Valuation of Stock Loans under Exponential Phase-type Lévy Models

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Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Abstract

Stock loan, or security lending, is currently a very popular financial service provided by many financial institutions. It is a collateral loan where stocks are used as collateral. The borrower may repay the principal with interest and regain the stock, or make no repayment and surrender the stock. This thesis is concerned with the stock loan valuation problem, in which the underlying stock price is modeled as an exponential Lévy Models of phase-type. The valuation problem can be formulated as an optimal stopping problem of a perpetual American option with a time varying exercise price. As the phase-type jump diffusion forms a dense class in Lévy processes, our solution can approximate the solution under general Lévy models arbitrarily close.

摘要

股票抵押貸款(Stock loans)是現今金融業界非常流行的投資產品。它是一種以股票作為抵押品的貸款。借貸人可以歸還本金及利息以續回股票，或可以放棄用以抵押的股票而不作還款。此論文是探討有關股票在指數性相位型Lévy過程模型下股票抵押貸款的定價問題。此定價問題可視為有時變性履約價的永久美式期權之最優化停止問題。由於相位型跳躍擴散過程是所有Lévy過程中的稠密子集，因此我們的定價模型可以用作估算一般Lévy模型所引伸之價格。

Contents

Chapter 1

Introduction

A stock loan is a loan issued by financial institutions (the lender) to its clients (the borrower) which is collateralized with stocks. Recently, stock loans become a very popular product in over-the-counter market. As reported by International Securities Lending Association, the global market size of these products exceeded £1 trillion [1].

Under the terms of the contract, the borrower has the right to repay the loan at anytime, or to simply default the loan with the loss of the collateral. With this in mind, the borrower's right can be regarded as a perpetual American option, which represents the right for the borrower to exercise the option at anytime, without a time limit. The value of this perpetual American option is therefore of central importance to the problem of stock loan valuation.

The value of this perpetual American option can be expressed as an ordinary perpetual American call option with a possibly negative interest rate. This creates the major challenge of stock loan pricing. Consider the case of geometric Brownian motion (GBM) for the stock price. The optimal exercise rule of a perpetual American call option is to exercise at the first time that the stock price rises to cross a constant

[1] This number is quoted from the article: An Introduction to Securities Lending, Executive Summary, Page 8, issued by Australian Securities Lending Association Limited at 1 August 2005.

level. This constant level is called the optimal exercise boundary. If interest rate is positive, the stock price will cross any fixed boundary almost surely. The perpetual American call option can then be valued directly with a variational inequality (VI). In contrast, when interest rate is negative, the problem becomes complicated. Given any fixed boundary level greater than the current stock price, there is a positive probability that the stock price will never cross this level.

Xia and Zhou (2007) are pioneers of solving the stock loan problem. They value the stock loan under the classical GBM model using a purely probabilistic approach. Zhang and Zhou (2009) then extended the framework to a regime switching model and solved the problem using variational inequalities. Dai and Xu (2009) studied the optimal redeeming strategy of stock loans with finite maturity under GBM. Yam et al. (2010) considered the callable feature of the stock loans.

Although most studies on stock loan adopt the GBM model for stock price, empirical evidences (e.g. Andersen et al. (2002), Pan (2002) and Eraker et al (2003)) show that jump diffusion model would be a better model for asset prices to capture the heavy tails of the empirical distribution. Therefore, a jump diffusion model with flexible jump distribution is worth considering for stock loan valuation.

Merton (1976) is the first one to propose jump diffusion for asset price modeling using a Gaussian jump distribution. Another notable jump diffusion model is the double-exponential jump diffusion proposed by Kou (2002). The generalization of jump diffusion model is the exponential Lévy model, such as the variance-gamma model (Madan et al., 1998), CGMY model (Carr et al., 1999) and normal inverse Gaussian model (Barndorff-Nielssen, 2000).

Sun (2010) recently considered the stock loan valuation problem under the double-exponential jump diffusion model in the first chapter of her thesis. While it is a good start, the asset return distribution is not flexible enough to capture the empirical distribution implied by market data. For this reason, we incorporate the phase-type jump diffusion to stock loans valuation.

The phase-type distribution is dense over the class of all positive valued distributions. By making use of this fact, Asmussen et al. (2007) show that the class of phase-type jump diffusion model is dense over all exponential Lévy model. In other words, the option price derived from phase-type jump diffusion models can be used to approximate the corresponding price under a general exponential Lévy model. In particular, Asmussen et al. (2007) approximate the CGMY model by the phase-type jump diffusion. In fact, the phase-type jump diffusion model embraces the Kou (2002) model and the mixed-exponential jump diffusion model (Cai and Kou, 2011) as its special cases.

Asmussen et al. (2004) solved the price of the perpetual American put option with positive interest rate under phase-type jump diffusion models. They used the technique of Wiener-Hopf factorization to derive the optimal exercise boundary. Then the pricing problem is reduced to the evaluation of the corresponding expectation at the given exercise boundary.

While Wiener-Hopf factorization is useful to solve American option pricing problem involving Lévy processes and, in particular, the phase-type Lévy model, it relies heavily on the positive interest rate or, in the limiting case, zero interest rate. The method no longer works for a negative effective interest rate in the stock loan valuation problem. We take the variational inequality approach as in Zhang and Zhou (2009).

Under the phase-type jump diffusion model, we show that the price of the perpetual American option satisfies an ordinary integro-differential equation (OIDE). The solution of this OIDE is closely linked to the root characteristics of a Cramér-Lundberg equation (C-L equation). The root characteristics of the C-L equation is first studied in a special case of the phase-type distribution, the hyperexponential distribution. By making use of the properties of this special case, we extend our result to a fairly general class of phase-type models.

The rest of the thesis is organized as follow. Chapter 2 introduces the elements of our problem. Chapter 3 presents some general properties of stock loans. Chapter 4 presents the methodologies of valuation. Chapter 5 discusses a possible extension to incorporate stochastic volatility. Chapter 6 concludes.

Chapter 2

Problem Formulation

In this chapter, we describe the formulation of stock loan valuation under the phase-type Lévy model. We introduce the phase-type distribution, its use in the phase-type jump diffusion model, and the formulation of stock loan as a perpetual American call option pricing problem.

2.1 Phase-type distribution

2.1.1 A generalization of the exponential distribution

Consider a continuous time Markov process with 1 transient state and 1 absorption state. The intensity matrix is given by

$$\begin{pmatrix} -\theta & \theta \\ 0 & 0 \end{pmatrix},$$

where $\theta > 0$. Let Y be the absorption time of this Markov process. Then the distribution of Y is the exponential distribution. The cumulative distribution function is

$$F_Y(y) = 1 - e^{-\theta y}. \tag{2.1}$$

A finite mixture of the exponential distribution is called hyperexponential distribution. This can be expressed as the absorption time of a continuous time Markov process with m transient state, 1 absorption state with an intensity matrix of the form

$$\begin{pmatrix} -\theta_1 & \cdots & 0 & \theta_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\theta_m & \theta_m \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

The cumulative distribution function is

$$F_Y(y) = \sum_{i=1}^{m} \alpha_i \left(1 - e^{-\theta_i y}\right), \tag{2.2}$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$. α_i is the probability for the process to start at state i. It can also be expressed using matrix notation,

$$F_Y(y) = 1 - \boldsymbol{\alpha} e^{\mathbf{T}y} \mathbf{1}, \tag{2.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $\mathbf{T} = \text{diag}(-\theta_1, \ldots, -\theta_m)$, $\mathbf{1} = (1, \ldots, 1)^T$.

A further generalization allows the transient states to be communicative. The resulting distribution becomes the phase-type distribution described in the next section.

2.1.2 Properties of the phase-type distribution

The phase-type distribution is the absorption time of a finite state continuous time Markov process with m transient states and 1 absorption state.

Let $\mathbf{T}$ be the intensity matrix of the transient states and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be an initial probability vector. The phase-type distribution is parameterized by $(m, \mathbf{T}, \boldsymbol{\alpha})$. The full intensity matrix of the Markov process can be written as

$$\boldsymbol{S} = \begin{pmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{0} & 0 \end{pmatrix},$$

where $\mathbf{t} = -\mathbf{T}\mathbf{1}$. The cumulative distribution function is given by

$$F_Y(y) = 1 - \boldsymbol{\alpha} e^{\mathbf{T}y}\mathbf{1}. \tag{2.4}$$

The density function is given by

$$f_Y(y) = \boldsymbol{\alpha} e^{\mathbf{T}y}\mathbf{t}. \tag{2.5}$$

Finally, the generating function is given by

$$M(t) = \mathbb{E}[e^{tY}] = \boldsymbol{\alpha}(-t\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}. \tag{2.6}$$

The class of phase-type distribution is very rich. When $\mathbf{T}$ is a diagonal matrix, the distribution reduces to a hyperexponential distribution. As shown in Johnson and Taafe (1988), the class of phase-type distribution is dense in the field of all distributions on $(0, \infty)$.

2.2 Phase-type jump diffusion model

2.2.1 Jump diffusion model

If the price of an asset S_t follows jump diffusion model, then the change in price consists of three components: drift, Brownian motion and a jump process. The stochastic dynamics can be expressed in the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \left(\nu + \frac{1}{2}\sigma^2\right) dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t} \left(e^{Y_i} - 1\right)\right), \tag{2.7}$$

where $\{W_t\}_{t\geq 0}$ is the standard Brownian motion, $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity λ, $\{Y_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables.

For a Poisson process N_t, when h is small, we have:

1. $\Pr\left(N_{t+h} - N_t = 0\right) = 1 - \lambda h + o(h)$;

2. $\Pr\left(N_{t+h} - N_t = 1\right) = \lambda h + o(h)$;

3. $\Pr\left(N_{t+h} - N_t \geq 2\right) = o(h)$,

where $o(h)$ is the asymptotic order symbol such that $\lim_{h\to 0} \frac{o(h)}{h} = 0$. Hence, (2.7) can be alternatively written as

$$\frac{dS_t}{S_t} = \left(\nu + \frac{1}{2}\sigma^2\right) dt + \sigma dW_t + \left(e^{Y_i} - 1\right) dN_t. \tag{2.8}$$

An application of Itō's formula gives

$$d\ln S_t = \nu dt + \sigma dW_t + Y_i dN_t, \tag{2.9}$$

which implies that

$$S_t = S_0 \exp\left(\nu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right). \tag{2.10}$$

For $r > 0$, we have

$$\mathbb{E}\left[e^{-rt} S_t\right] = S_0 \exp\left(-r + \nu + \frac{1}{2}\sigma^2 + \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)\right).$$

If we set $\nu = r - \sigma^2/2 - \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)$, then $\{e^{-rt}S_t\}_{t\geq 0}$ is a martingale. This motivates the definition in the next section.

2.2.2 The stock price model

The stock price process is defined on a risk-neutral probability space $\left(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}\right)$. We write

$$S_t = \exp(X_t), \tag{2.11}$$

$$X_t = x + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{2.12}$$

where $\mu = r - \sigma^2/2 - \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)$. The distribution of Y_i, $i \in \mathbb{N}$, is a two-sided phase-type distribution and the density function is given by

$$f_Y(y) = p\boldsymbol{\alpha}^+ e^{\mathbf{T}^+ y}\mathbf{t}^+ I_{\{y\geq 0\}} + (1-p)\boldsymbol{\alpha}^- e^{-\mathbf{T}^- y}\mathbf{t}^- I_{\{y<0\}}. \tag{2.13}$$

Note that the financial market is incomplete under the jump diffusion setting. That means that not all contingent claims can be perfectly hedged and the martingale measure is not unique. In other words, there are infinitely many equivalent martingale measures. Our choice $\mathbb{P}$ is the one that preserves the phase-type structure of the log-price X_t as proposed by Assmusen et al. (2004).

2.3 Stock Loans

Stock loan is a collateral loan where stocks are used as collateral. The borrower will receive the loan principle (q), pay the service charge (c) and have the right to repay the principal with interest (continuously compounded with rate γ) and regain the stock anytime in the future. The transactions can be summarized as follows:

- The borrower receives money $q - c$ as well as V_0, a perpetual American option with time varying strike price $qe^{\gamma t}$.
- The bank receives S_0 (one unit of stock) as collateral.

By equating the benefits of both parties, it is seen that the service charge is

$$c = q + V_0 - S_0. \tag{2.14}$$

We have the following representation of the value of the perpetual American option:

$$V_0 = V(x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(S_\tau - qe^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} | S_0 = e^x\right], \tag{2.15}$$

where $\mathcal{T}_u$, $u \geq 0$, is the set of all stopping time taking values in the time interval (u, ∞). By taking the transformation $\tilde{S}_t = S_t e^{-\gamma t}$, the value can be written as

$$V(x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-(r-\gamma)\tau}\left(\tilde{S}_\tau - q\right)^+ I_{\{\tau<\infty\}} | \tilde{S}_0 = e^x\right], \tag{2.16}$$

which is the value of a perpetual American option with constant strike price and a possibly negative effective interest rate $\tilde{r} = r - \gamma$.

From now on, we will stick with the transformed stock price process $\tilde{S}_t$ as the underlying stock of the American option. We also define $\tilde{X}_t$ to be the transformed log-price. Their dynamics are given by

$$\tilde{S}_t = \exp(\tilde{X}_t), \tag{2.17}$$

$$\tilde{X}_t = x + \tilde{\mu}t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{2.18}$$

where $\tilde{\mu} = r - \gamma - \sigma^2/2 - \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)$.

Chapter 3

General Properties of Stock Loans

3.1 Preliminary results

We establish some properties of the perpetual American option as a function of the stock price value. Take $S = e^x$ and write $v(S) = V(\ln S) = V(x)$.

Lemma 3.1 *$v(S)$, as a deterministic function of the initial stock price S, satisfies the following properties:*

1. *$(S - q)^+ \leq v(S) \leq S$ for all $S > 0$;*

2. *$v(S)$ is convex, continuous and nondecreasing in S on $(0, \infty)$.*

Proof For the first item, observe that

$$v(S) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(S_\tau - qe^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid S_0 = S\right]. \tag{3.1}$$

By taking $\tau = 0$, we get $(S - q)^+ \leq v(S)$. On the other hand, since $(S - qe^{\gamma\tau})^+ \leq S$, we have

$$
\begin{aligned}
v(S) &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(S_\tau - qe^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid S_0 = S\right] \\
&\leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0} \mathbb{E}\left[e^{-r\tau} S_\tau I_{\{\tau<\infty\}} \mid S_0 = S\right] \\
&\leq S.
\end{aligned}
$$

Next, it is obvious that $v(\cdot)$ is a nondecreasing function. Convexity of $v(\cdot)$ is a direct consequence of the convexity of $\max\{\cdot, 0\}$ function and the essential supremum operator. As the function value is finite, convexity of $v(\cdot)$ implies its continuity.

□

The next lemma is an essential step to solve the optimal stopping problem.

Lemma 3.2 *Define* $k = \inf\{S > 0 : S - q \geq v(S)\} \geq q$, *where* $\inf\emptyset = \infty$. *Then* $\{S > 0 : S - q \geq v(S)\} = [k, \infty)$.

Proof If $k = \infty$, the result is obvious. For the case that $k \in [q, \infty)$, we have $v(k) = k - q$ by the continuity of v. We claim that $v(S) = S - q$ for $S \geq k$. Otherwise, there exists $k_0 \geq k$ such that $v(k_0) > k_0 - q$ because of Lemma 3.1. By convexity, we have

$$
\frac{v(S) - v(k)}{S - k} \geq \frac{v(k_0) - v(k)}{k_0 - k} > 1.
$$

for any $S \geq k_0$. As a consequence $v(S) \geq \frac{v(k_0)-v(k)}{k_0-k}(S - k) + k - q$ which implies $v(S) > S$ for sufficiently large value of S. This is a contradiction to Lemma 3.1.

□

Using similar methods of Xia and Zhou (2007), we now prove that the optimal stopping time is a first hitting time. In other words, it is optimal to exercise the perpetual American option at the first time when the transformed log-price exceeds a predetermined level. Such a level is called the optimal exercise boundary.

Theorem 3.1 *If $\tilde{X}_t$ follows a Lévy process, then the optimal stopping time is of the form*

$$\tau_b = \inf\left\{t \geq 0 : \tilde{X}_t \geq b\right\}, \tag{3.2}$$

where b is a constant.

Proof The stock loan value at time t can be written as

$$\begin{aligned}
V_t &= v(S_t) \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\left(S_t e^{X_\tau - X_t} - qe^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid \mathcal{F}_t\right] \\
&= e^{\gamma t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\left(e^{-\gamma t} S_t e^{X_\tau - X_t} - qe^{\gamma(\tau-t)}\right)^+ I_{\{\tau<\infty\}} \mid \mathcal{F}_t\right] \\
&= e^{\gamma t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(xe^{X_\tau} - qe^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid \mathcal{F}_0\right]_{x=e^{-\gamma t}S_t} \\
&= e^{\gamma t} v(e^{-\gamma t} S_t).
\end{aligned}$$

Hence, the optimal stopping time (cf. Karatzas and Shreve, 1998, Chapter 2.5) is

$$\begin{aligned}
\tau^* &= \inf\left\{t \geq 0 : S_t - qe^{\gamma t} \geq v(S_t)\right\} \\
&= \inf\left\{t \geq 0 : S_t - qe^{\gamma t} \geq e^{\gamma t} v(e^{-\gamma t} S_t)\right\} \\
&= \inf\left\{t \geq 0 : S_t e^{-\gamma t} - q \geq v(e^{-\gamma t} S_t)\right\} \\
&= \inf\left\{t \geq 0 : e^{-\gamma t} S_t \geq k\right\} \\
&= \inf\left\{t \geq 0 : \tilde{X}_t \geq \ln k\right\},
\end{aligned}$$

where k is the value defined in Lemma 3.2.

□

We denote the optimal exercise boundary by b^* and the optimal stopping time by τ_{b^*}. Theorem 3.1 greatly reduces the dimensionality of the optimization problem. The original optimization problem has to search over all possible stopping time. Yet, the optimal stopping time is in the form of a first hitting time and we only need to search for an optimal exercise boundary, which is a one-dimensional optimization problem. In other words, the value function is given by

$$V(x) = \sup_{b \geq \max\{\ln q, x\}} V_b(x) = \sup_{b \geq \max\{\ln q, x\}} \mathbb{E}\left[e^{-\tilde{r}\tau_b}\left(e^{\tilde{X}_{\tau_b}} - q\right)^+ I_{\{\tau_b < \infty\}} | \tilde{X}_0 = x\right]. \quad (3.3)$$

3.2 Characterization of the function $V(x)$

We want to show that $V(x)$ is a solution of an integro-differential equation (OIDE) and derive its functional form. Before going into that, we first introduce the Cramér-Lundberg equation (C-L equation)

$$G(\beta) = \frac{\sigma^2}{2}\beta^2 + \tilde{\mu}\beta + \lambda p \boldsymbol{\alpha}^+(-\beta\mathbf{I} - \mathbf{T}^+)^{-1}\mathbf{t}^+ + \lambda(1-p)\boldsymbol{\alpha}^-(\beta\mathbf{I} - \mathbf{T}^-)^{-1}\mathbf{t}^- - \lambda = \tilde{r}. \quad (3.4)$$

We use the symbol $\mathcal{B}^+$ to denote the collection of roots to the C-L equation with real part larger than or equal to 1 and $\mathcal{B}^-$ to denote the collection of those roots with negative real part. The root characteristics of the C-L equation play a central role in our problem as we will see in later sections. As a starting point, observe the following properties regarding this equation:

1. $\{e^{-rt}S_t\}_{t\geq 0} = \left\{e^{-\tilde{r}t}\tilde{S}_t\right\}_{t\geq 0}$ is a martingale implies that $1 \in \mathcal{B}^+$.

2. The function $G(\beta)$ satisfies

$$\mathbb{E}\left[e^{\beta\tilde{X}_t}\right] = e^{G(\beta)t} \tag{3.5}$$

for β belongs to some bounded interval covering $[0,1]$.

If $G'(1) \geq 0$, it will be shown that $V(x) = e^x$ and that $q = c$. That means that the bank has no intention to make such a stock loan contract with the given loan interest rate γ and current stock price S_0. Therefore, we will focus on the more interesting case $G'(1) < 0$. The case for $G'(1) \geq 0$ is postponed to section 4.3.

It is worth noting that $G'(1) < 0$ implies $\gamma > r$. In other words, the effective interest rate $\tilde{r} = r - \gamma$ is indeed negative. To see this, recall

$$G(\beta) = \frac{\sigma^2}{2}\beta^2 + \left(r - \gamma - \frac{\sigma^2}{2} - \lambda\left(\mathbb{E}\left[e^{Y_1}\right] - 1\right)\right)\beta + \lambda\left(\mathbb{E}\left[e^{\beta Y_1}\right] - 1\right). \tag{3.6}$$

Hence,

$$G'(1) = r - \gamma + \frac{\sigma^2}{2} + \lambda\mathbb{E}\left[Y_1 e^{Y_1} - e^{Y_1} + 1\right]. \tag{3.7}$$

Since $ye^y - e^y + 1 \geq 0$ for all $y \in \mathbb{R}$, $G'(1) < 0$ implies

$$\gamma > r + \frac{\sigma^2}{2} + \lambda\mathbb{E}\left[Y_1 e^{Y_1} - e^{Y_1} + 1\right] \geq r.$$

We are now ready to present the result which characterizes the function $V(x)$. It is easy to see that this new result embraces the stock loan valuation under double-exponential jump diffusion model (Sun, 2010) as its special case.

Theorem 3.2 *$V(x)$ satisfies the following integro-differential equation*

$$\begin{cases} (\mathcal{L}-\tilde{r})\,V(x) &= 0 \qquad\quad x < b^* \\ V(x) &= e^x - q \quad x \geq b^* \end{cases}, \tag{3.8}$$

where $\mathcal{L}h(x) = \frac{\sigma^2}{2}\frac{d^2h}{dx^2}(x) + \tilde{\mu}\frac{dh}{dx}(x) + \lambda\int_{-\infty}^{\infty}(h(x+y)-h(x))\,f_Y(y)dy$. *Furthermore, the solution takes the form*

$$V(x) = \begin{cases} \sum\limits_{\beta_j\in\mathcal{B}+} \omega_j e^{\beta_j x} & x < b^* \\ e^x - q & x \geq b^* \end{cases} \tag{3.9}$$

for some ω_j, $j \in \{i \mid \beta_i \in \mathcal{B}^+\}$ to be determined according to the model.

