Finite Volume Approximation of the Maxwell's Equations in Nonhomogeneous Media

CHUNG Tsz Shun Eric

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摘要

本文的重點是研究含有多種不同物理特性的三維區域內電磁場方程組的數值解法。這類方程通常稱爲交接面問題，數值求解這一類問題的困難主要在於原微分方程組的解在整個物理區域上正則性很低，所以一般數值方法精度比較差。我們將在本文中提出一種有限體積法。我們將充分利用原微分方程的局部光滑性以及特殊處理電磁場在不同介質的交接面上的物理條件，從而使這種有限體積法在非常無結構的網格上至少可達到一階收斂，並且在均勻正則的網格上可達到二階收斂。更爲重要的是，我們的有限體積法自動滿足兩個物理上極爲重要的散度定律。大量的數值例子已充分証明了這種新的數值方法的穩定性及有效性。就我們所知，這是第一個用於求解電磁場交接面問題的二階收斂的數值方法。也是第一次嚴格地全面地給出了有限體積法求解交接面問題的收斂性分析。

ABSTRACT

In this thesis, we consider the Maxwell's equations in a three-dimensional polyhedral domain composed of two dialectic materials with different physical parameters. A finite volume method is derived to solve the problem, and a new approach is proposed to handle the physical charateristics of the electromagnetic fields on the interface between the two different materials. The approximate electromagnetic fields are shown to satisfy the two divergence constraints in the discrete level. Convergence analysis will be given for both semi-discrete and fully-discrete problems. In the case of general polyhedral domains, our proposed method is first order convergent in space. The convergence is one order higher when the domain is a cuboid, though the true solution of Maxwell's system lacks enough global regularity in the entire physical domain due to the presence of the discontinuities of the physical coefficients across the interface. For both cases, the convergence in time is always second order. Numerical examples will also be given to consolidate our theortical results. To our knowledge, this is the first finite volume method with second order convergence for solving the Maxwell's equations in non-homogeneous media.

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Eric, Chung Tsz Shun

The Chinese University of Hong Kong

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Contents

Chapter 1

Introduction

The Maxwell's equations are a set of physical laws that govern all the electric- and magnetic-related systems which we see in our daily life, in the industrial and engineering applications. The solutions of the Maxwell's equations are hence widely needed in the study and design of these systems. Some examples include the systems making use of the electromagnetic wave guide, radiation and wave scattering, and so on. For some complex electromagnetic systems, which may involve many different physical media, the solutions of Maxwell's equations in non-homogeneous media are frequently required.

1.1 Applications of Maxwell's equations

First, we present two applications involving solution of the Maxwell's equations.

(I) *Target identification.*

The solution of Maxwell's equations can be useful in reconstructing the shape of a target. A trial electromagnetic pulse reflected from a known target is compared to that reflected from the desired target. The error can then be obtained. Further iterations then proceed by changing the shape of the known target to reduce the error.

(II) *Aerospace design.*

The materials used in the aerospaces are usually multilayered. In order to design an aeorspace which is hard to detect, the modeling of the electromagnetic properties of the multilayered material is required.

1.2 Introduction to Maxwell's equations

In this section we introduce the Maxwell's equations in a non-homogeneous domain. For simplicity, we consider a domain occupied by two different dialectic materials. The results of this thesis can be extended to the case that a domain is occupied by many different materials. Let Ω be a domain in $\mathbb{R}^3$ with boundary $\partial\Omega$ and unit outward normal vector $\mathbf{n}$. Let ϵ be the electric permittivity and μ be the magnetic permeability of the medium occupied by Ω. For fixed $T > 0$, the Maxwell's equations are:

$$\epsilon\frac{\partial \mathbf{E}}{\partial t} - \mathbf{curl}\,\mathbf{H} = \mathbf{J} \quad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} + \mathbf{curl}\,\mathbf{E} = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{1.2}$$

$$\text{div}(\epsilon\mathbf{E}) = \rho \quad \text{in} \quad \Omega \times (0,T), \tag{1.3}$$

$$\text{div}(\mu\mathbf{H}) = 0 \quad \text{in} \quad \Omega \times (0,T). \tag{1.4}$$

Here $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$ denote the electric and magnetic fields respectively. $\mathbf{J}(x,t)$ is the known applied current and $\rho(x,t)$ is the charge density. We remark here that (1.1) is called the Maxwell-Ampere law which states that any change in electric field would produce magnetic field. (1.2) is called the Faraday's law which states that any change in magnetic field would produce electric field. (1.3) and (1.4) are called the Gauss's law which describe the charge properties of electric and magnetic fields respectively.

Let Ω_1 be another domain such that $\bar{\Omega}_1 \subset \Omega$, and let $\Gamma = \partial\Omega_1$ with unit outward normal vector $\mathbf{m}$. We also let $\Omega_2 = \Omega\backslash\bar{\Omega}_1$. We assume that Ω_1 and Ω_2

are occupied by two different dialectic materials so that the parameters ϵ and μ are discontinuous across the interface Γ. We consider only the case that the parameters are two piecewise constant functions in Ω defined as

$$\epsilon = \begin{cases} \epsilon_1 & \text{in} \quad \Omega_1 \\ \epsilon_2 & \text{in} \quad \Omega_2 \end{cases}, \quad \mu = \begin{cases} \mu_1 & \text{in} \quad \Omega_1 \\ \mu_2 & \text{in} \quad \Omega_2 \end{cases}$$

where ϵ_i, μ_i $(i = 1, 2)$ are positive constants. Our numerical method, which will be presented later, is also applicable when the two parameters ϵ_i, μ_i $(i = 1, 2)$ are smooth functions.

We suppose that the Maxwell's equations (1.1)-(1.4) satisfy a perfect conductor boundary condition

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{1.5}$$

and initial conditions

$$\mathbf{E}(x, 0) = \mathbf{E}_0(x) \quad \text{and} \quad \mathbf{H}(x, 0) = \mathbf{H}_0(x) \quad \forall x \in \Omega, \tag{1.6}$$

such that the functions $\mathbf{E}_0(x)$ and $\mathbf{H}_0(x)$ satisfy

$$\text{div}(\epsilon\mathbf{E}_0) = \rho(x, 0) \quad \text{and} \quad \text{div}(\mu\mathbf{H}_0) = 0. \tag{1.7}$$

The boundary condition (1.5) for the electric field $\mathbf{E}$ implies the following boundary condition for the magnetic field $\mathbf{H}$:

$$\mathbf{curl}\,\mathbf{H} \times \mathbf{n} = -\mathbf{J} \times \mathbf{n} \quad \text{on} \quad \partial\Omega \times (0, T). \tag{1.8}$$

We further assume that the following continuity equation holds:

$$\frac{\partial \rho}{\partial t} = \text{div}\mathbf{J}, \tag{1.9}$$

which represents the conservation of electric charge. Throughout the paper, the jump of any function A across the interface Γ is defined as

$$[A] := A_2|_\Gamma - A_1|_\Gamma$$

where $A_i = A|_{\Omega_i}$ for $i = 1, 2$. It is known physically that the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ must satisfy the following jump conditions across the interface Γ:

$$[\mathbf{E} \times \mathbf{m}] = 0 \quad , \quad [\epsilon \mathbf{E} \cdot \mathbf{m}] = \rho_\Gamma, \tag{1.10}$$

$$[\mathbf{H} \times \mathbf{m}] = \mathbf{J}_\Gamma \quad , \quad [\mu \mathbf{H} \cdot \mathbf{m}] = 0, \tag{1.11}$$

where $\rho_\Gamma(x, t)$ is the surface charge density while $\mathbf{J}_\Gamma(x, t)$ is the surface current density. In addition, we will adopt the following constitutive relations

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{1.12}$$

$$\mathbf{B} = \mu \mathbf{H}, \tag{1.13}$$

where $\mathbf{D}$ and $\mathbf{B}$ are the electric flux density and the magnetic flux density respectively.

1.3 Historical outline of numerical methods

In this section, we give a brief outline of some existing numerical methods and related aspects in numerical solution of the Maxwell's equations.

To our knowledge, the first numerical methods was established by Yee [24] in 1966. In [24], a standard finite difference method is employed to approximate both the spatial and time derivatives in the curl Maxwell's equations (1.1)-(1.2) in homogeneous domain. However, the convergence analysis for this method is open for a long time, and in 1992, Monk and Süli [18] provide a proof for the second order convergence of Yee's scheme on nonuniform grids.

In order to handle complicated geometry of domains, finite element and finite volume methods are introduced. In Monk [17] and Raviart [22], a fully discrete finite element method is used to solve the decoupled time-dependent Maxwell's equations in homogeneous domain. In addition, the second order convergence analysis for the stationary problem is provided. In Ciarlet and Zou [8], a convergence analysis for the fully discrete time dependent problem is given. In Chen and Yee [3], a finite volume method is used to solve the Maxwell's equations in homogeneous domain, and in Nicolaides and Wang [20], convergence analysis for both semi-discrete and fully-discrete schemes are also provided.

However, the aforemensioned methods are concerned with only homogeneous medium cases. For many real applications, one is often encountered with the solution of the Maxwell's equations in non-homogeneous media. Several attempts have been made to handle the interface Maxwell's problems [3] [4] [23]. For example, Chen and Yee [3] studied an FDTD/FVTD hybrid method for the interface problem, assuming both the tangential components of the electric and magnetic fields are continuous across the interface and the electric field is tangentially piecewise constant on the interface. Chen, Du and Zou [4] proposed an edge finite element method for solving the Maxwell's system with very general inhomogeneous interface conditions and developed a general framework for its convergence analysis.

1.4 A new approach

The previously mensioned finite volume methods can only handle limited cases, namely, homogeneous domains and non-homogeneous domains with special interface conditions. In this section and the following chapters, we present a new finite volume approach to solve the Maxwell's equations in non-homogeneous media (cf. Chung and Zou [6]).

One of the improvements over the existing methods of our proposed method is that it can deal with inhomogeneous interface conditions, whereas the existing methods can only handle homogeneous interface conditions. In terms of implementation, our proposed method suggests a simple approach to handle the interface conditions. On the other hand, the numerical solution to the Maxwell's system found by our method can be proved to satisfy the two divergence constraints in discrete sense, which ensures physically consistent solution. In literature, we seldom find any argument discussing if the numerical solution of a method satisfies the divergence constraints.

In spite of the derivation of numerical scheme, the main part of this thesis is on the convergence analysis of the method. We will give the convergence analysis of both semi-discrete and fully discrete schemes. As for any interface problem, the true solution of the Maxwell's system has very low global regularity, namely, it is only in the space $H^1(\Omega)$. This fact greatly produces tremendous difficulty in dealing with the convergence analysis. However, despite low global regularity of solutions, it can be shown that, as for homogeneous domains in [20], our proposed method, with respect to spatial variables, is first order convergent for polyhedral domains and second order convergent for rectangular domains. Under a CFL stability condition, the fully discrete scheme is second order convergent in time.

In this thesis, we only consider the case when the domain is a polyhedron. In many real applications, however, we always encounter with smooth domains and any other irregular domains. For those cases, though they cannot be handled by our method directly, but the theory in this thesis can be further generalized to solve those problems without essential difficulties.

The thesis is organised as follows. In Chapter 2, some Sobolev space theory and related functional analytic tools will be presented. In Chapter 3, we will discuss the finite volume discretization of the domain. The discrete analog of divergence and curl operators will be defined. Then, we will prove discrete forms of some famous theorems in vector field theory and functional analysis. In Chapter 4 and Chapter 5, we will derive, respectively, the spatial and fully discretization of the Maxwell's equations. In addition, we show how the semi-discrete and fully discrete solutions satisfy the two divergence constraints in discrete level. A complete convergence analysis for both schemes will also be given. In Chapter 6, two numerical examples will be shown to consolidate our theory.

Chapter 2

Mathematical Backgrounds

In this chapter, we present some mathematical notations and basic mathematical tools that will be used in our subsequent numerical analysis.

2.1 Sobolev spaces

Let m be a nonnegative integer and $1 \le p < \infty$, we define the Sobolev space

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \quad ; \quad \partial^\alpha u \in L^p(\Omega) \quad \forall |\alpha| \le m\},$$

which is equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} = (\sum_{0\le|\alpha|\le m} \|\partial^\alpha u\|^p_{L^p(\Omega)})^{1/p}$$

and the semi-norm

$$|u|_{W^{m,p}(\Omega)} = (\sum_{|\alpha|=m} \|\partial^\alpha u\|^p_{L^p(\Omega)})^{1/p}.$$

Here $\partial^\alpha u$ denotes the α-th order weak derivative of u. When $p = 2$, we write $H^m(\Omega) = W^{m,2}(\Omega)$ which is indeed a Hilbert space. Let $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$. We say $\mathbf{u} \in W^{m,p}(\Omega)^3$ if and only if $u_i \in W^{m,p}(\Omega)$ for $i = 1, 2, 3$. We extend the norm in $W^{m,p}(\Omega)^3$ in the usual way, namely

$$\|\mathbf{u}\|_{W^{m,p}(\Omega)^3} = (\sum_{i=1}^{3} \|u_i\|^p_{W^{m,p}(\Omega)})^{1/p},$$

and extend similarly for the semi-norm. Note that same definitions are adopted for Ω_1 and Ω_2.

By $L^p(0,T;X)$ we mean the set of all strongly measurable functions $u(t,\cdot)$ from $[0,T]$ into the Banach space X such that

$$\int_0^T \|u(t)\|_X^p dt < \infty \quad \text{for} \quad 1 \leq p < \infty,$$

where the integral is understood in the Bochner sense. Similar to $W^{m,p}(\Omega)$, we define

$$W^{m,p}(0,T;X) = \{u \in L^p(0,T;X) \quad ; \quad \frac{\partial^\alpha u}{\partial t^\alpha} \in L^p(0,T;X) \quad \forall |\alpha| \leq m\},$$

with norm

$$\|u\|_{W^{m,p}(0,T;X)} = (\sum_{0\leq|\alpha|\leq m} \|\frac{\partial^\alpha u}{\partial t^\alpha}\|_X^p)^{1/p}.$$

When $p=2$, we write $W^{m,p}(0,T;X)$ as $H^m(0,T;X)$. Similarly, $\mathbf{u} \in L^p(0,T;X)^3$ if and only if $u_i \in L^p(0,T;X)$, for $i=1,2,3$. The norm and semi-norm in $L^p(0,T;X)^3$ are defined in a similar fashion.

Furthermore, $\mathbf{u} \in C^m(\Omega)^3$ if and only if $u_i \in C^m(\Omega)$ for $i=1,2,3$ where $C^m(\Omega)$ denotes the space of m times differentiable functions in Ω with norm

$$\|u_i\|_{C^m(\Omega)} = \sum_{\alpha=0}^m \sup_\Omega |\partial^\alpha u_i|.$$

Similarly, $C^m(0,T;X)$ denotes the space of m times differentiable functions from $[0,T]$ into X with norm

$$\|u\|_{C^m(0,T;X)} = \sum_{\alpha=0}^m \sup_{0\leq t\leq T} \|\partial^\alpha u(t)\|_X.$$

2.2 Tools from functional analysis

In this section, we quote without proof some well known results in literature. These results are very useful tools for the convergence analysis of our finite volume method which will be presented later.

The fisrt one is called the Bramble-Hilbert lemma.

Theorem 2.1 (Bramble-Hilbert lemma) *Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz continuous boundary. Let f be a continuous linear functional on the space $W^{k+1,p}(\Omega)$, for some integer $k \geq 0$ and some real number $p \in [0,\infty]$, such that,*

$$f(P) = 0,$$

for any polynomial P of degree less than or equal to k. Then for all $v \in W^{k+1,p}(\Omega)$,

$$|f(v)| \leq K(\Omega)\|f\|^*_{W^{k+1,p}(\Omega)}|v|_{W^{k+1,p}(\Omega)},$$

for some constant $K(\Omega)$ depends only on Ω and $\|\cdot\|^*_{W^{k+1,p}(\Omega)}$ denotes the norm in the dual space of $W^{k+1,p}(\Omega)$.

The second one is called the Sobolev embedding theorem. We only present part of it. For a full version, we refer readers to Adams [1].

Theorem 2.2 (Sobolev embedding theorem) *Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz continuous boundary. Suppose that $mp > n$. Then*

$$W^{j+m,p}(\Omega) \hookrightarrow C^j(\bar{\Omega}),$$

for any integer $j \geq 0$.

In the above theorem, the notation $W^{j+m,p}(\Omega) \hookrightarrow C^j(\bar{\Omega})$ means the following: for any $u \in W^{j+m,p}(\Omega)$, we have $u \in C^j(\bar{\Omega})$ and there exist a constant $K(\Omega)$ depends only on Ω and indepenent of u such that

$$\|u\|_{C^j(\bar{\Omega})} \leq K(\Omega)\|u\|_{W^{j+m,p}(\Omega)}.$$

When $\Omega = (a,b)$, we have the following special case.

Theorem 2.3 *Let $\Omega = (a,b)$. Then for $1 \leq p < \infty$,*

$$W^{1+j,p}(a,b) \hookrightarrow C^j([a,b]),$$

for any integer $j \geq 0$.

Chapter 3

Discretization of Vector Fields

In this chapter we will present the finite volume discretization of the two important differential operators: div and **curl**. They are cruial for our subsequent derivation of numerical approximations. In addition, we will provide discrete analog of some famous theorems in vector field theory.

3.1 Domain triangulation

We now discuss the triangulation of the domain Ω. It is actually the Voronoi-Delanuay triangulation which has some useful properties that allow us to derive the numerical schemes in the subsequent chapters. Most notations used below are borrowed from Nicolaides, Wang and Wu [19] [20] [21]. For details of Voronoi-Delanuay triangulation, see Fortune [12].

Assume that both Ω and Ω_1 are polyhedra. We triangulate Ω by using standard finite element type tetrahedra which we call primal elements. This triangulation of Ω is not arbitrary in the sense that primal faces, that is the face of primal element, should align with the interface Γ. That means the two triangulations in Ω_1 and Ω_2 match each other on Γ as well as they are combined into a standard triangulation of the whole domain Ω. A primal element with as least one face lying on Γ is called an interface primal element. Similarly, a primal face and a primal edge lying on Γ is called an interface primal face and an interface primal edge respectively. We denote by h the maximum side length of all primal elements. We assume that the ratios of any two edges of an individual primal element are uniformly bounded from above and below as h tends to 0. This is equivalent to say that all dihedral angles of each tetrahedron are acute.

The dual elements are formed by connecting adjacent circumcenters of primal elements. In the case of a primal element with face on the boundary, connect the circumcenter to the boundary face. It is easy to see that the dual elements are convex polyhedra with faces being convex polygons. However, there are some dual faces belonging to both Ω_1 and Ω_2. Owing to this fact, some definitions and convergence analysis related to dual elements are more complicated and are not similar to those for primal elements. We call the dual elements, dual faces and dual edges with non-empty intersection with both Ω_1 and Ω_2 the inteface dual elements, interface dual faces and interface dual edges respectively. With these definitions, we conclude with the following properties concerning the primal and dual meshes. First, primal edges are orthogonal to and in one-to-one correspondence with dual faces. Secondly, dual edges are also orthogonal to and in one-to-one correspondence with primal faces. These orthogonalities are the key to the derivation of our numerical schemes.

3.2 Mesh dependent norms

Let N and L be the number of primal elements and dual elements respectively. Let F be the number of primal faces (or dual edges) and M be the number of primal edges (or dual faces). Assume that these quantities, as well as primal nodes and dual nodes, are numbered sequentially in some way. The individual elements, faces, edges, nodes of the primal mesh are denoted by τ_i, κ_j, σ_k and

ν_l respectively. Those quantities relating dual mesh are denoted by primed form such as τ_i'. A direction is assigned to each primal and dual edge by the rule that positive direction maens it points from lower node number to higher node number. Direction is also assigned to each primal and dual face such that it is the same as the corresponding dual and primal edge. We also denote F_1 the number of interior primal faces (or dual edges) and M_1 the number of interior primal edges (or dual faces).

Let s_j be the area of κ_j and h_j' be the length of σ_j'. We define

$$\bar{h}_j' = \begin{cases} \mu_1^{-1} h_j' & \text{if } \sigma_j' \in \Omega_1 \\ \mu_2^{-1} h_j' & \text{if } \sigma_j' \in \Omega_2 \\ (\mu_1^{-1} a_j + \mu_2^{-1}(1 - a_j)) h_j' & \text{otherwise,} \end{cases}$$

where $0 < a_j < 1$ denotes the ratio of the length of the portion of σ_j' that belongs to Ω_1. For any u and v in $\mathbb{R}^{F_1}$, we introduce an mesh and parameter dependent inner product defined by

$$(u, v)_W := \sum_{\kappa_j \in \Omega} u_j v_j s_j \bar{h}_j' = (Su, D'v) = (D'u, Sv), \tag{3.1}$$

where $S := \text{diag}(s_j)$ and $D' := \text{diag}(\bar{h}_j')$ are $F_1 \times F_1$ diagonal matrices, $(\cdot, \cdot)$ denotes the standard Euclidean inner product. With this inner product, the associated norm is defined as

$$\|u\|_W := (u, u)_W^{\frac{1}{2}}. \tag{3.2}$$

Clearly, this norm is equivalent to the standard discrete L^2-norm. Now, let s_j' be the area of κ_j', h_j be the length of σ_j and let

$$\bar{s}_j' = \begin{cases} \epsilon_1 s_j' & \text{if } \kappa_j' \in \Omega_1 \\ \epsilon_2 s_j' & \text{if } \kappa_j' \in \Omega_2 \\ (\epsilon_1 b_j + \epsilon_2(1 - b_j)) s_j' & \text{otherwise,} \end{cases}$$

where $0 < b_j < 1$ denotes the ratio of the area of the portion of κ'_j that belongs to Ω_1. Similarly, we define an mesh and parameter dependent inner product in $\mathbb{R}^{M_1}$ by

$$(u,v)_{W'} := \sum_{\kappa'_j \in \Omega} u_j v_j \bar{s}'_j h_j = (S'u, Dv) = (Du, S'v), \tag{3.3}$$

where $S' := \text{diag}(\bar{s}'_j)$ and $D := \text{diag}(h_j)$ are $M_1 \times M_1$ diagonal matrices. The associated norm is

$$\|u\|_{W'} := (u,u)^{\frac{1}{2}}_{W'}, \tag{3.4}$$

which is again equivalent to the discrete L^2-norm.

Denote by M_Γ the number of inteface primal edges. Define a $M_\Gamma \times M_\Gamma$ diagonal matrix $D_\Gamma := \text{diag}(h_j)$ with components corresponding to interface primal edges. Now, for any vectors $u, v \in \mathbb{R}^{M_\Gamma}$, we define the following inner product

$$(u,v)_{W_\Gamma} := \sum_{\sigma_j \in \Gamma} h_j^2 u_j v_j = (D_\Gamma u, D_\Gamma v), \tag{3.5}$$

with the associated norm

$$\|u\|_{W_\Gamma} := (u,u)^{\frac{1}{2}}_{W_\Gamma}. \tag{3.6}$$

3.3 Discrete circulation operators

In this section, we present the finite volume discretization of the **curl** operator. Furthermore, discrete forms of some famous theorems are provided.

Let $\sigma_j \in \partial\kappa_i$. We say σ_j is oriented positively along $\partial\kappa_i$ if the direction of σ_j agrees with the direction of $\partial\kappa_i$ formed by the right hand rule with the thumb pointing to the direction of σ'_i. Otherwise, we say σ_j is oriented negatively along $\partial\kappa_i$. For each interior primal face κ_i, we define discrete circulation by

$$(Cu)_{\kappa_i} := \sum_{\sigma_j \in \partial\kappa_i} u_j \tilde{h}_j, \tag{3.7}$$

where

$$\tilde{h}_j = \begin{cases} h_j & \text{if } \sigma_j \text{ is oriented positively along } \partial\kappa_i \\ -h_j & \text{if } \sigma_j \text{ is oriented negatively along } \partial\kappa_i. \end{cases}$$

Similarly, for each interior dual face κ_i' the discrete circulation is defined as

$$(C'u)_{\kappa_i'} := \sum_{\sigma_j' \in \partial\kappa_i'} u_j \tilde{h}_j', \tag{3.8}$$

where

$$\tilde{h}_j' = \begin{cases} \bar{h}_j' & \text{if } \sigma_j' \text{ is oriented postively along } \partial\kappa_i' \\ -\bar{h}_j' & \text{if } \sigma_j' \text{ is oriented negatively along } \partial\kappa_i'. \end{cases}$$

Clearly, C and C' are linear operators mapping from $\mathbb{R}^M$ to $\mathbb{R}^{F_1}$ and $\mathbb{R}^{F_1}$ to $\mathbb{R}^{M_1}$ respectively. We remark that (3.7) and (3.8) are discrete analog of the integrals

$$\int_{\kappa_i} \mathbf{curl}\,\mathbf{E} \cdot \mathbf{n}_i \, d\sigma \quad \text{and} \quad \int_{\kappa_i'} \mathbf{curl}\,\mathbf{H} \cdot \mathbf{n}_i \, d\sigma$$

by virtue of the Stokes' theorem where $\mathbf{n}_i$ represents the unit normal vector to both primal and dual faces.

With the definition of the discrete circulation operator C, we define the following inner product

$$(u, v)_V := \sum_{\kappa_i \in \Omega} (Cu)_i (Cv)_i s_i^{-1} \bar{h}_i' = (S^{-1}Cu, D'Cv) = (D'Cu, S^{-1}Cu) \tag{3.9}$$

for any vectors $u, v \in \mathbb{R}^M$ and its associated norm

$$||u||_V := (u, u)_V^{\frac{1}{2}}. \tag{3.10}$$

Clearly, this norm is equivalent to the discrete H^1-norm.

For each strictly interior dual edge σ_j', that is, both end points of σ_j' lie in Ω, we define a row vector whose ith component is the sign of the orientation of σ_j' relative to the ith strictly interior dual face. Collecting these vectors, we have a

$F_1 \times M_1$ matrix G defined as

$$(G)_{ji} := \begin{cases} 1 & \text{if } \sigma'_j \text{ is oriented positively along } \partial\kappa'_i \\ -1 & \text{if } \sigma'_j \text{ is oriented negatively along } \partial\kappa'_i \\ 0 & \text{if } \sigma'_j \text{ does not meet } \partial\kappa'_i. \end{cases}$$

Let $w \in \mathbb{R}^M$ be a vector whose kth component is the value assigned on the kth primal edge. Let $w_1 \in \mathbb{R}^{M_1}$ be the restriction of w to the interior primal edges. Denote by $w|_{\partial\Omega}$ the components of w on the boundary. Likewise, denote by $v \in \mathbb{R}^{F_1}$ the vector whose jth component represents a value on the jth interior dual edge.

Lemma 3.1 *With the above definitions of w, w_1 and v together with $w|_{\partial\Omega} = 0$, we have*

$$Cw = GDw_1, \tag{3.11}$$

and

$$C'v = G^T D'v. \tag{3.12}$$

Proof. To see (3.11) is true, we consider the ith component to both sides. Note that the ith component corresponds to the primal face κ_i. By definition (3.7) and the fact that $w|_{\partial\Omega} = 0$, we have

$$\begin{aligned} (Cw)_{\kappa_i} &= \sum_{\sigma_j \in \partial\kappa_i} w_j \tilde{h_j} \\ &= \sum_{j=1}^{M_1} c_j w_j h_j, \end{aligned}$$

where

$$c_j = \begin{cases} 1 & \text{if } \sigma_j \text{ is oriented postively along } \partial\kappa_i \\ -1 & \text{if } \sigma_j \text{ is oriented negatively along } \partial\kappa_i \\ 0 & \text{if } \sigma_j \text{ does not meet } \partial\kappa_i \end{cases}$$

and σ_j's are interior primal edges. For the right hand side, we have

$$(GDw_1)_{\kappa_i} = \sum_{j=1}^{M_1} g_j h_j w_j,$$

where

$$g_j = \begin{cases} 1 & \text{if } \sigma'_i \text{ is oriented postively along } \partial\kappa'_j \\ -1 & \text{if } \sigma'_i \text{ is oriented negatively along } \partial\kappa'_j \\ 0 & \text{if } \sigma'_i \text{ does not meet } \partial\kappa'_j \end{cases}$$

is the ith row of the matrix G. By the orthogonality between primal and dual meshes, we conclude that c_j and g_j are the same which implies (3.11). The proof for (3.12) can be done by similar techniques.

□

Now, we have the following result.

Lemma 3.2 *Using the same definitions in Lemma 3.1, we have*

$$(Cw, D'v) = (C'v, Dw_1). \tag{3.13}$$

Proof. Applying Lemma 3.1, we have

$$\begin{aligned} (C'v, Dw_1) &= (G^T D'v, Dw_1) && \text{by (3.12)} \\ &= (D'v, GDw_1) \\ &= (D'v, Cw) && \text{by (3.11).} \end{aligned}$$

□

We remark that (3.13) is the discrete form of the Green's formula

$$\int_\Omega \mathbf{curl}\,\mathbf{E}\cdot\mathbf{B}\,dx = \int_\Omega \mathbf{curl}\,\mathbf{B}\cdot\mathbf{E}\,dx$$

which holds when $\mathbf{E}\times\mathbf{n} = 0$ on $\partial\Omega$.

Concerning with the matrix G, we have the following lemma.

Lemma 3.3 *Let L be the number of interior primal nodes. Then*

$$\text{rank}(G) = E_1 - L.$$

Proof. Recall that each column of G corresponds a dual face. Take any interior dual element. Without loss of generality, let $\kappa_1', \kappa_2', \cdots, \kappa_l'$ be dual faces lying on the boundary of the chosen dual element. Then consider a linear combination of the corresponding columns of G:

$$a_1c_1 + a_2c_2 + \cdots + a_lc_l,$$

where c_i denotes the i-th column of G. We choose a_i in the following way. If the direction of κ_j' is pointing outward to the dual element, choose $a_j = 1$. Otherwise, choose $a_j = -1$. Clearly, we have

$$a_1c_1 + a_2c_2 + \cdots + a_lc_l = 0.$$

Since each interior dual element corresponds to a interior primal node, we have the desired result.

□

We emphasis here that all the above results are valid when both Ω and Ω_1 are rectangular domains. In this case, both primal and dual elements are cuboids while both primal and dual faces are rectangles. From Lemma 3.3, we know that the matrix G^TG is positive semi-definite. So, $\lambda(G^TG) \geq 0$ where $\lambda(G^TG)$ represents eigenvalues of G^TG. Denote $\lambda_{\min}^{+}(G^TG)$ be the smallest positive eigenvalue of G^TG. Then, we have the following result.

