Rough Isometry and Analysis on Manifold

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Abstract

In this thesis we will survey some analytic results related to rough isometries between Riemannian manifolds. The concept of rough isometry is first introduced by Kanai. Since to be roughly isometric is an equivalent relation, we would expect that there are some invariants share with roughly isometric manifolds. Kanai showed that the volume growth rate, the isoperimetric inequality and the existence of positive Green function are examples of these invariants. He also proved that a manifold has Liouville property of positive harmonic functions if it is roughly isometric to a Euclidean space. After Kanai, Holopainen used similar method to prove that the Liouville D_p-property is another roughly isometric invariant. In all results above, the assumption on positive injectivity radius on manifolds was used. Later Coulhon and Saloff-Coste used a different method to study rough isometries. They didn't use the assumption on positive injectivity radius. They showed that some Sobolev inequalities, which are equivalent to some isoperimetric inequalities, are uniformly roughly isometric invariant. They also proved that the Poincaré inequality is also preserved under uniform rough isometries. Combine with a previous result given by Grigor'yan and Saloff-Coste independently, which gave an equivalent statement of parabolic Harnack inequality, they found that the parabolic Harnack inequality is a uniformly roughly isometric invariant.

Contents

Chapter 1

Introduction

In this chapter, we give the definition of rough isometries between two metric spaces and discuss some basic properties of them.

1.1 Rough Isometries

Let X be a metric space. For a point x in X, $B_r(x)$ denotes the open r-ball around x: Moreover for a subset Y of X we denote by $B_r(Y)$ the r-neighborhood of Y; $B_r(Y) = \{x \in X : d(x, Y) < r\}$. A subset Y of X is called ε-*full* in X for $\varepsilon > 0$ if $X = B_\varepsilon(Y)$, and is said to be *full* if it is ε-*full* for some $\varepsilon > 0$.

Definition 1.1 *A map $\varphi : X_1 \to X_2$ between two metric spaces X_1 and X_2, not necessarily continuous, is called a rough isometry, if*

1. *the image of φ is full in X_2,*

2. *there exists constants $a \geq 1$ and $b \geq 0$ such that*

$$a^{-1}d(x, y) - b \leq d(\varphi(x), \varphi(y)) \leq ad(x, y) + b, \ \forall x, y \in X_1.$$

We can easily show that if $\varphi : X \to Y$ and $\psi : Y \to Z$ are rough isometries then so is the composition $\psi \circ \varphi : X \to Z$ and we have a mapping $\varphi^{-1} : Y \to X$ such that both $d(\varphi^{-1} \circ \varphi(x), x)$ and $d(\varphi \circ \varphi^{-1}(y), y)$ are bounded in $x \in X$ and in $y \in Y$, respestively. In fact, for each $y \in Y$, choose $x \in X$ so that $d(\varphi(x), y) < \varepsilon$, where we assume that the image of φ is ε-full in Y, and put $\varphi^{-1}(y) = x$. We call φ^{-1} a *rough inverse* of φ. Two metric space is said to be *roughly isometric* if there is a rough isometry between them. Therefore we have

Proposition 1.1 *To be roughly isometric defines an equivalent relation.*

We give some basic examples of rough isometries.

Example 1.1 *An arbitrary mapping between two compact metric spaces is a rough isometry. Therefore any two compact metric spaces are roughly isometric.*

Example 1.2 *If X and Y are roughly isometric, then X and $Y \times K$ are roughly isometric where K is an arbitrary compact metric space. In other words, rough isometries neglect "compact factors".*

Definition 1.2 *A diffeomorphism φ of two Riemannian manifolds X onto Y is called a quasi-isometry if there is a constant $a \geq 1$ such that*

$$a^{-1}|\xi| \leq |d\varphi(\xi)| \leq a|\xi|, \ \forall \xi \in TX.$$

Another kind of isometries is so called *pseudo-isometries* introduced by Mostow [Mo].

Definition 1.3 *A pseudo-isometry of X into Y is a continuous map satisfying*

$$a^{-1}d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq ad(x_1, x_2), \ \forall x_1, x_2 \in X$$

with suitable constants $a \geq 1$ and $b \geq 0$.

Example 1.3 *Both quasi-isometries and pseudo-isometries are rough isometries.*

Since we have not assumed a rough isometry even to be continuous, the local geometry of a manifold does not brought into another manifold by a rough isometry. So we need some additional conditions on rough isometries and Riemannian manifolds which governs local geometries of the manifolds. On a Riemannian manifold, there is a natural Riemannian measure. So we can consider a Riemannian manifold as a metric measure space and we have the following definition.

Definition 1.4 *Let X_1 and X_2 be two metric measure spaces. A rough isometry φ from X_1 to X_2 is said to be uniform if there exists constant $C > 0$ such that*

$$C^{-1}V_1(x,1) \leq V_2(\varphi(x),1) \leq CV_1(x,1), \ \forall x \in X_1,$$

where $V_1(x,1)$ and $V_2(\varphi(x),1)$ are measure of $B_1(x)$ in X_1 and $B_2(\varphi(x))$ in X_2.

In general, uniform rough isometries does not define an equivalence relation. But it define an equivalence relation in manifolds satisfy the local volume doubling property.

Definition 1.5 *A complete Riemannian manifold X is said to be satisfies the local volume doubling property if $\forall r > 0$, $\exists C_r > 0$ such that $\forall x \in M$, we have*

$$V(x,2r) \leq C_r V(x,r)$$

where $V(x,r)$ denotes the volume of the ball with center x and radius r.

Another local geometric property which will be used later is the local Poincaré inequality.

Definition 1.6 *A complete Riemannian manifold X is said to satisfy the local Poincaré inequality if $\forall \sigma \geq 1,\ r > 0,\ \exists C_{\sigma,r} > 0$ such that $\forall f \in C^{\infty}(M),\ x \in M$, we have*

$$(\int_{B(x,r)} |f(y) - f_r(x)|^{\sigma} dy)^{1/\sigma} \leq C_{\sigma,r}(\int_{B(x,2r)} |\nabla f(y)|^{\sigma} dy)^{1/\sigma}$$

where $f_r(x) = \frac{1}{V(x,r)} \int_{B(x,r)} f(y)dy$.

From [B] and the Bishop comparison theorem [BC], we have

Lemma 1.1 *If the Ricci curvature of a complete Riemannian manifold is bounded below, then both local volume doubling property and local Poincaré inequality will be satisfied.*

Lemma 1.2 *If two complete Riemannian manifold have bounded below Ricci curvature and positive injectivity radii, then every rough isometries between them are uniform.*

Proof:
Suppose X_1 and X_2 are manifolds of dimensions n_1 and n_2 with positive injectivity radius and Ricci curvature bounded below from $-(n_i - 1)K^2$, $i = 1, 2$, where $K > 0$. Let φ be a mapping from X_1 to X_2, then from Bishop comparison theorem, we have

$$V_1(x, 1) \leq V_K(1),\ \forall x \in X_1.$$

where $V_K(1)$ is the volume of a geodesic ball of radius 1 in the simply connected complete Riemannian n_1-manifold of constant sectional curvature $-K^2$.
On the other hand,

$$V_2(\varphi(x), r) \geq v_0 r^{n_2}$$

where $r = \min\{1, \mathrm{inj}(X_2)/2 > 0\}$ and v_0 is a positive constant depending only on dimension of X_2 [Cr].
Therefore

$$V_1(x, 1) \leq \frac{V_K(1)}{v_0 r^{n_2}} V_2(\varphi(x), r) \leq \frac{V_K(1)}{v_0 r^{n_2}} V_2(\varphi(x), 1).$$

Since the above inequality is true for any φ, the proof of the lemma is completed by exchanging X_1 and X_2. □

1.2 Discrete approximation of Riemannian manifolds

In [Mi], Milnor gives examples of pairs of Riemannian manifolds roughly isometric to each other, and suggests the method of discrete or combinatorial approximation of geometries of Riemannian manifolds. Suppose that Γ is a finitely generated group with finite generator system A. For an element $\gamma \neq 1$ of Γ, let $|\gamma|_A$ be the smallest positive integer k such that γ is represented by a product of k elements of $A \cup A^{-1}$, and put $|1|_A = 0$. This $|\cdot|_A$ is called the *word norm* of Γ with respect to A, and satisfies the following conditions for all $\beta, \gamma \in \Gamma$:

1. $|\gamma|_A \geq 0$, and $|\gamma|_A = 0$ iff $\gamma = 1$,
2. $|\gamma^{-1}|_A = |\gamma|_A$,
3. $|\beta\gamma|_A \leq |\beta|_A + |\gamma|_A$.

Also the word norms corresponding to two finite generator systems A and B are equivalent; i.e., there is a constant $a \geq 1$ such that $a^{-1}|\gamma|_A \leq |\gamma|_B \leq a|\gamma|_A$ for all γ. Now suppose moreover that Γ acts freely and properly discontinuously on a complete Riemannian manifold X as isometry and that X/Γ is compact. Fix a point o in X and put $\|\gamma\| = \rho(o, \gamma o)$ for $\gamma \in \Gamma$. Then obviously the following hold for all $\beta, \gamma \in \Gamma$:

1. $\|\gamma\| \geq 0$, and $\|\gamma| = 0$ iff $\gamma = 1$,
2. $\|\gamma^{-1}\| = \|\gamma\|$,
3. $\|\beta\gamma\| \leq \|\beta\| + \|\gamma\|$.

In this situation, Milnor [Mi] has shown the inequalities

$$a^{-1}|\gamma|_A - b \leq \|\gamma\| \leq a|\gamma|_A, \ \forall \gamma \in \Gamma,$$

where $a \geq 1$ and $b \geq 0$ are suitable constants. Now put $d_A(\beta,\gamma) = |\beta^{-1}\gamma|_A$. Then d_A is a left-invariant metric on Γ, called the *word metric* of Γ with respect to A, and the map $\varphi : \Gamma \to X$, $\gamma \to \gamma o$ is a rough isometry (with respect to the word metric d_A of Γ and the Riemannian metric ρ of X) satisfying the inequality

$$a^{-1}d_A(\beta,\gamma) - b \leq \rho(\varphi(\zeta),\varphi(\gamma)) \leq a d_A(\beta,\gamma), \ \forall \beta,\gamma \in \Gamma.$$

Thus we can conclude the following propposition since to be roughly isometric is an equivalence relation.

Proposition 1.2 *If a discrete group Γ acts freely and properly discontinuously on complete Riemannian manifolds X and Y isometries in such a way that both X/Γ and Y/Γ are compact, then X is roughly isometric to Y.*

Also Milnor has shown that the volume growth rate of X is dominated by that of Γ. In fact he proved

$$c^{-1}\#\{\gamma \in \Gamma : |\gamma|_A \leq a'^{-1}r - b\} \leq V(o,r) \leq c\#\{\gamma \in \Gamma : |\gamma|_A \leq a'r + b'\},$$

where $a' \geq 1$, $b' \geq 0$ and $c \geq 1$ are constants, and, for a set S, $\#S$ denotes the cardinality of S. This fact suggests that geometry of the Riemannian manifold X may be approximated by the combinatorial geometry of the discrete group Γ.

To establish our theorems of invariance of geometric properties of manifolds under rough isometry, we approximate a Riemannian manifold by a combinatorial structure, which we call a net. In case of Milnor's work, the orbit Γo of the action of Γ on X may be considered as a net in our sense, and we have already seen that the geometry of the discrete group Γ reflects that of the Riemannian manifold X. This is also the case with a net in a complete Riemannian manifold. Moreover a net in a complete Riemannian manifold has a canonical metric of combinatorial nature, which corresponds to the word metric in the case of a finitely generated group, and we will see that the net is roughly isometric to the manifold.

Now the scheme of the proofs of our theorems of invariance of geometric properties under rough isometries is stated in the following form. Suppose that the complete Riemannian manifolds X and Y are roughly isometric to each other.

1. A rough isometry between X and Y induces a rough isometry between nets P in X and Q in Y.

2. A discrete approximation lemma suggests that the geometries of P and Q coincide with those of X and Y, respectively.

3. Two roughly isometric nets P and Q have the same geometry.

If the above statements are proved, then it is easy to see that X and Y have the same geometry. We will see that the first statement is always true. The third statement is, in general, easy to prove. So most of our work will be concentrated in the proofs of discrete approximation lemmas.
Now we give the definition of nets.

Definition 1.7 *Let P be a countable set. A family $N = \{N(p) : p \in P\}$ is called a net structure of P if the following conditions hold for all $p, q \in P$:*

1. *$N(p)$ is a finite subset of P,*

2. *$q \in N(p)$ iff $p \in N(q)$.*

For a point $p \in P$, sometimes we denote $q \in N(p)$ by $q \sim p$ and q is called a neighbor of p. By a net we mean a countable set with a net structure.

Connecting by a segment each pair of two points which are neighbors of each other, we see immediately that a net is essentially nothing but a countable 1-dimensional locally finite simplicial complex without orientation, or equivalently, locally finite countable graph.

Definition 1.8 *Suppose that P is a net.*

1. *A sequence $\mathbf{p} = (p_0, \cdots, p_l)$ of points in P is called a path from p_0 to p_l of length l if each p_k is a neighbor of p_{k-1}.*

2. *P is called connected if any two points in P are connected by a path.*

3. *For points p and q of a connected net P, $d(p,q)$ denotes the minimum of the length of paths from p to q. Obviously this d satisfied the axioms of metric. We call this d the combinatorial metric of P.*

4. *P is said to be uniform if $sup\{\#N(p) : p \in P\} < \infty$, where, for a set S, $\#S$ denotes the cardinality of it.*

Lemma 1.3

1. *If P is a uniform connected nets, d be the combinatorial metric of P, then, for all $r \geq 0$ and for all finite subsets S of P, the inequality*

$$\#\{p \in P : d(p,S) \leq r\} \leq \lambda^r \#S$$

holds, where $\lambda \geq 1$ is a constant independent of r and S.

2. *Suppose that P and Q are connected nets, P uniform, and that $\varphi : P \to Q$ is a rough isometry with respect to the combinatorial metrics of P and Q. Then there is a constant μ such that*

$$\#S \leq \mu \#\varphi(S)$$

for any finite subset S of P.

Proof:
The first statement is obvious from the definition.
We prove the second statement. $\forall p, q \in P$ such that $d(p,q) \geq a(b+1)$, where a and b are constants in the definition of rough isometry,

$$d(\varphi(p), \varphi(q)) \geq a^{-1} d(p,q) - b \geq 1.$$

Thus $\varphi(p) \neq \varphi(q)$.
For any finite $S \subset P$, there exists $r \in \mathbb{N}$, $p_1, \cdots, p_r \in S$ such that $d(p_i, p_j) \geq a(b+1)$, $\forall 1 \leq i, j \leq r$, $i \neq j$ and $\forall q \in S$, $d(q, p_i) \leq a(b+1)$ for some $1 \leq i \leq r$. Then $\varphi(p_i) \neq \varphi(p_j)$ for any $1 \leq i, j \leq r$, $i \neq j$.
Therefore by the first statement

$$\#\varphi(S) \geq r \geq \frac{\#S}{\lambda^{a(b+1)}},$$

which complete the proof of lemma. □

Suppose that X is a complete Riemannian manifold, and let ρ be the Riemannian metric. A subset P of X is said to be *ε-separated* for $\varepsilon > 0$, if $\rho(p, q) \geq \varepsilon$ whenever p and q are distinct points of P, and an ε-separated subset is called *maximal* if it is maximal with respect to the order relation of inclusion. Obviously a maximal ε-separated subset of X is ε-full in X. Let P be a maximal ε-separated subset of X. We define a net structure $N = \{N(p) : p \in P\}$ of P by $N(p) = \{q \in P : 0 < \rho(p, q) \leq 2\varepsilon\}$.

Definition 1.9 *A maximal ε-separated subset of a complete Riemannian manifold X with the net structure described above is called an ε-net in X.*

It is easy to see that an ε-net in a complete Riemannian manifold is connected if the manifold is connected. In our later discussions, all manifolds and nets are assumed to be connected unless otherwise indicated.

Lemma 1.4 *Let X be a complete Riemannian manifold satisfying local volume doubling property, and let P be an ε-separated subset of X. Then we have*

$$\#\{p \in P : x \in B_r(p)\} \leq \nu$$

for all r>0 and for all $x \in X$, where ν depending only on ε, r and constants in local volume doubling property. Consequently every ε-net in a complete Riemannian manifold satisfies local volume doubling property is uniform.

Proof:
Fix $r > 0$ and $x \in X$, and put $P_x = \{p \in P : x \in B_r(p)\}$. Obviously $B_{\varepsilon/2}(p) \subset B_{r+\varepsilon/2}(x) \subset B_{2r+\varepsilon/2}(p)$ holds for all p$\in P_x$. Also by local volume doubling property we have $V(p, \varepsilon/2) \geq CV(p, 2r + \varepsilon/2)$, where C depends on r and ε. Hence with the fact that $B_{\varepsilon/2}(p)$'s are disjoint, we conclude

$$\begin{aligned} V(x, r+\varepsilon/2) &\geq \sum_{p\in P_x} V(p, \varepsilon/2) \\ &\geq C(\varepsilon, r) \sum_{p\in P_x} V(p, 2r+\varepsilon/2) \\ &\geq C(\varepsilon, r) V(x, r+\varepsilon/2) \# P_x \end{aligned}$$

i.e., $\# P_x \leq C^{-1}$. □

The following lemma will be a fundamental tool in later discussions, because this lemma makes it possible to interpret the geometry of a Riemannian manifold into the combinatorial geometry of an ε-net in the manifold.

Lemma 1.5 *Let X be a complete Riemannian manifold satisfies local volume doubling property, and P an ε-net in X. Then inclusion of P with combinatorial metric d into X with Riemannian metric ρ is a rough isometry. In fact we have*

$$\frac{1}{2\varepsilon}\rho(p_1, p_2) \leq d(p_1, p_2) \leq a\rho(p_1, p_2) + b, \ \forall p_1, p_2 \in P, \tag{1.1}$$

where $a \geq 1$ and $b \geq 0$ are constants depending only on ε and the constants in local volume doubling property. Consequently P is roughly isometric to X.

Proof:
The first inequality in (1.1) trivially holds (without the assumption on local volume doubling property). we prove the second inequality in (1.1). Suppose that p_1, p_2 are arbitrary distinct points of P. Let γ be a minimizing geodesic from p_1 to p_2 with unit speed. Put $P_\gamma = \{q \in P : B_\varepsilon(q) \cap \gamma \neq \emptyset\}$. Obviously $\{B_\varepsilon(q) : q \in P_\gamma\}$ covers γ, and $d(p_1, p_2) \leq \# P_\gamma$. Moreover take the positive integer k so that $k - 1 < d(p_1, p_2)/2\varepsilon \leq k$, and let

$x_0(=p_1), x_1, \cdots, x_{k-1}, x_k(=p_2)$ be the points on γ such that $d(x_{j-1}, x_j) = d(p_1,p_2)/k$ for $j = 1, \cdots, k$. Then $q \in B_\varepsilon(\gamma) \subset \cup_{j=0}^k B_{2\varepsilon}(x_j)$ for all $q \in P_\gamma$, and therefore $P_\gamma \subset \cup_{j=0}^k \{q \in P : x_j \in B_{2\varepsilon(q)}\}$. Hence from Lemma 1.4 we have $\#P_\gamma \leq \sum_{j=0}^k \#\{q \in P : x_j \in B_{2\varepsilon}(q)\} \leq \nu(k+1) < \nu(d(p_1,p_2)/2\varepsilon + 2)$. Thus we conclude $d(p_1,p_2) < \nu(\rho(p_1,p_2)/2\varepsilon + 2)$. □

The above lemma especially suggests that any two nets in a complete Riemannian manifold satisfies the local volume doubling property are roughly isometric to each other.

Definition 1.10 *Suppose m is a strictly positive function on an uniform net P and*

$$c_m = \sup_{\substack{x,y \\ x\sim y}} \frac{m(x)}{m(y)} < \infty.$$

Then (P,m) is called a ponderable net.

Lemma 1.6 *Let P be a ponderable net. If we put $V(x,n) = \sum_{y\in B(x,n)} m(y)$, $B(x,n) = \{y \in P : d(y,x) < n\}$ then*

$$m(x) \leq V(x,n) \leq m(x) C^n N^n, \ \forall x \in P, n \in \mathbb{N}^*.$$

Moreover, (P,d,m) is ponderable if and only if it satisfies the local volume doubling property.

Definition 1.11 *Put for all $E \subset P$*

$$\|f\|_{p,E} = (\sum_E |f(x)|^p m(x))^{1/p},$$

write $\|\cdot\|_p = \|\cdot\|_{p,P}$,

and the gradient of a function f on P is defined to be

$$\delta f(x) = (\sum_{y\sim x} |f(y) - f(x)|^2)^{1/2}.$$

Lemma 1.7 *Let (X, ρ, V) be a complete Riemannian manifold satisfying local volume doubling property and (P, d, m) be an ε-net of X, where $m(x) = V(x, \varepsilon)$. Then*

1. *P is ponderable.*

2. *The inclusion from P to X is an uniform rough isometry.*

Proof:
We only need to proved that inclusion from P to X, which is a rough isometry by Lemma 1.1, is uniform. We may assume that $\varepsilon \leq 1$, then

$$m(B(x, 1)) = m(x) = V(x, \varepsilon) \leq V(x, 1)$$

and

$$V(x, 1) \leq CV(x, \varepsilon) = Cm(x) = Cm(B(x, 1)),$$

for any $x \in P$. Therefore the inclusion is uniform. □

Definition 1.12 *Put for all $E \subset P$*

$$\|f\|_{p,E} = (\sum_E |f(x)|^p m(x))^{1/p},$$

write $\|\cdot\|_p = \|\cdot\|_{p,P}$,

and the gradient of a function f on P is defined to be

$$\delta f(x) = (\sum_{y \sim x} |f(y) - f(x)|^2)^{1/2}.$$

Definition 1.13 *Suppose ψ is a function on X we associate a function $\tilde{\psi}$ on P by*

$$\tilde{\psi}(x) = \psi_\varepsilon(x) = \frac{1}{V(x, \varepsilon)} \int_{B(x,\varepsilon)} \psi(\xi) d\xi, \ x \in P.$$

Lemma 1.8 *Supoose M is a Riemannian manifold satisfies the local volume doubling property. Then there exists C,C' such that for all* $x \in P$, $n \in \mathbb{N}$, $1 \leq p \leq +\infty$ *and function* $\psi \in C_0^\infty(M)$, *we have*

$$\|\tilde{\psi}\|_{p,B(x,n)} \leq C\|\psi\|_{p,B(x,C'n)}.$$

In particular,

$$\|\tilde{\psi}\|_{p,P} \leq C\|\psi\|_{p,M}.$$

Proof:

$$\begin{aligned}
\|\tilde{\psi}\|^p_{p,B(x,n)} &= \sum_{y\in B(x,n)} \left(\frac{1}{V(y,\varepsilon)}\int_{B(y,\varepsilon)} \psi(\xi)d\xi\right)^p V(y,\varepsilon) \\
&\leq \sum_{y\in B(x,n)} \frac{1}{V(y,\varepsilon)}\left(\int_{B(y,\varepsilon)} \psi^p(\xi)d\xi\right)V(y,\varepsilon) \\
&\leq C\int_{B(x,(2n+1)\varepsilon)} \psi^p(\xi)d\xi.
\end{aligned}$$

Here Lemma 1.4 is used. □

Let $(\theta_x)_{x\in P}$ be a C^∞ partition of unity of M such that $\theta_x \geq \frac{1}{N}$ on $\bar{B}(x,\varepsilon/2)$, $\theta_x = 0$ on $B(x,3\varepsilon/2)^c$ and satisfies $\|\nabla\theta_x\|_\infty \leq C$, $\forall x \in P$.