Proof Consider the following function as a candidate solution:

$$u(x) = \begin{cases} \sum\limits_{\beta_j\in\mathcal{B}+} \omega_j e^{\beta_j x} & x < b^* \\ e^x - q & x \geq b^* \end{cases}.$$

It is reasonable to assume that ω_j, $j \in \{i \mid \beta_i \in \mathcal{B}^+\}$ should be chosen such that $u(\cdot)$ satisfies the conditions described in Lemma 3.1. In particular, we should have $(e^x - q)^+ \leq u(x) \leq e^x$ for all $x \in \mathbb{R}$. It also satisfies the OIDE

$$\begin{cases} (\mathcal{L}-\tilde{r})\,u(x) &= 0 \qquad\quad x < b^* \\ u(x) &= e^x - q \quad x \geq b^* \end{cases}. \tag{3.10}$$

However, it may not be continuously differentiable at b^*. Hence, we construct a sequence of function $\{u_n(x)\}_{n=1}^{\infty}$ such that

1. $u_n(x) \to u(x)$ as $n \to \infty$ for all x;

2. $u_n(x)$ is twice continuous differentiable for all $n \in \mathbb{N}$;

3. For $x \leq b^*$ or $x \geq b^* + \frac{1}{n}$, $u_n(x) \equiv u(x)$;

4. For $b^* \leq x \leq b^* + \frac{1}{n}$, $0 \leq u_n(x) \leq M_1$, where M_1 is a positive constant.

For any $x < b^*$, we have

$$(\mathcal{L} - \tilde{r})\, u_n(x) = \lambda \int_{b^*-x}^{b^*-x+1/n} \left[u_n(x+y) - u(x+y)\right] f_Y(y)dy. \tag{3.11}$$

Note that

$$|u_n(x) - u(x)| \leq \max_{x \in (b^*, b^*+1/n} |u_n(x)| + \max_{x \in (b^*, b^*+1/n} |u(x)| \leq M_2,$$

where $M_2 = M_1 + e^{b^*+1}$. Then we have

$$\begin{aligned} |\mathcal{L}u_n(x) - \tilde{r}u_n(x)| &\leq \lambda p \boldsymbol{\alpha}^+ \mathbf{t}^+ \int_{b^*-x}^{b^*-x+1/n} \left[u_n(x+y) - u(x+y)\right] dy \qquad (3.12) \\ &\leq \frac{\lambda p \boldsymbol{\alpha}^+ \mathbf{t}^+ M_2}{n} \to 0 \text{ uniformly for all } x < b^* \text{ , as } n \to \infty. \end{aligned}$$

Next, by applying Itō's formula to $\left\{e^{-\tilde{r}t} u_n(\tilde{X}_t)\right\}_{t \geq 0}$, we can obtain a sequence of local martingale $\left\{M_t^{(n)}\right\}_{t \geq 0}$ for $n \in \mathbb{N}$ as follows:

$$M_t^{(n)} = e^{-\tilde{r}(t \wedge \tau_{b^*})} u_n(\tilde{X}_{t \wedge \tau_{b^*}}) - u_n(x) - - \int_0^{t \wedge \tau_{b^*}} e^{-\tilde{r}s} \left[(\mathcal{L} - \tilde{r})\, u_n(\tilde{X}_s)\right] ds. \tag{3.13}$$

We claim that it is a true martingale for any $n \in \mathbb{N}$. Note that for any $t \geq 0$,

$$
\begin{aligned}
&|e^{-\tilde{r}(t\wedge\tau_{b^*})}u_n(\tilde{X}_{t\wedge\tau_{b^*}})| \\
\leq\ & |e^{-\tilde{r}t}u_n(\tilde{X}_t)I_{\{t<\tau_{b^*}\}}| + |e^{-\tilde{r}t}u_n(\tilde{X}_{\tau_{b^*}})I_{\{t\geq\tau_{b^*},\tilde{X}_{\tau_{b^*}}<b+1/n\}}| + |e^{-\tilde{r}t}u_n(\tilde{X}_{\tau_{b^*}})I_{\{t\geq\tau_{b^*},\tilde{X}_{\tau_{b^*}}\geq b+1/n\}}| \\
\leq\ & |e^{-\tilde{r}t}u(\tilde{X}_t)I_{\{t<\tau_{b^*}\}}| + M_1e^{-\tilde{r}t} + e^{-\tilde{r}t}(e^{\tilde{X}_{\tau_{b^*}}} - q)I_{\{\tau_{b^*}<\infty\}}. \qquad (3.14)
\end{aligned}
$$

From the definition in (3.13) and noting (3.12) and (3.14), we establish the following inequality,

$$
|M_t^{(n)}| \leq |e^{-\tilde{r}t}u(\tilde{X}_t)I_{\{t<\tau_{b^*}\}}|+|u_n(x)|+M_1e^{-\tilde{r}t}+e^{-\tilde{r}t}(e^{\tilde{X}_{\tau_{b^*}}}-q)I_{\{\tau_{b^*}<\infty\}}-\frac{\lambda p\boldsymbol{\alpha}^+\mathbf{t}^+M_2\left(e^{-\tilde{r}t}-1\right)}{n\tilde{r}}. \qquad (3.15)
$$

For the first term in the right hand side of (3.15), we have for any fixed $T > 0$

$$
\begin{aligned}
&\mathbb{E}_x\left[\sup_{t\in[0,T]} e^{-\tilde{r}t}u(\tilde{X}_t)I_{\{t<\tau_{b^*}\}}|\right] \\
\leq\ & e^{-\tilde{r}T}\mathbb{E}_x\left[e^{\sup_{t\in[0,T]}\tilde{X}_t}\right] \\
\leq\ & e^{-\tilde{r}T}\mathbb{E}_x\left[e^{x+\tilde{\mu}T+\sigma\sup_{t\in[0,T]}W_t+\sum_{i=1}^{N_T}Y_i^+}\right] \\
=\ & 2\Phi(\sigma\sqrt{T})\exp\left(-\tilde{r}T + x + \tilde{\mu}T + \frac{\sigma^2T}{2} + p\lambda T\boldsymbol{\alpha}^+\left(-\mathbf{I}-\mathbf{T}^+\right)^{-1}\mathbf{t}^+\right) \\
<\ & \infty,
\end{aligned}
$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. It is now easy to see that

$$
\mathbb{E}_x\sup_{t\in[0,T]}|M_t^{(n)}| < \infty, \qquad (3.16)
$$

which guarantees that $M_t^{(n)}$ is a true martingale for all n. Then, we know that for $x < b^*$

$$
\begin{aligned}
u(x) &= \lim_{n\to\infty} u_n(x) \\
&= \lim_{n\to\infty} \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u_n(\tilde{X}_{t\wedge\tau_{b^*}})\right] - \lim_{n\to\infty}\mathbb{E}_x M_t^{(n)} - \lim_{n\to\infty}\mathbb{E}_x\left[\int_0^{t\wedge\tau_{b^*}} e^{-\tilde{r}s}\left[(\mathcal{L}-\tilde{r})\,u_n(\tilde{X}_s)\right]ds\right] \\
&= \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})\right],
\end{aligned}
$$

with the last equality implied by the dominated convergence theorem (DCT). Now, let $t\to\infty$ and apply Fatou's lemma to get

$$
\begin{aligned}
u(x) &= \lim_{t\to\infty}\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})\right] \\
&= \lim_{t\to\infty}\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}<\infty\}}\right] + \lim_{t\to\infty}\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}=\infty\}}\right] \\
&\geq \mathbb{E}_x\left[e^{-\tilde{r}(\tau_{b^*})}u(\tilde{X}_{\tau_{b^*}})I_{\{\tau_{b^*}<\infty\}}\right] \\
&= \mathbb{E}_x\left[e^{-\tilde{r}(\tau_{b^*})}(e^{\tilde{X}_{\tau_{b^*}}}-q)I_{\{\tau_{b^*}<\infty\}}\right].
\end{aligned}
$$

On the other hand,

$$
\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})\right] = \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}(e^{\tilde{X}_{t\wedge\tau_{b^*}}}-q)I_{\{\tau_{b^*}\leq t\}}\right] + \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right]. \tag{3.17}
$$

Since

$$
e^{-\tilde{r}(t\wedge\tau_{b^*})}(e^{\tilde{X}_{t\wedge\tau_{b^*}}}-q)I_{\{\tau_{b^*}\leq t\}} \leq e^{-\tilde{r}(t\wedge\tau_{b^*})}(e^{\tilde{X}_{t\wedge\tau_{b^*}}}-q)I_{\{\tau_{b^*}<\infty\}}
$$

and

$$
\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}(e^{\tilde{X}_{t\wedge\tau_{b^*}}}-q)I_{\{\tau_{b^*}<\infty\}}\right] < \infty,
$$

DCT implies that the first term on the right hand side of (3.17) converges to

$$\mathbb{E}_x\left[e^{-\tilde{r}(\tau_{b^*})}(e^{\tilde{X}_{\tau_{b^*}}}-q)I_{\{\tau_{b^*}<\infty\}}\right]$$

as $t \to \infty$. For the second term, we claim that

$$\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right] \to 0$$

as $t \to \infty$. This can be shown by considering the following two cases:

Case 1: $G'(1) < 0$, there exists $\kappa_0 > 1$ such that $G(\kappa_0) - \tilde{r} < 0$. In addition, there exists $C_0 > 0$ such that $u(x) < C_0 e^{\kappa_0 x}$ for all $x < b^*$. Hence,

$$\begin{aligned}\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right] &< \mathbb{E}_x\left[C_0 e^{-\tilde{r}t+\kappa_0\tilde{X}_t}I_{\{\tau_{b^*}>t\}}\right]\\ &\leq \mathbb{E}_x\left[C_0 e^{-\tilde{r}t+\kappa_0\tilde{X}_t}\right]\\ &\to 0 \text{ as } t \to \infty.\end{aligned}$$

Case 2: $G'(1) \geq 0$, as $v(x) \leq e^x$ for all $x \in \mathbb{R}$, we have

$$\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right] \leq \mathbb{E}_x\left[e^{-\tilde{r}t+\tilde{X}_t}I_{\{\tau_{b^*}>t\}}\right]. \tag{3.18}$$

Consider the probability measure $\hat{\mathbb{P}}$ as follows:

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{-\tilde{r}t+\tilde{X}_t}. \tag{3.19}$$

It is shown in Appendix A of Asmussen et al. (2004) that $\{\hat{X}_t\}_{t>0}$ is also a phase-type jump diffusion process and the corresponding Lévy exponent is given by

$$\hat{G}(s) = G(1+s) - G(1). \tag{3.20}$$

Define $\hat{\tau}_{b^*} = \inf\{t \geq 0 : \hat{X}_t \geq b^*\}$. Then

$$\mathbb{E}_x\left[e^{-\tilde{r}t+\tilde{X}_t} I_{\{\tau_{b^*}>t\}}\right] = \hat{\mathbb{E}}_x\left[I_{\{\hat{\tau}_{b^*}>t\}}\right] \to \hat{\mathbb{E}}_x\left[I_{\{\hat{\tau}_{b^*}=\infty\}}\right] \tag{3.21}$$

as $t \to \infty$. Observe that $\hat{G}'(0) = G'(1) \geq 0$ and $\hat{G}(0) = 0$, under $\hat{\mathbb{P}}$ measure

$$\Pr(\hat{\tau}_{b^*} < \infty) = \lim_{\tilde{r}\to 0} \hat{\mathbb{E}}\left[e^{-\tilde{r}\hat{\tau}_{b^*}}\right] = 1. \tag{3.22}$$

This proves the claim.

Now, we can conclude that

$$u(x) = \mathbb{E}_x\left[e^{-\tilde{r}(\tau_{b^*})}(e^{\tilde{X}_{\tau_{b^*}}} - q) I_{\{\tau_{b^*}<\infty\}}\right] = V(x). \tag{3.23}$$

That means the candidate solution $u(\cdot)$ is indeed a solution. This completes the proof.

□

Using similar techniques presented by Zhang and Zhou (2009), we can simplify our solution by dropping the term with $\beta_j = 1$:

Proposition 3.3 *Under the condition that $G'(1) < 0$, suppose j_0 is the index such that $\beta_{j_0} = 1$, then we have $\omega_{j_0} = 0$. In other words, the value function takes the form*

$$V(x) = \begin{cases} \sum_{\beta_j \in \mathcal{B}^+ \setminus \{1\}} \omega_j e^{\beta_j x} & x < b^* \\ e^x - q & x \geq b^* \end{cases}.$$

Proof For $t \leq \tau_{b^*}$, we have

$$\mathbb{E}\left[e^{-\tilde{r}t}V(\tilde{X}_t) \mid \tilde{X}_0 = x\right] = V(x) + \int_0^t e^{-\tilde{r}s}\left(\mathcal{L} - \tilde{r}\right)V(\tilde{X}_s)ds = V(x).$$

Hence, for any $T > 0$,

$$\begin{aligned} V(x) &= \mathbb{E}\left[e^{-\tilde{r}\tau_{b^*}\wedge T}V(\tilde{X}_{\tau_{b^*}\wedge T}) \mid \tilde{X}_0 = x\right] \\ &\le \mathbb{E}\left[e^{-\tilde{r}\tau_{b^*}}V(\tilde{X}_{\tau_{b^*}})I_{\{\tau_{b^*}<T\}} \mid \tilde{X}_0 = x\right] + \mathbb{E}\left[e^{-\tilde{r}T}V(\tilde{X}_T)I_{\{\tau_{b^*}\ge T\}} \mid \tilde{X}_0 = x\right]. \end{aligned}$$

It is clear that the first term converges to

$$\mathbb{E}\left[e^{-\tilde{r}\tau_{b^*}}\left(e^{\tilde{X}_{\tau_{b^*}}} - q\right)I_{\{\tau_{b^*}<\infty\}} \mid \tilde{X}_0 = x\right]$$

as $T \to \infty$. To complete the proof, we require the second term to converge to zero as $T \to \infty$.

By Theorem 3.2, $V(x)$ is a linear combination of $e^{\beta_i x}$ for $x < b^*$, we consider the validity of

$$\mathbb{E}\left(e^{-\tilde{r}T}e^{\kappa\tilde{X}_T}\right) \to 0 \text{ as } T \to \infty$$

for different values of κ.

Note that $\mathbb{E}\left(e^{-\tilde{r}T}e^{\kappa\tilde{X}_T}\right) = e^{(G(\kappa)-\tilde{r})T}$. For $\kappa = 1$, the expectation becomes $e^0 = 1$ and does not converge to zero. Hence the term $\omega_{j_0}e^x$ should be dropped from the linear combination by setting the coefficient to zero.

On the other hand, since $G'(1) < 0$, there exists $\kappa_0 > 1$ such that $G(\kappa_0) - \tilde{r} < 0$. Furthermore, for any $\beta_i \in \mathcal{B}^+ \setminus \{1\}$, there exists $K_i > 0$ such that $e^{\beta_i x} \le K_i e^{\kappa_0 x}$ for $x \in (-\infty, b^*)$. We have $\mathbb{E}\left(e^{-\tilde{r}T}e^{\beta_i\tilde{X}_T}\right) \le K_i\mathbb{E}\left(e^{-\tilde{r}T}e^{\kappa_0\tilde{X}_T}\right) \to 0$ as $T \to \infty$.

□

We summarize our results in this chapter. For any given exponential phase-type Lévy model, the stock loan valuation is divided into two cases. If $G'(1) \geq 0$, the stock loan is not reasonable to exist. Otherwise, if $G'(1) < 0$, we solve roots from the C-L equation (3.4). The valuation formula of stock loan is given by Proposition 3.3 in which the optimal exercise boundary b^* can be determined by setting a differential to zero.

However, it is not an obvious task to study the root characteristics of the C-L equation (3.4) in general. The following chapter presents some important special cases for which the solutions are obtained in explicit form.

Chapter 4

Valuation

This chapter is devoted to the derivation of the valuation formula. We first solve the problem under hyperexponential jump diffusions, a special case of the phase-type jump diffusion. Although the hyperexponential jump diffusion model is studied by Cai (2009) for a first passage time problem and Cai and Kou (2011) for barrier and lookback option pricing, they only consider the case of positive interest rate and the optimal exercise boundary is yet to be investigated. By making use of the solution of the hyperexponential case, we extend our result to a fairly general class of phase-type jump diffusion models. Except the last section of this chapter, we assume that $G'(1) < 0$, where $G(\cdot)$ is defined in (3.4).

4.1 Hyperexponential jumps

Suppose $\mathbf{T}^+$ and $\mathbf{T}^-$ take the following form

$$\mathbf{T}^+ = \begin{pmatrix} -\eta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\eta_m \end{pmatrix}, \quad \mathbf{T}^- = \begin{pmatrix} -\theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\theta_n \end{pmatrix}, \tag{4.1}$$

where $\eta_i > 1$ for $i = 1, \dots, m$ and $\theta_k > 0$ for $k = 1, \dots, n$.

Then the phase-type jump distribution is reduced to a hyperexponential class. The following proposition summarizes the root characteristics of the C-L equation (3.4). A similar result is obtained by Cai (2009) for the case of non-negative interest rate.

Proposition 4.1 *The Cramér-Lundberg equation $G(\beta) = \tilde{r}$ has exactly n distinct negative real roots and $m+2$ distinct real roots which are greater than or equal to 1.*

Proof Under hyperexponential jump diffusion, we have

$$G(\beta) = \frac{\sigma^2}{2}\beta^2 + \tilde{\mu}\beta + \lambda p \sum_{i=1}^{m} \frac{\alpha_i^+ \eta_i}{\eta_i - \beta} + \lambda(1-p) \sum_{j=1}^{n} \frac{\alpha_i^- \theta_j}{\theta_j + \beta} - \lambda. \tag{4.2}$$

It is clear that:

1. $G(0) = 0$;
2. $G(\infty) = \infty$;
3. $G(-\infty) = \infty$;
4. $G(\eta_i-) = \infty$, $G(\eta_i+) = -\infty$ for $i = 1, \dots, m$;
5. $G(-\theta_j-) = -\infty$, $G(-\theta_j+) = \infty$ for $j = 1, \dots, n$;
6. $G(\beta)$ is continuous except the values η_i, $i = 1, \dots, m$ and $-\theta_j$, $j = 1, \dots, n$,

where $G(u\pm) = \lim_{x \to u\pm} G(x)$. Then we know that $G(\beta) = \tilde{r}$ has at least one root in each of the intervals

$$(-\infty, -\theta_n), (-\theta_n, -\theta_{n-1}), \dots, (-\theta_2, -\theta_1), (\eta_1, \eta_2), \dots, (\eta_{m-1}, \eta_m), (\eta_m, \infty).$$

Moreover, $G(\beta) = \tilde{r}$ has the same number of roots as the $m+n+2$ degree polynomial

$$(G(\beta) - \tilde{r}) \prod_{i=1}^{m} (\eta_i - \beta) \prod_{j=1}^{n} (\theta_j + \beta).$$

Therefore it has at most $m + n + 2$ real roots.

Also observe that $G(\beta)$ is decreasing on the interval $(-\theta_1, 0)$ and $G(0) = 0$, $G(-\theta_1+) = \infty$, there is no root in the interval $(-\theta_1, 0)$. Now recall that 1 is always a root and complex root always exists in pair, we deduce that there are two real roots in the interval $(0, \eta_1)$. Our assumption $G'(1) < 0$ implies that they are distinct and both of them greater than or equal to 1.

□

By Theorem 3.2, the solution is of the form

$$V_b(x) = \begin{cases} \sum_{j=1}^{m+1} \omega_j e^{\beta_j x} & x < b \\ e^x - q & x \geq b \end{cases}, \tag{4.3}$$

where $b \in \mathbb{R}$ is a constant, $G(\beta_i) - \tilde{r} = 0$ for all i and $1 < \beta_1 < \beta_2 < \ldots < \beta_{m+1}$.

