On Density Theorems, Connectedness Results and Error Bounds in Vector Optimization

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Abstract

A well-known theorem proved by Arrow, Barankin and Blackwell states that if the n-dimensional real Euclidean space is equipped with its natural ordering, then for any compact convex subset A, the set of the positive proper efficient points of A is dense in the set of the efficient points of A. In Chapter 1, we give a survey on several generalizations of this theorem. For a weakly compact convex subset in a normed space with partial ordering defined by a quasi-Bishop-Phelps cone, the density result stated above holds valid. Also we consider the density result in a general topological vector space equipped with a weakly closed convex cone which admits strictly positive continuous linear functionals. Moreover several density results in general dual space setting are presented. In Chapter 2, a density theorem of super efficient points are discussed.

In Chapter 3, we deal with the connectedness results in vector optimization as well as the contractibility of the set of the efficient points in a locally convex space and the path-connectedness of the set of the positive proper efficient points in a reflexive Banach space.

In last chapter, we present some recent results on error bounds. Topics included are:

1. error bound concerning lower semicontinuous functions in normed spaces.
2. sufficient condition for a proper weakly lower semicontinuous function on a reflexive Banach space to have an error bound with fractional exponent.
3. error bound for a quadratic function on the n-dimensional real Euclidean space.

摘要

一個由 Arrow、Barankin 及 Blackwell 所證明的著名定理指出：若n維實Euclid空間之元素依自然順序關係，那麼任何緊凸子集A的正常態有效點集在A的有效點集內稠密。在此論文中，我們綜述這個定理的一些推廣。設有一賦範線性空間，其上的偏序由一個擬 Bishop－Phelps 錐所定義，則對於任何在這賦範線性空間中的弱緊凸子集，上面提到的稠密性結果保留真確。同時我們也考慮在普遍拓樸線性空間上，其中偏序由一個弱閉凸錐所定義，而拓樸線性空間上亦存在嚴格正連續線性泛函之下的稠密性結果。此外，一些在普遍對偶空間中的稠密性結果亦有所介紹。在第二章中我們會討論對於超有效點集的稠密性結果。

第三章中我們處理向量最優化問題中的連通性、有關局部凸拓樸線性空間中有效點集的可縮性及自反Banach空間中正常態有效點集的道路連通性等等結果。

在最後一章，我們介紹一些有關誤差界的最近結果。論題包括：

一．有關賦範線性空間中的下半連續函數的誤差界。

二．自反Banach空間中真弱下半連續函數存在分數指數誤差界的充分條件。

三．有關n維實Euclid空間中二次函數的誤差界。

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Contents

0 Introduction **1**

1 Density Theorems in Vector Optimization **7**

1.1 Preliminary 7

1.2 The Arrow-Barankin-Blackwell Theorem in Normed Spaces 14

1.3 The Arrow-Barankin-Blackwell Theorem in Topological Vector Spaces 27

1.4 Density Results in Dual Space Setting 32

2 Density Theorem for Super Efficiency **45**

2.1 Definition and Criteria for Super Efficiency 45

2.2 Henig Proper Efficiency 53

2.3 Density Theorem for Super Efficiency 58

3 Connectedness Results in Vector Optimization **63**

3.1 Set-valued Maps 64

3.2 The Contractibility of the Efficient Point Sets 67

3.3 Connectedness Results in Vector Optimization Problems 83

4 Error Bounds In Normed Spaces **90**

4.1 Error Bounds of Lower Semicontinuous Functions in Normed Spaces 91

4.2 Error Bounds of Lower Semicontinuous Convex Functions in Reflexive Banach Spaces 100

4.3 Error Bounds with Fractional Exponents 105

4.4 An Application to Quadratic Functions 114

Bibliography **133**

Chapter 0

Introduction

In the study of vector optimization, one main aim is to identify the efficient points of a given set A in a partially ordered topological vector space $\mathcal{X}$ equipped with an ordering cone S. However, observed by Kuhn and Tucker and later by Geoffrion, some efficient points exhibit certain abnormal property. To overcome this drawback of efficient points, several kinds of proper efficiencies have been introduced; see Kuhn and Tucker [30], Hurwicz [31], Geoffrion [32], Borwein [33], Harley [34], Benson [35], Henig [36] and Borwein and Zhuang [1]; also see Guerraggio, Molho and Zaffaroni [37]. Some are related to the scalarization problem; some focus on trade-off ratios between the components of the objective function; some focus on the geometric approach and some on the stability property. In general, the proper efficiencies have been defined with respect to conical orderings.

Among these proper efficiencies the positive proper efficiency plays an important role in the vector optimization. It is closely related to the scalarization of a vector problem. Every positive proper efficient point is a minimum solution of a scalar optimization problem

$$\min_{a \in A} f(a)$$

where f is a continuous linear functional strictly positive on the ordering cone.

In 1953, Arrow, Barankin and Blackwell [15] proved the famous result: If $\mathbb{R}^n$

is equipped with its natural (componentwise) ordering and if A is a compact convex subset of $\mathbb{R}^n$, then the set of the positive proper efficient points in A is dense in the set of the efficient points in A: for any efficient point, there is a point sufficiently near which is supported by some strictly positive continuous linear functional. This theorem has important implications in both vector optimization and mathematical economics. In the study of multiple objective optimization, the main aim is to identify the set of efficient points. Also in the study of mathematical economics, a strictly positive continuous linear functional p can be regarded as a pricing system and a point a_0 supported by p can be regarded as an optimal alternative allocation of resources under a certain corresponding pricing system. Then this theorem tells us that "nearly every" efficient alternative allocation of resource with respect to the componentwise ordering can be sustained by a certain pricing system.

In the past 30 years, the Arrow-Barankin-Blackwell Theorem has been generalized in many directions by many authors. Harley [34] and Bitran and Magnanti [38] extended the theorem to $\mathbb{R}^n$ with arbitrary closed convex pointed ordering cone. Radner [40], Majumdar [41] and Peleg [42] proved the density results in the infinite dimensional normed vector lattices l^∞, L^∞ and l^p with $1 \leq p \leq \infty$ respectively. Chichilinisky and Kalman [43] gave the result in the framework of Hilbert spaces.

In the setting of general normed spaces, Borwein [39] and Satz [44] extended the above theorem of Arrow, Barankin and Blackwell in a normed space partially ordered by a convex cone with a weakly compact base and with a base norm respectively. Furthermore, Jahn [23] showed the density result holds in the setting that the normed space is partially ordered by a Bishop-Phelps cone and the set A is assumed to be weakly compact convex. Petschke [24] extended Jahn's result to the case in that the ordering cone has a closed bounded base. Petschke's result has been further generalized in two directions: Gallagher and Saleh [20] showed that Petschke's result remains valid in the setting of locally convex topological vector space, and, recently Ng and Zheng [14] gave a density result in a general normed space partially ordered by a quasi-Bishop-Phelps cone. In Section 1.2 we will give a survey on the density result based on the paper [14].

Note that, the results above require severe restriction on the ordering cone. In order to relax these requirements, several authors had their researches in another direction. Ferro [17] first gave the density result in a normed space partially ordered by a closed convex cone whose dual cone has nonempty algebraic interior and with the set A being compact convex; later he extended his result in which the ordering cone is only required to admit a strictly positive continuous linear functional (indeed it is a necessary condition for such a density theorem). Also Gallagher and Saleh [20] introduced the concept of D-cones and give a density result in the setting of normed spaces. Notice that in some sense the result of Gallagher and Saleh also extended the Ferro's later result in locally bounded spaces setting. Chen [25] gave a generalization to the theorem of Arrow, Barankin and Blackwell in the framework of a locally convex space, partially ordered by a closed convex cone with nonempty topological interior and with the corresponding dual cone admitting a nonempty quasi-interior. Recently, Zheng [9] generalized Ferro's result in a general topological vector space with a weakly closed convex ordering cone with nonempty quasi-interior. In Section 1.3 we will discuss this generalization of Arrow-Barankin-Blackwell Theorem with the rather weak requirement on the ordering cone in the framework of a general topological vector space.

Moreover, it is interesting and worthy to study the Arrow-Barankin-Blackwell Theorem in the dual spaces setting. This has also some economics applications. Majumdar [41] and Peleg [42] considered the space L^∞ with the allowable class of support functionals in the set of nonnegative elements in L^1 and the space l^∞ with the allowable class of support functionals in the set of nonnegative elements in l^1 respectively. Ferro [18] has also considered the l^∞ case. However, it was

Gallagher [21] who first studied the general case in the dual space setting. His result was further extended by Song [22]: if F is a compact convex subset in the dual space $\mathcal{X}^*$ of a normed space $\mathcal{X}$ which is partially ordered by a closed convex cone, then the set of points in F supported by the strictly positive elements of $\mathcal{X}$ (in the canonical embedding of $\mathcal{X}$ in $\mathcal{X}^{**}$) is dense in the set of the efficient points in F with respect to the dual cone. Recently Ng and Zheng [14] extended Song's result based on the density result given by Zheng [9]. All these will be studied in our Section 1.4.

The relationship between the set of the positive proper efficient points and the set of the efficient points has been studied by many authors. In this thesis we also discuss another kind of proper efficiency: the super efficiency. This concept was first introduced by Borwein and Zhuang [1] in the setting of normed spaces in 1993. In [1], some desirable properties, involving some characterization of super efficiency by scalar optimization and density property in the set of the efficient points, were presented. In 1997, Zheng [2] further generalized the concept of super efficiency in the setting of locally convex spaces, and also examined the relationship among super efficiency and other kinds of proper efficiency. It is remarkable that super efficiency is highly related to the Henig proper efficiency. Furthermore, in [3], Zheng gave a density result concerning the super efficiency: if a locally convex space is equipped with an ordering cone having a closed bounded base and if the set A is weakly compact, then the set of the super efficient points in A is dense in the set of the efficient points in A. Besides, this density result is also valid for the Henig proper efficiency instead of super efficiency. These density results will be presented in Chapter 2.

In general, one of the important directions in the study of vector optimization is to investigate the structures and the topological properties of the sets of the efficient points and of the proper efficient points. Other than the density property, connectedness of those sets is also of interest. One reason for studying connectedness is that, in a continuous maximization problem on the commodity space in the study of economics, normally a large number of Pareto-optimal alternatives are resulted, and then the connectedness property ensures a continuous moving from one efficient alternative to any other one along a "path" consisting of efficient alternatives only. This study on connectedness of the set of the efficient points has been carried out by a number of authors: Peleg [45], Smale [46], Naccacche [47], Schecter [48], Bitran and Magnanti [38], Choo and Atkins [49], Warburton [8], Gong [7], Luc [50, 51, 52] and Zheng [6]; also see the reference therein.

In particular, Gong [7] studied the connectedness of the efficient solution set in a convex vector optimization for set-valued maps in the normed space. A vector minimization problem for a set-valued map F is:

$$(VMP) \quad Min\{F(x) : x \in A\}, \quad \text{with respect to cone } S,$$

where A is a nonempty compact convex subset of a real Hausdorff topological vector space $\mathcal{X}$, S is a closed convex cone with a base in a real normed space $\mathcal{Y}$, and F is an upper semicontinuous S-convex set-valued map from A to $\mathcal{Y}$ with compact values. Let S^{+i} be the set of all strictly positive continuous linear functionals on $\mathcal{Y}$. For each $x \in A$ and each $h \in S^{+i}$, let

$$P(x, h) = \{y \in F(x) | h(y) = \min\{h(z) | z \in F(x)\}\};$$

that is, $P(x, h)$ is the solution set of the scalar minimization problem of the function h on $F(x)$. Let E, U, V respectively denote the sets:

E: the efficient solution set of (VMP),

U: the positive proper efficient outcome set of (VMP) and

V: the efficient outcome set of (VMP).

Under this setup, Gong's main results (Theorem 4.1 and Theorem 5.1 in [7]) are: E, U, V are connected if

$$P(x, h) \text{ is connected}, \quad \forall x \in A, \forall h \in S^{+i}. \tag{1}$$

However, in the Section 3.3 of this thesis, I will show that the condition (1) can be dropped and Gong's results still remain valid. In Chapter 3, we will also briefly report on the contractibility and path-connectedness results in [6].

The final part of this thesis is devoted to consider the error bounds in optimization problems. Error bounds take an important role in the sensitivity analysis of the mathematical programming and in convergence analysis of some algorithms. The error bound analysis concerns that if a function is given, what relation between the distance of an arbitrary point from the zero set and the value of function at that point is. In recent years the research on error bound has attracted many researchers and a large number of publications have appeared to report the progress of this study. For more detail, one can refer to excellent survey papers of Lewis and Pang [53] and Pang [54] and the references therein. In [10], the error bound for a proper lower semicontinuous function on a normed space is studied, several sufficient conditions and several necessary conditions have been given for validity of error bounds. Also some equivalent conditions for error bound were presented in the setting that the normed space is a reflexive Banach space and the function is lower semicontinuous convex. As an application, a computable Lipschitz bound constant was given. In [11], Ng and Zheng considered the error bounds with fractional exponents other than exponent one. Using the partial order induced by a proper weakly lower semicontinuous function on a reflexive Banach space, Ng and Zheng established several results concerning the sufficient conditions to guarantee the error bounds with fractional exponents hold for those functions. Moreover an application which aims at identifying the exponents of the error bound for a quadratic function on $\mathbb{R}^n$ was also given. In Chapter 4 we will give a systematic survey on [10] and [11].

Chapter 1

Density Theorems in Vector Optimization

1.1 Preliminary

Throughout this thesis the topological vector spaces all are assumed to be real Hausdorff; all the corresponding topological dual spaces are assumed to separate points in those topological vector spaces.

Let $\mathcal{X}$ be a topological vector space. A binary relation $\preceq$ is said to be a **pre-order** in $\mathcal{X}$ if it is (1)**reflexive** (for any $x \in \mathcal{X}$, $x \preceq x$) and (2)**transitive** (for any $x, y, z \in \mathcal{X}$, $x \preceq z$ whenever $x \preceq y$ and $y \preceq z$). Further $\preceq$ is a **partial order** in $\mathcal{X}$ if it is a pre-order and it is (3)**antisymmetric** ($x = y$ whenever $x \preceq y$ and $y \preceq x$).

A subset $S \subseteq \mathcal{X}$ is called a **cone** if $\alpha S \subseteq S$ for any $\alpha \geq 0$. Suppose S is a convex cone; that is,

$$S + S \subseteq S \quad \text{and} \quad \forall \alpha \geq 0,\, \alpha S \subseteq S;$$

then S specifies a pre-order $\preceq_S$ in $\mathcal{X}$ by: for all $x, y \in \mathcal{X}$,

$$x \preceq_S y \quad \text{if and only if} \quad y - x \in S. \tag{1.1}$$

In this case the cone S is called an **ordering cone** in $\mathcal{X}$. Throughout this thesis the cone S is assumed to be nontrivial; that is, $S \neq \emptyset$ and $S \neq \mathcal{X}$.

A cone S is said to be **pointed** if $S \cap -S = \{0\}$. Suppose S is a pointed convex cone. It is easy to show that $\preceq_S$ defined in (1.1) is a partial order in $\mathcal{X}$. In this case $(\mathcal{X}, \preceq_S)$ is said to be a **partially ordered topological vector space**. More detail can be found in [26] and [27]. Usually we will consider the setting with the pointed convex ordering cone.

For a cone S, $\Theta \subseteq S$ is said to be a **base** of S if

(i) Θ is convex,

(ii) $0 \notin cl(\Theta)$ and

(iii) $S \subseteq cone(\Theta)$,

where $cl(\Theta)$ is the closure of Θ in $\mathcal{X}$ and $cone(\Theta) := \{t\theta | t \geq 0, \theta \in \Theta\}$. The following remark is easy to verify.

Remark 1.1.1 *A cone S is pointed convex if S has a base.*

In vector optimization analysis, it is worthwhile to consider the continuous linear functionals on $\mathcal{X}$. Let $\mathcal{X}^*$ denote the topological dual space of $\mathcal{X}$. Suppose S is a cone in $\mathcal{X}$; we use the following notations:

$$S^+ := \{f \in \mathcal{X}^* | f(s) \geq 0, \quad \forall s \in S\} \qquad \text{and}$$

$$S^{+i} := \{f \in \mathcal{X}^* | f(s) > 0, \quad \forall s \in S \setminus \{0\}\}.$$

S^+ is called the **dual cone** of a cone S, which consists of all the **positive continuous linear functionals** on $\mathcal{X}$. S^{+i} is called the **quasi-interior** of the dual cone, which consists of all the **strictly positive continuous linear functionals** on $\mathcal{X}$.

The following proposition tells that the dual cone induces a pre-order in the topological dual space.

Proposition 1.1.1 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be a cone. Then S^+ is a weak-*-closed convex cone in $\mathcal{X}^*$.*

Proof: Since $\{f \in \mathcal{X}^* | f(s) \geq 0\}$ is weak-*-closed for each $s \in S$ and $S^+ := \bigcap_{s \in S} \{f \in \mathcal{X}^* | f(s) \geq 0\}$, S^+ is weak-*-closed. Also it is easy to verify that $S^+ + S^+ \subseteq S^+$ and $\alpha S^+ \subseteq S^+$ for all $\alpha \geq 0$. □

As a result, the dual cone S^+ specifies a pre-order in $\mathcal{X}^*$ by the association $f \preceq_{S^+} g$ if and only if $g - f \in S^+$. The following proposition discusses when the pre-order induced is further a partial order in the topological dual space.

Proposition 1.1.2 *Let $\mathcal{X}$ be a locally convex space and $S \subseteq \mathcal{X}$ be a convex cone. Then S^+ is a pointed cone in $\mathcal{X}^*$ if and only if $cl(S - S) = \mathcal{X}$.*

Proof: Suppose $cl(S - S) \neq \mathcal{X}$. Then there exists $x_0 \in \mathcal{X}$ such that $x_0 \notin cl(S - S)$. Note that $cl(S - S)$ is closed convex. By the Separation Theorem there is $f \in \mathcal{X}^* \setminus \{0\}$ such that $f(x_0) > \sup\{f(y) | y \in cl(S - S)\}$. If there exists $y_0 \in cl(S-S)$ such that $f(y_0) \neq 0$ then $\frac{f(x_0)}{f(y_0)} y_0 \in cl(S-S)$ and $f(\frac{f(x_0)}{f(y_0)} y_0) = f(x_0)$. It leads to a contradiction. Therefore we have $f(s) = 0$ for any $s \in S$. Hence $f \in S^+ \cap (-S^+)$ and so S^+ is not pointed. Conversely, suppose $cl(S - S) = \mathcal{X}$. Since $S^+ \cap (-S^+) = \{f \in \mathcal{X}^* | f(s) = 0, \forall s \in S\}$, it follows from $cl(S - S) = \mathcal{X}$ that $S^+ \cap (-S^+) = \{0\}$. □

A cone S is said to be **generating** if $S - S = \mathcal{X}$. By Proposition 1.1.1 and Proposition 1.1.2, the dual cone S^+ specifies a partial order in $\mathcal{X}^*$ if $\mathcal{X}$ is locally convex and S is generating convex.

The quasi-interior S^{+i} of the dual cone is not necessarily nonempty (for example, $\mathcal{X} = B[a, b]$, the set of all bounded functions on $[a, b]$, and

$$S = \{x \in B[a, b] | x(t) \geq 0, \forall t \in [a, b]\},$$

then $S^{+i} = \emptyset$. See [19]) and not necessarily identical to the topological interior of the dual cone S^+ (for example, $\mathcal{X} = l^p$ with $1 < p < +\infty$ and S is the nonnegative orthant, then $int(S^+) = \emptyset$ but $S^{+i} \neq \emptyset$. See [19]).

Since the assumption of nonempty quasi-interior of the dual cone (that is, $(\mathcal{X}, \preceq_S)$ admits strictly positive continuous linear functionals) is indispensable to guarantee the existence of the positive proper efficient points, which is the core concept in this chapter and which definition will be stated later, it is important to know when $S^{+i} \neq \emptyset$. The following propositions give some equivalent conditions for it.

Proposition 1.1.3 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be a convex cone. Then $S^{+i} \neq \emptyset$ if and only if there exists an open convex set U in $\mathcal{X}$ such that $0 \notin U$ and $S \subseteq cone(U)$.*

Proof: Suppose $f \in S^{+i}$. Define $U = \{x \in \mathcal{X} | f(x) > 1\}$. Clearly U is an open convex set in $\mathcal{X}$ such that $0 \notin U$ and $S \subseteq cone(U)$. Conversely, suppose there exists an open convex set U in $\mathcal{X}$ such that $0 \notin U$ and $S \subseteq cone(U)$. It follows from the Separation Theorem that there exists $f \in \mathcal{X}^* \setminus \{0\}$ such that $f(u) > f(0) = 0$ for all $u \in U$. This and $S \subseteq cone(U)$ imply that $f \in S^{+i}$ and thus S^{+i} is nonempty. □

Let $w\text{-}cl(\Theta)$ denote the closure of Θ with respect to the weak topology of the topological vector space. Then we have the next equivalent condition for $S^{+i} \neq \emptyset$.

Proposition 1.1.4 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be a convex cone. Then $S^{+i} \neq \emptyset$ if and only if S has a base Θ such that $0 \notin w\text{-}cl(\Theta)$.*

Proof: Suppose $f \in S^{+i}$. Define $\Theta = \{s \in S | f(s) = 1\}$. Clearly Θ is a base of S and $0 \notin w\text{-}cl(\Theta)$ (Because $w\text{-}cl(\Theta) \subseteq \{x \in \mathcal{X} | f(x) = 1\}$).

Conversely suppose S has a base Θ such that $0 \notin w\text{-}cl(\Theta)$. Since the weak topology is locally convex, by the Separation Theorem, there exists $f \in \mathcal{X}^* \setminus \{0\}$ such that $f(\theta) > f(0) = 0$ for all $\theta \in \Theta$. This implies that $f \in S^{+i}$ and thus S^{+i} is nonempty. □

Since every closed convex subset in a locally convex space is weakly closed, the following corollary is directly from Proposition 1.1.4.

Corollary 1.1.5 *Let $\mathcal{X}$ be a locally convex space and $S \subseteq \mathcal{X}$ be a convex cone. Then $S^{+i} \neq \emptyset$ if and only if S has a base.*

Now let us state the definition of the efficient points and the positive proper efficient points below. They are the core concepts in this chapter.

Definition 1.1.1 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be the ordering cone. Let A be a nonempty subset of $\mathcal{X}$. Let A be a nonempty subset of $\mathcal{X}$. An element $\bar{a} \in A$ is called an* **efficient point** *of A with respect to S if*

$$(A - \bar{a}) \cap -S \subseteq S. \tag{1.2}$$

The set of the efficient points of A with respect to S is denoted by $E(A, S)$.

Remark 1.1.2 *In case that S is pointed, (1.2) can be simplified to*

$$(A - \bar{a}) \cap -S = \{0\}. \tag{1.3}$$

In other word, $\bar{a} \in E(A, S)$ whenever $\nexists a \in A$ such that $a \preceq_S \bar{a}$ and $a \neq \bar{a}$.

Definition 1.1.2 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be the ordering cone with nonempty quasi-interior S^{+i}. Let A be a nonempty subset of $\mathcal{X}$. An element $a_0 \in A$ is called a* **positive proper efficient point** *of A with respect to S if*

$$\exists f \in S^{+i}, \forall a \in A, \quad f(a_0) \leq f(a). \tag{1.4}$$

The set of the positive proper efficient points of A with respect to S is denoted by $Pos(A, S)$.

The following proposition gives a way to identify whether a point $a_0 \in A$ is a positive proper efficient point or not.

Proposition 1.1.6 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be the ordering cone. Let A be a nonempty convex subset of $\mathcal{X}$. An element $a_0 \in A$ is a positive proper efficient point of A with respect to S if and only if there exists an open convex subset U of $\mathcal{X}$ such that*

(i) $S \subseteq cone(U)$ and

(ii) $cone(A - a_0) \cap -U = \emptyset$.

Proof: Suppose $a_0 \in Pos(A, S)$. Then these exists $f \in S^{+i}$ such that for all $a \in A$, $f(a_0) \leq f(a)$; hence $f(a - a_0) \geq 0$ for all $a \in A$. Therefore we have

$$cone(A - a_0) \subseteq \{x \in \mathcal{X} | f(x) \geq 0\}. \tag{1.5}$$

Let $U = \{x \in \mathcal{X} | f(x) > 1\}$. Clearly U is open convex and $S \subseteq cone(U)$. Since $-U = \{x \in \mathcal{X} | f(x) < -1\}$, it follows from (1.5) that $cone(A - a_0) \cap -U = \emptyset$. Conversely, suppose U is an open convex subset satisfying (i) and (ii). From (ii) and the Separation Theorem there are $f \in \mathcal{X}^* \setminus \{0\}$ and $\gamma \in \mathbb{R}$ such that

$$f(x) \geq \gamma > f(-u), \qquad \forall x \in cone(A - a_0), \forall u \in U. \tag{1.6}$$

Since $0 \in cone(A - a_0)$. $\gamma \leq 0$. This implies that $f(u) > -\gamma \geq 0$ for any $u \in U$; hence, from (i), $f \in S^{+i}$. Aiming to show $a_0 \in Pos(A)$, it suffices to show that $f(x) \geq 0$ for any $x \in cone(A - a_0)$. Suppose not; there is $x_0 \in cone(A - a_0)$ such that $f(x_0) < 0$. Then there is a sufficiently large $n_0 \in \mathbb{N}$ such that $f(n_0 x_0) < \gamma \leq 0$. It contradicts with (1.6). This proves the result. □

In the rest of this preliminary section, we discuss some elementary properties of the set of the efficient points and the set of the positive proper efficient points.

Proposition 1.1.7 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be the ordering cone such that $S^{+i} \neq \emptyset$. Let A be a nonempty subset of $\mathcal{X}$. Then $Pos(A, S) \subseteq E(A, S)$.*

Proof: Let us prove by contradiction. Suppose a_0 in A and $a_0 \notin E(A, S)$. In view of (1.2), there exists $a \in A$ such that $a - a_0 \in -S$ and $a - a_0 \notin S$. Therefore $a - a_0 \in (-S) \setminus \{0\}$; hence for any $f \in S^{+i}$, $f(a) = f(a_0) + f(a - a_0) < f(a_0)$. As a result, $a_0 \notin Pos(A, S)$. □

Proposition 1.1.8 *Let $\mathcal{X}$ be a topological vector space. Let S and K be closed convex pointed cones in $\mathcal{X}$ such that $S \setminus \{0\} \subseteq int(K)$. Let A be a nonempty closed convex subset of $\mathcal{X}$. Then $E(A, K) \subseteq Pos(A, S)$.*

Proof: Let $a_0 \in E(A, K)$. Since K is convex pointed, it follows from Remark 1.1.2 that $(A - a_0) \cap -K = \{0\}$. As $0 \notin -int(K)$, we have $(A - a_0) \cap -int(K) = \emptyset$. Since $(A - a_0)$ is closed convex and $-int(K)$ is open convex, from the Separation Theorem, there exists $f \in \mathcal{X}^* \setminus \{0\}$ such that

$$\forall a \in A, \forall s \in S \setminus \{0\}, \quad f(a) - f(a_0) > f(-s). \tag{1.7}$$

Firstly, as $a_0 \in A$, we have $f(s) > f(a_0) - f(a_0) = 0$ for any $s \in S \setminus \{0\}$. This implies that $f \in S^{+i}$. Secondly, by (1.7), it is easy to show that $f(a) - f(a_0) \geq 0$ for any $a \in A$. Combining two results we have $a_0 \in Pos(A, S)$. □

Proposition 1.1.9 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be a convex pointed cone. Let A and B be nonempty subsets of $\mathcal{X}$ such that $A \subseteq B \subseteq A + S$. Then $E(A, S) = E(B, S)$.*

Proof: For any $\bar{a} \in E(A, S)$, suppose that $\bar{a} \notin E(B, S)$. Then there is $b' \in B$ such that $b' \preceq_S \bar{a}$ and $b' \neq \bar{a}$. Since $B \subseteq A + S$, b' can be written as $a' + s'$ for some $a' \in A$ and some $s' \in S$. Therefore $a' \preceq_S b' \preceq_S \bar{a}$ and $a' \neq \bar{a}$; it contradicts with $\bar{a} \in E(A, S)$. Conversely, suppose $\bar{b} \in E(B, S)$. Since $(A - \bar{b}) \cap -S \subseteq (B - \bar{b}) \cap -S = \{0\}$, it is sufficient to show that $\bar{b} \in A$. Suppose not; that is, there exist $a'' \in A$ and $s'' \in S \setminus \{0\}$ such that $\bar{b} = a'' + s''$. However we have $a'' \in A \subseteq B$, $a'' \preceq_S \bar{b}$ and $a'' \neq \bar{b}$. It contradicts with $\bar{b} \in E(B, S)$. □

Proposition 1.1.10 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be an ordering cone. Let A and B be nonempty subsets of $\mathcal{X}$ such that $A \subseteq B \subseteq A + S$. Then $Pos(A, S) = Pos(B, S)$.*

Proof: Suppose $a_0 \in Pos(A, S)$: there exists $f \in S^{+i}$ such that (1.4) holds. Therefore $f(a_0) \leq f(a) + f(s) = f(a + s)$ for any $a \in A$ and any $s \in S$. It follows from $B \subseteq A + S$ that $Pos(A, S) \subseteq Pos(B, S)$. Conversely, suppose $b_0 \in Pos(B, S)$. There exists $f \in S^{+i}$ such that $f(b_0) \leq f(b)$ for $b \in B$. It follows from $A \subseteq B$ that $f(b_0) \leq f(a)$ for $a \in A$. It is sufficient to show that $b_0 \in A$. Suppose not: that is, there exist $a' \in A$ and $s' \in S \setminus \{0\}$ such that $b_0 = a' + s'$. Then for each $g \in S^{+i}$, $g(a') < g(a') + g(s') = g(b_0)$. It contradicts with $b_0 \in Pos(B, S)$. Therefore $b_0 \in Pos(A, S)$. □

1.2 The Arrow-Barankin-Blackwell Theorem in Normed Spaces

In the preceding section we have known that the set of the positive proper efficient points is contained in the set of the efficient points. In the following sections, we turn to a well-known problem, first considered by Arrow, Barankin and Blackwell, which concerns the conditions required to guarantee the density of the set of positive proper efficient points in the set of the efficient points. In 1953, Arrow, Barankin and Blackwell [15] proved the following remarkable result: Given a nonempty compact convex subset A in $\mathbb{R}^n$ equipped with the natural order (that is, with an ordering cone of $\mathbb{R}^n_+ := \{(x_r)_{1\leq r\leq n} \in \mathbb{R}^n | x_r \geq 0, 1 \leq r \leq n\}$.), $Pos(A, \mathbb{R}^n_+)$ is dense in $E(A, \mathbb{R}^n_+)$. This result has been further generalized by a number of authors.

In this section, we discuss several density results in the setting of normed spaces. In 1988, J. Jahn [23] proved that, in a normed space $\mathcal{X}$ which is partially ordered by a Bishop-Phelps cone S, for a weakly compact convex subset A, $Pos(A, S)$ is norm dense in $E(A, S)$. This result was further generalized by M. Petschke in 1990. The following is Petschke's result.

Theorem 1.2.1 ([24] Corollary 4.2) *Let $\mathcal{X}$ be a normed space partially ordered by a convex cone S with a closed bounded base. Let $A \subseteq \mathcal{X}$ be a weakly compact convex subset. Then $E(A, S) \subseteq cl(Pos(A, S))$.*

Actually the requirement that the ordering cone has a closed bounded base is a great restriction. For example, the nonnegative orthants in l^p and L^p with $1 \leq p \leq +\infty$ do not have any closed bounded bases (indeed, no bounded bases); see [19]. Therefore it is natural to try to relax this limitation.

On the other side, as we state out in preceding section, the condition of nonempty quasi-interior of the dual cone is indispensable. Several authors gave some density results under this necessary condition on the ordering cone. In particular, F. Ferro proved the following result in 1993.

Theorem 1.2.2 ([18] Theorem 2.2) *Let $\mathcal{X}$ be a normed space partially ordered by a closed convex pointed cone S such that $S^{+i} \neq \emptyset$. Let $A \subseteq \mathcal{X}$ be a compact convex subset. Then $E(A, S) \subseteq cl(Pos(A, S))$.*

Comparing with Theorem 1.2.1, Theorem 1.2.2 requires a weaker restriction on the ordering cone S (indeed, necessary to guarantee the existence of positive proper efficient points) but a stronger restriction on the set A. This is revealed that there may be a trade-off between the restriction on the set A and the restriction on the ordering cone S.

In this section, we will present the result given by K. F. Ng and X. Y. Zheng [14], in which Theorem 1.2.1 is generalized by the way that the ordering cone is a quasi-Bishop-Phelps cone and other requirements remain. Also a density result with no compactness assumption on the set A is studied.

Before our discussion, let us start with a definition.

Definition 1.2.1 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex pointed cone. Let τ be a locally convex topology on $\mathcal{X}$ weaker than (or equal to) the norm topology such that every τ-compact subset of $\mathcal{X}$ is norm bounded. A sequence $\{S_n\}_{n\in\mathbb{N}}$ of τ-closed convex pointed cones in $\mathcal{X}$ is called an τ-enlargement of S if*

(i) $\forall n \in \mathbb{N}$, $S \setminus \{0\} \subseteq int(S_n)$, and

(ii) for each bounded sequence $\{c_n\}_{n\in\mathbb{N}}$ with $c_n \in S_n$ for each $n \in \mathbb{N}$, $dist(c_n, S) \to 0$, where $dist(c_n, S) = \inf\{\|c_n - s\| | s \in S\}$ for each $n \in \mathbb{N}$.

In this section we use ω denote the weak topology of $\mathcal{X}$.

Proposition 1.2.3 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex pointed cone with a base Θ. Suppose that $\delta = \inf\{\|\theta\| | \theta \in \Theta\} > 1$. For each $n \in \mathbb{N}$, define*

$$S_n = cl(cone(\Theta + \frac{1}{n}B(\mathcal{X}))), \tag{1.8}$$

where $B(\mathcal{X})$ is the closed unit ball of $\mathcal{X}$. Then $\{S_n\}_{n\in\mathbb{N}}$ is a ω-enlargement of S in $\mathcal{X}$.

Proof: Clearly every weakly compact subset in $\mathcal{X}$ is bounded. Next, S_n is a weakly closed convex pointed cone with a base $cl(\Theta + \frac{1}{n}B(\mathcal{X}))$ (see Theorem 1.1 in [1]). Moreover, for each $n \in \mathbb{N}$, since $\Theta \subseteq int(S_n)$, $S \setminus \{0\} \subseteq int(S_n)$. Now suppose $\{c_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{X}$ with $c_n \in S_n$ for each $n \in \mathbb{N}$. In view of (1.8), for each $n \in \mathbb{N}$, there exist $t_n \geq 0$, $\theta_n \in \Theta$, $x_n \in B(\mathcal{X})$ and $y_n \in B(\mathcal{X})$ such that

$$c_n = t_n(\theta_n + \frac{1}{n}x_n) + \frac{1}{n}y_n.$$

Note that $c_n - \frac{1}{n}y_n = t_n(\theta_n + \frac{1}{n}x_n)$; hence

$$\|c_n\| \geq t_n\|\theta_n + \frac{1}{n}x_n\| - \frac{1}{n}\|y_n\| \geq t_n(\delta - \frac{1}{n}) - \frac{1}{n}.$$

This and boundedness of $\{c_n\}_{n\in\mathbb{N}}$ imply that $\{t_n\}_{n\in\mathbb{N}}$ is a bounded scalar sequence. Since

$$dist(c_n, S) \leq \|c_n - t_n\theta_n\| \leq \frac{t_n}{n}\|x_n\| + \frac{1}{n}\|y_n\|$$

and $\frac{t_n}{n}\|x_n\| + \frac{1}{n}\|y_n\| \to 0$ when $n \to +\infty$, we have $dist(c_n, S) \to 0$. Combining all these, $\{S_n\}_{n\in\mathbb{N}}$ is a ω-enlargement of S in $\mathcal{X}$. □

Before we state the definition of quasi-Bishop-Phelps cones, we first recall the definition of Bishop-Phelps cones and the concept that a cone is representable as a Bishop-Phelps cone.

