Some Recent Advances in Numerical Solutions of Electromagnetic Problems

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Abstract

This dissertation identifies a unified formulation for most existing split and unsplit perfectly matched layers (PMLs) for solving the time-dependent Maxwell's equations.

Several finite difference schemes are proposed for efficiently solving the 1D time-dependent Maxwell's system, these schemes including the first order scheme, modified Yee scheme and Lax-Wendroff scheme. Convergence and stability are rigorously demonstrated for the finite difference schemes.

Numerical experiments are presented, which validate the effectiveness of the finite difference schemes proposed in this dissertation.

摘要

这篇论文主要针对目前文献上已有的, 处理时间相关的 Maxwell 方程的分裂与不分裂的完全匹配层方法(Perfectly Matched Layer),找到了一个统一的形式.

对于一维时间相关的 Maxwell 系统的多种完全匹配层方法, 提出了一些行之有效的差分格式, 包括一阶格式, 改进的 Yee 格式和 Lax-Wendroff 格式. 并严格地论证了这些差分格式的收敛性和稳定性.

数值实验进一步验证了本文所提出差分格式的效果.

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Chapter 1

Introduction

1.1 The Generalized PML Theory

1.1.1 Background

With the rapidly increasing interest in solving Maxwells equations in the time domain comes the need to develop effective and reliable mathematical methods to truncate the infinite computational domain. This question is closely related to the solution of wave problems and has received tremendous attention in the past two decades (c.f. [BT] & [EM]). It is still one of the central and largely open challenging problems to engineers and applied mathematicians. The development of such methods is essentially necessary due to the increasing use of high-order accurate methods to avoid ruining the accurate interior solutions by artificial reflections from the computational boundary. A new and exciting advance in this direction was achieved about 10 years ago by Berenger in [B1]. This new approach suggests to use an absorbing layer, instead of the commonly used absorbing boundary, designed in such a way that all waves entering the layer, regardless of their frequency and angle of incidence, would be absorbed completely and without reflections into the computational domain. Such layers, often called perfectly matched layers (PMLs), seemed to overcome the reflection problems and their derivation, first presented for the two-dimensional Maxwells equations, were then generated to three-dimensional systems [B2], the equations of acoustics [Hu], and the equations of linear elasticity [CL], and so on [AGH].

Since the first PML invented by Berenger, numerous PMLs have been constructed. In [B1], Berenger introduced a novel boundary condition for truncating two-dimensional finite difference-time domain meshes. His PML technique is based on using a layer of lossy material to absorb outgoing radiation from the computational domain. Inside the PML layer, the Cartesian field components are split into two subcomponents (i.e. $H_x = H_{xy} + H_{xz}$). Late in 1994, another formulation of PML was given by Chew and Weedon late in 1994 [CW], and is also closely related to the one by Rappaport [R]. Their approach is based on introducing complex coordinate stretching variables, and Maxwell's equations were modified to add additional degrees of freedom. The modifications allow the specification of a lossy material layer such that a planar interface between the PML material and free space is reflectionless for all frequencies, polarizations, and angles of incidence [SKLL].

In 1996, a new PML formulation was proposed by Sacks et al. [SKLL], based on a properly constructed anisotropic medium. This approach appears more attractive in view of the fact that there is no need for the Chew-Weedon modification of the spatial derivative operators via coordinate stretching, thus Maxwell's equations maintain their usual physical form, except for the strange properties of the anisotropic medium. However, as it will be seen below, these two approaches are mathematically identical, provided that the electric and magnetic fields presented in the Chew-Weedon stretching -coordinate formulation are properly defined [ZC1]. Also in 1996, Veihl and Mittra [VM] proposed another alternative formulation of the Berenger's scheme where the spltting of the field components is again avoided. Instead, time- and field-dependent sources are introduced. As suggested in [ZC1], an unsplit-field implementation of PMLs in the time domain can also be effected directly from the anisotropic medium formulation of the PML [ZC2]. In the same year, Zhao and Cangellaris improved Veihl and Mittra's work in two of their important contributions [ZC1] & [ZC2]. In 1997, Ziolkowski constructed an absorber that deals with ultrafast pulses, the time-derivative Lorentz material model for the polarization and magnetization fields [Z].

1.1.2 Derivation

In this subsection, we shall review the main steps in the derivation of the perfectly matched layers in three dimensions, following the ideas of [ZC2]. In this way the two-dimensional TE (scalar magnetic field) and TM (scalar electric field) polarizations will be simultaneously treated. A more detailed presentation can be found in [ZC2]. Using Fourier transform in time ($e^{i\omega t}$ dependence) and the frequency-domain anisotropic constitutive relations $\mathbf{D} = \overline{\varepsilon} \cdot \mathbf{E}$ and $\mathbf{B} = \overline{\mu} \cdot \mathbf{H}$, Maxwell's equations (with $J = 0$) and the divergence-free conditions are reduced to

$$\begin{cases} j\omega\overline{\mu} \cdot \mathbf{H} &= -\nabla \times \mathbf{E}, \\ j\omega\overline{\varepsilon} \cdot \mathbf{E} &= \nabla \times \mathbf{H}, \\ \nabla \cdot (\overline{\varepsilon} \cdot \mathbf{E}) &= 0, \\ \nabla \cdot (\overline{\mu} \cdot \mathbf{H}) &= 0, \end{cases} \tag{1.1.1}$$

where $\overline{\varepsilon}, \overline{\mu}$ are the permittivity and permeability tensors of the medium, which are assumed [SKLL] to be of the form

$$\begin{cases} \overline{\varepsilon} = \varepsilon \mathrm{diag}\{a_x, a_y, a_z\} = \varepsilon\Lambda, \\ \overline{\mu} = \mu \mathrm{diag}\{a_x, a_y, a_z\} = \mu\Lambda, \end{cases} \tag{1.1.2}$$

with ε and μ being real numbers that satisfy $\varepsilon \geq \varepsilon_0$ and $\mu \geq \mu_0$. In (1.1.2), the entries of the diagonal matrix

$$\Lambda = \mathrm{diag}\{a_x, a_y, a_z\} \tag{1.1.3}$$

are, in general, complex dimensionless constants.

By the definition, the scaled fields can be written as

$$\begin{cases} \hat{\mathbf{E}} &= \{\hat{E}_x, \hat{E}_y, \hat{E}_z\}^T &= G^{-1}\{E_x, E_y, E_z\}^T \\ \hat{\mathbf{H}} &= \{\hat{H}_x, \hat{H}_y, \hat{H}_z\}^T &= G^{-1}\{H_x, H_y, H_z\}^T, \end{cases} \tag{1.1.4}$$

and

$$G = \mathrm{diag}\{g_x, g_y, g_z\} \tag{1.1.5}$$

with g_x, g_y, g_z being, in general, complex constants. Using the notation $\overline{\mathbf{G}}$ and $\overline{\mathbf{\Lambda}}$ to denote the tensors with matrix representations G and Λ, respectively, we may rewrite (1.1.1) in terms of the scaled fields as follows:

$$\begin{cases} j\omega\mu\overline{\mathbf{\Lambda}} \cdot \overline{\mathbf{G}} \cdot \hat{\mathbf{H}} &= -\nabla \times (\overline{\mathbf{G}} \cdot \hat{\mathbf{E}}) \\ j\omega\varepsilon\overline{\mathbf{\Lambda}} \cdot \overline{\mathbf{G}} \cdot \hat{\mathbf{E}} &= \nabla \times (\overline{\mathbf{G}} \cdot \hat{\mathbf{H}}) \\ \nabla \cdot (\varepsilon\overline{\mathbf{\Lambda}} \cdot \overline{\mathbf{G}} \cdot \hat{\mathbf{E}}) &= 0 \\ \nabla \cdot (\varepsilon\overline{\mathbf{\Lambda}} \cdot \overline{\mathbf{G}} \cdot \hat{\mathbf{H}}) &= 0. \end{cases} \tag{1.1.6}$$

The choice of the scaling factors, g_x, g_y, g_z, according to the equations

$$\left(\frac{g_x}{g_y}\right)^2 = \frac{a_y}{a_x}, \quad \left(\frac{g_y}{g_z}\right)^2 = \frac{a_z}{a_y}, \quad \left(\frac{g_z}{g_x}\right)^2 = \frac{a_x}{a_z} \tag{1.1.7}$$

allows us to rewrite (1.1.6) in the form

$$\begin{cases} j\omega\mu\hat{\mathbf{H}} &= -\nabla_{\mathbf{a}} \times \hat{\mathbf{E}} \\ j\omega\varepsilon\hat{\mathbf{E}} &= \nabla_{\mathbf{a}} \times \hat{\mathbf{H}} \\ \nabla_{\mathbf{a}} \cdot (\varepsilon\hat{\mathbf{E}}) &= 0 \\ \nabla_{\mathbf{a}} \cdot (\mu\hat{\mathbf{H}}) &= 0, \end{cases} \tag{1.1.8}$$

where

$$\nabla_{\mathbf{a}} \equiv \frac{1}{\sqrt{a_y a_z}}\partial_x\,\hat{\mathbf{x}} + \frac{1}{\sqrt{a_z a_x}}\partial_y\,\hat{\mathbf{y}} + \frac{1}{\sqrt{a_x a_y}}\partial_z\,\hat{\mathbf{z}},$$

and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are unit axial vectors in the cartesian coordinate system. The system (1.1.8) is very similar to the modified Maxwell system with complex coordinate stretching used in [CW]. Indeed, using the notation

$$s_x = \sqrt{a_y a_z}\,, \quad s_y = \sqrt{a_z a_x}\,, \quad s_z = \sqrt{a_x a_y}\,, \tag{1.1.9}$$

the system (1.1.8) becomes mathematically equivalent to the modified Maxwell system in frequency domain in [CW]. However, there is an important difference between the two: (1.1.8) is for the scaled fields while the equivalent one in [CW] was proposed assuming that the fields are physical fields. The nonphysicality of the fields in [CW] manifests itself as the requirement for an arbitrary field-splitting in order to produce the time-domain equations for the absorbing layer.

Assuming plane wave propagation has following form for the scaled fields in the anisotropic medium, namely

$$\begin{cases} \hat{\mathbf{E}} & = G^{-1} \cdot \mathbf{E} & = \hat{\mathbf{E}}_0 e^{-j\,\mathbf{k}\cdot\mathbf{r}} \\ \hat{\mathbf{H}} & = G^{-1} \cdot \mathbf{H} & = \hat{\mathbf{H}}_0 e^{-j\,\mathbf{k}\cdot\mathbf{r}} \end{cases} \tag{1.1.10}$$

where $\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}$, and $\hat{\mathbf{E}}_0$ and $\hat{\mathbf{H}}_0$ are the scaled complex-constant field amplitudes. Using (1.1.10) in the system (1.1.8), and noting that the medium is assumed to be homogeneous, one obtains

$$\begin{cases} \mathbf{k}_S \times \hat{\mathbf{E}} &= \omega\mu\hat{\mathbf{H}} \\ \mathbf{k}_S \times \hat{\mathbf{H}} &= -\omega\varepsilon\hat{\mathbf{E}} \\ \mathbf{k}_S \cdot \hat{\mathbf{E}} &= 0 \\ \mathbf{k}_S \cdot \hat{\mathbf{H}} &= 0 \end{cases} \tag{1.1.11}$$

where

$$\mathbf{k}_S \equiv \frac{k_x}{s_x}\hat{\mathbf{x}} + \frac{k_y}{s_y}\hat{\mathbf{y}} + \frac{k_z}{s_z}\hat{\mathbf{z}}. \tag{1.1.12}$$

Eliminating $\hat{\mathbf{H}}$ between the first two equations of (1.1.11) results in

$$\mathbf{k}_S \times \mathbf{k}_S \times \hat{\mathbf{E}} = -\omega^2\mu\varepsilon\hat{\mathbf{E}}$$

and, finally, along with (1.1.12), we get

$$\omega^2\mu\varepsilon = \frac{k_x^2}{s_x^2} + \frac{k_y^2}{s_y^2} + \frac{k_z^2}{s_z^2}. \tag{1.1.13}$$

The relation (1.1.13) is the dispersion relation for the anisotropic medium currently considered. Obviously, (1.1.13) is satisfied by

$$\begin{cases} k_x = ks_x \sin\theta\cos\phi \\ k_y = ks_y \sin\theta\sin\phi \\ k_z = ks_z \cos\theta, \end{cases} \tag{1.1.14}$$

where $k = \omega\sqrt{\mu\varepsilon}$. Clearly, the propagation characteristics of the wave along $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ can be controlled by varying the variables s_x, s_y and s_z or, effectively (see (1.1.9)), by varying the properties of the anisotropic medium.

1.1.3 Reflection Properties

In this subsection, we discuss the reflection properties, following the ideas of [PZC]. A relationship between the tensors of two anisotropic media separated by a planar interface can be established for the interface to be reflectionless for all frequencies and all angles of incidence. We shall present below this relationship without any proof.

Without loss of generality, the planar interface is taken to coincide with the $x = 0$ plane in a cartesian coordinate system. The space $x > 0$ (Medium 1) is filled with a homogeneous medium with tensors $\varepsilon_1\Lambda_1, \mu_1\Lambda_1$, where $\Lambda_1 = \text{diag}\{a_{1x}, a_{1y}, a_{1z}\}$ and the corresponding values of s_{1x}, s_{1y}, and s_{1z} are given by (1.1.9). The space $x < 0$ (Medium 2) is filled with a homogeneous medium with tensors $\varepsilon_2\Lambda_2, \mu_2\Lambda_2$, where $\Lambda_2 = \text{diag}\{a_{2x}, a_{2y}, a_{2z}\}$, and the corresponding values of s_{2x}, s_{2y}, and s_{2z} are given by (1.1.9). A plane wave propagating from Medium 1 toward Medium 2 is assumed to be obliquely incident on the interface at $x = 0$. Its polarization is assumed to be arbitrary. Then the reflective ratios are given by

$$R^{TE} = \frac{k_{1s}\mu_2 s_{2x} - k_{2s}\mu_1 s_{1x}}{k_{1s}\mu_2 s_{2x} + k_{2s}\mu_1 s_{1x}} \tag{1.1.15}$$

$$R^{TM} = \frac{k_{1s}\varepsilon_2 s_{2x} - k_{2s}\varepsilon_1 s_{1x}}{k_{1s}\varepsilon_2 s_{2x} + k_{2s}\varepsilon_1 s_{1x}} \tag{1.1.16}$$

(c.f. [CW]). Therefore, for the material interface to be *reflectionless* the following relationships between the properties of the two anisotropic media are necessary

$$\varepsilon_1 = \varepsilon_2, \qquad \mu_1 = \mu_2, \tag{1.1.17}$$

$$s_{1y} = s_{2y}, \qquad s_{1z} = s_{2z}. \tag{1.1.18}$$

The relations in (1.1.18) are instrumental for achieving the reflectionless interface. In terms of the entries of the tensors Λ_1 and Λ_2 and making use of (1.1.9), we derive

$$\frac{a_{2x}}{a_{1x}} = \frac{a_{1y}}{a_{2y}} = \frac{a_{1z}}{a_{2z}}. \tag{1.1.19}$$

For clarity, we summarize the results we have derived above. Let i denote one of the axes in a righthand Cartesian coordinate system xyz, and j, k the other two axes in the system. The jk-plane interface between two anisotropic media characterized by the tensor $\varepsilon_1 \text{diag}\{a_{1i}, a_{1j}, a_{1k}\}$, $\mu_1 \text{diag}\{a_{1i}, a_{1j}, a_{1k}\}$ and $\varepsilon_2 \text{diag}\{a_{2i}, a_{2j}, a_{2k}\}$, $\mu_2 \text{diag}\{a_{2i}, a_{2j}, a_{2k}\}$, respectively, will be reflectionless for all frequencies and any angle of incidence if $\varepsilon_1 = \varepsilon_2, \mu_1 = \mu_2$ and the elements of the tensors satisfy the relation

$$\frac{a_{2i}}{a_{1i}} = \frac{a_{1j}}{a_{2j}} = \frac{a_{1k}}{a_{2k}}. \tag{1.1.20}$$

The use of this result in the construction of absorbing PMLs for the numerical grid truncation is discussed in the next section.

1.2 Unified Formulation

1.2.1 Face-, Edge- and Corner-PMLs

In this subsection, we review the classification of various PML regions, following the ideas of [ZC2]. Consider a rectangular volume Ω, in a linear, homogeneous and isotropic medium of permittivity ε and permeability μ. We shall take a

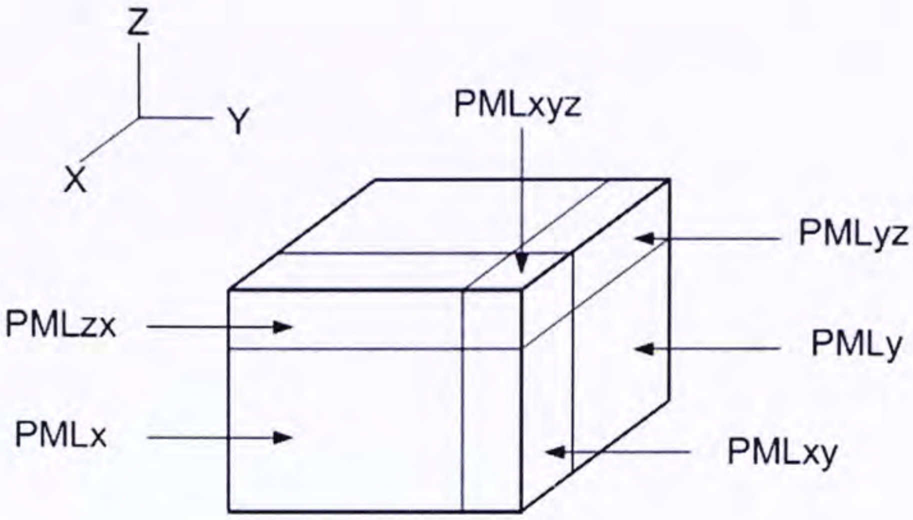

Figure 1.1: Classification of the various PML regions.

finite method to model electromagnetic interactions insider this volume, and it will be done by surrounding the volume by a PML developed on the basis of the aforementioned theory with the additional property that they dissipate the waves propagating through them. For the purposes of numerical computation, a PML should be of finite thickness. If sufficient attenuation is effected by these absorbing PMLs, zero field values may be assumed at the end of the PMLs, thus effecting simple Dirichlet boundary conditions at the ends of the domain of numerical computation without giving rise to spurious reflections.

Fig 1.1 gives one fraction of the volume Ω with a PML of finite thickness attached on its outer surface. We shall now review three types of PMLs: face-PMLs, edge-PMLs and corner PMLs.

Face-PMLs

These PMLs are rectangular volumes, placed next to the six faces. Layers PML_y and PML_z in Fig 1.1 fall in this category. Let us analysis the layer PML_x. This layer is expected to be perfectly matched to the homogeneous medium inside Ω. Let Medium 1 be the medium inside Ω, thus, $\varepsilon_1 = \varepsilon$, $\mu_1 = \mu$, and $a_{1x} = a_{1y} = a_{1z} = 1$. Let Medium 2 be the layer PML_x. According to the results in the previous section, layer PML_x will be perfectly matched to the interior medium if $\varepsilon_2 = \varepsilon$, $\mu_2 = \mu$ and, from (1.1.20) we should have

$$a_{2x} = \frac{1}{w_x}, \quad a_{2y} = a_{2z} = w_x,$$

or, in matrix form,

$$\Lambda^x = \text{diag}\{\frac{1}{w_x}, w_x, w_x\}. \tag{1.2.1}$$

Similarly, we can workout the matrices Λ^y and Λ^z:

$$\Lambda^y = \text{diag}\{w_y, \frac{1}{w_y}, w_y\}, \qquad \Lambda^z = \text{diag}\{w_z, w_z, \frac{1}{w_z}\}. \tag{1.2.2}$$

Edge-PMLs

These PMLs are rectangular volumes, placed next to the twelve edges. In Fig 1.1 one can see three of these edge-PMLs, PML_{xy}, PML_{yz} and PML_{zx}. Let us look at the layer PML_{xy}. This PML must be constructed in such a way that it is matched to both face-PMLs PML_x and PML_y. In view of (1.1.20) and the fact that the $[\Lambda]$-matrices of the the face-PMLs PML_x and PML_y are, respectively, $\text{diag}\{\frac{1}{w_x}, w_x, w_x\}$ and $\text{diag}\{w_y, \frac{1}{w_y}, w_y\}$, this edge-PML should have the parameters ε and μ, and the elements of its Λ-matrix should satiesfy the relations

$$\frac{w_x^{-1}}{a_x} = \frac{a_y}{w_x} = \frac{w_x}{a_z}, \qquad \frac{a_x}{w_y} = \frac{w_y^{-1}}{a_y} = \frac{w_y}{a_z}.$$

It is easy to see that these relations imply the following Λ-matrix for this edge-PML:

$$\Lambda^{xy} = \text{diag}\{\frac{w_y}{w_x}, \frac{w_x}{w_y}, w_x w_y\}, \tag{1.2.3}$$

which can be written as

$$\Lambda^{xy} = \Lambda^x \Lambda^y.$$

This last relation is very useful since it presents a simple means for us to construct the tensors of other edge-PMLs.

Corner-PMLs

These PMLs are rectangular volumes, placed next to the eight corners, see, e.g., PML_{xyz} in Fig 1.1. Their construction is based on the observation that they should be matched to the three edge-PMLs, PML_{xy}, PML_{yz} and PML_{zx}. Application of (1.1.20) at the three relevant interfaces leads to the following expression for the Λ-matrix of the corner-PML

$$\Lambda^{xyz} = \mathrm{diag}\{\frac{w_y w_z}{w_x}, \frac{w_z w_x}{w_y}, \frac{w_x w_y}{w_z}\}, \tag{1.2.4}$$

or simply,

$$\Lambda^{xyz} = \Lambda^x \Lambda^y \Lambda^z.$$

1.2.2 Unified PML Equations in 3D

Next, we focus on the edge-PMLs to derive the PML equations (the corner-PMLs is similar). Substituting (1.2.3) into (1.1.1), we obtain

$$\nabla \times \mathbf{E} = -j\omega\mu \begin{pmatrix} \frac{w_y}{w_x} & 0 & 0 \\ 0 & \frac{w_x}{w_y} & 0 \\ 0 & 0 & w_x w_y \end{pmatrix} \mathbf{H},$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon \begin{pmatrix} \frac{w_y}{w_x} & 0 & 0 \\ 0 & \frac{w_x}{w_y} & 0 \\ 0 & 0 & w_x w_y \end{pmatrix} \mathbf{E},$$

which gives a 3D edge-PMLs in the frequency domain as follows:

$$\begin{cases} -j\omega\mu\frac{w_y}{w_x}H_x = & (\nabla \times \mathbf{E})_x = & (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) \\ -j\omega\mu\frac{w_x}{w_y}H_y = & (\nabla \times \mathbf{E})_y = & (\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) \\ -j\omega\mu w_x w_y H_z = & (\nabla \times \mathbf{E})_z = & (\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) \\ j\omega\varepsilon\frac{w_y}{w_x}E_x = & (\nabla \times \mathbf{H})_x = & (\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}) \\ j\omega\varepsilon\frac{w_x}{w_y}E_y = & (\nabla \times \mathbf{H})_y = & (\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}) \\ j\omega\varepsilon w_x w_y E_z = & (\nabla \times \mathbf{H})_z = & (\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}). \end{cases} \tag{1.2.5}$$

1.2.3 Unified PML Equations in 2D

In this subsection, we will establish the equations of a PML medium for the two-dimension TE (traverse electric) mode case. And the TM (traverse magnetic) mode case is similar. In the Cartesian coordinates let us consider a problem without variation along z, but with the electric field lying in (x, y) plane. The electromagnetic field involves three components only, E_x, E_y, H_z, and the Maxwell equations reduce to a set of three equations. From the fourth, fifth and third equation of (1.2.5), we derive these three equations in the frequency domain

$$\left\{\begin{array}{rl} j\omega\varepsilon\frac{w_y}{w_x}E_x = & \frac{\partial H_z}{\partial y} \\ j\omega\varepsilon\frac{w_x}{w_y}E_y = & -\frac{\partial H_z}{\partial x} \\ -j\omega\mu w_x w_y H_z = & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}. \end{array}\right. \tag{1.2.6}$$

We claim that all the existing PML formulations can be derived from (1.2.6) through the relations:

$$w_x = 1 + \frac{\sigma_x}{j\omega\varepsilon}, \quad w_y = 1 + \frac{\sigma_y}{j\omega\varepsilon}, \quad w_z = 1 + \frac{\sigma_z}{j\omega\varepsilon}, \tag{1.2.7}$$

i.e. the formulation (1.2.6) is a unified formulation of all PMLs..

1.2.4 Examples of PML Formulations

In this subsection, we shall take the 5 most popular existing PML formulations as examples to demonstrate that all of them can be all derived from the unified system (1.2.6). In all this derivations, we shall assume that all fields are zero for $t \leq 0$ in the PML regions, which are necessary for inverse Fourier transform back to the time domain.

Formulation 1: Zhao-Cangellaris's formulation [ZC2].

A different mathematical formulation was investigated by Zhao-Cangellaris in [ZC2] for Maxwell's equations using some properly constructed anistropic media and no any splitting of the fields. The edge-PML formulation in the frequency domain can be written as follows:

$$\left\{\begin{aligned} j\omega\varepsilon w_y E_x &= w_x \frac{\partial H_z}{\partial y} \\ j\omega\varepsilon w_x E_y &= -w_y \frac{\partial H_z}{\partial x} \\ -j\omega\mu w_x w_y H_z &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}, \end{aligned}\right. \tag{1.2.8}$$

which leads directly to the following edge-PML formulation in the time domain:

$$\left\{\begin{aligned} \varepsilon\frac{\partial E_x}{\partial t} + \sigma_y E_x &= \frac{\partial H_z}{\partial y} + \frac{\sigma_x}{\varepsilon}\int_0^t \frac{\partial H_z(\tau)}{\partial y} d\tau \\ \varepsilon\frac{\partial E_y}{\partial t} + \sigma_x E_y &= -\frac{\partial H_z}{\partial x} - \frac{\sigma_y}{\varepsilon}\int_0^t \frac{\partial H_z(\tau)}{\partial x} d\tau \\ \frac{\partial H_z}{\partial t} + \frac{\sigma_x+\sigma_y}{\varepsilon} H_z + & \frac{\sigma_x\sigma_y}{\varepsilon^2}\int_0^t H_z(\tau)d\tau = -\frac{1}{\mu}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right). \end{aligned}\right. \tag{1.2.9}$$

Clearly, by multiplying the first and second equation of the unified formulation (1.2.6) by w_x and w_y respectively, then we deduce the Zhao-Cangellaris's formulation (1.2.8) directly from (1.2.6). Furthermore, using the edge-PML (1.2.8), we can obtain the following face-PML formulation in the frequency domain, by taking $\sigma_y = 0$, i.e. $w_y = 1$:

$$\left\{\begin{aligned} j\omega\varepsilon E_x &= w_x \frac{\partial H_z}{\partial y} \\ j\omega\varepsilon w_x E_y &= -\frac{\partial H_z}{\partial x} \\ -j\omega\mu w_x H_z &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}. \end{aligned}\right. \tag{1.2.10}$$

The corresponding face-PML formulation in the time domain can then be written as:

$$\left\{\begin{aligned} \varepsilon\frac{\partial E_x}{\partial t} &= \frac{\partial H_z}{\partial y} + \frac{\sigma_x}{\varepsilon}\int_0^t \frac{\partial H_z(\tau)}{\partial y} d\tau \\ \varepsilon\frac{\partial E_y}{\partial t} + \sigma_x E_y &= -\frac{\partial H_z}{\partial x} \\ \frac{\partial H_z}{\partial t} + \frac{\sigma_x}{\varepsilon} H_z &= -\frac{1}{\mu}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right). \end{aligned}\right. \tag{1.2.11}$$

Formulation 2: Gedney's formulation [G].

In his paper, Gedney proposed a mixed finite element method, based on the anisotropic uniaxial formulation of the PML, to simulate wave propagation on unbounded domains. His derivatives led to the following edge-PML formulation in the frequency domain:

$$\begin{cases} j\omega w_y D_x = & \frac{\partial H_z}{\partial y} \\ E_x = & \frac{1}{\varepsilon} w_x D_x \\ j\omega w_x D_y = & -\frac{\partial H_z}{\partial x} \\ E_y = & \frac{1}{\varepsilon} w_y D_y \\ -j\omega w_x B_z = & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \\ w_y H_z = & \frac{1}{\mu} B_z, \end{cases} \tag{1.2.12}$$

which implies readily the formulation in the time domain:

$$\begin{cases} \frac{\partial D_x}{\partial t} = & -\frac{1}{\varepsilon}\sigma_y D_x + \frac{\partial H_z}{\partial y} \\ \frac{\partial E_x}{\partial t} = & \frac{1}{\varepsilon}\frac{\partial D_x}{\partial t} + \frac{\sigma_x}{\varepsilon^2} D_x \\ \frac{\partial D_y}{\partial t} = & -\frac{1}{\varepsilon}\sigma_x D_y - \frac{\partial H_z}{\partial x} \\ \frac{\partial E_y}{\partial t} = & \frac{1}{\varepsilon}\frac{\partial D_y}{\partial t} + \frac{\sigma_y}{\varepsilon^2} D_y \\ \frac{\partial B_z}{\partial t} = & -\sigma_x B_z - (\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) \\ \frac{\partial H_z}{\partial t} = & -\sigma_y H_z + \frac{1}{\mu}\frac{\partial B_z}{\partial t}. \end{cases} \tag{1.2.13}$$

If define

$$D_x = \varepsilon \frac{1}{w_x} E_x, \quad D_y = \varepsilon \frac{1}{w_y} E_y, \quad B_z = \mu w_y H_z$$

in the unified formulation (1.2.6), we find that formulation (1.2.12) is equivalent to (1.2.6). Furthermore, using the edge-PML (1.2.12), we can obtain the following face-PML formulation in the frequency domain, by taking $\sigma_y = 0$, i.e. $w_y = 1$:

$$\begin{cases} j\omega D_x = & \frac{\partial H_z}{\partial y} \\ E_x = & \frac{1}{\varepsilon} w_x D_x \\ j\omega w_x D_y = & -\frac{\partial H_z}{\partial x} \\ E_y = & \frac{1}{\varepsilon} D_y \\ -j\omega w_x B_z = & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \\ H_z = & \frac{1}{\mu} B_z. \end{cases} \tag{1.2.14}$$

The corresponding face-PML formulation in the time domain can then be stated as:

$$\begin{cases} \frac{\partial D_x}{\partial t} = & \frac{\partial H_z}{\partial y} \\ \frac{\partial E_x}{\partial t} = & \frac{1}{\varepsilon}\frac{\partial D_x}{\partial t} + \frac{\sigma_x}{\varepsilon^2} D_x \\ \frac{\partial D_y}{\partial t} = & -\frac{1}{\varepsilon}\sigma_x D_y - \frac{\partial H_z}{\partial x} \\ \frac{\partial E_y}{\partial t} = & \frac{1}{\varepsilon}\frac{\partial D_y}{\partial t} \\ \frac{\partial B_z}{\partial t} = & -\sigma_x B_z - (\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) \\ \frac{\partial H_z}{\partial t} = & \frac{1}{\mu}\frac{\partial B_z}{\partial t}. \end{cases} \tag{1.2.15}$$

Formulation 3: Becache-Joly's formulation [BJ].