Definition 1.14 *Suppose* f *is a function on* P, *we associate a function* $\hat{f}$ *on* M *by*

$$\hat{f}(y) = \sum_{x\in P} f(x)\theta_x(y),\ y \in M.$$

Similar to Lemma 1.8, we have

Lemma 1.9 *Supoose M is a Riemannian manifold satisfies the local volume doubling property. Then there exists C,C' such that for all* $z \in M$, $r > 0$, $1 \leq p \leq +\infty$ *and function* $f \in C_0(P)$, *we have*

$$\|\hat{f}\|_{p,B(z,r)} \leq C\|f\|_{p,B(\bar{z},[C'r])},$$

where [a] denotes the integral part of a and $\bar{z}$ is a point on P with $\rho(z,\bar{z}) \le \varepsilon$. In particular,

$$\|\hat{f}\|_{p,M} \le C\|f\|_{p,P}.$$

If $f \ge 0$, then for all $x \in P, n \in \mathbb{N}$, we have

$$\|f\|_{p,B(x,n)} \le C\|\hat{f}\|_{p,B(x,C'n)}$$

and

$$\|f\|_{p,P} \le C\|\hat{f}\|_{p,M}.$$

Lemma 1.10 *Suppose M satisfies the local volume doubling property and local Poincaré inequality. Then for all $\varepsilon > 0$, $1 \le p < \infty$ there exists $C = C(\varepsilon, p)$ such that for any $x, y \in M$, $\rho(x,y) \le 2\varepsilon$, we have*

$$|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^p V(x,\varepsilon) \le C \int_{B(x,6\varepsilon)} |\nabla\psi(\xi)|^p d\xi, \ \forall \psi \in C_0^\infty(M)$$

Lemma 1.11 *Suppose M satisfies both local volume doubling property and local Poincaré inequality, then for all $p \ge 1$, there exists constants C,C' s.t. for all $x \in P$, $n \in \mathbb{N}$ and function $\psi \in C_0^\infty(M)$, we have*

$$\|\delta\tilde{\psi}\|_{p,B(x,n)} \le C\|\nabla\psi\|_{p,B(x,C'n)}.$$

In particular,

$$\|\delta\tilde{\psi}\|_{p,P} \le C\|\nabla\psi\|_{p,M}$$

Similarly, for all $z \in M$, $\bar{z} \in P$ such that $\rho(z,\bar{z}) \le \varepsilon$, $r > 0$ and function $f \in C_0(P)$,

$$\|\nabla\hat{f}\|_{p,B(z,r)} \le C\|\delta f\|_{p,B(\bar{z},[C'r])}$$

and

$$\|\nabla\hat{f}\|_{p,M} \le C\|\delta f\|_{p,P}.$$

Proof:
Suppose $\psi \in C_0^\infty(M)$

$$\begin{aligned}
\|\delta\tilde{\psi}\|_{p,B(x,n)} &= \Big(\sum_{y\in B(x,n)} (\delta\tilde{\psi}(y))^p m(y)\Big)^{1/p} \\
&= \Big(\sum_{y\in B(x,n)} \sum_{z\sim y} (\tilde{\psi}(z)-\tilde{\psi}(y))^2)^{p/2} m(y)\Big)^{1/p} \\
&\leq \Big(\sum_{y\in B(x,n)} N^{p/2} \sup_{z\sim y} |\tilde{\psi}(z)-\tilde{\psi}(y)|^p m(y)\Big)^{1/p} \\
&\leq C\Big(\sum_{y\in B(x,n)} \int_{B(y,6\varepsilon)} |\nabla\psi(\xi)|^p d\xi\Big)^{1/p} \\
&\leq C\Big(\int_{B(x,C'n)} |\nabla\psi(\xi)|^p d\xi\Big)^{1/p}.
\end{aligned}$$

Here we have used Lemma 1.10 and the first inequality in Lemma 1.11 is proved.
For the second inequality, since $\sum_{y\in P} \nabla\theta_y = 0$, we have

$$\nabla\hat{f}(x) = \sum_{y\in P} (f(y)-f(x))\nabla\theta_y(x), \ \forall x \in P,$$

and for all $x \in P, z \in B(x,\varepsilon)$,

$$\begin{aligned}
|\nabla\hat{f}(z)| &\leq C\sup\{|f(y)-f(x)|; d(y,x) \leq 2\} \\
&\leq C \sum_{d(z,x)\leq 2} \delta f(z).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{B(z,r)} |\nabla\hat{f}(\xi)|^p d\xi &\leq \sum_{x\in B(\bar{z},[C'r])} \int_{B(x,\varepsilon)} |\nabla\hat{f}(\xi)|^p d\xi \\
&\leq C \sum_{x\in B(\bar{z},[C'r])} \sum_{d(z,x)\leq 2} |\delta f(z)|^p m(x).
\end{aligned}$$

This ends the proof of the lemma. □

Chapter 2

Basic Properties of Rough Isometries

2.1 Volume growth rate

Since the rough isometricity between Riemannian manifold is an equivalence relation, we may expect that a rough isometry preserves some invariants of manifolds. In this section we show that the volume growth rates of geodesic balls in Riemannian manifolds are invariant under rough isometries.

Definition 2.1 *Let X be a complete Riemannian manifold, and o a point in X. Then X is said to be of polynomial growth of order k if*

$$\inf\{s > 0 : \limsup_{r\to\infty} r^{-s} V(o,r) < \infty\} = k.$$

X is said to be of exponential growth if

$$\limsup_{r\to\infty} r^{-1} \log V(o,r) > 0$$

holds.

Obviously these definitions do not depend on the choice of a point o in X. It is known that a complete Riemannian n-manifold of non-negative Ricci curvature is of polynomial growth of order $\leq$n, and that a simply connected complete Riemannian manifold of negative sectional curvature bounded away from zero is of exponential growth. For other examples of computations of volume growth rates, see [Mi]. In [CY], some relations between the volume growth rate and the other attributes of a Riemannian manifold are discussed. The purpose of this section is to prove the following theorem due to Kanai [K1].

Theorem 2.1 *Suppose that X and Y are complete Riemannian manifolds satisfying the local volume doubling property, and that X is uniformly roughly isometric to Y. Then X is of polynomial growth of order k (respectively of exponential growth) if so is Y.*

Corollary 2.1 *The hyperbolic spaces are not roughly isometric to the Euclidean spaces.*

We will prove Theorem 2.1 by showing that the volume growth rate of a manifold is approximated by that of an ε-net in the manifold.

Definition 2.2 *Let P be a ponderable net, and o a point in P. Then P is said to be of polynomial growth of order k if*

$$\inf\{s > 0 : \limsup_{n\to\infty} n^{-s} m(\{p \in P : d(o,p) < n\}) < \infty\} = k.$$

P is said to be of exponential growth if

$$\limsup_{n\to\infty} n^{-1} \log m(\{p \in P : d(o,p) < n\}) > 0$$

holds.

Lemma 2.1 *Let P and Q be ponderable nets uniformly roughly isometric to each other. Then there exists constant C such that for all* $r > 0$

$$m(\{p \in P : d(o,p) < r\}) \leq Cm(\{q \in Q : d(o',q) < ar + b\}),$$

where $o \in P$, $o' = \varphi(o)$ *and* a,b *are constants in the definition of rough isometry.*

Proof:
Let $\varphi : P \to Q$ be a rough isometry satisfying

$$a^{-1}d(p_1,p_2) - b \leq d(\varphi(p_1),\varphi(p_2)) \leq ad(p_1,p_2) + b, \ \forall\, p_1,p_2 \in P \tag{2.1}$$

Fix $o \in P$, and put $o' = \varphi(o)$. Then, with (2.1), we have

$$\begin{aligned} m(\{p \in P : d(o,p) < r\}) &= \sum_{\substack{p \in P \\ d(o,p)<r}} m(p) \\ &\leq C \sum_{\substack{p \in P \\ d(o,p)<r}} m(\varphi(p)) \\ &\leq C \sum_{\substack{q \in Q \\ d(o',q)<ar+b}} m(q) \\ &= Cm(\{q \in Q : d(o',q) < ar + b\}) \end{aligned}$$

and this implies the lemma. □

Lemma 2.2 *Let P and Q be ponderable nets uniformly roughly isometric to each other. Then P is of polynomial growth of order k (respectively of exponential growth) if and only if so is Q.*

Proof:
Applying Lemma 2.1, we have

$$\limsup_{n\to\infty} n^{-s} m(\{p \in P : d(o,p) < n\}) < \infty$$

if

$$\limsup_{n\to\infty} n^{-s} m(\{q \in Q : d(o', q) < an + b\}) < \infty$$

if

$$\limsup_{n\to\infty} n^{-s} m(\{q \in Q : d(o', q) < n\}) < \infty,$$

where $o \in P$ and $o' = \varphi(o)$.
Therefore the order of volume growth of P is not greater than that of Q.
The result follows by reversing P and Q. □

Lemma 2.3 *Suppose that X is a complete Riemannian manifold satisfying the local volume doubling property, and that P is an ε-net in X. Then there exists constant C and C' such that for all $x \in X$, $n \in \mathbb{N}$ and r sufficiently large, we have*

$$m(\{p \in P : d(x,p) < n\}) \leq CV(x, C'n)$$

and

$$V(x,r) \leq Cm(\{p \in P : d(\bar{x}, p) < C'r\}),$$

where $\bar{x} \in P$ such that $\rho(\bar{x}, x) < \varepsilon$.

Proof:
Form Lemma 1.8 and Lemma 1.9, we have

$$m(\{p \in P : d(x,p) < n\}) = \|1\|_{1,B(x,n)} \leq C\|\hat{1}\|_{1,B(x,C'n)} \leq CV(x, C'n)$$

and

$$V(x,r) = \|\hat{1}\|_{1,B(x,r)} \leq C\|1\|_{1,B(\bar{x},[C'r])} \leq Cm(\{p \in P : d(\bar{x}, p) < C'r\})$$

for large r. □

The following lemma claims that the volume growth rate of a manifold is approximated combinatorially.

Lemma 2.4 *Suppose that X is a complete Riemannian manifold satisfying the local volume doubling property, and that P is an ε-net in X. Then X is of polynomial growth of order k (respectively of exponential growth) if and only if P is of polynomial growth of order k (respectively of exponential growth).*

Proof:
Applying Lemma 2.3, for $p \in P$, we have

$$\limsup_{n\to\infty} n^{-s} m(\{q \in P : d(p,q) < n\}) < \infty$$

if and only if

$$\limsup_{r\to\infty} r^{-s} V(p,r) < \infty.$$

And this implies the lemma. □

Now Theorem 2.1 follows immediately from Lemma 1.7, Lemma 2.2, Lemma 2.4, Lemma 1.4 and Lemma 1.5. In fact, take X and Y as in Theorem 2.1, let P and Q be nets in X and Y, respectively. First note that both P and Q are ponderable. Then a uniform rough isometry between X and Y induces a uniform rough isometry between P and Q as Lemma 1.5 suggests, and therefore P and Q have the same growth rate. On the other hand, Lemma 2.4 says that the growth rates of P and Q coincide with those of X and Y, respectively. Hence we conclude that X and Y have the same volume growth rate and Theorem 2.1 is proved.

Lemma 2.5 *Let P and Q be ponderable nets uniformly roughly isometric to each other. Then*

$$\sum_{n=1}^{\infty} \frac{n}{m(\{p \in P : d(o,p) < n\})} < \infty$$

if and only if

$$\sum_{n=1}^{\infty} \frac{n}{m(\{q \in Q : d(o',q) < n\})} < \infty.$$

Proof:
Suppose

$$\sum_{1}^{\infty} \frac{n}{m(\{p \in P : d(o,p) < n\})} < \infty$$

then for positive integer $k > a + b$

$$\begin{aligned}
\sum_{n=k}^{\infty} \frac{n}{m(\{q \in Q : d(o', q) < n\})} &\leq \sum_{n=k}^{\infty} \frac{Cn}{m(\{p \in P : d(o,p) < a^{-1}(n-b)\})} \\
&\leq Ca \sum_{l=1}^{\infty} \frac{n}{m(\{p \in P : d(o,p) < l\})} \\
&< \infty
\end{aligned}$$

The proof is completed by reversing P and Q. □

Lemma 2.6 *Suppose that X is a complete Riemannian manifold satisfying the local volume doubling property, and that P is an ε-net in X, $p \in P$. Then*

$$\int_{1}^{\infty} \frac{t}{V(p,t)} dt < \infty$$

if and only if

$$\sum_{n=1}^{\infty} \frac{n}{m(\{q \in P : d(p,q) < n\})} < \infty.$$

Proof:
Suppose

$$\sum_{n=1}^{\infty} \frac{n}{m(\{q \in P : d(p,q) < n\})} < \infty.$$

From Lemma 2.3, take positive integer $k > C'$, we have

$$\begin{aligned}
\int_{k}^{\infty} \frac{t}{V(p,t)} dt &\leq \sum_{n=k}^{\infty} \frac{n+1}{V(p,n)} \\
&\leq \sum_{n=k}^{\infty} \frac{C(n+1)}{m(\{q \in P : d(p,q) < C'^{-1} n\})} \\
&\leq 2CC'^2 \sum_{n=1}^{\infty} \frac{n}{m(\{q \in P : d(p,q) < n\})} \\
&< \infty.
\end{aligned}$$

Suppose

$$\int_1^\infty \frac{t}{V(p,t)}dt < \infty.$$

Again from Lemma 2.3, for sufficiently large $k > C' + 1$

$$\begin{aligned}
\sum_{n=k}^{\infty} \frac{n}{m(\{q \in P : d(p,q) < n\})} &\le \int_k^\infty \frac{t}{m(\{q \in P : d(p,q) < t-1\})}dt \\
&\le \int_k^\infty \frac{Ct}{V(p, C'^{-1}(t-1))}dt \\
&\le 2CC'^2 \int_1^\infty \frac{t}{V(p,t)}dt \\
&< \infty.
\end{aligned}$$

This complete the proof of lemma. □

Using the same scheme of the proof of Theorem 2.1 but replacing Lemma 2.2 and Lemma 2.4 by Lemma 2.5 and Lemma 2.6, the following theorem is proved.

Theorem 2.2 *Suppose that X and Y are complete Riemannian manifolds satisfying the local volume doubling property, and that X is uniformly roughly isometric to Y. Then the volume growth condition*

$$\int_1^\infty \frac{t}{V(p,t)}dt < \infty$$

holds on X if and only if so does Y.

2.2 Sobolev Inequalities

Definition 2.3 *Suppose $1 \le p \le q < +\infty$. We say that M satisfies the Sobolev inequality $(S_{p,q})$ if*

$$S_{p,q} = S_{p,q}(M) = \inf\{\frac{\|\nabla\psi\|_p}{\|\psi\|_q} : \psi \in C_0^\infty(M), \psi \neq 0\} > 0.$$

In this case,

$$S_{p,q}\|\psi\|_q \leq \|\nabla\psi\|_p, \ \forall\psi \in C_0^\infty(M).$$

In this section we show that the validity of some Sobolev inequalities is inherited through uniform rough isometries under certain conditions on manifolds. For this we are going to consider another Sobolev inequalities of the following forms.

Definition 2.4 *M is said to be satisfies* $(S_{p,q}^\infty)$ *if*

$$S_{p,q}^\infty = S_{p,q}^\infty(M) = \inf\{\frac{\|\nabla\psi\|_p}{\|S\psi\|_q} : \psi \in C_0^\infty(M), \psi \neq 0\} > 0.$$

where S is an operator on $C_0^\infty(M)$ *defined by*

$$S\psi = \sum_{x\in P} \psi_\varepsilon(x)\theta_x,$$

and θ_x *is a partition of unity as stated before Lemma 1.9 and* ψ_ε *is defined in Definition 1.13.*

Lemma 2.7 *Suppose M satisfies the local volume doubling property and local Poincaé inequality. Then S operates on* L^p, $1 \leq p \leq +\infty$, *and*

$$\|S\psi\|_p \leq C_1\|\psi\|_p, \ \forall\psi \in C_0^\infty(M).$$

Moreover,

$$\|\psi - S\psi\|_p \leq C_2\|\nabla\psi\|_p, \ \forall\psi \in C_0^\infty(M).$$

Proof:
The first inequality holds by Lemma 1.8 and Lemma 1.9 since $S\psi = \hat{\tilde{\psi}}$.

For the second inequality

$$
\begin{aligned}
& \|\psi - S\psi\|_p^p \\
= & \int |\psi - \sum_{x\in P} \psi_\varepsilon(x)\theta_x|^p \\
= & \int |\sum_{x\in P} (\psi(y) - \psi_\varepsilon(x))\theta_x(y)|^p dy \\
\leq & N \sum_{z\in P} \int_{B(z,\varepsilon)} (\sum_{x\sim z} |\psi(y) - \psi_\varepsilon(x)|)^p dy \\
\leq & N^p \sum_{z\in P} \sum_{x\sim z} \int_{B(z,\varepsilon)} |\psi(y) - \psi_\varepsilon(x)|^p dy \\
\leq & 2^{p-1} N^p \sum_{z\in P} (N \int_{B(z,\varepsilon)} |\psi(y) - \psi_\varepsilon(z)|^p dy + \sum_{x\sim z} |\psi_\varepsilon(z) - \psi_\varepsilon(x)|^p V(z,\varepsilon))
\end{aligned}
$$

The result follows by applying local Poincaré inequality and Lemma 1.10. □

The following proposition tells the relation between $(S_{p,p})$ and $(S^{\infty}_{p,p})$.

Proposition 2.1 *Suppose M satisfies local volume doubling property and local Poincaé inequality, then $(S_{p,p})$ is equivalent to $(S^{\infty}_{p,p})$.*

Proof:
The inequality $S_{p,p}(M) \leq CS^{\infty}_{p,p}(M)$ follows easily from the first part of Lemma 2.7.
Now for any $\psi \in C_0^{\infty}(M)$,

$$
\begin{aligned}
\|\psi\|_p & \leq \|\psi - S\psi\|_p + \|S\psi\|_p \\
& \leq C\|\nabla\psi\|_p + \|S\psi\| \\
& \leq (C + S^{\infty}_{p,p}(M)^{-1})\|\nabla\psi\|_p
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\|\nabla\psi\|_p}{\|\psi\|_p} & \geq (C + S^{\infty}_{p,p}(M)^{-1})^{-1} \\
S_{p,p} & \geq \frac{S^{\infty}_{p,p}}{1 + CS^{\infty}_{p,p}(M)}
\end{aligned}
$$

And the result follows. □

Definition 2.5 *Suppose P is a net,*

$$S_{p,q} = S_{p,q}(P) = \inf\{\frac{\|\delta f\|_p}{\|f\|_q} : f \in c_0(P), f \neq 0\}$$

We say that P satisfies the Sobolev inequality $(S_{p,q})$ if $S_{p,q}(P) > 0$. In this case,

$$S_{p,q}\|f\|_q \leq \|\delta f\|_p, \ \forall f \in c_0(P)$$

The following proposition is refered to [K3].