Definition 1.2.2 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone.*

1. *S is a Bishop-Phelps cone if there exists a nonzero continuous linear functional f (that is, $f \in \mathcal{X}^* \setminus \{0\}$) such that*

$$S = \{x \in \mathcal{X} | \|x\| \leq f(x)\}.$$

2. *S is said to be representable as a Bishop-Phelps cone if there exist a nonzero continuous linear functional f and an equivalent norm $\|\cdot\|_e$ such that*

$$S = \{x \in \mathcal{X} | \|x\|_e \leq f(x)\}.$$

The following is a main result of Petschke [24].

Theorem 1.2.4 ([24] Theorem 3.2) *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a nontrivial cone (that is $S \neq \{0\}$ and $S \neq \mathcal{X}$). Then S is representable as a Bishop-Phelps cone if and only if S is a convex cone with a closed bounded base.*

Remark 1.2.1 *See [26]. A convex cone with a closed bounded base is a closed convex pointed cone.*

Remark 1.2.2 *It is easy to show that a closed convex cone with a bounded base has a closed bounded base.*

Remark 1.2.3 *In view of Remark 1.2.2, we know that, given that S is a closed convex cone, S is representable as a Bishop-Phelps cone if S has a bounded base.*

In view of these, the quasi-Bishop-Phelps cone is defined as follows:

Definition 1.2.3 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone. S is a quasi-Bishop-Phelps cone if there exists a compact subset G of $\mathcal{X}^*$ such that*

$$S \subseteq \{x \in \mathcal{X} | \|x\| \leq \sup\{f(x) | f \in G\}\}.$$

The next proposition follows from Remark 1.2.3.

Proposition 1.2.5 *Let $\mathcal{X}$ be a normed space. Let S be a closed convex cone. If S has a bounded base then S is a quasi-Bishop-Phelps cone.*

Proposition 1.2.6 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex pointed cone. The following statements are equivalent.*

(A) S is a quasi-Bishop-Phelps cone.

(B) $0 \notin \omega\text{-}cl(U_S)$, where $U_S = \{s \in S | \|s\| = 1\}$.

(C) For any bounded net $\{s_\lambda\}_{\lambda\in\Lambda}$ *in* S, $s_\lambda \xrightarrow{\omega} 0$ *if and only if* $s_\lambda \to 0$.

Proof: It is clear that $(C) \Rightarrow (B)$.

Firstly, let's show that $(B) \Rightarrow (A)$. On the condition of (B), there exist $g_1, \cdots, g_m$ in $\mathcal{X}^*$ such that

$$U_S \cap \{x \in \mathcal{X} \mid |g_i(x)| < 1, \forall i \in \{1, \cdots, m\}\} = \emptyset.$$

Fix $x \in U_S$. Therefore there is $1 \le i_0 \le m$ such that $|g_{i_0}(x)| \ge 1$. Without loss of generality we may assume $g_{i_0}(x) \ge 1$. So $\|x\| \le \max_{1\le i\le m}\{g_i(x)\}$ and thus

$$S \subseteq \{x \in \mathcal{X} \mid \|x\| \le \max_{1\le i\le m}\{g_i(x)\}\}.$$

Hence S is a quasi-Bishop-Phelps cone.

Secondly, let's show that $(A) \Rightarrow (C)$. Suppose $\{s_\lambda\}_{\lambda\in\Lambda}$ is a bounded net in S with $s_\lambda \xrightarrow{\omega} 0$. Without loss of generality we can assume that $\|s_\lambda\| \le 1$ for each $\lambda \in \Lambda$. Let G be a compact subset of $\mathcal{X}^*$ such that

$$S \subseteq \{x \in \mathcal{X} \mid \|x\| \le \sup\{f(x) \mid f \in G\}\}.$$

For arbitrary $\epsilon > 0$, there exist $f_1, \cdots, f_n$ in G such that

$$G \subseteq \bigcup_{i=1}^{n} (f_i + \frac{\epsilon}{2} B(\mathcal{X}^*)),$$

where $B(\mathcal{X}^*)$ is the closed unit ball of $\mathcal{X}^*$. For each $\lambda \in \Lambda$, since $\|s_\lambda\| \le 1$, we have

$$\|s_\lambda\| \le \sup\{f(s_\lambda) \mid f \in G\} \le \max_{1\le i\le n}\{f_i(s_\lambda)\} + \frac{\epsilon}{2}. \tag{1.9}$$

It follows from $s_\lambda \xrightarrow{\omega} 0$ that there exists $\lambda_0 \in \Lambda$ such that for each $\lambda \in \Lambda$ with $\lambda > \lambda_0$, $\max_{1\le i\le n}\{f_i(s_\lambda)\} \le \frac{\epsilon}{2}$. This and (1.9) imply that $s_\lambda \to 0$. □

Recall that every weakly convergent sequence in $\mathcal{X}$ is bounded. Also notice that the weak topology ω of $\mathcal{X}$ is metrizable on every bounded subset of $\mathcal{X}$ if $\mathcal{X}^*$ is separable. Then Proposition 1.2.6 implies the following corollary.

Corollary 1.2.7 *Let $\mathcal{X}$ be a normed space such that $\mathcal{X}^*$ is separable. Then a closed convex pointed cone S of $\mathcal{X}$ is a quasi-Bishop-Phelps cone if and only if each sequence in S which weakly converges to 0 is convergent to 0.*

In view of preceding corollary, we have the following definition.

Definition 1.2.4 *Let $\mathcal{X}$ be a normed space. Then a closed convex cone $S \subseteq \mathcal{X}$ is said to have property (W) if and only if each sequence in S which weakly converges to 0 is convergent to 0.*

Therefore the following implication is clear.

Remark 1.2.4 *S has a bounded base $\Rightarrow$ S is a quasi-Bishop-Phelps cone $\Rightarrow$ S has property (W).*

See [14]; there is an example that S has a property (W) but S has no bounded base.

Now we present the density results in the normed space.

Lemma 1.2.8 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone with an τ-enlargement $\{S_n\}_{n\in\mathbb{N}}$. Let $A \subseteq \mathcal{X}$ be an τ-compact convex subset. Then for each $\bar{a} \in E(A,S)$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n \in E(A,S_n)$ for each $n \in \mathbb{N}$ such that $dist(a_n, \bar{a} - S) \to 0$. Hence $E(A,S) \subseteq cl(Pos(A,S) + S)$.*

Proof: Let $A_n = (\bar{a} - S_n) \cap A$. Then $E(A_n, S_n) \subseteq E(A, S_n)$ (Because for any $x \in E(A_n, S_n)$, we have $x \in \bar{a} - S_n$ and thus $x - S_n \subseteq \bar{a} - S_n$; so $A_n \cap (x - S_n) = \{x\} \Rightarrow A \cap (\bar{a} - S_n) \cap (x - S_n) = \{x\} \Rightarrow A \cap (x - S_n) = \{x\}$). Since A is τ-compact and S_n is τ-closed, A_n is again τ-compact. Using argument involved Zorn's Lemma we have $E(A_n, S_n) \neq \emptyset$. By Proposition 1.1.8, $E(A_n, S_n) \subseteq E(A, S_n) \subseteq Pos(A, S)$. Therefore we can pick $a_n \in E(A_n, S_n) \subseteq Pos(A, S)$ for each $n \in \mathbb{N}$. As $a_n \in A_n \subseteq \bar{a} - S_n$, we have $\bar{a} - a_n \in S_n$. Since $\{S_n\}_{n\in\mathbb{N}}$ is a τ-enlargement of S and

$\{a_n\}_{n\in\mathbb{N}}$ is bounded (because A is bounded), we have $dist(\bar{a} - a_n, S) \to 0$. In the other word, $dist(a_n, \bar{a} - S) \to 0$. As a consequence, $E(A, S) \subseteq cl(Pos(A, S) + S)$. □

Theorem 1.2.9 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone with a base; let $A \subseteq \mathcal{X}$ be a weakly compact convex subset. Then for each $\bar{a} \in E(A, S)$, there exist a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $Pos(A, S)$ and a sequence $\{s_n\}_{n\in\mathbb{N}}$ in S such that $a_n \xrightarrow{\omega} \bar{a}$ and $||a_n + s_n - \bar{a}|| \to 0$.*

Proof: Suppose S has a base Θ. Without loss of generality we can assume that $\delta = \inf\{\|\theta\| | \theta \in \Theta\} > 1$. By Proposition 1.2.3 there exists a ω-enlargement $\{S_n\}_{n\in\mathbb{N}}$ of S. In view of Lemma 1.2.8, for each $n \in \mathbb{N}$, there exist $a_n \in Pos(A, S)$ and $s_n \in S$, such that

$$\|a_n + s_n - \bar{a}\| \to 0. \tag{1.10}$$

Hence the second conclusion follows. Next, since A is weakly compact, by Eberlein-Smulian Theorem, there is a subsequence of $\{a_n\}_{n\in\mathbb{N}}$ weakly convergent to some $a_0 \in A$. Without loss of generality, we can assume that $a_n \xrightarrow{\omega} a_0$. This and (1.10) imply that $s_n \xrightarrow{\omega} \bar{a} - a_0$. Since S is weakly closed, $\bar{a} - a_0 \in S$. As $\bar{a} \in E(A, S)$, we conclude that $a_0 = \bar{a}$. So the first conclusion $a_n \xrightarrow{\omega} \bar{a}$ follows. □

Remark 1.2.5 *Since the norm topology and the weak topology coincide on every compact subset of $\mathcal{X}$, the first conclusion of Theorem 1.2.9 implies Theorem 1.2.2.*

The following main result extends the theorem of Petschke.

Theorem 1.2.10 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone with a base and property (W); let $A \subseteq \mathcal{X}$ be a weakly compact convex subset. Then $E(A, S) \subseteq cl(Pos(A, S))$.*

Proof: Let $\bar{a} \in E(A, S)$. Following the notation in the proof of Theorem 1.2.9, we have $a_n \xrightarrow{\omega} \bar{a}$ and $||a_n + s_n - \bar{a}|| \to 0$. These two imply that $s_n \xrightarrow{\omega} 0$. Since S has property (W), $s_n \to 0$ and thus $a_n \to \bar{a}$. Since $\bar{a}$ is arbitrarily chosen, we have $E(A, S) \subseteq cl(Pos(A, S))$. □

The following theorem is directly from Theorem 1.2.10 and Remark 1.2.4.

Theorem 1.2.11 *Let $S \subseteq \mathcal{X}$ be a closed convex cone with a base; let $A \subseteq \mathcal{X}$ be a weakly compact convex subset. Assume that S is a quasi-Bishop-Phelps cone. Then $E(A, S) \subseteq cl(Pos(A, S))$.*

Next we discuss the density theorem that there is no compactness assumption on the set A. In order to reach this result, some notations and lemmas are needed.

First let us consider a general locally convex space $\mathcal{X}$. Suppose S is a cone with a base Θ. We define

$$int_\Theta(S^+) = \{f \in S^+ | \inf\{f(\theta) | \theta \in \Theta\} > 0\}.$$

Remark 1.2.6 *By the Separation Theorem, $int_\Theta(S^+) \neq \emptyset$.*

Remark 1.2.7 $S^+ + int_\Theta(S^+) \subseteq int_\Theta(S^+) \subseteq S^{+i}$.

Next proposition states that $int_\Theta(S^+)$ and $int(S^+)$ coincide when Θ is a bounded base in a normed space, where $int(S^+)$ is the norm interior of S^+ in $\mathcal{X}^*$.

Proposition 1.2.12 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a cone with a bounded base Θ. Then $int_\Theta(S^+) = int(S^+)$.*

Proof: Let $\delta = \inf\{\|\theta\| | \theta \in \Theta\} > 0$. Firstly, for any $f \in int(S^+)$, there exists $\epsilon > 0$ such that $f + g \in S^+$ for any $g \in \mathcal{X}^*$ with $\|g\| \leq \epsilon$. Take $\theta_\epsilon \in \Theta$ such that

$$f(\theta_\epsilon) < \inf\{f(\theta) | \theta \in \Theta\} + \frac{\epsilon\delta}{2}. \tag{1.11}$$

By Hahn-Banach Theorem, there exists $g_\epsilon \in \mathcal{X}^*$ such that $\|g_\epsilon\| = \epsilon$ and $g_\epsilon(\theta_\epsilon) = -\epsilon\|\theta_\epsilon\|$, so $f + g_\epsilon \in S^+$. Therefore $f(\theta_\epsilon) + g_\epsilon(\theta_\epsilon) \geq 0$; that is, $f(\theta_\epsilon) \geq \epsilon\|\theta_\epsilon\| \geq \epsilon\delta$. This and (1.11) imply that

$$\inf\{f(\theta)|\theta \in \Theta\} > f(\theta_\epsilon) - \frac{\epsilon\delta}{2} \geq \frac{\epsilon\delta}{2} > 0.$$

This results that $f \in int_\Theta(S^+)$.

Secondly, suppose $f \in int_\Theta(S^+)$. Let $\inf\{f(\theta)|\theta \in \Theta\} = \alpha > 0$. Since Θ is bounded, there exists $\epsilon > 0$ such that whenever $\theta \in \Theta$ and $g \in \mathcal{X}^*$ with $\|g\| \leq \epsilon$, $|g(\theta)| \leq \frac{\alpha}{2}$; furthermore,

$$(f + g)(\theta) \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} > 0.$$

This tells that $f \in int(S^+)$. □

Corollary 1.2.13 *In a normed space $\mathcal{X}$ with a convex cone S, $int(S^+) \neq \emptyset$ if and only if S has a bounded base.*

Lemma 1.2.14 ([3] Lemma 3.1(i)) *Let $\mathcal{X}$ be a locally convex space and $S \subseteq \mathcal{X}$ be an ordering cone with a bounded base Θ. Suppose $\{x_n\}_{n\in\mathbb{N}}$ a decreasing sequence with respect to S; that is, $x_1 \succeq_S x_2 \succeq_S \cdots \succeq_S x_n \succeq_S \cdots$. Also suppose that $f \in int_\Theta(S^+)$ such that $\{f(x_n)\}_{n\in\mathbb{N}}$ is bounded below. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$.*

Proof: Since $f \in int_\Theta(S^+)$, let $\alpha = \inf\{f(\theta)|\theta \in \Theta\} > 0$. For any neighborhood V of 0 in $\mathcal{X}$, since Θ is bounded, there exists $t_0 > 0$ such that

$$t\Theta \subseteq V, \qquad \forall t \in [0, t_0]. \tag{1.12}$$

Since $f \in int_\Theta(S^+) \subseteq S^{+i}$ and $\{x_n\}_{n\in\mathbb{N}}$ is decreasing with respect to S, we have $\{f(x_n)\}_{n\in\mathbb{N}}$ is decreasing. This and that $\{f(x_n)\}_{n\in\mathbb{N}}$ is bounded below imply that

$\{f(x_n)\}_{n\in\mathbb{N}}$ is convergent. Therefore there is $n_0 \in \mathbb{N}$ such that

$$f(x_m - x_n) < \alpha t_0, \qquad \forall n, m, n \geq m \geq n_0. \tag{1.13}$$

Also for any $n \geq m \geq n_0$, $x_m - x_n \succeq_S 0$, so there are $\lambda_{mn} \geq 0$ and $\theta_{mn} \in \Theta$ such that

$$x_m - x_n = \lambda_{mn}\theta_{mn}.$$

This and (1.13) imply that

$$\alpha t_0 > f(x_m - x_n) = \lambda_{mn} f(\theta_{mn}) \geq \lambda_{mn}\alpha \geq 0;$$

hence $t_0 > \lambda_{mn} \geq 0$. So, by (1.12), we have $x_m - x_n \in V$. As a consequence, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$. □

We say that A has a **domination property** with respect to S if, for any $a \in A$, there is $\bar{a} \in E(A, S)$ such that $\bar{a} \preceq_S a$. The following lemma gives a sufficient condition for the domination property of A.

Lemma 1.2.15 ([3] Theorem 3.1) *Let $\mathcal{X}$ be a locally convex space and $S \subseteq \mathcal{X}$ be a closed convex cone with a bounded base Θ. Let $A \subseteq \mathcal{X}$ be a sequentially complete subset. Suppose that there is $f \in int_\Theta(S^+)$ such that f is bounded below on A. Then A has the domination property.*

Proof: For any $x_0 \in A$,

$$\inf\{f(x)|x \in A \cap (x_0 - S)\} \geq \inf\{f(x)|x \in A\} > -\infty;$$

therefore for any $\epsilon > 0$ given, there is $x_\epsilon \in A \cap (x_0 - S)$ such that

$$f(x_\epsilon) < \inf\{f(x)|x \in A \cap (x_0 - S)\} + \epsilon.$$

In this way, for any $a_0 \in A$, we can construct a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A such that

(i) $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence with respect to S, and

(ii) $f(a_n) < \inf\{f(x)|x \in A \cap (a_{n-1} - S)\} + \frac{1}{n}$, for all $n \in \mathbb{N}$.

Since (i) and that f is bounded below on A, by Lemma 1.2.14, $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$. From the sequential completeness of A, there is $\bar{a} \in A$ such that $a_n \to \bar{a}$. Since S is a closed convex cone, one has $a_n \succeq_S \bar{a}$ for all $n \in \mathbb{N}$ and $\bar{a} \preceq_S a_0$. It suffices to show that $\bar{a} \in E(A, S)$. Suppose not: there exists $\hat{a} \in A$ such that $\hat{a} \neq \bar{a}$ and $\hat{a} \preceq_S \bar{a}$. As $f \in int_\Theta(S^+) \subseteq S^{+i}$, $f(\hat{a}) < f(\bar{a})$. On the other hand, since $a_n \succeq_S \bar{a} \succeq_S \hat{a}$, we have $\hat{a} \in A \cap (a_n - S)$ for all $n \in \mathbb{N}$. Therefore by (ii), for any $n \in \mathbb{N}$,

$$f(\hat{a}) \geq \inf\{f(x)|x \in A \cap (a_n - S)\} > f(a_{n+1}) - \frac{1}{n+1}.$$

When $n \to +\infty$, we have $f(\hat{a}) \geq f(\bar{a})$. This contradicts the assertion before. As a result, $\bar{a} \in E(A, S)$. □

Here is another main result in this section. Note that there is no compactness assumption on the set A.

Theorem 1.2.16 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone with a weakly compact base; let $A \subseteq \mathcal{X}$ be a closed convex subset and $\bar{a} \in E(A, S)$. If there exists $r > 0$ such that $A \cap B(\bar{a}, r)$ is complete, then $\bar{a} \in cl(Pos(A, S))$. Hence $E(A, S) \subseteq cl(Pos(A, S))$ if A is complete itself.*

Proof: Let Θ be a weakly compact base of S. Without loss of generality we may assume that $\inf\{\|\theta\||\theta \in \Theta\} > 1$. Then for each $n \in \mathbb{N}$, we let

$$S_n = cone(\Theta + \frac{1}{n}B(\mathcal{X})).$$

By Remark 1.2.1, S_n is a closed convex cone with a closed bounded base $\Theta + \frac{1}{n}B(\mathcal{X})$. Let $A_r := A \cap B(\bar{a}, r)$. By assumption A_r is complete; since $\bar{a} - S_n$

is closed, $A_r \cap (\bar{a} - S_n)$ is also complete. Firstly, we show that $E(A_r \cap (\bar{a} - S_n), S_n) \neq \emptyset$. In view of Lemma 1.2.15, it is sufficient to show that there exists $f \in int_{\Theta+\frac{1}{n}B(\mathcal{X})}((S_n)^+)$ such that f is bounded below on $A_r \cap (\bar{a} - S_n)$. By the Separation Theorem, $int_{\Theta+\frac{1}{n}B(\mathcal{X})}((S_n)^+) \neq \emptyset$. Pick $f \in int_{\Theta+\frac{1}{n}B(\mathcal{X})}((S_n)^+)$. Since $A_r \cap (\bar{a} - S_n) \subseteq B(\bar{a}, r)$, f is bounded below on $A_r \cap (\bar{a} - S_n)$. Therefore $E(A_r \cap (\bar{a} - S_n), S_n) \neq \emptyset$.

Pick $a_n \in E(A_r \cap (\bar{a} - S_n), S_n)$ for each $n \in \mathbb{N}$. Then there exist $\theta_n \in \Theta$, $b_n \in B(\mathcal{X})$ and $t_n \geq 0$ such that

$$a_n = \bar{a} - t_n\big(\theta_n + (\frac{1}{n})b_n\big). \tag{1.14}$$

Note that $A_r \cap (\bar{a} - S_n)$ is bounded and $\inf\{\|x\| | x \in \Theta + \frac{1}{n}B(\mathcal{X})\} > 0$. Then $\{t_n\}_{n\in\mathbb{N}}$ is bounded. Without loss of generality we can assume that $t_n \to t \geq 0$ and $\{\theta_n\}_{n\in\mathbb{N}}$ weakly convergent to some $\theta \in \Theta$ (Note that Θ is weakly compact.). Then it follows from (1.14) that $\{a_n\}_{n\in\mathbb{N}}$ weakly convergent to $\bar{a} - t\theta$. Note that $\bar{a} - t\theta \in A \cap (\bar{a} - S)$. Since $\bar{a} \in E(A, S)$, we can conclude that $t = 0$. As $\Theta + \frac{1}{n}B(\mathcal{X})$ is bounded, it again follows from (1.14) and $t = 0$ that $a_n \to \bar{a}$.

What remains is to show that $a_n \in Pos(A, S)$ for sufficiently large $n \in \mathbb{N}$. Since $a_n \to \bar{a}$, without loss of generality we may assume that $\|a_n - \bar{a}\| \leq \frac{r}{2}$ for each n. Therefore

$$A \cap B(a_n, \frac{r}{2}) \subseteq A \cap B(\bar{a}, r) = A_r. \tag{1.15}$$

From Proposition 1.1.8,

$$a_n \in E(A_r \cap (\bar{a} - S_n), S_n) \subseteq E(A_r, S_n) \subseteq Pos(A_r, S).$$

Then there exists $f_n \in S^{+i}$ such that for all $x \in A_r$,

$$f_n(a_n) \leq f_n(x). \tag{1.16}$$

And for any $x \in A \setminus A_r$, according to (1.15), $\|x - a_n\| > \frac{r}{2}$. By the convexity of A,

$$a_n + \frac{r}{2\|x - a_n\|}(x - a_n) \in A \cap B(a_n, \frac{r}{2}) \subseteq A_r.$$

Thus by (1.16),

$$f_n(a_n) \leq f(a_n + \frac{r}{2\|x - a_n\|}(x - a_n));$$

that is, $f_n(a_n) \leq f_n(x)$. Combining this with (1.16), we have $f_n(a_n) \leq f_n(a)$ for all $a \in A$; that is, $a_n \in Pos(A, S)$. □

1.3 The Arrow-Barankin-Blackwell Theorem in Topological Vector Spaces

In this section, we will discuss the result given by X. Y. Zheng [9], in which Arrow-Barankin-Blackwell Theorem is generalized in the setting of general topological vector spaces. This follows along the proofs of the main result of Ferro [18] in the setting of normed spaces and of the main result of Chen [25] in the setting of locally convex spaces.

Before the main result is discussed, we introduce several notations and lemmas needed.

Let $\mathcal{X}$ be a topological vector space, $\mathcal{X}^*$ be the corresponding topological dual space, we assume that $\mathcal{X}^*$ separates the points of $\mathcal{X}$.

Let $\mathbf{N(0)}$ denote the family of all neighborhoods of 0 in $\mathcal{X}$. And for each $V \in \mathbf{N(0)}$, V^o denotes the **polar** of V, that is,

$$V^o = \{f \in \mathcal{X}^* | |f(x)| \leq 1, \forall x \in V\}.$$

Clearly if V_1, $V_2 \in \mathbf{N(0)}$ with $V_1 \supseteq V_2$, then $V_1^o \subseteq V_2^o$. Let $\mathcal{D} = \{(n, V) | n \in \mathbb{N}, V \in \mathbf{N(0)}\}$. We define a binary relation $\preceq$ in $\mathcal{D}$ as follows:

$$(n_1, V_1) \preceq (n_2, V_2) \quad \text{if and only if} \quad n_1 \leq n_2 \text{ and } V_1 \supseteq V_2.$$

Then we have the following lemmas.

Lemma 1.3.1 *$(\mathcal{D}, \preceq)$ is a directed set.*

Proof: For any (n_1, V_1), $(n_2, V_2) \in \mathcal{D}$ given, take $n = \max\{n_1, n_2\}$ and $V = V_1 \cap V_2$. Clearly $(n, V) \in \mathcal{D}$ and $(n_i, V_i) \preceq (n, V)$ for $i = 1, 2$. □

Lemma 1.3.2 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be a weakly closed convex cone. Then $x \in S$ if and only if $f(x) \geq 0$ for each $f \in S^+$.*

Proof: The necessity is clear from the definition of S^+. Now suppose $x \notin S$. Note that the weak topology is locally convex. Since $\{x\}$ is weakly compact convex and S is weakly closed convex, by the Separation Theorem, there exists $f \in \mathcal{X}^* \setminus \{0\}$ such that $f(x) < f(s)$ for all $s \in S$. It is easy to show, that $f(s) \geq 0$ for all $s \in S$ (that is, $f \in S^+$), and, that $f(x) < 0$. These prove the sufficiency. □

Lemma 1.3.3 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be a weakly closed convex cone. Suppose $\bar{p} \in S^{+i}$. Set*

$$B(n, V) = (\frac{1}{n})\bar{p} + V^o \cap S^+, \quad \forall (n, V) \in \mathcal{D}, \tag{1.17}$$

and,

$$A(n, V) = \bigcup_{t>0} tB(n, V), \quad \forall (n, V) \in \mathcal{D}. \tag{1.18}$$

Then we have

1. *$A(n_1, V_1) \subseteq A(n_2, V_2)$ if $(n_1, V_1) \preceq (n_2, V_2)$, $(n_i, V_i) \in \mathcal{D}$ for $i = 1, 2$, and*
2. *$x \in S$ if and only if $f(x) \geq 0$ for each $f \in \bigcup_{(n,V)\in\mathcal{D}} A(n, V)$.*

Proof:

1. Let $(n_1, V_1) \preceq (n_2, V_2)$ where $(n_i, V_i) \in \mathcal{D}$ for $i = 1, 2$. For each $f \in A(n_1, V_1)$, it follows from (1.17) and (1.18) that there exist $t > 0$ and $p_1 \in V_1^o \cap S^+$ such that
$$f = t\big((\frac{1}{n_1})\bar{p} + p_1\big). \tag{1.19}$$
Since $(n_1, V_1) \preceq (n_2, V_2)$, we have $0 \leq \frac{n_1}{n_2} \leq 1$ and $V_1^o \subseteq V_2^o$. Clearly $(\frac{n_1}{n_2})p_1 \in V_1^o \cap S^+ \subseteq V_2^o \cap S^+$. Therefore this and (1.19) imply that
$$f = t\big((\frac{1}{n_1})\bar{p} + p_1\big) = (\frac{n_2}{n_1})t\big((\frac{1}{n_2})\bar{p} + (\frac{n_1}{n_2})p_1\big) \in (\frac{n_2}{n_1})tB(n_2, V_2) \subseteq A(n_2, V_2).$$
As a result, $A(n_1, V_1) \subseteq A(n_2, V_2)$.

2. By Proposition 1.1.1, S^+ is weak-*-closed; therefore, in view of Lemma 1.3.2, it suffices to show that $S^+ = cl(\bigcup_{(n,V)\in\mathcal{D}} A(n, V))$. Firstly, it is clear that $\bigcup_{(n,V)\in\mathcal{D}} A(n, V) \subseteq S^+$. Secondly, let $f \in S^+$. There exists $V_f \in \mathbf{N}(\mathbf{0})$ such that $f \in V_f \cap S^+$. As $f + (\frac{1}{n})\bar{p} \in A(n, V_f) \subseteq \bigcup_{(n,V)\in\mathcal{D}} A(n, V)$ for each $n \in \mathbb{N}$, we have $f \in cl(\bigcup_{(n,V)\in\mathcal{D}} A(n, V))$ and it ends the proof.

□

Remark 1.3.1 *Both the sets $B(n, V)$ and $A(n, V)$ defined for each $(n, V) \in \mathcal{D}$ are subsets of the quasi-interior S^{+i}.*

And the lemma below is a well-known theorem. We state it without the proof.

Lemma 1.3.4 ([16]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological vector spaces and let A and B be compact convex subsets in $\mathcal{X}$ and in $\mathcal{Y}$ respectively. Let Φ be a real-valued function on $A \times B$. Suppose that, for each $b \in B$, $\Phi(\cdot, b)$ is a continuous convex function on A, and that, for each $a \in A$, $\Phi(a, \cdot)$ is a continuous concave function on B. Then there is a pair $(a_0, b_0) \in A \times B$ such that*
$$\Phi(a_0, b) \leq \Phi(a_0, b_0) \leq \Phi(a, b_0)$$
for all $a \in A$ and for all $b \in B$.

Now we present the main result of this section.

Theorem 1.3.5 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be a weakly closed convex cone such that $S^{+i} \neq \emptyset$. Suppose that A is a nonempty compact convex subset of $\mathcal{X}$. Then $E(A, S) \subseteq cl(Pos(A, S))$.*

Proof: Fix $\bar{a} \in E(A, S)$. With a translation we can assume that $\bar{a} = 0 \in E(A, S)$. Let $\bar{p} \in S^{+i}$; $B(n, V)$ and $A(n, V)$ are defined as in Lemma 1.3.3 for each $(n, V) \in \mathcal{D}$. By Alaoglu Theorem (Theorem 3.15 in [4]), V^o is a weak-*-compact convex subset in $\mathcal{X}^*$ for each $V \in \mathbf{N(0)}$. Together with that S^+ is weak-*-closed convex, we have that $B(n, V) = (\frac{1}{n})\bar{p} + V^o \cap S^+$ is weak-*-compact convex in $\mathcal{X}^*$ for each $(n, V) \in \mathcal{D}$. Define the function $\Phi : A \times B(n, V) \to \mathbb{R}$ by

$$\Phi(a, f) = f(a), \qquad \forall a \in A, \forall f \in B(n, V).$$

Clearly Φ satisfies the hypothesis stated in Lemma 1.3.4, when $\mathcal{X}^*$ is equipped with the weak-* topology. Therefore there exist $a_{(n,V)} \in A$ and $f_{(n,V)} \in B(n, V)$ such that

$$f(a_{(n,V)}) \leq f_{(n,V)}(a_{(n,V)}) \leq f_{(n,V)}(a), \qquad \forall a \in A, \forall f \in B(n, V). \tag{1.20}$$

Since $f_{(n,V)} \in B(n, V) \subseteq S^{+i}$, it follows from (1.20) that $a_{(n,V)} \in Pos(A, S)$. As $0 \in A$, again from (1.20) we have

$$f(a_{(n,V)}) \leq 0, \qquad \forall (n, V) \in \mathcal{D}, \forall f \in B(n, V). \tag{1.21}$$

By Lemma 1.3.1, $\{a_{(n,V)}\}_{(n,V)\in\mathcal{D}}$ is a net in $Pos(A, S)$. Since A is compact, there exists a subnet of $\{a_{(n,V)}\}_{(n,V)\in\mathcal{D}}$ convergent to some $a_0 \in A$. Therefore $a_0 \in cl(Pos(A, S))$. It is sufficient to show that $a_0 = 0 \in E(A, S)$ to end the proof. Without loss of generality, we may assume that $\{a_{(n,V)}\}_{(n,V)\in\mathcal{D}}$ converges to a_0. Therefore

$$g(a_{(n,V)}) \to g(a_0), \qquad \forall g \in \bigcup_{(n,V)\in\mathcal{D}} A(n, V). \tag{1.22}$$

Fix $g \in \bigcup_{(n,V)\in\mathcal{D}} A(n,V)$. Then there is $(n_g, V_g) \in \mathcal{D}$ such that $g \in A(n_g, V_g)$. According to (1.22), for any arbitrary $\epsilon > 0$, there is $(n_{g,\epsilon}, V_{g,\epsilon}) \in \mathcal{D}$ such that, $(n_g, V_g) \preceq (n_{g,\epsilon}, V_{g,\epsilon})$, and

$$g(a_0) \leq g(a_{(n_{g,\epsilon},V_{g,\epsilon})}) + \epsilon, . \tag{1.23}$$

By (1) of Lemma 1.3.3, $g \in A(n_g, V_g) \subseteq A(n_{g,\epsilon}, V_{g,\epsilon})$. Therefore there are $t > 0$ and $f \in B(n_{g,\epsilon}, V_{g,\epsilon})$ such that $g = tf$. By (1.21),

$$g(a_{(n_{g,\epsilon},V_{g,\epsilon})}) = tf(a_{(n_{g,\epsilon},V_{g,\epsilon})}) \leq 0.$$

Since ϵ is arbitrary, this and (1.23) imply that $g(a_0) \leq 0$. Since g is an arbitrary element in $\bigcup_{(n,V)\in\mathcal{D}} A(n,V)$, we have $g(a_0) \leq 0$ for all $g \in \bigcup_{(n,V)\in\mathcal{D}} A(n,V)$. By (2) of Lemma 1.3.3, $a_0 \in -S$. Since $0 \in E(A,S)$, it should be that $a_0 = 0$. As a result, the theorem follows. □

Considering the case that the topological vector space is equipped with its weak topology, it is easy to deduce the theorem below using similar argument in the proof of Theorem 1.3.5.

Theorem 1.3.6 *Let $\mathcal{X}$ be a topological vector space and $S \subseteq \mathcal{X}$ be a weakly closed convex cone such that $S^{+i} \neq \emptyset$. Suppose that A is a nonempty weakly compact convex subset of $\mathcal{X}$. Then $E(A,S) \subseteq w\text{-}cl(Pos(A,S))$.*

Since every closed convex subset is weakly closed in a locally convex space, we have the following corollary.

Corollary 1.3.7 *Let $\mathcal{X}$ be a locally convex space and $S \subseteq \mathcal{X}$ be a closed convex cone such that $S^{+i} \neq \emptyset$. Suppose that A is a nonempty compact (weakly compact, respectively) convex subset of $\mathcal{X}$. Then $E(A,S) \subseteq cl(Pos(A,S))$ ($E(A,S) \subseteq w\text{-}cl(Pos(A,S))$, respectively).*

1.4 Density Results in Dual Space Setting

Let $\mathcal{X}$ be a topological vector space and $\mathcal{X}^*$ be the corresponding topological dual space. Let A be a nonempty subset of $\mathcal{X}$. A point $a_0 \in A$ is said to be supported by a linear function ϕ if $\phi(a_0) \leq \phi(a)$ for all $a \in A$. In this case, a_0 is called the support point of the set A and ϕ is called the support functional of the set A. In general, we can consider a subset F of $\mathcal{X}^*$: A point $a_0 \in A$ is said to a F-support point, denoted by $a_0 \in F\text{-}supp(A)$, if there exists $f_0 \in F$ such that $f_0(a_0) \leq f_0(a)$ for all $a \in A$. In contrast to the sections before, you may conclude that all the positive proper efficient points are the support points with the set of support functionals consisting of all the strictly positive continuous linear functionals; that is, the set $Pos(A, S)$ is indeed coinciding $S^{+i}\text{-}supp(A)$.

