Two Nonlinear Output Regulation Problems

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摘 要

上世纪 90 年代以来，非线性输出调节问题就一直是最活跃的控制问题之一。简单地说，输出调节问题就是设计一个反馈控制器来让闭环系统稳定，并且让输出在有一组扰动信号的情况下，渐进地跟踪一组参考信号。参考信号和干扰信号都是由一组命名为外部系统的常微分方程产生。基于黄和陈提出来的将输出调节问题转化为镇定性问题的体系结构[28]，我们在此论文中考虑了两个问题：用输出反馈解决一类非线性标准型系统的半全局鲁棒输出调节问题，用输出调节方法解决转动-移动激励器系统的抗干扰问题和鲁棒抗干扰问题。

此论文主要由两个部分组成。第一部分是关于半全局鲁棒输出调节问题。我们在解决这个问题的过程中主要克服了三个难题。首先，当输出调节问题被转化成为包括原系统和内模的增广系统的镇定性问题后，这个增广系统的镇定性问题不能套用任何已有的镇定性结果直接解决。通过利用 Lyapunov 分析和 Teel, Praly 两人提出的半全局回溯技术[53]，我们克服了这个难题。其次，通过引入陈和黄提出的非线性内模[8]，我们去除了调节方程的解必须是多项式的限制。最后，采用 Khalil 和 Esfandiari 提出的高增益观察器[41]，我们设计出了一个输出反馈控制器。

此论文的第二部分是用测量输出反馈控制解决转动-移动激励器系统的抗干扰问题和鲁棒抗干扰问题。

（1） 抗干扰问题。众所周知的是，转动-移动激励器系统是一个有着非双曲零动态方程的非最小相位非线性系统。基于黄的处理有着非双曲零动态方程的非最小相位非线性系统的工作[20][24]，以及求解转动-移动激励器系统的调节方程的解的工作[27]，我们设计控制器解决了转动-传动激励器系统的抗干扰问题。

（2） 鲁棒抗干扰问题。我们用基于鲁棒输出调节方法解决了这个问题。在这个过程中主要克服了两个难题：第一个是设计非线性内模来包含非多项式的非线性，第二个是用黄提出的参数优化技术[22]优化控制参数以得到令人满意的瞬态响应。

Abstract

The nonlinear output regulation problem has been one of the most active control problems since 1990s. Briefly, output regulation is to design a feedback control law for a plant, such that the closed-loop system is internally stable, and the output asymptotically tracks a class of reference inputs in the presence of a class of disturbances. Both the reference inputs and disturbances are generated by an autonomous differential equation called exosystem. Based on the existing framework proposed by Huang and Chen [28], which translated the robust output regulation problem into a robust stabilization problem, we considered two problems in this thesis: the semiglobal robust output regulation problem for a class of nonlinear systems in normal form via output feedback control; and, the disturbance rejection and the robust disturbance rejection problem for the Rotational / translational Actuator(RTAC) system by output regulation method.

This thesis mainly consists of two parts. The first part is about the semiglobal robust output regulation problem. We solved this problem by overcoming three difficulties. First, the output regulation problem can be translated into a stabilization problem of an augmented systems composed of the original plant and the internal model. But the stabilization problem of the augmented system cannot be treated directly by any existing stabilization result. Using the Lyapunov's direct method and the semiglobal backstepping technique by Teel and Praly [55], we have solved it. Second, we have eliminated the polynomial assumption imposed on the solution of the regulator equations by taking advantage of the nonlinear internal model by Chen and Huang [8]. Third, we obtain an output feedback controller by making use of the high gain observer by Khalil and Esfandiari [43].

The second part is about the disturbance rejection and the robust disturbance rejection problem of the RTAC system by the measurement output feedback control.

i: *Disturbance rejection.* It is well known that the RTAC system is a nonminimum phase nonlinear system with nonhyperbolic zero dynamics. Based on the work handling nonminimum phase systems with nonhyperbolic zero dynamics by Huang [20], [24], and the work of solving the regulator equations of the RTAC systems by Huang [27], we get a design to solve the disturbance rejection of the RTAC system.

ii: *Robust disturbance rejection.* We have obtained a design based on the robust output regulation method by overcoming two major obstacles. First, devise a nonlinear internal model to account for non-polynomial nonlinearities. Second, use the parameter optimization technique by Huang [22] to get more desirable transient response.

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Contents

Chapter 1

Introduction

More and more researchers and designers are getting interested in nonlinear control in many areas, such as process control, robotics, aircrafts control, biomedical engineering, etc. Generally, physical systems inherently contain nonlinearities, some of which do not allow linear approximation, such as saturation, dead-zones, backlash, etc. In these situations, linear control based on linear approximation cannot compensate for these nonlinearities. In some other situations that the linearization is applicable, the linear control cannot get good enough transient performance or large enough range of operation to satisfy practical requirements. Also, the traditional linear systems are founded on the superposition principle, but this principle does not apply to nonlinear systems. Usually, different classes of nonlinear systems require different control techniques. Hence, there exist great possibilities in the research of nonlinear systems in the future.

In this thesis we will address the semiglobal robust output regulation problem, and further investigate a benchmark nonlinear control problem (robust disturbance rejection of the Rotational-Translational Actuator(RTAC) system) by the output regulation method which can be regarded as a local robust output regulation problem. In this chapter, let us give some introduction about the background of nonlinear control, output regulation, semiglobal stabilization and the benchmark problem which are preliminary knowledge of the problems to be solved.

This chapter is organized as follows. In Section 1.1, introduction about the nonlinear control systems is given. In Section 1.2, the development of output regulation and its recent research directions are given. In Section 1.3, some progress on the research topic of semiglobal stabilization is given, since the solvability of the semiglobal robust output regulation problem to be considered is finally converted into the solvability of a semiglobal robust stabilization problem. Section 1.4 gives an introduction to a benchmark nonlinear cotnrol problem, including description of the RTAC system, and the progress on the research of the disturbance rejection problem and the robust disturbance rejection problem of the RTAC system. Section 1.5 closes this chapter with the contribution of this thesis.

1.1 Nonlinear Control Systems

A general multivariable nonlinear control system is described by the following two equations

$$\begin{aligned} \dot{\xi} &= f(\xi, u) \\ y &= h(\xi, u) \end{aligned} \tag{1.1}$$

where $\xi \in R^n$ is the plant state, $u \in R^m$ the plant input, $y \in R^p$ the plant output, and $f : R^n \times R^m \to R^n$, $h : R^n \times R^m \to R^p$. The components of ξ, u, y, f, h are denoted, respectively, by

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix},$$

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad h = \begin{bmatrix} h_1 \\ \vdots \\ h_p \end{bmatrix}.$$

For many nonlinear control systems, the function $f(\xi, u)$ is linear in the input u, and the function $h(\xi, u)$ does not depend on the input u explicitly. In this case, we can write, with some abuse of the notation, $h(\xi, u) = h(\xi)$ and $f(\xi, u) = f(\xi) + g(\xi)u$ for some functions $f : R^n \to R^n$, $g : R^n \to R^{n \times m}$, and $h : R^n \to R^p$. Therefore, (1.1) can be further simplified as follows

$$\begin{aligned} \dot{\xi} &= f(\xi) + g(\xi)u \\ y &= h(\xi). \end{aligned} \tag{1.2}$$

We call (1.2) an *affine nonlinear control system*. Note that $g(\xi)$ can be expanded as $g(\xi) = [g_1(\xi), \cdots, \cdots, g_m(\xi)]$ where $g_i : R^n \to R^n$ for $i = 1, \cdots, m$.

The class of nonlinear control laws takes the following form

$$\begin{aligned} u &= k(\xi, z_c, y) \\ \dot{z}_c &= g(\xi, z_c, y), \quad z_c(0) = z_{c0} \end{aligned} \tag{1.3}$$

where $z_c \in R^{n_c}$ is the compensator state, and $k : R^n \times R^{n_c} \times R^p \to R^m$, $g : R^n \times R^{n_c} \times R^p \to R^{n_c}$. The control law (1.3) is a very general form of control law. It includes the static state feedback control law

$$u = k(\xi)$$

as a special case when z_c does not appear in the function k, and it includes the dynamic output feedback control law

$$\begin{aligned} u &= k(z_c, y) \\ \dot{z}_c &= g(z_c, y) \end{aligned}$$

as a special case when ξ does not appear in the functions k and g.

Under the above controller (1.3), the closed-loop system is like this,

$$\begin{aligned} \dot{x}_c &= f_c(x_c), \quad x_c(0) = x_{c0} \\ y &= h_c(x_c) \end{aligned}$$

where $x_c = \begin{bmatrix} \xi \\ z_c \end{bmatrix}$.

In the following, we will give a brief description about the normal form and the conditions under which the general nonlinear control systems can be transformed into this norm form.

Definition 1.1 For each $i = 1, \cdots, p$, the i^{th} output y_i of the system (1.2) is said to have a *relative degree* r_i at a point ξ_0 if

(i)

$$L_g L_f^k h_i(\xi) \stackrel{def}{=} \left[L_{g_1} L_f^k h_i(\xi), L_{g_2} L_f^k h_i(\xi), \cdots, L_{g_m} L_f^k h_i(\xi)\right] = 0_{1\times m} \tag{1.4}$$

for all $k < r_i - 1$, and for all ξ in an open neighborhood of ξ_0, where $L_f h_i(\xi) = \frac{\partial h_i}{\partial \xi} f(\xi)$, $L_f^k h_i(\xi) = \frac{\partial L_f^{k-1} h_i(\xi)}{\partial \xi} f(\xi)$,

(ii)

$$L_g L_f^{r_i - 1} h_i(\xi_0) \neq 0_{1\times m}. \tag{1.5}$$

The system (1.2) is said to have a *vector relative degree* $\{r_1, \cdots, r_p\}$ at a point ξ_0 if

(i) For all $1 \le i \le p$, the i^{th} output $h_i(\xi)$ has a relative degree r_i at ξ_0, and

(ii) The $p \times m$ matrix

$$D(\xi) = \begin{bmatrix} L_g L_f^{r_1-1} h_1(\xi) \\ L_g L_f^{r_2-1} h_2(\xi) \\ \vdots \\ L_g L_f^{r_p-1} h_p(\xi) \end{bmatrix} \tag{1.6}$$

has full row rank at $\xi = \xi_0$.

∎

If $r = r_1 + \cdots + r_p < n$, then there exists a diffeomorphic coordinate transformation

$$\begin{bmatrix} z \\ x \end{bmatrix} = T(\xi)$$

such that system (1.2) can be transformed to a *normal form* as follows,

$$\begin{aligned} \dot{z} &= \bar{f}_0(z, x) \\ \dot{x}_1^i &= x_2^i \\ &\cdots \\ \dot{x}_{r_i-1}^i &= x_{r_i}^i \\ \dot{x}_{r_i}^i &= \bar{f}_{r_i}(z, x) + \bar{g}(z, x)u \\ y_i &= x_1^i \end{aligned} \tag{1.7}$$

where $z \in R^{n-r}$, $x_j^i \in R$, $x^i = (x_1^i, \cdots, x_{r_i}^i)$, $x = (x^1, \cdots, x^p)$, $i = 1, \cdots, p$, $j = 1, \cdots, r_i$, and $\bar{f}_0 : R^n \to R^{n-r}$, $\bar{f}_{r_i} : R^n \to R$, $\bar{g} : R^n \to R^m$. For convenience, we will replace $\bar{f}_0$, $\bar{f}_r$ and $\bar{g}$ with f_0, f_r and g respectively in the following sections.

Let v be the q-dimensional exogenous signal representing the reference inputs and/or the disturbances, which can be generated by an *exosystem* of the form

$$\dot{v} = A_1 v, \ \ v(0) = v_0, \ \ t \leq 0.$$

And let $w \in R^N$ be the system uncertainties. Then the system (1.2) can be generalized into the following system

$$\begin{aligned} \dot{\xi} &= f(\xi, v, w) + g(\xi, v, w)u \\ y &= h(\xi, v, w). \end{aligned} \tag{1.8}$$

1.2 Output Regulation

Over the past decades, the *output regulation problem* (also known as *servomechanism problem*) has been one of most fundamental problems in control theory. Briefly, the output regulation problem is to design a control law for a plant, such that the closed-loop system is internally stable, and the output of the closed-loop system asymptotically tracks a class of reference inputs in the presence of a class of disturbances. Both the reference inputs and disturbances are generated by an autonomous differential equation called exosystem.

For the class of linear systems, this problem has been thoroughly studied in the 1970s by Davison [9], Francis [13], and Francis and Wonham [15]. The research in this period has generated the salient controller synthesis technique known as internal model principle. That is, any regulator solving the output regulation problem should incorporate an internal model of the exosystem. The internal model principle converts the linear output regulation problem into an eigenvalue placement problem for an augmented linear system.

For the class of nonlinear systems, the output regulation problem was first treated for the special case in which the exogenous signals are constant by Desoer and Lin [10], Francis and Wonham [15], and Huang and Rugh [31]. The same problem with time varying exogenous signals was first studied by Isidori and Byrnes [38], and Huang and Rugh [32], [33]. Particularly, Isidori and Byrnes [38] linked the solvability of the output regulation problem to that of the regulator equations, which pushed the research of output regulation to a new stage.

Since the plant inevitably contains some type of uncertainties, it is desirable to further require the controller be able to maintain the property of asymptotic tracking and disturbance rejection in the closed-loop system regardless of the uncertainties. The problem of designing such controllers for the plant is called *robust output regulation problem*. The nonlinear robust output regulation problem was studied by quite a few people, such as Huang and Lin [29], [30], Huang [19], [21], Byrnes, et al [1], Delli Priscoli [50] and Khalil [39]. Various solvability conditions have been given which impose assumptions on the solution of the regulator equations. In particular, Huang and Lin [29], [30] found that, in the presence of small parameter uncertainties and when the exogenous input is time varying, the solution of the output regulation problem requires the internal model be not only able to generate inputs corresponding to the trajectories of the exosystem, but also a number of their higher order nonlinear deformations.

At the beginning of the research on robust output regulation, only local asymptotic stability of the closed-loop system is guaranteed, and the asymptotic regulation of the error output of the closed-loop system can be guaranteed only when the initial state of the plant, the controller, and the exosystem, and the uncertain parameter are sufficiently small. In practice, it is desirable to design control laws that render the global asymptotic stability of the equilibrium of the closed-loop system, and asymptotic regulation of the error output of the closed-loop system for any initial state of the plant, the controller, and the exosystem, and arbitrarily large uncertain parameter. Such problem is called *global robust output regulation problem*. Some people have addressed this problem for nonlinear systems with special structures, such as Chen and Huang [7], Huang and Chen [28], Khallil [40], and Serrari and Isidori [51]. Especially in [28], a systematic approach was developed that converted the robust output regulation problem for a given plant into a robust stabilization problem of an augmented system composed of the given plant and an internal model. In particular, by utilizing the nonlinear internal model, Huang and Chen removed the assumption that the solution of the regulator equations is a trigonometric polynomial of t.

The results of global robust output regulation are conceptually appealing, but the solvability of this problem needs strong assumptions, such as input-to-state stability assumption. And the plants are limited to some special forms, such as lower triangular form and output feedback form. In order to make the solvability conditions less restrictive, many people considered the *semiglobal robust output regulation problem.* Compared to its global counterpart, semiglobal robust output regulation problem only requires the initial state of the plant, the controller, the exosystem, and the uncertain parameter to be in any given compact sets, which makes the solvability conditions less restrictive. This problem has been studied by Isidori [35], Serrani and Isidori, et al [52], [53], Khalil [39], [40], Mahmoud and Khalil [49], to certain degree.

1.3 Semiglobal Stabilization

Stabilization problem can be regarded as the special case of the output regulation problem where the output is regulated to zero. The fundamental stabilization techniques contain Lyapunov-based methods (backstepping, adaptive control), passivity-based technique, neural-network-based technique, and small gain technique based on input-to-state stability, etc. Generally, the solvability of the output regulation problem is translated into the solvability of a stabilization problem, that is why we emphasize stabilization techniques while we will mainly consider output regulation problems in this thesis. Recently Huang and Chen [28] proposed a general framework which systematically translated the robust output regulation problem into a stabilization problem. In this thesis, all the solvability of the robust output regulation problem are based on this framework. In this section, we will introduce the semiglobal stabilization problem and semiglobal robust stabilization problem.

So far, most results on *semiglobal stabilization* are concerned with the systems in normal form [34],

$$\begin{aligned}
\dot{z} &= f_0(z,x) \\
\dot{x}_1 &= x_2 \\
&\cdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= f_r(z,x) + g(z,x)u \\
y &= h(z,x).
\end{aligned} \tag{1.9}$$

However, in Section 9.3 [34], a counter-example is given to show that the semiglobal stabilization problem of (1.9) may not be solvable. So some researchers consider a more special form with the z subsystem modified into $\dot{z} = f_0(z, x_1)$ [34], [46], [54].

Currently, results of the *semiglobal robust stabilization* problem are also limited to systems in normal form [35], [40], [47], and [55] denoted as

$$\begin{aligned}
\dot{z} &= f_0(z, x, v, w) \\
\dot{x}_1 &= x_2 \\
&\cdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= f_r(z, x, v, w) + g(z, x, v, w)u \\
y &= h(z, x, v, w),
\end{aligned} \tag{1.10}$$

where v and w denote the exogenous signals and plant uncertainties respectively as described in Section 1.1. To make the semiglobal robust stabilization problem of (1.10) solvable, two assumptions are needed [40], [47], [55],

i: $g_i(z, x, v, w)$ has known sign and $|g_i(z, x, v, w)| > b_i$, where b_i is a positive real number.

ii: The subsystem $\dot{z} = f_0(z, x, v, w)$ is assumed to be input-to-state stable, and $\dot{z} = f_0(z, 0, v, w)$ is assumed to be locally exponentially stable uniformly in v and w.

To replace assumption ii with a less restrictive one, some people consider a more special system with the z subsystem modified to $\dot{z} = f_0(z, x_1, v, w)$ [52]. In this situation, the following assumption is used to replace assumption ii : $\dot{z} = f_0(z, x_1, v, w)$ is assumed to be globally asymptotically stable and locally exponentially stable uniformly in v and w.

1.4 A Benchmark Nonlinear Control Problem

The following problem provides a benchmark for examining nonlinear control design techniques within the framework of a nonlinear fourth-order dynamical system.

The Rotational/Translational Actuator(RTAC) system depicted in Figure 1.1 is introduced in [3]. It was originally studied as a simplified model of a dual-spin spacecraft to investigate the resonance capture phenomenon. Then, it has been studied to investigate the utility of a rotational proof-mass actuator for stabilizing translational motion. The system consists of a translational cart of mass M connected to a fixed wall by a linear spring of a stiffness k. The cart is constrained to have one-dimensional travel. The proof-mass actuator attached to the cart has mass m and centroidal moment of inertia I about its center of mass, which is located a distance e from the point about which the proof-mass rotates. Its motion occurs in a horizontal plane so that no gravitational forces need to be considered. N denotes the control torque applied to the proof mass, and F is the disturbance force on the cart.

The problem of designing a feedback control law to achieve asymptotic disturbance rejection / attenuation while maintaining good transient response in the closed-loop system is known as a nonlinear benchmark problem [3], and has been an intensive research subject since 1995 [3], [4], [5], [11], [18], [44], [48], [56], and [58]. In particular, the above problem has been formulated as an output regulation problem in [23], and it is shown that the RTAC system is a nonminimum phase nonlinear system with nonhyperbolic zero dynamics. It is well known that the nonhyperbolicity of the zero dynamics is a major obstacle to the applicability of the output regulation theory since the solvability of the regulator equations associated with the problem cannot be determined by the center manifold theory [38]. Nevertheless, an approximation solution based on the power series solution of regulator equations has been given in [23].

Based on the work handling nonminimum phase systems with nonhyperbolic zero dynamics by Huang [20], [24], and the work of solving the regulator equations of the RTAC systems by Huang [27], we get a design to solve the disturbance rejection. With regard to the fact that the framework dealing with robust output regulation problem proposed by Huang and Chen in [28] can handle the systems whose regulator equations have non-polynomial solution, we address the robust asymptotic disturbance rejection using this framework. Moreover, we use the parameter optimization technique by Huang [22] to get better transient response.

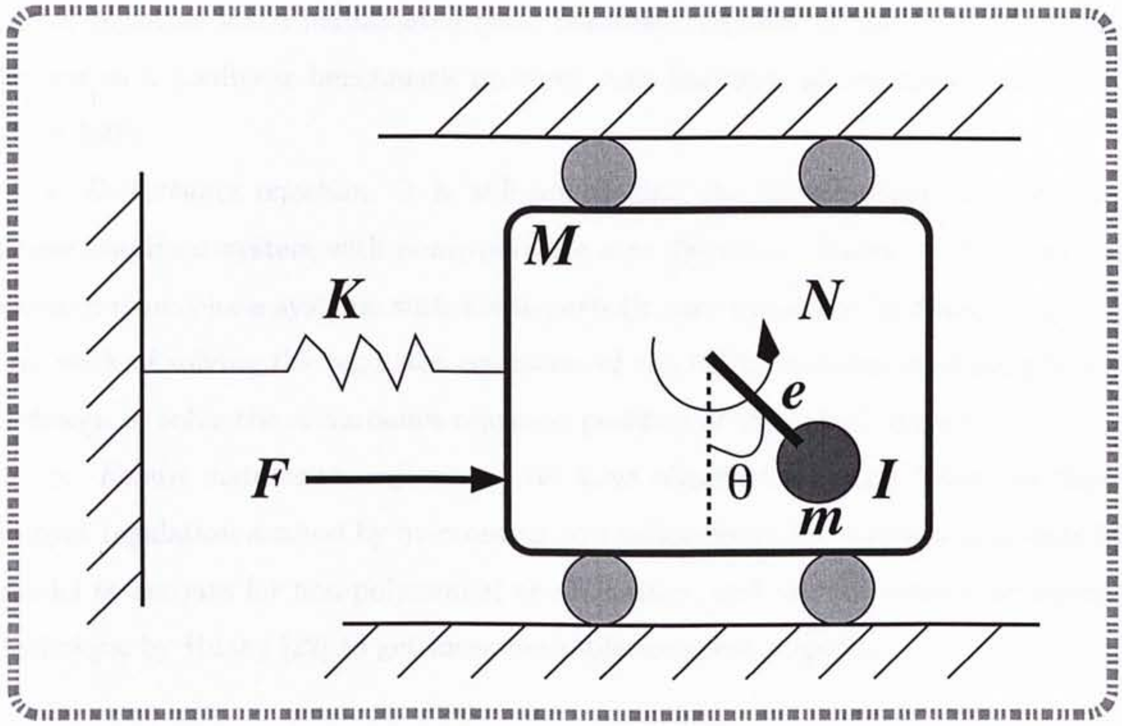

Figure 1.1: Rotational/translational actuator

1.5 Contribution of this Thesis

In the first part of this thesis, we address the semiglobal robust output regulation problem for a class of nonlinear systems in normal form via output feedback control. We solve this problem by overcoming three difficulities. First, the output regulation problem can be translated into a stabilization problem of an augmented systems composed of the original plant and the internal model. But the augmented system is of the form that cannot be treated directly by any existing stabilization result. Using the Lyapunov's direct method and the semiglobal backstepping technique by Teel and Praly [55], we solve it. Second, we eliminate the polynomial assumption imposed on the solution of the regulator equations by taking advantage of the nonlinear internal model by Chen and Huang [8]. Third, we get an output feedback controller by taking use of the high gain observer by Khalil and Esfandiari [43].

In the second part of this thesis, we consider the disturbance rejection and the robust disturbance rejection problem of the RTAC system by the measurement output feedback control. The problem of designing a feedback control law to achieve asymptotic disturbance rejection while maintaining good transient response in the closed-loop system is known as a nonlinear benchmark problem, and has been an intensive research subject since 1995.

i: *Disturbance rejection.* It is well known that the RTAC system is a nonminimum phase nonlinear system with nonhyperbolic zero dynamics. Based on the work handling nonminimum phase systems with nonhyperbolic zero dynamics by Huang [20], [24], and the work of solving the regulator equations of the RTAC systems by Huang [27], we get a design to solve the disturbance rejection problem of the RTAC system.

ii: *Robust disturbance rejection.* We have obtained a design based on the robust output regulation method by overcoming two major obstacles: devise a nonlinear internal model to account for non-polynomial nonlinearities, and use the parameter optimization technique by Huang [22] to get more desirable transient response.

Chapter 2

Semiglobal Robust Output Regulation of a Class of Nonlinear Systems via Output Feedback Control

The semiglobal robust output regulation problem is a challenging problem due to the following two obstacles lying in the existing literatures: the solution of the regulator equations should be polynomials, and the semiglobal robust stabilization problem is only solvable for a small class of nonlinear systems. In this chapter, we establish the solvability conditions of the semiglobal robust output regulation problem for a special class of nonlinear SISO systems in the normal form. We take three steps to deal with this problem based on a recently developed general framework for handling the robust output regulation problem. First, convert the robust output regulation problem of the given plant into a robust stabilization problem of an augmented system composed of the original plant and a well defined internal model. Second, solve the semiglobal robust stabilization problem of the augmented system via partial state feedback. Third, solve the semiglobal robust stabilization problem of the augmented system via output feedback. Taking advantage of the nonlinear internal model, we have obtained a result that does not rely on the polynomial assumption of the solution of the regulator equations needed in the existing literatures. Also we weaken the input-to-state stability assumption imposed on the zero dynamics.

This chapter is organized as follows: Section 2.1 gives an introduction. Section 2.2 introduces the semiglobal backstepping technique. Section 2.3 aims to convert the robust output regulation problem into a robust stabilization problem. Section 2.4 addresses the solvability of the semiglobal robust stabilization problem via partial state feedback. Section 2.5 resorts to the saturated high gain observer to estimate the state such that the stabilization problem can be solved via output feedback control. Section 2.6 gives an example. Section 2.7 closes this chapter with some remarks.

2.1 Introduction

The output regulation problem (also known as servomechanism problem) has been one of the most active control problems since 1970s. Briefly, the output regulation problem is to design a feedback control law for a plant, such that the closed-loop system is internally stable, and the output of the closed-loop system asymptotically tracks a class of reference inputs in the presence of a class of disturbances. Both the reference inputs and disturbances are generated by an autonomous differential equation called exosystem. For the class of linear systems, this problem was thoroughly studied in the 1970s by Davison [9], Francis [13], and Francis and Wonham [15]. The research in this period has generated salient controller synthesis technique known as internal model principle. The internal model principle converts the output regulation problem into an eigenvalue placement problem for an augmented linear system. For the class of nonlinear systems, the problem has been extensively pursued since early 1990s. Recently, more attentions have been paid to the global or semiglobal robust output regulation problem [28], [35], [40], and [52]. In this chapter, we will consider the solvability of the semiglobal robust output regulation problem for the following single-input single-output systems:

$$
\begin{aligned}
\dot{z} &= f_0(z, x_1, v, w) \\
\dot{x}_1 &= x_2 \\
&\cdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= f_r(z, x_1, \cdots, x_r, v, w) + g(v, w)u \\
\dot{v} &= A_1 v \\
e &= x_1 - q(v, w)
\end{aligned} \tag{2.1}
$$

where $z \in R^n$, $x_i \in R$, $i = 1, \cdots, r$, are the plant states, $u \in R$ is the control input, $e \in R$ is the measurable error output, $v \in V \subset R^q$ with V compact the exogenous signal representing the disturbance and/or the reference input generated by the neutrally stable autonomous system, $w \in W \subset R^N$ with W compact a vector of unknown constant parameters. f_0, f_r, g, q are sufficiently smooth functions satisfying $f_0(0, 0, 0, w) = 0$, $f_r(0, 0, \cdots, 0, 0, w) = 0$ and $g(v, w) \geq b_0 > 0$ for all $v \in V$, $w \in W$.

The class of control laws considered here are described by

$$
\begin{aligned}
u &= k(z_c, e) \\
\dot{z}_c &= f_z(z_c, e)
\end{aligned} \tag{2.2}
$$

where z_c is the compensator state vector of dimension n_c to be specified later.