For $x < b$, we have $(\mathcal{L} - \tilde{r})\, V_b(x) = 0$. Therefore,

$$
\begin{aligned}
0 &= \mathcal{L}V_b(x) - \tilde{r}V_b(x) \\
&= \frac{\sigma^2}{2}\frac{d^2V_b}{dx^2}(x) + \tilde{\mu}\frac{dV_b}{dx}(x) + \lambda\int_{-\infty}^{\infty}\left(V_b(x+y) - V_b(x)\right)f_Y(y)dy - \tilde{r}V_b(x) \\
&= \sum_{j=1}^{m+1}\omega_j e^{\beta_j x}\left(G(\beta_j) - \tilde{r}\right) - \lambda\int_{b-x}^{\infty}\sum_{j=1}^{m+1}\omega_j e^{\beta_j(x+y)}f_Y(y)dy \\
&\quad +\lambda\int_{b-x}^{\infty}\left(e^{x+y} - q\right)f_Y(y)dy \\
&= -\lambda\sum_{j=1}^{m+1}\omega_j e^{\beta_j x}\sum_{i=1}^{m}p\alpha_i^{+}\frac{\eta_i}{\eta_i - \beta_j}e^{-(\eta_i-\beta_j)(b-x)} + \lambda e^{x}\sum_{i=1}^{m}p\alpha_i^{+}\frac{\eta_i}{\eta_i - 1}e^{-(\eta_i-1)(b-x)} \\
&\quad -\lambda q\sum_{i=1}^{m}p\alpha_i e^{-\eta_i(b-x)} \\
&= \lambda\sum_{i=1}^{m}p\alpha_i^{+}e^{\eta_i x}\left(\frac{\eta_i}{\eta_i - 1}e^{-(\eta_i-1)b} - qe^{-\eta_i b} - \sum_{j=1}^{m+1}\omega_j\frac{\eta_i}{\eta_i - \beta_j}e^{-(\eta_i-\beta_j b}\right). \qquad (4.4)
\end{aligned}
$$

It is clear that $(\omega_1, \ldots, \omega_{m+1})$ should be chosen such that all the values inside the brackets in the summand of (4.4) equal to zero. That is

$$
\sum_{j=1}^{m+1}\omega_j\frac{\eta_i}{\eta_i - \beta_j}e^{-(\eta_i-\beta_j)b} = \frac{\eta_i}{\eta_i - 1}e^{-(\eta_i-1)b} - qe^{-\eta_i b}, \qquad (4.5)
$$

for $i = 1, \ldots, m$. Moreover, the function $V_b(\cdot)$ should also be continuous at b. This gives

$$
\sum_{j=1}^{m+1}\omega_j e^{\beta_j b} = e^b - q. \qquad (4.6)
$$

Now we obtained a $(m+1) \times (m+1)$ system of linear equations

$$\begin{cases} \sum_{j=1}^{m+1} \omega_j \frac{\eta_i}{\eta_i-\beta_j} e^{-(\eta_i-\beta_j)b} & = & \frac{\eta_i}{\eta_i-1} e^{-(\eta_i-1)b} - qe^{-\eta_i b} \quad \text{for } i = 1, \ldots, m \\ \sum_{j=1}^{m+1} \omega_j e^{\beta_j b} & = & e^b - q \end{cases} . \tag{4.7}$$

4.1.1 Solution of the linear system

We are intended to solve (4.7). Take $\tilde{\omega_j} = \omega_j e^{\beta_j b}$ for $j = 1, \ldots, m+1$, $\eta_{m+1} = 0$, $R_i = \frac{e^b}{\eta_i - 1}$ for $i = 1, \ldots, m$ and $R_{m+1} = q - e^b$. Then the linear system becomes

$$\sum_{j=1}^{m+1} \tilde{\omega_j} \frac{\beta_j}{\eta_i - \beta_j} = R_i \text{ for } i = 1, \ldots, m+1. \tag{4.8}$$

Using partial fraction, we have

$$\sum_{j=1}^{m+1} \frac{D_j \beta_j}{x - \beta_j} = \sum_{i=1}^{m+1} R_i \prod_{k=1}^{m+1} \frac{\eta_i - \beta_k}{x - \beta_k} \prod_{l=1, l \neq i}^{m+1} \frac{x - \eta_l}{\eta_i - \eta_l}, \tag{4.9}$$

where D_j, $j = 1, \ldots, m+1$ are the partial fraction coefficients. Multiplying (4.9) by $(x - \beta_k)$ on both sides and set $x = \beta_k$, we obtain

$$D_k \beta_k = \frac{\sum_{i=1}^{m+1} \left(R_i \prod_{j=1}^{m+1} (\eta_i - \beta_j) \prod_{l=1, l \neq i}^{m+1} \frac{\beta_k - \eta_l}{\eta_i - \eta_l} \right)}{\prod_{j=1, j \neq k}^{m+1} (\beta_k - \beta_j)}. \tag{4.10}$$

Hence,

$$D_k = \frac{\sum_{i=1}^{m+1} \left(R_i \prod_{j=1}^{m+1} (\eta_i - \beta_j) \prod_{l=1, l \neq i}^{m+1} \frac{\beta_k - \eta_l}{\eta_i - \eta_l} \right)}{\beta_k \prod_{j=1, j \neq k}^{m+1} (\beta_k - \beta_j)}. \tag{4.11}$$

When $x = \eta_i$ in (4.9), we have

$$\sum_{j=1}^{m+1} \frac{D_j\beta_j}{\eta_i - \beta_j} = R_i \prod_{k=1}^{m+1} \frac{\eta_i - \beta_k}{\eta_i - \beta_k} \prod_{l=1,l\neq i}^{m+1} \frac{\eta_i - \eta_l}{\eta_i - \eta_l} = R_i.$$

Hence, $\tilde{\omega_j} = D_j$, $j = 1, \ldots, m+1$, and we conclude that

$$\omega_j = \frac{\sum\limits_{i=1}^{m+1} \left(R_i \prod\limits_{k=1}^{m+1} (\eta_i - \beta_k) \prod\limits_{l=1,l\neq i}^{m+1} \frac{\beta_j - \eta_l}{\eta_i - \eta_l} \right)}{\beta_j e^{\beta_j b} \prod\limits_{k=1,k\neq j}^{m+1} (\beta_j - \beta_k)}, \tag{4.12}$$

where $R_i = \frac{e^b}{\eta_i - 1}$ for $i = 1, \ldots, m$ and $R_{m+1} = q - e^b$.

4.1.2 Solution of the optimal exercise boundary

After obtaining the coefficients in (4.3), the remaining unknown is the optimal exercise boundary b^*. Note that b^* is the value which maximizes the candidate solution $V_b(x)$. The following identity is useful for that purpose.

Lemma 4.1 *If $\{\beta_k\}_{k=1}^{m+1}$ and $\{\eta_i\}_{i=1}^{m+1}$ are all distinct, we have*

$$\prod_{k=1,k\neq j}^{m+1} (\beta_k - 1) = \sum_{k=1}^{m+1} \prod_{k=1,k\neq j}^{m+1} (\beta_k - \eta_i) \prod_{l=1,l\neq i}^{m+1} \left(\frac{\eta_l - 1}{\eta_l - \eta_i} \right). \tag{4.13}$$

Proof Consider the following polynomial

$$P_j(x) = \prod_{k=1,k\neq j}^{m+1} (\beta_k - 1 - x) \quad \text{for } j = 1, \ldots, m+1, \tag{4.14}$$

which are of degree m. Observe that

$$P_j(\eta_i - 1) = \prod_{k=1,k\neq j}^{m+1} (\beta_k - \eta_i). \tag{4.15}$$

By Lagrange interpolation,

$$\begin{aligned} L_j(x) &= \sum_{i=1}^{m+1} P_j(\eta_i - 1) \prod_{l=1,l\neq i}^{m+1} \left(\frac{\eta_l - 1 - x}{\eta_l - \eta_i} \right) \\ &= \sum_{i=1}^{m+1} \prod_{k=1,k\neq j}^{m+1} (\beta_k - \eta_i) \prod_{l=1,l\neq i}^{m+1} \left(\frac{\eta_l - 1 - x}{\eta_l - \eta_i} \right) \end{aligned} \tag{4.16}$$

is a polynomial of degree m which past through all the points in the set

$$\{(\eta_i - 1, P_j(\eta_i - 1))\}_{i=1}^{m+1}.$$

As $P_j(x)$ is a polynomial of degree m and it matches the value of $L_j(x)$ at $m+1$ points, we have

$$P_j(x) = L_j(x) \quad \forall x \in \mathbb{R}.$$

By putting $x = 0$, the result follows.

□

Our objective is to maximize the function

$$\begin{aligned} V_b(x) &= \begin{cases} \sum_{j=1}^{m+1} \omega_j e^{\beta_j x} & x < b \\ e^x - q & x \geq b \end{cases} \\ &= \begin{cases} \sum_{j=1}^{m+1} \tilde{\omega_j} e^{\beta_j (x-b)} & x < b \\ e^x - q & x \geq b \end{cases} \end{aligned}$$

over $b \in \mathbb{R}$. It suffices to maximize the value function on the interval $(-\infty, b)$. On $(-\infty, b)$, we have

$$\frac{d}{db}V_b(x) = \sum_{j=1}^{m+1} e^{\beta_j(x-b)} \left(\frac{d}{db}\tilde{\omega_j} - \tilde{\omega_j}\beta_j \right).$$

Some simple algebras show that

$$\frac{d}{db}\tilde{\omega_j} - \tilde{\omega_j}\beta_j = \frac{(1-\beta_j)\, e^b \left[\sum_{j=1}^{m} \frac{1}{\eta_i - 1} \prod_{k=1}^{m+1} (\eta_i - \beta_k) \prod_{l=1, l\neq i}^{m+1} \frac{\beta_j - \eta_l}{\eta_i - \eta_l} + \prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\beta_j - \eta_l}{\eta_l} \right]}{\beta_j \prod_{k=1, k\neq j}^{m+1} (\beta_j - \beta_k)} + \frac{q \prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\beta_j - \eta_l}{\eta_l}}{\prod_{k=1, k\neq j}^{m+1} (\beta_j - \beta_k)}.$$

Therefore $\frac{d}{db}\tilde{\omega_j} - \tilde{\omega_j}\beta_j = 0$ if and only if

$$\begin{aligned}
\frac{e^b}{q} &= \frac{\beta_j}{\beta_j - 1} \frac{\prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\beta_j - \eta_l}{\eta_l}}{\sum_{j=1}^{m} \frac{1}{\eta_i - 1} \prod_{k=1}^{m+1} (\eta_i - \beta_k) \prod_{l=1, l\neq i}^{m+1} \frac{\beta_j - \eta_l}{\eta_i - \eta_l} + \prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\beta_j - \eta_l}{\eta_l}} \\
&= \frac{1}{\beta_j - 1} \frac{\prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\eta_l - 1}{\eta_l}}{\sum_{j=1}^{m+1} \prod_{k=1, k\neq j}^{m+1} (\beta_k - \eta_i) \prod_{l=1, l\neq i}^{m+1} \left(\frac{\eta_l - 1}{\eta_l - \eta_i} \right)} \\
&= \prod_{k=1}^{m+1} \frac{\beta_k}{\beta_k - 1} \prod_{l=1}^{m} \frac{\eta_l - 1}{\eta_l},
\end{aligned}$$

where the last equality is an application of Lemma 4.1. Hence $\frac{d}{db}V_b(x) = 0$ at $b = b^*$ where

$$b^* = \ln \left(q \prod_{k=1}^{m+1} \frac{\beta_k}{\beta_k - 1} \prod_{l=1}^{m} \frac{\eta_l - 1}{\eta_l} \right). \tag{4.17}$$

It is then easy to see that $V_b(x)$ is maximized at b^*.

4.2 Phase-type jumps

We are now ready to extend the previous results into phase-type jump diffusion models. Suppose $\mathbf{T}^+$ and $\mathbf{T}^-$ are symmetric (and hence diagonalizable) matrix, and have distinct eigenvalues. Then, there exists orthogonal matrix $\mathbf{Q}^+$ and $\mathbf{Q}^-$ such that

$$\mathbf{T}^+ = (\mathbf{Q}^+)^T \mathbf{\Lambda}^+ \mathbf{Q}^+ \quad \text{and} \quad \mathbf{T}^- = (\mathbf{Q}^-)^T \mathbf{\Lambda}^- \mathbf{Q}^- , \tag{4.18}$$

where

$$\mathbf{\Lambda}^+ = \begin{pmatrix} -\eta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\eta_m \end{pmatrix}, \quad \mathbf{\Lambda}^- = \begin{pmatrix} -\theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\theta_n \end{pmatrix}.$$

We have the following result regarding the roots of the Cramér-Lundberg equation (3.4).

Theorem 4.2 *The Cramér-Lundberg equation $G(\beta) = \tilde{r}$ has exactly $m+1$ roots in the complex domain $\mathcal{D}_+ = \{z \in \mathbb{C} | Re(z) > 1\}$ and exactly n roots in the complex domain $\mathcal{D}_- = \left\{z \in \mathbb{C} | Re(z) < \max_i\{-\theta_i\}\right\}$.*

Proof

$$\begin{aligned} \text{Let } f_0(z) &= \tilde{\mu} z + \tfrac{\sigma^2}{2} z^2 + \lambda p \left(\boldsymbol{\alpha}^+ (-z\mathbf{I} - \mathbf{\Lambda}^+)^{-1} (-\mathbf{\Lambda}^+)\mathbf{1} - 1\right) \\ &\quad + \lambda(1-p)\left(\boldsymbol{\alpha}^- (-z\mathbf{I} - \mathbf{\Lambda}^-)^{-1} (-\mathbf{\Lambda}^-)\mathbf{1} - 1\right) - \tilde{r}, \\ f_1(z) &= \tilde{\mu} z + \tfrac{\sigma^2}{2} z^2 + \lambda p \left(\boldsymbol{\alpha}^+ (\mathbf{Q}^+)^T (-z\mathbf{I} - \mathbf{\Lambda}^+)^{-1} (-\mathbf{\Lambda}^+)\mathbf{Q}^+\mathbf{1} - 1\right) \\ &\quad + \lambda(1-p)\left(\boldsymbol{\alpha}^- (\mathbf{Q}^-)^T (-z\mathbf{I} - \mathbf{\Lambda}^-)^{-1} (-\mathbf{\Lambda}^-)\mathbf{Q}^-\mathbf{1} - 1\right) - \tilde{r}, \\ f_t(z) &= [f_0(z)]^{(1-t)} [f_1(z)]^t \quad \text{for } t \in (0,1). \end{aligned}$$

Note that $f_t(z)$ have m poles $\eta_1, \ldots, \eta_m$ in $\mathcal{D}_+$ for all $t \in [0,1]$. From the hyperexponential case (Proposition 4.1), we know that $f_0(t)$ have $m+1$ zeros in $\mathcal{D}_+$. We want to construct a boundary strip $\mathcal{C}_+$ of $\mathcal{D}_+$ such that $f_t(z)$ have no zero on it.

Since $|f_t(z)| \to \infty$ as $|z| \to \infty$ for $t = 0, 1$, there exists $R \in \mathbb{R}$ such that all roots of $f_t(z) = 0,\ t \in (0,1)$ are in the region

$$\mathcal{D}_R = \{z \in \mathbb{C} : Re(z) \geq 0,\ |z| \leq R\}. \tag{4.19}$$

On the other hand, as $G'(1) < 0$, there exists $\kappa_1 \in \mathbb{R},\ \kappa_1 > 1$ (In fact, we can take κ_1 arbitrarily close to 1), such that $f_t(\kappa_1) = Re(f_t(\kappa_1)) < 0$. For $t = 0, 1,\ \nu \in \mathbb{R}$ we have

$$\begin{aligned} e^{Re(f_t(\kappa_1+i\nu))} &= \left|e^{f_t(\kappa_1+i\nu)}\right| \\ &= \left|\mathbb{E}\left(e^{(\kappa_1+i\nu)X_1-\bar{r}}\right)\right| \\ &\leq \mathbb{E}\left(e^{(\kappa_1)X_1-\bar{r}}\right) \\ &= e^{Re(f_t(\kappa_1))} \\ &< 1. \end{aligned}$$

Hence, we have $Re(f_t(\kappa_1 + i\nu)) < 0\ \ \forall \nu \in \mathbb{R}$. This gives the boundary strip

$$\mathcal{C}_+ = \{z \in \mathbb{C} : |z| = R,\ Re(z) \geq \kappa_1\} \cup \{z \in \mathbb{C} : Re(z) = \kappa_1,\ -R \leq Im(z) \leq R\}. \tag{4.20}$$

By the continuity of $f_t(z)$ and the Argument Principle, we deduce that

$$n_t = \frac{1}{2\pi i} \oint_{\mathcal{C}_+} \frac{f_t'(z)}{f_t(z)} dz \tag{4.21}$$

is integer valued and continuous over $t \in [0,1]$. Hence $n_0 = n_1$, i.e. $f_1(z)$ have $m+1$ zeros in $\mathcal{D}_+$. This completes the proof of the first part of the statement.

To show the second part of the statement, we repeat the above arguments with the following boundary strip,

$$\mathcal{C}_- = \{z \in \mathbb{C} : |z| = R,\ Re(z) \le \kappa_2\} \cup \{z \in \mathbb{C} : Re(z) = \kappa_2,\ -R \le Im(z) \le R\}, \tag{4.22}$$

where $\kappa_2 \in \mathbb{R}$ and $\kappa_2 < \max_i\{-\theta_i\}$ is chosen arbitrarily close to $\max_i\{-\theta_i\}$.

□

According to Theorem 3.2, if there are no multiple roots with positive real part in the C-L equation (3.4), then the solution is of the form

$$V_b(x) = \begin{cases} \sum_{j=1}^{m+1} \omega_j e^{\beta_j x} & x < b \\ e^x - q & x \ge b \end{cases}. \tag{4.23}$$

Using this solution form, we compute that

$$\begin{aligned} \mathcal{L}V_b(x) - \tilde{r}V_b(x) &= \lambda p \boldsymbol{\alpha}^+ \Big[(\mathbf{Q}^+)^T e^{\boldsymbol{\Lambda}^+(b-x)} \Big((-\boldsymbol{\Lambda}^+ - \mathbf{I})^{-1} e^{b\mathbf{I}} + q{\boldsymbol{\Lambda}^+}^{-1} \\ &\quad - \sum_{j=1}^{m+1} \omega_j \left(-\boldsymbol{\Lambda}^+ - \beta_j \mathbf{I}\right)^{-1} e^{\beta_j b \mathbf{I}} \Big) \left(-\boldsymbol{\Lambda}^+\right) \mathbf{Q}^+ \Big] \mathbf{1}, \end{aligned} \tag{4.24}$$

which is equal to 0 on $(-\infty, b)$. Hence we obtain the system of linear equations

$$\left\{\begin{array}{rcll}\sum_{j=1}^{m+1}\omega_j\frac{\eta_i}{\eta_i-\beta_j}e^{-(\eta_i-\beta_j)b} & = & \frac{\eta_i}{\eta_i-1}e^{-(\eta_i-1)b}-qe^{-\eta_i b} & \text{for } i=1,\ldots,m\\ \sum_{j=1}^{m+1}\omega_j e^{\beta_j b} & = & e^b-q & \end{array}\right. , \tag{4.25}$$

which is the same as the linear system (4.7) in the hyperexponential case. Therefore, the coefficients are given by

$$\omega_j=\frac{\sum_{i=1}^{m+1}\left(R_i\prod_{k=1}^{m+1}(\eta_i-\beta_k)\prod_{l=1,l\neq i}^{m+1}\frac{\beta_j-\eta_l}{\eta_i-\eta_l}\right)}{\beta_j e^{\beta_j b}\prod_{k=1,k\neq j}^{m+1}(\beta_j-\beta_k)}, \tag{4.26}$$

and the optimal exercise boundary is given by

$$b^*=\ln\left(q\prod_{k=1}^{m+1}\frac{\beta_k}{\beta_k-1}\prod_{l=1}^{m}\frac{\eta_l-1}{\eta_l}\right). \tag{4.27}$$

4.3 The case for $G'(1)\geq 0$

By substituting $\beta_1=1$ into (4.27), we get $b^*=\infty$. Again, by setting $\beta_1=1$ in (4.26), we observe that

$$\omega_j\to 0 \text{ for } j\neq 1$$

and

$$\omega_1\to\frac{\sum_{k=1}^{m+1}\prod_{k=2}^{m+1}(\beta_k-\eta_i)\prod_{l=1,l\neq i}^{m+1}\left(\frac{\eta_l-1}{\eta_l-\eta_i}\right)}{\prod_{k=2}^{m+1}(\beta_k-1)}=1$$

as $b\to\infty$, where the last equality is a result of Lemma 4.1 when $j=1$. Noting (4.23), we have

$$V(x) \geq \sup_{b \geq \max\{\ln q, \ln S\}} V_b(x) = e^x.$$

On the other hand, we know that $V(x) \leq e^x$ from Lemma 3.1. As a result, we have $V(x) = e^x$.

Chapter 5

Future Research Direction

This chapter discusses the possible generalization of the stock loan problem to stochastic volatility. We adopt the approach in Fouque et al. (2003) and consider a fast mean-reverting stochastic volatility model. The stock loan pricing formula is derived in the form of asymptotic expansion.

5.1 The fast mean-reverting stochastic volatility model

Consider a pair of process $(S_t^\varepsilon, Y_t^\varepsilon)$ which satisfies

$$dS_t^\varepsilon = rS_t^\varepsilon dt + f(Y_t^\varepsilon)S_t^\varepsilon dW_t, \tag{5.1}$$

$$dY_t^\varepsilon = \left[\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dZ_t, \tag{5.2}$$

where S_t^ε is the stock price process, $f(Y_t^\varepsilon)$ is a positive valued function representing the volatility, Y_t^ε is a Ornstein-Uhlenbeck (OU) process with mean reverting speed $\frac{1}{\varepsilon}$, $\varepsilon > 0$ is a small parameter, (W_t, Z_t) are Brownian motions with correlation $\rho \in (-1, 1)$ and

$$\Lambda(y) = \frac{\rho(\mu - r)}{f(y)} + c(y)\sqrt{1-\rho^2} \tag{5.3}$$

is the market price of risk. Let $X_t^\varepsilon = \ln\left(e^{-\gamma t} S_t^\varepsilon\right)$. Itō's formula gives

$$dX_t^\varepsilon = \left(r - \gamma - \frac{f(Y_t^\varepsilon)^2}{2}\right) dt + f(Y_t^\varepsilon) dW_t. \tag{5.4}$$

Although we do not introduce jumps in (5.1) and (5.2), the method used in this chapter is possible to generalize to phase-type Lévy process with stochastic volatility in the future.

5.2 Asymptotic expansion of stock loan

We are interested in the stock loan on S_t^ε defined in (5.1) and (5.2). As a starting point, we use $V^\varepsilon(x,y)$ to denote the price of the perpetual American option corresponding to the stock loan (see (2.16)), i.e.

$$V^\varepsilon(x,y) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0} \mathbb{E}\left[e^{-(r-\gamma)\tau}\left(e^{X_\tau^\varepsilon} - q\right)^+ I_{\{\tau<\infty\}} | X_0^\varepsilon = x, Y_0^\varepsilon = y\right]. \tag{5.5}$$

Using similar arguments presented in previous chapters, $V^\varepsilon(x,y)$ is known to be a solution to the partial differential equation

$$\begin{cases} \mathcal{L}^\varepsilon V^\varepsilon(x,y) &= 0 \qquad \text{for } x < b^\varepsilon(y) \\ V^\varepsilon(b^\varepsilon(y),y) &= e^{b^\varepsilon(y)} - q \quad \text{for } x \geq b^\varepsilon(y) \end{cases}, \tag{5.6}$$

where $b^\varepsilon(y)$ is the optimal exercise boundary,

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2, \tag{5.7}$$

with

$$\mathcal{L}_0 = (m-y)\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2}; \tag{5.8}$$

$$\mathcal{L}_1 = \sqrt{2}\nu\rho f(y)\frac{\partial^2}{\partial x\partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}; \tag{5.9}$$

$$\mathcal{L}_2 = \frac{1}{2}f(y)^2\frac{\partial^2}{\partial x^2} + \left(\tilde{r} - \frac{f(y)^2}{2}\right)\frac{\partial}{\partial x} - \tilde{r}\cdot. \tag{5.10}$$

Note that the operator $\mathcal{L}_0$ is the infinitesimal generator of the OU process Y_t defined by

$$dY_t = (m - Y_t)dt + \sqrt{2}\nu dZ_t, \tag{5.11}$$

which has the invariant distribution $\mathcal{N}(m, \nu^2)$.

