_i\sigma(\frac{K_i z_i}{\lambda_i}) + \mu_i z_{i+1} + K_{i-1}\sigma'(\frac{K_{i-1}z_{i-1}}{\lambda_{i-1}})\dot{z}_{i-1} + v_i \\
&\vdots \\
\dot{z}_n &= -\mu_n\lambda_n\sigma(\frac{K_n z_n}{\lambda_n}) + K_{n-1}\sigma'(\frac{K_{n-1}z_{n-1}}{\lambda_{n-1}})\dot{z}_{n-1} + v_n.
\end{aligned} \quad (7.14)$$

Suppose that, for some positive numbers $v_{i,M}$, $i = 1, ..., n$, the design parameters λ_i and K_i $(i = 1, ..., n)$ can be chosen so that the following inequalities hold:

$$\begin{aligned}
\frac{\lambda_{i+1}}{K_{i+1}} &< \frac{\lambda_i}{4}, \quad i = 1, ..., n-1 \\
v_{1,M} &< \mu_1^L\frac{\lambda_1}{4}, \\
v_{i,M} + 4K_{i-1}\mu_{i-1}^U\lambda_{i-1} &< \mu_i^L\frac{\lambda_j}{4}, \quad i = 2, ..., n
\end{aligned} \quad (7.15)$$

and

$$6\frac{K_{i-1}}{K_i\mu_i^L}\gamma^i < 1, \quad i = 2, ..., n. \tag{7.16}$$

Then system (7.14) is RAG with restriction $v_{j,M}$ on the inputs v_j, $j = 1, ..., n$ and has linear gain functions. In particular, an asymptotic bound on the state variable z_i is given as

$$\|z_i\|_a \le \max\{\tilde{\gamma}_{v_1}^i \|v_1\|_a, ..., \tilde{\gamma}_{v_n}^i \|v_n\|_a\}, \tag{7.17}$$

where

$$\begin{aligned} \tilde{\gamma}_{v_1}^1 &= \frac{2}{\mu_1^L K_1}, \quad \tilde{\gamma}_{v_j}^1 = \frac{2\gamma_{v_j}^2}{K_1}, \quad j = 2, ..., n-1, \\ \tilde{\gamma}_{v_j}^i &= \begin{cases} \frac{6K_{i-1}}{K_i\mu_i^L}\gamma_{v_j}^{i-1}, & j = 1, ..., i-1, \\ \frac{3(j-i+1)}{\mu_j^L \prod_{\ell=i}^{j} K_\ell}, & j = i, ..., n-1. \end{cases} \end{aligned} \tag{7.18}$$

for $i = 2, .., n$ and

$$\tilde{\gamma}_{v_n}^n = \frac{3}{K_n\mu_n^L}, \quad \tilde{\gamma}_{v_n}^j = \tilde{\gamma}_{v_n}^n \prod_{\ell=j}^{n-1} \frac{3}{K_\ell}, \quad j = 1, ..., n-1. \tag{7.19}$$

∎

Remark 7.1 Consider the sets $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n$ defined as

$$\Omega_i = \{z \in \mathbb{R}^n \ : \ \|z_j\| \le \frac{\lambda_j}{K_j}, \quad j \ge i\}. \tag{7.20}$$

The following two facts are given in the proof of Lemma C.2.1 [23]:

(i) All the Ω_i, $i = 1, ..., n$ are positively invariant;

(ii) Every trajectory starting in $\mathbb{R}^n \backslash \Omega_n$ enters in finite time the set Ω_n and every trajectory starting in $\Omega_i \backslash \Omega_{i-1}$ for $i = 2, ..., n$, enters in finite time the set Ω_{i-1}. ∎

The following proposition is also established in [45].

Proposition 7.1 Suppose the sets $\{(\lambda_i^\star, K_i^\star) : i = 1, ..., n\}$, $\{v_{i,M}^\star : i = 1, ..., n\}$ are such that (7.15) and (7.16) hold. Then, for any $\varepsilon > 0$, the choice

$$(\lambda_i, K_i) = (\varepsilon^i \lambda_i^\star, \varepsilon K_i^\star), \quad i = 1, ..., n \tag{7.21}$$

fulfills (7.15) and (7.16), with $v_{i,M}$ given by

$$v_{i,M} = \varepsilon^i v_{i,M}^\star, \quad i = 1, ..., n. \tag{7.22}$$

∎

χ_i **subsystem**

Proposition 7.2 Consider the following nonlinear system

$$\dot{x} = Mx + g(u, d(t)) \tag{7.23}$$

where M is Hurwitz. The function $g(u, d(t))$ is C^1 satisfying $g(0, d(t)) = 0$ and $d(t)$ belongs to a compact set. Then system (7.23) is RAG with restriction Δ on the input u and without restriction on the initial state $x(t_0)$ and has a linear gain function.

Proof: Since M is Hurwitz, there exists a symmetric positive definite matrix P such that

$$PM + M^T P = -I.$$

Let $V(x) = \frac{1}{2}x^T Px$. The derivative of $V(x)$ along system (7.23) satisfies

$$\begin{aligned} \frac{\partial V(x)}{\partial x}\{Mx + g(u, d(t))\} &= -x^T x + 2x^T Pg(u, d(t)) \\ &\leq -\frac{1}{2}\|x\|^2 + 2\|Pg(u, d(t))\|^2. \end{aligned} \tag{7.24}$$

Since the function $g(u, d(t))$ is C^1 satisfying $g(0, d(t)) = 0$ and $d(t)$ belongs to a compact set it holds that

$$\|Pg(u, d(t))\| \leq \|u\|\rho(u)$$

for some smooth function $\rho(u)$.

If $\|u_{[t_0,\infty)}\| \leq \Delta$, there exists a positive number a such that $\rho(\|u_{[t_0,\infty)}\|) \leq a$.

Therefore, we obtain the following implication

$$\|x\| \geq \chi(\|u\|) = 2a\|u\| \Rightarrow \frac{\partial V(x)}{\partial x}\{Mx + g(u, d(t))\} < 0.$$

Note that $p_{\min}\|x\|^2 = \underline{\alpha}(\|x\|) \leq V(x) = \frac{1}{2}x^T Px \leq \bar{\alpha}(\|x\|) = p_{\max}\|x\|^2$, where $p_{\min}(p_{\max})$ is the minimal (maximal) eigenvalue of P. Therefore, system (7.23) is RISS with restriction Δ on u and has a linear gain function

$$\underline{\alpha}^{-1} \circ \bar{\alpha} \circ \chi(r) = 2\sqrt{\frac{p_{\max}}{p_{\min}}}ar.$$

Now it follows from $\|u\|_a \leq \Delta$ that $\rho(\|u\|_a) \leq a$. Therefore, system (7.23) is RAG with restriction Δ on the input u and has a linear gain function $2\sqrt{\frac{p_{\max}}{p_{\min}}}ar$. ∎

Lemma 7.2 Suppose the sets $\{(\lambda_i^\star, K_i^\star) \ : \ i = 1, ..., n\}$, $\{v_{i,M}^\star \ : \ i = 1, ..., n\}$ are such that (7.15) and (7.16) hold. Then for $i = 1, ..., n-1$, χ_i subsystem is RAG with respect to col($\dot{\tilde{x}}_{i+1}, ..., \dot{\tilde{x}}_n, \dot{\chi}_{i+1}, ..., \dot{\chi}_n$) and has linear gain functions, i.e., there exists positive numbers $L_{i,j}$ and $L'_{i,j}$ for $j = i+1, ..., n$ such that

$$\|\chi_i\|_a \leq \sum_{j=i+1}^{n} L_{i,j}\|\dot{\tilde{x}}_j\|_a + \sum_{j=i+1}^{n} L'_{i,j}\|\dot{\chi}_j\|_a \tag{7.25}$$

for $i = 1, ..., n-1$.

Proof: As far as the gains between v_j and $\dot{\tilde{x}}_i$ are concerned, note that

$$\dot{\tilde{x}}_i = \mu_i(z_{i+1} - \lambda_i\sigma(\frac{K_i z_i}{\lambda_i})) + v_i, \tag{7.26}$$

and thus, the following estimate holds:

$$\|\dot{\tilde{x}}_i\| \leq \mu_i^U(\|z_{i+1}\| + 2K_i\|z_i\|) + \|v_i\|. \tag{7.27}$$

For $j = 1, ..., n-1$, $i = j+1, ..., n$, the following inequalities are established in [45]

$$\begin{aligned} \|z_{i+1}\|_a &\leq \bar{\gamma}_{v_j}^{i+1}\|v_j\|_a \leq \varepsilon^{i-j}h_{i,j}\|v_j\|_a \\ \|z_i\|_a &\leq \bar{\gamma}_{v_j}^{i}\|v_j\|_a \leq \varepsilon^{i-j-1}h'_{i,j}\|v_j\|_a \end{aligned}$$

for some positive numbers $h_{i,j}$, $h'_{i,j}$. Thus, it follows that for $j = 1, ..., n-1$, $i = j+1, ..., n$, there exists positive number $\Gamma_{i,j}$ such that

$$\|\dot{\tilde{x}}_i\|_a \leq \Gamma_{i,j}\varepsilon^{i-j}\|v_j\|_a. \tag{7.28}$$

Since M_n is Hurwitz, the following asymptotic estimate

$$\|\dot{\chi}_n\|_a = \|\chi_n\|_a = 0$$

holds.

Assume that there exists positive numbers $L_{i,j}$, $L'_{i,j}$ and Δ_i for $i = \ell, ..., n-1$, $j = i+1, ..., n$ such that

$$\|\chi_i\|_a \leq \sum_{j=i+1}^{n} L_{i,j}\|\dot{\tilde{x}}_j\|_a + \sum_{j=i+1}^{n} L'_{i,j}\|\dot{\chi}_j\|_a \tag{7.29}$$

and $\|\dot{\chi}_i\|_a \leq \Delta_i$. It follows from (7.28) that

$$\bar{g}_{\ell-1}(\|\dot{\tilde{x}}_\ell\|_a, ..., \|\dot{\tilde{x}}_n\|_a, \|\dot{\chi}_\ell\|_a, ..., \|\dot{\chi}_n\|_a, \mu, v) \leq \hat{a}_i$$

for positive number $\hat{a}_i$. It follows from Proposition 7.2 that there exists positive numbers $L_{\ell-1,j}$ and $L'_{\ell-1,j}$ for $j = \ell, ..., n$ such that

$$\|\chi_{\ell-1}\|_a \leq \sum_{j=\ell}^{n} L_{\ell-1,j}\|\dot{\tilde{x}}_j\|_a + \sum_{j=\ell}^{n} L'_{\ell-1,j}\|\dot{\chi}_j\|_a. \tag{7.30}$$

Moreover, $\|\dot{\chi}_{\ell-1}\|_a \leq \Delta_{\ell-1}$ for some positive number $\Delta_{\ell-1}$. By induction, it completes the proof.

∎

7.3.2 Fulfillment of the restrictions

Lemma 7.3 Suppose the sets $\{(\lambda_i^\star, K_i^\star) \ : \ i = 1, ..., n\}$, $\{v_{i,M}^\star \ : \ i = 1, ..., n\}$ are such that (7.15) and (7.16) hold. Choosing ε sufficiently small such that

(1) The following estimate holds for $j = 1, ..., n-1$ and $i = j, ..., n-1$

$$\|\dot{\chi}_i\|_a \leq \varepsilon^{i+1-j} h_{i,j} \|v_j\|_a \tag{7.31}$$

for some positive number $h_{i,j}$;

(2) The restrictions $v_{i,M}$ on v_i, $i = 1, ..., n$ are satisfied in finite time, namely there exists a time T and positive number ε such that for all $t \geq T$

$$\|\dot{\chi}_i(t)\| \ \leq \ \vartheta_i \varepsilon^{i+1}, \tag{7.32}$$

$$\|g_i(\dot{\bar{x}}_{i+1}(t), ..., \dot{\bar{x}}_n(t), \dot{\chi}_i(t), ..., \dot{\chi}_n(t), \mu, v)\| \ < \ v_{i,M} \tag{7.33}$$

for some positive number ϑ_i $(i = 1, ..., n)$.

Proof: $i = n$: Since M_n is Hurwitz, there exists positive number T_n such that

$$\begin{aligned} \|g_n(\chi_n(t))\| \ &< \ \min\{v_{n,M}, \mu_n^U \lambda_n\} \\ \|\dot{\chi}_n(t)\| \ &< \ \varepsilon^{n+1} \end{aligned} \tag{7.34}$$

for all $t \geq T_n$. Therefore (7.32) and (7.33) are true for $i = n$.

$i = n-1$: Since $\dot{\bar{x}}_n = \mu_n u + g_n(\chi_n)$ and $\|u\| \leq \lambda_n$, it holds that

$$\|\dot{\bar{x}}_n(t)\| \leq \mu_n^U \lambda_n + \mu_n^U \lambda_n = 2\mu_n^U \lambda_n^\star \varepsilon^n \tag{7.35}$$

for all $t \geq T_n$.

