# BAYESIAN APPROACH TO VARIABLE SAMPLING PLANS FOR THE WEIBULL DISTRIBUTION WITH CENSORING

**By**

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The undersigned certify that we have read a thesis, entitled ' Optimal variable sampling plan for the Weibull distribution with censoring' submitted to the Graduate School by Chen Jian Wei (陳建偉) in partial fulfilment of the requirements for the degree of Master of Philosophy in Statistics. We recommend that it be accepted.

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## DECLARATION

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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# ABSTRACT


In this thesis, two topics of the Bayesian variable sampling plan with censoring are studied. At first, we generalize the Lam's work (1990, 1994) to the Weibull distribution with Type II censoring and introduce a more general polynomial loss function. For single and double sampling plans, the explicit expressions of the Bayes risks are derived respectively and a finite algorithm for obtaining the optimal sampling plans is proposed. Secondly, a model of single sampling plan for general life distribution with Type I censoring is developed. However, we focus our attention on the case of the Weibull distribution with both parameters unknown. As an illustration, some numerical examples are considered. It shows that our models are simple and efficient. Finally, the single sampling plan is compared with the double sampling plan. Furthermore, we also make a comparison between the single sampling plan with Type II censoring and that with Type I censoring.

# CONTENTS

# CHAPTER 1.

# Introduction

## 1.1 Introduction

In the study of sampling inspection problem, there are many schemes for choosing a sampling plan such as the operating characteristic curve (OC curve ) schemes, the defence sampling schemes, Dodge & Romig's schemes and the decision theory schemes (see e.g. Wetherill (1977)). From the economical point of view, the decision approach is probably a more scientific and reasonable method. This approach has been studied by many statisticians, including Wetherill and Campling (1966), Guenther (1971), Fertig & Mann (1974),Wetherill & Köllerstö-rm (1979). However, most of them have just considered single sampling plans with linear loss function only. Hald (1968, 1981) studied a Bayesian single variable sampling plan with a polynomial loss function, but the optimal sample size he obtained is usually not an integer. Then, Lam (1988a,b) developed a model for single sampling plans with a polynomial loss function, and the quality of an item is measured by a normal random variable. By the Bayesian approach, Lam suggested a finite algorithm so that an optimal sampling plan with integer sample size can be found in finite steps of searching.

In the problem of variable sampling plan, usually we take a random sample from a batch first and compare the observations with the standard value. The batch will be accepted if the observations are close to the standard value, otherwise the batch will be rejected. This is the single sampling plan. In practice, it is more realistic to accept the batch when the observations are very close to the standard value and reject it when the observations are far from the standard value. In the intermediate case, a second random sample is taken. Based on the observations of these two random samples, a decision of acceptance or rejection will be made. This is the double sampling plan. The double sampling plan is useful in sampling inspection. There are various schemes for choosing a double sampling plan, including the OC curve double schemes, Dodge and Romig's double schemes, Stein's double sampling plans and the decision theory double schemes. Furthermore, Pfanzagl(1963), Wetherill and Campling (1966) also gave respectively the comparison between the single and the double sampling plans obtained by the decision theory approach. Recently, Lam & Lam (1995) have studied Bayesian double sampling plan for the normal distribution.

The lifetime data analysis is a very important topic in the sampling inspection. Although the normal distribution fits very well in many situations, in life testing problems it is more realistic to use the exponential, the Weibull, the gamma and the log-normal distributions. Moreover, lifetime data are often censored. For example, in measuring the life time of light bulbs, or the life of electronic components as well as the survival times of patients who suffer from cancer, the data may be censored. Two kinds of censoring are commonly applied. If the items are unduly expensive, we may put $n$ items on inspection and terminate it when a preassigned number of items have failed. This is Type II censoring. If the inspection cost increases heavily with time, we may put $n$ items on inspection and terminate it at a preassigned time. This is Type I censoring. Sometimes, the random censoring is also used.

So far, a great deal of research work has focused on the single sampling plan for the exponential distribution with Type II censoring. Among these are Guenther, Patil and Uppuluri (1976), Engelhardt and Bain (1978), Kocherlakota and Balakrishnan(1986). Lam(1990,1994) and Lam & Choy (1995) studied the same problem for a polynomial loss function, in which the quality of an item is measured by an exponential random variable and is subject to Type II, Type I and random censoring respectively. However, the loss functions studied in these papers do not take into account the cost of testing time. The optimal sampling plans subject to censoring actually are equivalent to the sampling plans without censoring.

## 1.2 Bayesian approach to variable sampling plan for the exponential distribution

Lam (1990) suggested a model of variable sampling plan for the exponential distribution with Type II censoring. To start with, suppose that a batch of $N$ is presented for an acceptance inspection and the lifetime $X$ of an item follows a exponential distribution $Exp(\lambda)$ with the density function

$$f(x|\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{1.2.1}$$

Furthermore, $\lambda$ is unknown but has a conjugate gamma prior distribution $\Gamma(\alpha, \beta)$ with the density function $g(\lambda) = \beta^{\alpha}\lambda^{\alpha-1}e^{-\beta\lambda}/\Gamma(\alpha)$ for $\lambda > 0$ and 0 otherwise, where $\alpha > 0$ and $\beta > 0$ are known.

Let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be the order statistics of $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$. Since the sampling is subject to Type II censoring at the $r$th failure time, the true observations are as follows:

$$Y_i = \begin{cases} X_{(i)}, & i = 1, \ldots, r, \\ X_{(r)}, & i = r+1, \ldots, n. \end{cases}$$

It is well known that the maximum likelihood estimator(MLE) of the average lifetime $\theta = 1/\lambda$ for $X$ is given by $\hat{\theta}_r = (\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)})/r$.

Using the observed data $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)$ and MLE for the average lifetime $E(X|\lambda)$, Lam(1990) study the following one-sided decision function:

$$\delta(\underset{\sim}{X}) = \begin{cases} d_0, & \hat{\theta}_r \geq T, \\ d_1, & otherwise, \end{cases} \tag{1.2.2}$$

where $d_0$ represents the decision of accepting the batch, $d_1$ denotes the decision of rejecting the batch, and $T$ is the minimum acceptance time. The loss function is a polynomial

$$L(\lambda, \delta(\underline{X})) = \begin{cases} nC_s + C_0 + C_1\lambda + \cdots + C_k\lambda^k, & \delta(\underline{X}) = d_0, \\ \\ nC_s + C_r, & \delta(\underline{X}) = d_1, \end{cases} \tag{1.2.3}$$

where $C_0, \ldots, C_k, C_s$ and $C_r$ are constants with a natural constraint:

$$C_0 + C_1\lambda + \cdots + C_k\lambda^k \geq 0, \qquad \text{for} \quad \lambda > 0.$$

Here, $C_s$ is the inspection cost per item, $C_r$ is the cost due to rejecting the batch.

To explain the reasons, let $\mu_0$ be the standard value specified by the national standard or the contract. Suppose the quality of an item in the batch is measured by the lifetime $X$. If $X \geq \mu_0$, we can sell at the normal price so that it can be accepted without additional loss. If $X < \mu_0$, it could be sold at a reduced price. An extra cost which can be assumed to be proportional to $\mu_0 - X$ is incurred, i.e., the item is accepted with cost $C(\mu_0 - X)$, where $C$ is a proportional constant. Then the cost function when the batch is accepted is given by

$$\begin{aligned} N \int_0^{\mu_0} C(\mu_0 - x)\lambda e^{-\lambda x} dx &= NC \left[ \mu_0(1 - e^{-\lambda\mu_0}) - \frac{1}{\lambda} \int_0^{\lambda\mu_0} y e^{-y} dy \right] \\ &= a_0 + a_1\lambda + a_2\lambda^2 + \ldots \end{aligned} \tag{1.2.4}$$

where $a_0, \; a_1, \; a_2 \ldots$ are constants and independent of $\lambda$.

Now, suppose that the quality of an item is measured by the reliability $R(t_0)$ and the standard value of $R(t_0)$ is $p_0$, where $p_0$ and $t_0$ are also specified by the national standard or the contract. Thus, if $R(t_0) \geq p_0$, we can accept the item without additional loss; otherwise, if $R(t_0) < p_0$, as mentioned above, we may accept it with an extra cost $C'(p_0 - R(t_0))$, where $C'$ is a proportional constant. Consequently, the cost function when the batch is accept is given by

$$NC'(p_0 - R(t_0)) = NC' \left[ p_0 - (1 - e^{-\lambda\mu_0}) \right] = a'_0 + a'_1\lambda + a'_2\lambda^2 + \ldots \tag{1.2.5}$$

where $a_0', a_1', a_2' \ldots$ are also constants and independent of $\lambda$.

In either (1.2.4) or (1.2.5), the exact costs due to acceptance of the batch are all power series of $\lambda$. Thus, a polynomial loss function should be a good approximation to the exact cost function. For $0 \le x < 1$, let

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt = \int_0^{x/(1-x)} \frac{y^{a-1}}{(1+y)^{a+b}}dy,$$

be the incomplete Beta function and write $B_1(a,b) = B(a,b)$. Then the incomplete Beta ratio is defined by $I_x(a,b) = B_x(a,b)/B(a,b)$ which is an increasing function of $x$.

An explicit expression for the Bayes risk can be derived for general degree $k$. As a demonstration for $k = 2$, Lam (1990) showed that the Bayes risk for a single sampling plan $(n, r, T)$ with Type II censoring is given by

$$\begin{aligned} R(n,r,T) = nC_s + C_0 + C_1\tfrac{\alpha}{\beta} + C_2\tfrac{\alpha(\alpha+1)}{\beta^2} - (C_0 - C_r)I_s(r,\alpha) \\ -\tfrac{\alpha}{\beta}C_1 I_s(r,\alpha+1) - \tfrac{\alpha(\alpha+1)}{\beta^2}C_2 I_s(r,\alpha+2), \end{aligned} \tag{1.2.6}$$

where $s = rT/(rT+\beta)$. Furthermore, Lam also showed the following result:

The optimal sampling size $n_0$ satisfies the following inequality

$$n_0 \le \min\left\{[C_r/C_s], [C_0\beta^2 + C_1\alpha\beta + C_2\alpha(\alpha+1)]/(C_s\beta), \right\}. \tag{1.2.7}$$

where $[a]$ is the integer part of $a$.

According to this result, a finite algorithm for finding an optimal sampling plan was suggested by Lam (1990).

To do this, we start with $n = 1$ and for each $r = 1, \ldots, n$, find $R^*(n,r) = \min\limits_T R(n,r,T)$. Then move $n$ to $n+1$, and continue. This procedure is repeated until $n$ reaches the above bound given by (1.2.7). Therefore, the minimum Bayes risk is determined by comparison. The corresponding sampling plan is the optimal sampling plan. However, the following numerical results show that the optimal policy always has the form $(n_0, n_0, T_0)$.

Table 1.1 *The minimum Bayes risk and optimal sampling plans with Type II censoring as $C_s = 0.5$ is fixed*

| $\alpha$ | $\beta$ | $C_0$ | $C_1$ | $C_2$ | $C_r$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ |
|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 1.0 | 1.0 | 1.0 | 1.0 | 10.0 | 8.1308 | 2 | 2 | 0.3668 |
| 2.0 | 1.0 | 3.0 | 3.0 | 3.0 | 30.0 | 22.0544 | 4 | 4 | 0.3669 |
| 2.5 | 1.2 | 10.0 | 10.0 | 10.0 | 100.0 | 71.5240 | 8 | 8 | 0.3669 |
| 2.0 | 1.0 | 40.0 | -5.0 | 20.0 | 200.0 | 120.0104 | 11 | 11 | 0.3226 |
| 2.3 | 1.0 | 50.0 | 20.0 | 30.0 | 400.0 | 249.3763 | 16 | 16 | 0.3088 |
| 2.5 | 1.2 | 50.0 | 20.0 | 30.0 | 400.0 | 233.3674 | 16 | 16 | 0.3149 |

From the table, we can see that the optimal sampling policy with Type II censoring is equivalent to the optimal sampling policy without Type II censoring. It seems that the censoring does not make sense. Why?