Semiglobal Robust Output Regulation Problem (SGRORP): Given any compact sets $Z_0 \in R^n$, $X_0 \in R^r$, $Z_c \in R^{n_c}$, $V \in R^q$ and $W \in R^N$ containing the origins of their respective Euclidian spaces, find a controller of the form (2.2), such that the closed-loop system composed of (2.1) and (2.2) with its state being denoted by $x_c = \text{col}(z, x_1, \cdots, x_r, z_c)$ has the following properties,

P1: for all $x_c(0) \in Z_0 \times X_0 \times Z_c$, and for all $v \in V$, $w \in W$, the solution of the closed-loop system exists and is bounded for all $t > 0$; and,

P2: the tracking error $e(t)$ approaches zero asymptotically, i.e.,

$$\lim_{t \to \infty} e(t) = 0.$$

Remark 2.1 System (2.1) is a special case of the so-called normal form of the SISO affine nonlinear systems. Under the assumption that the affine nonlinear systems

$$\dot{\zeta} = f(\zeta, v, w) + g(\zeta, v, w)u \tag{2.3}$$

$$y = h(\zeta, v, w) \tag{2.4}$$

has a relative degree r univformly in (v, w), then there exists a diffeomorphic coordinate transformation

$$\begin{bmatrix} z \\ x \end{bmatrix} = T(\zeta, v, w)$$

such that the system (2.3) can be transformed to the normal form

$$\begin{aligned} \dot{z} &= f_0(z, x_1, \cdots, x_r, v, w) \\ \dot{x}_1 &= x_2 \\ &\cdots \\ \dot{x}_{r-1} &= x_r \\ \dot{x}_r &= f_r(z, x_1, \cdots, x_r, v, w) + g(z, x_1, \cdots, x_r, v, w)u. \end{aligned} \tag{2.5}$$

∎

Remark 2.2 When the plant (2.1) is extended to a more general form with the z subsystem as $\dot{z} = f_0(z, x_j, v, w)$, $1 \leq j \leq r$, the semiglobal robust output regulation problem of the system can still be solved. For convenience, we use $\dot{z} = f_0(z, x_1, v, w)$ to describe the design process in this chapter. ∎

The semiglobal robust output regulation problem for the plant (2.1) via output feedback control has been studied by Serrani, Isidori and Marconi [52] under the assumption that the solution of the regulator equations of the system satisfy certain immersion condition which essentially requires the solution of the regulator equations be a polynomial of $v(t)$ or a trigonometric polynomial of t [25]. The global robust output regulation of the same class of systems by state feedback control is studied by Huang and Chen [28] where a systematic approach is developed that converts the robust output regulation problem for a given plant into a robust stabilization problem of an augmented system composed of the given plant and an internal model. In particular, by utilizing the nonlinear internal model, Huang and Chen has removed the assumption that the solution of the regulator equations is a trigonometric polynomial of t. However, the paper requires that the zero dynamics be input-to-state stable which is more restrictive than the assumption needed in [52]. In this chapter, we will combine the framework proposed in [28] with the stabilization technique used in [52], [55] to solve the semiglobal robust output regulation problem for the plant (2.1). By taking advantage of the nonlinear internal model, we can eliminate the polynomial assumption on the solution of the regulator equations needed in [52] and also weaken the input-to-state stability assumption on the zero dynamics of the plant. It should be noted that the semiglobal robust output regulation problem is also studied for a class of nonlinear systems more general than (2.1) by Isidori [35] and Khalil [40]. However, these two papers also assume that the zero dynamics of the plant are input-to-state stable.

2.2 Semiglobal Backstepping Technique

The main stabilization technique we used here is the *semiglobal backstepping* technique introduced by Teel and Praly in [55]. For convenience, let us review the ULP assumption and the semiglobal backstepping technique as follows.

ULP Assumption (*Uniform Lyapunov Property*) For the C^1 system

$$\dot{z} = h(z, \mu(t)), \tag{2.6}$$

where $z \in R^n$, $\mu(t) = \text{col}(v(t), w) \in V \times W$ as defined in Section 2.1, there exists an open set $\mho_1$ in R^n, a nonnegative real number $\underline{c} < 1$, a real number $c \geq 1$, and a C^1 function $V(z) : \mho_1 \to [0, \infty)$ such that the set $\{z : V(z) \leq c+1\}$ is a compact subset of $\mho_1$, and along the trajectory of (2.6)

$$\dot{V}(z) \leq -\Phi_1(z)$$

where $\Phi_1(z)$ is continuous on $\mho_1$ and positive definite on the set $\mho_z = \{z : \underline{c} < V(z) \leq c+1\}$. (Notes: The number 1 in this section is arbitrary and can be replaced by any other positive real constant).

Before we gives the Semiglobal Backstepping Proposition, let us first states the lemma [55] as follows, which is important for the proof of the Proposition.

Lemma 2.1 Let S be a compact set in a product space $R^n \times R^m$, and denote by S_z and S_x its respective projections, i.e., $S \in S_z \times S_x$. Let $\chi(z)$ be a continuous real function on S_z which is positive definite on the set $\{(z,x) : x = 0\} \bigcap S$. Let $\psi(x)$ be a continuous real function on S_x which is positive definite on $S_x/0$. Let $\phi(z,x,\mu)$ be a continuous real function on $S \times (V \times W)$ which satisfies

$$\phi(z,x,\mu(t)) = 0 \;\; \forall (z,x,\mu) \in (\{(z,x) : x = 0\} \cap S) \times (V \times W).$$

Let k be a function of class-K_∞. Under these conditions, there exists a positive real number K_* such that, for all $K > K_*$,

$$-\chi(z) - k(K)\psi(x) + \varphi(z,x,u) < 0 \;\; \forall (z,x,\mu) \in S \times (V \times W).$$

∎

Remark 2.3 In this thesis, $k(K)$ can be selected as K. This lemma guarantees the existence of a high gain K_*, but the K_* is hard to be calculated out by this lemma. ∎

Proposition 2.1 (*Semiglobal Backstepping*) (Lemma 2.2 of [55]) Consider the C^1 nonlinear control system

$$\begin{aligned} \dot{z} &= f_0(z,x,\mu(t)) \\ \dot{x} &= f(z,x,\mu(t)) + g(z,x,\mu(t))u \end{aligned} \tag{2.7}$$

where $x \in R$, $z \in R^n$, $\mu(t) = \mathrm{col}(v(t), w) \in V \times W$, the sign of $g(z,x,\mu(t))$ is constant, and the magnitude of g is bounded away from zero by a strictly positive real number b, i.e.,

$$|g(z,x,\mu(t))| \geq b \quad \forall (z,x,\mu(t)) \in R^n \times R \times (V \times W). \tag{2.8}$$

Suppose the subsystem $\dot{z} = f_0(z,0,\mu(t))$ satisfies the ULP assumption. Given $\sigma \geq 1$, define the function

$$V_a(z,x) = c\frac{V(z)}{c+1-V(z)} + \sigma\frac{x^2}{\sigma+1-x^2} \tag{2.9}$$

and the set

$$\mho_2 = \{z : V(z) \leq c+1\} \times \{x : x^2 \leq \sigma + 1\} \tag{2.10}$$

Under these conditions, $V_a(z,x) : \mho_2 \to [0,\infty)$ is proper on $\mho_2$. Further, if

$$u = -K\text{sgn}(g)x, \tag{2.11}$$

then for each strictly positive real number ρ, there exists a positive real number K_* such that, for each $K \geq K_*$, the derivative of $V_a(z,x)$ along the trajectory of (2.7) satisfies

$$\dot{V}_a \leq -\Phi_2(z,x) \tag{2.12}$$

where $\Phi_2(z,x)$ is continuous on $\mho_2$ and positive definite on the set $\{(z,x) : \underline{c} + \rho \leq V_a(z,x) \leq c^2 + \sigma^2 + 1\}$. ∎

Remark 2.4 Proof of Proposition 2.1 can be found in [55]. $\Phi_2(z,x)$ can be in the form of $\Phi_2(z,x) = \frac{c(c+1)}{2(c+1-V)}\Phi_1(z) + \frac{\mu}{\mu+1}Kbx^2$. When the assumptions of Proposition 2.1 are all satisfied, and we choose $V(z)$ as a positive definite function, then we can conclude that the trajectory of the system (2.7) starting from the set $\{(z,x) : V_a(z,x) \leq c^2 + \sigma^2 + 1\}$ will enter the set $\{(z,x) : V_a(z,x) \leq \underline{c} + \rho\}$ and remain in it thereafter. By choosing $\underline{c}$ and ρ arbitrarily small, the trajectory can be arbitrarily close to the origin. If the equilibrium is the origin, then it is stable. However, the asymptotic stability property can not be guaranteed without the additional assumption that the system (2.6) is locally exponentially stable. ∎

2.3 Output Regulation Converted to Stabilization

In this section, we will convert the semiglobal robust output regulation problem of the plant (2.1) into a semiglobal robust stabilization problem for an augmented system composed of the original plant and the internal model based on the general framework recently proposed in [28].

At the outset, let us make some standard assumptions.

A1. There exists sufficiently smooth function $\mathbf{z}(v,w)$ with $\mathbf{z}(0,0) = 0$ satisfying, for all $v \in V$, $w \in W$, the following equation

$$\frac{\partial \mathbf{z}(v,w)}{\partial v}A_1 v = f_0(\mathbf{z}(v,w), q(v,w), v, w). \tag{2.13}$$

Remark 2.5 Under assumption A1, the solution of the regulator equations of system (2.1) exists and can be solved as follows,

$$
\begin{aligned}
\mathbf{x}_1(v,w) &= q(v,w) \\
\mathbf{x}_i(v,w) &= \frac{\partial \mathbf{x}_{i-1}(v,w)}{\partial v}A_1 v,\ i=2,\cdots,r \\
\mathbf{u}(v,w) &= \frac{1}{g(v,w)}\left(\frac{\partial \mathbf{x}_r(v,w)}{\partial v}A_1 v - f_r(\mathbf{z}(v,w),\mathbf{x}_1(v,w),\cdots,\mathbf{x}_r(v,w),v,w)\right) \quad (2.14)
\end{aligned}
$$

We will denote the solution of the regulator equations of (2.1) by $\mathbf{z}(v,w), \mathbf{x}(v,w), \mathbf{u}(v,w)$, with $\mathbf{x}(v,w) = col(\mathbf{x}_1(v,w),\cdots,\mathbf{x}_r(v,w))$. ∎

Remark 2.6 The solvability of the above regulator equations is only a necessary condition for the solvability of the output regulation problem [38]. To guarantee the solvability of the robust output regulation problem, additional conditions have to be imposed on the solution of the regulator equations [1], [25] and [30]. Despite the different appearances of these conditions, they amount to requiring the system admit a linear internal model [28], which in turn essentially requires the system only contain polynomial nonlinearities. Recently, a much less restrictive condition is given as stated below in Assumption A2 [28]. To introduce Assumption A2, let us first note that, if $\pi(v,w)$ is a polynomial function of v or a trigonometric polynomial function of t along the trajectory of the exosystem, then there exist an integer r and real numbers $a_1,\cdots,a_r$ such that $\pi(v,w)$ satisfies a differential equation of the following form [25]:

$$
\frac{d^r\pi(v(t),w)}{dt^r} - a_1\pi(v(t),w) - a_2\frac{d\pi(v(t),w)}{dt} - \cdots - a_r\frac{d^{(r-1)}\pi(v(t),w)}{dt^{(r-1)}} = 0 \qquad (2.15)
$$

for all trajectories $v(t)$ of the exosystem and all $w \in R^N$. We call the monic polynomial $P(\lambda) = \lambda^r - a_r\lambda^{r-1} - \cdots - a_2\lambda - a_1$ a zeroing polynomial of $\pi(v,w)$ if $\pi(v,w)$ satisfies (2.15). $P(\lambda)$ is called a minimal zeroing polynomial of $\mathbf{u}(v,w)$ if $P(\lambda)$ is a zeroing polynomial of $\pi(v,w)$ of least degree. Let $\pi_i(v,w)$, $i=1,\cdots,I$, for some positive integer I, be I polynomials in v. They are called pairwise coprime if their minimal zeroing polynomials $P_1(\lambda),\cdots,P_I(\lambda)$ are pairwise coprime. ∎

A2. There exist pairwise coprime polynomials $\pi_1(v,w),\cdots,\pi_I(v,w)$ with $r_1,\cdots,r_I$ being the degrees of their minimal zeroing polynomials $P_1(s),\cdots,P_I(s)$, and sufficiently smooth function $\Gamma : R^{r_1+\cdots+r_I} \to R$ vanishing at the origin such that, for all trajectories $v(t)$ of the exosystem, and $w \in R^N$,

$$\begin{aligned}\mathbf{u}(v,w) &= \Gamma\Big(\pi_1(v,w), \dot{\pi}_1(v,w)\cdots, \frac{d^{(r_1-1)}\pi_1(v,w)}{dt^{(r_1-1)}}, \cdots, \pi_I(v,w), \dot{\pi}_I(v,w), \cdots, \\ &\quad \frac{d^{(r_I-1)}\pi_I(v,w)}{dt^{(r_I-1)}}\Big), \end{aligned} \tag{2.16}$$

and the pair$\{E, \Phi\}$ is observable, where E is the gradient of Γ at the origin, and $\Phi = \text{diag}\{\Phi_1, \cdots, \Phi_I\}$ with $\Phi_j, j = 1, \cdots, I$, being the companion matrix of the polynomial $P_j(s)$, where $P_j(s) = s^{r_j} - a_1 s^0 - a_2 s^1 - \cdots - a_{r_j} s^{r_j-1}$.

Remark 2.7 It is shown in [28] that under assumptions A1 and A2, let

$$\begin{aligned}\theta(v,w) &= T col\Big(\pi_1(v,w), \dot{\pi}_1(v,w)\cdots, \frac{d^{(r_1-1)}\pi_1(v,w)}{dt^{(r_1-1)}}, \cdots, \pi_I(v,w), \dot{\pi}_I(v,w), \cdots, \\ &\quad \frac{d^{(r_I-1)}\pi_I(v,w)}{dt^{(r_I-1)}}\Big) \\ \alpha(\theta) &= T\Phi T^{-1}\theta \\ \beta(\theta) &= \Gamma(T^{-1}\theta), \end{aligned} \tag{2.17}$$

where T is any nonsingular matrix. Then the triple $\{\theta, \alpha, \beta\}$ is such that

$$\begin{aligned}\frac{d\theta\,(v,w)}{dt} &= \alpha\,(\theta\,(v,w)) \\ \mathbf{u}(v,w) &= \beta\,(\theta\,(v,w)). \end{aligned} \tag{2.18}$$

The triple $\{\theta, \alpha, \beta\}$ is called a linearly observable steady state generator of plant (2.1) with output u. The notion of the steady state generator leads to a dynamic system as follows

$$\dot{\eta} = M\eta + N(u - \beta(\eta) + ET^{-1}\eta) \tag{2.19}$$

where the pair $\{M, N\}$ is controllable with M Hurwitz, and T satisfies the Sylvester equation $T\Phi - MT = NE$. The system (2.19) is called an internal model of (2.1) with output u. The plant and the internal model define an augmented system. Under the following coordinate and input transformation

$$\begin{aligned}
\bar{\eta} &= \eta - \theta(v, w) \\
\bar{z} &= z - \mathbf{z}(v, w) \\
\bar{x}_1 &= x_1 - \mathbf{x}_1(v, w) = e \\
\bar{x}_i &= x_i - \mathbf{x}_i(v, w), \ \ i = 2, \cdots, r \\
\bar{u} &= u - \beta(\eta),
\end{aligned} \tag{2.20}$$

the augmented system takes the following form

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}_0(\bar{z}, \bar{x}_1, v, w) \\
\dot{\bar{x}}_i &= \bar{x}_{i+1}, \ \ i = 1, \cdots, r-1 \\
\dot{\bar{\eta}} &= (M + NET^{-1})\bar{\eta} + N\bar{u} \\
\dot{\bar{x}}_r &= \bar{f}_r(\eta, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) + g(v, w)\bar{u}
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
\bar{f}_0(\bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) &= f_0(\bar{z} + \mathbf{z}(v, w), \bar{x}_1 + \mathbf{x}_1(v, w), \cdots, \bar{x}_r + \mathbf{x}_r(v, w), v, w) \\
&\quad - f_0(\mathbf{z}(v, w), \mathbf{x}_1(v, w), \cdots, \mathbf{x}_r(v, w), v, w) \\
\bar{f}_r(\eta, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) &= -\frac{\partial \mathbf{x}_r(v, w)}{\partial v} A_1 v + f_r(\bar{z} + \mathbf{z}(v, w), \bar{x}_1 + \mathbf{x}_1(v, w), \cdots, \\
&\quad \bar{x}_r + \mathbf{x}_r(v, w), v, w) + g(v, w)\beta(\eta)
\end{aligned}$$

It is further shown in [28] that if a control law of the form

$$\begin{aligned}
\bar{u} &= \bar{k}(\xi, e) \\
\dot{\xi} &= \omega(\xi, e),
\end{aligned} \tag{2.22}$$

where ξ is the compensator state vector of dimension n_ξ to be specified later, can stabilize the augmented system (2.21), then the following controller

$$\begin{aligned}
u &= \bar{k}(\xi, e) + \beta(\eta) \\
\dot{\xi} &= \omega(\xi, e)
\end{aligned} \tag{2.23}$$

solves the robust output regulation problem of the original plant (2.1). Therefore, in what follows, it suffices to study the semiglobal robust stabilization problem of the augmented system (2.21). ∎

To make the above stabilization problem more tractable, let us perform on (2.21) one more coordinate transformation as follows

$$\tilde{\eta} = \bar{\eta} - g^{-1}(v,w)N\bar{x}_r$$

Then we can get

$$\begin{aligned}
\dot{\tilde{\eta}} &= \gamma(\tilde{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) \\
&= M\tilde{\eta} - g^{-1}(v,w)N\Big(-\frac{\partial \mathbf{x}_r(v,w)}{\partial v}A_1 v + f_r(\bar{z} + \mathbf{z}(v,w), \bar{x}_1 + \mathbf{x}_1(v,w), \cdots, \\
&\quad \bar{x}_r + \mathbf{x}_r(v,w), v, w)\Big) - \frac{\partial g^{-1}(v,w)}{\partial v}A_1 v N\bar{x}_r + g^{-1}(v,w)MN\bar{x}_r + N(ET^{-1}\bar{\eta} - \beta(\eta)) \\
&= M\tilde{\eta} - N\big(\beta^{[2]}(\tilde{\eta} + g^{-1}(v,w)N\bar{x}_r + \theta) - \beta^{[2]}(g^{-1}(v,w)N\bar{x}_r + \theta)\big) \\
&\quad + \phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w)
\end{aligned} \tag{2.24}$$

where

$$\begin{aligned}
&\phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) \\
&= g^{-1}(v,w)N\Big(\frac{\partial \mathbf{x}_r(v,w)}{\partial v}A_1 v - f_r(\bar{z} + \mathbf{z}(v,w), \bar{x}_1 + \mathbf{x}_1(v,w), \cdots, \bar{x}_r + \mathbf{x}_r(v,w), v, w)\Big) \\
&\quad - \frac{\partial g^{-1}(v,w)}{\partial v}A_1 v N\bar{x}_r + g^{-1}(v,w)MN\bar{x}_r - N\beta(\theta) \\
&\quad - N\big(\beta^{[2]}(g^{-1}(v,w)N\bar{x}_r + \theta) - \beta^{[2]}(\theta)\big)
\end{aligned}$$

Under this transformation, system (2.21) is converted into the following standard form,

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}_0(\bar{z}, \bar{x}_1, v, w) \\
\dot{\bar{x}}_i &= \bar{x}_{i+1}, \quad i = 1, \cdots, r-1 \\
\dot{\tilde{\eta}} &= \gamma(\tilde{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) \\
\dot{\bar{x}}_r &= \bar{f}_r(\tilde{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) + g(v,w)\bar{u}
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
\bar{f}_r(\tilde{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) &= -\frac{\partial \mathbf{x}_r(v,w)}{\partial v}A_1 v + f_r(\bar{z} + \mathbf{z}(v,w), \bar{x}_1 + \mathbf{x}_1(v,w), \cdots, \\
&\quad \bar{x}_r + \mathbf{x}_r(v,w), v, w) + g(v,w)\beta(\tilde{\eta} + g^{-1}(v,w)N\bar{x}_r + \theta)
\end{aligned}$$

Now, we can precisely state the semiglobal robust stabilization problem of the augmented system (2.25) as follows.

Semiglobal Robust Stabilization Problem (SGRSP): Given any compact sets $\bar{Z}_0 \subset R^n$, $\bar{X}_0 \subset R^r$, $\Pi_{\tilde{\eta}} \subset R^{n_{\tilde{\eta}}}$, $\Xi_0 \subset R^{n_\xi}$, $V \subset R^q$, and $W \subset R^N$ containing the origins of respective Euclidian spaces, where $n_{\tilde{\eta}} = r_1 + \cdots + r_I$ is the dimension of $\tilde{\eta}$, find a controller of the form (2.22) such that the equilibrium point $(\bar{z}, \bar{x}, \tilde{\eta}, \xi) = (0, 0, 0, 0)$ of the closed-loop system composed of (2.25) and (2.22) is asymptotically stable with $\bar{Z}_0 \times \bar{X}_0 \times \Pi_{\tilde{\eta}} \times \Xi_0$ being contained in its basin of attraction.

2.4 Solvability of the Semiglobal Robust Stabilization Problem via Partial State Feedback

In this section, we will consider the solvability of the semiglobal robust stabilization problem of the augmented system (2.25) via partial state feedback control of the form $\bar{u} = k(\bar{x}_1, \cdots, \bar{x}_r)$. Since the robust stabilization problem of (2.25) can not be handled by any existing stabilization result directly, it is a challenging problem. We will compose the Lyapunov's direct method and the semiglobal backstepping technique [55] to handle it in the following.

In what follows, let us make some coordinate transformation,

$$\tau = \bar{x}_r + k^{r-1} b_0 \bar{x}_1 + k^{r-2} b_1 \bar{x}_2 + \cdots + k b_{r-2} \bar{x}_{r-1}, \tag{2.26}$$

where $k > 0$ is a parameter to be determined later, and the polynomial $\lambda^{r-1} + b_{r-2}\lambda^{r-2} + \cdots + b_1\lambda + b_0$ is Hurwitz by carefully choosing positive numbers $b_0,\ b_1,\ \cdots,\ b_{r-2}$. This technique proposed in [46] is widely used to solve stabilization problems. For convenience, denote $u_r = -k^{r-1} b_0 \bar{x}_1 - k^{r-2} b_1 \bar{x}_2 - \cdots - k b_{r-2} \bar{x}_{r-1}$.

Under the above transformation, the system (2.25) can be transformed into a new system that has the same form as (2.7). Then, it is possible to stabilize (2.25) using semiglobal backstepping technique as stated in Section 2.3.

Let $Z = col(\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1})$. Then, the system (2.25) can be rewritten as follows,

$$\begin{aligned} \dot{Z} &= F(Z, v, w) + G\tau \\ \dot{\tilde{\eta}} &= \gamma(\tilde{\eta}, Z, \tau, v, w) \\ \dot{\tau} &= \chi(Z, \tilde{\eta}, \tau, v, w) + g(v, w)\bar{u} \end{aligned} \tag{2.27}$$

where $F(Z,v,w) = \begin{bmatrix} \bar{f}_0(\bar{z},\bar{x}_1,v,w) \\ \bar{x}_2 \\ \vdots \\ -k^{r-1}b_0\bar{x}_1 - k^{r-2}b_1\bar{x}_2 - \cdots - kb_{r-2}\bar{x}_{r-1} \end{bmatrix}$, $G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$,

$$\begin{aligned}
\gamma(\tilde{\eta},Z,\tau,v,w) &= M\tilde{\eta} - N\big(\beta^{[2]}(\tilde{\eta}+g^{-1}(v,w)N(\tau+u_r)+\theta) \\
&\quad -\beta^{[2]}(g^{-1}(v,w)N(\tau+u_r)+\theta)\big) + \phi(\bar{z},\bar{x}_1,\cdots,\bar{x}_{r-1},\tau+u_r,v,w), \\
\chi(Z,\tilde{\eta},\tau,v,w) &= f_r\big(\bar{z}+\mathbf{z}(v,w),\bar{x}_1+\mathbf{x}_1(v,w),\cdots,(\tau+u_r)+\mathbf{x}_r(v,w),v,w\big) \\
&\quad +g(v,w)\beta(\tilde{\eta}+g^{-1}(v,w)N(\tau+u_r)+\theta) - \dot{\mathbf{x}}_r(v,w) \\
&\quad +k^{r-1}b_0\bar{x}_2 + k^{r-2}b_1\bar{x}_3 + \cdots + kb_{r-2}(\tau+u_r).
\end{aligned}$$

Let $\bar{y} = \tau$ be a new output for the system (2.27), then the zero dynamics of (2.27) with respect to the output $\bar{y}$ are given by

$$\begin{aligned}
\dot{Z} &= F(Z,v,w) \\
\dot{\tilde{\eta}} &= \gamma(\tilde{\eta},Z,0,v,w)
\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}
\gamma(\tilde{\eta},Z,0,v,w) &= M\tilde{\eta} - N\big(\beta^{[2]}(\tilde{\eta}+g^{-1}(v,w)Nu_r+\theta) - \beta^{[2]}(g^{-1}(v,w)Nu_r+\theta)\big) \\
&\quad +\phi(\bar{z},\bar{x}_1,\cdots,\bar{x}_{r-1},u_r,v,w)
\end{aligned}$$

Before we solve the semiglobal robust stabilization problem of the system (2.27), let us first make two more assumptions as follows,

A3. The equilibrium $\bar{z} = 0$ of the system

$$\dot{\bar{z}} = \bar{f}_0(\bar{z},0,v,w) \tag{2.29}$$

is globally asymptotically stable uniformly with respect to $v(t) \in V,\ w \in W$, and there exists a C^1 positive definite proper function $V_0(\bar{z})$ satisfying:

$$\begin{aligned}
\underline{a}_0||\bar{z}||^2 &\leq V_0(\bar{z}) \leq \bar{a}_0||\bar{z}||^2 \\
\frac{\partial V_0(\bar{z})}{\partial \bar{z}}\bar{f}_0(\bar{z},0,v,w) &\leq -a_0||\bar{z}||^2,\ \forall v(t) \in V,\ w \in W \\
||\frac{\partial V_0(\bar{z})}{\partial \bar{z}}|| &\leq \hat{a}_0||\bar{z}||
\end{aligned}$$

where $||\bar{z}|| \in [0,\delta]$, and $\underline{a}_0,\ \bar{a}_0,\ a_0,\ \hat{a}_0,\ \delta$ are some positive numbers.

Remark 2.8 Assumption A3 also implies that the origin is an exponential stable equilibrium point of the system $\dot{\bar{z}} = \bar{f}_0(\bar{z}, 0, v, w)$ uniformly in (v, w). ∎

A4. Let P satisfies $PM + M^T P = -I$, there exists a positive number r satisfying

$$-2\tilde{\eta}^T PN(\beta^{[2]}(\tilde{\eta} + d) - \beta^{[2]}(d)) \leq (1 - r)\tilde{\eta}^T \tilde{\eta} \tag{2.30}$$

where $d = g^{-1}(v, w)Nu_r + \theta$, and $\beta^{[2]}(\cdot)$ is the nonlinear part of $\beta(\cdot)$.

Remark 2.9 This inequality is used to restrict the growth rate of the nonlinearity of $\beta(\cdot)$. When the the function $\beta(\cdot)$ is linear or is globally Lipschitz with a sufficiently small Lipschitz constant, this inequality is satisfied automatically [28]. ∎

Remark 2.10 In [52], the internal model is in the form $\dot{\xi} = \Phi\xi + N\theta$. The eigenvalues of Φ are on the imaginary axis, so a low gain tuning parameter ϵ is taken into account to help to make the system stable. In this paper, after some coordinates transformation, our internal model is transformed into $\dot{\tilde{\eta}} = M\tilde{\eta} - N\big(\beta^{[2]}(\tilde{\eta} + g^{-1}(v, w)N(\tau + u_r) + \theta) - \beta^{[2]}(g^{-1}(v, w)N(\tau + u_r) + \theta)\big) + \phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1}, \tau + u_r, v, w)$, where M is Hurwitz. If we impose assumption A4 on the nonlinear part of the above equation, then the $\tilde{\eta}$ subsystem will have some desirable property for the stabilization of the overall system. ∎

Lemma 2.2 Under assumption A3, given any $R > 0$, there exists a real number $k_{*1} > 0$ such that when $k > k_{*1}$, the equilibrium $Z = 0$ of the system

$$\dot{Z} = F(Z, v, w) \tag{2.31}$$

can be made uniformly asymptotically stable, with domain of attraction containing any given compact set $B_R^{n+r-1} = \{Z : ||Z|| \leq R\}$.

Proof: We separate the proof of this lemma into three steps. First, the equilibrium $(\bar{z}, \bar{x}) = (\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1})$ is locally asymptotically stable. Second, all trajectories starting from B_R^{n+r-1} are bounded. Third, the trajectories are eventually convergent.

Step i: Please see Section 4.4 in [34].