Proposition 2.2 *Let P and Q be ponderable connected nets. If P is uniformly roughly isometric to Q, then for any $1 \leq p < +\infty$ and $1 \leq q < +\infty$, P satisfies $S_{p,q}$ if and only if so does Q.*

Proof:
Let $\varphi : P \to Q$ be a uniform rough isometry such that $Q = \cup_{x \in P} B(\varphi(x), \tau)$. Suppose that v is an arbitrary non-negative function on Q with finite support. Let y and y' be points of Q with $d(y, y') = 1$. Then we have

$$\begin{aligned}
&|v_\tau(y') - v_\tau(y)| \\
= \ & |\frac{1}{m(B(y',\tau))} \sum_{r' \in B(y',\tau)} v(r')m(r') - \frac{1}{m(B(y,\tau)} \sum_{r \in B(y,\tau)} v(r)m(r)| \\
= \ & |\frac{1}{m(B(y,\tau)) \cdot m(B(y',\tau))} \sum_{\substack{r \in B(y,\tau) \\ r' \in B(y',\tau)}} (v(r') - v(r))m(r)m(r')| \\
\leq \ & \frac{1}{m(B(y,\tau)) \cdot m(B(y',\tau))} \sum_{\substack{r \in B(y,\tau) \\ r' \in B(y',\tau)}} |v(r') - v(r)|m(r)m(r') \\
\leq \ & \sum_{\substack{r \in B(y,\tau) \\ r' \in B(y',\tau)}} |v(r') - v(r)|.
\end{aligned}$$

Moreover, for $r \in B(y, \tau)$ and $r' \in B(y', \tau)$, connecting them by a length-minimizing path $\gamma = (y_0, \cdots, y_l)$ with $y_0 = r$ and $y_l = r'$, and of length

$l \leq 2\tau + 1$, we obtain

$$\begin{aligned} |v(r') - v(r)| &\leq |v(y_0) - v(y_1)| + \cdots + |v(y_{l-1}) - v(y_l)| \\ &\leq \delta v(y_0) + \cdots + \delta v(y_{l-1}) \\ &\leq \sum_{y'' \in B(y,\tau)} \delta v(y') \end{aligned}$$

since $d(y_i, y) \leq \tau$ for $i = 0, \cdots, l-1$, and therefore we get

$$|v_\tau(y') - v_\tau(y)| \leq \sum_{y'' \in B(y,\tau)} \delta v(y'').$$

Thus, for any $y \in Q$, we have

$$\delta v_\tau(y) = (\sum_{y' \in N_y} (v_\tau(y') - v_\tau(y))^2)^{1/2} \leq C \sum_{r \in B(y,\tau)} \delta v(r),$$

and

$$(\delta v_\tau)^p(y) \leq C(\sum_{r \in B(y,\tau)} \delta v(r))^p \leq C \sum_{r \in B(y,\tau)} (\delta v)^p(r)$$

by the Hölder inequality and the uniformness of Q. This yields

$$\sum_{y \in Q} (\delta v_\tau)^p(y) m(y) \leq C \sum_{y \in Q} \sum_{r \in B(y,\tau)} (\delta v)^p(r) m(y) \leq C \sum_{y \in Q} (\delta v)^p(y) m(y),$$

since $m(y)$ and $m(r)$ are comparable. That is

$$(\sum_{y \in Q} (\delta v_\tau)^p(y) m(y))^{1/p} \leq C(\sum_{y \in Q} (\delta v)^p(y) m(y))^{1/p}. \tag{2.2}$$

Now define a finitely supported non-negative function u on P by $u = v_\tau \circ \varphi$. Note that for $x, x' \in P$ with $d(x, x') = 1$, there is a constant l_0 such that $d(\varphi(x), \varphi(x')) \leq l_0$, since φ is a uniform rough isometry. Therefore, connecting $\varphi(x)$ and $\varphi(x')$ by a path $\gamma = (y_0, \cdots, y_l)$ in Q with $y_0 = \varphi(x)$ and $y_l = \varphi(y')$, and of length $l \leq l_0$, we have

$$\begin{aligned} |u(x) - u(x')| &= |v_\tau(y_0) - v_\tau(y_l)| \\ &\leq |v_\tau(y_0) - v_\tau(y_1)| + \cdots + |v_\tau(y_{l-1}) - v_\tau(y_l)| \\ &\leq \delta v_\tau(y_0) + \cdots + \delta v_\tau(y_{l-1}) \\ &\leq \sum_{y \in B(\varphi(x), l_0 - 1)} \delta v_\tau(y), \end{aligned}$$

and this implies, as above,

$$\begin{aligned}(\delta u)^p(x) &\le C(\sum_{y\in B(\varphi(x),l_0-1)} \delta v_\tau(x))^p \\ &\le C \sum_{y\in B(\varphi(x),l_0-1)} (\delta v_\tau)^p(y)\end{aligned}$$

Hence we obtain

$$(\sum_{x\in P}(\delta u)^p(x)m(x))^{1/p} \le C(\sum_{y\in Q}(\delta v_\tau)^p(y)m(y))^{1/p}. \tag{2.3}$$

Moreover, we have

$$\begin{aligned}u^q(x) &= (\frac{1}{m(B(\varphi(x))}\sum_{y\in B(\varphi(x),\tau)} v(y)m(y))^q \\ &\ge c \sum_{y\in B(\varphi(x),\tau)} v^q(y),\end{aligned}$$

and consequently we get

$$\sum_{x\in P} u^q(x) \ge c\sum_{y\in Q} v^q(y)$$

since $Q = \cup_{x\in P} B(\varphi(x),\tau)$. This shows

$$(\sum_{x\in P} u^q(x)m(x))^{1/q} \ge c(\sum_{y\in Q} v^q(y)m(y))^{1/q}. \tag{2.4}$$

By (2.2), (2.3) and (2.4) we conclude

$$\begin{aligned}\frac{(\sum_{y\in Q}(\delta v)^p(y)m(y))^{1/p}}{(\sum_{y\in Q} v^q(y)m(y))^{1/q}} &\ge c\frac{(\sum_{x\in P}(\delta u)^p(x)m(x))^{1/p}}{(\sum_{x\in P} u^q(x)m(x))^{1/q}} \\ &\ge cS_{p,q}(P)\end{aligned}$$

for an arbitrary non-negative function v on Q with finite support. Moreover because $\delta v \ge \delta|v|$ for any function v on Q, we obtain $S_{p,q}(Q) \ge cS_{p,q}(P)$. This complete the proof of the proposition. □

Lemma 2.8 *Suppose M satisfies the local volume doubling property and local Poincaré inequality, then there exists C,C' such that for all* $z \in M$, $r > 0$ *and* $\psi \in C_0^\infty(M)$,

$$\|S\psi\|_{p,B(z,r)} \leq C\|\tilde{\psi}\|_{p,B(\bar{z},[C',r])}.$$

where $\bar{z} \in P$ *such that* $\rho(z,\bar{z}) \leq \varepsilon$. *Moreover if* $\psi \geq 0$, *then for all* $x \in P$, $n \in \mathbb{N}$,

$$\|\tilde{\psi}\|_{p,B(x,n)} \leq C\|S\psi\|_{p,B(x,C'n)}.$$

If $f \in c_0(X)$ *and* $f \geq 0$,

$$\|f\|_{p,B(x,n)} \leq C\|S\hat{f}\|_{p,B(x,C'n)}$$

Proof:
Observe that $S\psi = \hat{\tilde{\psi}}$, the first two inequality follow easily from Lemma 1.8 and Lemma 1.9
Now if $f \in c_0(P)$ and $f \geq 0$, we have for each $x \in P$,

$$\begin{aligned}
\tilde{\hat{f}}(x) &= \frac{1}{V(x,\varepsilon)}\int_{B(x,\varepsilon)} \hat{f}(y)dy \\
&= \frac{1}{V(x,\varepsilon)}\int_{B(x,\varepsilon)} \sum_{z\in P} f(z)\theta_z(y)dy \\
&\geq \frac{V(x,\varepsilon/2)}{NV(x,\varepsilon)} f(x) \\
&\geq cf(x)
\end{aligned}$$

for some constant c since $\theta_x(y) \geq 1/N$ for all $y \in B(x,\varepsilon/2)$.
The last inequality then follows from Lemma 1.9. □

The following approximation lemma is an immediate consequence of Lemma 1.9 and Lemma 2.8.

Lemma 2.9 *Suppose M satisfies the local volume doubling property and local Poincaé inequality,* $1 \leq p \leq q < +\infty$. *If P is an* ε*-net on M, then* $(S_{p,q}^\infty)$

on M is equivalent to $(S_{p,q})$ on P. More precisely, there exists constants c, C such that

$$cS_{p,q}(P) \leq S^{\infty}_{p,q}(M) \leq CS_{p,q}(P).$$

Using the same argument as in the last section, we have proved the main theorem of this section [CS].

Theorem 2.3 *Let X and Y be complete Riemannian manifolds satisfying the local volume doubling property and local Poincaré inequality which are uniformly roughly isometric to each other, then $(S^{\infty}_{p,q})$ on X is equivalent to $(S^{\infty}_{p,q})$ on Y for any $1 \leq p \leq q < +\infty$. Moreover $(S_{p,p})$ on X is equivalent to $(S_{p,p})$ on Y.*

2.3 Poincaré Inequality

We have defined the local version of Pincaré inequality in Section 1.1. We now define the global version.

Definition 2.6 *We say that a complete Riemannian manifold M satisfies the Poincaré inequality at infinity if there exists C and for all r_0 and $\sigma \geq 1$, there exists C_{σ,r_0} such that $\forall x \in M$, $\forall r \geq r_0$ and $\forall \psi \in C_0^{\infty}(M)$, we have*

$$\left(\int_{B(x,r)} |\psi(y) - \psi_r(x)|^{\sigma} dy\right)^{1/\sigma} \leq C_{\sigma,r_0} r \left(\int_{B(x,Cr)} |\nabla\psi(y)|^{\sigma} dy\right)^{1/\sigma},$$

where

$$\psi_r(x) = \frac{1}{V(x,r)} \int_{B(x,r)} \psi(\xi) d\xi.$$

It is known that a manifold with Ricci curvature non-negative satisfies the Poincaré inequality [B]. To prove that the Poincaré inequality at infinity is preserved under uniform rough isometry, we will use the same method as before. We must first define the Poincaré inequality on a net.

Definition 2.7 *We say that a ponderable net (P,m) satisfies the Poincaré inequality if there exists a constant* $C \geq 1$ *and, for all* $\sigma \geq 1$*, there exists* C_σ *such that, for any* $x \in P$*,* $n \in \mathbb{N}^*$ *and function f on P,*

$$(\sum_{y\in B(x,n)} |f(y) - f_n(x)|^\sigma m(y))^{1/\sigma} \leq C_\sigma n (\sum_{y\in B(x,Cn)} |\delta f(y)|^\sigma m(y))^{1/\sigma},$$

where

$$f_n(x) = \frac{1}{V(x,n)} \sum_{y\in B(x,n)} f(y)m(y).$$

Lemma 2.10 *Suppose* (P_1, m_1) *and* (P_2, m_2) *are two uniformly roughly isometric ponderable nets. If* (P_1, m_1) *satisfies the Poincaré inequality, so is* (P_2, m_2).

Proof:
We only prove the case $\sigma = 1$. Suppose $\Phi : P_1 \to P_2$ be a uniform rough isometry, and $k \in \mathbb{N}^*$ such that $[\Phi(P_1)]_k = P_2$. If f is a function with finite support on P_2, $f_k \circ \Phi$ is a function on P_1. Since P_1 satisfies the Poincaré inequality, $\forall x \in P_1,\ n \in \mathbb{N}^*$,

$$\begin{aligned} &\sum_{y\in B(x,n)} |(f_k \circ \Phi)(y) - (f_k \circ \Phi)_n(x)| m_1(y) \\ \leq\ & Cn \sum_{y\in B(x,C'n)} \delta(f_k \circ \Phi)(y) m_1(y). \end{aligned} \tag{2.5}$$

Obviously,

$$\sum_{y\in B(x,Cn)} \delta(f_k \circ \Phi)(y) m_1(y) \leq C_1 \sum_{z\in B(x,C_1'n)} \delta f_k(z) m_2(z), \tag{2.6}$$

since for C_1' sufficiently large, $B(\Phi(x), C_1'n)$ contains $\Phi(B(x, C'n))$ and that $m_2(\Phi(y)) \approx m_1(y)$. Now

$$\begin{aligned} |\delta f_k(z)|^2 &= \sum_{y\sim z} |f_k(z) - f_k(y)|^2 \\ &= \sum_{y\sim z} |\frac{1}{V(z,k)} \sum_{t\in B(z,k)} f(t)m_2(t) \\ &\quad -\frac{1}{V(y,k)} \sum_{s\in B(y\in B(y,k)} f(s)m_2(s)|^2 \\ &\leq 2\sum_{y\sim z}(\frac{1}{V(z,k)} \sum_{t\in B(z,k)} |f(t) - f(z)|^2 m_2 \\ &\quad +\frac{1}{V(y,k)} \sum_{s\in B(y,k)} |f(s) - f(z)|^2 m_2(s)) \\ &\leq \frac{2C_2}{V(z,k)} \sum_{t\in B(z,k)} |f(t) - f(z)|^2 m_2(t) \\ &\quad +\frac{C_3}{V(z,k)} \sum_{s\in B(z,k+1)} |f(s) - f(z)|^2 m_2(s). \end{aligned}$$

Using the fact that $B(y,k) \subset B(z,k+1)$ and that $V(z,k) \leq C_3V(y,k)$. Since $m_2(t) \approx V(z,k)$, $\forall z \in P_2$, $t \in B(z,k+1)$, we obtain

$$|\delta f_k(z)|^2 \leq C_4 \sum_{t\in B(z,k+1)} |f(t) - f(z)|^2.$$

Moreover

$$|f(t) - f(z)|^2 \leq (k+1)\sum_{i=1}^{j-1} |f(t_i) - f(t_{i+1})|^2 \leq (k+1) \sum_{y\in B(z,k+1)} |\delta f(y)|^2,$$

where $t = t_1, \cdots, t_i, \cdots, t_j = z$ is a minimal path from t to z. We get

$$|\delta f_k(z)|^2 \leq C_4(k+1)C_2^{k+1} \sum_{y\in B(z,k+1)} |\delta f(y)|^2,$$

therefore

$$\delta f_k(z) \leq C_5(\sum_{y\in B(z,k+1)} |\delta f(y)|^2)^{1/2} \leq C_6 \sum_{y\in B(z,k+1)} \delta f(y).$$

Finally

$$\sum_{z\in B(\Phi(x),C_1'n)} \delta f_k(z)m_2(z) \le C_7 \sum_{y\in B(\Phi(x),C_2'n)} \delta f(y)m_2(y). \tag{2.7}$$

Combine (2.5), (2.6) and (2.7), we have

$$\begin{aligned} &\sum_{y\in B(x,n)} |(f_k\circ\Phi)(y)-(f_k\circ\Phi)_n(x)|m_1(y) \\ \le\ & C_8 n \sum_{y\in B(\Phi(x),C_2'n)} \delta f(y)m_2(y). \end{aligned} \tag{2.8}$$

Now

$$\begin{aligned} &\sum_{z\in B(\Phi(x),n)} |f(z)-(f_k\circ\Phi)_n(x)|m_2(z) \\ \le\ & \sum_{z\in B(\Phi(x),n)} |f(z)-f_k\circ\Phi\circ\Phi^{-1}(z)|m_2(z) \\ &+ \sum_{z\in B(\Phi(x),n)} |f_k\circ\Phi\circ\Phi^{-1}(z)-(f_k\circ\Phi)_n(x)|m_2(z). \end{aligned} \tag{2.9}$$

where Φ^{-1} is a rough inverse of Φ and that $d_2(x,\Phi\circ\Phi^{-1}(x)) \le k$. Since $\Phi^{-1}(B(\Phi(x),n),n)) \subset B(x,C'3n)$ for sufficiently large C_3', the second term of right hand side of (2.9) satisfies

$$\begin{aligned} &\sum_{z\in B(\Phi(x),n)} |f_k\circ\Phi\circ\Phi^{-1}(z)-(f_k\circ\Phi)_n(x)|m_2(z) \\ \le\ & C_9 \sum_{z\in B(\Phi(x),n)} |f_k\circ\Phi\circ\Phi^{-1}(z)-(f_k\circ\Phi)_n(x)|m_1(\Phi^{-1}(z)) \\ \le\ & C_9 \sum_{y\in B(x,C_3'n)} |f_k\circ\Phi(y)-(f_k\circ\Phi)_n(x)|m_1(y) \\ \le\ & C_{10} n \sum_{y\in B(\Phi(x),C_4'n)} \delta f(y)m_2(y), \end{aligned}$$

where (2.8) has been used.

For the first term of right hand side of (2.9), write $\Phi\circ\Phi^{-1}(z) = \bar{z}$, then

$d_2(z, \bar{z}) \leq k$. We have

$$\begin{aligned}
&\sum_{z\in B(\Phi(x),n)} |f(z) - f_k \circ \Phi \circ \Phi^{-1}(z)|m_2(z) \\
= &\sum_{z\in B(\Phi(x),n)} |f(z) - f_k(\bar{z})|m_2(z) \\
\leq &\sum_{z\in B(\Phi(x),n)} (\frac{1}{V(\bar{z},k)} \sum_{t\in B(\bar{z},k)} |f(z) - f(t)|m_2(t))m_2(z) \\
\leq &\; C_{11} \sum_{z\in B(\Phi(x),n)} (\sum_{y\in B(z,2k)} |\delta f(y)|^2)^{1/2} m_2(z) \\
\leq &\; C_{12} \sum_{z\in B(\Phi(x),C_5'n)} \delta f(z)m_2(z),
\end{aligned}$$

by considering a minimal chain from t to z.
We have proved that

$$\sum_{z\in B(\Phi(x),n)} |f(z) - (f_k \circ \Phi)_n(x)|m_2(z) \leq C_{13}n \sum_{z\in B(\Phi(x),C_6'n)} \delta f(z)m_2(z),$$

and therefore

$$\begin{aligned}
\sum_{z\in B(\Phi(x),n)} |f(z) - f_n(z)|m_2(z) &\leq 2 \inf_{\alpha\in\mathbb{R}} \sum_{z\in B(\Phi(x),n)} |f(z) - \alpha|m_2(z) \\
&\leq 2 \sum_{z\in B(\Phi(x),n)} |f(z) - (f_k \circ \Phi)_n(x)|m_2(z) \\
&\leq 2C_{14}n \sum_{z\in B(\Phi(x),C_6'n)} \delta f(z)m_2(z),
\end{aligned}$$

which proved the Poincaré inequality on P_2. □

We next prove the following approximation lemma.

Lemma 2.11 *Suppose M satisfies the local volume doubling property and local Poincaré inequality and P be an ε-net on M. Then M satisfies the Poincaré inequality at infinity if and only if P satisfies the Poincaré inequality.*

Proof:
We prove the case $\sigma = 1$ only. Suppose that P satisfies the Poincaré inequality. For any $\psi \in C_0^\infty(M)$, $x \in M$, $r \geq \varepsilon$ and $\alpha \in \mathbb{R}$

$$\begin{aligned}
&\int_{B(x,r)} |\psi(y) - \alpha| dy \\
\leq\ & \int_{B(x,r)} \sum_{z \in P \cap B(x,r+\varepsilon)} |\psi(y) - \alpha| \chi_{B(z,\varepsilon)}(y) dy \\
\leq\ & \sum_{z \in P \cap B(x,r+\varepsilon)} \int_{B(z,\varepsilon)} |\psi(y) - \tilde{\psi}(z)| dy + \sum_{z \in P \cap B(x,r+\varepsilon)} m(z) |\tilde{\psi}(z) - \alpha|.
\end{aligned}$$

From the local Poincaé inequality, the first term above is bounded by

$$\begin{aligned}
& C_1 \sum_{z \in P \cap B(x,r+\varepsilon)} \int_{B(z,C_1'\varepsilon)} |\nabla \psi(y)| dy \\
\leq\ & C_2 \int_{B(x,C_2'r)} |\nabla \psi(y)| dy
\end{aligned}$$

For $x_0 \in P$ such that $\rho(x, x_0) \leq \varepsilon$, and $n \in \mathbb{N}^*$ such that $n \approx r$ and $P \cap B_M(x, r+\varepsilon) \subset B_P(x_0, n) \subset B_M(x, C"r)$, choose $\alpha = \psi_n(x_0)$. Since P satisfies the Poincaré inequality, the second term is then bounded by

$$C_3 n \sum_{y \in B(x_0, C_3'n)} |\delta \tilde{\psi}(y)| \leq C_4 r \int_{B(x, C_4''r)} |\nabla \psi(y)| dy,$$

where Lemma 1.11 is used.
Obviously that

$$\begin{aligned}
\int_{B(x,r)} |\psi(y) - \psi_r(x)| dy &\leq 2 \inf_{\alpha \in \mathbb{R}} \int_{B(x,r)} |\psi(y) - \alpha| dy \\
&\leq C_5 r \int_{B(x,C_5'r)} |\nabla \psi(y)| dy,
\end{aligned}$$

for any $r \geq \varepsilon$. Combine with the local Poincaé inequality, we get the Poincaré inequality at infinity.
Suppose M satisfies the Poincaré inequality at infinity. For any function f with finite support on P, $x \in P$, $n \in \mathbb{N}^*$, and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}
& \sum_{y \in B(x,n)} |f(y) - \alpha| m(y) \\
\leq\ & C\Big(\sum_{y \in B(x,n)} \int_{B(y,\varepsilon/2)} |f(y) - \hat{f}(z)| dz + \sum_{y \in B(x,n)} \int_{B(y,\varepsilon/2)} |\hat{f}(z) - \alpha| dz \Big).
\end{aligned}$$

Choose $\alpha = (\hat{f})_n(x)$, apply Poincaré inequality at infinity on M and Lemma 1.11, the second term above is bounded by

$$\begin{aligned}\int_{B(x,C'n)} |\hat{f}(z) - \alpha| dz &\leq C_1 n \int_{B(x,C_1'n)} |\nabla \hat{f}(z)| dz \\ &\leq C_2 n \sum_{y \in B(x,C_2'n)} \delta f(y) m(y).\end{aligned}$$

Note that $\theta_t(z) = 0$ when $d(z,t) \geq 3\varepsilon/2$, we have

$$\begin{aligned}\int_{B(y,\varepsilon/2)} |f(y) - \hat{f}(z)| dz &= \int_{B(y,\varepsilon/2)} |f(y) - \sum_{t \sim y} f(t)\theta_t(z)| dz \\ &= \int_{B(y,\varepsilon/2)} |\sum_{t \sim y} (f(y) - f(t))\theta_t(z)| dz \\ &\leq C_3 \sum_{t \sim y} |f(y) - f(t)| m(y) \\ &\leq C_4 \delta f(y) m(y).\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{y \in B(x,n)} |f(y) - f_n(x)| m(y) &\leq 2 \inf_{\alpha \in \mathbb{R}} \sum_{y \in B(x,n)} |f(y) - \alpha| m(y) \\ &\leq Cn \sum_{y \in B(x,C_3'n)} \delta f(y) m(y)\end{aligned}$$

and the Poincaré inequality on P is proved. □

Combining Lemma 2.10 and Lemma 2.11, we obtain the following theorem immediately [CS].

Theorem 2.4 *Suppose X and Y are two uniformly roughly isometric complete Riemannian manifolds satisfying the local volume doubling property and local Poincaré inequality, then X satisfies the Poincaré inequality at infinity if so does Y.*

Chapter 3

Parabolic Harnack Inequality

3.1 Parabolic Harnack Inequality

In this section we are going to prove that the parabolic Harnack inequality is invariant under uniform rough isometries.

Definition 3.1 *We say that M satisfies the parabolic Harnack inequality at distance less than R* (**PH(R)**) *if there exists $C > 0$ such that, for all $x \in M$, $s \in \mathbb{R}$, and all $r \in (0, R)$, any positive solution u of $(\Delta + \partial_t)u = 0$ in $Q = (s, s + r^2) \times B(x, r)$ satisfies*

$$\sup_{Q_-} u \leq C \inf_{Q_+} u$$

where

$$Q_- = (s + r^2/6, s + r^2/3) \times B(x, r/2)$$

and

$$Q_+ = (s + 2r^2/3, s + r^2) \times B(x, r/2).$$

We say that M satisfies the parabolic Harnack inequality if $(\mathbf{PH}(\infty))$ *is satisfied.*

A remarkable result tells that $(\mathbf{PH(R)})$ is closely related to the following two properties.

Definition 3.2

1. *We say that M satisfies the volume doubling property at distant less then R* $(\mathbf{D(R)})$ *if there exists $C > 0$ such that for all $r \in (0, R)$, $x \in M$,*

$$V(x, 2r) \leq CV(x, r).$$

2. *We say that M satisfies the Poincaré inequality at distant less then R* $(\mathbf{P(R)})$ *if there exists $C > 0$ such that for all $r \in (0, R)$, $x \in M$, $\psi \in C_0^\infty(M)$,*

$$\int_{B(x,r)} |\psi - \psi_r|^2 \leq Cr^2 \int_{B(x,2r)} |\nabla\psi|^2.$$

Note that if M satisfies $(\mathbf{D(R)})$ and $(\mathbf{P(R)})$ for some $R > 0$, then $(\mathbf{P}(\infty))$ is equivalent to the Poincaré inequality at infinity with $\sigma = 2$ defined in the last section.