Furthermore, note that $\mathcal{X}^*$ is a topological vector space itself. We can consider the same concepts with the support points in $\mathcal{X}^*$ and the support functionals in $\mathcal{X}^{**}$. However, since there is a canonical embedding of $\mathcal{X}$ in $\mathcal{X}^{**}$, we can consider the support functional restricted in the predual instead of the dual. In this case, let F be a nonempty subset of $\mathcal{X}^*$ and A be a nonempty subset of $\mathcal{X}$. A point $f_0 \in F$ is said to a A-weak-*-support point of F if there exists $a_0 \in A$ such that $f_0(a_0) \leq f(a_0)$ for all $f \in F$. Without any ambiguity caused, and for short, we use $A\text{-}supp(F)$ to denote the set of all A-weak-*-support points of F and we directly use the element $x \in \mathcal{X}$ to denote a weak-* continuous linear functional $\hat{x} : \mathcal{X}^* \to \mathbb{R}$, where

$$\hat{x}(f) := f(x), \quad \forall f \in \mathcal{X}^*.$$

In this section, let $\mathcal{X}$ be a normed space and let S be a convex cone in $\mathcal{X}$. By Proposition 1.1.1, the dual cone S^+ is a weak-*-closed convex cone in the topological dual space $\mathcal{X}^*$. Hence a pre-order $\preceq_{S^+}$ is induced in the way that for any f, $g \in \mathcal{X}^*$, $f \preceq_{S^+} g$ if and only if $g - f \in S^+$. In the view of Definition 1.1.1, $\bar{f} \in F$ is said to be an efficient point of F with respect to S^+, denoted by

$\bar{f} \in E(F, S^+)$, if

$$(F - \bar{f}) \cap (-S^+) \subseteq S^+.$$

Moreover, if S is a convex cone such that $cl(S - S) = \mathcal{X}$, then by Proposition 1.1.2 S^+ is a pointed cone in $\mathcal{X}^*$. In this case, $\bar{f} \in E(F, S^+)$ if and only if

$$(F - \bar{f}) \cap (-S^+) = \{0\}.$$

Suppose S is a convex cone. Analogous to S^{+i} the set of strictly positive continuous linear functionals on $\mathcal{X}$, we define

$$S_p := \{s \in S | g(s) > 0, \forall g \in S^+ \setminus \{0\}\}.$$

S_p consists of all the strictly positive elements in the ordering cone S. It is easy to verify that S_p-$supp(F) \subseteq E(F, S^+)$. Therefore, analogous to the Arrow-Barankin-Blackwell Theorem, it is worthwhile to study when S_p-$supp(F)$ is dense in $E(F, S^+)$. In this section we will present two density results in this dual setting. The first one is first proved by W. Song [22] and the second one is by K. F. Ng and X. Y. Zheng [14].

Before our discussion on the density result, as we know that the condition $S_p \neq \emptyset$ is indispensable to such a density result in the dual space setting, we first study some propositions concerning the set S_p.

Throughout this section, we use ω^* denote the weak-* topology of $\mathcal{X}$.

Lemma 1.4.1 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone. Suppose $x \in \mathcal{X}$. If $g(x) \geq 0$ for all $g \in S^+$, then $x \in S$.*

Proof: Suppose $x \notin S$. By the Separation Theorem, there is $f \in \mathcal{X}^* \setminus \{0\}$ such that

$$\inf\{f(s) | s \in S\} > f(x). \tag{1.24}$$

It is easy to verify that

$$\inf\{f(s)|s \in S\} = 0; \tag{1.25}$$

thus $f \in S^+$. However, by (1.24) and (1.25), $f(x) < 0$; it contradicts with the hypothesis. So we have $x \in S$. □

Proposition 1.4.2 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone. Then $S_p \neq \emptyset$ if and only if S^+ has a weak-*-closed base.*

Proof: Suppose $s_0 \in S_p$. Then let $\Psi = \{g \in S^+|g(s_0) = 1\}$. Clearly Ψ is a base of S^+. Since S^+ is weak-*-closed, Ψ is a weak-*-closed base of S^+. Conversely, suppose S^+ has a weak-*-closed base Ψ. Consider that $\mathcal{X}^*$ is equipped with its weak-* topology, which is locally convex indeed. Then by the Separation Theorem, there exists $x_0 \in \mathcal{X} \setminus \{0\}$ such that $\psi(x_0) > 0(x_0) = 0$ for all $\psi \in \Psi$. Clearly

$$g(x_0) > 0, \qquad \forall g \in S^+ \setminus \{0\}. \tag{1.26}$$

Then $g(x_0) \geq 0$ for all $g \in S^+$. By Lemma 1.4.1, we have $x_0 \in S$. Furthermore, in view of (1.26), $x_0 \in S_p$ and thus $S_p \neq \emptyset$. □

It is natural to study the relationship between $int(S)$, which is the norm-interior of S, and S_p, which is the set of strictly positive elements.

Proposition 1.4.3 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a convex cone. Then $int(S) \subseteq S_p$.*

Proof: Let $s \in int(S)$. Suppose $s \notin S_p$: that is, $g_s(s) = 0$ for some $g_s \in S^+ \setminus \{0\}$. Since $s \in int(S)$, there is a neighborhood V of 0 in $\mathcal{X}$ such that $s + V \subseteq S$. As

$g_s \in S^+$, we have

$$g_s(v) = g_s(s+v) \geq 0, \qquad \forall v \in V.$$

As a result, $g_s = 0$, which contradicts with $g_s \in S^+ \setminus \{0\}$. Hence $int(S) \subseteq S_p$. □

Also the following result concerns when norm interior of a closed convex cone is nonempty.

Proposition 1.4.4 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone. Then $int(S) \neq \emptyset$ if and only if S^+ has a weak-*-compact base.*

Proof: Suppose $s_0 \in int(S)$. Let $\Psi = \{g \in S^+ | g(s_0) = 1\}$. Clearly Ψ is a weak-*-closed base of S^+. Since $s_0 \in int(S)$, $s_0 - S$ is a neighborhood of 0 in $\mathcal{X}$. By Alaoglu Theorem, $\{x^* \in \mathcal{X}^* | x^*(s_0 - s) \leq 1, \forall s \in S\}$ is weak-*-compact. For any $\psi \in \Psi$ and any $s \in S$,

$$\psi(s_0 - s) = \psi(s_0) - \psi(s) \leq \psi(s_0) = 1.$$

Therefore $\Psi \subseteq \{x^* \in \mathcal{X}^* | x^*(s_0 - s) \leq 1, \forall s \in S\}$ and thus Ψ is weak-*-compact. Conversely, suppose S^+ has a weak-*-compact base Ψ. Let us consider the case that $\mathcal{X}^*$ is equipped with a weak-* topology. Then by the Separation Theorem, there is $x_0 \in \mathcal{X} \setminus \{0\}$ such that

$$\inf\{\psi(x_0) | \psi \in \Psi\} > 0.$$

Without loss of generality, we let $\inf\{\psi(x_0) | \psi \in \Psi\} > 1$. Since Ψ is weak-*-compact, $V := \{x \in \mathcal{X} | \psi(x) \leq 1, \forall \psi \in \Psi\}$ is a norm-neighborhood of 0 in $\mathcal{X}$. Then for any $v \in V$ and any $\psi \in \Psi$,

$$\psi(x_0 - v) \geq 0;$$

by Lemma 1.4.1, $x_0 - v \in S$. Therfore $x_0 - V \subseteq S$ and hence $x_0 \in int(S)$. □

The following proposition gives a very remarkable result.

Proposition 1.4.5 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone. Suppose $int(S) \neq \emptyset$. Then $S_p = int(S)$.*

Proof: In view of Proposition 1.4.3, it suffices to show that $S_p \subseteq int(S)$. Suppose $s_0 \in S \setminus (int(S) \cup \{0\})$. Since $int(S)$ is nonempty open convex, by the Separation Theorem, there exists $f \in \mathcal{X}^* \setminus \{0\}$ such that

$$f(s_0) \leq f(s), \qquad \forall s \in S.$$

Since s_0 and 0 are in S, we have $f(s_0) = 0 = \inf\{f(s)|s \in S\}$. This implies that $f \in S^+ \setminus \{0\}$. This and $f(s_0) = 0$ imply that $s_0 \notin S_p$. □

In 1997, W. Song [22] gave the following density result in the dual space setting. The proof below is due to K. F. Ng and X. Y. Zheng [14].

Theorem 1.4.6 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone such that $S_p \neq \emptyset$. Suppose F is a compact (weak-*-compact respectively) convex subset of $\mathcal{X}^*$. Then*

$$E(F, S^+) \subseteq cl(S_p\text{-}supp(F))$$

$$(E(F, S^+) \subseteq \omega^*\text{-}cl(S_p\text{-}supp(F)) \quad \textit{respectively}).$$

Proof: Since the norm topology and the weak-* topology are identical on every compact subset of $\mathcal{X}^*$, we need only to show that the theorem holds valid in the case when F is weak-*-compact. Let $\mathcal{X}^*$ is equipped with its weak-*topology. Note that S^+ is a weak-*-closed convex cone in $\mathcal{X}^*$. As $S_p \neq \emptyset$, we pick some $s_0 \in S_p$. Let $\Psi = \{g \in S^+|g(s_0) = 1\}$. By Proposition 1.4.2 Ψ is a weak-*-closed base of S^+. Also S^+ is pointed. Considering all these, S^+ is a closed convex cone with a base and F is a compact convex set in the locally convex space $(\mathcal{X}^*, \omega^*)$.

Since S_p is the set of the strictly positive weak-*-continuous linear functionals on $\mathcal{X}^*$, by Corollary 1.3.7, we have

$$E(F, S^+) \subseteq \omega^*\text{-}cl(S_p\text{-}supp(F)).$$

This proves the theorem. □

Now we discuss another density result by K. F. Ng and X. Y. Zheng [14] in the dual space setting. The concepts of the enlargement and the quasi-*-Bishop-Phelps cone are needed. Now we give the definition of quasi-*-Bishop-Phelps cones.

Definition 1.4.1 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a convex cone. The dual cone S^+ of S is called a quasi-*-Bishop-Phelps cone if there exists a compact subset K of $\mathcal{X}$ such that*

$$S^+ \subseteq \{f \in \mathcal{X}^* | \|f\| \le \sup\{f(x) | x \in K\}\}.$$

The following proposition tells us that given that S is closed convex, then S^+ is a quasi-*-Bishop-Phelps cone if $int(S)$ is nonempty.

Proposition 1.4.7 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a closed convex cone. Then $int(S) \neq \emptyset$ if and only if there exists $x_0 \in \mathcal{X} \setminus \{0\}$ such that*

$$S^+ \subseteq \{f \in \mathcal{X}^* | \|f\| \le f(x_0)\}.$$

Proof: Suppose $int(S) \neq \emptyset$. Then there exist $x_0 \in S \setminus \{0\}$ and $r > 0$ such that $B(x_0, r) := \{x \in \mathcal{X} | \|x - x_0\| \le r\} \subseteq S$. Therefore for all $g \in S^+$ and for all $x \in B(x_0, r)$, $g(x) \ge 0$. This implies that

$$g(x_0) \ge \sup\{g(x) | x \in B(0, r)\} = r\|g\|, \qquad \forall g \in S^+.$$

Hence, $S^+ \subseteq \{f \in \mathcal{X}^* | \|f\| \leq f(\frac{x_0}{r})\}$. Conversely, suppose that $S^+ \subseteq \{f \in \mathcal{X}^* | \|f\| \leq f(x_0)\}$ for some $x_0 \in \mathcal{X} \setminus \{0\}$. Then for all $g \in S^+$ and for all $x \in B(x_0, 1)$,

$$g(x_0 - x) \leq \|g\| \leq g(x_0);$$

hence $g(x) \geq 0$. By Lemma 1.4.1, $B(x_0, 1) \subseteq S$; thus $x_0 \in int(S)$. □

Using the similar argument in the proof of Proposition 1.2.6, we have the following proposition which states some equivalent conditions for a dual cone being a quasi-*-Bishop-Phelps cone.

Proposition 1.4.8 *Let $\mathcal{X}$ be a normed space and $S \subseteq \mathcal{X}$ be a convex cone. The following statements are equivalent.*

(A) The dual cone S^+ is a quasi--Bishop-Phelps cone.*

(B) $0 \notin \omega^*\text{-}cl(U_{S^+})$, where $U_{S^+} = \{g \in S^+ | \|g\| = 1\}$.

(C) For any bounded net $\{f_\lambda\}_{\lambda \in \Lambda}$ in S^+, $f_\lambda \xrightarrow{\omega^*} 0$ if and only if $f_\lambda \to 0$.

Recall that every weak-*-convergent sequence in $\mathcal{X}^*$ is bounded with respect to the norm topology of $\mathcal{X}^*$ if $\mathcal{X}$ is a Banach space. Also the weak-* topology is metrizable if $\mathcal{X}$ is separable. Then we have the following corollary.

Corollary 1.4.9 *Let $\mathcal{X}$ be a separable Banach space and $S \subseteq \mathcal{X}$ be a convex cone. Then the dual cone S^+ is a quasi-*-Bishop-Phelps cone if and only if every sequence in S^+ which is weak-*-convergent to 0 is also convergent to 0.*

In order to reach our another main density result in the dual space of a Banach space in this section, several lemmas are first introduced. The lemma below is without proof stated.

Lemma 1.4.10 ([28] Theorem 1) *Let $\mathcal{X}$ be a Banach Space and let F be a weak-*-closed convex subset of $\mathcal{X}^*$. Then for any boundary point $f \in F$, there exist a sequence $\{f_n\}_{n\in\mathbb{N}}$ in F and, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ with $\|x_n\| = 1$ for each $n \in \mathbb{N}$, such that*

$$\|f_n - f\| \to 0$$

and

$$\forall n \in \mathbb{N}, \qquad f_n(x_n) = \inf\{f(x_n) | f \in F\}.$$

Lemma 1.4.11 *Let $\mathcal{X}$ be a Banach space and $S \subseteq \mathcal{X}$ be a closed convex cone. Suppose S^+ has a weak-*-closed base Ψ such that $\inf\{\|\psi\| | \psi \in \Psi\} > 1$. For each $n \in \mathbb{N}$, define*

$$S^+(n) := cone(\Psi + \frac{1}{n}B(\mathcal{X}^*)), \tag{1.27}$$

where $B(\mathcal{X}^*) := \{x^* \in \mathcal{X}^* | \|x^*\| \leq 1\}$. Then $S^+(n)$ is a weak--closed cone with a base $\Psi + \frac{1}{n}B(\mathcal{X}^*)$ in $\mathcal{X}^*$.*

Proof: By Krein-Smulian Theorem (see Theorem 12.1 in [29]), for each $n \in \mathbb{N}$, it is sufficient to show the weak-*-closedness of $S^+(n)$ by showing that $S^+(n) \cap dB(\mathcal{X}^*)$ is weak-*-closed for each $d > 0$. Fix $n \in \mathbb{N}$ and $d > 0$. Suppose $\{g_\lambda\}_{\lambda\in\Lambda}$ is a net in $S^+(n) \cap dB(\mathcal{X}^*)$ such that $g_\lambda \xrightarrow{\omega^*} g$ for some $g \in \mathcal{X}^*$. Clearly $g \in dB(\mathcal{X}^*)$. For each $\lambda \in \Lambda$, there exist $\psi_\lambda \in \Psi$, $h_\lambda \in B(\mathcal{X}^*)$ and $t_\lambda \geq 0$ such that $g_\lambda = t_\lambda(\psi_\lambda + (\frac{1}{n})h_\lambda)$. By Alaoglu Theorem, $B(\mathcal{X}^*)$ is weak-*-compact. Without loss of generality we can assume that $\{h_\lambda\}_{\lambda\in\Lambda}$ is weak-*-convergent to some $h \in B(\mathcal{X}^*)$. Also since $\inf\{\|f\| | f \in \Psi + \frac{1}{n}B(\mathcal{X}^*)\} > 0$ and $\{g_\lambda\}_{\lambda\in\Lambda} \subseteq dB(\mathcal{X}^*)$, we have $\{t_\lambda\}_{\lambda\in\Lambda}$ is a bounded scalar sequence. Again without loss of generality we can assume that $\{t_\lambda\}_{\lambda\in\Lambda}$ converges to some $t \geq 0$. There are two cases.

1. If $t = 0$:

$$t_\lambda \psi_\lambda = g_\lambda - t_\lambda (\frac{1}{n}) h_\lambda \xrightarrow{\omega^*} g.$$

   Since $\{t_\lambda \psi_\lambda\}_{\lambda \in \Lambda} \subseteq S^+$ and S^+ is weak-*-closed, $g \in S^+ \subseteq S^+(n)$.

2. If $t > 0$:

$$\psi_\lambda = (\frac{1}{t_\lambda}) g_\lambda - (\frac{1}{n}) h_\lambda \xrightarrow{\omega^*} (\frac{1}{t}) g - (\frac{1}{n}) h =: \psi.$$

   Since $\{\psi_\lambda\}_{\lambda \in \Lambda} \subseteq \Psi$ and Ψ is weak-*-closed, $(\frac{1}{t}) g_\lambda - (\frac{1}{n}) h \in \Psi$. Therefore $g = t\big(\psi + (\frac{1}{n}) h\big) \in S^+(n)$.

As a result, $S^+(n)$ is a weak-*-closed cone with a base $\Psi + \frac{1}{n} B(\mathcal{X}^*)$ in $\mathcal{X}^*$. □

Lemma 1.4.12 *With the same setting in Lemma 1.4.11, $\{S^+(n)\}_{n \in \mathbb{N}}$ is a ω^*-enlargement of S^+ in $\mathcal{X}^*$.*

Proof: Firstly, every weak-*-compact subset of $\mathcal{X}^*$ is bounded. Secondly, by Lemma 1.4.11, $S^+(n)$ is a weak-*-closed cone in $\mathcal{X}^*$ for each $n \in \mathbb{N}$. Thirdly, since $\Psi \subseteq int(S^+(n))$, $S^+ \setminus \{0\} \subseteq int(S^+(n))$. Fourthly, for each bounded sequence $\{g_n\}_{n \in \mathbb{N}}$ with $g_n \in S^+(n)$ for each n, using the similar argument in the proof of Proposition 1.2.3, we have $dist(g_n, S^+) \to 0$. □

Lemma 1.4.13 *Let $\mathcal{X}$ be a Banach space and $S \subseteq \mathcal{X}$ be a closed convex cone such that S^+ has a weak-*-closed base Ψ with $\inf\{\|\psi\| | \psi \in \Psi\} > 1$. For each $n \in \mathbb{N}$, define $S^+(n)$ as in (1.27). Suppose $F \subseteq \mathcal{X}^*$ be a weak-*-compact convex subset. Then for each $n \in \mathbb{N}$, each $\phi_n \in E(F, S^+(n))$ and arbitrary $\epsilon > 0$, there exist $f_n \in S_p$-supp(F) and $g_n \in S^+(n)$ such that $\|f_n - \phi_n + g_n\| < \epsilon$.*

Proof: Fix $n \in \mathbb{N}$, $\phi_n \in E(F, S^+(n))$ and $\epsilon > 0$. Then

$$(F - \phi_n) \cap -S^+(n) = \{0\};$$

hence $(F - \phi_n) \cap -int(S^+(n)) = \emptyset$. Note that $int(S^+(n)) \neq \emptyset$. It follows from the Separation Theorem that there exists $\varphi \in \mathcal{X}^{**} \setminus \{0\}$ such that

$$\inf\{\varphi(f) | f \in (F - \phi_n)\} \geq \sup\{-\varphi(f) | f \in S^+(n)\}.$$

Therefore

$$\inf\{\varphi(f) | f \in (F - \phi_n)\} - \sup\{-\varphi(f) | f \in S^+(n)\} \geq 0;$$

thus

$$\inf\{\varphi(f) | f \in (F - \phi_n + S^+(n))\} \geq 0.$$

Since $0 \in (F - \phi_n + S^+(n))$, $\inf\{\varphi(f) | f \in (F - \phi_n + S^+(n))\} = 0$. Hence 0 is a boundary point of $F - \phi_n + S^+(n)$. It follows from the weak-*-compactness of F and Lemma 1.4.11 that $F - \phi_n + S^+(n)$ is weak-*-closed. It is also clear that $F - \phi_n + S^+(n)$ is convex. In view of Lemma 1.4.10, there exist $g_0 \in F - \phi_n + S^+(n)$ and $x_0 \in \mathcal{X}$ with $\|x_0\| = 1$, such that

$$\|g_0\| < \min\{\frac{1}{2n}, \epsilon\} \tag{1.28}$$

and

$$\forall g \in F - \phi_n + S^+(n), \qquad g_0(x_0) \leq g(x_0). \tag{1.29}$$

Now pick $f_n \in F$ and $g_n \in S^+(n)$ such that $g_0 = f_n - \phi_n + g_n$. So the conclusion $\|f_n - \phi_n + g_n\| = \|g_0\| < \epsilon$ follows from (1.28). What remains is to show that $f_n \in S_p$-*supp*(F). Let $f \in F$ be arbitrary. Since $f - \phi_n + g_n \in F - \phi_n + S^+(n)$, by (1.29), we have

$$f_n(x_0) \leq f(x_0).$$

That is, $f_n(x_0) = \inf\{f(x_0) | f \in F\}$. Therefore it suffices to show that $x_0 \in S_p$. Pick $h_0 \in B(\mathcal{X}^*)$ such that $h_0(x_0) = -1$. Let $\psi \in \Psi$ be arbitrary. Since $\psi + (\frac{1}{n})h_0 \in \Psi + (\frac{1}{n})B(\mathcal{X}^*) \subseteq F - \phi_n + S^+(n)$, by (1.28) and (1.29), we have

$$(\psi + (\frac{1}{n})h_0)(x_0) \geq g_0(x_0) \geq -\|g_0\| > -\frac{1}{2n}.$$

Therefore $\psi(x_0) \geq \frac{1}{2n} > 0$. This and Lemma 1.4.1 implies that $x_0 \in S_p$. □

Theorem 1.4.14 *Let $\mathcal{X}$ be a Banach space and $S \subseteq \mathcal{X}$ be a closed convex cone such that S_p is nonempty and S^+ is a quasi-*-Bishop-Phelps cone. Suppose $F \subseteq \mathcal{X}^*$ is a weak-*-closed convex subset. Then $E(F, S^+) \subseteq cl(S_p\text{-}supp(F))$.*

Proof: Suppose $\bar{f} \in E(F, S^+)$. Let $F_0 = F \cap (\bar{f} + B(\mathcal{X}^*))$. Then F_0 is a weak-*-compact convex subset of $\mathcal{X}^*$ and $\bar{f} \in E(F_0, S^+)$. Since $S_p \neq \emptyset$, by Proposition 1.4.2, S^+ has a weak-*-closed base Ψ. Without loss of generality we may assume that $\inf\{\|\psi\| | \psi \in \Psi\} > 1$. For each $n \in \mathbb{N}$, we define

$$S^+(n) := cone(\Psi + \frac{1}{n}B(\mathcal{X}^*)).$$

By Lemma 1.4.12, $\{S^+(n)\}_{n\in\mathbb{N}}$ is a ω^*-enlargement of S^+ in $\mathcal{X}^*$.
Note that F_0 is weak-*-compact convex. It follows from Lemma 1.2.8 that there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}}$, with $\phi_n \in E(F_0, S^+(n))$ for each $n \in \mathbb{N}$, such that $dist(\phi_n, \bar{f} - S^+) \to 0$. Therefore there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ in S^+ such that

$$\|\phi_n - (\bar{f} - h_n)\| \to 0. \tag{1.30}$$

In view of Lemma 1.4.13, there are sequences $\{f_n\}_{n\in\mathbb{N}}$ in S_p-$supp(F_0)$ and $\{g_n\}_{n\in\mathbb{N}}$ in $S^+(n)$ such that for each $n \in \mathbb{N}$,

$$\|f_n - \phi_n + g_n\| < \frac{1}{n}. \tag{1.31}$$

Since

$$\|f_n + g_n - \bar{f} + h_n\| \leq \|f_n - \phi_n + g_n\| + \|\phi_n - \bar{f} + g_n\|,$$

following from (1.30) and (1.31),

$$\|f_n + g_n - \bar{f} + h_n\| \to 0. \tag{1.32}$$

Note that $\{f_n\}_{n\in\mathbb{N}} \subseteq F_0$ is a bounded sequence. This and (1.32) imply that $\{g_n + h_n\}_{n\in\mathbb{N}}$ is bounded. Furthermore, $g_n + h_n \in S^+(n) + S^+ \subseteq S^+$ for each $n \in \mathbb{N}$. By the fact that $\{S^+(n)\}_{n\in\mathbb{N}}$ is a ω^*-enlargement of S^+ in $\mathcal{X}^*$, we have

$$dist(g_n + h_n, S^+) \to 0.$$

Then there exists $\{k_n\}_{n\in\mathbb{N}}$ in S^+ such that $\|g_n + h_n - k_n\| \to 0$. From this and (1.32) we now have

$$\|f_n + k_n - \bar{f}\| \to 0. \tag{1.33}$$

Since F_0 is weak-*-compact, without loss of generality we can assume that $f_n \xrightarrow{\omega^*} f_0 \in F_0$. It follows from (1.33) that $k_n \xrightarrow{\omega^*} \bar{f} - f_0$. Note that $\{k_n\}_{n\in\mathbb{N}} \subseteq S^+$ and S^+ is weak-*-closed; so $\bar{f} - f_0 \in S^+$. Since $\bar{f} \in E(F, S^+)$, we should have $f_0 = \bar{f}$; thus $k_n \xrightarrow{\omega^*} 0$. From the boundedness of $\{f_n\}_{n\in\mathbb{N}}$ and (1.33), $\{k_n\}_{n\in\mathbb{N}}$ is also bounded. Therefore by the assumption that S^+ is a quasi-*-Bishop-Phelps cone and by Proposition 1.4.8, $k_n \to 0$. Therefore by (1.33) again, $f_n \to \bar{f}$. Using the argument in proof of Theorem 1.2.16, we have $f_n \in S_p\text{-}supp(F)$ for sufficiently large n. This shows that $\bar{f} \in cl(S_p\text{-}supp(F))$ and ends the proof. □

The corollary below follows from Theorem 1.4.14 and Proposition 1.4.7.

Corollary 1.4.15 *Let $\mathcal{X}$ be a Banach space and $S \subseteq \mathcal{X}$ be a closed convex cone with a nonempty interior. Suppose $F \subseteq \mathcal{X}^*$ be a weak-*-closed convex subset. Then $E(F, S^+) \subseteq cl(S_p\text{-}supp(F))$.*

Remark 1.4.1 *Comparing Theorem 1.4.6 and Theorem 1.4.14, we will also find the interesting "trade-off". Theorem 1.4.6 requires a rather weak restriction on the ordering cone (indeed, $S_p \neq \emptyset$ is the necessary condition in the density theorem) but a relatively strong restriction on the set F (F is compact convex). On the other hand, Theorem 1.4.14 requires a relatively strong limitation on the ordering cone (in the way it admits a quasi-*-Bishop-Phelps dual cone) but a relatively weak assumption on the set F (indeed F is only needed to be weak-*-closed convex). It is interesting to develop more results in the dual space setting to verify whether such a "trade-off" gives the two different streams of density results.*

Chapter 2

Density Theorem for Super Efficiency

In the preceding chapter we have presented several results concerning the density of the set of the positive proper efficient points in the set of the efficient points. In this chapter we will discuss another kind of proper efficiency: the super efficiency. We will present some facts about the super efficiency and a theorem concerning the density of the set of the super efficient points in the set of the efficient points will be given. The results reported in this chapter are originally given by J. M. Borwein and D. Zhuang [1] in the setting of normed vector spaces. The extensions to the setting of locally convex topological vector spaces are due to X. Y. Zheng [2, 3]. In this chapter we give a systematic survey of these authors especially [2] and [3].

2.1 Definition and Criteria for Super Efficiency

Definition 2.1.1 *Let $\mathcal{Y}$ be a normed vector space, $S \subseteq \mathcal{Y}$ be an ordering cone, and A be a nonempty subset of $\mathcal{Y}$. A point $\hat{a}$ in A, is called a* **super efficient**

point *of A with respect to S, if, there exists a real number $M > 0$ such that*

$$cl[cone(A - \hat{a})] \cap (B - S) \subseteq MB, \tag{2.1}$$

where B is the closed unit ball in $\mathcal{Y}$. The set of all super efficient points of A with respect to S is denoted by $SE(A, S)$.

Remark 2.1.1 *It will be shown below that the condition (2.1) can be replaced by*

$$cone(A - \hat{a}) \cap (B - S) \subseteq MB. \tag{2.2}$$

Moreover this condition implies that $\hat{a}$ is an efficient point. Indeed if $a \in A$ and $a \preceq_S \hat{a}$, then $n(a - \hat{a}) \in -S$ and hence $n(a - \hat{a})$ belongs to L.H.S. of (2.2) for each $n \in \mathbb{N}$; thus (2.2) implies that $a - \hat{a} = 0$ and $a = \hat{a}$.

The following definition is clearly consistent to Definition 2.1.1.

Definition 2.1.2 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone, and A be a nonempty subset of $\mathcal{X}$. A point $\hat{a}$ of A is called a* **super efficient point** *of A with respect to S, if, for any neighborhood V of 0 in $\mathcal{X}$, there exists a neighborhood U of 0 in $\mathcal{X}$ such that*

$$cl[cone(A - \hat{a})] \cap (U - S) \subseteq V. \tag{2.3}$$

The set of all super efficient points of A with respect to S is denoted by SE(A,S).

The proposition below shows us that (2.3) can be replaced by (2.4) (dropping the closure on the L.H.S.).

Proposition 2.1.1 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone, and A be a nonempty subset of $\mathcal{X}$. Let $\hat{a} \in A$. Then, $\hat{a} \in SE(A, S)$, if and only if, for any neighborhood V of 0 in $\mathcal{X}$, there exists a neighborhood U of 0 in $\mathcal{X}$ such that*

$$cone(A - \hat{a}) \cap (U - S) \subseteq V. \tag{2.4}$$

Proof: The necessity part is obvious.

For any neighborhood V of 0, there is V_1, a neighborhood of 0, such that $V_1 + V_1 \subseteq V$. Therefore there is U_1, a neighborhood of 0, such that

$$cone(A - \hat{a}) \cap (U_1 - S) \subseteq V_1.$$

Also there is U_2, a neighborhood of 0, such that $U_2 - U_2 \subseteq (U_1 \cap V_1)$. It follows that,

$$\begin{aligned} cl[cone(A-\hat{a})] \cap (U_2 - S) &\subseteq (cone(A-\hat{a}) + U_2) \cap (U_2 - S) \\ &\subseteq \big(cone(A-\hat{a}) \cap (U_2 - U_2 - S)\big) + U_2 \\ &\subseteq V_1 + U_2 \\ &\subseteq V. \end{aligned}$$

□

Thus the following two remarks are clear.

Remark 2.1.2 *(2.3) in Definition 2.1.2 can be replaced by (2.4).*

Remark 2.1.3 *(2.1) in Definition 2.1.1 can be replaced by (2.2).*

Other than using the setting above, we can also identify the super efficient points by virtue of the family of continuous seminorms defining the topology in $\mathcal{X}$.

Proposition 2.1.2 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone, A be a nonempty subset of $\mathcal{X}$ and let $\hat{a} \in A$. Then, $\hat{a} \in SE(A, S)$, if and only if, for any continuous seminorm p on $\mathcal{X}$, there is a continuous seminorm q on $\mathcal{X}$ such that, $p(a - \hat{a}) \leq q(x)$ whenever $a \in A$, $x \in \mathcal{X}$ with $a - \hat{a} \preceq_S x$.*

Proof: First, we show the sufficiency. For any neighborhood V of 0 given, without loss of generality, we may assume that V is closed, circled and convex. Let

$$p(x) = \inf\{t \geq 0 | x \in tV\}, \quad x \in \mathcal{X}. \tag{2.5}$$

By Theorem 1.35 in [4], p is a continuous seminorm on $\mathcal{X}$ and

$$V = \{x \in \mathcal{X} | p(x) \leq 1\}. \tag{2.6}$$

By assumption, there is a continuous seminorm q on $\mathcal{X}$ such that, $p(a - \hat{a}) \leq q(x)$ whenever $a \in A$, $x \in \mathcal{X}$ with $a - \hat{a} \preceq_S x$. Let

$$U = \{x \in \mathcal{X} | q(x) \leq 1\}. \tag{2.7}$$

Then U is a neighborhood of 0. Assume that $x \in cone(A - \hat{a}) \cap (U - S)$ and $x \neq 0$: There are $t > 0$, $a \in A$ and $u \in U$ such that $x = t(a - \hat{a}) \preceq_S u$; that is, $(a - \hat{a}) \preceq_S u/t$. By assumption it follows that $p(a - \hat{a}) \leq q(u/t)$. Multiplying by t and making use of (2.7) we have

$$p(x) = q(u) \leq 1.$$

Hence by (2.6), $x \in V$; it implies that $cone(A - \hat{a}) \cap (U - S) \subseteq V$. By Proposition 2.1.1 , $\hat{a} \in SE(A, S)$.

Conversely, suppose that $\hat{a} \in SE(A, S)$. For any continuous seminorm p on $\mathcal{X}$, let

$$V = \{x \in \mathcal{X} | p(x) \leq 1\};$$

thus

$$p(x) = \inf\{t \geq 0 | x \in tV\}, \quad x \in \mathcal{X}. \tag{2.8}$$

By Definition 2.1.2, there is a neighborhood U of 0 such that

$$cl[cone(A - \hat{a})] \cap (U - S) \subseteq V. \tag{2.9}$$

Without loss of generality, we can assume that U is closed, circled and convex. Let

$$q(x) = \inf\{t \geq 0 | x \in tU\}, \quad x \in \mathcal{X};$$

thus, q is a continuous seminorm on $\mathcal{X}$ and $U = \{x \in \mathcal{X} | q(x) \leq 1\}$. Let $a \in A$, and let $x \in \mathcal{X}$ with $(a - \hat{a}) \preceq_S x$: there is $s \in S$ such that $a - \hat{a} = x - s$. We prove that $p(a - \hat{a}) \leq q(x)$ by considering two cases.

1. Suppose $q(x) = 0$. Then we must have that $p(a - \hat{a}) = 0$. Indeed if not, then by (2.8), there is $t_0 > 0$ such that $a - \hat{a} \notin t_0 V$, that is, $(a - \hat{a})/t_0 \notin V$. However, $(a - \hat{a})/t_0 \in cone(A - \hat{a})$ and $(a - \hat{a})/t_0 = (x - s)/t_0 \in U - S$ (because $q(x/t_0) = q(x)/t_0 = 0$). This contradicts with (2.9), and establishes our claim.

2. Suppose that $q(x) > 0$. Then we have $x/q(x) \in U$. Note that $(a - \hat{a})/q(x) \in cone(A - \hat{a})$ and $(a - \hat{a})/q(x) = (x - s)/q(x) \in U - S$. By (2.9), $(a - \hat{a})/q(x) \in V$. Therefore $p\big((a - \hat{a})/q(x)\big) \leq 1$ and thus $p(a - \hat{a}) \leq q(x)$.

□

We have observed that every super efficient point is an efficient point. This can also be seen immediately from the preceding proposition.

Corollary 2.1.3 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone, and A be a nonempty subset of $\mathcal{X}$. Then $SE(A, S) \subseteq E(A, S)$.*

Proof: Suppose $\hat{a} \in SE(A, S)$ and that $a \in A$ with $a - \hat{a} \preceq_S 0$ for some $a \in A$. Let p be a continuous seminorm on $\mathcal{X}$ and take a continuous seminorm q according to Proposition 2.1.2. Letting $x = 0$, we have $p(a - \hat{a}) \leq q(x) = 0$. Since p is arbitrary, we can conclude that $a = \hat{a}$. That is, $\hat{a} \in E(A, S)$.

Making use of the fact that for any continuous seminorm r on a normed vector space $\mathcal{Y}$, there is a real number $M > 0$ such that for any $y \in \mathcal{Y}$, $r(y) \leq M\|y\|$, we can reach the following corollary.

Corollary 2.1.4 *Let $\mathcal{Y}$ be a normed vector space, $S \subseteq \mathcal{Y}$ be an ordering cone, A be a nonempty subset of $\mathcal{Y}$ and $\hat{a} \in A$. Then, $\hat{a} \in SE(A, S)$, if and only if, there is a real number $M > 0$ such that $\|a - \hat{a}\| \leq M\|y\|$ whenever $a \in A$, $y \in \mathcal{Y}$ with $(a - \hat{a}) \preceq_S y$.*

In the discussion in this chapter, the notations below will be frequently used. Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a base Θ, and

$$int_\Theta(S^+) = \{f \in S^+ | \inf\{f(\theta) | \theta \in \Theta\} > 0\}.$$

Thus $int_\Theta(S^+) \subseteq S^{+i}$. By the Separation Theorem, we have $int_\Theta(S^+) \neq \emptyset$. Also we let $\mathbf{N}(\mathbf{0})$ denoting the family of all neighborhoods of 0 in $\mathcal{X}$.