Lemma 3.4 *Assume that both Ω and Ω_1 are rectangular domains. Then there exist a constant K independent of h such that*

$$\lambda_{\min}^{+}(G^TG) \geq Kh^2. \tag{3.14}$$

With Lemma 3.4, we have the following

Lemma 3.5 *Assume that both Ω and Ω_1 are rectangular domains. Let $u \in \mathbb{R}^M$ with $u|_{\partial\Omega} = 0$ and $Cu \neq 0$. Then there exist a constant K independent of h such that*

$$\|u\|_{W'} \leq K\|u\|_V. \tag{3.15}$$

Proof. By the definition of V-norm, we have

$$\|u\|_V^2 = (D'S^{-1}Cu, Cu) \geq Kh^{-1}(Cu, Cu).$$

By Lemma 3.2 and $Cu \neq 0$, we have

$$0 < (Cu, Cu) = (G^TGDu, Du),$$

and consequently $G^TGDu \neq 0$. Let λ_j, $j = 1, 2, \cdots, M_1$, be the eigenvalues of G^TG and let ω_j be the corresponding eigenvectors. Since G^TG is positive semi-definite, let M^* be such that

$$\lambda_j = 0, \quad \text{for} \quad 1 \leq j \leq M^*$$
$$\lambda_j > 0, \quad \text{for} \quad M^* + 1 \leq j \leq M_1.$$

Notice that we can choose ω_j, for $j = 1, 2, \cdots, M_1$, such that they form an orthonormal basis for $\mathbb{R}^{M_1}$. So, we can express Du into the following form

$$Du = \sum_{j=1}^{M_1} (Du, \omega_j)\omega_j.$$

By the fact that $G^TGDu \neq 0$ and $\{\omega_j\}_{j=1}^{M^*}$ spans the null space of G^TG, we have

$$(Du, \omega_j) = 0, \quad \text{for} \quad 1 \leq j \leq M^*,$$

and consequently,

$$Du = \sum_{j=M^*+1}^{M_1} (Du, \omega_j)\omega_j.$$

Hence, we obtain

$$\begin{aligned}(Cu, Cu) &= (G^T G Du, Du)\\ &= (G^T G \sum_{j=M^*+1}^{M_1} (Du, \omega_j)\omega_j, \sum_{j=M^*+1}^{M_1} (Du, \omega_j)\omega_j)\\ &= (\sum_{j=M^*+1}^{M_1} (Du, \omega_j) G^T G \omega_j, \sum_{j=M^*+1}^{M_1} (Du, \omega_j)\omega_j).\end{aligned}$$

Since $G^T G\omega_j = \lambda_j \omega_j$, we finally obtain

$$\begin{aligned}(Cu, Cu) &= (\sum_{j=M^*+1}^{M_1} (Du, \omega_j)\lambda_j\omega_j, \sum_{j=M^*+1}^{M_1} (Du, \omega_j)\omega_j)\\ &\geq \min_{M^*+1\leq j\leq M_1} \lambda_j(\sum_{j=M^*+1}^{M_1} (Du, \omega_j)\omega_j, \sum_{j=M^*+1}^{M_1} (Du, \omega_j)\omega_j)\\ &= \min_{M^*+1\leq j\leq M_1} \lambda_j (Du, Du).\end{aligned}$$

Hence,

$$\begin{aligned}\|u\|_V^2 &\geq Kh^{-1}\lambda_{\min}^{+}(G^T G)(Du, Du)\\ &= Kh^{-1}\lambda_{\min}^{+}(G^T G)(DS'^{-1}S'u, Du)\\ &\geq Kh^{-2}\lambda_{\min}^{+}(G^T G)\|u\|_{W'}^2.\end{aligned}$$

Then (3.15) follows from (3.14).

□

We remark here that (3.15) is the discrete analog of the following Poincare's inequality:

$$\int_\Omega |u|^2\, dx \leq K \int_\Omega |\nabla u|^2\, dx,$$

for any $u \in H_0^1(\Omega)$.

3.4 Discrete flux operators

In this section, we give the finite volume discretization of the divergence operator. We also provide the discrete versions of some famous results in vector field theory.

Let τ_i be a primal element and $\kappa_j \in \partial\tau_i$ be a primal face. We say κ_j is oriented positively along $\partial\tau_i$ if the dual edge σ'_j on κ_j is directed toward the outside of τ_i. Otherwise we say κ_j is oriented negatively along $\partial\tau_i$. For each primal element τ_i we define a discrete flux by

$$(\mathcal{D}u)_i := \sum_{\kappa_j \in \partial\tau_i} u_j \tilde{s}_j, \qquad \forall u \in \mathbb{R}^{F_1} \tag{3.16}$$

where no components of u related to the boundary faces are involved, and $\tilde{s}_j$ is given by

$$\tilde{s}_j = \begin{cases} s_j & \text{if } \kappa_j \text{ is oriented positively along } \partial\tau_i \\ -s_j & \text{if } \kappa_j \text{ is oriented negatively along } \partial\tau_i. \end{cases}$$

The mapping $\mathcal{D}$ is the discrete version of the divergence operator by noting that

$$\int_{\tau_i} \operatorname{div} \mathbf{u}\, dx = \int_{\partial\tau_i} \mathbf{u} \cdot \mathbf{n}\, ds.$$

Similarly, for each dual element τ'_i, we define a discrete flux by

$$(\mathcal{D}'u)_i := \sum_{\kappa'_j \in \partial\tau'_i} u_j \tilde{s}'_j, \quad \forall u \in \mathbb{R}^{M_1} \tag{3.17}$$

where

$$\tilde{s}'_j = \begin{cases} \bar{s}'_j & \text{if } \kappa'_j \text{ is oriented postively along } \partial\tau'_i \\ -\bar{s}'_j & \text{if } \kappa'_j \text{ is oriented negatively along } \partial\tau'_i. \end{cases}$$

We next present a discrete analog of the identity $\operatorname{div}(\mathbf{curl}\,\mathbf{u}) = 0$ for the discrete divergence operators $\mathcal{D}$ and $\mathcal{D}'$. To do so, we introduce two matrices B_1 and B'_1. B_1 is a $F_1 \times N$ matrix given by

$$(B_1)_{ji} := \begin{cases} 1 & \text{if } \kappa_j \text{ is oriented postively along } \partial\tau_i \\ -1 & \text{if } \kappa_j \text{ is oriented negatively along } \partial\tau_i \\ 0 & \text{if } \kappa_j \text{ does not meet } \partial\tau_i, \end{cases}$$

while B_1' is a $M_1 \times L$ matrix given by

$$(B_1')_{ji} := \begin{cases} 1 & \text{if } \kappa_j' \text{ is oriented postively along } \partial\tau_i' \\ -1 & \text{if } \kappa_j' \text{ is oriented negatively along } \partial\tau_i' \\ 0 & \text{if } \kappa_j' \text{ does not meet } \partial\tau_i'. \end{cases}$$

Lemma 3.6 *We have*

$$\mathcal{D} = B_1^T S \quad , \quad \mathcal{D}' = (B_1')^T S' \tag{3.18}$$

and

$$B_1^T C = 0 \quad , \quad (B_1')^T C' = 0. \tag{3.19}$$

Proof. For any $u \in \mathbb{R}^{F_1}$, we have

$$(\mathcal{D}u)_i = \sum_{\kappa_j \in \partial\tau_i} u_j \tilde{s_j} = \sum_{j=1}^{F_1} d_j u_j s_j$$

where

$$d_j = \begin{cases} 1 & \text{if } \kappa_j \text{ is oriented positively along } \partial\tau_i \\ -1 & \text{if } \kappa_j \text{ is oriented negatively along } \partial\tau_i \\ 0 & \text{if } \kappa_j \text{ does not meet } \partial\tau_i. \end{cases}$$

Clearly, the vector formed by d_j's is the i-th column of the matrix B_1 and hence $\mathcal{D} = B_1^T S$. The relation $\mathcal{D}' = (B_1')^T S'$ can be proved similarly.

For (3.19), we observe that the i-th row of B_1^T is the direction of κ_j with respect to τ_i. Let $w \in \mathbb{R}^M$ with $w|_{\partial\Omega} = 0$. Then in the i-th component of $B_1^T C w$, each w_j which is involved in that component appears exactly twice with two opposite signs, hence $(B_1^T C w)_i = 0$. Similar argument can be applied to show $(B_1')^T C' = 0$.

□

Here we quote a lemma from Nicolaides and Wu [21]. We know that for any vector field $\mathbf{F}$ with $\mathbf{curl}\,\mathbf{F} = 0$, there must exist a scalar potential p such that $\mathbf{F} = \nabla p$. The following lemma shows the discrete analog of this fact, namely, for any vector v with discrete circulation free, there must exist a scalar potential.

Lemma 3.7 *We have*

(a) Let $v \in \mathbb{R}^{F_1}$. *Then there exist* $\phi \in \mathbb{R}^N$ *such that* $v = B_1\phi$ *if and only if* $G^T v = 0$.

(b) Let $v \in \mathbb{R}^{M_1}$. *Then there exist* $\phi \in \mathbb{R}^L$ *such that* $v = B_1'\phi$ *if and only if* $Gv = 0$.

Proof. The proof for part (a) can be found in Nicolaides and Wu [21]. Part (b) can be proved in an analogous way.

□

Chapter 4

Spatial Discretization of the Maxwell's Equations

In this chapter we present the spatial discretization of the Maxwell's equations (1.1)-(1.4) by finite volume method. We will give the semi-discrete approximation of (1.1)-(1.2) and show from this semi-discrete scheme that how (1.3) and (1.4) are satisfied in the discrete level. In addition to this consistency property, we will also give the convergence analysis of our finite volume method.

4.1 Derivation

First we introduce the following average quantities. Consider the magnetic flux density $\mathbf{B}$. We define its primal face average $B_f \in \mathbb{R}^{F_1}$ by

$$(B_f)_i := \frac{1}{s_i} \int_{\kappa_i} \mathbf{B} \cdot \mathbf{n}_i \, d\sigma,$$

for each primal face κ_i. We define its dual edge average $B'_e \in \mathbb{R}^{F_1}$ by the following fashion. For each non-interface dual edge σ'_i, we define

$$(B'_e)_i := \frac{1}{h'_i} \int_{\sigma'_i} \mathbf{B} \cdot \mathbf{n}_i \, dl.$$

For each interface dual edge σ_i', we let $\sigma_i^1 = \sigma_i' \cap \Omega_1$ and $\sigma_i^2 = \sigma_i' \cap \Omega_2$ be the portion of σ_i' in Ω_1 and Ω_2 respectively. Then we define

$$(B_e')_i := \alpha_i \frac{1}{h_i^1} \int_{\sigma_i^1} \mathbf{B} \cdot \mathbf{n}_i \, dl + (1-\alpha_i) \frac{1}{h_i^2} \int_{\sigma_i^2} \mathbf{B} \cdot \mathbf{n}_i \, dl,$$

where $\alpha_i := \mu_1^{-1} h_i^1 (\bar{h}_i')^{-1}$ and h_i^r represents the length of σ_i^r for $r = 1, 2$. Notice that the dual edge average for interface dual edge is defined as the weighted average of the edge averages of the corresponding portions of σ_i' in Ω_1 and Ω_2. The reason for the choice of the weight α_i will be apparent in the derivation of the semi-discrete scheme.

Now, we turn to the electric field $\mathbf{E}$. For each primal edge σ_i, we define the primal edge average $E_e \in \mathbb{R}^{M_1}$ by

$$(E_e)_i := \frac{1}{h_i} \int_{\sigma_i} \mathbf{E} \cdot \mathbf{n}_i \, dl.$$

Similar to the dual edge average for $\mathbf{B}$, we define the dual face average $E_f' \in \mathbb{R}^{M_1}$ of $\mathbf{E}$ by the following fashion. For each non-interface dual face κ_i', we define

$$(E_f')_i := \frac{1}{s_i'} \int_{\kappa_i'} \mathbf{E} \cdot \mathbf{n}_i \, d\sigma.$$

For each interface dual face, let $\kappa_i^1 = \kappa_i' \cap \Omega_1$ and $\kappa_i^2 = \kappa_i' \cap \Omega_2$ be the portions of κ_i' in Ω_1 and Ω_2, and s_i^1 and s_i^2 be the area of them respectively. We define

$$(E_f')_i := \beta_i \frac{1}{s_i^1} \int_{\kappa_i^1} \mathbf{E} \cdot \mathbf{n}_i \, d\sigma + (1-\beta_i) \frac{1}{s_i^2} \int_{\kappa_i^2} \mathbf{E} \cdot \mathbf{n}_i \, d\sigma,$$

where $\beta_i := \epsilon_1 s_i^1 (\bar{s}_i')^{-1}$. Clearly, the dual face average of $\mathbf{E}$ for an interface dual face is defined as the weighted average of the face averages of the portions of κ_i' in Ω_1 and Ω_2. Here, the choice of β_i will become apparent in the later derivation.

In the finite volume scheme, we approximate the edge averages of $\mathbf{E}$ on each primal edge and the face averages of $\mathbf{B}$ on each primal face. Now, integrating both sides of (1.2) on a primal face κ_j, we have

$$\frac{d}{dt} \int_{\kappa_j} \mathbf{B} \cdot \mathbf{n}_j \, d\sigma + \int_{\kappa_j} \mathbf{curl}\, \mathbf{E} \cdot \mathbf{n}_j \, d\sigma = 0.$$

By the Stokes' theorem,

$$\frac{d}{dt}\int_{\kappa_j} \mathbf{B}\cdot\mathbf{n}_j\, d\sigma + \sum_{\sigma_i\in\partial\kappa_j}\int_{\sigma_i}\mathbf{E}\cdot\mathbf{t}_i\, dl = 0,$$

where the directions $\mathbf{t}_i's$ are defined by the right hand rule on the face κ_j. Then

$$\frac{d}{dt}((B_f)_j s_j) + \sum_{\sigma_i\in\partial\kappa_j}(E_e)_i\tilde{h}_i = 0$$

which can be written as

$$\frac{d}{dt}((B_f)_j s_j) + (CE_e)_{\kappa_j} = 0. \tag{4.1}$$

Integrating both sides of (1.1) on a non-interface dual face $\kappa_j' \in \Omega_r$, $r = 1, 2$, we get

$$\frac{d}{dt}\int_{\kappa_j'} \epsilon_r\mathbf{E}\cdot\mathbf{n}_j\, d\sigma - \sum_{\sigma_i'\in\partial\kappa_j'}\int_{\sigma_i'}\frac{1}{\mu_i}\mathbf{B}\cdot\mathbf{t}_i\, dl = \int_{\kappa_j'}\mathbf{J}\cdot\mathbf{n}_j\, d\sigma,$$

where the directions $\mathbf{t}_i's$ are defined by the right hand rule on the face κ_j'. From the definitions of $\tilde{h}_j'$ and $\bar{s}_j'$, we have

$$\frac{d}{dt}((E_f')_j\bar{s}_j') - (C'B_e')_{\kappa_j'} = \int_{\kappa_j'}\mathbf{J}\cdot\mathbf{n}_j\, d\sigma. \tag{4.2}$$

Now, suppose that κ_j' is an interface dual face. Let $\kappa_j' = \kappa_j^1 \cup \kappa_j^2$ where κ_j^i is the part of κ_j' that lies in Ω_i. (see Figure 1)

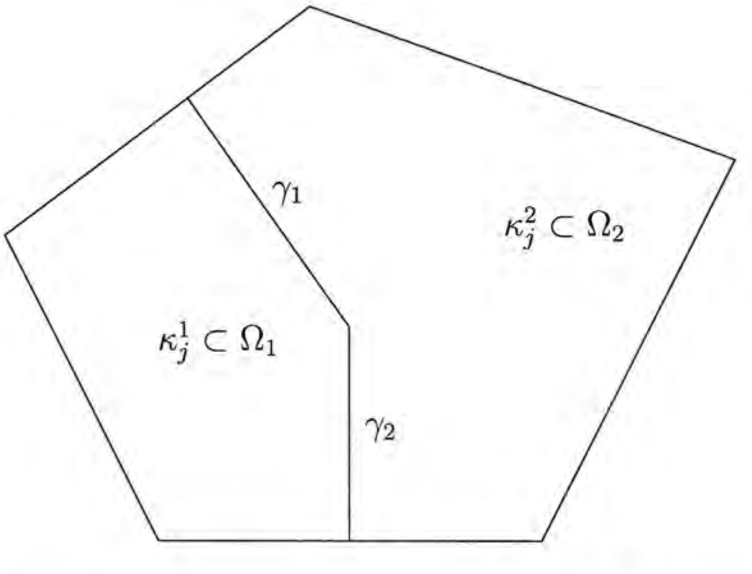

Figure 1: An interface dual face κ_j' with normal n_j

Similarly integrating both sides of (1.1) on κ_j',

$$\sum_{r=1}^{2} \frac{d}{dt} \int_{\kappa_j^r} \epsilon_r \mathbf{E} \cdot \mathbf{n}_j \, d\sigma - \sum_{r=1}^{2} \int_{\kappa_j^r} \mathbf{curl}\, \mathbf{H} \cdot \mathbf{n}_j \, d\sigma = \int_{\kappa_j'} \mathbf{J} \cdot \mathbf{n}_j \, d\sigma.$$

Applying Stokes' theorem and the relation $\mathbf{B} = \mu \mathbf{H}$, we have

$$\begin{aligned} &\frac{d}{dt} \int_{\kappa_j^1} \epsilon_1 \mathbf{E} \cdot \mathbf{n}_j \, d\sigma + \frac{d}{dt} \int_{\kappa_j^2} \epsilon_2 \mathbf{E} \cdot \mathbf{n}_j \, d\sigma \\ &\quad - \sum_{\sigma_i' \in \partial \kappa_j^1} \int_{\sigma_i'} \frac{1}{\mu_1} \mathbf{B} \cdot \mathbf{t}_i^1 \, dl - \sum_{\sigma_i' \in \partial \kappa_j^2} \int_{\sigma_i'} \frac{1}{\mu_2} \mathbf{B} \cdot \mathbf{t}_i^2 \, dl = \int_{\kappa_j'} \mathbf{J} \cdot \mathbf{n}_j \, d\sigma, \end{aligned} \tag{4.3}$$

where we remark that there are some edges in $\partial \kappa_j^r$, $r = 1, 2$, that belong to Γ and are not edges of our primal and dual meshes, in this case, γ_1 and γ_2 (see Figure 1). Furthermore, the directions $\mathbf{t}_i^1$ and $\mathbf{t}_i^2$ are defined by the right hand rule with respect to κ_j^1 and κ_j^2 respectively. From figure 1, we see that

$$\begin{aligned} \sum_{\sigma_i' \in \partial \kappa_j^1} \int_{\sigma_i'} \frac{1}{\mu_1} \mathbf{B} \cdot \mathbf{t}_i^1 \, dl = &\sum_{\sigma_i' \in \partial \kappa_j^1 \setminus \Gamma} \int_{\sigma_i'} \frac{1}{\mu_1} \mathbf{B} \cdot \mathbf{t}_i^1 \, dl \\ &+ \int_{\gamma_1} \frac{1}{\mu_1} \mathbf{B}_1 \cdot \mathbf{t}_i^1 \, dl + \int_{\gamma_2} \frac{1}{\mu_1} \mathbf{B}_1 \cdot \mathbf{t}_i^1 \, dl \end{aligned}$$

and

$$\begin{aligned} \sum_{\sigma_i' \in \partial \kappa_j^2} \int_{\sigma_i'} \frac{1}{\mu_2} \mathbf{B} \cdot \mathbf{t}_i^2 \, dl = &\sum_{\sigma_i' \in \partial \kappa_j^2 \setminus \Gamma} \int_{\sigma_i'} \frac{1}{\mu_2} \mathbf{B} \cdot \mathbf{t}_i^2 \, dl \\ &+ \int_{\gamma_1} \frac{1}{\mu_2} \mathbf{B}_2 \cdot \mathbf{t}_i^2 \, dl + \int_{\gamma_2} \frac{1}{\mu_2} \mathbf{B}_2 \cdot \mathbf{t}_i^2 \, dl \end{aligned}$$

where we recall that $\mathbf{B}_i = \mathbf{B}|_{\Omega_i}$ for $i = 1, 2$. Here, and in the sequel, we will use $\mathbf{B}$ without the subscript i if no confusion is caused. Notice that $\mathbf{t}_i^1$ and $\mathbf{t}_i^2$ are the same if σ_i' is an interface dual edge. When $\mathbf{t}_i^1$ and $\mathbf{t}_i^2$ represent directions on γ_i, $i = 1, 2$, they have opposite directions. Assume that the directions of γ_1 and γ_2 are the direction of $\mathbf{t}_i^2$. Summing up the two equations, the right hand side is given by

$$\begin{aligned} \sum_{r=1}^{2} \sum_{\sigma_i' \in \Omega_r} \int_{\sigma_i'} \frac{1}{\mu_r} \mathbf{B} \cdot \mathbf{t}_i \, dl + &\sum_{\sigma_i' \cap \Gamma \neq \phi} \Big(\int_{\sigma_i' \cap \Omega_1} \frac{1}{\mu_1} \mathbf{B}_1 \cdot \mathbf{t}_i \, dl + \int_{\sigma_i' \cap \Omega_2} \frac{1}{\mu_2} \mathbf{B}_2 \cdot \mathbf{t}_i \, dl \Big) \\ &+ \int_{\gamma_1} (\mathbf{H}_2 - \mathbf{H}_1) \cdot \mathbf{t}_1 \, dl + \int_{\gamma_2} (\mathbf{H}_2 - \mathbf{H}_1) \cdot \mathbf{t}_2 \, dl, \end{aligned}$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ are the directions of γ_1 and γ_2 respectively. On γ_1, we have

$$(\mathbf{H}_2 - \mathbf{H}_1) \cdot \mathbf{t}_1 = ((\mathbf{H}_2 - \mathbf{H}_1) \times \mathbf{m}_1) \cdot \mathbf{n}_j = \mathbf{J}_\Gamma \cdot \mathbf{n}_j. \tag{4.4}$$

Similarly, on γ_2,

$$(\mathbf{H}_2 - \mathbf{H}_1) \cdot \mathbf{t}_2 = ((\mathbf{H}_2 - \mathbf{H}_1) \times \mathbf{m}_2) \cdot \mathbf{n}_j = \mathbf{J}_\Gamma \cdot \mathbf{n}_j, \tag{4.5}$$

where $\mathbf{m}_1$ and $\mathbf{m}_2$ are the unit normal vector on the interface Γ at γ_1 and γ_2 respectively. Collecting these results, (4.3) becomes,

$$\begin{aligned} \frac{d}{dt}\int_{\kappa_j^1} \epsilon_1 \mathbf{E} \cdot \mathbf{n}_j \, d\sigma + \frac{d}{dt}\int_{\kappa_j^2} \epsilon_2 \mathbf{E} \cdot \mathbf{n}_j \, d\sigma - \sum_{r=1}^{2} \sum_{\sigma_i' \in \Omega_r} \int_{\sigma_i'} \frac{1}{\mu_r} \mathbf{B} \cdot \mathbf{t}_i \, dl \\ - \sum_{\sigma_i' \cap \Gamma \neq \phi} (\int_{\sigma_i^1} \frac{1}{\mu_1} \mathbf{B}_1 \cdot \mathbf{t}_i \, dl + \int_{\sigma_i^2} \frac{1}{\mu_2} \mathbf{B}_2 \cdot \mathbf{t}_i \, dl) \\ = \int_{\kappa_j'} \mathbf{J} \cdot \mathbf{n}_j \, d\sigma + \sum_{r=1}^{2} \int_{\gamma_r} \mathbf{J}_\Gamma \cdot \mathbf{n}_j \, dl. \end{aligned}$$

By the definition of face and edge averages for those faces and edges relating the interface, we obtain

$$\frac{d}{dt}((E_f')_j \bar{s}_j') - (C' B_e')_{\kappa_j'} = \int_{\kappa_j'} \mathbf{J} \cdot \mathbf{n}_j \, d\sigma. + \sum_{r=1}^{2} \int_{\gamma_r} \mathbf{J}_\Gamma \cdot \mathbf{n}_j \, dl \tag{4.6}$$

We remark that the other interface dual faces are handled similarly.

Now let $E \in \mathbb{R}^{M_1}$ and $B \in \mathbb{R}^{F_1}$ be the approximations of the primal edge and face averages of the true solution $\mathbf{E}$ and $\mathbf{B}$ to (1.1)-(1.4) respectively. (4.1) suggests the following approximation

$$s_j \frac{dB_j}{dt} + (CE)_j = 0. \tag{4.7}$$

We suppose that the values of the dual face average and the corresponding primal edge average are approximately equal as h tends to zero. We also suppose the same result holds for primal faces and dual edges. Then (4.2) suggests the following approximation

$$\bar{s}_j' \frac{dE_j}{dt} - (C'B)_j = \int_{\kappa_j'} \mathbf{J} \cdot \mathbf{n}_j \, d\sigma. \tag{4.8}$$

For components related to interface dual faces, we suppose that the face averages on κ_j^1 and κ_j^2 are nearly the same as the corresponding edge average on σ_j. Likewise, the edge averages on σ_j^1 and σ_j^2 are approximately the same as the face average on κ_j. Now (4.6) suggest the following approximation

$$\bar{s}_j' \frac{dE_j}{dt} - (C'B)_j = \int_{\kappa_j'} \mathbf{J} \cdot \mathbf{n}_j \, d\sigma + \sum_{r=1}^{2} \int_{\gamma_r} \mathbf{J}_\Gamma \cdot \mathbf{n}_j \, dl. \tag{4.9}$$

Hence we have the following semi-discrete approximation : Find $E \in \mathbb{R}^{M_1}$ and $B \in \mathbb{R}^{F_1}$ such that

$$S' \frac{dE}{dt} - C'B = \tilde{J} \tag{4.10}$$

$$S \frac{dB}{dt} + CE = 0 \tag{4.11}$$

where $\tilde{J} \in \mathbb{R}^{M_1}$ is defined as

$$\tilde{J}_j := \int_{\kappa_j'} \mathbf{J} \cdot \mathbf{n}_j \, d\sigma \tag{4.12}$$

for each non-interface dual face and

$$\tilde{J}_j := \int_{\kappa_j'} \mathbf{J} \cdot \mathbf{n}_j \, d\sigma + \sum_{r=1}^{2} \int_{\gamma_r} \mathbf{J}_\Gamma \cdot \mathbf{n}_j \, dl \tag{4.13}$$

for each interface dual face. We supplement the system (4.10)-(4.11) with the initial condition:

$$E(0) = E_e(0), \qquad B(0) = B_f(0), \tag{4.14}$$

where $E_e(0)$ and $B_f(0)$ are the primal edge average of $\mathbf{E}$ and primal face average of $\mathbf{B}$ at time $t = 0$.

Theorem 4.1 *The semi-discrete scheme (4.10)-(4.11) has a unique solution.*

Proof. The uniqueness follows from the fact that (4.10)-(4.11) is a system of linear ordinary differential equations with constant coefficients.

□

4.2 Consistency theory

In previous section, we have derived a semi-discrete approximation for (1.1)-(1.2). We are now in a position to present the consistency theory for our finite volume method.

Let us consider the continuous Maxwell's equations (1.1)-(1.4). Taking the divergence to both sides of (1.1) and (1.2), we have

$$\frac{\partial}{\partial t}\mathrm{div}(\epsilon\mathbf{E}) = \mathrm{div}\mathbf{J},$$
$$\frac{\partial}{\partial t}\mathrm{div}(\mu\mathbf{H}) = 0.$$

By the initial condition (1.7) and the continuity equation (1.9), we have

$$\mathrm{div}(\epsilon\mathbf{E}) = \rho,$$
$$\mathrm{div}(\mu\mathbf{H}) = 0,$$

for all $t \in (0, T)$. Hence, any solution (**E**,**B**) satisfying (1.1)-(1.2), with the continuity equation being hold for any time and the initial function ($\mathbf{E}_0$,$\mathbf{B}_0$) satisfies the divergence constraints (1.3)-(1.4), must satisfy the same divergence constraints (1.3)-(1.4).

It is clear that the semi-discrete approximation (4.10)-(4.11) is the discrete analog of the continuous Maxwell's equations (1.1)-(1.2). In order to ensure the finite volume solution to (4.10)-(4.11) represents the solution which also satisfies the divergence constraints, it is required to show that the finite volume solution (E, B) satisfies some discrete analog of the divergence constraints.

The following theorem shows how the finite volume solution B satisfies the divergence constraint in the discrete level.

Theorem 4.2 *Suppose B is the solution of the semi-discrete scheme (4.10)-(4.11). Then*

$$\mathcal{D}B = 0 \qquad \text{for any time } t \geq 0. \tag{4.15}$$

Proof. We observe that

$$\begin{aligned}\frac{d}{dt}(\mathcal{D}B) &= \frac{d}{dt}(B_1^T SB) && \text{by (3.18)}\\ &= -B_1^T CE && \text{by (4.11)}\\ &= 0 && \text{by (3.19)}.\end{aligned}$$

Also, by the initial condition $\text{div}(\mu \mathbf{H}_0) = 0$, we obtain for any primal element τ_i the following

$$\begin{aligned}\int_{\tau_i} \text{div}(\mu \mathbf{H}_0)\, dx = 0\\ \sum_{\kappa_j \in \partial\tau_i} \int_{\kappa_j} \mathbf{B}_0 \cdot \mathbf{n}_j\, d\sigma = 0\\ (\mathcal{D}B_f(0))_i = 0.\end{aligned}$$

Note that $B = B_f$ at time $t = 0$. Hence $\mathcal{D}B = 0$ for any time $t \geq 0$.

□

The next theorem will display how the finite volume solution E satisfies the divergence constraint in the discrete level.