Since χ_{n-1} subsystem is RAG with linear gain functions, the following estimate

$$\|\chi_{n-1}\|_a \leq L_{n-1,n} \|\dot{\bar{x}}_n\|_a \tag{7.36}$$

holds for some positive number $L_{n-1,n}$. Hence, there exists a time $T_{n-1} > T_n$ such that

$$\|\chi_{n-1}(t)\| \leq 2L_{n-1,n} \mu_n^U \lambda_n^\star \varepsilon^n \tag{7.37}$$

for all $t \geq T_{n-1}$. Since $\bar{g}_{n-1}(\dot{\bar{x}}_n, \dot{\chi}_n, \mu, v)$ is locally Lipschitz in $\mathrm{col}(\dot{\bar{x}}_n, \dot{\chi}_n)$, for sufficiently small ε in (7.34) and (7.35), there exists positive number ϑ_{n-1} such that

$$\|\dot{\chi}_{n-1}(t)\| \leq \vartheta_{n-1}\varepsilon^n \tag{7.38}$$

for all $t \geq T_{n-1}$. Thus (7.32) is true for $i = n-1$.

Since

$$\|\chi_{n-1}\|_a \leq L_{n-1,n}\|\dot{\bar{x}}_n\|_a,$$

it follows from (7.28) that for $j = 1, ..., n-1$,

$$\|\chi_{n-1}\|_a \leq L_{n-1,n}\Gamma_{n,j}\varepsilon^{n-j}\|v_j\|_a. \tag{7.39}$$

Since $\bar{g}_{n-1}(\dot{\bar{x}}_n, \dot{\chi}_n, \mu, v)$ is locally Lipschitz in $\mathrm{col}(\dot{\bar{x}}_n, \dot{\chi}_n)$, for sufficiently small ε in (7.34) and (7.35), there exists positive number $h_{n-1,j}$ for $j = 1, ..., n-1$ such that

$$\|\dot{\chi}_{n-1}\|_a \leq \varepsilon^{n-j}h_{n-1,j}\|v_j\|_a. \tag{7.40}$$

Thus (7.31) is true for $i = n-1$.

Since $g_{n-1}(\dot{\bar{x}}_n, \dot{\chi}_n, \mu, v)$ is locally Lipschitz in $\mathrm{col}(\dot{\bar{x}}_n, \dot{\chi}_n)$, for sufficiently small ε in (7.34) and (7.35), there exists positive numbers $G_{n-1,n}$ and $G'_{n-1,n}$ such that

$$\begin{aligned} \|g_{n-1}(\dot{\bar{x}}_n(t), \dot{\chi}_n(t), \mu, v)\| &\leq G_{n-1,n}\|\dot{\bar{x}}_n(t)\| + G'_{n-1,n}\|\dot{\chi}_n(t)\| \\ &\leq 2G_{n-1,n}\mu_n^U\lambda_n^\star\varepsilon^n + G'_{n-1,n}\varepsilon^{n+1} \end{aligned}$$

for all $t \geq T_{n-1}$. Hence, (7.33) is true for $j = n-1$ if

$$2G_{n-1,n}\mu_n^U\lambda_n^\star\varepsilon^n + G'_{n-1,n}\varepsilon^{n+1} \leq v_{n-1,M}^\star\varepsilon^{n-1} \tag{7.41}$$

which can be satisfied taking ε sufficiently small.

$i = \ell$: Now suppose that (7.31), (7.32) and (7.33) is true for $i = \ell+1, ..., n$, that is

$$\|g_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\chi}_i, ..., \dot{\chi}_n, \mu, v)\| < v_{i,M}, \quad i = \ell+1, ..., n, \tag{7.42}$$

$$\|\dot{\chi}_i(t)\| \leq \vartheta_i\varepsilon^{i+1}, \quad i = \ell+1, ..., n, \quad t \geq T_{\ell+1} \tag{7.43}$$

$$\|\dot{\chi}_i\|_a \leq \varepsilon^{i+1-j}h_{i,j}\|v_j\|_a, \quad j = 1, ..., n-1, \ \ i = \ell+1, ..., n-1. \tag{7.44}$$

To show that (7.32) holds for $i = \ell$, recall that

$$\dot{\bar{x}}_i = \mu_i(z_{i+1} - \lambda_i\sigma(\frac{K_i z_i}{\lambda_i})) + g_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\chi}_i, ..., \dot{\chi}_n, \mu, v). \tag{7.45}$$

If $T_{\ell+1}$ is sufficiently large, $z \in \Omega_\ell$. We obtain the following extimate

$$\begin{aligned} \|\dot{\tilde{x}}_i\| &\le \mu_i^U(\frac{\lambda_{i+1}}{K_{i+1}} + \lambda_i) + v_{i,M} \le \frac{5}{4}\mu_i^U\lambda_i + v_{i,M} \\ &= (\frac{5}{4}\mu_i^U\lambda_i^\star + v_{i,M}^\star)\varepsilon^i \stackrel{def}{=} \delta_i\varepsilon^i \end{aligned} \tag{7.46}$$

for all $i = \ell+1, ..., n-1$.

Since χ_ℓ subsystem is RAG with linear gain function, the following asymptotic estimate

$$\|\chi_\ell\|_a \le \sum_{j=\ell+1}^{n} L_{\ell,j}\|\dot{\tilde{x}}_j\|_a + \sum_{j=\ell+1}^{n} L'_{\ell,j}\|\dot{\chi}_j\|_a \tag{7.47}$$

holds for some positive numbers $L_{\ell,j}$ and $L'_{\ell,j}$ for $j = \ell+1, ..., n$. Thus, there exists a time $T_\ell > T_{\ell+1}$ such that

$$\|\chi_\ell(t)\| \le \sum_{j=\ell+1}^{n} L_{\ell,j}\delta_j\varepsilon^j + \sum_{j=\ell+1}^{n} L'_{\ell,j}\vartheta_j\varepsilon^{j+1} \le \vartheta_\ell\varepsilon^{\ell+1} \tag{7.48}$$

for all $t \ge T_\ell$.

Since $g_\ell(\dot{\tilde{x}}_{\ell+1}, ..., \dot{\tilde{x}}_n, \dot{\chi}_\ell, ..., \dot{\chi}_n, \mu, v)$ is locally Lipschitz in $\mathrm{col}(\dot{\tilde{x}}_{\ell+1}, ..., \dot{\tilde{x}}_n, \dot{\chi}_\ell, ..., \dot{\chi}_n)$, for sufficiently small ε, it follows from (7.44) that $z \in \Omega_\ell$ implies

$$\begin{aligned} &\|g_\ell(\dot{\tilde{x}}_{\ell+1}(t), ..., \dot{\tilde{x}}_n(t), \dot{\chi}_\ell(t), ..., \dot{\chi}_n(t), \mu, v)\| \\ \le\ & \sum_{j=\ell+1}^{n} G_{\ell,j}\|\dot{\tilde{x}}_j(t)\| + \sum_{j=\ell}^{n} G'_{\ell,j}\|\dot{\chi}_j(t)\| \\ \le\ & \sum_{j=\ell+1}^{n} G_{\ell,j}\delta_j\varepsilon^j + \sum_{j=\ell}^{n} G'_{\ell,j}\vartheta_j\varepsilon^{j+1} \end{aligned} \tag{7.49}$$

for all $t \ge T_\ell$. Hence (7.33) holds for $i = \ell$ if

$$\sum_{j=\ell+1}^{n} G_{\ell,j}\delta_j\varepsilon^j + \sum_{j=\ell}^{n} G'_{\ell,j}\vartheta_i\varepsilon^{j+1} < v_{\ell,M}^\star\varepsilon^\ell \tag{7.50}$$

which can be satisfied taking ε sufficiently small.

It follows from (7.47) and the expression of χ_ℓ subsystem that there exist positive numbers $\bar{L}_{\ell,j}$ and $\bar{L}'_{\ell,j}$ for $j = \ell+1, ..., n$ such that

$$\|\dot{\chi}_\ell\|_a \le \sum_{i=\ell+1}^{n} \bar{L}_{\ell,i}\|\dot{\tilde{x}}_i\|_a + \sum_{i=\ell+1}^{n} \bar{L}'_{\ell,i}\|\dot{\chi}_i\|_a. \tag{7.51}$$

Substituting (7.43) and (7.46) into (7.51) gives that (7.32) holds for $i = \ell$.

Substituting (7.44) and (7.28) into (7.51) gives that there exists positive number $h_{\ell,j}$ $(j = 1, ..., n-1)$ such that

$$\|\dot{\chi}_\ell\|_a \le \varepsilon^{\ell+1-j}h_{\ell,j}\|v_j\|_a. \tag{7.52}$$

Hence (7.31) holds for $i = \ell$

By induction, it completes the proof. ∎

7.3.3 Small gain conditions

System (7.2) can be viewed as the feedback interconnection of system (7.10) viewing v_i as input and $\dot{\tilde{x}}_i, \dot{\chi}_i$ as outputs and static mapping $g_i(\dot{\tilde{x}}_{i+1}, ..., \dot{\tilde{x}}_n, \dot{\chi}_i, ..., \dot{\chi}_n, \mu, v)$ viewing $\dot{\tilde{x}}_i, \dot{\chi}_i$ as inputs and $g_i(\dot{\tilde{x}}_{i+1}, ..., \dot{\tilde{x}}_n, \dot{\chi}_i, ..., \dot{\chi}_n, \mu, v)$ as output. These two systems are subject to the interconnection

$$\begin{aligned} v_i &= g_i(\dot{\tilde{x}}_{i+1}, ..., \dot{\tilde{x}}_n, \dot{\chi}_i, ..., \dot{\chi}_n, \mu, v), \quad i = 1, ..., n-1, \\ v_n &= g_n(\chi_n) = g_n(M_n^{-1}\dot{\chi}_n). \end{aligned} \tag{7.53}$$

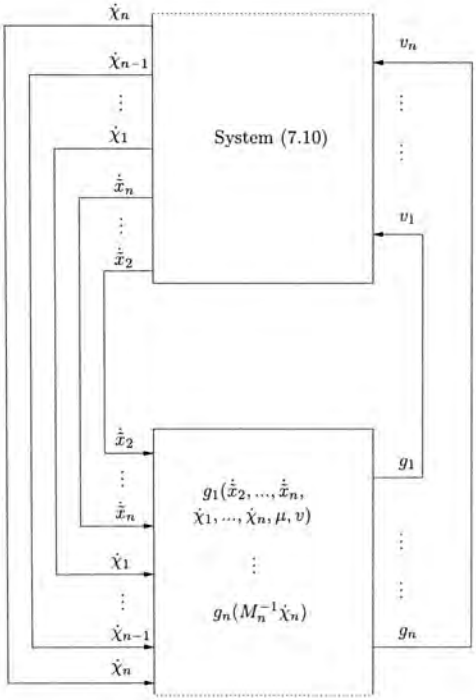

Figure 7.1: Inter-connection of system 7.2

Recall that

$$\|g_j(\dot{\tilde{x}}_{j+1}, ..., \dot{\tilde{x}}_n, \dot{\chi}_j, ..., \dot{\chi}_n, \mu, v)\| \le \sum_{i=j+1}^{n} G_{j,i}\|\dot{\tilde{x}}_i\| + \sum_{k=j}^{n} G'_{j,k}\|\dot{\chi}_k\|,$$

$$\|\dot{\tilde{x}}_i\|_a \le \Gamma_{i,j}\varepsilon^{i-j}\|v_j\|_a, \quad \|\dot{\chi}_k\|_a \le h_{k,j}\varepsilon^{k+1-j}\|v_j\|_a, \quad \|\dot{\chi}_n\|_a = \|\chi_n\|_a = 0 \tag{7.54}$$

where $j = 1, ..., n-1$, $i = j+1, ..., n$ and $k = j, ..., n-1$.

As pointed out in [45], the goal that system (7.2) is globally attractive can be achieved if the gain between g_i and v_j can be rendered arbitrarily small. Therefore, the overall system is globally attractive if the small gain condition

$$\sum_{i=j+1}^{n} G_{j,i}\Gamma_{i,j}\varepsilon^{i-j} + \sum_{k=j}^{n-1} G'_{j,k}h_{k,j}\varepsilon^{k+1-j} < 1 \tag{7.55}$$

is satisfied for $j = 1, ..., n-1$. Clearly, (7.55) can be satisfied taking ε sufficiently small.

7.3.4 Uniform Global Asymptotic Stability of Closed Loop System

Note that the Jacobian linearization of each subsystem in (7.13) is Hurwitz. Appealing to Lemma 4.7 [31] in combination of the triangular structure of (7.13), we can obtain that the linearization of (7.13) is UGAS. Therefore, the system (7.13) under the controller (7.12) is GAS by using Theorem 7.1.

7.4 Global Robust Output Regulation

Consider the following feedforward system,

$$\begin{aligned}
\dot{x}_1 &= a_1x_1 + c_2x_2 + f_1(\dot{x}_2, ..., \dot{x}_n, v, w) \\
&\vdots \\
\dot{x}_i &= a_ix_i + c_{i+1}x_{i+1} + f_i(\dot{x}_{i+1}, ..., \dot{x}_n, v, w) \\
&\vdots \\
\dot{x}_{n-1} &= a_{n-1}x_{n-1} + c_nx_n + f_{n-1}(\dot{x}_n, v, w) \\
\dot{x}_n &= a_nx_n + bu + f_n(v, w) \qquad (7.56) \\
e &= x_n - q_d(v, w) \\
\dot{v} &= S \cdot v \qquad (7.57)
\end{aligned}$$

where, $x = \text{col}(x_1, ..., x_n)$ with $x_i \in \mathbb{R}$ $(i = 1, ..., n)$ are the plant states, $u \in \mathbb{R}$ is the control input, $e \in \mathbb{R}$ is the tracking error, $v \in \mathbb{R}^q$ is the exogenous signal representing the disturbance and/or the reference input, and $w \in \mathbb{R}^N$ is the uncertain parameter. The coefficients $a_i = 0$ $(i = 1, ..., n-1)$, $b > 0$ and $c_i \neq 0$ $(i = 2, ..., n)$. All the functions are sufficiently smooth with $f_i(0, ..., 0, 0, w)$ for $i = 1, ..., n$ and $q_d(0, w) = 0$ for all $w \in \mathbb{R}^{n_w}$. The exosystem is neutrally stable, i.e., the eigenvalues of S are simple and have zero real parts.