Becache and Joly further developed the Zhao-Cangellaris's work and introduced the following edge-PML equations in the frequency domain:

$$\begin{cases} j\omega D_x = & \frac{\partial H_z}{\partial y} \\ w_y E_x = & \frac{1}{\varepsilon} w_x D_x \\ j\omega D_y = & -\frac{\partial H_z}{\partial x} \\ w_x E_y = & \frac{1}{\varepsilon} w_y D_y \\ -j\omega B_z = & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \\ w_x w_y H_z = & \frac{1}{\mu} B_z, \end{cases} \tag{1.2.16}$$

which implies readily the formulation in the time domain:

$$\begin{cases} \frac{\partial D_x}{\partial t} = & \frac{\partial H_z}{\partial y} \\ \frac{\partial E_x}{\partial t} + \frac{\sigma_y}{\varepsilon} E_x = & \frac{1}{\varepsilon}\frac{\partial D_x}{\partial t} + \frac{\sigma_x}{\varepsilon^2} D_x \\ \frac{\partial D_y}{\partial t} = & -\frac{\partial H_z}{\partial x} \\ \frac{\partial E_y}{\partial t} + \frac{\sigma_x}{\varepsilon} E_y = & \frac{1}{\varepsilon}\frac{\partial D_y}{\partial t} + \frac{\sigma_y}{\varepsilon^2} D_y \\ \frac{\partial B_z}{\partial t} = & -(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) \\ \frac{\partial^2 H_z}{\partial t^2} + \frac{\sigma_x+\sigma_y}{\varepsilon}\frac{\partial H_z}{\partial t} & +\sigma_x\sigma_y H_z = \frac{1}{\mu}\frac{\partial^2 B_z}{\partial t^2}. \end{cases} \tag{1.2.17}$$

If define

$$\varepsilon w_y E_x = w_x D_x, \quad \varepsilon w_x E_y = w_y D_y, \quad \mu w_x w_y H_z = B_z$$

in the unified formulation (1.2.6), we find that formulation (1.2.16) is equivalent to (1.2.6). Moreover, using the edge-PML (1.2.16), we can obtain the following face-PML formulation in the frequency domain, by taking $\sigma_y = 0$, i.e. $w_y = 1$:

$$\left\{\begin{aligned} j\omega D_x =& \ \tfrac{\partial H_z}{\partial y} \\ E_x =& \ \tfrac{1}{\varepsilon} w_x D_x \\ j\omega D_y =& \ -\tfrac{\partial H_z}{\partial x} \\ w_x E_y =& \ \tfrac{1}{\varepsilon} D_y \\ -j\omega B_z =& \ \tfrac{\partial E_y}{\partial x} - \tfrac{\partial E_x}{\partial y} \\ w_x H_z =& \ \tfrac{1}{\mu} B_z. \end{aligned}\right. \tag{1.2.18}$$

The corresponding time domain system is then stated as follows:

$$\left\{\begin{aligned} \tfrac{\partial D_x}{\partial t} =& \ \tfrac{\partial H_z}{\partial y} \\ \tfrac{\partial E_x}{\partial t} =& \ \tfrac{1}{\varepsilon}\tfrac{\partial D_x}{\partial t} + \tfrac{\sigma_x}{\varepsilon^2} D_x \\ \tfrac{\partial D_y}{\partial t} =& \ -\tfrac{\partial H_z}{\partial x} \\ \tfrac{\partial E_y}{\partial t} + \tfrac{\sigma_x}{\varepsilon} E_y =& \ \tfrac{1}{\varepsilon}\tfrac{\partial D_y}{\partial t} \\ \tfrac{\partial B_z}{\partial t} =& \ -(\tfrac{\partial E_y}{\partial x} - \tfrac{\partial E_x}{\partial y}) \\ \tfrac{\partial H_z}{\partial t} + \tfrac{\sigma_x}{\varepsilon} H_z =& \ \tfrac{1}{\mu}\tfrac{\partial B_z}{\partial t}. \end{aligned}\right. \tag{1.2.19}$$

Formulation 4: Ziolkowski's formulation [Z].

Ziolkowski introduced a new concept "Lorentz material" for an absorbing layer, and derived the following edge-PML system:

$$\left\{\begin{aligned} j\omega\varepsilon w_y E_x =& \ \tfrac{\partial H_z}{\partial y} - J_x \\ j\omega\varepsilon J_x =& \ -\sigma_x \tfrac{\partial H_z}{\partial y} \\ j\omega\varepsilon w_x E_y =& \ -\tfrac{\partial H_z}{\partial x} - J_y \\ j\omega\varepsilon J_y =& \ \sigma_y \tfrac{\partial H_z}{\partial x} \\ j\omega H_z + \tfrac{1}{\varepsilon}(\sigma_x + \sigma_y) H_z =& \ -\tfrac{1}{\mu}(\tfrac{\partial E_y}{\partial x} - \tfrac{\partial E_x}{\partial y}) - K \\ j\omega\varepsilon^2 K =& \ \sigma_x \sigma_y H_z, \end{aligned}\right. \tag{1.2.20}$$

which gives then the time domain equations:

$$\begin{cases} \varepsilon\frac{\partial E_x}{\partial t} + \sigma_y E_x = & \frac{\partial H_z}{\partial y} - J_x \\ \varepsilon\frac{\partial J_x}{\partial t} = & -\sigma_x\frac{\partial H_z}{\partial y} \\ \varepsilon\frac{\partial E_y}{\partial t} + \sigma_x E_y = & -\frac{\partial H_z}{\partial x} - J_y \\ \varepsilon\frac{\partial J_y}{\partial t} = & \sigma_y\frac{\partial H_z}{\partial x} \\ \frac{\partial H_z}{\partial t} + \frac{1}{\varepsilon}(\sigma_x+\sigma_y)H_z = & -\frac{1}{\mu}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) - K \\ \varepsilon^2\frac{\partial K}{\partial t} = & \sigma_x\sigma_y H_z. \end{cases} \tag{1.2.21}$$

If define

$$\sigma_x\frac{\partial H_z}{\partial y} = -j\omega\varepsilon J_x, \quad \sigma_y\frac{\partial H_z}{\partial x} = j\omega\varepsilon J_y, \quad \sigma_x\sigma_y H_z = j\omega\varepsilon^2 K$$

in the unified formulation (1.2.6), we find that formulation (1.2.20) is also equivalent to (1.2.6). And moreover, using the edge-PML (1.2.20) we can obtain the following face-PML formulation in the frequency domain, by taking $\sigma_y = 0$, i.e. $w_y = 1$:

$$\begin{cases} j\omega\varepsilon w_y E_x = & \frac{\partial H_z}{\partial y} - J_x \\ j\omega\varepsilon J_x = & -\sigma_x\frac{\partial H_z}{\partial y} \\ j\omega\varepsilon w_x E_y = & -\frac{\partial H_z}{\partial x} \\ j\omega H_z + \frac{1}{\varepsilon}\sigma_x H_z = & -\frac{1}{\mu}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}), \end{cases} \tag{1.2.22}$$

and the corresponding time domain equations:

$$\begin{cases} \varepsilon\frac{\partial E_x}{\partial t} + \sigma_y E_x = & \frac{\partial H_z}{\partial y} - J_x \\ \varepsilon\frac{\partial J_x}{\partial t} = & -\sigma_x\frac{\partial H_z}{\partial y} \\ \varepsilon\frac{\partial E_y}{\partial t} + \sigma_x E_y = & -\frac{\partial H_z}{\partial x} \\ \frac{\partial H_z}{\partial t} + \frac{1}{\varepsilon}\sigma_x H_z = & -\frac{1}{\mu}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}). \end{cases} \tag{1.2.23}$$

Formulation 5: Berenger's formulation [B1].

In his first innovative work on PMLs, Berenger used the techniques to split the transverse electric wave equation and introduce a damping factor $\sigma(x)$ in each equation in those places where the normal derivative operator ∂_x appears. The PML system in the time domain derived by Berenger can be stated as:

$$\left\{\begin{array}{rl} \varepsilon\frac{\partial E_x}{\partial t} = & \frac{\partial H_z}{\partial y} \\ \varepsilon\frac{\partial E_y}{\partial t} + \sigma E_y = & -\frac{\partial H_z}{\partial x} \\ \frac{\partial H^x}{\partial t} + \frac{\sigma}{\varepsilon}H^x = & -\frac{1}{\mu}\frac{\partial E_y}{\partial x} \\ \frac{\partial H^y}{\partial t} = & \frac{1}{\mu}\frac{\partial E_x}{\partial y}. \end{array}\right. \tag{1.2.24}$$

If we introduce a new variable $\widetilde{E}_x$ by

$$\widetilde{E}_x = w_x E_x, \tag{1.2.25}$$

then we can derive

$$\left\{\begin{array}{rl} \varepsilon\frac{\partial E_x}{\partial t} = & \frac{\partial H_z}{\partial y} \\ \varepsilon\frac{\partial E_y}{\partial t} + \sigma E_y = & -\frac{\partial H_z}{\partial x} \\ \frac{\partial H_z}{\partial t} + \frac{\sigma}{\varepsilon}H_z = & -\frac{1}{\mu}(\frac{\partial E_y}{\partial x} - \frac{\partial \widetilde{E}_x}{\partial y}) \\ \varepsilon\frac{\partial E_x}{\partial t} + \sigma E_x = & \frac{\partial \widetilde{E}_x}{\partial t}, \end{array}\right. \tag{1.2.26}$$

which yields the PML equations in frequency domain:

$$\left\{\begin{array}{rl} j\omega\varepsilon E_x = & \frac{\partial H_z}{\partial y} \\ j\omega\varepsilon w_x E_y = & -\frac{\partial H_z}{\partial x} \\ j\omega\mu w_x H_z = & -(\frac{\partial E_y}{\partial x} - \frac{\partial \widetilde{E}_x}{\partial y}) \\ w_x E_x = & \widetilde{E}_x. \end{array}\right. \tag{1.2.27}$$

Recalling the unified formulation (1.2.10), we find that if we let $(\widetilde{E}_x, E_y, H_z)$ satisfy the unified formulation (1.2.10), then we obtain the following system:

$$\left\{\begin{array}{rl} j\omega\varepsilon\widetilde{E}_x = & w_x\frac{\partial H_z}{\partial y} \\ j\omega\varepsilon w_x E_y = & -\frac{\partial H_z}{\partial x} \\ -j\omega\mu w_x H_z = & \frac{\partial E_y}{\partial x} - \frac{\partial \widetilde{E}_x}{\partial y}, \end{array}\right. \tag{1.2.28}$$

which is equivalent to (1.2.27).

1.3 Inhomogeneous Initial Conditions

The homogeneous initial conditions are used for the derivations of PML equations in most existing works. In this section, we shall demonstrate that the PML equations with inhomogeneous initial conditions can be converted into PML systems with homogenous initial conditions and some extra lower-order terms. All our subsequent considerations take place in the PML regions. Consider the TE_z model with inhomogeneous initial conditions

$$\begin{cases} \varepsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y}, & E_x(x,y,0) = E_{0x}(x,y), \\ \varepsilon \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}, & E_y(x,y,0) = E_{0y}(x,y), \\ \mu \frac{\partial H_z}{\partial t} = -(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}), & H_z(x,y,0) = H_{0z}(x,y), \end{cases} \tag{1.3.1}$$

and taking

$$\begin{cases} E_x(x,y,t) = \widetilde{E}_x(x,y,t) + E_{0x}(x,y), \\ E_y(x,y,t) = \widetilde{E}_y(x,y,t) + E_{0y}(x,y), \\ H_z(x,y,t) = \widetilde{H}_z(x,y,t) + H_{0z}(x,y), \end{cases}$$

then $(\widetilde{E}_x, \widetilde{E}_y, \widetilde{H}_z)$ satisfy the following equations:

$$\begin{cases} \varepsilon \frac{\partial \widetilde{E}_x}{\partial t} = \frac{\partial \widetilde{H}_z}{\partial y} + \frac{\partial H_{0z}}{\partial y}, & \widetilde{E}_x(x,y,0) = 0, \\ \varepsilon \frac{\partial \widetilde{E}_y}{\partial t} = -\frac{\partial \widetilde{H}_z}{\partial x} - \frac{\partial H_{0z}}{\partial x}, & \widetilde{E}_y(x,y,0) = 0, \\ \mu \frac{\partial \widetilde{H}_z}{\partial t} = -(\frac{\partial \widetilde{E}_y}{\partial x} - \frac{\partial \widetilde{E}_x}{\partial y}) - (\frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y}), & \widetilde{H}_z(x,y,0) = 0. \end{cases} \tag{1.3.2}$$

Using (1.2.8), we see that the PML equations in the frequency domain are

$$\begin{cases} j\omega\varepsilon w_y \widetilde{E}_x = w_x \frac{\partial \widetilde{H}_z}{\partial y} + 2\pi\delta(w) w_x \frac{\partial H_{0z}}{\partial y} \\ j\omega\varepsilon w_x \widetilde{E}_y = -w_y \frac{\partial \widetilde{H}_z}{\partial x} - 2\pi\delta(w) w_y \frac{\partial H_{0z}}{\partial x} \\ j\omega\mu w_x w_y \widetilde{H}_z = -(\frac{\partial \widetilde{E}_y}{\partial x} - \frac{\partial \widetilde{E}_x}{\partial y}) - 2\pi\delta(w)(\frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y}). \end{cases} \tag{1.3.3}$$

By the inverse Fourier transform

$$(\frac{\delta(\omega)}{j\omega})^\vee = \frac{1}{2\pi} t,$$

we obtain the corresponding PML equations in the time domain

$$\begin{cases} \varepsilon\frac{\partial\widetilde{E}_x}{\partial t}+\sigma_y\widetilde{E}_x= & \frac{\partial\widetilde{H}_z}{\partial y}+\frac{\sigma_x}{\varepsilon}\int_0^t\frac{\partial\widetilde{H}_z(\tau)}{\partial y}d\tau+\frac{\partial H_{0z}}{\partial y}+\frac{\sigma_x}{\varepsilon}\frac{\partial H_{0z}}{\partial y}t \\ \varepsilon\frac{\partial\widetilde{E}_y}{\partial t}+\sigma_x\widetilde{E}_y= & -\frac{\partial\widetilde{H}_z}{\partial x}-\frac{\sigma_x}{\varepsilon}\int_0^t\frac{\partial\widetilde{H}_z(\tau)}{\partial x}d\tau-\frac{\partial H_{0z}}{\partial x}-\frac{\sigma_y}{\varepsilon}\frac{\partial H_{0z}}{\partial x}t \\ \frac{\partial\widetilde{H}_z}{\partial t}+\frac{\sigma_x+\sigma_y}{\varepsilon}\widetilde{H}_z & +\frac{\sigma_x\sigma_y}{\varepsilon^2}\int_0^t\widetilde{H}_z(\tau)d\tau=-\frac{1}{\mu}(\frac{\partial\widetilde{E}_y}{\partial x}-\frac{\partial\widetilde{E}_x}{\partial y})-\frac{1}{\mu}(\frac{\partial E_{0y}}{\partial x}-\frac{\partial E_{0x}}{\partial y}) \end{cases} \tag{1.3.4}$$

with homogenous initial conditions.

Similarly, for Formulation 2 with inhomogeneous conditions

$$\begin{cases} \frac{\partial D_x}{\partial t}= & \frac{\partial H_z}{\partial y}, & D_x(x,y,0)=D_{0x}(x,y), \\ \frac{\partial E_x}{\partial t}= & \frac{1}{\varepsilon}\frac{\partial D_x}{\partial t}, & E_x(x,y,0)=E_{0x}(x,y), \\ \frac{\partial D_y}{\partial t}= & -\frac{\partial H_z}{\partial x}, & D_y(x,y,0)=D_{0y}(x,y), \\ \frac{\partial E_y}{\partial t}= & \frac{1}{\varepsilon}\frac{\partial D_y}{\partial t}, & E_y(x,y,0)=E_{0y}(x,y), \\ \frac{\partial B_z}{\partial t}= & -(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}), & B_z(x,y,0)=B_{0z}(x,y), \\ \frac{\partial H_z}{\partial t}= & \frac{1}{\mu}\frac{\partial B_z}{\partial t}, & H_z(x,y,0)=H_{0z}(x,y), \end{cases} \tag{1.3.5}$$

we set

$$\begin{cases} D_x(x,y,t)= & \widetilde{D}_x(x,y,t)+D_{0x}(x,y), \\ E_x(x,y,t)= & \widetilde{E}_x(x,y,t)+E_{0x}(x,y), \\ D_y(x,y,t)= & \widetilde{D}_y(x,y,t)+D_{0y}(x,y), \\ E_y(x,y,t)= & \widetilde{E}_y(x,y,t)+E_{0y}(x,y), \\ B_z(x,y,t)= & \widetilde{B}_z(x,y,t)+B_{0z}(x,y), \\ H_z(x,y,t)= & \widetilde{H}_z(x,y,t)+H_{0z}(x,y), \end{cases}$$

then using (1.2.13), we derive the PML equations for $(\widetilde{D}_x, \widetilde{E}_x, \widetilde{D}_y, \widetilde{E}_y, \widetilde{B}_z, \widetilde{H}_z)$:

$$\begin{cases} \frac{\partial\widetilde{D}_x}{\partial t}= & -\frac{1}{\varepsilon}\sigma_y\widetilde{D}_x+\frac{\partial\widetilde{H}_z}{\partial y}+\frac{\partial H_{0z}}{\partial y} \\ \frac{\partial\widetilde{E}_x}{\partial t}= & \frac{1}{\varepsilon}\frac{\partial\widetilde{D}_x}{\partial t}+\frac{\sigma_x}{\varepsilon^2}\widetilde{D}_x \\ \frac{\partial\widetilde{D}_y}{\partial t}= & -\frac{1}{\varepsilon}\sigma_x\widetilde{D}_y-\frac{\partial\widetilde{H}_z}{\partial x}-\frac{\partial H_{0z}}{\partial x} \\ \frac{\partial\widetilde{E}_y}{\partial t}= & \frac{1}{\varepsilon}\frac{\partial\widetilde{D}_y}{\partial t}+\frac{\sigma_y}{\varepsilon^2}\widetilde{D}_y \\ \frac{\partial\widetilde{B}_z}{\partial t}= & -\sigma_x\widetilde{B}_z-(\frac{\partial\widetilde{E}_y}{\partial x}-\frac{\partial\widetilde{E}_x}{\partial y})-(\frac{\partial E_{0y}}{\partial x}-\frac{\partial E_{0x}}{\partial y}) \\ \frac{\partial\widetilde{H}_z}{\partial t}= & -\sigma_y\widetilde{H}_z+\frac{1}{\mu}\frac{\partial\widetilde{B}_z}{\partial t} \end{cases} \tag{1.3.6}$$

with homogenous initial conditions.

Chapter 2

Numerical Analysis of PMLs

Perfectly Matched layer (PML) was introduced by Berenger in 1994 for numerically solving Maxwell equations in infinite domains. This technique has been widely and successfully applied in solving different types of wave propagation problems [Ha]. Numerous PML formulations have been proposed, and several survey articles have appeared in the literature in which authors have mainly focused on the derivations of PML equations, main advantages and shortcomings of various different types of PML equations and their applications in engineering problems. To our knowledge, there have been very few results that are concerned with the convergence, stability and error estimates of this powerful techniques. Most discussions on convergence of PMLs are based on Helmholtz type equations in the frequency domain, due to its simplicity and the availability of exact solution representations outside of disk or one side of a straight line in 2D. These can be found in the work of Collino and Monk [CM], Chew and Weedon [CW], Chen and Wu [CWu] and Petropoulos [P]. It seems that there is still no discussion on stability, convergence and error estimates in the time domain for PML equations, which are in fact listed as open problems in the survey papers (c.f. [CM] & [Ha]).

It is the purpose of this chapter to present a complete discussion about the numerical analysis for the 1D PML equations in the time domain. Though our techniques are only applicable for 1D, they also provide some insights on 2D or 3D problems.

The plan of this chapter is as follows. In section 2.1 we introduce several PMLs for continuous wave equations, Maxwell equations, acoustic equations and 1-D hyperbolic systems, and give the exponential decays estimates for each continuous PMLs. Section 2.2 deals with the various up-winding schemes corresponding to the continuous PMLs proposed in section 2.1, and presents the exponential decays estimates for each discrete schemes. Finally, in section 2.3 and section 2.4 we propose the modified Yee scheme and Lax-Wendroff scheme respectively, which both are second order in spatial and also have exponential decays.

2.1 Continuous PMLs

In this section we consider two types of perfectly matched layers for hyperbolic systems, which include Maxwell's equations, acoustic equations in 1-D, and so on. Some similarities and differences are compared. This section is organized as follows: At the beginning, we introduce infinite and finite PMLs for wave equations, Maxwell equations and acoustic equations respectively. And then, we propose the PML equations for general 1D hyperbolic systems.

First, let us recall from [Ha] a general formulation of PMLs for the hyperbolic system:

$$\frac{\partial u}{\partial t} + A(y)\frac{\partial u}{\partial x} + \sum_j B_j(y)\frac{\partial u}{\partial y_j} + C(y)u = 0, \tag{2.1.1}$$

where $u \in R^n$, $A(y) \in R^{n\times n}$, $B_j(y) \in R^{n\times n}$, and the physical domain is located at $x < 0$, the interface is at $x = 0$, and the absorbing region is $x > 0$. Then the PML equation for $x > 0$ is given by

$$\frac{\partial u}{\partial t} + A(y)(\frac{\partial u}{\partial x} + \sigma\mu u) + \sum_j B_j(y)\frac{\partial u}{\partial y_j} + C(y)u + w = 0, \tag{2.1.2}$$

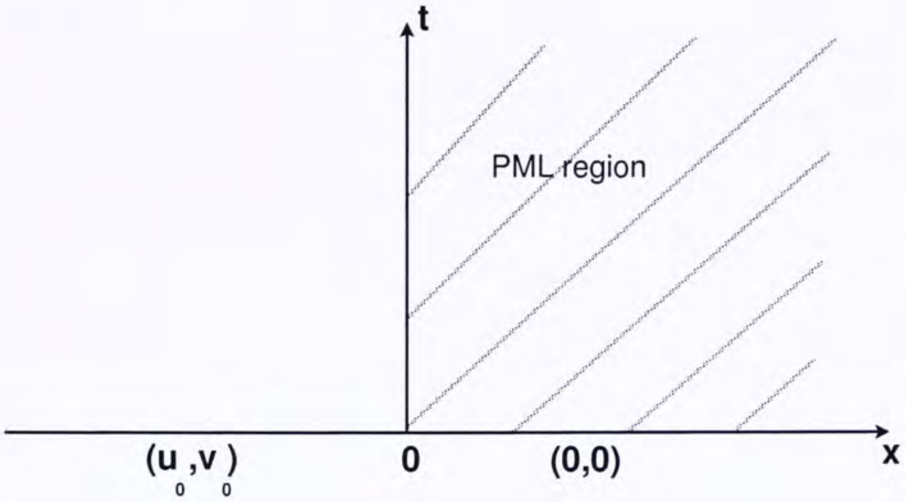

Figure 2.1: Infinite Perfectly Matched Layers.

$$Rw + \sigma w + \sigma A(y)(\frac{\partial u}{\partial x} + \sigma \mu u) = 0, \tag{2.1.3}$$

where

$$R = \frac{\partial}{\partial t} + \sum_j \beta_j \frac{\partial}{\partial y_j} + \alpha,$$

and β_j, α are constants. This system in fact contains a artificial variable w.

2.1.1 PMLs for Wave Equations

We start with the following one dimensional system of wave equations:

$$\begin{cases} u_t = v_x, \quad v_t = u_x, & x \in R, \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in R, \end{cases} \tag{2.1.4}$$

where u_0 and v_0 are known functions, and u_0 and v_0 have support only for $x < 0$, i.e. we use $x = 0$ as the interface. Taking $U = u + v, \ \ V = u - v$, we obtain from (2.1.4) that

$$\begin{cases} U_t = U_x, & x \in R, \quad t > 0, \\ V_t + V_x = 0, & x \in R, \quad t > 0 \end{cases} \tag{2.1.5}$$

and $U(x,0) = u_0 + v_0, V(x,0) = u_0 - v_0$. By observation of the system (2.1.5), we see that the characteristic curves for U and V are, respectively, given by

$$x + t = \xi \quad \text{and} \quad x - t = \xi, \quad \xi \in R.$$

Therefore U and V satisfy

$$U(x,t) = u_0(x+t) + v_0(x+t), \quad V(x,t) = u_0(x-t) - v_0(x-t)$$

and we can calculate easily that

$$u(x,t) = \frac{1}{2}(u_0(x+t) + v_0(x+t) + u_0(x-t) - v_0(x-t)),$$

$$v(x,t) = \frac{1}{2}(u_0(x+t) + v_0(x+t) - u_0(x-t) + v_0(x-t)).$$

Since it is obviously seen that U vanishes in the positive real axis, which in turn implies by the characteristic method that U vanishes identically in the first quadrant of the $(x,t)-$plane, i.e., $U(x,t) = 0$ for $x \geq 0$, $t \geq 0$. For V, noticing that there exist some constant C_0 only dependent on u_0 and v_0 such that $V \leq C_0$ on the whole real axis, we can analogously deduce as above that $V \leq C_0$ in the upper plane of R^2.

Let $W = (u,v)^T$, then (2.1.4) can be written as

$$W_t + AW_x = 0,$$

where $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. By (2.1.2)-(2.1.3), we see that Hagstrom's PML for $x > 0$ is given by

$$\begin{cases} u_t + \sigma u = v_x + \sigma v, & x \geq 0, \quad t > 0, \\ v_t + \sigma v = u_x + \sigma u, & x \geq 0, \quad t > 0, \\ u(x,0) = v(x,0) = 0, & x \geq 0, \end{cases} \tag{2.1.6}$$

where

$$\sigma(x) = \begin{cases} \sigma(x), & x > 0, \ \sigma'(x) \geq 0, \\ 0, & x \leq 0. \end{cases}$$

Note that for $x < 0$, (2.1.6) is reduced to (2.1.4) with $\sigma(x) = 0, \ \ x \leq 0$.

We shall demonstrate below that for any $\sigma(x)$ with $\sigma(0) = 0$, and $\sigma'(x) \geq 0$, the solution of (2.1.4) and (2.1.6) are identical for $x \leq 0$, and the solution of (2.1.6) decay to zero exponentially as $x \to +\infty$.

For simplicity, let (u^σ, v^σ) denote the solution of (2.1.6), with (u^0, v^0) being the solution of the original wave equation (2.1.4).

Lemma 2.1.1 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$, then the system (2.1.6) has a unique solution (u^σ, v^σ), and*

$$u^\sigma = u^0, \quad v^\sigma = v^0 \quad for \quad x \leq 0, \quad t > 0 \tag{2.1.7}$$

and there exists some constant $C_0 > 0$ independent of σ, x and t, such that

$$|u^\sigma(x,t)| \leq C_0 e^{-2\int_0^x \sigma(\xi)d\xi}, \; |v^\sigma(x,t)| \leq C_0 e^{-2\int_0^x \sigma(\xi)d\xi}, \quad \forall\, x \geq 0, \; t > 0. \tag{2.1.8}$$

Proof. Taking $U^\sigma = u^\sigma + v^\sigma, \;\; V^\sigma = u^\sigma - v^\sigma$, we obtain from (2.1.6) that

$$\begin{cases} U_t^\sigma = U_x^\sigma, & x \in R, \quad t > 0, \\ V_t^\sigma + 2\sigma V^\sigma + V_x^\sigma = 0, & x \in R, \quad t > 0 \end{cases} \tag{2.1.9}$$

and $U^\sigma(x,0) = u_0 + v_0, V^\sigma(x,0) = u_0 - v_0$. In order to eliminate σ from the second equation above, we set

$$M^\sigma = V^\sigma e^{2\int_0^x \sigma(\xi)d\xi}, \tag{2.1.10}$$

then it follows from (2.1.9) that

$$\begin{cases} U_t^\sigma = U_x^\sigma, \quad M_t^\sigma + M_x^\sigma = 0, & x \in R, \quad t > 0, \\ U^\sigma(x,0) = u_0 + v_0, \quad M^\sigma(x,0) = u_0 - v_0, & x \in R. \end{cases} \tag{2.1.11}$$

Therefore U^σ and M^σ satisfy

$$U^\sigma(x,t) = u_0(x+t) + v_0(x+t), \quad M^\sigma(x,t) = u_0(x-t) - v_0(x-t)$$

and

$$V^\sigma(x,t) = e^{-2\int_0^x \sigma(\xi)d\xi}(u_0(x-t) - v_0(x-t)).$$

We can calculate easily that

$$u^\sigma = \frac{1}{2}\left[(u_0(x+t) + v_0(x+t)) + e^{-2\int_0^x \sigma(\xi)d\xi}(u_0(x-t) - v_0(x-t))\right],$$

$$v^\sigma = \frac{1}{2}\left[(u_0(x+t)+v_0(x+t)) - e^{-2\int_0^x \sigma(\xi)d\xi}(u_0(x-t)-v_0(x-t))\right],$$

which implies the existence of (u^σ, v^σ).

Clearly the above transformation (2.1.10) is also valid for $\sigma = 0$, i.e. $M^0 = u - v$, $U^0 = u + v$, and

$$\begin{cases} U_t^0 = U_x^0, \quad M_t^0 + M_x^0 = 0, & x \in R, \quad t > 0 \\ U^0(x,0) = u_0 + v_0, \quad M^0(x,0) = u_0 - v_0. & x \in R. \end{cases} \tag{2.1.12}$$

By the uniqueness of the hyperbolic system, we see clearly that

$$U^\sigma = U^0, \quad M^\sigma = M^0, \quad x \in R, \quad t > 0, \tag{2.1.13}$$

which in turn implies

$$u^\sigma(0,t) = u^0(0,t), \quad v^\sigma(0,t) = v^0(0,t), \quad t \geq 0.$$

Therefore we have derived $u^\sigma = u^0$, $v^\sigma = v^0$, for $x \leq 0$, $t \geq 0$.

Next, by observation of the system (2.1.12), we see that the characteristic curves for U^0 and M^0 are, respectively, given by

$$x + t = \xi \quad \text{and} \quad x - t = \xi, \quad \xi \in R.$$

Since it is obviously seen that U^0 vanishes in the positive real axis, which in turn implies by the characteristic method that U^0 vanishes identically in the first quadrant of the $(x,t)-$plane, i.e., $U^\sigma(x,t) = 0$ for $x \geq 0$, $t > 0$. For M^0, noticing that there exist some constant C_0 only dependent on u_0 and v_0 such that $M^0 \leq C_0$ on the whole real axis, we can analogously deduce as above that $M^0 \leq C_0$ in the upper plane of R^2. Now, by (2.1.13), we have that $U^\sigma = 0, M^\sigma \leq C_0$ for $x \geq 0, t > 0$. By the definition of M^σ (2.1.10), further we obtain

$$|V^\sigma| \leq C_0 e^{-\int_0^x \sigma(\xi)d\xi}, \quad x \geq 0, \ t > 0,$$

together with $U^\sigma = u^\sigma + v^\sigma$, $V^\sigma = u^\sigma - v^\sigma$, we obtain the decay result (2.1.8).

♯

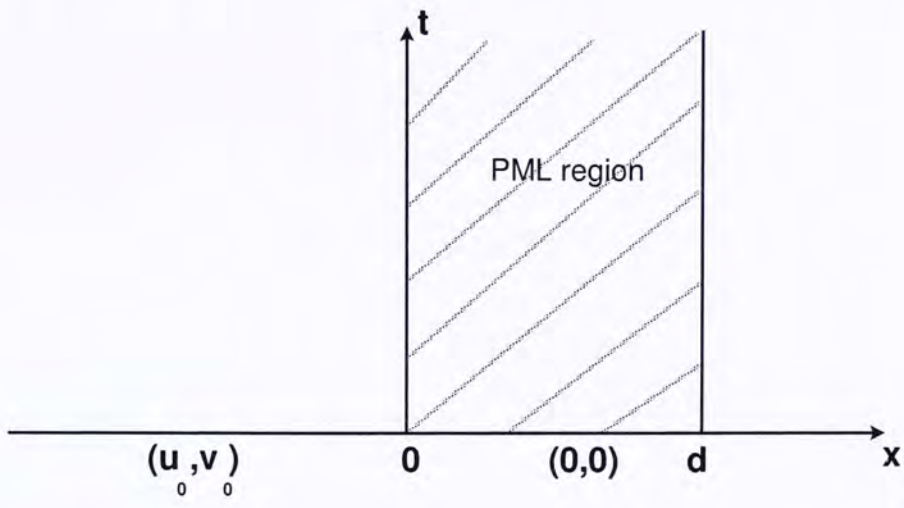

Figure 2.2: Finite Perfectly Matched Layers.