In [J], D. Jerison shows that $(\mathbf{D(R)})$ and $(\mathbf{P(R)})$ imply the stronger Poincaré inequality

$$\int_{B(x,r)} |\psi - \psi_r|^2 \leq Cr^2 \int_{B(x,r)} |\nabla\psi|^2, \ \forall x \in M, \ 0 < r < R.$$

The main result of this section is the following [G][S3].

Theorem 3.1 *The following properties are equivalent.*

1. *M satisfies* $(\mathbf{PH(R)})$.

2. M satisfies **(D(R))** *and* **(P(R))**.

If the above theorem is proved, then Theorem 2.4 in sections 2.3 will implies that **(PH(∞))** is invariant under uniform rough isometries. The following results is the key that allows the use of Moser's iterative method [M2] when **(D(R))** and **(P(R))** are satisfied.

Lemma 3.1 *Suppose M satisfies* **(D(R))**, *then for any* $x \in M$, $0 < s \leq r \leq R$,

$$V(x,r) \leq 2V(x,s)(r/s)^{\nu_0}$$

where ν_0 *depending only on the constant in* **(D(R))**.

Proof:
Take integer n such that $2^{n-1} < r/s \leq 2^n$, then

$$V(x,r) \leq V(x,2^n s) \leq C^n V(x,s) \leq 2V(x,s)(r/s)^{\nu_0}$$

where $\nu_0 = \log C/\log 2$, C is the constant in **(D(R))**. □

Lemma 3.2 *Suppose M satisfies* **(D(R))**. *Then there exists* $C > 0$ *such that for any* $y \in M$, $0 < s \leq r \leq R$,

$$\|f_s\|_2 \leq CV^{-1/2}(r/s)^{\nu_0/2}\|f\|_1, \ \forall f \in C_0^\infty(B)$$

where

$$f_s(x) = \frac{1}{V(x,s)} \int_{B(x,s)} f(z)dz,$$

$B = B(y,r)$, $V(y,r)$ *and* ν_0 *be the constant in Lemma 3.1.*

Proof:
Consider if $d(x, z) < s$, then $V(z, s) \leq V(x, 2s) \leq CV(x, s)$. Therefore

$$\begin{aligned}
\|f_s\|_1 &= \int_M |f_s(x)|dx \\
&\leq \int_M \frac{1}{V(x,s)} \int_M \chi_{B(x,s)}(z)|f(z)|dzdx \\
&= \int_M \frac{1}{V(x,s)} \int_M \chi_{B(z,s)}(x)|f(z)|dzdx \\
&\leq C\int_M \frac{1}{V(z,s)} \int_M \chi_{B(z,s)}(x)|f(z)|dzdx \\
&= C\int_M |f(z)|dz.
\end{aligned}$$

Suppose $B \cap B(x, s) \neq \emptyset$, then from Lemma 3.1

$$\begin{aligned}
V(x,s)^{-1} &\leq CV(x, 2r+s)^{-1}(2r/s+1)^{\nu_0} \\
&\leq CV^{-1}(2r/s+1)^{\nu_0}
\end{aligned}$$

and

$$\begin{aligned}
\|f_s\|_\infty &\leq \|\frac{1}{V(x,s)}\int \chi_{B(x,s)}(z)|f(z)|dz\|_\infty \\
&\leq CV^{-1}(2r/s+1)^{\nu_0}\|f\|_1.
\end{aligned}$$

Thus

$$\begin{aligned}
\|f_s\|_2^2 &\leq \|f_s\|_1\|f_s\|_\infty \\
&\leq CV^{-1}(2r/s+1)^{\nu_0}\|f\|_1^2
\end{aligned}$$

and this complete the proof of the lemma. □

Lemma 3.3 *Suppose M satisfies* **(D(R))** *and* **(P(R))**. *Then there exists constant C such that for any $0 < s < R/4$, $f \in C_0^\infty(M)$, we have*

$$\|f - f_s\|_2 \leq Cs\|\nabla f\|_2.$$

Proof:
Fix $0 < s < R/4$. Let $\{B_j, j \in J\}$ be a collection of balls of radius $s/2$ such that $B_i \cap B_j = \emptyset$ if $i \neq j$ and $M = \cup_{i\in J} 2B_i$, where $tB = B(x, tr)$. Such a collection always exists. Now

$$\begin{aligned}
\|f - f_s\|_2^2 &\leq \sum_{i\in J} \int_{2B_i} |f(x) - f_s(x)|^2 \\
&\leq 2\sum_{i\in J} (\int_{2B_i} |f(x) - f_{4B_i}|^2 + |f_{4B_i} - f_s(x)|^2).
\end{aligned}$$

Using **(D(R))** and **(P(R))**, we have

$$\begin{aligned}
\int_{2B_i} |f(x) - f_{4B_i}|^2 &\leq \int_{4B_i} |f(x) - f_{4B_i}|^2 \\
&\leq Cs^2 \int_{8B_i} |\nabla f|^2
\end{aligned}$$

and

$$\begin{aligned}
\int_{2B_i} |f_{4B_i} - f_s(x)|^2 &\leq \int_{2B_i} \int \chi_{B(x,s)} V(x,s)^{-1} |f_{4B_i} - f(z)|^2 dz dx \\
&\leq CV_i^{-1} \int_{2B_i} \int_{4B_i} |f_{4B_i} - f(z)|^2 dz dx \\
&\leq Cs^2 \int_{8B_i} |\nabla f|^2.
\end{aligned}$$

Hence, with the help of Lemma 1.4, we obtain

$$\begin{aligned}
\|f - f_s\|_2^2 &\leq Cs^2 \sum_{i\in J} \int_{8B_i} |\nabla f|^2 \\
&\leq Cs^2 \|\nabla f\|_2^2.
\end{aligned}$$

This ends the proof of Lemma 3.3. □

Lemma 3.4 *Given $\nu > 2$, the three following properties are equivalent.*

1. $\|e^{-t\Delta} f\|_\infty \leq C_1 t^{-\nu/2} \|f\|_1$, $\forall 0 < t < t_0$.

2. $\|f\|_{2\nu/(\nu-2)}^2 \leq C_2(\|\nabla f\|_2^2 + t_0^{-1}\|f\|_2^2)$.

3. $\|f\|_2^{2+4/\nu} \leq C_3(\|\nabla f\|_2^2 + t_0^{-1}\|f\|_2^2)\|f\|_1^{4/\nu}$.

Moreover, 3. implies 1. with $C_1 = (\nu C C_3)^{\nu/2}$ *and 1. implies 2. with* $C_2 = CC_1^{2/\nu}$, *where* C *is some numerical constant.*

The proof of 1. implies 2. follows from [V3]. The equivalence with 3. follows from [CKS].

Theorem 3.2 *[S2] Let* M *satisfies* **(D(R))** *and* **(P(R))**. *Then, there exists* $\nu > 2$ *and* $C > 0$ *such that, for all* $x \in M$, $0 < r < R$, *the Sobolev inequality* **(S(R))**

$$(\int |\psi|^{2\nu/(\nu-2)})^{(\nu-2)/\nu} \leq CV(x,r)^{-2/\nu} r^2 \int (|\nabla\psi|^2 + r^{-2}|\psi|^2)$$

is satisfied for any $\psi \in C_0^\infty(B(x,r))$ *where* $V = V(x,r)$.

Proof:
Fix $x \in M$, $0 < r < r_0$, ν_0 as in Lemma 3.2 and set $nu = \max\{3, \nu_0\}$. For any $f \in C_0^\infty(B(x,r))$.
If $0 < s \leq r/4$, then by Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} \|f\|_2 &\leq \|f - f_s\|_2 + \|f_2\|_2 \\ &\leq C(s\|\nabla f\|_2 + V^{-1/2}(r/s)^{\nu/2}\|f\|_1). \end{aligned}$$

If $s > r/4$, then

$$\|f\|_2 \leq Csr^{-1}\|f\|_2.$$

Therefore

$$\|f\|_2 \leq C(s(\|\nabla f\|_2 + r^{-1}\|f\|_2) + V^{-1/2}(r/s)^{\nu/2}\|f\|_1)$$

for any $s > 0$.
Optimizing over $s > 0$ yields

$$\|f\|_2^{2+4/\nu} \leq CV^{-2/\nu}r^2(\|\nabla f\|_2^2 + r^{-2}\|f\|_2^2)\|f\|_1^{4/\nu}$$

and the theorem follows from Lemma 3.4. □

We next employ the Moser's iterative technique to prove a mean value inequality.

Theorem 3.3 *Assume that* **(D(R))** *and* **(S(R))** *are satisfied. Let* $\delta \in (0,1)$. *Then, any non-negative function* u *such that* $(\partial_t + \Delta)u \leq 0$ *in* $Q = (s, s+r^2) \times B(x,r)$ *satisfies*

$$\sup_{Q_\delta} u^2 \leq C\delta^{-\gamma}(r^2V)^{-1}\|u\|_{2,Q}^2.$$

Here $\gamma > 0$ *depends only on the constant appearing in* **(D(R))**. *The constant* C *is independent of* u, δ, s *and of the ball* $B(x,r)$ *of radius* $0 < r < R$.

Proof:
For any non-negative function $\phi \in C_0^\infty(B(x,r))$, we have

$$\int_{B(x,r)} (\phi\partial_t u + \nabla\phi \cdot \nabla u) \leq 0. \tag{3.1}$$

For $\phi = \psi^2 y$ with $\psi \in C_0^\infty(B(x,r))$, some manipulation involving the inequality $|ab| \leq \frac{1}{2}(\varepsilon a^2 + b^2/\varepsilon)$ and inequality (3.1) yields

$$\int_{B(x,r)} (2\psi^2 u\partial_t u + |\nabla(\psi u)|^2) \leq A\|\nabla\psi\|_\infty^2 \int_{supp(\psi)} u^2$$

Where A is a numerical constant which will change from line to line. If χ is a smooth function of the time variable t, we easily get

$$\partial_t(\int_B (\chi\psi u)^2) + \chi^2 \int_B |\nabla(\psi u)|^2 \leq A\chi(\chi\|\nabla\psi\|_\infty^2 + \|\chi'\|_\infty) \int_{supp(\psi)} u^2$$

where $B = B(x,r)$. we choose ψ and χ such that

$$0 \leq \psi \leq 1,\ \mathrm{supp}(\psi) \subset (1-\sigma)B,$$
$$\psi = 1 \text{ in } (1-\sigma')B,\ |\nabla\psi| \leq (\tau r)^{-1},$$
$$0 \leq \chi \leq 1,\ \chi = 0 \text{ in } (-\infty, s+\sigma r^2),$$
$$\chi = 1 \text{ in } (s+\sigma' r^2, +\infty),\ |\chi'| \leq (\tau r^2)^{-1},$$

where $0 < \sigma < \sigma' < 1$ and $\tau = \sigma' - \sigma$. Setting $I_\sigma = (s+\sigma r^2, s+r^2)$ and integrating our inequality over (s,t) with $t \in I_{\sigma'}$, we obtain

$$\sup_{I'_\sigma}\{\int_{(1-\sigma')B} u^2\} + \int\int_{Q_{\sigma'}} |\nabla u|^2 \leq A(r\tau)^{-2} \int\int_{Q_\sigma} u^2. \tag{3.2}$$

Let $E(B) = CV^{-2/\nu}r^2$ be the Sobolev constant for the ball B given by **(S(R))** and recall the $q = \nu/(\nu-2)$ where $\nu > 2$ is the parameter appearing in **(S(R))**. Thanks to the Hölder inequality

$$\int w^{2(1+2/\nu)} \leq (\int w^{2q})^{1/q}(\int w^2)^{2/\nu},$$

(S(R)) gives

$$\int w^{2(1+2/\nu)} \leq (\int w^2)^{2/\nu} E(B) \int (|\nabla w|^2 + r^{-2}|w|^2),$$

for all $w \in C_0^\infty(B)$. Returning to the subsolution u, the above inequality and (3.2) yield

$$\int\int_{Q_{\sigma'}} u^{2\theta} \leq E(B)(A(r\tau)^{-2}\int\int_{Q_\sigma} u^{2p})^\theta \tag{3.3}$$

with $\theta = 1 + 2/\nu$. For all $p \geq 1$, u^p is also a non-negative subsolution of our equation. Therefore, (3.3) yields

$$\int\int_{Q_{\sigma'}} u^{2p\theta} \leq E(B)(A(r\tau)^{-2}\int\int_{Q_\sigma} u^{2p})^\theta \tag{3.4}$$

We now set $\tau_i = 2^{-1-i}$ so that $\sum_1^\infty \tau_i = 1/2$. We also set $\sigma_0 = 0, \sigma_{i+1} = \sigma_i + \tau_i = \sum_1^i \tau_j$. Applying (3.4) with $p = p_i = \theta^i$, $\sigma = \sigma_i$, $\sigma' = \sigma_{i+1}$, we get

$$\int\int_{Q_{\sigma_{i+1}}} u^{2\theta^{i+1}} \leq E(B)(A^{i+1}r^{-2}\int\int_{Q_{\sigma_i}} u^{2\theta^i})^\theta.$$

Hence,

$$(\int\int_{Q_{\sigma_{i+1}}} u^{2\theta^{i+1}})^{\theta^{-1-i}} \leq A^{\sum(j+1)\theta^{-1-j}} E(B)^{\sum\theta^{-1-j}} r^{-2\sum\theta^{-j}} \int\int_Q u^2$$

where all summation are taken from 1 to $i+1$. Letting i tend to infinity, we obtain

$$\sup_{Q_{1/2}} u^2 \leq AE(B)^{\nu/2} r^{-2-\nu} \|u\|_{2,Q}^2. \tag{3.5}$$

This ends the proof of Theorem 3.3 when $\delta = 1/2$. The full statement follows by using an easy covering argument and **(D(R))**. □

Corollary 3.1 *Assume that* **(D(R))** *and* **(P(R))** *are satisfied. Fix* $0 < p < +\infty$ *and* $\delta \in (0,1)$. *Then, any non-negative* u *such that* $(\partial_t + \Delta)u \leq 0$ *in* $Q = (s, s+r^2) \times B$ *satisfies*

$$\sup_{Q_\delta} u^p \leq C\delta^{-\gamma}(r^2V)^{-1}\|u\|_{p,Q}^p.$$

Here $\gamma > 0$ *depends only on the constant appearing in* **(D(R))**. *The constant* C *is independent of* u, δ, s *and of the ball of radius* $0 < r < R$.

Proof:
If $p \geq 2$ and $\delta \leq 1/2$, this statement follows from Theorem 3.3 and Jensen's inequality. The case $p \geq 2$ and $0 < \delta < 1/2$ is then obtained by a covering argument.
For $0 < p < 2$, the proof is more intricate. Fix $0 < \sigma < 1/2$ and set $\tau = \sigma/4$. Theorem 3.3 yields

$$\sup_{Q_\sigma} u \leq F(B)\tau^{-\gamma/2}\|u\|_{2,Q_{\sigma-\tau}}$$

where $F(B)^2 = C(r^2V)^{-1}$. Moreover, $\|u\|_2 \leq \|u\|_\infty^{1-p/2}\|u\|_p^{p/2}$. Hence, setting $J = F(B)\|u\|_{p,Q}^{p/2}$, we get

$$\sup_{Q_\sigma} \leq \tau^{\gamma/2} J(\sup_{Q_{\sigma-\tau}} u)^{1-p/2}.$$

We now fix $\delta \in (0,1/2)$ and set $\sigma_0 = \delta$, $\sigma_{i+1} = \sigma_i - \tau_{i+1}$ with $\tau_{i+1} = \sigma_i/4$ for all $i \geq 0$. This gives $\sigma_i = (3/4)^{-1-i}\delta$ and $\tau_{i+1} = 4^{-1}(3/4)^{-1-i}\delta$. we obtain

$$\sup_{Q_{\sigma_{i-1}}} \leq A^{\gamma^i}\delta^{-\gamma/2}J(\sup_{Q_{\sigma-\tau}} u)^{1-p/2}$$

and

$$\sup_{Q_\sigma} \leq A^{\gamma\sum(j+1)(1-p/2)^j}(\delta^{-\gamma/2}J)^{\sum(1-p/2)^j}(\sup_{Q_{\sigma-\tau}} u)^{(1-p/2)^i}$$

where all the summations run from 0 to $i-1$ and A is a numerical constant which may change from line to line. When i tends to infinity, this yields

$$\sup_{Q_\delta} \leq A^{\gamma/p^2}(\delta^{-\gamma/2}F(B))^{2/p}\|u\|_{p,Q}$$

which, raised to the power p, is the desired inequality.

Theorem 3.4 *Assume that* **(D(R))** *and* **(S(R))** *hold. For $0 < \delta < 1$ and $0 < p < +\infty$, any positive function u such that $(\partial_t + \Delta)u \geq 0$ in $Q = (s, s+r^2) \times B$ satisfies*

$$\sup_{Q_\sigma} u^{-p} \leq C\delta^{-\gamma}(Vr^2)^{-1}\|u^{-1}\|_{p,Q}^p. \tag{3.6}$$

Here, $\gamma > 0$ depends only on the constant appearing in **(D(R))**. *The constant C is independent of δ, p, u, s and of the ball B of radius $0 < r < R$.*

Proof:
For any negative $\phi \in C_0^\infty(B)$, we have

$$\int_B (\phi\partial_t u + \nabla\phi \cdot \nabla u) \geq 0. \tag{3.7}$$

Setting

$$\phi = -\alpha u^{\alpha-1}\psi^2, \ w = u^{\alpha/2}$$

with $-\infty < \alpha < 0$, we obtain

$$-\int(\psi^2\partial_t(w^2) + 4\alpha^{-1}(\alpha-1)|\nabla w|^2 + 4w\psi\nabla w \cdot \nabla\psi) \geq 0.$$

Note that $1 < \alpha^{-1}(\alpha - 1) < +\infty$. Using the elementary inequality $|xy| \leq \frac{1}{2}(x^2 + y^2)$, yields

$$\int \partial_t(w\psi)^2 + 2\int |\nabla w|^2 \leq A\|\nabla\psi\|_\infty^2 \int_{supp(\psi)} w^2$$

where A is numerical constant. The arguments used to prove (3.3) apply here as well, and they give

$$\int\int_{Q_{\sigma'}} u^{\alpha\theta} \leq E(B)(A(\tau r)^{-2}\int\int_{Q_\sigma} u^\alpha)^\theta \tag{3.8}$$

for $0 < \sigma < \sigma' < 1$ and $\tau = \sigma' - \sigma$. Here, $\theta = 1 + 2/\nu$, and $E(B) = CV^{-2/\nu}r^2$ is the Sobolev constant for the ball B given by **(S(R))**.
Now, an argument very similar to the one used to derive (3.5) from (3.2) leads fromm (3.8) to the desired inequality (3.6). □

In order to state the next result, we set

$$Q'_\delta = (s, s + (1-\delta)r^2) \times (1-\delta)B.$$

Theorem 3.5 *Assume that* **(D(R))** *and* **(S(R))** *are satisfied. Fix* $0 < p_0 < 1 + 2/\nu$ *where* ν *is the parameter appearing in* **(S(R))**. *For all* $0 < \delta < 1$ *and all* $0 < p \leq p_0$, *any positive function* u *such that* $(\partial_t + \Delta)u \geq 0$ *in* $Q = (s, s + r^2) \times B$ *satisfies*

$$\|u\|^p_{p_0, Q'_\delta} \leq (C\delta^{-(2+\nu)}(Vr^2)^{-1})^{1-p/p_0}\|u\|^p_{p,Q}. \tag{3.9}$$

Here, the constant C *is independent of* δ, p, u, s *and of the ball* B *of radius* $0 < r < R$ *but depends on* p_0.

Proof:
In (3.7), we set

$$\phi = \alpha u^{\alpha-1}\psi^2, \ w = u^{\alpha/2}$$

with $0 < \alpha < p_0(1 + 2/\nu)^{-1} < 1$. We get

$$\int (\psi^2\partial_t(w^2) + 4\alpha^{-1}(\alpha - 1)|\nabla w|^2 + 4w\psi\nabla w \cdot \nabla\psi) \geq 0.$$

Set $\varepsilon = 1 - p_0(1+2/\nu)^{-1}$. Note that $\alpha - 1$ is negative and that $\alpha^{-1}|\alpha - 1| \geq \varepsilon$. Using the inequality $|xy| \leq \frac{1}{2}(\varepsilon x^2 + \varepsilon^{-1}y^2)$ yields

$$-\int \partial_t(w\psi)^2 + 2\int |\nabla w|^2 \leq A\|\nabla\psi\|^2_\infty \int_{supp(\psi)} w^2 \tag{3.10}$$

where A is a constant which depend only on ε.
Again, we follow the argument used to prove (3.3), but this time we must take into account the minus sign in front of the first integral. This minus sign leads us to reverse the time and this explains why we are working with the set Q'_σ instead of Q_σ. From (3.10), we obtain

$$\int\int_{Q'_{\sigma'}} u^{\alpha\theta} \leq E(B)(A(\tau r)^{-2}\int\int_{Q'_\sigma} u^\alpha)^\theta \tag{3.11}$$

for $0 < \sigma < \sigma' < 1$ and $\tau = \sigma' - \sigma$, and where $\theta = 1 + 2/\nu$, $E(B) = SV^{-2/\nu}r^2$. Now, define $p_i = p_0\theta^{-i}$ and note that, thanks to the Hölder inequality, it is enough to prove (3.9) for $p = p_i$, $i = 0, 1, \cdots$. Thus, fix i and apply (3.11) with $\alpha = p_i\theta^j$, $j = 0, \cdots, i-1$, and $\sigma_0 = 0$, $\sigma_j = \sigma_{j-1} + \tau_j$, $\tau_j = \delta 2^{-j}$ where $0 < \delta < 1$ is a fixed parameter. Observe that $\alpha_j \leq p_0(1 + 2/\nu)^{-1}$ so that (3.11)can indeed be applied. It follows that

$$\int\int_{Q'_{\sigma_j}} u^{q_0\theta^j} \leq E(B)(A^j(\delta r)^{-2} \int\int_{Q'_{\sigma_{j-1}}} u^{q_0\theta^{j-1}})^\theta$$

for $j = 1, \cdots i$, and thus

$$\int\int_{Q'_{\sigma_i}} u^{p_0} \leq E(B)^{\sum \theta^j} A^{\sum(i-j)\theta^{j+1}} (\delta r)^{-2\sum \theta^{j+1}} (\int\int_Q u^{p_i})^{\theta^i}$$

where the summation run from 0 to $i-1$. Since $\sigma_i = \delta \sum_1^i 2^{-i} < \delta$, we finally get

$$(\int\int_{Q'_\delta} u^{p_0})^{p_i/p_0} \leq (CE(B)^{\nu/2}(\delta r)^{-(2+\nu)})^{1-p_i/p_0} \int\int_Q u^{p_i}.$$

Here we have used $\sum_0^{i-1} \theta^j = (\theta^i - 1)/(\theta - 1)$, $\sum_0^{i-1}(i-j)\theta^{j+1} \leq C_\theta(\theta^i - 1)$, for $i \geq 1$, and $\theta^i = p_0/p_i$, $\theta = 1 + 2/\nu$. Replacing $E(B)$ by its value in terms of r, V, gives the desired inequality (3.9). □

Lemma 3.5 *Fix $\delta, \tau \in (0, 1)$. Assume that* **(D(R))** *and* **(P(R))** *are satisfied. For any positive function u such that $(\partial_t + \Delta)u \geq 0$ in $Q + (s, s+r^2) \times B$, there is a constant $c = c(u, \tau)$ such that, for all $\lambda > 0$,*

$$\bar{\mu}(\{(t, z) \in K_+ : \log u < -\lambda - c\}) \leq Cr^2V\lambda^{-1}$$

and

$$\bar{\mu}(\{(t, z) \in K_- : \log u > \lambda - c\}) \leq Cr^2V\lambda^{-1}$$

where $\bar{\mu}$ is the product measure on $\mathbb{R} \times M$, $K_+ = (s + \tau r^2, s + r^2) \times (1 - \delta)B$ and $K_- = (s, s + \tau r^2) \times (1 - \delta)B$. Here the constant C is independent of $\lambda > 0, u, s$ and of the ball B of radius $0 < r < R$.