Step ii: Let $\xi_i = \frac{\bar{x}_i}{k^{i-1}}$, $i = 1, \cdots, r-1$ and denote $\xi = (\xi_1, \cdots, \xi_{r-1})$, then we can transform the system (2.31) into the following system:

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}_0(\bar{z}, \xi_1, v, w) \\ \dot{\xi} &= kA\xi \end{aligned} \tag{2.32}$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -b_0 & -b_1 & -\cdots & -b_{r-3} & -b_{r-2} \end{bmatrix}.$$

Since A is Hurwitz, there exists a real symmetric positive definite matrix P_0 satisfying the following equation $A^T P_0 + P_0 A = -I$. Pick a Lyapunov function $V_0(\bar{z})$ satisfying assumption A3. Let $V_a(\bar{z}, \xi) = V_0(\bar{z}) + \xi^T P_0 \xi$.

Since the function $\bar{f}_0$ is smooth, it can be written as follows,

$$\bar{f}_0(\bar{z}, \xi_1, v, w) = \bar{f}_0(\bar{z}, 0, v, w) + p(\bar{z}, \xi_1, v, w)\xi_1, \tag{2.33}$$

where $p(\bar{z}, \xi_1, v, w)$ is a smooth function. The derivative of V_a along the trajectory of (2.32) is given as follows,

$$\begin{aligned} \dot{V}_a(\bar{z}, \xi) &= \frac{\partial V_a}{\partial \bar{z}} \bar{f}_0(\bar{z}, \xi_1, v, w) + \frac{\partial V_a}{\partial \xi} kA\xi \\ &= \frac{\partial V_0}{\partial \bar{z}} \left(\bar{f}_0(\bar{z}, 0, v, w) + p(\bar{z}, \xi_1, v, w)\xi_1\right) + k\xi^T (A^T P_0 + P_0 A)\xi \\ &= \frac{\partial V_0}{\partial \bar{z}} \bar{f}_0(\bar{z}, 0, v, w) + \frac{\partial V_0}{\partial \bar{z}} p(\bar{z}, \xi_1, v, w)\xi_1 - k||\xi||^2. \end{aligned} \tag{2.34}$$

When $k \geq 1$, we have

$$\begin{aligned} B_R^{n+r-1} &\subset \{(\bar{z}, \bar{x}) : ||\bar{z}|| \leq R,\ |\bar{x}_i| \leq R,\ i = 1, \cdots, r-1\} \\ &\subseteq \{(\bar{z}, \xi) : ||\bar{z}|| \leq R,\ |\xi_i| \leq R,\ i = 1, \cdots, r-1\} \stackrel{def}{=} D_R^{n+r-1} \end{aligned} \tag{2.35}$$

Since $V_a(\bar{z}, \xi)$ is continuous with respect to $\bar{z}$ and ξ on the compact set D_R^{n+r-1}, it has a maximal value denoted as c_{max}. The compact set $\Omega_c \stackrel{def}{=} \{(\bar{z}, \xi) : V_a(\bar{z}, \xi) \leq c_{max}\}$ satisfies the following property $\Omega_c \supseteq D_R^{n+r-1}$.

In the following we will show that $\dot{V}_a$ is negative definite along the trajectory of (2.32) in Ω_c in two cases.

(a) $\xi = 0$.

$$\dot{V}_a(\bar{z}, \xi) = \frac{\partial V_0}{\partial \bar{z}} \bar{f}_0(\bar{z}, 0, v, w).$$

Assumption A3 guarantees that it is negative definite. By continuity, there exists a neighborhood D_0 containing the set $\{(\bar{z}, \xi) : ||\bar{z}|| \leq R,\ ||\xi|| = 0\}$ such that $\dot{V}_a(\bar{z}, \xi)$ is negative definite for all $(\bar{z}, \xi) \in D_0$.

(b) $\xi \neq 0$. Set $k_{*1} = b_1/b_2$, where $b_1 = \max_{(\bar{z},\xi)\in\Omega_c/D_0} |\frac{\partial V}{\partial \bar{z}} p(\bar{z}, \xi_1, v, w)\xi_1|$, $b_2 = \min_{(\bar{z},\xi)\in\Omega_c/D_0} ||\xi||^2$. When $k > \max\{1, k_{*1}\}$, $\dot{V}_a$ is negative definite in Ω_c.

Hence, every trajectory starting from Ω_c will remain in Ω_c. That is to say, the trajectory is bounded.

Step iii: Since A is Hurwitz, $\lim_{t\to\infty} \xi(t) = 0$. That is to say, the trajectory starting from D_R^{n+r-1} will eventually enter the set D_0. Next, we will use the idea from LaSalle's Invariance Principle to prove that the trajectory will eventually converge to the origin.

Since the solution $\xi(t)$ is bounded, the positive limit set of $\xi(t)$ is nonempty, and denoted as L^+. Moreover, $\xi(t)$ approaches L^+ as $t \to \infty$. Obviously, $L^+ \subset D_0$.

As stated in step i, the equilibrium $(0, 0)$ is locally asymptotically stable, so there exists a neighborhood of $(0, 0)$ denoted as $\mathcal{N}_1$ such that any trajectory starting from this neighborhood will approaches the equilibrium $(0, 0)$ asymptotically. Let $(\bar{z}(0), 0)$ be an arbitrary point in L^+. If we can prove that any trajectory starting from this point will enter $\mathcal{N}_1$ in finite time, then we complete the proof.

Let $\Phi(t, (\bar{z}, \xi))$ be the transition function of the system (2.32). The assumption A3 guarantees $\lim_{t\to\infty} \bar{z}(t) = 0$ when $\bar{z}(0)$ is sufficiently small, so there exists a $t_1 > 0$ such that $\Phi(t_1, (\bar{z}(0), 0)) \in \mathcal{N}_1$. By continuity, there exists a neighborhood $\mathcal{N}_2$ of $(\bar{z}(0), 0)$ such that any trajectory starting from the neighborhood will enter $\mathcal{N}_1$, i.e., $\Phi(t_1, (\bar{z}, \xi)) \in \mathcal{N}_1,\ \ \forall(\bar{z}, \xi) \in \mathcal{N}_2$. By definition of positive limit point, there exists a $t_2 > 0$ such that $(\bar{z}(t_2), \xi(t_2)) \in \mathcal{N}_2$. Hence, $\Phi(t_2, (\bar{z}(0), 0)) \in \mathcal{N}_1$, that is to say, the trajectory from any point in L^+ will enter $\mathcal{N}_1$, and will converge asymptotically to the equilibrium thereafter. With regard to the relation between the systems (2.31) and (2.32), we conclude that the equilibrium of (2.31) is uniformly asymptotically stable, with domain of attraction containing any given compact set $B_R^{n+r-1} = \{Z : ||Z|| \leq R\}$.

∎

Remark 2.11 When we design the parameter k_{*1}, we would like it to be as small as possible. The selection of the open set D_0 can determine the value of k_{*1}. The larger the set D_0 is, the smaller the parameter k_{*1} can be. ∎

Lemma 2.3 Under assumptions A3 and A4, for any compact set $\Pi_{\tilde{\eta}}$ ($\tilde{\eta}(0) \in \Pi_{\tilde{\eta}}$), and any compact set Π_Z ($Z(0) \in \Pi_Z$), when $k > k_{*1}$ where k_{*1} is determined in Lemma 2.2, the system (2.28) is uniformly asymptotically stable, with its domain of attraction containing the compact set $\Pi_Z \times \Pi_{\tilde{\eta}}$.

Proof: Consider the system

$$\begin{aligned} \dot{\tilde{\eta}} &= \gamma(\tilde{\eta}, Z, 0, v, w) \\ &= M\tilde{\eta} - N\big(\beta^{[2]}(\tilde{\eta} + g^{-1}(v,w)Nu_r + \theta) - \beta^{[2]}(g^{-1}(v,w)Nu_r + \theta)\big) \\ &\quad +\phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1}, u_r, v, w) \end{aligned} \tag{2.36}$$

Let $V_1(\tilde{\eta}) = \frac{2}{r}\tilde{\eta}^T P\tilde{\eta}$, where P is defined in assumption A4. Then there exists two positive definite matrices $\underline{P}$ and $\bar{P}$ such that $2\tilde{\eta}^T\underline{P}\tilde{\eta} \le V_1(\tilde{\eta}) \le 2\tilde{\eta}^T\bar{P}\tilde{\eta}$. The derivative of $V_1(\tilde{\eta})$ along the trajectory of (2.36) is as follows,

$$\begin{aligned} \frac{dV_1(\tilde{\eta})}{dt} &= \frac{2}{r}\tilde{\eta}^T M^T P\tilde{\eta} - \frac{2}{r}\left(\beta^{[2]}(\tilde{\eta}+d) - \beta^{[2]}(d)\right)^T N^T P\tilde{\eta} + \frac{2}{r}\phi^T P\tilde{\eta} \\ &\quad +\frac{2}{r}\tilde{\eta}^T PM\tilde{\eta} - \frac{2}{r}\tilde{\eta}^T PN\left(\beta^{[2]}(\tilde{\eta}+d) - \beta^{[2]}(d)\right) + \frac{2}{r}\tilde{\eta}^T P\phi \\ &= -\frac{2}{r}\tilde{\eta}^T\tilde{\eta} - \frac{4}{r}\tilde{\eta}^T PN\left(\beta^{[2]}(\tilde{\eta}+d) - \beta^{[2]}(d)\right) + \frac{4}{r}\tilde{\eta}^T P\phi. \end{aligned}$$

Under assumption A4, there exists a positive number $r < 1$ satisfying

$$-2\tilde{\eta}^T PN(\beta^{[2]}(\tilde{\eta}+d) - \beta^{[2]}(d)) \le (1-r)\tilde{\eta}^T\tilde{\eta},$$

where $d = g^{-1}(v,w)Nu_r + \theta$. Then,

$$\begin{aligned} \frac{dV_1(\tilde{\eta})}{dt} &\le -2\tilde{\eta}^T\tilde{\eta} + \frac{4}{r}\tilde{\eta}^T P\phi \\ &\le -2\tilde{\eta}^T\tilde{\eta} + \tilde{\eta}^T\tilde{\eta} + 4||r^{-1}P\phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1}, u_r, v, w)||^2 \\ &= -||\tilde{\eta}||^2 + 4||r^{-1}P\phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1}, u_r, v, w)||^2. \end{aligned}$$

Find $R > 0$ such that $\Pi_Z \times \Pi_{\tilde{\eta}} \in B_R^{n+r+n_{\tilde{\eta}}-1} = \{(Z,\tilde{\eta}) : ||(Z,\tilde{\eta})|| \le R\}$. Then, $\Pi_Z \in B_R^{n+r-1}$. By Lemma 2.2 we can find $k_{*1} > 0$ such that, when $k > k_{*1}$, the Z subsystem can be made uniformly asymptotically stable. When k is fixed, denote $||r^{-1}P\phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1}, u_r, v, w)|| = \bar{\phi}(Z, v, w)$. Obviously, $\bar{\phi}(0, v, w) = 0$ and $\bar{\phi}(Z, v, w)$ is C^1, then we have

$$||r^{-1}P\phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1}, u_r, v, w)|| = \bar{\phi}(Z, v, w) \le ||Z||a(Z, v, w),$$

where $a(Z,v,w) \geq 1$ is a real valued function.

When $(Z,v,w) \in \Pi_Z \times V \times W$, $a(Z,v,w)$ has a maximum a_{max}. Then,

$$\frac{dV_1(\tilde{\eta})}{dt} \leq -||\tilde{\eta}||^2 + 4a_{max}^2||Z||^2$$

Thus $V_1(\tilde{\eta})$ is an ISS Lyapunov function for the system (2.36). Then we can conclude that (2.36) is input-to-state stable with $\tilde{\eta}$ as state and Z as input. When the Z subsystem is made uniformly asymptotically stable, we can get $\tilde{\eta}(t) \to 0$ as $t \to \infty$. ∎

Theorem 2.1 Under assumptions A3 and A4, for any $\varepsilon > 0$, there exists $K_{*1} > 0$ such that, for any $k > k_{*1}$, $K > K_{*1}$ where k_{*1} is determined in Lemma 2.2, the trajectory $(Z, \tilde{\eta}, \tau)$ of the closed-loop system composed of (2.27) and the controller $\bar{u} = -K\tau$ starting from initial conditions $(Z(0), \tilde{\eta}(0)) \in \Pi_Z \times \Pi_{\tilde{\eta}}$ and initial condition $\tau(0) \in \Pi_\tau$, where Π_τ is determined by k and Π_Z, is bounded, enters in finite time the set $B_\varepsilon^{n+n_{\tilde{\eta}}+r}$ and remains in $B_\varepsilon^{n+n_{\tilde{\eta}}+r}$ thereafter, where $B_\varepsilon^{n+n_{\tilde{\eta}}+r} = \{(Z,\tilde{\eta},\tau) : |(Z,\tilde{\eta},\tau)| \leq \varepsilon\}$.

Proof: Let $Z_1 = col(Z, \tilde{\eta})$. The system (2.27) can be transformed into the following system,

$$\begin{aligned} \dot{Z}_1 &= F_1(Z_1, \tau, v, w) \\ \dot{\tau} &= \chi(Z_1, \tau, v, w) + g(v,w)\bar{u} \end{aligned} \tag{2.37}$$

where $F_1(Z_1, \tau, v, w) = \begin{bmatrix} F(Z,v,w) + G\tau \\ \gamma(\tilde{\eta}, Z, \tau, v, w) \end{bmatrix}$.

The closed loop system composed of (2.37) and the controller $\bar{u} = -K\tau$ is as follows,

$$\begin{aligned} \dot{Z}_1 &= F_1(Z_1, \tau, v, w) \\ \dot{\tau} &= \chi(Z_1, \tau, v, w) - g(v,w)K\tau \end{aligned} \tag{2.38}$$

With respect to the output $\bar{y} = \tau$, the zero dynamics of (2.37) is as follows,

$$\dot{Z}_1 = F_1(Z_1, 0, v, w) = \begin{bmatrix} F(Z,v,w) \\ \gamma(\tilde{\eta}, Z, 0, v, w) \end{bmatrix} \tag{2.39}$$

We take two steps to solve this theorem. Step i is to show that the zero dynamics (2.39) of the plant (2.37) satisfies the ULP assumption. By semiglobal backstepping technique we can reach the desired conclusion, but it is hard to compute the design parameter K_{*1}.

Step ii is to use the Lyapunov's direct method to get a systematic method of computing K_{*1}.

Step i: Please see [52]. For convenience, it is also listed in the following.

It has been shown in Lemma 2.3 that the equilibrium of the zero dynamics (2.39) is uniformly asymptotically stable, with domain of attraction which includes the compact set. $\Pi_Z \times \Pi_{\bar{\eta}}$. In other words, $\Pi_Z \times \Pi_{\bar{\eta}}$ is a subset of the domain of attraction Δ of (2.39). Pick $k \geq k_{*1}$, in the flow of (2.39) with initial conditions $v(0) = 0$, there is a diffeomorphism to an open ball around the origin which is diffeomorphic to $R^{n+r+n_{\bar{\eta}}-1}$ [57]. Denote the diffeomorphism with

$$\begin{aligned} \Psi : \Delta &\rightarrow R^{n+r+n_{\bar{\eta}}-1} \\ Z_1 &\rightarrow \tilde{Z}_1 \end{aligned}$$

with $\Psi(0) = 0$. Then Ψ maps $\dot{Z}_1 = F_1(Z_1, 0, v, w)$ defined on Δ into $\dot{\tilde{Z}}_1 = \tilde{F}_1(\tilde{Z}_1, v, w)$ defined on $R^{n+r+n_{\bar{\eta}}-1}$, for which the equilibrium $\tilde{Z}_1$ is globally uniformly asymptotically stable for every $(v, w) \in V \times W$. Hence, by the converse theorem, there exists a smooth, positive definite and proper function $\tilde{V}(\tilde{Z}_1)$, such that

$$\frac{\partial \tilde{V}(\tilde{Z}_1)}{\partial \tilde{Z}_1} \tilde{F}(\tilde{Z}_1, v, w) < 0, \ \forall \tilde{Z}_1 \neq 0.$$

Under the mapping Ψ, the image $\tilde{\Pi}$ of $\Pi_Z \times \Pi_{\bar{\eta}}$ is a compact set. Therefore, there exists a number $c_1 > 1$ such that $\{\tilde{Z}_1 : \tilde{V}(\tilde{Z}_1) \leq c_1\} \supset \tilde{\Pi}$. Let $V(Z_1) = \tilde{V}(\Psi(Z_1))$, and define the set Ω_{c_1} by $\Omega_{c_1} = \{Z_1 : V(Z_1) \leq c_1\}$, such that $\Pi_Z \times \Pi_{\bar{\eta}} \subset \Omega_{c_1} \subset \Omega_{c_1+1} \subset \Delta$. In particular, $V(Z_1)$ is proper on Δ such that,

$$\frac{\partial V(Z_1)}{\partial Z_1} F_1(Z_1, 0, v, w) \leq -\beta(|Z_1|), \ \forall Z_1 \in \Omega_{c_1+1}, \ Z_1 \neq 0, \ \forall (v, w) \in V \times W$$

for some class-K function $\beta(\cdot)$. Thus, the ULP assumption is satisfied. By the semiglobal backstepping technique there exists a $K_{*1} > 0$ satisfying our need. But the computation method given in the technique is hard to use. We will use the Lyapunov's direct method to get a proper K_{*1} as follows.

Step ii: Find $R > 0$ such that $\Pi_Z \times \Pi_{\bar{\eta}} \subset B_R^{n+n_{\bar{\eta}}+r-1}$ where $B_R^{n+n_{\bar{\eta}}+r-1} \stackrel{def}{=} \{Z_1 : ||Z_1|| \leq R\}$. Let $c = \max_{||Z_1|| \leq R} V(Z_1)$ and $\Omega_c = \{Z_1 : V(Z_1) \leq c\}$, then $\Pi_Z \times \Pi_{\bar{\eta}} \subset B_R^{n+n_{\bar{\eta}}+r-1} \subset \Omega_c \subset \Omega_{c+1}$. Then

$$|\bar{x}_i| \leq R,$$

and

$$|\tau| = |\bar{x}_r + k^{r-1}b_0\bar{x}_1 + k^{r-2}b_1\bar{x}_2 + \cdots + kb_{r-2}\bar{x}_{r-1}| \leq |1 + k^{r-1}b_0 + k^{r-2}b_1 + \cdots + kb_{r-2}|R.$$

Pick k satisfying $k > k_{*1}$ where k_{*1} is defined in Lemma 2.2. Letting

$$\sigma = |1 + k^{r-1}b_0 + k^{r-2}b_1 + \cdots + kb_{r-2}|^2 R^2$$

gives $\tau^2 \leq \sigma$. Without loss of generality, assume $R > 1$, then $\sigma > 1$.

Define a Lyapunov function candidate for the closed loop system (2.38) as follows,

$$V_b(Z_1, \tau) = c\frac{V(Z_1)}{c + 1 - V(Z_1)} + \sigma\frac{\tau^2}{\sigma + 1 - \tau^2} \tag{2.40}$$

in the set $\{Z_1 : V(Z_1) < c + 1\} \times \{\tau : \tau^2 < \sigma + 1\}$.

Assuming $V_b(Z_1, \tau) \leq c^2 + \sigma^2 + 1$ gives,

$$\begin{aligned} c\frac{V(Z_1)}{c + 1 - V(Z_1)} &\leq c^2 + \sigma^2 + 1 \\ \sigma\frac{\tau^2}{\sigma + 1 - \tau^2} &\leq c^2 + \sigma^2 + 1. \end{aligned} \tag{2.41}$$

The above two inequalities imply

$$\begin{aligned} V(Z_1) &\leq (c + 1)\frac{c^2 + \sigma^2 + 1}{c^2 + \sigma^2 + 1 + c} \\ \tau^2 &\leq (\sigma + 1)\frac{c^2 + \sigma^2 + 1}{c^2 + \sigma^2 + 1 + \sigma}. \end{aligned} \tag{2.42}$$

Thus,

$$V(Z_1) < c + 1, \;\; \tau^2 < \sigma + 1, \tag{2.43}$$

also, we can get the following inequalities,

$$\begin{aligned} \frac{c}{c + 1} \leq \frac{c(c + 1)}{(c + 1 - V(Z_1))^2} &\leq \frac{(c^2 + \sigma^2 + 1 + c)^2}{c(c + 1)} \\ \frac{\sigma}{\sigma + 1} \leq \frac{\sigma(\sigma + 1)}{(\sigma + 1 - \tau^2)^2} &\leq \frac{(c^2 + \sigma^2 + 1 + \sigma)^2}{\sigma(\sigma + 1)}. \end{aligned}$$

The derivative of $V_b(Z_1, \tau)$ along the system (2.38) is as follows,

$$\begin{aligned} \dot{V}_b(Z_1, \tau) = {} & \frac{c(c + 1)}{(c + 1 - V(Z_1))^2}\frac{\partial V}{\partial Z_1}F_1(Z_1, \tau, v, w) \\ & + \frac{\sigma(\sigma + 1)}{(\sigma + 1 - \tau^2)^2}2\tau(\chi(Z_1, \tau, v, w) - g(v, w)K\tau). \end{aligned} \tag{2.44}$$

The function $F_1(Z_1, \tau, v, w)$ can be put in the following form

$$F_1(Z_1, \tau, v, w) = F_1(Z_1, 0, v, w) + p_1(Z_1, \tau, v, w)\tau,$$

where $p_1(Z_1, \tau, v, w)$ is a smooth function.
Define the set $\Theta_{c^2+\sigma^2+1}$ by

$$\Theta_{c^2+\sigma^2+1} \overset{def}{=} \{(Z_1, \tau) : V_b(Z_1, \tau) \leq c^2 + \sigma^2 + 1\}.$$

If $(Z_1, \tau) \in \Theta_{c^2+\sigma^2+1}$, then,

$$\begin{aligned}
\dot{V}_b(Z_1, \tau) &= \frac{c(c+1)}{(c+1-V(Z_1))^2}\frac{\partial V}{\partial Z_1}(F_1(Z_1, 0, v, w) + p_1(Z_1, \tau, v, w)\tau) \\
&\quad + \frac{\sigma(\sigma+1)}{(\sigma+1-\tau^2)^2}2\tau(\chi(Z_1, \tau, v, w) - g(v, w)K\tau) \\
&\leq \frac{c(c+1)}{(c+1-V(Z_1))^2}\frac{\partial V}{\partial Z_1}F_1(Z_1, 0, v, w) - 2\frac{\sigma}{\sigma+1}b_0K\tau^2 \\
&\quad + \Big(\frac{(c^2+\sigma^2+1+c)^2}{c(c+1)}|\frac{\partial V}{\partial Z_1}p_1(Z_1, \tau, v, w)\tau| \\
&\quad + \frac{(c^2+\sigma^2+1+\sigma)^2}{\sigma(\sigma+1)}2|(\chi(Z_1, \tau, v, w)||\tau|\Big) \qquad (2.45)
\end{aligned}$$

Choose any arbitrarily small number $\rho > 0$ satisfying $\Theta_\rho \subset B_\varepsilon^{n+n_{\tilde{\eta}}+r}$. Define two sets S, S_0 by

$$\begin{aligned}
S &= \{(Z_1, \tau) : \rho \leq V_b(Z_1, \tau) \leq c^2 + \sigma^2 + 1\} \\
S_0 &= \{(Z_1, \tau) : \tau = 0\} \cap S.
\end{aligned}$$

Since $V_b(Z_1, \tau) \leq c^2 + \sigma^2 + 1$ gives $V(Z_1) < c + 1$, and $\tau^2 < \sigma + 1$, then

$$S_0 \subset \{(Z_1, \tau) : V(Z_1) < c + 1, \tau = 0\}.$$

By the ULP assumption, $\dot{V}_b(Z_1, \tau) < 0$ when $(Z_1, \tau) \in S_0$.

By continuity, there exists some open set S_1 such that $S_1 \supset S_0$ and $\dot{V}_b(Z_1, \tau) < 0$ when $(Z_1, \tau) \in S_1$. Consider the compact set $S_2 = S \setminus S_1$. Since $\tau \neq 0$ at each point of S_2, then there exists $m > 0$ such that

$$\tau^2 > m, \ \forall (Z_1, \tau) \in S_2.$$

Also, there exists $M > 0$ such that $\forall (Z_1, \tau) \in S_2, \ \forall (v, w) \in V \times W$,

$$\Big(\frac{(c^2+\sigma^2+1+c)^2}{c(c+1)}|\frac{\partial V}{\partial Z_1}p_1(Z_1, \tau, v, w)\tau| + \frac{(c^2+\sigma^2+1+\sigma)^2}{\sigma(\sigma+1)}2|(\chi(Z_1, \tau, v, w)||\tau|\Big) \leq M.$$

Hence,

$$\dot{V}_b(Z_1,\tau) \le -2\frac{\sigma}{\sigma+1}b_0 K m + M.$$

Letting $K_{*1} = \frac{(\sigma+1)M}{2\sigma b_0 m}$ gives when $K > K_{*1}$, $\dot{V}_b(Z_1,\tau) < 0$. Therefore, the trajectory of (2.37) starting from any given compact set $\Pi_Z \times \Pi_{\tilde{\eta}} \times \Pi_\tau$ is bounded, enters in finite time any given arbitrarily small compact set $B_\varepsilon^{n+n_{\tilde{\eta}}+r}$, and remains in it thereafter. ∎

Lemma 2.4 Under assumptions A3 and A4, there exists $k_{*2} > 0$ and $K_{*2} > 0$ such that when $k > k_{*2}$ and $K > K_{*2}$, the closed loop system (2.38) is locally exponentially stable.

Proof: Let $\xi_i = \frac{\bar{x}_i}{k^{i-1}}$, $i = 1, \cdots, r-1$, denote $\xi = (\xi_1, \cdots, \xi_{r-1})$, then we can transform the system (2.38) into the following system:

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}_0(\bar{z}, \xi_1, v, w) \\
\dot{\xi} &= kA\xi + \frac{B\tau}{k^{r-2}} \\
\dot{\tilde{\eta}} &= \gamma(\tilde{\eta}, \bar{z}, \xi, \tau, v, w) \\
\dot{\tau} &= \chi(\tilde{\eta}, \bar{z}, \xi, \tau, v, w) - g(v,w)K\tau
\end{aligned} \tag{2.46}$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -b_0 & -b_1 & -\cdots & -b_{r-3} & -b_{r-2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Since A is Hurwitz, there exists a real symmetric positive definite matrix P_0 satisfying the following equation $A^T P_0 + P_0 A = -I$.

Pick a Lyapunov candidate

$$V_a = \epsilon_1 V_0(\bar{z}) + \xi^T P_0 \xi + \frac{2}{r}\tilde{\eta}^T P \tilde{\eta} + \frac{1}{2}\tau^2,$$

where ϵ_1 is a positive number to be determined, $V_0(\bar{z})$ is defined in assumption A3, r and P are defined in assumption A4.

Since the function $\bar{f}_0$ is smooth, it can be written as follows,

$$\bar{f}_0(\bar{z}, \xi_1, v, w) = \bar{f}_0(\bar{z}, 0, v, w) + p(\bar{z}, \xi_1, v, w)\xi_1, \tag{2.47}$$

where $p(\bar{z},\xi_1,v,w)$ is a smooth function. Since the function $\chi(\tilde{\eta},\bar{z},\xi,\tau,v,w)$ is smooth, we have

$$|\chi(\tilde{\eta},\bar{z},\xi,\tau,v,w)| \leq ||(\tilde{\eta},\bar{z},\xi)||a(\tau,v,w) + |\tau|b(\tilde{\eta},\bar{z},\xi,v,w),$$

where $a(\tau,v,w) \geq 1$ and $b(\tilde{\eta},\bar{z},\xi,v,w) \geq 1$ are smooth functions (see [42]).