Consider the following asymptotic expansions for $V^\varepsilon(x,y)$ and $b^\varepsilon(y)$:

$$V^\varepsilon(x,y) = V_0(x,y) + \sqrt{\varepsilon}V_1(x,y) + \varepsilon V_2(x,y) + \varepsilon^{\frac{3}{2}}V_3(x,y) + \dots\ , \tag{5.12}$$

$$b^\varepsilon(y) = b_0(y) + \sqrt{\varepsilon}b_1(y) + \varepsilon b_2(y) + \varepsilon^{\frac{3}{2}}b_3(y) + \dots\ . \tag{5.13}$$

We aim to compute the first two leading order terms of the above expansions. i.e.

$$V_0(x,y) + \sqrt{\varepsilon}V_1(x,y) \tag{5.14}$$

and

$$b_0(y) + \sqrt{\varepsilon}b_1(y). \tag{5.15}$$

Substituting (5.12) into (5.6) gives

$$
\begin{aligned}
\mathcal{L}^{\varepsilon} V^{\varepsilon} &= \frac{1}{\varepsilon}\mathcal{L}_0 V_0 + \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_1 V_0 + \mathcal{L}_0 V_1\right) + \left(\mathcal{L}_2 V_0 + \mathcal{L}_1 V_1 + \mathcal{L}_0 V_2\right) \\
&\quad + \sqrt{\varepsilon}\left(\mathcal{L}_2 V_1 + \mathcal{L}_1 V_2 + \mathcal{L}_0 V_3\right) + o(\sqrt{\varepsilon}) \\
&= 0. \qquad (5.16)
\end{aligned}
$$

This implies all the terms of the expansion in (5.16) should be equal to zero.

We use $\langle \cdot \rangle$ to denote the expectation with respect to the invariant distribution $\mathcal{N}(m, \nu^2)$:

$$
\langle h \rangle = \frac{1}{\nu\sqrt{2\pi}} \int_{-\infty}^{\infty} h(y) e^{-\frac{(y-m)^2}{2\nu^2}} dy. \qquad (5.17)
$$

In the following analysis, we have to solve the following Poisson equation:

$$
\mathcal{L}_0 g + h = 0. \qquad (5.18)
$$

In order to admit a solution $g(\cdot)$ with reasonable growth towards infinity, the equation requires the following Fredholm solvability condition

$$
\langle h \rangle = 0. \qquad (5.19)
$$

5.2.1 The zeroth order term

Consider the zeroth order term in (5.16)

$$
\mathcal{L}_0 V_0 = 0. \qquad (5.20)
$$

As $\mathcal{L}_0$ is a differential operator with respect to y, (5.20) implies that $V_0(x, y)$ is independent of y.

For the first order term in (5.16)

$$\mathcal{L}_1 V_0 + \mathcal{L}_0 V_1 = 0, \tag{5.21}$$

since V_0 is independent of y, the equation is reduced to

$$\mathcal{L}_0 V_1 = 0. \tag{5.22}$$

This implies, again, that $V_1(x, y)$ is independent of y.

For the second order term in (5.16):

$$\mathcal{L}_2 V_0 + \mathcal{L}_1 V_1 + \mathcal{L}_0 V_2 = 0, \tag{5.23}$$

because $\mathcal{L}_1 V_1 = 0$, (5.23) is reduced to the Poisson equation in V_2

$$\mathcal{L}_0 V_2 + \mathcal{L}_2 V_0 = 0. \tag{5.24}$$

The solvability condition implies

$$\langle \mathcal{L}_2 V_0 \rangle = \langle \mathcal{L}_2 \rangle V_0 = 0, \tag{5.25}$$

where $\langle \mathcal{L}_2 \rangle$ is the operator $\mathcal{L}_2$ with $f(y)^2$ replaced by $\bar{\sigma}^2 = \langle f^2 \rangle$, i.e.

$$\langle \mathcal{L}_2 \rangle V_0 = \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 V_0}{\partial x^2} + \left(\tilde{r} - \frac{\bar{\sigma}^2}{2} \right) \frac{\partial V_0}{\partial x} - \tilde{r} V_0 = 0.$$

Recall the expansion of the optimal exercise boundary:

$$b^{\varepsilon}(y) = b_0 + \sqrt{\varepsilon} b_1(y) + o(\sqrt{\varepsilon}). \tag{5.26}$$

Expanding both sides of the boundary condition in (5.6) according to the exercise boundary gives

$$\begin{aligned} V^{\varepsilon}(b^{\varepsilon}(y),y) &= V_0(b_0,y)+\sqrt{\varepsilon}\left(V_1(b_0+\sqrt{\varepsilon}b_1(y),y)+b_1(y)\frac{\partial V_0}{\partial x}(b_0,y)\right) \\ &\quad +o(\sqrt{\varepsilon}); \qquad (5.27) \\ e^{b^{\varepsilon}(y)}-q &= e^{b_0}-q+\sqrt{\varepsilon}b_1(y)e^{b_0}+o(\sqrt{\varepsilon}). \qquad (5.28) \end{aligned}$$

Equating the zeroth order terms, we have

$$V_0(b_0,y)=e^{b_0}-q. \tag{5.29}$$

This suggests that V_0 is the solution under a constant volatility model. The solution is given in Xia and Zhou (2007):

- If $-2\tilde{r}/\bar{\sigma}^2>1$,

$$V_0(x)=\begin{cases} \frac{(\beta-1)^{\beta-1}}{\beta^{\beta}}q^{1-\beta}e^{\beta x} & \text{for } x<b_0 \\ e^x-q & \text{for } x\geq b_0 \end{cases}. \tag{5.30}$$

  where $\beta=-\frac{2\tilde{r}}{\bar{\sigma}^2}$, $b_0=\ln(\frac{\beta q}{\beta-1})$.

- If $-2\tilde{r}/\bar{\sigma}^2\leq 1$,

$$V_0(x)=e^x. \tag{5.31}$$

For the purpose of illustrating asymptotic expansion, we focus on the more interesting case where $-2\tilde{r}/\bar{\sigma}^2>1$.

5.2.2 The first order term

The solution of the Poisson equation (5.24) can be written as

$$V_2 = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right)V_0. \tag{5.32}$$

On the other hand, the third order term in (5.16) gives

$$\mathcal{L}_2 V_1 + \mathcal{L}_1 V_2 + \mathcal{L}_0 V_3 = 0, \tag{5.33}$$

which is a Poisson equation in V_3. Solvability condition implies that

$$\begin{aligned}
\langle\mathcal{L}_2\rangle V_1 &= -\langle\mathcal{L}_1 V_2\rangle \\
&= \left\langle \mathcal{L}_1\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right)\right\rangle V_0 \\
&= \left\langle \mathcal{L}_1\mathcal{L}_0^{-1}\left(f(y)^2 - \langle f^2\rangle\right)\right\rangle \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)V_0 \\
&= \left(v_2\frac{\partial^3}{\partial x^3} + (v_1 - 3v_2)\frac{\partial^2}{\partial x^2} + (2v_2 - v_1)\frac{\partial}{\partial x}\right)V_0, \qquad (5.34)
\end{aligned}$$

where

$$\begin{aligned}
v_1 &= \frac{\nu}{\sqrt{2}}(2\rho\langle f\phi'\rangle - \langle\Lambda\phi'\rangle\,; \qquad (5.35)\\
v_2 &= \frac{\rho\nu}{\sqrt{2}}\langle f\phi'\rangle\,, \qquad (5.36)
\end{aligned}$$

and $\phi(y)$ is a solution to the Poisson following equation

$$\mathcal{L}_0\phi(y) = f(y)^2 - \langle f^2\rangle\,. \tag{5.37}$$

The boundary and smooth fit condition (cf. Villeneuve, 2007) of V^ε are

$$V^\varepsilon(b^\varepsilon(y), y) = e^{b^\varepsilon(y)} - q \tag{5.38}$$

and

$$\left.\frac{\partial V^{\varepsilon}}{\partial x}\right|_{(x,y)=(b^{\varepsilon}(y),y)} = e^{b^{\varepsilon}(y)} \tag{5.39}$$

respectively. Expanding both sides of the boundary condition in (5.6) according to the exercise boundary gives

$$\begin{aligned} V^{\varepsilon}(b^{\varepsilon}(y),y) &= V_0(b_0,y)+\sqrt{\varepsilon}\left(V_1(b_0+\sqrt{\varepsilon}b_1(y),y)+b_1(y)\frac{\partial V_0}{\partial x}(b_0,y)\right) \\ &\quad +o(\sqrt{\varepsilon}); \\ e^{b^{\varepsilon}(y)}-q &= e^{b_0}-q+\sqrt{\varepsilon}b_1(y)e^{b_0}+o(\sqrt{\varepsilon}). \end{aligned}$$

Equating the terms in $\sqrt{\varepsilon}$ order, we get

$$V_1(b_0+\sqrt{\varepsilon}b_1(y),y)+b_1(y)\frac{\partial V_0}{\partial x}(b_0,y)=b_1(y)e^{b_0}, \tag{5.40}$$

which implies

$$V_1(b_0+\sqrt{\varepsilon}b_1(y),y)=0. \tag{5.41}$$

Expanding both sides of the smooth fit condition according to the exercise boundary gives

$$\begin{aligned} &\frac{\partial V^{\varepsilon}}{\partial x}(b^{\varepsilon}(y),y) \\ = &\begin{cases} \frac{\partial V_0}{\partial x}(b_0,y)+\sqrt{\varepsilon}\left(\frac{\partial V_1}{\partial x}(b_0+\sqrt{\varepsilon}b_1(y),y)+b_1(y)\frac{\partial^2 V_0}{\partial x^2}(b_0+,y)\right)+o(\sqrt{\varepsilon}) & \text{if } b_1(y)>0 \\ \frac{\partial V_0}{\partial x}(b_0,y)+\sqrt{\varepsilon}\left(\frac{\partial V_1}{\partial x}(b_0+\sqrt{\varepsilon}b_1(y),y)+b_1(y)\frac{\partial^2 V_0}{\partial x^2}(b_0-,y)\right)+o(\sqrt{\varepsilon}) & \text{if } b_1(y)\le 0 \end{cases} \end{aligned} \tag{5.42}$$

and

$$e^{b^\varepsilon(y)} = e^{b_0} + \sqrt{\varepsilon} b_1(y) e^{b_0} + o(\sqrt{\varepsilon}). \tag{5.43}$$

Collecting terms of $\mathcal{O}(\sqrt{\varepsilon})$,

$$\frac{\partial V_1}{\partial x}(b_0 + \sqrt{\varepsilon} b_1(y), y) = b_1(y) e^{b_0}. \tag{5.44}$$

Summarizing all these, we obtain the PDE of V_1:

$$\begin{cases} \langle \mathcal{L}_2 \rangle V_1 & = \left(v_2 \frac{\partial^3}{\partial x^3} + (v_1 - 3v_2) \frac{\partial^2}{\partial x^2} + (2v_2 - v_1) \frac{\partial}{\partial x} \right) V_0, \quad \text{for } x < b_0 + \sqrt{\varepsilon} b_1(y); \\ V_1(b_0 + \sqrt{\varepsilon} b_1(y), y) & = 0; \\ \frac{\partial V_1}{\partial x}(b_0 + \sqrt{\varepsilon} b_1(y), y) & = \begin{cases} b_1(y) e^{b_0} - b_1(y) \frac{\partial^2 V_0}{\partial x^2}(b_0 +, y) & \text{if } b_1(y) > 0; \\ b_1(y) e^{b_0} - b_1(y) \frac{\partial^2 V_0}{\partial x^2}(b_0 -, y) & \text{if } b_1(y) \leq 0. \end{cases} \end{cases} \tag{5.45}$$

We solve this PDE by dividing it into two cases.

Case 1: $b_1(y) < 0$. For $x < b_0 + \sqrt{\varepsilon} b_1(y)$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle V_1 &= \left(v_2 \frac{\partial^3}{\partial x^3} + (v_1 - 3v_2) \frac{\partial^2}{\partial x^2} + (2v_2 - v_1) \frac{\partial}{\partial x} \right) V_0 \\ &= a_1 \beta \left(v_2 \beta^2 + (v_1 - 3v_2)\beta + (2v_2 - v_1) \right) e^{\beta x}, \end{aligned} \tag{5.46}$$

where $a_1 = \frac{(\beta-1)^{\beta-1}}{\beta^\beta} q^{1-\beta}$. To construct a particular solution for V_1, consider the solution form

$$V_1^p(x) = c_1 x e^{\beta x}. \tag{5.47}$$

By substituting this into the left hand side of (5.46), we get

$$\begin{aligned}
&\frac{1}{2}\bar{\sigma}^2\frac{\partial^2 V_1^p}{\partial x^2}+\left(\tilde{r}-\frac{\bar{\sigma}^2}{2}\right)\frac{\partial V_1^p}{\partial x}-\tilde{r}V_1^p \\
=\ &\frac{1}{2}\bar{\sigma}^2\left(2c_1\beta e^{\beta x}+c_1\beta^2 x e^{\beta x}\right)+\left(\tilde{r}-\frac{\bar{\sigma}^2}{2}\right)\left(c_1 e^{\beta x}+c_1\beta x e^{\beta x}\right)-\tilde{r}c_1 x e^{\beta x} \\
=\ &c_1\bar{\sigma}^2\beta e^{\beta x}+c_1\left(\tilde{r}-\frac{\bar{\sigma}^2}{2}\right)e^{\beta x},
\end{aligned}$$

where the last equality holds with

$$\frac{1}{2}\bar{\sigma}^2\beta^2+\left(\tilde{r}-\frac{\bar{\sigma}^2}{2}\right)\beta-\tilde{r}=0.$$

This implies

$$c_1=\frac{a_1\beta\left(v_2\beta^2+(v_1-3v_2)\beta+(2v_2-v_1)\right)}{\bar{\sigma}^2\beta+\left(\tilde{r}-\frac{\bar{\sigma}^2}{2}\right)}. \tag{5.48}$$

It is clear that the homogeneous solution is of the form

$$V_1^h(x)=c_2e^{\beta x}+c_3e^x. \tag{5.49}$$

We claim that $c_3=0$. To see this, define

$$E^\varepsilon(t,x,y)=\mathbb{E}\left[e^{-\tilde{r}(T-t)}\left(S_T^\varepsilon\right)^\kappa|S_t^\varepsilon=x,Y_t^\varepsilon=y\right] \tag{5.50}$$

and consider the following expansion

$$E^\varepsilon(t,x,y)=E_0(t,x,y)+\sqrt{\varepsilon}E_1(t,x,y)+o(\sqrt{\varepsilon}). \tag{5.51}$$

As argued in Zhang and Zhou (2009), if $c_3\neq 0$, we should have

$$E_0(t,x,y) + \sqrt{\varepsilon} E_1(t,x,y) \to 0 \text{ as } T \to \infty \tag{5.52}$$

for $\kappa = 1$. Following a similar analysis for $V^\varepsilon(x,y)$, we know that E_0 is the expectation evaluated with Black-Scholes model and this is solved in Zhang and Zhou (2009) that

$$E_0(t,x,y) = e^{\kappa x + (\kappa-1)(\kappa-\beta)\frac{\bar{\sigma}^2(T-t)}{2}}. \tag{5.53}$$

E_1 is given by

$$\begin{aligned} E_1(t,x,y) &= -(T-t)\left(v_2 \frac{\partial^3}{\partial x^3} + (v_1 - 3v_2)\frac{\partial^2}{\partial x^2} + (2v_2 - v_1)\frac{\partial}{\partial x}\right) E_0 \\ &= -(T-t)\kappa \left(v_2 \kappa^2 + (v_1 - 3v_2)\kappa + (2v_2 - v_1)\right) e^{\kappa x + (\kappa-1)(\kappa-\beta)\frac{\bar{\sigma}^2(T-t)}{2}}. \end{aligned} \tag{5.54}$$

We refer to Fouque et al. (2003) for details. It is now easy to see that

$$E_0(t,x,y) + \sqrt{\varepsilon} E_1(t,x,y) \to 0 \text{ as } T \to \infty$$

does not hold for $\kappa = 1$. This implies $c_3 = 0$ and proves the claim.

For $x < b_0 + \sqrt{\varepsilon} b_1(y)$, a general solution of V_1 is the sum of the homogeneous solution and the particular solution:

$$V_1(x) = c_1 x e^{\beta x} + c_2 e^{\beta x}. \tag{5.55}$$

Substituting this into the boundary condition of (5.45) yields

$$0 = V_1(b_0 + b_1(y)\sqrt{\varepsilon}) = c_1(b_0 + b_1(y)\sqrt{\varepsilon}) e^{\beta(b_0 + b_1(y)\sqrt{\varepsilon})} + c_2 e^{\beta(b_0 + b_1(y)\sqrt{\varepsilon})}, \tag{5.56}$$

and

$$c_2 = -c_1(b_0 + b_1(y)\sqrt{\varepsilon}). \tag{5.57}$$

Evaluating both sides using the smooth fit condition in (5.45) gives

$$\begin{aligned}
\frac{\partial V_1}{\partial x}(b_0 + \sqrt{\varepsilon}b_1(y), y) &= c_1 e^{\beta(b_0+b_1(y)\sqrt{\varepsilon})} + \left[c_1(b_0 + b_1(y)\sqrt{\varepsilon}) + c_2\right] \beta e^{\beta(b_0+b_1(y)\sqrt{\varepsilon})} \\
&= c_1 e^{\beta(b_0+b_1(y)\sqrt{\varepsilon})} \\
&= c_1 e^{\beta b_0}(1 + b_1(y)\beta\sqrt{\varepsilon}) + o(\sqrt{\varepsilon})
\end{aligned}$$

and

$$b_1(y)e^{b_0} - b_1(y)\frac{\partial^2 V_0}{\partial x^2}(b_0-, y) = b_1(y)(e^{b_0} - a_1\beta^2 e^{\beta b_0}) = b_1(y)(1-\beta)e^{b_0}.$$

Neglecting the $o(\sqrt{\varepsilon})$ terms and equating both sides, we have

$$b_1(y) = \frac{c_1 e^{\beta b_0}}{(1-\beta)e^{b_0} - c_1\sqrt{\varepsilon}e^{\beta b_0}} \tag{5.58}$$

which is independent of y. To summarize,

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} + c_2 e^{\beta x} & \text{for } x < b_0 + \sqrt{\varepsilon}b_1 \\ 0 & \text{for } x \geq b_0 + \sqrt{\varepsilon}b_1 \end{cases}, \tag{5.59}$$

where c_1 is given in (5.48), c_2 in (5.57) and b_1 in (5.58).

Case 2: $b_1(y) \geq 0$. For $x < b_0$

$$\langle \mathcal{L}_2 \rangle V_1 = a_1\beta\left(v_2\beta^2 + (v_1 - 3v_2)\beta + (2v_2 - v_1)\right)e^{\beta x}, \tag{5.60}$$

and for $b_0 \leq x < b_0 + \sqrt{\varepsilon} b_1(y)$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle V_1 &= \left(v_2 \frac{\partial^3}{\partial x^3} + (v_1 - 3v_2) \frac{\partial^2}{\partial x^2} + (2v_2 - v_1) \frac{\partial}{\partial x} \right) V_0 \\ &= 0. \end{aligned} \tag{5.61}$$

By similar arguments in the previous case, we can write

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} + \hat{c}_2 e^{\beta x} & \text{for } x < b_0 \\ \hat{d}_2 e^{\beta x} & \text{for } b_0 \leq x < b_0 + \sqrt{\varepsilon} b_1(y) \\ 0 & \text{for } x \geq b_0 + \sqrt{\varepsilon} b_1(y) \end{cases} . \tag{5.62}$$

Continuity at $b_0 + \sqrt{\varepsilon} b_1(y)$ implies $\hat{d}_2 = 0$. Continuity at b_0 implies

$$c_1 b_0 e^{\beta b_0} + \hat{c}_2 e^{\beta b_0} = 0,$$

or

$$\hat{c}_2 = -c_1 b_0. \tag{5.63}$$

Since the optimal exercise boundary does not change in this case, we set $b_1 = 0$. Then,

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} - c_1 b_0 e^{\beta x} & \text{for } x < b_0 \\ 0 & \text{for } x \geq b_0 \end{cases} . \tag{5.64}$$

In summary, if

$$\frac{c_1 e^{\beta b_0}}{(1-\beta) e^{b_0} - c_1 \sqrt{\varepsilon} e^{\beta b_0}} < 0, \tag{5.65}$$

then

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} + c_2 e^{\beta x} & \text{for } x < b_0 + \sqrt{\varepsilon} b_1(y) \\ 0 & \text{for } x \geq b_0 + \sqrt{\varepsilon} b_1(y) \end{cases},$$

where c_1 is given in (5.48), c_2 in (5.57) and b_1 in (5.58). Otherwise,

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} - c_1 b_0 e^{\beta x} & \text{for } x < b_0 \\ 0 & \text{for } x \geq b_0 \end{cases},$$

where c_1 is given in (5.48), $b_1 = 0$.

Chapter 6

Conclusion

This thesis provides a theoretical treatment of stock loans valuation under exponential phase-type Lévy models. Using the variational inequality approach, we characterized the value function of a stock loan under general exponential phase-type Lévy models and derived an explicit solution of the stock loan value and optimal exercise policy for a fairly general class of phase-type jump diffusion models. We emphasis again that our result could be applied to approximate the corresponding price under a general exponential Lévy model arbitrarily close.

We also discussed a possible extension to stochastic volatility model for stock loans. We adopted a fast mean-reverting stochastic volatility model and analyzed the price behavior using the technique of asymptotic expansion. A possible future research direction is to prove the order of convergence of this approximation and to combine the phase-type Lévy model with the stochastic volatility asymptotic analysis.

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Valuation of Stock Loans under Exponential Phase-type Lévy Models

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in
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Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Abstract

Stock loan, or security lending, is currently a very popular financial service provided by many financial institutions. It is a collateral loan where stocks are used as collateral. The borrower may repay the principal with interest and regain the stock, or make no repayment and surrender the stock. This thesis is concerned with the stock loan valuation problem, in which the underlying stock price is modeled as an exponential Lévy Models of phase-type. The valuation problem can be formulated as an optimal stopping problem of a perpetual American option with a time varying exercise price. As the phase-type jump diffusion forms a dense class in Lévy processes, our solution can approximate the solution under general Lévy models arbitrarily close.