2.1.2 Finite PMLs for Wave Equations

The discussion in Sect. 2.1.1 demonstrate that the infinite PML equations produce non-reflection matched layers, but it is not suitable for computational purpose. We shall now consider the finite PML. In what follows, we consider the interval $(-\infty, d)$ instead of interval $(-d, d)$ for convenience.

The finite Hagstrom's PML equations for (2.1.4) are given by

$$\begin{cases} u_t^{\sigma,d} + \sigma u^{\sigma,d} = v_x^{\sigma,d} + \sigma v^{\sigma,d}, & x \in (-\infty, d), \quad t > 0, \\ v_t^{\sigma,d} + \sigma v^{\sigma,d} = u_x^{\sigma,d} + \sigma u^{\sigma,d}, & x \in (-\infty, d), \quad t > 0, \\ u^{\sigma,d}(x, 0) = u_0(x), & x \in (-\infty, d), \\ v^{\sigma,d}(x, 0) = v_0(x), & x \in (-\infty, d), \\ u^{\sigma,d}(d, t) + v^{\sigma,d}(d, t) = 0, & t \geq 0. \end{cases} \tag{2.1.14}$$

Lemma 2.1.2 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$, then the system (2.1.14) has a unique solution $(u^{\sigma,d}, v^{\sigma,d})$ and there exists some constant $C_0 > 0$ such that*

$$|u^{\sigma,d}(x,t)| \leq C_0 e^{-2\int_0^x \sigma(\xi)d\xi}, \ |v^{\sigma,d}(x,t)| \leq C_0 e^{-2\int_0^x \sigma(\xi)d\xi}, \quad \forall\ x \geq 0,\ t > 0.$$

Proof. Taking $U^{\sigma,d}(x,t) = u^{\sigma,d}(x,t) + v^{\sigma,d}(x,t)$, $V^{\sigma,d}(x,t) = u^{\sigma,d}(x,t) - v^{\sigma,d}(x,t)$, we get

$$\begin{cases} U_t^{\sigma,d} = U_x^{\sigma,d}, & x \in (-\infty, d), \quad t > 0, \\ V_t^{\sigma,d} + 2\sigma V^{\sigma,d} + V_x^{\sigma,d} = 0, & x \in (-\infty, d), \quad t > 0, \\ U^{\sigma,d}(x,0) = u_0 + v_0, & x \in (-\infty, d), \\ V^{\sigma,d}(x,0) = u_0 - v_0, & x \in (-\infty, d), \\ U^{\sigma,d}(d,t) = 0, & t \geq 0. \end{cases}$$

We introduce notation

$$\begin{cases} \mathcal{E}_u = U^\sigma - U^{\sigma,d}, \\ \mathcal{E}_v = V^\sigma - V^{\sigma,d}, \end{cases}$$

where U^σ and V^σ satisfy (2.1.9), then we have

$$\begin{cases} \frac{\partial \mathcal{E}_u}{\partial t} = \frac{\partial \mathcal{E}_u}{\partial x}, & x \in (-\infty, d), \quad t > 0, \\ \frac{\partial \mathcal{E}_v}{\partial t} + 2\sigma \mathcal{E}_v + \frac{\partial \mathcal{E}_v}{\partial x} = 0, & x \in (-\infty, d), \quad t > 0, \\ \mathcal{E}_u(x,0) = \mathcal{E}_v(x,0) = 0, & x \in (-\infty, d) \\ \mathcal{E}_u(d,t) = U^\sigma(d,t), & t \geq 0. \end{cases}$$

By the Maximum Principle, we have

$$|\mathcal{E}_u(x,t)| \leq \max_{x,\ t}\{|\mathcal{E}_u(x,0)|, |\mathcal{E}_u(d,t)|\} \leq \max_t\{|U^\sigma(d,t)|\},$$

$$|\mathcal{E}_v(x,t)| \leq \max_x\{|\mathcal{E}_v(x,0)|\} = 0,$$

together with Lemma 2.1.1 which shows that

$$|u^\sigma(d,t)| \leq C_0 e^{-2\int_0^d \sigma(\xi)d\xi}, \quad |v^\sigma(d,t)| \leq C_0 e^{-2\int_0^d \sigma(\xi)d\xi},$$

$$|U^\sigma(d,t)| \leq C_0 e^{-2\int_0^d \sigma(\xi)d\xi}, \quad |V^\sigma(d,t)| \leq C_0 e^{-2\int_0^d \sigma(\xi)d\xi},$$

implies

$$|\mathcal{E}_u(x,t)| \leq C_0 e^{-2\int_0^d \sigma(\xi)d\xi}, \quad |\mathcal{E}_v(x,t)| = 0.$$

From Lemma 2.1.1, we also obtain

$$|U^\sigma(x,t)| \leq C_0 e^{-2\int_0^x \sigma(\xi)d\xi}, \quad |V^\sigma(x,t)| \leq C_0 e^{-2\int_0^x \sigma(\xi)d\xi},$$

together with $U^{\sigma,d} = U^\sigma - \mathcal{E}_u$ and $V^{\sigma,d} = V^\sigma - \mathcal{E}_v$, we have

$$|U^{\sigma,d}(x,t)| \le C_0 e^{-2\int_0^x \sigma(\xi)d\xi}, \quad |V^{\sigma,d}(x,t)| \le C_0 e^{-2\int_0^x \sigma(\xi)d\xi}.$$

which imply immediately

$$|u^{\sigma,d}(x,t)|,\ |v^{\sigma,d}(x,t)| \le C_0 e^{-2\int_0^x \sigma(\xi)d\xi}, \quad x \ge 0, \quad t > 0.$$

♯

From the proof of Lemma 2.1.2 , we note that

$$|\mathcal{E}_u(x,t)| \le C_0 e^{-2\int_0^d \sigma(\xi)d\xi}, \quad |\mathcal{E}_v(x,t)| = 0,$$

i.e.

$$|[u^\sigma(x,t) + v^\sigma(x,t)] - [u^{\sigma,d}(x,t) + v^{\sigma,d}(x,t)]| = |U^\sigma - U^{\sigma,d}| = |\mathcal{E}_u| \le C_0 e^{-2\int_0^d \sigma(\xi)d\xi},$$

$$|[u^\sigma(x,t) - v^\sigma(x,t)] - [u^{\sigma,d}(x,t) - v^{\sigma,d}(x,t)]| = |V^\sigma - V^{\sigma,d}| = |\mathcal{E}_v| = 0,$$

therefore we have the following corollary:

Corollary 2.1.1 *Assume that $\sigma(x) \in L^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \ge 0$, then for Hagstrom's PML, we have*

$$\begin{aligned} |u^\sigma(x,t) - u^{\sigma,d}(x,t)| &\le C_0 e^{-2\int_0^d \sigma(\xi)d\xi}, \quad x \in (-\infty, d),\ t > 0, \\ |v^\sigma(x,t) - v^{\sigma,d}(x,t)| &\le C_0 e^{-2\int_0^d \sigma(\xi)d\xi}, \quad x \in (-\infty, d),\ t > 0. \end{aligned} \tag{2.1.15}$$

Remark 2.1.1 *(2.1.15) implies that at the interface $x = 0$, the difference between the infinite PML fields $u^\sigma(0,t)$, $v^\sigma(0,t)$ and finite PML fields $u^{\sigma,d}(0,t)$ and $v^{\sigma,d}(0,t)$ are very small, exponentially small in terms of the width $d > 0$ of the finite PML and the PML parameter $\sigma(x)$.*

2.1.3 Berenger's PMLs for Maxwell Equations

In this subsection, we consider another kind of PMLs equation different from Hagstrom's PML in subsection 2.1.1: the Berenger's PML equations. Applying the theory developed in [B1] to the Maxwell system:

$$\frac{\partial E}{\partial t} = -\frac{\partial H}{\partial x}, \quad \frac{\partial H}{\partial t} = -\frac{\partial E}{\partial x}, \quad x \in R, \quad t > 0, \tag{2.1.16}$$

we derive the PML equation:

$$\begin{cases} \frac{\partial E}{\partial t} + \sigma E = -\frac{\partial H}{\partial x}, & x \in R, \quad t > 0 \\ \frac{\partial H}{\partial t} + \sigma H = -\frac{\partial E}{\partial x}, & x \in R, \quad t > 0. \end{cases} \tag{2.1.17}$$

The physical domain is $x < 0$, and the PML domain is $x > 0$, $x = 0$ is the interface, the initial data $E(x, 0)$ and $H(x, 0)$ are assumed to have the support for $x \leq 0$, i.e.

$$E_0(x) = H_0(x) = 0, \quad x \geq 0.$$

For simplicity, let (E^σ, H^σ) denote the solution of (2.1.17), with (E^0, H^0) being the solution of the original system (2.1.16). Then we have

Lemma 2.1.3 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$, then there exists a unique solution (E^σ, H^σ) to (2.1.17), and*

$$E^\sigma = E^0, \quad H^\sigma = H^0 \quad for \quad x \leq 0, \quad t > 0 \tag{2.1.18}$$

and for some constant, the following holds

$$|E^\sigma(x,t)| \leq C_0 e^{-\int_0^x \sigma(\xi) d\xi}, \ |H^\sigma(x,t)| \leq C_0 e^{-\int_0^x \sigma(\xi) d\xi}, \quad \forall \ x \geq 0, \ t > 0. \tag{2.1.19}$$

Proof. Taking $U^\sigma = E^\sigma + H^\sigma$ and $V^\sigma = E^\sigma - H^\sigma$, we obtain from (2.1.17) that

$$\begin{cases} U_t^\sigma + \sigma U^\sigma + U_x^\sigma = 0, \\ V_t^\sigma + \sigma V^\sigma - V_x^\sigma = 0, \end{cases} \tag{2.1.20}$$

and $U^\sigma(x, 0) = E_0(x) + H_0(x)$,$V^\sigma(x, 0) = E_0(x) - H_0(x)$. In order to eliminate σ from the above equations, we set

$$\begin{cases} M^\sigma = U^\sigma e^{\int_0^x \sigma(\xi) d\xi}, \\ N^\sigma = V^\sigma e^{-\int_0^x \sigma(\xi) d\xi}, \end{cases} \tag{2.1.21}$$

then M^σ and N^σ satisfy

$$\begin{cases} M_t^\sigma + M_x^\sigma = 0, \quad N_t^\sigma = N_x^\sigma & x \in R, \quad t > 0, \\ M^\sigma(x, 0) = E_0 + H_0, \quad N^\sigma(x, 0) = E_0 - H_0, & x \in R. \end{cases} \tag{2.1.22}$$

This yields

$$M^\sigma(x,t) = E_0(x-t) + H_0(x-t), \quad N^\sigma(x,t) = E_0(x+t) - H_0(x+t)$$

and

$$U^\sigma(x,t) = e^{-2\int_0^x \sigma(\xi)d\xi}(E_0(x-t) + H_0(x-t)),$$
$$V^\sigma(x,t) = e^{2\int_0^x \sigma(\xi)d\xi}(E_0(x+t) - H_0(x+t)).$$

From these relations, we can calculate easily that

$$E^\sigma = \frac{1}{2}\left[e^{-2\int_0^x \sigma(\xi)d\xi}(E_0(x-t) + H_0(x-t)) + e^{2\int_0^x \sigma(\xi)d\xi}(E_0(x+t) - H_0(x+t))\right],$$

$$H^\sigma = \frac{1}{2}\left[e^{-2\int_0^x \sigma(\xi)d\xi}(E_0(x-t) + H_0(x-t)) - e^{2\int_0^x \sigma(\xi)d\xi}(E_0(x+t) - H_0(x+t))\right],$$

which implies the existence of (E^σ, H^σ).

From (2.1.22), we have $M^\sigma = M^0, N^\sigma = N^0$, or $E^\sigma(0,t) = E^0(0,t), H^\sigma(0,t) = H^0(0,t)$ for all $t \geq 0$, therefore $E^\sigma = E^0,\ x \leq 0,\ t \geq 0$. Because $V^\sigma(x,0) = 0$, $x \geq 0$, by the same arguments as the final part of the proof in Lemma 2.1.1, we have deduce $V^\sigma = 0$ for $x \geq 0, t \geq 0$ and $|M^\sigma| \leq C_0$, and further we obtain

$$|U^\sigma| \leq C_0 e^{-\int_0^x \sigma(\xi)d\xi}, \quad x \geq 0,\ t \geq 0,$$

together with $V^\sigma = 0$ for $x \geq 0, t \geq 0$, implies (2.1.19).

♯

Remark 2.1.2 *The PML equations (2.1.6) and (2.1.17) are perfect without any reflection, but with different exponential decay, i.e. the fields given by (2.1.6) decay twice faster than the fields given by the Berenger's PML (2.1.17).*

2.1.4 Finite Berenger's PMLs for Maxwell Equations

Similar to the subsection 2.1.2, here we shall consider the Berenger's PMLs for Maxwell Equations with finite width $d > 0$ corresponding to the Berenger's PML equations last subsection. Let $E^{\sigma,d}$ and $H^{\sigma,d}$ be the solution of the following system:

$$\begin{cases} E_t^{\sigma,d} + \sigma E^{\sigma,d} = -H_x^{\sigma,d}, & x \in (-\infty, d\ ), \quad t > 0, \\ H_t^{\sigma,d} + \sigma H^{\sigma,d} = -E_x^{\sigma,d}, & x \in (-\infty, d\ ), \quad t > 0, \\ E^{\sigma,d}(x,0) = E_0, \quad H^{\sigma,d}(x,0) = H_0, & x \in (-\infty, d), \\ E^{\sigma,d}(d,t) - H^{\sigma,d}(d,t) = 0, & t \geq 0. \end{cases} \tag{2.1.23}$$

Then we have

Lemma 2.1.4 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$, then the system (2.1.23) has a unique solution $(E^{\sigma,d}, H^{\sigma,d})$ and there exists some constant $C_0 > 0$ such that*

$$|E^{\sigma,d}(x,t)| \leq C_0 e^{-\int_0^x \sigma(\xi)d\xi}, \ |H^{\sigma,d}(x,t)| \leq C_0 e^{-\int_0^x \sigma(\xi)d\xi}, \quad x \geq 0, \quad t > 0.$$

Proof. Taking $U^{\sigma,d}(x,t) = E^{\sigma,d}(x,t) + H^{\sigma,d}(x,t)$, $V^{\sigma,d}(x,t) = E^{\sigma,d}(x,t) - H^{\sigma,d}(x,t)$, we get

$$\begin{cases} U_t^{\sigma,d} + \sigma U^{\sigma,d} + U_x^{\sigma,d} = 0, & x \in (-\infty, d\ ), \quad t > 0, \\ V_t^{\sigma,d} + \sigma V^{\sigma,d} - V_x^{\sigma,d} = 0, & x \in (-\infty, d\ ), \quad t > 0, \\ U^{\sigma,d}(x,0) = u_0 + v_0, & x \in (-\infty, d), \\ V^{\sigma,d}(x,0) = u_0 - v_0, & x \in (-\infty, d), \\ V^{\sigma,d}(d,t) = 0, & t \geq 0. \end{cases}$$

We introduce notation

$$\begin{cases} \mathcal{E}_u = U^\sigma - U^{\sigma,d}, \\ \mathcal{E}_v = V^\sigma - V^{\sigma,d}, \end{cases}$$

where U^σ and V^σ satisfy (2.1.20), then we get

$$\begin{cases} \frac{\partial \mathcal{E}_u}{\partial t} + 2\sigma \mathcal{E}_u + \frac{\partial \mathcal{E}_u}{\partial x} = 0, & x \in (-\infty, d), \quad t > 0, \\ \frac{\partial \mathcal{E}_v}{\partial t} + 2\sigma \mathcal{E}_v - \frac{\partial \mathcal{E}_v}{\partial x} = 0, & x \in (-\infty, d), \quad t > 0, \\ \mathcal{E}_u(x,0) = \mathcal{E}_v(x,0) = 0, & x \in (-\infty, d), \\ \mathcal{E}_v(d,t) = V^\sigma(d,t), & t > 0. \end{cases}$$

By the Maximum Principle, we have

$$|\mathcal{E}_u(x,t)| \le \max_x\{|\mathcal{E}_u(x,0)|\} = 0,$$

$$|\mathcal{E}_v(x,t)| \le \max_{x,\ t}\{|\mathcal{E}_v(x,0)|, |\mathcal{E}_v(d,t)|\} \le \max_t\{|V^\sigma(d,t)|\},$$

together with Lemma 2.1.3 which indicates that

$$|E^\sigma(d,t)| \le C_0 e^{-\int_0^d \sigma(\xi)d\xi}, \quad |H^\sigma(d,t)| \le C_0 e^{-\int_0^d \sigma(\xi)d\xi},$$

$$|U^\sigma(d,t)| \le C_0 e^{-\int_0^d \sigma(\xi)d\xi}, \quad |V^\sigma(d,t)| \le C_0 e^{-\int_0^d \sigma(\xi)d\xi},$$

implies

$$|\mathcal{E}_u(x,t)| = 0, \quad |\mathcal{E}_v(x,t)| \le C_0 e^{-\int_0^d \sigma(\xi)d\xi}.$$

From Lemma 2.1.3, we also obtain

$$|U^\sigma(x,t)| \le C_0 e^{-\int_0^x \sigma(\xi)d\xi}, \quad |V^\sigma(x,t)| \le C_0 e^{-\int_0^x \sigma(\xi)d\xi},$$

together with$U^{\sigma,d} = U^\sigma - \mathcal{E}_u$ and $V^{\sigma,d} = V^\sigma - \mathcal{E}_v$, we have

$$|U^{\sigma,d}(x,t)| \le C_0 e^{-\int_0^x \sigma(\xi)d\xi}, \quad |V^{\sigma,d}(x,t)| \le C_0 e^{-\int_0^x \sigma(\xi)d\xi}.$$

Now, it is easy to see

$$|E^{\sigma,d}(x,t)|,\ |H^{\sigma,d}(x,t)| \le C_0 e^{-\int_0^x \sigma(\xi)d\xi}, \quad x \ge 0, \quad t > 0.$$

♯

From the proof of Lemma 2.1.4 , we note that

$$|\mathcal{E}_u(x,t)| = 0, \quad |\mathcal{E}_v(x,t)| \le C_0 e^{-\int_0^d \sigma(\xi)d\xi},$$

i.e.

$$\begin{aligned}
|[E^\sigma(x,t) + H^\sigma(x,t)] - [E^{\sigma,d}(x,t) + H^{\sigma,d}(x,t)]| &= |U^\sigma - U^{\sigma,d}| \\
&= |\mathcal{E}_u| = 0, \\
|[E^\sigma(x,t) - H^\sigma(x,t)] - [E^{\sigma,d}(x,t) - H^{\sigma,d}(x,t)]| &= |V^\sigma - V^{\sigma,d}| \\
&= |\mathcal{E}_v| \le C_0 e^{-\int_0^d \sigma(\xi)d\xi},
\end{aligned}$$

which lead the following corollary:

Corollary 2.1.2 *Assume that $\sigma(x) \in L^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$, then for the Berenger's PML, we have*

$$\begin{aligned} &|E^\sigma(x,t) - E^{\sigma,d}(x,t)| \leq C_0 e^{-\int_0^d \sigma(\xi)d\xi}, \quad x \in (-\infty, d),\ t > 0, \\ &|H^\sigma(x,t) - H^{\sigma,d}(x,t)| \leq C_0 e^{-\int_0^d \sigma(\xi)d\xi}, \quad x \in (-\infty, d),\ t > 0. \end{aligned} \tag{2.1.24}$$

2.1.5 PMLs for Acoustic Equations

In this subsection, we shall introduce the PML equations for the acoustic equations. Now consider the following 1-D acoustic equations:

$$\begin{cases} u_t + U_0 u_x = -\frac{1}{\rho_0} p_x, & x \in R, \quad t > 0, \\ p_t + U_0 p_x = -\rho_0 c_0^2 u_x, & x \in R, \quad t > 0, \end{cases} \tag{2.1.25}$$

where ρ_0 is density, c_0 is speed of light and U_0 is the mean speed in x-direction. One can rewrite (2.1.25) as

$$\begin{pmatrix} u \\ p \end{pmatrix}_t + \begin{pmatrix} U_0 & \frac{1}{\rho_0} \\ \rho_0 c_0^2 & U_0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}_x = 0, \quad x \in R, \quad t > 0. \tag{2.1.26}$$

Using (2.1.2), its Hagstrom's PML is given by [Ha]

$$\begin{pmatrix} u \\ p \end{pmatrix}_t + \sigma \begin{pmatrix} u \\ p \end{pmatrix} + \begin{pmatrix} U_0 & \frac{1}{\rho_0} \\ \rho_0 c_0^2 & U_0 \end{pmatrix} \left(\begin{pmatrix} u \\ p \end{pmatrix}_x + \sigma \begin{pmatrix} u \\ p \end{pmatrix} \right) = 0 \quad x \in R, \ t > 0, \tag{2.1.27}$$

where $x = 0$ is the interface between the computational domain and the PML.

Lemma 2.1.5 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$, then Hagstrom's PML (2.1.27) and the original system (2.1.26) generate the same solution for all $x \leq 0, t > 0$, and there exists some constant $C_0 > 0$ such that*

$$|u(x,t)| + |p(x,t)| \leq C_0 e^{-(\frac{1+U_0+c_0}{U_0+c_0}) \int_0^x \sigma(\xi)d\xi}, \quad \forall\, x \geq 0, \quad t > 0. \tag{2.1.28}$$

Proof. Let

$$A = \begin{pmatrix} U_0 & \frac{1}{\rho_0} \\ \rho_0 c_0^2 & U_0 \end{pmatrix}, \ R = \frac{1}{\sqrt{1+\rho_0^2 c_0^2}} \begin{pmatrix} 1 & 1 \\ -\rho_0 c_0 & \rho_0 c_0 \end{pmatrix}, \ U = R^{-1} \begin{pmatrix} u \\ p \end{pmatrix}. \tag{2.1.29}$$

We see from (2.1.27) that

$$U_t + \sigma U + \Lambda(U_x + \sigma U) = 0,$$

where $\Lambda = \text{diag}(U_0 - c_0, U_0 + c_0)$ formed by the two eigenvalues of A, and R is the matrix consisting of eigenvectors associated with eigenvalues $U_0 + c_0$ and $U_0 - c_0$, or componentwise,

$$\begin{cases} U_t^1 + \sigma(1 + U_0 - c_0)U^1 + (U_0 - c_0)U_x^1 = 0, \\ U_t^2 + \sigma(1 + U_0 + c_0)U^2 + (U_0 + c_0)U_x^2 = 0, \\ \begin{pmatrix} U^1(x,0) \\ U^2(x,0) \end{pmatrix} = R^{-1} \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix}. \end{cases} \tag{2.1.30}$$

Since $U^1(x,0) = 0$ for $x \geq 0$, it follows $U^1(x,t) = 0$ for $x \geq 0$, $t \geq 0$ due to $U_0 - c_0 < 0$. For $U^2(x,t)$, by completely the same arguments as the final part of the proof in Lemma 2.1.1, it is clearly that

$$|U^2(x,t)| \leq C_0 e^{-(\frac{1+U_0+c_0}{U_0+c_0})\int_0^x \sigma(\xi)d\xi}, \quad x \geq 0, \quad t > 0.$$

thus, by $\begin{pmatrix} u \\ p \end{pmatrix} = R \begin{pmatrix} U^1 \\ U^2 \end{pmatrix}$, we find that

$$|u(x,t)| + |p(x,t)| \leq C_0 e^{-(\frac{1+U_0+c_0}{U_0+c_0})\int_0^x \sigma(\xi)d\xi}, \quad \forall\, x \geq 0, \quad t > 0.$$

♯

Remark 2.1.3 *Clearly we have $U_0 < c_0$, since c_0 is speed of light.*

2.1.6 Berenger's PMLs for Acoustic Equations

In this subsection, we give two kind of Berenger's PML for equation (2.1.26), which are equivalent under the coordinate transformation. The first type Berenger's PML for equation (2.1.26) is given by

$$\begin{pmatrix} u \\ p \end{pmatrix}_t + \sigma \begin{pmatrix} u \\ p \end{pmatrix} + \begin{pmatrix} U_0 & \frac{1}{\rho_0} \\ \rho_0 c_0^2 & U_0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}_x = 0, \quad x \in R, \quad t > 0, \tag{2.1.31}$$

where $x = 0$ is interface. We shall derive the following exponential decay results:

Lemma 2.1.6 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$, then the Berenger's PML (2.1.31) and the original system (2.1.26) generate the same solution for all $x \leq 0, t > 0$, and there exists some constant $C_0 > 0$ such that*

$$|u(x,t)| + |p(x,t)| \leq C_0 e^{-(\frac{1}{U_0+c_0}) \int_0^x \sigma(\xi) d\xi}, \quad \forall\, x \geq 0, \quad t > 0. \tag{2.1.32}$$

Proof. Let A, U and R be the same as in (2.1.29). Then we can write (2.1.31) as

$$U_t + \sigma U + \Lambda U_x = 0,$$

where $\Lambda = \text{diag}(U_0 - c_0, U_0 + c_0)$ formed by the two eigenvalues of A, and R is the matrix consisting of eigenvectors associated with eigenvalues $U_0 + c_0$ and $U_0 - c_0$, or componentwise,

$$\begin{cases} U_t^1 + \sigma U^1 + (U_0 - c_0) U_x^1 = 0, \\ U_t^2 + \sigma U^2 + (U_0 + c_0) U_x^2 = 0, \\ \begin{pmatrix} U^1(x,0) \\ U^2(x,0) \end{pmatrix} = R^{-1} \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix}. \end{cases} \tag{2.1.33}$$

Since $U^1(x,0) = 0$ for $x \geq 0$, it follows that $U^1(x,t) = 0$ for $x \geq 0,\ t \geq 0$ due to $U_0 - c_0 < 0$. For $U^2(x,t)$, by the same arguments as the final part of the proof in Lemma 2.1.1, it is clearly that

$$|U^2(x,t)| \leq C_0 e^{-(\frac{1}{U_0+c_0}) \int_0^x \sigma(\xi) d\xi}, \quad x \geq 0, \quad t > 0.$$

thus, by $\begin{pmatrix} u \\ p \end{pmatrix} = R \begin{pmatrix} U^1 \\ U^2 \end{pmatrix}$, we conclude that

$$|u(x,t)| + |p(x,t)| \leq C_0 e^{-(\frac{1}{U_0+c_0}) \int_0^x \sigma(\xi) d\xi}, \quad \forall\, x \geq 0, \quad t > 0.$$

♯

Remark 2.1.4 *The decay rates of scheme (2.1.27) is $e^{-\int_0^x \sigma(\xi) d\xi}$ time faster than the decay rate of the scheme (2.1.31).*

Using coordinate transform, one may derive another form of the Berenger PML for equations (2.1.26), which is given by [TY]

$$\begin{cases} u_t - \frac{M}{\rho_0 c_0} p_t + (1-M^2)^{\frac{1}{2}} c_0 \sigma u = -\frac{1-M^2}{\rho_0} p_x, \\ p_t - \rho_0 c_0 M u_t + (1-M^2)^{\frac{1}{2}} c_0 \sigma p = -\rho_0 c_0^2 (1-M^2) u_x, \end{cases} \tag{2.1.34}$$

where $M = \frac{U_0}{c_0} < 1$. For this system, we have

Lemma 2.1.7 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$, then the Berenger's PML (2.1.34) and the original system (2.1.26) generate the same solution for all $x \leq 0, t > 0$, and there exists some constant $C_0 > 0$ such that*

$$|u(x,t)| + |p(x,t)| \leq C_0 e^{-\frac{1}{(1-M^2)^{1/2}} \int_0^x \sigma(\xi) d\xi}, \quad \forall\ x \geq 0, \quad t > 0. \tag{2.1.35}$$

Proof. Let $W = (u,p)^T$, then one may rewrite (2.1.34) as

$$\begin{pmatrix} 1 & -\frac{M}{\rho_0 c_0} \\ -\rho_0 c_0 M & 1 \end{pmatrix} W_t + (1-M^2)^{\frac{1}{2}} c_0 \sigma W + \begin{pmatrix} 0 & \frac{1-M^2}{\rho_0} \\ \rho_0 c_0^2 (1-M^2) & 0 \end{pmatrix} W_x = 0$$

or simply it is written as

$$SW_t + (1-M^2)^{\frac{1}{2}} c_0 \sigma W + DW_x = 0 \tag{2.1.36}$$

where

$$S = \begin{pmatrix} 1 & -\frac{M}{\rho_0 c_0} \\ -\rho_0 c_0 M & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \frac{1-M^2}{\rho_0} \\ \rho_0 c_0^2 (1-M^2) & 0 \end{pmatrix}.$$

The eigenvalues of S are given by $1+M$ and $1-M$, and eigenvalues of D by $c_0(1-M^2)$ and $c_0(M^2-1)$. And S and D can be diagonalized simultaneously by R given by (2.1.29), i.e. by letting $R^{-1}W = Z$, we find

$$\begin{cases} (1+M)Z_t^1 + (1-M^2)^{\frac{1}{2}} c_0 \sigma Z^1 - c_0(1-M^2) Z_x^1 = 0, \\ (1-M)Z_t^2 + (1-M^2)^{\frac{1}{2}} c_0 \sigma Z^2 + c_0(1-M^2) Z_x^2 = 0. \end{cases}$$

By completely the same arguments as the final part of the proof in Lemma 2.1.1, since $Z^1(x,0) = 0,\ x \geq 0$, then $Z^1(x,t) = 0$, for $x \geq 0,\ t \geq 0$ and

$$|Z^2(x,t)| \leq C_0 e^{-\frac{1}{(1-M^2)^{1/2}} \int_0^x \sigma(\xi) d\xi}, \quad x \geq 0, \quad t > 0.$$

Thus, by $\begin{pmatrix} u \\ p \end{pmatrix} = R \begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix}$, we find that

$$|u(x,t)| + |p(x,t)| \le C_0 e^{-\frac{1}{(1-M^2)^{1/2}} \int_0^x \sigma(\xi) d\xi}, \quad \forall\, x \ge 0, \quad t > 0.$$

♯

2.1.7 PMLs for 1-D Hyperbolic Systems

In the last subsection of the part, we propose the PML equations for the general 1D hyperbolic systems:

$$\begin{cases} u_t + Au_x = 0, & x \in R, \quad t > 0, \\ u(x,0) = u_0(x), & x \in R, \end{cases} \tag{2.1.37}$$

where $u_0(x) = 0$ for $x \ge 0$, and A is a hyperbolic matrix, i.e. there exist constants $\lambda_1^-, \cdots, \lambda_k^-, \lambda_{k+1}^+, \cdots, \lambda_s^+$ such that

$$R^{-1}AR = \mathrm{diag}(\lambda_1^-, \cdots, \lambda_k^-, \lambda_{k+1}^+, \cdots, \lambda_s^+) = \Lambda, \tag{2.1.38}$$

where $\lambda_l^- < 0$ and $\lambda_l^+ > 0$.