Remark 7.2 Since we will also consider the case when $a_i < 0$ $(i = 1, ..., n-1)$, we will keep a_i in system (7.56). Moreover, it is noted that system (7.56) is not so special as it looks like. For example, the following systems can be converted into the form of (7.56).

$$\begin{aligned}
\dot{x}_1 &= c_2 x_2 + f_1(x_3, ..., x_n, u, v, w) \\
&\vdots \\
\dot{x}_i &= c_{i+1} x_{i+1} + f_i(x_{i+2}, ..., x_n, u, v, w) \\
&\vdots \\
\dot{x}_{n-2} &= c_{n-1} x_{n-1} + f_{n-2}(x_n, u, v, w) \\
\dot{x}_{n-1} &= c_n x_n + f_{n-1}(u, v, w) \\
\dot{x}_n &= bu + f_n(v, w) \\
e &= x_n - q_d(v, w) \\
\dot{v} &= S \cdot v. \qquad (7.58)
\end{aligned}$$

∎

The objective of this section is to present the solvability conditions of the state feedback global robust servomechanism problem or alternatively the global robust output regulation problem for system (7.56). The problem can be precisely described as follows.

The class of dynamic state feedback control law considered here can be described by

$$\begin{aligned}
u &= k(x, z_c, e) \\
\dot{z}_c &= f_z(x, z_c, e) \qquad (7.59)
\end{aligned}$$

where z_c is the compensator state vector of dimension n_c to be specified later.

Global robust output regulation problem: Design a control law of the form (7.59) such that

(i) For any $x(0)$ and $z_c(0)$, the trajectories of the closed-loop system exist and are bounded for all $t \geq 0$.

(ii) The tracking error $e(t)$ of the trajectories described in (i) approaches zero asymptotically, i.e., $lim_{t\to\infty} e(t) = 0$.

Conditions under which the robust servomechanism problem for general nonlinear systems can be converted into the above robust regulation problem are given in [18]. For the class of feedforward systems (7.56), these conditions can be given as follows:

A 7.4 There exist sufficiently smooth functions $\mathbf{x}(v, w) = \text{col}(\mathbf{x}_1(v, w), \cdots, \mathbf{x}_n(v, w))$ and

$\mathbf{u}(v,w)$, with $\mathbf{x}(0,0)=0$ and $\mathbf{u}(0,0)=0$, such that, for $i=1,...,n-1$

$$\begin{aligned}
\mathbf{x}_n(v,w) &= q_d(v,w) \\
\dot{\mathbf{x}}_n(v,w) &= a_n\mathbf{x}_n(v,w)+b\mathbf{u}(v,w)+f_n(v,w) \\
\dot{\mathbf{x}}_i(v,w) &= a_i\mathbf{x}_i(v,w)+c_{i+1}\mathbf{x}_{i+1}(v,w)+f_i(\dot{\mathbf{x}}_{i+1}(v,w),...,\dot{\mathbf{x}}_n(v,w),v,w).
\end{aligned}$$

Remark 7.3 If the solution for the regulator equation is $\mathbf{x}(v,w)=0$ and $\mathbf{u}(v,w)=-b^{-1}f_n(v,w)$, the output regulation problem for (7.56) reduces to the input suppression problem studied in [45]. ∎

However, the solvability of the above regulator equations is insufficiently for solving the robust output regulation, some additional conditions have to be imposed on the solution of the regulator equations.

A 7.5 Let $\pi_n=\mathbf{u}(v,w)$ and $\pi_i=\mathbf{x}_i(v,w)$ for $i=1,..,n-1$. There exists positive number r_i and real numbers $\kappa_{i,1},...,\kappa_{i,r_i}$ such that

$$\frac{d^{r_i}\pi_i(v(t),w)}{dt^{r_i}}-\kappa_{i,1}\pi_i(v(t),w)-\kappa_{i,2}\dot{\pi}_i(v(t),w)-\cdots-\kappa_{i,r_i}\frac{d^{r_i-1}\pi_i(v(t),w)}{dt^{r_i-1}}=0$$

for all trajectories $v(t)\in V$ of the exosystem and all $w\in W$.

Remark 7.4 It is shown in Corollary 6.13 [19] that under assumption 7.5, system (7.56) has a steady-state generator with output $\mathrm{col}(x_1,...,x_{n-1},u)$ described as follows.

For $i=1,...,n$, $j=1,...,I_i$, let

$$\begin{aligned}
\theta_i(v,w) &= T_i\mathrm{col}\ (\theta_i^1(v,w),...,\theta_i^{I_i}(v,w)), \\
\theta_i^j(v,w) &= \mathrm{col}\ (\pi_i^j(v,w),\dot{\pi}_i^j(v,w),...,\frac{d^{r_i^j-1}\pi_i^j(v,w)}{dt^{r_i^j-1}}),
\end{aligned}$$

and $\theta(v,w)=\mathrm{col}(\theta_1(v,w),...,\theta_n(v,w))$, where T_i is any nonsingular matrix of dimension r_i. Then, it is ready to verify that θ satisfies

$$\dot{\theta}=\alpha(\theta),\quad \mathrm{col}\ (x_1,...,x_{n-1},u)=\beta(\theta) \tag{7.60}$$

where $\alpha(\theta)=T\Phi T^{-1}\theta$ with $T=\text{block diag}(T_1,...,T_n)$, $\Phi=\text{block diag}(\Phi_1,...,\Phi_n)$ with $\Phi_i=\text{block diag}(\Phi_i^1,...,\Phi_i^{I_i})$, and $\beta(\theta)=\mathrm{col}(\beta_1(\theta_1),...,\beta_n(\theta_n))$ with $\beta_i(\theta_i)=\Psi_iT_i^{-1}\theta_i$,

$$\Phi_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \kappa_{i,1} & \kappa_{i,2} & \kappa_{i,3} & \cdots & \kappa_{i,r_i} \end{bmatrix}, \quad \Psi_i = [1 \quad 0 \quad \cdots \quad 0].$$

The triple (θ, α, β) is called the steady-state generator of system (7.56) with output $\text{col}(x_1, ..., x_{n-1}, u)$. It can be seen that the steady-state generator is a dynamic system that can reproduce the partial solution of the regulator equations. With the steady-state generator ready, we can further define the internal model as follows.

For $i = 1, ..., n$, let (M_i, N_i) be any controllable pair with M_i a *Hurwitz* matrix of dimension r_i and N_i a column vector. Then there exists a nonsingular matrix T_i satisfying the *Sylvester* equation $T_i\Phi_i - M_iT_i = N_i\Psi_i$. Define, for $i = 1, ..., n-2$

$$\begin{aligned} \dot{\eta}_i &= M_i\eta_i + N_ix_i + L_i(x_i - \Psi_iT_i^{-1}\eta_i) + H_i(x_{i+1} - \Psi_{i+1}T_{i+1}^{-1}\eta_{i+1}) \\ \dot{\eta}_{n-1} &= M_{n-1}\eta_{n-1} + N_{n-1}x_{n-1} + L_{n-1}(x_{n-1} - \Psi_{n-1}T_{n-1}^{-1}\eta_{n-1}) + H_{n-1}e \\ \dot{\eta}_n &= M_n\eta_n + N_nu + L_n(u - \Psi_nT_n^{-1}\eta_n) + H_ne. \end{aligned} \tag{7.61}$$

where $\eta_i \in \mathbb{R}^{r_i}$ and L_i, H_i $(i = 1, ..., n)$ can be any matrixes with appropriate dimensions. The collection of (7.61) is called the internal model of (7.56) with output $\text{col}(x_1, ..., x_{n-1}, u)$. ∎

Remark 7.5 If $L_i = H_i = 0$ for $i = 1, .., n$, the internal model (7.61) reduces to the internal model candidate introduced in [18]. The purpose of involving L_i and H_i in (7.61) is to make the augmented system in the standard feedforward form with the property that the linearization of each dynamic uncertainty is Hurwitz. ∎

Here we make an extra assumption as follows:

A 7.6 For $i = 1, ..., n-1$, Φ_i is invertible.

Remark 7.6 When the steady state $\mathbf{x}(v, w) = 0$, Assumption 7.6 can be relaxed. Assume $\pi(v, w)$ be a degree k polynomial in v. As pointed out in Remark 6.15 [19], if $P(\lambda) =$

$\lambda^r - \kappa_1 - \kappa_2\lambda - \cdots - \kappa_r\lambda^{r-1}$ is the minimal polynomial of the matrix S_{kf} where

$$S_{kf} = \begin{bmatrix} S^{[1]} & 0 & \cdots & 0 \\ 0 & S^{[2]} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & S^{[k]} \end{bmatrix},$$

$P(\lambda)$ is a zeroing polynomial of $\pi(v,w)$. If $v^{[i]} = 0$ (i is even) in $\mathbf{x}_i(v,w)$, $S^{[i]} = 0$ (i is even) in S_{kf}. It follows from Theorem 5.16 [19] that Φ_i is invertible. ∎

For convenience, the remainder of design procedure will be split into three steps.

Step 1.

Now attaching the internal model (7.61) to system (7.56) yields the augmented system with the state variables $(x_1, \eta_1, ..., x_n, \eta_n)$. Performing on the augmented system the coordinate and input transformation:

$$\begin{aligned} \bar{x}_n &= x_n - \mathbf{x}_n(v,w) \\ \bar{x}_i &= x_i - \Psi_i T_i^{-1}\eta_i, \quad i = 1, ..., n-1 \\ \bar{u} &= u - \Psi_n T_n^{-1}\eta_n \\ \bar{\eta} &= \eta - \theta \end{aligned}$$

defines the augmented system in new coordinates and inputs as follows, for $i = 1, ..., n-2$

$$\begin{aligned} \dot{\bar{x}}_i &= (a_i - \Psi_i T_i^{-1} L_i - \Psi_i T_i^{-1} N_i)\bar{x}_i + (c_{i+1} - \Psi_i T_i^{-1} H_i)\bar{x}_{i+1} \\ &+ \Psi_i T_i^{-1}(a_i I - M_i - N_i \Psi_i T_i^{-1})\bar{\eta}_i + c_{i+1}\Psi_{i+1}T_{i+1}^{-1}\bar{\eta}_{i+1} \\ &+ \hat{f}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\bar{\eta}}_{i+1}, ..., \dot{\bar{\eta}}_{n-1}, v, w) \\ \dot{\bar{\eta}}_i &= (M_i + N_i \Psi_i T_i^{-1})\bar{\eta}_i + (L_i + N_i)\bar{x}_i + H_i \bar{x}_{i+1} \\ &\vdots \\ \dot{\bar{x}}_{n-1} &= (a_{n-1} - \Psi_{n-1}T_{n-1}^{-1}L_{n-1} - \Psi_{n-1}T_{n-1}^{-1}N_{n-1})\bar{x}_{n-1} \\ &+ \Psi_{n-1}T_{n-1}^{-1}(a_{n-1}I - M_{n-1} - N_{n-1}\Psi_{n-1}T_{n-1}^{-1})\bar{\eta}_{n-1} \\ &+ (c_n - \Psi_{n-1}T_{n-1}^{-1}H_{n-1})\bar{x}_n + \hat{f}_{n-1}(\dot{\bar{x}}_n, v, w) \\ \dot{\bar{\eta}}_{n-1} &= (M_{n-1} + N_{n-1}\Psi_{n-1}T_{n-1}^{-1})\bar{\eta}_{n-1} + (L_{n-1} + N_{n-1})\bar{x}_{n-1} + H_{n-1}\bar{x}_n \\ \dot{\bar{x}}_n &= a_n \bar{x}_n + b\bar{u} + b\Psi_n T_n^{-1}\bar{\eta}_n \\ \dot{\bar{\eta}}_n &= (M_n + N_n \Psi_n T_n^{-1})\bar{\eta}_n + (L_n + N_n)\bar{u} + H_n \bar{x}_n \end{aligned} \tag{7.62}$$

where, for $i = 1, ..., n-2$,

$$\hat{f}_{n-1}(\dot{\bar{x}}_n, v, w) = f_{n-1}(\dot{\bar{x}}_n + \dot{\mathbf{x}}_n(v,w), v, w)$$

$$\begin{aligned} & \hat{f}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\bar{\eta}}_{i+1}, ..., \dot{\bar{\eta}}_{n-1}, v, w) \\ = \;& f_i(\dot{\bar{x}}_n + \dot{\mathbf{x}}_n(v,w), \dot{\bar{x}}_{n-1} + \Psi_{n-1}T_{n-1}^{-1}(\dot{\bar{\eta}}_{n-1} + \dot{\theta}_{n-1}), ..., \dot{\bar{x}}_{i+1} + \Psi_{i+1}T_{i+1}^{-1}(\dot{\bar{\eta}}_{i+1} + \dot{\theta}_{i+1}), v, w) \\ + \;& a_i \Psi_i T_i^{-1}\theta_i + c_{i+1}\Psi_{i+1}T_{i+1}^{-1}\theta_{i+1}. \end{aligned}$$

It is easy to check that $\hat{f}_i(0, ..., 0, v, w) = 0$ for $i = (1, ..., n-1)$, that is, the origin $\mathrm{col}(\bar{x}, \bar{\eta}) = 0$ is the equilibrium point of the unforced augmented system (7.62) for all $v(t)$ of the exosystem and any $w \in \mathbb{R}^N$.