Along the trajectories of (2.46), and when $\bar{z} \in [0,\delta]$, $\tilde{\eta}$, ξ, v and w are in their compact sets respectively,

$$\begin{aligned}
\dot{V}_a &= \epsilon_1 \frac{\partial V_0}{\partial \bar{z}}\left(\bar{f}_0(\bar{z},0,v,w) + p(\bar{z},\xi_1,v,w)\xi_1\right) + k\xi^T(A^T P_0 + P_0 A)\xi \\
&\quad + \frac{\tau}{k^{r-2}}(B^T P_0 \xi + \xi^T P_0 B) - \frac{2}{r}\tilde{\eta}^T\tilde{\eta} - \frac{4}{r}\tilde{\eta}^T PN\left(\beta^{[2]}(\tilde{\eta}+d) - \beta^{[2]}(d)\right) \\
&\quad + \frac{4}{r}\tilde{\eta}^T P\phi(\bar{z},\xi,\tau+u_r,v,w) + \tau\chi(\tilde{\eta},\bar{z},\xi,\tau,v,w) - g(v,w)K\tau^2 \\
&\leq -a_0\epsilon_1||\bar{z}||^2 + \epsilon_1\hat{a}_0 p_m||\bar{z}|||\xi_1| - k||\xi||^2 + \frac{p_{0m}}{k^{r-2}}|\tau|||\xi|| \\
&\quad -||\tilde{\eta}||^2 + 4||r^{-1}P\phi(\bar{z},\xi,\tau+u_r,v,w)||^2 + a_m|\tau|||(\tilde{\eta},\bar{z},\xi)|| + b_m|\tau|^2 - b_0K|\tau|^2 \\
&\leq -a_0\epsilon_1||\bar{z}||^2 + \epsilon_1\hat{a}_0 p_m\mu_1||\bar{z}||^2 + \epsilon_1\frac{\hat{a}_0 p_m}{\mu_1}|\xi|^2 - k||\xi||^2 + \frac{p_{0m}}{k^{r-2}}(|\tau|^2 + ||\xi||^2) - ||\tilde{\eta}||^2 \\
&\quad + ||(\bar{z},\xi,\tau)||^2(c(\bar{z},\xi,\tau,v,w))^2 + a_m u_2||(\tilde{\eta},\bar{z},\xi)||^2 + \frac{a_m}{u_2}|\tau|^2 + b_m|\tau|^2 - b_0K|\tau|^2 \\
&\leq -(a_0\epsilon_1 - \epsilon_1\hat{a}_0 p_m\mu_1 - c_m^2 - a_m u_2)||\bar{z}||^2 - (k - \epsilon_1\frac{\hat{a}_0 p_m}{\mu_1} - \frac{p_{0m}}{k^{r-2}} - c_m^2 - a_m u_2)||\xi||^2 \\
&\quad -(1 - a_m u_2)||\tilde{\eta}||^2 - (b_0 K - \frac{p_{0m}}{k^{r-2}} - c_m^2 - \frac{a_m}{u_2} - b_m)|\tau|^2
\end{aligned}$$

where p_m is the maximum value of $p(\bar{z},\xi_1,v,w)$, p_{0m} is determined by P_0 and B, $c(\bar{z},\xi,\tau,v,w) \geq 1$, a_m is the maximum value of $a(\tau,v,w)$, b_m is the maximum value of $b(\tilde{\eta},\bar{z},\xi,v,w)$, c_m is the maximum value of $c(\bar{z},\xi,\tau,v,w)$, μ_1 and μ_2 are two positive numbers to be determined later, $d = g^{-1}(v,w)N(\tau+u_r) + \theta$.

When μ_1 and μ_2 is small enough, k and K is large enough, ϵ_1 is proper chosen, the coefficients of $||\bar{z}||^2$, $||\xi||^2$, $||\tilde{\eta}||^2$ and $|\tau|^2$ can all be made negative. That is to say, there exist $\mu_{1*} > 0$, $\mu_{2*} > 0$, $k_{*2} > 0$, $K_{*2} > 0$ and $\epsilon_{1*} > 0$ such that when $\mu_1 < \mu_{1*}$, $\mu_2 < \mu_{2*}$, $k > k_{*2}$, $K > K_{*2}$ and $\epsilon_1 > \epsilon_{1*}$, the closed loop system (2.46) is locally exponentially stable, so is (2.38).

∎

Remark 2.12 With the local exponential stability property of the closed loop system (2.38), we can build the following theorem on the basis of Lemma 2.4 and Theorem 2.1 [36]. ∎

Theorem 2.2 Under assumptions A3 and A4, when $k > k_*$ and $K > K_*$ where $k_* = \max\{k_{*1}, k_{*2}\}$ and $K_* = \max\{K_{*1}, K_{*2}\}$ the equilibrium $(Z, \tilde{\eta}, \tau) = (0, 0, 0)$ of (2.38) is asymptotically stable, with its domain of attraction containing the set $\Pi_Z \times \Pi_{\tilde{\eta}} \times \Pi_\tau$. The partial state feedback controller takes the following form

$$\bar{u} = -K(\bar{x}_r + k^{r-1}b_0\bar{x}_1 + k^{r-2}b_1\bar{x}_2 + \cdots + kb_{r-2}\bar{x}_{r-1}). \tag{2.48}$$

∎

2.5 Design of the Output Feedback Regulator

The controller (2.48) depends on $\bar{x}_1, \cdots, \bar{x}_r$, where $\bar{x}_1 = e$ is measurable and therefore can be used in the feedback controller, but the other states $\bar{x}_2, \cdots, \bar{x}_r$ can not be used. In order to obtain an output feedback controller depending only on $\bar{x}_1$. We can use the saturated high-gain observer [12] to generate the estimates of $\bar{x}_2, \cdots, \bar{x}_r$, and to ensure the semiglobal stabilization of the interconnected system by careful choice of the design parameters. Denote $\tilde{x} = \text{col}(\tilde{x}_1, \cdots, \tilde{x}_r)$, then the observer is as follows,

$$\dot{\tilde{x}} = A\tilde{x} + Be \tag{2.49}$$

where $\lambda^r + c_{r-1}\lambda^{r-1} + \cdots + c_1\lambda + c_0$ is hurwitz, $l > 0$ is to be determined later,

$$A = \begin{bmatrix} -lc_{r-1} & 1 & 0 & \cdots & 0 \\ -l^2c_{r-2} & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -l^{r-1}c_1 & 0 & 0 & \cdots & 1 \\ -l^rc_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} lc_{r-1} \\ l^2c_{r-2} \\ \cdots \\ l^{r-1}c_1 \\ l^rc_0 \end{bmatrix}.$$

As in [12], in order to get rid of the peaking phenomenon, a saturation function $\text{sat}(\cdot, \cdot)$ is used to saturate the control,

$$\text{sat}(a, a_0) = \begin{cases} a & \text{if } |a| \le a_0, \\ \frac{a}{|a|}a_0 & \text{if } |a| > a_0. \end{cases}$$

Then the new controller is as follows,

$$\begin{aligned} \bar{u} &= -K\text{sat}(\tilde{\tau}, \tilde{\tau}_0) \\ \dot{\tilde{x}} &= A\tilde{x} + Be \end{aligned} \tag{2.50}$$

where $\tilde{\tau} = \tilde{x}_r + k^{r-1}b_0\tilde{x}_1 + k^{r-2}b_1\tilde{x}_2 + \cdots + kb_{r-2}\tilde{x}_{r-1}$, and $\tilde{\tau}_0$ is the set value to saturate the control.

With the new controller (2.50), the closed loop system can be written as follows,

$$\begin{aligned}
\dot{Z}_1 &= F_1(Z_1, \tau, v, w) \\
\dot{\tau} &= \chi(Z_1, \tau, v, w) - g(v, w)K\text{sat}(\tilde{\tau}, \tilde{\tau}_0) \\
\dot{\tilde{x}} &= A\tilde{x} + Be.
\end{aligned} \tag{2.51}$$

Theorem 2.3 Under assumptions A3 and A4, for any $\varepsilon > 0$ there exist $k_* > 0$, $K_* > 0$, $l_* > 0$ and $\tilde{\tau}_0 > 0$ such that, for any $k_* > 0$, $K > K_*$ and $l > l_*$, the trajectory of the closed loop system (2.51) starting from the compact set $\Pi_Z \times \Pi_{\bar{\eta}} \times \Pi_\tau \times \Xi_0$ is bounded, enters in finite time the set $B_\varepsilon^{n+n_{\bar{\eta}}+2r}$ and remains in $B_\varepsilon^{n+n_{\bar{\eta}}+2r}$ thereafter, where $B_\varepsilon^{n+n_{\bar{\eta}}+2r} = \{(Z, \bar{\eta}, \tau, \tilde{x}) : |(Z, \bar{\eta}, \tau, \tilde{x})| \leq \varepsilon\}$.
Proof: Let $\xi_i = l^{r-i}(\bar{x}_i - \tilde{x}_i)$, $i = 1, \cdots, r$, and $\xi = col(\xi_1, \cdots, \xi_r)$. Then the closed loop system (2.51) can be rewritten as follows,

$$\begin{aligned}
\dot{Z}_1 &= F_1(Z_1, \tau, v, w) \\
\dot{\tau} &= \chi(Z_1, \tau, v, w) - g(v, w)K\tau + \phi_1(\tau, \xi) \\
\dot{\xi} &= l\bar{A}\xi + \bar{B}\phi_2(Z_1, \tau, \xi, v, w)
\end{aligned} \tag{2.52}$$

where

$$\begin{aligned}
\phi_1(\tau, \xi) &= g(v, w)K(\tau - \text{sat}(\tilde{\tau}, \tilde{\tau}_0)) \\
\phi_2(Z_1, \tau, \xi, v, w) &= \bar{f}_r(\bar{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) - g(v, w)K\text{sat}(\tilde{\tau}, \tilde{\tau}_0)
\end{aligned}$$

$$\bar{A} = \begin{bmatrix} c_{r-1} & 1 & 0 & \cdots & 0 \\ c_{r-2} & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ c_1 & 0 & 0 & \cdots & 1 \\ c_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \text{ and } \bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

By Theorem 2.2, there exists $K_* > 0$ such that when $K > K_*$, the derivative of $V_b(Z_1, \tau)$ along the trajectory of (2.37) satisfies the following inequality,

$$\dot{V}_b(Z_1, \tau) \leq -\alpha_2(V_b(Z_1, \tau)), \ \forall (Z_1, \tau) \in S$$

where $\alpha_2(\cdot)$ is a class K_∞ function and $S = \{(Z_1, \tau) : \rho \le V_b(Z_1, \tau) \le c^2 + \sigma^2 + 1\}$. Choose K satisfying $K > K_*$, and choose the saturation level as $\tilde{\tau}_0 = \max_{(Z_1,\tau)\in\Theta_{c^2+\sigma^2+1}} |\tau|$. Notice that, for all $(Z_1, \tau, \xi) \in \Theta_{c^2+\sigma^2+1} \times R^r$,

$$\begin{aligned} |\phi_1(\tau, \xi)| &\le d_1 \\ |\phi_1(\tau, \xi)| &\le \gamma(|\xi|) \\ ||\bar{B}\phi_2(Z_1, \tau, \xi, v, w)|| &\le d_2 \end{aligned}$$

where d_1, d_2 are positive numbers, and $\gamma(\cdot)$ is a continuous function satisfying $\gamma(0) = 0$. The derivative of $V_b(Z_1, \tau)$ along the trajectory of (2.52) satisfies the following inequality,

$$\begin{aligned} \dot{V}_b(Z_1, \tau) &= \frac{c(c+1)}{(c+1-V(Z_1))^2} \frac{\partial V}{\partial Z_1} F_1(Z_1, \tau, v, w) \\ &\quad + \frac{\sigma(\sigma+1)}{(\sigma+1-\tau^2)^2} 2\tau\big(\chi(Z_1, \tau, v, w) - g(v, w)K\tau + \phi_1(\tau, \xi)\big) \\ &\le -\alpha_2(V_b(Z_1, \tau)) + 2\frac{(c^2+\sigma^2+1+\sigma)^2}{\sigma(\sigma+1)} |\tau||\phi_1(\tau, \xi)| \\ &\le -\alpha_2(V_b(Z_1, \tau)) + 2M_1|\phi_1(\tau, \xi)| \\ &\le -\alpha_2(V_b(Z_1, \tau)) + 2M_1 d_1 \end{aligned} \tag{2.53}$$

where $M_1 = \max_{\tau\in\Theta_{c^2+\sigma^2+1}} \frac{(c^2+\sigma^2+1+\sigma)^2}{\sigma(\sigma+1)} |\tau|$.

Let P_2 satisfies $P_2\bar{A} + \bar{A}^T P_2 = -I$, and let $V_\xi(\xi) = \xi^T P_2 \xi$. Then,

$$\begin{aligned} \dot{V}_\xi(\xi) &= l\xi^T(\bar{A}^T P_2 + P_2\bar{A})\xi + ({\phi_2}^T \bar{B}^T P_2 \xi + \xi^T P_2 \bar{B}\phi_2) \\ &\le -l||\xi||^2 + 2|\xi^T P_2| d_2 \\ &\le -(l - \frac{d_3}{\varpi})||\xi||^2 + \varpi {d_2}^2 \\ &\le -(l - \frac{d_3}{\varpi}) d_4 V_\xi(\xi) + \varpi {d_2}^2 \end{aligned}$$

where ϖ is any positive number, and d_3, d_4 are two positive number determined by P_2. Letting $d_5 = (l - \frac{d_3}{\varpi})d_4$ gives

$$\dot{V}_\xi(\xi) \le -d_5 V_\xi(\xi) + 4\varpi {d_2}^2. \tag{2.54}$$

If l is large enough, d_5 is a positive number.

Claim that there exists a time $T > 0$ (independent of l) such that, for every initial state $(Z_1(0), \tau(0)) \in (\Pi_Z \times \Pi_{\bar{\eta}}) \times \Pi_\tau$, the solution of the closed loop system (2.52) is defined for all $t \in [0, T]$, and $(Z_1(t), \tau(t)) \in \Theta_{c^2+\sigma^2+1}$, $\forall t \in [0, T]$. By (2.53), we have

$$V_b(Z_1(t), \tau(t)) - V_b(Z_1(0), \tau(0)) \le 2M_1 d_1 t.$$

Let $T < \frac{1}{2M_1 d_1}$, then

$$V_b(Z_1(t), \tau(t)) \le V_b(Z_1(0), \tau(0)) + 1 < c^2 + \sigma^2 + 1.$$

That is,

$$(Z_1(t), \tau(t)) \in \Theta_{c^2+\sigma^2+1}, \ \forall t \in [0, T].$$

Claim that for any positive number ε_1 there exists a number $l_* > 0$ such that if $l > l_*$, then $||\xi(T)|| \le \varepsilon_1$. Moreover, if $(Z_1(0), \tau(0)) \in (\Pi_Z \times \Pi_{\tilde{\eta}}) \times \Pi_\tau$, then for all $t > T$, $(Z_1(t), \tau(t)) \in \Theta_{c^2+\sigma^2+1}$ and $||\xi(t)|| \le \varepsilon_1$. The proof is similar to Lemma 3 [37], and it is also given as follows. By comparison arguments, (2.54) gives

$$||\xi(t)||^2 \le d_6\big(e^{-d_5 t}||\xi(0)||^2 + \frac{1 - e^{-d_5 t}}{d_5}\varpi d_2{}^2\big),$$

where $d_6 > 0$ is a real number depending only on P_2.

Choose ϖ to satisfy $2d_6 \varpi d_2{}^2 \le \varepsilon_1{}^2$, and choose $l_{*1} > 1$ to satisfy $d_5 > 1$ when $l > l_{*1}$, then

$$||\xi(t)||^2 \le d_6 e^{-d_5 t}||\xi(0)||^2 + \frac{\varepsilon_1{}^2}{2}. \tag{2.55}$$

Because

$$\lim_{l\to\infty} d_6 e^{-d_5 T}||\xi(0)||^2 = \lim_{l\to\infty} d_6 e^{-(l - \frac{d_3}{\varpi})d_4 T}||\xi(0)||^2 \le \lim_{l\to\infty} d_6 e^{-(l-\frac{d_3}{\varpi})d_4 T} l^r ||\bar{x}(0) - \tilde{x}(0)||^2 = 0,$$

$$\lim_{l\to\infty} d_6 e^{-d_5 T}||\xi(0)||^2 = 0.$$

Since $\bar{x}(0)$ and $\tilde{x}(0)$ range over a compact set, there exists a real number $l_{*2} > 0$ such that

$$d_6 e^{-d_5 T}||\xi(0)||^2 \le \frac{\varepsilon_1{}^2}{2}. \tag{2.56}$$

Then, $||\xi(T)|| < \varepsilon_1$ for $l > l_* = \max(l_{*1}, l_{*2})$. By (2.55), obviously, we have $||\xi(t)|| \le \varepsilon_1$. When $(Z_1, \tau) \in S$ and $||\xi(t)|| \le \varepsilon_1$,

$$\dot{V}_b(Z_1, \tau) \le -\alpha_2(V_b(Z_1, \tau)) + 2M_1\gamma(\varepsilon_1). \tag{2.57}$$

Choose ε_1 so that

$$\alpha_2^{-1}(4M_1\gamma(\varepsilon_1)) < \rho, \tag{2.58}$$

then $\dot{V}_b(Z_1, \tau) \le -\frac{\varrho}{2}$. Therefore, $(Z_1(t), \tau(t)) \in \Theta_{c^2+\sigma^2+1}$.

Claim that the trajectory of the closed loop system is bounded, enters the set $B_\varepsilon^{n+n_{\bar{\eta}}+2r}$ in finite time and remains in it thereafter. We use contradiction to prove it. Assume $V_b(Z_1, \tau)$ is always decreasing and converges to a nonnegative limit $V_{b\infty}$, and $V_{b\infty} \ge \rho$. Let L^+ denote the positive limit set of the trajectory of $(Z_1(t), \tau(t))$. Then $V_b(Z_1, \tau) = V_{b\infty}$ at every point of L^+. Pick initial condition in L^+, then $\dot{V}_b(Z_1, \tau) = 0$. From the inequality (2.57), $0 \le -\alpha_2(V_{b\infty}) + 2M_1\gamma(\varepsilon_1)$, then $\rho < V_{b\infty} < \alpha_2^{-1}(2M_1\gamma(\varepsilon_1))$ which is a contradiction against inequality (2.58). Therefore, the trajectory enters Θ_ρ. Since $\dot{V}_b(Z_1, \tau)$ is negative at each point of the boundary of Θ_ρ, the trajectory remains in Θ_ρ after entering it. Since $\tilde{x}_i = \bar{x}_i - \frac{\xi_i(t)}{l^{r-i}}$, when $l > 1$ and t is large enough, we have $||\tilde{x}(t)|| < \varepsilon$.

∎

Since the closed loop system under partial state feedback (2.38) is locally exponentially stable. Hence, the trajectory of the closed loop systems (2.52) is convergent [41]. As a result, the semiglobal robust output regulation problem of (2.1) is solvable by the output feedback control law of the following form,

$$\begin{aligned} u &= \beta(\eta) - K\mathrm{sat}(\tilde{\tau}, \tilde{\tau}_0) \\ \dot{\eta} &= M\eta + N(u - \beta(\eta) + ET^{-1}\eta) \\ \dot{\tilde{x}} &= A\tilde{x} + Be \end{aligned} \tag{2.59}$$

where $\tilde{\tau} = \tilde{x}_r + k^{r-1}b_0\tilde{x}_1 + k^{r-2}b_1\tilde{x}_2 + \cdots + kb_{r-2}\tilde{x}_{r-1}$, and $\tilde{\tau}_0$ is the set value to saturate the control.

2.6 An example

Consider the plant as follows,

$$\begin{aligned} \dot{z} &= -z + x_1 + zx_1 - v_1 z + v_2 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= wz - x_1 + 0.1\sin^2(wx_1) + u \\ e &= x_1 - v_1 \end{aligned} \tag{2.60}$$

with the exosystem

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -v_1. \end{aligned} \tag{2.61}$$

where $v(t) \in V = \{v_1^2 + v_2^2 \leq 1\}$, $w \in W = \{|w| \leq 1\}$, $z(0) \in Z_0 = \{|z(0)| \leq 1\}$, $x(0) \in X_0 = \{|x(0)| \leq 1\}$. The objective is to design a controller to solve the output regulation problem.

The solution of the regulator equation is

$$\begin{aligned} \mathbf{z}(v,w) &= v_1 \\ \mathbf{x}_1(v,w) &= v_1 \\ \mathbf{x}_2(v,w) &= v_2 \\ \mathbf{u}(v,w) &= -wv_1 - 0.1\sin^2(wv_1). \end{aligned}$$

Let $\pi_1(v,w) = wv_1$. The minimal zeroing polynomial of $\pi_1(v,w)$ is $P_1(s) = s^2 + 1$, $\Phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $E = \begin{bmatrix} -1 & 0 \end{bmatrix}$. The pair $\{E, \Phi\}$ is observable, thus the generator is linearly observable.

Choose $M = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $N = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which are controllable. Solving the Sylvester equation $MT + NE = T\Phi$ gives $T = \begin{bmatrix} -0.5 & 0.5 \\ -0.8 & 0.4 \end{bmatrix}$. Under the above design,

$$\theta = T \begin{bmatrix} \pi_1(v,w) \\ \dot{\pi}_1(v,w) \end{bmatrix} = T \begin{bmatrix} wv_1 \\ wv_2 \end{bmatrix} = \begin{bmatrix} -0.5wv_1 + 0.5wv_2 \\ -0.8wv_1 + 0.4wv_2 \end{bmatrix},$$

and

$$\beta(\theta) = -2\theta_1 + 2.5\theta_2 - 0.1\sin^2(2\theta_1 - 2.5\theta_2).$$

The internal model is as follows,

$$\dot{\eta} = M\eta + N(u - \beta(\eta) + ET^{-1}\eta). \tag{2.62}$$

Performing on the plant and the internal model the following coordinates transformation:

$$
\begin{aligned}
\bar{\eta} &= \eta - \theta \\
\bar{z} &= z - \mathbf{z}(v,w) = z - v_1 \\
\bar{x}_1 &= x_1 - \mathbf{x}_1(v,w) = x_1 - v_1 = e \\
\bar{x}_2 &= x_2 - \mathbf{x}_2(v,w) = x_2 - v_2 \\
\bar{u} &= u - \beta(\eta)
\end{aligned} \tag{2.63}
$$

gives

$$
\begin{aligned}
\dot{\bar{z}} &= -\bar{z} + \bar{x}_1 + \bar{z}\bar{x}_1 + v_1\bar{x}_1 \\
\dot{\bar{\eta}} &= (M + NET^{-1})\bar{\eta} + N\bar{u} \\
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= w\bar{z} - \bar{x}_1 + 0.1\sin^2(w\bar{x}_1 + wv_1) + wv_1 + \beta(\eta) + \bar{u}.
\end{aligned} \tag{2.64}
$$

Letting $\tilde{\eta} = \bar{\eta} - N\bar{x}_2$ gives

$$
\dot{\tilde{\eta}} = M\tilde{\eta} - N\big(\beta^{[2]}(\tilde{\eta} + N\bar{x}_2 + \theta) - \beta^{[2]}(N\bar{x}_2 + \theta)\big) + \phi(\bar{z}, \bar{x}_1, \bar{x}_2, v, w) \tag{2.65}
$$

where

$$
\begin{aligned}
\phi(\bar{z}, \bar{x}_1, \bar{x}_2, v, w) &= -N\big(w\bar{z} - \bar{x}_1 + 0.1\sin^2(w\bar{x}_1 + wv_1) - 0.1\sin^2(wv_1)\big) + MN\bar{x}_2 \\
&\quad -N\big(\beta^{[2]}(N\bar{x}_2 + \theta) - \beta^{[2]}(\theta)\big).
\end{aligned}
$$

Let $\tau = \bar{x}_2 + kb_0\bar{x}_1$ where $b_0 > 0$ such that $\lambda + b_0$ is Hurwitz. Here, $u_r = -kb_0\bar{x}_1$. Letting $Z = \mathrm{col}(\bar{z}, \bar{x}_1)$ gives

$$
\begin{aligned}
\dot{Z} &= F(Z, v, w) + G\tau \\
\dot{\tilde{\eta}} &= M\tilde{\eta} - N\big(\beta^{[2]}(\tilde{\eta} + N(\tau - kb_0\bar{x}_1) + \theta) - \beta^{[2]}(N(\tau - kb_0\bar{x}_1) + \theta)\big) \\
&\quad +\phi(\bar{z}, \bar{x}_1, \tau - kb_0\bar{x}_1, v, w) \\
\dot{\tau} &= w\bar{z} - (1 + k^2b_0^2)\bar{x}_1 + 0.1\sin^2(w\bar{x}_1 + wv_1) + wv_1 + \beta(\eta) + kb_0\tau + \bar{u},
\end{aligned} \tag{2.66}
$$

where $F(Z, v, w) = \begin{bmatrix} -\bar{z} + \bar{x}_1 + \bar{z}\bar{x}_1 + v_1\bar{x}_1 \\ -kb_0\bar{x}_1 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$
\begin{aligned}
\phi(\bar{z}, \bar{x}_1, (\tau - kb_0\bar{x}_1), v, w) &= -N\big(w\bar{z} - \bar{x}_1 + 0.1\sin^2(w\bar{x}_1 + wv_1) - 0.1\sin^2(wv_1)\big) \\
&\quad +MN(\tau - kb_0\bar{x}_1) - N\big(\beta^{[2]}(N(\tau - kb_0\bar{x}_1) + \theta) - \beta^{[2]}(\theta)\big).
\end{aligned}
$$

If we define a new output for the above system as $\bar{y} = \tau$, then the zero dynamics with respect to the output $\bar{y}$ are given by

$$\begin{aligned} \dot{Z} &= F(Z,v,w) = \begin{bmatrix} -\bar{z} + \bar{x}_1 + \bar{z}\bar{x}_1 + v_1\bar{x}_1 \\ -kb_0\bar{x}_1 \end{bmatrix} \\ \dot{\tilde{\eta}} &= M\tilde{\eta} - N\big(\beta^{[2]}(\tilde{\eta} - Nkb_0\bar{x}_1 + \theta) - \beta^{[2]}(-Nkb_0\bar{x}_1 + \theta)\big) \\ &\quad +\phi(\bar{z}, \bar{x}_1, -kb_0\bar{x}_1, v, w), \end{aligned} \tag{2.67}$$

where

$$\begin{aligned} \phi(\bar{z}, \bar{x}_1, -kb_0\bar{x}_1, v, w) &= -N\big(w\bar{z} - \bar{x}_1 + 0.1\sin^2(w\bar{x}_1 + wv_1) - 0.1\sin^2(wv_1)\big) \\ &\quad -MNkb_0\bar{x}_1 - N\big(\beta^{[2]}(-Nkb_0\bar{x}_1 + \theta) - \beta^{[2]}(\theta)\big). \end{aligned}$$

Since $\mathbf{u}(\mathbf{v},\mathbf{w}) = -wv_1 - 0.1\sin^2(wv_1)$ is not a polynomial, and the z subsystem $\dot{\bar{z}} = -\bar{z} + \bar{x}_1 + \bar{z}\bar{x}_1 + v_1\bar{x}_1$ is not input to state stable, the semiglobal output regulation of this example can not be handled by any existing techniques.

To verify assumption A3, note that the equilibrium $\bar{z} = 0$ of the following system

$$\dot{\bar{z}} = \bar{f}_0(\bar{z}, 0, v, w) = -\bar{z}$$

is globally asymptotically stable and locally exponentially stable, uniformly with respect to $v(t) \in V$, $w \in W$, and there exists a Lyapunov function $V_0(\bar{z}) = \bar{z}^2$ such that,

$$\frac{\partial V_0(\bar{z})}{\partial \bar{z}} \bar{f}_0(\bar{z}, 0, v, w) = -\bar{z}^2.$$

To verify assumption A4, letting $PM + M^TP = -I$ and solving the Lyapunov equation gives $P = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.25 \end{bmatrix}$. Further

$$\begin{aligned} & -2\tilde{\eta}^T \begin{bmatrix} 0.5 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} |{\beta_1}^{[2]}(\tilde{\eta} + d) - {\beta_1}^{[2]}(d)| \\ = \;& 0.1|(\tilde{\eta}_1 + \tilde{\eta}_2)\left(\sin^2(2\tilde{\eta}_1 - 2.5\tilde{\eta}_2 + 2d_1 - 2.5d_2) - \sin^2(2d_1 - 2.5d_2)\right)| \\ \le \;& |0.1(\tilde{\eta}_1 + \tilde{\eta}_2)(2\tilde{\eta}_1 - 2.5\tilde{\eta}_2)| \\ \le \;& 0.275|\tilde{\eta}|^2. \end{aligned}$$

Thus, $-2\tilde{\eta}^T PN({\beta_1}^{[2]}(\tilde{\eta} + d) - {\beta_1}^{[2]}(d)) \le (1 - r)\tilde{\eta}^T\tilde{\eta}$ is satisfied when $0 < r < 0.725$.

To get k_* according to the algorithm in Lemma 2.2. Taking $V_a(\bar{z}, \bar{x}_1) = V_0(\bar{z}) + \frac{1}{2}\bar{x}_1^2$ and $b_0 = 1$, then

$$\dot{V}_a(\bar{z}, \bar{x}_1) = -2\bar{z}^2 + 2\bar{z}(1 + \bar{z} + v_1)\bar{x}_1 - k\bar{x}_1^2.$$

When $\bar{x}_1 = 0$,

$$\dot{V}_a(\bar{z}, \bar{x}_1) = -2\bar{z}^2 \leq 0.$$

By continuity, there is an open set D_0 containing the set $\{(\bar{z}, \bar{x}_1) : |\bar{z}| \leq 1,\ \bar{x}_1 = 0\}$ such that $\dot{V}_a$ is negative definite.