The following two propositions study the relation between $SE(A, S)$ and $int_\Theta(S^+)$.

Proposition 2.1.5 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a bounded base Θ and A be a subset of $\mathcal{X}$. Assume that there are $f \in int_\Theta(S^+)$ and $\hat{a} \in A$ such that*

$$f(\hat{a}) = \inf\{f(a) | a \in A\}. \tag{2.10}$$

Then $\hat{a} \in SE(A, S)$.

Proof: Suppose not: that is, there is $V_0 \in \mathbf{N}(\mathbf{0})$ such that for any $U \in \mathbf{N}(\mathbf{0})$, $cone(A - \hat{a}) \cap (U - S) \not\subseteq V_0$. Therefore for each U, there are $u_U \in U, \theta_U \in \Theta, \lambda_U \geq 0$ such that

$$u_U - \lambda_U \theta_U \in cone(A - \hat{a}), \tag{2.11}$$

and,

$$u_U - \lambda_U \theta_U \notin V_0. \tag{2.12}$$

Take $V_1 \in \mathbf{N}(\mathbf{0})$ with $V_1 - V_1 \subseteq V_0$. Since $\{u_U\}_{U \in \mathbf{N}(\mathbf{0})}$ converges to 0, without loss of generality, we can assume that $u_U \in V_1$ for each U. Therefore by (2.12) and $V_1 - V_1 \subseteq V_0$, we have $\lambda_U \theta_U \notin V_1$. By assumption that Θ is bounded, there is a real number $\lambda_0 > 0$ such that for $0 \leq \mu < \lambda_0$, $\mu\Theta \subseteq V_1$. Hence for each U, $\lambda_U \geq \lambda_0$. On the other hand, by (2.10), one has

$$f(x) \geq 0, \qquad \text{for any } x \in cone(A - \hat{a}).$$

This and (2.11) imply that $f(u_U - \lambda_U \theta_U) \geq 0$, so together with the fact that for each U, $\lambda_U \geq \lambda_0$, we have for each U,

$$f(u_U) \geq f(\lambda_U \theta_U) \geq \lambda_0 \inf\{f(\theta) | \theta \in \Theta\}.$$

However, as $\{u_U\}_{U \in \mathbf{N}(\mathbf{0})}$ converges to 0, it follows that

$$0 \geq \lambda_0 \inf\{f(\theta) | \theta \in \Theta\},$$

which contradicts $f \in int_\Theta(S^+)$. □

And the following proposition provides a partial converse for the preceding one.

Proposition 2.1.6 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a bounded base Θ and A be a convex subset of $\mathcal{X}$. Assume that $\hat{a} \in SE(A, S)$. Then there is $f \in int_\Theta(S^+)$ such that $f(\hat{a}) = \inf\{f(a) | a \in A\}$.*

Proof: Because of the assumption that Θ is a base of S, there is a convex neighborhood V of 0 such that $(-\Theta) \cap 2V = \emptyset$; so,

$$(-\Theta - V) \cap V = \emptyset. \tag{2.13}$$

Also by the assumption that $\hat{a} \in SE(A, S)$, there is an open convex neighborhood U of 0 such that $U \subseteq V$ and $cone(A - \hat{a}) \cap (-U - S) \subseteq V$. By (2.13),

$$cone(A - \hat{a}) \cap (-U - \Theta) \subseteq V \bigcap (-U - \Theta) = \emptyset.$$

Note that $-U-\Theta$ is open convex and $cone(A-\hat{a})$ is convex; then by the Separation Theorem, there is $f \in X^* \setminus \{0\}$ such that

$$\inf\{f(x)|x \in cone(A - \hat{a})\} \geq \sup\{f(x)|x \in -U - \Theta\}. \tag{2.14}$$

It is easy to show that

$$\inf\{f(x)|x \in cone(A - \hat{a})\} = 0; \tag{2.15}$$

consequently we have

$$f(\hat{a}) = \inf\{f(a)|a \in A\}.$$

What is remained is to show that $f \in int_\Theta(S^+)$. By (2.14) and (2.15),

$$0 \geq \sup\{f(x)|x \in -U - \Theta\}. \tag{2.16}$$

Since $f \neq 0$ and U is a neighborhood of 0, there is $u \in U$ such that $f(u) < 0$. By (2.16), for any $\theta \in \Theta$, $0 \geq -f(u) - f(\theta)$, that is $f(\theta) \geq -f(u)$; thus

$$0 < -f(u) \leq \inf\{f(\theta)|\theta \in \Theta\}.$$

This implies $f \in int_\Theta(S^+)$. □

Remark 2.1.4 *Note that $int_\Theta(S^+) \subseteq S^{+i}$, this tells that every super efficient point of a convex subset with respect to a bounded-based cone is again a positive proper efficient point.*

In this sense, super efficiency is a very strong kind of proper efficiency. Under certain setting it is found that the set of the super efficient points is contained in the set of other sorts of proper efficient points, such as the positive proper ones. Therefore the density result in this chapter actually extends theorem of Arrow, Barankin and Blackwell.

2.2 Henig Proper Efficiency

First, we give a definition of **Henig dilating cones** in locally convex spaces.

Definition 2.2.1 *Let $\mathcal{X}$ be a locally convex space and $S \subseteq \mathcal{X}$ be an ordering cone with a base Θ. Suppose V is a convex neighborhood of 0 in $\mathcal{X}$ such that $0 \notin cl(\Theta + V)$. Let*

$$S_V(\Theta) = cone(\Theta + V).$$

Clearly $(\Theta + V)$ is convex and $0 \notin cl(\Theta + V)$. Therefore the following remark is obvious.

Remark 2.2.1 *$S_V(\Theta)$ is a convex pointed cone with a base $(\Theta + V)$.*

Using the concept of Henig dilating cone, we first introduce the Henig proper efficiency with respect to a base of a cone in a locally convex space.

Definition 2.2.2 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a base Θ and A be a subset in $\mathcal{X}$. $\hat{a} \in A$ is a* **Henig proper efficient point of A with respect to Θ**, *if there is a convex neighborhood V of 0 in $\mathcal{X}$, with $0 \notin cl(\Theta + V)$, such that*

$$cl[cone(A - \hat{a})] \cap -S_V(\Theta) = \{0\}. \tag{2.17}$$

The set of all Henig proper efficient points of A with respect to Θ is denoted by $HE(A, \Theta)$.

Suppose that Θ is a base of an ordering cone S in a locally convex space $\mathcal{X}$. By the Separation Theorem, we have $int_\Theta(S^+) \neq \emptyset$. Take some $f_\Theta \in \mathcal{X}^*$, with

$$f_\Theta \in int_\Theta(S^+);$$

let

$$\alpha_\Theta = \inf\{f_\Theta(\theta) | \theta \in \Theta\} > 0.$$

Define

$$V_\Theta = \{x \in \mathcal{X} | |f_\Theta(x)| < \frac{\alpha_\Theta}{2}\}.$$

Also let $\mathbf{N}_\Theta(\mathbf{0})$ denote the family of all convex neighborhoods of 0 contained in V_Θ. In the following discussion, the notations above will be used without further remark.

For any convex neighborhood V of 0 with $0 \notin cl(\Theta + V)$, there is a convex neighborhood V_0 of 0 with $V_0 \subseteq V_\Theta$ (that is, $V_0 \in \mathbf{N}_\Theta(\mathbf{0})$) such that $S_{V_0}(\Theta) \subseteq S_V(\Theta)$ (Simply take $V_0 = V \bigcap V_\Theta$.). Therefore, we can give a simple way below to check the Henig proper efficiency with respect to a base Θ.

Proposition 2.2.1 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a base Θ, A be a subset of $\mathcal{X}$ and $\hat{a} \in A$. Then $\hat{a} \in HE(A, \Theta)$, if and only if, there is $V \in \mathbf{N}_\Theta(\mathbf{0})$ such that $\hat{a} \in E(A, S_V(\Theta))$, that is*

$$cone(A - \hat{a}) \cap -S_V(\Theta) = \{0\}. \qquad (2.18)$$

Proof: The necessity part is obvious. Suppose that there is a convex neighborhood V of 0 with $V \subseteq V_\Theta$ such that (2.18) holds. Then

$$cone(A - \hat{a}) \cap (-\Theta - V) = \emptyset.$$

Let V_0 be a convex neighborhood of 0 such that $V_0 + V_0 \subseteq V$; therefore we have

$$(cone(A - \hat{a}) + V_0) \cap (-\Theta - V_0) = \emptyset,$$

which implies that $cl[cone(A - \hat{a})] \cap -S_{V_0}(\Theta) = \{0\}$, proving (2.17).

In the following definition of the Henig proper efficient points with respect to cone S, we use $B(S)$ to denote the family of all bases of an ordering cone S in a locally convex space.

Definition 2.2.3 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a base and A be a subset in $\mathcal{X}$. $\hat{a} \in A$ is a* **Henig proper efficient point of A with respect to** S, *if*

$$\hat{a} \in \bigcap_{\Theta \in B(S)} HE(A, \Theta).$$

The set of the Henig proper efficient points of A with respect to S is denoted by $HE(A, S)$; that is

$$HE(A, S) = \bigcap_{\Theta \in B(S)} HE(A, \Theta).$$

In the two propositions below, we study some properties of the Henig proper efficiency.

Proposition 2.2.2 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a base and A be a subset in $\mathcal{X}$. Then for any $\Theta_1, \Theta_2 \in B(S)$, there are $\Theta', \Theta" \in B(S)$ such that,*

1. $HE(A, \Theta_1) \cup HE(A, \Theta_2) \subseteq HE(A, \Theta')$, *and,*

2. $HE(A, \Theta") \subseteq HE(A, \Theta_1) \cap HE(A, \Theta_2)$.

Proof: Since $\Theta_i \in B(S)$, by the Separation Theorem, there are $f_i \in int_{\Theta}(S^+)$ such that

$$\alpha_i = \inf\{f_i(\theta) | \theta \in \Theta_i\} > 0, \qquad i = 1, 2.$$

1. Take $\Theta' = \Theta_1 + \Theta_2$. To begin with we show that $\Theta' \in B(S)$. Firstly, for any $\theta' \in \Theta'$, there are $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$ such that $\theta' = \theta_1 + \theta_2$. We have

$$f_1(\theta') = f_1(\theta_1 + \theta_2) \geq f_1(\theta_1) \geq \alpha_1 > 0,$$

so $0 \notin cl(\Theta')$. Secondly, we clearly have $cone(\Theta') \subseteq S$. Thirdly for each $s \in S \backslash \{0\}$, there are $\lambda_i > 0$, $\theta_i \in \Theta_i$, for $i = 1, 2$, such that $s = \lambda_1 \theta_1 = \lambda_2 \theta_2$.

Therefore

$$s = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}(\theta_1 + \theta_2) \in cone(\Theta').$$

These show that $\Theta' \in B(S)$. In the next we aim to show that $HE(A, \Theta_1) \cup HE(A, \Theta_2) \subseteq HE(A, \Theta')$. By symmetry it suffices to show that $HE(A, \Theta_1) \subseteq HE(A, \Theta')$. For any $\hat{a} \in HE(A, \Theta_1)$, by Proposition 2.2.1, there is $V \in \mathbf{N}_{\Theta_1}(\mathbf{0})$ such that $cone(A - \hat{a}) \cap -S_V(\Theta_1) = \{0\}$. To show the result, it is sufficient to show that $S_V(\Theta') \subseteq S_V(\Theta_1)$. Suppose $s \in S_V(\Theta')$, there are $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$, $\lambda \geq 0$ and $v \in V$ such that $s = \lambda(\theta_1 + \theta_2 + v)$. Since $\theta_2 \in cone(\Theta_1)$, there exist $\mu > 0$ and $\theta_3 \in \Theta_1$ such that $\theta_2 = \mu\theta_3$. Let

$$\theta_4 = \frac{\theta_1 + \mu\theta_3}{1 + \mu}.$$

By the convexity of Θ_1 and V, $\theta_4 \in \Theta_1$ and $\frac{v}{1+\mu} \in V$, hence

$$s = \lambda(\theta_1 + \theta_2 + v) = \lambda(1 + \mu)(\theta_4 + \frac{v}{1 + \mu}) \in S_V(\Theta_1).$$

Therefore $S_V(\Theta') \subseteq S_V(\Theta_1)$ and the result follows.

2. Take $\Theta" = co(\Theta_1 \cup \Theta_2)$, where $co(\Theta_1 \cup \Theta_2)$ is the convex hull of $(\Theta_1 \cup \Theta_2)$ in $\mathcal{X}$. Firstly, by the convexity of Θ_1 and Θ_2, for each $\theta" \in \Theta"$, there are $0 \leq \lambda \leq 1$, $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$ such that $\theta" = \lambda\theta_1 + (1 - \lambda)\theta_2$. Therefore

$$\begin{aligned} (f_1 + f_2)(\theta") &\geq \lambda f_1(\theta_1) + (1 - \lambda) f_2(\theta_2) \\ &\geq \lambda\alpha_1 + (1 - \lambda)\alpha_2 \\ &\geq \min\{\alpha_1, \alpha_2\} \\ &> 0, \end{aligned}$$

hence $0 \notin cl(\Theta")$. Secondly, clearly $cone(\Theta") \subseteq S \subseteq cone(\Theta_1) \subseteq cone(\Theta")$. These imply that $\Theta" \in B(S)$. Note that for $i = 1, 2$, $\Theta_i \subseteq \Theta"$ and thus for any $V \in \mathbf{N}_{\Theta_i}(\mathbf{0}) \cap \mathbf{N}_{\Theta"}(\mathbf{0})$, $S_V(\Theta_i) \subseteq S_V(\Theta")$. It follows that for $i = 1, 2$, $HE(A, \Theta") \subseteq HE(A, \Theta_i)$ and consequently the result follows.

□

Proposition 2.2.3 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a bounded base Θ_0 and A be a subset of $\mathcal{X}$. Then for any $\Theta \in B(S)$, $HE(A, \Theta_0) \subseteq HE(A, \Theta)$.*

Proof: Let $\Theta \in B(S)$. By the Separation Theorem, there is $f_\Theta \in int_\Theta(S^+)$ such that

$$\alpha_\Theta := \inf\{f_\Theta(\theta) | \theta \in \Theta\} > 0. \tag{2.19}$$

Since Θ_0 is bounded and f_Θ is positive, there is $\lambda > 0$ such that

$$\lambda\Theta_0 \subseteq \{x \in \mathcal{X} | 0 < f_\Theta(x) < \alpha_\Theta\}. \tag{2.20}$$

Clearly $\lambda\Theta_0 \in B(S)$. Take $V \in \mathbf{N}_\Theta(\mathbf{0}) \cap \mathbf{N}_{\lambda\Theta_0}(\mathbf{0})$. For each $s \in S_V(\Theta)$, there is $\mu \geq 0$, $\theta \in \Theta$ and $v \in V$ such that $s = \mu(\theta + v)$. Since $\theta \in S$ and $\lambda\Theta_0 \in B(S)$, there are $\tau > 0$ and $\theta_0 \in \lambda\Theta_0$ such that $\theta = \tau\theta_0$. Hence $f_\Theta(\theta) = \tau f_\Theta(\theta_0)$. By (2.19), $f_\Theta(\theta) \geq \alpha_\Theta$; by (2.20), $0 < f_\Theta(\theta_0) < \alpha_\Theta$. Therefore $\tau \geq 1$ and hence $\frac{v}{\tau} \in V$ as V is a convex neighborhood of 0. Hence,

$$s = \mu\tau(\theta_0 + \frac{v}{\tau}) \in S_V(\lambda\Theta_0).$$

Therefore

$$S_V(\Theta) \subseteq S_V(\lambda\Theta_0) = S_{V/\lambda}(\Theta_0);$$

thus

$$HE(A, \Theta_0) \subseteq HE(A, \Theta).$$

Corollary 2.2.4 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a bounded base Θ_0 and A be a subset of $\mathcal{X}$. Then $HE(A, S) = HE(A, \Theta_0)$.*

2.3 Density Theorem for Super Efficiency

In this section, some propositions concerning relations between the Henig proper efficiency and the super efficiency are presented. Finally a density theorem for the super efficiency is discussed.

Proposition 2.3.1 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a base and A be a nonempty subset of $\mathcal{X}$. Then $SE(A,S) \subseteq HE(A,S)$.*

Proof: Using the relation that

$$HE(A,S) = \bigcap_{\Theta \in B(S)} HE(A,\Theta),$$

we only need to show that for each $\Theta \in B(S)$, $SE(A,S) \subseteq HE(A,\Theta)$. Given Θ a base of S, because $0 \notin cl(\Theta)$, there is a convex neighborhood V of 0 such that $(-\Theta) \cap 2V = \emptyset$; so,

$$(-\Theta - V) \cap V = \emptyset. \tag{2.21}$$

For any $\hat{a} \in SE(A,S)$, there is $U \in \mathbf{N}(\mathbf{0})$ such that $cone(A - \hat{a}) \cap (-U - S) \subseteq V$. Without loss of generality we can assume that $U \subseteq V$ and thus by (2.21),

$$cone(A - \hat{a}) \cap (-U - \Theta) \subseteq V \cap (-U - \Theta) = \emptyset.$$

It follows that

$$cone(A - \hat{a}) \cap -S_U(\Theta) = \{0\}.$$

That is, $\hat{a} \in HE(A,\Theta)$ and so $SE(A,S) \subseteq HE(A,S)$. □

The Proposition 2.3.1 actually gives a generalization to Proposition 3.2 in Borwein and Zhuang [1] of the setting of normed spaces. Besides, the proposition next generalizes Proposition 3.3 in Borwein and Zhuang [1] from the setting of normed spaces to locally convex spaces.

Proposition 2.3.2 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a bounded base and A be a nonempty subset of $\mathcal{X}$. Then $SE(A, S) = HE(A, S)$.*

Proof: Let Θ be a bounded base of S. By Corollary 2.2.4 and Proposition 2.3.1, it suffices to prove that $HE(A, \Theta) \subseteq SE(A, S)$. Let $\hat{a} \in HE(A, \Theta)$; then by Proposition 2.2.1, there is $V_0 \in \mathbf{N}_\Theta(\mathbf{0})$ such that $\hat{a} \in E(A, S_{V_0}(\Theta))$.
Suppose $\hat{a} \notin SE(A, S)$: There is $V_1 \in \mathbf{N}(\mathbf{0})$ such that for any $U \in \mathbf{N}(\mathbf{0})$,

$$cone(A - \hat{a}) \cap (U - S) \nsubseteq V_1;$$

that is, for each $U \in \mathbf{N}(\mathbf{0})$, there are $a_U \in A$, $u_U \in U$, $\theta_U \in \Theta$, $\lambda_U \geq 0$ and $\mu_U \geq 0$ such that

$$\lambda_U(a_U - \hat{a}) = u_U - \mu_U \theta_U$$

and

$$u_U - \mu_U \theta_U \notin V_1. \tag{2.22}$$

Note that $\lambda_U(a_U - \hat{a}) \notin V_1$, so $\lambda_U > 0$ and $a_U \neq \hat{a}$. Take $V_2 \in \mathbf{N}(\mathbf{0})$ with $V_2 - V_2 \subseteq V_1$. As the net $\{u_U\}_{U \in \mathbf{N}(\mathbf{0})}$ converges to 0, without loss of generality we can assume that $\{u_U\}_{U \in \mathbf{N}(\mathbf{0})} \subseteq V_2$. By (2.22) and since $V_2 - V_2 \subseteq V_1$, we have $\mu_U \theta_U \notin V_2$ for each $U \in \mathbf{N}(\mathbf{0})$. Since Θ is bounded, there is a real number $\mu_0 > 0$ such that for any τ with $0 \leq \tau < \mu_0$, $\tau\Theta \subseteq V_2$. Hence for each U, $\mu_U > \mu_0$. This and the fact that $\{u_U\}_{U \in \mathbf{N}(\mathbf{0})}$ converges to 0 lead to that $\{\frac{u_U}{\mu_U}\}_{U \in \mathbf{N}(\mathbf{0})}$ again converges to 0. Therefore there is $U_0 \in \mathbf{N}(\mathbf{0})$ such that $-\frac{u_{U_0}}{\mu_{U_0}} \in V_0$ and hence

$$a_{U_0} - \hat{a} = \frac{\mu_{U_0}}{\lambda_{U_0}}\left(\frac{u_{U_0}}{\mu_{U_0}} - \theta_{U_0}\right) = -\frac{\mu_{U_0}}{\lambda_{U_0}}\left(-\frac{u_{U_0}}{\mu_{U_0}} + \theta_{U_0}\right) \in -S_{V_0}(\Theta).$$

Since $\hat{a} \in E(A, S_{V_0}(\Theta))$ and $a_{U_0} \in A$, we have $a_{U_0} = \hat{a}$, contradicting to an earlier assertion. Consequently $\hat{a} \in HE(A, \Theta)$ and thus $SE(A, S) = HE(A, S)$. □

We are now ready for the main result which asserts that the set of the super efficient points is dense in the set of the efficient points; this generalizes many previous density results before.

First we introduce two lemmas in order to show the theorem.

Lemma 2.3.3 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a closed bounded base Θ, A be a weakly compact subset of $\mathcal{X}$ and $\bar{a} \in E(A, S)$. Then for any $V \in \mathbf{N}(\mathbf{0})$, there is $U_V \in \mathbf{N}_\Theta(\mathbf{0})$ such that*

$$(A - \bar{a}) \cap -cl(S_{U_V}(\Theta)) \subseteq V.$$

Proof: Suppose that this is false: there exists $V_0 \in \mathbf{N}(\mathbf{0})$ such that for any $U \in \mathbf{N}_\Theta(\mathbf{0})$,

$$(A - \bar{a}) \cap -cl(S_U(\Theta)) \nsubseteq V_0.$$

Then for any $U \in \mathbf{N}_\Theta(\mathbf{0})$, there exists $a_U \in A$ such that

$$a_U - \bar{a} \in -cl(S_U(\Theta)), \tag{2.23}$$

and

$$a_U - \bar{a} \notin V_0. \tag{2.24}$$

By (2.23), $(a_U - \bar{a} + U) \cap -S_U(\Theta) \neq \emptyset$ and hence there are $u_U \in U$, $v_U \in U$, $\theta_U \in \Theta$ and $\lambda_U \geq 0$ such that

$$a_U - \bar{a} + u_U = -\lambda_U(\theta_U + v_U). \tag{2.25}$$

Note that both $\{u_U\}_{U \in \mathbf{N}_\Theta(\mathbf{0})}$ and $\{v_U\}_{U \in \mathbf{N}_\Theta(\mathbf{0})}$ converge to 0. Suppose that $\lambda_U \to 0$: by (2.25) and the boundedness of Θ, we have $a_U - \bar{a}$ convergent to 0, contradicting to (2.24). Therefore $\lambda_U \nrightarrow 0$. Then without loss of generality we can rewrite (2.25) as

$$-\theta_U = \frac{1}{\lambda_U}(a_U - \bar{a} + u_U) + v_U. \tag{2.26}$$

Again, suppose $\{\lambda_U\}_{U\in\mathbf{N}_\Theta(\mathbf{0})}$ is unbounded: Since A is weakly compact, A is bounded. So the right hand side of (2.26) converges to 0, contradicting the fact that Θ is a base. Therefore without loss of generality, assume that $\{\lambda_U\}_{U\in\mathbf{N}_\Theta(\mathbf{0})}$ converges to some λ, with $\lambda > 0$. Since A is weakly compact, we may also assume that $\{a_U\}_{U\in\mathbf{N}_\Theta(\mathbf{0})}$ weakly converges to some a, where $a \in A$. Then from (2.26), $\{-\theta_U\}_{U\in\mathbf{N}_\Theta(\mathbf{0})}$ weakly converges to $\frac{1}{\lambda}(a-\bar{a})$. Note that every closed convex subset in a locally convex space is weakly closed, so Θ is weakly closed and thus

$$\frac{1}{\lambda}(a-\bar{a}) \in -\Theta;$$

that is, $a \preceq_S \bar{a}$. By (2.24), $a \neq \bar{a}$. This contradicts with $\bar{a} \in E(A,S)$ and hence the result follows.

Lemma 2.3.4 *Let $\mathcal{X}$ be a locally convex space, $K \subseteq \mathcal{X}$ be a pointed ordering cone, A be a weakly compact subset of $\mathcal{X}$ and $\bar{a} \in A$. Let $B = (A-\bar{a}) \cap -cl(K)$. If there is $\hat{a} \in A$ such that $\hat{a}-\bar{a} \in E(B,K)$, then $\hat{a} \in E(A,K)$.*

Proof: By that $\hat{a}-\bar{a} \in E(B,K)$ and K is pointed,

$$\{\hat{a}-\bar{a}\} = B \cap (\{\hat{a}-\bar{a}\} - K) \tag{2.27}$$

Consider the set $A \cap (\hat{a}-K)$, we aim to show that $\{\hat{a}\} = A \bigcap(\hat{a}-K)$. For each $a \in A \cap (\hat{a}-K)$, there is $k \in K$ such that $a = \hat{a}-k$. Hence $a-\bar{a} = \hat{a}-\bar{a}-k$. Since $a-\bar{a} \in (A-\bar{a}) \subseteq B$ and, $a-\bar{a} = (\hat{a}-\bar{a})-k \in \{\hat{a}-\bar{a}\}-K$, one has $a-\hat{a} \in B \cap (\{\hat{a}-\bar{a}\}-K)$. By (2.27), $a-\bar{a} = \hat{a}-\bar{a}$, that is $a = \hat{a}$. As a result, $\hat{a} \in E(A,K)$.

Theorem 2.3.5 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be an ordering cone with a closed bounded base and A be a weakly compact subset of $\mathcal{X}$. Then $E(A,S) \subseteq cl[SE(A,S)]$.*

Proof: Let $\bar{a} \in E(A, S)$. In order to prove the theorem, we aim to show that for any neighborhood V of 0 given,

$$(\bar{a} + V) \cap SE(A, S) \neq \emptyset.$$

Let f_Θ, α_Θ and V_Θ as the settings as before. By Lemma 2.3.3, it follows that there is a convex neighborhood U of 0 with $U \subseteq V_\Theta$, such that

$$(A - \bar{a}) \cap -cl(S_U(\Theta)) \subseteq V.$$

We define

$$A_\Theta = (A - \bar{a}) \cap -cl(S_U(\Theta)).$$

As U and Θ are convex, so $-cl(S_U(\Theta))$ is a closed convex subset and thus a weakly closed subset in $\mathcal{X}$. Together with $(A - \bar{a})$ being weakly compact, one can get that A_Θ is weakly compact. As f_Θ is weakly continuous, there is $\hat{a} \in A$ with $\hat{a} - \bar{a} \in A_\Theta \subseteq V$, such that,

$$f_\Theta(\hat{a} - \bar{a}) = \inf\{f(x) | x \in A_\Theta\}. \tag{2.28}$$

On the other hand, from the definitions of f_Θ and V_Θ and that $U \subseteq V_\Theta$, we have

$$f_\Theta(x) > \frac{\alpha_\Theta}{2}, \qquad \forall x \in \Theta + U.$$

Therefore one has $f_\Theta \in (S_U(\Theta))^{+i}$. By (2.28),

$$\hat{a} - \bar{a} \in Pos(A_\Theta, S_U(\Theta)) \subseteq E(A_\Theta, S_U(\Theta)).$$

Using Lemma 2.3.4 with substituting $B = A_\Theta$ and $K = S_U(\Theta)$,

$$\hat{a} \in E(A, S_U(\Theta)).$$

It follows from Proposition 2.2.1, $\hat{a} \in HE(A, \Theta)$. Since Θ is a bounded base of S, by Proposition 2.3.2, $\hat{a} \in SE(A, S)$. Note that $\hat{a} - \bar{a} \in V$, and hence

$$\hat{a} \in (\bar{a} + V) \cap SE(A, S) \neq \emptyset.$$

That is, the theorem follows. □

Chapter 3

Connectedness Results in Vector Optimization

In the study of vector optimization, one of the most important problems is to investigate the topological properties of the efficient outcome sets and the efficient solution sets. In contrast to the density properties discussed in previous two chapters, we will present some connectedness results in vector optimization in this chapter. Given a vector minimization problem for a set-valued map F:

$$(VMP) \quad Min\{F(x) : x \in A\}, \quad \text{with respect to cone } S,$$

the corresponding efficient outcome set is denoted by $Min(F(A), S)$,

$$Min(F(A), S) := E(F(A), S); \tag{3.1}$$

the corresponding efficient solution set is denoted by $Min(A, S, F)$,

$$Min(A, S, F) := \bigcup_{y \in Min(F(A),S)} F^{-1}(y). \tag{3.2}$$

X. H. Gong [7] gave the following connectedness results (definitions of these not yet defined will be given in Section 3.1):

Theorem 3.0.1 *Let A be a nonempty compact convex subset of a real Hausdorff topological vector space $\mathcal{X}$. Let S be a closed convex cone in a real normed space $\mathcal{Y}$ with a base, and let F be an upper semicontinuous S-convex set-valued map from A to $\mathcal{Y}$ with compact values. Furthermore, assume that*

$$P(F(x),h) \text{ is connected,} \qquad \forall h \in S^{+i}, \forall x \in A, \tag{3.3}$$

where

$$P(F(x),h) = \{y \in F(x) | h(y) = \min\{h(z) | z \in F(x)\}\}.$$

Then $\bigcup_{h \in S^{+i}} G(h)$ and $Min(F(A), S)$ are connected, where

$$G(h) = P(F(A),h) = \{y \in F(A) | h(y) = \min\{h(z) | z \in F(A)\}\}.$$

Theorem 3.0.2 *With the same assumptions in Theorem 3.0.1, one also has $Min(A, S, F)$ is connected, where*

$$Min(A, S, F) = \{a \in A | F(a) \cap Min(F(A), S) \neq \emptyset\}.$$

Inspired by the work X. Y. Zheng done in [6], I will show that even the condition (3.3) is dropped Theorem 3.0.1 holds valid. Furthermore as the proof of Theorem 3.0.2 involves the use of Theorem 3.0.1, (3.3) is also not essential to Theorem 3.0.2.

Besides, some other connectedness results, especially the ones about contractibility, given by X. Y. Zheng [6], are presented in this chapter.

3.1 Set-valued Maps

In this preliminary section, after giving some basic definitions involving set-valued maps we establish some useful theorems concerning these maps. These results are frequently used in this chapter. More details can be found in J. P. Audin and I. Ekeland [5].

Definition 3.1.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets. A* **set-valued map** *F from $\mathcal{X}$ to $\mathcal{Y}$, denoted as $F : \mathcal{X} \rightrightarrows \mathcal{Y}$, is a map associating each $x \in \mathcal{X}$ with a* **subset** *$F(x)$ in $\mathcal{Y}$, where $F(x)$ is called the* **image** *or* **value** *of F at x.*

We say that the set-valued map F is closed-valued (respectively compact, bounded and any properties of a subset and so on) if all images of F are closed (respectively compact, bounded and any properties of a subset and so on) in $\mathcal{Y}$.

For any subset A of $\mathcal{X}$, we use the following notation for the image of A under F:

$$F(A) = \bigcup_{x \in A} F(x).$$

A set-valued map F is said to be **proper** if there exists $x_0 \in \mathcal{X}$ such that $F(x_0) \neq \emptyset$. The **domain** of F is defined by

$$Dom(F) := \{x \in \mathcal{X} | F(x) \neq \emptyset\}.$$

Clearly F is proper if $Dom(F) \neq \emptyset$. If $Dom(F) = \mathcal{X}$, we say F is **strict**.

The **image** of F is defined by

$$Im(F) := \bigcup_{x \in \mathcal{X}} F(x).$$

Remark 3.1.1 $Im(F) = \bigcup_{x \in \mathcal{X}} F(x) = \bigcup_{x \in Dom(F)} F(x)$.

Another important concept is the **graph** of F, which is defined by

$$Graph(F) = \{(x, y) \in \mathcal{X} \times \mathcal{Y} | y \in F(x)\}.$$

Remark 3.1.2 *$Dom(F)$ and $Im(F)$ are the projections of $Graph(F)$ on $\mathcal{X}$ and $\mathcal{Y}$ respectively.*

We say that the set-valued map F is closed (respectively compact, bounded and any properties of a subset and so on) if $Graph(F)$ is closed (respectively compact, bounded and any properties of a subset and so on) in $\mathcal{X} \times \mathcal{Y}$.

The **inverse map** F^{-1} is a set-valued map from $\mathcal{Y}$ to $\mathcal{X}$ such that for any $y \in \mathcal{Y}$,

$$F^{-1}(y) = \{x \in \mathcal{X} | y \in F(x)\}.$$

Recall that in a vector space $\mathcal{Y}$ which is equipped an ordering cone S, a subset A of $\mathcal{Y}$ is said to be S-convex if $A + S$ is convex.

Definition 3.1.2 *Let A be a convex subset of a vector space $\mathcal{X}$ and S be an ordering cone in a vector space $\mathcal{Y}$. A set-valued map F from $\mathcal{X}$ to $\mathcal{Y}$ is said to be S-convex on A if, for any a_1, $a_2 \in A$ and $t \in [0, 1]$,*

$$tF(a_1) + (1-t)F(a_2) \subseteq F(ta_1 + (1-t)a_2) + S.$$

Remark 3.1.3 *$F(A)$ is S-convex in $\mathcal{Y}$ if F is S-convex on a convex subset A.*

Now consider some topological concepts for set-valued maps.

Definition 3.1.3 *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. A set-valued map F from $\mathcal{X}$ to $\mathcal{Y}$ is said to be* **upper semicontinuous** *at $x_0 \in \mathcal{X}$ if, for each neighborhood V containing $F(x_0)$ in $\mathcal{Y}$, there exists a neighborhood U of x_0 in $\mathcal{X}$ such that $F(U) \subseteq V$. F is said to be upper semicontinuous if F is upper semicontinuous at each $x \in \mathcal{X}$.*

Definition 3.1.4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. A set-valued map F from $\mathcal{X}$ to $\mathcal{Y}$ is said to be* **lower semicontinuous** *at $x_0 \in \mathcal{X}$ if, for each neighborhood V in $\mathcal{Y}$ with $V \cap F(x_0) \neq \emptyset$, there exists a neighborhood U of x_0 in $\mathcal{X}$ such that for each $x \in U$, $V \cap F(x) \neq \emptyset$. F is said to be lower semicontinuous if F is lower semicontinuous at each $x \in \mathcal{X}$.*

Remark 3.1.4 *See [5], F is lower semicontinuous at $x_0 \in \mathcal{X}$ if and only if for any net $\{x_\mu\}$ converging to x_0 and any $y_0 \in F(x_0)$, there exists a net $\{y_\mu\}$, with $y_\mu \in F(x_\mu)$ for each μ, converging to y_0.*

Let $(\mathcal{X}, \tau_\mathcal{X})$ and $(\mathcal{Y}, \tau_\mathcal{Y})$ be locally convex spaces; let $\omega_\mathcal{X}$ and $\omega_\mathcal{Y}$ be the weak topologies of $(\mathcal{X}, \tau_\mathcal{X})$ and $(\mathcal{Y}, \tau_\mathcal{Y})$ respectively. We say that a set-valued map F is a strong-weak (respectively weak-strong, weak-weak) upper semicontinuous if F is upper semicontinuous with respect to $(\mathcal{X}, \tau_\mathcal{X})$ and $(\mathcal{Y}, \omega_\mathcal{Y})$ (respectively $(\mathcal{X}, \omega_\mathcal{X})$ and $(\mathcal{Y}, \tau_\mathcal{Y})$, $(\mathcal{X}, \omega_\mathcal{X})$ and $(\mathcal{Y}, \omega_\mathcal{Y})$). Similar terminology applies to the lower semicontinuities. Also F is said to be strong-weak closed if $Graph(F)$ is closed with respect to $\tau_\mathcal{X} \times \omega_\mathcal{Y}$ and so on.