摘要

股票抵押貸款(Stock loans)是現今金融業界非常流行的投資產品。它是一種以股票作為抵押品的貸款。借貸人可以歸還本金及利息以續回股票，或可以放棄用以抵押的股票而不作還款。此論文是探討有關股票在指數性相位型Lévy過程模型下股票抵押貸款的定價問題。此定價問題可視為有時變性履約價的永久美式期權之最優化停止問題。由於相位型跳躍擴散過程是所有Lévy過程中的稠密子集，因此我們的定價模型可以用作估算一般Lévy模型所引伸之價格。

Contents

Chapter 1

Introduction

A stock loan is a loan issued by financial institutions (the lender) to its clients (the borrower) which is collateralized with stocks. Recently, stock loans become a very popular product in over-the-counter market. As reported by International Securities Lending Association, the global market size of these products exceeded £1 trillion [1].

Under the terms of the contract, the borrower has the right to repay the loan at anytime, or to simply default the loan with the loss of the collateral. With this in mind, the borrower's right can be regarded as a perpetual American option, which represents the right for the borrower to exercise the option at anytime, without a time limit. The value of this perpetual American option is therefore of central importance to the problem of stock loan valuation.

The value of this perpetual American option can be expressed as an ordinary perpetual American call option with a possibly negative interest rate. This creates the major challenge of stock loan pricing. Consider the case of geometric Brownian motion (GBM) for the stock price. The optimal exercise rule of a perpetual American call option is to exercise at the first time that the stock price rises to cross a constant

[1]This number is quoted from the article: An Introduction to Securities Lending, Executive Summary, Page 8, issued by Australian Securities Lending Association Limited at 1 August 2005.

level. This constant level is called the optimal exercise boundary. If interest rate is positive, the stock price will cross any fixed boundary almost surely. The perpetual American call option can then be valued directly with a variational inequality (VI). In contrast, when interest rate is negative, the problem becomes complicated. Given any fixed boundary level greater than the current stock price, there is a positive probability that the stock price will never cross this level.

Xia and Zhou (2007) are pioneers of solving the stock loan problem. They value the stock loan under the classical GBM model using a purely probabilistic approach. Zhang and Zhou (2009) then extended the framework to a regime switching model and solved the problem using variational inequalities. Dai and Xu (2009) studied the optimal redeeming strategy of stock loans with finite maturity under GBM. Yam et al. (2010) considered the callable feature of the stock loans.

Although most studies on stock loan adopt the GBM model for stock price, empirical evidences (e.g. Andersen et al. (2002), Pan (2002) and Eraker et al (2003)) show that jump diffusion model would be a better model for asset prices to capture the heavy tails of the empirical distribution. Therefore, a jump diffusion model with flexible jump distribution is worth considering for stock loan valuation.

Merton (1976) is the first one to propose jump diffusion for asset price modeling using a Gaussian jump distribution. Another notable jump diffusion model is the double-exponential jump diffusion proposed by Kou (2002). The generalization of jump diffusion model is the exponential Lévy model, such as the variance-gamma model (Madan et al., 1998), CGMY model (Carr et al., 1999) and normal inverse Gaussian model (Barndorff-Nielssen, 2000).

Sun (2010) recently considered the stock loan valuation problem under the double-exponential jump diffusion model in the first chapter of her thesis. While it is a good start, the asset return distribution is not flexible enough to capture the empirical distribution implied by market data. For this reason, we incorporate the phase-type jump diffusion to stock loans valuation.

The phase-type distribution is dense over the class of all positive valued distributions. By making use of this fact, Asmussen et al. (2007) show that the class of phase-type jump diffusion model is dense over all exponential Lévy model. In other words, the option price derived from phase-type jump diffusion models can be used to approximate the corresponding price under a general exponential Lévy model. In particular, Asmussen et al. (2007) approximate the CGMY model by the phase-type jump diffusion. In fact, the phase-type jump diffusion model embraces the Kou (2002) model and the mixed-exponential jump diffusion model (Cai and Kou, 2011) as its special cases.

Asmussen et al. (2004) solved the price of the perpetual American put option with positive interest rate under phase-type jump diffusion models. They used the technique of Wiener-Hopf factorization to derive the optimal exercise boundary. Then the pricing problem is reduced to the evaluation of the corresponding expectation at the given exercise boundary.

While Wiener-Hopf factorization is useful to solve American option pricing problem involving Lévy processes and, in particular, the phase-type Lévy model, it relies heavily on the positive interest rate or, in the limiting case, zero interest rate. The method no longer works for a negative effective interest rate in the stock loan valuation problem. We take the variational inequality approach as in Zhang and Zhou (2009).

Under the phase-type jump diffusion model, we show that the price of the perpetual American option satisfies an ordinary integro-differential equation (OIDE). The solution of this OIDE is closely linked to the root characteristics of a Cramér-Lundberg equation (C-L equation). The root characteristics of the C-L equation is first studied in a special case of the phase-type distribution, the hyperexponential distribution. By making use of the properties of this special case, we extend our result to a fairly general class of phase-type models.

The rest of the thesis is organized as follow. Chapter 2 introduces the elements of our problem. Chapter 3 presents some general properties of stock loans. Chapter 4 presents the methodologies of valuation. Chapter 5 discusses a possible extension to incorporate stochastic volatility. Chapter 6 concludes.

Chapter 2

Problem Formulation

In this chapter, we describe the formulation of stock loan valuation under the phase-type Lévy model. We introduce the phase-type distribution, its use in the phase-type jump diffusion model, and the formulation of stock loan as a perpetual American call option pricing problem.

2.1 Phase-type distribution

2.1.1 A generalization of the exponential distribution

Consider a continuous time Markov process with 1 transient state and 1 absorption state. The intensity matrix is given by

$$\begin{pmatrix} -\theta & \theta \\ 0 & 0 \end{pmatrix},$$

where $\theta > 0$. Let Y be the absorption time of this Markov process. Then the distribution of Y is the exponential distribution. The cumulative distribution function is

$$F_Y(y) = 1 - e^{-\theta y}. \tag{2.1}$$

A finite mixture of the exponential distribution is called hyperexponential distribution. This can be expressed as the absorption time of a continuous time Markov process with m transient state, 1 absorption state with an intensity matrix of the form

$$\begin{pmatrix} -\theta_1 & \cdots & 0 & \theta_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\theta_m & \theta_m \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

The cumulative distribution function is

$$F_Y(y) = \sum_{i=1}^{m} \alpha_i \left(1 - e^{-\theta_i y}\right), \tag{2.2}$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$. α_i is the probability for the process to start at state i. It can also be expressed using matrix notation,

$$F_Y(y) = 1 - \boldsymbol{\alpha} e^{\mathbf{T}y}\mathbf{1}, \tag{2.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $\mathbf{T} = \text{diag}(-\theta_1, \ldots, -\theta_m)$, $\mathbf{1} = (1, \ldots, 1)^T$.

A further generalization allows the transient states to be communicative. The resulting distribution becomes the phase-type distribution described in the next section.

2.1.2 Properties of the phase-type distribution

The phase-type distribution is the absorption time of a finite state continuous time Markov process with m transient states and 1 absorption state.

Let $\mathbf{T}$ be the intensity matrix of the transient states and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be an initial probability vector. The phase-type distribution is parameterized by $(m, \mathbf{T}, \boldsymbol{\alpha})$. The full intensity matrix of the Markov process can be written as

$$\boldsymbol{S} = \begin{pmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{0} & 0 \end{pmatrix},$$

where $\mathbf{t} = -\mathbf{T}\mathbf{1}$. The cumulative distribution function is given by

$$F_Y(y) = 1 - \boldsymbol{\alpha} e^{\mathbf{T}y}\mathbf{1}. \tag{2.4}$$

The density function is given by

$$f_Y(y) = \boldsymbol{\alpha} e^{\mathbf{T}y}\mathbf{t}. \tag{2.5}$$

Finally, the generating function is given by

$$M(t) = \mathbb{E}[e^{tY}] = \boldsymbol{\alpha}(-t\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}. \tag{2.6}$$

The class of phase-type distribution is very rich. When $\mathbf{T}$ is a diagonal matrix, the distribution reduces to a hyperexponential distribution. As shown in Johnson and Taafe (1988), the class of phase-type distribution is dense in the field of all distributions on $(0, \infty)$.

2.2 Phase-type jump diffusion model

2.2.1 Jump diffusion model

If the price of an asset S_t follows jump diffusion model, then the change in price consists of three components: drift, Brownian motion and a jump process. The stochastic dynamics can be expressed in the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \left(\nu + \frac{1}{2}\sigma^2\right) dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t} \left(e^{Y_i} - 1\right)\right), \tag{2.7}$$

where $\{W_t\}_{t\geq 0}$ is the standard Brownian motion, $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity λ, $\{Y_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables.

For a Poisson process N_t, when h is small, we have:

1. $\Pr\left(N_{t+h} - N_t = 0\right) = 1 - \lambda h + o(h)$;

2. $\Pr\left(N_{t+h} - N_t = 1\right) = \lambda h + o(h)$;

3. $\Pr\left(N_{t+h} - N_t \geq 2\right) = o(h)$,

where $o(h)$ is the asymptotic order symbol such that $\lim_{h\to 0} \frac{o(h)}{h} = 0$. Hence, (2.7) can be alternatively written as

$$\frac{dS_t}{S_t} = \left(\nu + \frac{1}{2}\sigma^2\right) dt + \sigma dW_t + \left(e^{Y_i} - 1\right) dN_t. \tag{2.8}$$

An application of Itō's formula gives

$$d\ln S_t = \nu dt + \sigma dW_t + Y_i dN_t, \tag{2.9}$$

which implies that

$$S_t = S_0 \exp\left(\nu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right). \tag{2.10}$$

For $r > 0$, we have

$$\mathbb{E}\left[e^{-rt} S_t\right] = S_0 \exp\left(-r + \nu + \frac{1}{2}\sigma^2 + \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)\right).$$

If we set $\nu = r - \sigma^2/2 - \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)$, then $\{e^{-rt} S_t\}_{t\geq 0}$ is a martingale. This motivates the definition in the next section.

2.2.2 The stock price model

The stock price process is defined on a risk-neutral probability space $\left(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}\right)$. We write

$$\begin{aligned} S_t &= \exp(X_t), & (2.11) \\ X_t &= x + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, & (2.12) \end{aligned}$$

where $\mu = r - \sigma^2/2 - \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)$. The distribution of Y_i, $i \in \mathbb{N}$, is a two-sided phase-type distribution and the density function is given by

$$f_Y(y) = p\boldsymbol{\alpha}^+ e^{\mathbf{T}^+ y}\mathbf{t}^+ I_{\{y\geq 0\}} + (1-p)\boldsymbol{\alpha}^- e^{-\mathbf{T}^- y}\mathbf{t}^- I_{\{y<0\}}. \tag{2.13}$$

Note that the financial market is incomplete under the jump diffusion setting. That means that not all contingent claims can be perfectly hedged and the martingale measure is not unique. In other words, there are infinitely many equivalent martingale measures. Our choice $\mathbb{P}$ is the one that preserves the phase-type structure of the log-price X_t as proposed by Assmusen et al. (2004).

2.3 Stock Loans

Stock loan is a collateral loan where stocks are used as collateral. The borrower will receive the loan principle (q), pay the service charge (c) and have the right to repay the principal with interest (continuously compounded with rate γ) and regain the stock anytime in the future. The transactions can be summarized as follows:

- The borrower receives money $q - c$ as well as V_0, a perpetual American option with time varying strike price $qe^{\gamma t}$.
- The bank receives S_0 (one unit of stock) as collateral.

By equating the benefits of both parties, it is seen that the service charge is

$$c = q + V_0 - S_0. \tag{2.14}$$

We have the following representation of the value of the perpetual American option:

$$V_0 = V(x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(S_\tau - qe^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} | S_0 = e^x\right], \tag{2.15}$$

where $\mathcal{T}_u$, $u \geq 0$, is the set of all stopping time taking values in the time interval (u, ∞). By taking the transformation $\tilde{S}_t = S_t e^{-\gamma t}$, the value can be written as

$$V(x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-(r-\gamma)\tau}\left(\tilde{S}_\tau - q\right)^+ I_{\{\tau<\infty\}} | \tilde{S}_0 = e^x\right], \tag{2.16}$$

which is the value of a perpetual American option with constant strike price and a possibly negative effective interest rate $\tilde{r} = r - \gamma$.

From now on, we will stick with the transformed stock price process $\tilde{S}_t$ as the underlying stock of the American option. We also define $\tilde{X}_t$ to be the transformed log-price. Their dynamics are given by

$$\tilde{S}_t = \exp(\tilde{X}_t), \tag{2.17}$$

$$\tilde{X}_t = x + \tilde{\mu}t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{2.18}$$

where $\tilde{\mu} = r - \gamma - \sigma^2/2 - \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)$.

Chapter 3

General Properties of Stock Loans

3.1 Preliminary results

We establish some properties of the perpetual American option as a function of the stock price value. Take $S = e^x$ and write $v(S) = V(\ln S) = V(x)$.

Lemma 3.1 *$v(S)$, as a deterministic function of the initial stock price S, satisfies the following properties:*

1. *$(S-q)^+ \leq v(S) \leq S$ for all $S > 0$;*

2. *$v(S)$ is convex, continuous and nondecreasing in S on $(0, \infty)$.*

Proof For the first item, observe that

$$v(S) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(S_\tau - qe^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid S_0 = S\right]. \tag{3.1}$$

By taking $\tau = 0$, we get $(S-q)^+ \leq v(S)$. On the other hand, since $(S - qe^{\gamma\tau})^+ \leq S$, we have

$$
\begin{aligned}
v(S) &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(S_\tau - qe^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid S_0 = S\right] \\
&\leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-r\tau} S_\tau I_{\{\tau<\infty\}} \mid S_0 = S\right] \\
&\leq S.
\end{aligned}
$$

Next, it is obvious that $v(\cdot)$ is a nondecreasing function. Convexity of $v(\cdot)$ is a direct consequence of the convexity of $\max\{\cdot, 0\}$ function and the essential supremum operator. As the function value is finite, convexity of $v(\cdot)$ implies its continuity.

□

The next lemma is an essential step to solve the optimal stopping problem.

Lemma 3.2 *Define* $k = \inf\{S > 0 : S - q \geq v(S)\} \geq q$, *where* $\inf \emptyset = \infty$. *Then* $\{S > 0 : S - q \geq v(S)\} = [k, \infty)$.

Proof If $k = \infty$, the result is obvious. For the case that $k \in [q, \infty)$, we have $v(k) = k - q$ by the continuity of v. We claim that $v(S) = S - q$ for $S \geq k$. Otherwise, there exists $k_0 \geq k$ such that $v(k_0) > k_0 - q$ because of Lemma 3.1. By convexity, we have

$$
\frac{v(S) - v(k)}{S - k} \geq \frac{v(k_0) - v(k)}{k_0 - k} > 1.
$$

for any $S \geq k_0$. As a consequence $v(S) \geq \frac{v(k_0)-v(k)}{k_0-k}(S - k) + k - q$ which implies $v(S) > S$ for sufficiently large value of S. This is a contradiction to Lemma 3.1.

□

Using similar methods of Xia and Zhou (2007), we now prove that the optimal stopping time is a first hitting time. In other words, it is optimal to exercise the perpetual American option at the first time when the transformed log-price exceeds a predetermined level. Such a level is called the optimal exercise boundary.

Theorem 3.1 *If $\tilde{X}_t$ follows a Lévy process, then the optimal stopping time is of the form*

$$\tau_b = \inf\left\{t \geq 0 : \tilde{X}_t \geq b\right\}, \tag{3.2}$$

where b is a constant.

Proof The stock loan value at time t can be written as

$$\begin{aligned}
V_t &= v(S_t) \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\left(S_t e^{X_\tau - X_t} - q e^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid \mathcal{F}_t\right] \\
&= e^{\gamma t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\left(e^{-\gamma t} S_t e^{X_\tau - X_t} - q e^{\gamma(\tau-t)}\right)^+ I_{\{\tau<\infty\}} \mid \mathcal{F}_t\right] \\
&= e^{\gamma t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(x e^{X_\tau} - q e^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid \mathcal{F}_0\right]_{x=e^{-\gamma t} S_t} \\
&= e^{\gamma t} v(e^{-\gamma t} S_t).
\end{aligned}$$

Hence, the optimal stopping time (cf. Karatzas and Shreve, 1998, Chapter 2.5) is

$$\begin{aligned}
\tau^* &= \inf\left\{t \geq 0 : S_t - q e^{\gamma t} \geq v(S_t)\right\} \\
&= \inf\left\{t \geq 0 : S_t - q e^{\gamma t} \geq e^{\gamma t} v(e^{-\gamma t} S_t)\right\} \\
&= \inf\left\{t \geq 0 : S_t e^{-\gamma t} - q \geq v(e^{-\gamma t} S_t)\right\} \\
&= \inf\left\{t \geq 0 : e^{-\gamma t} S_t \geq k\right\} \\
&= \inf\left\{t \geq 0 : \tilde{X}_t \geq \ln k\right\},
\end{aligned}$$

where k is the value defined in Lemma 3.2.

□

We denote the optimal exercise boundary by b^* and the optimal stopping time by τ_{b^*}. Theorem 3.1 greatly reduces the dimensionality of the optimization problem. The original optimization problem has to search over all possible stopping time. Yet, the optimal stopping time is in the form of a first hitting time and we only need to search for an optimal exercise boundary, which is a one-dimensional optimization problem. In other words, the value function is given by

$$V(x) = \sup_{b \geq \max\{\ln q, x\}} V_b(x) = \sup_{b \geq \max\{\ln q, x\}} \mathbb{E}\left[e^{-\tilde{r}\tau_b} \left(e^{\tilde{X}_{\tau_b}} - q \right)^+ I_{\{\tau_b < \infty\}} | \tilde{X}_0 = x \right]. \quad (3.3)$$

3.2 Characterization of the function $V(x)$

We want to show that $V(x)$ is a solution of an integro-differential equation (OIDE) and derive its functional form. Before going into that, we first introduce the Cramér-Lundberg equation (C-L equation)

$$G(\beta) = \frac{\sigma^2}{2}\beta^2 + \tilde{\mu}\beta + \lambda p \boldsymbol{\alpha}^+(-\beta\mathbf{I} - \mathbf{T}^+)^{-1}\mathbf{t}^+ + \lambda(1-p)\boldsymbol{\alpha}^-(\beta\mathbf{I} - \mathbf{T}^-)^{-1}\mathbf{t}^- - \lambda = \tilde{r}. \quad (3.4)$$

We use the symbol $\mathcal{B}^+$ to denote the collection of roots to the C-L equation with real part larger than or equal to 1 and $\mathcal{B}^-$ to denote the collection of those roots with negative real part. The root characteristics of the C-L equation play a central role in our problem as we will see in later sections. As a starting point, observe the following properties regarding this equation:

1. $\{e^{-rt}S_t\}_{t\geq 0} = \left\{e^{-\tilde{r}t}\tilde{S}_t\right\}_{t\geq 0}$ is a martingale implies that $1 \in \mathcal{B}^+$.

2. The function $G(\beta)$ satisfies

$$\mathbb{E}\left[e^{\beta \tilde{X}_t}\right] = e^{G(\beta)t} \tag{3.5}$$

for β belongs to some bounded interval covering $[0, 1]$.

If $G'(1) \geq 0$, it will be shown that $V(x) = e^x$ and that $q = c$. That means that the bank has no intention to make such a stock loan contract with the given loan interest rate γ and current stock price S_0. Therefore, we will focus on the more interesting case $G'(1) < 0$. The case for $G'(1) \geq 0$ is postponed to section 4.3.

It is worth noting that $G'(1) < 0$ implies $\gamma > r$. In other words, the effective interest rate $\tilde{r} = r - \gamma$ is indeed negative. To see this, recall

$$G(\beta) = \frac{\sigma^2}{2}\beta^2 + \left(r - \gamma - \frac{\sigma^2}{2} - \lambda\left(\mathbb{E}\left[e^{Y_1}\right] - 1\right)\right)\beta + \lambda\left(\mathbb{E}\left[e^{\beta Y_1}\right] - 1\right). \tag{3.6}$$

Hence,

$$G'(1) = r - \gamma + \frac{\sigma^2}{2} + \lambda\mathbb{E}\left[Y_1 e^{Y_1} - e^{Y_1} + 1\right]. \tag{3.7}$$

Since $ye^y - e^y + 1 \geq 0$ for all $y \in \mathbb{R}$, $G'(1) < 0$ implies

$$\gamma > r + \frac{\sigma^2}{2} + \lambda\mathbb{E}\left[Y_1 e^{Y_1} - e^{Y_1} + 1\right] \geq r.$$

We are now ready to present the result which characterizes the function $V(x)$. It is easy to see that this new result embraces the stock loan valuation under double-exponential jump diffusion model (Sun, 2010) as its special case.