Theorem 4.3 *Suppose E is the solution of the semi-discrete scheme (4.10)-(4.11). Then*

$$\mathcal{D}'E = \tilde{\rho} + \xi \qquad \text{for any time } t \geq 0, \tag{4.16}$$

where $\tilde{\rho}$ and ξ are vectors in $\mathbb{R}^L$ with

$$\tilde{\rho}_j := \int_{\tau_j'} \rho\, dx + \int_{\tau_j' \cap \Gamma} \rho_\Gamma\, d\sigma \quad \text{and} \quad \xi := \mathcal{D}'(E_e - E_f')(0). \tag{4.17}$$

Proof. We observe that

$$\begin{aligned}\frac{d}{dt}(\mathcal{D}'E) &= \frac{d}{dt}((B_1')^T S'E) && \text{by (3.18)}\\ &= (B_1')^T C'B + (B_1')^T \tilde{J}_1 + (B_1')^T \tilde{J}_2 && \text{by (4.10)}\\ &= (B_1')^T \tilde{J}_1 + (B_1')^T \tilde{J}_2 && \text{by (3.19)},\end{aligned}$$

where

$$(\tilde{J}_1)_i := \int_{\kappa_i'} \mathbf{J} \cdot \mathbf{n}_i \, d\sigma$$
$$(\tilde{J}_2)_i := \sum_{r=1}^{2} \int_{\gamma_r} \mathbf{J}_\Gamma \cdot \mathbf{n}_i \, dl.$$

We remark here that $(\tilde{J}_2)_i = 0$ for each non-interface dual face κ_i'. Also, $((B_1')^T \tilde{J}_2)_j = 0$ for any non-interface dual element τ_j'. Integrating both sides of the continuity equation (1.9) on a non-interface dual element τ_j', we obtain

$$\int_{\tau_j'} \frac{\partial \rho}{\partial t} \, dx = \int_{\tau_j'} \operatorname{div}\mathbf{J} \, dx,$$
$$= \sum_{\kappa_i' \in \partial \tau_j'} \int_{\kappa_i'} \mathbf{J} \cdot \mathbf{n}_i \, d\sigma,$$

where $\mathbf{n}_i$ is the unit outward normal vector of τ_j' on the boundary face κ_i'. By the definition of the matrix B_1', we have

$$\frac{d}{dt} \int_{\tau_j'} \rho \, dx = ((B_1')^T \tilde{J}_1)_j.$$

So, for each non-interface dual element τ_j', we obtain

$$\frac{d}{dt} (\mathcal{D}' E)_j = \frac{d}{dt} \int_{\tau_j'} \rho \, dx. \tag{4.18}$$

Similarly, integrating both sides of the continuity equation (1.9) on an interface dual element τ_j', we get

$$\int_{\tau_j'} \frac{\partial \rho}{\partial t} \, dx = \sum_{\kappa_i' \in \partial \tau_j'} \int_{\kappa_i'} \mathbf{J} \cdot \mathbf{n}_i \, d\sigma - \int_{\tau_j' \cap \Gamma} [\mathbf{J} \cdot \mathbf{m}] \, d\sigma.$$

From (1.1), we have

$$\frac{d}{dt} \int_{\tau_j' \cap \Gamma} [\epsilon \mathbf{E} \cdot \mathbf{m}] \, d\sigma - \int_{\tau_j' \cap \Gamma} [\mathbf{curl}\, \mathbf{H} \cdot \mathbf{m}] \, d\sigma = \int_{\tau_j' \cap \Gamma} [\mathbf{J} \cdot \mathbf{m}] \, d\sigma,$$
$$\frac{d}{dt} \int_{\tau_j' \cap \Gamma} \rho_\Gamma \, d\sigma - \int_{\tau_j' \cap \Gamma} [\mathbf{curl}\, \mathbf{H} \cdot \mathbf{m}] \, d\sigma = \int_{\tau_j' \cap \Gamma} [\mathbf{J} \cdot \mathbf{m}] \, d\sigma.$$

By the equations (4.4) and (4.5),

$$\int_{\tau_j'\cap\Gamma}[\mathbf{curl}\,\mathbf{H}\cdot\mathbf{m}]\,d\sigma = \sum_{\gamma_r'\in\partial(\tau_j'\cap\Gamma)}\int_{\gamma_r'}[\mathbf{H}\cdot\mathbf{t}_r]\,dl = \sum_{\gamma_r'\in\partial(\tau_j'\cap\Gamma)}\int_{\gamma_r'}\mathbf{J}_\Gamma\cdot\mathbf{n}_r\,dl.$$

By the definition of the matrix B_1', we note that

$$\sum_{\gamma_r'\in\partial(\tau_j'\cap\Gamma)}\int_{\gamma_r'}\mathbf{J}_\Gamma\cdot\mathbf{n}_r\,dl = ((B_1')^T\tilde{J}_2)_j.$$

Hence, for each interface dual element τ_j', we obtain

$$\frac{d}{dt}(\mathcal{D}'E)_j = \frac{d}{dt}\int_{\tau_j'}\rho\,dx + \frac{d}{dt}\int_{\tau_j'\cap\Gamma}\rho_\Gamma\,d\sigma. \tag{4.19}$$

Also, by the initial condition $\mathrm{div}(\epsilon\mathbf{E}_0) = \rho(0)$, we obtain for any dual element τ_j' the following

$$\int_{\tau_j'}\mathrm{div}(\epsilon\mathbf{E}_0)\,dx = \int_{\tau_j'}\rho(0)\,dx$$

$$\sum_{\kappa_i'\in\partial\tau_j'}\int_{\kappa_i'}\epsilon\mathbf{E}_0\cdot\mathbf{n}_i\,d\sigma = \int_{\tau_j'}\rho(0)\,dx + \int_{\tau_j'\cap\Gamma}\rho_\Gamma(0)\,d\sigma$$

$$(\mathcal{D}'E_f'(0))_j = \tilde{\rho}_j(0).$$

Integrating both sides of (4.18) and (4.19) on $[0,t]$,

$$\mathcal{D}'E(t) = \mathcal{D}'E(0) + \tilde{\rho}(t) - \tilde{\rho}(0),$$

and finally the following

$$\mathcal{D}'E(t) = \tilde{\rho}(t) + \mathcal{D}'(E_e - E_f')(0),$$

since $E = E_e$ at time $t = 0$.

□

4.3 Convergence theory

We devote this section to the convergence analysis of our semi-discrete finite volume method. We further divide this section into two parts. In the first part, we give a proof of the semi-discrete approximation for the case when both Ω and Ω_1 are polyhedral domains. We have shown that the approximation is first order convergent. In the second part, we consider a special case when both Ω and Ω_1 are rectangular domains. In this case, we can prove that the approximation is actually second order convergent.

4.3.1 Polyhedral domain

Before we go on with the convergence analysis on our finite volume method, we need the following technical lemma which is essential in the later analysis. In fact, it is the Bramble-Hilbert lemma we cited in Chapter 2. However, we have a sharper estimate on the constant $K(\Omega)$.

Lemma 4.1 *Let τ_i be a tetrahedral primal element. Suppose that f is a bounded linear functional on the space $W^{1,p}(\tau_i)$ such that $f(c) = 0$ for any constant function $c \in \mathbb{R}^1$. Then for any $v \in W^{1,p}(\tau_i)$,*

$$|f(v)| \leq Kh^{1-\frac{3}{p}}|v|_{W^{1,p}(\tau_i)} \tag{4.20}$$

holds for some generic constant K.

Proof. We prove this lemma by first considering the standard tetrahedral element $\hat{\tau}_i$ with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Clearly, there is an affine transformation $\hat{T}$ that maps τ_i onto $\hat{\tau}_i$. We denote by $\hat{v}$ the transformed function. Then, by the Bramble-Hilbert lemma, there is a constant K independent of τ_i such that

$$|f(\hat{v})| \leq K|\hat{v}|_{W^{1,p}(\hat{\tau}_i)}. \tag{4.21}$$

Define a 3×3 matrix A as follows:

$$A := \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix}.$$

Then A defines an affine transformation that maps $\hat{\tau}_i$ onto a tetrahedral element $\tilde{\tau}_i$ with vertices $(0,0,0)$, $(h,0,0)$, $(0,h,0)$ and $(0,0,h)$. Let $\tilde{v}$ be the corresponding transformed function. Also, we denote the coordinate systems in $\hat{\tau}_i$ and $\tilde{\tau}_i$ as $(\hat{x},\hat{y},\hat{z})$ and $(\tilde{x},\tilde{y},\tilde{z})$ respectively. Then, by the chain rule, we have

$$\frac{\partial \hat{v}}{\partial \hat{x}} = \frac{\partial \hat{v}}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \tilde{y}}\frac{\partial \tilde{x}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \tilde{z}}\frac{\partial \tilde{x}}{\partial \hat{z}} = h\frac{\partial \tilde{v}}{\partial \tilde{x}}.$$

Similarly, we have

$$\frac{\partial \hat{v}}{\partial \hat{y}} = h\frac{\partial \tilde{v}}{\partial \tilde{y}}, \qquad \frac{\partial \hat{v}}{\partial \hat{z}} = h\frac{\partial \tilde{v}}{\partial \tilde{z}}.$$

Therefore,

$$\begin{aligned} |\hat{v}|^p_{W^{1,p}(\hat{\tau}_i)} &= \int_{\hat{\tau}_i} |\frac{\partial \hat{v}}{\partial \hat{x}}|^p + |\frac{\partial \hat{v}}{\partial \hat{y}}|^p + |\frac{\partial \hat{v}}{\partial \hat{z}}|^p \, d\hat{x}d\hat{y}d\hat{z}, \\ &= h^p \int_{\tilde{\tau}_i} (|\frac{\partial \tilde{v}}{\partial \tilde{x}}|^p + |\frac{\partial \tilde{v}}{\partial \tilde{y}}|^p + |\frac{\partial \tilde{v}}{\partial \tilde{z}}|^p)\, |A|^{-1} \, d\tilde{x}d\tilde{y}d\tilde{z}, \\ &= h^{p-3}|\tilde{v}|^p_{W^{1,p}(\tilde{\tau}_i)}. \end{aligned}$$

Then, (4.21) becomes

$$|f(\tilde{v})| \leq Kh^{1-\frac{3}{p}}|\tilde{v}|_{W^{1,p}(\tilde{\tau}_i)}. \tag{4.22}$$

Now, we can find an affine transformation $\hat{Q} : \tilde{\tau}_i \to \tau_i$ independent of h such that $\hat{Q}A\hat{T} = I$, which is the identity transformation. By applying $\hat{Q}$ to (4.22) and the chain rule to the right hand side of (4.22), we have

$$|f(v)| \leq Kh^{1-\frac{3}{p}}|v|_{W^{1,p}(\tau_i)}.$$

□

We now proceed to develop the convergence theory for the semi-discrete approximation. To do so, substracting (4.7) from (4.1), we have

$$S\frac{d}{dt}(B - B_f) + C(E - E_e) = 0. \tag{4.23}$$

Similarly, subtracting (4.8) from (4.2) and (4.9) from (4.6) we obtain

$$S'\frac{d}{dt}(E - E'_f) - C'(B - B'_e) = 0. \tag{4.24}$$

By the boundary condition $\mathbf{E} \times \mathbf{n} = 0$ on $\partial\Omega$, we have

$$(E - E_e)|_{\partial\Omega} = 0. \tag{4.25}$$

Now, multiplying (4.23) by $D'(B - B'_e)$ and (4.24) by $D(E - E_e)$, then adding the two equations together, we have

$$\begin{aligned}&(S(\dot{B} - \dot{B}_f), D'(B - B_e)) + (S'(\dot{E} - \dot{E}'_f), D(E - E_e))\\ &= (C'(B - B'_e), D(E - E_e)) - (C(E - E_e), D'(B - B'_e)),\end{aligned}$$

where the dot represents derivative in time. By (4.25) and lemma 3.2,

$$(C'(B - B'_e), D(E - E_e)) - (C(E - E_e), D'(B - B'_e)) = 0,$$

and consequently

$$(\dot{B} - \dot{B}_f, B - B'_e)_W + (\dot{E} - \dot{E}'_f, E - E_e)_{W'} = 0. \tag{4.26}$$

Now, we rewrite (4.26) as

$$\begin{aligned}&(\dot{B} - \dot{B}'_e, B - B'_e)_W + (\dot{E} - \dot{E}_e, E - E_e)_{W'}\\ &= (\dot{E}'_f - \dot{E}_e, E - E_e)_{W'} + (\dot{B}_f - \dot{B}'_e, B - B'_e)_W.\end{aligned}$$

Applying integration by parts with respect to time in the above equation, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|B - B'_e\|_W^2 + \|E - E_e\|_{W'}^2)\\ &\qquad = (\dot{E}'_f - \dot{E}_e, E - E_e)_{W'} + (\dot{B}_f - \dot{B}'_e, B - B'_e)_W.\end{aligned} \tag{4.27}$$

The following theorem is devoted to show that our semi-discrete finite volume approximation of the Maxwell's equations is first order convergent.

Theorem 4.4 *Assume that* $(\mathbf{E},\mathbf{B}) \in (W^{1,1}(0,T;W^{1,p}(\Omega_i))^3)^2$, *for* $i = 1,2$, *satisfies (1.1)-(1.4) and* $p > 2$. *Let (E,B) be the solution of (4.10)-(4.11) on non-uniform grids with maximum grid size h. Then*

$$\begin{aligned}&\max_{0\le t\le T}(\|(E-E_e)(t)\|_{W'} + \|(B-B_f)(t)\|_W)\\ &\le Kh\sum_{i=1}^{2}(\|\mathbf{E}\|_{W^{1,1}(0,T;W^{1,p}(\Omega_i))^3} + \|\mathbf{B}\|_{W^{1,1}(0,T;W^{1,p}(\Omega_i))^3}).\end{aligned} \tag{4.28}$$

Proof. We prove this theorem by using (4.27). For each non-interface interior primal edge σ_i, we have

$$(\dot{E}'_f - \dot{E}_e)_i = \frac{1}{s'_i}\int_{\kappa'_i} \dot{\mathbf{E}}\cdot\mathbf{n}_i\, d\sigma - \frac{1}{h_i}\int_{\sigma_i}\dot{\mathbf{E}}\cdot\mathbf{n}_i\, dl,$$

where $\mathbf{n}_i$ is the unit normal vector to the dual face κ'_i. According to Sobolev embedding theorem, for $p > 2$, we have

$$W^{1,p}(\tau'_j\cup\tau'_l) \hookrightarrow L^1(\kappa'_i), \qquad W^{1,p}(\tau'_j\cup\tau'_l) \hookrightarrow L^1(\sigma_i),$$

where τ'_j and τ'_l are two dual element sharing the same dual face κ'_i. Hence, $(\dot{E}'_f - \dot{E}_e)_i$ is a bounded linear functional on $W^{1,p}(\tau'_j\cup\tau'_l)^3$ and vanishes for any constant functions. By Lemma 4.1,

$$|(\dot{E}'_f - \dot{E}_e)_i| \le Kh^{1-\frac{3}{p}}|\dot{\mathbf{E}}|_{W^{1,p}(\tau'_j\cup\tau'_l)^3},$$

for some generic constant K.

Now, for each interface primal edge σ_i, we have

$$\begin{aligned}(\dot{E}'_f - \dot{E}_e)_i &= ((\beta\dot{E}_{f_1} + (1-\beta)\dot{E}_{f_2}) - \dot{E}_e)_i\\ &= \beta(\dot{E}_{f_1} - \dot{E}_e)_i + (1-\beta)(\dot{E}_{f_2} - \dot{E}_e)_i.\end{aligned}$$

Notice that $(\dot{E}_{f_1} - \dot{E}_e)_i$ and $(\dot{E}_{f_2} - \dot{E}_e)_i$ are bounded linear functional on $W^{1,p}((\tau'_j\cup\tau'_l)\cap\Omega_1)^3$ and $W^{1,p}((\tau'_j\cup\tau'_l)\cap\Omega_2)^3$ respectively and both of them vanish for any constant functions. Lemma 4.1 then yields

$$\begin{aligned}|(\dot{E}_{f_1} - \dot{E}_e)_i| &\le Kh^{1-\frac{3}{p}}|\dot{\mathbf{E}}|_{W^{1,p}((\tau'_j\cup\tau'_l)\cap\Omega_1)^3},\\ |(\dot{E}_{f_2} - \dot{E}_e)_i| &\le Kh^{1-\frac{3}{p}}|\dot{\mathbf{E}}|_{W^{1,p}((\tau'_j\cup\tau'_l)\cap\Omega_2)^3}.\end{aligned}$$

Hence,

$$
\begin{aligned}
\|\dot{E}'_f - \dot{E}_e\|^2_{W'} &= \sum_{i=1}^{M_1} \bar{s}_i h_i |(\dot{E}'_f - \dot{E}_e)_i|^2, \\
&= \sum_{\sigma_i \cap \Gamma = \phi} \bar{s}_i h_i |(\dot{E}'_f - \dot{E}_e)_i|^2 + \sum_{\sigma_i \cap \Gamma \neq \phi} \bar{s}_i h_i |(\dot{E}'_f - \dot{E}_e)_i|^2, \\
&\leq K h^{5-\frac{6}{p}} \sum_{i=1}^{M_1} (|\dot{\mathbf{E}}|^2_{W^{1,p}((\tau'_j \cup \tau'_l) \cap \Omega_1)^3} + |\dot{\mathbf{E}}|^2_{W^{1,p}((\tau'_j \cup \tau'_l) \cap \Omega_2)^3}), \\
&\leq K h^{5-\frac{6}{p}} \sum_{r=1}^{2} (\sum_{i=1}^{M_1} |\dot{\mathbf{E}}|^p_{W^{1,p}((\tau'_j \cup \tau'_l) \cap \Omega_1)^3} + |\dot{\mathbf{E}}|^p_{W^{1,p}((\tau'_j \cup \tau'_l) \cap \Omega_2)^3})^{\frac{2}{p}} (\sum_{i=1}^{M_1} 1)^{1-\frac{2}{p}}.
\end{aligned}
$$

By the fact that

$$
h^3 \sum_{i=1}^{M_1} 1 \leq K,
$$

we conclude that

$$
\|\dot{E}'_f - \dot{E}_e\|_{W'} \leq K h \sum_{r=1}^{2} |\dot{\mathbf{E}}|_{W^{1,p}(\Omega_r)^3}. \tag{4.29}
$$

Similarly, we have

$$
\|\dot{B}_f - \dot{B}'_e\|_{W} \leq K h \sum_{r=1}^{2} |\dot{\mathbf{B}}|_{W^{1,p}(\Omega_r)^3}. \tag{4.30}
$$

Integrating both sides of (4.27) on the interval $(0, t)$ and by the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\|(B - B'_e)(t)\|^2_W + \|(E - E_e)(t)\|^2_{W'} \\
&\leq 2 \int_0^t (\|(B - B'_e)(s)\|_W \|(\dot{B}_f - \dot{B}'_e)(s)\|_W \\
&\qquad + \|(E - E_e)(s)\|_{W'} \|(\dot{E}'_f - \dot{E}_e)(s)\|_{W'})\, ds, \\
&\leq 2 \max_{0 \leq t \leq T} (\|(B - B'_e)(t)\|_W + \|(E - E_e)(t)\|_{W'}) \\
&\qquad \times \int_0^T \|(\dot{B}_f - \dot{B}'_e)(s)\|_W + \|(\dot{E}'_f - \dot{E}_e)(s)\|_{W'}\, ds.
\end{aligned}
$$

Then by (4.29) and (4.30), we have

$$\max_{0\le t\le T}(\|(E-E_e)(t)\|_{W'}+\|(B-B'_e)(t)\|_W)$$
$$\le Kh\sum_{i=1}^{2}(|\mathbf{E}|_{W^{1,1}(0,T;W^{1,p}(\Omega_i))^3}+|\mathbf{B}|_{W^{1,1}(0,T;W^{1,p}(\Omega_i))^3}).$$

In order to complete the proof, we first observe that

$$\|(B-B_f)(t)\|_W \le \|(B-B'_e)(t)\|_W+\|(B'_e-B_f)(t)\|_W.$$

So, it remains to estimate $\|(B'_e-B_f)(t)\|_W$. From (4.30), we know that

$$\|B_f-B'_e\|_W \le Kh\sum_{r=1}^{2}|\mathbf{B}|_{W^{1,p}(\Omega_r)^3}.$$

Hence,

$$\max_{0\le t\le T}\|(B_f-B'_e)(t)\|_W \le Kh\sum_{r=1}^{2}\max_{0\le t\le T}|\mathbf{B}(t)|_{W^{1,p}(\Omega_r)^3},$$
$$\le Kh\sum_{r=1}^{2}\|\mathbf{B}\|_{W^{1,1}(0,T;W^{1,p}(\Omega_i))^3},$$

where the last step follows from Sobolev embedding theorem.

□

4.3.2 Rectangular domain

We now give the convergence analysis on our semi-discrete finite volume approximation of the Maxwell's equations when both Ω and Ω_1 are rectangular domains. It is clear that all the derivations we developed for the finite volume scheme with the polydedral domain can be carried over to the rectangular domain case.

First, we need the following technical lemma which is a sharp estimate of the Bramble-Hilbert lemma.

Lemma 4.2 *Let τ_i be a cubic primal element. Suppose that f is a bounded linear functional on the space $W^{2,p}(\tau_i)$ such that $f(c) = 0$ for any linear function $c \in P_1(\tau_i)$. Then for any $v \in W^{2,p}(\tau_i)$,*

$$|f(v)| \leq Kh^{2-\frac{3}{p}}|v|_{W^{2,p}(\tau_i)} \tag{4.31}$$

holds for some generic constant K. Moreover, if f vanishes at all quadratic functions, then

$$|f(v)| \leq Kh^{3-\frac{3}{p}}|v|_{W^{3,p}(\tau_i)}. \tag{4.32}$$

Proof. We prove this lemma by first considering the standard cubic element $\hat{\tau}_i = [0,1]^3$. Clearly, there is an affine transformation $\hat{T}$ that maps τ_i onto $\hat{\tau}_i$. We let $\hat{v}$ be the transformed function. Then, by the Bramble-Hilbert lemma, there is a constant K independent of τ_i such that

$$|f(\hat{v})| \leq K|\hat{v}|_{W^{2,p}(\hat{\tau}_i)}. \tag{4.33}$$

Define a 3×3 matrix A as follows:

$$A := \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix}.$$

Then A defines an affine transformation that maps $\hat{\tau}_i$ onto a cubic element $\tilde{\tau}_i = [0,h]^3$. Let $\tilde{v}$ be the corresponding transformed function. Also, we denote the coordinate systems in $\hat{\tau}_i$ and $\tilde{\tau}_i$ as $(\hat{x}, \hat{y}, \hat{z})$ and $(\tilde{x}, \tilde{y}, \tilde{z})$ respectively. Then, by the chain rule, we have

$$\frac{\partial \hat{v}}{\partial \hat{x}} = \frac{\partial \hat{v}}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \tilde{y}}\frac{\partial \tilde{x}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \tilde{z}}\frac{\partial \tilde{x}}{\partial \hat{z}} = h\frac{\partial \tilde{v}}{\partial \tilde{x}}.$$

Applying the chain rule again, we have

$$\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} = h^2 \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}.$$

Similarly, we have

$$\frac{\partial^2 \hat{v}}{\partial \hat{y}^2} = h^2 \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}, \qquad \frac{\partial^2 \hat{v}}{\partial \hat{z}^2} = h^2 \frac{\partial^2 \tilde{v}}{\partial \tilde{z}^2}.$$

Therefore,

$$\begin{aligned}
|\hat{v}|^p_{W^{2,p}(\hat{\tau}_i)} &= \int_{\hat{\tau}_i} |\frac{\partial^2 \hat{v}}{\partial \hat{x}^2}|^p + |\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}|^p + |\frac{\partial^2 \hat{v}}{\partial \hat{z}^2}|^p \, d\hat{x} d\hat{y} d\hat{z}, \\
&= h^{2p} \int_{\tilde{\tau}_i} (|\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}|^p + |\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}|^p + |\frac{\partial^2 \tilde{v}}{\partial \tilde{z}^2}|^p) \, |A|^{-1} \, d\tilde{x} d\tilde{y} d\tilde{z}, \\
&= h^{2p-3} |\tilde{v}|^p_{W^{2,p}(\tilde{\tau}_i)}.
\end{aligned}$$

Then, (4.33) becomes

$$|f(\tilde{v})| \leq K h^{2-\frac{3}{p}} |\tilde{v}|_{W^{1,p}(\tilde{\tau}_i)}. \tag{4.34}$$

Now, we can find an affine transformation $\hat{Q} : \tilde{\tau}_i \to \tau_i$ independent of h such that $\hat{Q} A \hat{T} = I$, which is the identity transformation. By applying $\hat{Q}$ to (4.34) and the chain rule to the right hand side of (4.34), we have

$$|f(v)| \leq K h^{2-\frac{3}{p}} |v|_{W^{2,p}(\tau_i)}.$$

(4.32) can be proved in a similar way.

□

We are now in a position to establish our convergence theory. Differentiating both sides of (4.10) with respect to t, we get

$$S' \frac{d^2 E}{dt^2} - C' \frac{dB}{dt} = \frac{d\tilde{J}}{dt},$$

and by (4.11),

$$S' \frac{d^2 E}{dt^2} + C' S^{-1} C E = \frac{d\tilde{J}}{dt}. \tag{4.35}$$

Rewrite (4.35) into the following form

$$S' \frac{d^2}{dt^2}(E - E_e) + C' S^{-1} C (E - E_e) = \frac{d\tilde{J}}{dt} - S' \frac{d^2 E_e}{dt^2} - C' S^{-1} C E_e,$$

and by (4.1), we then derive

$$S'\frac{d^2}{dt^2}(E-E_e)+C'S^{-1}C(E-E_e)=\frac{d}{dt}(\tilde{J}-S'\frac{dE_e}{dt}+C'B_f). \tag{4.36}$$

Namely, $E-E_e$ satisfies the second order ordinary differential equation (4.36) with the following initial condition

$$(E-E_e)(0)=0. \tag{4.37}$$

Multiplying both sides of (4.36) by $D(\dot{E}-\dot{E}_e)$, we obtain

$$(S'(\ddot{E}-\ddot{E}_e),D(\dot{E}-\dot{E}_e))+(C'S^{-1}C(E-E_e),D(\dot{E}-\dot{E}_e))=(\frac{df}{dt},D(\dot{E}-\dot{E}_e)),$$

where

$$f:=\tilde{J}-S'\frac{dE_e}{dt}+C'B_f.$$

By (3.13),

$$(S'(\ddot{E}-\ddot{E}_e),D(\dot{E}-\dot{E}_e))+(D'S^{-1}C(E-E_e),C(\dot{E}-\dot{E}_e))=(\frac{df}{dt},D(\dot{E}-\dot{E}_e)).$$

By Integration by parts with respect to t, we have

$$\frac{1}{2}\frac{d}{dt}\|\dot{E}-\dot{E}_e\|_{W'}^2+\frac{1}{2}\frac{d}{dt}\|E-E_e\|_V^2=(\frac{df}{dt},D(\dot{E}-\dot{E}_e)). \tag{4.38}$$

Now, we will use Lemmata 4.3-4.5 to establish our convergence theory for the semi-discrete approximation. First, for any primal face κ_i, without loss of generality, we assume

$$\kappa_i=\{(x,y,z):x=x_i,y_i\le y\le y_{i+1},z_i\le z\le z_{i+1}\}.$$

Let τ_j and τ_k be two primal elements sharing the face κ_i and let C_j and C_k, with $C_j<C_k$, be the center of τ_j and τ_k respectively. Then define

$$\Lambda_i:=\{(x,y,z):C_j\le x\le C_k,y_i\le y\le y_{i+1},z_i\le z\le z_{i+1}\}.$$

Now, let $\Pi_h\mathbf{B}$ be the standard piecewise linear element interpolation of the function $\mathbf{B}$. That is, for all x in Λ_i, we have

$$\Pi_h\mathbf{B}(x) = \sum_{s=1}^{8} \mathbf{B}(\nu_{l_s})\phi_{l_s}(x),$$

where ν_{l_s} denotes the nodal points of Λ_i and $\phi_{l_s}(x)$ is a linear function satisfying

$$\phi_{l_s}(\nu_{l_r}) = 0, \qquad \text{for } r \neq s$$

$$\phi_{l_s}(\nu_{l_r}) = 1, \qquad \text{for } r = s.$$

We remark here that for an interface primal face κ_i, the corresponding Λ_i has two parts, one part in Ω_1 and the other in Ω_2. Let $\Lambda_i^r = \Lambda_i \cap \Omega_r$, for $r = 1, 2$, which is a cuboid. Then we define $\Pi_h\mathbf{B}$ in each of the two parts in a similar fashion. Clearly, $\Pi_h\mathbf{B}$ is a tri-linear function in each Λ_i or Λ_i^r, for $r = 1, 2$.

Then we have

Lemma 4.3 *f can be written into the following form:*

$$f = \bar{J} + C'(B_f - \Pi_h B_f) - \bar{H} + \bar{J}_\Gamma + g, \tag{4.39}$$

where

$$\bar{J}_j := s_j'(J_f' - J_e)_j, \quad (\bar{J}_\Gamma)_j := \sum_{r=1}^{2} \int_{\gamma_r} (\mathbf{J}_\Gamma - \Pi_h \mathbf{J}_\Gamma) \cdot \mathbf{n}\, dl. \tag{4.40}$$

For $\sigma_j \cap \Gamma = \phi$,

$$\bar{H}_j := s_j' \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, (\Pi_h \mathbf{H} - \mathbf{H}) \cdot \mathbf{n}\, dl \tag{4.41}$$

and for $\sigma_j \cap \Gamma \neq \phi$,

$$\bar{H}_j := \sum_{k=1}^{2} s_j^k \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, (\Pi_h \mathbf{H}_k - \mathbf{H}_k) \cdot \mathbf{n}\, dl. \tag{4.42}$$

In addition, $\bar{J}_\Gamma$ is a vector having components corresponding to all interface primal edges and g is a vector having components corresponding to the interface primal edges which lie on edges of Ω_1.

Proof. We divide the proof into three parts.