We recall that the Hagstrom's PML equations corresponding to (2.1.37) are as follows:

$$\begin{cases} u_t + \sigma u + A(u_x + \sigma u) = 0, & x \in R, \quad t > 0, \\ u(x,0) = u_0(x), & x \in R, \end{cases} \tag{2.1.39}$$

where $\sigma(x) = 0,\ x \le 0,\ \sigma'(x) \ge 0$. Let $w = R^{-1}u$, we find that

$$\begin{cases} w_t + \sigma w + \Lambda(w_x + \sigma w) = 0, & x \in R, \quad t > 0, \\ u(x,0) = R^{-1}u_0(x), & x \in R, \end{cases} \tag{2.1.40}$$

or

$$\begin{cases} w_t^l + \sigma(1 + \lambda_l^-)w^l + \lambda_l^- w_x^l = 0, & l = 1, \cdots, k, \\ w_t^l + \sigma(1 + \lambda_l^+)w^l + \lambda_l^+ w_x^l = 0, & l = k+1, \cdots, s \end{cases}$$

and $w^l(x,0) = 0$ for $x \geq 0$. Because $\lambda_l^- < 0$ and $\lambda_l^+ > 0$, by completely the same arguments as the final part of the proof in Lemma 2.1.1, we have that $w^l(x,t) = 0,\ x \geq 0,\ t \geq 0, l = 1, \cdots, k$ and

$$|w^l(x,t)| \leq C_0 e^{(-\frac{1+\lambda_l^+}{\lambda_l^+})\int_0^x \sigma(\xi)d\xi}, \qquad x \geq 0,\ t \geq 0,\ l = k+1, \cdots, s.$$

Lemma 2.1.8 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$. Let u^0 be the solution of (2.1.37) and u be the solution of (2.1.39), then we have*

(*i*) $u(x,t) = u^0(x,t),\ x \leq 0,\ t > 0.$

(*ii*) $|u(x,t)| \leq C_0 e^{-\mu \int_0^x \sigma(\xi)d\xi},\ x \geq 0,\ t > 0,$ *where* $\mu = \min\limits_{l \geq k+1} \left\{ \frac{1+\lambda_l^+}{\lambda_l^+} \right\}.$

Now we consider the Berenger's PML for equation (2.1.37):

$$\begin{cases} u_t + \sigma u + A u_x = 0, & x \in R, \quad t > 0, \\ u(x,0) = u_0(x), & x \in R, \end{cases} \tag{2.1.41}$$

where $\sigma(x) = 0, x \leq 0, \sigma'(x) \geq 0$. Letting $w = R^{-1}u$, it is easy to check that

$$\begin{cases} w_t + \sigma w + \Lambda w_x = 0, & x \in R, \quad t > 0, \\ u(x,0) = R^{-1}u_0(x), & x \in R, \end{cases} \tag{2.1.42}$$

or it can be written componentwise as follows:

$$\begin{cases} w_t^l + \sigma w^l + \lambda_l^- w_x^l = 0, & l = 1, \cdots, k, \\ w_t^l + \sigma w^l + \lambda_l^+ w_x^l = 0, & l = k+1, \cdots, s. \end{cases}$$

clearly, $w^l(x,0) = 0$ for $x \geq 0$. Because $\lambda_l^- < 0$ and $\lambda_l^+ > 0$, by completely the same arguments as the final part of the proof in Lemma 2.1.1, we derive that $w^l(x,t) = 0,\ x \geq 0,\ t \geq 0,\ l = 1, \cdots, k$ and

$$|w^l(x,t)| \leq C_0 e^{(-\frac{1}{\lambda_l^+})\int_0^x \sigma(\xi)d\xi}, \qquad x \geq 0,\ t \geq 0,\ l = k+1, \cdots, s.$$

This with $u = Rw$ leads to the following lemma:

Lemma 2.1.9 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$. Let u^0 be the solution of (2.1.37) and u be the solution of (2.1.41), then we have*

(i) $u(x,t) = u^0(x,t),\ \forall\ x \leq 0,\ t > 0.$

(ii) $|u(x,t)| \leq C_0 e^{-\mu \int_0^x \sigma(\xi) d\xi},\ \forall\ x \geq 0,\ t > 0,$ *where* $\mu = \min\limits_{l \geq k+1} \left\{ \frac{1}{\lambda_l^+} \right\}.$

Remark 2.1.5 *The decay rates of scheme (2.1.39) is $e^{-\int_0^x \sigma(\xi)d\xi}$ time faster than the decay rate of scheme (2.1.41).*

2.2 Discrete PMLs

In this section, we shall propose some modified first order up-winding schemes for the spatial discretization of those various continuous PML equations established in section 2.1. Convergence and stability analysis are made to those schemes, which together with the exponentially decay results obtained in this section, manifest the favorable aspects of these schemes theoretically.

This section is organized as follows: we deduce the infinite and finite discrete PMLs for the wave equation in the first two subsections and subsection 2.2.3 and 2.2.4 give the infinite and finite discrete Berenger's PMLs for the wave equation respectively. At the end of this section, we shall give the discrete PMLs for general 1D hyperbolic systems.

2.2.1 Discrete PMLs for Wave Equations

In this subsection, we shall derive two infinite discrete PML equations corresponding to the continuous PML equations. The first one is the so called up-winding scheme, and the second one is non-reflective in perfectly matched layers, which is an improvement of the first one. Recall the wave equation (2.1.4):

$$\begin{cases} u_t = v_x, \quad v_t = u_x, & x \in R,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in R \end{cases}$$

and its Hagstrom's PML equation (2.1.6):

$$\begin{cases} u_t + \sigma u = v_x + \sigma v, & x \in R, \quad t > 0, \\ v_t + \sigma v = u_x + \sigma u, & x \in R, \quad t > 0, \\ u(x,0) = u_0, \quad v(x,0) = v_0, & x \in R. \end{cases}$$

Let $U = u^\sigma + v^\sigma, V = u^\sigma - v^\sigma$, we obtain

$$\begin{cases} U_t = U_x, & x \in R, \quad t > 0, \\ V_t + 2\sigma V + V_x = 0, & x \in R, \quad t > 0. \end{cases} \tag{2.2.1}$$

We first study the up-winding scheme for discretizing the system. To do so, we consider finite time $0 < t < T$ and divide the time interval $[0, T]$ into equally-distributed subintervals:

$$0 = t_0 < t_1 < \cdots < t_M = T,$$

with $t_n = n\tau$ and $\tau = \frac{T}{M}$ being the time step size. For the space interval, we also partition in uniformly as follows:

$$-\infty < \cdots < x_{-j} < \cdots < x_0 < \cdots < x_j < \cdots \infty,$$

with $x_j = jh$ and h being the space mesh size. Let (U_j^n, V_j^n) be finite difference solution of (2.2.1), that is

$$\begin{gathered} \frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0, \\ \frac{V_j^{n+1} - V_j^n}{\Delta t} + 2\overline{\sigma}_j V_j^n + \frac{V_j^n - V_{j-1}^n}{\Delta x} = 0, \\ -\infty < j < \infty, \ 0 \le n \le M, \end{gathered} \tag{2.2.2}$$

where $u_j^{\sigma,n}$ and $v_j^{\sigma,n}$ are defined by

$$\begin{cases} U_j^n = u_j^{\sigma,n} + v_j^{\sigma,n}, \\ V_j^n = u_j^{\sigma,n} - v_j^{\sigma,n}. \end{cases}$$

We need to study the stability of (2.2.2) and the decay properties for $j \ge 0$. Clearly, if $r = \frac{\Delta t}{\Delta x} < 1$ (CFL-condition), then

$$\max_{j \in Z} |U_j^{n+1}| \le \max_{j \in Z} |U_j^n| \le \max_{j \in Z} |U_j^0|.$$

A simple calculation show that $\{V_j^n\}$ satisfy

$$V_j^{n+1} = (1 - 2\overline{\sigma}_j \Delta t - r)V_j^n + rV_{j-1}^n,$$

therefore, we get

$$V_j^{n+1} = [1 - r(2\overline{\sigma}_j \Delta x + 1)]V_j^n + rV_{j-1}^n$$

for some $\overline{\sigma}_j \approx \sigma(x_j)$. Now if we set $(2\overline{\sigma}_j \Delta x + 1)r \leq 1$, then

$$\begin{aligned} |V_j^{n+1}| &\leq \{1 - r(2\overline{\sigma}_j \Delta x + 1) + r\} \max_{j \in Z} |V_j^n| \\ &\leq (1 - 2r\overline{\sigma}_j \Delta t) \max_{j \in Z} |V_j^n| \leq \max_{j \in Z} |V_j^n| \leq \max_{j \in Z} |V_j^0|, \end{aligned}$$

thus the scheme (2.2.2) is stable.

Lemma 2.2.1 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$. Let (U_j^n, V_j^n) be the solution of (2.2.2) and $\overline{\sigma}_j$ be given by*

$$\overline{\sigma}_j = \frac{e^{2\int_{x_{j-1}}^{x_j} \sigma(\xi) d\xi} - 1}{2\Delta x}. \tag{2.2.3}$$

When the mesh size satisfies $r = \frac{\Delta t}{\Delta x} < 1$, then we have

$$\max_{j \in Z} |U_j^n| \leq \max_{j \in Z} |U_j^0|, \quad 1 \leq n \leq M.$$

Takeing $m_j^n = V_j^n e^{2\int_0^{x_j} \sigma(\xi) d\xi}$, then we obtain

$$\max_{j \in Z} |m_j^n| \leq \max_{j \in Z} |m_j^0|,$$

which implies that

$$|V_j^n| \leq |m_j^n| e^{-2\int_0^{x_j} \sigma(\xi) d\xi} \leq C_0 e^{-2\int_0^{x_j} \sigma(\xi) d\xi}, \quad \forall\ j \geq 0.$$

Proof. In order to show that V_j^n decays exponentially for $j \geq 0$, we multiply the second equation of (2.2.2) by $e^{2\int_0^{x_j} \sigma(\xi) d\xi}$, and set

$$m_j^n = V_j^n e^{2\int_0^{x_j} \sigma(\xi) d\xi}, \quad \text{for all } j, n. \tag{2.2.4}$$

Then $\{m_j^n\}$ satisfy

$$\frac{m_j^{n+1} - m_j^n}{\Delta t} + 2\overline{\sigma}_j m_j^n + \frac{m_j^n - m_{j-1}^n e^{2\int_{x_{j-1}}^{x_j} \sigma(\xi)d\xi}}{\Delta x} = 0,$$

which implies

$$\begin{aligned} m_j^{n+1} &= (1 - 2\overline{\sigma}_j \Delta t - r)m_j^n + r m_{j-1}^n e^{2\int_{x_{j-1}}^{x_j} \sigma(\xi)d\xi} \\ &= (1 - r(2\overline{\sigma}_j \Delta x + 1))m_j^n + r m_{j-1}^n e^{2\int_{x_{j-1}}^{x_j} \sigma(\xi)d\xi}. \end{aligned} \tag{2.2.5}$$

Noting

$$\overline{\sigma}_j = \frac{e^{2\int_{x_{j-1}}^{x_j} \sigma(\xi)d\xi} - 1}{2\Delta x},$$

together with (2.2.5), we can deduce

$$|m_j^{n+1}| \le \max_{j\in Z} |m_j^n| \qquad \text{for all } j \in Z,$$

and therefore

$$\max_{j\in Z} |m_j^{n+1}| \le \max_{j\in Z} |m_j^n| \le \cdots \le \max_{j\in Z} |m_j^0|. \tag{2.2.6}$$

Then from (2.2.3) and (2.2.6), we obtain

$$|V_j^n| \le |m_j^n| e^{-2\int_0^{x_j} \sigma(\xi)d\xi} \le C_0 e^{-2\int_0^{x_j} \sigma(\xi)d\xi}, \quad \forall\, j \ge 0.$$

♯

By Lemma 2.2.1, we see that the finite difference solution obtained by (2.2.2) decays exponentially as $j \to +\infty$. But V_0^{n+1}, U_0^{n+1} for $n \ge 0$, will not agree with the original finite difference solution obtained without PML. In order to have a perfectly matched finite difference solution, i.e., non-reflective finite difference solution at $j = 0$, we first set $m = Ve^{2\int_0^x \sigma(\xi)d\xi}$, then

$$m_t + m_x = 0, \quad \forall x \in R, \quad t \ge 0.$$

Clearly, the up-winding scheme for this system is given by

$$\frac{m_j^{n+1} - m_j^n}{\Delta t} + \frac{m_j^n - m_{j-1}^n}{\Delta x} = 0, \qquad j \in Z. \tag{2.2.7}$$

Define $V_j^n = m_j^n e^{-2\int_0^{x_j} \sigma(\xi)d\xi}$, and substituting into (2.2.7), we obtain the finite difference equation for V_j^n:

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + 2\overline{\sigma}_{j-1} V_{j-1}^n + \frac{V_j^n - V_{j-1}^n}{\Delta x} = 0, \qquad j \in Z, \tag{2.2.8}$$

where

$$\overline{\sigma}_{j-1} = \frac{1 - e^{-2\int_{x_{j-1}}^{x_j} \sigma(\xi)d\xi}}{2\Delta x} = e^{2\int_{x_{j-1}}^{x_j} \sigma(\xi)d\xi}\overline{\sigma}_j. \tag{2.2.9}$$

Lemma 2.2.2 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$. Let (U_j^n, V_j^n) be the finite difference solution of*

$$\begin{cases} \frac{U_j^{n+1}-U_j^n}{\Delta t} - \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0, \\ \frac{V_j^{n+1}-V_j^n}{\Delta t} + 2\overline{\sigma}_{j-1} V_j^n + \frac{V_j^n - V_{j-1}^n}{\Delta x} = 0, \\ \qquad -\infty < j < \infty, \ 0 \leq n \leq M, \end{cases} \tag{2.2.10}$$

and $\overline{\sigma}_{j-1}$ be given in (2.2.9). When the mesh size satisfies $r = \frac{\Delta t}{\Delta x} < 1$, then we have

$$\max_{j\in Z} |U_j^n| \leq \max_{j \in Z} |U_j^0|, \quad 1 \leq n \leq M.$$

Taking $m_j^n = V_j^n e^{2\int_0^{x_j} \sigma(\xi)d\xi}$, then we obtain

$$\max_{j\in Z} |m_j^n| \leq \max_{j\in Z} |m_j^0|,$$

which implies that

$$|V_j^n| \leq |m_j^n| e^{-2\int_0^{x_j} \sigma(\xi)d\xi} \leq C_0 e^{-2\int_0^{x_j} \sigma(\xi)d\xi}, \quad j \geq 0.$$

Thus the scheme (2.2.10) is stable and its solution decays exponentially. Moreover, the scheme (2.2.10) is non-reflective.

Proof. The first part of the lemma is obvious by the construction of the scheme. Next, we want to show the scheme (2.2.10) is also non-reflective. Consider (2.1.4) and let $U^0 = u^0 + v^0$ and $V^0 = u^0 - v^0$, then we obtain

$$\begin{cases} U_t^0 = U_x^0, & x \in R, \quad t > 0, \\ V_t^0 + V_x^0 = 0, & x \in R, \quad t > 0. \end{cases}$$

The corresponding up-winding scheme is

$$\frac{U_j^{0,n+1}-U_j^{0,n}}{\Delta t} - \frac{U_{j+1}^{0,n}-U_j^{0,n}}{\Delta x} = 0,$$
$$\frac{V_j^{0,n+1}-V_j^{0,n}}{\Delta t} + \frac{V_j^{0,n}-V_{j-1}^{0,n}}{\Delta x} = 0,$$
$$-\infty < j < \infty,\ 0 \le n \le M,$$

thus U_j^n and $U_j^{0,n}$ satisfy the same difference equation and $U_0^n = U_0^{0,n}$. By the construction, since V_j^n satisfies (2.2.8) and $V_j^n e^{2\int_0^{x_j}\sigma(\xi)d\xi} = m_j^n$, then m_j^n satisfies (2.2.7). Therefore m_j^n and $V_j^{0,n}$ satisfy the same difference equation and $V_0^n = m_0^n = V_0^{0,n}$. So, we have derived that $u_0^{\sigma,n} = u_0^{0,n}$ and $v_0^{\sigma,n} = v_0^{0,n}$, which means the scheme (2.2.10) is non-reflective.

♯

Thanks to the Taylor series expansions, we can derive the convergence of scheme (2.2.10).

Lemma 2.2.3 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \ge 0$. Let (U,V) be the exact solution of (2.2.1), and (U_j^n, V_j^n) be the finite difference solution of (2.2.10) and $\overline{\sigma}_{j-1}$ be given by (2.2.9). If*

$$\max_{j\in Z} |U_j^0 - U(x_j,0)| = \max_{j\in Z} |V_j^0 - V(x_j,0)| = O(\Delta x + \Delta t),$$

and the mesh size satisfies $r = \frac{\Delta t}{\Delta x} < 1$, then for any $n \ge 0$, we have

$$\max_{j\in Z} |U_j^n - U(x_j,t_n)| \le O(\Delta x + \Delta t), \tag{2.2.11}$$
$$\max_{j\in Z} |V_j^n - V(x_j,t_n)| \le O(\Delta x + \Delta t). \tag{2.2.12}$$

Proof. For the first equation of (2.2.1) at $x = x_j$, $t = t_n$, we have

$$\begin{aligned} 0 &= U_t(x_j,t_n) - U_x(x_j,t_n) \\ &= \left[\frac{U(x_j,t_{n+1}) - U(x_j,t_n)}{\Delta t} - \frac{\Delta t}{2}U_{tt}(x_j,\eta_n)\right] \\ &- \left[\frac{U(x_{j+1},t_n) - U(x_j,t_n)}{\Delta x} - \frac{\Delta x}{2}U_{xx}(\xi_j,t_n)\right] \end{aligned} \tag{2.2.13}$$

due to the Taylor expansion, where $t_n \le \eta_n \le t_{n+1}$, $x_j \le \xi_j \le x_{j+1}$.

Letting $e_j^n = U_j^n - U(x_j, t_n)$, subtracting (2.2.13) from (2.2.10), we obtain

$$\frac{e_j^{n+1} - e_j^n}{\Delta t} - \frac{e_{j+1}^n - e_j^n}{\Delta x} = -\frac{\Delta t}{2} U_{tt}(x_j, \eta_n) + \frac{\Delta x}{2} U_{xx}(\xi_j, t_n),$$

which gives

$$e_j^{n+1} \;=\; (1-r)e_j^n + re_{j+1}^n + \Delta t\left[\|U_{xx}\|_\infty \Delta x + \|U_{tt}\|_\infty \Delta t\right].$$

Since the mesh size satisfies $r = \frac{\Delta t}{\Delta x} < 1$, we deduce

$$|e^{n+1}| \le \|e^n\|_\infty + \Delta t\, O(\Delta x + \Delta t)$$

together with $\|e^0\|_\infty = O(\Delta x + \Delta t)$, we derive

$$\max_{j \in Z} |U_j^n - U(x_j, t_n)| = \|e^n\|_\infty \le O(\Delta x + \Delta t).$$

Consider the second equation of (2.2.1) at $x = x_j$ and $t = t_n$, we have

$$0 = V_t(x_j, t_n) + 2\sigma(x_j)V(x_j, t_n) + V_x(x_j, t_n).$$

Thanks to the Taylor series expansions, the above equation can be written as

$$\begin{aligned} 0 \;=\; & \left[\frac{V(x_j, t_{n+1}) - V(x_j, t_n)}{\Delta t} - \frac{\Delta t}{2} V_{tt}(x_j, \eta_n)\right] \\ + \; & \left[\frac{V(x_{j+1}, t_n) - V(x_j, t_n)}{\Delta x} - \frac{\Delta x}{2} V_{xx}(\xi_j, t_n)\right] \\ + \; & 2\left[\overline{\sigma}_{j-1} + (\sigma(x_j)^2 + \frac{\sigma'(x_j)}{2})\Delta x + O((\Delta x)^2)\right]\left[V(x_{j-1}, t_n) - V_x(\zeta_j, t_n)\Delta x\right], \end{aligned} \tag{2.2.14}$$

where $t_n \le \eta_n \le t_{n+1}$, $x_j \le \zeta_j \le x_{j+1}$. Letting $e_j^n = V(x_j, t_n) - V_j^n$, subtracting (2.2.10) from (2.2.14), we obtain

$$\begin{aligned} &\frac{e_j^{n+1} - e_j^n}{\Delta t} + 2\overline{\sigma}_{j-1} e_{j-1}^n + \frac{e_j^n - e_{j-1}^n}{\Delta x} = \frac{\Delta t}{2} V_{tt}(x_j, \eta_n) + \frac{\Delta x}{2} V_{xx}(\xi_j, t_n) \\ &+2\left[V_x(\zeta_j, t_n)\overline{\sigma}_{j-1} - (\sigma(x_j)^2 + \frac{\sigma'(x_j)}{2})V(x_{j-1}, t_n)\right]\Delta x + O((\Delta x)^2), \end{aligned}$$

and noting that $V(x,t)$, $V_x(x,t)$, $V_{xx}(x,t)$, $V_{tt}(x,t)$, $\sigma(x)$ and $\sigma'(x)$ are bounded, we deduce

$$e_j^{n+1} = (1-r)e_j^n + (r - 2\overline{\sigma}_{j-1}\Delta t)e_{j-1}^n + \Delta t\; O(\Delta x + \Delta t).$$

When $r \leq 1$ and Δx is small enough, we have

$$|e^{n+1}| \leq \|e^n\|_\infty + \Delta t O(\Delta x + \Delta t),$$

together with $\|e^0\|_\infty = O(\Delta x + \Delta t)$, we derive

$$\max_j |V_j^n - V(x_j, t_n)| = \|e^n\|_\infty \leq O(\Delta x + \Delta t).$$

Therefore we obtain the following convergence estimate:

$$\max_j |u_j^{\sigma,n} - u(x_j, t_n)| \leq O(\Delta x + \Delta t),$$
$$\max_j |v_j^{\sigma,n} - v(x_j, t_n)| \leq O(\Delta x + \Delta t).$$

♯

2.2.2 Finite Discrete PMLs for Wave Equations

The discussions in last subsection demonstrate that the second infinite discrete PML equations produce non-reflective matched layers, but it is not suitable for computational purpose. we shall now consider finite discrete PML. The up-winding scheme for infinite Hagstrom's PML (2.1.9):

$$\begin{cases} U_t^\sigma - U_x^\sigma = 0, & x \in R,\ t > 0, \\ V_t^\sigma + 2\sigma V^\sigma + V_x^\sigma = 0, & x \in R,\ t > 0 \end{cases}$$

is given by

$$\begin{cases} \frac{U_j^{\sigma,n+1} - U_j^{\sigma,n}}{\Delta t} - \frac{U_{j+1}^{\sigma,n} - U_j^{\sigma,n}}{\Delta x} = 0, \\ \frac{V_j^{\sigma,n+1} - V_j^{\sigma,n}}{\Delta t} + 2\overline{\sigma}_{j-1} V_{j-1}^{\sigma,n} + \frac{V_j^{\sigma,n} - V_{j-1}^{\sigma,n}}{\Delta x} = 0, \\ \quad - \infty < j < \infty,\ 0 \leq n \leq M, \end{cases} \tag{2.2.15}$$

where

$$\begin{cases} U_j^{\sigma,n} = u_j^{\sigma,n} + v_j^{\sigma,n}, \\ V_j^{\sigma,n} = u_j^{\sigma,n} - v_j^{\sigma,n}, \end{cases}$$

and $\overline{\sigma}_{j-1}$ be given in (2.2.9). The up-winding scheme for finite Hagstrom's PML:

$$\begin{cases} U_t^{\sigma,d} - U_x^{\sigma,d} = 0, & x \in (-\infty, d), \quad t > 0, \\ V_t^{\sigma,d} + 2\sigma V^{\sigma,d} + V_x^{\sigma,d} = 0, & x \in (-\infty, d), \quad t > 0 \end{cases}$$

is given by

$$\begin{cases} \frac{U_j^{\sigma,d,n+1}-U_j^{\sigma,d,n}}{\Delta t} - \frac{U_{j+1}^{\sigma,d,n}-U_j^{\sigma,d,n}}{\Delta x} = 0, \\ \frac{V_j^{\sigma,d,n+1}-V_j^{\sigma,d,n}}{\Delta t} + 2\overline{\sigma}_{j-1}V_{j-1}^{\sigma,d,n} + \frac{V_j^{\sigma,d,n}-V_{j-1}^{\sigma,d,n}}{\Delta x} = 0, \\ \quad -\infty < j < N,\ 0 \le n \le M, \end{cases} \tag{2.2.16}$$

where

$$\begin{cases} U_j^{\sigma,d,n} = u_j^{\sigma,d,n} + v_j^{\sigma,d,n}, \\ V_j^{\sigma,d,n} = u_j^{\sigma,d,n} - v_j^{\sigma,d,n}, \end{cases}$$

and $U_N^{\sigma,d,n} = 0$ for $0 \le n \le M$.

Lemma 2.2.4 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \ge 0$. Let $U_j^{\sigma,d,n}$ and $V_j^{\sigma,d,n}$ be the difference solution (2.2.16), then we have*

$$|U_j^{\sigma,d,n}| \le C_0 e^{-2\int_0^{x_j}\sigma(\xi)d\xi}, \quad |V_j^{\sigma,d,n}| \le C_0 e^{-2\int_0^{x_j}\sigma(\xi)d\xi}, \quad n \ge 0, \quad N \ge j \ge 0. \tag{2.2.17}$$

Proof. Taking $\mathcal{E}_j^{u,n} = U_j^{\sigma,n} - U_j^{\sigma,d,n}$, $\mathcal{E}_j^{v,n} = V_j^{\sigma,n} - V_j^{\sigma,d,n}$, then we obtain

$$\begin{cases} \frac{\mathcal{E}_j^{u,n+1}-\mathcal{E}_j^{u,n}}{\Delta t} - \frac{\mathcal{E}_{j+1}^{u,n}-\mathcal{E}_j^{u,n}}{\Delta x} = 0, \\ \frac{\mathcal{E}_j^{v,n+1}-\mathcal{E}_j^{v,n}}{\Delta t} + 2\overline{\sigma}_{j-1}\mathcal{E}_{j-1}^{v,n} + \frac{\mathcal{E}_j^{v,n}-\mathcal{E}_{j-1}^{v,n}}{\Delta x} = 0, \\ \mathcal{E}_j^{u,0} = \mathcal{E}_j^{v,0} = 0, \quad j \le N, \qquad \mathcal{E}_N^{u,n} = U_N^{\sigma,n}. \end{cases}$$

By the Maximum Principle, we have

$$\begin{aligned} |\mathcal{E}_j^{u,n}| &\le \max_{-\infty \le j \le N,\ 0 \le n \le M} \{|\mathcal{E}_j^{u,0}|, |\mathcal{E}_N^{u,n}|\} \le \max_{0 \le n \le M} \{|U_N^{\sigma,n}|\} \\ |\mathcal{E}_j^{v,n}| &\le \max_{-\infty \le j \le N} \{|\mathcal{E}_j^{v,0}|\} = 0, \end{aligned}$$

together with Lemma 2.2.2 that

$$|U_N^{\sigma,n}| \le C_0 e^{-2\int_0^{x_N}\sigma(\xi)d\xi}, \quad |V_N^{\sigma,n}| \le C_0 e^{-2\int_0^{x_N}\sigma(\xi)d\xi},$$

implies

$$|\mathcal{E}_j^{u,n}| \le C_0 e^{-2\int_0^{x_N}\sigma(\xi)d\xi}, \quad |\mathcal{E}_j^{v,n}| = 0.$$

By Lemma 2.2.2, we also get

$$|U_j^{\sigma,n}| \le C_0 e^{-2\int_0^{x_j}\sigma(\xi)d\xi}, \quad |V_j^{\sigma,n}| \le C_0 e^{-2\int_0^{x_j}\sigma(\xi)d\xi},$$

together with $U_j^{\sigma,d,n} = U_j^{\sigma,n} - \mathcal{E}_j^{u,n}$ and $V_j^{\sigma,d,n} = V_j^{\sigma,n} - \mathcal{E}_j^{v,n}$, we can deduce

$$|U_j^{\sigma,d,n}| \le C_0 e^{-2\int_0^{x_j}\sigma(\xi)d\xi}, \quad |V_j^{\sigma,d,n}| \le C_0 e^{-2\int_0^{x_j}\sigma(\xi)d\xi}.$$

♯

From the proof of Lemma 2.2.4 , we note that

$$|u_j^{\sigma,d,n}|,\ |v_j^{\sigma,d,n}| \le C_0 e^{-2\int_0^{x_j}\sigma(\xi)d\xi}, \quad M \ge n \ge 0, \quad N \ge j \ge 0, \tag{2.2.18}$$

and

$$|\mathcal{E}_j^{u,n}| \le C_0 e^{-2\int_0^{x_N}\sigma(\xi)d\xi}, \quad |\mathcal{E}_j^{v,n}| = 0,$$

then we have the following corollary:

Corollary 2.2.1 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \ge 0$. For Hagstrom's PML, we have*

$$\begin{aligned} |u_j^{\sigma,n} - u_j^{\sigma,d,n}| &\le C_0 e^{-2\int_0^{x_N}\sigma(\xi)d\xi}, \\ |v_j^{\sigma,d,n} - v_j^{\sigma,d,n}| &\le C_0 e^{-2\int_0^{x_N}\sigma(\xi)d\xi}, \quad M \ge n \ge 0, \quad N \ge j \ge 0. \end{aligned} \tag{2.2.19}$$

2.2.3 Discrete Berenger's PMLs for Wave Equations

In this and next subsection, we propose the infinite and finite discrete Berenger's PMLs for wave equations, which have the same exponential decays, stability and convergence as the discrete Hagstrom's PMLs.

Now we consider the following Berenger's PML for the equations (2.1.4):

$$\begin{cases} u_t^\sigma + \sigma u^\sigma = v_x^\sigma, & x \in R, \quad t > 0 \\ v_t^\sigma + \sigma v^\sigma = u_x^\sigma, & x \in R, \quad t > 0 \\ u^\sigma(x,0) = v^\sigma(x,0) = 0, & x \in R, \end{cases}$$

and let $U = u^\sigma + v^\sigma, V = u^\sigma - v^\sigma$, then we obtain

$$\begin{cases} U_t + \sigma U - U_x = 0, & x \in R, \quad t > 0, \\ V_t + \sigma V + V_x = 0, & x \in R, \quad t > 0. \end{cases} \tag{2.2.20}$$

In order to have perfectly matched finite difference solution, i.e. non-reflective finite difference solution at $j = 0$, we first set $M = Ue^{-\int_0^x \sigma(\xi)d\xi}$ and $N = Ve^{\int_0^x \sigma(\xi)d\xi}$, then

$$\begin{cases} M_t - M_x = 0, & x \in R, \quad t > 0, \\ N_t + N_x = 0, & x \in R, \quad t > 0, \end{cases} \tag{2.2.21}$$

and its up-winding scheme is given by

$$\begin{cases} \frac{M_j^{n+1} - M_j^n}{\Delta t} - \frac{M_{j+1}^n - M_j^n}{\Delta x} = 0, \\ \frac{N_j^{n+1} - N_j^n}{\Delta t} + \frac{N_j^n - N_{j-1}^n}{\Delta x} = 0. \end{cases} \tag{2.2.22}$$

Then define U_j^n and V_j^n by $U_j^n = M_j^n e^{\int_0^{x_j} \sigma(\xi)d\xi}$ and $V_j^n = N_j^n e^{-\int_0^{x_j} \sigma(\xi)d\xi}$, and substituting into (2.2.22), we obtain the finite difference equations for U_j^n and V_j^n:

$$\begin{cases} \frac{U_j^{n+1} - U_j^n}{\Delta t} + \overline{\sigma}_j U_{j+1}^n - \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0, \\ \frac{V_j^{n+1} - V_j^n}{\Delta t} + \overline{\sigma}_{j-1} V_{j-1}^n + \frac{V_j^n - V_{j-1}^n}{\Delta x} = 0, \end{cases} \tag{2.2.23}$$

where

$$\overline{\sigma}_j = \frac{1 - e^{-\int_{x_j}^{x_{j+1}} \sigma(\xi)d\xi}}{\Delta x}. \tag{2.2.24}$$

Lemma 2.2.5 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$. Let (U_j^n, V_j^n) be the finite difference solution of (2.2.23) and $\overline{\sigma}_{j-1}$ be given in (2.2.24). Taking $N_j^n = V_j^n e^{2\int_0^{x_j} \sigma(\xi)d\xi}$, then we have*

$$\max_{j \in Z} |N_j^n| \leq \max_{j \in Z} |N_j^0|,$$

which implies that

$$|V_j^n| \leq |N_j^n| e^{-2\int_0^{x_j} \sigma(\xi)d\xi} \leq C_0 e^{-2\int_0^{x_j} \sigma(\xi)d\xi}, \quad j \geq 0.$$

So the scheme (2.2.23) is stable and its solutions decay exponentially. Moreover, the scheme (2.2.23) is non-reflective.