When $\bar{x}_1 \neq 0$ and $(\bar{z}, \bar{x}_1) \in \Omega_c / D_0$, we have,

$$\begin{aligned} \dot{V}_a(\bar{z}, \bar{x}_1) &\leq -2\bar{z}^2 + \bar{z}^2 + 2(1 + v_1)\bar{z}\bar{x}_1 - k\bar{x}_1^2 \\ &\leq -\bar{z}^2 + 4|\bar{z}||\bar{x}_1| - k\bar{x}_1^2 \\ &\leq 4\bar{x}_1^2 - k\bar{x}_1^2. \end{aligned}$$

Taking $k_{*1} = 4$, when $k > k_{*1}$, $\dot{V}_a$ is negative definite.

Since

$$\begin{aligned} \bar{f}_0(\bar{z}, \bar{x}_1, v, w) &= -\bar{z} + \bar{x}_1 + \bar{z}\bar{x}_1 + v_1\bar{x}_1 \\ &= -\bar{z} + (1 + \bar{z} + v_1)\bar{x}_1 \\ &= \bar{f}_0(\bar{z}, 0, v, w) + p(\bar{z}, \bar{x}_1, v, w)\bar{x}_1 \end{aligned}$$

where

$$p(\bar{z}, \bar{x}_1, v, w) = 1 + \bar{z} + v_1,$$

then

$$p_{max} = \max_{(\bar{z}, \bar{x}_1) \in \Omega_c, v \in V, w \in W}(||p(\bar{z}, \bar{x}_1, v, w)||) = 3.$$

Since

$$\begin{aligned} \dot{V}_a(\bar{z}, \bar{x}_1) &\leq -2\bar{z}^2 + 2(\bar{z} + 1 + v_1)\bar{z}\bar{x}_1 - k\bar{x}_1^2 \\ &\leq -2\bar{z}^2 + 6|\bar{z}||\bar{x}_1| - k\bar{x}_1^2 \\ &\leq -\bar{z}^2 + (9 - k)\bar{x}_1^2, \end{aligned}$$

taking $k > k_{*2} = 10$,

$$\dot{V}_a(\bar{z}, \bar{x}_1) \leq -||Z||^2.$$

Therefore,

$$k_* = \max\{1,\ k_{*1},\ k_{*2}\} = 10.$$

Let

$$\begin{aligned} V(Z_1) &= V_a(\bar{z}, \bar{x}_1) + \epsilon_2 V_1(\tilde{\eta}) \\ &= \bar{z}^2 + \frac{1}{2}\bar{x}_1^2 + \epsilon_2 \frac{2}{r}\tilde{\eta}^T P \tilde{\eta} \end{aligned}$$

where ϵ_2 is a positive real number to be determined later.

Recall that

$$\begin{aligned} \phi(\bar{z}, \bar{x}_1, -kb_0\bar{x}_1, v, w) &= -N\big(w\bar{z} - \bar{x}_1 + 0.1\sin^2(w\bar{x}_1 + wv_1) - 0.1\sin^2(wv_1)\big) \\ &\quad -MNkb_0\bar{x}_1 - N\big(\beta^{[2]}(-Nkb_0\bar{x}_1 + \theta) - \beta^{[2]}(\theta)\big). \end{aligned}$$

The derivative of $V_1(\tilde{\eta})$ along the $\tilde{\eta}$ subsystem of (2.67) is as follows,

$$\frac{dV_1(\tilde{\eta})}{dt} \leq -||\tilde{\eta}||^2 + 4||r^{-1}P\phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1}, u_r, v, w)||^2.$$

Since

$$\begin{aligned} & ||r^{-1}P\phi(\bar{z}, \bar{x}_1, \cdots, \bar{x}_{r-1}, u_r, v, w)|| \\ \leq\ & \frac{1}{r}\Big(||PN(w\bar{z} - \bar{x}_1)|| + ||PN\big(0.1\sin^2(w\bar{x}_1 + wv_1) - 0.1\sin^2(wv_1)\big)|| \\ & +||PMNkb_0\bar{x}_1|| + ||PN\big(\beta^{[2]}(-Nkb_0\bar{x}_1 + \theta) - \beta^{[2]}(\theta)\big)||\Big) \\ \leq\ & \frac{1}{r}\Big(\sqrt{2}(|\bar{z}| + |\bar{x}_1|) + 0.1\sqrt{2}|\bar{x}_1| + 5\sqrt{5}|\bar{x}_1| + 3\sqrt{2}|\bar{x}_1|\Big) \\ \leq\ & \frac{1}{r}(1.4|\bar{z}| + 17|\bar{x}_1|) \\ \leq\ & \frac{24}{r}||Z||, \end{aligned}$$

taking $r = 0.7$, then

$$a_{max} = \frac{24}{r} = 34.$$

The derivative of $V(Z_1)$ along (2.67) is as follows,

$$\begin{aligned}\dot{V}(Z_1) &\le -||(\bar{z},\bar{x}_1)||^2 - \epsilon_2||\tilde{\eta}||^2 + 4\epsilon_2 a_{max}^2||(\bar{z},\bar{x}_1)||^2 \\ &\le -\frac{1}{2}||(\bar{z},\bar{x}_1)||^2 - \frac{1}{8a_{max}^2}||\tilde{\eta}||^2 \\ &\le -a||(\bar{z},\bar{x}_1,\tilde{\eta})||^2\end{aligned}$$

where $\epsilon_2 = \frac{1}{8a_{max}^2} = 0.00011$ and $a = \min\{\frac{1}{2}, \frac{1}{8a_{max}^2}\} = 0.00011$. Hence,

$$V(Z_1) = \bar{z}^2 + \frac{1}{2}\bar{x}_1^2 + 0.00031\tilde{\eta}^T P\tilde{\eta}$$

Taking $R = \sqrt{2}$ such that $\Pi_z \times \Pi_{\tilde{\eta}} \in B_R^4$. Let $c = \max_{||Z_1|| \le R} V(Z_1) = 2$, then $\sigma = 8$, and

$$\begin{aligned}V_b(Z_1,\tau) &= c\frac{V(Z_1)}{c+1-V(Z_1)} + \sigma\frac{\tau^2}{\sigma+1-\tau^2} \\ &= 2\frac{V(Z_1)}{3-V(Z_1)} + 8\frac{\tau^2}{9-\tau^2}\end{aligned}$$

Obviously,

$$p_1(Z_1,\tau,v,w)\tau = \begin{bmatrix} 0 \\ \tau \\ MN\tau - N\Big(\beta^{[2]}(\tilde{\eta} + N(\tau - 10\bar{x}_1) + \theta) - \beta^{[2]}(\tilde{\eta} - 10N\bar{x}_1 + \theta)\Big)\end{bmatrix},$$

$$\chi(Z_1,\tau,v,w) = w\bar{z} - 101\bar{x}_1 + 0.1\sin^2(w\bar{x}_1 + wv_1) + wv_1 + \beta(\eta) + 10\tau$$

Letting $\rho = 0$ and $S_1 = \{(Z_1,\tau) : |\tau| < 0.5\} \cap S$ gives

$$\dot{V}_b(Z_1,\tau) \le -\frac{16}{9}K\frac{1}{4} + 7.3.$$

Letting $K_* = 20$ gives

$$\dot{V}_b(Z_1,\tau) < 0.$$

Hence, the controller can be listed as follows,

$$\begin{aligned} u &= \beta(\eta) - K\mathrm{sat}(\tilde{\tau},\tilde{\tau}_0) \\ \dot{\eta} &= M\eta + N(u - \beta(\eta) + ET^{-1}\eta) \\ \dot{\tilde{x}}_1 &= -lc_1\tilde{x}_1 + \tilde{x}_2 + lc_1 e \\ \dot{\tilde{x}}_2 &= -l^2c_0\tilde{x}_1 + l^2c_0 e \end{aligned} \tag{2.68}$$

where $\tilde{\tau} = \tilde{x}_2 + kb_0\tilde{x}_1$, $K = 20$, $k = 10$, $b_0 = 1$, $l = 100$, $c_0 = 1$, $c_1 = 2$, $\tilde{\tau}_0 = 1$.

The performance of the controller is simulated with initial conditions $z(0) = 1$, $x_1(0) = 0$, $x_2(0) = -1$, $v_1(0) = 1$, $v_2(0) = 0$, $w = 1, \eta(0) = 0$, $\tilde{x}_1(0) = \tilde{x}_2(0) = 0$, and is shown in Figure 2.1, 2.2 and 2.3.

2.7 Concluding Remarks

In this chapter we established the solvability conditions of the semiglobal robust output regulation problem for a class of nonlinear SISO systems in normal form. We solved this problem by mainly getting rid of three drawbacks. First, the output regulation problem can be translated into a stabilization problem of an augmented systems composed of the original plant and the internal model. But the stabilization problem of the augmented system can not be treated directly by any existing stabilization result. Using the Lyapunov's direct method and the semiglobal backstepping technique by Teel and Praly [55], we solve it. Second, we eliminate the polynomial assumption imposed on the solution of the regulator equations by taking advantage of the nonlinear internal model by Chen and Huang [8]. Third, we get an output feedback controller by taking use of the high gain observer by Khalil and Esfandiari [43].

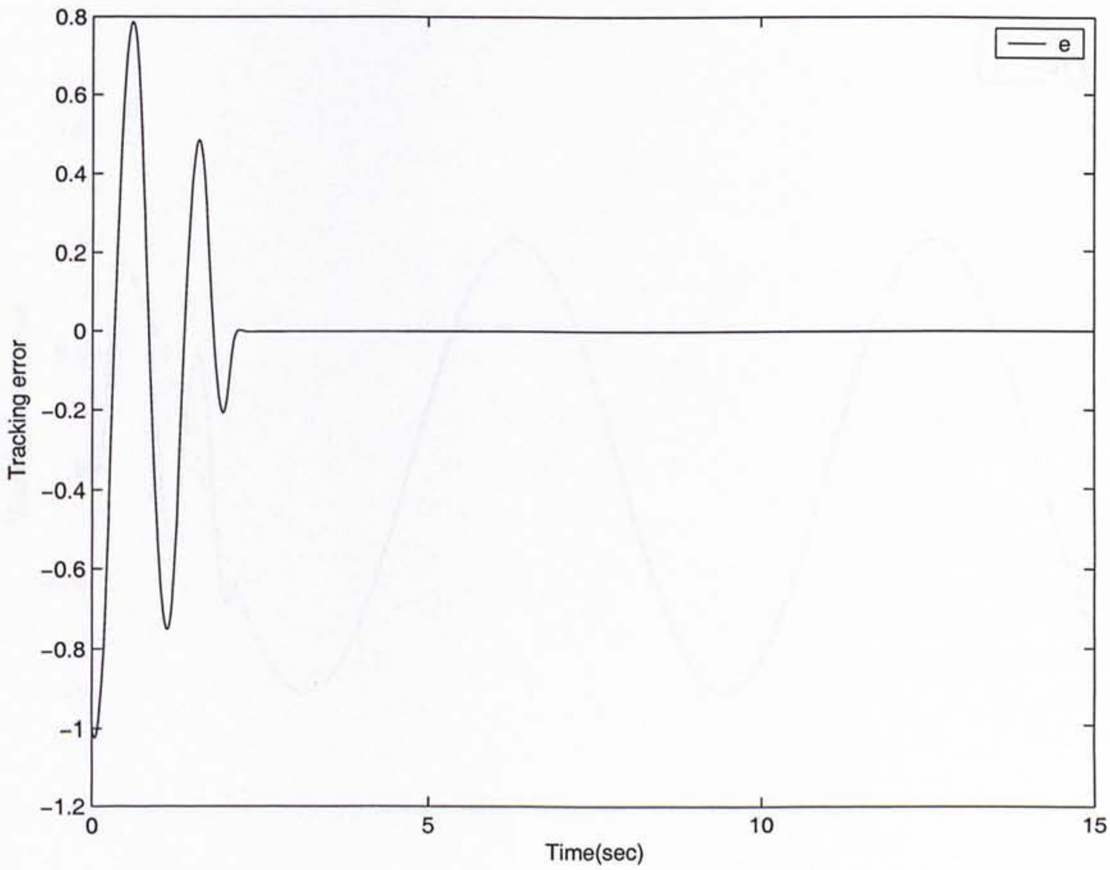

Figure 2.1: Profile of the tracking error of the system

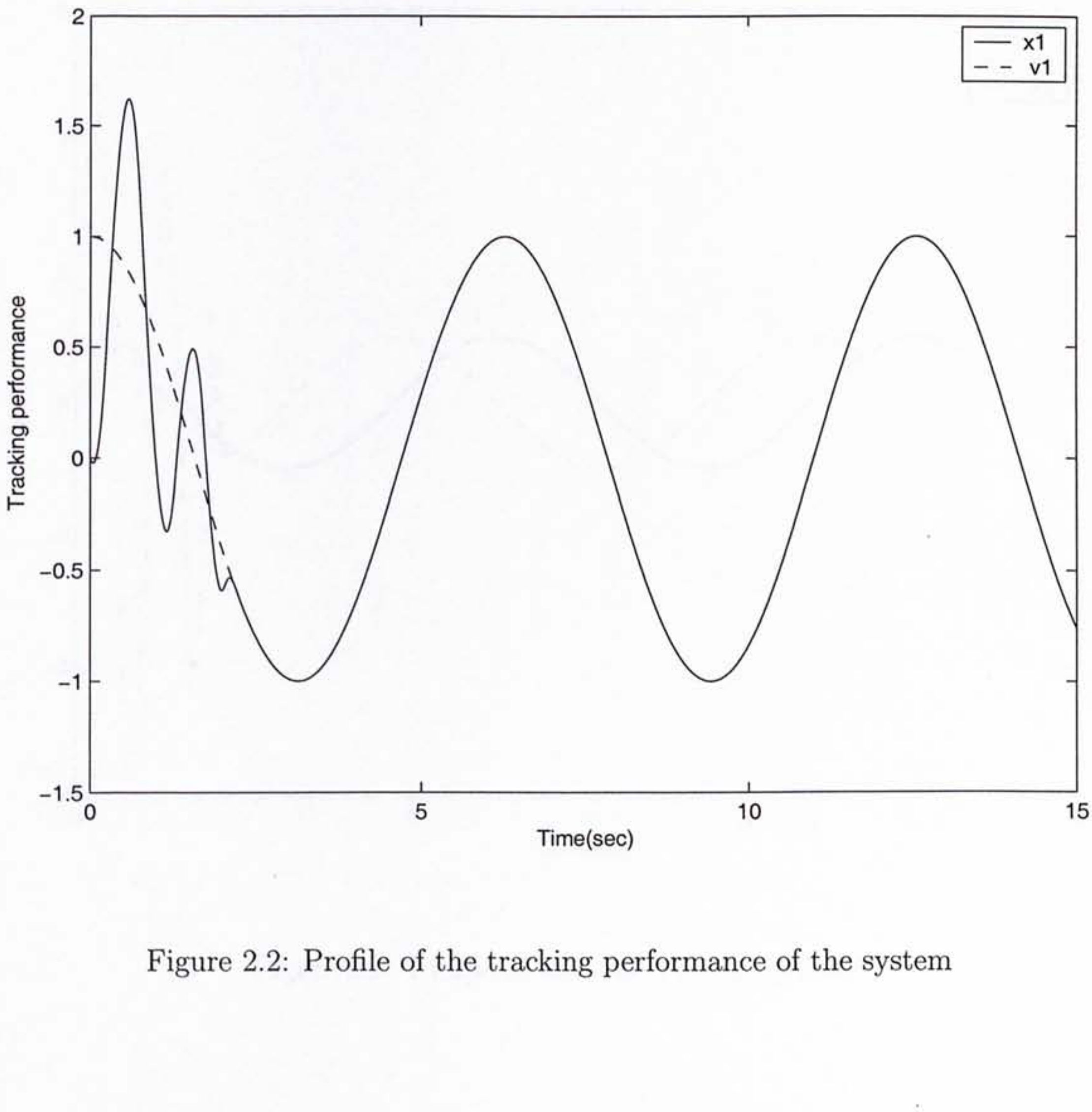

Figure 2.2: Profile of the tracking performance of the system

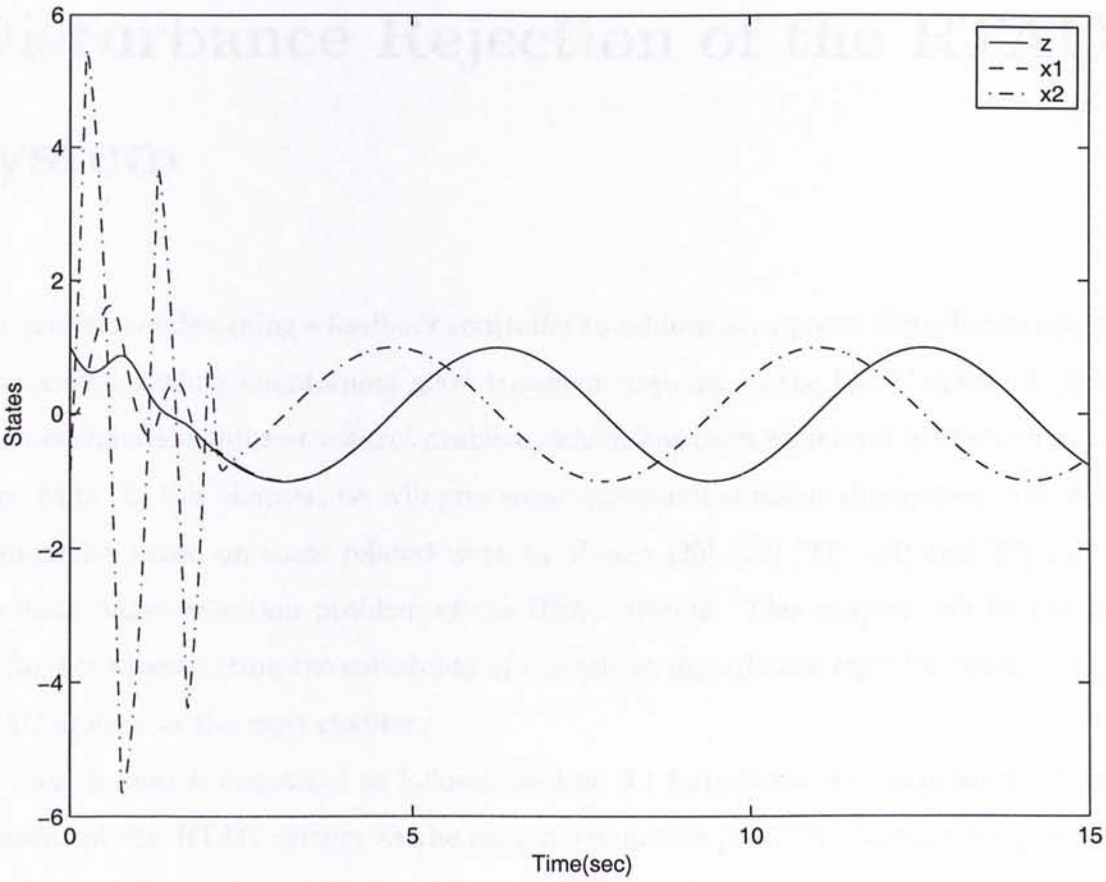

Figure 2.3: Profile of the states of the system

Chapter 3

Disturbance Rejection of the RTAC system

The problem of designing a feedback controller to achieve asymptotic disturbance rejection / attenuation while maintaining good transient response in the RTAC system is known as a benchmark nonlinear control problem, which has been an intensive research subject since 1995. In this chapter, we will give some introduction about this system, and design a controller based on some related work by Huang [20], [22] [23], [24] and [27] to solve the disturbance rejection problem of the RTAC system. This chapter will be the basis for further investigating the solvability of the robust disturbance rejection problem of the RTAC system in the next chapter.

This chapter is organized as follows. Section 3.1 formulates the disturbance rejection problem of the RTAC system as the output regulation problem. Section 3.2 presents a solution for this problem. Section 3.3 gives design process of the control parameters and simulation results. Section 3.4 closes the chapter with some concluding remarks.

3.1 Disturbance Rejection Problem Formulated into Output Regulation Problem

The motion equations of RTAC are derived in [3] and are given below,

$$\begin{aligned} \ddot{\zeta} + \zeta &= \epsilon(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) + F \\ \ddot{\theta} &= -\epsilon\ddot{\zeta}\cos\theta + u \end{aligned} \tag{3.1}$$

where ξ is the one-dimensional displacement of the cart, θ is the angular position of the proof body, and F and u are the disturbance and control input. The coupling between the translational and rotational motion is captured by the parameter ϵ which is defined by

$$\epsilon = \frac{me}{\sqrt{(I + me^2)(M + m)}}$$

where ϵ is the eccentricity of the proof body.

Letting $x = \text{col}[x_1\ x_2\ x_3\ x_4] = \text{col}[\zeta\ \ \dot{\zeta}\ \ \theta\ \ \dot{\theta}]$ and $y = \zeta$ yields the following state space representation of (3.1),

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u + g_2(x)F \\ y &= x_1 \end{aligned} \tag{3.2}$$

where

$$f(x) = \begin{bmatrix} x_2 \\ \frac{-x_1 + \epsilon x_4^2 \sin x_3}{1 - \epsilon^2 \cos^2 x_3} \\ x_4 \\ \frac{\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3)}{1 - \epsilon^2 \cos^2 x_3} \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 0 \\ \frac{-\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} \\ 0 \\ \frac{1}{1 - \epsilon^2 \cos^2 x_3} \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ \frac{1}{1 - \epsilon^2 \cos^2 x_3} \\ 0 \\ \frac{-\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} \end{bmatrix} \tag{3.3}$$

where $1 - \epsilon^2 \cos^2 x_3 \neq 0$ for all x_3 since $0 < \epsilon < 1$.

The basic objective is to design a partial state (x_1 and x_3) feedback controller such that, under a sinusoidal disturbance $F(t) = A_m \sin \omega t$ where A_m is unknown, for all sufficiently small initial state of the plant and the control law, and all sufficiently small A_m, the solution of the closed-loop system exists and is bounded for all $t \geq 0$, and the cart position x_1 asymptotically approaches 0. This problem has been formulated as an output regulation problem in [23] and is repeated here. Introduce the following system

$$\dot{v} = A_1 v, \ \ t \geq 0, \ \ v(0) = v_0 \tag{3.4}$$

with

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \ v(0) = \begin{bmatrix} 0 \\ A_m \end{bmatrix}. \tag{3.5}$$

Clearly, the solution of (3.4) satisfies $v_1(t) = A_m \sin \omega t$. We will call (3.4) as an exosystem in the sequel. Let $f(x, u, v) = f(x) + g_1(x)u + g_2(x)v_1$ and $h(x, u, v) = x_1$. Then we can define a composite system as follows

$$\begin{aligned} \dot{x} &= f(x, u, v) \\ \dot{v} &= A_1 v \\ y &= h(x, u, v). \end{aligned} \tag{3.6}$$

Thus the disturbance rejection problem described above can be formulated as looking for a controller of the form

$$\begin{aligned} \dot{z} &= g_z(z, x_1, x_3) \\ u &= k(z) \end{aligned} \tag{3.7}$$

where $z \in R^{n_z}$ for some integer n_z is the state of the controller, k and g_z are sufficiently smooth functions satisfying $k(0) = 0$ and $g_z(0, 0, 0) = 0$, such that for all sufficiently small initial state $x(0)$, $z(0)$, and $v(0)$, the trajectories of the closed-loop system composed of (3.6) and (3.7) are bounded and y approaches zero asymptotically.

Since x_1 and x_3 are considered as measurable output, the controller described by (3.7) is called *measurement output feedback control.*

In reality, the value of ϵ is not precisely known. If the controller is required to maintain the above asymptotic disturbance rejection property in the presence of the variation of the parameter ϵ, then the problem of designing such a controller is called the robust output regulation problem, which will be addressed in the next chapter.

3.2 Solvability of the Output Regulation Problem via Measurement Output Feedback Control

In this section, we will solve the disturbance rejection problem of the RTAC system formulated in Section 3.1 via output regulation method. For this purpose, consider the composite system consisting of the RTAC system and the exosystem as follows

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} &= \begin{bmatrix} x_2 \\ \frac{-x_1+\epsilon x_4^2 \sin x_3 + v_1 - \epsilon(\cos x_3)u}{1-\epsilon^2\cos^2 x_3} \\ x_4 \\ \frac{\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3) - \epsilon(\cos x_3)v_1 + u}{1-\epsilon^2\cos^2 x_3} \\ \omega v_2 \\ -\omega v_1 \end{bmatrix} \\
e &= x_1.
\end{aligned} \tag{3.8}$$

It is known from the standard output regulation theory that the above problem is solvable only if the regulator equations associated with the composite system (3.6), i.e, the following equations,

$$\begin{aligned}
\frac{\partial \mathbf{x}(v)}{\partial v} A_1 v &= f(\mathbf{x}(v), \mathbf{u}(v), v) \\
0 &= h(\mathbf{x}(v), \mathbf{u}(v), v)
\end{aligned} \tag{3.9}$$

are solvable for a pair of sufficiently smooth functions $\mathbf{x}(v)$ and $\mathbf{u}(v)$ satisfying $\mathbf{x}(0) = 0$ and $\mathbf{u}(0) = 0$. The solvability of the regulator equations is related to the zero dynamics (3.10) of the composite system (3.6).

Differentiating the error output e twice gives

$$\begin{aligned}
\dot{e} &= \dot{x}_1 = x_2 \\
\ddot{e} &= \dot{x}_2 = \frac{-x_1 + \epsilon x_4^2 \sin x_3 + v_1 - \epsilon(\cos x_3)u}{1 - \epsilon^2 \cos^2 x_3}.
\end{aligned}$$

Thus the composite system has a well defined relative degree 2 at the origin with

$$\begin{aligned}
D_a(x, v) &= \frac{-\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} \\
E_a(x, v) &= \frac{-x_1 + \epsilon x_4^2 \sin x_3 + v_1}{1 - \epsilon^2 \cos^2 x_3} \\
H_a(x, v) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\end{aligned}$$

Since

$$\text{rank}\frac{\partial H_a(x,v)}{\partial(x_1,x_2)} = 2$$

then we have the partition $x = \text{col}(x^1, x^2)$ with $x^1 = \text{col}(x_1, x_2)$ and $x^2 = \text{col}(x_3, x_4)$ and the following functions

$$\begin{aligned}
x^1 &= \sigma(x^2, v) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
u_e(x,v) &= -\frac{E_a(x,v)}{D_a(x,v)} = -\frac{-x_1 + \epsilon x_4^2 \sin x_3 + v_1}{-\epsilon \cos x_3} \\
u_e(x,v)|_{x^1=\sigma(x^2,v)} &= -\frac{\epsilon x_4^2 \sin x_3 + v_1}{-\epsilon \cos x_3}
\end{aligned}$$

as well as the zero dynamics of (3.8)

$$\begin{aligned}
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= x_4^2 \tan x_3 + \frac{v_1}{\epsilon \cos x_3} \\
\dot{v}_1 &= \omega v_2 \\
\dot{v}_2 &= -\omega v_1.
\end{aligned} \tag{3.10}$$

The first two equations of (3.10) with $v_1 = 0$ are

$$\begin{aligned}
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= x_4^2 \tan x_3
\end{aligned} \tag{3.11}$$

which is actually the zero dynamics of the RTAC system when the disturbance F is set to zero. It is known that if the equilibrium of (3.11) is hyperbolic, then the regulator equations associated with the composite system admit a solution [25], [38]. However, the Jacobian matrix of the zero dynamics at $(0,0)$ is

$$J = \begin{bmatrix} 0 & 1/\epsilon \\ 0 & 0 \end{bmatrix}.$$

The system is clearly not hyperbolic. Thus, the existing theory cannot determine the solvability of the regulator equations. Nevertheless, by taking advantage of the special structure of (3.10), we can actually solve (3.9) as follows.

First, expand (3.9) as follows:

$$\begin{aligned}
\frac{\partial \mathbf{x}_1(v)}{\partial v_1}\omega v_2 - \frac{\partial \mathbf{x}_1(v)}{\partial v_2}\omega v_1 &= \mathbf{x}_2(v) \\
\frac{\partial \mathbf{x}_2(v)}{\partial v_1}\omega v_2 - \frac{\partial \mathbf{x}_2(v)}{\partial v_2}\omega v_1 &= \frac{-\mathbf{x}_1(v) + \epsilon \mathbf{x}_4^2(v)\sin \mathbf{x}_3(v)}{1-\epsilon^2\cos^2 \mathbf{x}_3(v)} \\
&+ \frac{-\epsilon\cos \mathbf{x}_3(v)}{1-\epsilon^2\cos^2 \mathbf{x}_3(v)}\mathbf{u}(v) + \frac{1}{1-\epsilon^2\cos^2 \mathbf{x}_3(v)} v_1 \\
\frac{\partial \mathbf{x}_3(v)}{\partial v_1}\omega v_2 - \frac{\partial \mathbf{x}_3(v)}{\partial v_2}\omega v_1 &= \mathbf{x}_4(v) \\
\frac{\partial \mathbf{x}_4(v)}{\partial v_1}\omega v_2 - \frac{\partial \mathbf{x}_4(v)}{\partial v_2}\omega v_1 &= \frac{\epsilon\cos \mathbf{x}_3(v)(\mathbf{x}_1(v) - \epsilon \mathbf{x}_4^2(v)\sin \mathbf{x}_3(v))}{1-\epsilon^2\cos^2 \mathbf{x}_3(v)} \\
&+ \frac{1}{1-\epsilon^2\cos^2 \mathbf{x}_3(v)}\mathbf{u}(v) + \frac{-\epsilon\cos \mathbf{x}_3(v)}{1-\epsilon^2\cos^2 \mathbf{x}_3(v)} v_1 \\
0 &= \mathbf{x}_1(v)
\end{aligned}$$

where $\mathbf{x}(v) = \mathrm{col}(\mathbf{x}_1(v),\ \mathbf{x}_2(v),\ \mathbf{x}_3(v),\ \mathbf{x}_4(v))$.