(i) For any non-interface primal edge σ_j, from (1.1), we get

$$\frac{d}{dt}\int_{\sigma_j} \epsilon \mathbf{E}\cdot\mathbf{n}\,dl - \int_{\sigma_j} \mathbf{curl}\,\mathbf{H}\cdot\mathbf{n}\,dl = \int_{\sigma_j} \mathbf{J}\cdot\mathbf{n}\,dl.$$

Dividing both sides by h_j gives

$$\epsilon\frac{dE_e}{dt} - \frac{1}{h_j}\int_{\sigma_j} \mathbf{curl}\,\mathbf{H}\cdot\mathbf{n}\,dl = \frac{1}{h_j}\int_{\sigma_j} \mathbf{J}\cdot\mathbf{n}\,dl.$$

Hence,

$$\begin{aligned} f_j &= \int_{\kappa'_j} \mathbf{J}\cdot\mathbf{n}\,d\sigma - s'_j\frac{1}{h_j}\int_{\sigma_j}\mathbf{J}\cdot\mathbf{n}\,dl + (C'B_f)_j - s'_j\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\mathbf{H}\cdot\mathbf{n}\,dl \\ &= s'_j(J'_f - J_e)_j + (C'B_f)_j - s'_j\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\mathbf{H}\cdot\mathbf{n}\,dl, \end{aligned}$$

where

$$(J'_f)_j := \frac{1}{s'_j}\int_{\kappa'_j}\mathbf{J}\cdot\mathbf{n}\,d\sigma \quad \text{and} \quad (J_e)_j := \frac{1}{h_j}\int_{\sigma_j}\mathbf{J}\cdot\mathbf{n}\,dl.$$

We further write f_j into the following form

$$\begin{aligned} f_j &= s'_j(J'_f - J_e)_j + (C'(B_f - \Pi_h B_f))_j + (C'\Pi_h B_f)_j \\ &\quad - s'_j\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\Pi_h\mathbf{H}\cdot\mathbf{n}\,dl + s'_j\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,(\Pi_h\mathbf{H} - \mathbf{H})\cdot\mathbf{n}\,dl. \end{aligned}$$

To calculate the term

$$(C'\Pi_h B_f)_j - s'_j\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\Pi_h\mathbf{H}\cdot\mathbf{n}\,dl,$$

we consider the following figure:

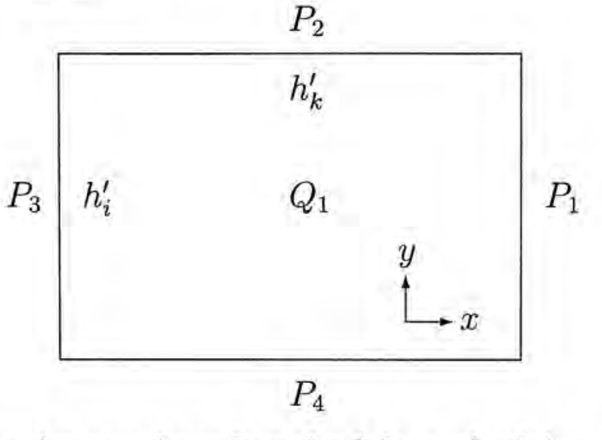

Figure 2: An non-interface dual face κ'_j with normal $\mathbf{n}$

Since $\Pi_h\mathbf{B}$ is a linear function, we have

$$\begin{aligned}
&(C'\Pi_h B_f)_j - s'_j \frac{1}{h_j}\int_{\sigma_j} \mathbf{curl}\,\Pi_h\mathbf{H}\cdot\mathbf{n}\,dl \\
&= h'_k\Pi_h\mathbf{H}_x(P_4) - h'_k\Pi_h\mathbf{H}_x(P_2) + h'_i\Pi_h\mathbf{H}_y(P_1) - h'_i\Pi_h\mathbf{H}_y(P_3) \\
&\quad - s'_j\mathbf{curl}\,\Pi_h\mathbf{H}\cdot\mathbf{n}(Q_1) \\
&= h'_k\Pi_h\mathbf{H}_x(P_4) - h'_k\Pi_h\mathbf{H}_x(P_2) + h'_i\Pi_h\mathbf{H}_y(P_1) - h'_i\Pi_h\mathbf{H}_y(P_3) \\
&\quad - s'_j(\frac{\partial\Pi_h\mathbf{H}_y}{\partial x}(Q_1) - \frac{\partial\Pi_h\mathbf{H}_x}{\partial y}(Q_1)) \\
&= 0,
\end{aligned}$$

where, in the above calculation, we have assumed, without loss of generality, that the normal direction $\mathbf{n}$ is the same as the z-axis direction. Hence, we have

$$f_j = s'_j(J'_f - J_e)_j + (C'(B_f - \Pi_h B_f))_j + s'_j\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,(\Pi_h\mathbf{H} - \mathbf{H})\cdot\mathbf{n}\,dl.$$

(ii) For each interface primal edge σ_j not lying on the edges of Ω_1, we have from (1.1) the following two equations

$$\begin{aligned}
\frac{d}{dt}\int_{\sigma_j}\epsilon_1\mathbf{E}_1\cdot\mathbf{n}\,dl - \int_{\sigma_j}\mathbf{curl}\,\mathbf{H}_1\cdot\mathbf{n}\,dl &= \int_{\sigma_j}\mathbf{J}\cdot\mathbf{n}\,dl, \\
\frac{d}{dt}\int_{\sigma_j}\epsilon_2\mathbf{E}_2\cdot\mathbf{n}\,dl - \int_{\sigma_j}\mathbf{curl}\,\mathbf{H}_2\cdot\mathbf{n}\,dl &= \int_{\sigma_j}\mathbf{J}\cdot\mathbf{n}\,dl.
\end{aligned}$$

Dividing both sides by h_j and by the interface condition (1.10), we get

$$\begin{aligned}
\epsilon_1\frac{dE_e}{dt} - \frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\mathbf{H}_1\cdot\mathbf{n}\,dl &= \frac{1}{h_j}\int_{\sigma_j}\mathbf{J}\cdot\mathbf{n}\,dl, \\
\epsilon_2\frac{dE_e}{dt} - \frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\mathbf{H}_2\cdot\mathbf{n}\,dl &= \frac{1}{h_j}\int_{\sigma_j}\mathbf{J}\cdot\mathbf{n}\,dl.
\end{aligned}$$

Multiplying s_j^1 to the first equation and s_j^2 to the second equation and adding the resulting two equations together, we obtain

$$\bar{s}'_j\frac{dE_e}{dt} - \sum_{k=1}^{2} s_j^k\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\mathbf{H}_k\cdot\mathbf{n}\,dl = s'_j\frac{1}{h_j}\int_{\sigma_j}\mathbf{J}\cdot\mathbf{n}\,dl.$$

So,

$$f_j = s'_j(J'_f - J_e)_j + \sum_{r=1}^{2} \int_{\gamma_r} \mathbf{J}_\Gamma \cdot \mathbf{n}\, dl + (C'B_f)_j - \sum_{k=1}^{2} s_j^k \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, \mathbf{H}_k \cdot \mathbf{n}\, dl.$$

Since σ_j does not lie on the edges of Ω_1, γ_1 and γ_2 combine and form only one line, which is denoted by γ_j. Hence,

$$f_j = s'_j(J'_f - J_e)_j + \int_{\gamma_j} \mathbf{J}_\Gamma \cdot \mathbf{n}\, dl + (C'B_f)_j - \sum_{k=1}^{2} s_j^k \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, \mathbf{H}_k \cdot \mathbf{n}\, dl.$$

We further write f_j into the following form

$$\begin{aligned} f_j = {} & s'_j(J'_f - J_e)_j + \int_{\gamma_j} \Pi_h \mathbf{J}_\Gamma \cdot \mathbf{n}\, dl + (C'\Pi_h B_f)_j - \sum_{k=1}^{2} s_j^k \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, \Pi_h \mathbf{H}_k \cdot \mathbf{n}\, dl \\ & + \int_{\gamma_j} (\mathbf{J}_\Gamma - \Pi_h \mathbf{J}_\Gamma) \cdot \mathbf{n}\, dl + (C'(B_f - \Pi_h B_f))_j \\ & + \sum_{k=1}^{2} s_j^k \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, (\Pi_h \mathbf{H}_k - \mathbf{H}_k) \cdot \mathbf{n}\, dl. \end{aligned}$$

To calculate the term

$$\int_{\gamma_j} \Pi_h \mathbf{J}_\Gamma \cdot \mathbf{n}\, dl + (C'\Pi_h B_f)_j - \sum_{k=1}^{2} s_j^k \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, \Pi_h \mathbf{H}_k \cdot \mathbf{n}\, dl,$$

we consider the following figure:

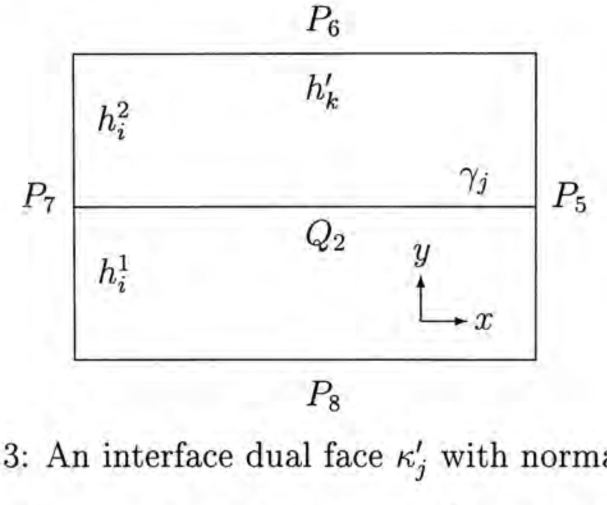

Figure 3: An interface dual face κ'_j with normal $\mathbf{n}$

In the figure above, the lower rectangle represents κ_j^1 while the upper rectangle represents κ_j^2. Notice that, since we are considering a uniform mesh, Q_2 is the mid-point of γ_j. Since $\Pi_h\mathbf{B}_k$, for $k=1,2$, is a linear function, we have

$$\int_{\gamma_j} \Pi_h\mathbf{J}_\Gamma\cdot\mathbf{n}\,dl = h'_k(\Pi_h\mathbf{H}_{2x}-\Pi_h\mathbf{H}_{1x})(Q_2),$$

$$\begin{aligned}(C'\Pi_hB_f)_j &= h'_k\Pi_h\mathbf{H}_{1x}(P_8) - h'_k\Pi_h\mathbf{H}_{2x}(P_6) + h_i^1\Pi_h\mathbf{H}_{1y}(P_5) + h_i^2\Pi_h\mathbf{H}_{2y}(P_5)\\ &\quad - h_i^1\Pi_h\mathbf{H}_{1y}(P_7) - h_i^2\Pi_h\mathbf{H}_{2y}(P_7),\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=1}^{2} s_j^k\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\Pi_h\mathbf{H}_k\cdot\mathbf{n}\,dl\\ &= s_j^1\mathbf{curl}\,\Pi_h\mathbf{H}_1\cdot\mathbf{n}(Q_2) + s_j^2\mathbf{curl}\,\Pi_h\mathbf{H}_k\cdot\mathbf{n}(Q_2)\\ &= s_j^1\left(\frac{\partial\Pi_h\mathbf{H}_{1y}}{\partial x}-\frac{\partial\Pi_h\mathbf{H}_{1x}}{\partial y}\right)(Q_2) + s_j^2\left(\frac{\partial\Pi_h\mathbf{H}_{2y}}{\partial x}-\frac{\partial\Pi_h\mathbf{H}_{2x}}{\partial y}\right)(Q_2)\end{aligned}$$

Collecting all terms, we have

$$\int_{\gamma_j}\Pi_h\mathbf{J}_\Gamma\cdot\mathbf{n}\,dl + (C'\Pi_hB_f)_j - \sum_{k=1}^{2}s_j^k\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\Pi_h\mathbf{H}_k\cdot\mathbf{n}\,dl = 0.$$

Hence, we obtain

$$\begin{aligned}f_j &= s'_j(J'_f-J_e)_j + \int_{\gamma_j}(\mathbf{J}_\Gamma-\Pi_h\mathbf{J}_\Gamma)\cdot\mathbf{n}\,dl + (C'(B_f-\Pi_hB_f))_j\\ &\quad+\sum_{k=1}^{2}s_j^k\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,(\Pi_h\mathbf{H}_k-\mathbf{H}_k)\cdot\mathbf{n}\,dl.\end{aligned}$$

(iii) For each interface primal edge σ_j lying on the edges of Ω_1, following the proof similiar to (ii), we have the following

$$\begin{aligned}f_j &= s'_j(J'_f-J_e)_j + \sum_{r=1}^{2}\int_{\gamma_r}\Pi_h\mathbf{J}_\Gamma\cdot\mathbf{n}\,dl + (C'\Pi_hB_f)_j - \sum_{k=1}^{2}s_j^k\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\Pi_h\mathbf{H}_k\cdot\mathbf{n}\,dl\\ &\quad+\sum_{r=1}^{2}\int_{\gamma_2}(\mathbf{J}_\Gamma-\Pi_h\mathbf{J}_\Gamma)\cdot\mathbf{n}\,dl + (C'(B_f-\Pi_hB_f))_j\\ &\quad+\sum_{k=1}^{2}s_j^k\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,(\Pi_h\mathbf{H}_k-\mathbf{H}_k)\cdot\mathbf{n}\,dl.\end{aligned}$$

To calculate the term

$$\sum_{r=1}^{2}\int_{\gamma_r}\Pi_h\mathbf{J}_\Gamma\cdot\mathbf{n}\,dl + (C'\Pi_hB_f)_j - \sum_{k=1}^{2}s_j^k\frac{1}{h_j}\int_{\sigma_j}\mathbf{curl}\,\Pi_h\mathbf{H}_k\cdot\mathbf{n}\,dl,$$

we consider the following figure:

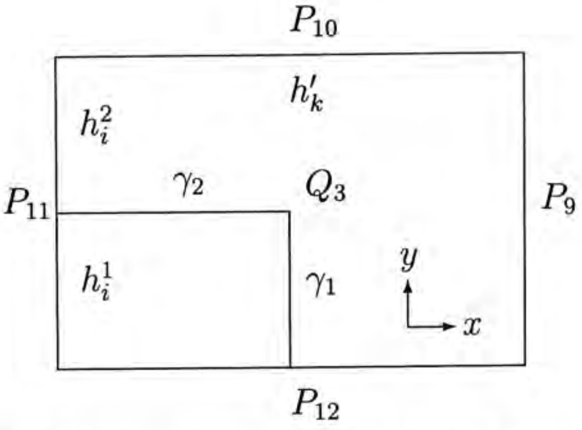

Figure 4: An interface dual face κ'_j with normal $\mathbf{n}$

In the figure above, the smaller part represents κ_j^1 and the remaining part represents κ_j^2. Since $\Pi_h \mathbf{B}_k$, for $k = 1, 2$, is a linear function, we have

$$\sum_{r=1}^{2} \int_{\gamma_r} \Pi_h \mathbf{J}_\Gamma \cdot \mathbf{n}\, dl = -h_i^1(\Pi_h \mathbf{H}_{2y} - \Pi_h \mathbf{H}_{1y})(R_1) + h_k^1(\Pi_h \mathbf{H}_{2x} - \Pi_h \mathbf{H}_{1x})(R_2),$$

and

$$\begin{aligned}(C'\Pi_h B_f)_j &= h_k^1 \Pi_h \mathbf{H}_{1x}(P_{12}) + h_k^2 \Pi_h \mathbf{H}_{1x}(P_{12}) - h'_k \Pi_h \mathbf{H}_{2x}(P_{10}) + h'_i \Pi_h \mathbf{H}_{1y}(P_9) \\ &\quad - h_i^1 \Pi_h \mathbf{H}_{1y}(P_{11}) - h_i^2 \Pi_h \mathbf{H}_{2y}(P_{11}),\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=1}^{2} s_j^k \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, \Pi_h \mathbf{H}_k \cdot \mathbf{n}\, dl \\ &= s_j^1 \mathbf{curl}\, \Pi_h \mathbf{H}_1 \cdot \mathbf{n}(Q_3) + s_j^2 \mathbf{curl}\, \Pi_h \mathbf{H}_k \cdot \mathbf{n}(Q_3) \\ &= s_j^1 (\frac{\partial \Pi_h \mathbf{H}_{1y}}{\partial x} - \frac{\partial \Pi_h \mathbf{H}_{1x}}{\partial y})(Q_3) + s_j^2 (\frac{\partial \Pi_h \mathbf{H}_{2y}}{\partial x} - \frac{\partial \Pi_h \mathbf{H}_{2x}}{\partial y})(Q_3)\end{aligned}$$

where R_1 and R_2 denote the mid-points of γ_1 and γ_2 respectively. Furthermore, we have

$$\sum_{r=1}^{2} \int_{\gamma_r} \Pi_h \mathbf{J}_\Gamma \cdot \mathbf{n}\, dl = -h_i^1(\Pi_h \mathbf{H}_{2y} - \Pi_h \mathbf{H}_{1y})(Q_3) + h_k^1(\Pi_h \mathbf{H}_{2x} - \Pi_h \mathbf{H}_{1x})(Q_3) + g_j,$$

where

$$g_j := h_i^1(\Pi_h \mathbf{H}_{2y} - \Pi_h \mathbf{H}_{1y})(Q_3) - h_k^1(\Pi_h \mathbf{H}_{2x} - \Pi_h \mathbf{H}_{1x})(Q_3)$$
$$- h_i^1(\Pi_h \mathbf{H}_{2y} - \Pi_h \mathbf{H}_{1y})(R_1) + h_k^1(\Pi_h \mathbf{H}_{2x} - \Pi_h \mathbf{H}_{1x})(R_2).$$

By the definition of $\mathbf{J}_\Gamma$, we have

$$\begin{aligned} g_j &= h_i^1(\Pi_h \mathbf{J}_\Gamma \cdot \mathbf{n}(Q_3) - \Pi_h \mathbf{J}_\Gamma \cdot \mathbf{n}(R_1)) \\ &\quad + h_k^1(\Pi_h \mathbf{J}_\Gamma \cdot \mathbf{n}(R_2) - \Pi_h \mathbf{J}_\Gamma \cdot \mathbf{n}(Q_3)), \end{aligned} \tag{4.43}$$

where we remark that $\mathbf{n}$ is the normal vector of the dual face κ'_j. Collecting all the terms, we obtain

$$\sum_{r=1}^{2} \int_{\gamma_r} \Pi_h \mathbf{J}_\Gamma \cdot \mathbf{n}\, dl + (C' \Pi_h B_f)_j - \sum_{k=1}^{2} s_j^k \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, \Pi_h \mathbf{H}_k \cdot \mathbf{n}\, dl = g_j,$$

and finally the following

$$f_j = s'_j(J'_f - J_e)_j + \sum_{r=1}^{2} \int_{\gamma_r} (\mathbf{J}_\Gamma - \Pi_h \mathbf{J}_\Gamma) \cdot \mathbf{n}\, dl + (C'(B_f - \Pi_h B_f))_j$$
$$+ \sum_{k=1}^{2} s_j^k \frac{1}{h_j} \int_{\sigma_j} \mathbf{curl}\, (\Pi_h \mathbf{H}_k - \mathbf{H}_k) \cdot \mathbf{n}\, dl + g_j.$$

Summarizing the results obtained in (i)-(iii), Lemma 4.3 follows.

□

Now, we give some estimates for $E - E_e$. We will consider the two cases: $C(E - E_e)(t) \neq 0$ for a.e. $t \in [0, T]$; and $C(E - E_e)(t) = 0$ for $t \in (t_1, t_2)$ with $(t_1, t_2) \subset [0, T]$. First we show

Lemma 4.4 *Assume that $C(E - E_e) \neq 0$ for a.e. $t \in [0, T]$. Then there exist a constant K independent of h such that*

$$\|E - E_e\|_{W_\Gamma} \leq K \|E - E_e\|_V. \tag{4.44}$$

Proof. By (3.15) in Lemma 3.5, we have for any $u \in \mathbb{R}^M$ with $u|_{\partial\Omega} = 0$,

$$(S'u, Du) \leq K(D'S^{-1}Cu, Cu). \tag{4.45}$$

Consider the following auxillary problem: Find $\tilde{u} \in \mathbb{R}^M$ such that

$$\begin{cases} C\tilde{u} = -S(\dot{B} - \dot{B}_f), & \text{for all interior primal face.} \\ \tilde{u} = E - E_e, & \text{for all interface primal edge.} \end{cases} \tag{4.46}$$

Clearly, by (4.23), the problem (4.46) has a solution $\tilde{u} = E - E_e$. Now, we solve the problem (4.46) in the following way. For each $\tilde{u}_j$ corresponding to an primal edge σ_j in Ω_2, we take $\tilde{u}_j = (E - E_e)_j$, i.e., the component of $E - E_e$ corresponding to σ_j. Then, with the components corresponding to Ω_2 and Γ fixed, we rewrite (4.46) into the following linear system

$$G_1 D\tilde{u} = b, \tag{4.47}$$

where b is a vector containing all the related known components and G_1 is the restriction of G to Ω_1. We remark here that in the system (4.47), the number of equations is in general greater than the number of unknowns. However, since (4.46) has a solution, the system (4.47) is consistent.

Since the matrix G_1 has the same structure as the matrix G, by Lemma 3.3, there are $O(N^3)$ free variables in the system (4.47). We choose these free variables to be all equal to some interface components with the condition that each component appears $O(N)$ times. We can do this since there are $O(N^2)$ interface components. Then, after fixing free variables, the other components can be uniquely determined by solving the system (4.47).

Putting $\tilde{u}$ into the equation (4.45), we have

$$(S'\tilde{u}, D\tilde{u}) \leq K(D'S^{-1}C\tilde{u}, C\tilde{u}). \tag{4.48}$$

For the left hand side, we have

$$(S'\tilde{u}, D\tilde{u}) \geq (S'\bar{u}, D\bar{u}),$$

where $\bar{u}$ denotes a vector having the same interface components and free components as $\tilde{u}$ and having the remaining components vanish. So, we have

$$(S'\bar{u}, D\bar{u}) \geq K\|E - E_e\|_{W_\Gamma}^2.$$

For the right hand side, since $\tilde{u}$ is the solution to the system (4.47), we have

$$(D'S^{-1}C\tilde{u}, C\tilde{u}) = (D'(\dot{B} - \dot{B}_f), S(\dot{B} - \dot{B}_f)).$$

Multiplying both sides of (4.23) by $D'(\dot{B} - \dot{B}_f)$, we have

$$\begin{aligned} &(S(\dot{B} - \dot{B}_f), D'(\dot{B} - \dot{B}_f)) \\ = &-(C(E - E_e), D'(\dot{B} - \dot{B}_f)) \\ \leq &K(D'S^{-1}C(E - E_e), C(E - E_e))^{\frac{1}{2}}(S(\dot{B} - \dot{B}_f), D'(\dot{B} - \dot{B}_f))^{\frac{1}{2}}, \end{aligned}$$

where we have applied the Cauchy-Schwarz inequality in the last step. So, we derive

$$(S(\dot{B} - \dot{B}_f), D'(\dot{B} - \dot{B}_f)) \leq K(D'S^{-1}C(E - E_e), C(E - E_e))$$

which completes the proof.

□

We remark here that (4.44) is the discrete analog of the following trace theorem

$$\int_\Gamma |u|^2\, ds \leq K(\int_\Omega |u|^2\, dx + \int_\Omega |\nabla u|^2\, dx),$$

for any $u \in H^1(\Omega)$ and of the Poincare's inequality

$$\int_\Gamma |u|^2\, ds \leq K \int_\Omega |\nabla u|^2\, dx,$$

for any $u \in H_0^1(\Omega)$.

In addition to Lemma 4.4, we have the following

Lemma 4.5 *Assume that $C(E - E_e) \neq 0$. Then there exist a constant K independent of h such that*

$$\max_{\sigma_j \in \Gamma} |E - E_e|_j \leq K \|E - E_e\|_V, \tag{4.49}$$

$$\max_{\sigma_j \in \Omega} |E - E_e|_j \leq K \|E - E_e\|_V. \tag{4.50}$$

Proof. The proof of this lemma is similar to that of Lemma 4.4. For (4.49), we take all the $O(N^3)$ free variables in (4.47) as $\max_{\sigma_j \in \Gamma} |E - E_e|_j$. For (4.50), we take all the $O(N^3)$ free variables in (4.47) as $\max_{\sigma_j \in \Omega} |E - E_e|_j$. Then the result follows.

□

Now, we proceed with the convergence analysis on the semi-discrete approximation (4.1)-(4.2). The following theorem gives the V-norm estimate for $E - E_e$.

Theorem 4.5 *Suppose that $\mathbf{B} \in W^{2,1}(0,T;H^3(\Omega_r))^3$, for $r = 1, 2$, is the true solution of (1.1)-(1.4), and that $\mathbf{J} \in W^{2,1}(0,T;H^2(\Omega))^3$ and $\mathbf{J}_\Gamma \in W^{2,1}(0,T;H^3(\Gamma))^3$. Let E be the solution of (4.10)-(4.11) on uniform grid. Then*

$$\begin{aligned}&\max_{0 \leq t \leq T} \|(E - E_e)(t)\|_V \\ \leq & K h^2 \Big(\sum_{r=1}^{2} \|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^3} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}\Big).\end{aligned} \tag{4.51}$$

Proof. Integrating both sides of (4.38) from 0 to t and by the initial condition (4.37), we have

$$\|(\dot{E} - \dot{E}_e)(t)\|_{W'}^2 + \|(E - E_e)(t)\|_V^2 = 2 \int_0^t \Big(\frac{df}{dt}, D(\dot{E} - \dot{E}_e)\Big)(s)\, ds. \tag{4.52}$$

Integration by parts then yields,

$$\int_0^t \Big(\frac{df}{dt}, D(\dot{E} - \dot{E}_e)\Big)(s)\, ds = \Big(\frac{df}{dt}, D(E - E_e)\Big)(t) - \int_0^t \Big(\frac{d^2 f}{dt^2}, D(E - E_e)\Big)(s)\, ds.$$

By (4.39), we know that

$$
\begin{aligned}
&(\frac{df}{dt}, D(E-E_e))(t) \\
&= (\frac{d}{dt}(\bar{J} + C'(B_f - \Pi_h B_f) - \bar{H} + \bar{J}_\Gamma + g), D(E-E_e))(t) \\
&= (\frac{d}{dt}(\bar{J} - \bar{H} + \bar{J}_\Gamma + g), D(E-E_e))(t) + (D'(\dot{B}_f - \Pi_h \dot{B}_f), C(E-E_e))(t).
\end{aligned}
$$

Notice that the theorem is trivially true at time t if $C(E-E_e)(t) = 0$. So, without loss of generality, we assume that $C(E-E_e)(t) \neq 0$ for all $0 < t < T$. Now, we estimate the above equation term by term.

By the Cauchy-Schwarz inequality, we have

$$
(\frac{d\bar{J}}{dt}, D(E-E_e))(t) \leq \|S'^{-1}\frac{d\bar{J}}{dt}\|_{W'}\|E-E_e\|_{W'},
$$

and by Lemma 3.5, we have

$$
(\frac{d\bar{J}}{dt}, D(E-E_e))(t) \leq \|S'^{-1}\frac{d\bar{J}}{dt}\|_{W'}\|E-E_e\|_{V},
$$

By the Sobolev embedding theorem, $(\dot{J}'_f - \dot{J}_e)_j$ defines a bounded linear functional on the space $H^2(\tau'_i \cup \tau'_k)^3$ where τ'_i and τ'_k are two dual elements sharing the same face κ'_j. Clearly, $(\dot{J}'_f - \dot{J}_e)_j$ vanishes for any linear functions since we are considering uniform mesh. By the Bramble-Hilbert lemma,

$$
|(\dot{J}'_f - \dot{J}_e)_j| \leq Kh^{\frac{1}{2}}|\dot{\mathbf{J}}|_{H^2(\tau'_i \cup \tau'_k)^3}.
$$

From (4.40), we have

$$
\begin{aligned}
\|S'^{-1}\frac{d\bar{J}}{dt}\|^2_{W'} &= \sum_{j=1}^{M_1} \bar{s}'_j h_j \bar{s}_j'^{-2} s_j'^2 |(\dot{J}'_f - \dot{J}_e)_j|^2 \\
&\leq Kh^4 \sum_{j=1}^{M_1} |\dot{\mathbf{J}}|^2_{H^2(\tau'_i \cup \tau'_k)^3} \\
&= Kh^4 |\dot{\mathbf{J}}|^2_{H^2(\Omega)^3}
\end{aligned}
$$

By the Cauchy-Schwarz inequality and Lemma 3.5, we have

$$
-(\frac{d\bar{H}}{dt}, D(E-E_e))(t) \leq K\|S'^{-1}\frac{d\bar{H}}{dt}\|_{W'}\|E-E_e\|_{V}.
$$

Corresponding to each non-interface primal edge σ_j, we have

$$\frac{1}{h_j}\int_{\sigma_j} \mathbf{curl}\,(\Pi_h\dot{\mathbf{H}} - \dot{\mathbf{H}})\cdot\mathbf{n}\,dl$$

defines a bounded linear functional on the space $H^3(\tau_i'\cup\tau_k')^3$ and vanishes for any quadratic functions. So,

$$|\frac{1}{h_j}\int_{\sigma_j} \mathbf{curl}\,(\Pi_h\dot{\mathbf{H}} - \dot{\mathbf{H}})\cdot\mathbf{n}\,dl| \le Kh^{\frac{1}{2}}|\dot{\mathbf{H}}|_{H^3(\tau_i'\cup\tau_k')^3}.$$

Similarly, corresponding to each interface primal edge σ_j, for $k=1,2$, we have

$$|\frac{1}{h_j}\int_{\sigma_j} \mathbf{curl}\,(\Pi_h\dot{\mathbf{H}}_k - \dot{\mathbf{H}}_k)\cdot\mathbf{n}\,dl| \le Kh^{\frac{1}{2}}|\dot{\mathbf{H}}|_{H^3((\tau_i'\cup\tau_k')\cap\Omega_k)^3}.$$

Hence, we obtain

$$\begin{aligned}
\|S'^{-1}\frac{d\bar{H}}{dt}\|_{W'}^2 &= \sum_{j=1}^{M_1} \bar{s}_j' h_j \bar{s}_j'^{-2}|\frac{d\bar{H}_j}{dt}|^2 \\
&\le Kh^4\sum_{k=1}^{2}\sum_{j=1}^{M_1}|\dot{\mathbf{H}}|^2_{H^3((\tau_i'\cup\tau_k')\cap\Omega_k)^3} \\
&= Kh^4\sum_{k=1}^{2}|\dot{\mathbf{H}}|^2_{H^3(\Omega_k)^3}.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$(\frac{d\bar{J}_\Gamma}{dt}, D(E-E_e))(t) \le \|D^{-1}\frac{d\bar{J}_\Gamma}{dt}\|_{W_\Gamma}\|E-E_e\|_{W_\Gamma}.$$

By Lemma 4.4, we have

$$(\frac{d\bar{J}_\Gamma}{dt}, D(E-E_e))(t) \le K\|D^{-1}\frac{d\bar{J}_\Gamma}{dt}\|_{W_\Gamma}\|E-E_e\|_{V}.$$

By Sobolev embedding theorem, the term

$$\sum_{r=1}^{2}\int_{\gamma_r}(\dot{\mathbf{J}}_\Gamma - \Pi_h\dot{\mathbf{J}}_\Gamma)\cdot\mathbf{n}\,dl$$

defines a bounded linear functional on the space $H^2(\kappa_i\cup\kappa_l)^3$, where κ_i and κ_l are two interface primal faces sharing the same edge σ_j, and vanishes for any linear functions. By the Bramble-Hilbert lemma, we have

$$|\sum_{r=1}^{2}\int_{\gamma_r}(\dot{\mathbf{J}}_\Gamma - \Pi_h\dot{\mathbf{J}}_\Gamma)\cdot\mathbf{n}\,dl| \le Kh^2|\dot{\mathbf{J}}_\Gamma|_{H^2(\kappa_i\cup\kappa_l)^3}.$$

So, we obtain

$$\begin{aligned}\|D^{-1}\frac{d\bar{J}_\Gamma}{dt}\|_{W_\Gamma}^2 &= \sum_{\sigma_j\in\Gamma} h_j^2 h_j^{-2}|\frac{d(\bar{J}_\Gamma)_j}{dt}|^2\\ &\le Kh^4\sum_{\sigma_j\in\Gamma}|\dot{\mathbf{J}}_\Gamma|^2_{H^2(\kappa_i\cup\kappa_l)^3}\\ &= Kh^4|\dot{\mathbf{J}}_\Gamma|^2_{H^2(\Gamma)^3}.\end{aligned}$$

Notice that g has only non-zero components corresponding to interface primal edge lying on edges of Ω_1, there are only $O(N)$ non-zero components in g. By the Cauchy-Schwarz inequality, we have

$$(\frac{dg}{dt}, D(E-E_e))(t) \le Kh^{\frac{1}{2}}\|\frac{dg}{dt}\|_2 \max_{\sigma_j\in\Gamma}|E-E_e|_j,$$

and by Lemma 4.5, we have

$$(\frac{dg}{dt}, D(E-E_e))(t) \le Kh^{\frac{1}{2}}\|\frac{dg}{dt}\|_2\|E-E_e\|_V.$$

From (4.43), we know that (see Figure 4)

$$\dot{g}_j = h_i^1(\Pi_h\dot{\mathbf{J}}_\Gamma\cdot\mathbf{n}(Q_3) - \Pi_h\dot{\mathbf{J}}_\Gamma\cdot\mathbf{n}(R_1)) + h_k^1(\Pi_h\dot{\mathbf{J}}_\Gamma\cdot\mathbf{n}(R_2) - \Pi_h\dot{\mathbf{J}}_\Gamma\cdot\mathbf{n}(Q_3)).$$

Now, we estimate $\dot{g}_j$. First

$$|\dot{g}_j| \le Kh^2\|\dot{\mathbf{J}}_\Gamma\|_{C^1(\Gamma)^3}.$$

So,

$$\|\frac{dg}{dt}\|_2^2 = \sum_{\sigma_j}|\dot{g}_j|^2 \le Kh^3\|\dot{\mathbf{J}}_\Gamma\|^2_{C^1(\Gamma)^3},$$

where the above summation is taken over all the primal edges σ_j lying on the edges of Ω_1 and has $O(N)$ terms.