Therefore, it is necessary to generalize Lam's(1990,1994) model for other life distribution, such as the Weibull distribution. Furthermore, in this generalized model, we should make sure that the censoring will make sense.

## 1.3 Outline of the thesis

The Weibull distribution is also very popular and important in life testing, clinical trial and survival analysis. There is little research work done on the optimal sampling plan for this distribution with the censoring. In this thesis, we shall generalize Lam's (1990,1994) model to the Weibull distribution case and introduce a more general polynomial loss function so that the censoring will make sense. Then, we develop a general method of sampling plan for any life distribution with Type I censoring in different ways.

In Chapters 2 and 3, a model of variable sampling plan for the Weibull distribution with a known shape parameter and the Type II censoring is studied. The explicit expressions of the Bayes risk for single and double sampling plan are derived respectively and a finite algorithm and a discretization for obtaining the optimal sampling plan are proposed. Meanwhile, we also show that the sampling plan with Type II censoring is the same as the sampling plan without censoring if the cost of testing time is not taken into account in the loss function. Some numerical examples and the sensitivity analysis are also discussed.

In Chapter 4, we develop a general model for single sampling plan with Type I censoring in different ways. The Bayes risks for the cases of the Weibull distribution with both unknown parameters, the two-parameter exponential distribution and the gamma distribution are derived respectively. A finite algorithm and a discretization method for finding an approximately optimal sampling plan are suggested. Some numerical examples for Weibull distribution with both unknown parameters are studied in detail. It shows that our models are quite simple and efficient.

Finally, in Chapter 5, our sampling plan is compared with the OC curve sampling plan. We then explain why our sampling is more economical. Moreover, the comparison between the single and the double sampling plans as well as single sampling plans with Type II and that with Type I censoring are also discussed.

# CHAPTER 2.

# Single Variable Sampling Plan With Type II Censoring

In this chapter, we study a model of the single variable sampling plan for the Weibull distribution $W(m, \lambda)$ with known shape parameter $m$ and Type II censoring. In Sections 2 and 3, the model of single sampling plan with the polynomial loss function is formulated and the explicit expressions of the Bayes risk is obtained. Furthermore, a finite algorithm for finding an optimal sampling plan is suggested. In Section 4, as an illustration, the model with quadratic loss function is studied. Some numerical examples and the sensitivity analysis are also studied.

## 2.1 Model

We consider a single variable sampling plan by making the following assumptions.

**Assumption 1.** Suppose that a batch of $N$ items is presented for inspection. Let the lifetime of an item be $X$ which has a Weibull distribution $W(m, \lambda)$ with the density function

$$f(x|m,\lambda) = \begin{cases} \lambda m x^{m-1} e^{-\lambda x^m}, & x \geq 0, \\ 0, & x < 0. \end{cases} \qquad (2.1.1)$$

Furthermore, assume that the shape parameter $m$ is known and the scale parameter $\lambda$ is unknown but has a conjugate gamma prior distribution $\Gamma(\alpha, \beta)$ with density function $g(\lambda) = \beta^{\alpha}\lambda^{\alpha-1}e^{-\beta\lambda}/\Gamma(\alpha)$ for $\lambda > 0$ and 0 otherwise, where $\alpha > 0$ and $\beta > 0$ are known, and $\alpha m > 1$.

**Assumption 2.** A random sample $\underset{\sim}{X} = (X_1, \ldots, X_n)$ of size $n$ is taken from the batch. Let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be the order statistics of $X_1, X_2, \ldots, X_n$. The sampling is subject to Type II censoring at the $r$th failure time.

Note here that the shape parameter $m$ of the Weibull distribution is known. This assumption has been widely used in the literature. For example, in the accelerated testing, both the Arrhenius-Weibull model and the power-Weibull model assume that the shape parameter of the Weibull distribution is known (see, e.g. Nelson (1990)). On the other hand, this assumption is also used by other papers (see, e.g. Kingston (1982), Tseng (1990, 1994)).

Since the sample is censored at the $r$th failure time, the true observations are as follows:

$$Y_i = \begin{cases} X_{(i)}, & i = 1, \ldots, r, \\ \\ X_{(r)}, & i = r+1, \ldots, n. \end{cases} \tag{2.1.2}$$

Obviously, the average lifetime of $X$ is given by

$$E(X|\lambda) = (\frac{1}{\lambda})^{1/m}\Gamma(1 + \frac{1}{m}). \tag{2.1.3}$$

It is well known that the maximum likelihood estimator(MLE) of the parameter $\theta = 1/\lambda$ is given by

$$\hat{\theta}_r = (\sum_{i=1}^{r} X_{(i)}^{m} + (n-r)X_{(r)}^{m})/r. \tag{2.1.4}$$

Hence, the MLE of the average lifetime $E(X|\lambda)$ is $\hat{\theta}_r^{1/m}\Gamma(1+\frac{1}{m})$. Then, we have the following lemma.

**Lemma 2.1** *Suppose that the random variable $X$ has the Weibull distribution $W(m,\lambda)$ with known $m$, then $\hat{\theta}_r$ has the gamma distribution $\Gamma(r, r\lambda)$.*

Since $X^m$ has the exponential distribution $Exp(\lambda) = W(1,\lambda)$, the proof of Lemma 2.1 is straightforward (see, e.g, Sinha(1986)).

**Lemma 2.2** *Suppose that the random variable $X$ has the Weibull distribution $W(m,\lambda)$ with known $m$, and $\lambda$ has the gamma prior distribution $\Gamma(\alpha,\beta)$, then*

$$E(X_{(i)}|\lambda) = (\tfrac{1}{\lambda})^{1/m}C(n,i), \qquad i=1,\ldots,n, \tag{2.1.5}$$

*where* $C(n,i) = \Gamma(1+\frac{1}{m})\begin{pmatrix} n \\ i-1 \end{pmatrix}\sum\limits_{k=0}^{i-1}(-1)^k\begin{pmatrix} i-1 \\ k \end{pmatrix}\frac{n-i+1}{(n-i+k+1)^{1/m+1}}$, *and* $X_{(i)}$ *is the ith order statistic.*

**Proof:** Because the density function $f_i(x|\lambda)$ of the $i$th order statistic $X_{(i)}$ is given by

$$f_i(x|\lambda) = \tfrac{n!}{(i-1)!(n-i)!}F^{i-1}(x)[1-F(x)]^{n-i}f(x). \tag{2.1.6}$$

From (2.1.6), we have

$$\begin{aligned} E(X_{(i)}|\lambda) &= \tfrac{n!}{(i-1)!(n-i)!}\int_0^\infty xF^{i-1}(x)[1-F(x)]^{n-i}f(x)dx \\ &= \tfrac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{i-1}(-1)^k\begin{pmatrix} i-1 \\ k \end{pmatrix}\int_0^\infty m\lambda x^m e^{-\lambda x^m[(n-i)+k+1]}dx \end{aligned} \tag{2.1.7}$$

By taking transformation $y = \lambda x^m[(n-i)+k+1]$, we have

$$E(X_{(i)}|\lambda) = \tfrac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{i-1}(-1)^k\begin{pmatrix} i-1 \\ k \end{pmatrix}\frac{1}{\lambda^{1/m}(n-i+k+1)^{m/1+1}}\int_0^\infty y^{1/m}e^{-y}dy$$

$$= (\tfrac{1}{\lambda})^{\frac{1}{m}}\Gamma(1+\tfrac{1}{m}) \begin{pmatrix} n \\ i-1 \end{pmatrix} \sum_{k=0}^{i-1}(-1)^k \begin{pmatrix} i-1 \\ k \end{pmatrix} \frac{n-i+1}{(n-i+k+1)^{1/m+1}}. \qquad (2.1.8)$$

This completes the proof of Lemma 2.2. Furthermore, it follows from (2.1.5) that

$$E(X_{(i)}) = E(\tfrac{1}{\lambda})^{1/m}C(n,i) = \frac{\beta^{1/m}\Gamma(\alpha-\frac{1}{m})}{\Gamma(\alpha)}C(n,i), \qquad i=1,\ldots,n. \qquad (2.1.9)$$

•

## 2.2 Loss function and finite algorithm

Here, we study a single sampling plan $(n, r, T)$ in which $n$ is the sample size, $r$ is the number of failed items after testing, and $T$ is the minimum acceptance time. Based on the observed data $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)$ and the maximum likelihood estimator (MLE) for the average lifetime $E(X|\lambda)$, it is reasonable to apply the following one-sided decision function

$$\delta(\underset{\sim}{X}) = \begin{cases} d_0, & \hat{\theta}_r^{\frac{1}{m}}\Gamma(1+\frac{1}{m}) \geq T, \\ \\ d_1, & otherwise, \end{cases} \qquad (2.2.1)$$

where $d_0$ represents the decision of accepting the batch and $d_1$ denotes the decision of rejecting the batch. In fact, if the quality of an item is measured by its lifetime $X$, an one-sided decision function $\delta(\underset{\sim}{X})$ of form (2.2.1) should be adopted. On the other hand, if the quality of an item is measured by the reliability $R(t_0) = P(X > t_0) = exp(-\lambda t_0^m)$, then the MLE of $R(t_0)$ is $\hat{R}(t_0) = exp[-t_0^m/\hat{\theta}_r]$. Because of the fact, the larger the value $\hat{\theta}_r$, the better the quality, the decision function $\delta(\underset{\sim}{X})$ should be of the form (2.2.1).

By using decision function (2.2.1), the following polynomial loss function is studied

$$L(\lambda, \delta(\underset{\sim}{X})) = \begin{cases} A(n,r) + X_{(r)}a_t + C_0 + C_1\lambda + \cdots + C_k\lambda^k, & \delta(\underset{\sim}{X}) = d_0, \\ \\ A(n,r) + X_{(r)}a_t + C_r, & \delta(\underset{\sim}{X}) = d_1, \end{cases} \qquad (2.2.2)$$

where $A(n,r) = nC_s - (n-r)r_s$, $C_0, \ldots, C_k, C_s, r_s, a_t$ and $C_r$ are constants with a natural constraint:

$$C_0 + C_1\lambda + \cdots + C_k\lambda^k \geq 0, \qquad for \quad \lambda > 0. \qquad (2.2.3)$$

In the polynomial loss function (2.2.2), $C_s$ is the sampling cost per item including the inspection cost per item and the normal price of an item, $r$ is the number of failed items. After testing, there are $(n-r)$ unfailed items which can be sold at a reduced price $r_s$, and $(n-r)r_s$ is the total salvage value of the sample. Hence, $A(n,r)$ in (2.2.2) represents the net sampling cost with a natural constraint: $C_s > r_s$.

Moreover, $a_t$ is the rate of time-consuming cost. Since $X_{(r)}$ represents the testing time, $X_{(r)}a_t$ is the time-consuming cost.

Finally, $C_r$ is the cost due to rejecting the batch, while the polynomial $C_0 + C_1\lambda +, \cdots + C_k\lambda^k$, as we explained in Chapter 1, is a good approximation to the exact cost function when the batch is accepted.

In conclusion, the loss function in our model has a very general form which includes the sampling cost, the time-consuming cost and the decision loss. It is a generalization of the loss function studied in Lam's papers (1988a,b, 1990, 1994), in the sense that the salvage value of an unfailed item and the time-consuming cost are introduced here. Both of them are important especially for destructive inspection and in life testing so that the model is more realistic.