Proof:
First we note that we can assume that u is a supersolution in $(s, s+r^2) \times B'$

where B' is a concentric ball larger than $B = B(x, r)$, We set $w = -\log u$. Then, for any non-negative function $\psi \in C_0^\infty(B)$, we have

$$\partial_t \int \psi^2 w \leq \int \psi^2 u^{-1} \Delta u = \int (-\psi^2 |\nabla w|^2 + 2\psi \nabla w \cdot \nabla \psi).$$

Using again $|ab| \leq \frac{1}{2}(\frac{1}{2}a^2 + 2b^2)$, we get

$$\partial_t \int \psi^2 w + \frac{1}{2} \int |\nabla w|^2 \psi^2 \leq A \|\nabla \psi\|_\infty^2 \mu(supp(\psi)). \tag{3.12}$$

Here, we choose $\psi(z) = (1 - \rho(x, z)/r)_+$ where x is the center of B and r its radius. Apply the weighted Poincaré inequality with weight ψ^2 ([J], [SS]) to the function w, reads

$$\int |w - W|^2 \psi^2 \leq A r^2 \int |\nabla w|^2 \psi^2$$

with

$$W = (\int w\psi^2)/(\int \psi^2).$$

This and (3.12) give

$$\partial_t W + (A_1 r^2 V)^{-1} \int_{(1-\delta)B} |w - W|^2 \leq A_2 r^{-2}$$

for some constant $A_1, A_2 > 0$. We rewrite the last inequality as

$$\partial_t \bar{W} + (A_1 r^2 V)^{-1} \int_{(1-\delta)B} |\bar{w} - \bar{W}|^2 \leq 0 \tag{3.13}$$

where

$$\bar{w}(t, z) = w(t, z) - A_2 r^{-2}(t - s')$$

with $s' = s + \tau r^2$.
Now, we set $c(u) = \bar{W}(s')$, and

$$\Omega_t^+(\lambda) = \{z \in \delta B : \bar{w}(t, z) > c + \lambda\}$$

$$\Omega_t^-(\lambda) = \{z \in \delta B : \bar{w}(t, z) < c - \lambda\}.$$

Then, if $t > s'$,

$$\bar{w}(t, z) - \bar{W}(t) \geq \lambda + c - \bar{W}(t) > \lambda$$

in $\Omega_t^+(\lambda)$, because $c = \bar{W}(s')$ and $\delta_t \bar{W} \leq 0$. Using this in (3.13), we get

$$\partial_t \bar{W}(t) + (A_1 r^2 V)^{-1} |\lambda + c - \bar{W}(t)|^{-1} \geq \mu(\Omega_t^+(\lambda)).$$

Integrating from s' to $s + r^2$, we obtain

$$\bar{\mu}(\{(t,z) \in K_+ : \bar{w}(t,z) > c + \lambda\}) \leq A_1 r^2 V \lambda^{-1}$$

and, returning to $-\log u = w = \bar{w} + A_2 r^{-2}(t - s')$,

$$\bar{\mu}(\{(t,z) \in K_+ : \log u(t,z) + A_2 r^{-2}(t-s') < -\lambda - c\}) \leq A_1 r^2 V \lambda^{-1}.$$

Finally,

$$\begin{aligned} & \bar{\mu}(\{(t,z) \in K_+ : \log u(t,z) < -\lambda - c\}) \\ \leq\ & \bar{\mu}(\{(t,z) \in K_+ : \log u(t,z) + A_2 r^{-2}(t-s') < -(\lambda/2) - c\}) \\ & + \bar{\mu}(\{(t,z) \in K_+ : A_2 r^{-2}(t-s') > \lambda/2\}) \\ \leq\ & A_3 r^2 V \lambda^{-1}. \end{aligned}$$

This proved the first inequality in Lemma 3.5. Working with Ω_t^- instead of Ω_t^+, we obtain the second inequality by a similar argument. □

Consider a collection of measurable subsets U_σ, $0 < \sigma \leq 1$, of some fixed measure space endowed with a measure ν, such that $U_\sigma \subset U_{\sigma'}$ if $\sigma' \leq \sigma$. In our application, the space will be $\mathbb{R} \times M$ with measure $\bar{\mu}$ and U_σ will be Q_σ or Q'_σ.

Lemma 3.6 *Fix $0 < \delta \leq 1$. Let γ, C, $p_0 < p_1 \leq +\infty$ be positive constants. Let f be a positive measurable function on $U_0 = U$ which satisfies*

$$\|f\|_{p_0, U_{\sigma'}} \leq (C(\sigma' - \sigma)^{-\gamma} \nu(U))^{1/p - 1/p_0} \|f\|_{p, U_\sigma},$$

for all σ, σ', p such that $0 < \sigma < \sigma' \leq \delta \leq 1$ and $0 < p \leq p_1 < p_0$. Assume further that f satisfies

$$\nu(\log f > \lambda) \leq C\nu(U)$$

for all $\lambda > 0$. *Then,*

$$\|f\|_{p_0,U_\sigma} \leq A\nu(U)^{1/p_0}$$

where A *depends only on* γ, δ, C *and on a positive lower bound on* $1/p_1 - 1/p_0$.

Proof:
We can clearly assume that $\nu(U) = 1$. We set

$$\psi = \psi(\sigma) = \log(\|f\|_{p_0,U_\sigma}), \ for\ 0 < \sigma \leq \delta.$$

Decomposing U_σ into the sets where $\log f > \psi/2$, we get

$$\begin{aligned} \|f\|_{p,U_\sigma} &\leq \|f\|_{p_0,U_\sigma}\nu(\log f > \psi/2)^{1/p-1/p_0} + e^{\psi/2} \\ &\leq e^{\psi}(C/\psi)^{1/p-1/p_0} + e^{\psi/2}. \end{aligned} \tag{3.14}$$

Here, we have used successively the Hölder inequality and the second hypothesis of the lemma. We want to choose p so that the two terms in the right-hand side of (3.14) are equal, and $0 < p \leq p_1$. This is possible if

$$(1/p - 1/p_0)^{-1} = (2/\psi)\log(\psi/2C) \leq (1/p_1 - 1/p_0)^{-1},$$

and this last inequality is certainly satisfied when

$$\psi \geq A_1 C \tag{3.15}$$

where A_1 depends only on a positive lower bound on $1/p_1 - 1/p_0$. Assuming that (3.15) hols and that p has been chosen as above, we obtain

$$\|f\|_{p,U_\sigma} \leq 2e^{\psi/2}. \tag{3.16}$$

The first hypothesis of the lemma and (3.16) yield

$$\begin{aligned} \psi(\sigma') &\leq \log(2C(\sigma' - \sigma)^{-\gamma})^{1/p-1/p_0}e^{\psi/2}) \\ &= (1/p - 1/p_0)\log(2C(\sigma' - \sigma)^{-\gamma}) + \psi/2 \end{aligned}$$

for $0 < \sigma < \sigma' \leq \delta$. By the choice of p made above,

$$\psi(\sigma') \leq \frac{\psi}{2}(\frac{\log(2C(\sigma' - \sigma)^{-\gamma})}{\log(\psi/2C)} + 1).$$

On the other hand, if

$$\psi \geq 8C^3(\sigma' - \sigma)^{-2\gamma}, \tag{3.17}$$

we have

$$\psi(\sigma') \leq \frac{3}{4}\psi.$$

On the other hand, if one of the hypothesis (3.15), (3.17) made on ψ are not satisfied, we have

$$\psi(\sigma') \leq \psi \leq A_1 C + 8C^3(\sigma' - \sigma)^{-2\gamma}.$$

In all cases, we obtain

$$\psi(\sigma') \leq \frac{3}{4}\psi(\sigma) + A_2(\sigma' - \sigma)^{-2\gamma} \tag{3.18}$$

where A_2 depends only on C and on a positive lower bound on $1/p_1 - 1/p_0$. For any sequence

$$0 \leq \sigma_i < \sigma_{i-1} < \cdots < \sigma_0 = \delta,$$

iteration of (3.18) yields,

$$\psi(\sigma_0) \leq (3/4)^i \psi(\sigma_i) + A_2 \sum_0^\infty (3/4)^j (\sigma_{i+1} - \sigma_i)^{-2\gamma},$$

and, when i tends to infinity,

$$\psi(\delta) \leq A_2 \sum_0^\infty (3/4)^j (\sigma_{i+1} - \sigma_i)^{-2\gamma}.$$

The desired bound follows if we set $\sigma_i = \delta(1+j)^{-1}$. □

The result of Theorem 3.2, 3.4, 3.5 and Lemma 3.5, 3.6 yield a weak Harnack inequality for supersolution.

Theorem 3.6 *Assume that* **(D(R))** *and* **(P(R))** *are satisfied. By Theorem 3.2, there exists* $\nu > 2$ *such that* **(S(R))** *is satisfied. Fix* $p_0 \in (0, 1 + 2/\nu))$. *Fix* $0 < \varepsilon < \eta < \sigma < 1$ *and* $0 < \zeta < 1$ *and set*

$$Q_- = (s + \varepsilon r^2, s + \eta r^2) \times B(x, \zeta r) \text{ and } Q_+ = (s + \sigma r^2, s + r^2) \times B(x, \zeta r).$$

Then, any positive function u such that $(\partial_t+\Delta)u \geq 0$ in $Q=(s,s+r^2)\times B$, satisfies

$$\|u\|_{p_0,Q_-} \leq C(r^2V)^{1/p_0} \inf_{Q_+} u.$$

Here the constant C is independent of u,s and of the ball B of radius $0<r<R$.

Proof:
Fix u and let $c=c(u)$ be constant given by Lemma 3.5 where we have picked $\tau \in (\eta,\sigma)$, $\delta \in (0,\zeta)$. Lemma 3.5 and Theorem 3.5 (resp. Theorem 3.4) show that one can apply Lemma 3.6 to $e^c u$ (resp. $e^{-c}u^{-1}$). This yields

$$e^c\|u\|_{p_0,Q_-} \leq C(r^2V)^{1/p_0} \text{ (resp. } e^{-c}\sup_{Q_+} u^{-1} \leq C).$$

The statement of Theorem 3.6 follows. □

Combining Theorem 3.6 and Corollary 3.1, we get 1. ⇒ 2. of Theorem 3.1. To complete the proof of Theorem 3.1, we have

Theorem 3.7 **(PH(R))** *implies* **(D(R))**.

Proof:
Assume that **(PH(R))** holds on M, $\partial_t+\Delta$ admits a positive fundamental solution $(t,x,y)\to h_t(x,y)$. The function h_t can be interpreted as the kernel of the heat diffusion semigroup $H_t=e^{-t\Delta}$. Since Δ is self-adjoint, h_t is a symmetric kernel. Applying **(PH(R))** to h_{r^2} with $0<r<R$, we obtain

$$V(x,r)h_{r^2}(x,x) \leq C\int_{B(x,r)} h_{2r^2}(x,y)dy \leq C.$$

This gives

$$h_{r^2}(x,x) \leq CV(x,r)^{-1},\ \forall x\in M,\ 0<r<R.$$

Fix $x\in M$, $0<r<R$, set $B=B(x,r)$ and consider the function u defined by

$$\begin{cases} u(s,z) &= h_s\chi_B(z) & \text{if } s>0, \\ u(s,z) &= 1 & \text{if } s\geq 0. \end{cases}$$

The function u is a non-negative solution of $\partial_t + \Delta$ in $(-\infty, +\infty) \times B(x,r)$ in a weak sense. Applying **(PH(R))** twice, we get

$$\begin{aligned} 1 &= u(-r^2/4, x) \\ &\leq Cu(r^2/2, x) \\ &= \int_B h_{r^2/2}(x,y)dy \\ &\leq C^2 V(x,r) h_{r^2}(x,x) \end{aligned}$$

and this proves that there exists c, C such that

$$cV(x,r)^{-1} \leq h_{r^2}(x,x) \leq CV(x,r)^{-1}$$

for all $x \in M$ and $0 < r < R$. Clearly, this and **(PH(R))** imply **(D(R))**. □

Theorem 3.8 **(PH(R))** *implies* **(P(R))**.

Proof:
We follow an argument of Kusuoka and Stroock [KD]. Fix $x \in M$, $0 < r < R$ and set $B = B(x,r)$, $V = V(x,r)$. Denote by $H_{B,t}$ the heat kernel associated with Δ having Neumann boundary condition on the boundary of B and $h_{B,t}$ be the kernel of $H_{B,t}$, that is,

$$H_{B,t} f(y) = \int_B h_{B,t}(z,y) f(z) dz.$$

Assume that **(PH(R))** holds. By Theorem 3.7, **(D(R))** is also satisfied. Reasoning as in Theorem 3.7, we see that **(PH(R))** yields

$$h_{B,r^2}(z,y) \geq cV^{-1}$$

for all $z, y \in \frac{1}{2}B$. Thus, for $y \in \frac{1}{2}B$,

$$\begin{aligned} H_{B,r^2}(f - H_{B,r^2} f(y))^2(y) &\geq \frac{c}{V} \int_{\frac{1}{2}B} |f(z) - H_{B,r^2} f(y)|^2 dz \\ &\geq \frac{c}{V} \int_{\frac{1}{2}B} |f - f_{\frac{1}{2}B}|^2 \end{aligned}$$

and

$$\int_B H_{B,r^2}(f - H_{B,r^2} f(y))^2(y) dy \geq c' \int_{\frac{1}{2}B} |f - f_{\frac{1}{2}B}|^2.$$

Since

$$\begin{aligned}
\int_B H_{B,r^2}(f - H_{B,r^2}f(y))^2(y)dy &= \|f\|_{2,B}^2 - \|H_{B,r^2}f\|_{2,B}^2 \\
&= -\int_0^{r^2} \partial_s \|H_{B,s}f\|_2^2 ds \\
&\leq 2r^2 \int_B |\nabla f|^2,
\end{aligned}$$

the desired Poincaré inequality is proved. □

The following theorem is an immediate consequence of Theorem 3.1.

Theorem 3.9 *Suppose X and Y are two uniformly roughly isometric manifolds satisfying the volume doubling property and Poincaré inequality at distance less than R for all $R > 0$, then X satisfies the parabolic Harnack inequality if and only if so does Y.*

Since the parabolic Harnack inequality holds on every manifolds with non-negative Ricci curvature, we have the following theorem [CS].

Theorem 3.10 *Suppose M is a complete Riemannian manifold with Ricci curvature bounded below. If M is uniformly roughly isometric to a manifold with non-negative Ricci curvature, then M satisfies the parabolic Harnack inequality. In particular, every positive harmonic function on M is constant.*

Chapter 4

Parabolicity and Liouville D_p-property

4.1 Parabolicity

A complete manifold M is said to be *parabolic* if all positive superharmonic functions on M are constant. Also there is an equivalent definition. Let $p_t(x,y)(t>0, x, y \in M)$ be the minimal positive fundamental solution of the heat equation $(\partial/\partial t - \Delta)u = 0$ for function u on $(0,\infty) \times M$. Then M is non-parabolic if and only if the Green function $g(x,y) = \int_0^\infty p_t(x,y)dt$ exists. As is well known, the Euclidean n-space is non-parabolic if and only if $n \geq 3$. the following proposition was first pointed out by Ahlfors for dimension 2, and later by Varopoulos [V2] for all dimensions.

Proposition 4.1 *Suppose M is a non-parabolic complete Riemannian manifold, then there exists $p \in M$, such that, the volume $V(p,t)$ of geodesic ball centered at p of radius t satisfies the growth condition*

$$\int_1^\infty \frac{t}{V(p,t)} dt < \infty.$$

Observe that this volume growth condition holds at one point if and only if it holds at all points of M. The obvious question is to determine if this condition is also sufficient. Unfortunately, an example of Greene [V2] indicated that this is not true in general. But this condition is also sufficient for non-parabolicity in manifolds with some curvature assumption [V1].

Proposition 4.2 *Suppose M is a complete Riemannian manifold with non-negative Ricci curvature, $p \in M$, then M is non-parabolic if and only if*

$$\int_1^\infty \frac{t}{V(p,t)} dt < \infty.$$

To prove that parabolicity is preserved under uniform rough isometry for manifolds satisfying certain kind of conditions, we employ a criterion of parabolicity.

Definition 4.1 *Let M be a complete Riemannian manifold , and Ω a non-empty bounded domain in M with smooth boundary. The capacity of Ω is defined by*

$$cap(\Omega) = \inf\{\int_M |\nabla u|^2 dx : u \in C_0^\infty(M), u|_\Omega = 1\}.$$

Then we get [FS]

Proposition 4.3 *M is non-parabolic if and only if $cap(\Omega) > 0$.*

Proof:
First we prove the "only if" part. Suppose that M is non-parabolic. Fix a point p in Ω and put $v(x) = \log g(p, x)$, where g denotes the Green function. Since $g(p, \cdot)$ is harmonic except at p, we have $\Delta v = -|\nabla v|^2$ on $M \setminus \Omega$. Thus for an arbitrary $u \in C_0^\infty(M)$ with $u = 1$ on Ω, we get, by Green's formula,

that

$$\begin{aligned}
\int_{M\setminus\Omega} u^2|\nabla v|^2 dx &= -\int_{M\setminus\Omega} u^2 \Delta v dx \\
&= -\int_{\partial\Omega} \frac{\partial v}{\partial \nu} dx + 2\int_{M\setminus\Omega} u < \nabla u, \nabla v > dx \\
&\leq -\int_{\partial\Omega} \frac{\partial v}{\partial \nu} dx + 2\int_{M\setminus\Omega} |u||\nabla u||\nabla v| dx \\
&\leq -\int_{\partial\Omega} \frac{\partial v}{\partial \nu} dx + \int_{M\setminus\Omega} |\nabla u|^2 dx + \int_{M\setminus\Omega} u^2 |\nabla v|^2 dx
\end{aligned}$$

that is,

$$\int_M |\nabla u|^2 dx \geq \int_{\partial\Omega} \frac{\partial v}{\partial \nu} dx,$$

where $\partial/\partial\nu$ denotes the "inward" normal derivative on the boundary of Ω. Since for small $\varepsilon > 0$

$$\int_{B(p,\varepsilon)} \frac{\partial g(p,x)}{\partial \nu} dx = 1,$$

where $\frac{\partial}{\partial \nu}$ is the inward normal derivative on $\partial B(p, \varepsilon)$, therefore

$$\int_{B(p,\varepsilon)} \frac{\partial v}{\partial \nu} dx = \int_{B(p,\varepsilon)} \frac{1}{g(p,x)} \frac{\partial g(p,x)}{\partial \nu} dx$$

tends to 0 as ε goes to 0.
This shows that

$$cap(\Omega) \geq \int_{\partial\Omega} \frac{\partial v}{\partial \nu} dx = -\int_{\Omega} \Delta v dx = \int_{\Omega} |\nabla v|^2 dx > 0.$$

Next we show the "if" part. Assume $cap(\Omega) > 0$, take an increasing sequence of bounded domains Ω_k in M with smooth boundaries so that they cover M and each of them contains $\bar{\Omega}$. Then for each k there is a function $u_k \in C^\infty(\bar{\Omega}_k \setminus \Omega)$ which is harmonic on $\Omega_k \setminus \bar{\Omega}$ and satisfies the Dirichlet condition $u_k = 1$ on $\partial\Omega$ and $u_k = 0$ on $\partial\Omega_k$. Note that

$$cap(\Omega) = \lim_{k\to\infty} \int_{\Omega_k\setminus\Omega} |\nabla u_k|^2 dx.$$

By the Harnack inequality and the Schauder estimate, we can find a subsequence $\{u_j\}$ of $\{u_k\}$ which converges, with respect to the $C^{2,\alpha}$-norm on any compact subset in $M \setminus \Omega$, to a positive function $u \in C^\infty(M \setminus \Omega)$ harmonic on $M \setminus \bar{\Omega}$ and with $u = 1$ on $\partial\Omega$ [GT]. Obviously the extension of u by $u = 1$ on Ω is a positive superharmonic function on M, and therefore, to prove the non-parabolicity of M, it is sufficient to show that u is not constant. By Green's formula, we get

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} dx = \lim_{j\to\infty} \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} dx = \lim_{j\to\infty} \int_{\Omega_j \setminus \Omega} |\nabla u_j|^2 dx = cap(\Omega) > 0,$$

and this implies that u is non-constant. □

To use the discrete approximation method, we have to define the parabolicity on a net.

Definition 4.2 *A function u on a net P with measure m is said to be superharmonic if $Lu \leq 0$ where L is a linear operator acting on functions u on P defined by*

$$Lu(p) = \frac{1}{\sum_{q\sim p}(m(q) + m(p))} \sum_{q\sim p}(u(q) - u(p))(m(q) + m(p)), \; p \in P,$$

A net P is said to be parabolic if every positive superharmonic function on P is constant.