By the definition of V-norm, we have

$$(D'(\dot{B}_f - \Pi_h\dot{B}_f), C(E-E_e))(t) \le K\|\dot{B}_f - \Pi_h\dot{B}_f\|_W\|E-E_e\|_V.$$

Notice that $\dot{B}_f - \Pi_h \dot{B}_f$ defines a bounded linear functional on $H^2(\tau_i \cup \tau_k)^3$, where τ_i and τ_k are two primal elements sharing the same face κ_j, and vanishes for any linear functions. So,

$$|(\dot{B}_f - \Pi_h \dot{B}_f)_j| \leq K h^{\frac{1}{2}} |\dot{\mathbf{B}}|_{H^2(\tau_i \cup \tau_k)^3}.$$

Hence, we obtain

$$\begin{aligned} \|\dot{B}_f - \Pi_h \dot{B}_f\|_W^2 &= \sum_{j=1}^{F_1} s_j \bar{h}_j |(\dot{B}_f - \Pi_h \dot{B}_f)_j|^2 \\ &\leq K h^4 \sum_{j=1}^{F_1} |\dot{\mathbf{B}}|^2_{H^2(\tau_i \cup \tau_k)^3} \\ &= K h^4 \sum_{r=1}^{2} |\dot{\mathbf{B}}|^2_{H^2(\Omega_r)^3}. \end{aligned}$$

Collecting the above results, we obtain

$$(\frac{df}{dt}, D(E - E_e))(t) \leq K h^2 (\sum_{r=1}^{2} |\dot{\mathbf{B}}|_{H^3(\Omega_r)^3} + |\dot{\mathbf{J}}|_{H^2(\Omega)^3} + \|\dot{\mathbf{J}}_\Gamma\|_{H^3(\Gamma)^3}) \|E - E_e\|_V.$$

Following a similar proof, we have

$$(\frac{d^2 f}{dt^2}, D(E - E_e))(t) \leq K h^2 (\sum_{r=1}^{2} |\ddot{\mathbf{B}}|_{H^3(\Omega_r)^3} + |\ddot{\mathbf{J}}|_{H^2(\Omega)^3} + \|\ddot{\mathbf{J}}_\Gamma\|_{H^3(\Gamma)^3}) \|E - E_e\|_V.$$

Then, from (4.52), we have

$$\begin{aligned} &\|(E - E_e)(t)\|_V^2 \\ &\leq K h^2 \max_{0 \leq t \leq T} \|(E - E_e)(t)\|_V \\ &\quad \times (\sum_{r=1}^{2} \|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^3} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}). \end{aligned}$$

By the Young's inequality, we obtain the desired result.

□

The following theorem gives our main result in this section.

Theorem 4.6 *Suppose that* $\mathbf{B} \in W^{2,1}(0,T;H^3(\Omega_r))^3$ *and* $\mathbf{E} \in W^{1,1}(0,T;H^3(\Omega_r))^3$, *for* $r = 1,2$, *is the true solution of (1.1)-(1.4), and that* $\mathbf{J} \in W^{2,1}(0,T;H^2(\Omega))^3$ *and* $\mathbf{J}_\Gamma \in W^{2,1}(0,T;H^3(\Gamma))^3$. *Let* (E,B) *be the solution of (4.10)-(4.11) on uniform grid. Then*

$$\begin{aligned}
&\max_{0\le t\le T}(\|(E-E_e)(t)\|_{W'} + \|(B-B_f)(t)\|_W) \\
\le& Kh^2\Big(\sum_{r=1}^{2}\|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \sum_{r=1}^{2}\|\mathbf{E}\|_{W^{1,1}(0,T;H^3(\Omega_r))^3} \\
&+ \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^3} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}\Big).
\end{aligned} \tag{4.53}$$

Proof. Multiplying both sides of (4.23) by $D'(B-B_f)(t)$, we have

$$(S\frac{d}{dt}(B-B_f), D'(B-B_f))(t) + (C(E-E_e), D'(B-B_f))(t) = 0.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|(B-B_f)(t)\|_W^2 \le K\|(E-E_e)(t)\|_V\|(B-B_f)(t)\|_W.$$

Integrating from 0 to t, we obtain

$$\begin{aligned}
\|(B-B_f)(t)\|_W^2 &\le K\int_0^t \|(E-E_e)(s)\|_V^2\,ds + \int_0^t \|(B-B_f)(s)\|_W^2\,ds \\
&\le K\max_{0\le t\le T}\|(E-E_e)(t)\|_V^2 + \int_0^t \|(B-B_f)(s)\|_W^2\,ds.
\end{aligned}$$

Applying the Gronwall's inequality, we obtain

$$\|(B-B_f)(t)\|_W^2 \le K\max_{0\le t\le T}\|(E-E_e)(t)\|_V^2.$$

Then the desired result for $B - B_f$ follows from Theorem 4.5.

Now, let

$$I := \{t \in [0,T] : C(E-E_e)(t) \ne 0\}.$$

Then for any $t \in I$, by Lemma 3.5, we have

$$\|(E - E_e)(t)\|_{W'} \leq K\|(E - E_e)(t)\|_V$$

which with Theorem 4.5 yields

$$\begin{aligned}&\max_{t \in I} \|(E - E_e)(t)\|_{W'} \\ \leq & Kh^2 (\sum_{r=1}^{2} \|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^3} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}).\end{aligned} \tag{4.54}$$

For $t \in [0,T]\backslash I$, it suffices to prove the following Lemma 4.6.

□

Lemma 4.6 *Suppose that* $\mathbf{B} \in W^{2,1}(0,T;H^3(\Omega_r))^3$ *and* $\mathbf{E} \in W^{1,1}(0,T;H^3(\Omega_r))^3$, *for* $r = 1, 2$, *is the true solution of (1.1)-(1.4), and that* $\mathbf{J} \in W^{2,1}(0,T;H^2(\Omega))^3$ *and* $\mathbf{J}_\Gamma \in W^{2,1}(0,T;H^3(\Gamma))^3$. *Let* E *be the solution of (4.10)-(4.11) on uniform grid with* $C(E - E_e)(t) = 0$ *for all* $t_1 < t < t_2$. *Then*

$$\begin{aligned}&\max_{t_1<t<t_2} \|(E - E_e)(t)\|_{W'} \\ \leq & Kh^2 (\sum_{r=1}^{2} \|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \sum_{r=1}^{2} \|\mathbf{E}\|_{W^{1,1}(0,T;H^3(\Omega_r))^3} \\ & + \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^3} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}).\end{aligned} \tag{4.55}$$

Proof. For any $t \in (t_1, t_2)$, since $C(E - E_e)(t) = 0$, by Lemma 3.7, there exist $\phi \in \mathbb{R}^L$ such that

$$D(E - E_e) = B_1'\phi.$$

Then

$$\begin{aligned}(S'\frac{d}{dt}(E - E_e), D(E - E_e)) &= (S'\frac{d}{dt}(E - E_e), B_1'\phi) \\ &= (\frac{d}{dt}\mathcal{D}'(E - E_e), \phi) \qquad \text{by (3.18).}\end{aligned}$$

By Theorem 5.3, we have

$$(S'\frac{d}{dt}(E-E_e), D(E-E_e)) = (\frac{d}{dt}(\tilde{\rho}-\mathcal{D}'E_e), \phi).$$

Now, we define a vector $E_p \in \mathbb{R}^{M_1}$ in the following fashion. For any primal edge with non-empty intersection with edges of Ω_1 and normal to Γ,

$$(E_p)_j := \beta_j(\mathbf{E}\cdot\mathbf{n})(Q_j^1) + (1-\beta_j)(\mathbf{E}\cdot\mathbf{n})(Q_j^2),$$

where $\mathbf{n}$ is the direction of the primal edge and Q_j^r denotes the mid-points of the face κ_j^r, for $r = 1, 2$. Here we recall that $\kappa_j^r = \kappa_j' \cap \Omega_r$. For a face in Ω_2, we divide the face in the same way as its neighbouring face in Ω_1. For the other primal edges,

$$(E_p)_j := (\mathbf{E}\cdot\mathbf{n})(P_j),$$

where P_j is the mid-point of the primal edge. So,

$$(S'\frac{d}{dt}(E-E_e), D(E-E_e)) = (\frac{d}{dt}(\tilde{\rho}-\mathcal{D}'E_p), \phi) + (\frac{d}{dt}\mathcal{D}'(E_p-E_e), \phi),$$

and consequently,

$$(S'\frac{d}{dt}(E-E_e), D(E-E_e)) = (\frac{d}{dt}(\tilde{\rho}-\mathcal{D}'E_p), \phi) + (S'(\dot{E}_p-\dot{E}_e), D(E-E_e)).$$

For any dual element τ_j', we denote by $\Pi_h\mathbf{E}$ the standard finite element linear interpolation of the function $\mathbf{E}$ on τ_j'. The formula is the same as $\Pi_h\mathbf{B}$. For an interface dual element, since it has a non-empty intersection with both Ω_1 and Ω_2, we define $\Pi_h\mathbf{E}$ in each of the two parts of the dual element. For each non-interface dual element τ_j',

$$\tilde{\rho}_j = \int_{\tau_j'} \rho\, dx = \int_{\tau_j'} \mathrm{div}(\epsilon\mathbf{E})\, dx.$$

We rewrite $\tilde{\rho}_j$ as

$$\tilde{\rho}_j = \int_{\tau_j'} \mathrm{div}(\epsilon(\mathbf{E}-\Pi_h\mathbf{E}))\, dx + \int_{\tau_j'} \mathrm{div}(\epsilon\Pi_h\mathbf{E})\, dx,$$

and by the divergence theorem,

$$\tilde{\rho}_j = (\mathcal{D}'(E_f' - \Pi_h E_f'))_j + \int_{\tau_j'} \operatorname{div}(\epsilon \Pi_h \mathbf{E})\, dx.$$

Now, for each interface dual element τ_j',

$$\tilde{\rho}_j = \int_{\tau_j'} \rho\, dx + \int_{\tau_j' \cap \Gamma} \rho_\Gamma\, d\sigma = \int_{\tau_j'} \operatorname{div}(\epsilon \mathbf{E})\, dx + \int_{\tau_j' \cap \Gamma} [\epsilon \mathbf{E} \cdot \mathbf{m}]\, d\sigma.$$

We rewrite it as

$$\begin{aligned}\tilde{\rho}_j = &\int_{\tau_j'} \operatorname{div}(\epsilon(\mathbf{E} - \Pi_h \mathbf{E}))\, dx + \int_{\tau_j' \cap \Gamma} [\epsilon(\mathbf{E} - \Pi_h \mathbf{E}) \cdot \mathbf{m}]\, d\sigma \\ &+ \int_{\tau_j'} \operatorname{div}(\epsilon \Pi_h \mathbf{E})\, dx + \int_{\tau_j' \cap \Gamma} [\epsilon \Pi_h \mathbf{E} \cdot \mathbf{m}]\, d\sigma,\end{aligned}$$

and by the divergence theorem,

$$\tilde{\rho}_j = (\mathcal{D}'(E_f' - \Pi_h E_f'))_j + \int_{\tau_j'} \operatorname{div}(\epsilon \Pi_h \mathbf{E})\, dx + \int_{\tau_j' \cap \Gamma} [\epsilon \Pi_h \mathbf{E} \cdot \mathbf{m}]\, d\sigma.$$

Hence, we obtain

$$\begin{aligned}&(S' \frac{d}{dt}(E - E_e), D(E - E_e)) \\ =&(\frac{d}{dt}\mathcal{D}'(E_f' - \Pi_h E_f'), \phi) + (S'(\dot{E}_p - \dot{E}_e), D(E - E_e)) + (\frac{dR}{dt}, \phi),\end{aligned}$$

where

$$R_j := \int_{\tau_j'} \operatorname{div}(\epsilon \Pi_h \mathbf{E})\, dx + \int_{\tau_j' \cap \Gamma} [\epsilon \Pi_h \mathbf{E} \cdot \mathbf{m}]\, d\sigma - (\mathcal{D}' E_p)_j.$$

Since $\Pi_h \mathbf{E}$ is a linear function in each dual element, by a direct computation, we have $R = 0$. Finally, we obtain

$$\frac{1}{2}\frac{d}{dt}\|E - E_e\|_{W'}^2 = (S'(\dot{E}_f' - \Pi_h \dot{E}_f'), D(E - E_e)) + (S'(\dot{E}_p - \dot{E}_e), D(E - E_e)).$$

Integrating from t_1 to t, and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\|(E - E_e)(t)\|_{W'}^2 \\ =&\|(E - E_e)(t_1)\|_{W'}^2 + 2\int_{t_1}^{t} \|(\dot{E}_f' - \Pi_h \dot{E}_f')(s)\|_{W'} \|(E - E_e)(s)\|_{W'}\, ds \\ &+ 2\int_{t_1}^{t} \|(\dot{E}_p - \dot{E}_e)(s)\|_{W'} \|(E - E_e)(s)\|_{W'}\, ds.\end{aligned}$$

By (4.54), we have

$$\begin{aligned}&\|(E-E_e)(t_1)\|_{W'}\\ \leq& Kh^2(\sum_{r=1}^{2}\|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3}+\|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^3}+\|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}).\end{aligned}$$

For each non-interface dual face, the term $\dot{E}'_f - \Pi_h\dot{E}'_f$ defines a bounded linear functional vanishes for any linear functions, so

$$|(\dot{E}'_f - \Pi_h\dot{E}'_f)_j| \leq Kh^{\frac{1}{2}}|\dot{\mathbf{E}}|_{H^2(\tau'_i\cup\tau'_k)^3}.$$

For each interface dual face κ'_j, we have

$$(\dot{E}'_f - \Pi_h\dot{E}'_f)_j = \beta_j\frac{1}{s^1_j}\int_{\kappa^1_j}(\dot{\mathbf{E}}-\Pi_h\dot{\mathbf{E}})\cdot\mathbf{n}\,d\sigma + (1-\beta_j)\frac{1}{s^2_j}\int_{\kappa^2_j}(\dot{\mathbf{E}}-\Pi_h\dot{\mathbf{E}})\cdot\mathbf{n}\,d\sigma.$$

So, we have

$$|(\dot{E}'_f - \Pi_h\dot{E}'_f)_j| \leq Kh^{\frac{1}{2}}\sum_{r=1}^{2}|\dot{\mathbf{E}}|_{H^2((\tau'_i\cup\tau'_k)\cap\Omega_r)^3}.$$

Consequently,

$$\begin{aligned}\|\dot{E}'_f - \Pi_h\dot{E}'_f\|^2_{W'} &= \sum_{j=1}^{M_1}\bar{s}'_j h_j|(\dot{E}'_f - \Pi_h\dot{E}'_f)_j|^2\\ &\leq Kh^4\sum_{r=1}^{2}\sum_{j=1}^{M_1}|\dot{\mathbf{E}}|^2_{H^2((\tau'_i\cup\tau'_k)\cap\Omega_r)^3}\\ &= Kh^4\sum_{r=1}^{2}|\dot{\mathbf{E}}|^2_{H^2(\Omega_r)^3}.\end{aligned}$$

From the definition of E_p, for any primal edge σ_j with non-empty intersection with edges of Ω_1 and normal to Γ, we have

$$\begin{aligned}&(\dot{E}_p-\dot{E}_e)_j\\ =&\beta_j(\dot{\mathbf{E}}\cdot\mathbf{n})(Q^1_j)+(1-\beta_j)(\dot{\mathbf{E}}\cdot\mathbf{n})(Q^2_j)-\frac{1}{h_j}\int_{\sigma_j}\dot{\mathbf{E}}\cdot\mathbf{n}\,dl\\ =&\beta_j((\dot{\mathbf{E}}\cdot\mathbf{n})(Q^1_j)-\frac{1}{h_j}\int_{\sigma_j}\dot{\mathbf{E}}\cdot\mathbf{n}\,dl)+(1-\beta_j)((\dot{\mathbf{E}}\cdot\mathbf{n})(Q^2_j)-\frac{1}{h_j}\int_{\sigma_j}\dot{\mathbf{E}}\cdot\mathbf{n}\,dl).\end{aligned}$$

Notice that

$$\begin{aligned}
&(\dot{\mathbf{E}}\cdot\mathbf{n})(Q_j^1) - \frac{1}{h_j}\int_{\sigma_j} \dot{\mathbf{E}}\cdot\mathbf{n}\,dl \\
=&(\dot{\mathbf{E}}\cdot\mathbf{n})(Q_j^1) - (\dot{\mathbf{E}}\cdot\mathbf{n})(P_j) - \frac{1}{h_j}\int_{\sigma_j} (\dot{\mathbf{E}}\cdot\mathbf{n} - (\dot{\mathbf{E}}\cdot\mathbf{n})(P_j))\,dl \\
\leq& Kh\|\dot{\mathbf{E}}\|_{C^1(\Omega_1)^3}.
\end{aligned}$$

Similarly, we have

$$(\dot{\mathbf{E}}\cdot\mathbf{n})(Q_j^2) - \frac{1}{h_j}\int_{\sigma_j} \dot{\mathbf{E}}\cdot\mathbf{n}\,dl \leq Kh\|\dot{\mathbf{E}}\|_{C^1(\Omega_2)^3}.$$

Since the number of primal edges with non-empty intersection with edges of Ω_1 and normal to Γ is $O(N)$, we obtain

$$\begin{aligned}
\sum_{\sigma_j} \bar{s}'_j h_j |(\dot{E}_p - \dot{E}_e)_j|^2 &\leq Kh^5 \sum_{r=1}^{2} \|\dot{\mathbf{E}}\|_{C^1(\Omega_r)^3} \sum_{\sigma_j} 1 \\
&\leq Kh^4 \sum_{r=1}^{2} \|\dot{\mathbf{E}}\|_{H^3(\Omega_r)^3}.
\end{aligned}$$

For the other components of $\dot{E}_p - \dot{E}_e$, by the definition of E_p, we have

$$(\dot{E}_p - \dot{E}_e)_j = (\dot{\mathbf{E}}\cdot\mathbf{n})(P_j) - \frac{1}{h_j}\int_{\sigma_j} \dot{\mathbf{E}}\cdot\mathbf{n}\,dl.$$

Since, $(\dot{E}_p - \dot{E}_e)_j$ defines a bounded linear functional which vanishes for any linear functions, by similar steps as above, we obtain

$$\sum_{\sigma_j} \bar{s}'_j h_j |(\dot{E}_p - \dot{E}_e)_j|^2 \leq Kh^4 \sum_{r=1}^{2} |\dot{\mathbf{E}}|^2_{H^2(\Omega_r)^3}.$$

Consequently,

$$\|\dot{E}_p - \dot{E}_e\|_{W'} \leq Kh^2 \sum_{r=1}^{2} |\dot{\mathbf{E}}|^2_{H^3(\Omega_r)^3}.$$

Collecting the above results, we have proved the desired estimate.

□

We remark here that Theorem 4.6 shows our semi-discrete finite volume approximation of the Maxwell's equations is second order convergent for rectangular domains. Furthermore, the above estimates are optimal since the W and W' norms are the discrete analog of L^2-norm.

Chapter 5

Fully Discretization of the Maxwell's Equations

In this chapter, a fully discretization of the Maxwell's equations, that is discretization in both space and time, will be presented. For the fully discrete finite volume approximation of Maxwell's equations, we will prove that the solution to this discrete approximation satisfies the divergence constraints in discrete sense. Furthermore, a convergence analysis will be given in both the following cases: first, the domains Ω and Ω_1 are two polyhedra; second, the domains Ω and Ω_1 are two cuboids. In the second case, we can prove that the convergence rate is one order higher, that is, it is second order convergent. Also, the convergence in time is second order for both cases.

5.1 Derivation

In this section, we will derive the fully discrete approximation of Maxwell's equations by our finite volume method. Our approach is to discretize the time derivatives in (4.10)-(4.11) by finite differences. Let us recall the definition of finite difference. For any smooth function $u(t)$, we can approximate its first order derivative at a point t by the following formula

$$\frac{du(t)}{dt} \approx \frac{u(t+\tau) - u(t-\tau)}{2\tau}, \tag{5.1}$$

for small τ. It is called the central difference approximation of first order derivatives. It can be shown, by using a Taylor expansion, that this approximation is second order.

Let N_T be the number of subintervals of $[0, T]$ and Δt be the length of each subinterval. Denote $t_n := n\Delta t$, for $0 \leq n \leq N_T - 1$. In our finite volume method, we approximate the true solution $\mathbf{E}(t)$ at times t_n with the approximation represented by E^n while the true solution $\mathbf{B}(t)$ at times $t_{n+\frac{1}{2}}$ with the approximation represented by $B^{n+\frac{1}{2}}$. This method is the so called leapfrog scheme. The initial condition $B^{\frac{1}{2}}$ is computed by using Taylor's expansion and the Maxwell's equations (1.1)-(1.2).

For (4.10), we apply the central difference approximation to the derivative in time at time $t = t_{n+\frac{1}{2}}$, then

$$S' \frac{E^{n+1} - E^n}{\Delta t} - C' B^{n+\frac{1}{2}} = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} \tilde{J} \, dt,$$

where we use the average value of $\tilde{J}$ on the subinterval $[n\Delta t, (n+1)\Delta t]$ to approximate the value of $\tilde{J}$ at $t = t_{n+\frac{1}{2}}$. Clearly, this approximation is second order accurate. Similarly, for (4.11), we apply the central difference approximation to the time derivative at time $t = t_{n+1}$, so we get

$$S \frac{B^{n+\frac{3}{2}} - B^{n+\frac{1}{2}}}{\Delta t} + C E^{n+1} = 0.$$

Now, we have the fully discrete scheme: Given $(E^n, B^{n+\frac{1}{2}})_{0 \leq n \leq N_T - 1}$, the next approximation $(E^{n+1}, B^{n+\frac{3}{2}})$ is calculated by solving the following equations

$$S'(E^{n+1} - E^n) - \Delta t C' B^{n+\frac{1}{2}} = \tilde{J}^{n+\frac{1}{2}} \tag{5.2}$$

$$S(B^{n+\frac{3}{2}} - B^{n+\frac{1}{2}}) + \Delta t C E^{n+1} = 0 \tag{5.3}$$

where

$$\tilde{J}^{n+\frac{1}{2}} := \int_{n\Delta t}^{(n+1)\Delta t} \tilde{J}\, dt,$$

We also supplement (5.2)-(5.3) with an initial condition

$$E^0 = E_e(0), \qquad B^{\frac{1}{2}} = B_f(t_{\frac{1}{2}}). \tag{5.4}$$

Theorem 5.1 *The fully discrete scheme (5.2)-(5.3) has a unique solution.*

Proof. The reason for the uniqueness follows from the fact that (5.2)-(5.3) is an explicit finite difference method for solving system of linear first order ordinary differential equations.

□

5.2 Consistency theory

As explained in last chapter, it is important to know whether the solution of the fully discrete approximation of Maxwell's equations satisfies the divergence constraints in some discrete sense. Otherwise, the solution is not representing true phenomenon since both the magnetic and electric fields must satisfy the divergence constraints.

In the following theorem, we have shown that the solution $B^{n+\frac{1}{2}}$ to (5.2)-(5.3) satisfies the divergence constraint in discrete level.

Theorem 5.2 *Let $B^{n+\frac{1}{2}}$, for $0 \le n \le N_T - 1$, be the solution to the fully discrete scheme (5.2)-(5.3), then $B^{n+\frac{1}{2}}$ is divergence-free in the discrete level, i.e.,*

$$\mathcal{D}B^{n+\frac{1}{2}} = 0, \qquad 0 \le n \le N_T - 1. \tag{5.5}$$

Proof. By Lemma 3.6 and (5.3), we have

$$\mathcal{D}(B^{n+\frac{3}{2}} - B^{n+\frac{1}{2}}) = B_1^T S(B^{n+\frac{3}{2}} - B^{n+\frac{1}{2}}) = -\Delta t B_1^T C E^{n+1} = 0.$$

Taking the divergence in both sides of (1.2), we obtain

$$\frac{\partial}{\partial t}\mathrm{div}(\mu\mathbf{H}) = 0,$$

that implies $\mathrm{div}(\mu\mathbf{H}) = 0$ at time $t = \frac{1}{2}\Delta t$ by noting (1.7). Integrating this equation over a primal element τ_i and using the Stokes' theorem lead to (at $t = \frac{1}{2}\Delta t$)

$$\sum_{\kappa_j \in \partial\tau_i} \int_{\kappa_j} \mathbf{B} \cdot \mathbf{n}_j \, d\sigma = 0.$$

By the definition of B_f, this can be written as

$$(\mathcal{D}B_f^{\frac{1}{2}})_i = 0$$

for any i. So $\mathcal{D}B_f^{\frac{1}{2}} = 0$. Using $B^{\frac{1}{2}} = B_f^{\frac{1}{2}}$, we conclude that

$$\mathcal{D}B^{n+\frac{1}{2}} = 0, \qquad 0 \le n \le N_T - 1.$$

□

The next theorem shows how the solution E^n to (5.2)-(5.3) satisfies the divergence constraint in discrete level.