By using Bayesian approach, our objective is to determine an optimal sampling plan $(n_0, r_0, T_0)$ for minimizing the Bayes risk $R(n,r,T) = E[L(\lambda, \delta(\underset{\sim}{X}))]$. Now, an explicit expression of the Bayes risk is given by

$$
\begin{aligned}
R(n,r,T) &= E\left[L(\lambda,\delta(\underset{\sim}{X}))\right] = E\left\{E\left[L(\lambda,\delta(\underset{\sim}{X}))\right]|\lambda\right\} \\
&= E\Big[A(n,r) + E(X_{(r)}|\lambda)a_t + C_r \\
&\qquad + (C_0 - C_r + C_1\lambda + \cdots + C_k\lambda^k)P(\hat{\theta}_r^{\frac{1}{m}}\Gamma(1+\tfrac{1}{m}) \geq T)\Big] \\
&= n(C_s - r_s) + rr_s + C_r + \sum_{l=0}^{k} C_l^* E\left[\lambda^l P(\hat{\theta}_r \geq T_m)\right] + a_t E(X_{(r)}) \\
&= n(C_s - r_s) + rr_s + C_r + \sum_{l=0}^{k} C_l^* \tfrac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(1 - I_s(r,\alpha+l)) + a_t \tfrac{\beta^{1/m}\Gamma(\alpha-\frac{1}{m})}{\Gamma(\alpha)} C(n,r),
\end{aligned}
\tag{2.2.4}
$$

where $s = rT_m/(rT_m+\beta) = rT^m/(rT^m+\beta\Gamma(1+\frac{1}{m})^m)$ with $T_m = T^m\Gamma(1+\frac{1}{m})^{-m}$, while $C_l^*$ is $C_0 - C_r$ if $l = 0$, and is $C_l$, otherwise. Note that because of $\alpha m > 1$, $R(n,r,T)$ is well defined. Then by using Lemma A1, the proof of (2.2.4) is straightforward.

On the basis of (2.2.4), a simple algorithm for determination of an optimal sampling plan can be implemented in the following way.

**1.** Fix $n$. For $r = 0, \ldots, n$, minimize $R(n,r,T)$ with respect to $T$ and denote the minimum value by $R^*(n,r)$.

**2.** Increase $n$ by one. Repeat the above procedure and continue.

**3.** By comparison, the smallest Bayes risk $R(n_0, r_0, T_0) = \min_{r \leq n} R^*(n,r)$ is the minimum Bayes risk and the corresponding sampling plan is an optimal sampling plan.

The following theorem justifies that the algorithm is finite, i.e., we can find an optimal sampling plan in finite steps of searching.

**Theorem 2.1.** *The optimal sampling size $n_0$ satisfies the following inequality*

$$n_0 \le \min\left\{[C_r/(C_s - r_s)], [\sum_{l=0}^{k} C_l \frac{\Gamma(\alpha + l)}{\Gamma(\alpha)\beta^l}/(C_s - r_s)], [R^*(n,r)/(C_s - r_s)]\right\}. \tag{2.2.5}$$

*where $[a]$ is the integer part of $a$.*

**Proof:** Let $(n_0, r_0, T_0)$ be an optimal sampling plan. Then the minimum Bayes risk must satisfy the following inequality

$$R(n_0, r_0, T_0) \le \min\{R(0,0,\infty), R(0,0,0), R^*(n,r)\} \tag{2.2.6}$$

where $(0,0,\infty)$ is the plan of rejecting the batch without sampling, and $(0,0,0)$ is the plan of accepting the batch without sampling.

Now $R(0,0,\infty) = C_r$ and $R(0,0,0) = \sum_{l=0}^{k} C_l E(\lambda^l) = \sum_{l=0}^{k} C_l \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}$.

From (2.2.2) and (2.2.3), we have

$$R(n_0, r_0, T_0) \ge n_0 C_s - (n_0 - r) r_s + E(X_{(r)}) a_s \ge n_0 (C_s - r_s). \tag{2.2.7}$$

The proof is completed by the combination of (2.2.6) and (2.2.7).

In fact, (2.2.5) gives an adaptive upper bound for the optimal sample size $n_0$. Obviously, step 1 is the key step of the algorithm. This amounts to solving the equation $\frac{dR}{dT} = 0$ which can be reduced to an algebraic equation

$$\begin{cases} \sum_{l=0}^{k} C_l^* \Gamma(\alpha + r + l)(rT_m + \beta)^{k-l} = 0, \\ \\ T^m = T_m \Gamma(1 + \frac{1}{m})^m. \end{cases} \tag{2.2.8}$$

Because the equation (2.2.8) is a simple algebraic equation of degree $k$, this algorithm is not only a finite algorithm but also a simple algorithm.

**Theorem 2.2.** *Let* $(n_0, r_0, T_0)$ *be an optimal single sampling plan. If* $a_t = 0$, *then* $r_0 = n_0$.

**Proof:** Suppose $r_0 < n_0$. Since $(n_0, r_0, T_0)$ is an optimal sampling plan and $a_t = 0$, it follows from (2.2.4) and the fact that $C_s > r_s$, we have

$$\begin{aligned} R(n_0, r_0, T_0) &= n_0(C_s - r_s) + r_0 r_s + C_r + \sum_{l=0}^{k} C_l^* \frac{\Gamma(\alpha + l)}{\Gamma(\alpha)\beta^l}(1 - I_s(r_0, \alpha + l)) \\ &= (n_0 - r_0)(C_s - r_s) + R(r_0, r_0, T_0) \\ &> R(r_0, r_0, T_0). \end{aligned}$$

This means that $(n_0, r_0, T_0)$ is not an optimal sampling plan. Hence $r_0 = n_0$.

Theorem 2.2 reveals the importance of introducing the time-consuming cost, otherwise, the sampling plan with censoring will be the same as the sampling plan without censoring.

## 2.3 Numerical examples and sensitivity analysis

To illustrate the model and the algorithm for the determination of an optimal sampling plan and the minimum Bayes risk discussed, we assume that the degree $k$ in the loss function (2.2.4) is 2. Then, from (2.2.4), the Bayes risk of the single sampling plan $(n, r, T)$ is given by

$$R(n,r,T) = n(C_s - r_s) + rr_s + C_r + (C_0 - C_r)(1 - I_s(r,\alpha))$$

$$+\tfrac{C_1\alpha}{\beta}(1 - I_s(r,\alpha+1)) + \tfrac{C_2\alpha(\alpha+1)}{\beta^2}(1 - I_s(r,\alpha+2)) + a_t\frac{\beta^{1/m}\Gamma(\alpha-\frac{1}{m})}{\Gamma(\alpha)}C(n,r). \quad (2.3.1)$$

Now, for the purpose of comparison, we take $m = 2.5, \alpha = 2.5, \beta = 1, C_0 = 5, C_1 = 5, C_2 = 5, C_r = 50, C_s = 0.5, r_s = 0.2$ and $a_t = 2$ as the standard values of the parameters and coefficients. The numerical results are tabulated in Tables 2.1-2.10. In each table, only one parameter or one coefficient can be varied and others are fixed. The values of the varying parameters or that of the varying coefficients are given in column 1. The optimal sampling plan $(n_0, r_0, T_0)$ and the corresponding minimum Bayes risks $R^*$ are given by column 2-5 respectively. Note that the sampling plan $(0,0,0)$ represents the plan accepting a batch without sampling, and $(0,0,\infty)$ denotes the plan rejecting a batch without sampling.

In practice, the parameters and coefficients are usually unknown and should be estimated. Since the estimated values are usually inaccurate, the corresponding 'optimal sampling plan' will not be the true optimal sampling plan. We can then call the 'optimal sampling plan' as the estimated sampling plan. Therefore, it is necessary to investigate how far is the estimated sampling plan different from the optimal sampling plan. Sensitivity analysis studies the behavior of the optimal solution as the values of parameters and/or coefficients change. Now, the standard values are assumed to be the true value of the parameters and coefficients. In Tables 2.1-2.10, the true parameters or coefficients and the corresponding minimum Bayes risks are marked by '*'. The efficiency of a sampling plan is defined as the ratio of the minimum Bayes risk to the Bayes risk under the sampling plan (see Hald (1981)). By using an estimated sampling plan, the true (but not the minimum ) Bayes risk $R$ and their efficiencies are given in column 6 and 7 of each table respectively. For example, in Table 2.2, if $\alpha$ is inaccurately estimated as 2.7, the estimated Bayes risk is 44.4253 and the estimated sampling plan is $(5, 4, 0.6402)$. By using the estimated plan, as the true $\alpha$ is 2.5, the true Bayes risk (but not the minimum)is 42.9624. As the true minimum Bayes risk is 42.7690, the efficiency of the estimated sampling plan is then equal to $R^*/R$ =42.7690/42.9624=0.9952. Sensitivity analysis for the other values of $\alpha$, other parameters or coefficients can be conducted in a similar way. Therefore, we can conclude that the model for the single variable sampling plan with Type II censoring is insensitive to the parameters and coefficients.

As the exponential distribution $Exp(\lambda)$ is a special Weibull distribution $W(1, \lambda)$. Therefore, all the results in this chapter can be applied to the case for the exponential distribution with Type II censoring. This chapter is a generalization of Lam's earlier work (1990). Here, we study a more general loss function which is more realistic so that the trivial case, the complete sampling, is avoided.

Robustness study investigates the effect when the assumption on the distribution is violated. In Table 2.1, as $m$ varies, the lifetime $X$ will have different Weibull distributions $W(m, \lambda)$. Therefore, Table 2.1 not only shows that the optimal solution is insensitive to the parameter $m$, but also justifies that the optimal solution is also robust as the distribution of $X$ varies.

Table 2.1. *The minimum Bayes risk and optimal sampling plans as $m$ varies*

| $m$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 0.5 | 42.7448 | 6 | 4 | 0.3745 | 58.2410 | 0.7343 |
| 1.0 | 42.7464 | 6 | 4 | 0.4306 | 53.0987 | 0.8055 |
| 2.0 | 42.7640 | 6 | 4 | 0.5988 | 43.0523 | 0.9934 |
| 2.5* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 3.0 | 42.7473 | 6 | 4 | 0.6828 | 43.0368 | 0.9938 |
| 3.5 | 42.7435 | 6 | 4 | 0.7109 | 43.4583 | 0.9841 |
| 4.0 | 42.7626 | 5 | 4 | 0.7389 | 43.9506 | 0.9731 |

Table 2.2. *The minimum Bayes risk and optimal sampling plans as $\alpha$ varies*

| $\alpha$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 1.5 | 31.5645 | 6 | 3 | 0.5510 | 44.6133 | 0.9587 |
| 2.0 | 37.7265 | 6 | 4 | 0.5882 | 43.2518 | 0.9888 |
| 2.2 | 39.8156 | 6 | 4 | 0.6259 | 42.7729 | 0.9991 |
| 2.5* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 2.7 | 44.4253 | 5 | 4 | 0.6402 | 42.9624 | 0.9952 |
| 3.0 | 47.6809 | 6 | 5 | 0.5912 | 43.3188 | 0.9873 |
| 3.5 | 50.0000 | 0 | 0 | $\infty$ | 50.0000 | 0.8451 |

Table 2.3. *The minimum Bayes risk and optimal sampling plans as $\beta$ varies*

| $\beta$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 0.6 | 50.0000 | 0 | 0 | $\infty$ | 50.0000 | 0.8451 |
| 0.8 | 46.0045 | 4 | 3 | 0.6584 | 42.8888 | 0.9972 |
| 0.9 | 44.4266 | 5 | 4 | 0.6259 | 42.8000 | 0.9993 |
| 1.0* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 1.1 | 40.9868 | 6 | 4 | 0.6278 | 42.8531 | 0.9983 |
| 1.4 | 35.6874 | 5 | 3 | 0.5636 | 44.0669 | 0.9705 |
| 2.0 | 50.0000 | 0 | 0 | 0.0000 | 61.2500 | 0.6900 |