By a mere inspection, the regulator equations can be partially solved as follows

$$\begin{aligned}
\mathbf{x}_1(v) &= 0 \\
\mathbf{x}_2(v) &= 0 \\
\mathbf{u}(v) &= \mathbf{x}_4^2(v)\tan \mathbf{x}_3(v) + \frac{v_1}{\epsilon\cos \mathbf{x}_3(v)}
\end{aligned} \tag{3.12}$$

with $\mathbf{x}_3(v)$ and $\mathbf{x}_4(v)$ satisfying

$$\begin{aligned}
\frac{\partial \mathbf{x}_3(v)}{\partial v} A_1 v &= \mathbf{x}_4(v) \\
\frac{\partial \mathbf{x}_4(v)}{\partial v} A_1 v &= \mathbf{x}_4^2(v)\tan \mathbf{x}_3(v) + \frac{v_1}{\epsilon\cos \mathbf{x}_3(v)}.
\end{aligned} \tag{3.13}$$

Equations (3.13) can be viewed as the invariant manifold equation associated with the zero dynamics (3.10).

It suffices to solve (3.13) in order to solve (3.9). To this end, note that equations (3.13) hold if and only if, for all sufficiently small trajectories $v(t)$ of the exosystem,

$$\frac{d\mathbf{x}_3(v)}{dt} = \mathbf{x}_4(v) \tag{3.14}$$

$$\frac{d\mathbf{x}_4(v)}{dt} = \mathbf{x}_4^2(v)\frac{\sin \mathbf{x}_3(v)}{\cos \mathbf{x}_3(v)} + \frac{v_1}{\epsilon\cos \mathbf{x}_3(v)}. \tag{3.15}$$

(3.15) can be written as

$$\cos \mathbf{x}_3(v)\frac{d\mathbf{x}_4(v)}{dt} - \mathbf{x}_4^2(v)\sin \mathbf{x}_3(v) = -\frac{1}{\epsilon\omega}\frac{dv_2}{dt}. \tag{3.16}$$

Using the identity

$$\frac{d((\cos x_3)x_4)}{dt} = -(\sin x_3)x_4\frac{dx_3}{dt} + (\cos x_3)\frac{dx_4}{dt} = -(\sin x_3)x_4^2 + (\cos x_3)\frac{dx_4}{dt}$$

in (3.16) gives

$$\frac{d((\cos \mathbf{x}_3(v))\mathbf{x}_4(v))}{dt} = -\frac{1}{\epsilon\omega}\frac{dv_2}{dt}$$

which further leads to, upon noting $\mathbf{x}_4(0) = 0$,

$$\mathbf{x}_4(v) = \frac{-v_2}{\epsilon\omega\cos \mathbf{x}_3(v)} = \frac{-1}{\epsilon\omega^2\cos \mathbf{x}_3(v)}\frac{dv_1}{dt}. \tag{3.17}$$

Combining (3.14) and (3.17) gives

$$\frac{d\sin \mathbf{x}_3(v)}{dt} = -\frac{1}{\epsilon\omega^2}\frac{dv_1}{dt}$$

which yields, upon noting $\mathbf{x}_3(0) = 0$,

$$\sin \mathbf{x}_3(v) = -\frac{v_1}{\epsilon\omega^2} \tag{3.18}$$

which further yields

$$\mathbf{x}_3(v) = \arcsin\frac{-v_1}{\epsilon\omega^2}. \tag{3.19}$$

Substituting (3.19) into (3.17) gives

$$\mathbf{x}_4(v) = \frac{1}{\epsilon\cos \mathbf{x}_3(v)} = \frac{-v_2}{\epsilon\omega}\frac{1}{\sqrt{1-(\frac{v_1}{\epsilon\omega^2})^2}} \tag{3.20}$$

where $-\epsilon\omega^2 < v_1 < \epsilon\omega^2$.

Next, a simple calculation gives

$$\frac{\partial f(0,0,0)}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/(1-\epsilon^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon/(1-\epsilon^2) & 0 & 0 & 0 \end{bmatrix},$$

$$\frac{\partial f(0,0,0)}{\partial u} = \begin{bmatrix} 0 \\ -\epsilon/(1-\epsilon^2) \\ 0 \\ 1/(1-\epsilon^2) \end{bmatrix}, \quad \frac{\partial f(0,0,0)}{\partial v} = \begin{bmatrix} 0 & 0 \\ \frac{1}{1-\epsilon^2} & 0 \\ 0 & 0 \\ \frac{-\epsilon}{1-\epsilon^2} & 0 \end{bmatrix}.$$

It can be easily verified that the pair $\left(\frac{\partial f(0,0,0)}{\partial x}, \frac{\partial f(0,0,0)}{\partial u}\right)$ is controllable for all $\epsilon > 0$, but the pair

$$\left(\left[\begin{array}{cc} \frac{\partial h}{\partial x}(0,0,0) & \frac{\partial h}{\partial v}(0,0,0) \end{array}\right], \left[\begin{array}{cc} \frac{\partial f}{\partial x}(0,0,0) & \frac{\partial f}{\partial v}(0,0,0) \\ 0 & A_1 \end{array}\right]\right)$$

is not detectable. Thus the asymptotic disturbance rejection problem can be solvable by the state feedback but not the output feedback control [38].

Nevertheless, since the angular position of the proof-mass actuator x_3 is also measurable. We can define a measurement output as $y_m = h_m(x, u, v) = \mathrm{col}(x_1, x_3)$. Then it can be verified that the following pair

$$\left(\left[\begin{array}{cc} \frac{\partial h_m}{\partial x}(0,0,0) & \frac{\partial h_m}{\partial v}(0,0,0) \end{array}\right], \left[\begin{array}{cc} \frac{\partial f(0,0,0)}{\partial x} & \frac{\partial f(0,0,0)}{\partial v} \\ 0 & A_1 \end{array}\right]\right)$$

is detectable. Thus the problem can be solvable by a dynamic measurement output feedback control.

Let K_x be such that $\frac{\partial f(0,0,0)}{\partial x} + \frac{\partial f(0,0,0)}{\partial u} K_x$ is Hurwitz, and L be such that

$$\left[\begin{array}{cc} \frac{\partial f}{\partial x}(0,0,0) & \frac{\partial f}{\partial v}(0,0,0) \\ 0 & A_1 \end{array}\right] - L \left[\begin{array}{cc} \frac{\partial h_m}{\partial x}(0,0,0) & \frac{\partial h_m}{\partial v}(0,0,0) \end{array}\right] \tag{3.21}$$

is Hurwitz, and $z = \mathrm{col}(z_1, z_2)$ with $z_1 \in \Re^4$ and $z_2 \in \Re^2$. Then a dynamic measurement output feedback controller that solves the output regulation problem for RTAC system can be given as follows:

$$\begin{aligned} u &= k(z_1, z_2) = \mathbf{u}(z_2) + K_x(z_1 - \mathbf{x}(z_2)) \\ \dot{z} &= g(z, y_m) \\ &= \left[\begin{array}{c} f(z_1) + g_1(z_1)k(z_1, z_2) + g_2(z_1)[1,\ 0]z_2 \\ A_1 z_2 \end{array}\right] + L\left[y_m - h_m(z_1, k(z_1, z_2), z_2)\right]. \end{aligned}$$

3.3 Parameters Design and Simulation Results

To evaluate the performance of this controller by computer simulation, let us give the specific gains K_x and L for the case where $\epsilon = 0.20$, and $\omega = 3$. First, letting $K_x = [-16.52\ -83.52\ -15.4\ -20.7]$ places the eigenvalues of $\frac{\partial f(0,0,0)}{\partial x} + \frac{\partial f(0,0,0)}{\partial u} K_x$ at $[(-0.848 \pm 2.52j),\ (-1.25 \pm 0.828j)]$. The above eigenvalues are based on **ITAE** (**i**ntegral of the **t**ime multiplied by the absolute value of the error) prototype design with cutoff frequency equal to 1 [14].

Next, letting the eigenvalues of (3.21) be given by

$$\begin{bmatrix} -0.1871 \pm 3.0918j & -0.7065 \pm 1.1866j & -1.3627 & -12.6325 \end{bmatrix}$$

gives

$$L = \begin{bmatrix} 3.4152 & -3.0473 \\ 1.9628 & 5.5501 \\ -3.4819 & 11.6188 \\ -4.5875 & 1.6509 \\ -3.3591 & -1.0914 \\ -1.0312 & -1.7223 \end{bmatrix}.$$

Simulation has been run for the initial state $x(0) = \text{col}(0.1, 0, 0, 0)$, $z(0) = 0$, and various values of the amplitude A_m. With $\omega = 3$, Figure 3.1 shows the profile of the displacement x_1 of the closed-loop system. It can be seen that this controller is able to completely eliminate the affect of the disturbance on the output as the time tends to infinity.

Next we take a look at what will happen if the parameter ϵ undergoes perturbations. Figures 3.4 shows the profiles of the displacement x_1 of the closed-loop system under the same controller with the parameter ϵ being equal to 0.18, 0.20 and 0.22, respectively. It can be seen that when the parameter ϵ deviates from its nominal value 0.20, the displacement x_1 displays a sizable non-decaying oscillation. Thus we have seen that the performance of this controller is not robust with respect to parameter variations. It is desirable to have a regulator that can maintain its performance in the presence of small parameter variations. Such a regulator is called a robust regulator, and will be introduced in the next chapter.

3.4 Concluding Remarks

In this chapter, we considered the disturbance rejection problem of the RTAC system via the output regulation problem. Using the explicit solution of the regulator equations [27], we got a controller to solve the disturbance rejection problem.

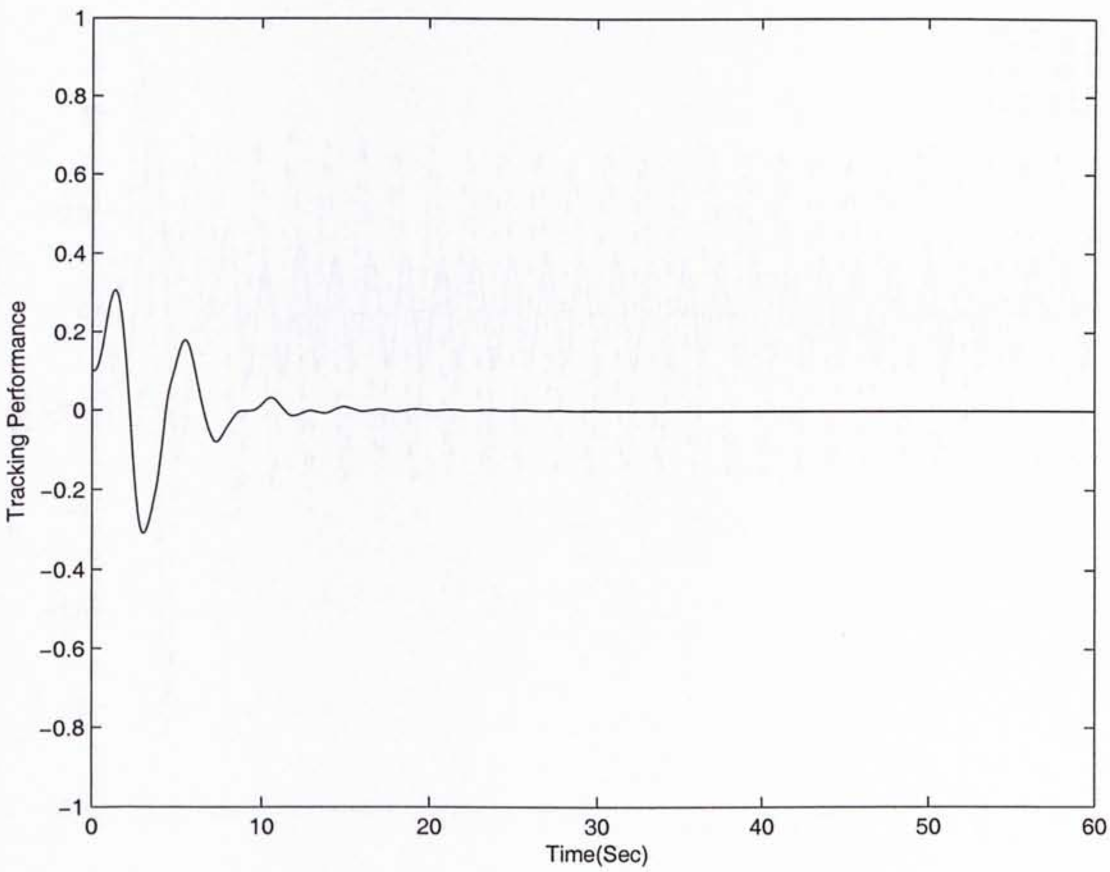

Figure 3.1: The profile of the displacement x_1 with $\epsilon = 0.2$, $\omega = 3$ and $A_m = 0.5$.

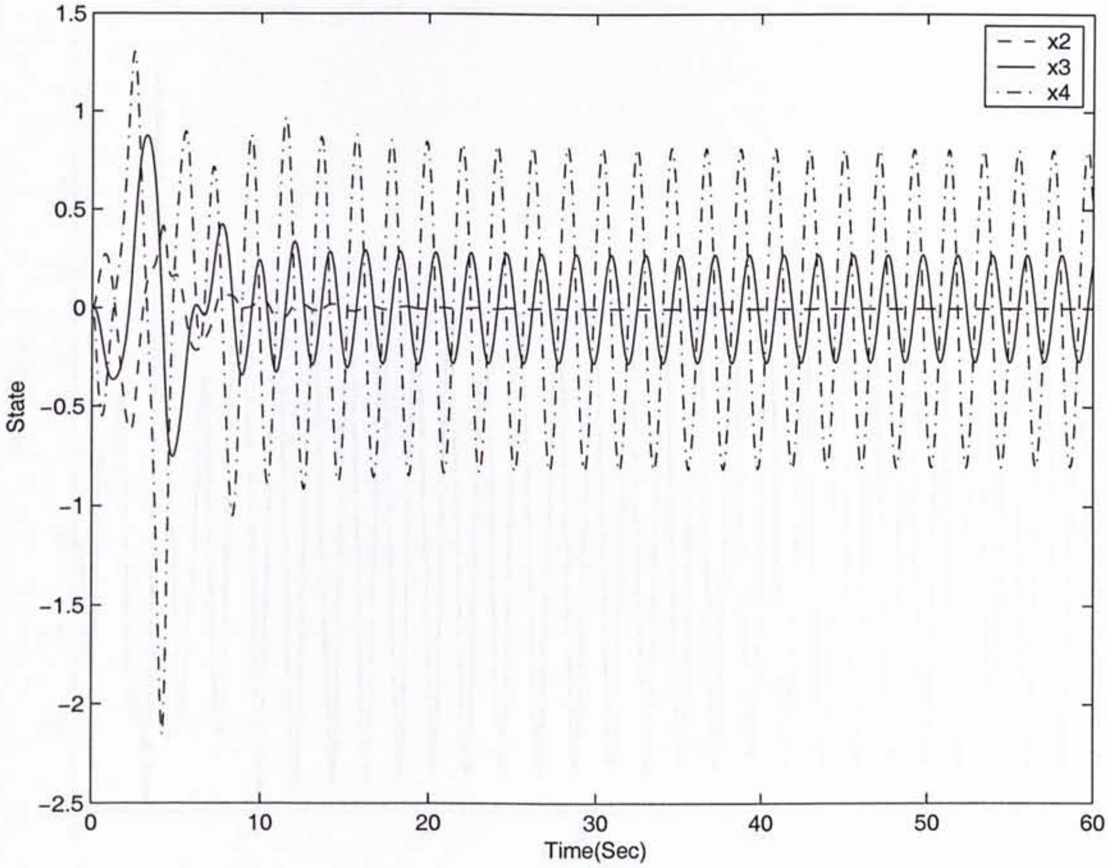

Figure 3.2: The profiles of the state variables (x_2, x_3, x_4) with $\epsilon = 0.2$, $\omega = 3$ and $A_m = 0.5$.

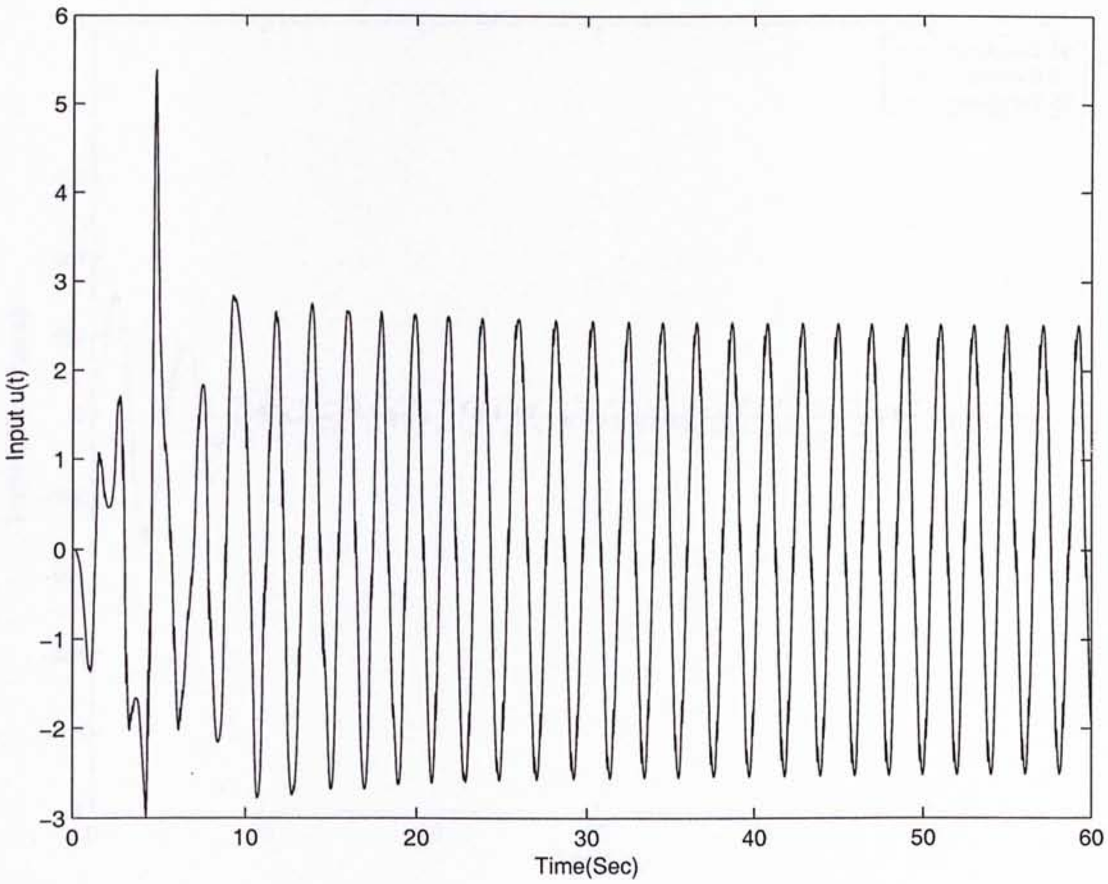

Figure 3.3: The profile of the control input u with $\epsilon = 0.2$, $\omega = 3$ and $A_m = 0.5$.

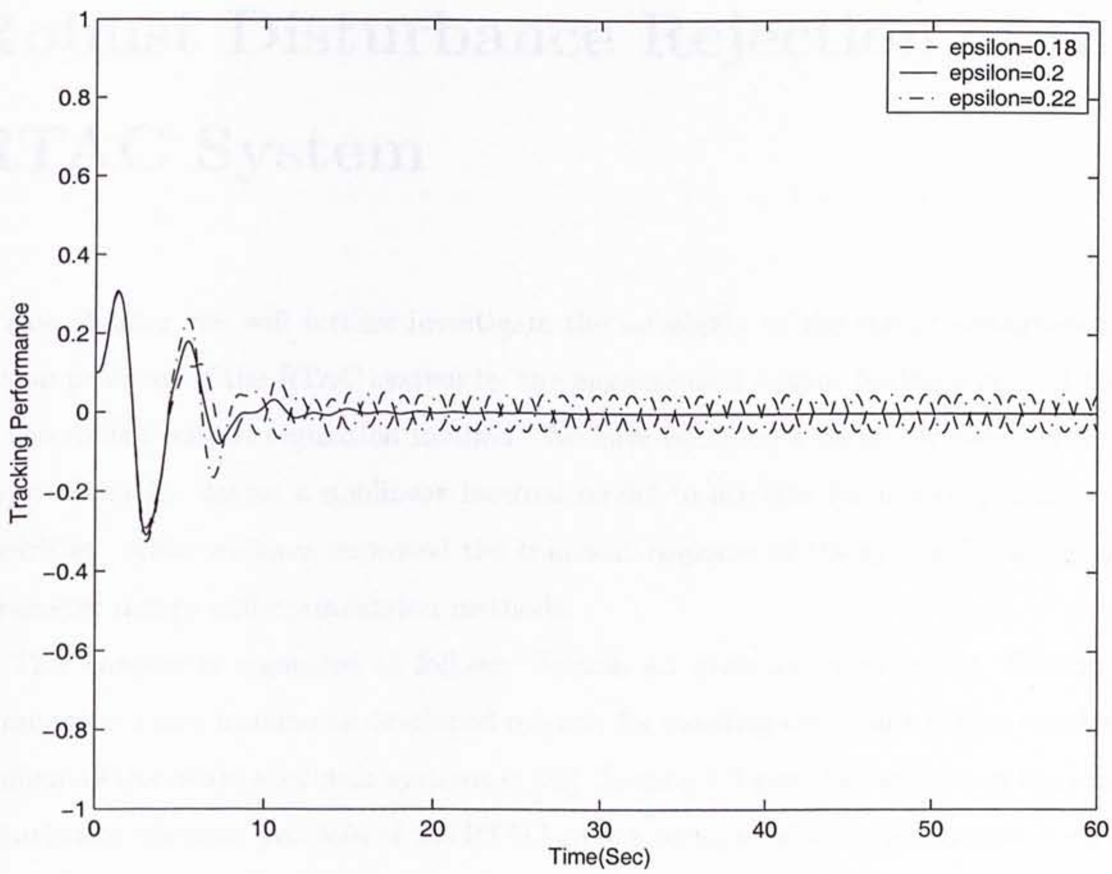

Figure 3.4: The profiles of the displacement x_1 when ϵ undergoes perturbation.

Chapter 4

Robust Disturbance Rejection of the RTAC System

In this chapter, we will further investigate the solvability of the robust disturbance rejection problem of the RTAC system by the measurement output feedback control based on the robust output regulation method. We have obtained a design by overcoming the major obstacle: devise a nonlinear internal model to account for non-polynomial nonlinearities. Also, we have improved the transient response of the system by using some parameter design and optimization methods.

This chapter is organized as follows: Section 4.1 gives an introduction. Section 4.2 summarizes a new framework developed recently for handling the robust output regulation problem of uncertain nonlinear systems in [28]. Section 4.3 gives the solution of the robust disturbance rejection problem of the RTAC system using a measurement output feedback control. Section 4.4 describes the parameter optimization method and the ITAE prototype design method. Section 4.5 shows effect of the optimization method and the simulation results. Section 4.6 closes the chapter by some remarks.

4.1 Introduction

In this chapter, we will further take into account the model uncertainty of the RTAC system, and solve the robust disturbance rejection problem of the RTAC system based on a new framework for handling the robust output regulation problem developed recently in [28]. In order to solve this problem, we need to overcome the major obstacle: the complexity of the solution of the regulator equations. The current robust output regulation theory can only handle systems whose regulator equations admit a solution which is a polynomial in the exogenous signals [1], [19], [21] and [29]. This limitation is caused by the employment of the linear internal model. However, as will be seen in Section 4.2, that the solution of the regulator equations of the RTAC system involves non-polynomial nonlinearities such as sinusoidal functions or non-rational functions. By employing a nonlinear internal model technique developed very recently, we have also circumvented this difficulty. Finally, we note that the current robust output regulation theory only offers the full information feedback and error output feedback control strategies. However, for RTAC system, it is unrealistic to assume the availability of the information on the unknown disturbance. On the other hand, certain observability condition does not hold to warrant an error output feedback control. To deal with this dilemma, a measurement output feedback control is introduced which only utilizes two measurable variables, namely, the displacement of the cart, and the angular position of the proof body. Comparing with all previous work on the benchmark control problem, the major novelty of the approach of this chapter is that it results in a measurement output feedback controller that can completely eliminate the influence of a sinusoidal disturbance to the output of the RTAC system in the presence of the model uncertainty.

4.2 A General Framework for Robust Output Regulation

As we have seen in Chapter 3 that the controller designed based on the output regulation theory performs poorly when the true value of the parameter ϵ is unknown. Designing a controller that can maintain its performance in the presence of parameter uncertainties is called the robust output regulation problem which has been studied in [1], [19], [21], and [29], to just name a few. Such a controller should not depend on ϵ since ϵ is unknown. Let us denote the nominal value of ϵ by ϵ_0, Then we can write $\epsilon = \epsilon_0 + w$ where w is an unknown parameter modelling the deviation of the true value of ϵ from its nominal value ϵ_0. To emphasize the reliance of the solution of the regulator equations on the unknown parameter w, we use $\mathbf{u}(v,w)$ and $\mathbf{x}(v,w)$ to denote the solution of the regulator equations, i.e.,

$$
\begin{aligned}
\mathbf{x}_1(v,w) &= 0 \\
\mathbf{x}_2(v,w) &= 0 \\
\mathbf{x}_3(v,w) &= \arcsin\left(\frac{-v_1}{(\epsilon_0+w)\omega^2}\right) \\
\mathbf{x}_4(v,w) &= \frac{-v_2}{(\epsilon_0+w)\omega}\left(\frac{1}{\sqrt{1-(\frac{-v_1}{(\epsilon_0+w)\omega^2})^2}}\right) \\
\mathbf{u}(v,w) &= \mathbf{x}_4{}^2(v,w)\tan\mathbf{x}_3(v,w)+\frac{v_1}{(\epsilon_0+w)\cos\mathbf{x}_3(v,w)}.
\end{aligned}
$$

The robust output regulation problem is handled by the so-called internal model principle which is completely different from the technique for handling the output regulation problem. As a result, the solvability of the robust regulation problem is much more challenging than that of the output regulation problem. In fact, the existing results on the solvability of the robust output regulation problem not only require the solvability of the regulator equations, but also require that the solution of the regulator equations be a polynomial of the exogenous signal $v(t)$ [1], [19], [21], and [25]. This requirement is imposed due to the employment of linear internal models. It can be seen that the solution of the regulation equations of the RTAC system is clearly not polynomial in $v(t)$, and therefore the existing approach cannot solve the robust output regulation problem of the RTAC system. Recently, a general framework is developed for handling the robust output regulation problem. This framework can convert, under a set of conditions, the robust output regulation problem for a given plant into a robust stabilization problem of an augmented system. In order to apply this framework to the RTAC system, we need to verify that the RTAC system indeed satisfies the conditions of the conversion, and the augmented system is stabilizable by the measurement output feedback control. For these purposes, let us summarize in this section the framework developed in [28] .

Consider a plant described by

$$
\begin{aligned}
\dot{x} &= f(x,u,v,w),\ x(0)=x_0 \\
y &= h(x,u,v,w),\ t\geq 0
\end{aligned}
\tag{4.1}
$$

and an exosystem described by

$$\dot{v} = a(v), \; v(0) = v_0 \tag{4.2}$$

where x is the n-dimensional plant state, u the m-dimensional plant input, y the p-dimensional plant output representing the tracking error, v the q-dimensional exogenous signal representing the disturbance and/or the reference input, and w the N-dimensional plant uncertain parameter whose nominal value is 0. The functions f, h and a are sufficiently smooth satisfying $f(0,0,0,w) = 0$ and $h(0,0,0,w) = 0$ for all w, and $a(0) = 0$.

Let us first list two standard assumptions.