Step 2.

By Corollary 7.4 [19], the global robust output regulation problem for system (7.56) will be solved if the equilibrium point of (7.62) can be rendered to be globally asymptotically stable for all trajectories $v(t) \in V$ of the exosystem and all $w \in W$. However, since $M_i + N_i\Psi_i T_i^{-1} = T_i^{-1}\Phi_i T_i$ and all eigenvalues of the matrix Φ_i have zero real part, $\bar{\eta}_i$ subsystem in (7.62) does not satisfy Assumption 7.2. Therefore, the design procedure in Section 3 can not be directly applied to system (7.62). To circumvent this difficulty, we further performance on (7.62) another coordinate transformation

$$z_i = \bar{\eta}_i - P_i\bar{x}_i, \quad i = 1, ..., n-1,$$

which yields, for $i = 1, ..., n-2$,

$$\begin{aligned}
\dot{z}_i &= \{M_i + N_i\Psi_iT_i^{-1} - P_i\Psi_iT_i^{-1}(a_iI - M_i - N_i\Psi_iT_i^{-1})\}z_i \\
&+ \{(M_i + N_i\Psi_iT_i^{-1})P_i + L_i + N_i - P_i(a_i - \Psi_iT_i^{-1}L_i - \Psi_iT_i^{-1}N_i) \\
&- P_i\Psi_iT_i^{-1}(a_iI - M_i - N_i\Psi_iT_i^{-1})P_i\}\bar{x}_i \\
&+ \{H_i - P_i(c_{i+1} - \Psi_iT_i^{-1}H_i) - c_{i+1}P_i\Psi_{i+1}T_{i+1}^{-1}P_{i+1}\}\bar{x}_{i+1} - c_{i+1}P_i\Psi_{i+1}T_{i+1}^{-1}z_{i+1} \\
&- P_i\hat{f}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\bar{\eta}}_{i+1}, ..., \dot{\bar{\eta}}_{n-1}, v, w) \\
&= \{M_i + N_i\Psi_iT_i^{-1} - P_i\Psi_iT_i^{-1}(a_iI - M_i - N_i\Psi_iT_i^{-1})\}z_i \\
&+ (I + P_i\Psi_iT_i^{-1})\{L_i + N_i + (-a_iI + M_i + N_i\Psi_iT_i^{-1})P_i\}\bar{x}_i \\
&+ \{H_i - P_i(c_{i+1} - \Psi_iT_i^{-1}H_i) - c_{i+1}P_i\Psi_{i+1}T_{i+1}^{-1}P_{i+1}\}\bar{x}_{i+1} - c_{i+1}P_i\Psi_{i+1}T_{i+1}^{-1}z_{i+1} \\
&- P_i\hat{f}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{\bar{\eta}}_{i+1}, ..., \dot{\bar{\eta}}_{n-1}, v, w) \\
\dot{z}_{n-1} &= \{M_{n-1} + N_{n-1}\Psi_{n-1}T_{n-1}^{-1} - P_{n-1}\Psi_{n-1}T_{n-1}^{-1}(a_{n-1}I - M_{n-1} - N_{n-1}\Psi_{n-1}T_{n-1}^{-1})\}z_{n-1} \\
&+ (I + P_{n-1}\Psi_{n-1}T_{n-1}^{-1})\{L_{n-1} + N_{n-1} + (-a_{n-1}I + M_{n-1} + N_{n-1}\Psi_{n-1}T_{n-1}^{-1})P_{n-1}\}\bar{x}_{n-1} \\
&+ \{(I + P_{n-1}\Psi_{n-1}T_{n-1}^{-1})H_{n-1} - c_nP_{n-1}\}\bar{x}_n - P_{n-1}\hat{f}_{n-1}(\dot{\bar{x}}_n, v, w). \qquad (7.63)
\end{aligned}$$

In Remark 7.7, we will prove that the pair $(-\Psi_iT_i^{-1}(a_iI - M_i - N_i\Psi_iT_i^{-1}), M_i + N_i\Psi_iT_i^{-1})$ is observable. Therefore, we choose P_i to render $\overline{M}_i = M_i + N_i\Psi_iT_i^{-1} - P_i\Psi_iT_i^{-1}(a_iI - M_i - N_i\Psi_iT_i^{-1})$ Hurwitz. In standard feedforward system, there is no $\bar{x}_i$ in subsystem z_i. Thus, choose $L_i = -N_i - (-a_iI + M_i + N_i\Psi_iT_i^{-1})P_i$ for $i = 1, ..., n-1$.

Remark 7.7 Denote $A = M_i + N_i\Psi_iT_i^{-1} = T_i^{-1}\Phi_iT_i$ and $C = \Psi_iT_i^{-1}$. It is easy to show that (C, A) is observable since (Ψ_i, Φ_i) is observable. Letting $C_2 = a_iC - CA$ gives that $C_2A^j = a_iCA^j - CA^{j+1}$ for $j = 0, ..., r_i - 2$. It follows from $A = T_i^{-1}\Phi_iT_i$ that A is similar to Φ_i. It holds that

$$C_2A^{r_i-1} = a_iCA^{r_i-1} - CA^{r_i} = a_iCA^{r_i-1} - C(\kappa_{i,1}I + \kappa_{i,2}A + \cdots + \kappa_{i,r_i}A^{r_i}).$$

Therefore,

$$\begin{bmatrix} C_2 \\ C_2A \\ \vdots \\ C_2A^{r_i-2} \\ C_2A^{r_i-1} \end{bmatrix} = \begin{bmatrix} a_i & -1 & 0 & \cdots & 0 \\ 0 & a_i & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_i & -1 \\ -\kappa_{i,1} & -\kappa_{i,2} & \cdots & -\kappa_{i,r_{i-1}} & a_i - \kappa_{i,r_i} \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r_i-2} \\ CA^{r_i-1} \end{bmatrix}$$

$$= (a_iI - \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \kappa_{i,1} & \kappa_{i,2} & \cdots & \kappa_{i,r_{i-1}} & \kappa_{i,r_i} \end{bmatrix}) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r_i-2} \\ CA^{r_i-1} \end{bmatrix}$$

$$= (a_iI - \Phi_i) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r_i-2} \\ CA^{r_i-1} \end{bmatrix}$$

Since all eigenvalues of Φ_i have zero real part, $a_iI - \Phi_i$ is nonsingular for all $a_i \leq 0$. Therefore, the pair (C_2, A) is observable.

∎

Performing on (7.62) the coordinate transformation

$$z_n = \bar{\eta}_n - b^{-1}N_n\bar{x}_n$$

yields

$$\dot{z}_n = M_nz_n + (b^{-1}(M_n - a_nI)N_n + H_n)\bar{x}_n + L_n\bar{u}.$$

Let

$$H_n = -b^{-1}(M_n - a_nI)N_n + b^{-1}L_n(a_n + \Psi_nT_n^{-1}N_n). \tag{7.64}$$

Letting $\tilde{u} = \bar{u} + b^{-1}(a_n + \Psi_nT_n^{-1}N_n)\bar{x}_n$ makes the last two equations in (7.62) in the form

$$\begin{aligned} \dot{\bar{x}}_n &= b\tilde{u} + b\Psi_nT_n^{-1}z_n \\ \dot{z}_n &= M_nz_n + L_n\tilde{u}. \end{aligned}$$

Let $L_n = 0$.

Therefore, in terms of coordinate $\text{col}(z_1, ..., z_n, \bar{x}_1, ..., \bar{x}_n)$, system (7.62) is in the following feedforward form:

$$\begin{aligned}
\dot{\bar{x}}_i &= a_i\bar{x}_i + \Psi_i T_i^{-1}(a_i I - M_i - N_i\Psi_i T_i^{-1})z_i + \{-\Psi_i T_i^{-1}H_i + c_{i+1}(1 + \Psi_{i+1}T_{i+1}^{-1}P_{i+1})\}\bar{x}_{i+1} \\
&+ c_{i+1}\Psi_{i+1}T_{i+1}^{-1}z_{i+1} + \bar{f}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{z}_{i+1}, ..., \dot{z}_{n-1}, v, w) \\
\dot{z}_i &= \overline{M}_i z_i + \{(I + P_i\Psi_i T_i^{-1})H_i - c_{i+1}P_i(1 + \Psi_{i+1}T_{i+1}^{-1}P_{i+1})\}\bar{x}_{i+1} - c_{i+1}P_i\Psi_{i+1}T_{i+1}^{-1}z_{i+1} \\
&- P_i\bar{f}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{z}_{i+1}, ..., \dot{z}_{n-1}, v, w), \quad i = 1, ..., n-2 \\
\dot{\bar{x}}_{n-1} &= a_{n-1}\bar{x}_{n-1} + \Psi_{n-1}T_{n-1}^{-1}(a_{n-1}I - M_{n-1} - N_{n-1}\Psi_{n-1}T_{n-1}^{-1})z_{n-1} \\
&+ (c_n - \Psi_{n-1}T_{n-1}^{-1}H_{n-1})\bar{x}_n + \bar{f}_{n-1}(\dot{\bar{x}}_n, v, w) \\
\dot{z}_{n-1} &= \overline{M}_{n-1}z_{n-1} + \{(I + P_{n-1}\Psi_{n-1}T_{n-1}^{-1})H_{n-1} - c_nP_{n-1}\}\bar{x}_n - P_{n-1}\bar{f}_{n-1}(\dot{\bar{x}}_n, v, w) \\
\dot{\bar{x}}_n &= b\tilde{u} + b\Psi_n T_n^{-1}z_n \\
\dot{z}_n &= M_n z_n
\end{aligned} \tag{7.65}$$

where, for $i = 1, ..., n-2$,

$$\begin{aligned}
\bar{f}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{z}_{i+1}, ..., \dot{z}_{n-1}, v, w) &= \hat{f}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{z}_{n-1} + P_{n-1}\dot{\bar{x}}_{n-1}, ..., \dot{z}_{i+1} + P_{i+1}\dot{\bar{x}}_{i+1}, v, w), \\
\bar{f}_{n-1}(\dot{\bar{x}}_n, v, w) &= \hat{f}_{n-1}(\dot{\bar{x}}_n, v, w).
\end{aligned}$$

Remark 7.8 If $a_i < 0$ $(i = 1, ..., n-1)$, due to the triangular structure of (7.65), the controller $\tilde{u} = -\bar{x}_n$ solves the global robust stabilization problem for system (7.65). ∎

Step 3.

Note that (7.65) is not in the familiar feedforward form (7.2) yet. Therefore, we need to perform extra transformation. Since $I + P_i\Psi_i T_i^{-1}$ is nonsingular, it follows from A.12 in [30] that $1 + \Psi_i T_i^{-1}P_i \neq 0$. Similarly, we can obtain that $1 - \Psi_i T_i^{-1}(M_i + N_i\Psi_i T_i^{-1})\overline{M}_i^{-1}P_i \neq 0$ since $I - P_i\Psi_i T_i^{-1}(M_i + N_i\Psi_i T_i^{-1})\overline{M}_i^{-1} = (I + P_i\Psi_i T_i^{-1})^{-1}$ is nonsingular.