Table 2.4. *The minimum Bayes risk and optimal sampling plans as $C_0$ varies*

| $C_0$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 1.0 | 40.7458 | 5 | 4 | 0.6268 | 42.7956 | 0.9993 |
| 2.5 | 41.5065 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 4.0 | 42.2646 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 5.0* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 5.5 | 43.0130 | 6 | 4 | 0.6548 | 42.7863 | 0.9995 |
| 7.0 | 43.6933 | 6 | 4 | 0.6548 | 42.7863 | 0.9995 |
| 10.0 | 45.024 | 4 | 3 | 0.6828 | 43.0337 | 0.9938 |

Table 2.5. *The minimum Bayes risk and optimal sampling plans as $C_1$ varies*

| $C_1$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| -5.0 | 32.2220 | 5 | 4 | 0.5147 | 46.3753 | 0.9222 |
| -1.0 | 37.3022 | 6 | 4 | 0.5707 | 43.7044 | 0.9786 |
| 4.0 | 41.9807 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 5.0* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 5.5 | 43.1199 | 6 | 4 | 0.6548 | 42.7863 | 0.9996 |
| 7.0 | 44.1205 | 6 | 4 | 0.6548 | 42.7863 | 0.9996 |
| 10.0 | 45.7538 | 4 | 3 | 0.7109 | 43.3467 | 0.9867 |

Table 2.6. *The minimum Bayes risk and optimal sampling plans as $C_2$ varies*

| $C_2$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 1.0 | 26.2500 | 0 | 0 | 0 | 61.2500 | 0.6900 |
| 2.5 | 37.1554 | 6 | 4 | 0.5147 | 46.2856 | 0.9240 |
| 4.5 | 41.9668 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 5.0* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 5.5 | 43.4239 | 6 | 4 | 0.6548 | 42.7863 | 0.9995 |
| 7.0 | 45.0164 | 5 | 4 | 0.7109 | 43.4334 | 0.9847 |
| 10.0 | 46.9435 | 4 | 3 | 0.7949 | 44.8040 | 0.9546 |

Table 2.7. *The minimum Bayes risk and optimal sampling plans as $C_s$ varies*

| $C_s$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 0.21 | 41.0901 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 0.3 | 41.5690 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 0.4 | 42.1690 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 0.5* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 0.6 | 43.2795 | 4 | 3 | 0.6548 | 42.8795 | 0.9974 |
| 0.7 | 43.6795 | 4 | 3 | 0.6548 | 42.8795 | 0.9974 |
| 1.0 | 44.6663 | 2 | 2 | 0.6548 | 43.6663 | 0.9745 |

Table 2.8. *The minimum Bayes risk and optimal sampling plans as $r_s$ varies*

| $r_s$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 0.0 | 42.9811 | 5 | 4 | 0.6268 | 42.7956 | 0.9994 |
| 1.0 | 42.8891 | 5 | 4 | 0.6268 | 42.7956 | 0.9994 |
| 1.5 | 42.8391 | 5 | 4 | 0.6268 | 42.7656 | 0.9994 |
| 0.2* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 0.25 | 42.6690 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 0.3 | 42.5690 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 0.4 | 42.4584 | 6 | 3 | 0.6528 | 43.0531 | 0.9934 |

Table 2.9. *The minimum Bayes risk and optimal sampling plans as $a_t$ varies*

| $a_t$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 0.0 | 41.1387 | 5 | 5 | 0.6268 | 43.4182 | 0.9850 |
| 1.0 | 42.1626 | 6 | 5 | 0.6278 | 42.8863 | 0.9972 |
| 1.5 | 42.4813 | 5 | 4 | 0.6268 | 42.7956 | 0.9994 |
| 2.0* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 2.5 | 43.0018 | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 3.0 | 43.2274 | 6 | 4 | 0.6548 | 42.7863 | 0.9996 |
| 5.0 | 43.8931 | 6 | 3 | 0.6548 | 43.0585 | 0.9933 |

Table 2.10. *The minimum Bayes risk and optimal sampling plans as $C_r$ varies*

| $C_r$ | $R(n_0, r_0, T_0)$ | $n_0$ | $r_0$ | $T_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 10.0 | 10.0000 | 0 | 0 | $\infty$ | 50.0000 | 0.8451 |
| 30.0 | 29.5913 | 3 | 2 | 0.8229 | 45.1140 | 0.9480 |
| 40.0 | 36.9064 | 4 | 3 | 0.7109 | 43.0237 | 0.9822 |
| 50.0* | 42.7690* | 6 | 4 | 0.6268 | 42.7690 | 1.0000 |
| 60.0 | 47.4605 | 6 | 4 | 0.5987 | 43.0539 | 0.9930 |
| 70.0 | 51.2840 | 6 | 5 | 0.5427 | 45.0404 | 0.9495 |
| 100.0 | 58.8646 | 6 | 5 | 0.4866 | 48.8833 | 0.8749 |

# CHAPTER 3.

## Double Variable Sampling Plan With Type II Censoring

In this chapter, we generalize the model of Chapter 2 to the case of double variable sampling plan. In Section 3.2, we consider the model of double sampling plan in which the variable still has $W(m, \lambda)$ with known shape parameter $m$ and Type II censoring. In Section 3.3, under the assumption that the loss function is a more general polynomial loss function, the explicit expressions of the Bayes risk for a double sampling plan is obtained. Furthermore, a finite algorithm for finding an optimal sampling plan is suggested. In Section 3.4, a discretization method for finding an approximately optimal sampling plan is proposed and some numerical examples are studied. Finally, in Section 3.5, the sensitivity analysis is also discussed.

### 3.1 Model

Under the assumptions made for a single sampling plan in Chapter 2, we consider a model for the double sampling plan. First of all, a random sample $\underset{\sim}{X}_1 = (X_1, \ldots, X_{n_1})$ of size $n_1$ is taken from the batch. Let $X_{(1)}^{(1)} \leq X_{(2)}^{(1)} \leq \ldots \leq X_{(n_1)}^{(1)}$ be the order statistics of $X_1, X_2, \ldots, X_{n_1}$. Since the sample is subject to

Type II censoring at the $r_1$th failure time, the MLE of the average life time $E(X|\lambda)$ is $\hat{\theta}_1^{1/m}\Gamma(1+\frac{1}{m})$, with $\hat{\theta}_1 = \left(\sum_{i=1}^{r_1} X_{(i)}^{(1)m} + (n_1 - r_1)X_{(r_1)}^{(1)\,m}\right)/r_1$. Then it is reasonable to accept the batch, if $\hat{\theta}_1^{1/m}\Gamma(1+\frac{1}{m}) \geq T_0$ and reject the batch, if $\hat{\theta}_1^{1/m}\Gamma(1+\frac{1}{m}) < T_1$, with $T_0 > T_1$. In the intermediate case, a second sample of size $n_2$, $\underset{\sim}{X}_2 = (X_{n_1+1}, \ldots, X_n)$ will be taken, here $n = n_1 + n_2$ and $\underset{\sim}{X}_2$ is independent of $\underset{\sim}{X}_1$.

The second sample is also subject to Type II censoring at the $r_2$th failure time. Let $X_{(1)}^{(2)} \leq X_{(2)}^{(2)} \leq \ldots \leq X_{(n_2)}^{(2)}$ be the order statistics of $X_{n_1+1}, X_{n_2+2}, \ldots, X_{n_1+n_2}$. Then, let $r = r_1 + r_2$ and define

$$\hat{\theta} = \left(\sum_{i=1}^{r_1} X_{(i)}^{(1)m} + (n_1 - r_1)X_{(r_1)}^{(1)\,m} + \sum_{j=1}^{r_2} X_{(j)}^{(2)m} + (n_2 - r_2)X_{(r_2)}^{(2)\,m}\right)/r. \quad (3.1.1)$$

Afterwards, the batch is accepted if $\hat{\theta}^{1/m}\Gamma(1+\frac{1}{m}) \geq T_2$. and rejected otherwise. This is plausible because of the following Lemmas.

**Lemma 3.1.** *Suppose that the random variable* X *has a Weibull distribution* $W(m, \lambda)$. *Then*

1. $\hat{\theta}^{1/m}\Gamma(1+\frac{1}{m})$ *is the MLE of the average lifetime* $E(X|\lambda)$,
2. $\hat{\theta}$ *has the gamma distribution* $\Gamma(r, r\lambda)$,

*where* $\hat{\theta}$ *is defined by* (3.1.1) *and* $r = r_1 + r_2$.

**Proof.** For $i = 1, 2$, define

$$Y_j^{(i)} = \begin{cases} X_{(j)}^{(i)}, & j = 1, \ldots, r_i, \\ X_{(r_i)}^{(i)}, & j = r_i + 1, \ldots, n_i. \end{cases}$$

Obviously, the likelihood function of $\underset{\sim}{Y} = (Y_{(1)}^{(1)}, \ldots, Y_{(n_1)}^{(1)}; Y_{(1)}^{(2)}, \ldots, Y_{(n_2)}^{(2)})$ is determined by

$$L(\underset{\sim}{Y}) = \frac{n_1!}{(n_1-r_1)!}\frac{n_2!}{(n_2-r_2)!}(\lambda m)^{r_1+r_2}\left(\prod_{i=1}^{r_1} X_{(i)}^{(1)^{m-1}}\right)\left(\prod_{j=1}^{r_2} X_{(j)}^{(2)^{m-1}}\right)$$

$$exp\left\{-\lambda[\sum_{i=1}^{r_1} X_{(i)}^{(1)^m} + (n_1-r_1)X_{(r_1)}^{(1)\ m} + \sum_{j=1}^{r_2} X_{(j)}^{(2)^m} + (n_2-r_2)X_{(r_2)}^{(2)\ m}]\right\}.$$

Therefore, it is easy to check that the MLE of parameter $\theta = 1/\lambda$ is given by (3.1.1). Consequently, $\hat{\theta}^{1/m}\Gamma(1+\frac{1}{m})$ is the MLE of the average lifetime $E(X|\lambda) = (\frac{1}{\lambda})^{1/m}\Gamma(1+\frac{1}{m})$. Now, write $\hat{\theta}_2 = \left[\sum_{i=1}^{r_2}(n_2-i+1)(X_{(i)}^{(2)^m} - X_{(i-1)}^{(2)\ m})\right]/r_2$. Then (3.1.1) becomes

$$\hat{\theta} = \frac{r_1}{r}\hat{\theta}_1 + \frac{r_2}{r}\hat{\theta}_2. \tag{3.1.2}$$

It follows from Lemma 2.1 that $\hat{\theta}_1 \sim \Gamma(r_1, r_1\lambda)$ and $\hat{\theta}_2 \sim \Gamma(r_2, r_2\lambda)$. Since $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, we have $\hat{\theta} \sim \Gamma(r, r\lambda)$. This completes the proof of Lemma 3.1.

By the same argument as in Lemma 2.2, we have the following results.