Theorem 3.2 *$V(x)$ satisfies the following integro-differential equation*

$$\begin{cases} (\mathcal{L}-\tilde{r})\,V(x) &= 0 \qquad x<b^* \\ V(x) &= e^x-q \quad x\geq b^* \end{cases}, \tag{3.8}$$

where $\mathcal{L}h(x)=\frac{\sigma^2}{2}\frac{d^2h}{dx^2}(x)+\tilde{\mu}\frac{dh}{dx}(x)+\lambda\int_{-\infty}^{\infty}(h(x+y)-h(x))\,f_Y(y)dy$*. Furthermore, the solution takes the form*

$$V(x)=\begin{cases} \sum\limits_{\beta_j\in\mathcal{B}^+}\omega_j e^{\beta_j x} & x<b^* \\ e^x-q & x\geq b^* \end{cases} \tag{3.9}$$

for some ω_j*,* $j\in\{i\mid\beta_i\in\mathcal{B}^+\}$ *to be determined according to the model.*

Proof Consider the following function as a candidate solution:

$$u(x)=\begin{cases} \sum\limits_{\beta_j\in\mathcal{B}^+}\omega_j e^{\beta_j x} & x<b^* \\ e^x-q & x\geq b^* \end{cases}.$$

It is reasonable to assume that ω_j, $j\in\{i\mid\beta_i\in\mathcal{B}^+\}$ should be chosen such that $u(\cdot)$ satisfies the conditions described in Lemma 3.1. In particular, we should have $(e^x-q)^+\leq u(x)\leq e^x$ for all $x\in\mathbb{R}$. It also satisfies the OIDE

$$\begin{cases} (\mathcal{L}-\tilde{r})\,u(x) &= 0 \qquad x<b^* \\ u(x) &= e^x-q \quad x\geq b^* \end{cases}. \tag{3.10}$$

However, it may not be continuously differentiable at b^*. Hence, we construct a sequence of function $\{u_n(x)\}_{n=1}^{\infty}$ such that

1. $u_n(x)\to u(x)$ as $n\to\infty$ for all x;

2. $u_n(x)$ is twice continuously differentiable for all $n\in\mathbb{N}$;

3. For $x \leq b^*$ or $x \geq b^* + \frac{1}{n}$, $u_n(x) \equiv u(x)$;

4. For $b^* \leq x \leq b^* + \frac{1}{n}$, $0 \leq u_n(x) \leq M_1$, where M_1 is a positive constant.

For any $x < b^*$, we have

$$(\mathcal{L} - \tilde{r})\, u_n(x) = \lambda \int_{b^*-x}^{b^*-x+1/n} \left[u_n(x+y) - u(x+y)\right] f_Y(y)dy. \tag{3.11}$$

Note that

$$|u_n(x) - u(x)| \leq \max_{x\in(b^*,b^*+1/n} |u_n(x)| + \max_{x\in(b^*,b^*+1/n} |u(x)| \leq M_2,$$

where $M_2 = M_1 + e^{b^*+1}$. Then we have

$$\begin{aligned} |\mathcal{L}u_n(x) - \tilde{r}u_n(x)| &\leq \lambda p\boldsymbol{\alpha}^+\mathbf{t}^+ \int_{b^*-x}^{b^*-x+1/n} \left[u_n(x+y) - u(x+y)\right] dy \qquad (3.12) \\ &\leq \frac{\lambda p\boldsymbol{\alpha}^+\mathbf{t}^+ M_2}{n} \to 0 \text{ uniformly for all } x < b^* \text{ , as } n \to \infty. \end{aligned}$$

Next, by applying Itō's formula to $\left\{e^{-\tilde{r}t}u_n(\tilde{X}_t)\right\}_{t\geq 0}$, we can obtain a sequence of local martingale $\left\{M_t^{(n)}\right\}_{t\geq 0}$ for $n \in \mathbb{N}$ as follows:

$$M_t^{(n)} = e^{-\tilde{r}(t\wedge\tau_{b^*})}u_n(\tilde{X}_{t\wedge\tau_{b^*}}) - u_n(x) - - \int_0^{t\wedge\tau_{b^*}} e^{-\tilde{r}s}\left[(\mathcal{L} - \tilde{r})\, u_n(\tilde{X}_s)\right] ds. \tag{3.13}$$

We claim that it is a true martingale for any $n \in \mathbb{N}$. Note that for any $t \geq 0$,

$$
\begin{aligned}
&|e^{-\tilde{r}(t\wedge\tau_{b^*})}u_n(\tilde{X}_{t\wedge\tau_{b^*}})| \\
\leq\ & |e^{-\tilde{r}t}u_n(\tilde{X}_t)I_{\{t<\tau_{b^*}\}}| + |e^{-\tilde{r}t}u_n(\tilde{X}_{\tau_{b^*}})I_{\{t\geq\tau_{b^*},\tilde{X}_{\tau_{b^*}}<b+1/n\}}| + |e^{-\tilde{r}t}u_n(\tilde{X}_{\tau_{b^*}})I_{\{t\geq\tau_{b^*},\tilde{X}_{\tau_{b^*}}\geq b+1/n\}}| \\
\leq\ & |e^{-\tilde{r}t}u(\tilde{X}_t)I_{\{t<\tau_{b^*}\}}| + M_1e^{-\tilde{r}t} + e^{-\tilde{r}t}(e^{\tilde{X}_{\tau_{b^*}}} - q)I_{\{\tau_{b^*}<\infty\}}. \qquad (3.14)
\end{aligned}
$$

From the definition in (3.13) and noting (3.12) and (3.14), we establish the following inequality,

$$
|M_t^{(n)}| \leq |e^{-\tilde{r}t}u(\tilde{X}_t)I_{\{t<\tau_{b^*}\}}|+|u_n(x)|+M_1e^{-\tilde{r}t}+e^{-\tilde{r}t}(e^{\tilde{X}_{\tau_{b^*}}}-q)I_{\{\tau_{b^*}<\infty\}}-\frac{\lambda p\boldsymbol{\alpha}^+\mathbf{t}^+M_2\left(e^{-\tilde{r}t}-1\right)}{n\tilde{r}}. \qquad (3.15)
$$

For the first term in the right hand side of (3.15), we have for any fixed $T > 0$

$$
\begin{aligned}
& \mathbb{E}_x\left[\sup_{t\in[0,T]} e^{-\tilde{r}t}u(\tilde{X}_t)I_{\{t<\tau_{b^*}\}}|\right] \\
\leq\ & e^{-\tilde{r}T}\mathbb{E}_x\left[e^{\sup_{t\in[0,T]}\tilde{X}_t}\right] \\
\leq\ & e^{-\tilde{r}T}\mathbb{E}_x\left[e^{x+\tilde{\mu}T+\sigma\sup_{t\in[0,T]}W_t+\sum_{i=1}^{N_T}Y_i^+}\right] \\
=\ & 2\Phi(\sigma\sqrt{T})\exp\left(-\tilde{r}T+x+\tilde{\mu}T+\frac{\sigma^2T}{2}+p\lambda T\boldsymbol{\alpha}^+\left(-\mathbf{I}-\mathbf{T}^+\right)^{-1}\mathbf{t}^+\right) \\
<\ & \infty,
\end{aligned}
$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. It is now easy to see that

$$
\mathbb{E}_x \sup_{t\in[0,T]} |M_t^{(n)}| < \infty, \qquad (3.16)
$$

which guarantees that $M_t^{(n)}$ is a true martingale for all n. Then, we know that for $x < b^*$

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= \lim_{n\to\infty} \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u_n(\tilde{X}_{t\wedge\tau_{b^*}})\right] - \lim_{n\to\infty} \mathbb{E}_x M_t^{(n)} - \lim_{n\to\infty} \mathbb{E}_x\left[\int_0^{t\wedge\tau_{b^*}} e^{-\tilde{r}s}\left[(\mathcal{L}-\tilde{r})\,u_n(\tilde{X}_s)\right]ds\right] \\ &= \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})\right], \end{aligned}$$

with the last equality implied by the dominated convergence theorem (DCT). Now, let $t \to \infty$ and apply Fatou's lemma to get

$$\begin{aligned} u(x) &= \lim_{t\to\infty} \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})\right] \\ &= \lim_{t\to\infty} \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}<\infty\}}\right] + \lim_{t\to\infty} \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}=\infty\}}\right] \\ &\geq \mathbb{E}_x\left[e^{-\tilde{r}(\tau_{b^*})}u(\tilde{X}_{\tau_{b^*}})I_{\{\tau_{b^*}<\infty\}}\right] \\ &= \mathbb{E}_x\left[e^{-\tilde{r}(\tau_{b^*})}(e^{\tilde{X}_{\tau_{b^*}}} - q)I_{\{\tau_{b^*}<\infty\}}\right]. \end{aligned}$$

On the other hand,

$$\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})\right] = \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}(e^{\tilde{X}_{t\wedge\tau_{b^*}}} - q)I_{\{\tau_{b^*}\leq t\}}\right] + \mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right]. \tag{3.17}$$

Since

$$e^{-\tilde{r}(t\wedge\tau_{b^*})}(e^{\tilde{X}_{t\wedge\tau_{b^*}}} - q)I_{\{\tau_{b^*}\leq t\}} \leq e^{-\tilde{r}(t\wedge\tau_{b^*})}(e^{\tilde{X}_{t\wedge\tau_{b^*}}} - q)I_{\{\tau_{b^*}<\infty\}}$$

and

$$\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}(e^{\tilde{X}_{t\wedge\tau_{b^*}}} - q)I_{\{\tau_{b^*}<\infty\}}\right] < \infty,$$

DCT implies that the first term on the right hand side of (3.17) converges to

$$\mathbb{E}_x\left[e^{-\tilde{r}(\tau_{b^*})}(e^{\tilde{X}_{\tau_{b^*}}} - q)I_{\{\tau_{b^*}<\infty\}}\right]$$

as $t \to \infty$. For the second term, we claim that

$$\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right] \to 0$$

as $t \to \infty$. This can be shown by considering the following two cases:

Case 1: $G'(1) < 0$, there exists $\kappa_0 > 1$ such that $G(\kappa_0) - \tilde{r} < 0$. In addition, there exists $C_0 > 0$ such that $u(x) < C_0 e^{\kappa_0 x}$ for all $x < b^*$. Hence,

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right] &< \mathbb{E}_x\left[C_0 e^{-\tilde{r}t+\kappa_0\tilde{X}_t}I_{\{\tau_{b^*}>t\}}\right] \\
&\leq \mathbb{E}_x\left[C_0 e^{-\tilde{r}t+\kappa_0\tilde{X}_t}\right] \\
&\to 0 \text{ as } t \to \infty.
\end{aligned}$$

Case 2: $G'(1) \geq 0$, as $v(x) \leq e^x$ for all $x \in \mathbb{R}$, we have

$$\mathbb{E}_x\left[e^{-\tilde{r}(t\wedge\tau_{b^*})}u(\tilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right] \leq \mathbb{E}_x\left[e^{-\tilde{r}t+\tilde{X}_t}I_{\{\tau_{b^*}>t\}}\right]. \tag{3.18}$$

Consider the probability measure $\hat{\mathbb{P}}$ as follows:

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{-\tilde{r}t+\tilde{X}_t}. \tag{3.19}$$

It is shown in Appendix A of Asmussen et al. (2004) that $\{\hat{X}_t\}_{t>0}$ is also a phase-type jump diffusion process and the corresponding Lévy exponent is given by

$$\hat{G}(s) = G(1+s) - G(1). \tag{3.20}$$

Define $\hat{\tau}_{b^*} = \inf\{t \geq 0 : \hat{X}_t \geq b^*\}$. Then

$$\mathbb{E}_x\left[e^{-\tilde{r}t+\tilde{X}_t} I_{\{\tau_{b^*}>t\}}\right] = \hat{\mathbb{E}}_x\left[I_{\{\hat{\tau}_{b^*}>t\}}\right] \to \hat{\mathbb{E}}_x\left[I_{\{\hat{\tau}_{b^*}=\infty\}}\right] \tag{3.21}$$

as $t \to \infty$. Observe that $\hat{G}'(0) = G'(1) \geq 0$ and $\hat{G}(0) = 0$, under $\hat{\mathbb{P}}$ measure

$$\Pr(\hat{\tau}_{b^*} < \infty) = \lim_{\tilde{r} \to 0} \hat{\mathbb{E}}\left[e^{-\tilde{r}\hat{\tau}_{b^*}}\right] = 1. \tag{3.22}$$

This proves the claim.

Now, we can conclude that

$$u(x) = \mathbb{E}_x\left[e^{-\tilde{r}(\tau_{b^*})}(e^{\tilde{X}_{\tau_{b^*}}} - q) I_{\{\tau_{b^*}<\infty\}}\right] = V(x). \tag{3.23}$$

That means the candidate solution $u(\cdot)$ is indeed a solution. This completes the proof.

□

Using similar techniques presented by Zhang and Zhou (2009), we can simplify our solution by dropping the term with $\beta_j = 1$:

Proposition 3.3 *Under the condition that $G'(1) < 0$, suppose j_0 is the index such that $\beta_{j_0} = 1$, then we have $\omega_{j_0} = 0$. In other words, the value function takes the form*

$$V(x) = \begin{cases} \sum_{\beta_j \in \mathcal{B}^+ \setminus \{1\}} \omega_j e^{\beta_j x} & x < b^* \\ e^x - q & x \geq b^* \end{cases}.$$

Proof For $t \leq \tau_{b^*}$, we have

$$\mathbb{E}\left[e^{-\tilde{r}t}V(\tilde{X}_t) \mid \tilde{X}_0 = x\right] = V(x) + \int_0^t e^{-\tilde{r}s}\left(\mathcal{L} - \tilde{r}\right)V(\tilde{X}_s)ds = V(x).$$

Hence, for any $T > 0$,

$$\begin{aligned} V(x) &= \mathbb{E}\left[e^{-\tilde{r}\tau_{b^*}\wedge T}V(\tilde{X}_{\tau_{b^*}\wedge T}) \mid \tilde{X}_0 = x\right] \\ &\leq \mathbb{E}\left[e^{-\tilde{r}\tau_{b^*}}V(\tilde{X}_{\tau_{b^*}})I_{\{\tau_{b^*}<T\}} \mid \tilde{X}_0 = x\right] + \mathbb{E}\left[e^{-\tilde{r}T}V(\tilde{X}_T)I_{\{\tau_{b^*}\geq T\}} \mid \tilde{X}_0 = x\right]. \end{aligned}$$

It is clear that the first term converges to

$$\mathbb{E}\left[e^{-\tilde{r}\tau_{b^*}}\left(e^{\tilde{X}_{\tau_{b^*}}} - q\right)I_{\{\tau_{b^*}<\infty\}} \mid \tilde{X}_0 = x\right]$$

as $T \to \infty$. To complete the proof, we require the second term to converge to zero as $T \to \infty$.

By Theorem 3.2, $V(x)$ is a linear combination of $e^{\beta_i x}$ for $x < b^*$, we consider the validity of

$$\mathbb{E}\left(e^{-\tilde{r}T}e^{\kappa\tilde{X}_T}\right) \to 0 \text{ as } T \to \infty$$

for different values of κ.

Note that $\mathbb{E}\left(e^{-\tilde{r}T}e^{\kappa\tilde{X}_T}\right) = e^{(G(\kappa)-\tilde{r})T}$. For $\kappa = 1$, the expectation becomes $e^0 = 1$ and does not converge to zero. Hence the term $\omega_{j_0}e^x$ should be dropped from the linear combination by setting the coefficient to zero.

On the other hand, since $G'(1) < 0$, there exists $\kappa_0 > 1$ such that $G(\kappa_0) - \tilde{r} < 0$. Furthermore, for any $\beta_i \in \mathcal{B}^+\setminus\{1\}$, there exists $K_i > 0$ such that $e^{\beta_i x} \leq K_i e^{\kappa_0 x}$ for $x \in (-\infty, b^*)$. We have $\mathbb{E}\left(e^{-\tilde{r}T}e^{\beta_i\tilde{X}_T}\right) \leq K_i\mathbb{E}\left(e^{-\tilde{r}T}e^{\kappa_0\tilde{X}_T}\right) \to 0$ as $T \to \infty$.

□

We summarize our results in this chapter. For any given exponential phase-type Lévy model, the stock loan valuation is divided into two cases. If $G'(1) \geq 0$, the stock loan is not reasonable to exist. Otherwise, if $G'(1) < 0$, we solve roots from the C-L equation (3.4). The valuation formula of stock loan is given by Proposition 3.3 in which the optimal exercise boundary b^* can be determined by setting a differential to zero.

However, it is not an obvious task to study the root characteristics of the C-L equation (3.4) in general. The following chapter presents some important special cases for which the solutions are obtained in explicit form.

Chapter 4

Valuation

This chapter is devoted to the derivation of the valuation formula. We first solve the problem under hyperexponential jump diffusions, a special case of the phase-type jump diffusion. Although the hyperexponential jump diffusion model is studied by Cai (2009) for a first passage time problem and Cai and Kou (2011) for barrier and lookback option pricing, they only consider the case of positive interest rate and the optimal exercise boundary is yet to be investigated. By making use of the solution of the hyperexponential case, we extend our result to a fairly general class of phase-type jump diffusion models. Except the last section of this chapter, we assume that $G'(1) < 0$, where $G(\cdot)$ is defined in (3.4).

4.1 Hyperexponential jumps

Suppose $\mathbf{T}^+$ and $\mathbf{T}^-$ take the following form

$$\mathbf{T}^+ = \begin{pmatrix} -\eta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\eta_m \end{pmatrix}, \quad \mathbf{T}^- = \begin{pmatrix} -\theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\theta_n \end{pmatrix}, \tag{4.1}$$

where $\eta_i > 1$ for $i = 1, \ldots, m$ and $\theta_k > 0$ for $k = 1, \ldots, n$.

Then the phase-type jump distribution is reduced to a hyperexponential class. The following proposition summarizes the root characteristics of the C-L equation (3.4). A similar result is obtained by Cai (2009) for the case of non-negative interest rate.

Proposition 4.1 *The Cramér-Lundberg equation $G(\beta) = \tilde{r}$ has exactly n distinct negative real roots and $m + 2$ distinct real roots which are greater than or equal to 1.*

Proof Under hyperexponential jump diffusion, we have

$$G(\beta) = \frac{\sigma^2}{2}\beta^2 + \tilde{\mu}\beta + \lambda p \sum_{i=1}^{m} \frac{\alpha_i^+ \eta_i}{\eta_i - \beta} + \lambda(1-p) \sum_{j=1}^{n} \frac{\alpha_i^- \theta_j}{\theta_j + \beta} - \lambda. \tag{4.2}$$

It is clear that:

1. $G(0) = 0$;
2. $G(\infty) = \infty$;
3. $G(-\infty) = \infty$;
4. $G(\eta_i -) = \infty$, $G(\eta_i +) = -\infty$ for $i = 1, \ldots, m$;
5. $G(-\theta_j -) = -\infty$, $G(-\theta_j +) = \infty$ for $j = 1, \ldots, n$;
6. $G(\beta)$ is continuous except the values η_i, $i = 1, \ldots, m$ and $-\theta_j$, $j = 1, \ldots, n$,

where $G(u\pm) = \lim_{x \to u\pm} G(x)$. Then we know that $G(\beta) = \tilde{r}$ has at least one root in each of the intervals

$$(-\infty, -\theta_n), (-\theta_n, -\theta_{n-1}), \ldots, (-\theta_2, -\theta_1), (\eta_1, \eta_2), \ldots, (\eta_{m-1}, \eta_m), (\eta_m, \infty).$$

Moreover, $G(\beta) = \tilde{r}$ has the same number of roots as the $m+n+2$ degree polynomial

$$(G(\beta) - \tilde{r}) \prod_{i=1}^{m} (\eta_i - \beta) \prod_{j=1}^{n} (\theta_j + \beta).$$

Therefore it has at most $m + n + 2$ real roots.

Also observe that $G(\beta)$ is decreasing on the interval $(-\theta_1, 0)$ and $G(0) = 0$, $G(-\theta_1+) = \infty$, there is no root in the interval $(-\theta_1, 0)$. Now recall that 1 is always a root and complex root always exists in pair, we deduce that there are two real roots in the interval $(0, \eta_1)$. Our assumption $G'(1) < 0$ implies that they are distinct and both of them greater than or equal to 1.

□

By Theorem 3.2, the solution is of the form

$$V_b(x) = \begin{cases} \sum_{j=1}^{m+1} \omega_j e^{\beta_j x} & x < b \\ e^x - q & x \geq b \end{cases}, \tag{4.3}$$

where $b \in \mathbb{R}$ is a constant, $G(\beta_i) - \tilde{r} = 0$ for all i and $1 < \beta_1 < \beta_2 < \ldots < \beta_{m+1}$.