If we put

$$\nu(p) = \sum_{q\sim p} \frac{m(q) + m(p)}{2}, \quad \text{and} \quad \pi(p,q) = \begin{cases} \nu(p)^{-1} & \text{if } q \in N_p \\ 0 & \text{otherwise.} \end{cases}$$

Then L can be written as

$$Lu(p) = \sum_{q\in N_p} \pi(p,q)u(q)\frac{m(q) + m(p)}{2} - u(p), \quad p \in P.$$

For each $k = 0, 1, \cdots$, define a function $\pi_k : P \times P \to \mathbb{R}$ inductively by

$$\pi_0(p,q) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}, \quad \pi_{k+1}(p,q) = \sum_{r\in P} \pi_k(p,r)\pi(r,q)\frac{m(r) + m(p)}{2}.$$

This corresponds to the heat kernel of a Riemannian manifold. Moreover, the *Green function* g of P is defined by

$$g(p,q) = \sum_{k=0}^{\infty} \pi_k(p,q),$$

if it exists. Since we have been assuming that the net P is connected, it is easy to see that $g(p,q) < \infty$ for all $p,q \in P$ if $g(p_0,q_0) < \infty$ for some $p_0, q_0 \in P$. Moreover if $g < \infty$ then for each fixed $q \in P$, we have

$$Lg_q(p) = \begin{cases} -1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}, \tag{4.1}$$

where $g_q(p) = g(p,q)$. We prove a discrete counterpart of Itô's theorem [I].

Lemma 4.1 *P is non-parabolic if and only if $g < \infty$.*

Proof:
The "if" part is trivial from (4.1). We prove the "only if" part. Let u be a non-constant positive superharmonic function on P, and put $f = -Lu \geq 0$. We may assume $f \not\equiv 0$. (In fact, in the case when $Lu = 0$, take a real number a between $\inf u$ and $\sup u$, and define a function u' on P by $u'(p) = u(p)$ if $u(p) \leq a$, and $u'(p) = a$ if $u(p) > a$. This u' is a non-constant positive superharmonic function on P with $Lu' \not\equiv 0$.) Then we get

$$\begin{aligned}
& \sum_{j=0}^{k} \sum_{q \in P} \pi_j(p,q) f(q) \\
= & -\sum_{j=0}^{k} \sum_{q \in P} \pi_j(p,q) \Big(\sum_{r \in P} \pi(q,r) u(r)(m(r) + m(q)) - u(q)\Big) \\
= & -\sum_{j=0}^{k} \sum_{q \in P} (\pi_{j+1}(p,q) - \pi_j(p,q)) u(q) \\
= & \ u(p) - \sum_{q \in P} \pi_{k+1}(p,q) u(q) \\
\leq & \ u(p),
\end{aligned}$$

and this show that

$$\sum_{q \in P} g(p,q) f(q) = \sum_{j=0}^{\infty} \sum_{q \in P} \pi_j(p,q) f(q)$$

is absolutely summable. Thus we conclude $g < \infty$. $\square$

For functions u and v on P, we define function $< \delta u, \delta v >$ on P by

$$< \delta u, \delta v > (p) = \sum_{q \sim p} (u(q) - u(p))(v(q) - v(p)).$$

Note that,

$$\delta u(p) = \sqrt{< \delta u, \delta u > (p)},$$

where δu is defined as in Section 1.2. Then we get the Green's formula on nets.

Lemma 4.2 *Let u and v be functions on P, and assume that at least one of them has finite support. Then the following identity holds:*

$$\sum_{p \in P} (\nu u L v(p) + < \delta u, \delta v > (p) m(p)) = 0.$$

Proof:

$$\begin{aligned}
& \sum_{p \in P} \nu u L v(p) \\
= \quad & \sum_{p \in P} u(p) \sum_{q \sim p} (v(q) - v(p))(m(q) + m(p)) \\
= \quad & \sum_{p \in P} \sum_{q \sim p} u(p)v(q)(m(q) + m(p)) - \sum_{p \in P} \sum_{q \sim p} u(p)v(p)(m(q) + m(p)) \\
= \quad & \sum_{p \in P} \sum_{q \sim p} (u(q)v(p) + u(p)v(q) - u(q)v(q) - u(p)v(p))m(p) \\
= \quad & - \sum_{p \in P} < \delta u, \delta v > (p) m(p)
\end{aligned}$$

where the finite support assumption has been used in the finiteness of each terms. $\square$

We are now in a position to give a discrete version of Proposition 4.3. For a finite subset S of P, the *capacity* of S is defined by

$$cap(S) = \inf\{\sum_{p \in P} \delta^2 u(p) m(p) : u \in c_0(P),\ u = 1 \ on \ S\}.$$

Lemma 4.3 *Let S be a non-empty finite subset of a net P. Then P is non-parabolic if and only if $cap(S) > 0$.*

Proof:

Take an increasing sequence of finite subsets S_k of P so that $S \subset S_k$, $P = \cup S_k$, and, for each k, let u_k be a function on P which minimizes the quantity $\sum_{p\in P}\delta^2 v(p)$ among all functions v on P with $v = 1$ on S and $v = 0$ on $P \setminus S_k$. Obviously $0 \leq u_k \leq 1$, $u_k = 1$ on S, $u_k = 0$ on $P \setminus S_k$, and $cap(S) = \lim_{k\to\infty} \sum_{p\in P}\delta^2 u_k(p)$. Moreover we can see that $Lu_k = 0$ on $S_k \setminus S$ as follows. Let w be an arbitrary function on P such that its support lies in $S_k \setminus S$, and put $u_{k,t} = u_k + tw$, $t \in (-1, 1)$. Then $\sum_{p\in P}\delta^2 u_{k,t}(p)$ is minimized at $t = 0$, and hence, by Lemma 4.2, we get

$$0 = \frac{1}{2}\frac{d}{dt}|_{t=0} \sum_{p\in P} \delta^2 u_{k,t}(p)m(p) = \sum_{p\in P} < \delta u_k, \delta w > (p) = - \sum_{p\in S_k\setminus S} \nu w L u_k(p).$$

Since this must hold for any w, we have $Lu_k = 0$ on $S_k \setminus S$. Now we can find a subsequence $\{u_j\}$ of $\{u_k\}$ which converges pointwise to a function u on P. It is easy to see that u is positive superharmonic function such that $u = 1$ on S and $Lu = 0$ on $P \setminus S$. In addition, by Lemma 4.2, we have

$$\begin{aligned} -\sum_{p\in S}\nu(p)Lu(p) &= -\lim_{j\to\infty}\sum_{p\in S}\nu(p)Lu_j(p) \\ &= \lim_{j\to\infty}\sum_{p\in P}\delta^2 u_j(p)m(p) \\ &= cap(S). \end{aligned}$$

Now the "if" part of the lemma follows directly, since u is non-constant if $cap(S) > 0$.

We prove the "only if" part. Assume P is non-parabolic. Then, by Lemma 4.1, the Green function g exists. Note that it is sufficient to show that $cap(S) > 0$ only for S consisting only one element of P, say q. By the choice of u and the maximum principle, we get $g(p,q)/g(q,q) \geq u(p)$ for all $q \in P$. If u were identically equal to 1, then $g(p,q) \geq g(q,q)$ in contradiction to (4.1). Hence u is non-constant, and consequently $cap(\{q\}) > 0$. □

Corollary 4.1 *Suppose that P and Q are ponderable nets uniformly roughly isometric to each other. Then P is parabolic if so is Q.*

Proof:
Let $\varphi : P \to Q$ be a uniform rough isometry, and suppose that P is non-parabolic. Then, for a non-empty finite subset S of P, we have $cap(S) > 0$ by Lemma 4.3. We show that $cap(\varphi(S)) > 0$, which implies the non-parabolicity of Q. Let v be an arbitrary function on Q of finite support with $v = 1$ on $\varphi(S)$, and put $u = v \circ \varphi$. Obviously $u = 1$ on S, and hence, it suffices to show that

$$\|\delta u\|_2 \leq C\|\delta v\|_2$$

with some constant C independent of v. By the definition, there exists a constant C_1 such that for all $p, p' \in P$ with $d(p,p') = l$ there is a length-minimizing path $\gamma = (q_0, \cdots, q_l)$ in Q from $q_0 = \varphi(p)$ to $q_l = \varphi(p')$ of length $l \leq C_1$. From this we get

$$(u(p') - u(p))^2 \leq C_1((v(q_0) - v(q_1))^2 + \cdots + (v(q_{l-1}) - v(q_l))^2)$$

and hence, with the uniformness of P, we get

$$\delta^2 u(p) \leq C_2 \sum_{d(q,\varphi(p))<C_1} \delta^2 v(q).$$

Again, from the uniformness assumption on P and Q, we obtain a constant C as required. □

Lemma 4.4 *Suppose M satisfies the local volume doubling property and local Poincaré inequality and P be an ε-net in M. Then M is parabolic if and only if P is parabolic.*

Proof:
First we show that P is non-parabolic if so is M. Assume that M is non-parabolic, and take a non-empty bounded domain Ω in M with smooth boundary. Then by Proposition 4.3, Ω has a positive capacity. We will show that the finite subset $S = \{p \in P : B_{2\varepsilon}(p) \cap \Omega \neq \emptyset\}$ of P also has a positive capacity, which implies the non-parabolicity of P by Lemma 4.3.
Suppose that f is an arbitrary function on P of finite support with $f = 1$ on S, then $\hat{f}$ is a function on M with compact support and $\hat{f} = 1$ on Ω (Section 1.2). Therefore by Lemma 1.11 we have

$$cap(\Omega) \leq \|\nabla \hat{f}\|_2^2 \leq C\|\delta f\|_2^2$$

for some constant C independent of f. This proves $cap(S) > 0$.
Next we show the non-parabolicity of M under the assumption that P is non-parabolic. Fix a non-empty finite subset S of P. Then, by Lemma 4.3, $cap(S) > 0$. Also let Ω be a bounded domain in M with smooth boundary such that $B_\varepsilon(p) \subset \Omega$ for $p \in S$. For an arbitrary function $\psi \in C_0^\infty(M)$ with $\psi = 1$ on Ω, $\tilde{\psi}$ is a function on P with finite support and $\tilde{\psi} = 1$ on S. Therefore by Lemma 1.11 we have

$$cap(S) \leq \|\delta\tilde{\psi}\|_2^2 \leq C\|\nabla\psi\|_2^2$$

for some constant C independent of ψ. This shows that $cap(\Omega) > 0$, and consequently, implies the non-parabolicity of M as Proposition 4.3 suggests. □

Combining Corollary 4.1 and Lemma 4.4, we get the following theorem immediately [K2].

Theorem 4.1 *Suppose that X and Y are complete Riemannian manifolds satisfying the local volume doubling property and local Poincaré inequality and uniformly roughly isometric to each other. Then X is non-parabolic if so is Y.*

We have the following theorem as a corollary.

Theorem 4.2 *Suppose M is a complete Riemannian manifold satisfying local volume doubling property and local Poincaré inequality. If M is uniformly roughly isometric to a manifold with non-negative Ricci curvature, then M is non-parabolic if and only if*

$$\int_1^\infty \frac{t}{V(p,t)}dt < \infty.$$

Proof:
The "only if" part follows from Proposition 4.1.
Suppose M is uniformly roughly isometric to a manifold X with Ricci curvature non-negative. If

$$\int_1^\infty \frac{t}{V(p,t)}dt < \infty$$

holds on M, then this volume growth condition also holds on X by Theorem 2.2. By Proposition 4.2, X is non-parabolic. It then follows that M is non-parabolic by Theorem 4.1. □

4.2 Liouville D_p-property

Let G be an open subset of a Riemannian n-manifold M^n. A function $u \in C(G) \cap W^1_{p,loc}(G)$, with $1 < p < \infty$, is called *p-harmonic in G* if it is a weak solution of

$$- \operatorname{dvi}(|\nabla u|^{p-2}\nabla u) = 0, \tag{4.2}$$

that is,

$$\int_G < |\nabla u|^{p-2}\nabla u, \nabla\phi >= 0$$

for every $\phi \in C_0^\infty(G)$. Equation (4.2) is the Euler-Lagrange equation of the variational integral

$$\int_G |\nabla u|^p.$$

We say that a Riemannian n-manifold M^n has the *Liouville D_p-property* if every p-harmonic function u on M^n with

$$\int_{M^n} |\nabla u|^p < +\infty$$

is constant. In this section we study the invariance of the Liouville D_p-property under rough isometries between Riemannian manifolds. All results in this section can be generalized to so call $\mathcal{A}$-harmonic functions [H3] and all proofs are just the same.
First we are going to study some properties of p-harmonic functions. We have the following Caccioppoli-type inequality [H2].

Lemma 4.5 *Let u be a positive p-harmonic function in G, and let $v = u^{q/p}$ where $q \in \mathbb{R} \setminus \{0, p-1\}$. Then for every non-negative $\eta \in C_0^\infty(G)$,*

$$\int_G \eta^p |\nabla v|^p \leq |\frac{q}{q-p+1}|^p \int_G v^p |\nabla\eta|^p.$$

Proof:
Write $\kappa = q - p + 1$. Since $v = u^{q/p}$, we have

$$\nabla v = (q/p)u^{(q-p)/p}\nabla u,$$

and

$$|\nabla v|^p = |q/p|^p u^{q-p}|\nabla u|^p.$$

Let $\eta \in C_0^\infty(G)$ be non-negative and let $\varphi = u^\kappa \eta^p$. Then

$$\nabla\varphi = pu^\kappa\eta^{p-1}\nabla\eta + \kappa\eta^p u^{q-p}\nabla u,$$

Let $\eta \in W^1_{p,0}(G)$. Since u is p-harmonic in G and the support of φ is compact subset of G,

$$\int_G <|\nabla u|^{p-2}\nabla u, \nabla\varphi>= 0,$$

and so

$$0 = p\int_G <|\nabla u|^{p-2}\nabla u, u^\kappa\eta^{p-1}\nabla\eta> +\kappa\int_G |\nabla u|^p\eta^p u^{q-p}.$$

Therefore we obtain

$$\begin{aligned}
\int_G \eta^p|\nabla v|^p &= \int_G \eta^p|q/p|^p u^{q-p}|\nabla u|^p \\
&\leq \frac{p|q/p|^p}{|\kappa|}\int_G |\nabla u|^{p-1}\eta^{p-1}u^\kappa|\nabla\eta| \\
&= |\frac{q}{\kappa}|\int_G u^{q/p}|\nabla\eta|\eta^{p-1}u^{\kappa-q/p}|\nabla u|^{p-1}|q/p|^{p-1} \\
&\leq |\frac{q}{\kappa}|(\int_G u^q|\nabla\eta|^p)^{1/p}(\int_G \eta^p u^{q-p}|\nabla u|^p|q/p|^p)^{(p-1)/p} \\
&= |\frac{q}{\kappa}|(\int_G v^p|\nabla\eta|^p)^{1/p}(\int_G \eta^p|\nabla v|^p)^{(p-1)/p}
\end{aligned}$$

The lemma is proved. □

Lemma 4.6 *Let u be a weak positive supersolution of (4.2) in G. Then for any compact subset $K \subset G$ and $\varphi \in C_0^\infty(G)$ with $\varphi = 1$ on K, we have*

$$\int_K |\nabla \log u|^p \leq C\int_G |\nabla\varphi|^p$$

where C is independent of K and φ.

Proof:
We may assume that $\operatorname{ess\,inf} u > 0$. Pick a non-negative $\varphi \in C_0^\infty(G)$ with $\varphi = 1$ in K. Since the function $\eta = \varphi^p u^{1-p} \in W^1_{p,0}(G)$ is non-negative, then

$$\begin{aligned} 0 &\leq \int_G < |\nabla u|^{p-2}\nabla u, \nabla\eta > \\ &= \int_G < |\nabla u|^{p-2}\nabla u, p\varphi^{p-1}u^{1-p}\nabla\varphi - (p-1)u^{-p}\varphi^p\nabla u > . \end{aligned}$$

Now Hölder's inequality yields

$$\begin{aligned} \int_{supp(\varphi)} |\nabla u|^p u^{-p}\varphi^p &\leq C\int_{supp(\varphi)} |\nabla u|^{p-1}u^{1-p}|\nabla\varphi|\varphi^{p-1} \\ &\leq C(\int_{supp(\varphi)} |\nabla u|^p u^{-p}\varphi^p)^{(p-1)/p}(\int_G |\nabla\varphi|^p)^{1/p}. \end{aligned}$$

Since $\varphi = 1$ on K, we have

$$(\int_K (|\nabla u|u^{-1})^p)^{1/p} \leq C(\int_G |\nabla\varphi|^p)^{1/p}.$$

This complete the proof of the lemma. □

In order to get a local Harnack inequality for p-harmonic function which will be needed later, we impose the following property to a manifold.

Definition 4.3 *We say that a Riemannian manifold M has bounded geometry if the following two conditions hold on M*

1. *The Ricci curvature of M is uniformly bounded from below by*

$$-(n-1)K^2, \text{ with } K > 0,$$

2. *The injectivity radius of M, denoted by inj(M), is positive.*

The well known comparison theorem [BC] and [CGT] says that for Riemannian manifold satisfying condition 1 above, we have estimates

$$|B(x,r)| \leq V_K(r) \text{ and } \frac{|B(x,R)|}{|B(x,r)|} \leq \frac{V_K(R)}{V_K(r)}$$

for the volumes of geodesic balls for all $x \in M$ and $R \geq r > 0$. Here $V_K(r)$ is the volume of a geodesic ball of radius r in the simply connected complete Riemannian n-manifold of constant sectional curvature $-K^2$. Volumes of small geodesic balls in M have a lower bound

$$|B(x,r)| \geq v_0 r^n$$

for all $x \in M$ and for all $r \leq \text{inj}(M)/2$, where v_0 is a positive constant depending only on n. This estimate is proved by C.B. Croke [Cr]. It is also proved by Croke [Cr] (see also [CGT]) that for every domain $\Omega \subset B(x,r)$ with smooth boundary, $r \leq \text{inj}(M)/2$,

$$\text{vol}(\Omega)^{(n-1)/n} \leq C\,\text{area}(\partial\Omega),$$

where C depends only on n. Hence

$$\begin{aligned} |\Omega|^{(m-1)/m} &\leq C|\Omega|^{1/n-1/m}\text{area}(\partial\Omega) \\ &\leq C|B(x,r)|^{1/n-1/m}\text{area}(\partial\Omega) \end{aligned}$$

if $m \geq n$. It is well-known [C] that this isoperimetric inequality implies that

$$\Big(\int_{B(x,r)} |u|^{m/(m-1)}\Big)^{(m-1)/m} \leq C|B(x,r)|^{1/n-1/m} \int_{B(x,r)} |\nabla u|$$

for all $u \in C_0^\infty(B(x,r))$. By applying this and Hölder's inequality to functions $v = |u|^\gamma$, where $u \in C_0^\infty(B(x,r))$ and γ is suitable, and approximating, we obtain a Sobolev estimate.

Lemma 4.7 *Suppose that M is a complete Riemannian n-manifold, with inj$(M) > 0$, and that $q \leq p < m$, where $m \geq n$. Then there exists a constant $C = C(n,m,p)$ such that*

$$\Big(\int_{B(x,r)} |u|^{pm/(m-p)}\Big)^{(m-p)/m} \leq C|B(x,r)|^{p/n-p/m} \int_{B(x,r)} |\nabla u|^p$$

for every $u \in W^1_{p,0}(B(x,r))$ and $r \leq$ inj$(M)/2$.

We also have the following local Poincaré inequality for Riemannian manifolds with Ricci curvature bounded below [B] [C].

Lemma 4.8 *Suppose M is a complete Riemannian manifold with Ricci curvature bounded below by* $-(n-1)K^2$*, with* $K > 0$*, then there exists constant* c_n *depends only on n such that*

$$\int_{B(x,r)} |u - u_r| \leq r e^{c_n(1+Kr)} \int_{B(x,r)} |\nabla u|,$$

for every $u \in W_1^1(B(x,r))$.

We can now prove the local Harnack inequality for p-harmonic function [H3].