**Lemma 3.2** *Suppose that the random variable $X$ has the Weibull distribut-ion $W(m,\lambda)$ with known $m$, and $\lambda$ has the gamma prior distribution $\Gamma(\alpha,\beta)$, then,*

$$E(X_{(r_j)}^{(j)}) = \frac{\beta^{1/m}\Gamma(\alpha-\frac{1}{m})}{\Gamma(\alpha)}C(n_j,r_j),\quad r_j = 1,\ldots,n_j;\ \ j=1,2, \tag{3.1.3}$$

*where $C(n_j,r_j)$ is the same as that in (2.1.5) and $X_{(r_j)}^{(j)}$ is the $r_j$th order statistic in the $j$th sample.*

## 3.2 Loss function and Bayes risk

Based on Lemma 3.1, it is reasonable to study the sampling plans $(n_1, r_1, n_2, r_2, T_0, T_1, T_2)$ and apply the following decision function.

$$\delta(\underset{\sim}{X}) = \begin{cases} d_0, & (\hat{\theta}_1 \geq T_{0m}) \bigcup (T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} \geq T_{2m}), \\ d_1, & (\hat{\theta}_1 < T_{1m}) \bigcup (T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} < T_{2m}), \end{cases} \tag{3.2.1}$$

where $T_{im} = T_i^m \Gamma(1 + \frac{1}{m})^{-m}, \quad i = 0, 1, 2.$

As in the case of single sampling plan, the loss function is the sum of the sampling cost, the time-consuming cost and the decision cost in the first and second samples. Thus, the loss function should have the form:

$$L(\lambda, \delta(\underset{\sim}{X})) = \begin{cases} A(n_1, r_1) + X_{(r_1)}^{(1)} a_t + \sum\limits_{l=0}^{k} C_l \lambda^l, & \hat{\theta}_1 \geq T_{0m}, \\ A(n_1, r_1) + X_{(r_1)}^{(1)} a_t + C_r, & \hat{\theta}_1 < T_{1m}, \\ A(n_1, r_1) + X_{(r_1)}^{(1)} a_t + A(n_2, r_2) + X_{(r_2)}^{(2)} a_t + \sum\limits_{l=0}^{k} C_l \lambda^l, & T_{1m} \leq \hat{\theta}_1 < T_{0m}, \\ & \hat{\theta} \geq T_{2m}, \\ A(n_1, r_1) + X_{(r_1)}^{(1)} a_t + A(n_2, r_2) + X_{(r_2)}^{(2)} a_t + C_r, & T_{1m} \leq \hat{\theta}_1 < T_{0m}, \\ & \hat{\theta} < T_{2m}. \end{cases} \tag{3.2.2}$$

where $A(n_j, r_j), C_0, \cdots, C_k, C_s, r_s, a_t$ and $C_r$ are the same as that in (2.2.2).

Furthermore, it is reasonable to assume that $T_0, T_1$ and $T_2$ satisfy the condition

$$T_1 \leq T_2 \leq T_0, \tag{3.2.3}$$

or $T_{1m} \leq T_{2m} \leq T_{0m}$. It is a natural condition. If otherwise $T_2 > T_0$ or $T_{2m} > T_{0m}$ say, we should reject the batch if $T_{0m} < \hat{\theta} < T_{2m}$. Then it follows from (3.2.1) that

$$T_{0m} < \frac{r_1}{r}\hat{\theta}_1 + \frac{r_2}{r}\hat{\theta}_2 < \frac{r_1}{r}T_{0m} + \frac{r_2}{r}\hat{\theta}_2,$$

i.e., $\hat{\theta}_2 > T_{0m}$. This means that we should reject the batch even if $T_{0m} < \hat{\theta} < T_{2m}$

and $\hat{\theta}_2 > T_{0m}$. Note that $T_{0m}$ is the minimum acceptance bound for the first sample. It is unimaginable that when the average lifetime of the combined sample and that of the second sample (the sample size $n_2$ may be equal to $n_1$) are both greater than $T_{0m}$, we still reject the batch! Therefore, we should assume that $T_2 \leq T_0$ or $T_{2m} \leq T_{0m}$. Similarly, we should also assume that $T_2 \geq T_1$ or $T_{2m} \geq T_{1m}$. Hence, (3.2.3) follows.

Obviously, if $n_2 = 0$ and $T_0 = T_1$, the double sampling plan will reduce to a single sampling plan. In other words, a double sampling plan $(n_1, r_1, 0, 0, T_0, T_0, T_2)$ represents a single sampling plan.

Let

$$s(u) = (rT_{2m} - r_1 u)/(rT_{2m} + \beta); \quad s_i = (r_1 T_{im})/(r_1 T_{im} + \beta);$$

$$T_m(u) = \tfrac{r}{r_2} T_{2m} - \tfrac{r_1}{r_2} u; \quad T_{*m} = \min\{T_{0m}, \tfrac{r}{r_1} T_{2m}\};$$
$$r_j = 1, \ldots, n_j; \quad j = 1, 2, \quad i = 0, 1. \tag{3.2.4}$$

Since $T_{0m} \geq T_{1m}$, then $s_0 \geq s_1$ and $I_{s_0}(r, \alpha) \geq I_{s_1}(r, \alpha)$. It follows from Lemmas A1-A4 and the independence of $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ that the Bayes risk for the sampling plan $(n_1, r_1, n_2, r_2, T_0, T_1, T_2)$ is given by

$$R(n_1, r_1, n_2, r_2, T_0, T_1, T_2) = E\left[L(\lambda, \delta(\underset{\sim}{X}))\right] = I_0 + I_1 + I_2 + I_3 + I_4, \tag{3.2.5}$$

where

$$I_0 = E\left[A(n_1, r_1) + X^{(1)}_{(r_1)} a_t + (A(n_2, r_2) + X^{(2)}_{(r_2)} a_t) I_{(T_{1m} \leq \hat{\theta}_1 < T_{0m})}\right]$$

$$= n_1(C_s - r_s) + rr_s + [n_2(C_s - r_s) + rr_s] E[P(T_{1m} \leq \hat{\theta}_1 < T_{0m})]$$
$$+ E(X^{(1)}_{(r_1)}) a_t + E[E(X^{(2)}_{(r_2)} | \lambda) E(I_{(T_{1m} \leq \hat{\theta}_1 < T_{0m})} | \lambda)] a_s$$

$$= n_1(C_s - r_s) + rr_s + [n_2(C_s - r_s) + rr_s](I_{s_0}(r_1,\alpha) - I_{s_1}(r_1,\alpha))$$
$$+E[(\tfrac{1}{\lambda})^{\frac{1}{m}} C(n_1,r_1)]a_t + a_t E[(\tfrac{1}{\lambda})^{\frac{1}{m}} C(n_2,r_2) P(T_{1m} \le \hat{\theta}_1 < T_{0m})]$$

$$= n_1(C_s - r_s) + rr_s + [n_2(C_s - r_s) + r_2 r_s](I_{s_0}(r_1,\alpha) - I_{s_1}(r_1,\alpha))$$
$$+a_t \tfrac{\beta^{1/m}\Gamma(\alpha - \frac{1}{m})}{\Gamma(\alpha)}[C(n_1,r_1) + C(n_2,r_2)(I_{s_0}(r_1,\alpha - \tfrac{1}{m}) - I_{s_1}(r_1,\alpha - \tfrac{1}{m}))],$$

$$I_1 = E\left[(C_0 + C_1\lambda + \ldots + C_k\lambda^k) P(\hat{\theta}_1 \ge T_{0m})\right] = \sum_{l=0}^{k} C_l \tfrac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(1 - I_{s_0}(r_1,\alpha+l)),$$

$$I_2 = E\left[C_r P(\hat{\theta}_1 < T_{1m})\right] = C_r I_{s_1}(r_1,\alpha),$$

$$I_3 = E\left[(C_0 + C_1\lambda + \ldots + C_k\lambda^k) P(T_{1m} \le \hat{\theta}_1 < T_{0m}, \hat{\theta} \ge T_{2m})\right],$$
$$= \sum_{l=0}^{k} C_l \left\{ \tfrac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(I_{s_0}(r_1,\alpha+l) - I_{s_1}(r_1,\alpha+l)) - \tfrac{r_1^{r_1}\beta^\alpha \Gamma(r_1+\alpha+l)}{\Gamma(r_1)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \tfrac{u^{r_1-1} I_{s(u)}(r_2, r_1+\alpha+l)}{(r_1 u + \beta)^{r_1+\alpha+l}} du \right\}$$

$$I_4 = E\left[C_r P(T_{1m} \le \hat{\theta}_1 < T_{0m}, \hat{\theta} < T_{2m})\right]$$

$$= C_r \tfrac{r_1^{r_1}\beta^\alpha \Gamma(r_1+\alpha)}{\Gamma(r_1)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \tfrac{u^{r_1-1}}{(r_1 u + \beta)^{r_1+\alpha}} I_{s(u)}(r_2, r_1+\alpha) du. \tag{3.2.6}$$

and $I_B$ stands for the indicator function of event $B$. Consequently, from (3.2.5) the Bayes risk for the double sampling plan $(n_1, n_2, r_1, r_2, T_0, T_1, T_2)$ is given by

$R(n_1, r_1, n_2, r_2, T_0, T_1, T_2)$

$$= n_1(C_s - r_s) + r_1 r_s + C_r + [n_2(C_s - r_s) + r_2 r_s](I_{s_0}(r_1,\alpha) - I_{s_1}(r_1,\alpha))$$

$$+a_t \tfrac{\beta^{1/m}\Gamma(\alpha - \frac{1}{m})}{\Gamma(\alpha)}[C(n_1,r_1) + C(n_2,r_2)(I_{s_0}(r_1,\alpha - \tfrac{1}{m}) - I_{s_1}(r_1,\alpha - \tfrac{1}{m}))]$$

$$+ \sum_{l=0}^{k} C_l^* \tfrac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(1 - I_{s_1}(r_1,\alpha+l)) - \sum_{l=0}^{k} C_l^* \tfrac{r_1^{r_1}\beta^\alpha \Gamma(r_1+\alpha+l)}{\Gamma(r_1)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \tfrac{u^{r_1-1} I_{s(u)}(r_2, r_1+\alpha+l)}{(r_1 u + \beta)^{r_1+\alpha+l}} du, \tag{3.2.7}$$

where $C_l^*(l = 1, \ldots, k)$ are the same as that in (2.2.6).

Accordingly, as in the single sampling plan, a simple algorithm for the determination of an optimal double sampling plan can be implemented in the following way.

**1.** Fix a pair of integers $n_1$ and $n_2$, and for each of $r_1 = 0, \ldots, n_1; r_2 = 0, \ldots, n_2$, evaluate the minimum value of the Bayes risk $R^*(n_1, r_1, n_2, r_2)$, and hence $R^*(n_1, n_2) = \min_{r_1, r_2} R^*(n_1, r_1, n_2, r_2)$.

**2.** The procedure is repeated for another pair of integers, $n_1$ and $n_2 + 1$ say, and continue.

**3.** By comparison, choose the smallest Bayes risk $\min_{n_1, n_2} R^*(n_1, n_2)$. Then the corresponding sampling plan $(n_{10}, n_{20}, r_{10}, r_{20}, T_{00}, T_{10}, T_{20})$ is an optimal double sampling plan.

Obviously, if both the optimal sample sizes $n_{10}$ and $n_{20}$ are bounded above, our algorithm will be a finite algorithm. This is true because of the following Theorems 3.1 and 3.2, of which Theorem 3.2 is in fact a modification of Theorem 2.1. It shows that the optimal size $n_{10}$ of the first sample is bounded above.

**Theorem 3.1.** *The optimal sample size $n_{10}$ of the first sample satisfies the following inequality*

$$n_{10} \leq \min \left\{ [C_r/(C_s - r_s)], [\sum_{l=0}^{k} C_l \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}/(C_s - r_s)], [R^*(n_1, n_2)/(C_s - r_s)] \right\}, \tag{3.2.8}$$

*where $[a]$ is the integer part of $a$.*

Theorem 3.2 justifies that the optimal size $n_{20}$ of the second sample is also bounded above.

**Theorem 3.2.** *The optimal sample size* $n_{20}$ *of the second sample satisfies the following inequality*

$$n_{20} \leq [C_r/(C_s - r_s)], \tag{3.2.9}$$

*where* $[a]$ *is the integer part of* $a$.