For $x < b$, we have $(\mathcal{L} - \tilde{r})\, V_b(x) = 0$. Therefore,

$$
\begin{aligned}
0 &= \mathcal{L}V_b(x) - \tilde{r}V_b(x) \\
&= \frac{\sigma^2}{2}\frac{d^2V_b}{dx^2}(x) + \tilde{\mu}\frac{dV_b}{dx}(x) + \lambda\int_{-\infty}^{\infty}\left(V_b(x+y) - V_b(x)\right) f_Y(y)dy - \tilde{r}V_b(x) \\
&= \sum_{j=1}^{m+1}\omega_j e^{\beta_j x}\left(G(\beta_j) - \tilde{r}\right) - \lambda\int_{b-x}^{\infty}\sum_{j=1}^{m+1}\omega_j e^{\beta_j(x+y)} f_Y(y)dy \\
&\quad +\lambda\int_{b-x}^{\infty}\left(e^{x+y} - q\right) f_Y(y)dy \\
&= -\lambda\sum_{j=1}^{m+1}\omega_j e^{\beta_j x}\sum_{i=1}^{m} p\alpha_i^+ \frac{\eta_i}{\eta_i - \beta_j}e^{-(\eta_i-\beta_j)(b-x)} + \lambda e^x\sum_{i=1}^{m} p\alpha_i^+\frac{\eta_i}{\eta_i - 1}e^{-(\eta_i-1)(b-x)} \\
&\quad -\lambda q\sum_{i=1}^{m} p\alpha_i e^{-\eta_i(b-x)} \\
&= \lambda\sum_{i=1}^{m} p\alpha_i^+ e^{\eta_i x}\left(\frac{\eta_i}{\eta_i - 1}e^{-(\eta_i-1)b} - qe^{-\eta_i b} - \sum_{j=1}^{m+1}\omega_j\frac{\eta_i}{\eta_i - \beta_j}e^{-(\eta_i-\beta_j b}\right). \qquad (4.4)
\end{aligned}
$$

It is clear that $(\omega_1, \ldots, \omega_{m+1})$ should be chosen such that all the values inside the brackets in the summand of (4.4) equal to zero. That is

$$
\sum_{j=1}^{m+1}\omega_j\frac{\eta_i}{\eta_i - \beta_j}e^{-(\eta_i-\beta_j)b} = \frac{\eta_i}{\eta_i - 1}e^{-(\eta_i-1)b} - qe^{-\eta_i b}, \qquad (4.5)
$$

for $i = 1, \ldots, m$. Moreover, the function $V_b(\cdot)$ should also be continuous at b. This gives

$$
\sum_{j=1}^{m+1}\omega_j e^{\beta_j b} = e^b - q. \qquad (4.6)
$$

Now we obtained a $(m+1) \times (m+1)$ system of linear equations

$$\begin{cases} \sum_{j=1}^{m+1} \omega_j \frac{\eta_i}{\eta_i-\beta_j} e^{-(\eta_i-\beta_j)b} &= \frac{\eta_i}{\eta_i-1} e^{-(\eta_i-1)b} - qe^{-\eta_i b} \qquad \text{for } i=1,\ldots,m \\ \sum_{j=1}^{m+1} \omega_j e^{\beta_j b} &= e^b - q \end{cases} . \tag{4.7}$$

4.1.1 Solution of the linear system

We are intended to solve (4.7). Take $\tilde{\omega_j} = \omega_j e^{\beta_j b}$ for $j = 1, \ldots, m+1$, $\eta_{m+1} = 0$, $R_i = \frac{e^b}{\eta_i - 1}$ for $i = 1, \ldots, m$ and $R_{m+1} = q - e^b$. Then the linear system becomes

$$\sum_{j=1}^{m+1} \tilde{\omega_j} \frac{\beta_j}{\eta_i - \beta_j} = R_i \ \text{ for } i = 1, \ldots, m+1. \tag{4.8}$$

Using partial fraction, we have

$$\sum_{j=1}^{m+1} \frac{D_j \beta_j}{x - \beta_j} = \sum_{i=1}^{m+1} R_i \prod_{k=1}^{m+1} \frac{\eta_i - \beta_k}{x - \beta_k} \prod_{l=1, l\neq i}^{m+1} \frac{x - \eta_l}{\eta_i - \eta_l}, \tag{4.9}$$

where D_j, $j = 1, \ldots, m+1$ are the partial fraction coefficients. Multiplying (4.9) by $(x - \beta_k)$ on both sides and set $x = \beta_k$, we obtain

$$D_k \beta_k = \frac{\sum_{i=1}^{m+1} \left(R_i \prod_{j=1}^{m+1} (\eta_i - \beta_j) \prod_{l=1, l\neq i}^{m+1} \frac{\beta_k - \eta_l}{\eta_i - \eta_l} \right)}{\prod_{j=1, j\neq k}^{m+1} (\beta_k - \beta_j)}. \tag{4.10}$$

Hence,

$$D_k = \frac{\sum_{i=1}^{m+1} \left(R_i \prod_{j=1}^{m+1} (\eta_i - \beta_j) \prod_{l=1, l\neq i}^{m+1} \frac{\beta_k - \eta_l}{\eta_i - \eta_l} \right)}{\beta_k \prod_{j=1, j\neq k}^{m+1} (\beta_k - \beta_j)}. \tag{4.11}$$

When $x = \eta_i$ in (4.9), we have

$$\sum_{j=1}^{m+1} \frac{D_j\beta_j}{\eta_i - \beta_j} = R_i \prod_{k=1}^{m+1} \frac{\eta_i - \beta_k}{\eta_i - \beta_k} \prod_{l=1,l\neq i}^{m+1} \frac{\eta_i - \eta_l}{\eta_i - \eta_l} = R_i.$$

Hence, $\tilde{\omega_j} = D_j$, $j = 1, \ldots, m+1$, and we conclude that

$$\omega_j = \frac{\sum\limits_{i=1}^{m+1} \left(R_i \prod\limits_{k=1}^{m+1} (\eta_i - \beta_k) \prod\limits_{l=1,l\neq i}^{m+1} \frac{\beta_j - \eta_l}{\eta_i - \eta_l} \right)}{\beta_j e^{\beta_j b} \prod\limits_{k=1,k\neq j}^{m+1} (\beta_j - \beta_k)}, \tag{4.12}$$

where $R_i = \frac{e^b}{\eta_i - 1}$ for $i = 1, \ldots, m$ and $R_{m+1} = q - e^b$.

4.1.2 Solution of the optimal exercise boundary

After obtaining the coefficients in (4.3), the remaining unknown is the optimal exercise boundary b^*. Note that b^* is the value which maximizes the candidate solution $V_b(x)$. The following identity is useful for that purpose.

Lemma 4.1 *If $\{\beta_k\}_{k=1}^{m+1}$ and $\{\eta_i\}_{i=1}^{m+1}$ are all distinct, we have*

$$\prod_{k=1,k\neq j}^{m+1} (\beta_k - 1) = \sum_{k=1}^{m+1} \prod_{k=1,k\neq j}^{m+1} (\beta_k - \eta_i) \prod_{l=1,l\neq i}^{m+1} \left(\frac{\eta_l - 1}{\eta_l - \eta_i} \right). \tag{4.13}$$

Proof Consider the following polynomial

$$P_j(x) = \prod_{k=1,k\neq j}^{m+1} (\beta_k - 1 - x) \quad \text{for } j = 1, \ldots, m+1, \tag{4.14}$$

which are of degree m. Observe that

$$P_j(\eta_i - 1) = \prod_{k=1,k\neq j}^{m+1} (\beta_k - \eta_i). \tag{4.15}$$

By Lagrange interpolation,

$$
\begin{aligned}
L_j(x) &= \sum_{i=1}^{m+1} P_j(\eta_i - 1) \prod_{l=1,l\neq i}^{m+1} \left(\frac{\eta_l - 1 - x}{\eta_l - \eta_i} \right) \\
&= \sum_{i=1}^{m+1} \prod_{k=1,k\neq j}^{m+1} (\beta_k - \eta_i) \prod_{l=1,l\neq i}^{m+1} \left(\frac{\eta_l - 1 - x}{\eta_l - \eta_i} \right) \qquad (4.16)
\end{aligned}
$$

is a polynomial of degree m which past through all the points in the set

$$\{(\eta_i - 1, P_j(\eta_i - 1))\}_{i=1}^{m+1}.$$

As $P_j(x)$ is a polynomial of degree m and it matches the value of $L_j(x)$ at $m+1$ points, we have

$$P_j(x) = L_j(x) \quad \forall x \in \mathbb{R}.$$

By putting $x = 0$, the result follows.

□

Our objective is to maximize the function

$$
\begin{aligned}
V_b(x) &= \begin{cases} \sum_{j=1}^{m+1} \omega_j e^{\beta_j x} & x < b \\ e^x - q & x \geq b \end{cases} \\
&= \begin{cases} \sum_{j=1}^{m+1} \tilde{\omega_j} e^{\beta_j (x-b)} & x < b \\ e^x - q & x \geq b \end{cases}
\end{aligned}
$$

over $b \in \mathbb{R}$. It suffices to maximize the value function on the interval $(-\infty, b)$. On $(-\infty, b)$, we have

$$\frac{d}{db}V_b(x) = \sum_{j=1}^{m+1} e^{\beta_j(x-b)}\left(\frac{d}{db}\tilde{\omega_j} - \tilde{\omega_j}\beta_j\right).$$

Some simple algebras show that

$$\begin{aligned}\frac{d}{db}\tilde{\omega_j} - \tilde{\omega_j}\beta_j &= \frac{(1-\beta_j)\,e^b\left[\sum\limits_{j=1}^{m}\frac{1}{\eta_i-1}\prod\limits_{k=1}^{m+1}(\eta_i-\beta_k)\prod\limits_{l=1,l\neq i}^{m+1}\frac{\beta_j-\eta_l}{\eta_i-\eta_l}+\prod\limits_{k=1}^{m+1}\beta_k\prod\limits_{l=1}^{m}\frac{\beta_j-\eta_l}{\eta_l}\right]}{\beta_j\prod\limits_{k=1,k\neq j}^{m+1}(\beta_j-\beta_k)} \\ &\quad+\frac{q\prod\limits_{k=1}^{m+1}\beta_k\prod\limits_{l=1}^{m}\frac{\beta_j-\eta_l}{\eta_l}}{\prod\limits_{k=1,k\neq j}^{m+1}(\beta_j-\beta_k)}.\end{aligned}$$

Therefore $\frac{d}{db}\tilde{\omega_j} - \tilde{\omega_j}\beta_j = 0$ if and only if

$$\begin{aligned}\frac{e^b}{q} &= \frac{\beta_j}{\beta_j-1}\frac{\prod\limits_{k=1}^{m+1}\beta_k\prod\limits_{l=1}^{m}\frac{\beta_j-\eta_l}{\eta_l}}{\sum\limits_{j=1}^{m}\frac{1}{\eta_i-1}\prod\limits_{k=1}^{m+1}(\eta_i-\beta_k)\prod\limits_{l=1,l\neq i}^{m+1}\frac{\beta_j-\eta_l}{\eta_i-\eta_l}+\prod\limits_{k=1}^{m+1}\beta_k\prod\limits_{l=1}^{m}\frac{\beta_j-\eta_l}{\eta_l}} \\ &= \frac{1}{\beta_j-1}\frac{\prod\limits_{k=1}^{m+1}\beta_k\prod\limits_{l=1}^{m}\frac{\eta_l-1}{\eta_l}}{\sum\limits_{j=1}^{m+1}\prod\limits_{k=1,k\neq j}^{m+1}(\beta_k-\eta_i)\prod\limits_{l=1,l\neq i}^{m+1}\left(\frac{\eta_l-1}{\eta_l-\eta_i}\right)} \\ &= \prod_{k=1}^{m+1}\frac{\beta_k}{\beta_k-1}\prod_{l=1}^{m}\frac{\eta_l-1}{\eta_l},\end{aligned}$$

where the last equality is an application of Lemma 4.1. Hence $\frac{d}{db}V_b(x) = 0$ at $b = b^*$ where

$$b^* = \ln\left(q\prod_{k=1}^{m+1}\frac{\beta_k}{\beta_k-1}\prod_{l=1}^{m}\frac{\eta_l-1}{\eta_l}\right). \tag{4.17}$$

It is then easy to see that $V_b(x)$ is maximized at b^*.

4.2 Phase-type jumps

We are now ready to extend the previous results into phase-type jump diffusion models. Suppose $\mathbf{T}^+$ and $\mathbf{T}^-$ are symmetric (and hence diagonalizable) matrix, and have distinct eigenvalues. Then, there exists orthogonal matrix $\mathbf{Q}^+$ and $\mathbf{Q}^-$ such that

$$\mathbf{T}^+ = (\mathbf{Q}^+)^T \mathbf{\Lambda}^+ \mathbf{Q}^+ \quad \text{and} \quad \mathbf{T}^- = (\mathbf{Q}^-)^T \mathbf{\Lambda}^- \mathbf{Q}^- \ , \tag{4.18}$$

where

$$\mathbf{\Lambda}^+ = \begin{pmatrix} -\eta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\eta_m \end{pmatrix}, \quad \mathbf{\Lambda}^- = \begin{pmatrix} -\theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\theta_n \end{pmatrix}.$$

We have the following result regarding the roots of the Cramér-Lundberg equation (3.4).

Theorem 4.2 *The Cramér-Lundberg equation $G(\beta) = \tilde{r}$ has exactly $m+1$ roots in the complex domain $\mathcal{D}_+ = \{z \in \mathbb{C} | Re(z) > 1\}$ and exactly n roots in the complex domain $\mathcal{D}_- = \left\{z \in \mathbb{C} | Re(z) < \max_i\{-\theta_i\}\right\}$.*

Proof

$$\begin{aligned} \text{Let } f_0(z) &= \tilde{\mu} z + \tfrac{\sigma^2}{2} z^2 + \lambda p \left(\boldsymbol{\alpha}^+ (-z\mathbf{I} - \mathbf{\Lambda}^+)^{-1} (-\mathbf{\Lambda}^+)\mathbf{1} - 1\right) \\ &\quad + \lambda(1-p) \left(\boldsymbol{\alpha}^- (-z\mathbf{I} - \mathbf{\Lambda}^-)^{-1} (-\mathbf{\Lambda}^-)\mathbf{1} - 1\right) - \tilde{r}, \\ f_1(z) &= \tilde{\mu} z + \tfrac{\sigma^2}{2} z^2 + \lambda p \left(\boldsymbol{\alpha}^+ (\mathbf{Q}^+)^T (-z\mathbf{I} - \mathbf{\Lambda}^+)^{-1} (-\mathbf{\Lambda}^+)\mathbf{Q}^+\mathbf{1} - 1\right) \\ &\quad + \lambda(1-p) \left(\boldsymbol{\alpha}^- (\mathbf{Q}^-)^T (-z\mathbf{I} - \mathbf{\Lambda}^-)^{-1} (-\mathbf{\Lambda}^-)\mathbf{Q}^-\mathbf{1} - 1\right) - \tilde{r}, \\ f_t(z) &= [f_0(z)]^{(1-t)} [f_1(z)]^t \quad \text{for} \quad t \in (0,1). \end{aligned}$$

Note that $f_t(z)$ have m poles $\eta_1, \ldots, \eta_m$ in $\mathcal{D}_+$ for all $t \in [0,1]$. From the hyperexponential case (Proposition 4.1), we know that $f_0(t)$ have $m+1$ zeros in $\mathcal{D}_+$. We want to construct a boundary strip $\mathcal{C}_+$ of $\mathcal{D}_+$ such that $f_t(z)$ have no zero on it.

Since $|f_t(z)| \to \infty$ as $|z| \to \infty$ for $t = 0, 1$, there exists $R \in \mathbb{R}$ such that all roots of $f_t(z) = 0,\ t \in (0,1)$ are in the region

$$\mathcal{D}_R = \{z \in \mathbb{C} : Re(z) \geq 0,\ |z| \leq R\}. \tag{4.19}$$

On the other hand, as $G'(1) < 0$, there exists $\kappa_1 \in \mathbb{R},\ \kappa_1 > 1$ (In fact, we can take κ_1 arbitrarily close to 1), such that $f_t(\kappa_1) = Re(f_t(\kappa_1)) < 0$. For $t = 0, 1,\ \nu \in \mathbb{R}$ we have

$$\begin{aligned} e^{Re(f_t(\kappa_1 + i\nu))} &= \left|e^{f_t(\kappa_1+i\nu)}\right| \\ &= \left|\mathbb{E}\left(e^{(\kappa_1+i\nu)X_1 - \bar{r}}\right)\right| \\ &\leq \mathbb{E}\left(e^{(\kappa_1)X_1 - \bar{r}}\right) \\ &= e^{Re(f_t(\kappa_1))} \\ &< 1. \end{aligned}$$

Hence, we have $Re(f_t(\kappa_1 + i\nu)) < 0\ \ \forall \nu \in \mathbb{R}$. This gives the boundary strip

$$\mathcal{C}_+ = \{z \in \mathbb{C} : |z| = R,\ Re(z) \geq \kappa_1\} \cup \{z \in \mathbb{C} : Re(z) = \kappa_1,\ -R \leq Im(z) \leq R\}. \tag{4.20}$$

By the continuity of $f_t(z)$ and the Argument Principle, we deduce that

$$n_t = \frac{1}{2\pi i} \oint_{\mathcal{C}_+} \frac{f_t'(z)}{f_t(z)} dz \tag{4.21}$$

is integer valued and continuous over $t \in [0,1]$. Hence $n_0 = n_1$, i.e. $f_1(z)$ have $m+1$ zeros in $\mathcal{D}_+$. This completes the proof of the first part of the statement.

To show the second part of the statement, we repeat the above arguments with the following boundary strip,

$$\mathcal{C}_- = \{z \in \mathbb{C} : |z| = R,\ Re(z) \le \kappa_2\} \cup \{z \in \mathbb{C} : Re(z) = \kappa_2,\ -R \le Im(z) \le R\}, \tag{4.22}$$

where $\kappa_2 \in \mathbb{R}$ and $\kappa_2 < \max_i\{-\theta_i\}$ is chosen arbitrarily close to $\max_i\{-\theta_i\}$.

□

According to Theorem 3.2, if there are no multiple roots with positive real part in the C-L equation (3.4), then the solution is of the form

$$V_b(x) = \begin{cases} \sum_{j=1}^{m+1} \omega_j e^{\beta_j x} & x < b \\ e^x - q & x \ge b \end{cases}. \tag{4.23}$$

Using this solution form, we compute that

$$\begin{aligned} \mathcal{L}V_b(x) - \tilde{r}V_b(x) &= \lambda p \boldsymbol{\alpha}^+ \Big[(\mathbf{Q}^+)^T e^{\boldsymbol{\Lambda}^+(b-x)} \Big(\left(-\boldsymbol{\Lambda}^+ - \mathbf{I}\right)^{-1} e^{b\mathbf{I}} + q{\boldsymbol{\Lambda}^+}^{-1} \\ &\quad - \sum_{j=1}^{m+1} \omega_j \left(-\boldsymbol{\Lambda}^+ - \beta_j \mathbf{I}\right)^{-1} e^{\beta_j b \mathbf{I}} \Big) \left(-\boldsymbol{\Lambda}^+\right) \mathbf{Q}^+ \Big] \mathbf{1}, \end{aligned} \tag{4.24}$$

which is equal to 0 on $(-\infty, b)$. Hence we obtain the system of linear equations

$$\begin{cases} \sum_{j=1}^{m+1} \omega_j \frac{\eta_i}{\eta_i-\beta_j} e^{-(\eta_i-\beta_j)b} &= \frac{\eta_i}{\eta_i-1} e^{-(\eta_i-1)b} - q e^{-\eta_i b} \quad \text{for } i=1,\dots,m \\ \sum_{j=1}^{m+1} \omega_j e^{\beta_j b} &= e^b - q \end{cases}, \tag{4.25}$$

which is the same as the linear system (4.7) in the hyperexponential case. Therefore, the coefficients are given by

$$\omega_j = \frac{\sum_{i=1}^{m+1} \left(R_i \prod_{k=1}^{m+1} (\eta_i - \beta_k) \prod_{l=1, l\neq i}^{m+1} \frac{\beta_j - \eta_l}{\eta_i - \eta_l} \right)}{\beta_j e^{\beta_j b} \prod_{k=1, k\neq j}^{m+1} (\beta_j - \beta_k)}, \tag{4.26}$$

and the optimal exercise boundary is given by

$$b^* = \ln \left(q \prod_{k=1}^{m+1} \frac{\beta_k}{\beta_k - 1} \prod_{l=1}^{m} \frac{\eta_l - 1}{\eta_l} \right). \tag{4.27}$$

4.3 The case for $G'(1) \geq 0$

By substituting $\beta_1 = 1$ into (4.27), we get $b^* = \infty$. Again, by setting $\beta_1 = 1$ in (4.26), we observe that

$$\omega_j \to 0 \text{ for } j \neq 1$$

and

$$\omega_1 \to \frac{\sum_{k=1}^{m+1} \prod_{k=2}^{m+1} (\beta_k - \eta_i) \prod_{l=1, l\neq i}^{m+1} \left(\frac{\eta_l - 1}{\eta_l - \eta_i} \right)}{\prod_{k=2}^{m+1} (\beta_k - 1)} = 1$$

as $b \to \infty$, where the last equality is a result of Lemma 4.1 when $j = 1$. Noting (4.23), we have

$$V(x) \geq \sup_{b \geq \max\{\ln q, \ln S\}} V_b(x) = e^x.$$

On the other hand, we know that $V(x) \leq e^x$ from Lemma 3.1. As a result, we have $V(x) = e^x$.

Chapter 5

Future Research Direction

This chapter discusses the possible generalization of the stock loan problem to stochastic volatility. We adopt the approach in Fouque et al. (2003) and consider a fast mean-reverting stochastic volatility model. The stock loan pricing formula is derived in the form of asymptotic expansion.

5.1 The fast mean-reverting stochastic volatility model

Consider a pair of process $(S_t^\varepsilon, Y_t^\varepsilon)$ which satisfies

$$
\begin{aligned}
dS_t^\varepsilon &= rS_t^\varepsilon dt + f(Y_t^\varepsilon)S_t^\varepsilon dW_t, & (5.1)\\
dY_t^\varepsilon &= \left[\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dZ_t, & (5.2)
\end{aligned}
$$

where S_t^ε is the stock price process, $f(Y_t^\varepsilon)$ is a positive valued function representing the volatility, Y_t^ε is a Ornstein-Uhlenbeck (OU) process with mean reverting speed $\frac{1}{\varepsilon}$, $\varepsilon > 0$ is a small parameter, (W_t, Z_t) are Brownian motions with correlation $\rho \in (-1, 1)$ and

$$\Lambda(y) = \frac{\rho(\mu - r)}{f(y)} + c(y)\sqrt{1-\rho^2} \tag{5.3}$$

is the market price of risk. Let $X_t^\varepsilon = \ln\left(e^{-\gamma t} S_t^\varepsilon\right)$. Itō's formula gives

$$dX_t^\varepsilon = \left(r - \gamma - \frac{f(Y_t^\varepsilon)^2}{2}\right) dt + f(Y_t^\varepsilon) dW_t. \tag{5.4}$$

Although we do not introduce jumps in (5.1) and (5.2), the method used in this chapter is possible to generalize to phase-type Lévy process with stochastic volatility in the future.

5.2 Asymptotic expansion of stock loan

We are interested in the stock loan on S_t^ε defined in (5.1) and (5.2). As a starting point, we use $V^\varepsilon(x,y)$ to denote the price of the perpetual American option corresponding to the stock loan (see (2.16)), i.e.

$$V^\varepsilon(x,y) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-(r-\gamma)\tau}\left(e^{X_\tau^\varepsilon} - q\right)^+ I_{\{\tau<\infty\}} | X_0^\varepsilon = x, Y_0^\varepsilon = y\right]. \tag{5.5}$$

Using similar arguments presented in previous chapters, $V^\varepsilon(x,y)$ is known to be a solution to the partial differential equation

$$\begin{cases} \mathcal{L}^\varepsilon V^\varepsilon(x,y) = 0 & \text{for } x < b^\varepsilon(y) \\ V^\varepsilon(b^\varepsilon(y), y) = e^{b^\varepsilon(y)} - q & \text{for } x \geq b^\varepsilon(y) \end{cases}, \tag{5.6}$$

where $b^\varepsilon(y)$ is the optimal exercise boundary,

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2, \tag{5.7}$$

with

$$\begin{aligned}
\mathcal{L}_0 &= (m-y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2}; && (5.8)\\
\mathcal{L}_1 &= \sqrt{2}\nu\rho f(y)\frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}; && (5.9)\\
\mathcal{L}_2 &= \frac{1}{2}f(y)^2\frac{\partial^2}{\partial x^2} + \left(\tilde{r} - \frac{f(y)^2}{2}\right)\frac{\partial}{\partial x} - \tilde{r}\,\cdot\,. && (5.10)
\end{aligned}$$

Note that the operator $\mathcal{L}_0$ is the infinitesimal generator of the OU process Y_t defined by

$$dY_t = (m - Y_t)dt + \sqrt{2}\nu dZ_t, \tag{5.11}$$

which has the invariant distribution $\mathcal{N}(m, \nu^2)$.