Finally we list out two useful theorems; their proofs can be found in Chapter 3, Section 1 of [5].

Theorem 3.1.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff topological spaces, and F be an upper semicontinuous set-valued map from $\mathcal{X}$ to $\mathcal{Y}$ with closed values. Then F is closed.*

Theorem 3.1.2 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff topological spaces, and F be a closed set-valued map from $\mathcal{X}$ to $\mathcal{Y}$. Suppose that $\mathcal{Y}$ is compact. Then F is upper semicontinuous.*

3.2 The Contractibility of the Efficient Point Sets

We start our discussion by giving the definition of contractibility of a topological space.

Definition 3.2.1 *Let $\mathcal{Z}$ be a topological space. $\mathcal{Z}$ is said to be* **contractible** *if there is a continuous map $H : \mathcal{Z} \times [0,1] \to \mathcal{Z}$ and a point $z_0 \in \mathcal{Z}$ such that $H(z,0) = z$ and $H(z,1) = z_0$ for each $z \in \mathcal{Z}$.*

Providing that $\mathcal{Z}$ is contractible, $H(z,\cdot)$ gives a continuous path from z to z_0 for each $z \in \mathcal{Z}$. Therefore the following remark is obvious.

Remark 3.2.1 *$\mathcal{Z}$ is path connected if $\mathcal{Z}$ is contractible.*

Definition 3.2.2 *Let $\mathcal{X}$ be a locally convex space and $\mathcal{Z} \subseteq \mathcal{X}$. $\mathcal{Z}$ is said to be weakly contractible if $\mathcal{Z}$ is contractible with respect to the weak topology of $\mathcal{X}$.*

Let $\mathcal{X}$ be a locally convex space and $S \subseteq \mathcal{X}$ be a closed convex pointed cone. A function g on $\mathcal{X}$ is said to be (strictly) increasing if $g(x_2) > g(x_1)$ whenever $x_1, x_2 \in \mathcal{X}$ with $x_1 \prec_S x_2$ (that is, $x_2 - x_1 \in S \setminus \{0\}$). A function g on $\mathcal{X}$ is said to be st-convex if g is convex and $g(\frac{x_1+x_2}{2}) < \frac{g(x_1)+g(x_2)}{2}$ whenever $x_1, x_2 \in \mathcal{X} \setminus \{0\}$ with $x_1 \neq \alpha x_2$ for any constant $\alpha > 0$.

Given a continuous convex function g on $\mathcal{X}$, the subdifferential ∂g is a set-valued map from $\mathcal{X}$ to its topological dual space $\mathcal{X}^*$ with for each $x \in \mathcal{X}$,

$$\partial g(x) = \{f \in X^* | f(h) \leq g(x+h) - g(x), \text{ for all } h \in X\}.$$

Definition 3.2.3 *Let $\mathcal{X}$ be a locally convex space. A continuous st-convex function g on $\mathcal{X}$ is said to have property (St) if, $\partial g(\mathcal{X})$ is bounded below on each bounded subset of $\mathcal{X}$, that is, for each bounded subset A of $\mathcal{X}$, there is a constant $M > 0$ such that for each $a \in A$ and each $f \in \partial g(\mathcal{X})$, $f(a) \geq -M$. Furthermore, $\mathcal{X}$ is said to have the property (St), if there exists a continuous st-convex function g_0 on $\mathcal{X}$ such that g_0 has property (St).*

In order to reach the main result about the contractibility of the efficient point sets in this section, we first state several lemmas.

Lemma 3.2.1 *Let $\mathcal{X}$ be a locally convex space and $S \subseteq \mathcal{X}$ be a closed convex cone with a bounded base. Suppose that g_0 is a continuous st-convex function on $\mathcal{X}$ and g_0 has property (St). Then there exists $f_0 \in S^{+i}$ such that $g_0 + f_0$ is (strictly) increasing on $\mathcal{X}$.*

Proof: Let Θ be a bounded base of S. By the Separation Theorem, there exists $f' \in X^* \setminus \{0\}$ such that

$$\alpha = \inf\{f'(\theta) | \theta \in \Theta\} > 0.$$

Note that $f' \in S^{+i}$. As Θ is bounded and g_0 has property (St), there exists a constant $M > 0$ such that for each $f \in \partial g_0(\mathcal{X})$ and each $\theta \in \Theta$, $f(\theta) \geq -M$. Define $f_0 = (\frac{M}{\alpha} + 1)f'$; clearly $f_0 \in S^{+i}$. Aim to show that $g_0 + f_0$ is (strictly) increasing on $\mathcal{X}$. Letting $x \in \mathcal{X}$ and $s \in S \setminus \{0\}$, since Θ is a base of S, there exist $t > 0$ and $\theta \in \Theta$ such that $s = t\theta$. Besides, we pick $f_1 \in \partial g_0(\mathcal{X})$ and thus $g_0(x+s) - g_0(x) \geq f_1(s)$ and $f_1(\theta) \geq -M$. Hence,

$$\begin{aligned}
(g_0 + f_0)(x+s) - (g_0 + f_0)(x) &= g_0(x+s) - g_0(x) + f_0(s) \\
&\geq f_1(s) + f_0(s) \\
&= tf_1(\theta) + t(\frac{M}{\alpha} + 1)f'(\theta) \\
&\geq t(-M) + t(\frac{M}{\alpha} + 1)\alpha \\
&= \alpha t \\
&> 0.
\end{aligned}$$

As a result, $g_0 + f_0$ is (strictly) increasing. □

The following lemma asserts that if A is weakly compact and S-convex, then the set of global minimizers of a (strictly) increasing continuous convex function on the set $(x - S) \cap (A + S)$ is a nonempty convex subset of the efficient point set of A for each $x \in A + S$.

Lemma 3.2.2 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be a closed convex pointed cone and $A \subseteq \mathcal{X}$ be a weakly compact S-convex subset. Let g be a (strictly) increasing continuous convex function on $\mathcal{X}$. Let $G : A + S \rightrightarrows \mathcal{X}$, $l : A + S \to \mathbb{R}$ and $L : A + S \rightrightarrows \mathcal{X}$ be defined by*

$$G(x) := (x - S) \cap (A + S), \quad \forall x \in A + S, \tag{3.4}$$

$$l(x) := \inf\{g(y) | y \in G(x)\}, \quad \forall x \in A + S, \tag{3.5}$$

and

$$L(x) := \{y \in G(x) | g(y) = l(x)\}, \quad \forall x \in A + S. \tag{3.6}$$

Then for each $x \in A + S$, $L(x)$ is a nonempty convex subset of $E(A, S)$.

Proof: Let $x \in A + S$. Firstly we note that

$$L(x) = \{y \in G(x) | g(y) \leq l(x)\}.$$

Since g is convex and $G(x)$ is convex, $\{y \in G(x) | g(y) \leq l(x)\}$ is convex; that is, $L(x)$ is convex.

Secondly, we show that $L(x)$ is nonempty. As $(x - S) \cap (A + S)$ may not be weakly compact in $\mathcal{X}$, we define another set-valued map G' on $A + S$ by

$$G'(z) := (z - S) \cap A, \quad \text{for each } z \in A + S.$$

And we define corresponding l' and L' on $A + S$ by replacing G by G' in (3.5) and (3.6) respectively. We note that for each $z \in A + S$, $G'(z)$ is nonempty and weakly compact (as $(z - S)$ is weakly closed and A is weakly compact). On the other hand, as g is a continuous convex function on a locally convex space, g is weakly lower semicontinuous. As a result, $l'(z)$ is well-defined and $L'(z)$ is nonempty for each $z \in A + S$. Next we aim to show that $L'(x) = L(x)$.

Suppose $z \in L(x)$; then there are s_1, $s_2 \in S$ and $a \in A$ such that

$$z = a + s_1 = x - s_2.$$

Hence,

$$a = z - s_1 \quad \text{and} \quad a = x - s_1 - s_2 \in (x - S) \cap A = G'(x) \subseteq G(x). \tag{3.7}$$

Since g is (strictly) increasing and $z \in L(x)$, if $s_1 \neq 0$, it follows that

$$g(a) = g(z - s_1) < g(z) = \inf\{g(y) | y \in G(x)\} \leq g(a).$$

A contradiction occurs; thus $z = a$. From $G'(x) \subseteq G(x)$, we have $l'(x) \geq l(x)$ and therefore by (3.6) and (3.7),

$$g(z) = l(x) \leq l'(x) \leq g(a) = g(z);$$

that is, $z \in L'(x)$. Conversely , suppose that $z \in L'(x)$ and we aim to show that $z \in L(x)$. Indeed , if not, there exists $z_1 \in G(x)$ such that $g(z_1) < g(z)$. Then, there exist $s_1,\ s_2 \in S$ and $a \in A$ such that

$$z_1 = a + s_1 = x - s_2.$$

Hence

$$a = z_1 - s_1 \quad \text{and} \quad a = x - s_1 - s_2 \in (x - S) \cap A = G'(x).$$

As g is (strictly) increasing, so $g(z_1 - s_1) \leq g(z_1)$; thus,

$$g(a) = g(z_1 - s_1) \leq g(z_1) < g(z).$$

It contradicts the facts that $a \in G'(x)$ and $z \in L'(x)$, hence $z \in L(x)$. Combining the above two claims, $L(x) = L'(x)$ and thus $L(x)$ is nonempty.

Lastly we want to show that $L'(x)$ is a subset of $E(A, S)$. Suppose $z \in L'(x)$ such that $z \notin E(A, S)$, there is $s \in S \setminus \{0\}$ such that $z - s \in A$. Note that $z - s \in (x - S) \cap A = G'(x)$. However by the (strictly) increasingness of g, $g(z - s) < g(z)$, it contradicts that $z \in L'(x)$. Consequently, $L(x)$ is a nonempty convex subset of $E(A, S)$. □

Theorem 3.2.3 *Let $\mathcal{X}$ be a topological space, $\mathcal{Y}$ be a locally convex space, G be a lower semicontinuous strong-weak closed set-valued map from $\mathcal{X}$ to $\mathcal{Y}$, and g be a continuous convex function on $\mathcal{Y}$. Let $l(x) = \inf\{g(y) | y \in G(x)\}$ and $L(x) = \{y \in G(x) | g(y) = l(x)\}$ for each $x \in \mathcal{X}$. If $L(\mathcal{X})$ is relatively weakly compact, then l is continuous on $Dom(L)$ and L is strong-weak upper semicontinuous on $Dom(L)$.*

Proof: Firstly let us show that l is continuous. Let $x \in Dom(L)$ and $\{x_\mu\}_{\mu\in I}$ be a net in $Dom(L)$ with $\{x_\mu\}_{\mu\in I}$ convergent to x. Aim to show that $\lim_\mu l(x_\mu) = l(x)$. Let $\{y_\mu\}_{\mu\in I}$ be a net in $\mathcal{Y}$ such that $y_\mu \in L(x_\mu)$ for each $\mu \in I$. Since $L(\mathcal{X})$ is relative weakly compact, without loss of generality, there is $y \in \mathcal{Y}$ to which $\{y_\mu\}_{\mu\in I}$ weakly converges. Since g is continuous convex in a locally convex space, g is weakly lower semicontinuous and thus

$$g(y) \leq \liminf_{\mu} g(y_\mu) = \liminf_{\mu} l(x_\mu). \tag{3.8}$$

Also since G is strong-weak closed, we have $y \in G(x)$ and thus $l(x) \leq g(y)$. Combining with (3.8), we have

$$l(x) \leq \liminf_{\mu} l(x_\mu). \tag{3.9}$$

On the other hand, take a point $z \in L(x) \subseteq G(x)$. Since G is lower semicontinuous, there exists $\{z_\mu\}_{\mu\in I}$ with $z_\mu \in G(x_\mu)$ for each $\mu \in I$ such that $\{z_\mu\}_{\mu\in I}$ converges to z. Therefore for each $\mu \in I$,

$$g(z_\mu) \geq l(x_\mu). \tag{3.10}$$

From the continuity of g,

$$\limsup_{\mu} g(z_\mu) = g(z) = l(x).$$

From this and (3.10), we have

$$l(x) \geq \limsup_{\mu} l(x_\mu).$$

Together with (3.9), l is continuous on $Dom(L)$.

Secondly, we want to show that L is strong-weak upper semicontinuous on $Dom(L)$. Suppose that this is false: there exists $a \in Dom(L)$ at which L is not strong-weak upper semicontinuous; that is, there exist a weakly open neighborhood V of $L(a)$ in $\mathcal{Y}$, a net $\{a_\lambda\}_{\lambda\in\Lambda}$ in $Dom(L)$ converging to a, and, $\{b_\lambda\}_{\lambda\in\Lambda}$ with each

$b_\lambda \in L(a_\lambda)$ such that $b_\lambda \notin V$ for each $\lambda \in \Lambda$. Since $L(\mathcal{X})$ is relative weakly compact, without loss of generality, we can assume that $\{b_\lambda\}_{\lambda\in\Lambda}$ weakly converges to some $b \in \mathcal{Y}$. From that G is strong-weak closed, we have $b \in G(a)$. Also g is continuous convex, so it is weakly lower semicontinuous. Together with that l is continuous,

$$g(b) \leq \lim_\lambda g(b_\lambda) = \lim_\lambda l(a_\lambda) = l(a).$$

This and $b \in G(a)$ imply that $b \in L(a) \subseteq V$. However, it is impossible as V is a weakly open and $b_\lambda \notin V$ for each $\lambda \in \Lambda$. □

Lemma 3.2.4 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be a closed convex pointed cone and A be a weakly compact S-convex subset of $\mathcal{X}$ such that the interior of $A + S$ is nonempty. Let $Z = int(A + S) \cup E(A, S)$ and for each $z \in Z$, $G(z) = (z - S) \cap (A + S)$. Then G is lower semicontinuous.*

Proof: Firstly for each $z_0 \in E(A, S)$, $G(z_0) = \{z_0\}$. As $z \in G(z)$ for each $z \in Z$, G is lower semicontinuous on $E(A, S)$. Secondly for each $z_0 \in int(A + S)$ given, for each $y \in G(z_0)$ and each open neighborhood V of y in $\mathcal{X}$ given, we aim to show that there exists a neighborhood U of z_0 in $\mathcal{X}$ such that for each $z \in U$, $V \cap G(z) \neq \emptyset$. As $z_0 \in int(A+S)$, there exists a neighborhood W_1 of 0 in $\mathcal{X}$ such that

$$z_0 + W_1 \subseteq A + S. \tag{3.11}$$

Since $\lim_{t\to 1}(ty + (1-t)z_0) = y$, there exists $0 < t_0 < 1$ such that

$$t_0 y + (1 - t_0)z_0 \in V. \tag{3.12}$$

Since A is S-convex, as $y \in A + S$ and by (3.11),

$$t_0 y + (1 - t_0)(z_0 + W_1) \subseteq A + S;$$

that is,

$$t_0y + (1 - t_0)z_0 + (1 - t_0)W_1 \subseteq A + S.$$

Hence $t_0y+(1-t_0)z_0$ is an interior point of $A+S$. By (3.12) and that V is open, $t_0y + (1 - t_0)z_0$ is also an interior point of V. As a result,

$$t_0y + (1 - t_0)z_0 \in int(V \cap (A + S)).$$

Therefore there is a neighborhood W_2 of 0 in $\mathcal{X}$ such that

$$t_0y + (1 - t_0)z_0 + W_2 \subseteq V \cap (A + S). \tag{3.13}$$

Since $y \in G(z_0) \subseteq (z_0 - S)$, one has $t_0(z_0 - y) \in S$; also as $z_0 = t_0y + (1 - t_0)z_0 + t_0(z_0 - y)$, one has

$$z_0 \in t_0y + (1 - t_0)z_0 + W_2 + S.$$

Now we take $U = t_0y + (1 - t_0)z_0 + W_2 + S$, which is, indeed, a neighborhood of z_0. For any $z \in U$, there is $s \in S$ such that

$$z - s \in t_0y + (1 - t_0)z_0 + W_2.$$

By (3.13), $z - s \in V \cap (A + S)$. Combining with $z - s \in (z - S)$,

$$z - s \in V \cap (A + S) \cap (z - S) = V \cap G(z).$$

Therefore $V \cap G(z) \neq \emptyset$ for each $z \in U$. □

Lemma 3.2.5 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be a closed convex cone and A be a weakly compact S-convex subset of $\mathcal{X}$. $G(x) := (x - S) \cap (A + S)$ for each $x \in A + S$. Then G is strong-weak closed.*

Proof: Let $\{(x_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ be a net in $Graph(G)$ such that

$$x_\lambda \to x_0 \in A+S, \quad y_\lambda \xrightarrow{\omega_{\mathcal{X}}} y_0 \in A+S, \tag{3.14}$$

where $\omega_{\mathcal{X}}$ is the weak topology of $\mathcal{X}$. Note that any $\lambda \in \Lambda$, $y_\lambda \in G(x_\lambda) = (x_\lambda - S) \cap (A+S)$. Therefore

$$x_\lambda - y_\lambda \in S. \tag{3.15}$$

Note that S is closed convex in a locally convex space so S is weakly closed, By (3.14) and (3.15), $x_0 - y_0 \in S$. Also as A is weakly compact and S is weakly closed, $A+S$ is weakly closed. By (3.14), $y_0 \in A+S$. As a result,

$$y_0 \in (x_0 - S) \cap (A+S);$$

that is

$$y_0 \in G(x_0).$$

□

Now we are ready to present the main result in this section.

Theorem 3.2.6 *Let $\mathcal{X}$ be a locally convex space with property (St), $S \subseteq \mathcal{X}$ be a closed convex cone with a bounded base, A be a weakly compact S-convex subset of $\mathcal{X}$ such that the relative interior of $A+S$ is nonempty. Then $E(A,S)$ is weakly contractible.*

Proof: Taking a subspace if necessary, we can assume that $int(A+S)$ is nonempty. By property (St) of $\mathcal{X}$, there is a continuous st-convex function g_1 on $\mathcal{X}$ having property (St). Pick $f_1 \in \partial g_1(0)$ and define g_0 by

$$g_0(x) = g_1(x) - g_1(0) - f_1(x), \quad \text{for each } x \in \mathcal{X}.$$

Note that g_0 is nonnegative continuous st-convex on $\mathcal{X}$ with $g_0(0) = 0$. As $\partial g_0(\mathcal{X}) = \partial g_1(\mathcal{X}) - f_1$, g_0 has property (St). By Lemma 3.2.1, there is $f_0 \in S^{+i}$ such that $g_0 + f_0$ is (strictly) increasing. Define $g = g_0 + f_0$, so g is an (strictly) increasing continuous st-convex function. Since A is weakly compact, f_0 is bounded on A. Therefore it is easy to show that there exists $s_0 \in S \setminus A$ such that

$$f_0(s_0) + \inf\{f_0(a) | a \in A\} > 0. \tag{3.16}$$

Let $A_0 = A + s_0$; by (3.16),

$$\inf\{f_0(x) | x \in A_0\} > 0. \tag{3.17}$$

Clearly $E(A, S) = E(A_0, S) - s_0$ and $int(A_0 + S) \neq \emptyset$; therefore it suffices to show that $E(A_0, S)$ is weakly contractible.

Let $Z = int(A_0 + S) \cup E(A_0, S)$. For each $z \in Z$, let

$$G(z) := (z - S) \cap (A_0 + S),$$

$$l(z) := \inf\{g(y) | y \in G(z)\},$$

and

$$L(z) := \{y \in G(z) | g(y) = l(z)\}.$$

By Lemma 3.2.2, for each $z \in Z$, $L(z)$ is a nonempty convex subset of $E(A_0, S)$. Note that, by (3.17), $0 \notin L(z)$. We further claim that L is a point-valued map on Z. For each $z \in Z$, let y_1, $y_2 \in L(z)$. Since $L(z)$ is convex, $\frac{y_1+y_2}{2} \in L(z)$. Hence

$$g(y_1) = g(\frac{y_1 + y_2}{2}) = g(y_2) = l(z).$$

As f_0 is linear, one has

$$g_0(\frac{y_1 + y_2}{2}) = \frac{g_0(y_1) + g_0(y_2)}{2}.$$

By st-convexity of g_0, $y_1 = \alpha y_2$ for some constant $\alpha > 0$. Suppose $0 < \alpha \leq 1$. By convexity of g_0 and $g_0(0) = 0$,

$$g_0(\alpha y_2) \leq (1-\alpha)g_0(0) + \alpha g_0(y_2) = \alpha g_0(y_2).$$

Therefore

$$\begin{aligned} g(y_2) = g(y_1) = g(\alpha y_2) &= f_0(\alpha y_2) + g_0(\alpha y_2) \\ &\leq \alpha f_0(y_2) + \alpha g_0(y_2) \\ &= \alpha g(y_2). \end{aligned}$$

Note that $g(y_2) = g_0(y_2) + f_0(y_2)$. Since g_0 is nonnegative, $g_0(y_2) \geq 0$; as $y_2 \in A_0 + S$, it follows from (3.17) and $f_0 \in S^{+i}$ that $g(y_2) > 0$. So we should have $\alpha = 1$. And it is similar in case that $\alpha \geq 1$ (consider $y_2 = \frac{1}{\alpha} y_1$). Therefore $y_1 = y_2$ and L is a point-valued map.

By Lemma 3.2.4, G is lower semicontinuous, and by Lemma 3.2.5, G is strong-weak closed. Since A_0 is weakly compact and $L(Z) \subseteq E(A_0, S) \subseteq A_0$, $L(Z)$ is relatively weakly compact. According to Theorem 3.2.3, L is a strong-weak continuous pointed-valued map on Z.

Finally, we show the weak contractibility of $E(A_0, S)$. Pick $a_0 \in int(A_0 + S)$; so $L(a_0) \in E(A_0, S)$. Define a map $H : E(A_0, S) \times [0,1] \to E(A_0, S)$ by

$$H(a,t) := L(ta_0 + (1-t)a), \quad \forall a \in E(A_0, S), \forall t \in [0,1].$$

Clearly for each $a \in E(A_0, S)$, $H(a, 0) = a$ and $H(a, 1) = L(a_0)$. By the strong-weak continuity of L, H is weakly continuous. It follows that $E(A_0, S)$ is weakly contractible. □

The strong topology of a locally convex space and its weak topology coincide on each compact subset. Therefore the following corollary is directly from Theorem 3.2.6.

Corollary 3.2.7 *Let $\mathcal{X}$ be a locally convex space with property (St), $S \subseteq \mathcal{X}$ be a closed convex cone with a bounded base, A be a compact S-convex subset of $\mathcal{X}$ such that the relative interior of $A+S$ is nonempty. Then $E(A,S)$ is contractible.*

After discussing on the contractibility results in the setting of locally convex spaces, we present some results in the setting of normed spaces.

Let $\mathcal{X}$ be a normed space. $\|\cdot\|$ is called to be strictly convex if $\|\frac{x_1+x_2}{2}\| < 1$ whenever x_1, $x_2 \in \mathcal{X}$ with $\|x_1\| = \|x_2\| = 1$ and $x_1 \neq x_2$. $\|\cdot\|$ is called to be locally uniformly convex if, for any $\epsilon > 0$ and any $x \in \mathcal{X}$ with $\|x\| = 1$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| > 1 - \delta$ whenever $y \in \mathcal{X}$ with $\|y\| = 1$ and $\|x - y\| > \epsilon$. Clearly $\|\cdot\|$ is strictly convex if $\|\cdot\|$ is locally uniformly convex.

Note that $\|\cdot\|$ is strictly convex if and only if $\|\cdot\|$ is a st-convex function. Hence every strictly convex normed space has property (St) as

$$\partial\|\cdot\|(X) \subseteq \{f \in X^* | \|f\| \leq 1\}$$

is bounded below on every bounded subset of $\mathcal{X}$.

Before showing a contractibility result in some normed spaces, we first state the following well known lemma. For sake of convenience, we provide its proof.

Lemma 3.2.8 *Let $\mathcal{X}$ be a normed space with an equivalent norm $\|\cdot\|$ which is locally uniformly convex. Assume that $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ with $\|x_n\| \to \|x_0\|$ and $\{x_n\}_{n\in\mathbb{N}}$ weakly convergent to x_0 for some $x_0 \in \mathcal{X}$. Then $x_n \to x_0$.*

Proof: If $x_0 = 0$, then clearly $\|x_n\| \to \|x_0\|$ implies $x_n \to x_0$.
Suppose $x_0 \neq 0$; then $x_n \to x_0$ if and only if $x_n/\|x_n\| \to x_0/\|x_0\|$. Therefore we may, without loss of generality, assume that $\|x_n\| = \|x_0\| = 1$ for each $n \in \mathbb{N}$. Take $f \in \mathcal{X}^*$ such that $\|f\| = 1$ and $f(x_0) = 1$. Therefore

$$f(x_n + x_0) \leq \|x_n + x_0\| \leq 2;$$

as $\{x_n\}_{n\in\mathbb{N}}$ is weakly convergent to x_0, we have

$$f(x_n + x_0) \to f(x_0 + x_0) = 2.$$

Consequently $\|x_n + x_0\| \to 2$ and hence, following from the locally uniform convexity of $\|\cdot\|$, $x_n \to x_0$.

Theorem 3.2.9 *Let $\mathcal{X}$ be a normed space with a locally uniformly convex equivalent norm, $S \subseteq \mathcal{X}$ be a closed convex cone with a bounded base, and $A \subseteq \mathcal{X}$ be a weakly compact S-convex subset such that $A+S$ has a nonempty relative interior. Then $E(A, S)$ is contractible.*

Proof: Let $\|\cdot\|$ be a locally uniformly convex equivalent norm of $\mathcal{X}$. Without loss of generality, assume $int(A + S) \neq \emptyset$. For all $x \in \mathcal{X}$, define

$$g_0(x) = \|x\|.$$

Note that g_0 is a nonnegative continuous st-convex function with $g_0(0) = 0$ and g_0 has property (St). By Lemma 3.2.1, there is $f_0 \in S^{+i}$ such that $g = g_0 + f_0$ is (strictly) increasing. Pick a point $s_0 \in S$ such that

$$\inf\{g(x) | x \in s_0 + A\} > 0.$$

Let $A_0 = s_0 + A$. Clearly $E(A, S) = E(A_0, S) - s_0$ and $int(A_0 + S) \neq \emptyset$. Let $Z = int(A_0 + S) \cup E(A_0, S)$. For each $z \in Z$, define

$$G(z) := (z - S) \cap (A_0 + S),$$

$$l(z) := \inf\{g(y) | y \in G(z)\},$$

and

$$L(z) := \{y \in G(z) | g(y) = l(z)\}.$$

Corresponding to the similar setting in proof for Theorem 3.2.6, we have L is a point-valued strong-weak continuous map from Z to $E(A_0, S)$ and for any $a \in E(A_0, S)$, $L(a) = a$. Next we aim to show that L is further a strong-strong continuous. If it is true, then $E(A_0, S)$ is contractible. Take $z \in Z$. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in Z such that $z_n \to z$. Since L is strong-weak continuous,

$$L(z_n) \xrightarrow{\omega_{\mathcal{X}}} L(z), \tag{3.18}$$

where $\omega_{\mathcal{X}}$ is the weak topology of $\mathcal{X}$. By Theorem 3.2.3, l is continuous on Z; so $l(z_n) \to l(z)$. Hence

$$g(L(z_n)) = l(z_n) \to l(z) = g(L(z)).$$

As $g = f_0 + g_0$, we have

$$f_0(L(z_n)) + \|L(z_n)\| \to f_0(L(z)) + \|L(z)\|.$$

It follows from $f_0 \in \mathcal{X}^*$ that $f_0(L(z_n)) \to f_0(L(z))$. Therefore

$$\|L(z_n)\| \to \|L(z)\|.$$

In view of Lemma 3.2.8, this and (3.18) imply

$$L(z_n) \to L(z).$$

Therefore L is strong-strong continuous; as a consequence, $E(A_0, S)$ is contractible. □

In contrast to Corollary 3.2.7, the subset A needs not be compact in Theorem 3.2.9. Also it is well-known that a normed space has an equivalent locally uniformly convex norm if it is either separable or reflexive.

Finally we study a result concerning the path connectedness of the positive proper efficient point sets. It is from the contractibility results.

Theorem 3.2.10 *Let $\mathcal{X}$ be a reflexive Banach space, $S \subseteq \mathcal{X}$ be a closed convex cone with a bounded base, and $A \subseteq \mathcal{X}$ be a bounded subset such that $A + S$ is closed convex. Then $Pos(A, S)$ is path connected.*

Proof: Let A_0 be the closed convex hull of A. As $A + S$ is closed convex,

$$A \subseteq A_0 \subseteq A + S.$$

Hence by Proposition 1.1.10, $Pos(A_0, S) = Pos(A, S)$. Therefore it suffices to show that $Pos(A_0, S)$ is path connected. Let a_1, $a_2 \in Pos(A_0, S)$; so there are f_1, $f_2 \in S^{+i}$ such that

$$f_j(a_j) = \inf\{f_j(a) | a \in A_0\}, \quad j = 1, 2. \tag{3.19}$$

Without loss of generality, let Θ be a closed bounded base of S. Hence Θ is weakly compact by the reflexivity of $\mathcal{X}$. Then,

$$\alpha_j = \min\{f_j(\theta) | \theta \in \Theta\} > 0, \quad j = 1, 2.$$

Let $\epsilon = \min\{\alpha_1, \alpha_2\}/(2\max\{\|f_1\|, \|f_2\|\})$ and B be the closed unit ball of $\mathcal{X}$. So, $\epsilon > 0$. Consider the subset $\Theta + \epsilon B$. By the construction of ϵ, it follows that for any $b \in \Theta + \epsilon B$,

$$f_j(b) > 0, \quad j = 1, 2. \tag{3.20}$$

Again, by the reflexivity of $\mathcal{X}$, ϵB is closed convex bounded so it is weakly compact. Therefore $\Theta + \epsilon B$ is also weakly compact. This and (3.20) imply that $0 \notin \Theta + \epsilon B$. Consider the Henig dilating cone

$$S_\epsilon(\Theta) := cone(\Theta + \epsilon B).$$

Note that $S_\epsilon(\Theta)$ is a closed convex cone with a closed bounded base $\Theta + \epsilon B$. By (3.20), f_1, $f_2 \in (S_\epsilon(\Theta))^{+i}$, hence, a_1, $a_2 \in Pos(A_0, S_\epsilon(\Theta)) \subseteq E(A_0, S_\epsilon(\Theta))$. Firstly, we claim that

$$E(A_0, S_\epsilon(\Theta)) \subseteq Pos(A_0, S). \tag{3.21}$$

For each $a \in E(A_0, S_\epsilon(\Theta))$, $(A_0 - a) \cap -S_\epsilon(\Theta) = \{0\}$. Therefore

$$(A_0 - a) \cap -int(S_\epsilon(\Theta)) = \emptyset.$$

By the Separation Theorem, there exists $f \in \mathcal{X}^* \setminus \{0\}$ such that

$$\inf\{f(x-a)|x \in A_0\} \geq \sup\{-f(x)|x \in S_\epsilon(\Theta)\}.$$

Since $0 \in S_\epsilon(\Theta)$ and $a \in A_0$,

$$\inf\{f(x-a)|x \in A_0\} = \sup\{-f(x)|x \in S_\epsilon(\Theta)\} = 0. \tag{3.22}$$

Therefore,

$$f(a) = \inf\{f(x)|x \in A_0\}. \tag{3.23}$$

Besides, there exists $b_0 \in B$ such that $f(b_0) > 0$. By (3.22), $f(\theta - \epsilon b_0) \geq 0$ for each $\theta \in \Theta$, so

$$f(\theta) \geq f(\epsilon b_0) > 0.$$

It follows that $f \in S^{+i}$. This and (3.23) imply that $a \in Pos(A_0, S)$.

Secondly, we claim that $E(A_0, S_\epsilon(\Theta))$ is contractible. Since $\Theta + \epsilon B$ is closed convex bounded, $S_\epsilon(\Theta)$ is a closed convex cone; A_0 is a weakly compact subset as it is closed convex bounded in a reflexive Banach space; $\Theta + \epsilon B$ has nonempty interior and therefore so does $A_0 + S_\epsilon(\Theta)$. Lastly without loss of generality, let

$$x_j + t_j(\theta_j + \epsilon b_j) \in A_0 + S_\epsilon(\Theta), \quad j = 1, 2,$$

where $x_j \in A_0$, $t_j > 0$, $\theta_j \in \Theta$, $b_j \in B$ for $j = 1$, 2. Let $\lambda \in (0,1)$, by the S-convexity of A and convexity of $\Theta + \epsilon B$,

$$\begin{aligned}
& \lambda[x_1 + t_1(\theta_1 + \epsilon b_1)] + (1-\lambda)[x_2 + t_2(\theta_2 + \epsilon b_2)] \\
= \; & \lambda x_1 + (1-\lambda)x_2 + \lambda t_1(\theta_1 + \epsilon b_1) + (1-\lambda)t_2(\theta_2 + \epsilon b_2) \\
\in \; & A_0 + S + [\lambda t_1 + (1-\lambda)t_2](\Theta + \epsilon B) \\
\subseteq \; & A_0 + S_\epsilon(\Theta).
\end{aligned}$$

Therefore A_0 is $S_\epsilon(\Theta)$-convex. Under these conditions, $E(A_0, S_\epsilon(\Theta))$ is contractible by Theorem 3.2.9. So there are a continuous map

$$H : E(A_0, S_\epsilon(\Theta)) \times [0,1] \to E(A_0, S_\epsilon(\Theta))$$

and a point $a_0 \in E(A_0, S_\epsilon(\Theta))$ such that for each $a \in E(A_0, S_\epsilon(\Theta))$, $H(a, 0) = a$ and $H(a, 1) = a_0$. Note that $E(A_0, S_\epsilon(\Theta)) \subseteq Pos(A_0, S)$; we define $\varphi : [0,1] \to Pos(A_0, S)$ by

$$\varphi(t) = \begin{cases} H(a_1, 2t), & 0 \le t \le \frac{1}{2} \\ H(a_2, 2-2t), & \frac{1}{2} < t \le 1 \end{cases} .$$

Then, φ provides a continuous path on $Pos(A_0, S)$ from a_1 to a_2, and therefore $Pos(A_0, S)$ is path connected. □

3.3 Connectedness Results in Vector Optimization Problems

In this section, we will move on the connectedness result in the vector optimization problem, that is, the connectedness of the efficient outcome sets and the efficient solution sets. We start by introducing two propositions. The following one is first proved by Warburton [8]. It tells that upper semicontinuity preserves connectedness.

Proposition 3.3.1 *Let $\mathcal{X}$ be a connected topological space and $\mathcal{Y}$ be a topological space. Assume that $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ is an upper semicontinuous set-valued map with connected values. Then $F(\mathcal{X})$ is connected.*

Proof: Suppose that $F(\mathcal{X})$ is not connected in $\mathcal{Y}$: there exist V_1 and V_2, both nonempty and open in $\mathcal{Y}$ such that

$$F(\mathcal{X}) = (F(\mathcal{X}) \cap V_1) \cup (F(\mathcal{X}) \cap V_2),$$

$$F(\mathcal{X}) \cap V_j \neq \emptyset, \quad j = 1, 2,$$

and,

$$F(\mathcal{X}) \cap V_1 \cap V_2 = \emptyset.$$

Consider the sets $F^{-1}(V_1)$ and $F^{-1}(V_2)$. Clearly $F^{-1}(V_1) \cup F^{-1}(V_2) = \mathcal{X}$ and $F^{-1}(V_j) \neq \emptyset$, for $j =$ 1,2. Firstly claim that

$$F^{-1}(V_1) \cap F^{-1}(V_2) = \emptyset. \tag{3.24}$$

Indeed, if not, let $x \in F^{-1}(V_1) \cap F^{-1}(V_2)$. Then $F(x) \cap V_j \neq \emptyset$, for $j =$ 1,2, this contradicts that $F(x)$ is connected in $\mathcal{Y}$. Secondly claim that $F^{-1}(V_1)$ is open in $\mathcal{X}$. Let $x \in F^{-1}(V_1)$; then $F(x) \subseteq V_1$, otherwise $F(x)$ is not connected indeed. Since F is upper semicontinuous, there exists an open neighborhood U of x in $\mathcal{X}$ such that $F(U) \subseteq V_1$, thus $U \subseteq F^{-1}(V_1)$. As x in arbitrary, we get that $F^{-1}(V_1)$ is open. Similarly $F^{-1}(V_2)$ is also open in $\mathcal{X}$. These and (3.24) imply that $\mathcal{X}$ is not connected. □

And the following proposition is well known. It tells that upper semicontinuity also preserves compactness. For sake of convenience, we provide its proof.