**proof.** For any $n_1, n_2 \geq 0, T_0 \geq T_1 \geq 0, T_2 \geq 0$, let $R_1^* = R(n_1, r_1, 0, 0, T_0, T_0, T_2)$; $R_2^* = R(n_1, r_1, n_2, r_2, T_0, T_1, T_2)$. Then, (2.24) yields

$$R_1^* = E[A(n_1, r_1) + X_{(r_1)}^{(1)} a_t] + E[\sum_{l=0}^{k} C_l \lambda^l P(\hat{\theta}_1 \geq T_{0m}) + C_r P(\hat{\theta}_1 < T_{0m})].$$

And (3.2.5) gives

$$\begin{aligned}
R_2^* &= E[A(n_1, r_1) + X_{(r_1)}^{(1)} a_t + (A(n_2, r_2) + X_{(r_2)}^{(2)} a_t) I_{(T_{1m} \leq \hat{\theta}_1 < T_{0m})}] \\
&\quad + E[\sum_{l=0}^{k} C_l \lambda^l P(\hat{\theta}_1 \geq T_{0m})] + E[C_r P(\hat{\theta}_1 < T_{1m}) \\
&\quad + C_r P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} < T_{2m})] + E[\sum_{l=0}^{k} C_l \lambda^l P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} \geq T_{2m})] \\
&= R_1^* + E\{[n_2(C_s - r_s) + r_2 r_s - C_r] P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} \geq T_{2m}) \\
&\quad + \sum_{l=0}^{k} C_l \lambda^l P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} \geq T_{2m}) + a_t E(X_{(r_2)}^{(2)} | \lambda) P(T_{1m} \leq \hat{\theta}_1 < T_{0m}) \\
&\quad + [n_2(C_s - r_s) + r_2 r_s] P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} < T_{2m})\}
\end{aligned} \tag{3.2.10}$$

If $n_2(C_s - r_s) + r_2 r_s > C_r$, then $R_2^* > R_1^*$, and $R_2^*$ will not be the minimum Bayes risk over all the double sampling plans. Hence, the optimal sample size $n_{20}$ must satisfy $n_{20}(C_s - r_s) \leq n_{20}(C_s - r_s) + r_{20} r_s \leq C_r$, and (3.2.9) follows.

Therefore, based on Theorems 3.1 and 3.2, our algorithm is a finite algorithm, i.e., an optimal sampling plan can be found in finite steps of searching.

## 3.3 Discretization method and numerical analysis

For single sampling plan with exponential distribution, Lam (1994) suggested a discretization method for minimizing the Bayes risk. Now, on the basis of the Theorems 3.1 and 3.2, a similar discretization method can be adopted for the present double sampling plan. To this end, we fix $n_1, n_2, r_1$ and $r_2$ first. Then we evaluate the Bayes risks at a sequence of particular sampling plans $(n_1, n_2, r_1, r_2, T_0, T_1, T_2)$, where $T_0, T_1$ and $T_2$ can take some discrete values. Thereafter, we can determine $\min_{T_0,T_1,T_2} R(n_1, r_1, n_2, r_2, T_0, T_1, T_2)$ by comparison. This method will be applicable if there exists a lower bound and a upper bound for the lifetime $X$. Although 0 can be taken as a lower bound, it is not clear in general whether an upper bound exists. An alternative way is to choose a number $T_U$ such that

$$P(0 \le X \le T_U) = 1 - v,$$

where $v$ is a preassigned number satisfying $0 < v < 1$. Because

$$P(0 \le X \le T_U) = \int_0^\infty \int_0^{T_U} \lambda m x^{m-1} e^{-\lambda x^m} \tfrac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} dx d\lambda$$

$$= \tfrac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty (1 - e^{-\lambda T_U^m}) \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda = 1 - \left(1 + \tfrac{T_U^m}{\beta}\right)^{-\alpha} = 1 - v,$$

we have

$$T_U = \left\{\beta(v^{-1/\alpha} - 1)\right\}^{1/m}. \quad (3.3.1)$$

Alternatively, we can choose a lower bound $T_L$ and an upper bound $T_U$ such that $P(T_L \le X \le T_U) = 1 - v$.

To illustrate the model and the discretization algorithm for the determination of an optimal double sampling plan and the minimum Bayes risk developed here, we still assume that the degree $k$ of the loss function (3.2.2) is 2. Then, the Bayes risk for the double sampling plan $(n_1, r_1, n_2, r_2, T_0, T_1, T_2)$ is given by

$$
\begin{aligned}
&R(n_1, r_1, n_2, r_2, T_0, T_1, T_2) \\
&= n_1(C_s - r_s) + r_1 r_s + C_r + [n_2(C_s - r_s) + r_2 r_s](I_{s_0}(r_1, \alpha) - I_{s_1}(r_1, \alpha)) \\
&+ a_t \tfrac{\beta^{1/m}\Gamma(\alpha - \frac{1}{m})}{\Gamma(\alpha)}[C(n_1, r_1) + C(n_2, r_2)(I_{s_0}(r_1, \alpha - \tfrac{1}{m}) - I_{s_1}(r_1, \alpha - \tfrac{1}{m}))] \\
&+ \sum_{l=0}^{2} C_l^* \tfrac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(1 - I_{s_1}(r_1, \alpha + l)) - \sum_{l=0}^{2} C_l^* \tfrac{r_1^{r_1}\beta^\alpha\Gamma(r_1+\alpha+l)}{\Gamma(r_1)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \tfrac{u^{r_1-1} I_{s(u)}(r_2, r_1+\alpha+l)}{(r_1 u+\beta)^{r_1+\alpha+l}} du.
\end{aligned}
\tag{3.3.2}
$$

Here, we shall study some numerical examples whose parameters and coefficients are: $m = 2.5, \alpha = 2.5, \beta = 1, C_0 = 5, C_1 = 5, C_2 = 5, C_r = 50, C_s = 0.5, r_s = 0.2$ and $a_s = 2.0$. These values will be taken as the standard values for the purpose of comparison. The numerical results are tabulated in Tables 3.1-3.10. In each table, only one parameter or one coefficient can be changed and the others are fixed. The values of the varying parameters or that of the varying coefficients are given in column 1. The optimal sampling plan $(n_{10}, r_{10}, n_{20}, r_{20}, T_{00}, T_{10}, T_{20})$ and the corresponding minimum Bayes risks $R^*$ (for the corresponding double sampling plan) are given by column 2 to column 9 respectively. Note that the sampling plan $(0, 0, 0, 0, 0, 0, 0)$ represents the plan of accepting a batch without sampling, and $(0, 0, 0, 0, \infty, \infty, \infty)$ denotes the plan of rejecting a batch without sampling.

In these examples, $T_U$ is chosen such that $P(0 \leq X \leq T_U) = 0.95$. Then, for each $n_1, r_2, n_2$ and $r_2$, a sequence of the double sampling plan $(n_1, r_1, n_2, r_2, T_0, T_1, T_2)$ is studied in the following way: $T_0 = iT_U/100,\ T_1 = jT_U/100,\ T_2 = kT_U/100,$ $i = 1, \ldots, 100;\quad j = 0, 1, \ldots, i;\quad k = j, \ldots, i$ Finally, the minimum Bayes risk is obtained and the corresponding optimal double sampling plan is determined accordingly by comparison.

As in Chapter 2, the standard values of the parameters and coefficients are chosen as the true values of them. In Tables 3.1-3.10, the true parameter or coefficient and the corresponding minimum Bayes risk are also marked by '*'. By using an estimated sampling plan, the true (but not the minimum ) Bayes risk $R$ and their efficiencies are given in columns 10 and 11 in each table respectively. For example, in Table 3.2, if $\alpha$ is inaccurately estimated as 2.2, the estimated Bayes risk is 39.2999 and the estimated samplings plan is $(5, 3, 6, 4, 0.6982, 0.5175, 0.6259)$. By using the estimated plan, as the true $\alpha$ is 2.5, the true Bayes risk ( but not the minimum) is 42.3229 . Since the true minimum Bayes risk is 42.2581, the efficiency of the estimated sampling plan is then equal to $R^*/R$=42.2581 /42.3229 =0.9984. Sensitivity analysis for the other values of $\alpha$, and other parameters and coefficients, can be conducted in a similar way.

From the tabulated results, the efficiencies of the estimated double sampling plan in most cases are greater than 0.95, even if the errors of a parameter or a coefficient are over 100%. Hence, we can conclude that the model is insensitive to the parameters and coefficients.

In Table 3.9, we can see that if $a_t = 0$, then the optimal double sampling plan $(3, 3, 5, 5, 0.7109, 0.4866, 0.6268)$. In this case, a complete sampling without use of Type II censoring is applied. In general, if the rate of time-consuming cost $a_t$ is negligible, the following Theorem 3.3 shows that this result will remain true.

**Theorem 3.3.** *Let* $(n_{10}, r_{10}, n_{20}, r_{20}, T_{00}, T_{10}, T_{20})$ *be an optimal double sampling plan.If* $a_t = 0$, *then* $r_{10} = n_{10}$, $r_{20} = n_{20}$.

**Proof:** Let $a_t = 0$ and $(n_{10}, r_{10}, n_{20}, r_{20}, T_{00}, T_{10}, T_{20})$ be a corresponding optimal double sampling plan. If $n_{10} > r_{10}$ or $n_{20} > r_{20}$, then, from (3.2.7) and the fact $C_s > r_s$, we have

$$R(n_{10}, r_{10}, n_{20}, r_{20}, T_{00}, T_{10}, T_{20})$$

$$= (n_{10} - r_{10})(C_s - r_s) + (n_{20} - r_{20})(C_s - r_s)(I_{s_0}(r_{10}, \alpha) - I_{s_1}(r_{10}, \alpha))$$

$$+ R(r_{10}, r_{10}, r_{20}, r_{20}, T_{00}, T_{10}, T_{20}) > R(r_{10}, r_{10}, r_{20}, r_{20}, T_{00}, T_{10}, T_{20})$$

Therefore, $(n_{10}, r_{10}, n_{20}, r_{20}, T_{00}, T_{10}, T_{20})$ will not be an optimal sampling plan. Hence, $n_{10} = r_{10}, n_{20} = r_{20}$. This completes the proof of the Theorem 3.3.

Table 3.1. *The minimum Bayes risk and optimal sampling plans as $m$ varies*

| $m$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 42.2559 | 4 | 3 | 4 | 3 | 0.5987 | 0.1503 | 0.3465 | 61.8656 | 0.6831 |
| 1.0 | 42.2653 | 4 | 3 | 4 | 3 | 0.5707 | 0.2905 | 0.4306 | 53.5580 | 0.7890 |
| 2.0 | 42.2625 | 4 | 3 | 6 | 4 | 0.6548 | 0.4586 | 0.5707 | 43.0361 | 0.9819 |
| 2.5* | 42.2581* | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 3.0 | 42.2871 | 4 | 3 | 6 | 4 | 0.7389 | 0.5988 | 0.6548 | 42.5036 | 0.9942 |
| 3.5 | 42.2820 | 4 | 3 | 4 | 3 | 0.7669 | 0.6268 | 0.7109 | 43.1883 | 0.9785 |
| 4.0 | 42.3054 | 4 | 3 | 4 | 3 | 0.7669 | 0.6548 | 0.7389 | 43.5126 | 0.9712 |

Table 3.2. *The minimum Bayes risk and optimal sampling plans as $\alpha$ varies*

| $\alpha$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.5 | 30.8493 | 6 | 2 | 6 | 2 | 0.5509 | 0.3729 | 0.5509 | 45.9448 | 0.9198 |
| 2.0 | 37.0581 | 4 | 2 | 6 | 4 | 0.6767 | 0.4554 | 0.5882 | 42.8229 | 0.9868 |
| 2.2 | 39.2999 | 5 | 3 | 6 | 4 | 0.6982 | 0.5175 | 0.6259 | 42.3229 | 0.9984 |
| 2.5* | 42.2581* | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 2.7 | 43.9602 | 4 | 3 | 4 | 3 | 0.7130 | 0.5431 | 0.6402 | 42.2812 | 0.9995 |
| 3.0 | 47.6809 | 6 | 5 | 0 | 0 | 0.5912 | 0.5912 | 0.0000 | 43.3188 | 0.9873 |
| 3.5 | 50.0000 | 0 | 0 | 0 | 0 | $\infty$ | $\infty$ | $\infty$ | 50.0000 | 0.8451 |