Since $I + P_i\Psi_i T_i^{-1}$ is nonsingular, choose H_i $(i = 1, ..., n-1)$ such that for $i = 1, ..., n-2$,

$$(I + P_i\Psi_i T_i^{-1})H_i - c_{i+1}P_i(1 + \Psi_{i+1}T_{i+1}^{-1}P_{i+1}) = 0$$

and

$$(I + P_{n-1}\Psi_{n-1}T_{n-1}^{-1})H_{n-1} - c_nP_{n-1} = 0.$$

It follows from z_{n-1} subsystem that

$$z_{n-1} = \overline{M}_{n-1}^{-1}\{\dot{z}_{n-1} + P_{n-1}\bar{f}_{n-1}(\dot{\bar{x}}_n, v, w)\}. \tag{7.66}$$

Substituting (7.66) into x_{n-1} subsystem gives that $(a_{n-1}=0)$

$$\begin{aligned}
\dot{\bar{x}}_{n-1} &= -\Psi_{n-1}T_{n-1}^{-1}(M_{n-1}+N_{n-1}\Psi_{n-1}T_{n-1}^{-1})\overline{M}_{n-1}^{-1}\{\dot{z}_{n-1}+P_{n-1}\bar{f}_{n-1}(\dot{\bar{x}}_n,v,w)\} \\
&+ (c_n-\Psi_{n-1}T_{n-1}^{-1}H_{n-1})\bar{x}_n+\bar{f}_{n-1}(\dot{\bar{x}}_n,v,w) \\
&= c_n\{1-\Psi_{n-1}T_{n-1}^{-1}(M_{n-1}+N_{n-1}\Psi_{n-1}T_{n-1}^{-1})\overline{M}_{n-1}^{-1}P_{n-1}\}\bar{x}_n+g_{n-1}(\dot{\bar{x}}_n,\dot{z}_{n-1},v,w) \\
&= \mu_{n-1}\bar{x}_n+g_{n-1}(\dot{\bar{x}}_n,\dot{z}_{n-1},v,w)
\end{aligned}$$

where

$$\begin{aligned}
& g_{n-1}(\dot{\bar{x}}_n,\dot{z}_{n-1},v,w) \\
&= \bar{f}_{n-1}(\dot{\bar{x}}_n,v,w)-\Psi_{n-1}T_{n-1}^{-1}(M_{n-1}+N_{n-1}\Psi_{n-1}T_{n-1}^{-1})\overline{M}_{n-1}^{-1}P_{n-1}\bar{f}_{n-1}(\dot{\bar{x}}_n,v,w) \\
&- \Psi_{n-1}T_{n-1}^{-1}(M_{n-1}+N_{n-1}\Psi_{n-1}T_{n-1}^{-1})\overline{M}_{n-1}^{-1}\dot{z}_{n-1}.
\end{aligned}$$

It follows from $\mu_{n-1}\neq 0$ that

$$\bar{x}_n=\frac{1}{\mu_{n-1}}(\dot{\bar{x}}_{n-1}-g_{n-1}(\dot{\bar{x}}_n,\dot{z}_{n-1},v,w)). \tag{7.67}$$

Substituting (7.66) into z_{n-2} subsystem of (7.65) gives that

$$\begin{aligned}
\dot{z}_{n-2} &= \overline{M}_{n-2}z_{n-2}-c_{n-1}P_{n-2}\Psi_{n-1}T_{n-1}^{-1}z_{n-1}-P_{n-2}\bar{f}_{n-2}(\dot{\bar{x}}_{n-1},\dot{\bar{x}}_n,\dot{z}_{n-1},v,w) \\
&= \overline{M}_{n-2}z_{n-2}+\tilde{f}_{n-2}(\dot{\bar{x}}_{n-1},\dot{\bar{x}}_n,\dot{z}_{n-1},v,w).
\end{aligned} \tag{7.68}$$

Substituting

$$z_{n-2} = \overline{M}_{n-2}^{-1}\dot{z}_{n-2}-\overline{M}_{n-2}^{-1}\tilde{f}_{n-2}(\dot{\bar{x}}_{n-1},\dot{\bar{x}}_n,\dot{z}_{n-1},v,w)$$

and (7.66) into x_{n-2} subsystem of (7.65) gives that $(a_{n-2}=0)$

$$\begin{aligned}
\dot{\bar{x}}_{n-2} &= -\Psi_{n-2}T_{n-2}^{-1}(M_{n-2}+N_{n-2}\Psi_{n-2}T_{n-2}^{-1})z_{n-2} \\
&+ \{-\Psi_{n-2}T_{n-2}^{-1}H_{n-2}+c_{n-1}(1+\Psi_{n-1}T_{n-1}^{-1}P_{n-1})\}\bar{x}_{n-1} \\
&+ c_{n-1}\Psi_{n-1}T_{n-1}^{-1}z_{n-1}+\bar{f}_{n-2}(\dot{\bar{x}}_{n-1},\dot{\bar{x}}_n,\dot{z}_{n-1},v,w) \\
&= -\Psi_{n-2}T_{n-2}^{-1}(M_{n-2}+N_{n-2}\Psi_{n-2}T_{n-2}^{-1})\{\overline{M}_{n-2}^{-1}\dot{z}_{n-2}-\overline{M}_{n-2}^{-1}\tilde{f}_{n-2}(\dot{\bar{x}}_{n-1},\dot{\bar{x}}_n,\dot{z}_{n-1},v,w)\} \\
&+ \{-\Psi_{n-2}T_{n-2}^{-1}H_{n-2}+c_{n-1}(1+\Psi_{n-1}T_{n-1}^{-1}P_{n-1})\}\bar{x}_{n-1} \\
&+ c_{n-1}\Psi_{n-1}T_{n-1}^{-1}\{\overline{M}_{n-1}^{-1}\dot{z}_{n-1}+\overline{M}_{n-1}^{-1}P_{n-1}\bar{f}_{n-1}(\dot{\bar{x}}_n,v,w)\}+\bar{f}_{n-2}(\dot{\bar{x}}_{n-1},\dot{\bar{x}}_n,\dot{z}_{n-1},v,w) \\
&= \mu_{n-2}\bar{x}_{n-1}+g_{n-2}(\dot{\bar{x}}_{n-1},\dot{\bar{x}}_n,\dot{z}_{n-1},\dot{z}_{n-2},v,w)
\end{aligned} \tag{7.69}$$

where

$$\mu_{n-2}=c_{n-1}\{1-\Psi_{n-2}T_{n-2}^{-1}(M_{n-2}+N_{n-2}\Psi_{n-2}T_{n-2}^{-1})\overline{M}_{n-2}^{-1}P_{n-2}\}(1+\Psi_{n-1}T_{n-1}^{-1}P_{n-1})\neq 0.$$

Due to the triangular structure of system (7.65), we rewrite it in the form:

$$\begin{aligned}
\dot{\bar{x}}_i &= \mu_i \bar{x}_{i+1} + g_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{z}_i, ..., \dot{z}_{n-1}, v, w) \\
\dot{z}_i &= \overline{M}_i z_i + \bar{g}_i(\dot{\bar{x}}_{i+1}, ..., \dot{\bar{x}}_n, \dot{z}_{i+1}, ..., \dot{z}_{n-1}, v, w), \quad i = 1, ..., n-2 \\
\dot{\bar{x}}_{n-1} &= \mu_{n-1}\bar{x}_n + g_{n-1}(\dot{\bar{x}}_n, \dot{z}_{n-1}, v, w) \\
\dot{z}_{n-1} &= \overline{M}_{n-1} z_{n-1} + \bar{g}_{n-1}(\dot{\bar{x}}_n, v, w) \\
\dot{\bar{x}}_n &= \mu_n \tilde{u} + g_n(z_n) \\
\dot{z}_n &= \overline{M}_n z_n
\end{aligned} \tag{7.70}$$

where $\mu_n = b$ and $g_n(z_n) = b\Psi_n T_n^{-1} z_n$.

The global robust stabilization result for (7.2) gives the solvability conditions of the robust global output regulation problem for system (7.56) as follows.

Theorem 7.2 Suppose system (7.56) satisfies $a_i = 0$ for $i = 2, ..., r$ and $c_i \neq 0$ for $i = 1, ..., r-1$. Then the global robust output regulation problem can be solved by a dynamic state feedback controller of the form,

$$\begin{aligned}
u &= \Psi_1 T_1 \eta_1 - b^{-1}(a_1 + \Psi_1 T_1^{-1} N_1)e \\
&- \lambda_n \sigma(K_n/\lambda_n(x_n + \Psi_n T_n^{-1}\eta_n + \lambda_{n-1}\sigma(K_{n-1}(x_{n-1} + \Psi_{n-1}T_{n-1}^{-1}\eta_{n-1} + \dots \\
&+ x_2 + \Psi_2 T_2^{-1}\eta_2 + \lambda_2\sigma(K_1 e/\lambda_1))/\lambda_{n-1}))), \\
\dot{\eta}_i &= M_i\eta_i + L_i(x_i - \Psi_i T_i^{-1}\eta_i) + N_i\Psi_i T_i^{-1}\eta_i + H_i(x_{i+1} - \Psi_{i+1}T_{i+1}^{-1}\eta_{i+1}), \quad i = 1, ..., n-2, \\
\dot{\eta}_{n-1} &= M_{n-1}\eta_{n-1} + L_{n-1}(x_{n-1} - \Psi_{n-1}T_{n-1}^{-1}\eta_{n-1}) + N_{n-1}\Psi_{n-1}T_{n-1}^{-1}\eta_{n-1} + H_{n-1}e \\
\dot{\eta}_n &= M_n\eta_n + L_n(u - \Psi_n T_n^{-1}\eta_n) + N_n\Psi_n T_n^{-1}\eta_n + H_n e.
\end{aligned} \tag{7.71}$$

∎

Example 7.1 Consider the following feedforward system

$$\begin{aligned}
\dot{x}_1 &= 3x_2 + (\dot{x}_2 - 2v_1 v_2^2 + v_1^3)u \\
\dot{x}_2 &= 2u + v_1^3 \\
y &= x_2 \\
e &= y - v_1^2 v_2
\end{aligned} \tag{7.72}$$

with the exosystem

$$\begin{aligned}
\dot{v}_1 &= v_2 \\
\dot{v}_2 &= -v_1.
\end{aligned} \tag{7.73}$$

These equations formulate the control problem of design a state feedback regulator to make the output y asymptotically track a sinusoidal signal of frequency 1 with arbitrarily large amplitude. It can be verified that this system satisfies Assumption 7.4, 7.5 and 7.6. In particular, the regulator equations associated with this system admit a globally defined solution as follows

$$\begin{aligned}\mathbf{x}_2(v,w) &= v_1^2 v_2 \\ \mathbf{x}_1(v,w) &= v_1^3 \\ \mathbf{u}(v,w) &= v_1^2 v_2. \end{aligned} \tag{7.74}$$

Let $g_o(x,u) = \text{col}(u, x_1)$, $\pi_1^1(v,w) = v_1^2 v_2$ and $\pi_2^1(v,w) = v_1^3$. Then, the minimal zeroing polynomial of $\pi_1^1(v,w)$ and $\pi_2^1(v,w)$ is $\lambda^4 + 10\lambda^2 + 9$.

The gradients and companion matrices are

$$\Psi_1 = \Psi_2 = [1 \quad 0 \quad 0 \quad 0], \quad \Phi_1 = \Phi_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -10 & 0 \end{bmatrix}.$$

For each $i = 1, 2$, the steady-state generator with output x_i is given by

$$\begin{aligned}\theta_i(v,w) &= T_i col(\pi_i^1(v,w), \dot{\pi}_i^1(v,w)) \\ \alpha_i(\theta_i) &= T_i \Phi_i T_i^{-1} \theta_i \\ \beta_i(\theta_i) &= \Psi_i T_i^{-1} \theta_i \end{aligned} \tag{7.75}$$

where T_i s any nonsingular matrix.

To design an internal model, let

$$M_1 = M_2 = \begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad N_1 = N_2 = \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}. \tag{7.76}$$

Solving the related Sylvester equation gives

$$T_1 = T_2 = \begin{bmatrix} 0.9981 & -0.1248 & 0.0135 & -0.0017 \\ 0.9788 & -0.2447 & 0.0376 & -0.0094 \\ 0.8615 & -0.4308 & 0.0615 & -0.0308 \\ 0.5500 & -0.5500 & 0.0500 & -0.0500 \end{bmatrix}. \tag{7.77}$$

Thus, for $i = 1, 2$

$$\beta_i(\theta_i) = \Psi_i T_i^{-1}\theta_i = [3.5305 \quad -4.4271 \quad 2.7083 \quad -0.9524]\theta_i \tag{7.78}$$

where,

$$\begin{aligned}
\theta_2(v,w) &= \begin{bmatrix} \theta_{21} \\ \theta_{22} \\ \theta_{23} \\ \theta_{24} \end{bmatrix} = T_2 \begin{bmatrix} \pi_2^1(v,w) \\ \dot{\pi}_2^1(v,w) \\ \ddot{\pi}_2^1(v,w) \\ \pi_2^{1(3)}(v,w) \end{bmatrix} \\
&= \begin{bmatrix} 0.5931v_1^3 - 0.0102v_2^3 - 0.3387v_1^2v_2 + 0.081v_1v_2^2 \\ 0.866v_1^3 - 0.0564v_2^3 - 0.5367v_1^2v_2 + 0.2256v_1v_2^2 \\ 0.677v_1^3 - 0.1848v_2^3 - 0.6465v_1^2v_2 + 0.369v_1v_2^2 \\ 0.4v_1^3 - 0.3v_2^3 - 0.6v_1^2v_2 + 0.3v_1v_2^2 \end{bmatrix} \\
\theta_1(v,w) &= \begin{bmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{13} \\ \theta_{14} \end{bmatrix} = T_1 \begin{bmatrix} \pi_1^1(v,w) \\ \dot{\pi}_1^1(v,w) \\ \ddot{\pi}_1^1(v,w) \\ \pi_1^{1(3)}(v,w) \end{bmatrix} \\
&= \begin{bmatrix} 0.1129v_1^3 + 0.027v_2^3 + 0.9036v_1^2v_2 - 0.2556v_1v_2^2 \\ 0.1789v_1^3 + 0.0752v_2^3 + 0.7156v_1^2v_2 - 0.3014v_1v_2^2 \\ 0.2152v_1^3 + 0.123v_2^3 + 0.431v_1^2v_2 - 0.2456v_1v_2^2 \\ 0.2v_1^3 + 0.1v_2^3 + 0.2v_1^2v_2 - 0.1v_1v_2^2 \end{bmatrix}.
\end{aligned}$$