Proposition 3.3.2 *Let $\mathcal{X}$ be a compact topological space and $\mathcal{Y}$ be a topological space. Assume that $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ is an upper semicontinuous set-valued map with compact values. Then $F(\mathcal{X})$ is compact.*

Proof: Let $\{O_i\}_{i \in I}$ be an open cover of $F(\mathcal{X})$ where I is some index set. For each $x \in \mathcal{X}$, let $\{O_i\}_{i \in I_x}$ be an open cover of $F(x)$, where $I_x \subseteq I$. As $F(x)$ is compact, there exists a finite subset C_x of I_x such that

$$F(x) \subseteq \bigcup_{i \in C_x} O_i.$$

Note that $\cup_{i \in C_x} O_i$ is a neighborhood of $F(x)$ in $\mathcal{Y}$; by the upper semicontinuity of F, there exists an open neighborhood $N(x)$ of x in $\mathcal{X}$ such that

$$F(N(x)) \subseteq \bigcup_{i \in C_x} O_i.$$

Note that $\{N(x)\}_{x \in \mathcal{X}}$ forms an open cover of $\mathcal{X}$. Since $\mathcal{X}$ is compact, there exist x_j, $1 \leq j \leq n$, such that $\mathcal{X} \subseteq \bigcup_{j=1}^{n} N(x_j)$. Therefore

$$F(\mathcal{X}) \subseteq \bigcup_{j=1}^{n} F(N(x_j)) \subseteq \bigcup_{j=1}^{n} \bigcup_{i \in C_{x_j}} O_i.$$

So, $\{O_i\}_{i \in I}$ has a finite subcover and hence $F(\mathcal{X})$ is compact.

The following result is by X. Y. Zheng.

Proposition 3.3.3 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be a closed convex cone with a base, and A be a weakly compact S-convex subset of $\mathcal{X}$. Then $Pos(A, S)$ is connected respect to the weak topology.*

Proof: For each bounded subset M of $\mathcal{X}$, let

$$P_M(f) := \sup\{|f(x)| | x \in M\}, \quad \text{for each } f \in X^*.$$

Then P_M is a seminorm on X^*. Let τ^* be the locally convex topology induced by the family $\{P_M | M \text{ is bounded in } \mathcal{X}\}$ on $\mathcal{X}^*$ and $\omega_{\mathcal{X}}$ be the weak topology of $\mathcal{X}$. Let $T : S^{+i} \rightrightarrows \mathcal{X}$ be defined by:

$$T(f) := \{a \in A | f(a) = \inf\{f(x) | x \in A\}\}, \quad \text{for each } f \in S^{+i}.$$

Clearly $T(S^{+i}) = Pos(A, S)$. Also note that if $f \in S^{+i}$,

$$\inf\{f(x) | x \in A\} = \inf\{f(x) | x \in A + S\}.$$

So, it is easy to verify that

$$T(f) = \{a \in A + S | f(a) = \inf\{f(x) | x \in A + S\}\},$$

and thus, for each $f \in S^{+i}$, $T(f)$ is convex and then connected with respect to the weak topology $\omega_{\mathcal{X}}$. Now we show that T is τ^*-$\omega_{\mathcal{X}}$ upper semicontinuous on S^{+i}. Suppose that this is not the case: there are $f_0 \in S^{+i}$, a weakly open neighborhood V_0 of $T(f_0)$ in $\mathcal{X}$, a net $\{f_\lambda\}_{\lambda\in\Lambda}$ with $f_\lambda \xrightarrow{\tau^*} f_0$ and a net $\{a_\lambda\}_{\lambda\in\Lambda}$ with each $a_\lambda \in T(f_\lambda)$, such that $a_\lambda \notin V_0$ for each $\lambda \in \Lambda$. Since for each $\lambda \in \Lambda$, $a_\lambda \in T(f_\lambda) \subseteq A$ and A is weakly compact, we can assume that, without loss of generality, $\{a_\lambda\}_{\lambda\in\Lambda}$ weakly converges to some $a_0 \in A$. This and that V_0 is weakly open imply that $a_0 \notin V_0$, hence $a_0 \notin T(f_0)$. Pick $a_1 \in T(f_0)$; so

$$f_0(a_1) < f_0(a_0). \tag{3.25}$$

Let $K := (A - a_0) \cup \{a_0, a_1\}$. Clearly K is a bounded subset. Since $f_\lambda \xrightarrow{\tau^*} f_0$, we have

$$\lim_\lambda P_K(f_\lambda - f_0) = 0. \tag{3.26}$$

As for each $\lambda \in \Lambda$, $(a_\lambda - a_0) \in K$, $a_0 \in K$ and $a_1 \in K$, by (3.26),

$$(f_\lambda - f_0)(a_\lambda - a_0) \to 0, \quad (f_\lambda - f_0)(a_0) \to 0, \text{ and } \quad (f_\lambda - f_0)(a_1) \to 0.$$

From this and $a_\lambda \xrightarrow{\omega_{\mathcal{X}}} a_0$,

$$f_\lambda(a_\lambda) - f_\lambda(a_1) \to f_0(a_0) - f_0(a_1). \tag{3.27}$$

Note that, for each $\lambda \in \Lambda$, $f_\lambda(a_\lambda) \le f_\lambda(a_1)$, this and (3.27) imply that

$$f_0(a_0) \le f_0(a_1).$$

It contradicts with (3.25). It results that T is τ^*-$\omega_{\mathcal{X}}$ upper semicontinuous on S^{+i}. Since T is with connected values, S^{+i} is τ^*-connected and $T(S^{+i}) = Pos(A, S)$, it follows from Proposition 3.3.1 that $Pos(A, S)$ in connected with respect to the weak topology of $\mathcal{X}$. □

By Corollary 1.3.7, $Pos(A, S)$ is dense in $E(A, S)$ with respect to the weak topology. Then we have the following corollary.

Corollary 3.3.4 *Let $\mathcal{X}$ be a locally convex space, $S \subseteq \mathcal{X}$ be a closed convex cone with a base, and A be a weakly compact S-convex subset of $\mathcal{X}$. Then $E(A, S)$ is connected respect to the weak topology.*

Using the technique used in the proof of preceding proposition, I give the following result concerning the connectedness of the efficient outcome set of a vector optimization problem in a normed space.

Theorem 3.3.5 *Let $\mathcal{X}$ be a topological vector space, $\mathcal{Y}$ be a normed space ordered by a closed convex cone S with a base, A be a compact convex subset of $\mathcal{X}$. Assume that F is an upper semicontinuous S-convex set-valued map from A to $\mathcal{Y}$ with compact values. Then $Pos(F(A), S)$ and $Min(F(A), S)$ are connected.*

Proof: From compactness and convexity of A, also upper semicontinuity and S-convexity of F, it follows from Proposition 3.3.2 and Remark 3.1.3 that $F(A)$ is compact S-convex. By Theorem 1.3.5 and Proposition 1.1.10,

$$Min(F(A), S) = E(F(A), S) \subseteq cl(Pos(F(A), S)).$$

So, it suffices to show that $Pos(F(A), S)$ is connected. Let $T : S^{+i} \rightrightarrows F(A)$ be a set-valued map defined by:

$$T(f) := \{y \in F(A) | f(y) = \min\{f(z) | z \in F(A)\}\}, \quad \text{for each } f \in S^{+i}.$$

Firstly, claim that T is with connected values. We can reach it by showing that $T(f)$ is convex for each $f \in S^{+i}$. As A is convex and F is S-convex, so $F(A)$ is S-convex. Note that for each $f \in S^{+i}$,

$$\min\{f(z) | z \in F(A)\} = \min\{f(z) | z \in F(A) + S\}.$$

Therefore, it is easy to verify that

$$T(f) = \{y \in F(A) + S | f(y) = \min\{f(z) | z \in F(A) + S\}\};$$

hence $T(f)$ is convex and then connected for each $f \in S^{+i}$.
Secondly we claim that T is upper semicontinuous. Since $F(A)$ is a compact subset in $\mathcal{Y}$, in view of Theorem 3.1.2, it is sufficient to show that T is closed. Let $\{(f_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ be a net in $Graph(T)$ such that $f_\lambda \to f_0 \in S^{+i}$ and $y_\lambda \to y_0 \in F(A)$. Since $y_\lambda \in T(f_\lambda)$, one has for any $y \in F(A)$ and any $\lambda \in \Lambda$ that

$$f_\lambda(y_\lambda) \leq f_\lambda(y).$$

Therefore we have

$$f_0(y_0) \leq f_0(y).$$

This leads to

$$y_0 \in T(f_0).$$

Therefore T is closed and T is upper semicontinuous. Since T is with connected values, S^{+i} is convex (thus connected) and $T(S^{+i}) = Pos(F(A), S)$, it follows from Proposition 3.3.1 that $Pos(F(A), S)$ is connected. □

In contrast to Theorem 3.0.1, Theorem 3.3.5 comments that (3.3) is not essential.

And now we present the connectedness result for the efficient solution sets by X. Y. Zheng.

Theorem 3.3.6 *Let $\mathcal{X}$ and $\mathcal{Y}$ be locally convex spaces, $S \subseteq \mathcal{Y}$ be a closed convex cone with a base, $A \subseteq \mathcal{X}$ be a compact convex subset. Assume that F is a strong-weak upper semicontinuous S-convex set-valued map from A to $\mathcal{Y}$ with weakly compact values. Then the efficient solution set $Min(A, S, F)$ is connected.*

Proof: By Proposition 3.3.2, $F(A)$ is weakly compact. By the S-convexity of F and convexity of A, $F(A)$ is S-convex. Then, by Corollary 3.3.4, $E(F(A), S)$ is connected with respect to the weak topology of $\mathcal{Y}$.

Consider the inverse set-valued map F^{-1} from $E(F(A), S)$ to A. Clearly

$$Min(A, S, F) = F^{-1}(E(F(A), S)).$$

In view of Proposition 3.3.1, we need only show that F^{-1} is a weak-strong upper semicontinuous set-valued map with connected values in $\mathcal{X}$. To do this, firstly, note that A is compact; by Theorem 3.1.2, we can get the result that F^{-1} is weak-strong upper semicontinuous by showing that F^{-1} is weak-strong closed. Clearly this condition is equivalent to that $Graph(F)$ is strong-weak closed. Now as F is strong-weak upper semicontinuous set-valued map with compact values in a Hausdorff space and by Theorem 3.1.1, $Graph(F)$ is strong-weak closed and thus F^{-1} is weak-strong upper semicontinuous.

Secondly, aim to prove that F^{-1} is with connected values. For each $y \in E(F(A), S)$, let a_1, $a_2 \in F^{-1}(y)$ and $\lambda \in [0, 1]$. By S-convexity of F,

$$\lambda F(a_1) + (1 - \lambda)F(a_2) \subseteq F(\lambda a_1 + (1 - \lambda)a_2) + S.$$

Since $y \in F(a_1)$ and $y \in F(a_2)$,

$$y = \lambda y + (1 - \lambda)y \subseteq F(\lambda a_1 + (1 - \lambda)a_2) + S.$$

Therefore there exist $z \in F(\lambda a_1 + (1 - \lambda)a_2)$ and $s \in S$ such that $y = z + s$. As A is convex, $z \in F(A)$. However, since $y \in E(F(A), S)$ and $z \in F(A)$, we conclude that $s = 0$ and hence

$$y \in F(\lambda a_1 + (1 - \lambda)a_2).$$

As a result, $\lambda a_1 + (1 - \lambda)a_2 \in F^{-1}(y)$ and consequently, $F^{-1}(y)$ is convex and thus connected. $\square$

Chapter 4

Error Bounds In Normed Spaces

In this chapter we will discuss the concept of the error bounds. Error bounds take an important role in the sensitivity analysis of the mathematical programming. What it concerns is that if a function is given, what relation between the distance of an arbitrary point from the zero set and the function value on that point is. Actually, in this chapter we consider the set of points with nonpositive values instead of the zero set. Also in contrast to most of previous researches on error bounds, we also study some results about the error bounds with fractional exponents other than exponent one. In this chapter we will give a systematic survey of the papers [10] and [11] by K. F. Ng and X. Y. Zheng.

Let $\mathcal{X}$ be a normed space, a function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is said to be **proper** if the set

$$dom(f) := \{x \in \mathcal{X} | f(x) < +\infty\}$$

is nonempty. Now we state the definition of the error bound as follows.

Definition 4.0.1 *Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Assume that the set*

$$S := \{x \in \mathcal{X} | f(x) \leq 0\}$$

is nonempty. Then f is said to have an **error bound**, *(or say that an error bound for f holds) if there exists a positive constant τ such that for each $x \in \mathcal{X}$,*

$$dist(x, S) \le \tau[f(x)]_+,$$

where $dist(x, S) = \inf\{\|x - y\| | y \in S\}$ and $[f(x)]_+ = \max\{f(x), 0\}$. In this case τ is said to be an error bound for f.

In this chapter, we always assume that S is nonempty.

Remark 4.0.1 *To verify that $\tau > 0$ is an error bound for f it is sufficient (and necessary) to check that $dist(x, S) \le \tau f(x)$ is true for each $x \in dom(f) \setminus S$.*

In this preliminary section, let us also state a lemma which is based on the famous Ekeland variational principle. It will be used later in our discussion.

Lemma 4.0.1 ([12] Theorem 2(ii)) *Let $(\mathcal{X}, d)$ be a complete metric space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function bounded from below. Let $Z := \{z \in \mathcal{X} | f(z) = \inf\{f(y) | y \in \mathcal{X}\}\}$ and $dist(x, Z) = \inf\{d(x, z) | z \in Z\}$ for each $x \in \mathcal{X}$. Let $\alpha > 0$. Suppose that for each $x \in \mathcal{X}$ with $f(x) > \inf\{f(y) | y \in \mathcal{X}\}$ there exists $x' \in \mathcal{X}$ such that*

1. *$x' \ne x$ and*

2. *$f(x') + \alpha d(x', x) \le f(x)$.*

Then, for each $x \in \mathcal{X}$, $f(x) - \inf\{f(y) | y \in \mathcal{X}\} \ge \alpha dist(x, Z)$.

4.1 Error Bounds of Lower Semicontinuous Functions in Normed Spaces

Recall that for each $x \in \mathcal{X}$ with $f(x) < +\infty$ and each $h \in \mathcal{X}$ with $\|h\| = 1$, the upper Dini-directional derivative and the lower Dini-directional derivative are denoted by

$$\overline{d}^{+} f(x)(h) := \limsup_{t \to 0^{+}} \frac{f(x+th) - f(x)}{t}$$

and

$$\underline{d}^{+} f(x)(h) := \liminf_{t \to 0^{+}} \frac{f(x+th) - f(x)}{t}$$

respectively. If $f(x) = +\infty$, we define $\underline{d}^{+} f(x)(h) := -\infty$. Note that $\underline{d}^{+} f(x)(h) \leq \overline{d}^{+} f(x)(h)$; if the equality holds, f is said to be right differentiable at x in the direction h.

Firstly, we discuss some sufficient conditions for a proper lower semicontinuous function f to have an error bound. We introduce the following mean-value type theorem.

Lemma 4.1.1 *Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $x \in dom(f)$, $h \in \mathcal{X}$ with $\|h\| = 1$ and $\lambda > 0$. Assume that there exists $\delta \in \mathbb{R}$ such that for each $t \in [0, \lambda)$,*

$$\underline{d}^{+} f(x+th)(h) \leq \delta.$$

Then

$$f(x + \lambda h) - f(x) \leq \lambda\delta.$$

Proof: For any $\epsilon > 0$, let

$$\lambda_{\epsilon} = \sup\{0 \leq t \leq \lambda | f(x+th) - f(x) \leq t(\delta + \epsilon)\}. \tag{4.1}$$

From the lower semicontinuity of f,

$$f(x + \lambda_{\epsilon} h) - f(x) \leq \lambda_{\epsilon}(\delta + \epsilon). \tag{4.2}$$

We show that $\lambda_{\epsilon} = \lambda$. Suppose not: that is, $\lambda_{\epsilon} < \lambda$. Then $\underline{d}^{+} f(x + \lambda_{\epsilon} h)(h) \leq \delta$, and so there exists $\lambda' \in (\lambda_{\epsilon}, \lambda)$ such that

$$f(x + \lambda' h) - f(x + \lambda_{\epsilon} h) \leq (\lambda' - \lambda_{\epsilon})(\delta + \epsilon).$$

This and (4.2) imply that

$$f(x+\lambda' h) - f(x) \leq \lambda'(\delta + \epsilon).$$

However, it contradicts with (4.1). As a result, $\lambda_\epsilon = \lambda$; thus

$$f(x+\lambda h) - f(x) \leq \lambda(\delta + \epsilon).$$

Then, the lemma follows from letting $\epsilon \to 0$. □

Lemma 4.1.2 *Let $(\mathcal{X}, d)$ be a metric space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Let $\tau > 0$ and $0 \leq \rho < 1$ be constants. Suppose that for each $x \in dom(f) \setminus S$ (that is, $0 < f(x) < +\infty$), there exists $x' \in \mathcal{X}$ with $0 \leq f(x') < +\infty$ such that*

$$dist(x', S) \leq \rho dist(x, S)$$

and

$$d(x, x') \leq \tau[f(x) - f(x')].$$

Then, for each $x \in \mathcal{X}$, $dist(x, S) \leq \tau[f(x)]_+$.

Proof: We may suppose that $x \in dom(f) \setminus S$ (otherwise the conclusion is trivially true). We write x_0 for x. In view of the assumption, we have only two different cases below.

1. There exists $\{x_1, x_2, ..., x_n\} \subseteq \mathcal{X}$ such that $\{x_1, x_2, ..., x_{n-1}\} \subseteq dom(f) \setminus S$, $f(x_n) = 0$ and for each $1 \leq i \leq n$,

$$d(x_{i-1}, x_i) \leq \tau[f(x_{i-1}) - f(x_i)].$$

Then

$$\begin{aligned} dist(x_0, S) &\leq d(x_0, x_n) \leq \sum_{i=1}^{n} d(x_{i-1}, x_i) \\ &\leq \sum_{i=1}^{n} \tau[f(x_{i-1}) - f(x_i)] = \tau[f(x_0) - f(x_n)] = \tau f(x_0). \end{aligned}$$

2. There exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $dom(f)\setminus S$ such that for each $k\in\mathbb{N}$

$$dist(x_k,S)\le \rho dist(x_{k-1},S)$$

and

$$d(x_{k-1},x_k)\le \tau[f(x_{k-1})-f(x_k)].$$

Then these imply that for each $k\in\mathbb{N}$,

$$\begin{aligned} dist(x_0,S) &\le d(x_0,x_k)+dist(x_k,S)\\ &\le \sum_{i=1}^{k} d(x_{i-1},x_i)+\rho^k dist(x_0,S)\\ &\le \sum_{i=1}^{k} \tau[f(x_{i-1})-f(x_i)]+\rho^k dist(x_0,S)\\ &= \tau[f(x_0)-f(x_k)]+\rho^k dist(x_0,S)\\ &\le \tau f(x_0)+\rho^k dist(x_0,S). \end{aligned}$$

Letting $k\to+\infty$, we have $dist(x_0,S)\le \tau f(x_0)$.

Combining two cases, the error bound result follows. □

By Lemma 4.1.2, we show the theorem below concerning the sufficient condition for a proper lower semicontinuous function on a normed space to have an error bound.

Theorem 4.1.3 *Let $\mathcal{X}$ be a normed space and $f:\mathcal{X}\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function. Let $0<\delta<+\infty$ and $0\le\rho<1$ be constants. Suppose that for each $x\in dom(f)\setminus S$, there exist $h_x\in\mathcal{X}$ with $\|h_x\|=1$ and $\lambda_x>0$ such that*

$$\underline{d}^+ f(x+th_x)(h_x)\le -\delta,\qquad \forall t\in[0,\lambda_x), \tag{4.3}$$

and

$$dist(x + \lambda_x h_x, S) \leq \rho dist(x, S). \tag{4.4}$$

Then, for each $x \in \mathcal{X}$, $dist(x, S) \leq \frac{1}{\delta}[f(x)]_+$.

Proof: Referring to Lemma 4.1.2, we aim to show that for each $x \in dom(f) \setminus S$, there exists x' such that $0 \leq f(x') < +\infty$, $dist(x', S) \leq \rho dist(x, S)$ and $d(x, x') \leq \frac{1}{\delta}[f(x) - f(x')]$. If $f(x + \lambda_x h_x) \geq 0$, we take $x' = x + \lambda_x h_x$. What remains is to show that $d(x, x + \lambda_x h_x) \leq \frac{1}{\delta}[f(x) - f(x + \lambda_x h_x)]$. By Lemma 4.1.1 and (4.3), $f(x + \lambda_x h_x) - f(x) \leq -\lambda_x \delta$, so,

$$d(x, x + \lambda_x h_x) = \|(x + \lambda_x h_x) - x\| = \lambda_x \leq \frac{1}{\delta}[f(x) - f(x + \lambda_x h_x)].$$

On the other hand, if $f(x + \lambda_x h_x) < 0$, let

$$t_x := \sup\{0 \leq t \leq \lambda_x | f(x + sh) > 0, \text{ for each } s \in [0, t]\}.$$

From the lower semicontinuity of f and definition of t_x we have

$$f(x + t_x h_x) \leq 0 \tag{4.5}$$

and

$$f(x + th_x) > 0 \quad \text{for all } t \in [0, t_x). \tag{4.6}$$

As S is closed, $dist(x, S) > 0$. It follows from (4.5), (4.6) and the lower semicontinuity of f that there exists $s_x \in (0, t_x)$ such that

$$dist(x + s_x h_x, S) \leq \|(x + s_x h_x) - (x + t_x h_x)\| = t_x - s_x \leq \rho dist(x, S)$$

and

$$0 < f(x + s_x h_x) < +\infty.$$

Now take $x' = x + s_x h_x$. What remains is to show that $d(x, x + s_x h_x) \leq \frac{1}{\delta}[f(x) - f(x + s_x h_x)]$; replacing λ_x by s_x and using the same argument in the first part, the result follows. □

Theorem 4.1.5 states that in the setting of Banach spaces, the condition (4.3) can be simplified and (4.4) can be dropped. In order to get this result, we discuss a lemma which generally follows from Lemma 4.0.1.

Lemma 4.1.4 *Let $(\mathcal{X}, d)$ be a complete metric space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $\tau > 0$ be a constant. Suppose that for each $x \in dom(f) \setminus S$, there exists $x' \in \mathcal{X}$ with $0 \leq f(x') < +\infty$ such that*

$$0 \neq d(x, x') \leq \tau[f(x) - f(x')].$$

Then, for each $x \in \mathcal{X}$, $dist(x, S) \leq \tau[f(x)]_+$.

Proof: We may assume that $\mathcal{X} \neq S$. Define $g(x) := \max\{f(x), 0\} = [f(x)]_+$ for each $x \in \mathcal{X}$. Note that g is a proper lower semicontinuous function bounded below. Without loss of generality, we consider $x \in dom(f) \setminus S$. By assumption, there exists $x' \in \mathcal{X}$ such that $0 \leq f(x') < +\infty$, $x' \neq x$ and $f(x') + \frac{1}{\tau}d(x, x') \leq f(x)$. It leads to that $g(x') + \frac{1}{\tau}d(x, x') \leq g(x)$. Then it follows from Lemma 4.0.1 and the fact $S = \{x \in \mathcal{X} | g(x) = \inf\{g(y) | y \in \mathcal{X}\}\}$ that for each $x \in \mathcal{X}$,

$$g(x) - \inf\{g(y) | y \in \mathcal{X}\} \geq \frac{1}{\tau} dist(x, S).$$

That is, $dist(x, S) \leq \tau[f(x)]_+$. □

Theorem 4.1.5 *Let $\mathcal{X}$ be a Banach space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $0 < \delta < +\infty$ be a constant. Suppose that for each $x \in dom(f) \setminus S$, there exists $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that*

$$\underline{d}^+ f(x)(h_x) \leq -\delta.$$

Then, for each $x \in \mathcal{X}$, $dist(x, S) \leq \frac{1}{\delta}[f(x)]_+$.

Proof: Let $x \in dom(f) \setminus S$ and $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that $\underline{d}^+ f(x)(h_x) \leq -\delta$. For any ϵ with $\delta > \epsilon > 0$,

$$\underline{d}^+ f(x)(h_x) \leq -\delta < -(\delta - \epsilon). \tag{4.7}$$

Since f is lower semicontinuous and $0 < f(x) < +\infty$, following from (4.7) there exists $\lambda > 0$ such that

$$0 \leq f(x + \lambda h_x) < +\infty$$

and

$$\frac{1}{\lambda}[f(x + \lambda h_x) - f(x)] \leq -(\delta - \epsilon).$$

These imply that

$$\|x - (x + \lambda h_x)\| = \lambda \leq \frac{1}{\delta - \epsilon}[f(x) - f(x + \lambda h_x)].$$

By Lemma 4.1.4, for each $x \in \mathcal{X}$, $dist(x, S) \leq \frac{1}{\delta - \epsilon}[f(x)]_+$. Letting $\epsilon \to 0$, $dist(x, S) \leq \frac{1}{\delta}[f(x)]$ for each $x \in \mathcal{X}$. □

Recall that cone of feasible directions of a convex set $C \subseteq \mathcal{X}$ at a point $x \in C$ is

$$\mathcal{F}_C(x) = \{v \in \mathcal{X} | x + tv \in C \text{ for some } t > 0\}.$$

The following corollary gives a general error bound result with constraints.

Corollary 4.1.6 *Let* $\mathcal{X}$ *be a Banach space and* $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous function. Let* $0 < \delta < +\infty$ *be a constant and* C *be a closed convex subset of* $\mathcal{X}$ *such that* $S_C := S \cap C \neq \emptyset$. *Suppose that for each* $x \in (dom(f) \setminus S) \cap C$, *there exists* $h_x \in \mathcal{F}_C(x)$ *with* $\|h_x\| = 1$ *such that* $\underline{d}^+ f(x)(h_x) \leq -\delta$. *Then, for each* $x \in C$, $dist(x, S_C) \leq \frac{1}{\delta}[f(x)]_+$.

Proof: Define $g(x) := f(x) + \delta_C(x)$, where δ_C is the indicator function of C, that is,

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C \end{cases}.$$

Note that g is a proper lower semicontinuous function. Also $S_C = \{x \in \mathcal{X} | g(x) \leq 0\}$; $dom(g) \setminus S_C = (dom(f) \setminus S) \cap C$; for each $x \in dom(g) \setminus S_C$ and each $h \in \mathcal{F}_C(x)$, $\underline{d}^+ g(x)(h) = \underline{d}^+ f(x)(h)$. Then by Theorem 4.1.5, for each $x \in \mathcal{X}$, we have $dist(x, S_C) \leq \frac{1}{\delta}[g(x)]_+$. Therefore, for each $x \in C$, $dist(x, S_C) \leq \frac{1}{\delta}[f(x)]_+$. □

Now we discuss the necessary conditions for an error bound for a proper lower semicontinuous function to hold. In favour of the discussion, we introduce the following notations: For $x \in \partial S$ (boundary of S), define

$$N_S^1(x) = \{h \in \mathcal{X} | \|h\| = 1 \text{ and } dist(x + \lambda h, S) = \lambda \text{ for some } \lambda > 0\}$$

and

$$\partial_N S = \{x \in \partial S | N_S^1(x) \neq \emptyset\}.$$

First we present a theorem concerning the local error bound.

Theorem 4.1.7 *Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $\tau > 0$ be a constant. Suppose that for each $x \in \partial S$, there is $\delta(x) > 0$ such that whenever $y \in \mathcal{X}$ with $\|y - x\| < \delta(x)$,*

$$dist(y, S) \leq \tau [f(y)]_+.$$

Then, for each $x \in \partial S$,

$$\inf\{\underline{d}^+ f(x)(h) | h \in N_S^1(x)\} \geq \frac{1}{\tau}.$$

Proof: Since $\inf\{\underline{d}^+ f(x)(h) | h \in N_S^1(x)\} = +\infty$ whenever $x \in \partial S \setminus \partial_N S$, we can only consider the case that $x \in \partial_N S$. For each $x \in \partial_N S$ and each $h \in N_S^1(x)$, by the definition of $N_S^1(x)$, there exists $\lambda > 0$ such that $dist(x + \lambda h, S) = \lambda$. Furthermore it is easy to verify that for each $t \in (0, \lambda)$, $dist(x + th, S) = t > 0$; therefore $x + th \notin S$ and $f(x + th) > 0$. Now from the assumption, for each $t \in (0, \min\{\delta(x), \lambda\})$,

$$t = dist(x + th, S) \leq \tau[f(x + th)]_+ = \tau[f(x + th) - f(x)].$$

Therefore, $\underline{d}^+ f(x)(h) \geq \frac{1}{\tau}$ and hence the result follows. □

The following corollary is a direct consequence of Theorem 4.1.7.

Corollary 4.1.8 *Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $\tau > 0$ be a constant. Suppose that for each $x \in \mathcal{X}$, $dist(x, S) \leq \tau[f(x)]_+$. Then, for each $x \in \partial S$,*

$$\inf\{\underline{d}^+ f(x)(h) | h \in N_S^1(x)\} \geq \frac{1}{\tau}.$$

A function f is said to satisfy Slater condition if there exists $x_0 \in \mathcal{X}$ such that $f(x_0) < 0$. In the following two corollaries, we present the results that in certain settings it is necessary for a function to satisfy Slater condition if an error bound holds.

Corollary 4.1.9 *Let $\mathcal{X}$ be a finite dimensional normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a differentiable function. Suppose that f has an error bound. Then f satisfies the Slater condition.*

Proof: Suppose that f does not satisfy the Slater condition. Then for each $x \in S$, $f(x) = \inf\{f(y) | y \in \mathcal{X}\}$ and so $\nabla f(x) = 0$. By Corollary 4.1.8, $N_S^1(x)$

should be empty for each $x \in \partial S$. Since $\mathcal{X}$ is a finite dimensional normed space, pick a point $z \in \mathcal{X} \setminus S$, there exists $x_0 \in \partial S$ such that $dist(z, S) = \|z - x_0\|$. This implies that

$$\frac{z - x_0}{\|z - x_0\|} \in N_S^1(x_0).$$

This contradiction tells us that f should satisfy the Slater condition.

Corollary 4.1.10 *Let $\mathcal{X}$ be a reflexive Banach space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a differentiable convex function. Suppose that f has an error bound. Then f satisfies the Slater condition.*

Proof: Since f is differentiable convex, S is closed convex in $\mathcal{X}$. Pick a point $z \in \mathcal{X} \setminus S$, by the reflexivity of $\mathcal{X}$, there exists $x_0 \in \partial S$ such that $dist(z, S) = \|z - x_0\|$. Using the similar argument in the proof of preceding corollary, the result follows.

4.2 Error Bounds of Lower Semicontinuous Convex Functions in Reflexive Banach Spaces

In this section, we will consider the error bound for a lower semicontinuous convex function in a reflexive Banach space. We will study some both sufficient and necessary conditions for an error bound to hold.

Recall that the right directional derivative and the left directional derivative of a function f are defined respectively by

$$d^+ f(x)(h) = \lim_{t \to 0+} \frac{f(x + th) - f(x)}{t}$$

and

$$d^-f(x)(h) = \lim_{t\to 0-} \frac{f(x+th)-f(x)}{t}.$$

They always exist for a convex function f. For each $x \in dom(f)$ and $h \in \mathcal{X}\setminus\{0\}$,

$$d^-f(x)(h) = -d^+f(x)(-h).$$

We define $d^+f(x)(h) = -\infty$ for all $h \in \mathcal{X}\setminus\{0\}$ in case that $f(x) = +\infty$. See [13], it is well known that as f is convex, for $0 \le t_1 < t_2$ with $x+t_1h$, $x+t_2h$ both in $dom(f)$,

$$d^+f(x+t_1h)(h) \le d^-f(x+t_2h)(h) \le d^+f(x+t_2h)(h). \tag{4.8}$$

Now we present the main result of this section.

Theorem 4.2.1 *Let $\mathcal{X}$ be a reflexive Banach space and $f : \mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous convex function. Let $\tau > 0$ be a constant. Then the following statements are equivalent.*

(A) τ is an error bound for f: $dist(x,S) \le \tau[f(x)]_+$ for each $x \in \mathcal{X}$.

(B) τ is a local error bound for f at each boundary point of S: for each $x \in \partial S$, there is an $\delta(x) > 0$ such that whenever $y \in \mathcal{X}$ with $\|y-x\| < \delta(x)$, $dist(y,S) \le \tau[f(y)]_+$.

(C) For each $x \in \partial S$,

$$\inf\{d^+f(x)(h) | h \in N_S^1(x)\} \ge \frac{1}{\tau}. \tag{4.9}$$

(D) For each $x \in \mathcal{X}\setminus S$, there exist $\lambda_x > 0$ and $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that

$$x + \lambda_x h_x \in S$$

and

$$d^+f(x+th_x)(h_x) \le -\frac{1}{\tau} \quad \textit{for each } t \in [0,\lambda_x).$$

(E) For each $x \in \mathcal{X} \setminus S$, there exists $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that

$$d^+ f(x)(h_x) \le -\frac{1}{\tau}.$$

Proof: $(A) \Rightarrow (B)$ and $(D) \Rightarrow (E)$ are trivial; $(B) \Rightarrow (C)$ is from Theorem 4.1.7 and $(E) \Rightarrow (A)$ is from Theorem 4.1.5. Therefore it is sufficient to show that $(C) \Rightarrow (D)$.

Take $x \in \mathcal{X} \setminus S$. Since $\mathcal{X}$ is reflexive and S is closed convex, there exists $x_0 \in \partial S$ such that $\|x - x_0\| = dist(x, S) > 0$. Let $\lambda_x = \|x - x_0\|$ and $h_x = \frac{1}{\lambda_x}(x_0 - x)$. Then $x_0 = x + \lambda_x h_x \in S$ and

$$-h_x \in N_S^1(x_0). \tag{4.10}$$

Now we consider two cases.

1. $0 < f(x) < +\infty$: By convexity of f and that x, $x + \lambda_x h_x$ both in $dom(f)$, we have $x + th_x \in dom(f)$ for each $t \in [0, \lambda_x)$. Also from (4.9) and (4.10), $d^+ f(x_0)(-h_x) \ge \frac{1}{\tau}$. This implies that $-d^- f(x_0)(h_x) \ge \frac{1}{\tau}$ and thus
$$d^- f(x_0)(h_x) \le -\frac{1}{\tau}.$$
See (4.8), we have that for each $t \in [0, \lambda_x)$,
$$d^+ f(x + th_x)(h_x) \le d^- f(x + \lambda_x h_x)(h_x) = d^- f(x_0)(h_x) \le -\frac{1}{\tau}.$$
Hence the statement (D) follows.

2. $f(x) = +\infty$: By the definition, we have $d^+ f(x)(h) = -\infty$ for all $h \in \mathcal{X}$ with $\|h\| = 1$. Therefore similar to the proof in above, the result follows from the convexity of f and (4.8).

□

Let $Eb(f) := \inf\{\tau > 0 | dist(x, S) \le \tau[f(x)]_+ \text{ for each } x \in \mathcal{X}\}$. Note that $Eb(f) = 0$ if and only if $\mathcal{X} = S$; $Eb(f) = +\infty$ if and only if f does not have any error bound. Therefore the following corollary is directly from Theorem 4.2.1.