Proof. By the construction of the scheme, we have

$$|V_j^n| \le |N_j^n| e^{-\int_0^{x_j}\sigma(\xi)d\xi} \le C_0 e^{-\int_0^{x_j}\sigma(\xi)d\xi}, \quad j \ge 0.$$

Since the scheme is up-wind and initial data $U_j^0 = 0, j \ge 0$, then $U_j^n = 0, j \ge 0$ for any n. Thus we have

$$|u_j^{\sigma,n}| \le C_0 e^{-\int_0^{x_j}\sigma(\xi)d\xi}, \quad |v_j^{\sigma,n}| \le C_0 e^{-\int_0^{x_j}\sigma(\xi)d\xi}.$$

Next, we want to show the scheme (2.2.23) is also non-reflective. Consider (2.1.4) and let $U^0 = u^0 + v^0$ and $V^0 = u^0 - v^0$, we obtain

$$\begin{cases} U_t^0 = U_x^0, & x \in R, \quad t > 0, \\ V_t^0 + V_x^0 = 0, & x \in R, \quad t > 0, \end{cases}$$

and the corresponding up-winding scheme is given by:

$$\begin{cases} \frac{U_j^{0,n+1} - U_j^{0,n}}{\Delta t} - \frac{U_{j+1}^{0,n} - U_j^{0,n}}{\Delta x} = 0, \\ \frac{V_j^{0,n+1} - V_j^{0,n}}{\Delta t} + \frac{V_j^{0,n} - V_{j-1}^{0,n}}{\Delta x} = 0. \end{cases}$$

By the construction, since U_j^n, V_j^n satisfies (2.2.23) and $U_j^n e^{\int_0^{x_j}\sigma(\xi)d\xi} = M_j^n$, $V_j^n e^{\int_0^{x_j}\sigma(\xi)d\xi} = N_j^n$, then M_j^n, N_j^n satisfies (2.2.22). Therefore M_j^n and $U_j^{0,n}$, N_j^n and $V_j^{0,n}$ satisfy the same difference equations, then $U_0^n = M_0^n = U_0^{0,n}$ and $V_0^n = N_0^n = V_0^{0,n}$. Thus we have $u_0^{\sigma,n} = u_0^{0,n}$ and $v_0^{\sigma,n} = v_0^{0,n}$, that means the scheme (2.2.23) is non-reflective.

♯

Similar to the proof of Lemma 2.2.3, we can derive the convergence of scheme (2.2.23).

Lemma 2.2.6 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \ge 0$. Let (U, V) be the exact solution of (2.2.1), and (U_j^n, V_j^n) be the finite difference solution of (2.2.23) and $\overline{\sigma}_{j-1}$ be given in (2.2.24). If*

$$\max_{j\in Z} |U_j^0 - U(x_j, 0)| = \max_{j\in Z} |V_j^0 - V(x_j, 0)| = O(\Delta x + \Delta t),$$

and the mesh size satisfies $r = \frac{\Delta t}{\Delta x} < 1$, *then for any* $n \geq 0$, *we have*

$$\max_{j\in Z} |U_j^n - U(x_j, t_n)| \leq O(\Delta x + \Delta t),$$

$$\max_{j\in Z} |V_j^n - V(x_j, t_n)| \leq O(\Delta x + \Delta t).$$

2.2.4 Finite Discrete Berenger's PMLs for Wave Equations

For the computational purpose similar to the subsection 2.2.2, here we propose the finite discrete Berenger's PMLs for wave equations. The up-winding scheme for infinite Berenger's PML (2.1.20):

$$\begin{cases} U_t^\sigma + \sigma U^\sigma + U_x^\sigma = 0, & x \in R,\ t > 0, \\ V_t^\sigma + \sigma V^\sigma - V_x^\sigma = 0, & x \in R,\ t > 0 \end{cases}$$

is given by

$$\begin{cases} \frac{U_j^{\sigma,n+1} - U_j^{\sigma,n}}{\Delta t} + \overline{\sigma}_j U_{j+1}^{\sigma,n} - \frac{U_{j+1}^{\sigma,n} - U_j^{\sigma,n}}{\Delta x} = 0, \\ \frac{V_j^{\sigma,n+1} - V_j^{\sigma,n}}{\Delta t} + \overline{\sigma}_{j-1} V_{j-1}^{\sigma,n} + \frac{V_j^{\sigma,n} - V_{j-1}^{\sigma,n}}{\Delta x} = 0, \\ \quad -\infty < j < \infty,\ 0 \leq n \leq M, \end{cases} \tag{2.2.25}$$

where

$$\begin{cases} U_j^{\sigma,n} = u_j^{\sigma,n} + v_j^{\sigma,n}, \\ V_j^{\sigma,n} = u_j^{\sigma,n} - v_j^{\sigma,n}, \end{cases}$$

and $\overline{\sigma}_{j-1}$ be given in (2.2.24). The up-winding scheme for finite Berenger's PML:

$$\begin{cases} U_t^{\sigma,d} + \sigma U^{\sigma,d} - U_x^{\sigma,d} = 0, & x \in (-\infty, d), & t > 0, \\ V_t^{\sigma,d} + \sigma V^{\sigma,d} + V_x^{\sigma,d} = 0, & x \in (-\infty, d), & t > 0 \end{cases}$$

is given by

$$\begin{cases} \frac{U_j^{\sigma,d,n+1} - U_j^{\sigma,d,n}}{\Delta t} + \overline{\sigma}_j U_{j+1}^{\sigma,d,n} - \frac{U_{j+1}^{\sigma,d,n} - U_j^{\sigma,d,n}}{\Delta x} = 0, \\ \frac{V_j^{\sigma,d,n+1} - V_j^{\sigma,d,n}}{\Delta t} + \overline{\sigma}_{j-1} V_{j-1}^{\sigma,d,n} + \frac{V_j^{\sigma,d,n} - V_{j-1}^{\sigma,d,n}}{\Delta x} = 0, \\ \quad -\infty < j < N,\ 0 \leq n \leq M, \end{cases} \tag{2.2.26}$$

where

$$\begin{cases} U_j^{\sigma,d,n} = u_j^{\sigma,d,n} + v_j^{\sigma,d,n}, \\ V_j^{\sigma,d,n} = u_j^{\sigma,d,n} - v_j^{\sigma,d,n}, \end{cases}$$

and $U_N^{\sigma,d,n} = 0$ for $0 \leq n \leq M$.

Lemma 2.2.7 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$. Let $U_j^{\sigma,d,n}$ and $V_j^{\sigma,d,n}$ be the difference solution (2.2.26), then we have*

$$|U_j^{\sigma,d,n}| \leq C_0 e^{-\int_0^{x_j} \sigma(\xi)d\xi}, \ |V_j^{\sigma,d,n}| \leq C_0 e^{-\int_0^{x_j} \sigma(\xi)d\xi}, \quad n \geq 0, \ N \geq j \geq 0. \quad (2.2.27)$$

Proof. Taking $\mathcal{E}_j^{u,n} = U_j^{\sigma,n} - U_j^{\sigma,d,n}$, $\mathcal{E}_j^{v,n} = V_j^{\sigma,n} - V_j^{\sigma,d,n}$, then we obtain

$$\begin{cases} \frac{\mathcal{E}_j^{u,n+1} - \mathcal{E}_j^{u,n}}{\Delta t} + \overline{\sigma}_j \mathcal{E}_{j+1}^{u,n} - \frac{\mathcal{E}_{j+1}^{u,n} - \mathcal{E}_j^{u,n}}{\Delta x} = 0 \\ \frac{\mathcal{E}_j^{v,n+1} - \mathcal{E}_j^{v,n}}{\Delta t} + \overline{\sigma}_{j-1} \mathcal{E}_{j-1}^{v,n} + \frac{\mathcal{E}_j^{v,n} + \mathcal{E}_{j-1}^{v,n}}{\Delta x} = 0 \end{cases}$$

with

$$\mathcal{E}_j^{u,0} = \mathcal{E}_j^{v,0} = 0, \quad j \leq N, \qquad \mathcal{E}_N^{u,n} = U_N^{\sigma,n}, \quad n \geq 0.$$

By the Maximum Principle, we have

$$\begin{aligned} |\mathcal{E}_j^{u,n}| &\leq \max_{-\infty \leq j \leq N, \ 0 \leq n \leq M} \{|\mathcal{E}_j^{u,0}|, |\mathcal{E}_N^{u,n}|\} \leq \max_{0 \leq n \leq M} \{|U_N^{\sigma,n}|\} \\ |\mathcal{E}_j^{v,n}| &\leq \max_{-\infty \leq j \leq N} \{|\mathcal{E}_j^{v,0}|\} = 0, \end{aligned}$$

together with Lemma 2.2.5 that

$$\begin{aligned} |u_N^{\sigma,n}| \leq C_0 e^{-\int_0^{x_N} \sigma(\xi)d\xi}, \quad |v_N^{\sigma,n}| \leq C_0 e^{-\int_0^{x_N} \sigma(\xi)d\xi}, \\ |U_N^{\sigma,n}| \leq C_0 e^{-\int_0^{x_N} \sigma(\xi)d\xi}, \quad |V_N^{\sigma,n}| \leq C_0 e^{-\int_0^{x_N} \sigma(\xi)d\xi}, \end{aligned}$$

implies

$$|\mathcal{E}_j^{u,n}| \leq C_0 e^{-\int_0^{x_N} \sigma(\xi)d\xi}, \quad |\mathcal{E}_j^{v,n}| = 0.$$

By Lemma 2.2.5, we also get

$$|U_j^{\sigma,n}| \leq C_0 e^{-\int_0^{x_j} \sigma(\xi)d\xi}, \quad |V_j^{\sigma,n}| \leq C_0 e^{-\int_0^{x_j} \sigma(\xi)d\xi},$$

together with $U_j^{\sigma,d,n} = U_j^{\sigma,n} - \mathcal{E}_j^{u,n}$ and $M_j^{\sigma,d,n} = M_j^{\sigma,n} - \mathcal{E}_j^{m,n}$, we can deduce

$$|U_j^{\sigma,d,n}| \leq C_0 e^{-\int_0^{x_j} \sigma(\xi)d\xi}, |V_j^{\sigma,d,n}| \leq C_0 e^{-\int_0^{x_j} \sigma(\xi)d\xi}.$$

♯

From the proof of Lemma 2.2.7 , we note that

$$|u_j^{\sigma,d,n}|,\ |v_j^{\sigma,d,n}| \leq C_0 e^{-\int_0^{x_j} \sigma(\xi) d\xi}, \quad M \geq n \geq 0, \quad N \geq j \geq 0 \tag{2.2.28}$$

and

$$|\mathcal{E}_j^{u,n}| \leq C_0 e^{-\int_0^{x_N} \sigma(\xi) d\xi}, \quad |\mathcal{E}_j^{v,n}| = 0,$$

then we have the following corollary:

Corollary 2.2.2 *Assume that* $\sigma(x) \in C^1(R)$ *with* $\sigma(0) = 0$ *and* $\sigma'(x) \geq 0$. *For Berenger's PML, we have*

$$\begin{aligned} |u_j^{\sigma,n} - u_j^{\sigma,d,n}| &\leq C_0 e^{-\int_0^{x_N} \sigma(\xi) d\xi}, \\ |v_j^{\sigma,d,n} - v_j^{\sigma,d,n}| &\leq C_0 e^{-\int_0^{x_N} \sigma(\xi) d\xi}, \quad M \geq n \geq 0, \quad N \geq j \geq 0. \end{aligned} \tag{2.2.29}$$

2.2.5 Discrete PMLs for 1-D Hyperbolic Systems

In the first part of this section, we give several infinite and finite discrete PMLs for wave equations, we shall propose the discrete PMLs for general 1D hyperbolic systems as the end of this section.

Consider the 1D hyperbolic system

$$u_t + A u_x = 0, \quad x \in R, \quad t > 0,$$

its Hagstrom's PML equation is given by:

$$u_t + \sigma u + A(u_x + \sigma u) = 0, \quad x \in R, \quad t > 0. \tag{2.2.30}$$

Suppose R is a matrix consisting of eigenvectors associated with eigenvalues of A, we can then transform the system (2.2.30) into a diagonal system by using the transformation $w = R^{-1}u$. The original hyperbolic system (2.2.30) becomes

$$w_t + \sigma w + \Lambda(w_x + \sigma w) = 0.$$

We set $\Lambda = \Lambda^- + \Lambda^+$, where $\Lambda^- = \text{diag}(\lambda_1^-, \cdots, \lambda_k^-, 0, \cdots, 0)$, $\Lambda^+ = \text{diag}$ $(0, \cdots, 0, \lambda_{k+1}^+, \cdots, \lambda_s^+)$, then we get $A = A^- + A^+$, where $A^- = R\Lambda^- R^{-1}$,

$A^+ = R\Lambda^+ R^{-1}$. Thus the up-winding scheme for (2.2.30) can be written as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \overline{\sigma}_j u_j^n + A^- \{ \frac{u_{j+1}^n - u_j^n}{\Delta x} + \overline{\sigma}_j u_{j+1}^n \} + A^+ \{ \frac{u_j^n - u_{j-1}^n}{\Delta x} + \overline{\sigma}_{j-1} u_{j-1}^n \} = 0. \tag{2.2.31}$$

Lemma 2.2.8 *Assume that $\sigma(x) \in C^1(R)$ with $\sigma(0) = 0$ and $\sigma'(x) \geq 0$. The scheme (2.2.31) is stable and its solutions decay exponentially.*

Remark 2.2.1 *The scheme (2.2.31) is not perfectly matched at $j = 0$, in general. Of course perfectly matched discrete version finite difference scheme can be constructed. But it seems not so necessary as the reflection magnitude at $j = 0$ (interface) is very small, i.e. $C_0 e^{-\mu \int_0^d \sigma(\xi) d\xi}$, where $\mu = \min\limits_{l \geq k+1} \left\{ \frac{1+\lambda_l^+}{\lambda_l^+} \right\}$.*

Remark 2.2.2 *Using an infinite PML, the true scattered field and the PML solution agree near the scatterer. Once the PML is truncated (as is necessary to obtain a finite computational domain) some error is introduced due to reflections from the truncation boundary. Discretizing the PML leads to a further perturbation of the solution due to dispersion and scattering from the numerical PML. All these errors can be controlled by a suitable design of the layer, which means by optimal choice of parameters $\sigma(x)$ and the length of the PML [CM] .*

2.3 Modified Yee schemes for PMLs

In section 2.2, we introduce some up-winding schemes for discretization of the continuous PML equations, but they are all first order in spatial. In order to obtain high order schemes and have the same decay properties, we propose two kinds of numerical schemes which both are second order in spatial in this and next section.

The first second order scheme is the modified Yee scheme. Yee scheme is the principal finite difference method used in the electromagnetic community, and has been developed and extended extensively. In this section, we introduce a modified Yee scheme, so the numerical scheme is stable and the numerical solution decays exponentially in the perfectly matched layers. We shall concentrate on studying the order of convergence in space (the time discretization is quite standard). For the space interval $[-\infty, d]$, we partition in uniformly as follows:

$$-\infty < \cdots < x_{-1} < 0 = x_0 < x_1 < \cdots < x_N = d,$$

with $x_j = jh$ and $h = \frac{d}{N}$ being the space mesh size. Its dual partition is given by:

$$-\infty < \cdots < x_{-\frac{1}{2}} < x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N-\frac{1}{2}},$$

with $x_{j+\frac{1}{2}} = x_j + \frac{h}{2}$, $j \le N-1$.

For vectors $\mathbf{u} = (\cdots, u_0, \cdots, u_{N-1})^T$ and $\mathbf{v} = (\cdots, v_{\frac{1}{2}}, \cdots, v_{N-\frac{1}{2}})^T$, we define the discrete norms as:

$$\|\mathbf{u}\|_{0,h}^2 \equiv \sum_{j=-\infty}^{N-1} hu_j^2, \qquad \|\mathbf{v}\|_{0,h^*}^2 \equiv \sum_{j=-\infty}^{N-1} hv_{j+\frac{1}{2}}^2, \tag{2.3.1}$$

and discrete inner-products as

$$< \mathbf{u}_1, \mathbf{u}_2 >_h \equiv \sum_{j=-\infty}^{N-1} hu_{1j}u_{2j}, \qquad < \mathbf{v}_1, \mathbf{v}_2 >_{h^*} \equiv \sum_{j=-\infty}^{N-1} hv_{1,j+\frac{1}{2}}v_{2,j+\frac{1}{2}}. \tag{2.3.2}$$

It will also be convenient to introduce

$$\|\mathbf{w}_1\|^2 \equiv \|\mathbf{u}_1\|_{0,h}^2 + \|\mathbf{v}_1\|_{0,h^*}^2, \qquad < \mathbf{w}_1, \mathbf{w}_2 > \equiv < \mathbf{u}_1, \mathbf{u}_2 >_h + < \mathbf{v}_1, \mathbf{v}_2 >_{h^*}, \tag{2.3.3}$$

for given vectors $\mathbf{w}_1^T = (\mathbf{u}_1^T, \mathbf{v}_1^T)$ and $\mathbf{w}_2^T = (\mathbf{u}_2^T, \mathbf{v}_2^T)$.

This section is organized as follows: we discuss the stability of the Yee scheme for wave equations in first subsection, and then we deduce the exponential decays, stability and convergence of the Yee scheme solution to the Berenger's PMLs in subsection 2.3.2 and 2.3.3 respectively. The last two subsections give the same properties of the Yee scheme solution to the Hagstrom's PMLs.

2.3.1 Stability of the Yee Scheme for Wave Equation

In this subsection, we shall deduce a stability result of the Yee scheme for the wave equation, which will be extensively used in the rest of this section. Recall the wave equation

$$\begin{cases} u_t = v_x, & x \in (-\infty, d), \quad t > 0, \\ v_t = u_x, & x \in (-\infty, d), \quad t > 0, \end{cases}$$

and its corresponding Yee scheme given by

$$\begin{cases} \dot{u}_j = \frac{v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}}}{h}, & j \leq N, \\ \dot{v}_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h}, & j \leq N-1, \end{cases} \tag{2.3.4}$$

where " $\cdot$ " denotes the derivative with respect to the time variable.

Lemma 2.3.1 *For the Yee scheme (2.3.4), we have the following stability estimate*

$$\|\mathbf{u}(t)\|_{0,h}^2 + \|\mathbf{v}(t)\|_{0,h^*}^2 \leq C_0, \qquad \forall\, t > 0.$$

Proof. Since $u_{-\infty} = u_N = 0$, we get

$$\begin{aligned} \sum_{j=-\infty}^{N-1} h\dot{u}_j u_j &= \sum_{j=-\infty}^{N-1} (v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}})u_j \\ &= (v_{N+\frac{1}{2}} - v_{N-\frac{1}{2}})u_N + \cdots + (v_{\frac{3}{2}} - v_{\frac{1}{2}})u_1 + (v_{\frac{1}{2}} - v_{-\frac{1}{2}})u_0 + \cdots \\ &= u_N v_{N+\frac{1}{2}} - (u_N - u_{N-1})v_{N-\frac{1}{2}} - \cdots - (u_1 - u_0)v_{\frac{1}{2}} - \cdots \\ &= -\sum_{j=-\infty}^{N-1} (u_{j+1} - u_j)v_{j+\frac{1}{2}} \end{aligned}$$

and similarly we can obtain

$$\sum_{j=-\infty}^{N-1} h\dot{v}_{j+\frac{1}{2}} v_{j+\frac{1}{2}} = \sum_{j=-\infty}^{N-1} (u_{j+1} - u_j)v_{j+\frac{1}{2}}.$$

From the above two equalities, we may deduce

$$\frac{d}{dt}\Big(\|\mathbf{u}(t)\|_{0,h}^2 + \|\mathbf{v}(t)\|_{0,h^*}^2\Big) = \sum_{j=-\infty}^{N} h\dot{u}_j u_j + \sum_{j=-\infty}^{N-1} h\dot{v}_{j+\frac{1}{2}} v_{j+\frac{1}{2}} = 0,$$

and integrate on [0,t], then we have

$$\|\mathbf{u}(t)\|_{0,h}^2 + \|\mathbf{v}(t)\|_{0,h^*}^2 \le \|\mathbf{u}(0)\|_{0,h}^2 + \|\mathbf{v}(0)\|_{0,h^*}^2 \le C_0 \qquad t > 0.$$

♯

2.3.2 Decay of the Yee Scheme Solution to the Berenger's PMLs

In this subsection, we shall derive a decay result of the Yee scheme solution to the Berenger's PML equations. Now, we consider the Berenger's PML equation

$$\begin{cases} u_t + \sigma u = v_x, & x \in (-\infty, d), \quad t > 0, \\ v_t + \sigma v = u_x, & x \in (-\infty, d), \quad t > 0 \end{cases} \tag{2.3.5}$$

and its corresponding Yee scheme given by

$$\begin{cases} \dot{u}_j + \overline{\sigma}_j u_j = \frac{v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}}}{h}, & j \le N, \\ \dot{v}_{j+\frac{1}{2}} + \overline{\sigma}_{j+\frac{1}{2}} v_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h}, & j \le N-1. \end{cases} \tag{2.3.6}$$

A decay result for this scheme will also be derived in the rest of this subsection. Firstly, we introduce the notations $\rho_j = e^{\int_{x_j}^{x_{j+\frac{1}{2}}} \sigma(\xi) d\xi}$ and $\alpha_j = \frac{\rho_j - \rho_j^{-1}}{h}$, and for our subsequent use, we give an estimate as follows:

Lemma 2.3.2 *If $\sigma(x)$ is a smooth enough real function and*

$$m_j = \frac{1}{2}\left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} + \sqrt{\frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4}} \right], \tag{2.3.7}$$

then we have

$$m_j = \beta_j + O(h).$$

Here $\sigma_j = \sigma(x_j), \sigma_{j+\frac{1}{2}} = \sigma(x_{j+\frac{1}{2}})$ etc. as the usual utilizations and

$$\beta_j = \frac{1}{96\sigma_j}(4\sigma_j^4 + 3(\sigma_j')^2 - 2\sigma_j \sigma_j'), \tag{2.3.8}$$

which is a constant independent of h.

Proof. By the Taylor series expansions, we can derive

$$\begin{aligned}\sigma_j + \sigma_{j+\frac{1}{2}} =& 2\sigma_j + \frac{\sigma_j'}{2}h + \frac{\sigma_j''}{8}h^2 + \frac{\sigma_j'''}{48}h^3 + \frac{\sigma_j^{(4)}}{384}h^4 + O(h^5),\\ (\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2 =& 4\sigma_j^2 + 2\sigma_j\sigma_j' h + \frac{1}{6}(2\sigma_j^4 + 3(\sigma_j')^2 + 2\sigma_j\sigma_j'')h^2 \\ &+ \frac{1}{24}(8\sigma_j^3\sigma_j' + 5\sigma_j'\sigma_j'' + \sigma_j\sigma_j''')h^3 + O(h^4).\end{aligned} \tag{2.3.9}$$

Substitute (2.3.9) into (2.3.7), and make use of the fact

$$\begin{aligned}\sqrt{a + bh + ch^2 + dh^3} &= \sqrt{a} + \frac{bh}{2\sqrt{a}} + (-\frac{b^2}{8a^{3/2}} + \frac{c}{2\sqrt{a}})h^2 \\ &+ (\frac{b^3}{16a^{5/2}} - \frac{bc}{4a^{3/2}} + \frac{d}{2\sqrt{a}})h^3 + O(h^4),\end{aligned}$$

we can get after straightforward but a little tedious algebraic calculations that

$$\begin{aligned} m_j &= \frac{1}{96\sigma_j}(4\sigma_j^4 + 3(\sigma_j')^2 - 2\sigma_j\sigma_j') \\ &+ \frac{1}{384\sigma_j^2}(12\sigma_j^4\sigma_j' - 3(\sigma_j')^3 + 6\sigma_j\sigma_j'\sigma_j'' - 2\sigma_j^2\sigma_j''')h + O(h^2).\end{aligned}$$

So the dominated constant term in m_j is

$$\beta_j = \frac{1}{96\sigma_j}(4\sigma_j^4 + 3(\sigma_j')^2 - 2\sigma_j\sigma_j'),$$

which is independent of h.

♯

Next, we give the sufficient conditions for the above Yee scheme (2.3.6) to have the decay result.

Theorem 2.3.1 *For the Yee scheme (2.3.6), if we take*

$$\overline{\sigma}_j = \sigma_j + m_j h^2, \quad \overline{\sigma}_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}} + m_j h^2,$$

with $m_j = c_j + O(h)$, *where*

$$c_j = \max\{0, \beta_j, \beta_{j-\frac{1}{2}}\} \geq 0, \tag{2.3.10}$$

and β_j is given by (2.3.8), then for any given $0 \le k \le N-1$, we have

$$\sum_{j=k}^{N-1} h(u_j(t))^2 \le C_0 e^{-2\int_0^{x_k} \sigma(\xi)d\xi},$$
$$\sum_{j=k}^{N-1} h(v_{j+\frac{1}{2}}(t))^2 \le C_0 e^{-2\int_0^{x_{k+\frac{1}{2}}} \sigma(\xi)d\xi}, \quad \forall\, t > 0. \tag{2.3.11}$$

Proof. Set $U_j = u_j e^{\int_0^{x_j} \sigma(\xi)d\xi}$, $V_{j+\frac{1}{2}} = v_{j+\frac{1}{2}} e^{\int_0^{x_{j+\frac{1}{2}}} \sigma(\xi)d\xi}$ and multiply the first equation of (2.3.6) by $e^{\int_0^{x_j} \sigma(\xi)d\xi}$, we get

$$\dot{U}_j + \overline{\sigma}_j U_j = \frac{1}{h}\left[V_{j+\frac{1}{2}} - V_{j-\frac{1}{2}}\right] + \frac{1}{h}\left[\rho_j^{-1} - 1\right] V_{j+\frac{1}{2}} + \frac{1}{h}\left[1 - \rho_{j-\frac{1}{2}}\right] V_{j-\frac{1}{2}}. \tag{2.3.12}$$

Next, multiply the both hand sides of (2.3.12) by U_j, we can further get

$$\dot{U}_j U_j + \overline{\sigma}_j U_j^2 = \frac{1}{h}\left[V_{j+\frac{1}{2}} - V_{j-\frac{1}{2}}\right] U_j + \frac{1}{h}\left[\rho_j^{-1} - 1\right] V_{j+\frac{1}{2}} U_j + \frac{1}{h}\left[1 - \rho_{j-\frac{1}{2}}\right] V_{j-\frac{1}{2}} U_j. \tag{2.3.13}$$

Completely the same manipulations as above for the second equation of (2.3.6) lead to

$$\begin{aligned} \dot{V}_{j+\frac{1}{2}} V_{j+\frac{1}{2}} &+ \overline{\sigma}_{j+\frac{1}{2}} V_{j+\frac{1}{2}}^2 = \frac{1}{h}\left[U_{j+1} - U_j\right] V_{j+\frac{1}{2}} \\ &+ \frac{1}{h}\left[\rho_{j+\frac{1}{2}}^{-1} - 1\right] U_{j+1} V_{j+\frac{1}{2}} + \frac{1}{h}\left[1 - \rho_j\right] U_j V_{j+\frac{1}{2}}. \end{aligned} \tag{2.3.14}$$

Now, summing up (2.3.13) and (2.3.14) over $j \le N-1$, and noting that $u_{-\infty} = u_N = 0$, $U_{-\infty} = U_N = 0$, we deduce

$$\begin{aligned}
&\sum_{j=-\infty}^{N-1} \dot{U}_j U_j + \sum_{j=-\infty}^{N-1} \dot{V}_{j+\frac{1}{2}} V_{j+\frac{1}{2}} + \sum_{j=-\infty}^{N-1} \overline{\sigma}_j U_j^2 + \sum_{j=-\infty}^{N-1} \overline{\sigma}_{j+\frac{1}{2}} V_{j+\frac{1}{2}}^2 \\
= &\sum_{j=-\infty}^{N-1} \left[\frac{\rho_j^{-1} - 1}{h}\right] V_{j+\frac{1}{2}} U_j + \sum_{j=-\infty}^{N-1} \left[\frac{1 - \rho_{j-\frac{1}{2}}}{h}\right] V_{j-\frac{1}{2}} U_j \\
&+ \sum_{j=-\infty}^{N-1} \left[\frac{\rho_{j+\frac{1}{2}}^{-1} - 1}{h}\right] U_{j+1} V_{j+\frac{1}{2}} + \sum_{j=-\infty}^{N-1} \left[\frac{1 - \rho_j}{h}\right] U_j V_{j+\frac{1}{2}} \\
= &\sum_{j=-\infty}^{N-1} \left[\frac{\rho_j^{-1} - \rho_j}{h}\right] U_j V_{j+\frac{1}{2}} + \sum_{j=-\infty}^{N-1} \left[\frac{\rho_{j+\frac{1}{2}}^{-1} - \rho_{j+\frac{1}{2}}}{h}\right] U_{j+1} V_{j+\frac{1}{2}} \\
\le &\frac{1}{2} \sum_{j=-\infty}^{N-1} \frac{1}{\overline{\sigma}_j} \left[\frac{\rho_j^{-1} - \rho_j}{h}\right]^2 V_{j+\frac{1}{2}}^2 + \frac{1}{2} \sum_{j=-\infty}^{N-1} \overline{\sigma}_j U_j^2 \\
&+ \frac{1}{2} \sum_{j=-\infty}^{N-1} \frac{1}{\overline{\sigma}_{j+\frac{1}{2}}} \left[\frac{\rho_{j+\frac{1}{2}}^{-1} - \rho_{j+\frac{1}{2}}}{h}\right]^2 U_{j+1}^2 + \frac{1}{2} \sum_{j=-\infty}^{N-1} \overline{\sigma}_{j+\frac{1}{2}} V_{j+\frac{1}{2}}^2 ,
\end{aligned}$$

which gives

$$\begin{aligned}
&\sum_{j=-\infty}^{N-1} \dot{U}_j U_j + \sum_{j=-\infty}^{N-1} \dot{V}_{j+\frac{1}{2}} V_{j+\frac{1}{2}} + \frac{1}{2} \sum_{j=-\infty}^{N-1} \overline{\sigma}_j U_j^2 + \frac{1}{2} \sum_{j=-\infty}^{N-1} \overline{\sigma}_{j+\frac{1}{2}} V_{j+\frac{1}{2}}^2 \\
\le &\frac{1}{2} \sum_{j=-\infty}^{N-1} \frac{1}{\overline{\sigma}_j} \left[\frac{\rho_j^{-1} - \rho_j}{h}\right]^2 V_{j+\frac{1}{2}}^2 + \frac{1}{2} \sum_{j=-\infty}^{N-1} \frac{1}{\overline{\sigma}_{j+\frac{1}{2}}} \left[\frac{\rho_{j+\frac{1}{2}}^{-1} - \rho_{j+\frac{1}{2}}}{h}\right]^2 U_{j+1}^2
\end{aligned} \tag{2.3.15}$$

In order to get the stability result, it is sufficient for us to have in (2.3.15) that

$$\left[\frac{\rho_j^{-1} - \rho_j}{h}\right]^2 \le \overline{\sigma}_j \overline{\sigma}_{j+\frac{1}{2}} \tag{2.3.16}$$

$$\left[\frac{\rho_{j+\frac{1}{2}}^{-1} - \rho_{j+\frac{1}{2}}}{h}\right]^2 \le \overline{\sigma}_j \overline{\sigma}_{j+\frac{1}{2}} \tag{2.3.17}$$

hold for $j \le N-1$. Since $\overline{\sigma}_j = \sigma_j + m_j h^2$, $\overline{\sigma}_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}} + m_j h^2$ as given in the theorem, then substitutions of both of them into (2.3.16) can reduce the inequality to

$$\alpha_j^2 \le (\sigma_j + m_j h^2)(\sigma_{j+\frac{1}{2}} + m_j h^2), \tag{2.3.18}$$

which can be further reformulated as a quadratic inequality for m_j,

$$m_j^2 + \frac{\sigma_j + \sigma_{j+\frac{1}{2}}}{h^2} m_j + \frac{\sigma_j \sigma_{j+\frac{1}{2}} - \alpha_j^2}{h^4} \geq 0. \tag{2.3.19}$$

The corresponding quadratic equation for (2.3.19) has two distinct solutions

$$m_j^{\pm} = \frac{1}{2} \left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} \pm \sqrt{\frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4}} \right],$$

due to the fact that

$$\Delta = \frac{(\sigma_j + \sigma_{j+\frac{1}{2}})^2}{h^4} - 4\frac{\sigma_j \sigma_{j+\frac{1}{2}} - \alpha_j^2}{h^4} = \frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4} > 0.$$

Here, m_j^- and m_j^+ denote the left root and the right root in the real axis, respectively. By $\sigma(x) \geq 0$, we have that

$$m_j^- = \frac{1}{2} \left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} - \sqrt{\frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4}} \right] < 0,$$

and by Lemma 2.3.2, we have for the other root

$$m_j^+ = \frac{1}{2} \left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} + \sqrt{\frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4}} \right],$$

that $m_j^+ = \beta_j + O(h)$, with β_j given by (2.3.8). Since we take in the theorem that

$$c_j = \max\{0, \beta_j, \beta_{j-\frac{1}{2}}\} \geq 0,$$

which implies that $m_j = c_j + O(h) \geq \max\{0, m_j^+\}$, and this subsequently ensure the inequality (2.3.19) holds for this m_j, i.e., (2.3.16) holds with such selection of m_j . Analogously, we can show that (2.3.17) also holds with this m_j.