Table 3.3. *The minimum Bayes risk and optimal sampling plans as $\beta$ varies*

| $\beta$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.6 | 50.0000 | 0 | 0 | 0 | 0 | $\infty$ | $\infty$ | $\infty$ | 50.0000 | 0.8451 |
| 0.8 | 45.8446 | 5 | 4 | 4 | 3 | 0.6584 | 0.5911 | 0.6359 | 42.4509 | 0.9954 |
| 0.9 | 43.9519 | 4 | 3 | 4 | 3 | 0.7154 | 0.5641 | 0.6398 | 42.3055 | 0.9989 |
| 1.0* | 42.2581* | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 1.1 | 40.5268 | 4 | 3 | 6 | 4 | 0.6895 | 0.5045 | 0.6278 | 42.2958 | 0.9991 |
| 1.4 | 35.0174 | 4 | 2 | 6 | 5 | 0.6028 | 0.3674 | 0.6028 | 43.7218 | 0.9665 |
| 2.0 | 22.1875 | 0 | 0 | 0 | 0 | 0.0000 | 0.0000 | 0.0000 | 61.2500 | 0.6900 |

Table 3.4. *The minimum Bayes risk and optimal sampling plans as $C_0$ varies*

| $C_0$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 40.1859 | 4 | 3 | 6 | 4 | 0.6828 | 0.5147 | 0.5988 | 42.3940 | 0.9968 |
| 2.5 | 40.9970 | 4 | 3 | 6 | 4 | 0.6828 | 0.5147 | 0.6268 | 42.2958 | 0.9991 |
| 4.0 | 41.7646 | 4 | 3 | 4 | 3 | 0.7109 | 0.5147 | 0.6268 | 42.2717 | 1.9996 |
| 5.0* | 42.2581* | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 5.5 | 42.5039 | 4 | 3 | 4 | 3 | 0.7108 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 7.0 | 43.2415 | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 10.0 | 44.6053 | 4 | 3 | 5 | 3 | 0.7388 | 0.5707 | 0.6548 | 42.4476 | 0.9955 |

Table 3.5. *The minimum Bayes risk and optimal sampling plans as $C_1$ varies*

| $C_1$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| -5.0 | 31.2508 | 5 | 3 | 6 | 5 | 0.5707 | 0.4026 | 0.5147 | 46.1565 | 0.9155 |
| -1.0 | 36.5450 | 4 | 3 | 6 | 5 | 0.6268 | 0.4306 | 0.5707 | 43.3985 | 0.9737 |
| 4.0 | 41.4962 | 4 | 3 | 6 | 4 | 0.6828 | 0.5147 | 0.6268 | 41.9858 | 0.9991 |
| 5.0* | 42.2581* | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 5.5 | 42.2684 | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 7.0 | 43.6530 | 4 | 3 | 5 | 3 | 0.7388 | 0.5707 | 0.6548 | 42.4127 | 0.9963 |
| 0.0 | 45.3614 | 4 | 3 | 5 | 3 | 0.7669 | 0.6268 | 0.6828 | 42.8780 | 0.9855 |

Table 3.6. *The minimum Bayes risk and optimal sampling plans as $C_2$ varies*

| $C_2$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 26.2500 | 0 | 0 | 0 | 0 | 0.0000 | 0.0000 | 0.0000 | 61.2500 | 0.6900 |
| 2.5 | 36.5381 | 3 | 2 | 6 | 5 | 0.5707 | 0.3465 | 0.5427 | 45.3618 | 0.9316 |
| 4.5 | 41.4897 | 4 | 3 | 6 | 4 | 0.6828 | 0.5147 | 0.5988 | 42.3939 | 0.9968 |
| 5.0* | 42.2581* | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 5.5 | 42.9692 | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6548 | 42.3463 | 0.9979 |
| 7.0 | 44.5328 | 4 | 3 | 5 | 3 | 0.7669 | 0.5987 | 0.6828 | 42.7819 | 0.9878 |
| 10.0 | 46.5386 | 4 | 3 | 5 | 3 | 0.8230 | 0.6828 | 0.7389 | 44.0028 | 0.9605 |

Table 3.7. *The minimum Bayes risk and optimal sampling plans as $C_s$ varies*

| $C_s$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.21 | 40.1548 | 6 | 3 | 6 | 4 | 0.7186 | 0.5147 | 0.6268 | 42.5078 | 0.9941 |
| 0.3 | 40.8758 | 6 | 4 | 6 | 4 | 0.7109 | 0.5427 | 0.6268 | 42.4495 | 0.9955 |
| 0.4 | 41.6231 | 5 | 3 | 6 | 4 | 0.7109 | 0.5147 | 0.6268 | 42.3307 | 0.9983 |
| 0.5* | 42.2581* | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 0.6 | 42.7751 | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.3307 | 1.0000 |
| 0.7 | 43.2547 | 2 | 2 | 4 | 3 | 0.7389 | 0.5147 | 0.6268 | 42.5776 | 0.9925 |
| 1.0 | 44.1494 | 2 | 2 | 2 | 2 | 0.7109 | 0.5427 | 0.6268 | 42.8499 | 0.9862 |

Table 3.8. *The minimum Bayes risk and optimal sampling plans as $r_s$ varies*

| $r_s$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 42.5182 | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 1.0 | 42.3889 | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 1.5 | 42.3243 | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 0.2* | 42.2581* | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 0.25 | 42.1789 | 4 | 3 | 5 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2582 | 0.9999 |
| 0.3 | 42.0587 | 5 | 3 | 5 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.3172 | 0.9986 |
| 0.4 | 41.7468 | 6 | 3 | 6 | 3 | 0.7109 | 0.5427 | 0.6268 | 43.5225 | 0.9937 |

Table 3.9. *The minimum Bayes risk and optimal sampling plans as $a_t$ varies*

| $a_t$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 40.5656 | 3 | 3 | 5 | 5 | 0.7109 | 0.4866 | 0.6268 | 42.5748 | 0.9926 |
| 1.0 | 41.6294 | 4 | 3 | 5 | 3 | 0.7109 | 0.5147 | 0.6268 | 42.2797 | 0.9995 |
| 1.5 | 41.9546 | 4 | 3 | 4 | 3 | 0.7108 | 0.5147 | 0.6268 | 42.2797 | 0.9995 |
| 2.0* | 42.2581* | 4 | 3 | 4 | 3 | 0.7108 | 0.5427 | 0.6268 | 42.2581 | 1.000 |
| 2.5 | 42.5318 | 5 | 3 | 5 | 3 | 0.7108 | 0.5427 | 0.6268 | 42.3173 | 0.9986 |
| 3.0 | 42.7463 | 5 | 3 | 5 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.3173 | 0.9986 |
| 5.0 | 43.5212 | 4 | 2 | 6 | 3 | 0.7389 | 0.5147 | 0.6268 | 42.5559 | 0.9930 |

Table 3.10. *The minimum Bayes risk and optimal sampling plans as $C_r$ varies*

| $C_r$ | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | True $R$ | Eff of $R$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.0 | 10.0000 | 0 | 0 | 0 | 0 | $\infty$ | $\infty$ | $\infty$ | 50.0000 | 0.8451 |
| 30.0 | 29.5644 | 3 | 2 | 3 | 1 | 0.8230 | 0.7669 | 0.7949 | 44.7485 | 0.9443 |
| 40.0 | 36.6067 | 3 | 2 | 5 | 3 | 0.7109 | 0.5987 | 0.6828 | 43.0237 | 0.9822 |
| 50.0* | 42.2581* | 4 | 3 | 4 | 3 | 0.7109 | 0.5427 | 0.6268 | 42.2581 | 1.0000 |
| 60.0 | 46.7629 | 4 | 3 | 6 | 5 | 0.6548 | 0.4586 | 0.5987 | 42.6877 | 0.9899 |
| 70.0 | 50.3313 | 4 | 3 | 6 | 5 | 0.6268 | 0.4306 | 0.5427 | 44.1948 | 0.9562 |
| 100.0 | 57.3211 | 6 | 4 | 6 | 6 | 0.5426 | 0.3745 | 0.4866 | 49.2185 | 0.8586 |

# CHAPTER 4.

## Single Variable Sampling Plan for General Life Distribution with Type I censoring

In Chapters 2 and 3, we discuss respectively the model of single and double sampling plan for the Weibull distribution with known shape parameter and Type II censoring. In practice, the shape parameter $m$ of the Weibull distribution $W(m, \lambda)$ is usually unknown. It is necessary to develop a model of single variable sampling plan for the Weibull distribution in which the scale parameter and shape parameter are both unknown. However, the methods introduced in previous chapters seem not workable for this problem.

In present chapter, we develop a model of single variable sampling plans for general life distribution with Type I censoring. In Section 2, assume that the loss function is a polynomial function, a general model of single sampling plan is formulated and the explicit expressions of the Bayes risks are obtained by using the Kaplan-Meier estimator. In Sections 3, 4 and 5, some special cases are discussed, including the single sampling plan for the Weibull distribution $W(m, \lambda)$ with both parameters unknown, the two-parameter exponential distribution and the gamma distribution. Finally, in Section 6, some numerical examples are studied in detail and the corresponding algorithm for finding an optimal sampling plan approximately is proposed.

## 4.1 Model

Suppose that a batch of $N$ items is presented for acceptance inspection or further processing. The lifetime of an item is a random variable $X$, which has a distribution $F(x|\underset{\sim}{\lambda})$, where $\underset{\sim}{\lambda} = (\lambda_1, \ldots, \lambda_m)$ is unknown and has a conjugate prior distribution $\pi(\underset{\sim}{\lambda})$.

Suppose that a random sample $\underset{\sim}{X} = (X_1, \ldots, X_n)$ of size $n$ is taken from the batch, giving the observation $\underset{\sim}{x} = (x_1, \ldots, x_n)$. Let $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ be the order statistics of $X_1, X_2, \ldots, X_n$. Since the sample is subject to Type I censoring, the true observations are as follows:

$$Z_i = \min\{X_i, t\}, \qquad \text{and} \qquad \delta_i = I_{\{X_i < t\}}.$$

Let $M = \max\{i : X_{(i)} \le t\}$ be the number of failures by time $t$. Clearly, $M$ has a binomial distribution $B(n, q)$ with

$$P(M = r|\underset{\sim}{\lambda}) = \begin{pmatrix} n \\ r \end{pmatrix} F(t|\underset{\sim}{\lambda})^r (1 - F(t|\underset{\sim}{\lambda}))^{n-r}, \quad r = 0, 1, \ldots, n. \tag{4.1.1}$$

where $F(t|\underset{\sim}{\lambda}) = 1 - S(t|\underset{\sim}{\lambda}) = P(X \le t|\underset{\sim}{\lambda})$ and $S(t|\underset{\sim}{\lambda})$ is the survival function. Obviously, $E(M|\underset{\sim}{\lambda}) = nF(t|\underset{\sim}{\lambda})$.

It is well known that under Type I censoring the Kaplan-Meier estimator of the survival function $S(t|\underset{\sim}{\lambda})$ is given by

$$\hat{S}_n(t) = \prod_{Z_{(i)} < t} \left( \frac{n-i}{n-i+1} \right)^{\delta_{(i)}} = \begin{cases} 1, & t \le X_{(1)}, \\ \prod_{X_{(i)} < t} \left( \frac{n-i}{n-i+1} \right), & \textit{otherwise}, \end{cases} \tag{4.1.2}$$

where for $i = 1, \ldots, n$, $Z_{(i)}$ is the $i$th order statistic of $Z_1, \ldots, Z_n$ and $\delta_{(i)}$ is the corresponding indicator function (see Kaplan and Meier (1958) for reference).