Theorem 4.3 *Suppose that M is a complete Riemannian n-manifold with bounded geometry. Then there exists, for each* $0 < r_0$*, a constant C such that*

$$\sup_{B(x,r/2)} u \leq C \inf_{B(x,r/2)} u,$$

for every positive p-harmonic function u in B(x,r) and $r \leq r_0$*, where C is independent of x and u.*

Proof:
Let $r \leq r_0$. suppose that u is a positive p-harmonic function in $B(x,r)$. Let $v = u^{q/p}$, where $q \in \mathbb{R} \setminus \{0, p-1\}$, let $m = \max\{n, p+1\}$, and write $\lambda = m/(m-p)$. The Sobolev estimate Lemma 4.7 and the Caccioppoli inequality Lemma 4.5 imply that

$$\begin{aligned}
(\int_{B(x,3r/4)} |\eta v|^{p\lambda})^{1/\lambda} &\leq C_1 |B(x,3r/4)|^{p/n-p/m} \int_{B(x,3r/4)} (\eta^p |\nabla v|^p + v^p |\nabla \eta|^p) \\
&\leq A((\frac{|q|}{|q-p+1|})^p + 1) \int_{B(x,3r/4)} v^p |\nabla \eta|^p \qquad (4.3)
\end{aligned}$$

for every non-negative $\eta \in C_0^\infty(B(x,3r/4))$, where

$$A = C_1 |B(x,3r/4)|^{p/n-p/m} \text{ and } C_1 = C_1(n,p).$$

Let $r/2 \leq t < t' \leq 3r/4$, and write $t_i = t + (t'-t)2^{-i}$ and $B_i = B(x,t_i)$ for every $i = 0, 1, \cdots$. Then $(t_i - t_{i+1})^{-p} = 2^{(i+1)p}(t'-t)^{-p}$, $B_0 = B(x,t')$, and $B(x,t) \subset B_i$ for every i. For each i, we choose a non-negative $\eta_i \in C_0^\infty(B(x,3r/4))$ such that $\eta_i = 1$ in B_{i+1}, $\eta_i = 0$ outside B_i, and $|\nabla\eta_i| \leq 2(t_i - t_{i+1})^{-1}$. Next we choose $q_0 \in \mathbb{R}\setminus\{0\}$ such that

$$|q_0\lambda^i - p + 1| \geq \frac{p(p-1)}{2m-p} \tag{4.4}$$

for every i. Applying (4.3) to η_i and to $q = q_0\lambda^i$ yields

$$(\int_{B_{i+1}} u^{q_0\lambda^{i+1}})^{1/\lambda} \leq A((\frac{|q_0\lambda^i|}{|q_0\lambda^i - p + 1|})^p + 1)\frac{2^{(i+1)p}}{(t'-t)^p}\int_{B_i} u^{q_0\lambda^i},$$

and so

$$(\int_{B_j}(u^{q_0})^{\lambda^i})^{1/\lambda^i} \leq A^{S_j}\prod_{i=0}^{j-1}(\frac{|q_0\lambda^i|^p}{|q_0\lambda^i - p + 1|^p} + 1)^{1/\lambda^i}\frac{2^{pS'_j}}{(t'-t)^{pS_j}}\int_{B_0} u^{q_0},$$

where $S_j = \sum_{i=0}^j \lambda^{-i}$ and $S'_j = \sum_{i=0}^j (I+1)\lambda^{-i}$. The condition (4.4) implies that the product above has an upper bound which depends only on n and p (note that $m = \max\{n, p+1\}$). Letting $j \to \infty$ we get $S_j \to m/p$ and

$$\sup_{B(x,t)} u^{q_0} \leq \frac{C_2 A^{m/p}|B(x,t')|}{(t'-t)^m} \fint_{B(x,t')} u^{q_0}, \tag{4.5}$$

where $C_2 = C_2(n,p)$ provided that (4.4) holds. The condition (4.4) holds for every $q_0 < 0$. Moreover, for every $q > 0$, there can be at most one i such that

$$|q\lambda^i - p + 1| < \frac{p(p-1)}{2m-p}.$$

Thus every interval $[q/\lambda, q]$ contains a number q_0 which satisfies (4.4) for all i. To get rid of (4.4), suppose that $q \neq 0$. If $q < 0$, we set $q_0 = q$, otherwise, we choose $q_0 \in [q/\lambda, q]$ such that (4.4) holds for every i. Next we choose $C_3 = \max\{C_2, (2C_1^{1/p} v_0^{m/n})^{-m}\}$. Then

$$\frac{C_3 A^{m/p}|B(x,t')|}{(t'-t)^m} \geq \frac{C_3 C_1^{m/p} v_0^{m/n}(r/2)^m}{(r/4)^m} \geq 1$$

since $\mathrm{inj}(M) > 0$. It follows from (4.5) that

$$\begin{aligned}\sup_{B(x,t)} u^q &= (\sup_{B(x,t)} u^{q_0})^{q/q_0} \\ &\leq (\frac{C_3 A^{m/p}|B(x,t')|}{(t'-t)^m})^{q/q_0} (\fint_{B(x,t')} u^{q_0})^{q/q_0} \\ &\leq \frac{C_3^\lambda A^{m\lambda/p}|B(x,t')|^\lambda}{(t'-t)^{m\lambda}} \fint_{B(x,t')} u^q. \end{aligned} \tag{4.6}$$

This holds for every $q \neq 0$ and $r/2 \leq t < t' \leq 3r/4$. Next we write $B(s) = B(x, r/2 + sr/4)$ for $0 \leq s \leq 1$. Since $A = C_1|B(x,3r/4)|^{p/n-p/m}$, we can write (4.6) as

$$\begin{aligned}\sup_{B(s)} u^q &\leq c(\frac{|B(x,3r/4)|}{r^n})^{m\lambda/n}(s'-s)^{-m\lambda} \fint_{B(s')} u^q \\ &\leq c(\frac{V_K(3r_0/4)}{r_0^n})^{m\lambda/n}(s'-s)^{-m\lambda} \fint_{B(s')} u^q.\end{aligned}$$

Here we used the volume comparison theorem to obtain first $|B(x,3r/4)| \leq V_K(3r/4)$ and then $V_K(3r/4)r^{-n} \leq V_K(3r_0/4)r_0^{-n}$. We have proved that

$$\sup_{B(s)} \leq (c(s'-s)^{m\lambda})^{-1/q} (\fint_{B(s')} u^q)^{1/q},$$

and

$$\inf_{B(s)} \geq (c(s'-s)^{m\lambda})^{1/q} (\fint_{B(s')} u^{-q})^{-1/q}$$

for all $q > 0$ and $0 \leq s < s' \leq 1$, where $c = c(n, p, K, r_0)$. By the refined version of the John-Nirenberg theorem [BG],

$$\sup_{B(x,r/2)} u \leq e^{c\,g(u)} \inf_{B(x,r/2)} u,$$

where

$$g(u) = \sup_{0\leq s\leq 1} \inf_{\alpha\in\mathbb{R}} \fint_{B(s)} |\log u - \alpha|$$

and $c = c(n, p, K, r_0)$. To estimate $g(u)$, we first use the local Poincaré inequality Lemma 4.8 and Hölder's inequality

$$
\begin{aligned}
g(u) &\leq \frac{1}{|B(x,r/2)|} \inf_{\alpha \in \mathbb{R}} \int_{B(x,3r/4)} |\log u - \alpha| \\
&\leq \frac{r e^{c_n(1+Kr)}}{|B(x,r/2)|} \int_{B(x,3r/4)} |\nabla \log u| \\
&\leq \frac{r e^{c_n(1+Kr)} |B(x,3r/4)|^{1-1/p}}{|B(x,r/2)|} \left(\int_{B(x,3r/4)} |\nabla \log u|^p\right)^{1/p}
\end{aligned}
$$

Furthermore, Lemma 4.6 implies that

$$
\int_{B(x,3r/4)} |\nabla \log u|^p \leq C \int_{B(x,r)} |\nabla \eta|^p \tag{4.7}
$$

for every $\eta \in C_0^\infty(B(x,r))$ such that $\eta = 1$ in $B(x, 3r/4)$. We obtain an upper bound $cr^{-p}|B(x,r)|$ for the right hand side of (4.7) by choosing η such that $|\nabla \eta| \leq 8/r$. Putting together these estimates yields

$$
\begin{aligned}
g(u) &\leq c e^{c_n(1+Kr)} \frac{|B(x,3r/4)|}{|B(x,r/2)|} \left(\frac{|B(x,r)|}{|B(x,3r/4)|}\right)^{1/p} \\
&\leq c e^{c_n(1+Kr_0)} \frac{V_K(3r/4)}{V_K(r/2)} \left(\frac{V_K(r)}{V_K(3r/4)}\right)^{1/p}
\end{aligned}
$$

Finally, we apply the volume comparison theorem to volume of n-balls in $\mathbb{R}^n$ to deduce first that $cr^n \leq V_K(r/2) (\leq V_K(3r/4))$, with $c = c(n)$, and then that

$$
\frac{V_K(3r/4)}{V_K(r/2)} \leq \frac{V_K(3r/4)}{cr^n} \leq \frac{V_K(3r_0/4)}{cr_0^n}.
$$

Similarly,

$$
\frac{V_K(r)}{V_K(3r/4)} \leq \frac{V_K(r_0)}{cr_0^n}.
$$

Hence $g(u)$ has an upper bound which depends only on n, p, K, and r_0. The theorem is proved. □

We have the following consequence of the local Harnack inequality.

Theorem 4.4 *Suppose that M is a complete Riemannian n-manifold with bounded geometry. Let*

$$r_0 = \min\{1, \frac{2}{3} inj(M)\}.$$

Then there exists a positive constant $c = c(n, p, K, r_0)$ *such that*

$$\rho(x, y) > c r_0 \max\{|\log \frac{u(x)}{u(y)}|, |\log \frac{1-u(x)}{1-u(y)}|\} - r_0,$$

whenever u is p-harmonic in M, with $\inf_M u = 0$ *and* $\sup_M u = 1$.

Proof:
Let x and y be two points in M. We may assume that $u(x) > u(y)$. Suppose first that $\rho(x, y) \geq r_0$. Let γ be a minimal geodesic from x to y, and let $l \geq 2$ be an integer such that $(l-1)r_0/2 < \rho(x, y) \leq l r_0/2$. Then there are points $x_0 = x, x_1, \cdots, x_l = y$ on γ such that $d(x_i, x_{i+1}) \leq r_0/2$ for all $i = 0, 1, \cdots, l-1$. Hence $B(x_i, r_0/2) \cap B(x_{i+1}, r_0/2) \neq \emptyset$ for all $i = 0, 1, \cdots, l-1$. The local Harnack inequality Theorem 4.3 implies that

$$\begin{aligned} u(x) \leq & \sup_{B(x_0, r_0/2)} u \leq C \inf_{B(x_0, r_0/2)} u \\ \leq & \ C \sup_{B(x_1, r_0/2)} u \leq C^2 \inf_{B(x_1, r_0/2)} u \leq \cdots \\ \leq & \ C^l \sup_{B(x_l, r_0/2)} u \leq C^{l+1} \inf_{B(x_l, r_0/2)} u \leq C^{l+1} u(y). \end{aligned}$$

Hence $l + 1 \geq (\log C)^{-1} \log(u(x)/u(y))$, and so

$$\rho(x, y) > c r_0 \log \frac{u(x)}{u(y)} - r_0,$$

with $c = (2 \log C)^{-1}$. If $\rho(x, y) < r_0$, there exists a point $z \in M$ such that $x, y \in B(z, r_0/2)$. Then $u(x) \leq C u(y)$ by Theorem 4.3, and so

$$c r_0 \log(u(x)/u(y)) - r_0 \leq -r_0/2.$$

The theorem follows by applying the same reasoning to the function $1 - u$.□

Manifolds which admit non-constant p-harmonic functions with bounded Dirichlet integral can be characterized by means of p-capacities. A *condenser* is a triple $(F_1, F_2; G)$, where F_1 and F_2 are disjoint, non-empty, and closed sets in $\bar{G}$.

Definition 4.4 *The p-capacity of a condenser $(F_1, F_2; G)$ is the number*

$$cap_p(F_1, F_2; G) = \inf_u \int_G |\nabla u|^p,$$

where the infimum is taken over all functions $u \in L^1_p(G)$ which are continuous in $G \cup F_1 \cup F_2$ with $u = 0$ in F_1 and $u = 1$ in F_2. Such a function is called admissible fo $(F_1, F_2; G)$. If the class of admissible functions is empty, we set $cap_p(F_1, F_2; G) = +\infty$.

Let $\{B_i\}_{i=1}^{\infty}$ be an exhaustion of M such that $B_i \subset\subset B_{i+1}$ for every i. We say that a set $A \subset M$ is unbounded if A has common points with $M \setminus B_i$ for every i.

Definition 4.5 *For an open set $\Omega \subset M$ and a compact set $F \subset \bar{\Omega}$, we define*

$$cap_p(F, \infty; \Omega) = \lim_{i \to \infty} cap_p(F, \bar{\Omega} \setminus B_i; \Omega).$$

Note that the limit exists and is independent of the exhaustion since the assumption $B_i \subset\subset B_{i+1}$ implies that

$$cap_p(F, \bar{\Omega} \setminus B_i; \Omega) \geq cap_p(F, \bar{\Omega} \setminus B_{i+1}; \Omega).$$

Definition 4.6 *An unbounded open set $\Omega \subset M$ is called p-hyperbolic if there exists a compact set $F \subset \bar{\Omega}$ such that $cap_p(F, \infty; \Omega) > 0$.*

We remark that any open set Ω' is p-hyperbolic if there exists a p-hyperbolic subst $\Omega \subset \Omega'$. We also observe that

$$cap_p(F, \bar{\Omega} \setminus D; \Omega) \geq cap_p(F, \infty; \Omega) > 0$$

for each open $D \subset\subset M$ if Ω is p-hyperbolic and F is as in the definition.

Definition 4.7 *An unbounded open set $\Omega \subset M$, with $\partial\Omega \neq \emptyset$, is called D_p-massive if there exists a p-harmonic function u in Ω which is continuous in $\bar{\Omega}$, with $u = 0$ in $\partial\Omega$, $\sup_\Omega u = 1$, and*

$$\int_\Omega |\nabla u|^p < +\infty.$$

It is clear from the definition that the sets $\{x : u(x) < a\}$ and $\{x : u(x) > b\}$, and even all components of these sets, are D_p-massive if u is a non-constant bounded p-harmonic function in M, with $|\nabla u| \in L^p(M)$, and $\inf u < a < b < \sup u$.
Next we explain the connection between D_p-massive and p-hyperbolic sets.

Lemma 4.9 *Every D_p-massive set is also p-hyperbolic.*

Proof:
Let Ω be D_p-massive, and let u be as in Definition 4.7. Suppose that $\{B_i\}_{i=1}^\infty$ is an exhaustion of M such that $B_i \subset\subset B_{i+1}$, and that $cap_p(F, \bar{\Omega} \setminus B_2; \Omega) > 0$, where $F = \bar{B}_1 \cap \partial\Omega \neq \emptyset$.
Next we choose admissible functions $w_i \in W_p^1(\Omega \cap B_i)$, $i \geq 2$, for condensers $(F, \bar{\Omega} \setminus B_i; \Omega)$ such that $0 \leq w_i \leq 1$,

$$\int_{\Omega \cap B_i} |\nabla w_i|^p \leq cap_p(F, \bar{\Omega} \setminus B_i; \Omega) + \frac{1}{i}, \tag{4.8}$$

and that $w_i = 1$ in all those components of $\Omega \cap B_i$ whose closures do not intersect F. We choose these functions in the following way. Suppose that w_2 is chosen. Let v_2 be the unique p-harmonic function in $\Omega \cap B_2$ such that $v_2 - w_2 \in W_{p,0}^1(\Omega \cap B_2)$. We set $v_2 = 1$ in $\Omega \setminus B_2$. Then

$$\int_{\Omega \cap B_2} |\nabla v_2|^p \leq \int_{\Omega \cap B_2} |\nabla w_2|^p$$

and $v_2 \geq u$ in Ω. Next we choose w_3. Then the set $A = \{x \in \Omega : w_3(x) > v_2(x)\}$ is a subset of $\Omega \cap B_2$. If $A \neq \emptyset$,

$$\int_A |\nabla v_2|^p \leq \int_A |\nabla w_3|^p,$$

since v_2 is p-harmonic in A. We redefine w_3 by setting $w_3 = v_2$ in A. Clearly (4.8) still holds. By continuing similarly, we get a decreasing sequence of functions $\{v_i\}$ such that v_i is p-harmonic in $\Omega \cap B_i$, $v_i \geq u_i$, and that

$$\int_{\Omega\cap B_i} |\nabla v_i|^p \leq \int_{\Omega\cap B_i} |\nabla w_i|^p.$$

To finish the proof, suppose that Ω is not p-hyperbolic. Then $cap_p(F, \bar{\Omega} \setminus B_i; \Omega) \to 0$, and so $\int_{\Omega\cap B_i} |\nabla v_i|^p \to 0$. Since $v_i \geq u$ and $\sup_\Omega u = 1$, the only possibility is that $v_i \to 1$. This is a contradiction since $\{v_i\}$ is decreasing. Hence Ω is p-hyperbolic. □

Note that the assumption $\int_\Omega |\nabla u|^p < +\infty$ was not needed in the proof. The converse of Lemma 4.9 is not true, that is, there are p-hyperbolic sets which are not D_p-massive. Indeed, let $p < n$ and let $\Omega \subset \mathbb{R}^n$ be the upper half space $\{x : x_n > 0\}$. By symmetry,

$$cap_p(\bar{B}^n(r) \cap \Omega, \infty; \Omega) = cap_p(\bar{B}^n(r), \infty; \mathbb{R}^n)/2.$$

It is well-known that $cap_p(\bar{B}^n(r), \infty; \mathbb{R}^n) = cr^{n-p} > 0$. Hence Ω is p-hyperbolic. On the other hand, Ω cannot be D_p-massive. Otherwise, the lower half space would be D_p-massive by symmetry. But this implies that $\mathbb{R}^n$ does not have the Liouville D_p-property which leads to a contradiction with [H1]. The exact relation between D_p-massive and p-hyperbolic sets is given by the following theorem.

Theorem 4.5 *An unbounded open set $\Omega \subset M$, with $\partial\Omega \neq \emptyset$, is D_p-massive if and only if there exists a p-hyperbolic $\Omega_1 \subset \Omega$ and a continuous function v in $\bar{\Omega}$ which is p-harmonic in $\Omega \setminus \bar{\Omega}_1$, with $v = 0$ in $\partial\Omega$, $v = 1$ in Ω_1, and $\int_\Omega |\nabla v|^p < +\infty$.*

Proof:
Suppose first that Ω is D_p-massive. Let u be as in Definition 4.7, and let $0 < \varepsilon < 1$. Then the set $\{x \in \Omega : u(x) > \varepsilon\}$ is D_p-massive, and hence p-hyperbolic. Furthermore, the function $v = \min\{u, \varepsilon\}/\varepsilon$ satisfies the assumption of the claim.

To prove the converse, let $\{B_i\}_{i=1}^{\infty}$ be an exhaustion of M, with $B_i \subset\subset B_{i+1}$. For $i \geq 2$, we write

$$\Omega_i = \Omega_1 \setminus \bar{B}_i, \ G_1 = \Omega \setminus \bar{\Omega}_1, \ G_i = \Omega \setminus \bar{\Omega}_i, \ G_i^k = G_i \cap B_k.$$

Let u_i^k be the unique p-harmonic function in G_i^k with boundary values $u_i^k - v \in W_{p,0}^1(G_i^k)$. We set $u_i^k = v$ in $\Omega \setminus G_i^k$. Now $0 \leq u_i^k \leq v$ and $u_{i+1}^k \leq u_i^k$ in Ω. Since the sequence $\{u_i^k\}_{k=1}^{\infty}$ is uniformly bounded, it is equicontinuous in G_i by the Hölder-continuity estimate [T]. By Ascoli's theorem, there exists a subsequence, still denoted by $\{u_i^k\}_{k=1}^{\infty}$, which converges locally uniformly in G_i to a function u_i. We set $u_i = v$ in $\Omega \setminus G_i$. Then u_i is p-harmonic in G_i and the sequence $\{u_i\}_{i=1}^{\infty}$ is decreasing. By Harnack's principle [HK], the limit function $u = \lim_{i\to\infty} u_i$ is p-harmonic in Ω. If we set $u = 0$ in $\partial\Omega$, then u is continuous in $\bar{\Omega}$ since $0 \leq u \leq v$ and $v \in C(\bar{\Omega})$, with $v = 0$ in $\partial\Omega$.
Next we shall show that u (multiplied by a suitable constant) satisfies the conditions in the definition of D_p-massiveness. First we observe that

$$\begin{aligned}
\int_\Omega |\nabla u_i^k|^p &= \int_{G_i^k} |\nabla u_i^k|^p + \int_{\Omega\setminus G_i^k} |\nabla v|^p \\
&\leq \int_{G_i^k} |\nabla v|^p + \int_{\Omega\setminus G_i^k} |\nabla v|^p \\
&= \int_\Omega |\nabla v|^p \\
&< +\infty.
\end{aligned}$$

Passing to a subsequence we conclude that there exists a vector field $X \in L^p(\Omega)$ such that $\nabla u_i^k \to X$ weakly in $L^p(\Omega)$ as $k \to \infty$. But the convergence of u_i^k implies that $X = \nabla u_i$. Now $u_i - v \in L_{p,0}^1(\Omega)$ since $u_i^k - v \in L_{p,0}^1(\Omega)$. This in turn implies that

$$\begin{aligned}
\int_\Omega |\nabla u_i|^p &= \int_{G_i} |\nabla u_i|^p + \int_{\Omega_i} |\nabla v|^p \\
&\leq \int_{G_i} |\nabla v|^p + \int_{\Omega_i} |\nabla v|^p \\
&= \int_\Omega |\nabla v|^p \\
&< +\infty.
\end{aligned}$$

By repeating the above reasoning, we get that $\int_\Omega |\nabla u|^p < +\infty$ and $u - v \in L_{p,0}^1(\Omega)$. It follows from Maz'ya's lemma [Ma], which obviously holds in our situation, that

$$|\nabla u_i|^{p-2}\nabla u_i \to |\nabla u|^{p-2}\nabla u$$

weakly in $L^{p/(p-1)}(\Omega)$. It remains to show that $u \not\equiv 0$. Since Ω_1 is p-hyperbolic, there exists a compact set $F \subset \bar{\Omega}_1$ such that $cap_p(F, \infty; \Omega_1) > 0$. Let $U \subset\subset M$ be a sufficiently large connected neighborhood of F so that $U \setminus \bar{\Omega}$ is non-empty. We write $\Omega_1' = \Omega_1 \cup U$ and $F_1 = \bar{U} \setminus \Omega$. Now $\bar{\Omega}_1'$ is also p-hyperbolic, and $cap_p(F_1, \infty; \Omega_1') > 0$ since F_1 and F lie in a same component of Ω_1'. For each i, u_i is admissible for the condenser $(\partial\Omega, \partial\Omega_i; G_i)$. Using this fact and well-known properties of capacities we get that

$$\begin{aligned}
\int_\Omega |\nabla u_i|^p &\geq cap_p(\partial\Omega, \partial\Omega_i; G_i) \\
&= cap_p(M \setminus \Omega, \bar{\Omega}_i; M) \\
&\geq cap_p(F_1, \bar{\Omega}_1' \setminus B_i; \Omega_1') \\
&\geq cap_p(F_1, \infty; \Omega_1') \\
&> 0
\end{aligned}$$

if i is large enough. Furthermore,

$$\begin{aligned}
\int_\Omega |\nabla u_i|^p &= \int_\Omega < |\nabla u_i|^{p-2}\nabla u_i, \nabla v > \\
&\to \int_\Omega < |\nabla u|^{p-2}\nabla u, \nabla v >,
\end{aligned}$$

and so ∇u cannot vanish identically in Ω. We conclude that u is non-constant. Multiplying u by a suitable constant, if necessary, we get a function which satisfies all the conditions in the definition of D_p-massiveness. The theorem is proved. □

An open set $G \subset\subset M$ is saided to be *regular* if, for all functions $h \in C(\bar{G} \cap W_p^1(G)$,

$$\lim_{x \to y} u(x) = h(y)$$

holds at every boundary point $y \in \partial G$ whenever u is the unique p-harmonic function in G with $u - h \in W_{p,0}^1(G)$. For example, all domains $\Omega \subset\subset M^n$ with C^1-boundaries are regular for all p.