Now, by (2.3.15), the stability result follows easily that

$$\sum_{j=-\infty}^{N-1} \dot{U}_j U_j + \sum_{j=-\infty}^{N-1} \dot{V}_{j+\frac{1}{2}} V_{j+\frac{1}{2}} \leq 0,$$

and therefore

$$\sum_{j=-\infty}^{N-1} h(U_j(t))^2 + \sum_{j=-\infty}^{N-1} h(V_{j+\frac{1}{2}}(t))^2 \leq \sum_{j=-\infty}^{N-1} h(U_j(0))^2 + \sum_{j=-\infty}^{N-1} h(V_{j+\frac{1}{2}}(0))^2 \leq C_0, \quad \forall\, t > 0,$$

which, together with the definitions of U_j and $V_{j+\frac{1}{2}}$, we draw the conclusions that,

$$\sum_{j=k}^{N} h(u_j(t))^2 \leq C_0 e^{-2\int_0^{x_k}\sigma(\xi)d\xi}, \quad \sum_{j=k}^{N} h(v_{j+\frac{1}{2}}(t))^2 \leq C_0 e^{-2\int_0^{x_{k+\frac{1}{2}}}\sigma(\xi)d\xi}, \quad \forall\, t > 0,$$

for any given $0 \leq k \leq N-1$. The proof is completed.

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2.3.3 Stability and Convergence of the Yee Scheme for the Berenger's PMLs

In this subsection, we shall derive the stability and convergence results of the Yee scheme for the Berenger's PML equations. Firstly, we discuss the stability. Consider the Berenger's PML equation (2.3.5)

$$\begin{cases} u_t + \sigma u = v_x, & x \in (-\infty, d), \quad t > 0, \\ v_t + \sigma v = u_x, & x \in (-\infty, d), \quad t > 0 \end{cases}$$

and its corresponding Yee scheme given by (2.3.6)

$$\begin{cases} \dot{u}_j + \overline{\sigma}_j u_j = \frac{v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}}}{h}, & j \leq N, \\ \dot{v}_{j+\frac{1}{2}} + \overline{\sigma}_{j+\frac{1}{2}} v_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h}, & j \leq N-1. \end{cases}$$

Next, we give the sufficient conditions for the above Yee scheme to be stable.

Theorem 2.3.2 *For the Yee scheme (2.3.6), if we take*

$$\overline{\sigma}_j = \sigma_j + m_j h^2, \quad \overline{\sigma}_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}} + m_j h^2$$

then the scheme (2.3.6) is stable.

Proof. By the choice of $\overline{\sigma}_j$ and $\overline{\sigma}_{j+\frac{1}{2}}$, the scheme (2.3.6) can be rewritten as

$$\begin{cases} \dot{u}_j + \sigma_j u_j = \frac{v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}}}{h} - m_j h^2 u_j, \\ \dot{v}_{j+\frac{1}{2}} + \sigma_{j+\frac{1}{2}} v_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h} - m_j h^2 v_{j+\frac{1}{2}}. \end{cases} \tag{2.3.20}$$

Multiplying the first and second equation of (2.3.20) by u_j and $v_{j+\frac{1}{2}}$ respectively, and summing up them over $j \le N-1$, we deduce

$$\begin{aligned} &\sum_{j=-\infty}^{N-1} h\dot{u}_j u_j + \sum_{j=-\infty}^{N-1} \sigma_j h u_j^2 + \sum_{j=-\infty}^{N-1} h\dot{v}_{j+\frac{1}{2}} v_{j+\frac{1}{2}} + \sum_{j=-\infty}^{N-1} \sigma_{j+\frac{1}{2}} h v_{j+\frac{1}{2}}^2 \\ = &\; -m_j h^2 \left(\sum_{j=-\infty}^{N-1} h u_j^2 + \sum_{j=-\infty}^{N-1} h v_{j+\frac{1}{2}}^2 \right) \le 0, \end{aligned}$$

which gives

$$\frac{d}{dt}\left(\|\mathbf{u}(t)\|_{0,h}^2 + \|\mathbf{v}(t)\|_{0,h^*}^2 \right) \le 0 .$$

Therefore we have the stability of semi-discrete solution,

$$\|\mathbf{u}(t)\|_{0,h}^2 + \|\mathbf{v}(t)\|_{0,h^*}^2 \le \|\mathbf{u}(0)\|_{0,h}^2 + \|\mathbf{v}(0)\|_{0,h^*}^2 \le C_0 , \qquad \forall\, t > 0. \quad (2.3.21)$$

The proof is completed.

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Setting $e_j^u(t) = u(x_j,t) - u_j(t)$ and $e_{j+\frac{1}{2}}^v(t) = v(x_{j+\frac{1}{2}},t) - v_{j+\frac{1}{2}}(t)$, we have the following theorem about convergence:

Theorem 2.3.3 *For the Yee scheme (2.3.6), we take*

$$\overline{\sigma}_j = \sigma_j + m_j h^2, \quad \overline{\sigma}_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}} + m_j h^2,$$

where $m_j = c_j + O(h)$ *is given by (2.3.7),*

$$c_j = \max\{0, \beta_j, \beta_{j-\frac{1}{2}}\} \ge 0,$$

and β_j *is given by (2.3.8). If* $\|\mathbf{e}^u(0)\|_{0,h} = \|\mathbf{e}^v(0)\|_{0,h^*} = O(h^2)$, *then we have*

$$\|\mathbf{e}^u(t)\|_{0,h}^2 + \|\mathbf{e}^v(t)\|_{0,h^*}^2 \le C h^4 T, \quad \forall\, 0 \le t \le T. \qquad (2.3.22)$$

Proof. Consider the continuous PML equations (2.3.5) at $x = x_j$, and by the Taylor series expansions, we derive

$$\left\{ \begin{aligned} \dot{u}(x_j,t) + \sigma_j u(x_j,t) &= \frac{v(x_{j+\frac{1}{2}},t) - v(x_{j-\frac{1}{2}},t)}{h} \\ &\quad -\frac{1}{24} v_{xxx}(x_j,t) h^2 + O(h^4), \\ \dot{v}(x_{j+\frac{1}{2}},t) + \sigma_{j+\frac{1}{2}} v(x_{j+\frac{1}{2}},t) &= \frac{u(x_{j+1},t) - u(x_j,t)}{h} \\ &\quad -\frac{1}{24} u_{xxx}(x_{j+\frac{1}{2}},t) h^2 + O(h^4), \end{aligned} \right. \qquad (2.3.23)$$

Subtracting (2.3.20) from (2.3.23), we derive

$$\begin{cases} \dot{e}_j^u + \sigma_j e_j^u = \frac{e^v_{j+\frac{1}{2}} - e^v_{j+\frac{1}{2}}}{h} - \left[\frac{1}{24} v_{xxx}(x_j, t) - m_j u_j\right] h^2 + O(h^4), \\ \dot{e}^v_{j+\frac{1}{2}} + \sigma_{j+\frac{1}{2}} e^v_{j+\frac{1}{2}} = \frac{e^u_{j+1} - e^u_j}{h} - \left[\frac{1}{24} u_{xxx}(x_{j+\frac{1}{2}}, t) - m_j v_{j+\frac{1}{2}}\right] h^2 + O(h^4). \end{cases} \tag{2.3.24}$$

Multiplying the first and second equation of (2.3.24) by e_j^u and $e^v_{j+\frac{1}{2}}$ respectively, and summing up them over $j \leq N-1$, we can deduce

$$\begin{aligned} & \sum_{j=-\infty}^{N-1} h\dot{e}_j^u e_j^u + \sum_{j=-\infty}^{N-1} \sigma_j h (e_j^u)^2 + \sum_{j=-\infty}^{N-1} h \dot{e}^v_{j+\frac{1}{2}} e^v_{j+\frac{1}{2}} + \sum_{j=-\infty}^{N-1} \sigma_{j+\frac{1}{2}} h (e^v_{j+\frac{1}{2}})^2 \\ & = -h^2 \sum_{j=-\infty}^{N-1} h \left[\frac{1}{24} v_{xxx}(x_j, t) - m_j u_j\right] e_j^u \\ & + h^2 \sum_{j=-\infty}^{N-1} h \left[\frac{1}{24} u_{xxx}(x_{j+\frac{1}{2}}, t) - m_j v_{j+\frac{1}{2}} e^v_{j+\frac{1}{2}}\right] + O(h^5), \end{aligned}$$

which gives

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\|\mathbf{e}^u(t)\|^2_{0,h} + \|\mathbf{e}^v(t)\|^2_{0,h^*}\right) \\ & \leq -h^2 \left\{ \sum_{j=-\infty}^{N-1} h[c_1(t)]_j e_j^u(t) + \sum_{j=-\infty}^{N-1} h[c_2(t)]_{j+\frac{1}{2}} e^v_{j+\frac{1}{2}}(t) \right\} + O(h^5), \end{aligned} \tag{2.3.25}$$

where $[c_1(t)]_j = \frac{1}{24} v_{xxx}(x_j, t) - m_j u_j$ and $[c_2(t)]_{j+\frac{1}{2}} = \frac{1}{24} u_{xxx}(x_{j+\frac{1}{2}}, t) - m_j v_{j+\frac{1}{2}}$.

Using notation (2.3.2), the inequality (2.3.25) is equivalent to

$$\frac{1}{2}\frac{d}{dt}(\|B\|^2) \leq -h^2 < A, B > +O(h^5) \leq h^2 \|A\| \|B\| + O(h^5),$$

which implies

$$\frac{d}{dt}\|B\| \leq h^2\|A\| + O(h^5), \tag{2.3.26}$$

where $A = (\mathbf{c}_1(t), \mathbf{c}_2(t))^T$ and $B = (\mathbf{e}^u(t), \mathbf{e}^v(t))^T$.

Integrating (2.3.26) over $[0, t]$, we obtain

$$\|B(t)\| \leq \|B(0)\| + h^2 \int_0^t \|A(\tau)\| \, d\tau, \quad \forall \, t > 0,$$

which is equivalent to

$$\begin{aligned}&\left\{\|\mathbf{e}^u(t)\|_{0,h}^2+\|\mathbf{e}^v(t)\|_{0,h^*}^2\right\}^{\frac{1}{2}}\\ \le\ &\left\{\|\mathbf{e}^u(0)\|_{0,h}^2+\|\mathbf{e}^v(0)\|_{0,h^*}^2\right\}^{\frac{1}{2}}+h^2\int_0^t\left[\|\mathbf{c}_1(\tau)\|_{0,h}^2+\|\mathbf{c}_2(\tau)\|_{0,h^*}^2\right]^{\frac{1}{2}}d\tau.\end{aligned}$$

By (2.3.21) we know that $\|\mathbf{c}_1(\tau)\|_{0,h}^2+\|\mathbf{c}_2(\tau)\|_{0,h^*}^2\le C_0$, together with $\|\mathbf{e}^u(0)\|_{0,h}=\|\mathbf{e}^v(0)\|_{0,h^*}=O(h^2)$, we draw the conclusion that:

$$\|\mathbf{e}^u(t)\|_{0,h}^2+\|\mathbf{e}^v(t)\|_{0,h^*}^2\le Ch^4T,\quad \forall\ 0\le t\le T.$$

The proof is completed.

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2.3.4 Decay of the Yee Scheme Solution to the Hagstrom's PMLs

Subsection 2.3.3 and 2.3.4 have discussed the exponential decays, stability and convergence of Yee scheme solution to the Berenger's PMLs, in this and next subsection we shall propose these properties of the Hagstrom's PMLs.

In this subsection, we shall derive a decay result of the Yee scheme solution to the Hagstrom's PML equations. Now, we consider the Hagstrom's PML equation

$$\begin{cases} u_t+\sigma u=v_x+\sigma v, & x\in(-\infty,d),\ t>0,\\ v_t+\sigma v=u_x+\sigma u, & x\in(-\infty,d),\ t>0\end{cases}\tag{2.3.27}$$

and its corresponding Yee scheme given by,

$$\begin{cases} \dot{u}_j+\overline{\sigma}_j u_j=\frac{v_{j+\frac{1}{2}}-v_{j-\frac{1}{2}}}{h}+\overline{\sigma}^*_j\frac{1}{2}(v_{j+\frac{1}{2}}+v_{j-\frac{1}{2}}), & j\le N,\\ \dot{v}_{j+\frac{1}{2}}+\overline{\sigma}_{j+\frac{1}{2}}v_{j+\frac{1}{2}}=\frac{u_{j+1}-u_j}{h}+\overline{\sigma}^*_{j+\frac{1}{2}}\frac{1}{2}(u_{j+1}+u_j), & j\le N-1.\end{cases}\tag{2.3.28}$$

We introduce notations $\rho_j\equiv e^{2\int_{x_j}^{x_{j+\frac{1}{2}}}\sigma(\xi)d\xi}$ and $\alpha_j\equiv\left[\frac{\rho_j^{-1}-\rho_j}{h}+\frac{1}{2}(\overline{\sigma}^*_j\rho_j^{-1}+\overline{\sigma}^*_{j+\frac{1}{2}}\rho_j)\right]$, and for our subsequent use, we give an estimate as follows:

Lemma 2.3.3 *If $\sigma(x)$ is a smooth enough real function and*

$$m_j = \frac{1}{2}\left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} + \sqrt{\frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4}}\right], \tag{2.3.29}$$

then we have

$$m_j = \beta_j + O(h).$$

Here $\sigma_j = \sigma(x_j), \sigma_{j+\frac{1}{2}} = \sigma(x_{j+\frac{1}{2}})$ etc. as the usual utilizations and

$$\beta_j = -\frac{1}{96\sigma_j}[16\sigma_j^4 + 24\sigma_j^2\sigma_j' - 3(\sigma_j')^2 + 4\sigma_j\sigma_j''], \tag{2.3.30}$$

which is a constant independent of h.

Proof. Thanks to the Taylor series expansions, we can derive

$$\sigma_j + \sigma_{j+\frac{1}{2}} = 2\sigma_j + \frac{\sigma_j'}{2}h + \frac{\sigma_j''}{8}h^2 + \frac{\sigma_j'''}{48}h^3 + \frac{\sigma_j^{(4)}}{384}h^4 + O(h^5)$$

together with

$$\begin{aligned}(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2 &= 4\sigma_j^2 + 2\sigma_j\sigma_j'h + \frac{1}{6}[-8\sigma_j^4 - 12\sigma_j^2\sigma_j' - 3(\sigma_j')^2 + \sigma_j\sigma_j'']h^2 \\ &+ \frac{1}{6}[-8\sigma_j^3\sigma_j' - 3\sigma_j^2\sigma_j'' - 6\sigma_j(\sigma_j')^2 + \sigma_j'\sigma_j'']h^3 + O(h^4),\end{aligned}$$

we obtain

$$\begin{aligned} m_j &= -\frac{1}{96\sigma_j}[16\sigma_j^4 + 24\sigma_j^2\sigma_j' - 3(\sigma_j')^2 + 4\sigma_j\sigma_j''] \\ &-\frac{1}{384\sigma_j^2}[48\sigma_j^4\sigma_j' + 24\sigma_j^2(\sigma_j')^2 + 3(\sigma_j')^3 + 24\sigma_j^3\sigma_j'' - 6\sigma_j\sigma_j'\sigma_j'' + 4\sigma_j^2\sigma_j''']h + O(h^2).\end{aligned}$$

Here we have used the fact that

$$\begin{aligned}\sqrt{a + bh + ch^2 + dh^3} &= \sqrt{a} + \frac{bh}{2\sqrt{a}} + (-\frac{b^2}{8a^{3/2}} + \frac{c}{2\sqrt{a}})h^2 \\ &+ (\frac{b^3}{16a^{5/2}} - \frac{bc}{4a^{3/2}} + \frac{d}{2\sqrt{a}})h^3 + O(h^4).\end{aligned}$$

Therefore the dominated constant term in m_j is

$$\beta_j = -\frac{1}{96\sigma_j}[16\sigma_j^4 + 24\sigma_j^2\sigma_j' - 3(\sigma_j')^2 + 4\sigma_j\sigma_j'']$$

which is is independent of h.

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Theorem 2.3.4 *For the Yee scheme (2.3.28), if we take*

$$\overline{\sigma}_j = \sigma_j + m_j h^2, \quad \overline{\sigma}_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}} + m_j h^2, \quad \overline{\sigma}^*_j = \sigma_j, \quad \overline{\sigma}^*_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}},$$

with $m_j = c_j + O(h)$, *where*

$$c_j = \max\{0, \beta_j, \beta_{j-\frac{1}{2}}\} \geq 0, \tag{2.3.31}$$

and β_j *is given by (2.3.30), then for any given* $0 \leq k \leq N-1$, *we have*

$$\begin{aligned} &\sum_{j=k}^{N-1} h(u_j(t))^2 \leq C_0 e^{-2\int_0^{x_k} \sigma(\xi)d\xi}, \\ &\sum_{j=k}^{N-1} h(v_{j+\frac{1}{2}}(t))^2 \leq C_0 e^{-2\int_0^{x_{k+\frac{1}{2}}} \sigma(\xi)d\xi}, \quad \forall\, t > 0. \end{aligned} \tag{2.3.32}$$

Proof. Set $U_j = u_j e^{2\int_0^{x_j} \sigma(\xi)d\xi}$, $V_{j+\frac{1}{2}} = v_{j+\frac{1}{2}} e^{2\int_0^{x_{j+\frac{1}{2}}} \sigma(\xi)d\xi}$ and multiply the first equation of (2.3.28) by $e^{\int_0^{x_j} \sigma(\xi)d\xi}$, we get

$$\begin{aligned} \dot{U}_j + \overline{\sigma}_j U_j &= \frac{1}{h}\left[V_{j+\frac{1}{2}} - V_{j-\frac{1}{2}}\right] + \frac{1}{h}\left[\rho_j^{-1} - 1\right] V_{j+\frac{1}{2}} \\ &+ \frac{1}{h}\left[1 - \rho_{j-\frac{1}{2}}\right] V_{j-\frac{1}{2}} + \overline{\sigma}^*_j \frac{1}{2}(V_{j+\frac{1}{2}}\rho_j^{-1} + V_{j-\frac{1}{2}}\rho_{j-\frac{1}{2}}). \end{aligned} \tag{2.3.33}$$

Next, multiply the both hand sides of (2.3.33) by U_j, we can further get

$$\dot{U}_j U_j + \overline{\sigma}_j U_j^2 = \frac{1}{h}\left[V_{j+\frac{1}{2}} - V_{j-\frac{1}{2}}\right] U_j + \frac{1}{h}\left[\rho_j^{-1} - 1\right] V_{j+\frac{1}{2}} U_j + \frac{1}{h}\left[1 - \rho_{j-\frac{1}{2}}\right] V_{j-\frac{1}{2}} U_j. \tag{2.3.34}$$

Completely the same manipulations as above for the second equation of (2.3.28) lead to

$$\begin{aligned} &\dot{V}_{j+\frac{1}{2}} V_{j+\frac{1}{2}} + \overline{\sigma}_{j+\frac{1}{2}} V^2_{j+\frac{1}{2}} = \tfrac{1}{h}\left[U_{j+1} - U_j\right] V_{j+\frac{1}{2}} + \tfrac{1}{h}\left[\rho^{-1}_{j+\frac{1}{2}} - 1\right] U_{j+1} V_{j+\frac{1}{2}} \\ &+\tfrac{1}{h}\left[1 - \rho_j\right] U_j V_{j+\frac{1}{2}} + \tfrac{1}{2}\overline{\sigma}^*_{j+\frac{1}{2}} \rho^{-1}_{j+\frac{1}{2}} V_{j+\frac{1}{2}} U_{j+1} + \tfrac{1}{2}\overline{\sigma}^*_{j+\frac{1}{2}} \rho_j V_{j+\frac{1}{2}} U_j. \end{aligned} \tag{2.3.35}$$

Now, summing up (2.3.34) and (2.3.35) over $j \leq N-1$, and noting that $u_{-\infty} =$

$u_N = 0$, $U_{-\infty} = U_N = 0$, we deduce

$$\sum_{j=-\infty}^{N-1} \dot{U}_j U_j + \sum_{j=-\infty}^{N-1} \dot{V}_{j+\frac{1}{2}} V_{j+\frac{1}{2}} + \sum_{j=-\infty}^{N-1} \overline{\sigma}_j U_j^2 + \sum_{j=-\infty}^{N-1} \overline{\sigma}_{j+\frac{1}{2}} V_{j+\frac{1}{2}}^2$$

$$= \sum_{j=-\infty}^{N-1} \left[\frac{\rho_j^{-1} - \rho_j}{h} + \frac{1}{2}(\overline{\sigma}_j^* \rho_j^{-1} + \overline{\sigma}_{j+\frac{1}{2}}^* \rho_j) \right] V_{j+\frac{1}{2}} U_j$$

$$+ \sum_{j=-\infty}^{N-1} \left[\frac{\rho_{j+\frac{1}{2}}^{-1} - \rho_{j+\frac{1}{2}}}{h} + \frac{1}{2}(\overline{\sigma}_{j+\frac{1}{2}}^* \rho_{j+\frac{1}{2}}^{-1} + \overline{\sigma}_{j+1}^* \rho_{j+\frac{1}{2}}) \right] V_{j+\frac{1}{2}} U_{j+1}$$

$$\leq \frac{1}{2} \sum_{j=-\infty}^{N-1} \frac{1}{\overline{\sigma}_j} \left[\frac{\rho_j^{-1} - \rho_j}{h} + \frac{1}{2}(\overline{\sigma}_j^* \rho_j^{-1} + \overline{\sigma}_{j+\frac{1}{2}}^* \rho_j) \right]^2 V_{j+\frac{1}{2}}^2 + \frac{1}{2} \sum_{j=-\infty}^{N-1} \overline{\sigma}_j U_j^2$$

$$+ \frac{1}{2} \sum_{j=-\infty}^{N-1} \frac{1}{\overline{\sigma}_{j+\frac{1}{2}}} \left[\frac{\rho_{j+\frac{1}{2}}^{-1} - \rho_{j+\frac{1}{2}}}{h} + \frac{1}{2}(\overline{\sigma}_{j+\frac{1}{2}}^* \rho_{j+\frac{1}{2}}^{-1} + \overline{\sigma}_{j+1}^* \rho_{j+\frac{1}{2}}) \right]^2 U_{j+1}^2 + \frac{1}{2} \sum_{j=-\infty}^{N-1} \overline{\sigma}_{j+\frac{1}{2}} V_{j+\frac{1}{2}}^2.$$

In order to get stability, we only need:

$$\left[\frac{\rho_j^{-1} - \rho_j}{h} + \frac{1}{2}(\overline{\sigma}_j^* \rho_j^{-1} + \overline{\sigma}_{j+\frac{1}{2}}^* \rho_j) \right]^2 \leq \overline{\sigma}_j \overline{\sigma}_{j+\frac{1}{2}} \tag{2.3.36}$$

$$\left[\frac{\rho_{j-\frac{1}{2}}^{-1} - \rho_{j-\frac{1}{2}}}{h} + \frac{1}{2}(\overline{\sigma}_{j-\frac{1}{2}}^* \rho_{j-\frac{1}{2}}^{-1} + \overline{\sigma}_j^* \rho_{j-\frac{1}{2}}) \right]^2 \leq \overline{\sigma}_j \overline{\sigma}_{j-\frac{1}{2}}, \tag{2.3.37}$$

hold for $j \leq N-1$. Since $\overline{\sigma}_j = \sigma_j + m_j h^2$, $\overline{\sigma}_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}} + m_j h^2$, $\overline{\sigma}_j^* = \sigma_j$ and $\overline{\sigma}_{j+\frac{1}{2}}^* = \sigma_{j+\frac{1}{2}}$ as given in the theorem, then substitutions of both of them into (2.3.36) can reduce the inequality to

$$m_j^2 + \frac{\sigma_j + \sigma_{j+\frac{1}{2}}}{h^2} m_j + \frac{\sigma_j \sigma_{j+\frac{1}{2}} - \alpha_j^2}{h^4} \geq 0. \tag{2.3.38}$$

The corresponding quadratic equation for (2.3.38) has two distinct solutions

$$m_j^{\pm} = \frac{1}{2} \left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} \pm \sqrt{\frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4}} \right],$$

due to the fact that

$$\Delta = \frac{(\sigma_j + \sigma_{j+\frac{1}{2}})^2}{h^4} - 4\frac{\sigma_j \sigma_{j+\frac{1}{2}} - \alpha_j^2}{h^4} = \frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4} > 0.$$

Here, m_j^- and m_j^+ denote the left root and the right root in the real axis, respectively. By $\sigma(x) \geq 0$, we have that

$$m_j^- = \frac{1}{2}\left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} - \sqrt{\frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4}}\right] < 0,$$

and by Lemma 2.3.3, we have for the other root

$$m_j^+ = \frac{1}{2}\left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} + \sqrt{\frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4}}\right],$$

that $m_j = \beta_j + O(h)$, with β_j is given by (2.3.30). Since we take in the theorem that

$$c_j = \max\{0, \beta_j, \beta_{j-\frac{1}{2}}\} \geq 0,$$

which implies that $m_j = c_j + O(h) \geq \max\{0, m_j^+\}$, and this subsequently ensure the inequality (2.3.38) holds for this m_j, i.e., (2.3.36) holds with such selection of m_j . Analogously, we can show that (2.3.37) also holds with this m_j.

Now, by (2.3.36) and (2.3.37), the stability result follows easily that

$$\sum_{j=-\infty}^{N-1} \dot{U}_j U_j + \sum_{j=-\infty}^{N-1} \dot{V}_{j+\frac{1}{2}} V_{j+\frac{1}{2}} \leq 0,$$

and therefore

$$\sum_{j=-\infty}^{N-1} h(U_j(t))^2 + \sum_{j=-\infty}^{N-1} h(V_{j+\frac{1}{2}}(t))^2 \leq \sum_{j=-\infty}^{N-1} h(U_j(0))^2 + \sum_{j=-\infty}^{N-1} h(V_{j+\frac{1}{2}}(0))^2 \leq C_0, \quad \forall\, t > 0,$$

which, together with the definition of U_j and $V_{j+\frac{1}{2}}$, we draw the conclusion that:

$$\sum_{j=k}^{N} h(u_j(t))^2 \leq C_0 e^{-2\int_0^{x_k} \sigma(\xi)d\xi}, \quad \sum_{j=k}^{N} h(v_{j+\frac{1}{2}}(t))^2 \leq C_0 e^{-2\int_0^{x_{k+\frac{1}{2}}} \sigma(\xi)d\xi}, \quad \forall\, t > 0,$$

for any given $0 \leq k \leq N-1$. The proof is completed.

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2.3.5 Stability and Convergence of the Yee Scheme for the Hagstrom's PMLs

In last subsection, we discuss the exponential decays, here we shall derive the stability and convergence results of the Yee scheme for the Hagstrom's PML equations. Firstly, we discuss the stability. Consider the Hagstrom's PML equation (2.3.27)

$$\begin{cases} u_t + \sigma u = v_x + \sigma v, & x \in (-\infty, d),\ t > 0, \\ v_t + \sigma v = u_x + \sigma u, & x \in (-\infty, d),\ t > 0 \end{cases}$$

and its corresponding Yee scheme given by (2.3.28),

$$\begin{cases} \dot{u}_j + \overline{\sigma}_j u_j = \frac{v_{j+\frac{1}{2}} - v_{j-\frac{1}{2}}}{h} + \overline{\sigma}^*_j \frac{1}{2}(v_{j+\frac{1}{2}} + v_{j-\frac{1}{2}}), & j \leq N, \\ \dot{v}_{j+\frac{1}{2}} + \overline{\sigma}_{j+\frac{1}{2}} v_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h} + \overline{\sigma}^*_{j+\frac{1}{2}} \frac{1}{2}(u_{j+1} + u_j), & j \leq N - 1. \end{cases}$$

Next, we give the sufficient conditions for the above Yee scheme to be stable.

Theorem 2.3.5 *For the Yee scheme (2.3.28), if we take*

$$\overline{\sigma}_j = \sigma_j + m_j h^2, \quad \overline{\sigma}_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}} + m_j h^2, \quad \overline{\sigma}^*_j = \sigma_j, \quad \overline{\sigma}^*_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}},$$

with $m_j = c_j + O(h)$, where

$$c_j = \max\{0, \gamma_j, \gamma_{j-\frac{1}{2}}\} \geq 0, \tag{2.3.39}$$

and γ_j is given by

$$\gamma_j = \frac{(\sigma'_j)^2}{32\sigma_j}, \quad \gamma_{j-\frac{1}{2}} = \frac{(\sigma'_{j-\frac{1}{2}})^2}{32\sigma_{j-\frac{1}{2}}}, \tag{2.3.40}$$

then the scheme (2.3.28) is stable. Moreover, if we set

$$c_j = \max\{0, \beta_j, \beta_{j-\frac{1}{2}}, \gamma_j, \gamma_{j-\frac{1}{2}}\} \geq 0, \tag{2.3.41}$$

where β_j given by (2.3.30), then the semi-discrete solution decays exponentially.