Theorem 5.3 *Let E^n, $0 \le n \le N_T - 1$, be the solution to the fully discrete scheme (5.2)-(5.3). Then we have*

$$\mathcal{D}'E^n = \tilde{\rho}^n + \mathcal{D}'(E_e - E_f')(0), \qquad 0 \le n \le N_T - 1 \tag{5.6}$$

where $\tilde{\rho}$ is a vector in $\mathbb{R}^L$ with

$$\tilde{\rho}_j^n := \int_{\tau_j'} \rho(x, t_n)\, dx + \int_{\tau_j' \cap \Gamma} \rho_\Gamma(x, t_n)\, d\sigma. \tag{5.7}$$

Proof. By Lemma 3.6 and (5.2), we have for $0 \le n \le N_T - 2$ that

$$\begin{aligned}\mathcal{D}'(E^{n+1} - E^n) &= (B_1')^T S'(E^{n+1} - E^n) = \Delta t (B_1')^T C' B^{n+\frac{1}{2}} + (B_1')^T \tilde{J}^{n+\frac{1}{2}} \\ &= (B_1')^T \tilde{J}^{n+\frac{1}{2}} = \int_{t_n}^{t_{n+1}} (B_1')^T \tilde{J}\, dt.\end{aligned}$$

Summing up all these equations over n, we obtain

$$\mathcal{D}'E^n = \mathcal{D}'E^0 + \int_0^{t_n} (B_1')^T \tilde{J}\, dt, \qquad 0 \le n \le N_T - 1. \tag{5.8}$$

Integrating the initial condition $\operatorname{div}(\epsilon \mathbf{E}(x,0)) = \rho(x,0)$ over a strictly interior dual element τ_i', we have

$$\sum_{\kappa_r' \in \partial \tau_i'} \int_{\kappa_r'} \epsilon \mathbf{E}(x,0) \cdot \mathbf{n}_r \, d\sigma = \int_{\tau_i'} \rho(x,0)\, dx, \tag{5.9}$$

which, by the definition of the face average, can be written as

$$(\mathcal{D}'(E_f')(0))_i = \int_{\tau_i'} \rho(x,0)\, dx. \tag{5.10}$$

We know $E^0 = E_e(0)$ for all primal edges corresponding to the dual faces of τ_i', then (5.10) is equivalent to

$$(\mathcal{D}'E^0)_i = \int_{\tau_i'} \rho(x,0)\, dx + e_i$$

where

$$e_i := (\mathcal{D}'(E_e - E_f')(0))_i.$$

For an interface dual element τ_j', that is $\tau_j' \cap \Gamma \neq \phi$, we can write

$$\int_{\tau_i'} \operatorname{div}(\epsilon \mathbf{E}(x,0))\, dx = \sum_{k=1}^{2} \int_{\tau_j' \cap \Omega_k} \operatorname{div}(\epsilon \mathbf{E}(x,0))\, dx = \int_{\tau_j'} \rho(x,0)\, dx.$$

By the divergence theorem and the jump condition $[\epsilon \mathbf{E} \cdot \mathbf{m}] = \rho_\Gamma$ on Γ, we obtain

$$(\mathcal{D}'E^0)_j = \int_{\tau_j'} \rho(x,0)\, dx + \int_{\tau_j' \cap \Gamma} \rho_\Gamma(x,0)\, d\sigma + e_j,$$

where

$$e_j := (\mathcal{D}'(E_e - E_f')(0))_j. \tag{5.11}$$

By the continuity equation (1.9), for any interface dual element τ_j', we have

$$\frac{\partial}{\partial t} \int_{\tau_j'} \rho\, dx = \int_{\tau_j'} \operatorname{div}\mathbf{J}\, dx = \sum_{k=1}^{2} \int_{\tau_j' \cap \Omega_k} \operatorname{div}\mathbf{J}\, dx$$

Applying the divergence theorem,

$$\frac{\partial}{\partial t}\int_{\tau_j'} \rho \, dx = \sum_{\kappa_r' \in \partial\tau_j'} \int_{\kappa_r'} \mathbf{J}\cdot\mathbf{n}_r \, d\sigma - \int_{\tau_j'\cap\Gamma} [\mathbf{J}\cdot\mathbf{m}] \, d\sigma.$$

From equation (1.1), we see

$$\begin{aligned}\int_{\tau_j'\cap\Gamma} [\mathbf{J}\cdot\mathbf{m}] \, d\sigma &= -\int_{\tau_j'\cap\Gamma} [\mathbf{curl}\,\mathbf{H}\cdot\mathbf{m}] \, d\sigma + \frac{\partial}{\partial t}\int_{\tau_j'\cap\Gamma} [\epsilon\mathbf{E}\cdot\mathbf{m}] \, d\sigma \\ &= -\int_{\tau_j'\cap\Gamma} [\mathbf{curl}\,\mathbf{H}\cdot\mathbf{m}] \, d\sigma + \frac{\partial}{\partial t}\int_{\tau_j'\cap\Gamma} \rho_\Gamma \, d\sigma.\end{aligned}$$

From figure 1 and the equations (4.4) and (4.5),

$$\int_{\tau_j'\cap\Gamma} [\mathbf{curl}\,\mathbf{H}\cdot\mathbf{m}] \, d\sigma = \sum_{\gamma_r' \in \partial(\tau_j'\cap\Gamma)} \int_{\gamma_r'} [\mathbf{H}\cdot\mathbf{t}_r] \, dl = \sum_{\gamma_r' \in \partial(\tau_j'\cap\Gamma)} \int_{\gamma_r'} \mathbf{J}_\Gamma\cdot\mathbf{n}_r \, d\sigma.$$

Combining the above results, we have

$$\frac{\partial}{\partial t}\int_{\tau_j'} \rho \, dx = ((B_1')^T\tilde{J})_j - \frac{\partial}{\partial t}\int_{\tau_j'\cap\Gamma} \rho_\Gamma \, d\sigma.$$

Integrating both sides over $[0, t_n]$ gives

$$\begin{aligned}&\int_{\tau_j'} \rho(x,t_n) \, dx - \int_{\tau_j'} \rho(x,0) \, dx \\ &\qquad = \int_0^{t_n} ((B')^T\tilde{J})_j \, dt + \int_{\tau_j'\cap\Gamma} \rho_\Gamma(x,0) \, d\sigma - \int_{\tau_j'\cap\Gamma} \rho_\Gamma(x,t_n) \, d\sigma.\end{aligned}$$

By a similiar argument, we can derive the same result for any strictly interior dual elements. Hence, we have proved (5.6).

□

We remark that the last term in (5.7) vanishes for any strictly interior dual element τ_i'. But for any interface dual element τ_j', we can integrate both sides of (1.3) over τ_j' and apply the divergence theorem to obtain

$$\sum_{\kappa_r' \in \partial\tau_j'} \int_{\kappa_r'} \epsilon\mathbf{E}\cdot\mathbf{n}_r \, d\sigma = \int_{\tau_j'} \rho \, dx + \int_{\tau_j'\cap\Gamma} \rho_\Gamma \, d\sigma.$$

Thus (5.6) is a fully discrete approximation of this integral version of the divergence constraint (1.3).

5.3 Convergence theory

In this section, we develop the convergence theory for the fully discrete approximation (5.2)-(5.3) of the Maxwell's equations. We divide this section into two parts. The first part deals with the case when both Ω and Ω_1 are polydedral domains. It can be shown that the convergence rate is $O(h)$. The second part deals with the case when both Ω and Ω_1 are rectangular domains and shows that the convergence rate is one order higher, that is, $O(h^2)$.

5.3.1 Polyhedral domain

Before the development of the convergence theory of our fully discrete finite volume approximation, we need the following technical lemma which is in fact the Bramble-Hilbert lemma but with a sharper estimate of the constant.

Lemma 5.1 *Suppose that f is a bounded linear functional on the space $W^{1,1}(0,\Delta t)$ and $f(c)=0$ for any constant functions $c \in \mathbb{R}^1$. Then there exist a constant K independent of Δt such that*

$$|f(v)| \leq K|v|_{W^{1,1}(0,\Delta t)}. \tag{5.12}$$

Proof. Define a linear transformation $\hat{T}:[0,\Delta t]\to[0,1]$ by $\hat{t}=(\Delta t)^{-1}t$. Denote $\hat{v}$ be the transformed function, that is, $\hat{v}(\hat{t})=\hat{v}((\Delta t)^{-1}t)=v(t)$. Then, by the Bramble-Hilbert lemma, there exist a generic constant K such that

$$|f(\hat{v})| \leq K|\hat{v}|_{W^{1,1}(0,1)}.$$

Notice that

$$\begin{aligned}|\hat{v}|_{W^{1,1}(0,1)} &= \int_0^1 |\frac{d\hat{v}}{d\hat{t}}|\, d\hat{t},\\ &= \int_0^{\Delta t} |\frac{dv}{dt}\frac{dt}{d\hat{t}}|\,(\Delta t)^{-1}\, dt,\end{aligned}$$

where in the last step, we have applied the inverse transformation $\hat{T}^{-1}$ to the integral. So, we obtain

$$|\hat{v}|_{W^{1,1}(0,1)} = \int_0^{\Delta t} |\frac{dv}{dt}| \, dt,$$

which implies the lemma.

□

We remark here that the above lemma can be generalized to the case that the space $W^{1,1}(0,\Delta t)$ is replaced by $W^{1,1}(n\Delta t,(n+1)\Delta t)$.

We are now in a position to give the convergence analysis for the fully discrete finite volume approximation. From (5.2)-(5.3), we obtain

$$S'((E^{n+1} - E_f^{'n+1}) - (E^n - E_f^{'n})) = \Delta t C'(B^{n+\frac{1}{2}} - B_e^{'n+\frac{1}{2}}) + \mathcal{M}^n \quad (5.13)$$

$$S((B^{n+\frac{3}{2}} - B_f^{n+\frac{3}{2}}) - (B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}})) = -\Delta t C(E^{n+1} - E_e^{n+1}) + \mathcal{N}^n \quad (5.14)$$

where, by a direct computation, we have

$$\mathcal{M}^n := \tilde{J}^{n+\frac{1}{2}} - S'(E_f^{'n+1} - E_f^{'n}) + \Delta t C' B_e^{'n+\frac{1}{2}} \quad (5.15)$$

$$\mathcal{N}^n := -S(B_f^{n+\frac{3}{2}} - B_f^{n+\frac{1}{2}}) - \Delta t C E_e^{n+1}. \quad (5.16)$$

Now, multiplying (5.13) by $D((E^n - E_e^n) + (E^{n+1} - E_e^{n+1}))$ and (5.14) by $D'((B^{n+\frac{1}{2}} - B_e^{'n+\frac{1}{2}}) + (B^{n+\frac{3}{2}} - B_e^{'n+\frac{3}{2}}))$, we have

$$\begin{aligned}
&(S'((E^{n+1} - E_f^{'n+1}) - (E^n - E_f^{'n})), D((E^n - E_e^n) + (E^{n+1} - E_e^{n+1}))) \\
&\quad + (S((B^{n+\frac{3}{2}} - B_f^{n+\frac{3}{2}}) - (B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}})), D'((B^{n+\frac{1}{2}} - B_e^{'n+\frac{1}{2}}) + (B^{n+\frac{3}{2}} - B_e^{'n+\frac{3}{2}}))) \\
&= \Delta t(C'(B^{n+\frac{1}{2}} - B_e^{'n+\frac{1}{2}}), D((E^n - E_e^n) + (E^{n+1} - E_e^{n+1}))) \\
&\quad - \Delta t(C(E^{n+1} - E_e^{n+1}), D'((B^{n+\frac{1}{2}} - B_e^{'n+\frac{1}{2}}) + (B^{n+\frac{3}{2}} - B_e^{'n+\frac{3}{2}}))) \\
&\quad + (\mathcal{M}^n, D((E^n - E_e^n) + (E^{n+1} - E_e^{n+1}))) \\
&\quad + (\mathcal{N}^n, D'((B^{n+\frac{1}{2}} - B_e^{'n+\frac{1}{2}}) + (B^{n+\frac{3}{2}} - B_e^{'n+\frac{3}{2}}))).
\end{aligned}$$

Adding all the equations from $n = 0, 1, ..., N_T - 1$, we obtain

$$
\begin{aligned}
&\|E^{N_T-1} - E_e^{N_T-1}\|_{W'}^2 + \|B^{N_T-\frac{1}{2}} - B_e^{'N_T-\frac{1}{2}}\|_W^2 \\
=&\Delta t(C'(B^{N_T-\frac{1}{2}} - B_e^{'N_T-\frac{1}{2}}), D(E^{N_T-1} - E_e^{N_T-1})) \\
&+ \sum_{i=0}^{N_T-1} ((E_f^{'i+1} - E_e^{i+1}) - (E_f^{'i} - E_e^{i}), (E^i - E_e^i) + (E^{i+1} - E_e^{i+1}))_{W'} \\
&+ \sum_{i=0}^{N_T-2} ((B_f^{i+\frac{3}{2}} - B_e^{'i+\frac{3}{2}}) - (B_f^{i+\frac{1}{2}} - B_e^{'i+\frac{1}{2}}), (B^{i+\frac{1}{2}} - B_e^{'i+\frac{1}{2}}) + (B^{i+\frac{3}{2}} - B_e^{'i+\frac{3}{2}}))_W \\
&+ \mathcal{A}_1 + \mathcal{A}_2,
\end{aligned}
\tag{5.17}
$$

where

$$
\begin{aligned}
\mathcal{A}_1 &:= \sum_{i=0}^{N_T-1} (\mathcal{M}^i, D((E^i - E_e^i) + (E^{i+1} - E_e^{i+1})), \\
\mathcal{A}_2 &:= \sum_{i=0}^{N_T-2} (\mathcal{N}^i, D'(B^{i+\frac{1}{2}} - B_e^{'i+\frac{1}{2}}) + (B^{i+\frac{3}{2}} - B_e^{'i+\frac{3}{2}}))).
\end{aligned}
$$

We give the error estimate of the fully discrete scheme in the following theorem.

Theorem 5.4 *Assume that* $(\mathbf{E}, \mathbf{B}) \in (H^2(0,T; W^{1,p}(\Omega_i))^3)^2$, *for* $i = 1, 2$ *and* $p > 2$, *satisfies (1.1)-(1.4) and* $\mathbf{J}_\Gamma \in H^2(0,T; W^{1,p}(\Gamma))^3$. *Let* $(E^n, B^{n+\frac{1}{2}}), 0 \leq n \leq N_T - 1$, *be the solution of (5.2)-(5.3) on non-uniform grids. Then, under the stability condition*

$$
c_m \Delta t < \frac{\min(h_{ij})}{\sqrt{M_3} M_2^{\frac{3}{2}}}, \tag{5.18}
$$

where M_2 *is the maximum of ratios of the maximum to minimum edge lengths over the union of adjacent elements,* M_3 *is the maximum number of dual edge over all dual faces, and*

$$
c_m^2 := \frac{1}{\min(\epsilon_1, \epsilon_2)\min(\mu_1, \mu_2)},
$$

we have

$$\max_{0\le i\le N_T-1}(\|E^i - E_e^i\|_{W'} + \|B^{i+\frac{1}{2}} - B_e^{i+\frac{1}{2}}\|_W) \le Kh(|\mathbf{J}_\Gamma|_{H^2(0,T;W^{1,p}(\Gamma)^3)} + \sum_{r=1}^{2}\|(\mathbf{E},\mathbf{B})\|_{H^2(0,T;W^{1,p}(\Omega_i)^3)^2}). \tag{5.19}$$

Proof. We consider the right hand side of (5.17). The proof consists of four parts.

(i) Firstly, we have

$$\begin{aligned}&\Delta t(C'(B^{N_T-\frac{1}{2}} - B_e^{'N_T-\frac{1}{2}}), D(E^{N_T-1} - E_e^{N_T-1}))\\ =&\Delta t(C'(SD')^{-\frac{1}{2}}((SD')^{\frac{1}{2}}(B^{N_T-\frac{1}{2}} - B_e^{'N_T-\frac{1}{2}})), (DS'^{-1})^{\frac{1}{2}}((DS')^{\frac{1}{2}}(E^{N_T-1} - E_e^{N_T-1}))\\ \le&\Delta t\|(DS'^{-1})^{\frac{1}{2}}C'(SD')^{-\frac{1}{2}}\|_2\|B^{N_T-\frac{1}{2}} - B_e^{'N_T-\frac{1}{2}}\|_W\|E^{N_T-1} - E_e^{N_T-1}\|_{W'}.\end{aligned}$$

From elementary linear algebra, we know that $\|(DS'^{-1})^{\frac{1}{2}}C'(SD')^{-\frac{1}{2}}\|_2$ is the largest singular value of the matrix $(DS'^{-1})^{\frac{1}{2}}C'(SD')^{-\frac{1}{2}}$. By the Gerschgorin's theorem,

$$\|(DS'^{-1})^{\frac{1}{2}}C'(SD')^{-\frac{1}{2}}\|_2 \le \frac{2\sqrt{M_3}}{\min(\epsilon_1,\epsilon_2)^{\frac{1}{2}}\min(\mu_1,\mu_2)^{\frac{1}{2}}}\max(\frac{\max_{ij}(h_{ij})^{\frac{3}{2}}}{\min_{ij}(h_{ij})^{\frac{5}{2}}}),$$

where $\max_{ij}$ and $\min_{ij}$ are taken over the union of adjacent elements. With the definitions of c_m, M_2 and M_3, we obtain

$$\begin{aligned}&\Delta t(C'(B^{N_T-\frac{1}{2}} - B_e^{'N_T-\frac{1}{2}}), D(E^{N_T-1} - E_e^{N_T-1}))\\ \le&\Delta t c_m\frac{2\sqrt{M_3}M_2^{\frac{3}{2}}}{\min(h_{ij})}\|B^{N_T-\frac{1}{2}} - B_e^{'N_T-\frac{1}{2}}\|_W\|E^{N_T-1} - E_e^{N_T-1}\|_{W'}\\ \le&c_m\Delta t\frac{\sqrt{M_3}M_2^{\frac{3}{2}}}{\min(h_{ij})}(\|B^{N_T-\frac{1}{2}} - B_e^{'N_T-\frac{1}{2}}\|_W^2 + \|E^{N_T-1} - E_e^{N_T-1}\|_{W'}^2).\end{aligned}$$

(ii) By the definition of integral, we have

$$\begin{aligned}|(E_f^{'i+1} - E_e^{i+1}) - (E_f^{'i} - E_e^i)| &= |\int_{i\Delta t}^{(i+1)\Delta t}\dot{E}_f' - \dot{E}_e\,dt\,|\\ &\le \int_{i\Delta t}^{(i+1)\Delta t}|\dot{E}_f' - \dot{E}_e|\,dt.\end{aligned}$$

Then

$$\begin{aligned}
\|(E_f'^{i+1} - E_e^{i+1}) - (E_f'^{i} - E_e^{i})\|_{W'}^2 &= \sum_{j=1}^{M_1} \bar{s}_j' h_j |(E_f'^{i+1} - E_e^{i+1})_j - (E_f'^{i} - E_e^{i})_j|^2 \\
&\le \sum_{j=1}^{M_1} \bar{s}_j' h_j \Big(\int_{i\Delta t}^{(i+1)\Delta t} |(\dot{E}_f' - \dot{E}_e)_j| \, dt \Big)^2 \\
&\le \Delta t \int_{i\Delta t}^{(i+1)\Delta t} \sum_{j=1}^{M_1} \bar{s}_j' h_j |(\dot{E}_f' - \dot{E}_e)_j|^2 \, dt \\
&= \Delta t \int_{i\Delta t}^{(i+1)\Delta t} \|\dot{E}_f' - \dot{E}_e\|_{W'}^2 \, dt.
\end{aligned}$$

By (4.29), we have

$$\begin{aligned}
\|(E_f'^{i+1} - E_e^{i+1}) - (E_f'^{i} - E_e^{i})\|_{W'}^2 &\le K h^2 \Delta t \int_{i\Delta t}^{(i+1)\Delta t} \sum_{r=1}^{2} |\dot{\mathbf{E}}(t)|_{W^{1,p}(\Omega_r)^3}^2 \, dt \\
&\le K h^2 \Delta t \sum_{r=1}^{2} \|\dot{\mathbf{E}}\|_{L^2(i\Delta t, (i+1)\Delta t; W^{1,p}(\Omega_r))^3}^2 .
\end{aligned}$$

Similarly, by (4.30), we have

$$\|(B_f^{i+\frac{3}{2}} - B_e'^{i+\frac{3}{2}}) - (B_f^{i+\frac{1}{2}} - B_e'^{i+\frac{1}{2}})\|_W^2 \le K h^2 \Delta t \sum_{r=1}^{2} \|\dot{\mathbf{B}}\|_{L^2((i+\frac{1}{2})\Delta t, (i+\frac{3}{2})\Delta t; W^{1,p}(\Omega_r))^3}^2 .$$

(iii) From (5.16),

$$\begin{aligned}
\mathcal{N}^i &= -\int_{(i+\frac{1}{2})\Delta t}^{(i+\frac{3}{2})\Delta t} (S\dot{B}_f) \, dt - \Delta t C E_e^{i+1} \\
&= \int_{(i+\frac{1}{2})\Delta t}^{(i+\frac{3}{2})\Delta t} (C E_e) \, dt - \Delta t C E_e^{i+1}.
\end{aligned}$$

Clearly $\mathcal{N}_l^i$, the l-th component of $\mathcal{N}^i$, is a bounded linear functional with variable $(CE_e)_l$ and $\mathcal{N}_l^i = 0$ for constant $(CE_e)_l$ in time, by Lemma 5.1, we have

$$|\mathcal{N}_l^i| \le K \Delta t \|C\dot{E}_e\|_{L^1((i+\frac{1}{2})\Delta t, (i+\frac{3}{2})\Delta t)}.$$

Notice that $(C\dot{E}_e)_l$ is a bounded linear functional with variable $\dot{\mathbf{E}}$ and vanishes for any constant functions in the union of two adjacent polyhedra. By the Bramble-

Hilbert lemma and a standand rescale change argument, we obtain

$$|(C\dot{E}_e)_l| \le Kh^{2-\frac{3}{p}}|\dot{\mathbf{E}}|_{(W^{1,p}(\tau_j \cup \tau_k))^3},$$

where τ_j and τ_k are two primal elements sharing the same face κ_l. Combining the results,

$$|\mathcal{N}_l^i| \le Kh^{2-\frac{3}{p}}\Delta t \int_{(i+\frac{1}{2})\Delta t}^{(i+\frac{3}{2})\Delta t} |\dot{\mathbf{E}}|_{(W^{1,p}(\tau_j \cup \tau_k))^3}\, dt.$$

By the Cauchy-Schwarz's inequality, we obtain

$$|\mathcal{N}_l^i|^2 \le Kh^{4-\frac{6}{p}}(\Delta t)^3 \int_{(i+\frac{1}{2})\Delta t}^{(i+\frac{3}{2})\Delta t} |\dot{\mathbf{E}}|^2_{(W^{1,p}(\tau_j \cup \tau_k))^3}\, dt.$$

Hence

$$\begin{aligned}
\|S^{-1}\mathcal{N}^i\|_W^2 &= \sum_{l=1}^{F_1} s_l \bar{h}_l'(s_l)^{-2}|\mathcal{N}_l^i|^2 \\
&\le Kh^{3-\frac{6}{p}}(\Delta t)^3 \int_{(i+\frac{1}{2})\Delta t}^{(i+\frac{3}{2})\Delta t} \sum_{l=1}^{F_1} |\dot{\mathbf{E}}|^2_{(W^{1,p}(\tau_j \cup \tau_k))^3}\, dt \\
&\le Kh^2\Delta t \sum_{r=1}^{2} \|\mathbf{E}\|^2_{H^1((i+\frac{1}{2})\Delta t,(i+\frac{3}{2})\Delta t; W^{1,p}(\Omega_r))^3}
\end{aligned}$$

where the last line follows from Holder's inequality. By the facts that

$$\sum_{i=1}^{N_T} a_i \le (N_T)^{\frac{1}{2}}(\sum_{i=1}^{N_T} a_i^2)^{\frac{1}{2}} \qquad \textit{and} \qquad N_T\Delta t = T,$$

we have

$$\sum_{i=1}^{N_T-2} \|S^{-1}\mathcal{N}^i\|_W \le Kh \sum_{i=1}^{2} \|\mathbf{E}\|_{H^1(0,T;W^{1,p}(\Omega_i))^3}.$$

(iv) Similiar to $\mathcal{N}^i$, we have

$$\begin{aligned}
\mathcal{M}_l^i &= \int_{i\Delta t}^{(i+1)\Delta t} (\tilde{J} - S'\dot{E}_f)_l\, dt + \Delta t(C'B_e'^{i+\frac{1}{2}})_l \\
&= -\int_{i\Delta t}^{(i+1)\Delta t} (C'B_e')_l\, dt + \Delta t(C'B_e'^{i+\frac{1}{2}})_l
\end{aligned}$$

and for any non-interface dual face κ_l', we can derive the following by the same argument as above:

$$|\mathcal{M}_l^i|^2 \leq Kh^{4-\frac{6}{p}}(\Delta t)^3 \int_{i\Delta t}^{(i+1)\Delta t} |\dot{\mathbf{B}}|^2_{(W^{1,p}(\tau_j' \cup \tau_k'))^3} \, dt.$$

Now, for any interface dual face κ_l', $\mathcal{M}_l^i$ is a bounded linear functional with variable $(C'B_e')_l$ and vanishes for any linear functions in time, so by Lemma 5.1, we have

$$|\mathcal{M}_l^i| \leq K(\Delta t)^2 \|C'\ddot{B}_e'\|_{L^1(i\Delta t,(i+1)\Delta t)}.$$

Notice that

$$\begin{aligned}(C'\ddot{B}_e')_l &= \sum_{\sigma_j' \in \partial\kappa_l'} \tilde{h}_j'(\ddot{B}_e')_j, \\ &= \sum_{\sigma_j' \in \partial\kappa_l^1} \tilde{h}_j'(\ddot{B}_e')_j + \sum_{\sigma_j' \in \partial\kappa_l^2} \tilde{h}_j'(\ddot{B}_e')_j + \frac{d^2}{dt^2}(\tilde{J}_\Gamma)_j,\end{aligned}$$

where

$$\tilde{J}_\Gamma := \sum_{r=1}^{2} \int_{\gamma_r} \mathbf{J}_\Gamma \cdot \mathbf{n} \, d\sigma.$$

Since the first term in the above equation vanishes for any constant functions, we have

$$\Big|\sum_{\sigma_j' \in \partial\kappa_l^1} \tilde{h}_j'(\ddot{B}_e')_j\Big| \leq Kh^{2-\frac{3}{p}} |\ddot{\mathbf{B}}|_{W^{1,p}((\tau_j' \cup \tau_k') \cap \Omega_1)^3},$$

where τ_j' and τ_k' are two dual elements sharing the same dual face κ_l'. Similarly, for the second term, we have

$$\Big|\sum_{\sigma_j' \in \partial\kappa_l^2} \tilde{h}_j'(\ddot{B}_e')_j\Big| \leq Kh^{2-\frac{3}{p}} |\ddot{\mathbf{B}}|_{W^{1,p}((\tau_j' \cup \tau_k') \cap \Omega_2)^3}.$$

The last term can be estimated in the following way

$$\Big|\frac{d^2}{dt^2}(\tilde{J}_\Gamma)_j\Big| \leq Kh \max_{\Gamma} |\ddot{\mathbf{J}}_\Gamma| \leq Kh \|\ddot{\mathbf{J}}_\Gamma\|_{W^{1,p}(\Gamma)^3}.$$

Combining the results,

$$|\mathcal{M}_l^i| \le Kh^{2-\frac{3}{p}}(\Delta t)^2 \int_{i\Delta t}^{(i+1)\Delta t} \sum_{r=1}^{2} |\ddot{\mathbf{B}}|_{W^{1,p}((\tau_j'\cup\tau_k')\cap\Omega_r)^3}\, dt + Kh(\Delta t)^2 \int_{i\Delta t}^{(i+1)\Delta t} \|\ddot{\mathbf{J}}_\Gamma\|_{W^{1,p}(\Gamma)^3}\, dt.$$

By the Cauchy-Schwarz's inequality, we obtain

$$|\mathcal{M}_l^i|^2 \le Kh^{4-\frac{6}{p}}(\Delta t)^5 \int_{i\Delta t}^{(i+1)\Delta t} \sum_{r=1}^{2} |\ddot{\mathbf{B}}|^2_{W^{1,p}((\tau_j'\cup\tau_k')\cap\Omega_r)^3}\, dt + Kh^2(\Delta t)^5 \int_{i\Delta t}^{(i+1)\Delta t} \|\ddot{\mathbf{J}}_\Gamma\|^2_{W^{1,p}(\Gamma)^3}\, dt.$$

Hence, by collecting the results for interface and non-interface components, we have

$$\begin{aligned}\|S'^{-1}\mathcal{M}^i\|^2_{W'} &= \sum_{l=1}^{M_1} \bar{s}_l' h_l (\bar{s}_l')^{-2} |\mathcal{M}_l^i|^2 \\ &\le Kh^{3-\frac{6}{p}}(\Delta t)^5 \int_{i\Delta t}^{(i+1)\Delta t} \sum_{l=1}^{M_1}\sum_{r=1}^{2} |\ddot{\mathbf{B}}|^2_{W^{1,p}((\tau_j'\cup\tau_k')\cap\Omega_r)^3}\, dt \\ &\quad + Kh(\Delta t)^5 N^2 \int_{i\Delta t}^{(i+1)\Delta t} \|\ddot{\mathbf{J}}_\Gamma\|^2_{W^{1,p}(\Gamma)^3}\, dt,\end{aligned}$$

since there are $O(N^2)$ interface components. So, by the Holder's inequality, we get

$$\|S'^{-1}\mathcal{M}^i\|^2_{W'} \le Kh^4(\Delta t) \sum_{r=1}^{2} \|\mathbf{B}\|^2_{H^2(i\Delta t,(i+1)\Delta t;W^{1,p}(\Omega_r))^3} + Kh^3(\Delta t)|\mathbf{J}_\Gamma|^2_{H^2(i\Delta t,(i+1)\Delta t;W^{1,p}(\Gamma))^3},$$

and consequently

$$\sum_{i=1}^{N_T-1} \|S'^{-1}\mathcal{M}^i\|_{W'} \le Kh^2 \sum_{r=1}^{2} \|\mathbf{B}\|_{H^2(0,T;W^{1,p}(\Omega_r))^3} + Kh^{\frac{3}{2}}|\mathbf{J}_\Gamma|_{H^2(0,T;W^{1,p}(\Gamma))^3}.$$

Finally, collecting the terms in (i)-(iv), we get the desired result.

□

5.3.2 Rectangular domain

We devote this section to the convergence analysis of our fully discrete finite volume method when both the domains Ω and Ω_1 are two cuboids. First, we have the following sharp form for the Bramble-Hilbert lemma.

Lemma 5.2 *Suppose that f is a bounded linear functional on the space $W^{2,1}(0,\Delta t)$ and $f(c)=0$ for any linear functions $c \in P_1(0,\Delta t)$. Then there exist a constant K independent of Δt such that*

$$|f(v)| \leq K\Delta t |v|_{W^{2,1}(0,\Delta t)}. \tag{5.20}$$

Moreover, if $f(c)=0$ for any quadratic functions $c \in P_2(0,\Delta t)$, then

$$|f(v)| \leq K(\Delta t)^2 |v|_{W^{3,1}(0,\Delta t)}. \tag{5.21}$$

Proof. Define a linear transformation $\hat{T} : [0,\Delta t] \to [0,1]$ by $\hat{t} = (\Delta t)^{-1} t$. Denote $\hat{v}$ be the transformed function, that is, $\hat{v}(\hat{t}) = v((\Delta t)^{-1}\hat{t}) = v(t)$. Then, by the Bramble-Hilbert lemma, there exist a generic constant K such that

$$|f(\hat{v})| \leq K|\hat{v}|_{W^{2,1}(0,1)}.$$

Notice that, by the chain rule

$$\begin{aligned} |\hat{v}|_{W^{2,1}(0,1)} &= \int_0^1 |\frac{d^2\hat{v}}{d\hat{t}^2}|\, d\hat{t}, \\ &= \int_0^{\Delta t} |(\Delta t)^2 \frac{d^2 v}{dt^2}|\, (\Delta t)^{-1}\, dt, \end{aligned}$$

where in the last step, we have applied the inverse transformation $\hat{T}^{-1}$ to the integral. So, we obtain

$$|\hat{v}|_{W^{2,1}(0,1)} = \Delta t \int_0^{\Delta t} |\frac{d^2 v}{dt^2}|\, dt,$$

which implies (5.20). (5.21) can be proved by a similar argument.