Then the internal model is as follows

$$\begin{aligned}
\dot{\eta}_1 &= M_1\eta_1 + L_1(x_1 - \Psi_1 T_1^{-1}\eta_1) + N_1x_1 + H_1e \\
\dot{\eta}_2 &= M_2\eta_2 + L_2(u - \Psi_2 T_2^{-1}\eta_2) + N_2u + H_2e.
\end{aligned} \tag{7.79}$$

Note that (7.72) can be converted into following form

$$\begin{aligned}
\dot{x}_1 &= 3x_2 + 0.5(\dot{x}_2 - 2v_1v_2^2 + v_1^3)(\dot{x}_2 - v_1^3) \\
\dot{x}_2 &= 2u + v_1^3 \\
y &= x_2 \\
e &= y - v_1^2v_2.
\end{aligned} \tag{7.80}$$

Using the canonical coordinate and input transformation

$$
\begin{aligned}
\bar{x}_2 &= x_2 - \mathbf{x}_2(v,w) \\
\bar{u} &= u - \Psi_2 T_2^{-1}\eta_2 \\
\bar{x}_1 &= x_1 - \Psi_1 T_1^{-1}\eta_1, \\
\bar{\eta}_i &= \eta_i - \theta_i(v,w), \quad i = 1,2
\end{aligned} \tag{7.81}
$$

put the augmented system (7.80) and (7.79) into the following

$$
\begin{aligned}
\dot{\bar{x}}_1 &= -[3.5305 \quad -4.4271 \quad 2.7083 \quad -0.9524]L_1\bar{x}_1 \\
&+ (3 - [3.5305 \quad -4.4271 \quad 2.7083 \quad -0.9524]H_1)\bar{x}_2 \\
&+ [-24.7135 \quad 48.6979 \quad -35.2083 \quad 13.3333]\bar{\eta}_1 \\
&+ 0.5\dot{\bar{x}}_2(\dot{\bar{x}}_2 + 2v_1v_2^2 - 2v_1^3) \\
\dot{\bar{\eta}}_1 &= \begin{bmatrix} 20.2440 & -35.4167 & 21.6667 & -7.6190 \\ 14.1220 & -21.7083 & 10.8333 & -3.8095 \\ 7.0610 & -8.8542 & 3.4167 & -1.9048 \\ 3.5305 & -4.4271 & 2.7083 & -1.9524 \end{bmatrix} \bar{\eta}_1 + (L_1 + N_1)\bar{x}_1 + H_1\bar{x}_2 \\
\dot{\bar{x}}_2 &= 2\bar{u} + 2[3.5305 \quad -4.4271 \quad 2.7083 \quad -0.9524]\bar{\eta}_2 \\
\dot{\bar{\eta}}_2 &= \begin{bmatrix} 20.2440 & -35.4167 & 21.6667 & -7.6190 \\ 14.1220 & -21.7083 & 10.8333 & -3.8095 \\ 7.0610 & -8.8542 & 3.4167 & -1.9048 \\ 3.5305 & -4.4271 & 2.7083 & -1.9524 \end{bmatrix} \bar{\eta}_2 + (L_2 + N_2)\bar{u} + H_2\bar{x}_2.
\end{aligned} \tag{7.82}
$$

Choose $P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ such that

$$
\begin{aligned}
\overline{M}_1 &= M_1 + N_1\Psi_1T_1^{-1} + P_1\Psi_1T_1^{-1}(M_1 + N_1\Psi_1T_1^{-1}) \\
&= \begin{bmatrix} 44.9576 & -84.1146 & 56.8750 & -20.9524 \\ 38.8356 & -70.4062 & 46.0417 & -17.1429 \\ 31.7746 & -57.5521 & 38.6250 & -15.2381 \\ 28.2440 & -53.1250 & 37.9167 & -15.2857 \end{bmatrix}
\end{aligned}
$$

which is Hurwitz.

Performing the coordinate transformation

$$\begin{aligned} z_1 &= \bar{\eta}_1 - P_1\bar{x}_1 \\ z_2 &= \bar{\eta}_2 - 0.5N_2\bar{x}_2 \end{aligned}$$

which yields

$$\begin{aligned}
\dot{\bar{x}}_1 &= (2.1094 - [3.5305 \quad -4.4271 \quad 2.7083 \quad -0.9524]L_1)\bar{x}_1 \\
&+ (3 - [3.5305 \quad -4.4271 \quad 2.7083 \quad -0.9524]H_1)\bar{x}_2 \\
&+ [-24.7135 \quad 48.6979 \quad -35.2083 \quad 13.3333]z_1 + 0.5\dot{\bar{x}}_2(\dot{\bar{x}}_2 + 2v_1v_2^2 - 2v_1^3) \\
\dot{z}_1 &= \begin{bmatrix} 44.9575 & -84.1146 & 56.8750 & -20.9523 \\ 38.8355 & -70.4062 & 46.0416 & -17.1428 \\ 31.7745 & -57.5521 & 38.6250 & -15.2381 \\ 28.2440 & -53.1250 & 37.9166 & -15.2857 \end{bmatrix} z_1 \\
&+ \left(\begin{bmatrix} 4.5305 & -4.4271 & 2.7083 & -0.9524 \\ 3.5305 & -3.4271 & 2.7083 & -0.9524 \\ 3.5305 & -4.4271 & 3.7083 & -0.9524 \\ 3.5305 & -4.4271 & 2.7083 & 0.0476 \end{bmatrix} (L_1 + N_1) - \begin{bmatrix} 3.2344 \\ 2.6719 \\ 2.3907 \\ 2.2501 \end{bmatrix} \right)\bar{x}_1 \\
&+ \left(\begin{bmatrix} 4.5305 & -4.4271 & 2.7083 & -0.9524 \\ 3.5305 & -3.4271 & 2.7083 & -0.9524 \\ 3.5305 & -4.4271 & 3.7083 & -0.9524 \\ 3.5305 & -4.4271 & 2.7083 & 0.0476 \end{bmatrix} H_1 - \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \right)\bar{x}_2 \\
&- \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} 0.5\dot{\bar{x}}_2(\dot{\bar{x}}_2 + 2v_1v_2^2 - 2v_1^3) \\
\dot{\bar{x}}_2 &= 2\bar{u} + 2[3.5305 \quad -4.4271 \quad 2.7083 \quad -0.9524]z_2 + 15\bar{x}_2 \\
\dot{z}_2 &= \begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} z_2 + L_2\bar{u} + \left(H_2 - 7.5 \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}\right)\bar{x}_3.
\end{aligned} \tag{7.83}$$

Letting

$$L_2 = 0,$$

$$L_1 = \begin{bmatrix} 4.5305 & -4.4271 & 2.7083 & -0.9524 \\ 3.5305 & -3.4271 & 2.7083 & -0.9524 \\ 3.5305 & -4.4271 & 3.7083 & -0.9524 \\ 3.5305 & -4.4271 & 2.7083 & 0.0476 \end{bmatrix}^{-1} \begin{bmatrix} 3.2344 \\ 2.6719 \\ 2.3907 \\ 2.2501 \end{bmatrix} - N_1 = - \begin{bmatrix} 6.875 \\ 3.4375 \\ 1.7187 \\ 0.8593 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 60 \\ 30 \\ 15 \\ 7.5 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 4.5305 & -4.4271 & 2.7083 & -0.9524 \\ 3.5305 & -3.4271 & 2.7083 & -0.9524 \\ 3.5305 & -4.4271 & 3.7083 & -0.9524 \\ 3.5305 & -4.4271 & 2.7083 & 0.0476 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.6135 \\ 1.6135 \\ 1.6135 \\ 1.6135 \end{bmatrix},$$

$$\tilde{u} = \bar{u} + 7.5\bar{x}_2$$

gives

$$\begin{aligned}
\dot{\bar{x}}_1 &= 2.1514\bar{x}_2 + [-24.7135 \quad 48.6979 \quad -35.2083 \quad 13.3333]z_1 + 0.5\dot{\bar{x}}_2(\dot{\bar{x}}_2 + 2v_1v_2^2 - 2v_1^3) \\
\dot{z}_1 &= \begin{bmatrix} 44.9575 & -84.1146 & 56.8750 & -20.9523 \\ 38.8355 & -70.4062 & 46.0416 & -17.1428 \\ 31.7745 & -57.5521 & 38.6250 & -15.2381 \\ 28.2440 & -53.1250 & 37.9166 & -15.2857 \end{bmatrix} z_1 - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} 0.5\dot{\bar{x}}_2(\dot{\bar{x}}_2 + 2v_1v_2^2 - 2v_1^3) \\
\dot{\bar{x}}_2 &= 2\tilde{u} + 2[3.5305 \quad -4.4271 \quad 2.7083 \quad -0.9524]z_2 \\
\dot{z}_2 &= \begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} z_2.
\end{aligned} \tag{7.84}$$

Following the steps established in Theorem 7.2, we obtain the following nested saturation controller to globally stabilizing system (7.84)

$$\tilde{u} = -\lambda_2\sigma(\frac{K_2}{\lambda_2}(\bar{x}_2 + \lambda_1\sigma(\frac{K_1}{\lambda_1}\bar{x}_1))) \tag{7.85}$$

where the saturation function is in form

$$\sigma(s) = sgn(s)(|s| + s^2 - |s|^3)$$

and $\lambda_1 = 0.099$, $\lambda_2 = 1.103$, $K_1 = 0.099$ and $K_2 = 44.551$.

Therefore, the controller

$$\begin{aligned}
u &= -\lambda_2\sigma(\frac{K_2}{\lambda_2}(e + \lambda_1\sigma(\frac{K_1}{\lambda_1}(x_1 + \Psi_1 T_1^{-1}\eta_1)))) - 7.5e + \Psi_2 T_2^{-1}\eta_2 \\
\dot{\eta}_1 &= M_1\eta_1 + L_1(x_1 - \Psi_1 T_1^{-1}\eta_1) + N_1 x_1 + H_1 e \\
\dot{\eta}_2 &= M_2\eta_2 + L_2(u - \Psi_2 T_2^{-1}\eta_2) + N_2 u2 + H_2 e
\end{aligned}$$

solves the global robust output regulation for system (7.72). The simulation result is shown in Fig. 7.2 and 7.3 for the case $x(0) = \text{col}(1,1)$ and $v(0) = \text{col}(1,1)$. ∎

7.5 Conclusion

In this chapter, we address the global robust output regulation problem for a class of feedforward systems, including the input disturbance suppression problem in [45] as a special case.

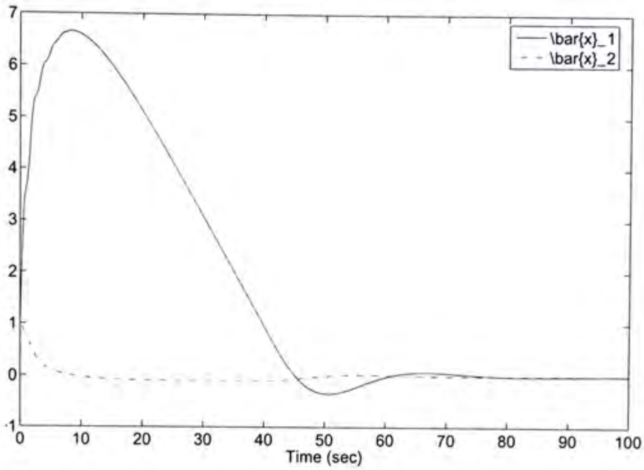

Figure 7.2: The states of the closed-loop system for Example 7.1

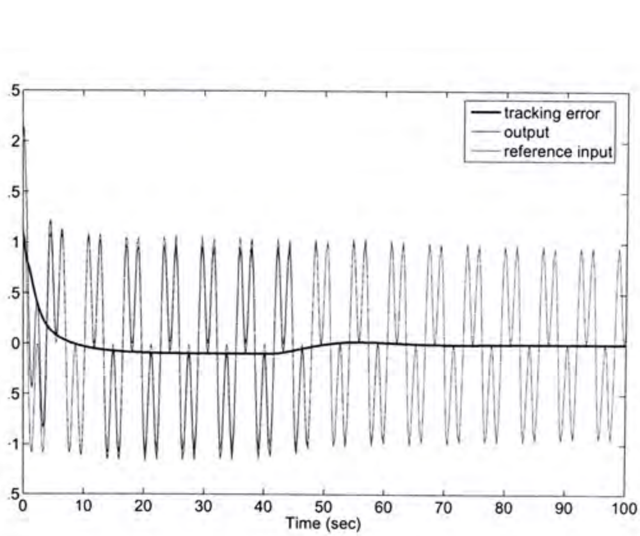

Figure 7.3: The tacking error for Example 7.1

Chapter 8

Conclusion

In this thesis, we have addressed two important problems in nonlinear control theory including robust stabilization and output regulation for feedforward systems. Some concluding remarks are given as follows.

In the first part of this thesis, we have established the four types of small gain theorem with restrictions for uncertain time-varying nonlinear systems, thus filling the gap between the small gain theorem with restrictions for time-varying systems and that for time invariant systems. Moreover, we have explored the connection between input-to-output formulation and Lyapunov function argument and known that each approach has its own advantage in dealing with nonlinear control problem. We have further gave a remark on various small gain conditions at the end of the first part.