Proof. By directly calculate, we can derive

$$\begin{aligned}
&\sum_{j=-\infty}^{N-1} h\dot{u}_j u_j + \sum_{j=-\infty}^{N-1} h\overline{\sigma}_j^* u_j^2 + \sum_{j=-\infty}^{N-1} h\dot{v}_{j+\frac{1}{2}} v_{j+\frac{1}{2}} + \sum_{j=-\infty}^{N-1} h\overline{\sigma}_{j+\frac{1}{2}}^* v_{j+\frac{1}{2}}^2 \\
= &\sum_{j=-\infty}^{N-1} \frac{\sigma_j}{2} h(v_{j+\frac{1}{2}} + v_{j-\frac{1}{2}})u_j + \sum_{j=-\infty}^{N-1} \frac{\sigma_{j+\frac{1}{2}}}{2} h(u_{j+1} + u_j)v_{j+\frac{1}{2}} \\
= &\sum_{j=-\infty}^{N-1} \frac{1}{2} h(\sigma_j + \sigma_{j+\frac{1}{2}})v_{j+\frac{1}{2}} u_j + \sum_{j=-\infty}^{N-1} \frac{1}{2} h(\sigma_{j+1} + \sigma_{j+\frac{1}{2}})v_{j+\frac{1}{2}} u_{j+1} \\
\leq &\frac{1}{2}\sum_{j=-\infty}^{N-1} \frac{h}{\overline{\sigma}_j}\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})^2}{4} v_{j+\frac{1}{2}}^2 + \frac{1}{2}\sum_{j=-\infty}^{N-1} \overline{\sigma}_j h u_j^2 \\
+ &\frac{1}{2}\sum_{j=-\infty}^{N-1} \frac{h}{\overline{\sigma}_{j+\frac{1}{2}}}\frac{(\sigma_{j+\frac{1}{2}} + \sigma_{j+1})^2}{4} v_{j+\frac{1}{2}}^2 + \frac{1}{2}\sum_{j=-\infty}^{N-1} \overline{\sigma}_{j+\frac{1}{2}} h v_{j+\frac{1}{2}}^2 .
\end{aligned} \tag{2.3.42}$$

Comparing the left and right hand sides, we see that for the stability, we only need:

$$(\sigma_j + \sigma_{j+\frac{1}{2}})^2 \leq 4\overline{\sigma}_j \overline{\sigma}_{j+\frac{1}{2}} \tag{2.3.43}$$

and

$$(\sigma_{j-\frac{1}{2}} + \sigma_j)^2 \leq 4\overline{\sigma}_{j-\frac{1}{2}} \overline{\sigma}_j . \tag{2.3.44}$$

Let us first consider (2.3.43), which is equivalent to

$$m_j^2 + \frac{\sigma_j + \sigma_{j+\frac{1}{2}}}{h^2} m_j - \frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2}{4h^4} \;\; \geq \;\; 0. \tag{2.3.45}$$

The corresponding quadratic equation for (2.3.45) has two distinct solutions

$$m_j^{\pm} \;\; = \;\; \frac{1}{2}\left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} \pm \frac{1}{h^2}\sqrt{2\sigma_j^2 + 2\sigma_{j+\frac{1}{2}}^2} \right].$$

due to the fact that

$$\Delta = \frac{(\sigma_j + \sigma_{j+\frac{1}{2}})^2}{h^4} - 4\frac{\sigma_j \sigma_{j+\frac{1}{2}} - \alpha_j^2}{h^4} = \frac{(\sigma_j - \sigma_{j+\frac{1}{2}})^2 + 4\alpha_j^2}{h^4} > 0.$$

Here, m_j^- and m_j^+ denote the left root and the right root in the real axis, respectively. By $\sigma(x) \geq 0$, we have that

$$m_j^- \;\; = \;\; \frac{1}{2}\left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} - \frac{1}{h^2}\sqrt{2\sigma_j^2 + 2\sigma_{j+\frac{1}{2}}^2} \right],$$

and thanks to the Taylor series expansions, we have for the other root

$$m_j^+ = \frac{1}{2}\left[-\frac{(\sigma_j + \sigma_{j+\frac{1}{2}})}{h^2} + \frac{1}{h^2}\sqrt{2\sigma_j^2 + 2\sigma_{j+\frac{1}{2}}^2}\right],$$

that $m_j = \gamma_j + O(h)$, with γ_j is given by

$$\gamma_j = \frac{(\sigma_j')^2}{32\sigma_j}, \quad \text{and} \quad \gamma_{j-\frac{1}{2}} = \frac{(\sigma'_{j-\frac{1}{2}})^2}{32\sigma_{j-\frac{1}{2}}}.$$

Since we take in the theorem that

$$c_j = \max\{0, \gamma_j, \gamma_{j-\frac{1}{2}}\} \geq 0,$$

which implies that $m_j = c_j + O(h) \geq \max\{0, m_j^+\}$, and this subsequently ensure the inequality (2.3.45) holds for this m_j, i.e., (2.3.43) holds with such selection of m_j . Analogously, we can show that (2.3.44) also holds with this m_j.

Now, by (2.3.42), the stability results follows easily that

$$\frac{d}{dt}\Big(\|\mathbf{u}(t)\|_{0,h}^2 + \|\mathbf{v}(t)\|_{0,h^*}^2\Big) \leq 0,$$

and therefore

$$\|\mathbf{u}(t)\|_{0,h}^2 + \|\mathbf{v}(t)\|_{0,h^*}^2 \leq \|\mathbf{u}(0)\|_{0,h}^2 + \|\mathbf{v}(0)\|_{0,h^*}^2 \leq C_0, \qquad t > 0. \qquad (2.3.46)$$

Moreover, if we set

$$c_j = \max\{0, \beta_j, \beta_{j-\frac{1}{2}}, \gamma_j, \gamma_{j-\frac{1}{2}}\} \geq 0,$$

where β_j given by (2.3.30), then we see that the semi-discrete solution decays by (2.3.32) and is stable by (2.3.46). The proof is completed.

♯

Next, we consider the convergence of the Yee scheme solution. Setting $e_j^u(t) = u(x_j, t) - u_j(t)$ and $e_{j+\frac{1}{2}}^v(t) = v(x_{j+\frac{1}{2}}, t) - v_{j+\frac{1}{2}}(t)$, we have the following theorem about convergence:

Theorem 2.3.6 *For the Yee scheme (2.3.28), if we take*

$$\overline{\sigma}_j = \sigma_j + m_j h^2, \quad \overline{\sigma}_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}} + m_j h^2, \quad \overline{\sigma}^*_j = \sigma_j, \quad \overline{\sigma}^*_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}},$$

where $m_j = c_j + O(h)$ is given by (2.3.29), c_j is by (2.3.41). Then we have

$$\|\mathbf{e}^u(t)\|^2_{0,h} + \|\mathbf{e}^v(t)\|^2_{0,h^*} \leq Ch^4 T, \quad \forall\ 0 \leq t \leq T, \tag{2.3.47}$$

provided $\|\mathbf{e}^u(0)\|_{0,h} = \|\mathbf{e}^v(0)\|_{0,h^*} = O(h^2)$.

Proof. Consider the continuous PML equations (2.3.27) at $x = x_j$. By the Taylor series expansions, we have

$$\left\{\begin{array}{l} \dot{u}(x_j,t) + (\sigma_j + m_j h^2)u(x_j,t) = \frac{v(x_{j+\frac{1}{2}},t) - v(x_{j-\frac{1}{2}},t)}{h} \\ \quad +\frac{\sigma_j}{2}(v(x_{j+\frac{1}{2}},t) + v(x_{j-\frac{1}{2}},t)) - \frac{1}{24}v_{xxx}(x_j,t)h^2 \\ \quad -\frac{1}{4}v_{xx}(x_j,t)h^2 + m_j h^2 u(x_j,t) + O(h^4), \\ \dot{v}(x_{j+\frac{1}{2}},t) + (\sigma_{j+\frac{1}{2}} + m_j h^2)v(x_{j+\frac{1}{2}},t) = \frac{u(x_{j+1},t) - u(x_j,t)}{h} \\ \quad +\frac{\sigma_{j+\frac{1}{2}}}{2}(u(x_{j+1},t) + u(x_j,t)) - \frac{1}{24}u_{xxx}(x_{j+\frac{1}{2}},t)h^2 \\ \quad -\frac{1}{4}u_{xx}(x_{j+\frac{1}{2}},t)h^2 + m_j h^2 v(x_{j+\frac{1}{2}},t) + O(h^4). \end{array}\right. \tag{2.3.48}$$

Subtracting (2.3.28) from (2.3.48), we derive

$$\left\{\begin{array}{lll} \dot{e}^u_j + (\sigma_j + m_j h^2)e^u_j & = & \frac{e^v_{j+\frac{1}{2}} - e^v_{j-\frac{1}{2}}}{h} + \frac{\sigma_j}{2}(e^v_{j+\frac{1}{2}} + e^v_{j-\frac{1}{2}}) \\ \multicolumn{3}{l}{\quad -\frac{1}{24}v_{xxx}(x_j,t)h^2 - \frac{1}{4}v_{xx}(x_j,t)h^2 + m_j h^2 u(x_j,t) + O(h^4),} \\ \dot{e}^v_{j+\frac{1}{2}} + (\sigma_{j+\frac{1}{2}} + m_j h^2)e^v_{j+\frac{1}{2}} & = & \frac{e^u_{j+1} - e^u_j}{h} + \frac{\sigma_{j+\frac{1}{2}}}{2}(e^u_{j+1} + e^u_j) \\ \multicolumn{3}{l}{\quad -\frac{1}{24}u_{xxx}(x_{j+\frac{1}{2}},t)h^2 - \frac{1}{4}u_{xx}(x_{j+\frac{1}{2}},t)h^2 + m_j h^2 v(x_{j+\frac{1}{2}},t) + O(h^4).} \end{array}\right. \tag{2.3.49}$$

Similarly to the proof of stability (2.3.46) of the semi-discrete solution, we obtain

$$\begin{aligned} &\sum_{j=-\infty}^{N-1} h\dot{e}^u_j e^u_j + \sum_{j=-\infty}^{N-1} h\dot{e}^v_{j+\frac{1}{2}} e^v_{j+\frac{1}{2}} \\ &\leq -h^2 \left\{ \sum_{j=-\infty}^{N-1} h \left[\frac{1}{24}v_{xxx}(x_j,t) + \frac{1}{4}v_{xx}(x_j,t) - m_j u_j\right] e^u_j \right. \\ &\quad \left. + \sum_{j=-\infty}^{N-1} h \left[\frac{1}{24}u_{xxx}(x_{j+\frac{1}{2}},t) + \frac{1}{4}u_{xx}(x_{j+\frac{1}{2}},t) - m_j v_{j+\frac{1}{2}}\right] e^v_{j+\frac{1}{2}} \right\} + O(h^5), \end{aligned}$$

which gives

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{e}^u(t)\|_{0,h}^2+\|\mathbf{e}^v(t)\|_{0,h^*}^2\right)$$
$$\le -h^2\left\{\sum_{j=-\infty}^{N-1} h[c_1(t)]_j e_j^u(t)+\sum_{j=-\infty}^{N-1} h[c_2(t)]_{j+\frac{1}{2}} e_{j+\frac{1}{2}}^v(t)\right\}+O(h^5) \quad (2.3.50)$$

where

$$[c_1(t)]_j = \frac{1}{24}v_{xxx}(x_j,t)+\frac{1}{4}v_{xx}(x_j,t)-m_j u_j,$$
$$[c_2(t)]_{j+\frac{1}{2}} = \frac{1}{24}u_{xxx}(x_{j+\frac{1}{2}},t)+\frac{1}{4}u_{xx}(x_{j+\frac{1}{2}},t)-m_j v_{j+\frac{1}{2}}.$$

Using notation (2.3.2), the inequality (2.3.50) is equivalent to

$$\frac{1}{2}\frac{d}{dt}(\|B\|^2) \;\le\; -h^2 < A,B > +O(h^5) \le h^2\|A\|\|B\|+O(h^5),$$

which implies

$$\frac{d}{dt}\|B\| \;\le\; h^2\|A\|+O(h^5), \qquad (2.3.51)$$

where $A=(\mathbf{c}_1(t),\mathbf{c}_2(t))^T$ and $B=(\mathbf{e}^u(t),\mathbf{e}^v(t))^T$.

Integrating (2.3.51) over $[0,t]$, we obtain

$$\|B(t)\| \le \|B(0)\|+h^2\int_0^t \|A(\tau)\|\,d\tau, \quad \forall\, t>0,$$

which is equivalent to

$$\left\{\|\mathbf{e}^u(t)\|_{0,h}^2+\|\mathbf{e}^v(t)\|_{0,h^*}^2\right\}^{\frac{1}{2}}$$
$$\le \left\{\|\mathbf{e}^u(0)\|_{0,h}^2+\|\mathbf{e}^v(0)\|_{0,h^*}^2\right\}^{\frac{1}{2}}+h^2\int_0^t \left[\|\mathbf{c}_1(\tau)\|_{0,h}^2+\|\mathbf{c}_2(\tau)\|_{0,h^*}^2\right]^{\frac{1}{2}}\,d\tau.$$

By (2.3.46) we know that $\|\mathbf{c}_1(\tau)\|_{0,h}^2+\|\mathbf{c}_2(\tau)\|_{0,h^*}^2 \le C_0$, together with $\|\mathbf{e}^u(0)\|_{0,h} = \|\mathbf{e}^v(0)\|_{0,h^*}=O(h^2)$, we draw the conclusion that:

$$\|\mathbf{e}^u(t)\|_{0,h}^2+\|\mathbf{e}^v(t)\|_{0,h^*}^2 \le Ch^4T, \quad \forall\, 0\le t\le T.$$

The proof is completed.

♯

2.4 Modified Lax-Wendroff Scheme for PMLs

In section 2.3, we proposed a modified Yee scheme which is second order for PMLs, and here, we shall derive another second order scheme: modified Lax-Wendroff scheme. The Lax-Wendroff scheme is a principal finite difference method used in the electromagnetic community. The modified Lax-Wendroff scheme present in this section is stable and decays exponentially in the perfectly matched layers as Yee scheme. Similarly to the discussion about the modified Yee scheme, we shall concentrate on studying the order of convergence in space.

This section is organized as follows: At the beginning, we shall show the exponential decays in parabolic equations by the Maximum Principle and its stability. In subsection 2.4.2, we use the same ideas to get exponential decays in hyperbolic equations with a viscous term and stability. At the end of this section, we derive the modified Lax-Wendroff scheme with exponential decays and show the stability of this scheme .

2.4.1 Exponential Decays in Parabolic Equations

In this subsection, we shall show how the parabolic equations decays Exponentially by the Maximum Principle. We first consider following the parabolic equation

$$u_t = u_x + \Delta t u_{xx}, \qquad x \in R,\ 0 < t < T, \tag{2.4.1}$$

with homogeneous initial conditions, and its corresponding semi-discrete scheme

$$\frac{u^{n+1} - u^n}{\Delta t} = u_x^n - \Delta t u_{xx}^n, \quad 0 \le n \le M - 1. \tag{2.4.2}$$

Lemma 2.4.1 *The solution of equation (2.4.1) has following decay property*

$$|u(x,t)| \le C_0 e^{-\frac{x}{\Delta t}}, \qquad x \ge 0. \tag{2.4.3}$$

Proof. Setting $w = ue^{\frac{x}{\Delta t}}$ in (2.4.1), we may derive

$$w_t = -w_x + \Delta t w_{xx}.$$

Using the Maximum Principle, we obtain

$$w(x,t) \le \max\{0, w(0,t)\} = \max\{0, u(0,t)\}.$$

Therefore, we deduce

$$|u(x,t)| = |w(x,t)|e^{-\frac{x}{\Delta t}} \le |u(0,t)|e^{-\frac{x}{\Delta t}} \le C_0 e^{-\frac{x}{\Delta t}}, \qquad \forall\, x \ge 0.$$

♯

Lemma 2.4.1 shows that the equation (2.4.1) is stable. For the semi-discrete scheme (2.4.2), we have the following lemma about the stability.

Theorem 2.4.1 *For any* $0 < \delta_2 < \frac{1}{2} < \delta_1 < 1$ *and* $\delta_1 + \delta_2 \le 1$, *if* $r = \frac{\Delta t}{\Delta x}$ *satisfies*

$$r \le C_0 \sqrt{\delta_2 (2 - \frac{1}{\delta_1})}, \tag{2.4.4}$$

then the scheme (2.4.2) is stable, where C_0 *is a constant depending only on mesh size* Δx.

Proof. Equation (2.4.2) is equivalent to

$$\frac{u^{n+1} - u^n}{\Delta t} - \Delta t u_{xx}^{n+1} = u_x^n - \Delta t [u^{n+1} - u^n]_{xx}. \tag{2.4.5}$$

Multiplying (2.4.5) by u^{n+1}, we can deduce

$$\begin{aligned}
& \frac{|u^{n+1}|^2 - |u^n|^2 + |u^{n+1} - u^n|^2}{2\Delta t} + \Delta t |u_x^{n+1}|^2 \\
= \; & < u_x^n, u^{n+1} > + \Delta t < (u^{n+1} - u^n)_x, u_x^{n+1} > \\
= \; & < u_x^{n+1}, u^{n+1} > - < (u^{n+1} - u^n)_x, u^{n+1} > + \Delta t < (u^{n+1} - u^n)_x, u_x^{n+1} > \\
= \; & < u^{n+1} - u^n, u_x^{n+1} > + \Delta t < (u^{n+1} - u^n)_x, u_x^{n+1} > \\
\le \; & |u^{n+1} - u^n||u_x^{n+1}| + \Delta t (C_0 \Delta x)^{-1} |u^{n+1} - u^n||u_x^{n+1}| \\
\le \; & \frac{1}{4\delta_1 \Delta t} |u^{n+1} - u^n|^2 + \delta_1 \Delta t |u_x^{n+1}|^2 + \Delta t [\frac{1}{4\delta_2} (C_0 \Delta x)^{-2} |u^{n+1} - u^n|^2 + \delta_2 |u_x^{n+1}|^2] \\
= \; & (\frac{\Delta t}{4\delta_2 C_0^2 (\Delta x)^2} + \frac{1}{4\delta_1 \Delta t}) |u^{n+1} - u^n|^2 + (\delta_1 + \delta_2) \Delta t |u_x^{n+1}|^2,
\end{aligned}$$

together with $\delta_1 + \delta_2 \leq 1$ and $r = \frac{\Delta t}{\Delta x} \leq C_0\sqrt{\delta_1(2 - \frac{1}{\delta_2})}$, we see that

$$|u^{n+1}| \leq |u^n|, \qquad \forall\, n \geq 0,$$

which implies that the scheme is stable.

♯

2.4.2 Exponential Decays in Hyperbolic Equations

Since the parabolic equation converges to the corresponding hyperbolic equation as coefficient of the viscous term goes to zero, and using the Maximum Principle of parabolic equation which we have used in last subsection, we would obtain the exponential decays in hyperbolic equations with a viscous term.

Next, we consider the following hyperbolic equation with a viscous term

$$V_t + 2\sigma V + V_x = \Delta t V_{xx}\,, \qquad x \in (-L, d),\ 0 < t < T, \tag{2.4.6}$$

where L is the length of the computational domain for $x < 0$ and d is the length of the PML.

Lemma 2.4.2 *When σ and L, d satisfy*

$$(L + d)\|\sigma\|_\infty \leq \frac{\sqrt{2}}{2}, \tag{2.4.7}$$

then for any $0 \leq d_1 \leq d$, the solution of (2.4.6) satisfies

$$\int_{d_1}^{d} V^2 dx \leq C_0 e^{-4\int_0^{d_1} \sigma(\xi)d\xi}. \tag{2.4.8}$$

Proof. Setting $w = Ve^{2\int_0^x \sigma(\xi)d\xi}$ in (2.4.6), we may derive

$$\begin{aligned} w_t + w_x &= \Delta t e^{2\int_0^x \sigma(\xi)d\xi} V_{xx} \\ &= \Delta t e^{2\int_0^x \sigma(\xi)d\xi} \{e^{-2\int_0^x \sigma(\xi)d\xi} w\}_{xx} \\ &= \Delta t e^{2\int_0^x \sigma(\xi)d\xi} \{(w_x - 2\sigma w)e^{-2\int_0^x \sigma(\xi)d\xi} w\}_x \\ &= \Delta t[(w_x - 2\sigma w)_x - 2\sigma(w_x - 2\sigma w)], \end{aligned}$$

i.e.,

$$w_t + w_x - \Delta t[(w_x - 2\sigma w)_x + 2\sigma(w_x - 2\sigma w)] = 0. \tag{2.4.9}$$

Multiplying (2.4.9) by w, we can calculate easily that

$$(w_t, w) + \Delta t(w_x, w_x) = \Delta t(4\sigma^2 w, w).$$

Since

$$w(x) = w(-L) + \int_{-L}^{x} w_x dx = \int_{-L}^{x} w_x dx \le \left[\int_{-L}^{x} 1dx\right]^{1/2} \left[\int_{-L}^{x} w_x^2 dx\right]^{1/2},$$

then

$$\frac{2}{(L+d)^2}\int_{-L}^{d} w^2 dx \le \int_{-L}^{d} w_x^2 dx,$$

together with condition $(L+d)\|\sigma\|_\infty \le \frac{\sqrt{2}}{2}$, we derive

$$\frac{2}{(L+d)^2}\int_{-L}^{d} w^2 dx \ge \int_{-L}^{d} 4\sigma^2(x) w^2 dx,$$

which gives $(w_t, w) \le 0$. Hence, we get

$$\int_0^d V^2 e^{4\int_0^x \sigma(\xi)d\xi} dx = \int_0^d w^2(x,t)dx \le C_0,$$

so, for any $0 \le d_1 \le d$, we have

$$\int_{d_1}^{d} V^2 dx \le C_0 e^{-4\int_0^{d_1} \sigma(\xi)d\xi}.$$

♯

Next, we discuss the stability of the semi-discrete scheme for equation (2.4.6). Since it is difficult for us to discuss the sufficient conditions for equation (2.4.6) directly, we consider the formulation (2.4.9) which is equivalent to equation (2.4.6). The formulation (2.4.9) can be rewritten as:

$$\begin{aligned} w_t &= -w_x + \Delta t(w_x - 2\sigma w)_x - 2\sigma\Delta t(w_x - \sigma w) \\ &= -w_x + \Delta t w_{xx} - 2\Delta t(\sigma w)_x - 2\Delta t\sigma w_x + 4\Delta t\sigma^2 w \end{aligned} \tag{2.4.10}$$

and its semi-discrete scheme is given by

$$\frac{w^{n+1}-w^n}{\Delta t} = -w_x^n + \Delta t w_{xx}^n - 2\Delta t(\sigma w^n)_x - 2\Delta t \sigma w_x^n + 4\Delta t \sigma^2 w^n, \ 0 \le n \le M-1. \tag{2.4.11}$$

We have the following theorem about the stability for the semi-discrete scheme (2.4.11):

Theorem 2.4.2 *For any* $0 < \varepsilon_1 < \frac{1}{2}$, $0 < \delta_2 < \frac{1}{2} - \varepsilon_1 < \frac{1}{2} < \delta_1 < 1 - \varepsilon_1$ *and* $\delta_1 + \delta_2 \le 1 - \varepsilon_1$, *when* $(L+d)\|\sigma\|_\infty < \sqrt{\frac{\varepsilon_1}{2}}$, Δx *small enough, and* $r = \frac{\Delta t}{\Delta x}$ *satisfies*

$$r \le C_0 \sqrt{\delta_2 (2 - \frac{1}{\delta_1})}, \tag{2.4.12}$$

then the scheme (2.4.11) is stable, where C_0 *is a constant depending only on mesh size* Δx.

Proof. Multiplying (2.4.11) by w^{n+1}, we have

$$\begin{aligned} & \frac{|w^{n+1}|^2 - |w^n|^2 + |w^{n+1}-w^n|^2}{2\Delta t} + \Delta t |w_x^{n+1}|^2 \\ = & \ \Delta t \left((w^{n+1} - w^n)_x, w_x^{n+1} \right) + (-(1+2\sigma\Delta t) w_x^n, w^{n+1}) \\ - & \ (\Delta t (2\sigma w^n)_x, w^{n+1}) + (4\Delta t \sigma^2 w^n, w^{n+1}) \\ \equiv & \ I + II + III + IV. \end{aligned}$$

For first three items, we can estimate as follows:

$$\begin{aligned}
I &= \Delta t((w^{n+1}-w^n)_x, w_x^{n+1})) \\
&\leq \Delta t|(w^{n+1}-w^n)_x||w_x^{n+1}| \\
&\leq \Delta t((C_0\Delta x)^{-1})|w^{n+1}-w^n||w_x^{n+1}|, \\
II &= (-(1+2\sigma\Delta t)w_x^n, w^{n+1}) \\
&\leq |(-(1+2\sigma\Delta t)w_x^{n+1}, w^{n+1})| + |((1+2\sigma\Delta t)(w^{n+1}-w^n)_x, w^{n+1})| \\
&\leq (1+2\|\sigma\|_\infty\Delta t)\left|w_x^{n+1}, w^{n+1}\right| + (1+2\|\sigma\|_\infty\Delta t)\left|(w^{n+1}-w^n), w^{n+1}\right| \\
&\leq (1+2\|\sigma\|_\infty\Delta t)|w^{n+1}-w^n||w_x^{n+1}|, \\
III &= |(-2\Delta t(\sigma w^n)_x, w^{n+1})| = |(2\Delta t\sigma w^n, w_x^{n+1})| \\
&\leq |(2\Delta t\sigma w^{n+1}, w_x^{n+1})| + |(2\Delta t\sigma(w^{n+1}-w^n), w_x^{n+1})| \\
&\leq 2\Delta t\|\sigma\|_\infty|(w^{n+1}, w_x^{n+1})| + (2\Delta t\sigma(w^{n+1}-w^n), w_x^{n+1}) \\
&\leq 2\Delta t\|\sigma\|_\infty|w^{n+1}-2^n||w_x^{n+1}|.
\end{aligned}$$

Since $|w|^2 \leq \frac{(L+d)^2}{2}|w_x|^2$, then for the last item , we have

$$\begin{aligned}
IV &= (4\Delta t\sigma^2 w^n, w^{n+1}) \\
&\leq 4\Delta t\|\sigma\|_\infty^2(w^n, w^{n+1}) \\
&\leq 4\Delta t\|\sigma\|_\infty^2[(w^{n+1}, w^{n+1}) + (w^n - w^{n+1}, w^{n+1})] \\
&\leq 4\Delta t\|\sigma\|_\infty^2[|w^{n+1}|^2 + |w^{n+1}-w^n||w^{n+1}|] \\
&\leq 2\Delta t\|\sigma\|_\infty^2(L+d)^2|w_x^{n+1}|^2 + 2\sqrt{2}\Delta t\|\sigma\|_\infty^2|w^{n+1}-w^n||w_x^{n+1}|.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
& \frac{|w^{n+1}|^2 - |w^n|^2 + |w^{n+1} - w^n|^2}{2\Delta t} + \Delta t |w_x^{n+1}|^2 \\
\leq\ & |w^{n+1} - w^n||w_x^{n+1}| + 2\Delta t \|\sigma\|_\infty^2 (L+d)^2 |w_x^{n+1}|^2 \\
+\ & \Delta t \Big[(C_0 \Delta x)^{-1} + 4\|\sigma\|_\infty + 2\sqrt{2}(L+d)\|\sigma\|_\infty^2 \Big] |w^{n+1} - w^n||w_x^{n+1}| \\
\leq\ & \frac{1}{4\delta_1 \Delta t}|w^{n+1} - w^n|^2 + \delta_1 \Delta t |w_x^{n+1}|^2 + 2\Delta t \|\sigma\|_\infty^2 (L+d)^2 |w_x^{n+1}|^2 \\
+\ & \frac{\Delta t}{4\delta_2}(C_0 \Delta x)^{-2} |w^{n+1} - w^n|^2 + \delta_2 \Delta t |w_x^{n+1}|^2 \\
=\ & \Big[\delta_1 + \delta_2 + 2(L+d)^2 |\sigma\|_\infty^2 \Big] |w_x^{n+1}|^2 + \Big[\frac{1}{4\delta_1 \Delta t} + \frac{(C_0\Delta x)^{-2}\Delta t}{4\delta_2} \Big] |w^{n+1} - w^n|^2.
\end{aligned}$$

By the conditions $0 < \varepsilon_1 < \frac{1}{2}$, $0 < \delta_2 < \frac{1}{2} - \varepsilon_1 < \frac{1}{2} < \delta_1 < 1 - \varepsilon_1$, $\delta_1 + \delta_2 \leq 1 - \varepsilon_1$, $(L+d)\|\sigma\|_\infty < \sqrt{\frac{\varepsilon_1}{2}}$, and $r \leq C_0 \sqrt{\delta_2 (2 - \frac{1}{\delta_1})}$, we draw the conclusion that

$$|w^{n+1}| \leq |w^n|, \qquad \forall\, n \geq 0,$$

which implies that the scheme is stable.

♯

2.4.3 Exponential Decays of Modified Lax-Wendroff Solutions

Finally, we shall show that the numerical solution of the modified Lax-Wendroff scheme decays exponentially. For ease of exposition, we introduce the following forward-, backward- and centered-difference operators,

$$\Delta^+ u_j = \frac{u_{j+1} - u_j}{h}, \qquad \Delta^- u_j = \frac{u_j - u_{j-1}}{h},$$

$$\Delta^+ \Delta^- u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}.$$

Next, for our discussions in the sequel, we firstly derive the following two properties of these difference operators.