Corollary 4.2.2 *Let $\mathcal{X}$ be a reflexive Banach space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$\frac{1}{Eb(f)} = -\sup_{x\in\mathcal{X}\setminus S} \inf_{h\in\mathcal{X},\|h\|=1} d^+f(x)(h) = \inf_{x\in\partial S} \inf_{h\in N_S^1(x)} d^+f(x)(h).$$

The following theorem gives the sufficient and necessary condition for an error bound for a continuous convex function to hold in a reflexive Banach space.

Theorem 4.2.3 *Let $\mathcal{X}$ be a reflexive Banach space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a continuous convex function. Then f has an error bound if and only if*

$$\delta := \inf_{x\in\mathcal{X}\setminus S} \inf\{\|x^*\| | x^* \in \partial f(x)\} > 0. \tag{4.11}$$

In this case, $dist(x, S) \leq \frac{1}{\delta}[f(x)]_+$ for each $x \in \mathcal{X}$.

Proof: Since f is continuous convex, for each $x \in \mathcal{X}$ and each $h \in \mathcal{X}$ with $\|h\| = 1$,

$$d^+f(x)(h) = \sup\{< x^*, h > | x^* \in \partial f(x)\}. \tag{4.12}$$

Suppose that there exists $\tau > 0$ such that for each $x \in \mathcal{X}$, $dist(x, S) \leq \tau[f(x)]_+$. In view of (E) of Theorem 4.2.1, for each $x \in \mathcal{X} \setminus S$, there exists $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that $d^+f(x)(h_x) \leq -\frac{1}{\tau}$. Therefore from (4.12),

$$\sup\{< x^*, h_x > | x^* \in \partial f(x)\} \leq -\frac{1}{\tau}.$$

Rewriting as follows, we have

$$\inf\{< x^*, -h_x > | x^* \in \partial f(x)\} \geq \frac{1}{\tau}.$$

Note that since $\| - h_x\| = 1$, $\|x^*\| \geq < x^*, -h_x >$; hence

$$\inf\{\|x^*\| | x^* \in \partial f(x)\} \geq \frac{1}{\tau}.$$

As it is valid for each $x \in \mathcal{X} \setminus S$, we have

$$\inf_{x \in \mathcal{X} \setminus S} \inf\{\|x^*\| | x^* \in \partial f(x)\} \geq \frac{1}{\tau} > 0.$$

Conversely, suppose that (4.11) holds. For each $x \in \mathcal{X} \setminus S$, $\partial f(x)$ is disjoint from $int(\delta(B(\mathcal{X}^*)))$, where $B(\mathcal{X}^*)$ is the unit ball of the dual space $\mathcal{X}^*$. By the Separation Theorem and the reflexivity of $\mathcal{X}$, there exists $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that

$$\sup\{< x^*, h_x > | x^* \in \partial f(x)\} \leq \inf\{< x^*, h_x > | x^* \in \delta B(\mathcal{X}^*)\}.$$

Note that, $\inf\{< x^*, h_x > | x^* \in \delta B(\mathcal{X}^*)\} = -\delta$, so $\sup\{< x^*, h_x > | x^* \in \partial f(x)\} \leq -\delta$. By (4.12),

$$d^+ f(x)(h_x) \leq -\delta.$$

Again, in view of Theorem 4.2.1, $\frac{1}{\delta}$ is an error bound for f. □

Corollary 4.2.4 *Let $\mathcal{X}$ be a reflexive Banach space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a continuous convex function. Suppose that*

$$\sup_{x \in \partial S} \inf_{h \in \mathcal{X}, \|h\|=1} d^+ f(x)(h) < 0.$$

Then an error bound for f holds.

Proof: We prove the corollary by showing that (B) of Theorem 4.2.1 holds. Let

$$\sup_{x \in \partial S} \inf_{h \in \mathcal{X}, \|h\|=1} d^+ f(x)(h) < -2\epsilon,$$

where ϵ is a positive constant. For any $x \in \partial S$, there exists $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that $d^+ f(x)(h_x) < -\epsilon$. See [13], $d^+ f(\cdot)(h_x)$ is upper semicontinuous. Therefore there exists $r > 0$ such that

$$d^+ f(y)(h_x) < -\epsilon \tag{4.13}$$

whenever $\|y-x\| < r$. Note that since f is continuous, there exists $\delta(x) > 0$ such that for each $y \in \mathcal{X}$ with $\|y-x\| < \delta(x)$, $f(y) - f(x) < \frac{r\epsilon}{2}$; that is,

$$f(y) < \frac{r\epsilon}{2}. \tag{4.14}$$

Now check that a local error bound holds locally at x. Let $y \in \mathcal{X} \setminus S$ with $\|y-x\| < \delta(x)$. From Lemma 4.1.1 and (4.13).

$$f(y + \frac{r}{2}h_x) - f(y) \le -(\frac{r}{2})\epsilon = -\frac{r\epsilon}{2}.$$

It follows from (4.14) that

$$f(y + \frac{r}{2}h_x) \le f(y) - \frac{r\epsilon}{2} < \frac{r\epsilon}{2} - \frac{r\epsilon}{2} = 0 < f(y).$$

Hence by Intermediate Value Theorem there exists $\lambda_y \in (0, \frac{r}{2})$ such that $f(y + \lambda_y h_x) = 0$. Again, by Lemma 4.1.1,

$$f(y + \lambda_y h_x) - f(y) \le -\lambda_y \epsilon,$$

and hence

$$f(y) \ge \lambda_y \epsilon = \epsilon \|y - (y + \lambda_y h_x)\| \ge \epsilon dist(y, S).$$

Therefore, for each $y \in \mathcal{X}$ with $\|y-x\| < \delta(x)$, $dist(y, S) \le \frac{1}{\epsilon}[f(y)]_+$. (B) of Theorem 4.2.1 holds and thus the result follows. □

4.3 Error Bounds with Fractional Exponents

In the previous sections, a proper function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ on a normed space $\mathcal{X}$ is said to have an error bound if there exists $\tau > 0$ such that for each $x \in \mathcal{X}$, $dist(x, S) \le \tau[f(x)]_+$, where $S := \{x \in \mathcal{X} | f(x) \le 0\}$. Now we introduce the concept of an error bound with exponent other than 1. Researches on this field have been brought out by K. F. Ng and X. Y. Zheng in [11]. The results reported in the following two sections are originally from [11]. In this section, we will present some sufficient conditions for error bounds with exponents to hold. The definition that an error bound with exponent holds is first stated as follows.

Definition 4.3.1 *Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Assume that the set*

$$S := \{x \in \mathcal{X} | f(x) \leq 0\}$$

is nonempty. Let $\gamma > 0$ be a constant. f is said to have an error bound with exponent γ if there exists a positive constant τ such that for each $x \in \mathcal{X}$,

$$dist(x, S) \leq \tau([f(x)]_+)^\gamma,$$

where $dist(x, S) := \inf\{\|x - y\| | y \in S\}$ and $[f(x)]_+ = \max\{f(x), 0\}$.

Recall that in this chapter we always assume that S is nonempty. Again obviously, without loss of generality, we can show an error bound with exponent γ holds for f by just checking whether $dist(x, S) \leq \tau[f(x)]^\gamma$ is true for each $x \in dom(f) \setminus S$ (that is, $0 < f(x) < \infty$).

We use the following notation for convenience.

Definition 4.3.2 *Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then*

$$Ex(f) := \{\gamma > 0 | f \text{ has an error bound with the exponent } \gamma\}.$$

Now we shall study the proposition below.

Proposition 4.3.1 *Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then $Ex(f)$ is an interval if $Ex(f)$ is nonempty.*

Proof: Suppose γ_1, γ_2 both in $Ex(f)$. Without loss of generality we let $\gamma_1 \leq \gamma_2$. There exist $\tau_1 > 0$ and $\tau_2 > 0$ such that for each $x \in \mathcal{X}\backslash S$, $dist(x, S) \leq \tau_i[f(x)]^{\gamma_i}$, $i = 1, 2$; hence

$$dist(x, S) \leq \max\{\tau_1, \tau_2\} \min\{[f(x)]^{\gamma_1}, [f(x)]^{\gamma_2}\}.$$

Note that for any $a > 0$ and for any $\gamma \in [\gamma_1, \gamma_2]$, $\min\{a^{\gamma_1}, a^{\gamma_2}\} \leq a^{\gamma}$. Consequently we have for each $\gamma \in [\gamma_1, \gamma_2]$,

$$dist(x, S) \leq \max\{\tau_1, \tau_2\}[f(x)]^{\gamma},$$

and thus this proposition follows. □

Let $\mathcal{X}$ be a topological space. A partial order "$\preceq$" in $\mathcal{X}$ is said to be closed if for each $x \in \mathcal{X}$, the subset $\{y \in \mathcal{X} | y \preceq x\}$ is closed. Also recall that for a subset A of $\mathcal{X}$ a point a_0 is said to be a minimal element of A with respect to $\preceq$ if for any $a \in A$, $a = a_0$ whenever $a \preceq a_0$. The following lemma follows from Zorn's Lemma.

Lemma 4.3.2 *Let $\mathcal{X}$ be a topological space equipped with a closed partial order "$\preceq$" and A be a compact subset of $\mathcal{X}$. Then for each $a \in A$, there is a minimal element a_0 of A with respect to $\preceq$ such that $a_0 \preceq a$.*

Proof: Let Σ be the family of all totally ordered subsets of A. Take $a \in A$. Define $\Sigma_a := \{D \in \Sigma | x \preceq a \text{ for each } x \in D\}$. By Zorn's Lemma, Σ_a has a maximal element D_0 with respect to the partial order of set inclusion. Note that the family $\{\{y \in \mathcal{X} | y \preceq x\} \cap A | x \in D_0\}$ has the finite intersection property. Since A is compact and $\{y \in \mathcal{X} | y \preceq x\}$ is closed for each $x \in D_0$, one has

$$\bigcap_{x \in D_0} (\{y \in \mathcal{X} | y \preceq x\} \cap A) \neq \emptyset.$$

Let $a_0 \in A \cap \bigcap_{x \in D_0}\{y \in \mathcal{X} | y \preceq x\}$. Therefore $a_0 \preceq x$ for each $x \in D_0$, in particular, $a_0 \preceq a$. Now we aim to show that a_0 is a minimal element of A with respect to $\preceq$. Suppose to the contrary, there is $a_1 \in A$ with $a_1 \preceq a_0$ and $a_1 \neq a_0$. Then we define $D_1 = D_0 \cup \{a_1\}$. Clearly $D_1 \in \Sigma_a$ and thus D_0 is no longer a maximal element in Σ_a with respect to the set inclusion. It contradicts the construction of the set D_0. As a result, a_0 is a minimal element of A with respect to $\preceq$ with $a_0 \preceq a$. □

Lemma 4.3.3 *Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper weakly lower semicontinuous function. Let $\gamma > 0$ and $\tau > 0$ be constants. Define a binary relation "$\preceq$" in $\mathcal{X}$ as follows: for any x, $y \in \mathcal{X}$,*

$$y \preceq x \qquad \text{if and only if} \qquad \|x - y\| \leq \tau(([f(x)]_+)^\gamma - ([f(y)]_+)^\gamma).$$

Then $\preceq$ is a weakly closed partial order in $\mathcal{X}$.

Proof: Clearly $\preceq$ is reflexive and antisymmetric. And the transitivity follows from the triangle inequality of norm $\|\cdot\|$. Now fix $x \in \mathcal{X}$, it is sufficient to show that $\{y \in \mathcal{X} | y \preceq x\}$ is closed with respect to the weak topology of $\mathcal{X}$. Note that

$$\{y \in \mathcal{X} | y \preceq x\} = \{y \in \mathcal{X} | \tau([f(y)]_+)^\gamma + \|x - y\| \leq \tau([f(x)]_+)^\gamma\}. \tag{4.15}$$

Since $\tau([f(\cdot)]_+)^\gamma$ and $\|x - \cdot\|$ are both weakly lower semicontinuous, $\tau([f(\cdot)]_+)^\gamma + \|x - \cdot\|$ is also weakly lower semicontinuous. This and (4.15) imply that $\{y \in \mathcal{X} | y \preceq x\}$ is weakly closed. □

The following is the main result of this section.

Theorem 4.3.4 *Let $\mathcal{X}$ be a reflexive Banach space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper weakly lower semicontinuous function. Let $\gamma > 0$ and $\tau > 0$ be constants. Suppose that for each $x \in dom(f) \setminus S$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathcal{X}$ such that*

1. $\|x_n - x\| \to 0$ *and*

2. *for sufficiently large* n, $0 < \|x - x_n\| \leq \tau([f(x)]^\gamma - [f(x_n)]^\gamma)$.

Then, for each $x \in \mathcal{X}$, $dist(x, S) \leq \tau([f(x)]_+)^\gamma$.

Proof: Define a binary relation "$\preceq$" in $\mathcal{X}$ the same as in Lemma 4.3.3. Therefore $\preceq$ is a weakly closed partial order in $\mathcal{X}$. Take $x \in dom(f) \setminus S$. Note that f is weakly lower semicontinuous, so S is closed and thus $dist(x, S) > 0$. Let $r \in (0, dist(x, S))$. Clearly

$$B(x, r) \cap S = \emptyset,$$

where $B(x, r) = \{y \in \mathcal{X} | \|y - x\| \leq r\}$. By the reflexivity of $\mathcal{X}$, $B(x, r)$ is weakly compact. Recall that $\preceq$ is weakly closed. In the view of Lemma 4.3.2, these imply that there is a minimal element x_0 of $B(x, r)$ such that $x_0 \preceq x$. Now we want to claim that $\|x - x_0\| = r$. Suppose to the contrary, that is, $\|x - x_0\| < r$, hence $r - \|x - x_0\| > 0$. Note that $x_0 \notin S$; since $x_0 \preceq x$ and $x \in dom(f)$, $x_0 \in dom(f)$. Therefore $x_0 \in dom(f) \setminus S$. Following from the assumptions in the statement of Theorem 4.3.4, there is $y_0 \in \mathcal{X}$ such that

$$0 < \|x_0 - y_0\| < r - \|x - x_0\| \tag{4.16}$$

and

$$0 < \|x_0 - y_0\| \leq \tau([f(x_0)]^\gamma - [f(y_0)]^\gamma). \tag{4.17}$$

From (4.16), $y_0 \neq x_0$ and $y_0 \in B(x, r)$. From (4.17), $y_0 \preceq x_0$. These contradict the fact that x_0 is a minimal element of $B(x, r)$. As a result, $\|x - x_0\| = r$. Furthermore, as $x_0 \preceq x$,

$$r = \|x_0 - x\| \leq \tau([f(x)]^\gamma - [f(x_0)]^\gamma) \leq \tau[f(x)]^\gamma.$$

Letting $r \to dist(x, S)$, we have $dist(x, S) \leq \tau[f(x)]^\gamma$. Consequently the error bound with exponent γ holds. □

Since a finite dimensional normed space is reflexive and its weak topology coincides with the norm topology, the following corollary is obvious.

Corollary 4.3.5 *Let $\mathcal{X}$ be a finite dimensional normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $\gamma > 0$ and $\tau > 0$ be constants. Suppose that for each $x \in dom(f) \setminus S$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that*

1. *$\|x_n - x\| \to 0$ and*
2. *for sufficiently large n, $0 < \|x - x_n\| \leq \tau([f(x)]^\gamma - [f(x_n)]^\gamma)$.*

Then, for each $x \in \mathcal{X}$, $dist(x, S) \leq \tau([f(x)]_+)^\gamma$.

The following corollary follows from the fact that every lower semicontinuous convex function is weakly lower semicontinuous.

Corollary 4.3.6 *Let $\mathcal{X}$ be a reflexive Banach space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let $\gamma > 0$ and $\tau > 0$ be constants. Suppose that for each $x \in dom(f) \setminus S$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that*

1. *$\|x_n - x\| \to 0$ and*
2. *for sufficiently large n, $0 < \|x - x_n\| \leq \tau([f(x)]^\gamma - [f(x_n)]^\gamma)$.*

Then, for each $x \in \mathcal{X}$, $dist(x, S) \leq \tau([f(x)]_+)^\gamma$.

For a proper lower semicontinuous convex function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, let $f^* : \mathcal{X}^* \to \mathbb{R} \cup \{+\infty\}$ be its conjugate function, that is, for each $x^* \in \mathcal{X}^*$,

$$f^*(x^*) = \sup\{< x^*, x > -f(x) | x \in \mathcal{X}\},$$

where $\mathcal{X}^*$ is the dual space of $\mathcal{X}$. As the conjugate problems will often be considered in some mathematical programming problems, it is worthy to study its error bound problem.

Theorem 4.3.7 *Let $\mathcal{X}$ be a normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Assume that the set*

$$S^* := \{x^* \in \mathcal{X}^* | f^*(x^*) \leq 0\}$$

is nonempty. Let $\gamma > 0$ and $\tau > 0$ be constants. Suppose that for each $x^* \in dom(f^*) \setminus S^*$, there exists a sequence $\{x_n^*\}_{n\in\mathbb{N}}$ in $\mathcal{X}^*$ such that

1. $\|x_n^* - x^*\| \to 0$ *and*

2. *for sufficiently large n, $0 < \|x^* - x_n^*\| \leq \tau([f^*(x^*)]^\gamma - [f^*(x_n^*)]^\gamma)$.*

Then, for each $x^* \in \mathcal{X}^*$, $dist(x^*, S^*) \leq \tau([f^*(x^*)]_+)^\gamma$.

Proof: Note that the conjugate function f^* is weak-* lower semicontinuous. The reason is: For each $a \in \mathbb{R}$,

$$\begin{aligned}\{p \in \mathcal{X}^* | f^*(p) \leq a\} &= \{p \in \mathcal{X}^* | \sup_{x\in\mathcal{X}}\{p(x) - f(x)\} \leq a\} \\ &= \bigcap_{x\in\mathcal{X}} \{p \in \mathcal{X}^* | p(x) - f(x) \leq a\}.\end{aligned}$$

Since $\{p \in \mathcal{X}^* | p(x) \leq a + f(x)\}$ is weak-*-closed for each $x \in \mathcal{X}$, f^* is weak-* lower semicontinuous. Additionally, every bounded weak-*-closed subset of $\mathcal{X}^*$ is weak-*-compact. Therefore the theorem follows by repeating the argument in proof of Theorem 4.3.4 with weak-* topology of $\mathcal{X}^*$ in place of weak topology of $\mathcal{X}$. □

Given a function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, $x \in \mathcal{X}$ and $h \in \mathcal{X}$ with $\|h\| = 1$. f is said to be directionally continuous in h at x if $\lim_{t\to 0^+} f(x + th) = f(x)$.

Theorem 4.3.8 *Let $\mathcal{X}$ be a reflexive Banach space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper weakly lower semicontinuous function. Let $\gamma > 0$ and $\delta > 0$ be constants.*

Suppose that for each $x \in dom(f) \setminus S$, there exists $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that f is directionally continuous in h_x at x, that is,

$$\lim_{t \to 0^+} f(x + th_x) = f(x), \tag{4.18}$$

and

$$\underline{d}^+ f(x)(h_x) \le -\delta[f(x)]^{1-\gamma}. \tag{4.19}$$

Then, f has an error bound with exponent γ.

Proof: Let $x \in dom(f) \setminus S$. Take $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that (4.18) and (4.19) hold. (4.18) and that $0 < f(x) < +\infty$ imply that there exists $\lambda > 0$ such that for all $t \in [0, \lambda]$, $0 < f(x + th_x) < +\infty$. Then for t sufficiently small, as f is directionally continuous in h_x at x,

$$[f(x + th_x)]^\gamma - [f(x)]^\gamma = \gamma[f(x)]^{\gamma-1}[f(x + th_x) - f(x)] + o(f(x + th_x) - f(x)),$$

and thus

$$\frac{[f(x + th_x)]^\gamma - [f(x)]^\gamma}{t} = \left[\gamma[f(x)]^{\gamma-1} + \frac{o(f(x + th_x) - f(x))}{f(x + th_x) - f(x)}\right] \frac{f(x + th_x) - f(x)}{t},$$

where $\frac{o(f(x+th_x)-f(x))}{f(x+th_x)-f(x)} \to 0$ if $t \to 0$. Therefore we have

$$\begin{aligned}
\liminf_{t \to 0^+} \frac{[f(x + th_x)]^\gamma - [f(x)]^\gamma}{t} &= \gamma[f(x)]^{\gamma-1} \liminf_{t \to 0^+} \frac{f(x + th_x) - f(x)}{t} \\
&= \gamma[f(x)]^{\gamma-1} \underline{d}^+ f(x)(h_x).
\end{aligned}$$

By (4.19),

$$\liminf_{t \to 0^+} \frac{[f(x + th_x)]^\gamma - [f(x)]^\gamma}{t} \le -\gamma\delta < -\frac{\gamma\delta}{2}.$$

Hence there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$, converging to 0, such that for each $n \in \mathbb{N}$, $t_n > 0$ and

$$\frac{[f(x + t_n h_x)]^\gamma - [f(x)]^\gamma}{t_n} \le -\frac{\gamma\delta}{2}.$$

Let $x_n = x + t_n h_x$, so $\|x_n - x\| \to 0$ and for each $n \in \mathbb{N}$,

$$0 < \|x - x_n\| = t_n \leq \frac{2}{\gamma\delta}([f(x)]^\gamma - [f(x_n)]^\gamma).$$

By Theorem 4.3.4, for each $x \in \mathcal{X}$,

$$dist(x, S) \leq \frac{2}{\gamma\delta}([f(x)]_+)^\gamma.$$

□

The following two corollaries imply that if the normed space $\mathcal{X}$ is finite dimensional or convexity of f is assumed, the condition that f is weakly lower semicontinuous can be relaxed. The first one is due to the fact that a finite dimensional normed space is reflexive and its weak topology is exactly its norm topology.

Corollary 4.3.9 *Let $\mathcal{X}$ be a finite dimensional normed space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $\gamma > 0$ and $\delta > 0$ be constants. Suppose that for each $x \in dom(f) \setminus S$, there exists $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that f is directionally continuous in h_x at x and $\underline{d}^+ f(x)(h_x) \leq -\delta[f(x)]^{1-\gamma}$. Then f has an error bound with exponent γ.*

Corollary 4.3.10 *Let $\mathcal{X}$ be a reflexive Banach space and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let $\gamma > 0$ and $\delta > 0$ be constants. Suppose that for each $x \in dom(f) \setminus S$, there exists $h_x \in \mathcal{X}$ with $\|h_x\| = 1$ such that $\underline{d}^+ f(x)(h_x) \leq -\delta[f(x)]^{1-\gamma}$. Then f has an error bound with exponent γ.*

Proof: Since f is lower semicontinuous convex, it is weakly lower semicontinuous. In view of Theorem 4.3.8, for any fixed $x \in dom(f) \setminus S$, it suffices to show that f is directionally continuous in h_x at x. Now $\underline{d}^+ f(x)(h_x) \leq -\delta[f(x)]^{1-\gamma} < 0$, there is $\lambda_0 > 0$ such that $\frac{f(x+\lambda_0 h_x)-f(x)}{\lambda_0} < 0$, so $x + \lambda_0 h_x \in dom(f)$. As f is convex, for each $t \in [0, \lambda_0]$,

$$f(x+th_x) = f[(1-\frac{t}{\lambda_0})x + \frac{t}{\lambda_0}(x+\lambda_0 h_x)] \leq (1-\frac{t}{\lambda_0})f(x) + \frac{t}{\lambda_0}f(x+\lambda_0 h_x).$$

Therefore,

$$\limsup_{t\to 0^+} f(x+th_x) \leq f(x).$$

Hence it follows from the lower semicontinuity of f that f is directionally continuous in h_x at x and the proof is complete. □

4.4 An Application to Quadratic Functions

In this section we use the results before to investigate the exponents the error bounds of a quadratic function on $\mathbb{R}^n$ with. Let us consider a general quadratic function f with

$$f(x) = x^T Q x + b^T x + c, \quad x \in \mathbb{R}^n, \tag{4.20}$$

where Q is a $n \times n$ symmetric matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$; y^T denotes the transpose of $y \in \mathbb{R}^n$. Let $ker(Q) = \{x \in \mathbb{R}^n | Qx = 0\}$ and $R(Q) = \{Qx | x \in \mathbb{R}^n\}$. We have $ker(Q) \perp R(Q)$ and $\mathbb{R}^n = ker(Q) + R(Q)$.

Furthermore, for sake of convenience, we introduce the following notations. We let

$$\mathcal{X}_1 = span(\{x \in \mathbb{R}^n | Qx = \alpha x \text{ for some } \alpha > 0\}),$$

$$\mathcal{X}_2 = span(\{x \in \mathbb{R}^n | Qx = \alpha x \text{ for some } \alpha < 0\}),$$

and

$$\mathcal{X}_3 = ker(Q).$$

Then we have $\mathcal{X}_i \perp \mathcal{X}_j$, for $i, j \in \{1, 2, 3\}$ with $i \neq j$; $Q\mathcal{X}_i = \mathcal{X}_i$, for $i = 1, 2$; $\mathbb{R}^n = \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3$ and for each $x \in \mathbb{R}^n$, there are unique $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$, such that $x = x_1 + x_2 + x_3$. And then for each $x \in \mathbb{R}^n$, we define a norm $\|\cdot\|_e$ by

$$\|x\|_e = (x_1^T Q x_1 - x_2^T Q x_2 + \|x_3\|^2)^{\frac{1}{2}}.$$

Hence,

$$\|x\|_e = (\|x_1\|_e^2 + \|x_2\|_e^2 + \|x_3\|^2)^{\frac{1}{2}}. \tag{4.21}$$

Note that $\|\cdot\|_e$ is an equivalent norm on $\mathbb{R}^n$. In this section we will make use of $\|\cdot\|_e$ instead of the original metric to show the error bound result for f, that is, showing the error bound hold with respect to $\|\cdot\|_e$. All the notations above will be used throughout this section.

In the first part of this section, we discuss the set $Ex(f)$ (defined in Definition 4.3.2 in page 106) of the quadratic function stated in (4.20) under the condition that $b \perp ker(Q)$

Consider an equation:

$$2Qx = -b. \tag{4.22}$$

Clearly it has a solution if $b \perp ker(Q)$ (as $b \in R(Q)$). Now we prove the following lemma.

Lemma 4.4.1 *For any two solutions θ_1 and θ_2 of (4.22), $f(\theta_1) = f(\theta_2)$.*

Proof: Now $2Q\theta_i = -b$, $i = 1, 2$. Let $e = \theta_2 - \theta_1$, then $Qe = 0$. Therefore,

$$\begin{aligned} f(\theta_1) &= \theta_1^T Q \theta_1 + b^T \theta_1 + c = \theta_1^T Q \theta_1 - 2\theta_1^T Q \theta_1 + c \\ &= -\theta_1^T Q \theta_1 + c = -(\theta_1 + e)^T Q (\theta_1 + e) + c \\ &= -\theta_2^T Q \theta_2 + c = f(\theta_2). \end{aligned}$$

□

Lemma 4.4.1 tells us that the constant r, $r := f(\theta)$ for some solution θ of (4.22), is well defined and independent of the choice of θ. Let θ be a solution of (4.22). Now we rewrite (4.20) as follows:

$$\begin{aligned} f(x) &= x^TQx + b^Tx + c \\ &= x^TQx + b^T(x-\theta) + \theta^TQ\theta - \theta^TQ\theta + b^T\theta + c \\ &= x^TQx - 2\theta^TQ(x-\theta) + \theta^TQ\theta - \theta^TQ\theta + b^T\theta + c \\ &= (x-\theta)^TQ(x-\theta) + 2\theta^TQ\theta - \theta^TQ\theta + b^T\theta + c \\ &= (x-\theta)^TQ(x-\theta) + f(\theta). \end{aligned}$$

Since we are considering the error bound problem, without loss of generality, instead of (4.20), we can consider the quadratic function f as

$$f(x) = x^TQx + r \tag{4.23}$$

where r is well-defined in view of Lemma 4.4.1. Furthermore, in virtue of $\|\cdot\|_e$ defined before, (4.23) can be further rewritten as

$$f(x) = \|x_1\|_e^2 - \|x_2\|_e^2 + r. \tag{4.24}$$

Now we turn to present the error bound results under the circumstance that $b \perp ker(Q)$. The following discussion is based on usage of (4.21) and (4.24). First of all, several lemmas are presented.

Lemma 4.4.2 *Suppose that Q has both positive and negative eigenvalues and $b \perp ker(Q)$. Then $Ex(f) \subseteq [\frac{1}{2}, +\infty)$.*

Proof: Pick α a positive eigenvalue of Q and let h_α be a corresponding eigenvector with $\|h_\alpha\|_e = 1$. Let $\mathcal{X}_1' = \{x_1 \in \mathcal{X}_1 | x_1 \perp h_\alpha\}$. Then $\mathcal{X}_1 = \mathcal{X}_1' + \mathbb{R}h_\alpha$ and

$Q\mathcal{X}_1' = \mathcal{X}_1'$. By (4.24), for any $x \in \mathbb{R}^n$ with $x = x_1 + x_2 + x_3$, $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$,

$$f(x) = \|x_1\|_e^2 - \|x_2\|_e^2 + r,$$

where $r := f(\theta)$ for some solution θ of (4.22). Consider $th_\alpha \in \mathbb{R}^n$ with $t > |r|^{\frac{1}{2}}$, then $f(th_\alpha) = t^2 + r > 0$. Moreover,

$$\begin{aligned}
& dist(th_\alpha, S) \\
= \; & dist(th_\alpha, \partial S) \\
= \; & \inf\{\left[\|x_1 - th_\alpha\|_e^2 + \|x_2\|_e^2 + \|x_3\|^2\right]^{\frac{1}{2}} : \|x_1\|_e^2 - \|x_2\|_e^2 + r = 0, \\
& \qquad x_i \in \mathcal{X}_i, i = 1, 2, 3\} \\
= \; & \inf\{\left[\|x_1 - th_\alpha\|_e^2 + \|x_1\|_e^2 + r\right]^{\frac{1}{2}} : \|x_1\|_e^2 + r \geq 0, x_1 \in \mathcal{X}_1\} \\
= \; & \inf\{\left[\|x_1' + \lambda h_\alpha - th_\alpha\|_e^2 + \|x_1' + \lambda h_\alpha\|_e^2 + r\right]^{\frac{1}{2}} : \|x_1' + \lambda h_\alpha\|_e^2 + r \geq 0, \\
& \qquad \lambda \in \mathbb{R}, x_1' \in \mathcal{X}_1'\} \\
= \; & \inf\{\left[2\|x_1'\|_e^2 + (\lambda - t)^2 + \lambda^2 + r\right]^{\frac{1}{2}} : \|x_1'\|_e^2 + \lambda^2 + r \geq 0, \\
& \qquad \lambda \in \mathbb{R}, x_1' \in \mathcal{X}_1'\}. \quad (4.25)
\end{aligned}$$

Let $g(\lambda, x_1') = 2\|x_1'\|_e^2 + (\lambda - t)^2 + \lambda^2 + r$ with $(\lambda, x_1') \in \mathbb{R} \times \mathcal{X}_1'$. We have

$$\frac{\partial g}{\partial \lambda} = 2(\lambda - t) + 2\lambda = 4\lambda - 2t \qquad (4.26)$$

and

$$\frac{\partial g}{\partial x_1'} = 4x_1'^T Q. \qquad (4.27)$$

In order to have minimal g, $\lambda = \frac{t}{2}$ and $x_1' = 0$. Now for sufficiently large t, $(\frac{t}{2}, 0) \in \{(\lambda, x_1') | \|x_1'\|_e^2 + \lambda^2 + r \geq 0, \lambda \in \mathbb{R}, x_1' \in \mathcal{X}_1'\}$, so $dist(th_\alpha, S) = [(\frac{t}{2} - t)^2 + (\frac{t}{2})^2 + r]^{\frac{1}{2}} = (\frac{t^2}{2} + r)^{\frac{1}{2}}$. Hence

$$\lim_{t \to +\infty} \frac{dist(th_\alpha, S)}{\sqrt{f(th_\alpha)}} = \lim_{t \to +\infty} \frac{(\frac{t^2}{2} + r)^{\frac{1}{2}}}{(t^2 + r)^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}.$$

This implies that $Ex(f) \subseteq [\frac{1}{2}, +\infty)$. It is because if $0 < \gamma < \frac{1}{2}$, $\lim_{t \to +\infty} \frac{dist(th_\alpha, S)}{[f(th_\alpha)]^\gamma} = +\infty$. This tells us that for any $\tau > 0$ given, we can find $t_0 > 0$ such that

$$dist(t_0 h_\alpha, S) > \tau[f(t_0 h_\alpha)]^\gamma.$$

□

Lemma 4.4.3 *Suppose that Q has both positive and negative eigenvalues, $b \perp ker(Q)$ and $r := f(\theta) = 0$ for some solution θ of (4.22). Then $Ex(f) \subseteq \{\frac{1}{2}\}$.*

Proof: In view of the proof of Lemma 4.4.2, by (4.25), (4.26) and (4.27), for any $t > 0$, $f(th_\alpha) = t^2 > 0$ and $dist(th_\alpha, S) = \frac{t}{\sqrt{2}}$. Hence, for each $t > 0$

$$dist(th_\alpha, S) = \frac{1}{\sqrt{2}}[f(th_\alpha)]^{\frac{1}{2}}.$$

This implies that $Ex(f) \subseteq \{\frac{1}{2}\}$ (Because if $0 < \gamma < \frac{1}{2}$, $\lim_{t\to+\infty} \frac{dist(th_\alpha,S)}{[f(th_\alpha)]^\gamma} = +\infty$ and if $\gamma > \frac{1}{2}$, $\lim_{t\to 0^+} \frac{dist(th_\alpha,S)}{[f(th_\alpha)]^\gamma} = +\infty$). □

Lemma 4.4.4 *Suppose that Q has both positive and negative eigenvalues, $b \perp ker(Q)$ and $r := f(\theta) < 0$ for some solution θ of (4.22). Then $Ex(f) \subseteq [\frac{1}{2}, 1]$.*

Proof: Again, let α, h_α $\mathcal{X}_1'$ be as in the proof of Lemma 4.4.2. Now for any $t > |r|^{\frac{1}{2}}$, we have $f(th_\alpha) = t^2 - |r|$. Following (4.25), for t sufficiently near $|r|^{\frac{1}{2}}$, note that $(\lambda, x_1') = (\frac{t}{2}, 0)$ is not in the set $\{(\lambda, x_1') | \|x_1'\|_e^2 + \lambda^2 - |r| \geq 0, \lambda \in \mathbb{R}, x_1' \in \mathcal{X}_1'\}$, therefore,

$$\begin{aligned}
& dist(th_\alpha, S) \\
= \ & \inf\{\big[2\|x_1'\|_e^2 + (\lambda - t)^2 + \lambda^2 - |r|\big]^{\frac{1}{2}} : \|x_1'\|_e^2 + \lambda^2 - |r| \geq 0, \lambda \in \mathbb{R}, x_1' \in \mathcal{X}_1'\} \\
= \ & \inf\{\big[2\|x_1'\|_e^2 + (\lambda - t)^2 + \lambda^2 - |r|\big]^{\frac{1}{2}} : \|x_1'\|_e^2 + \lambda^2 - |r| = 0, \lambda \in \mathbb{R}, x_1' \in \mathcal{X}_1'\} \\
= \ & \inf\{\big[2|r| - 2\lambda^2 + (\lambda - t)^2 + \lambda^2 - |r|\big]^{\frac{1}{2}} : |r| - \lambda^2 \geq 0, \lambda \in \mathbb{R}\} \\
= \ & \inf\{\big[(\lambda - t)^2 - \lambda^2 + |r|\big]^{\frac{1}{2}} : \lambda^2 \leq |r|, \lambda \in \mathbb{R}\} \\
= \ & \inf\{\big[-2\lambda t + t^2 + |r|\big]^{\frac{1}{2}} : \lambda^2 \leq |r|, \lambda \in \mathbb{R}\} \\
= \ & \big[t^2 - 2|r|^{\frac{1}{2}}t + |r|\big]^{\frac{1}{2}} \\
= \ & t - |r|^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$\lim_{t\to\sqrt{|r|}^+} \frac{dist(th_\alpha, S)}{f(th_\alpha)} = \lim_{t\to\sqrt{|r|}^+} \frac{t-|r|^{\frac{1}{2}}}{t^2-|r|} = \lim_{t\to\sqrt{|r|}^+} \frac{1}{t+\sqrt{|r|}} = \frac{1}{2\sqrt{|r|}},$$

and this implies that $Ex(f) \cap (1, +\infty) = \emptyset$. Combining with Lemma 4.4.2, one has $Ex(f) \subseteq [\frac{1}{2}, 1]$. □