In the second part of this thesis, we have solved the semi-global and global robust stabilization problem for feedforward systems. In [2], the authors studied the disturbance attenuation problem for a class of feedforward systems subject to input unmodeled dynamics. We have addressed the semi-global robust stabilization problem for this class of feedforward systems subject to both static uncertainties and dynamic uncertainties, and at the same time relax Hurwitzness of the linearization of the unmodeled dynamics to critical stability. We have further studied the global robust stabilization for the same class of feedforward systems in the presence of both static uncertainties and dynamic uncertainties assuming each (unforced) dynamic uncertainty is locally exponentially stable.

In the third part of this thesis, by appealing to the general framework for tackling the robust output regulation problem in [18], we have addressed the global robust output regulation problem for feedforward systems. In [45], the problem of asymptotically rejecting bounded disturbances which affect the input channel of a feedforward uncertain nonlinear system was solved. The robust output regulation problem under consideration in this thesis is more general and complicated than that studied in [45]. When the steady-state is equal to zero, the robust output regulation problem reduces to the input disturbance suppression problem studied in [45].

To conclude this thesis, we will give some future research perspectives. Currently, we have developed some tools to solve the robust stabilization and output regulation problems for feedfroward systems. However, all these results are applied to the robust perspective. It is interesting and necessary in practice to study the adaptive stabilization and output regulation problem for feedforward systems. The connection between input-to-output formulation and Lyapunov function argument established in the first part will benefit for us to solve these problems. And our tools of stabilization and output regulation for feedforward systems are based on state feedback. It is also interesting to explore the solvability conditions of stabilization and output regulation for feedforward systems using output feedback. Many practical nonlinear systems, such as the inverted pendulum on a cart, the vertical take-off and landing aircraft and translational oscillator with a rotational actuator, can be modeled in the feedforward form. Therefore it is interesting to explore the application of the theoretical results in this thesis to the practical nonlinear systems.

Appendix A

Appendix

A. 1 Lemma Let β be a class $\mathcal{KL}$ function, γ a class $\mathcal{K}$ function such that $\gamma(r) < r$, $\forall r > 0$, and $\mu \in (0, 1]$ a real number. For any nonnegative real numbers s and d, and any nonnegative real function $z(t) \in L^1_\infty$ satisfying

$$z(t) \leq \max\left\{\beta(s,t), \gamma\left(\|z_{[\mu t, t]}\|\right), d\right\}, \ \forall t \geq 0, \tag{A.1}$$

there exists a class $\mathcal{K}_\infty$ function $\hat{\beta}$ such that

$$z(t) \leq \max\left\{\hat{\beta}(s,t), d\right\}, \ \forall t \geq 0. \tag{A.2}$$

Proof: First, we choose a function $\bar{z}(t)$ as follows

$$\bar{z}(t) = \begin{cases} z(t) & \text{if } z(t) > d \\ 0 & \text{otherwise} \end{cases}, \ t \geq 0,$$

clearly, which is real nonnegative function and belongs to L^1_∞. Then we will show that

$$\bar{z}(t) \leq \max\left\{\beta(s,t), \gamma\left(\|\bar{z}_{[\mu t, t]}\|\right)\right\}, \ \forall t \geq 0. \tag{A.3}$$

To this end, we will consider the following two cases.

(i) $z(t) > d$: On one hand, from (A.1),

$$z(t) \leq \max\left\{\beta(s,t), \gamma\left(\|z_{[\mu t, t]}\|\right)\right\}.$$

On the other hand, $\|\bar{z}_{[\mu t, t]}\| = \|z_{[\mu t, t]}\|$. In fact, at the instant $t_1 \in [\mu t, t]$ when $z(t_1) \geq z(\tau)$, $\forall \tau \in [\mu t, t]$, we have $\|z_{[\mu t, t]}\| = z(t_1) \geq z(t) > d$. Thus, $\bar{z}(t_1) = z(t_1) \geq z(\tau) \geq \bar{z}(\tau)$, $\forall \tau \in [\mu t, t]$. That is, $\|\bar{z}_{[\mu t, t]}\| = \bar{z}(t_1) = z(t_1) = \|z_{[\mu t, t]}\|$.

As a result,

$$\bar{z}(t) = z(t) \leq \max\left\{\beta(s,t), \gamma\left(\|\bar{z}_{[\mu t, t]}\|\right)\right\}.$$

That is, (A.3) holds.

(ii) $z(t) \leq d$: The inequality (A.3) holds since $\bar{z}(t) = 0$.

Now, from (A.3), we have the following claim.

Claim: For any $r, \epsilon > 0$, there exists a nonnegative number $T_r(\epsilon)$ such that, if $\bar{z}(t)$ satisfies (A.3) with $s < r$, then $\bar{z}(t) < \epsilon$, $\forall t \geq T_r(\epsilon)$.

Proof: Since $\bar{z}(t) \in L^1_\infty$, denote $R = \left\|\bar{z}_{[0,\infty)}\right\|$, which is a finite nonnegative number. If $R = 0$, the proof is trivial. So, we suppose $R > 0$. And let $\delta_1 = R - \gamma(R) > 0$. For any $\delta_2 \in (0, \delta_1)$, there exists a finite $t_1 \geq 0$ such $\bar{z}(t_1) \geq R - \delta_2$. From (A.3), we have

$$R - \delta_2 \leq \bar{z}(t_1) \leq \max\left\{\beta(s,0), \gamma(R)\right\},$$

hence

$$R \leq \max\left\{\beta(s,0) + \delta_2, \gamma(R) + \delta_2\right\}.$$

And $R > \gamma(R) + \delta_2$ gives $R \leq \beta(s,0) + \delta_2 < \beta(r,0) + \delta_2$. Since δ_2 can be arbitrarily small, we have $R \leq \beta(r,0)$. As a result, $\bar{z}(t) \leq \beta(r,0)$, $t \geq 0$.

Next, there exist a real number $0 < \delta_3 < 1$ satisfying

$$\gamma(x) \leq \delta_3 x, \forall x \in [\epsilon, \beta(r,0)],$$

and a nonnegative integer n satisfying $\delta_3^n < \frac{\epsilon}{\beta(r,0)}$. Clearly, $\gamma^n(\beta(r,0)) < \epsilon$. Denote $t_i > 0$, $i = 1, \ldots, n$ be the first time instant such that

$$\beta(r, t_i) \leq \gamma^i(\beta(r,0)).$$

And define $\bar{t}_i$, $i = 0, \ldots, n$ as

$$\bar{t}_0 = 0,\ \bar{t}_i = \max\left\{t_i, \frac{1}{\mu}\bar{t}_{i-1}\right\},\ i = 1, \ldots, n.$$

Now, it can be proved by induction that, for $i = 0, \ldots, n$,

$$\bar{z}(t) \leq \gamma^i(\beta(r,0)), \forall t \geq \bar{t}_i. \tag{A.4}$$

Indeed, we have shown that (A.4) holds for $i = 0$. Suppose it holds for $n > i > 0$, then for $t \geq \bar{t}_{i+1}$, we have

$$\begin{aligned}
\bar{z}(t) &\leq \max\left\{\beta(s,t), \gamma\left(\|\bar{z}_{[\mu t, t]}\|\right)\right\} \\
&\leq \max\left\{\beta(r, t_{i+1}), \gamma\left(\gamma^i(\beta(r,0))\right)\right\} \\
&= \gamma^{i+1}(\beta(r,0)).
\end{aligned}$$

That is, (A.4) holds for $i = 0, \ldots, n$. Now, (A.4) with $i = n$ is

$$\bar{z}(t) \leq \gamma^n(\beta(r, 0)) < \epsilon, \forall t \geq T_r(\epsilon) \tag{A.5}$$

by choosing $T_r(\epsilon) \geq \bar{t}_n$. The proof of the claim is completed.

Using the above claim, we can find a class $\mathcal{KL}$ function $\hat{\beta}$ such that

$$\bar{z}(t) \leq \hat{\beta}(s, t),\ \forall t \geq 0.$$

By the definition of $\bar{z}(t)$, we note that (A.2) is satisfied.

The existence of $\hat{\beta}$ is given below, which is derived from the result in the proof of Lemma A.1 of [25], and the references within, such as Lemma 2.1.4 and Proposition 2.1.5 of [36], and Proofs of Lemma 3.1 and Proposition 2.5 of [37].

Step 1: From the proof of the claim, it is known that

$$\bar{z}(t) \leq \varphi(s) \tag{A.6}$$

with $\varphi(s) = \beta(s, 0)$.

Step 2: From the proof of the claim, it is clear that $T_r(\epsilon)$ always exists satisfying the claim and the following properties additionally.

(i) For each fixed $r > 0$, T_r maps $(0, \infty) \mapsto [0, \infty)$ satisfying $T_r(\epsilon) < \infty$ for any $\epsilon > 0$.

(ii) $T_r(\epsilon_1) \geq T_r(\epsilon_2)$, if $\epsilon_1 \leq \epsilon_2$.

So we can define for any $r, \epsilon > 0$,

$$\bar{T}_r(\epsilon) = \frac{2}{\epsilon} \int_{\epsilon/2}^{\epsilon} T_r(s) ds.$$

Since T_r is decreasing, $\bar{T}_r$ is well defined and is locally absolutely continuous. Also

$$\bar{T}_r(\epsilon) \geq \frac{2}{\epsilon} T_r(\epsilon) \int_{\epsilon/2}^{\epsilon} ds = T_r(\epsilon).$$

Furthermore,

$$\begin{aligned}
\frac{d\bar{T}_r(\epsilon)}{d\epsilon} &= -\frac{2}{\epsilon^2} \int_{\epsilon/2}^{\epsilon} T_r(s) ds + \frac{2}{\epsilon} \left[T_r(\epsilon) - \frac{1}{2} T_r(\epsilon/2) \right] \\
&= \frac{1}{\epsilon} \left[T_r(\epsilon) - \frac{2}{\epsilon^2} \int_{\epsilon/2}^{\epsilon} T_r(s) ds \right] + \frac{1}{\epsilon} \left[T_r(\epsilon) - T_r(\epsilon/2) \right] \\
&= \frac{1}{\epsilon} \left[T_r(\epsilon) - \bar{T}_r(\epsilon) \right] + \frac{1}{\epsilon} \left[T_r(\epsilon) - T_r(\epsilon/2) \right] \\
&\leq 0.
\end{aligned}$$

Hence, $\bar{T}_r$ decreases (not necessarily strictly). Finally, define

$$\tilde{T}_r(\epsilon) = \bar{T}_r(\epsilon) + \frac{r}{\epsilon}.$$

Then it follows that

for any fixed r, $\tilde{T}_r$ is continuous, maps $(0,\infty) \mapsto (0,\infty)$, and strictly decreasing.

Now, for each $r \in (0,\infty)$, denote $\psi_r = \tilde{T}_r^{-1}$. Then

$$\psi_r : (0,\infty) \mapsto (0,\infty)$$

is continuous and strictly decreasing. We also write $\psi_r(0) = +\infty$, which is consistent with the fact that

$$\lim_{t \to \infty 0^+} \psi_r(t) = +\infty.$$

It follows from the claim and the fact $\tilde{T}_r(\epsilon) \geq T_r(\epsilon)$ that, for any $r, \epsilon > 0$,

$$\bar{z}(t) < \epsilon, \forall t \geq \tilde{T}_r(\epsilon).$$

As $t = \tilde{T}_r(\psi_r(t))$ if $t > 0$, we have

$$\bar{z}(t) < \psi_r(t),\ \forall t > 0.$$

Furthermore, since $\psi_r(0) = \infty$, we obtain

$$\bar{z}(t) \leq \psi_r(t),\ \forall t \geq 0. \tag{A.7}$$

Step 3: Now for any $s \geq 0$ and $t \geq 0$, let

$$\bar{\psi}(s,t) = \min\left\{ \inf_{r \in (s,\infty)} \psi_r(t), \varphi(s) \right\}.$$

From the above two steps, we have

$$\bar{z}(t) \leq \bar{\psi}(s,t),\ \forall t \geq 0.$$

By its definition, for any fixed t, $\bar{\psi}(\cdot,t)$ is an increasing function (not necessarily strictly).Also because for any fixed $r \in (0,\infty)$, $\psi_r(t)$ decreases to 0 (this follows from the fact that $\psi_r : (0,\infty) \to (0,\infty)$ is continuous and strictly decreasing), it follows that

for any fixed s, $\bar{\psi}(s,t)$ decreases to 0 as $t \to \infty$.

Pick any function

$$\tilde{\psi} : [0,\infty) \times [0,\infty) \mapsto [0,a)$$

for some $a > 0$ (a can be $+\infty$) with the following properties.

(i) For any fixed $t \geq 0,$, $\tilde{\psi}(\cdot, t)$ is continuous and strictly increasing.

(ii) For any fixed $s \geq 0,$, $\tilde{\psi}(s, t)$ decreases to 0 as $t \to \infty$.

(iii) $\tilde{\psi}(s, t) \geq \bar{\psi}(s, t)$.