Lemma 2.4.3 *For any sequence $\{u_j\}_{j=-M}^{N-1}$, we have*

$$(i) \qquad \sum_{j=-M}^{N-1}(\Delta^+u_j)hv_j = -\sum_{j=-M}^{N-1}(\Delta^-v_j)hu_j, \tag{2.4.13}$$

$$(ii) \qquad \sum_{j=-M}^{N-1}hu_j^2 \leq [L+d]^2\sum_{j=-M}^{N-1}h\frac{(\Delta^+u_j)^2+(\Delta^-u_j)^2}{2}. \tag{2.4.14}$$

Proof. (i) Using $u_0 = u_N = 0$, we may derive

$$\begin{aligned}
\sum_{j=-L}^{N}\Delta^+u_jhv_j &= \sum_{j=-M}^{N-1}(u_{j+1}-u_j)hv_j = \sum_{j=-M}^{N-1}hu_{j+1}v_j - \sum_{j=-M}^{N-1}hu_jv_j \\
&= \sum_{j=-M}^{N-1}hu_jv_{j-1} - \sum_{j=-M}^{N-1}hu_jv_j = -\sum_{j=-M}^{N-1}h(v_j-v_{j-1}u_j) \\
&= -\sum_{j=-M}^{N-1}\Delta^-v_jhu_j.
\end{aligned}$$

(ii) Similarly to the proof of the Poincare inequality, we can deduce

$$\begin{aligned}
u_j &= u_{-L} + \sum_{j=-L}^{N-1}(u_{j+1}-u_j) = \sum_{j=-M}^{N-1}\frac{u_{j+1}-u_j}{h}\cdot h, \\
u_j^2 &\leq \sum_{j=-M}^{N-1}(\frac{u_{j+1}-u_j}{h})^2\cdot\sum_{j=-M}^{N-1}h^2 = (M+N)h\sum_{j=-M}^{N-1}h(\Delta^+u_j)^2,
\end{aligned}$$

which implies

$$\sum_{j=-M}^{N-1}hu_j^2 \leq [(M+N)h]^2\sum_{j=-M}^{N-1}h(\Delta^+u_j)^2.$$

In the same way, we can obtain

$$\begin{aligned}
\sum_{j=-M}^{N-1}hu_j^2 &\leq [(M+N)h]^2\sum_{j=-M}^{N-1}h(\Delta^-u_j)^2, \\
\sum_{j=-M}^{N-1}hu_j^2 &\leq [L+d]^2\sum_{j=-M}^{N-1}h\frac{(\Delta^+u_j)^2+(\Delta^-u_j)^2}{2}.
\end{aligned}$$

♯

Now, we shall show that the numerical solution of the modified Lax-Wendroff scheme decays exponentially. The modified Lax-Wendroff scheme for equation (2.4.6) is given by

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + 2\sigma_j V_j^n + \frac{V_{j+1}^n - V_{j-1}^n}{2h} = \Delta t \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{h^2}, \qquad (2.4.15)$$

which can rewritten as

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + 2\sigma_j V_j^n + \frac{1}{2}(\Delta^+ V_j^n + \Delta^- V_j^n) = \Delta t(\Delta^+\Delta^- V_j^n). \qquad (2.4.16)$$

Setting $W_j^n = e^{2\int_0^{x_j} \sigma(\xi)d\xi} V_j^n = \rho_j V_j^n$ in (2.4.16), it is easy to verify that

$$\begin{aligned} &\frac{W_j^{n+1} - W_j^n}{\Delta t} + \frac{W_{j+1}^n - W_{j+1}^n}{2h} + W_j^n[2\sigma_j + \frac{1}{2h}(e^{-2\int_{x_j}^{x_{j+1}} \sigma(\xi)d\xi} - e^{2\int_{x_{j-1}}^{x_j} \sigma(\xi)d\xi})] \\ = \; &\Delta t[\Delta^+\Delta^- W_j^n + (\Delta^- W_j^n)\rho_j(\Delta^+ \rho_j^{-1}) + \rho_j \Delta^+(W_j^n(\Delta^- \rho_j^{-1}))]. \qquad (2.4.17) \end{aligned}$$

Theorem 2.4.3 *Assume ε_1, ε_2, δ_1, δ_2 and δ_3 satisfy*

$$0 < \varepsilon_1, \varepsilon_2, \delta_1, \delta_2, \delta_3 < 1, \quad h^2 \Delta t < \varepsilon_2, \quad (L + d)\|\sigma\|_\infty < 2\sqrt{\varepsilon_1}$$

and

$$2(\delta_1 + \delta_2 + \delta_3) \le 1 - \varepsilon_1, \quad r \le C_0 \sqrt{\delta_2(1 - \varepsilon_2 - \frac{1}{4\delta_1})},$$

then the scheme (2.4.17) is stable, where C_0 is a constant depending only on mesh size Δx. Furthermore, for any given $0 \le k \le N - 1$, we have

$$\sum_{j=k}^{N-1} h(V_j^n)^2 \le C_0 e^{-4\int_0^{x_k} \sigma(\xi)d\xi}, \qquad \forall\; n \ge 0. \qquad (2.4.18)$$

Proof. Multiplying (2.4.17) by hW_j^{n+1} and summing up over $j = -M, 1, \cdots, N-1$, we can derive the estimates for the three terms on the left-hand side:

$$
\begin{aligned}
I &= \sum_{j=-M}^{N-1} \frac{W_j^{n+1} - W_j^n}{\Delta t} \cdot hW_j^{n+1} \\
&\geq \frac{\sum_{j=-M}^{N-1} h|W_j^{n+1}|^2 - \sum_{j=-M}^{N-1} h|W_j^n|^2 + \sum_{j=-M}^{N-1} h|W_j^{n+1} - W_j^n|^2}{2\Delta t}, \\
II &= \sum_{j=-M}^{N-1} \frac{W_{j+1}^n - W_{j+1}^n}{2h} \cdot hW_j^{n+1} = \sum_{j=-M}^{N-1} \frac{1}{2}(\Delta^+ W_j^n + \Delta^- W_j^n) hW_j^{n+1} \\
&= \sum_{j=-M}^{N-1} \frac{1}{2}(\Delta^+ W_j^{n+1} + \Delta^- W_j^{n+1}) h_j^{n+1} \\
&+ \sum_{j=-M}^{N-1} \frac{1}{2}\left[\Delta^+(W_j^n - W_j^{n+1}) + \Delta^-(W_j^n - W_j^{n+1})\right] hW_j^{n+1} \\
&= \sum_{j=-M}^{N-1} \frac{1}{2} h(-W_j^n + W_j^{n+1})(\Delta^+ W_j^{n+1} + \Delta^- W_j^{n+1}),
\end{aligned}
$$

$$
\begin{aligned}
III &= \sum_{j=-M}^{N-1} W_j^n [2\sigma_j + \frac{1}{2h}(e^{-2\int_{x_j}^{x_{j+1}} \sigma(\xi)d\xi} - e^{2\int_{x_{j-1}}^{x_j} \sigma(\xi)d\xi})] hW_j^{n+1} \\
&= \sum_{j=-M}^{N-1} AW_j^{n+1} hW_j^{n+1} + \sum_{j=-M}^{N-1} A(W_j^n - W_j^{n+1}) hW_j^{n+1},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= 2\sigma_j + \frac{1}{2h}(e^{-2\int_{x_j}^{x_{j+1}} \sigma(\xi)d\xi} - e^{2\int_{x_{j-1}}^{x_j} \sigma(\xi)d\xi}) \\
&= \frac{1}{3}(-4\sigma_j^3 + 2\sigma_j\sigma_j' - \sigma_j'')h^2 + O(h^4).
\end{aligned}
$$

For the first term on the right-hand side of equation (2.4.17), we know

$$
\begin{aligned}
&\Delta t \Delta^+ \Delta^- W_j^n hW_j^{n+1} \\
&= \Delta t(\Delta^+(\Delta^- W_j^{n+1}))W_j^{n+1} + \Delta t \Delta^+ \Delta^- (W_j^n - W_j^{n+1}) W_j^{n+1} \\
&\equiv IV_1 + IV_2,
\end{aligned}
$$

where

$$
\begin{aligned}
IV_1 &= \Delta t(\Delta^+(\Delta^- W_j^{n+1}))W_j^{n+1} \\
&= -\Delta t(\Delta^- W_j^{n+1})(\Delta^- W_j^{n+1}) \\
&= -\Delta t(\Delta^+ W_j^{n+1})(\Delta^+ W_j^{n+1}) \\
&= -\frac{1}{2}\Delta t((\Delta^- W_j^{n+1})^2 + (\Delta^+ W_j^{n+1})^2), \\
IV_2 &= \Delta t\Delta^+\Delta^-(W_j^n - W_j^{n+1})W_j^{n+1} \\
&= -\Delta t\Delta^-(w_j^n - W_j^{n+1})\Delta^- W_j^{n+1} \\
&\leq \frac{\Delta t}{h}[|W_j^n - W_j^{n+1}| + |W_{j-1}^n - W_{j-1}^{n+1}|]|\Delta^- W_j^{n+1}|.
\end{aligned}
$$

Then we see from above that

$$
\begin{aligned}
IV &= \sum_{j=-M}^{N-1} \Delta t\Delta^+\Delta^- W_j^n h W_j^{n+1} \\
&\leq \frac{\Delta t}{2}\sum_{j=-M}^{N-1}((\Delta^- W_j^{n+1})^2 + (\Delta^+ W_j^{n+1})^2) + \frac{2\Delta t}{h}\sum_{j=-M}^{N-1}|W_j^n - W_j^{n+1}||\Delta^- W_j^{n+1}|.
\end{aligned}
$$

For the second term of the RHS of equation (2.4.17), we know

$$
\begin{aligned}
& \Delta t(\Delta^- W_j^n)\rho_j(\Delta^+\rho_j^{-1})W_j^{n+1} \\
= {} & \Delta t B(\Delta^- W_j^n)W_j^{n+1} \\
= {} & \Delta t B(\Delta^- W_j^{n+1})W_j^{n+1} + \Delta t B(\Delta^-(W_j^n - W_j^{n+1}))W_j^{n+1}.
\end{aligned}
$$

Then we can further deduce

$$
\begin{aligned}
V &= \Delta t\sum_{j=-M}^{N-1} B(\Delta^- W_j^n)\rho_j(\Delta^+\rho_j^{-1})W_j^{n+1} \\
&\leq \Delta t\sum_{j=-M}^{N-1} B|\Delta^- W_j^{n+1}||W_j^{n+1}| + \Delta t\sum_{j=-M}^{N-1} B|W_j^n - W_j^{n+1}||\Delta^+ W_j^{n+1}|,
\end{aligned}
$$

where

$$
B = \rho_j(\Delta^+\rho_j^{-1}) = -2\sigma_j + (2\sigma_j^2 - \sigma_j')h + \frac{1}{3}(-4\sigma_j^3 + 6\sigma_j\sigma_j' - \sigma_j'')h^2 + O(h^3).
$$

For the third term of the RHS of equation (2.4.17), we know

$$\begin{aligned} & \Delta t\rho_j\Delta^+W_j^n(\Delta^-\rho_j^{-1})W_j^{n+1} \\ = \; & \Delta t\rho_j\Delta^+(W_j^{n+1}(\Delta^-\rho_j^{-1}))W_j^{n+1} + \Delta t\rho_j\Delta^+((W_j^n - W_j^{n+1})(\Delta^-\rho_j^{-1}))W_j^{n+1}, \end{aligned}$$

then we can derive

$$\begin{aligned} VI &= \Delta t\sum_{j=-M}^{N-1}\rho_j\Delta^+W_j^n(\Delta^-\rho_j^{-1})W_j^{n+1} \\ &\leq \Delta t\sum_{j=-M}^{N-1}\rho_j|W_j^{n+1}(\Delta^-\rho_j^{-1})||\Delta^-W_j^{n+1}| + \Delta t\sum_{j=-M}^{N-1}\rho_j|W_j^n - W_j^{n+1}|(\Delta^-\rho_j^{-1})|\Delta^-W_j^{n+} \\ &= \Delta t\sum_{j=-M}^{N-1}B|W_j^{n+1}||\Delta^-W_j^{n+1}| + \Delta t\sum_{j=-M}^{N-1}B|W_j^n - W_j^{n+1}||\Delta^-W_j^{n+1}|. \end{aligned}$$

From I to VI, we can obtain

$$\begin{aligned} & \frac{\sum_{j=-M}^{N-1}h|W_j^{n+1}|^2 - \sum_{j=-M}^{N-1}h|W_j^n|^2 + \sum_{j=-M}^{N-1}h|W_j^{n+1} - W_j^n|}{2\Delta t} \\ + \; & \Delta t\sum_{j=-M}^{N-1}\frac{h}{2}[(\Delta^+W_j^{n+1})^2 + (\Delta^-W_j^{n+1})^2] + \sum_{j=-M}^{N-1}Ah|W_j^{n+1}|^2 \\ \leq \; & \sum_{j=-M}^{N-1}\frac{h}{2}|W_j^{n+1} - W_j^n||\Delta^+W_j^{n+1} + \Delta^-W_j^{n+1}| \\ + \; & \sum_{j=-M}^{N-1}Ah|W_j^{n+1} - W_j^n||W_j^{n+1}| \qquad (2.4.19) \\ + \; & \Delta t\sum_{j=-M}^{N-1}(\frac{2}{h})h|W_j^{n+1} - W_j^n||\Delta^-W_j^{n+1}| + \Delta t\sum_{j=-M}^{N-1}2Bh|\Delta^-W_j^{n+1}||W_j^{n+1}| \\ + \; & \Delta t\sum_{j=-M}^{N-1}Bh|W_j^{n+1} - W_j^n||\Delta^+W_j^{n+1}| + \Delta t\sum_{j=-M}^{N-1}Bh|W_j^{n+1} - W_j^n||\Delta^-W_j^{n+1}|. \end{aligned}$$

Then by direct computing, we have:

$$
\begin{aligned}
& \frac{\sum_{j=-M}^{N-1} h|W_j^{n+1}|^2 - \sum_{j=-M}^{N-1} h|W_j^n|^2 + \sum_{j=-M}^{N-1} h|W_j^{n+1} - W_j^n|}{\Delta t} \\
+ & \Delta t \sum_{j=-M}^{N-1} h[(\Delta^+ W_j^{n+1})^2 + (\Delta^- W_j^{n+1})^2] + \sum_{j=-M}^{N-1} Ah|W_j^{n+1}|^2 \\
\leq & \frac{1}{4\delta_1 \Delta t} \sum_{j=-M}^{N-1} h|W_j^{n+1} - W_j^n|^2 + 2\delta_1 \Delta t \sum_{j=-M}^{N-1} h((\Delta^+ W_j^{n+1})^2 + (\Delta^- W_j^{n+1})^2) \\
+ & \frac{1}{4\delta_2} \Delta t \sum_{j=-M}^{N-1} [(\frac{4}{h} + 4B)]^2 h|W_j^{n+1} - W_j^n|^2 + 2\delta_2 \Delta t \sum_{j=-M}^{N-1} h((\Delta^+ W_j^{n+1})^2 + (\Delta^- W_j^{n+1})^2) \\
+ & \Delta t [\sum_{j=-M}^{N-1} 2\delta_3 h|\Delta^- W_j^{n+1}|^2 + (L+d)^2 \sum_{j=-M}^{N-1} \frac{B^2}{2\delta_3} h \frac{1}{2} ((\Delta^+ W_j^{n+1})^2 + (\Delta^- W_j^{n+1})^2)] \\
+ & \sum_{j=-M}^{N-1} Ah|W_j^{n+1} - W_j^n|^2.
\end{aligned}
$$

Since ε_1, ε_2, δ_1, δ_2 and δ_3 satisfy conditions

$$0 < \varepsilon_1, \varepsilon_2, \delta_1, \delta_2, \delta_3 < 1, \quad h^2 \Delta t < \varepsilon_2, \quad (L+d)\|\sigma\|_\infty < 2\sqrt{\varepsilon_1},$$

and

$$2(\delta_1 + \delta_2 + \delta_3) \leq 1 - \varepsilon_1, \quad r \leq C_0 \sqrt{\delta_2 (1 - \varepsilon_2 - \frac{1}{4\delta_1})},$$

which implies

$$2\delta_1 + 2\delta_2 + 2\delta_3 + \frac{\|\sigma\|_\infty^2 (L+d)^2}{4} \leq 1, \quad \frac{1}{4\Delta t} + \frac{4\Delta t}{\delta_2 h^2} \leq 1 - \varepsilon_2,$$

we then can derive

$$\sum_{j=-M}^{N-1} h|W_j^{n+1}|^2 \leq \sum_{j=-M}^{N-1} h|W_j^n|^2, \qquad \forall\, n \geq 0.$$

Finally, by the definition $V_j^n = e^{2\int_0^{x_j} \sigma(\xi) d\xi} W_j^n$, we see that: for any given $0 \leq k \leq N-1$,

$$\sum_{j=k}^{N-1} h(V_j^n)^2 \leq C_0 e^{-4\int_0^{x_k} \sigma(\xi) d\xi}, \qquad \forall\, n \geq 0.$$

♯

Chapter 3

Numerical Simulation

In this section, we shall show some numerical experiments to demonstrate the effectiveness of the PML theories developed in the thesis. We shall focus on the one-dimensional Maxwell's equations

$$\begin{cases} \dfrac{\partial E_y}{\partial t} = \dfrac{\partial H_z}{\partial x}, \\ \dfrac{\partial H_z}{\partial t} = \dfrac{\partial E_y}{\partial x}, \end{cases} \tag{3.0.1}$$

and its Berenger's PML equations

$$\begin{cases} \dfrac{\partial E_y}{\partial t} + \sigma E_y = \dfrac{\partial H_z}{\partial x}, \\ \dfrac{\partial H_z}{\partial t} + \sigma H_z = \dfrac{\partial E_y}{\partial x}. \end{cases} \tag{3.0.2}$$

The computational domain consists of a domain with $x \in [-50, 50]$ as illustrated in Fig 3.1. To terminate the computational domain in the x-direction, we add two additional layers, having $50 \leq |x| \leq 60$, in which the PML equations are solved. As the initial condition we use a magnetic pulse of the form

$$H_z(x, 0) = e^{-ln(2)\frac{x^2}{\delta^2}}, \qquad -60 \leq x \leq 60.$$

In all computations $\delta = 3$ and the initial electric field components are zero. Here, we choose the boundary conditions as

$$E(-60, t) = H(59.5, t) = 0, \qquad 0 \leq t \leq 70.$$

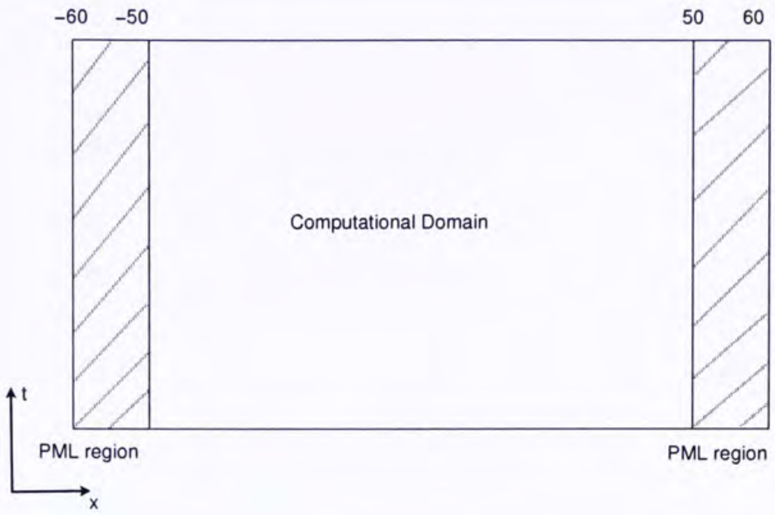

Figure 3.1: The geometry of the test case used throughout.

The time behavior of the computation is illustrated in Fig 3.2 and Fig 3.3, showing how the initial pulse spreads, enters the PML and is being effectively absorbed. In this particular case we have used the Yee scheme (2.3.6) with $\Delta x = 1$, $\Delta t = 0.01$ and choose an absorption profile given as

$$\sigma(x) = \begin{cases} \left(\dfrac{|x| - 50}{10}\right)^m & 50 \leq |x| \leq 60, \\ 0 & |x| \leq 50, \end{cases}$$

where m is the order of the profile. Typical values are $m = 2, 3, 4$ and we have chosen $m = 3$ in our computations [AGH].

The time behavior of the computation of Hagstrom's PML is illustrated in Fig 3.4 and Fig 3.5.

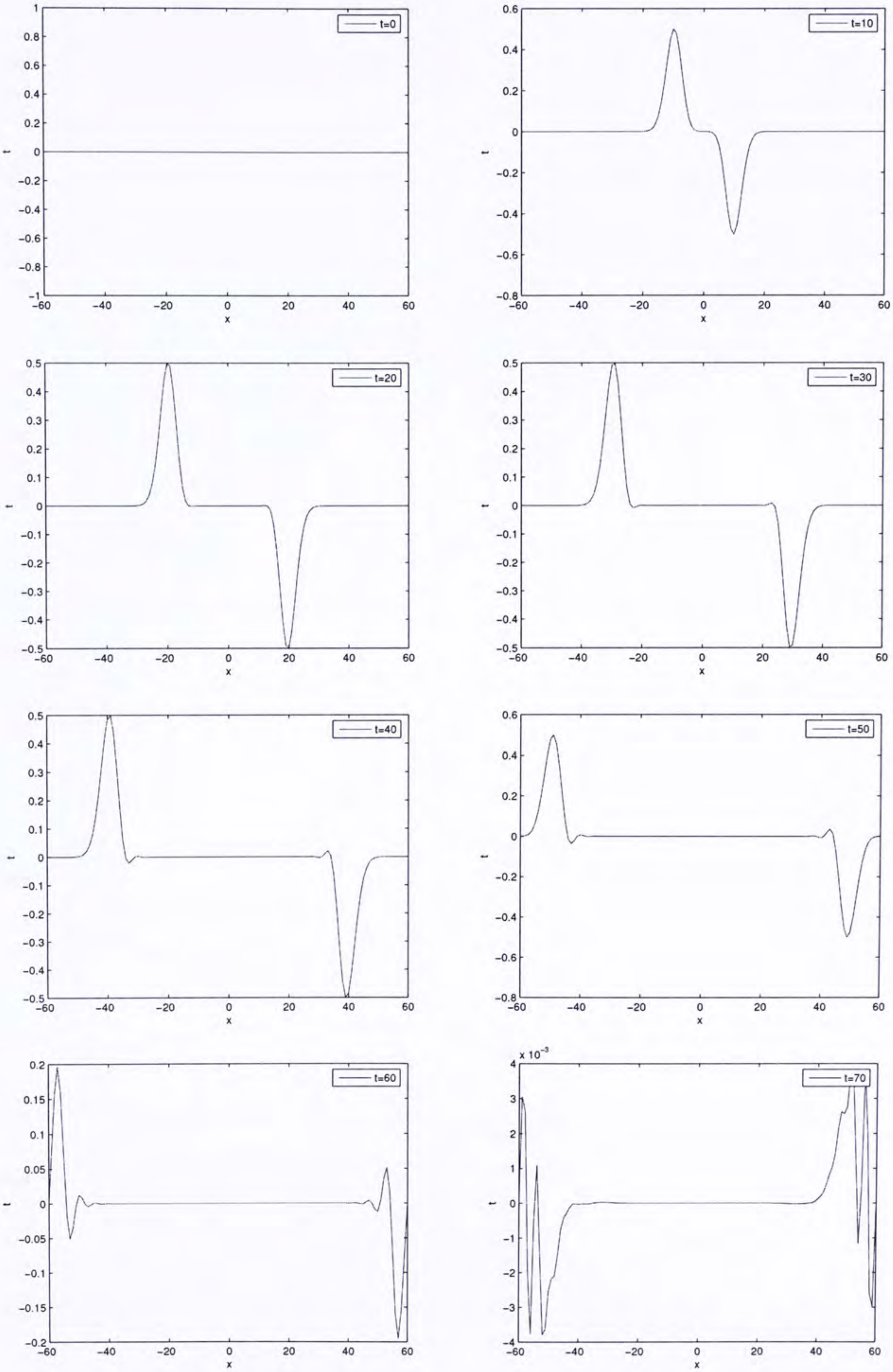

Figure 3.2: Short time dynamics of PML solutions subject to an initial magnetic pulse. The snapshots are given at T= $0, 10, \cdots, 70$ for the component E_y.

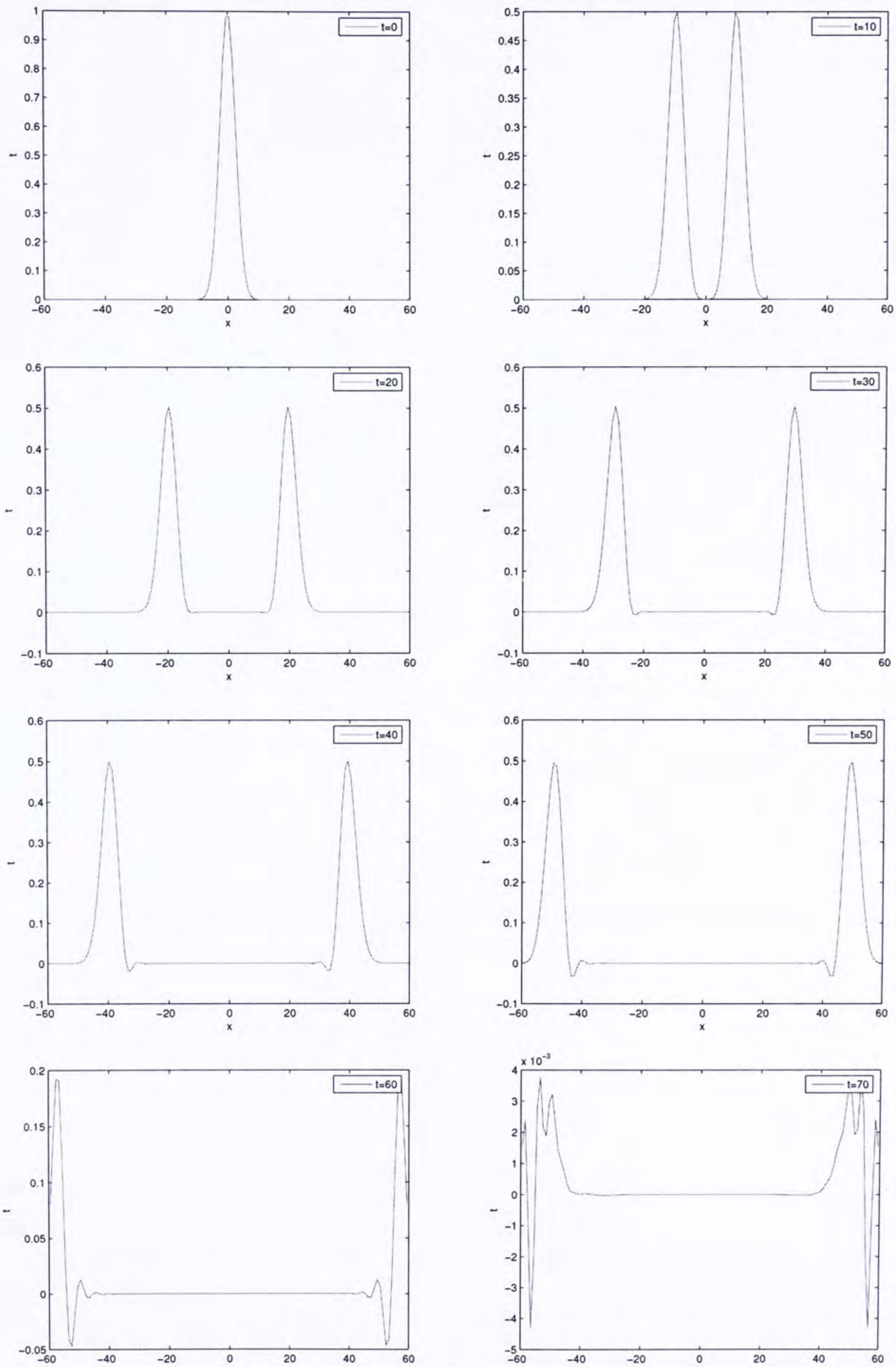

Figure 3.3: Short time dynamics of PML solutions subject to an initial magnetic pulse. The snapshots are given at T$= 0, 10, \cdots, 70$ for the component H_z.

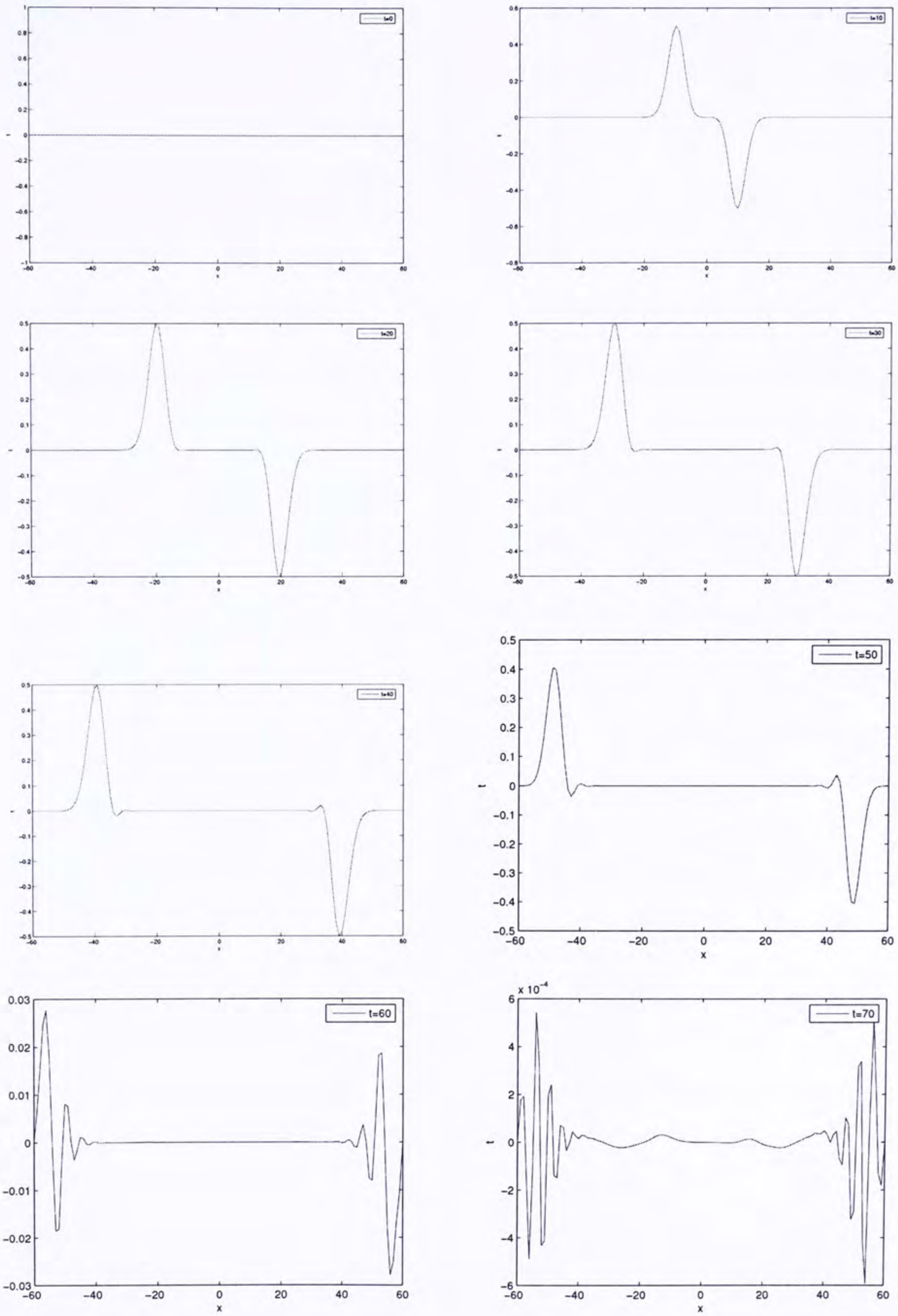

Figure 3.4: Short time dynamics of Hagstrom's PML solutions subject to an initial magnetic pulse. The snapshots are given at T= $0, 10, \cdots, 70$ for the component E_y.

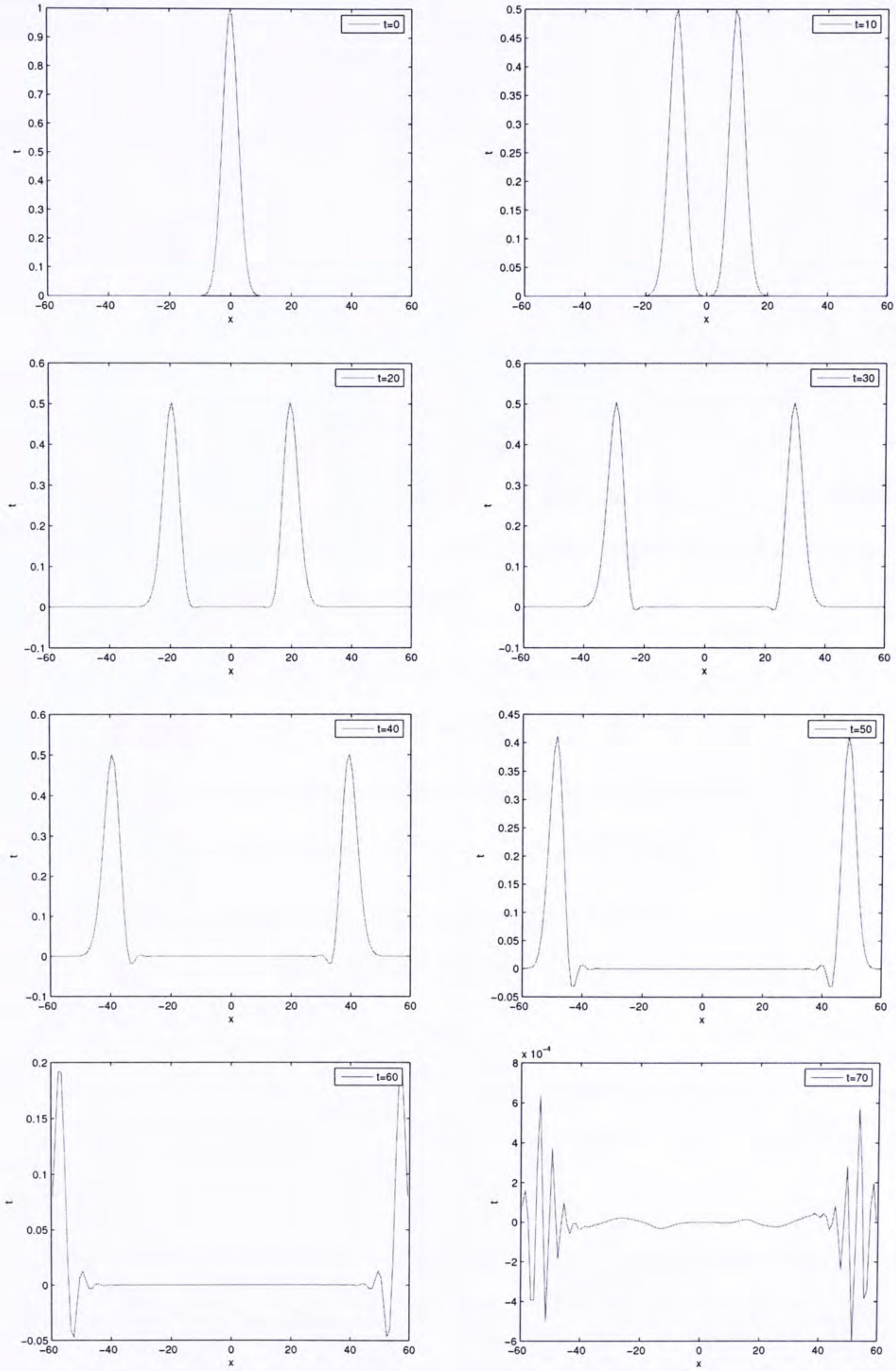

Figure 3.5: Short time dynamics of Hagstrom's PML solutions subject to an initial magnetic pulse. The snapshots are given at T= $0, 10, \cdots, 70$ for the component H_z.

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