Theorem 4.6 *A Riemannian manifold M admits a non-constant p-harmonic function u, with*

$$\int_M |\nabla u|^p < +\infty,$$

if and only if there exists two p-hyperbolic sets $\Omega_1, \Omega_2 \subset M$ *such that*

$$cap_p(\bar{\Omega}_1, \bar{\Omega}_2; M) < +\infty.$$

Proof:
If M does not have the Liouville D_p-property, there exists a non-constant bounded p-harmonic function u in M, with $\int_M |\nabla u|^p < +\infty$. Let $\inf u < a < b < \sup u$. Then the sets $\Omega_1 = \{x : u(x) < a\}$ and $\Omega_2 = \{x : u(x) > b\}$ are D_p-massive, hence p-hyperbolic. Moreover,

$$cap_p(\bar{\Omega}_1, \bar{\Omega}_2; M) \leq \frac{1}{(b-a)^p} \int_M |\nabla u|^p < +\infty,$$

since the function

$$v = \max\{0, \min\{\frac{u-a}{b-a}, 1\}\}$$

is admissible for the condenser $(\bar{\Omega}_1, \bar{\Omega}_2; M)$.
Suppose then that Ω_1 and Ω_2 are p-hyperbolic, with $cap_p(\bar{\Omega}_1, \bar{\Omega}_2; M) < +\infty$. Then there exists an admissible function w for the condenser $(\bar{\Omega}_1, \bar{\Omega}_2; M)$. By taking slightly larger open sets Ω_1' and Ω_2' with smooth boundaries and containing Ω_1 and Ω_2, respectively, such that $\Omega_1' \subset \{x : w(x) < 1/4\}$ and $\Omega_2' \subset \{x : w(x) > 3/4\}$, we obtain p-hyperbolic sets Ω_1' and Ω_2', with $cap_p(\bar{\Omega}_1', \bar{\Omega}_2'; M) < +\infty$. Now there exists a continuous function u in M which is p-harmonic in $M \setminus (\bar{\Omega}_1' \cup \bar{\Omega}_2')$ with $u = 0$ in $\bar{\Omega}_1'$, $u = 1$ in $\bar{\Omega}_2$, and $\int_M |\nabla u|^p < +\infty$. By Theorem 4.5, the sets $\{x : u(x) > b\}$ and $\{x : u(x) < a\}$ are disjoint D_p-massive sets for $0 < a < b < 1$. Call them G_1 and G_2. Let $\{B_i\}$ be an exhaustion of M such that B_i is regular for every i. Let u_j, $j = 1, 2$, be a p-harmonic function in G_j satisfying the conditions in Definition 4.7. We extend u_j to M by setting $u_j = 0$ in $M \setminus G_j$. Let $v_i \in C(\bar{B}_i)$ be p-harmonic in B_i such that $v_i = u_1$ in ∂B_i. Then

$$u_1 \leq v_i \leq 1 - u_2$$

in B_i. Furthermore,

$$\int_{B_i} |\nabla v_i|^p \le \int_{B_i} |\nabla u_1|^p \le \int_M |\nabla u_1|^p < +\infty.$$

Thus there exists a subsequence, denoted again by v_i, which converges locally uniformly in M to a p-harmonic v. Now $u_1 \le v \le 1 - u_2$ in M and $\int_M |\nabla v|^p < +\infty$. Since $\sup u_1 = \sup u_2 = 1$, v cannot be constant. The theorem is proved. □

The following lemma is a generalization of Lemma 1.11.

Lemma 4.10 *Suppose M satisfies the local volume doubling property and local Poincaé inequality, and that P is an ε-net in M. Then there exists constants C, C' such that for any $u \in C_0^\infty(M)$, $S' \subset P$,*

$$\|\delta \tilde{u}\|_{p,S'} \le C \|\nabla u\|_{p,S'_{C'\varepsilon}},$$

where $S'_{C'\varepsilon} = \{x \in M : \rho(x, S') \le C'\varepsilon\}$.
Similarly, for all $\Omega \subset M$ and function $f \in C_0(P)$,

$$\|\nabla \hat{f}\|_{p,\Omega} \le C \|\delta f\|_{p,\Omega_{C'\varepsilon} \cap P}.$$

We now define the p-hyperbolicity on nets.

Definition 4.8 *A subset S of a net P is p-hyperbolic, with $1 < p < \infty$, if there exists a finite non-empty set $E \subset S$ such that*

$$cap_p(E, \infty; S) = \inf_u \sum_{q \in S} |\nabla u(q)|^P m(q) > 0,$$

where the infimum is taken over all finitely supported functions u of $S \cup \partial S$, where $\partial S = \{q : \delta(q, S) = 1\}$, with $u = 1$ in E. Such functions are called admissible for $(E, \infty; S)$.

Lemma 4.11 *Suppose that $S' \subset P$ is connected subset, $\Omega = \{x \in M : d(x, S' \cup \partial S') < \varepsilon\}$, and that $S = \{q \in P : d(q, \Omega) < \varepsilon\}$. Then Ω is a domain and S is a connected subset.*

Proof:
Let x and y be any two points in Ω. Then there are points $q, q' \in S' \cup \partial S'$ such that $d(x, q) < \varepsilon$ and $d(y, q') < \varepsilon$. Since also $S' \subset \partial S'$ is connected, we can find a path in $S' \cup \partial S'$ from q to q'. Then the ε-neighborhood of this path is a connected subset of Ω which contains both x and y. This show that Ω is connected and therefore a domain since clearly Ω is open.
To show that S is connected, Let q and q' be any two points of S. Then there are points $x, y \in \Omega$ such that $d(x, q) < \varepsilon$ and $d(y, q') < \varepsilon$. Since Ω is a domain, there exists a rectifiable curve which connects x and y in Ω. It is easily see that the ε-neighborhood of this curve contains a path in P, and hence in S, from q to q'. Thus S is connected. □

Lemma 4.12 *Suppose M satisfies the local volume doubling property and local Poincaré inequality and C' be the constant in Lemma 4.10. Let S', $\Omega = S_{C'\varepsilon} = \{x \in M : \rho(x, S') < C'\varepsilon\}$. Then Ω is p-hyperbolic if S' is p-hyperbolic. Conversely, if a domain Ω in M is p-hyperbolic, then $S = \{p \in P : \rho(p, \Omega) < C'\varepsilon\}$ is p-hyperbolic.*

Proof:
Let $\{B_i\}$ be an exhaustion of M. Suppose first that S' is p-hyperbolic. Then there exists a finite non-empty set $E \subset S' \cup \partial S'$ such that $cap_p(E, \infty; S') > 0$. We set $F = \cup_{q \in E} \bar{B}(q, C'\varepsilon/2)$. Let $u \in C_0^\infty(M)$ such that $u = 1$ in F. Then $u = 0$ in $\bar{\Omega} \setminus B_i$ for some i. Observe that $1 - u$ is admissible for $(E, \bar{\Omega} \setminus B_i)$, $\tilde{u}$ is admissible for $(E, \infty; S')$, That is, $\tilde{u} = 1$ in E and it has finite support. By Lemma 4.10

$$\int_\Omega |\nabla u|^p \geq C \sum_{q \in S'} |\delta \tilde{u}(q)|^p m(q) \geq C cap_p(E, \infty; S').$$

Taking the infimum over all such functions u gives

$$cap_p(F, \infty; \Omega) \geq C cap_p(E, \infty; S') > 0,$$

and so Ω is p-hyperbolic.
For the proof of the second claim, we choose a compact set $F \subset \Omega$ such that $cap_p(F, \infty; \Omega) >)$. Let $E = \{q \in S \cup \partial S : \rho(q, F) < 2\varepsilon\}$. Then E is finite and non-empty. Let f be an admissible function for $(E, \infty; S)$. Since f has a finite support, $f = 0$ in $\Omega \setminus K$ for some compact set $K \subset M$. For each $x \in F$,

$$\hat{f}(x) = \sum_{q \in P_x} f(q)\theta_q(x),$$

where $P_x = \{p \in P : \rho(p, x) < 2\varepsilon\}$, since $P_x \subset E$ and $f(q) = 1$ in E. Hence $1 - \hat{f}$ is admissible for $(F, \bar{\Omega} \setminus B_i; \Omega)$ whenever $K \subset B_i$. By Lemma 4.10,

$$\sum_{q \in S} |\delta f(q)|^p m(q) \geq C \int_\Omega |\nabla \hat{f}|^p \geq C cap_p(F, \infty; \Omega) > 0.$$

Since this holds for all admissible function f we get

$$cap_p(E, \infty; S) \geq C cap_p(F, \infty; \Omega) > 0.$$

This ends the proof. □

The proof of the following lemma is similar to the proof of Corollary 4.1.

Lemma 4.13 *Let $\varphi : P_1 \to P_2$ be a uniform rough isometry between two ponderable nets, $1 \leq p \leq +\infty$. Then there exists constants C and C' such that if $S \subset P$ and $S' = \{q \in P_2 : d(q, S) < C'\}$, then S' is connected if S is connected. Furthermore, let u be a function of $S' \cup \partial S'$ and $v = u \circ \varphi$. Then*

$$\sum_{x \in S} |\delta v(x)|^p m_1(x) \leq C \sum_{q \in S'} |\delta u(q)|^p m_2(q).$$

Lemma 4.14 *Suppose X and Y be uniformly roughly isometric Riemannian manifolds satisfying the local volume doubling property and local Poincaé inequality. Let P and Q be ε-nets in X and Y, $\varphi : P \to Q$ be a uniform rough isometry. Then there exists constant C' such that if $\Omega \subset X$ is connected and p-hyperbolic and $S = \{q \in P : \rho(q, \Omega) < \varepsilon\}$, then the set $\Omega' = \{y \in Y : \rho(y, \varphi(S \cup \partial S)) < C'\}$, is a p-hyperbolic domain.*

Proof:
Clearly Ω' is open. To shown that it is connected, let x and y be any points of Ω'. Then there are points q and q' in $S \cup \partial S$ such that $x \in B(\varphi(q), C')$ and $y \in B(\varphi(q'), C')$. Both of these balls are contained in Ω'. Furthermore, since S is a connected subnet, so does $S \cup \partial S$. Thus there exists a path in $S \cup \partial S$, say $q_0 = q, q_1, \cdots, q_l = q'$, from q to q'. By definition of rough isometry, there exists constants a, b such that

$$\rho(\varphi(q_i), \varphi(q_{i+1})) \leq 2\varepsilon d(\varphi(q_i), \varphi(q_{i+1})) \leq 2\varepsilon(a+b),$$

and therefore $\cup_{i=0}^{l} B(\varphi(q_i), C')$ is a connected open subset of Ω' containing x and y if we choose $C' > 2\varepsilon(a+b)$. This implies that Ω' is a domian.
It remains to prove that Ω' is p-hyperbolic. First we observe that S is p-hyperbolic by Lemma 4.12. Thus there exists a finite set $E \subset S \cup \partial S$ such that $cap_p(E, \infty; S') > 0$. Let u be an admissible function in $S' \cup \partial S'$ for $(\varphi(E), \infty; S)$, that is, u has a finite support and $u = 1$ in $\varphi(E)$. For each $q \in S \cup \partial S$, we set $v(q) = u(\varphi(q))$. Then $v = 1$ in E. Since the support of v is finite, there is a point $\tilde{q} \in S$ and $\delta_0 > 0$ such that $v(q) = u(\varphi(q)) = 0$ if $d(\varphi(\tilde{q}), \varphi(q)) \geq \delta_0$. Since φ is a rough isometry, there exists $\delta_1 > 0$ such that, $d(\varphi(\tilde{q}), \varphi(q)) \geq \delta_0$, and so $v(q) = 0$, if $d(\tilde{q}, q) \geq \delta_1$. The uniformness of P implies that there can be only finitely many points $q \in P$ with $d(\tilde{q}, q) < \delta_1$. Hence the support of v is finite and v is admissible for $(E, \infty; S)$. Lemma 4.13 then implies that

$$\sum_{q \in S'} |\delta u(q)|^p m(q) \geq C \sum_{x \in S} |\delta v(x)|^p m(x) \geq C cap_p(E, \infty; S) > 0.$$

This is true for every admissible v. Hence $cap_p(\varphi(E), \infty; S') > 0$ and S' is p-hyperbolic. It follows from Lemma 4.12 that the C'-neighborhood of $S' \cup S'$ is p-hyperbolic. Hence, if we choose a larger C', Ω' is also p-hyperbolic as a larger set. □

We are now ready to prove the main theorem [H3].

Theorem 4.7 *Suppose X and Y are roughly isometric complete Riemannian manifolds with bounded geometry. Then X has the Liouville D_p-property if and only if so does Y.*

Proof:
First, note that X and Y satisfy the local volume doubling property, local Poincaré inequality and are unifomly roughly isometric by Lemma 1.1 and Lemma 1.2 in Section 1.1.
Fix $\varepsilon \leq \min\{\text{inj}(X)/2, \text{inj}(Y)/2\}$. Let P and Q be ε-nets in X and Y, respectively. Suppose that X does not have the Liouville D_p-property. By [H1], there exists a non-constant bounded p-harmonic function u in X with $\int_X |\nabla u|^p < +\infty$. We normalize u such that $\inf_X u = 0$ and $\sup_X u = 1$. It is sufficient to prove that Y also admits a non-constant p-harmonic function with L^p-integrable gradient. For each $a, b \in (0,1)$, we denote by Ω_a and Ω^b any component of sets $\{x \in X : u(x) < a\}$ and $\{x \in X : u(x) > b\}$, respectively. Then Ω_a andΩ^b are p-hyperbolic domains. Let $0 < s < 1/4$ and $3/4 < t < 1$. We write $S_s = \{q \in P : \rho(q, \Omega_s) < \varepsilon\}$ and $S^t = \{q \in P : \rho(q, \Omega^t) < \varepsilon\}$. Then the sets $D_s = \{x \in Y : \rho(x, \varphi(S_s \cup \partial S_s)) < C'\}$ and $D^t = \{x \in Y : \rho(x, \varphi(S^t \cup \partial S^t)) < C'\}$ are p-hyperbolic by Lemma 4.14. We claim that, for some $0 < s < 1/4$ and $3/4 < t < 1$, $cap_p(\bar{D}_s, \bar{D}^t; Y) < +\infty$ which then proves the theorem by Theorem 4.6. Let

$$v = \max\{0, \min\{2(u - 1/4), 1\}\}.$$

Now $v = 0$ in $\Omega_{1/4}$ and $v = 1$ in $\Omega^{3/4}$. Next we define function w on Q by $w = \tilde{v} \circ \psi$, where ψ is a rough inverse of φ. By Lemma 1.11 and 4.13, we have

$$\int_Y |\nabla \hat{w}|^p \leq C \int_X |\nabla v|^p \leq 2^p C \int_X |\nabla u|^p < +\infty.$$

It remains to show that $\hat{w}$ is admissible for $(\bar{D}_s, \bar{D}^t; Y)$ if s and t are properly chosen. Since ψ is a rough inverse of φ, there exists a constant c such that $\rho(x, \psi(\varphi(x))) \leq c$ for every $x \in P$. Let $q \in Q$ be such that $\rho(q, \bar{D}_s) \leq 2\varepsilon$. Then there is $y \in \bar{D}_s$, with $\rho(q, y) \leq 4\varepsilon$. Moreover, $\rho(y, \varphi(z)) \leq 2C'$ for some $z \in S_s \cup \partial S_s$, and so $\rho(q, \varphi(z)) \leq 2C' + 4\varepsilon$. Since ψ is a rough isometry, $\rho(\psi(q), \psi(\varphi(z))) \leq c'$ for some constant c'. Hence $\rho(\psi(q), z) \leq c + c'$. On the other hand, there is $z' \in S_s$ such that $\rho(z, z') \leq 2\varepsilon$. Finally, $\rho(z, x) \leq \varepsilon$ for some $x \in \Omega_s$. Hence, for every $y \in B(\psi(q), \varepsilon)$,

$$\rho(y, x) \leq c + c' + 4\varepsilon \equiv C, \tag{4.9}$$

where C is independent of q and x. Thus we can attach to each $q \in Q$, with $\rho(q, \bar{D}_s) \leq 2\varepsilon$, a point $x \in \Omega_s$ such that $\rho(y, x) \leq C$ whenever $y \in B(\psi(q), \varepsilon)$.

By Theorem 4.4, we can choose $0 < s < 1/4$ such that

$$\rho(\partial\Omega_s, \partial\Omega_{1/4}) \geq 2C. \tag{4.10}$$

It follows from (4.9) and (4.10) that $B(\psi(q), \varepsilon) \subset \Omega_{1/4}$ whenever $q \in Q$, with $\rho(q, \bar{D}_s) \leq 2\varepsilon$. But this implies that $w(q) = \tilde{v}(\psi(q)) = 0$ for such q, and so $\hat{w}(x) = 0$ for every $x \in \bar{D}_s$.
Similarly, we can choose $3/4 < t < 1$ such that

$$\rho(\partial\Omega^t, \partial\Omega^{3/4}) \geq 2C.$$

Then $B(\psi(q), \varepsilon) \subset \Omega^{3/4}$ if $q \in Q$ and $\rho(q, \bar{D}^t) \leq 2\varepsilon$. Hence $\hat{w}(x) = 1$ for every $x \in \bar{D}^t$. We have showed that w is admissible for $(\bar{D}_s, \bar{D}^t; Y)$ which then proves the theorem. □

Bibliography

[B] Buser P., *A note on the isoperimetric constant,* Ann. Sci. École Norm. Sup. 15, 1982, 213-230.

[BC] Bishop R. and Crittenden B., *Geometry of manifolds,* Academic Press, 1964.

[BG] Bombieri E. and Giusti E., *Harnack's inequality for elliptic differential equations on minimal surfaces,* Invent. Math. 15, 1972, 24-46.

[C] Chavel I., *Eigenvalues in Riemannian Geometry,* Academic Press, 1984.

[CGT] Cheeger J., Gromov, M. and Taylor, M., *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds,* J. Diff. Geom. 17, 1982, 15-53.

[CKS] Carlen E., Kusuoka S. and Stroock D., *Upper bounds for symmetric Markov transition functions,* Ann. Inst. H. Poincaré Non Linéaire 23, 1987, 245-287.

[Cr] Croke C.B., *Some isoperimetric inequalities and eigenvalue estimates,* Ann. Sci. École Norm. Sup. 13, 1980, 419-435.

[CS] Coulhon T. and Saloff-Coste L., *Variétés riemanniennes isométriques á l'infini,* Revista Mat. Iber. 11, 1995, 687-726.

[CY] Cheng S.Y. and Yau S.T., *Differential equations on Riemannian manifolds and their geometric applications,* Comm. Pure Appl. Math. 28, 1975, 333-352.

[FS] Fischer-Colbrie D. and Schoen R., *The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature,* Comm. Pure Appl. Math. 33, 1980, 199-211.

[G] Grigor'yan A., *The heat equation on non-compact Riemannian manifolds,* Math. USSR Sbornik 72, 1992, 47-77.

[Gr] Gromov M., *Hyperbolic manifolds, groups and actions,* Ann. of Math. Studies 97, Princeton Univ. Press, Princeton, 1981, 183-213.

[GT] Gilbarg D. and Trudinger N.S., *Elliptic Partial Differential Equation of Second Order,* 2nd edition, Springer, New York, 1983.

[H1] Holopainen I., *Nonlinear potential theory and quasiregular mappings on Riemannian manifolds,* Ann. Acad. Sci. Fenn. Ser. A I Math. Diss 74, 1990, 1-45.

[H2] Holopainen I., *Positive solutions of quasilinear elliptic equations on Riemannian manifolds,* Proc. London Math. Soc. 65, 1992, 651-672.

[H3] Holopainen I., *Rough isometries and p-harmonic functions with finite Dirichlet integral,* Rev. Mat. Iberoam. 10, 1994, 143-176.

[HK] Heinonen J. and Kilpeläinen T., *$\mathcal{A}$-superharmonic functions and supersolutions of degenerate elliptic equations,* Arkiv. Math. 26, 1988, 87-105.

[I] Itô S., *On existence of Green function and positive superharmonic function for linear elliptic operators of second order,* J. Math. Soc. Japan 16, 1964, 299-306.

[J] Jerison D., *The Poincaré inequality for vector fields satisfying the Hörmander's condition,* Duke Math. J. 53, 1986, 503-523.

[K1] Kanai M., *Rough isometries and combinatorial approximation of geometries of non-compact Riemannian manifolds.* J. Math. Soc. Japan 37, 1985, 391-413.

[K2] Kanai M., *Rough isometries and the parabolicity of Riemannian manifolds,* J. Math. Soc. Japan 38, 1986, 227-238.

[K3] Kanai M., *Analytic inequalities and rough isometries between non-compact Riemannian manifolds,* in Curvature and Topology of Riemannian Manifolds, Springer Lecture Notes 1201, 1986, 122-137.

[KD] Kusuoka S. and Stroock D., *Long time estimates for the heat kernel associated with uniformly subelliptic symmetric second order operator,* Ann. Math. 127, 1988, 165-189, 391-442.

[M1] Moser J., *On Harnack theorem for elliptic differential equations,* Comm. Pure Appl. Math. 14, 1961, 577-591.

[M2] Moser J., *A Harnack inequality for parabolic differential equations,* Comm. Pure Appl. Math. 16, 1964, 101-134. Correction in 20, 1967, 231-236.

[Ma] Maz'ya V.G., *In the continuity at a boundary point of solutions of quasi-linear elliptic equations,* Vestnik Leningrad Univ. 3, 1976, 225-242 (English translation).

[Mi] Milnor J., *A note on curvature and fundamental group,* J. Diff. Geom. 2, 1968, 1-7.

[Mo] Mostow G.D., *Strong rigidity of locally symmetric space,* Ann. of Math. Studies 78, Princeton Univ. Press, Princeton, 1973.

[PS] Phillips A. and Sullivan D., *Geometry of leaves,* Topology 20, 1981, 209-218.

[S1] Saloff-Coste L., *Uniformly elliptic operators on Riemannian manifolds,* J. Diff. Geom. 36, 1992, 417-450.

[S2] Saloff-Coste L., *A note on Poincaré, Sobolev, and Harnack inequalities,* Duke Math. J., I.M.R.N. 2, 1992, 27-38.

[S3] Saloff-Coste L., *Parabolic Harnack inequality for divergence form second order differential operators,* Potential Analysis 4, 1995, 429-467.

[SS] Saloff-Coste L. and Stroock D.W., *Opérateurs uniformément sous-elliptiques sur les groupes de Lie,* J. Funct. Anal. 98, 1991, 97-121.

[T] Trudinger N.S., *On Harnack type inequalities and their applications to quasilinear elliptic equations,* Comm. Pure Appl. Math. 20, 1967, 721-747.

[V1] Varopulos N., *The Poisson kernel on Positively Curved Manifolds,* J. Funct. Anal. 44, 1981, 359-380.

[V2] Varopulos N., *Potential theory and diffusion on Riemannian manifolds,* Conference on harmonic analysis in honor of Antoni Zygmund, Vol I, II, Wadsworth Math. Ser., Wadsworth, Belmont, Calif., 1983, 821-837.

[V3] Varoupoulos N., *Hardy-Littlewood theory for semigroups,* J. Funct. Anal. 63, 1985, 240-260.