Such a function $\tilde{\psi}$ always exists; for instance, it can be constructed as follows. Define first

$$\hat{\psi}(s, t) = \int_s^{s+1} \bar{\psi}(\varsigma, t) d\varsigma$$

Then $\hat{\psi}(\cdot, t)$ is an absolutely continuous function on every compact subset of $[0, \infty)$, and it satisfies

$$\hat{\psi}(s, t) \geq \bar{\psi}(s, t) \int_s^{s+1} d\varsigma = \bar{\psi}(s, t).$$

It follows that

$$\frac{\partial \hat{\psi}(s, t)}{\partial s} = \bar{\psi}(s + 1, t) - \bar{\psi}(s, t) \geq 0, \ a.e.,$$

and hence $\hat{\psi}(\cdot, t)$ is increasing. Also since for any fixed s,$\bar{\psi}(s, \cdot)$ decreases, so does $\hat{\psi}(s, \cdot)$. Note that

$$\bar{\psi}(s, t) \leq \bar{\psi}(s, 0) = \min \left\{ \inf_{r \in (s, \infty)} \psi_r(0), \varphi(s) \right\} = \varphi(s),$$

(recall that $\phi_r(0) = \infty$), so by Lebesgue dominated convergence theorem, for any fixed $s \geq 0$,

$$\lim_{t \to \infty} \hat{\psi}(s, t) = \int_s^{s+1} \lim_{t \to \infty} \bar{\psi}(\varsigma, t) d\varsigma = 0.$$

Now we see that the function $\hat{\psi}(s, t)$ satisfies all of the requirements for $\tilde{\psi}(s, t)$ except possibly for the strictly increasing property. We define $\tilde{\psi}$ as follows:

$$\tilde{\psi}(s, t) = \hat{\psi}(s, t) + \frac{s}{(s + 1)(t + 1)}.$$

Clearly it satisfies all the desired properties.

Finally, define

$$\hat{\beta}(s, t) = \sqrt{\varphi(s)} \sqrt{\tilde{\psi}(s, t)}.$$

Then it follows that $\beta(s, t)$ is a $\mathcal{KL}$ function, and

$$\bar{z}(t) \leq \sqrt{\varphi(s)} \sqrt{\bar{\psi}(s, t)} \leq \hat{\beta}(s, t), \ \forall t \geq 0,$$

which concludes the proof of the Lemma.

List of Figures

Bibliography

[1] D. Angeli, Y. Chitour and L. Marconi, *New results on robust stabilization via saturated feedback*, Proceedings of the 42nd IEEE Conference on Decision and Control, vol.5, pp. 4445-4450, December 2003.

[2] M. Arcak, A.R. Teel and P. Kokotovic, *Robust nonlinear control of feedforward systems with unmodeled dynamics*, Automatica, vol. 37, pp. 265-272, 2001.

[3] A. Astolfi, G. Kaliora and Z.P. Jiang, *Output feedback stabilization and approximate and restricted tracking for a class of cascaded systems*, IEEE Transactions on Automatic Control, vol. 50, pp. 1390-1396, 2005.

[4] C. I. Byrnes, F. Delli Priscoli and A. Isidori, *Output regulation of uncertain nonlinear systems*, Birkhauser, 1997.

[5] C.-T. Chen, *Linear system theory and design*, CBS College Publishing, 1984.

[6] Z. Chen and J. Huang, *Global robust servomechanism of lower triangular systems in general case*, Systems and Control Letters, vol. 52, pp. 209-220, 2004.

[7] Z. Chen and J. Huang, *Dissipativity, stabilization, and regulation of cascade-connected systems*, IEEE Transactions on Automatic Control, vol. 49, pp. 635-650, 2004.

[8] Z. Chen and J. Huang, *A simplified small gain theorem for time-varying nonlinear systems*, IEEE Transactions on Automatic Control, vol. 50, No. 9, 2005, to appear.

[9] Z. Chen and J. Huang, *Robust input-to-state stability and small gain theorem for nonlinear systems containing time-varying uncertainty*, Advanced Robust and Adaptive Control -Theory and Applications, Springer-Tsinghua, to appear.

[10] H.A. Edwards, Y.D. Lin and Y. Wang, *On input-to-state stability for time varying nonlinear systems*, Proceedings of the 39th IEEE Conference on Decision and Control, vol. 4, pp. 3501-3506, December 2000.

[11] G. F. Franklin, J.D. Powell and A. Emami-Nieini, *Feedback Control of Dynamic Systems*, Third Edition, Addison Wesley, 1994.

[12] F. Grognard, R. Sepulchre and G. Bastin, *Global stabilization of feedforward systems with exponentially unstable Jacobian linearization*, Systems and Control Letters, vol. 37, pp. 107-115, 1999.

[13] D.J. Hill, *A generalization of the small-gain theorem for nonlinear feedback systems*, Automatica, vol. 27, pp. 1043-1045, 1991.

[14] J. Huang, *Asymptotic tracking and disturbance rejection in uncertain nonlinear systems*, IEEE Transactions on Automatic Control, vol 40. pp. 1118-1122, 1995.

[15] J. Huang, *Asymptotic tracking of a nonminumum phase nonlinear system with non-hyperbolic zero dynamics*, IEEE Transactions on Automatic Control, vol.45, pp.542-546, 2000.

[16] J. Huang, *Remarks on robust output regulation problem for nonlinear systems*, IEEE Transactions on Automatic Control, vol.46, pp. 2028-2031, 2001.

[17] J. Huang, *On the solvability of the regulator equations for a class of nonlinear systems*, IEEE Transactions on Automatic Control, vol.48, pp. 880 - 885, 2003.

[18] J. Huang and Z. Chen, *A general framework for tackling the output regulation problem*, IEEE Transactions on Automatic Control, vol. 49, pp. 2203-2218, 2004.

[19] J. Huang, *Nonlinear output regulation: Theory and applications*, SIAM, Philadelphia, 2004.

[20] B. Igalls and E.D. Sontag, *Remarks on input to output stability*, Proceedings of the 38th IEEE Conference on Decision and Control, pp. 1226-1231, December 1999.

[21] B. Igalls, E.D. Sontag and Y. Wang, *Generalizations of asymtotic gain characterizations of ISS to input-to-output stability*, Proceedings of the 20th American Control Conference, pp. 2279-2284, December 2001.

[22] A. Isidori, *Nonlinear control systems*, New York: Springer-Verlag, vol. II, 1999.

[23] A. Isidori, L. Marconi, and A. Serrani, *Robust autonomous guidance: An internal model approach*, New York: Springer, 2003.

[24] H. Ito, *A constructive proof of ISS small-gain theorem using generalized scaling*, Proceedings of the 41th IEEE Conference on Decision and Control, pp.2286-2291, December 2002.

[25] Z.P. Jiang, A.R. Teel and L. Praly, *Small-gain theorem for ISS sysems and applications*, Mathematics of Control, Signals and Systems, vol. 7, pp. 95-120, 1994.

[26] Z.P. Jiang and I. Mareels, *A small-gain control method for nonliear cascaded systems with dynamic uncertainties*, IEEE Transactions on Automatic Control, vol. 42, pp. 292-308, 1997.

[27] Z. P. Jiang and Y. Wang, *Input-to-state stability for discrete-time nonlinear systems*, Automatica, vol. 37, pp.857-869, 2001.

[28] Z.P. Jiang and Y. Wang, *Small-gain theorem on input-to-output stability*, Proceedings of the Third International DCDIS Conference, pp. 220-224, 2003.

[29] G. Kaliora and A. Astolfi, *Nonlinear control of feedforward systems with bounded signals*, IEEE Transactions on Automatic Control, vol. 49, pp. 1975-1990, 2004.

[30] T. Kailath, *Linear systems*, Prentice-Hall, Inc., Upper Saddle River, 1980.

[31] H. Khalil, *Nonlinear systems*, Prince Hall, Upper Saddle River, New Jersey, 3rd edition, 2002.

[32] M. Krstic, J. Sun and P. Kokotovic, *Robust control of nonlinear systems with input unmodeled dynamics*, IEEE Transactions on Automatic Control, vol. 41, pp. 913-920, 1996.

[33] M. Krstic, *Feedback linearizability and explicit integrator forwarding controllers for classes of feedforward systems*, IEEE Transactions on Automatic Control, vol. 49, pp. 1668-1682, 2004.

[34] M. Krichman, E. D. Sontag and Y. Wang, *Input-output-to-state stability*, SIAM Journal on Control and Optimization, vol.39, pp.1874-1928, 2001.

[35] W. Y. Lan, Z. Chen and J. Huang, *Semi-global robust output regulation for a class of nonlinear systems using output feedback*, Proceedings of the 2005 American Control Conference, pp. 5127-5132, June 2005.

[36] Y. Lin, *Lyapunov function techniques for stabilization*, Ph.D. thesis, Rutgers University, 1992.

[37] Y. Lin, E.D. Sontag, and Y. Wang, *Recent results on Lyapunov theoretic techniques for nonlniear stability*, Report SYCON-93-09.

[38] Y. Lin, E.D. Sontag and Y. Wang, *Input to state stability for parameterized families of systems*, International Journal of Robust and Nonlinear Control, vol. 5, pp. 187-205, 1995.

[39] Y. Lin, E.D. Sontag and Y. Wang, *A smooth converse Lyapunov theorem for robust stability*, SIAM Jounal on Control and Optimization, vol. 34, pp. 124-160, 1996.

[40] Y. Lin, Y. Wang and D.Z. Cheng, *On nonuniform and semi-uniform input-to-state stbility for time varying systems*, Proceedings of the 16th IFAC World Congress, July 2005.

[41] Z. Lin and A. Saberi, *Semi-global stabilization of partially linear composite systems via feedback of the state of the linear part*, Systems and Control Letters, vol. 20, pp. 199-207, 1993.

[42] Z. Lin and A. Saberi, *Output regulation for linear systems subject to input saturation*, Automatica, vol. 32, No. 1, pp. 29-47, 1996.

[43] W. Liu, Y. Chitour and E.D. Sontag, *On finite gain stabilizability of linear systems subject to input saturation*, SIAM Jounal on Control and Optimization, vol. 31, pp. 1190-1219, 1996.

[44] L. Marconi and A. Isidori, *Robust global stabilization of a class of uncertain nonlinear systems*, Systems and Control Letters, vol. 41, pp. 281-290, 2000.

[45] L. Marconi, A. Isidori and A. Serrani, *Input disturbance suppression for a class of feedforward uncertain nonlinear systems*, Systems and Control Letters, vol. 45, pp. 227-236, 2002.

[46] F. Mazenc and L. Praly, *Adding integrations, saturated controls and stabilization for feedforward systems*, IEEE Transactions on Automatic Control, vol. 41, pp. 1559-1578, 1996.

[47] I. G. Polushin, A. Tayebi and H. J. Marquez *Control schemes for stable teleoperation with communication delay based on IOS small gain theorem*, Automatica, vol. 42, pp. 905-915, 2006.

[48] A. Rapaport and A. Astolfi, *A remark on the stability of interconnected nonlinear systems*, IEEE Transactions on Automatic Control, vol. 49, pp. 120-124, 2004.

[49] L. Praly and Y. Wang, *Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-ot-state stability*, Mathematics of Control, Signals and Systems, vol. 9, pp. 1-33, 1996.

[50] W. Rudin, Principles of mathematical analysis, Third Edition, McGRAW-HILL Book Company, 1976.

[51] R. Sepulchre, M. Jankovic and P. Kokotovic, *Constructive nonlinear control*, New York: Springer, 1997.

[52] J.J.E. Slotine and W. Li, *Applied nonlinear control*, Prentice-Hall, 1991.

[53] E.D. Sontag, *Smooth stabilization implies comprime factorization*, IEEE Transactions on Automatic Control, vol. 34, pp. 435-443, 1989.

[54] E.D. Sontag, *On the input-to-state stability property*, Journal of Control, vol. 1, pp. 24-36, 1995.

[55] E.D. Sontag and Y. Wang, *On the characteristics of the input-to-state stability property*, Systems and Control Letters, vol. 24, pp. 351-359, 1995.

[56] E.D. Sontag and A.R. Teel, *Changing supply function in input/state stable systems*, IEEE Transactions on Automatic Control, vol.40, pp.1476-1478, 1995.

[57] E.D. Sontag and Y. Wang, *New characterization of input-to-state stability*, IEEE Transactions on Automatic Control, vol. 41, pp. 1283-1294, 1996.

[58] E.D. Sontag and Y. Wang, *Notions of input to output stability*, Systems and Control Letters, vol. 38, pp. 351-359, 1999.

[59] E.D. Sontag and Y. Wang, *Lyapunov characterizations of input to output stability*, SIAM Journal on Control Optimization, vol. 39, No. 1, pp. 226-249, 2000.

[60] E.D. Sontag and B. Ingalls, *A small-gain theroem with applications to input/output systems, incremental stability, detectability, and interconnections*, Jounal of the Franklin Institute, pp. 211-229, 2002.

[61] A.R. Teel, *A nonlinear small gain theorem for the analysis of control systems with saturation*, IEEE Transactions on Automatic Control, vol. 41, pp. 1256-1270, 1996.

[62] M. Zhu and J. Huang, *Semi-global robust stabilization for a class of feedforward systems*, Proceedings of the 2006 American Control Conference, pp. 97 - 102, June 2006.

[63] M. Zhu and J. Huang, *Global robust stabilization for a class of feedforward systems with dynamic uncertainties*, Proceedings of the 6th World Congress on Intelligent Control and Automation, pp. 159 - 163, June 2006.

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[1] M. Zhu and J. Huang, *Semi-global robust stabilization for a class of feedforward systems*, Proceedings of the 2006 American Control Conference, pp. 97 - 102, June 2006.

[2] M. Zhu and J. Huang, *Global robust stabilization for a class of feedforward systems with dynamic uncertainties*, Proceedings of the 6th World Congress on Intelligent Control and Automation, pp. 159 - 163, June 2006.