Consider the following asymptotic expansions for $V^\varepsilon(x,y)$ and $b^\varepsilon(y)$:

$$\begin{aligned}
V^\varepsilon(x,y) &= V_0(x,y) + \sqrt{\varepsilon}V_1(x,y) + \varepsilon V_2(x,y) + \varepsilon^{\frac{3}{2}}V_3(x,y) + \dots\ , && (5.12)\\
b^\varepsilon(y) &= b_0(y) + \sqrt{\varepsilon}b_1(y) + \varepsilon b_2(y) + \varepsilon^{\frac{3}{2}}b_3(y) + \dots\ . && (5.13)
\end{aligned}$$

We aim to compute the first two leading order terms of the above expansions. i.e.

$$V_0(x,y) + \sqrt{\varepsilon}V_1(x,y) \tag{5.14}$$

and

$$b_0(y) + \sqrt{\varepsilon}b_1(y). \tag{5.15}$$

Substituting (5.12) into (5.6) gives

$$
\begin{aligned}
\mathcal{L}^{\varepsilon} V^{\varepsilon} &= \frac{1}{\varepsilon}\mathcal{L}_0 V_0 + \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_1 V_0 + \mathcal{L}_0 V_1\right) + \left(\mathcal{L}_2 V_0 + \mathcal{L}_1 V_1 + \mathcal{L}_0 V_2\right) \\
&\quad + \sqrt{\varepsilon}\left(\mathcal{L}_2 V_1 + \mathcal{L}_1 V_2 + \mathcal{L}_0 V_3\right) + o(\sqrt{\varepsilon}) \\
&= 0. \qquad (5.16)
\end{aligned}
$$

This implies all the terms of the expansion in (5.16) should be equal to zero.

We use $\langle\cdot\rangle$ to denote the expectation with respect to the invariant distribution $\mathcal{N}(m, \nu^2)$:

$$
\langle h \rangle = \frac{1}{\nu\sqrt{2\pi}} \int_{-\infty}^{\infty} h(y) e^{-\frac{(y-m)^2}{2\nu^2}} dy. \qquad (5.17)
$$

In the following analysis, we have to solve the following Poisson equation:

$$
\mathcal{L}_0 g + h = 0. \qquad (5.18)
$$

In order to admit a solution $g(\cdot)$ with reasonable growth towards infinity, the equation requires the following Fredholm solvability condition

$$
\langle h \rangle = 0. \qquad (5.19)
$$

5.2.1 The zeroth order term

Consider the zeroth order term in (5.16)

$$
\mathcal{L}_0 V_0 = 0. \qquad (5.20)
$$

As $\mathcal{L}_0$ is a differential operator with respect to y, (5.20) implies that $V_0(x, y)$ is independent of y.

For the first order term in (5.16)

$$\mathcal{L}_1 V_0 + \mathcal{L}_0 V_1 = 0, \tag{5.21}$$

since V_0 is independent of y, the equation is reduced to

$$\mathcal{L}_0 V_1 = 0. \tag{5.22}$$

This implies, again, that $V_1(x, y)$ is independent of y.

For the second order term in (5.16):

$$\mathcal{L}_2 V_0 + \mathcal{L}_1 V_1 + \mathcal{L}_0 V_2 = 0, \tag{5.23}$$

because $\mathcal{L}_1 V_1 = 0$, (5.23) is reduced to the Poisson equation in V_2

$$\mathcal{L}_0 V_2 + \mathcal{L}_2 V_0 = 0. \tag{5.24}$$

The solvability condition implies

$$\langle \mathcal{L}_2 V_0 \rangle = \langle \mathcal{L}_2 \rangle V_0 = 0, \tag{5.25}$$

where $\langle \mathcal{L}_2 \rangle$ is the operator $\mathcal{L}_2$ with $f(y)^2$ replaced by $\bar{\sigma}^2 = \langle f^2 \rangle$, i.e.

$$\langle \mathcal{L}_2 \rangle V_0 = \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 V_0}{\partial x^2} + \left(\tilde{r} - \frac{\bar{\sigma}^2}{2} \right) \frac{\partial V_0}{\partial x} - \tilde{r} V_0 = 0.$$

Recall the expansion of the optimal exercise boundary:

$$b^{\varepsilon}(y) = b_0 + \sqrt{\varepsilon} b_1(y) + o(\sqrt{\varepsilon}). \tag{5.26}$$

Expanding both sides of the boundary condition in (5.6) according to the exercise boundary gives

$$\begin{aligned} V^\varepsilon(b^\varepsilon(y), y) &= V_0(b_0, y) + \sqrt{\varepsilon}\left(V_1(b_0 + \sqrt{\varepsilon} b_1(y), y) + b_1(y)\frac{\partial V_0}{\partial x}(b_0, y)\right) \\ &\quad + o(\sqrt{\varepsilon}); \qquad (5.27) \\ e^{b^\varepsilon(y)} - q &= e^{b_0} - q + \sqrt{\varepsilon} b_1(y) e^{b_0} + o(\sqrt{\varepsilon}). \qquad (5.28) \end{aligned}$$

Equating the zeroth order terms, we have

$$V_0(b_0, y) = e^{b_0} - q. \qquad (5.29)$$

This suggests that V_0 is the solution under a constant volatility model. The solution is given in Xia and Zhou (2007):

- If $-2\tilde{r}/\bar{\sigma}^2 > 1$,

$$V_0(x) = \begin{cases} \frac{(\beta-1)^{\beta-1}}{\beta^\beta} q^{1-\beta} e^{\beta x} & \text{for } x < b_0 \\ e^x - q & \text{for } x \geq b_0 \end{cases}. \qquad (5.30)$$

  where $\beta = -\frac{2\tilde{r}}{\bar{\sigma}^2}$, $b_0 = \ln(\frac{\beta q}{\beta - 1})$.

- If $-2\tilde{r}/\bar{\sigma}^2 \leq 1$,

$$V_0(x) = e^x. \qquad (5.31)$$

For the purpose of illustrating asymptotic expansion, we focus on the more interesting case where $-2\tilde{r}/\bar{\sigma}^2 > 1$.

5.2.2 The first order term

The solution of the Poisson equation (5.24) can be written as

$$V_2 = -\mathcal{L}_0^{-1} \left(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle \right) V_0. \tag{5.32}$$

On the other hand, the third order term in (5.16) gives

$$\mathcal{L}_2 V_1 + \mathcal{L}_1 V_2 + \mathcal{L}_0 V_3 = 0, \tag{5.33}$$

which is a Poisson equation in V_3. Solvability condition implies that

$$\begin{aligned} \langle \mathcal{L}_2 \rangle V_1 &= -\langle \mathcal{L}_1 V_2 \rangle \\ &= \left\langle \mathcal{L}_1 \mathcal{L}_0^{-1} \left(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle \right) \right\rangle V_0 \\ &= \left\langle \mathcal{L}_1 \mathcal{L}_0^{-1} \left(f(y)^2 - \langle f^2 \rangle \right) \right\rangle \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) V_0 \\ &= \left(v_2 \frac{\partial^3}{\partial x^3} + (v_1 - 3v_2) \frac{\partial^2}{\partial x^2} + (2v_2 - v_1) \frac{\partial}{\partial x} \right) V_0, \end{aligned} \tag{5.34}$$

where

$$v_1 = \frac{\nu}{\sqrt{2}} (2\rho \langle f \phi' \rangle - \langle \Lambda \phi' \rangle ; \tag{5.35}$$

$$v_2 = \frac{\rho \nu}{\sqrt{2}} \langle f \phi' \rangle , \tag{5.36}$$

and $\phi(y)$ is a solution to the Poisson following equation

$$\mathcal{L}_0 \phi(y) = f(y)^2 - \langle f^2 \rangle . \tag{5.37}$$

The boundary and smooth fit condition (cf. Villeneuve, 2007) of V^ε are

$$V^\varepsilon(b^\varepsilon(y), y) = e^{b^\varepsilon(y)} - q \tag{5.38}$$

and

$$\left.\frac{\partial V^\varepsilon}{\partial x}\right|_{(x,y)=(b^\varepsilon(y),y)} = e^{b^\varepsilon(y)} \tag{5.39}$$

respectively. Expanding both sides of the boundary condition in (5.6) according to the exercise boundary gives

$$\begin{aligned} V^\varepsilon(b^\varepsilon(y),y) &= V_0(b_0,y)+\sqrt{\varepsilon}\left(V_1(b_0+\sqrt{\varepsilon}b_1(y),y)+b_1(y)\frac{\partial V_0}{\partial x}(b_0,y)\right) \\ &\quad +o(\sqrt{\varepsilon}); \\ e^{b^\varepsilon(y)}-q &= e^{b_0}-q+\sqrt{\varepsilon}b_1(y)e^{b_0}+o(\sqrt{\varepsilon}). \end{aligned}$$

Equating the terms in $\sqrt{\varepsilon}$ order, we get

$$V_1(b_0+\sqrt{\varepsilon}b_1(y),y)+b_1(y)\frac{\partial V_0}{\partial x}(b_0,y) = b_1(y)e^{b_0}, \tag{5.40}$$

which implies

$$V_1(b_0+\sqrt{\varepsilon}b_1(y),y) = 0. \tag{5.41}$$

Expanding both sides of the smooth fit condition according to the exercise boundary gives

$$\begin{aligned} &\frac{\partial V^\varepsilon}{\partial x}(b^\varepsilon(y),y) \\ &= \begin{cases} \frac{\partial V_0}{\partial x}(b_0,y)+\sqrt{\varepsilon}\left(\frac{\partial V_1}{\partial x}(b_0+\sqrt{\varepsilon}b_1(y),y)+b_1(y)\frac{\partial^2 V_0}{\partial x^2}(b_0+,y)\right)+o(\sqrt{\varepsilon}) & \text{if } b_1(y)>0 \\ \frac{\partial V_0}{\partial x}(b_0,y)+\sqrt{\varepsilon}\left(\frac{\partial V_1}{\partial x}(b_0+\sqrt{\varepsilon}b_1(y),y)+b_1(y)\frac{\partial^2 V_0}{\partial x^2}(b_0-,y)\right)+o(\sqrt{\varepsilon}) & \text{if } b_1(y)\le 0 \end{cases} \end{aligned} \tag{5.42}$$

and

$$e^{b^{\varepsilon}(y)} = e^{b_0} + \sqrt{\varepsilon} b_1(y) e^{b_0} + o(\sqrt{\varepsilon}). \tag{5.43}$$

Collecting terms of $\mathcal{O}(\sqrt{\varepsilon})$,

$$\frac{\partial V_1}{\partial x}(b_0 + \sqrt{\varepsilon} b_1(y), y) = b_1(y) e^{b_0}. \tag{5.44}$$

Summarizing all these, we obtain the PDE of V_1:

$$\begin{cases} \langle \mathcal{L}_2 \rangle V_1 & = \left(v_2 \frac{\partial^3}{\partial x^3} + (v_1 - 3v_2) \frac{\partial^2}{\partial x^2} + (2v_2 - v_1) \frac{\partial}{\partial x} \right) V_0, \quad \text{for } x < b_0 + \sqrt{\varepsilon} b_1(y); \\ V_1(b_0 + \sqrt{\varepsilon} b_1(y), y) & = 0; \\ \frac{\partial V_1}{\partial x}(b_0 + \sqrt{\varepsilon} b_1(y), y) & = \begin{cases} b_1(y) e^{b_0} - b_1(y) \frac{\partial^2 V_0}{\partial x^2}(b_0+, y) & \text{if } b_1(y) > 0; \\ b_1(y) e^{b_0} - b_1(y) \frac{\partial^2 V_0}{\partial x^2}(b_0-, y) & \text{if } b_1(y) \leq 0. \end{cases} \end{cases} \tag{5.45}$$

We solve this PDE by dividing it into two cases.

Case 1: $b_1(y) < 0$. For $x < b_0 + \sqrt{\varepsilon} b_1(y)$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle V_1 &= \left(v_2 \frac{\partial^3}{\partial x^3} + (v_1 - 3v_2) \frac{\partial^2}{\partial x^2} + (2v_2 - v_1) \frac{\partial}{\partial x} \right) V_0 \\ &= a_1 \beta \left(v_2 \beta^2 + (v_1 - 3v_2) \beta + (2v_2 - v_1) \right) e^{\beta x}, \end{aligned} \tag{5.46}$$

where $a_1 = \frac{(\beta-1)^{\beta-1}}{\beta^\beta} q^{1-\beta}$. To construct a particular solution for V_1, consider the solution form

$$V_1^p(x) = c_1 x e^{\beta x}. \tag{5.47}$$

By substituting this into the left hand side of (5.46), we get

$$\begin{aligned}
&\frac{1}{2}\bar{\sigma}^2\frac{\partial^2 V_1^p}{\partial x^2} + \left(\tilde{r} - \frac{\bar{\sigma}^2}{2}\right)\frac{\partial V_1^p}{\partial x} - \tilde{r}V_1^p \\
= & \frac{1}{2}\bar{\sigma}^2\left(2c_1\beta e^{\beta x} + c_1\beta^2 x e^{\beta x}\right) + \left(\tilde{r} - \frac{\bar{\sigma}^2}{2}\right)\left(c_1 e^{\beta x} + c_1\beta x e^{\beta x}\right) - \tilde{r}c_1 x e^{\beta x} \\
= & c_1\bar{\sigma}^2\beta e^{\beta x} + c_1\left(\tilde{r} - \frac{\bar{\sigma}^2}{2}\right)e^{\beta x},
\end{aligned}$$

where the last equality holds with

$$\frac{1}{2}\bar{\sigma}^2\beta^2 + \left(\tilde{r} - \frac{\bar{\sigma}^2}{2}\right)\beta - \tilde{r} = 0.$$

This implies

$$c_1 = \frac{a_1\beta\left(v_2\beta^2 + (v_1 - 3v_2)\beta + (2v_2 - v_1)\right)}{\bar{\sigma}^2\beta + \left(\tilde{r} - \frac{\bar{\sigma}^2}{2}\right)}. \tag{5.48}$$

It is clear that the homogeneous solution is of the form

$$V_1^h(x) = c_2 e^{\beta x} + c_3 e^{x}. \tag{5.49}$$

We claim that $c_3 = 0$. To see this, define

$$E^\varepsilon(t,x,y) = \mathbb{E}\left[e^{-\tilde{r}(T-t)}\left(S_T^\varepsilon\right)^\kappa | S_t^\varepsilon = x, Y_t^\varepsilon = y\right] \tag{5.50}$$

and consider the following expansion

$$E^\varepsilon(t,x,y) = E_0(t,x,y) + \sqrt{\varepsilon}E_1(t,x,y) + o(\sqrt{\varepsilon}). \tag{5.51}$$

As argued in Zhang and Zhou (2009), if $c_3 \neq 0$, we should have

$$E_0(t,x,y) + \sqrt{\varepsilon}E_1(t,x,y) \to 0 \text{ as } T \to \infty \tag{5.52}$$

for $\kappa = 1$. Following a similar analysis for $V^\varepsilon(x,y)$, we know that E_0 is the expectation evaluated with Black-Scholes model and this is solved in Zhang and Zhou (2009) that

$$E_0(t,x,y) = e^{\kappa x + (\kappa-1)(\kappa-\beta)\frac{\bar{\sigma}^2(T-t)}{2}}. \tag{5.53}$$

E_1 is given by

$$\begin{aligned} E_1(t,x,y) &= -(T-t)\left(v_2\frac{\partial^3}{\partial x^3} + (v_1 - 3v_2)\frac{\partial^2}{\partial x^2} + (2v_2 - v_1)\frac{\partial}{\partial x}\right)E_0 \qquad (5.54) \\ &= -(T-t)\kappa\left(v_2\kappa^2 + (v_1 - 3v_2)\kappa + (2v_2 - v_1)\right)e^{\kappa x + (\kappa-1)(\kappa-\beta)\frac{\bar{\sigma}^2(T-t)}{2}}. \end{aligned}$$

We refer to Fouque et al. (2003) for details. It is now easy to see that

$$E_0(t,x,y) + \sqrt{\varepsilon}E_1(t,x,y) \to 0 \text{ as } T \to \infty$$

does not hold for $\kappa = 1$. This implies $c_3 = 0$ and proves the claim.

For $x < b_0 + \sqrt{\varepsilon}b_1(y)$, a general solution of V_1 is the sum of the homogeneous solution and the particular solution:

$$V_1(x) = c_1 x e^{\beta x} + c_2 e^{\beta x}. \tag{5.55}$$

Substituting this into the boundary condition of (5.45) yields

$$0 = V_1(b_0 + b_1(y)\sqrt{\varepsilon}) = c_1(b_0 + b_1(y)\sqrt{\varepsilon})e^{\beta(b_0+b_1(y)\sqrt{\varepsilon})} + c_2 e^{\beta(b_0+b_1(y)\sqrt{\varepsilon})}, \tag{5.56}$$

and

$$c_2 = -c_1(b_0 + b_1(y)\sqrt{\varepsilon}). \tag{5.57}$$

Evaluating both sides using the smooth fit condition in (5.45) gives

$$\begin{aligned}
\frac{\partial V_1}{\partial x}(b_0 + \sqrt{\varepsilon}b_1(y), y) &= c_1 e^{\beta(b_0+b_1(y)\sqrt{\varepsilon})} + \left[c_1(b_0 + b_1(y)\sqrt{\varepsilon}) + c_2\right] \beta e^{\beta(b_0+b_1(y)\sqrt{\varepsilon})} \\
&= c_1 e^{\beta(b_0+b_1(y)\sqrt{\varepsilon})} \\
&= c_1 e^{\beta b_0}(1 + b_1(y)\beta\sqrt{\varepsilon}) + o(\sqrt{\varepsilon})
\end{aligned}$$

and

$$b_1(y)e^{b_0} - b_1(y)\frac{\partial^2 V_0}{\partial x^2}(b_0-, y) = b_1(y)(e^{b_0} - a_1\beta^2 e^{\beta b_0}) = b_1(y)(1-\beta)e^{b_0}.$$

Neglecting the $o(\sqrt{\varepsilon})$ terms and equating both sides, we have

$$b_1(y) = \frac{c_1 e^{\beta b_0}}{(1-\beta)e^{b_0} - c_1\sqrt{\varepsilon}e^{\beta b_0}} \tag{5.58}$$

which is independent of y. To summarize,

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} + c_2 e^{\beta x} & \text{for } x < b_0 + \sqrt{\varepsilon}b_1 \\ 0 & \text{for } x \geq b_0 + \sqrt{\varepsilon}b_1 \end{cases}, \tag{5.59}$$

where c_1 is given in (5.48), c_2 in (5.57) and b_1 in (5.58).

Case 2: $b_1(y) \geq 0$. For $x < b_0$

$$\langle \mathcal{L}_2 \rangle V_1 = a_1\beta\left(v_2\beta^2 + (v_1 - 3v_2)\beta + (2v_2 - v_1)\right)e^{\beta x}, \tag{5.60}$$

and for $b_0 \le x < b_0 + \sqrt{\varepsilon} b_1(y)$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle V_1 &= \left(v_2 \frac{\partial^3}{\partial x^3} + (v_1 - 3v_2) \frac{\partial^2}{\partial x^2} + (2v_2 - v_1) \frac{\partial}{\partial x} \right) V_0 \\ &= 0. \end{aligned} \tag{5.61}$$

By similar arguments in the previous case, we can write

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} + \hat{c}_2 e^{\beta x} & \text{for } x < b_0 \\ \hat{d}_2 e^{\beta x} & \text{for } b_0 \le x < b_0 + \sqrt{\varepsilon} b_1(y) \\ 0 & \text{for } x \ge b_0 + \sqrt{\varepsilon} b_1(y) \end{cases} . \tag{5.62}$$

Continuity at $b_0 + \sqrt{\varepsilon} b_1(y)$ implies $\hat{d}_2 = 0$. Continuity at b_0 implies

$$c_1 b_0 e^{\beta b_0} + \hat{c}_2 e^{\beta b_0} = 0,$$

or

$$\hat{c}_2 = -c_1 b_0. \tag{5.63}$$

Since the optimal exercise boundary does not change in this case, we set $b_1 = 0$. Then,

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} - c_1 b_0 e^{\beta x} & \text{for } x < b_0 \\ 0 & \text{for } x \ge b_0 \end{cases} . \tag{5.64}$$

In summary, if

$$\frac{c_1 e^{\beta b_0}}{(1-\beta) e^{b_0} - c_1 \sqrt{\varepsilon} e^{\beta b_0}} < 0, \tag{5.65}$$

then

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} + c_2 e^{\beta x} & \text{for } x < b_0 + \sqrt{\varepsilon} b_1(y) \\ 0 & \text{for } x \geq b_0 + \sqrt{\varepsilon} b_1(y) \end{cases},$$

where c_1 is given in (5.48), c_2 in (5.57) and b_1 in (5.58). Otherwise,

$$V_1(x) = \begin{cases} c_1 x e^{\beta x} - c_1 b_0 e^{\beta x} & \text{for } x < b_0 \\ 0 & \text{for } x \geq b_0 \end{cases},$$

where c_1 is given in (5.48), $b_1 = 0$.

Chapter 6

Conclusion

This thesis provides a theoretical treatment of stock loans valuation under exponential phase-type Lévy models. Using the variational inequality approach, we characterized the value function of a stock loan under general exponential phase-type Lévy models and derived an explicit solution of the stock loan value and optimal exercise policy for a fairly general class of phase-type jump diffusion models. We emphasis again that our result could be applied to approximate the corresponding price under a general exponential Lévy model arbitrarily close.

We also discussed a possible extension to stochastic volatility model for stock loans. We adopted a fast mean-reverting stochastic volatility model and analyzed the price behavior using the technique of asymptotic expansion. A possible future research direction is to prove the order of convergence of this approximation and to combine the phase-type Lévy model with the stochastic volatility asymptotic analysis.

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