□

We remark here that the above lemma can be generalized to the case that the space $W^{m,1}(0,\Delta t)$ is replaced by $W^{m,1}(n\Delta t,(n+1)\Delta t)$ for $m=2,3$.

From (5.2), we have the following two equations

$$S'(E^{n+1}-E^{n})-\Delta t C'B^{n+\frac{1}{2}}=\tilde{J}^{n+\frac{1}{2}},$$
$$S'(E^{n+2}-E^{n+1})-\Delta t C'B^{n+\frac{3}{2}}=\tilde{J}^{n+\frac{3}{2}}.$$

Subtracting, we have

$$S'(E^{n+2}-2E^{n+1}+E^{n})-\Delta t C'(B^{n+\frac{3}{2}}-B^{n+\frac{1}{2}})=\tilde{J}^{n+\frac{3}{2}}-\tilde{J}^{n+\frac{1}{2}},$$

and by (5.3), we obtain finally

$$S'(E^{n+2}-2E^{n+1}+E^{n})+(\Delta t)^2 C'S^{-1}CE^{n+1}=\tilde{J}^{n+\frac{3}{2}}-\tilde{J}^{n+\frac{1}{2}}. \tag{5.22}$$

For simplicity, we define $U^n:=E^n-E_e^n$. Then, we rewrite (5.22) into the following form

$$\begin{aligned}&S'(U^{n+2}-2U^{n+1}+U^{n})+(\Delta t)^2 C'S^{-1}CU^{n+1}\\&=\tilde{J}^{n+\frac{3}{2}}-\tilde{J}^{n+\frac{1}{2}}-S'(E_e^{n+2}-2E_e^{n+1}+E_e^{n})-(\Delta t)^2 C'S^{-1}CE_e^{n+1}.\end{aligned} \tag{5.23}$$

From (4.1), we know that

$$CE_e^{n+1}=-S\frac{d}{dt}B_f^{n+1},$$

so (5.23) becomes

$$\begin{aligned}&S'(U^{n+2}-2U^{n+1}+U^{n})+(\Delta t)^2 C'S^{-1}CU^{n+1}\\&=\tilde{J}^{n+\frac{3}{2}}-\tilde{J}^{n+\frac{1}{2}}-S'(E_e^{n+2}-2E_e^{n+1}+E_e^{n})+(\Delta t)^2 C'\frac{d}{dt}B_f^{n+1}.\end{aligned} \tag{5.24}$$

We further rewrite (5.24) into the following form

$$\begin{aligned}&S'(U^{n+2}-2U^{n+1}+U^{n})+(\Delta t)^2 C'S^{-1}CU^{n+1}\\&=\tilde{J}^{n+\frac{3}{2}}-\tilde{J}^{n+\frac{1}{2}}-S'(E_e^{n+2}-2E_e^{n+1}+E_e^{n})+\Delta t C'(B_f^{n+\frac{3}{2}}-B_f^{n+\frac{1}{2}})+\mathcal{Q}^{n+1},\end{aligned} \tag{5.25}$$

where

$$\mathcal{Q}^{n+1} := (\Delta t)^2 C' \frac{d}{dt} B_f^{n+1} - \Delta t C' (B_f^{n+\frac{3}{2}} - B_f^{n+\frac{1}{2}}). \tag{5.26}$$

Now we can rewrite (5.25) as

$$\begin{aligned} & S'(U^{n+2} - 2U^{n+1} + U^n) + (\Delta t)^2 C' S^{-1} C U^{n+1} \\ =& (\tilde{J}^{n+\frac{3}{2}} - S'(E_e^{n+2} - E_e^{n+1}) + \int_{(n+1)\Delta t}^{(n+2)\Delta t} C' B_f \, ds) \\ & - (\tilde{J}^{n+\frac{1}{2}} - S'(E_e^{n+1} - E_e^n) + \int_{n\Delta t}^{(n+1)\Delta t} C' B_f \, ds) + \mathcal{R}^{n+\frac{3}{2}} - \mathcal{R}^{n+\frac{1}{2}} + \mathcal{Q}^{n+1}, \end{aligned} \tag{5.27}$$

where

$$\mathcal{R}^{n+\frac{1}{2}} := \Delta t C' B_f^{n+\frac{1}{2}} - \int_{n\Delta t}^{(n+1)\Delta t} C' B_f \, ds.$$

Multiplying both sides of (5.27) by $D(U^{n+2} - U^n)$ and summing up all the equations from $n = 0$ to $n = j$, for any integer j with $0 \le j \le N_T - 3$, we obtain

$$\begin{aligned} & \|U^{j+2} - U^{j+1}\|_{W'}^2 + (\Delta t)^2 (C' S^{-1} C U^{j+2}, D U^{j+1}) \\ =& \|U^1 - U^0\|_{W'}^2 + \mathcal{A}_3 + \mathcal{A}_4, \end{aligned} \tag{5.28}$$

where

$$\begin{aligned} \mathcal{A}_3 :=& \sum_{n=0}^{j} ((\tilde{J}^{n+\frac{3}{2}} - S'(E_e^{n+2} - E_e^{n+1}) + \int_{(n+1)\Delta t}^{(n+2)\Delta t} C' B_f \, ds) \\ & - (\tilde{J}^{n+\frac{1}{2}} - S'(E_e^{n+1} - E_e^n) + \int_{n\Delta t}^{(n+1)\Delta t} C' B_f \, ds), D(U^{n+2} - U^n)), \\ \mathcal{A}_4 :=& \sum_{n=0}^{j} (\mathcal{R}^{n+\frac{3}{2}} - \mathcal{R}^{n+\frac{1}{2}} + \mathcal{Q}^{n+1}, D(U^{n+2} - U^n)). \end{aligned}$$

We further rewrite (5.28) into the following form

$$\begin{aligned} & \|U^{j+2} - U^{j+1}\|_{W'}^2 + (\Delta t)^2 \|U^{j+2}\|_V^2 \\ =& \|U^1 - U^0\|_{W'}^2 + (\Delta t)^2 (C' S^{-1} C U^{j+2}, D(U^{j+2} - U^{j+1})) + \mathcal{A}_3 + \mathcal{A}_4. \end{aligned} \tag{5.29}$$

Analogous to Lemma 4.4 and Lemma 4.5, we have the following:

Lemma 5.3 *Assume that $C(E^n - E_e^n) \neq 0$ for $2 \leq n \leq N_T - 1$. Then there exist a constant K independent of h such that*

$$\|E^n - E_e^n\|_{W_\Gamma} \leq K\|E^n - E_e^n\|_V. \tag{5.30}$$

Proof. By (3.15) in Lemma 3.5, we have for any $u \in \mathbb{R}^M$ with $u|_{\partial\Omega} = 0$ and $Cu \neq 0$,

$$(S'u, Du) \leq K(D'S^{-1}Cu, Cu). \tag{5.31}$$

Consider the following auxillary problem: Find $\tilde{u}^{n+1} \in \mathbb{R}^M$ such that

$$\begin{cases} C\tilde{u}^{n+1} = Sl^{n+1}, & \text{for all interior primal face} \\ \tilde{u}^{n+1} = E^{n+1} - E_e^{n+1}, & \text{for all interface primal edge,} \end{cases} \tag{5.32}$$

where

$$l^{n+1} := \frac{d}{dt}B_f^{n+1} - (\Delta t)^{-1}(B^{n+\frac{3}{2}} - B^{n+\frac{1}{2}}).$$

By (4.23) and (5.3), we have

$$C(E^{n+1} - E_e^{n+1}) = S\frac{d}{dt}B_f^{n+1} - (\Delta t)^{-1}S(B^{n+\frac{3}{2}} - B^{n+\frac{1}{2}}). \tag{5.33}$$

Hence, the problem (5.32) has a solution $\tilde{u}^{n+1} = E^{n+1} - E_e^{n+1}$. Now, we solve the problem (5.32) in the following way. For each $\tilde{u}_j^{n+1}$ corresponding to an primal edge σ_j in Ω_2, we take $\tilde{u}_j^{n+1} = (E^{n+1} - E_e^{n+1})_j$, where $(E^{n+1} - E_e^{n+1})_j$ is a component of $E^{n+1} - E_e^{n+1}$ corresponding to σ_j. Then, with the components corresponding to Ω_2 and Γ are already fixed, we rewrite (5.32) into the following linear system

$$G_1 D\tilde{u}^{n+1} = b^{n+1}, \tag{5.34}$$

where b^{n+1} is a vector containing all the related known components and G_1 is the restriction of G to Ω_1. We remark here that in system (5.34), number of equations is in general greater than number of unknowns. However, since (5.32) has a solution, the system (5.34) is consistent.

Since the matrix G_1 has the same structure as the matrix G, by Lemma 3.3, there are $O(N^3)$ free variables in the system (5.34). We choose these free variables are the interface components with the condition that each component appears $O(N)$ times. We can do this since there are $O(N^2)$ interface components. Then, after fixing free variables, the other components can be uniquely determined by solving the system (5.34).

Putting $\tilde{u}^{n+1}$ into the equation (5.31), we have

$$(S'\tilde{u}^{n+1}, D\tilde{u}^{n+1}) \leq K(D'S^{-1}C\tilde{u}^{n+1}, C\tilde{u}^{n+1}). \tag{5.35}$$

For the left hand side, we have

$$(S'\tilde{u}^{n+1}, D\tilde{u}^{n+1}) \geq (S'\bar{u}^{n+1}, D\bar{u}^{n+1}),$$

where $\bar{u}$ denotes a vector having the same interface components and free components as $\tilde{u}^{n+1}$ and having the other components vanish. So, we have

$$(S'\bar{u}^{n+1}, D\bar{u}^{n+1}) \geq K\|E^{n+1} - E_e^{n+1}\|_{W_\Gamma}^2.$$

For the right hand side, since $\tilde{u}^{n+1}$ is the solution to the system (5.34), we have

$$(D'S^{-1}C\tilde{u}^{n+1}, C\tilde{u}^{n+1}) = (D'l^{n+1}, Sl^{n+1}).$$

Multiplying both sides of (5.33) by $D'l^{n+1}$, we have

$$\begin{aligned}
&(Sl^{n+1}, D'l^{n+1}) \\
&= (C(E^{n+1} - E_e^{n+1}), D'l^{n+1}) \\
&\leq K(D'S^{-1}C(E^{n+1} - E_e^{n+1}), C(E^{n+1} - E_e^{n+1}))^{\frac{1}{2}}(Sl^{n+1}, D'l^{n+1})^{\frac{1}{2}}.
\end{aligned}$$

Hence, we obtain the desired estimate.

□

Lemma 5.4 *Assume that $C(E^n - E^n_e) \neq 0$ for $2 \leq n \leq N_T - 1$. Then there exist a constant K independent of h such that*

$$\max_{\sigma_j \in \Gamma} |E^n - E^n_e|_j \leq K\|E^n - E^n_e\|_V \tag{5.36}$$

$$\max_{\sigma_j \in \Omega} |E^n - E^n_e|_j \leq K\|E^n - E^n_e\|_V \tag{5.37}$$

Proof. (5.36) follows from the the proof of Lemma 5.3 by choosing all the free variables as $\max_{\sigma_j \in \Gamma} |E^n - E^n_e|_j$. Similarly, (5.37) follows from the the proof of Lemma 5.3 by choosing all the free variables as $\max_{\sigma_j \in \Omega} |E^n - E^n_e|_j$.

□

In the following theorem, we give the V-norm estimate for $E^n - E^n_e$.

Theorem 5.5 *Assume that $\mathbf{B} \in W^{2,1}(0,T;H^3(\Omega_i))^3 \cap W^{4,1}(0,T;H^2(\Omega_r))^3$, for $i = 1,2$, satisfies (1.1)-(1.4), $\mathbf{J} \in W^{2,1}(0,T;H^2(\Omega))^3$ and $\mathbf{J}_\Gamma \in W^{2,1}(0,T;H^3(\Gamma))^3$. Let E^n, $0 \leq n \leq N_T - 1$, be the solution of (5.2)-(5.3) on uniform grids. Then under the stability condition*

$$c_m \Delta t < \frac{\min(h_{ij})}{2M_2}, \tag{5.38}$$

where M_2 is the maximum of the ratios of the maximum to minimum edge lengths over the union of adjacent elements, and

$$c_m^2 := \frac{1}{\min(\epsilon_1, \epsilon_2)\min(\mu_1, \mu_2)},$$

we have

$$\begin{aligned} &\max_{0 \leq n \leq N_T - 1} \|E^n - E^n_e\|_V \\ \leq & Kh^2 \Big(\sum_{r=1}^{2} \|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \sum_{r=1}^{2} \|\mathbf{B}\|_{W^{4,1}(0,T;H^2(\Omega_r))^3} \\ &+ \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^2} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}\Big). \end{aligned} \tag{5.39}$$

Proof. By considering (5.29), we divide the proof into three parts.

(i) First, we have

$$\begin{aligned}
&(\Delta t)^2(C'S^{-1}CU^{j+2}, D(U^{j+2}-U^{j+1})) \\
&\le (\Delta t)^2(C'D'^{-\frac{1}{2}}S^{-\frac{1}{2}}(D'S^{-1})^{\frac{1}{2}}CU^{j+2}, S'^{-\frac{1}{2}}D^{\frac{1}{2}}(S'D)^{\frac{1}{2}}(U^{j+2}-U^{j+1})) \\
&\le (\Delta t)^2\|S'^{-\frac{1}{2}}D^{\frac{1}{2}}C'D'^{-\frac{1}{2}}S^{-\frac{1}{2}}\|_2\|U^{j+2}\|_V\|U^{j+2}-U^{j+1}\|_{W'}.
\end{aligned}$$

From elementary linear algebra, we know that $\|S'^{-\frac{1}{2}}D^{\frac{1}{2}}C'D'^{-\frac{1}{2}}S^{-\frac{1}{2}}\|_2$ is the largest singular value of the matrix $S'^{-\frac{1}{2}}D^{\frac{1}{2}}C'D'^{-\frac{1}{2}}S^{-\frac{1}{2}}$. By the Gerschgorin's theorem,

$$\|S'^{-\frac{1}{2}}D^{\frac{1}{2}}C'D'^{-\frac{1}{2}}S^{-\frac{1}{2}}\|_2 \le \frac{4}{\min(\epsilon_1,\epsilon_2)^{\frac{1}{2}}\min(\mu_1,\mu_2)^{\frac{1}{2}}}\max\left(\frac{\max_{ij}(h_{ij})^{\frac{3}{2}}}{\min_{ij}(h_{ij})^{\frac{5}{2}}}\right),$$

where $\max_{ij}$ and $\min_{ij}$ are taken over the union of adjacent elements. From the definitions of c_m and M_2, we obtain

$$\begin{aligned}
&(\Delta t)^2(C'S^{-1}CU^{j+2}, D(U^{j+2}-U^{j+1})) \\
&\le (\Delta t)^2\frac{4M_2c_m}{\min_{ij}(h_{ij})}\|U^{j+2}\|_V\|U^{j+2}-U^{j+1}\|_{W'} \\
&\le \Delta t\frac{2M_2c_m}{\min_{ij}(h_{ij})}(\|U^{j+2}-U^{j+1}\|_{W'}^2+(\Delta t)^2\|U^{j+2}\|_V^2).
\end{aligned}$$

(ii) We now estimate $\mathcal{A}_3$. By the definition of $\tilde{J}^{n+\frac{3}{2}}$, we have

$$\begin{aligned}
&\tilde{J}^{n+\frac{3}{2}} - S'(E_e^{n+2}-E_e^{n+1}) + \int_{(n+1)\Delta t}^{(n+2)\Delta t} C'B_f\, ds \\
&= \int_{(n+1)\Delta t}^{(n+2)\Delta t} (\tilde{J} - S'\frac{dE_e}{dt} + C'B_f)\, ds \\
&= \int_{(n+1)\Delta t}^{(n+2)\Delta t} f\, ds.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\tilde{J}^{n+\frac{1}{2}} - S'(E_e^{n+1}-E_e^{n}) + \int_{n\Delta t}^{(n+1)\Delta t} C'B_f\, ds \\
&= \int_{n\Delta t}^{(n+1)\Delta t} f\, ds.
\end{aligned}$$

Observe that

$$\begin{aligned}&\int_{(n+1)\Delta t}^{(n+2)\Delta t} f\ ds - \int_{n\Delta t}^{(n+1)\Delta t} f\ ds\\ =&(\int_{(n+1)\Delta t}^{(n+2)\Delta t} f\ ds - \Delta t f(t_{n+1})) + (\Delta t f(t_{n+1}) - \int_{n\Delta t}^{(n+1)\Delta t} f\ ds).\end{aligned}$$

By virtue of the Taylor expansion, there exist $\xi^{n+\frac{3}{2}} \in ((n+1)\Delta t, (n+2)\Delta t)$ such that

$$\int_{(n+1)\Delta t}^{(n+2)\Delta t} f\ ds - \Delta t f(t_{n+1}) = \frac{1}{2}(\Delta t)^2 \frac{d}{dt} f(\xi^{n+\frac{3}{2}}).$$

Similarly, there exist $\xi^{n+\frac{1}{2}} \in (n\Delta t, (n+1)\Delta t)$ such that

$$\Delta t f(t_{n+1}) - \int_{n\Delta t}^{(n+1)\Delta t} f\ ds = \frac{1}{2}(\Delta t)^2 \frac{d}{dt} f(\xi^{n+\frac{1}{2}}).$$

So, we rewrite $\mathcal{A}_3$ as

$$\mathcal{A}_3 = \sum_{n=0}^{j}(\frac{1}{2}(\Delta t)^2 \frac{d}{dt} f(\xi^{n+\frac{3}{2}}) + \frac{1}{2}(\Delta t)^2 \frac{d}{dt} f(\xi^{n+\frac{1}{2}}), D(U^{n+2} - U^n)).$$

Also, there exist $\eta^{n+1} \in (\xi^{n+\frac{1}{2}}, \xi^{n+\frac{3}{2}}) \subset (n\Delta t, (n+2)\Delta t)$ such that

$$\frac{1}{2}(\Delta t)^2 \frac{d}{dt} f(\xi^{n+\frac{3}{2}}) + \frac{1}{2}(\Delta t)^2 \frac{d}{dt} f(\xi^{n+\frac{1}{2}}) = (\Delta t)^2 \frac{d}{dt} f(\eta^{n+1}),$$

and consequently

$$\mathcal{A}_3 = \sum_{n=0}^{j}(\Delta t)^2 (\frac{d}{dt} f(\eta^{n+1}), D(U^{n+2} - U^n)).$$

By summation by parts, we have

$$\begin{aligned}\mathcal{A}_3 =&(\Delta t)^2(\frac{d}{dt} f(\eta_{j+1}), DU^{j+2}) + (\Delta t)^2(\frac{d}{dt} f(\eta_j), DU^{j+1})\\ &- \sum_{n=2}^{j}(\Delta t)^2(\frac{d}{dt} f(\eta_{n+1}) - \frac{d}{dt} f(\eta_{n-1}), DU^n).\end{aligned}$$

By Lemma 4.3, Lemma 5.3 and Lemma 5.4, we follow the same proof as in Theorem 4.5, then the following can be proved

$$(\frac{d}{dt} f(\eta_{j+1}), DU^{j+2}) \leq Kh^2(\sum_{r=1}^{2} |\dot{\mathbf{B}}|_{H^3(\Omega_r)^3} + |\dot{\mathbf{J}}|_{H^2(\Omega)^3} + |\dot{\mathbf{J}}_\Gamma|_{H^3(\Gamma)^3})(\eta_{j+1})\|U^{j+2}\|_V.$$

Consequently, we obtain

$$\begin{aligned}&(\frac{d}{dt}f(\eta_{j+1}), DU^{j+2})\\ \leq& Kh^2 \max_{2\leq n\leq N_T-1} \|U^n\|_V\\ &\times (\sum_{r=1}^{2} \|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^3} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}).\end{aligned}$$

Similarly,

$$\begin{aligned}&(\frac{d}{dt}f(\eta_{j}), DU^{j+1})\\ \leq& Kh^2 \max_{2\leq n\leq N_T-1} \|U^n\|_V\\ &\times (\sum_{r=1}^{2} \|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^3} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^2(\Gamma))^3}).\end{aligned}$$

By the definition of integral, we have

$$\frac{d}{dt}f(\eta^{n+1}) - \frac{d}{dt}f(\eta^{n-1}) = \int_{\eta^{n-1}}^{\eta^{n+1}} \frac{d^2 f}{dt^2}\, ds.$$

Following the proof in Theorem 4.5, we obtain

$$(\frac{d^2 f}{dt^2}, DU^n) \leq Kh^2(\sum_{r=1}^{2} |\ddot{\mathbf{B}}|_{H^3(\Omega_r)^3} + |\ddot{\mathbf{J}}|_{H^2(\Omega)^3} + |\ddot{\mathbf{J}}_\Gamma|_{H^3(\Gamma)^3})\|U^n\|_V.$$

So,

$$\begin{aligned}&-\sum_{n=2}^{j}(\frac{d}{dt}f(\eta^{n+1}) - \frac{d}{dt}f(\eta^{n-1}), DU^n)\\ \leq& Kh^2 \sum_{n=2}^{j} \int_{\eta^{n-1}}^{\eta^{n+1}} (\sum_{r=1}^{2} |\ddot{\mathbf{B}}|_{H^3(\Omega_r)^3} + |\ddot{\mathbf{J}}|_{H^2(\Omega)^3} + |\ddot{\mathbf{J}}_\Gamma|_{H^3(\Gamma)^3})\|U^n\|_V,\end{aligned}$$

and consequently,

$$\begin{aligned}&-\sum_{n=2}^{j}(\frac{d}{dt}f(\eta^{n+1}) - \frac{d}{dt}f(\eta^{n-1}), DU^n)\\ \leq& Kh^2 \max_{2\leq n\leq N_T-1} \|U^n\|_V \sum_{n=2}^{j} \int_{(n-2)\Delta t}^{(n+2)\Delta t} (\sum_{r=1}^{2} |\ddot{\mathbf{B}}|_{H^3(\Omega_r)^3} + |\ddot{\mathbf{J}}|_{H^2(\Omega)^3} + |\ddot{\mathbf{J}}_\Gamma|_{H^3(\Gamma)^3})\\ \leq& Kh^2 \max_{2\leq n\leq N_T-1} \|U^n\|_V\\ &\times (\sum_{r=1}^{2} |\mathbf{B}|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + |\mathbf{J}|_{W^{2,1}(0,T;H^2(\Omega))^3} + |\mathbf{J}_\Gamma|_{W^{2,1}(0,T;H^3(\Gamma))^3}).\end{aligned}$$

(iii) We rewrite $\mathcal{Q}^{n+1}$ as

$$\mathcal{Q}^{n+1} = (\Delta t)^2 G^T (D'\dot{B}_f^{n+1} - \frac{1}{\Delta t}\int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t} D'\dot{B}_f \, ds).$$

Hence, we have

$$(\mathcal{Q}^{n+1}, D(U^{n+2} - U^n)) = (\Delta t)^2 (D'(\dot{B}_f^{n+1} - \frac{1}{\Delta t}\int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t} \dot{B}_f \, ds), C(U^{n+2} - U^n)).$$

By using the Taylor expansion, there exist $\xi^{n+1} \in ((n+\frac{1}{2})\Delta t, (n+\frac{3}{2})\Delta t)$ such that

$$\dot{B}_f^{n+1} - \frac{1}{\Delta t}\int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t} \dot{B}_f \, ds = \frac{1}{24}(\Delta t)^2 \dddot{B}_f(\xi^{n+1}).$$

So, we have

$$(\mathcal{Q}^{n+1}, D(U^{n+2} - U^n)) = \frac{1}{24}(\Delta t)^4 (D'\dddot{B}_f(\xi^{n+1}), C(U^{n+2} - U^n)).$$

By summation by parts, we have

$$\begin{aligned}
&\sum_{n=0}^{j}(\mathcal{Q}^{n+1}, D(U^{n+2} - U^n)) \\
=&\frac{1}{24}(\Delta t)^4 (D'\dddot{B}_f(\xi^{j+1}), CU^{j+2}) + \frac{1}{24}(\Delta t)^4 (D'\dddot{B}_f(\xi^{j}), CU^{j+1}) \\
&- \frac{1}{24}(\Delta t)^4 \sum_{n=2}^{j}(D'(\dddot{B}_f(\xi^{n+1}) - \dddot{B}_f(\xi^{n-1})), CU^n).
\end{aligned}$$

Notice that

$$\|\dddot{B}_f\|_W \le K \sum_{r=1}^{2} |\dddot{\mathbf{B}}|_{C^0(\Omega_r)^3} \le K \sum_{r=1}^{2} |\dddot{\mathbf{B}}|_{H^2(\Omega_r)^3}.$$

So, we have

$$\begin{aligned}
&\frac{1}{24}(\Delta t)^4 (D'\dddot{B}_f(\xi^{j+1}), CU^{j+2}) \\
\le& K(\Delta t)^4 \|\dddot{B}_f(\xi^{j+1})\|_W \|U^{j+2}\|_V \\
\le& K(\Delta t)^4 \max_{2\le n\le N_T-1} \|U^n\|_V \sum_{r=1}^{2} |\dddot{\mathbf{B}}(\xi^{j+1})|_{H^2(\Omega_r)^3} \\
\le& Kh^2(\Delta t)^2 \max_{2\le n\le N_T-1} \|U^n\|_V \sum_{r=1}^{2} \|\mathbf{B}\|_{W^{4,1}(0,T;H^2(\Omega_r))^3},
\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{24}(\Delta t)^4(D'\ddot{B}_f(\xi^j), CU^{j+1})\\ \leq& Kh^2(\Delta t)^2 \max_{2\leq n\leq N_T-1} \|U^n\|_V \sum_{r=1}^{2} \|\mathbf{B}\|_{W^{4,1}(0,T;H^2(\Omega_r))^3}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&-\frac{1}{24}(\Delta t)^4 \sum_{n=2}^{j}(D'(\dddot{B}_f(\xi^{n+1}) - \dddot{B}_f(\xi^{n-1})), CU^n)\\ =&-\frac{1}{24}(\Delta t)^4 \sum_{n=2}^{j}\int_{\xi^{n-1}}^{\xi^{n+1}} (D'\frac{d^4}{dt^4}B_f, CU^n)\, ds\\ \leq& K(\Delta t)^4 \sum_{n=2}^{j}\int_{\xi^{n-1}}^{\xi^{n+1}} \|\frac{d^4}{dt^4}B_f\|_W \|U^n\|_V\, ds\\ \leq& Kh^2(\Delta t)^2 \max_{2\leq n\leq N_T-1} \|U^n\|_V \sum_{n=2}^{j}\int_{\xi^{n-1}}^{\xi^{n+1}} |\frac{\partial^4\mathbf{B}}{\partial t^4}|_{H^2(\Omega_r)^3}\, ds,\end{aligned}$$

and consequently,

$$\begin{aligned}&-\frac{1}{24}(\Delta t)^4 \sum_{n=2}^{j}(D'(\dddot{B}_f(\xi^{n+1}) - \dddot{B}_f(\xi^{n-1})), CU^n)\\ \leq& Kh^2(\Delta t)^2 \max_{2\leq n\leq N_T-1} \|U^n\|_V |\mathbf{B}|_{W^{4,1}(0,T;H^2(\Omega_r))^3}.\end{aligned}$$

(iv) We rewrite $\mathcal{R}^{n+\frac{3}{2}}$ as

$$\mathcal{R}^{n+\frac{3}{2}} = \Delta t G^T D'(B_f^{n+\frac{3}{2}} - \frac{1}{\Delta t}\int_{(n+1)\Delta t}^{(n+2)\Delta t} B_f\, ds).$$

Similarly, we have

$$\mathcal{R}^{n+\frac{1}{2}} = \Delta t G^T D'(B_f^{n+\frac{1}{2}} - \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} B_f\, ds).$$

So, we get

$$(\mathcal{R}^{n+\frac{3}{2}} - \mathcal{R}^{n+\frac{1}{2}}, D(U^{n+2} - U^n)) = \Delta t(D'I^{n+1}, C(U^{n+2} - U^n)),$$

where

$$I^{n+1} := (B_f^{n+\frac{3}{2}} - \frac{1}{\Delta t}\int_{(n+1)\Delta t}^{(n+2)\Delta t} B_f\, ds) - (B_f^{n+\frac{1}{2}} - \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} B_f\, ds).$$

By the Taylor expansion, there exist $\xi^{n+\frac{3}{2}} \in ((n+1)\Delta t, (n+2)\Delta t)$ such that

$$
\begin{aligned}
&B_f^{n+\frac{3}{2}} - \frac{1}{\Delta t}\int_{(n+1)\Delta t}^{(n+2)\Delta t} B_f \, ds \\
=& -\frac{1}{24}(\Delta t)^2 \ddot{B}_f^{n+\frac{3}{2}} - \frac{1}{\Delta t}\int_{(n+1)\Delta t}^{(n+2)\Delta t} \frac{1}{24}(s - t_{n+\frac{3}{2}})^4 \frac{\partial^4}{\partial t^4} B_f(\xi^{n+\frac{3}{2}}) \, ds.
\end{aligned}
$$

Also, there exist $\xi^{n+\frac{1}{2}} \in (n\Delta t, (n+1)\Delta t)$ such that

$$
\begin{aligned}
&B_f^{n+\frac{1}{2}} - \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} B_f \, ds \\
=& -\frac{1}{24}(\Delta t)^2 \ddot{B}_f^{n+\frac{1}{2}} - \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \frac{1}{24}(s - t_{n+\frac{1}{2}})^4 \frac{\partial^4}{\partial t^4} B_f(\xi^{n+\frac{1}{2}}) \, ds.
\end{aligned}
$$