Lemma 4.4.5 *Suppose that Q has both positive and negative eigenvalues, $b \perp ker(Q)$ and $r := f(\theta) > 0$ for some solution θ of (4.22). Then $Ex(f) \subseteq [\frac{1}{2}, 1]$.*

Proof: Pick α a negative eigenvalue of Q and let h_α be a corresponding eigenvector with $\|h_\alpha\|_e = 1$. Let $\mathcal{X}_2' = \{x_2 \in \mathcal{X}_2 | x_2 \perp h_\alpha\}$. Then $\mathcal{X}_2 = \mathcal{X}_2' + \mathbb{R}h_\alpha$ and $Q\mathcal{X}_2' = \mathcal{X}_2'$. By (4.24), for any $x \in \mathbb{R}^n$ with $x = x_1 + x_2 + x_3$, $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$,

$$f(x) = \|x_1\|_e^2 - \|x_2\|_e^2 + r.$$

Consider $th_\alpha \in \mathbb{R}^n$ with $t \in (0, \sqrt{r})$, then $f(th_\alpha) = -t^2 + r > 0$. Moreover, for each $t \in (0, \sqrt{r})$,

$$\begin{aligned}
& dist(th_\alpha, S) \\
= \; & dist(th_\alpha, \partial S) \\
= \; & \inf\{\big[\|x_1\|_e^2 + \|x_2 - th_\alpha\|_e^2 + \|x_3\|^2\big]^{\frac{1}{2}} : \|x_1\|_e^2 - \|x_2\|_e^2 + r = 0, \\
& \qquad x_i \in \mathcal{X}_i, i = 1, 2, 3\} \\
= \; & \inf\{\big[\|x_2\|_e^2 - r + \|x_2 - th_\alpha\|_e^2\big]^{\frac{1}{2}} : \|x_2\|_e^2 - r \geq 0, x_2 \in \mathcal{X}_2\} \\
= \; & \inf\{\big[\|x_2' + \lambda h_\alpha\|_e^2 - r + \|x_2' + \lambda h_\alpha - th_\alpha\|_e^2\big]^{\frac{1}{2}} : \|x_2' + \lambda h_\alpha\|_e^2 - r \geq 0, \\
& \qquad \lambda \in \mathbb{R}, x_2' \in \mathcal{X}_2'\} \\
= \; & \inf\{\big[2\|x_2'\|_e^2 + \lambda^2 - r + (\lambda - t)^2\big]^{\frac{1}{2}} : \|x_2'\|_e^2 + \lambda^2 \geq r, \lambda \in \mathbb{R}, x_2' \in \mathcal{X}_2'\} \\
= \; & \inf\{\big[2\|x_2'\|_e^2 + \lambda^2 - r + (\lambda - t)^2\big]^{\frac{1}{2}} : \|x_2'\|_e^2 + \lambda^2 = r, \lambda \in \mathbb{R}, x_2' \in \mathcal{X}_2'\} \\
= \; & \inf\{\big[-2\lambda t + t^2 + r\big]^{\frac{1}{2}} : r - \lambda^2 \geq 0, \lambda \in \mathbb{R}\} \\
= \; & \big[r - 2r^{\frac{1}{2}} t + t^2\big]^{\frac{1}{2}} \\
= \; & \sqrt{r} - t.
\end{aligned}$$

Hence,

$$\lim_{t \to \sqrt{r}^-} \frac{dist(th_\alpha, S)}{f(th_\alpha)} = \lim_{t \to \sqrt{r}^-} \frac{\sqrt{r} - t}{r - t^2} = \lim_{t \to \sqrt{r}^-} \frac{1}{\sqrt{r} + t} = \frac{1}{2\sqrt{r}}.$$

This implies that $Ex(f) \cap (1, +\infty) = \emptyset$. Combining with Lemma 4.4.2, one has $Ex(f) \subseteq [\frac{1}{2}, 1]$. □

Theorem 4.4.6 *Suppose that Q has both positive and negative eigenvalues, $b \perp ker(Q)$ and $r := f(\theta) = 0$ for some solution θ of (4.22). Then $Ex(f) = \{\frac{1}{2}\}$.*

Proof: Take any $x = x_1 + x_2 + x_3 \in \mathbb{R}^n \setminus S$, with $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$. By (4.24),

$$f(x) = \|x_1\|_e^2 - \|x_2\|_e^2 > 0,$$

hence $\|x_1\|_e \geq [f(x)]^{\frac{1}{2}} > 0$. Pick $h_x = \frac{-x_1}{\|x_1\|_e}$. Then

$$df(x)(h_x) = 2(x_1^T - x_2^T)Qh_x = -2\|x_1\|_e \leq -2[f(x)]^{\frac{1}{2}}.$$

By Corollary 4.3.9 and Lemma 4.4.3, $Ex(f) = \{\frac{1}{2}\}$. □

Theorem 4.4.7 *Suppose that Q has both positive and negative eigenvalues, $b \perp ker(Q)$ and $r := f(\theta) < 0$ for some solution θ of (4.22). Then $Ex(f) = [\frac{1}{2}, 1]$.*

Proof: Take any $x = x_1 + x_2 + x_3 \in \mathbb{R}^n \setminus S$, with $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$. By (4.24),

$$f(x) = \|x_1\|_e^2 - \|x_2\|_e^2 - |r| > 0;$$

hence $\|x_1\|_e > [f(x)]^{\frac{1}{2}} > 0$ and $\|x_1\|_e > |r|^{\frac{1}{2}} > 0$. Then for any $\gamma \in [\frac{1}{2}, 1]$, as $(2\gamma - 1) \in [0, 1]$,

$$\|x_1\|_e \geq |r|^{\frac{1}{2}(2\gamma-1)}[f(x)]^{\frac{1}{2}[1-(2\gamma-1)]} = |r|^{\gamma-\frac{1}{2}}[f(x)]^{1-\gamma} > 0. \qquad (4.28)$$

Pick $h_x = \frac{-x_1}{\|x_1\|_e}$. Then by (4.28),

$$df(x)(h_x) = 2(x_1^T - x_2^T)Qh_x = -2\|x_1\|_e \leq (-2|r|^{\gamma-\frac{1}{2}})[f(x)]^{1-\gamma}.$$

By Corollary 4.3.9 and Lemma 4.4.4, $Ex(f) = [\frac{1}{2}, 1]$. □

Theorem 4.4.8 *Suppose that Q has both positive and negative eigenvalues, $b \perp ker(Q)$ and $r := f(\theta) > 0$ for some solution θ of (4.22). Then $Ex(f) = [\frac{1}{2}, 1]$.*

Proof: Let $\gamma \in [\frac{1}{2}, 1]$. By Corollary 4.3.9 and Lemma 4.4.5, it suffices to show that $\gamma \in Ex(f)$ to complete the proof. Now take any $x = x_1 + x_2 + x_3 \in \mathbb{R}^n \setminus S$, with $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$.

Firstly, we deal with the case that $\|x_2\|_e \le \frac{\sqrt{r}}{2}$. We have $f(x) = \|x_1\|_e^2 - \|x_2\|_e^2 + r \ge \|x_1\|_e^2 + \frac{3}{4}r$. Let $y_2 \in \mathcal{X}_2$ with $\|y_2\|_e = \sqrt{r}$. Then $f(-y_2 + x_3) = 0$; hence

$$\begin{aligned} dist(x,S) &\le \|x - (-y_2 + x_3)\|_e \\ &= \|x_1 + (x_2 + y_2)\|_e \\ &= (\|x_1\|_e^2 + \|x_2 + y_2)\|_e^2)^{\frac{1}{2}} \\ &\le (\|x_1\|_e^2 + \frac{9}{4}r)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\frac{dist(x,S)}{[f(x)]^\gamma} \le \frac{(\|x_1\|_e^2 + \frac{9}{4}r)^{\frac{1}{2}}}{[\|x_1\|_e^2 + \frac{3}{4}r]^\gamma}.$$

Since $\gamma \ge \frac{1}{2}$, the right hand side is bounded; therefore there exists $\tau_1 > 0$ such that for all $x \in \mathbb{R}^n \setminus S$ with $\|x_2\|_e \le \frac{\sqrt{r}}{2}$, $dist(x,S) \le \tau_1[f(x)]^\gamma$.

Secondly, we consider the case that $\|x_2\|_e > \frac{\sqrt{r}}{2}$ and $x_1 = 0$. Pick $h_x = \frac{x_2}{\|x_2\|_e}$. So for each $t > 0$ with

$$f(x + th_x) = -(\|x_2\|_e + t)^2 + r > 0, \tag{4.29}$$

we have

$$\begin{aligned} df^\gamma(x + th_x)(h_x) &= \gamma[f(x+th_x)]^{\gamma-1}\nabla f(x+th_x)(h_x) \\ &= \gamma[f(x+th_x)]^{\gamma-1}[-2(x_2+th_x)^TQh_x] \\ &= -2\gamma[f(x+th_x)]^{\gamma-1}(\|x_2\|_e + t) \\ &= \frac{-2\gamma(\|x_2\|_e + t)}{[-(\|x_2\|_e + t)^2 + r]^{1-\gamma}}. \end{aligned}$$

By (4.29), $\|x_2\|_e + t < \sqrt{r}$; also $\|x_2\|_e + t > \frac{\sqrt{r}}{2}$; hence,

$$df^\gamma(x + th_x)(h_x) \le -2\gamma\frac{\frac{\sqrt{r}}{2}}{r^{1-\gamma}} = -\gamma r^{\gamma - \frac{1}{2}} \le -\frac{1}{\tau_2},$$

for some constant $\tau_2 > 0$. It follows from Mean Value Theorem that there exists $t_0 > 0$ such that

$$f(x + t_0 h_x) = 0,$$

$$f(x+th_x)>0, \quad \text{for each } t\in[0,t_0),$$

and there exists $\lambda_0 \in (0,t_0)$ such that

$$-[f(x)]^\gamma = [f(x+t_0h_x)]^\gamma - [f(x)]^\gamma = t_0 df(x+\lambda_0 h_x)(h_x) \leq -\frac{t_0}{\tau_2}.$$

Consequently,

$$dist(x,S) \leq \|x-(x+t_0h_x)\|_e = t_0 \leq \tau_2[f(x)]^\gamma$$

is valid for all $x=x_1+x_2+x_3 \in \mathbb{R}^n \setminus S$ with $\|x_2\|_e > \frac{\sqrt{r}}{2}$ and $x_1=0$.

Thirdly, we consider that $\|x_2\|_e > \frac{\sqrt{r}}{2}$ and $x_1 \neq 0$. Pick $h_x = \frac{1}{\sqrt{2}}(-\frac{x_1}{\|x_1\|_e} + \frac{x_2}{\|x_2\|_e})$. So for each $t \in (0, \sqrt{2}\|x_1\|_e]$ with

$$f(x+th_x) = (\|x_1\|_e - \frac{t}{\sqrt{2}})^2 - (\|x_2\|_e + \frac{t}{\sqrt{2}})^2 + r > 0,$$

we have

$$\begin{aligned} df^\gamma(x+th_x)(h_x) &= \gamma[f(x+th_x)]^{\gamma-1}\nabla f(x+th_x)(h_x) \\ &= 2\gamma[f(x+th_x)]^{\gamma-1}[(x_1 - \frac{tx_1}{\sqrt{2}\|x_1\|_e}) - (x_2 + \frac{tx_2}{\sqrt{2}\|x_2\|_e})]^T Q h_x \\ &= \sqrt{2}\gamma[f(x+th_x)]^{\gamma-1}[-\|x_1\|_e + \frac{t}{\sqrt{2}} - \|x_2\|_e - \frac{t}{\sqrt{2}}] \\ &= \frac{\sqrt{2}\gamma(-\|x_1\|_e + \frac{t}{\sqrt{2}} - \|x_2\|_e - \frac{t}{\sqrt{2}})}{[(\|x_1\|_e - \frac{t}{\sqrt{2}})^2 - (\|x_2\|_e + \frac{t}{\sqrt{2}})^2 + r]^{1-\gamma}}. \end{aligned}$$

Together with $\|x_2\|_e + \frac{t}{\sqrt{2}} > \frac{\sqrt{r}}{2}$ and $\|x_1\|_e \geq \frac{t}{\sqrt{2}}$, we have

$$\begin{aligned} df^\gamma(x+th_x)(h_x) &= \frac{-\sqrt{2}\gamma(\|x_1\|_e - \frac{t}{\sqrt{2}} + \|x_2\|_e + \frac{t}{\sqrt{2}})}{[(\|x_1\|_e - \frac{t}{\sqrt{2}})^2 - (\|x_2\|_e + \frac{t}{\sqrt{2}})^2 + r]^{1-\gamma}} \\ &\leq \frac{-\sqrt{2}\gamma[(\|x_1\|_e - \frac{t}{\sqrt{2}}) + \frac{\sqrt{r}}{2}]}{[(\|x_1\|_e - \frac{t}{\sqrt{2}})^2 + r]^{1-\gamma}}. \end{aligned} \tag{4.30}$$

The right hand side of (4.30) is bounded as $1-\gamma \leq \frac{1}{2}$. Therefore there exists $\tau_3 > 0$ such that for each $t \in (0, \sqrt{2}\|x_1\|_e]$ with $f(x+th_x) > 0$,

$$df^\gamma(x+th_x)(h_x) \leq -\frac{1}{\tau_3}.$$

We consider two cases:

1. There exists $t_0 \in (0, \sqrt{2}\|x_1\|_e]$ such that $f(x + t_0 h_x) = 0$: Again from Mean Value Theorem, $f(x + th_x) > 0$ for each $t \in [0, t_0)$ and there exists $\lambda_0 \in (0, t_0)$ such that

$$-[f(x)]^\gamma = [f(x + t_0 h_x)]^\gamma - [f(x)]^\gamma = t_0 df(x + \lambda_0 h_x)(h_x) \leq -\frac{t_0}{\tau_3}.$$

Hence,

$$dist(x, S) \leq \|x - (x + t_0 h_x)\|_e = t_0 \leq \tau_3 [f(x)]^\gamma$$

is valid for all $x = x_1 + x_2 + x_3 \in \mathbb{R}^n \setminus S$ with $\|x_2\|_e > \frac{\sqrt{r}}{2}$ and $x_1 \neq 0$.

2. For all $t \in (0, \sqrt{2}\|x_1\|_e]$, $f(x + th_x) > 0$: Again by Mean Value Theorem that there exists $\lambda_0 \in (0, \sqrt{2}\|x_1\|_e)$ such that

$$[f(x + \sqrt{2}\|x_1\|_e h_x)]^\gamma - [f(x)]^\gamma = \sqrt{2}\|x_1\|_e df(x + \lambda_0 h_x)(h_x) \leq -\frac{\sqrt{2}\|x_1\|_e}{\tau_3}. \quad (4.31)$$

Note that,

$$\begin{aligned} x + \sqrt{2}\|x_1\|_e h_x &= x_1 + x_2 + x_3 + \sqrt{2}\|x_1\|_e [\frac{1}{\sqrt{2}}(-\frac{x_1}{\|x_1\|_e} + \frac{x_2}{\|x_2\|_e})] \\ &= (1 + \frac{\|x_1\|_e}{\|x_2\|_e})x_2 + x_3 \\ &\in \mathcal{X}_2 + \mathcal{X}_3. \end{aligned}$$

As $x + \sqrt{2}\|x_1\|_e h_x \perp \mathcal{X}_1$, using the result in earlier part of this proof, we have

$$dist(x + \sqrt{2}\|x_1\|_e h_x, S) \leq \max\{\tau_1, \tau_2\}[f(x + \sqrt{2}\|x_1\|_e h_x)]^\gamma.$$

Therefore, this and (4.31) imply that for any $x = x_1 + x_2 + x_3 \in \mathbb{R}^n \setminus S$ with $\|x_2\|_e > \frac{\sqrt{r}}{2}$ and $x_1 \neq 0$,

$$\begin{aligned} dist(x, S) &\leq \sqrt{2}\|x_1\|_e + dist(x + \sqrt{2}\|x_1\|_e h_x, S) \\ &\leq \tau_3[[f(x)]^\gamma - [f(x + \sqrt{2}\|x_1\|_e h_x)]^\gamma] + \\ &\qquad \max\{\tau_1, \tau_2\}[f(x + \sqrt{2}\|x_1\|_e h_x)]^\gamma \\ &\leq \max\{\tau_1, \tau_2, \tau_3\}[f(x)]^\gamma. \end{aligned}$$

Combining all results above, take $\tau = \max\{\tau_1, \tau_2, \tau_3\}$. Then for any $x \in \mathbb{R}^n \setminus S$, we have

$$dist(x, S) \leq \tau[f(x)]^\gamma.$$

As a result, $[\frac{1}{2}, 1] \subseteq Ex(f)$. Together with Lemma 4.4.5, $Ex(f) = [\frac{1}{2}, 1]$. □

Theorem 4.4.9 *Suppose that Q has positive eigenvalues and has no negative eigenvalues. Suppose $b \perp ker(Q)$ and $r := f(\theta) = 0$ for some solution θ of (4.22). Then $Ex(f) = \{\frac{1}{2}\}$.*

Proof: Since Q has no negative eigenvalues, $S = \{x_2 + x_3 | x_2 \in \mathcal{X}_2, x_3 \in \mathcal{X}_3\}$. Therefore take any $x = x_1 + x_2 + x_3 \in \mathbb{R}^n \setminus S$ with $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$, we have $f(x) = \|x_1\|_e^2 > 0$ and $dist(x, S) = \|x_1\|_e > 0$, hence

$$\frac{dist(x, S)}{[f(x)]^{\frac{1}{2}}} = 1.$$

As a result, $Ex(f) = \{\frac{1}{2}\}$ □

Theorem 4.4.10 *Suppose that Q has positive eigenvalues and has no negative eigenvalues. Suppose $b \perp ker(Q)$ and $r := f(\theta) < 0$ for some solution θ of (4.22). Then $Ex(f) = [\frac{1}{2}, 1]$.*

Proof: For any $x = x_1 + x_2 + x_3 \in \mathbb{R}^n \setminus S$ with $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$,

$$f(x) = \|x_1\|_e^2 - |r| > 0;$$

hence $\|x_1\|_e > [f(x)]^{\frac{1}{2}} > 0$ and $\|x_1\|_e > |r|^{\frac{1}{2}} > 0$. Then for any $\gamma \in [\frac{1}{2}, 1]$, as $(2\gamma - 1) \in [0, 1]$,

$$\|x_1\|_e \geq |r|^{\frac{1}{2}(2\gamma-1)}[f(x)]^{\frac{1}{2}[1-(2\gamma-1)]} = |r|^{\gamma-\frac{1}{2}}[f(x)]^{1-\gamma} > 0. \tag{4.32}$$

Pick $h_x = \frac{-x_1}{\|x_1\|_e}$. Then by (4.32),

$$df(x)(h_x) = 2x_1^T Q h_x = -2\|x_1\|_e \leq (-2|r|^{\gamma-\frac{1}{2}})[f(x)]^{1-\gamma}.$$

From Corollary 4.3.9, $[\frac{1}{2}, 1] \subseteq Ex(f)$. On the other hand, note that $S = \{x \in \mathbb{R}^n | \|x_1\|_e^2 \leq |r|, x = x_1 + x_2 + x_3, x_i \in \mathcal{X}_i, i = 1, 2, 3\}$. Now we have

$$\lim_{x_1 \in \mathcal{X}_1, \|x_1\|_e \to +\infty} \frac{dist(x_1, S)}{\sqrt{[f(x_1)]}} = \lim_{x_1 \in \mathcal{X}_1, \|x_1\|_e \to +\infty} \frac{\|x_1\|_e - |r|^{\frac{1}{2}}}{(\|x_1\|_e^2 - |r|)^{\frac{1}{2}}} = 1$$

and

$$\lim_{x_1 \in \mathcal{X}_1, \|x_1\|_e \to \sqrt{|r|}^+} \frac{dist(x_1, S)}{f(x_1)} = \lim_{x_1 \in \mathcal{X}_1, \|x_1\|_e \to \sqrt{|r|}^+} \frac{\|x_1\|_e - |r|^{\frac{1}{2}}}{\|x_1\|_e^2 - |r|} = \frac{1}{2\sqrt{|r|}}.$$

These imply that $Ex(f)$ is disjoint from $(0, \frac{1}{2}) \cup (1, +\infty)$. Combining with the earlier result, $Ex(f) = [\frac{1}{2}, 1]$. $\square$

For the sake of considering all the cases, we state two propositions below. They are obviously true.

Proposition 4.4.11 *Suppose that Q has no negative eigenvalues, $b \perp ker(Q)$ and $r := f(\theta) > 0$ for some solution θ of (4.22). Then $S = \emptyset$.*

Proposition 4.4.12 *Suppose that Q has no positive eigenvalues and has negative eigenvalues. Suppose that $b \perp ker(Q)$ and $r := f(\theta) \leq 0$ for some solution θ of (4.22). Then $S = \mathbb{R}^n$ and $Ex(f) = (0, +\infty)$.*

Theorem 4.4.13 *Suppose that Q has no positive eigenvalues and has negative eigenvalues. Suppose $b \perp ker(Q)$ and $r := f(\theta) > 0$ for some solution θ of (4.22). Then $Ex(f) = (0, 1]$.*

Proof: For any $x = x_1 + x_2 + x_3 \in \mathbb{R}^n$ with $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$,

$$f(x) = -\|x_2\|_e^2 + r.$$

Note that $S = \{x \in \mathbb{R}^n | \|x_2\|_e \geq r\}$. Therefore take any $x \in \mathbb{R}^n \setminus S$, we have $dist(x, S) = \sqrt{r} - \|x_2\|_e$. Take any $\gamma \in (0, 1]$,

$$\frac{dist(x,S)}{[f(x)]^\gamma} = \frac{\sqrt{r} - \|x_2\|_e}{(r - \|x_2\|_e^2)^\gamma} = \frac{(\sqrt{r} - \|x_2\|_e)^{1-\gamma}}{(\sqrt{r} + \|x_2\|_e)^\gamma} \leq (\sqrt{r})^{1-2\gamma},$$

so $(0, 1] \subseteq Ex(f)$. On the other hand, since

$$\lim_{x_2 \in \mathcal{X}_2, \|x_2\|_e \to \sqrt{r}^-} \frac{dist(x_2, S)}{f(x_2)} = \lim_{x_2 \in \mathcal{X}_2, \|x_2\|_e \to \sqrt{r}^-} \frac{\sqrt{r} - \|x_2\|_e}{r - \|x_2\|_e^2} = \frac{1}{2\sqrt{r}},$$

$Ex(f)$ is disjoint from $(1, +\infty)$. Combining the results, one has $Ex(f) = (0, 1]$.
□

Now we turn to the situation that b is not orthogonal to $ker(Q)$. First, we let b_0 denote the projection of b on $ker(Q)$, that is, $b_0 \in \mathcal{X}_3$. In case that b is not orthogonal to $ker(Q)$, $b_0 \neq 0$ and $b - b_0 \perp ker(Q)$. Let θ_0 be a solution of the equation $2Qx = -(b - b_0)$ (such a θ_0 exists as $b - b_0 \perp ker(Q)$). And we also rewrite the equation (4.20) as follows: for any $x \in \mathbb{R}^n$

$$\begin{aligned} f(x) &= x^TQx + b^Tx + c \\ &= x^TQx + b^T(x - \theta_0) + b^T\theta_0 + c \\ &= x^TQx + (b_0^T - 2\theta_0^TQ)(x - \theta_0) + b^T\theta_0 + c \\ &= x^TQx - 2\theta_0^TQx + \theta_0^TQ\theta_0 + b_0^T(x - \theta_0) + \theta_0^TQ\theta_0 + b^T\theta_0 + c \\ &= (x - \theta_0)^TQ(x - \theta_0) + b_0^T(x - \theta_0) + f(\theta_0). \end{aligned}$$

Therefore in the following discussion we can consider the quadratic function:

$$f(x) = x^TQx + b_0^Tx + r_0, \quad x \in \mathbb{R}^n, \tag{4.33}$$

where $r_0 = f(\theta_0)$ is the value of f at some solution θ_0 of the equation $2Qx = -(b - b_0)$. Note that $f(\theta_0)$ is dependent on the choice of θ_0. In virtue of $\|\cdot\|_e$, (4.33) can be further rewritten as

$$f(x) = \|x_1\|_e^2 - \|x_2\|_e^2 + b_0^Tx_3 + r_0, \tag{4.34}$$

where $x = x_1 + x_2 + x_3 \in \mathbb{R}^n$, with $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$. Making use of this and $\|\cdot\|_e$, we discuss the exponents of the error bound of the quadratic function f with $b \not\perp ker(Q)$.

Lemma 4.4.14 *Suppose $b \not\perp ker(Q)$, then $\{1\} \subseteq Ex(f)$.*

Proof: Fix $x \in \mathbb{R}^n \setminus S$ with $x = x_1 + x_2 + x_3$, $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$. Now

$$f(x) = \|x_1\|_e^2 - \|x_2\|_e^2 + b_0^T x_3 + r_0.$$

Pick $h_x = \frac{-b_0}{\|b_0\|}$, then

$$df(x)(h_x) = (2x_1^T Q - 2x_2^T Q + b_0^T)h_x = b_0^T(\frac{-b_0}{\|b_0\|}) = -\|b_0\|.$$

by Corollary 4.3.9, $1 \in Ex(f)$. □

Theorem 4.4.15 *Suppose that Q has no negative eigenvalues and b is not orthogonal to $ker(Q)$. Then $Ex(f) = \{1\}$.*

Proof: By (4.34) and that Q has no negative eigenvalues, for each $x = x_1 + x_2 + x_3 \in \mathbb{R}^n$, with $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$,

$$f(x) = \|x_1\|_e^2 + b_0^T x_3 + r_0.$$

Let $\mathcal{X}_3' = \{x_3 \in \mathcal{X}_3 | x_3 \perp b_0\}$. Then $\mathcal{X}_3 = \mathcal{X}_3' + \mathbb{R}b_0$. For each $t > 0$, let $x_t = (t + \frac{-r_0}{\|b_0\|^2})b_0$, and so

$$f(x_t) = b_0^T(t + \frac{-r_0}{\|b_0\|^2})b_0 + r_0 = t\|b_0\|^2 > 0. \tag{4.35}$$

Therefore,

$$
\begin{aligned}
& dist(x_t, S) \\
= \;& dist(x_t, \partial S) \\
= \;& \inf\{\big[\|x_1\|_e^2 + \|x_2\|_e^2 + \|x_3 - x_t\|^2\big]^{\frac{1}{2}} : \|x_1\|_e^2 + b_0^T x_3 + r_0 = 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_i \in \mathcal{X}_i, i = 1,2,3\} \\
= \;& \inf\{\big[-b_0^T x_3 - r_0 + \|x_3 - x_t\|^2\big]^{\frac{1}{2}} : b_0^T x_3 + r_0 \leq 0, x_3 \in \mathcal{X}_3\} \\
= \;& \inf\{\big[-b_0^T(x_3 + \frac{r_0 b_0}{\|b_0\|^2}) + \|x_3 - x_t\|^2\big]^{\frac{1}{2}} : b_0^T(x_3 + \frac{r_0 b_0}{\|b_0\|^2}) \leq 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_3 \in \mathcal{X}_3\}. \qquad (4.36)
\end{aligned}
$$

Since $b_0 \in \mathcal{X}_3$, for any $x_3 \in \mathcal{X}_3$, $(x_3 + \frac{r_0 b_0}{\|b_0\|^2}) \in \mathcal{X}_3$ also. We can replace $(x_3 + \frac{r_0 b_0}{\|b_0\|^2})$ by $y_3 \in \mathcal{X}_3$ in (4.36). Note that $x_3 - x_t = y_3 - \frac{r_0 b_0}{\|b_0\|^2} - (t + \frac{-r_0}{\|b_0\|^2})b_0 = y_3 - tb_0$. Then (4.36) becomes:

$$
\begin{aligned}
& dist(x_t, S) \\
= \;& \inf\{\big[-b_0^T y_3 + \|y_3 - tb_0\|^2\big]^{\frac{1}{2}} : b_0^T y_3 \leq 0, y_3 \in \mathcal{X}_3\} \\
= \;& \inf\{\big[-b_0^T(y_3' + \lambda b_0) + \|y_3' - (\lambda - t)b_0\|^2\big]^{\frac{1}{2}} : b_0^T(y_3' + \lambda b_0) \leq 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda \in \mathbb{R}, y_3' \in \mathcal{X}_3'\} \\
= \;& \inf\{\big[-\lambda\|b_0\|^2 + -(\lambda - t)^2\|b_0\|^2 + \|y_3'\|\big]^{\frac{1}{2}} : \lambda \leq 0, y_3' \in \mathcal{X}_3'\} \\
= \;& \inf\{\big[(\lambda - t)^2 - \lambda\big]^{\frac{1}{2}}\|b_0\| : \lambda \leq 0\}.
\end{aligned}
$$

Now,

$$
\frac{d}{d\lambda}[(\lambda - t)^2 - \lambda]\Big|_{\lambda \leq 0} = [2(\lambda - t) - 1]\Big|_{\lambda \leq 0} \leq -2t - 1 < 0.
$$

Therefore,

$$
dist(x_t, S) = [(0 - t)^2 - 0]^{\frac{1}{2}}\|b_0\| = t\|b_0\|.
$$

Thus by (4.35) for each $t > 0$,

$$
dist(x_t, S) = \frac{1}{\|b_o\|} f(x_t).
$$

This imply that $Ex(f) \subseteq \{1\}$. Combining with Lemma 4.4.14, $Ex(f) = \{1\}$. □

Theorem 4.4.16 *Suppose that Q has negative eigenvalues and b is not orthogonal to $ker(Q)$. Then $Ex(f) = [\frac{1}{2}, 1]$.*

Proof: Fix $x \in \mathbb{R}^n \setminus S$ with $x = x_1 + x_2 + x_3$, where $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$. As $f(x) = \|x_1\|_e^2 - \|x_2\|_e^2 + b_0^T x_3 + r_0 > 0$,

$$\|x_1\|_e^2 + b_0^T x_3 + r_0 > \|x_2\|_e^2 \geq 0.$$

Define

$$y_x = \begin{cases} x_1 + (\|x_1\|_e^2 + b_0^T x_3 + r_0)^{\frac{1}{2}} \frac{x_2}{\|x_2\|_e} + x_3, & x_2 \neq 0 \\ x_1 + (\|x_1\|_e^2 + b_0^T x_3 + r_0)^{\frac{1}{2}} \hat{x_2} + x_3, & x_2 = 0 \end{cases},$$

where $\hat{x_2} \in \mathcal{X}_2$ with $\|\hat{x_2}\|_e = 1$. Note that $f(y_x) = 0$. Hence,

$$dist(x, S) \leq \|x - y_x\|_e = (\|x_1\|_e^2 + b_0^T x_3 + r_0)^{\frac{1}{2}} - \|x_2\|_e \leq [f(x)]^{\frac{1}{2}}.$$

This implies that $\frac{1}{2} \in Ex(f)$. It follows from Lemma 4.4.14 and Proposition 4.3.1 that

$$[\frac{1}{2}, 1] \subseteq Ex(f). \tag{4.37}$$

On the other hand, similar to the proof of Theorem 4.4.15, we let $\mathcal{X}_3' = \{x_3 \in \mathcal{X}_3 | x_3 \perp b_0\}$. Then $\mathcal{X}_3 = \mathcal{X}_3' + \mathbb{R} b_0$. For each $t > 0$, let $x_t = (t + \frac{-r_0}{\|b_0\|^2}) b_0$, and so

$f(x_t) = b_0^T(t + \frac{-r_0}{\|b_0\|^2})b_0 + r_0 = t\|b_0\|^2 > 0$. Therefore,

$$\begin{aligned}
& dist(x_t, S) \\
= \; & dist(x_t, \partial S) \\
= \; & \inf\{\big[\|x_1\|_e^2 + \|x_2\|_e^2 + \|x_3 - x_t\|^2\big]^{\frac{1}{2}} : \|x_1\|_e^2 - \|x_2\|_e^2 + b_0^T x_3 + r_0 = 0, \\
& \qquad x_i \in \mathcal{X}_i, i = 1, 2, 3\} \\
= \; & \inf\{\big[2\|x_1\|_e^2 + b_0^T x_3 + r_0 + \|x_3 - x_t\|^2\big]^{\frac{1}{2}} : \|x_1\|_e^2 + b_0^T x_3 + r_0 \geq 0, \\
& \qquad x_1 \in \mathcal{X}_1, x_3 \in \mathcal{X}_3\} \\
= \; & \inf\{\big[2\|x_1\|_e^2 + b_0^T(x_3 + \frac{r_0 b_0}{\|b_0\|^2)} + \|x_3 - x_t\|^2\big]^{\frac{1}{2}} : \|x_1\|_e^2 + b_0^T x_3 + r_0 \geq 0, \\
& \qquad x_1 \in \mathcal{X}_1, x_3 \in \mathcal{X}_3\} \\
= \; & \inf\{\big[2\|x_1\|_e^2 + b_0^T y_3 + \|y_3 - tb_0\|^2\big]^{\frac{1}{2}} : \|x_1\|_e^2 + b_0^T y_3 \geq 0, \\
& \qquad x_1 \in \mathcal{X}_1, y_3 \in \mathcal{X}_3\}. \qquad (4.38)
\end{aligned}$$

Let $g(x_1, y_3) = 2\|x_1\|_e^2 + b_0^T y_3 + \|y_3 - tb_0\|^2$ with $(x_1, y_3) \in \mathcal{X}_1 \times \mathcal{X}_3$. We have

$$\frac{\partial g}{\partial x_1} = 4x_1^T Q$$

and

$$\frac{\partial g}{\partial y_3} = b_0^T + 2(y_3^T - tb_0^T) = 2y_3^T - (2t-1)b_0^T.$$

In order to have minimal g, $x_1 = 0$ and $y_3 = (t - \frac{1}{2})b_0$. For t large enough, $(0, (t - \frac{1}{2})b_0)$ satisfies the constraint in (4.38). Therefore,

$$dist(x_t, S) = \big[b_0^T(t - \frac{1}{2})b_0 + \|\frac{1}{2}b_0\|^2\big]^{\frac{1}{2}} = (t - \frac{1}{4})^{\frac{1}{2}}\|b_0\|$$

and thus

$$\lim_{t \to +\infty} \frac{dist(x_t, S)}{\sqrt{f(x_t)}} = \lim_{t \to +\infty} \frac{(t - \frac{1}{4})^{\frac{1}{2}}\|b_0\|}{t^{\frac{1}{2}}\|b_0\|} = 1.$$

This implies that

$$Ex(f) \cap (0, \frac{1}{2}) = \emptyset. \qquad (4.39)$$

On the other hand, for t small enough, see (4.38), without loss of generality, let $x_1 = 0$ and $y_3 = \lambda b_0$ for some $\lambda > 0$. Therefore,

$$\begin{aligned} dist(x_t, S) &= \inf\{\left[\lambda\|b_0\|^2 + (\lambda - t)^2\|b_0\|^2\right]^{\frac{1}{2}} : \lambda \geq 0\} \\ &= \inf\{\left[\lambda + (\lambda - t)^2\right]^{\frac{1}{2}}\|b_0\| : \lambda \geq 0\}. \end{aligned}$$

Similar to the proof in Theorem 4.4.15, we have $dist(x_t, S) = t\|b_0\|$. Since,

$$\lim_{t \to 0^+} \frac{dist(x_t, S)}{f(x_t)} = \frac{1}{\|b_0\|},$$

$Ex(f) \cap (1, +\infty) = \emptyset$. Combining with (4.39) and (4.37),

$$Ex(f) = [\frac{1}{2}, 1].$$

□

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