A1: The equilibrium of the exosystem (4.2) at $v = 0$ is stable.

A2: There exist sufficiently smooth functions $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$ with $\mathbf{x}(0,0) = 0$ and $\mathbf{u}(0,0) = 0$ satisfying, for all $v \in V$, and $w \in W$ where V is an open neighborhood of the origin of $\Re^q$ and W an open neighborhood of the origin of $\Re^N$, the following equations

$$\begin{aligned} \frac{\partial \mathbf{x}(v,w)}{\partial v} a(v) &= f(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w) \\ 0 &= h(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w). \end{aligned} \tag{4.3}$$

Definition 4.1 Let $g_o : \Re^{n+m} \mapsto \Re^l$ be a mapping where $1 \leq l \leq n+m$. Under Assumptions A1 and A2, the nonlinear system (4.1) and (4.2) is said to have a *steady state generator* with output $g_o(x,u)$ if there exists a triple $\{\theta, \alpha, \beta\}$, where $\theta : \Re^{q+N} \mapsto \Re^s$, $\alpha : \Re^s \mapsto \Re^s$, and $\beta : \Re^s \mapsto \Re^l$ for some integer s are sufficiently smooth functions vanishing at the origin, such that, for all trajectories $v(t) \in V$ of (4.2) and all $w \in W$,

$$\begin{aligned} \frac{d\theta\left(v\left(t\right),w\right)}{dt} &= \alpha\left(\theta\left(v\left(t\right),w\right)\right) \\ g_o(\mathbf{x}(v(t),w), \mathbf{u}(v(t),w)) &= \beta\left(\theta\left(v\left(t\right),w\right)\right). \end{aligned} \tag{4.4}$$

If, in addition, the linearization of the pair $\{\beta(\theta), \alpha(\theta)\}$ at the origin is observable, then $\{\theta, \alpha, \beta\}$ is called a linearly observable steady state generator with output $g_o(x,u)$. ∎

Remark 4.1 Equations (4.3) are called regulator equations. If the mapping g_o takes the form $g_o(x,u) = \mathrm{col}(x,u)$, then the steady state generator is simply a dynamic system that can produce the solution of the regulator equations. In the sequel, we assume $g_o(x,u) = \mathrm{col}(x_{i_1}, x_{i_2}, \cdots, x_{i_d}, u)$ where $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ for some integer d satisfying $0 \leq d \leq n$, and, without loss of generality, we can always assume $i_j = j$ for $j = 1, \cdots, d$

since the index of the state variable can be relabelled to have this assumption satisfied. Existence of the steady state generator depends on some specific property of the solution of the regulator equations, and is discussed in detail in [28]. ∎

The definition of the steady state generator leads to a general characterization of the internal model as follows.

Definition 4.2 Under assumptions A1 and A2, suppose the system (4.1) and (4.2) has a steady state generator with output $g_o(x,u)$. Let $\gamma : \Re^{s+n+m} \mapsto \Re^s$ be a sufficiently smooth function vanishing at the origin. Then we call the following system

$$\dot{\eta} = \gamma(\eta, x, u) \tag{4.5}$$

an *internal model* with output $g_o(x,u)$ if, for all trajectories $v(t) \in V$ of (4.2) and all $w \in W$,

$$\gamma(\theta(v(t),w), \mathbf{x}(v(t),w), \mathbf{u}(v(t),w))) = \alpha(\theta(v(t),w)).$$

∎

Remark 4.2 It is clear that a steady state generator itself qualifies to be an internal model; therefore, some internal model for system (4.1) and (4.2) always exists if the system admits a steady state generator. However, the general characterization given in (4.5) offers more flexibility to render the augmented system defined below some desirable property to be elucidated in Remark 4.4 ∎

Remark 4.3 Attaching the internal model to the given plant yields the following augmented system

$$\dot{x} = f(x,u,v,w),\ \dot{\eta} = \gamma(\eta,x,u),\ y = h(x,u,v,w). \tag{4.6}$$

Performing on (4.6) the following coordinate and input transformation

$$\begin{aligned}
\bar{x}_i &= x_i - \beta_i(\eta),\ i = 1, \cdots, d \\
\bar{x}_i &= x_i - \mathbf{x}_i(v,w),\ i = d+1, \cdots, n \\
\bar{\eta} &= \eta - \theta(v,w) \\
\bar{u} &= u - \beta_u(\eta) = u - [\beta_{d+1}(\eta), \cdots, \beta_{d+m}(\eta)]^T
\end{aligned} \tag{4.7}$$

gives a new system denoted by

$$\dot{\bar{x}} = \bar{f}(\bar{x},\bar{\eta},\bar{u},v,w),\ \dot{\bar{\eta}} = \bar{\gamma}(\bar{x},\bar{\eta},\bar{u},v,w),\ y = \bar{h}(\bar{x},\bar{\eta},v,w) \tag{4.8}$$

where $\bar{x} = \mathrm{col}(\bar{x}_1,\cdots,\bar{x}_n)$. It can be verified that the system has the property

$$\begin{aligned} \bar{f}(0,0,0,v,w) &= 0 \\ \bar{\gamma}(0,0,0,v,w) &= 0 \\ \bar{h}(0,0,0,v,w) &= 0. \end{aligned} \tag{4.9}$$

∎

Theorem 4.1 Suppose system (4.1) and (4.2) satisfies Assumptions A1 and A2, and has a steady state generator with output $g_o(x,u) = \mathrm{col}(x_1,\cdots,x_d,u)$ and an internal model described by (4.5). Then if a controller of the form

$$\begin{aligned} \bar{u} &= k(\bar{x}_1,\cdots,\bar{x}_d,\xi) \\ \dot{\xi} &= g_\xi(\bar{x}_1,\cdots,\bar{x}_d,\xi,y) \end{aligned} \tag{4.10}$$

where $\xi \in R^{n_\xi}$ for some integer n_ξ, and k and g_ξ are sufficiently smooth functions vanishing at their respective origins stabilizes the equilibrium point $(\bar{x},\bar{\eta}) = (0,0)$ of (4.8), then the following controller

$$\begin{aligned} u &= \beta_u(\eta) + k(x_1-\beta_1(\eta),\cdots,x_d-\beta_d(\eta),\xi) \\ \dot{\eta} &= \gamma(x,\eta,u) \\ \dot{\xi} &= g_\xi(x_1-\beta_1(\eta),\cdots,x_d-\beta_d(\eta),\xi,y) \end{aligned} \tag{4.11}$$

solves the robust output regulation problem for the original plant (4.1) and the exosystem (4.2). ∎

Remark 4.4 In order to apply Theorem 4.1, we need to check whether or not the system (4.1) and (4.2) admits a steady state generator, among other things. An extensive discussion on the existence of the steady state generator is given in [28]. If the system does admit a steady state generator with output $g_o(x,u)$, then we can always find some internal model so that the robust output regulation problem of a given plant can be converted into a stabilization problem of the augmented plant. Clearly, whether or not the augmented system is stabilizable depends not only on the given plant but also on the particular internal model used. If the steady state generator is taken to be the internal model, i.e., $\dot{\eta} = \alpha(\eta)$, then the augmented system cannot be stabilized by any feedback control if the internal model itself is not a stable system. Therefore, it is important to find a particular internal model such that the stabilization problem of the augmented system is solvable. In Section , we will show that, the RTAC system does admit a steady state generator with output $g_o(x, u) = \mathrm{col}(x_1, x_3, u)$ and a specific internal model is available so that the augmented system is stabilizable by measurement output feedback control. ∎

4.3 Robust Asymptotic Disturbance Rejection of the RTAC System

Assuming the displacement x_1 of the cart, and the angular position x_3 of the proof body are measurable output variables, we will first show that the conditions of Theorem 4.1 is satisfied. Indeed, let $g_o(x, u) = \mathrm{col}(x_1, x_3, u)$, $\pi(v, w) = \frac{v_1}{(\epsilon_0 + w)}$, $\dot{\pi}(v, w) = \omega\frac{v_2}{(\epsilon_0 + w)}$, then

$$\begin{aligned} \mathbf{x}_1(v, w) &= 0 \stackrel{def}{=} \Gamma_{x_1}(\pi, \dot{\pi}) \\ \mathbf{x}_3(v, w) &= \arcsin\left(\frac{-v_1}{(\epsilon_0 + w)\omega^2}\right) = \arcsin\left(\frac{-\pi}{\omega^2}\right) \stackrel{def}{=} \Gamma_{x_3}(\pi, \dot{\pi}) \\ \mathbf{u}(v, w) &= \frac{\dot{\pi}^2}{\omega^4(1 - (\frac{\pi}{\omega^2})^2)} \frac{\frac{-\pi}{\omega^2}}{\sqrt{1 - (\frac{\pi}{\omega^2})^2}} + \frac{\pi}{\sqrt{1 - (\frac{\pi}{\omega^2})^2}} \stackrel{def}{=} \Gamma_u(\pi, \dot{\pi}). \end{aligned}$$

Now let $T \in \Re^{2\times 2}$ be any nonsingular matrix, and

$$\theta = T\begin{bmatrix} \pi \\ \dot{\pi} \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \tag{4.12}$$

$$\alpha(\theta) = T\Phi T^{-1}\theta, \quad \beta(\theta) = \Gamma(T^{-1}\theta) = \begin{bmatrix} \beta_{x_1}(\theta) \\ \beta_{x_3}(\theta) \\ \beta_u(\theta) \end{bmatrix} \tag{4.13}$$

where

$$\Gamma(\pi(v, w), \dot{\pi}(v, w)) = \begin{bmatrix} \Gamma_{x_1}(\pi(v, w), \dot{\pi}(v, w)) \\ \Gamma_{x_3}(\pi(v, w), \dot{\pi}(v, w)) \\ \Gamma_u(\pi(v, w), \dot{\pi}(v, w)) \end{bmatrix}.$$

Then it is ready to verify that the triple $\{\theta, \alpha(\theta), \beta(\theta)\}$ is a steady state generator of the RTAC system with output $g_o(x, u)$. Moreover, the steady state generator is linearly observable since the pair $(\Psi_u, \ \Phi)$ is observable where $\Psi_u = [1 \ \ 0]$ is the Jacobian of Γ_u at the origin.

Corresponding to the above steady state generator, we can define a dynamic system as follows. Let $M \in \Re^{2\times 2}$ be any Hulwitz matrix, and $N \in \Re^{2\times 1}$ be such that (M, N) is controllable. Then there is a unique nonsingular matrix T that is the solution of the Sylvester equation $T\Phi - MT = N\Psi_u$ since (Ψ_u, Φ) is observable (Theorem 7-10 of [6]). Let

$$\dot{\eta} = M\eta + N(u - \beta_u(\eta) + \Psi_u T^{-1}\eta) \tag{4.14}$$

where $\eta \in R^2$. Then,

$$\begin{aligned} & M\theta + N(\mathbf{u}(v, w) - \beta(\theta(v, w)) + \Psi_u T^{-1}\theta(v, w)) \\ = \ & M\theta + N\Psi_u T^{-1}\theta = T\Phi T^{-1}\theta = \alpha(\theta). \end{aligned} \tag{4.15}$$

Thus (4.14) is an internal model of (3.6) with output $g_o(x, u)$.

Let $M = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$ with $a_1 < 0$ and $a_2 < 0$, $N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. T is the solution of the Sylvester equation $T\Phi - MT = N\Psi_u$. Since M is Hurwitz, and (M, N) is controllable, the Sylvester equation has a unique nonsingular solution T as follows:

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} \frac{-1}{a_1 + \omega^2 + \frac{a_2^2\omega^2}{a_1+\omega^2}} & \frac{-a_2}{a_1+\omega^2} t_{11} \\ \frac{a_2\omega^2}{a_1+\omega^2} t_{11} & t_{11} \end{bmatrix}.$$

Denoting $\beta(\eta) = \mathrm{col}(\beta_{x_1}(\eta), \beta_{x_3}(\eta), \beta_u(\eta))$ and performing the following coordinate and input translation

$$\begin{aligned} \bar{x}_1 &= x_1 - \beta_{x_1}(\eta) \\ \bar{x}_2 &= x_2 - \mathbf{x}_2(v, w) \\ \bar{x}_3 &= x_3 - \beta_{x_3}(\eta) \\ \bar{x}_4 &= x_4 - \mathbf{x}_4(v, w) \\ \bar{\eta} &= \eta - \theta(v, w) \\ \bar{u} &= u - \beta_u(\eta) \end{aligned}$$

on the augmented system consisting of the RTAC system and the internal model (4.14) gives

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= \frac{-\bar{x}_1 + (\epsilon_0 + w)(\bar{x}_4 + \mathbf{x}_4(v,w))^2 \sin(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))}{1 - (\epsilon_0 + w)^2 \cos^2(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))} \\
&+ \frac{-(\epsilon_0 + w)\cos(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))}{1 - (\epsilon_0 + w)^2 \cos^2(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))}(\bar{u} + \beta_u(\bar{\eta}+\theta)) \\
&+ \frac{v_1}{1 - (\epsilon_0 + w)^2 \cos^2(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))} \\
\dot{\bar{x}}_3 &= \dot{x}_3 - \dot{\beta}_{x_3}(\bar{\eta}+\theta) = \bar{x}_4 + \mathbf{x}_4(v,w) - \frac{\partial \beta_{x_3}(\bar{\eta}+\theta)}{\partial \bar{\eta}}\dot{\bar{\eta}} - \frac{\partial \beta_{x_3}(\bar{\eta}+\theta)}{\partial \theta}\dot{\theta} \\
&= \bar{x}_4 + \frac{-v_2}{(\epsilon_0 + w)\omega} \frac{1}{\sqrt{1 - \left(\frac{-v_1}{(\epsilon_0+w)\omega^2}\right)^2}} \\
&+ \frac{[1\ 0]T^{-1}(M + N\Psi_u T^{-1})\bar{\eta} + [1\ 0]T^{-1}N\bar{u} + [1\ 0]\Phi T^{-1}\theta}{\omega^2\sqrt{1 - \left(\frac{[1\ 0]T^{-1}(\bar{\eta}+\theta)}{\omega^2}\right)^2}} \\
\dot{\bar{x}}_4 &= \frac{(\epsilon_0 + w)\cos(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))(\bar{x}_1 - (\epsilon_0 + w)(\bar{x}_4 + \mathbf{x}_4(v,w))^2 \sin(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta)))}{1 - (\epsilon_0 + w)^2 \cos^2(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))} \\
&+ \frac{1}{1 - (\epsilon_0 + w)^2 \cos^2(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))}(\bar{u} + \beta_u(\bar{\eta}+\theta)) + \\
&\frac{-(\epsilon_0 + w)\cos(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))}{1 - (\epsilon_0 + w)^2 \cos^2(\bar{x}_3 + \beta_{x_3}(\bar{\eta}+\theta))}v_1 + \frac{-v_1\omega + \frac{v_1 v_2^2 + v_1^3}{(\epsilon_0+w)^2\omega^3}}{(\epsilon_0 + w)\omega\left[1 - \frac{v_1^2}{(\epsilon_0+w)^2\omega^4}\right]^{\frac{3}{2}}} \\
\dot{\bar{\eta}} &= (M + N\Psi_u T^{-1})\bar{\eta} + N\bar{u}.
\end{aligned} \tag{4.16}
$$

By Theorem 4.1, it suffices to (locally) stabilize the equilibrium point at the origin of (4.16) with $v = 0$ and $w = 0$ by a controller depending on $\bar{x}_1$ and $\bar{x}_3$ only. To this end, linearizing the augmented system (4.16) with v and w being set to zero and noting $\Psi_u = [1, 0]$ gives

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= \frac{-1}{1-\epsilon_0^2}\bar{x}_1 + \frac{-\epsilon_0}{1-\epsilon_0^2}\bar{u} + \frac{-\epsilon_0}{1-\epsilon_0^2}\Psi_u T^{-1}\bar{\eta} \\
\dot{\bar{x}}_3 &= \bar{x}_4 + \frac{1}{\omega^2}\Psi_u T^{-1}(M + N\Psi_u T^{-1})\bar{\eta} + \frac{1}{\omega^2}\Psi_u T^{-1}N\bar{u} \\
\dot{\bar{x}}_4 &= \frac{\epsilon_0}{1-\epsilon_0^2}\bar{x}_1 + \frac{1}{1-\epsilon_0^2}\bar{u} + \frac{1}{1-\epsilon_0^2}\Psi_u T^{-1}\bar{\eta} \\
\dot{\bar{\eta}} &= (M + N\Psi_u T^{-1})\bar{\eta} + N\bar{u}.
\end{aligned}
$$

The above system can be put into the following matrix form as follows

$$\begin{aligned}\dot{\bar{x}} &= A\bar{x} + B_\eta\bar{\eta} + \bar{B}\bar{u} \\ \dot{\bar{\eta}} &= (M + N\Psi_u T^{-1})\bar{\eta} + N\bar{u} \end{aligned} \tag{4.17}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{1-\epsilon_0^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon_0}{1-\epsilon_0^2} & 0 & 0 & 0 \end{bmatrix}, B_\eta = \begin{bmatrix} 0_{1\times 2} \\ \frac{-\epsilon_0}{1-\epsilon_0^2}\Psi_u T^{-1} \\ \frac{1}{\omega^2}\Psi_u T^{-1}(M + N\Psi_u T^{-1}) \\ \frac{1}{1-\epsilon_0^2}\Psi_u T^{-1} \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ \frac{-\epsilon_0}{1-\epsilon_0^2} \\ \frac{1}{\omega^2}\Psi_u T^{-1} N \\ \frac{1}{1-\epsilon_0^2} \end{bmatrix}.$$

Moreover, let

$$y_m = C_m \begin{bmatrix} \bar{x} \\ \bar{\eta} \end{bmatrix}$$

where

$$C_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then it can be verified that the linear system with $\mathrm{col}(\bar{x}, \bar{\eta})$ as the state, $\bar{u}$ as the input, and y_m as the output is both stabilizable and detectable.

Now let K and L be such that the two matrices

$$\begin{bmatrix} A & B_\eta \\ 0 & M + N\Psi_u T^{-1} \end{bmatrix} + K \begin{bmatrix} \bar{B} \\ N \end{bmatrix} \tag{4.18}$$

and

$$\begin{bmatrix} A & B_\eta \\ 0 & M + N\Psi_u T^{-1} \end{bmatrix} - LC_m \tag{4.19}$$

are Hurwitz. Then a linear output feedback controller that stabilizes (4.16) can be given as follows:

$$\begin{aligned}\bar{u} &= K\xi \\ \dot{\xi} &= \begin{bmatrix} A & B_\eta \\ 0 & M + N\Psi_u T^{-1} \end{bmatrix}\xi + \begin{bmatrix} \bar{B} \\ N \end{bmatrix}\bar{u} + L\begin{bmatrix} \bar{x}_1 - \xi_1 \\ \bar{x}_3 - \xi_3 \end{bmatrix}. \end{aligned} \tag{4.20}$$

4.4 Algorithms to Design and Optimize the Parameters K_x and L

We use the ITAE Prototype method (See Appendix A.) and one parameter optimization method to design and optimize the parameters K_x and L.

Our optimization method is revised from the optimization technique proposed by Huang [22], which is based on the pole assignment result given by Bhattacharyya and De Souza [2]. The pole assignment result is as follows, given three matrices $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $A_c \in R^{n\times n}$ where $(A,\ B)$ is controllable, B is full rank, A_c is Hurwitz, and $\sigma(A)\bigcap\sigma(A_c) = \emptyset$, then for any matrix $G \in R^{m\times n}$, the following Sylvester equation

$$AT - TA_c = -BG \tag{4.21}$$

has the unique solution T and T is nonsingular.

Equation (4.21) can be transformed into the following equation

$$A + BGT^{-1} = TA_cT^{-1} \tag{4.22}$$

which shows that GT^{-1} can be regarded as the gain matrix. For convenience, we list the steps to get the gain matrix K satisfying $A + BK = A_c$ as follows.

Step 1: Form a Hurwitz matrix A_c which has the desirable eigenvalues.

Step 2: Pick an arbitrary nonzero matrix G of dimension $m \times n$.

Step 3: Solve (4.21) for T, and get $K = GT^{-1}$.

From the above steps, we can see that K is a function of G, i.e., $K(G) = GT^{-1}$. However, the gain matrix K gained by the above algorithm is not the best solution. In order to get better transient response, we will use a parameter optimization method to optimize the gain matrix K. First of all, let us define the performance function as follows

$$Q(G) = \frac{1}{2}(||K(G)||_F^2)^{1/2} = \frac{1}{2}\sqrt{\sum_{i,j=1}^{m,n} K_{ij}^2}.$$

where K_{ij} is the ith row and jth column element of the matrix K. Many gradient based parameter optimization techniques can be used to solve the problem. For convenience, we will take use of the following iterative steps [22] based on the steepest descent method to minimize the performance function $Q(G)$.

Step 1: Arbitrarily choose a nonzero $G_0 \in R^{m\times n}$ and let $k = 0$.

Step 2: Solve the following Sylvester equation for T_k

$$AT_k - T_k A_c = -BG_k.$$

Step 3: Let $K_k = G_k T_k^{-1}$ and $\Gamma_k = \text{grad}Q(G_k)$. Stop if $||\Gamma_k||$ is sufficiently small. Otherwise, goto Step 2.

In detail, the computation process is as follows. Let g_{ij} be the ith row and jth column element of the matrix G_k. Taking the partial derivative over g_{ij} of the two sides of the equation (4.21) gives

$$A\frac{\partial T}{\partial g_{ij}} - \frac{\partial T}{\partial g_{ij}}A_c = -BH \tag{4.23}$$

where H is a matrix with only the ith row and jth column element equal to 1 and others equal to 0.

Denote the pth row and qth column element of Γ_k as $(\Gamma_k)_{pq} = \frac{\partial Q(G_k)}{\partial g_{pq}}$, denote T_{lj} as the lth row and jth column element of the matrix T_k, and denote T_{lj}^{-1} as the lth row and jth column element of the matrix T_k^{-1}. The performance function $Q(G_k)$ can be given by

$$Q(G_k) = \frac{1}{2}\sum_{i,j=1}^{m,n}(\sum_{l=1}^{n} g_{il}T_{lj}{}^{-1})^2. \tag{4.24}$$

Taking the partial derivative over g_{ij} of the two sides of (4.24) gives

$$\frac{\partial Q(G_k)}{\partial g_{pq}} = \frac{1}{Q(G_k)}\sum_{i,j=1}^{m,n}\Big((\sum_{l=1}^{n} g_{il}T_{lj}{}^{-1})(\sum_{l=1}^{n} g_{il}\frac{\partial T_{lj}{}^{-1}}{\partial g_{pq}})\Big) + \sum_{j=1}^{n}\Big(T_{qj}^{-1}\sum_{l=1}^{n}(g_{pl}T_{lj}^{-1})\Big). \tag{4.25}$$

Since $T_k T_k^{-1} = I$, taking the partial derivative over g_{ij} of the two sides of this equation gives

$$\frac{\partial T_k}{\partial g_{ij}}T_k{}^{-1} + T_k\frac{\partial T_k{}^{-1}}{\partial g_{ij}} = 0.$$

Hence,

$$\frac{\partial T_k{}^{-1}}{\partial g_{ij}} = -T_k{}^{-1}\frac{\partial T_k}{\partial g_{ij}}T_k{}^{-1}$$

Step 4: Find s_k such that $Q(G_k - s_k\Gamma_k) = \min_{s\geq 0} Q(G_k - s_k\Gamma_k)$.

Step 5: Let $G_{k+1} = G_k + s_k\Gamma_k$, $k = k+1$, and goto Step 2.

4.5 Parameters design and Simulation Results

By Theorem 4.1, the controller that solves the robust output regulation problem of the original system is given as follows,

$$\begin{aligned} u &= K\xi + \beta_u(\eta) \\ \dot{\xi} &= \begin{bmatrix} A & B_\eta \\ 0 & M + N\Psi_u T^{-1} \end{bmatrix} \xi + \begin{bmatrix} \bar{B} \\ N \end{bmatrix} K\xi + L \begin{bmatrix} x_1 - \xi_1 \\ x_3 - \beta_{x_3}(\eta) - \xi_3 \end{bmatrix} \\ \dot{\eta} &= M\eta + N(u - \beta_u(\eta) + \Psi_u T^{-1}\eta). \end{aligned} \tag{4.26}$$

A specific controller has been synthesized with the various parameters as follows. $\omega = 3$, $\epsilon_0 = 0.2$, $\Phi = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}$, and $T = \begin{bmatrix} -0.0833 & -0.0278 \\ 0.2500 & -0.0833 \end{bmatrix}$.

Also, $K = \begin{bmatrix} 5.9374 & -3.4198 & -0.9555 & -2.5082 & 5.9333 & -1.7874 \end{bmatrix}$ which is such that the eigenvalues of the matrix (4.18) are

$$1.2 \times \begin{bmatrix} -0.3099 \pm 1.2634j & -0.5805 \pm 0.7828j & -0.7346 \pm 0.2873j \end{bmatrix}.$$

We can see the optimization effect from the following comparison results, also shown in figure 4.1. Without the optimization method, the performance function $Q(G) = 2094.7$, and the corresponding parameter

$$L = 1000 \times \begin{bmatrix} 0.0032 & -0.0574 \\ -0.0183 & -0.3872 \\ 0.0004 & 0.0156 \\ 0.0554 & 1.4210 \\ -0.0145 & 0.1807 \\ 0.1195 & 1.4713 \end{bmatrix}.$$

Under the optimization algorithm of iterating 500 times, the performance function $Q(G) =$

341.3, and the corresponding parameter

$$L = \begin{bmatrix} 10.5527 & -8.2521 \\ 31.4243 & -55.4722 \\ -0.7366 & 8.1973 \\ -125.6199 & 212.9719 \\ -37.5606 & 26.2688 \\ -69.5350 & 210.0203 \end{bmatrix}.$$

Computer simulation has been used to evaluate the performance of the closed-loop system with the initial state being $x(0) = \text{col}(0.1, 0, 0, 0)$, $\eta(0) = 0$, and $\xi(0) = 0$. Although both the two L's place the eigenvalues of the matrix (4.19) at

$$\begin{bmatrix} -1.50 \pm 1.50\jmath & -2.25 & -3.75 & -4.50 & -5.25 \end{bmatrix},$$

the transient response under optimized design parameter L is much better. Figures 4.2 and 4.3 show that the transient response under unoptimized L is even unstable, which is in sharp contrast with figure 4.4 and 4.5 under optimized L. As expected, the parameter variations do not affect the steady state response of the output, as can be seen in figure 4.6. This is in sharp contrast with the nonlinear servo-regulator designed in chapter 3 where the same amount of parameter variations significantly affect the steady state response of the output.

If we arbitrarily place the eigenvalues of the matrix (4.19) at some other values with negative real parts, in most of the cases, the transient response under unoptimized L is unstable, but stable under optimized L. If we put the eigenvalues of the matrix (4.19) at the carefully chosen values

$$\begin{bmatrix} -16.50 \pm 4.50\jmath & -6.75 & -11.25 & -15.75 & -13.50 \end{bmatrix},$$

then the transient response under unoptimized L happens to be stable, but the performance is still worse than that under optimized L, as can be seen in the comparison of figures (4.7) and (4.8).

4.6 Concluding Remarks

This chapter has presented a solution of the robust asymptotic disturbance rejection problem for the RTAC system through the measurement output feedback control. The solution is obtained by circumventing two major difficulties. Simulation shows superior performance of the robust output regulation method in comparison with the output regulation method. Also, we used the ITAE prototype method and a parameter optimization method based on the steepest gradient technique to design and optimize the control parameters L and K_x, which led to much better transient response.