Hence,

$$
I^{n+1} = I_1^{n+1} + I_2^{n+1},
$$

where

$$
\begin{aligned}
I_1^{n+1} :=& -\frac{1}{24}(\Delta t)^2 (\ddot{B}_f^{n+\frac{3}{2}} - \ddot{B}_f^{n+\frac{1}{2}}) \\
I_2^{n+1} :=& -\frac{1}{\Delta t}\int_{(n+1)\Delta t}^{(n+2)\Delta t} \frac{1}{24}(s - t_{n+\frac{3}{2}})^4 \frac{\partial^4}{\partial t^4} B_f(\xi^{n+\frac{3}{2}}) \, ds \\
&+ \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \frac{1}{24}(s - t_{n+\frac{1}{2}})^4 \frac{\partial^4}{\partial t^4} B_f(\xi^{n+\frac{1}{2}}) \, ds.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
&\sum_{n=0}^{j} (\mathcal{R}^{n+\frac{3}{2}} - \mathcal{R}^{n+\frac{1}{2}}, D(U^{n+2} - U^n)) \\
=&\Delta t \sum_{n=0}^{j} (D' I_1^{n+1}, C(U^{n+2} - U^n)) + \Delta t \sum_{n=0}^{j} (D' I_2^{n+1}, C(U^{n+2} - U^n)).
\end{aligned}
$$

By the mean value theorem, there exist $\eta^{n+1} \in ((n+\frac{1}{2})\Delta t, (n+\frac{3}{2})\Delta t)$ such that

$$
I_1^{n+1} = -\frac{1}{24}(\Delta t)^3 \dddot{B}_f(\eta^{n+1}).
$$

By summation by parts, we have

$$
\begin{aligned}
\sum_{n=0}^{j} (D' I_1^{n+1}, C(U^{n+2} - U^n)) =& (D' I_1^{j+1}, CU^{j+2}) + (D' I_1^{j}, CU^{j+1}) \\
&- \sum_{n=2}^{j+1} (D'(I_1^{n+1} - I_1^{n-1}), CU^n).
\end{aligned}
$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}(D'I_1^{j+1}, CU^{j+2}) &\leq K\|I_1^{j+1}\|_W\|U^{j+2}\|_V \\ &\leq K(\Delta t)^3 \max_{2\leq n\leq N_T-1}\|U^n\|_V\|\dddot{B}_f(\eta^{j+1})\|_W \\ &\leq Kh^2\Delta t \max_{2\leq n\leq N_T-1}\|U^n\|_V\sum_{r=1}^{2}\|\dddot{\mathbf{B}}(\eta^{j+1})\|_{H^2(\Omega_r)^3} \\ &\leq Kh^2\Delta t \max_{2\leq n\leq N_T-1}\|U^n\|_V\sum_{r=1}^{2}\|\mathbf{B}\|_{W^{4,1}(0,T;H^2(\Omega_r))^3}.\end{aligned}$$

Similarly, we have

$$(D'I_1^{j}, CU^{j+1}) \leq Kh^2\Delta t \max_{2\leq n\leq N_T-1}\|U^n\|_V\sum_{r=1}^{2}\|\mathbf{B}\|_{W^{4,1}(0,T;H^2(\Omega_r))^3}.$$

By the definition of integral,

$$(D'(I_1^{n+1}-I_1^{n-1}), CU^n) = \int_{\eta^{n-1}}^{\eta^{n+1}}(D'\frac{dI_1}{dt}, CU^n)\, ds.$$

So,

$$-\sum_{n=2}^{j}(D'(I_1^{n+1}-I_1^{n-1}), CU^n) \leq K\sum_{n=2}^{j}\int_{\eta^{n-1}}^{\eta^{n+1}}\|\frac{dI_1}{dt}\|_W\|U^n\|_V\, ds.$$

Consequently, we obtain

$$\begin{aligned}&-\sum_{n=2}^{j}(D'(I_1^{n+1}-I_1^{n-1}), CU^n) \\ \leq &K(\Delta t)^3 \max_{2\leq n\leq N_T-1}\|U^n\|_V\sum_{n=2}^{j}\int_{\eta^{n-1}}^{\eta^{n+1}}\|\frac{d^4}{dt^4}B_f\|_W\, ds \\ \leq &Kh^2\Delta t \max_{2\leq n\leq N_T-1}\|U^n\|_V\sum_{n=2}^{j}\int_{\eta^{n-1}}^{\eta^{n+1}}\sum_{r=1}^{2}\|\frac{\partial^4\mathbf{B}}{\partial t^4}\|_{H^2(\Omega_r)^3}\, ds \\ \leq &Kh^2\Delta t \max_{2\leq n\leq N_T-1}\|U^n\|_V\sum_{r=1}^{2}|\mathbf{B}|_{W^{4,1}(0,T;H^2(\Omega_r))^3}.\end{aligned}$$

By the definition of I_2^{n+1}, we have

$$
\begin{aligned}
&(D'I_2^{n+1}, C(U^{n+2} - U^n)) \\
=&-\frac{1}{\Delta t}\int_{(n+1)\Delta t}^{(n+2)\Delta t} \frac{1}{24}(s - t_{n+\frac{3}{2}})^4 (\frac{\partial^4}{\partial t^4} B_f(\xi^{n+\frac{3}{2}}), C(U^{n+2} - U^n))\, ds \\
&+\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \frac{1}{24}(s - t_{n+\frac{1}{2}})^4 (\frac{\partial^4}{\partial t^4} B_f(\xi^{n+\frac{1}{2}}), C(U^{n+2} - U^n))\, ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|(D'I_2^{n+1}, C(U^{n+2} - U^n))| \\
\leq&\frac{1}{24}(\Delta t)^3 \int_{(n+1)\Delta t}^{(n+2)\Delta t} \|\frac{\partial^4}{\partial t^4} B_f(s)\|_W \|U^{n+2} - U^n\|_V\, ds \\
&+\frac{1}{24}(\Delta t)^3 \int_{n\Delta t}^{(n+1)\Delta t} \|\frac{\partial^4}{\partial t^4} B_f(s)\|_W \|U^{n+2} - U^n\|_V\, ds \\
\leq&\frac{1}{24}(\Delta t)^3 \max_{2\leq n\leq N_T-1} \|U^n\|_V \int_{n\Delta t}^{(n+2)\Delta t} \|\frac{\partial^4}{\partial t^4} B_f(s)\|_W\, ds.
\end{aligned}
$$

Since

$$
\|\frac{\partial^4}{\partial t^4} B_f(s)\|_W \leq K \sum_{r=1}^{2} \|\frac{\partial^4 \mathbf{B}}{\partial t^4}\|_{H^2(\Omega_r)^3},
$$

we finally obtain

$$
\begin{aligned}
&|\sum_{n=0}^{j}(D'I_2^{n+1}, C(U^{n+2} - U^n))| \\
\leq&\frac{1}{24}(\Delta t)^3 \max_{2\leq n\leq N_T-1} \|U^n\|_V \sum_{n=0}^{j} \int_{n\Delta t}^{(n+2)\Delta t} \sum_{r=1}^{2} \|\frac{\partial^4 \mathbf{B}}{\partial t^4}\|_{H^2(\Omega_r)^3}\, ds \\
\leq&\frac{1}{24}(\Delta t)^3 \max_{2\leq n\leq N_T-1} \|U^n\|_V \sum_{r=1}^{2} \|\mathbf{B}\|_{W^{4,1}(0,T;H^2(\Omega_r))^3}.
\end{aligned}
$$

Collecting terms in (i)-(iv), we obtain the desired result.

□

We now give our main estimate in this section.

Theorem 5.6 *Assume that* $\mathbf{B} \in W^{2,1}(0,T;H^3(\Omega_i))^3 \cap W^{4,1}(0,T;H^2(\Omega_r))^3$, $\mathbf{E} \in W^{1,1}(0,T;H^3(\Omega_r))^3$, *for* $i = 1,2$, *satisfy (1.1)-(1.4),* $\mathbf{J} \in W^{2,1}(0,T;H^2(\Omega))^3$ *and* $\mathbf{J}_\Gamma \in W^{2,1}(0,T;H^3(\Gamma))^3$. *Let* $(E^n, B^{n+\frac{1}{2}})$, $0 \le n \le N_T - 1$, *be the solution of (5.2)-(5.3) on uniform grids. Then under the stability condition*

$$c_m \Delta t < \frac{\min(h_{ij})}{2M_2}, \tag{5.40}$$

where M_2 *is the maximum of the ratios of the maximum to minimum edge lengths over the union of adjacent elements, and*

$$c_m^2 := \frac{1}{\min(\epsilon_1,\epsilon_2)\min(\mu_1,\mu_2)},$$

we have

$$\begin{aligned}
&\max_{0\le n\le N_T-1}(\|E^n - E_e^n\|_{W'} + \|B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}}\|_W \\
\le & Kh^2\Big(\sum_{r=1}^{2}(\|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \|\mathbf{B}\|_{W^{4,1}(0,T;H^2(\Omega_r))^3} + \|\mathbf{E}\|_{W^{1,1}(0,T;H^3(\Omega_r))^3}) \\
&+ \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^2} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}\Big).
\end{aligned} \tag{5.41}$$

Proof. From (5.3), we have

$$S(B^{n+\frac{3}{2}} - B^{n+\frac{1}{2}}) + \Delta t C E^{n+1} = 0.$$

So,

$$S((B^{n+\frac{3}{2}} - B_f^{n+\frac{3}{2}}) - (B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}})) + \Delta t C(E^{n+1} - E_e^{n+1}) = \mathcal{P}^{n+1}, \tag{5.42}$$

where

$$\mathcal{P}^{n+1} := -SB_f^{n+\frac{3}{2}} + SB_f^{n+\frac{1}{2}} - \Delta t C E_e^{n+1}.$$

From (4.23), we rewrite $\mathcal{P}^{n+1}$ as

$$\mathcal{P}^{n+1} = S\Delta t(\dot{B}_f^{n+1} - \frac{1}{\Delta t}\int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t} \dot{B}_f \, ds).$$

Multiplying both sides of (5.42) by $D'((B^{n+\frac{3}{2}} - B_f^{n+\frac{3}{2}}) + (B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}}))$ and summing up all the equations from $n = 0$ to $n = j$, where $0 \leq j \leq N_T - 2$, we obtain

$$\begin{aligned}
&(S(B^{j+\frac{3}{2}} - B_f^{j+\frac{3}{2}}), D'(B^{j+\frac{3}{2}} - B_f^{j+\frac{3}{2}})) \\
=& -\Delta t \sum_{n=0}^{j} (C(E^{n+1} - E_e^{n+1}), D'((B^{n+\frac{3}{2}} - B_f^{n+\frac{3}{2}}) + (B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}}))) \\
&+ \sum_{n=0}^{j} (\mathcal{P}^{n+1}, D'((B^{n+\frac{3}{2}} - B_f^{n+\frac{3}{2}}) + (B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}}))).
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&| -\Delta t \sum_{n=0}^{j} (C(E^{n+1} - E_e^{n+1}), D'((B^{n+\frac{3}{2}} - B_f^{n+\frac{3}{2}}) + (B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}})))| \\
\leq& \Delta t \sum_{n=0}^{N_T-2} \|E^{n+1} - E_e^{n+1}\|_V (\|B^{n+\frac{3}{2}} - B_f^{n+\frac{3}{2}}\|_W + \|B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}}\|_W) \\
\leq& K \max_{0 \leq n \leq N_T-1} \|E^n - E_e^n\|_V \max_{0 \leq n \leq N_T-1} \|B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}}\|_W,
\end{aligned}$$

where the last step follows from the fact that

$$\Delta t \sum_{n=0}^{N_T-2} 1 \leq K.$$

Since

$$\dot{B}_f^{n+1} - \frac{1}{\Delta t} \int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t} \dot{B}_f \, ds$$

defines a bounded linear functional which vanishes for any linear functions, so

$$|\dot{B}_f^{n+1} - \frac{1}{\Delta t} \int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t} \dot{B}_f \, ds| \leq K(\Delta t) \int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t} |\dddot{B}_f| \, ds.$$

By the Cauchy-Schwarz inequality,

$$|\dot{B}_f^{n+1} - \frac{1}{\Delta t} \int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t} \dot{B}_f \, ds|^2 \leq K(\Delta t)^3 \int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t} |\dddot{B}_f|^2 \, ds.$$

Hence,

$$\begin{aligned}\|S^{-1}\mathcal{P}^{n+1}\|_W^2 &= \sum_{i=1}^{F_1} s_i\bar{h}_i'|s_i\mathcal{P}_i^{n+1}|^2\\ &\leq K(\Delta t)^5\int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t}\sum_{i=1}^{F_1} s_i\bar{h}_i'|\dddot{B}_f|_i^2\,ds\\ &\leq K(\Delta t)^5\int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t}\sum_{r=1}^{2}|\dddot{\mathbf{B}}|_{H^2(\Omega_r)^3}^2\,ds.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\sum_{n=0}^{N_T-2}\|S^{-1}\mathcal{P}^{n+1}\|_W &\leq (N_T-1)^{\frac{1}{2}}(\sum_{n=0}^{N_T-2}\|S^{-1}\mathcal{P}^{n+1}\|_W^2)^{\frac{1}{2}}\\ &\leq K(\Delta t)^2(\sum_{n=0}^{N_T-2}\int_{(n+\frac{1}{2})\Delta t}^{(n+\frac{3}{2})\Delta t}\sum_{r=1}^{2}|\dddot{\mathbf{B}}|_{H^2(\Omega_r)^3}^2\,ds)^{\frac{1}{2}}\\ &\leq Kh^2\sum_{r=1}^{2}|\mathbf{B}|_{H^3(0,T;H^2(\Omega_r))^3}.\end{aligned}$$

By Theorem 5.5, we have the estimate for $B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}}$.

Now, we give the estimate for $E^n - E_e^n$. For any $0 \leq n \leq N_T - 1$ with $C(E^n - E_e^n) \neq 0$, by Lemma 3.5, we have

$$\|E^n - E_e^n\|_{W'} \leq K\|E^n - E_e^n\|_V.$$

Hence, by Theorem 5.5, we obtain

$$\begin{aligned}&\|E^n - E_e^n\|_{W'}\\ \leq &Kh^2(\sum_{r=1}^{2}\|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \sum_{r=1}^{2}\|\mathbf{B}\|_{W^{4,1}(0,T;H^2(\Omega_r))^3}\\ &+\|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^2} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}).\end{aligned} \tag{5.43}$$

For any n with $C(E^n - E_e^n) = 0$, the proof is complete by proving the following Lemma 5.5.

□

Lemma 5.5 *Suppose that* $\mathbf{B} \in W^{2,1}(0,T;H^3(\Omega_r))^3$ *and* $\mathbf{E} \in W^{1,1}(0,T;H^3(\Omega_r))^3$, *for* $r = 1,2$, *is the true solution of (1.1)-(1.4), and that* $\mathbf{J} \in W^{2,1}(0,T;H^2(\Omega))^3$ *and* $\mathbf{J}_\Gamma \in W^{2,1}(0,T;H^3(\Gamma))^3$. *Let* E^n *be the solution of (5.2)-(5.3) on uniform grid with* $C(E^n - E_e^n) = 0$ *for all* $n_1 < n < n_2$. *Then*

$$\begin{aligned}
&\max_{n_1<n<n_2} \|E^n - E_e^n\|_{W'} \\
\leq& Kh^2\Big(\sum_{r=1}^{2} \|\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_r))^3} + \sum_{r=1}^{2} \|\mathbf{E}\|_{W^{1,1}(0,T;H^3(\Omega_r))^3} \\
&+ \|\mathbf{J}\|_{W^{2,1}(0,T;H^2(\Omega))^3} + \|\mathbf{J}_\Gamma\|_{W^{2,1}(0,T;H^3(\Gamma))^3}\Big).
\end{aligned} \tag{5.44}$$

Proof. For any $n_1 < n < n_2$ with $C(E^n - E_e^n) = 0$, by Lemma 3.7, there exist $\phi^n \in \mathbb{R}^L$ such that

$$D(E^n - E_e^n) = B_1'\phi^n.$$

With the definition of U^n, for any $n_1 < n < n_2 - 1$, we have

$$\begin{aligned}
(S'(U^{n+1} - U^n), D(U^{n+1} + U^n)) &= (S'(U^{n+1} - U^n), B_1'(\phi^{n+1} + \phi^n)) \\
&= (\mathcal{D}'(U^{n+1} - U^n), \phi^{n+1} + \phi^n).
\end{aligned}$$

By Theorem 5.3, we apply a similar procedure as in the proof of Lemma 4.6 to $\mathcal{D}'U^{n+1}$ and $\mathcal{D}'U^n$, we obtain

$$\begin{aligned}
&(S'(U^{n+1} - U^n), D(U^{n+1} + U^n)) \\
=&(S'((E_f' - \Pi_h E_f')(t_{n+1}) - (E_f' - \Pi_h E_f')(t_n)), D(U^{n+1} + U^n)) \\
&+ (S'((E_p^{n+1} - E_e^{n+1}) - (E_p^n - E_e^n)), D(U^{n+1} + U^n)).
\end{aligned}$$

For $n = n_1$, we have

$$\begin{aligned}
&(S'(U^{n_1+1} - U^{n_1}), D(U^{n_1+1} + U^{n_1})) \\
=&(S'(U^{n_1+1} - U^{n_1}), B_1'\phi^{n_1+1}) + (S'(U^{n_1+1} - U^{n_1}), DU^{n_1}) \\
=&(\mathcal{D}'(U^{n_1+1} - U^{n_1}), \phi^{n_1+1}) + (S'(U^{n_1+1} - U^{n_1}), DU^{n_1}).
\end{aligned}$$

By Theorem 5.3 and the proof of Lemma 4.6, we have

$$
\begin{aligned}
(S'(U^{n_1+1} - U^{n_1}), D(U^{n_1+1} + U^{n_1})) &= (S'(U^{n_1+1} - U^{n_1}), DU^{n_1}) \\
&\quad + (S'((E_f' - \Pi_h E_f')(t_{n_1+1}) - (E_f' - \Pi_h E_f')(t_{n_1})), DU^{n_1+1}) \\
&\quad + (S'((E_p^{n_1+1} - E_e^{n_1+1}) - (E_p^{n_1} - E_e^{n_1})), DU^{n_1+1}).
\end{aligned}
$$

Hence, for any $n_1 < j < n_2 - 1$, we obtain

$$
\begin{aligned}
&\sum_{n=n_1}^{j} (S'(U^{n+1} - U^n), D(U^{n+1} + U^n)) \\
&= \sum_{n=n_1+1}^{j} \Big\{ (S'((E_f' - \Pi_h E_f')(t_{n+1}) - (E_f' - \Pi_h E_f')(t_n)), D(U^{n+1} + U^n)) \\
&\quad + (S'((E_p^{n+1} - E_e^{n+1}) - (E_p^n - E_e^n)), D(U^{n+1} + U^n)) \Big\} \\
&\quad + (S'((E_f' - \Pi_h E_f')(t_{n_1+1}) - (E_f' - \Pi_h E_f')(t_{n_1})), DU^{n_1+1}) \\
&\quad + (S'((E_p^{n_1+1} - E_e^{n_1+1}) - (E_p^{n_1} - E_e^{n_1})), DU^{n_1+1}) \\
&\quad + (S'(U^{n_1+1} - U^{n_1}), DU^{n_1}).
\end{aligned}
$$

For $n_1 \leq n \leq j$, we observe that

$$
(E_f' - \Pi_h E_f')(t_{n+1}) - (E_f' - \Pi_h E_f')(t_n) = \int_{n\Delta t}^{(n+1)\Delta t} (\dot{E}_f' - \Pi_h \dot{E}_f')\, ds,
$$

and that

$$
(E_p^{n+1} - E_e^{n+1}) - (E_p^n - E_e^n) = \int_{n\Delta t}^{(n+1)\Delta t} (\dot{E}_p - \dot{E}_e)\, ds.
$$

By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\sum_{n=n_1}^{j} (S'(U^{n+1} - U^n), D(U^{n+1} + U^n)) \\
&\leq \max_{n_1 < n < n_2} \|U^n\|_{W'} \int_{n_1\Delta t}^{n_2\Delta t} (\|\dot{E}_f' - \Pi_h \dot{E}_f'\|_{W'} + \|\dot{E}_p - \dot{E}_e\|_{W'})\, ds \\
&\quad + (S'(U^{n_1+1} - U^{n_1}), DU^{n_1}).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&(S'U^{j+1}, DU^{j+1}) \\
&\leq \max_{n_1<n<n_2} \|U^n\|_{W'} \int_{n_1\Delta t}^{n_2\Delta t} (\|\dot{E}'_f - \Pi_h \dot{E}'_f\|_{W'} + \|\dot{E}_p - \dot{E}_e\|_{W'})\, ds \\
&\quad + (S'U^{n_1}, DU^{n_1}) + (S'U^{n_1+1}, DU^{n_1}).
\end{aligned}
$$

In the proof of Lemma 4.6, we already have the estimate for $\|\dot{E}'_f - \Pi_h \dot{E}'_f\|_{W'}$ and $\|\dot{E}_p - \dot{E}_e\|_{W'}$. Also, by (5.43), we have the estimate for $(S'U^{n_1}, DU^{n_1})$. For the remaining term, we estimate in the following way

$$
(S'U^{n_1+1}, DU^{n_1}) \leq \|U^{n_1+1}\|_{W'}\|U^{n_1}\|_{W'} \leq (\max_{n_1<n<n_2} \|U^n\|_{W'})\|U^{n_1}\|_{W'}.
$$

Since $CU^{n_1} \neq 0$, we have the desired estimate by (5.43).

□

We remark here that Theorem 5.6 shows our fully discrete finite volume approximation of the Maxwell's equations is second order in W and W'-norm for rectangular domains. So, it is an optimal error estimate.

Chapter 6

Numerical Tests

In this chapter, we apply the finite volume method (5.2)-(5.3) to solve the Maxwell's system (1.1)-(1.4) in nonhomogeneous media. It can be seen from the numerical examples below that the convergence of the scheme is indeed of second order for the considered Maxwell's equations with discontinuous physical coefficients.

6.1 Convergence test

Let $\Omega \times [0,T] = [0,1]^3 \times [0,1]$ and $\Omega_1 = [\frac{1}{3}, \frac{2}{3}]^3$. We triangulate the domain Ω into smaller equal cuboids with N_D being the number of grid points in each axis direction, and divide $[0,T]$ into N_T equal subintervals. We assume the media are equipped with the following discontinuous physical parameters:

$$\epsilon = \begin{cases} 0.1 & \text{in } \Omega_1 \\ 2 & \text{in } \Omega_2 \end{cases}, \qquad \mu = \begin{cases} 0.05 & \text{in } \Omega_1 \\ 1 & \text{in } \Omega_2 \end{cases}$$

To check the accuracy of the finite volume method (5.2)-(5.3), we construct the Maxwell's system (1.1)-(1.4) with its exact solutions given by

$$\mathbf{E} = \begin{bmatrix} -e^{\pi t}\cos(2\pi x)\sin(2\pi y)\sin(2\pi z) \\ -e^{\pi t}\sin(2\pi x)\cos(2\pi y)\sin(2\pi z) \\ -e^{\pi t}\sin(2\pi x)\sin(2\pi y)\cos(2\pi z) \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -0.05\cos(2\pi x)\sin(2\pi y)\sin(2\pi z) + x \\ -0.05\sin(2\pi x)\cos(2\pi y)\sin(2\pi z) - y \\ -0.05\sin(2\pi x)\sin(2\pi y)\cos(2\pi z) + 1 \end{bmatrix}$$

We note that both $\mathbf{E}$ and $\mathbf{B}$ are continuous in Ω, but $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$ and $\mathbf{D} = \epsilon\mathbf{E}$ are discontinuous across the interface. We can verify that the exact solution $(\mathbf{E}, \mathbf{B})$ satisfies the interface conditions

$$[\mathbf{E} \times \mathbf{m}] = 0 \quad , \quad [\mathbf{B} \cdot \mathbf{m}] = 0.$$

Solving the fully discrete finite volume system (5.2)-(5.3), we obtain the following result:

N_T	N_D	error	ratio
180	6	0.6166	—
360	12	0.1777	3.47
720	24	0.0475	3.74
1440	48	0.0123	3.86
2880	96	0.0031	3.97

Table 1: Convergence rate for the first example

where the errors are the discrete L^2-norm errors between the true solution $(\mathbf{E}, \mathbf{B})$ and the finite volume solution (E, B) with the norms calculated using (3.2) and (3.4), namely

$$\max_{0 \le n \le N_T - 1} \left\{ \|E^n - E_e^n\|_{W'} + \|B^{n+\frac{1}{2}} - B_f^{n+\frac{1}{2}}\|_W \right\}.$$

From the table above, we see that the convergence rate is approximately $O(h^2)$, that indicates the second order accuracy of the proposed finite volume method (5.2)-(5.3).

Our second example is concerned with the Maxwell's system (1.1)-(1.4) with the following true solutions

$$\mathbf{E_1} = \begin{bmatrix} -e^{\pi t}\cos(6\pi x)\sin(6\pi y)\sin(6\pi z) + \cos(2\pi x)\sin(2\pi y)\sin(2\pi z) \\ -e^{\pi t}\sin(6\pi x)\cos(6\pi y)\sin(6\pi z) + \sin(2\pi x)\cos(2\pi y)\sin(2\pi z) \\ -e^{\pi t}\sin(6\pi x)\sin(6\pi y)\cos(6\pi z) + \sin(2\pi x)\sin(2\pi y)\cos(2\pi z) \end{bmatrix}$$

$$\mathbf{E_2} = \begin{bmatrix} -(e^{\pi t}+1)\cos(6\pi x)\sin(6\pi y)\sin(6\pi z) + \cos(2\pi x)\sin(2\pi y)\sin(2\pi z) \\ -(e^{\pi t}+1)\sin(6\pi x)\cos(6\pi y)\sin(6\pi z) + \sin(2\pi x)\cos(2\pi y)\sin(2\pi z) \\ -(e^{\pi t}+1)\sin(6\pi x)\sin(6\pi y)\cos(6\pi z) + \sin(2\pi x)\sin(2\pi y)\cos(2\pi z) \end{bmatrix}$$

where $\mathbf{E}_i = \mathbf{E}|_{\Omega_i}$, for $i = 1, 2$, and $\mathbf{B}$ is the same as above. In this example, $\mathbf{H}$ field and the normal component of $\mathbf{E}$ is discontinuous across the interface Γ. Solving the system with the finite volume method (5.2)-(5.3), we obtain the following result:

N_T	N_D	error	ratio
360	12	1.6090	—
720	24	0.4851	3.32
1440	48	0.1312	3.70
2880	96	0.0341	3.85
5760	192	0.0087	3.92

Table 2: Convergence rate for the second example

We see that the convergence rate is $O(h^2)$, which again demonstrates the second order accuracy of the numerical method (5.2)-(5.3).

6.2 Electromagnetic scattering

We now present a numerical experiment for an electromagnetic scattering problem by our finite volume method. Assume that a plane wave source is given on the boundary $x = 0$. We choose the source as given by

$$E_y = \sin(4\pi(x - c_2 t)), \quad H_z = \epsilon_2 c_2 \sin(4\pi(x - c_2 t))$$

where $c_2 = (\epsilon_2 \mu_2)^{-\frac{1}{2}}$ is the speed of light in the medium occupied by Ω_2. Note that both the electric and magnetic fields propagate in the x-direction. The numerical solution of the electric field E_y is shown in the following figure:

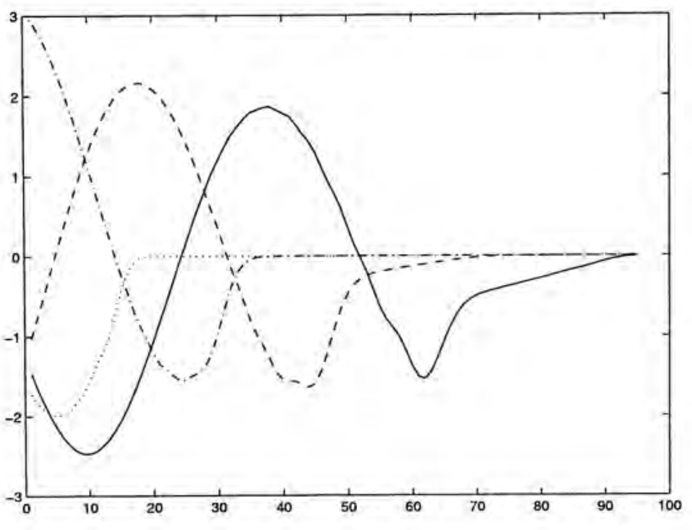

Figure 3: Numerical solution of E_y

where in figure 3 the dotted line, dash dot line, dash line and solid line represent respectively the snap shots of the electric field patterns at times $t = 0.25, 0.5, 0.75, 1$. In addition, the vertical axis denotes the amplitude of the field strength while the horizontal axis denotes the position in x-direction. We remark that the amplitudes of the waves have been doubled so that it looks clearer. The plot in figure 3 corresponds to the pattern of the electric field which does not pass through the inhomogeneous part of Ω, that is Ω_1. It shows that the electric field propagates smoothly in the x-direction.

In figure 4, we give the numerical solution of the magnetic flux density B_z and we have shown the snap shots of patterns of the magnetic flux density which passes through the inhomogeneous part of Ω, that is Ω_1. From the figure, we see that the wave propagates in the x-direction, but there are discontinuities when the wave passes through the interface between Ω_1 and Ω_2. We remark that the amplitudes of the waves have been doubled and all the notations in figure 4 are defined similarly as figure 3.

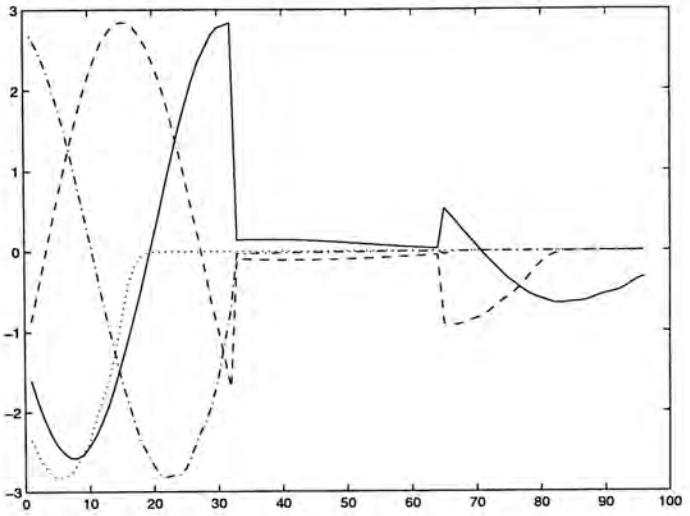

Figure 4: Numerical solution of B_z

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