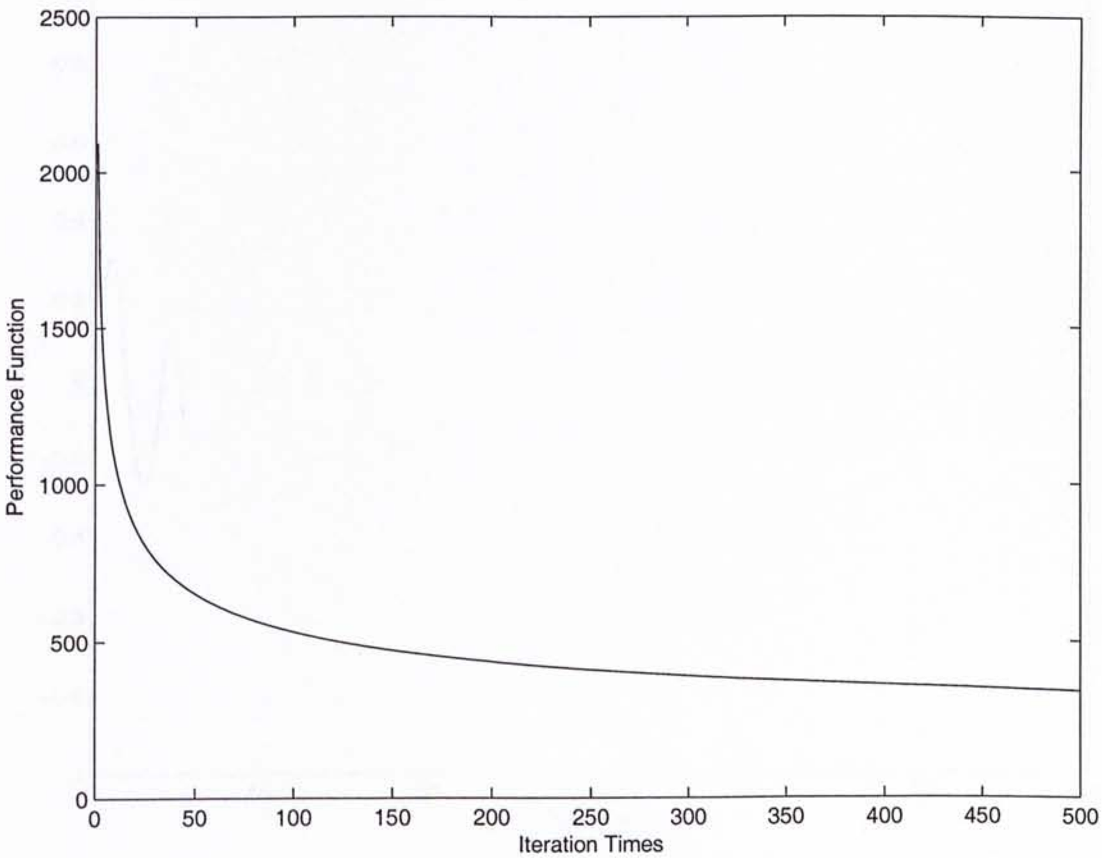

Figure 4.1: The profile of the performance function.

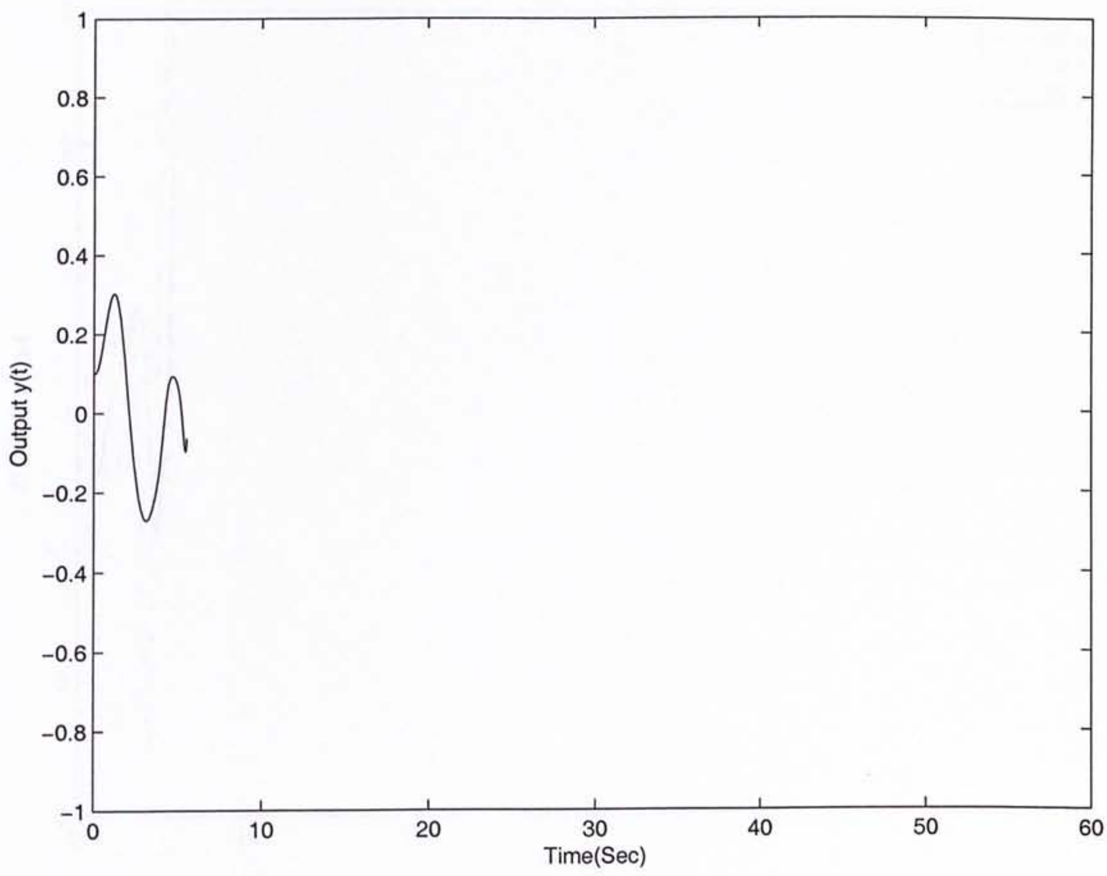

Figure 4.2: The profile of the displacement x_1 with $\epsilon = 0.2$, $\omega = 3$ and $A_m = 0.5$ under unoptimized parameter L .

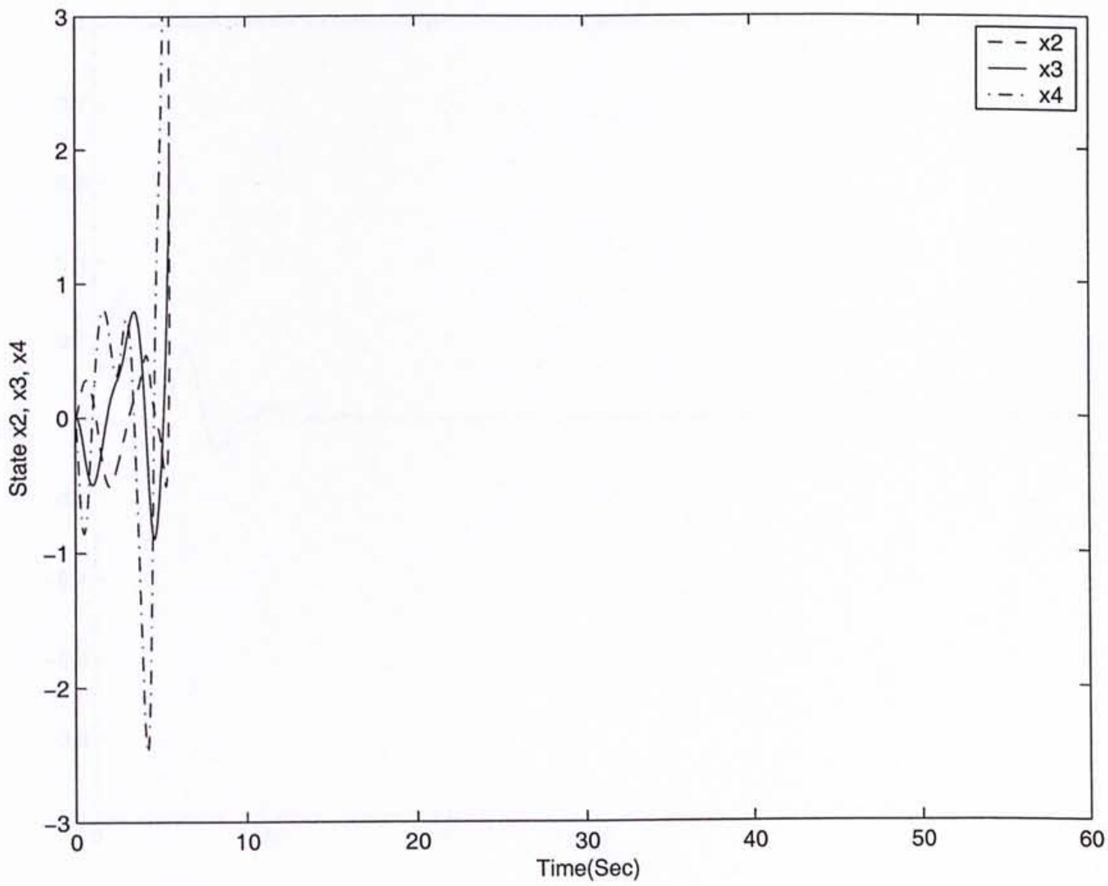

Figure 4.3: The profiles of the state variables (x_1, x_2, x_3) with $\epsilon = 0.2$, $\omega = 3$ and $A_m = 0.5$ under unoptimized parameter L.

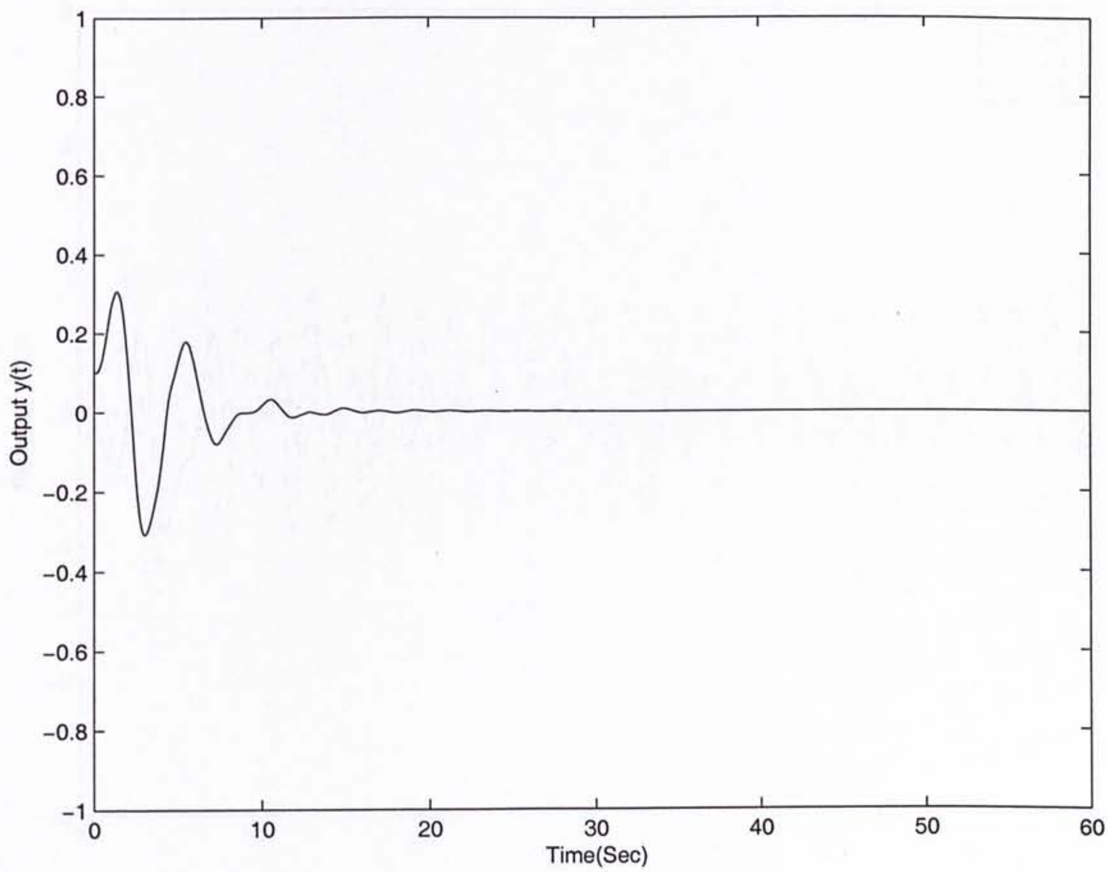

Figure 4.4: The profile of the displacement x_1 with $\epsilon = 0.2$, $\omega = 3$ and $A_m = 0.5$ under optimized parameter L.

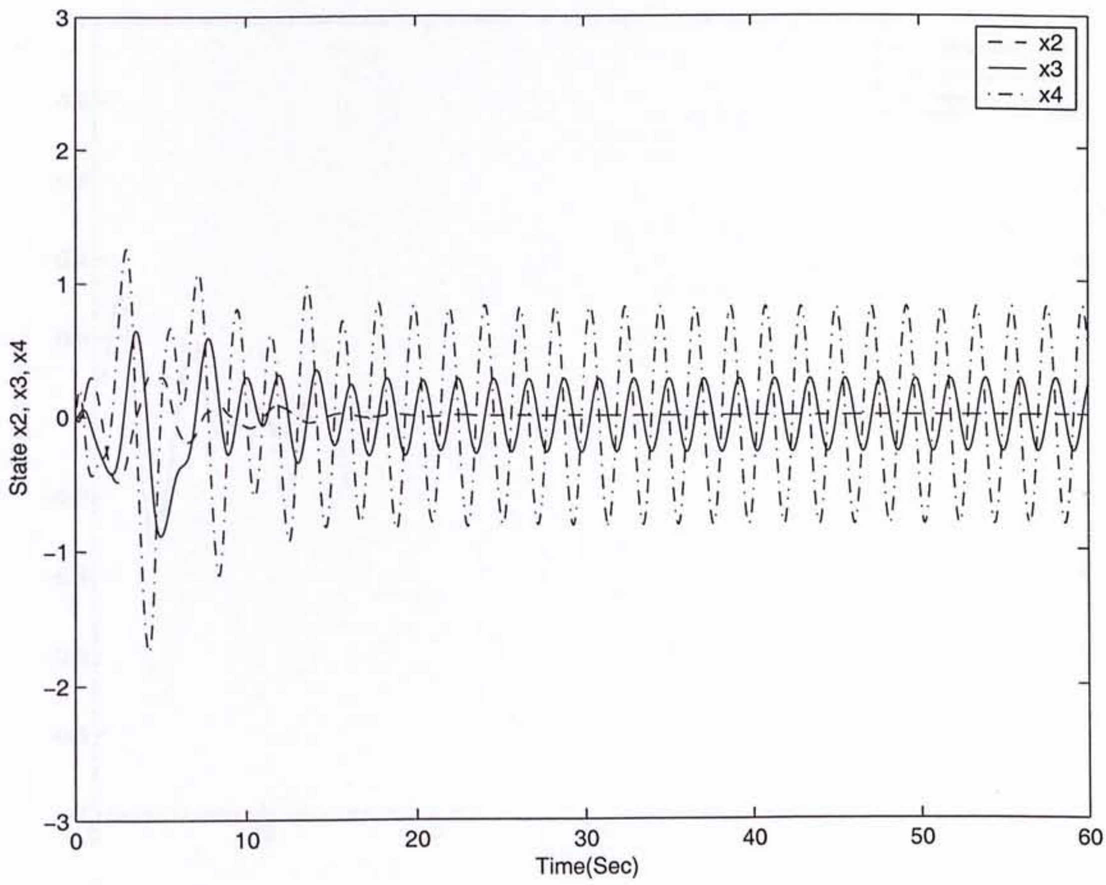

Figure 4.5: The profiles of the state variables (x_1, x_2, x_3) with $\epsilon = 0.2$, $\omega = 3$ and $A_m = 0.5$ under optimized parameter L.

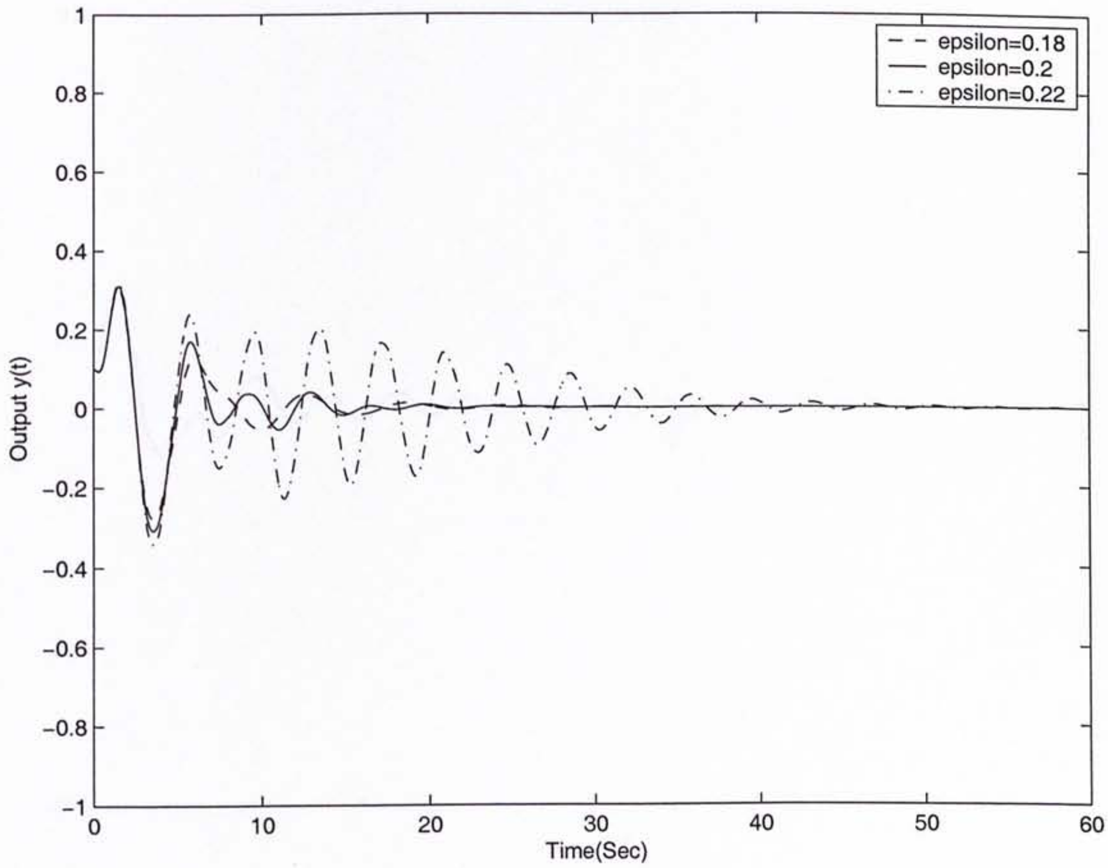

Figure 4.6: The profiles of the displacement x_1 with $\epsilon = 0.18, 0.2, 0.22$, $\omega = 3$ and $A_m = 0.5$ under optimized parameter L.

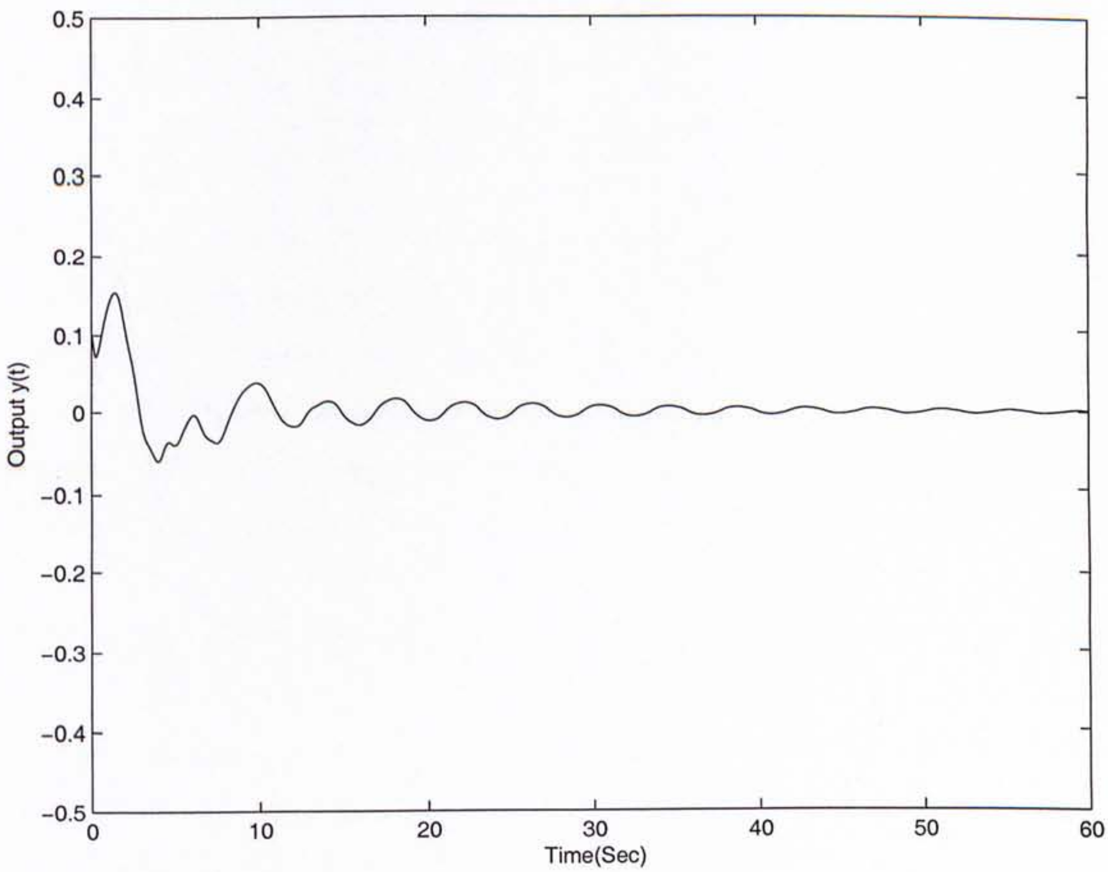

Figure 4.7: The profile of the displacement x_1 with $\epsilon = 0.2$, $\omega = 3$ and $A_m = 0.5$ under unoptimized parameter L.

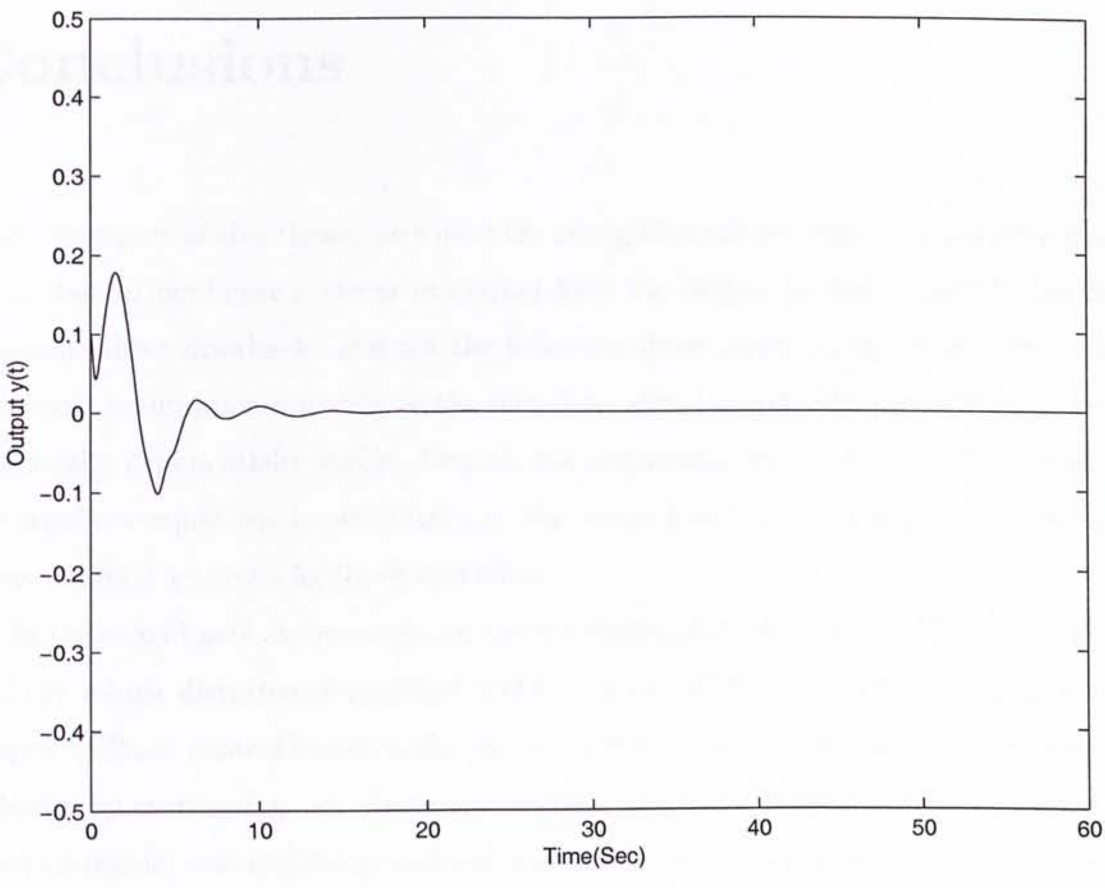

Figure 4.8: The profile of the displacement x_1 with $\epsilon = 0.2$, $\omega = 3$ and $A_m = 0.5$ under optimized parameter L.

Chapter 5

Conclusions

In the first part of this thesis, we solved the semiglobal robust output regulation problem for a class of nonlinear systems in normal form via output feedback control. We have overcome three drawbacks and got the following three results at the same time. First, only weak assumption imposed on the zero dynamics, i.e., globally asymptotically stable and locally exponentially stable. Second, the polynomial assumption on the solution of the regulator equations is weakened, i.e., the solution can be non-polynomial. Third, we have obtained a output feedback controller.

In the second part, we investigated the solvability of the disturbance rejection problem and the robust disturbance rejection problem of the RTAC system by the measurement output feedback control based on the robust output regulation method. We have obtained a design by overcoming two obstacles: devise a nonlinear internal model to account for non-polynomial nonlinearities, and improve the transient performance by using the parameter optimization design methods.

My future work contains the following problems:

1. Address the semiglobal robust output regulation of a wider class of systems, i.e, extend the z subsystem $\dot{z} = f_0(z, x_1, v, w)$ to $\dot{z} = f_0(z, x_j, v, w)$, or to a more general form $\dot{z} = f_0(z, x_1, \cdots, x_r, v, w)$. It is stated in [34] that when the z subsystem is in the form of $\dot{z} = f_0(z, x_1, \cdots, x_r, v, w)$, the whole system may be unstabilizable, so the output regulation problem of this system is challenging.

2. Consider global or semiglobal robust output regulation problem for uncertain nonlinear systems via output feedback control.

Biography

Guoqiang HU was born in Hubei Province, P.R. China. He received the Bachelor's degree of Engineering from University of Science and Technology of China in 2002, majoring in Automatic Control. Since 2002, he has been working towards his Master of Philosophy degree in the Department of Automation and Computer-Aided Engineering, the Chinese University of Hong Kong.

His research interests include nonlinear output regulation, stabilization, robust control and output feedback techniques.

The research related to this thesis has resulted in the following publications.

Publications:

1. J. Huang and G. Hu, "A Control Design for the Nonlinear Benchmark Problem via the Output Regulation Method", *Journal of Control Theory and Applications*, Vol. 2, No. 1, pp. 11-19, 2004.
2. J. Huang and G. Hu, "Robust Disturbance Rejection of the RTAC System", *Proceedings of the 23rd Chinese Control Conference*, pp. 1535-1539, Wuxi, China, Aug., 2004.
3. G. Hu and J. Huang, "Semiglobal Robust Output Regulation of a Class of Nonlinear Systems in Normal Form via Output Feedback Control", *Proceedings of the 5th World Congress on Intelligent Control and Automation*, pp. 783-787, Hangzhou, China, June, 2004.

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Appendix A.

ITAE Prototype Design

A convenient way to select the desirable pole locations for a closed-loop system is to make a member of a set of the so-called prototype polynomials as the characteristic polynomial of the closed-loop system. There are several sets of prototype polynomials one of which is shown in Table A.1.

This table is worked out by Graham and Lathrop [17] based on the criterion of minimizing integral of the time multiplied by the absolute value of the error (**ITAE**), that is,

$$P = \int_0^\infty t|e|dt.$$

In Table A.1, the nominal cutoff frequency is $\omega_0 = 1$ rad/sec. Pole locations for other values of ω_0 can be obtained by substituting s/ω_0 for s everywhere [16].

k	Pole Locations for $\omega_0 = 1$ rad/sec
1	$s+1$
2	$s+0.7071 \pm 0.7071j$
3	$(s+0.7081)(s+0.5210 \pm 1.068j)$
4	$(s+0.4240 \pm 1.2630j)(s+0.6260 \pm 0.4141j)$
5	$(s+0.8955)(s+0.3764 \pm 1.2920j)(s+0.5758 \pm 0.5339j)$
6	$(s+0.3099 \pm 1.2634i)(s+0.5805 \pm 0.7828j)(s+0.7346 \pm 0.2873j$
7	$(s+0.6816)(s+1.2123 \pm 1.0070j)(s+0.2492 \pm 1.0707j)(s+0.4214 \pm 0.5579j)$
8	$(s+2.0782)(s+0.6675)(s+0.2031 \pm 1.1774j)(s+0.3945 \pm 0.7479j)(s+0.6296 \pm 0.5567j)$

Table A.1: Pole locations of ITAE prototype design.