By the Kaplan-Meier estimator (4.1.2), we should study the single sampling plan $(n, t, p)$ with the following decision function:

$$\delta(\underline{X}) = \begin{cases} d_0, & \hat{S}_n(t) \geq p, \\ \\ d_1, & otherwise, \end{cases} \tag{4.1.3}$$

where $d_0$ represents the decision of accepting the batch and $d_1$ denotes the decision of rejecting the batch, while $p$ is the minimum acceptance reliability.

Using a similar argument to that developed in Chapter 2, the following polynomial loss function should be adopted.

$$L(\underline{\lambda}, \delta(\underline{X})) = \begin{cases} nC_s - (n-M)r_s + ta_t + L(\lambda_1, \ldots, \lambda_m), & \delta(\underline{X}) = d_0, \\ \\ nC_s - (n-M)r_s + ta_t + C_r, & \delta(\underline{X}) = d_1, \end{cases} \tag{4.1.4}$$

where the coefficients $C_s$, $r_s$, $a_t$ and $C_r$ are constants and $L(\lambda_1, \ldots, \lambda_m) = \sum_{0 \leq i_1 + \cdots + i_m \leq k} C_{i_1 \ldots i_m} \lambda_1^{i_1} \ldots \lambda_m^{i_m}$ and satisfies a natural constraint:

$$\sum_{0 \leq i_1 + \cdots + i_m \leq k} C_{i_1 \ldots i_m} \lambda_1^{i_1} \ldots \lambda_m^{i_m} \geq 0 \qquad \forall\ \lambda_j > 0, \ \ j = 1 \ldots m. \tag{4.1.5}$$

As before, the loss function is the sum of the sampling cost, time-consuming cost and the decision loss. The constraint (4.1.5) is natural because the left-hand side of (4.1.5) represents a part of loss due to accepting the batch and hence must be nonnegative.

In practice, $C_s$ represents the sampling cost per item including the inspection cost per item and the normal price of an item, $r_s$ is the salvage value per item of an unfailed item, then $nC_s - (n-M)r_s$ represents the net sampling cost.

Besides, $a_t$ is the rate of time-consuming cost, and $ta_s$ is the time-consuming cost.

Then, $C_r$ is the cost due to rejecting the batch. On the other hand, $L(\lambda_1, \ldots, \lambda_m)$ is an approximation to the exact cost function due to accepting the batch. To see this, suppose that a batch of $N$ items is presented for acceptance inspection and the quality of an item is measured by the reliability $S(t_s|\underset{\sim}{\lambda})$ of the item beyond time $t_s$, i.e. $S(t_s|\underset{\sim}{\lambda}) = P(X > t_s|\underset{\sim}{\lambda})$. Let the standard value of $S(t_s|\underset{\sim}{\lambda})$ be $p_s$. Thus, if $S(t_s|\underset{\sim}{\lambda}) \geq p_s$, we can accept the item without additional loss; otherwise, we may accept it with an extra cost $C(p_s - S(t_s|\underset{\sim}{\lambda}))$, where $C$ is a proportional constant. Thus, the cost function when the batched is accept is given by

$$\begin{aligned} & NC(p_s - S(t_s|\underset{\sim}{\lambda})) = NC\{p_s - (1 - F(t_s|\underset{\sim}{\lambda}))\} \\ & = C_{0\ldots0} + C_{10\ldots0}\lambda_1 + C_{010\ldots0}\lambda_2 + \cdots + C_{0\ldots01}\lambda_m + \cdots \end{aligned} \tag{4.1.6}$$

where $C_{0\ldots0}, C_{10\ldots0}, \ldots$ are constants and independent of $\underset{\sim}{\lambda}$.

The single sampling plan studied here is $(n, t, p)$ in which $n$ is the sample size, $t$ is the censoring time and $p$ is the minimum acceptance reliability. Then, our objective is to determine an optimal sampling plan $(n, t, p)$ for minimizing the Bayes risk.

$$\begin{aligned} R(n,t,p) &= E\left[L(\underset{\sim}{\lambda}, \delta(\underset{\sim}{X}))\right] = E\left\{E\left[L(\underset{\sim}{\lambda}, \delta(\underset{\sim}{X}))|\underset{\sim}{\lambda}\right]\right\} \\ &= E\Big\{nC_s - (n - E(M|\underset{\sim}{\lambda}))r_s + ta_t \\ &\quad + L(\lambda_1, \ldots, \lambda_m)P(\hat{S}_n(t) \geq p) + C_r P(\hat{S}_n(t) < p)\Big\}. \end{aligned} \tag{4.1.7}$$

Now,

$$\begin{aligned} P\left(\hat{S}_n(t) \geq p\right) &= P(M=0) + P\left(\prod_{X_{(i)}<t} \left(\tfrac{n-i}{n-i+1}\right) \geq p\right) \\ &= P(M=0) + P\left(\prod_{i=1}^{M} \left(\tfrac{n-i}{n-i+1}\right) \geq p\right) = P(M=0) + P\left(\tfrac{n-M}{n} \geq p\right) \end{aligned}$$

$$= P(M=0) + \sum_{r=1}^{[n(1-p)]} P(M=r) = \sum_{r=0}^{[n(1-p)]} \begin{pmatrix} n \\ r \end{pmatrix} F(t|\lambda)^r (1-F(t|\lambda))^{n-r}, \tag{4.1.8}$$

where, $[a]$ is the integer part of $a$. Hence, from (4.1.7) and (4.1.8), the Bayes risk for the single sampling plan with type I censoring is given by

$$R(n,t,p) = E\Big\{ nC_s - n(1-F(t|\lambda))r_s + ta_t + C_r + [(C_{0\dots 0} - C_r) + \sum_{1 \le i_1+\cdots+i_m \le k} C_{i_1\dots i_m} \lambda_1^{i_1} \dots \lambda_m^{i_m}] \sum_{r=0}^{[n(1-p)]} P(M=r) \Big\}$$

$$= n(C_s - r_s) + nE(F(t|\lambda))r_s + ta_t + C_r + \sum_{r=0}^{[n(1-p)]} \begin{pmatrix} n \\ r \end{pmatrix} \sum_{0 \le i_1+\cdots+i_m \le k} C^*_{i_1\dots i_m} E\left[ \lambda_1^{i_1} \dots \lambda_m^{i_m} F(t|\lambda)^r (1-F(t|\lambda))^{n-r} \right], \tag{4.1.9}$$

where, $C^*_{i_1\dots i_m}$ is $C_{0\dots 0} - C_r$ if all $i_j = 0$ for $j = 1, \dots, m$, and is $C_{i_1\dots i_m}$, otherwise.

On the basis of the expression (4.1.9), a simple algorithm for the determination of an optimal sampling plan can be stated in the following way:

**1.** Fix $n$, minimize $R(n,t,p)$ with respect to $t$ and $p$, let $R(n,t_n,p_n) = \min_{t,p} R(n,t,p)$.

**2.** Move $n$ to $n+1$, repeat the above procedure and continue.

**3.** By comparison, the smallest Bayes risk $\min_n R(n,t_n,p_n)$ is the minimum Bayes risk and the corresponding sampling plan is an optimal sampling plan.

The following theorem justifies that the algorithm is finite, i.e., we can find an optimal sampling plan in finite steps of searching. Actually, by the same argument as in Chapter 2, it is easy to show the following theorem.

**Theorem 4.1.** *For $n \ge 1$, let $R(n,t_n,p_n) = \min_{t,p} R(n,t,p)$. Then the optimal* *sampling size* $n_0$ *satisfies the following inequality*

$$n_0 \leq \min\left\{\frac{C_r}{C_s - r_s}, \frac{1}{C_s - r_s}\sum_{0\leq i_1+\cdots+i_m\leq k} C_{i_1\ldots i_m} E\left[\lambda_1^{i_1}\ldots\lambda_m^{i_m}\right], \frac{R(n,t_n,p_n)}{C_s - r_s}\right\}. \tag{4.1.10}$$

where $E\left[\lambda_1^{i_1}\ldots\lambda_m^{i_m}\right] = \int\cdots\int \lambda_1^{i_1}\ldots\lambda_m^{i_m} d\pi(\lambda_1,\ldots,\lambda_m)$.

**Proof.** Let $(n_0, t_0, p_0)$ be an optimal sampling plan. Then simitar to (2.2.6), we have

$$R(n_0, t_0, p_0) \leq \min\left\{R(0,0,1), R(0,0,0), R(n, t_n, p_n)\right\}. \tag{4.1.11}$$

where $(0,0,1)$ is the sampling plan of rejecting the batch without sampling, while $(0,0,0)$ is the sampling plan of accepting the batch without sampling. It is clear that $R(0,0,1) = C_r$ and $R(0,0,0) = \sum_{0\leq i_1+\cdots+i_m\leq k} C_{i_1\ldots i_m} E\left[\lambda_1^{i_1}\ldots\lambda_m^{i_m}\right]$. Furthmore, from (4.1.9), we have

$$R(n_0, t_0, p_0) \geq n_0(C_s - r_s) + nE(F(t|\underline{\lambda}))r_s + t_0 a_s \geq n_0(C_s - r_s). \tag{4.1.12}$$

The proof is completed by the combination of (4.1.11) and (4.1.12).

In fact, (4.1.10) gives an adaptive upper bound for the optimal sample size $n_0$. Therefore, our algorithm is a finite algorithm.

On the basis of the Theorem 4.1, a similar discretization method as introduced in Chapter 3 can be adopted for the present single sampling plan. First of all, for fixed $n$, we can evaluate the Bayes risks at a sequence of particular sampling plans $(n,t,p)$, where $t$ and $p$ take some discrete values. Then, we can determine $\min_{t,p} R(n,t,p)$ by comparison. This method will be applicable if there exists a lower bound and an upper bound for lifetime $X$. Although 0 can be taken as a lower bound, it is not clear whether an upper bound exits. Because

$$P(0 \leq X \leq T_U) = E(E(I_{(0\leq X\leq T_U)}|\underset{\sim}{\lambda})) = E(F(T_U|\underset{\sim}{\lambda})) = \int F(T_U|\underset{\sim}{\lambda})\pi(\underset{\sim}{\lambda})d\underset{\sim}{\lambda}. \quad (4.1.13)$$

Then we can solve the inequality $\int F(T_U|\underset{\sim}{\lambda})\pi(\underset{\sim}{\lambda})d\underset{\sim}{\lambda} \geq 1 - \upsilon$ for the upper bound $T_U$, where $\upsilon$ is a preassigned number satisfying $0 < \upsilon < 1$.

## 4.2 The case of the Weibull distribution

Suppose that the lifetime $X$ of an item in a batch follows a Weibull distribution $W(m, \lambda)$, i.e., $F(x|\underset{\sim}{\lambda})$ is a Weibull distribution with the density function

$$f(x|m,\lambda) = \begin{cases} \lambda m x^{m-1}e^{-\lambda x^m}, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad (4.2.1)$$

where $\underset{\sim}{\lambda} = (m, \lambda)$, the shape parameter $m$ and the scale parameter $\lambda$ are both unknown. Soland (1969) introduced a prior distribution for $(m, \lambda)$, such that $m$ has a prior discrete distribution $P(m = m_l) = p_l$, $(l = 1, \dots, L)$, where $p_l$ are known and $\sum_{l=1}^{L} p_l = 1$. The conditional prior distribution of $\lambda$ given $m_l$ is a gamma distribution $\Gamma(\alpha_l, \beta_l)$ with the density function $g(\lambda|m_l) = \beta_l^{\alpha_l}\lambda^{\alpha_l-1}e^{-\beta_l\lambda}/\Gamma(\alpha_l)$ for $\lambda > 0$, and 0 otherwise, where $\alpha_l > 0$ and $\beta_l > 0$ are known. Therefore, the joint prior distribution for $(m, \lambda)$ is given by

$$\pi(m_l, u) = P(m = m_l, \lambda \leq u) = p_l \int_0^u \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} t^{\alpha_l-1}e^{-\beta_l t}dt, \quad l = 1, \dots, L. \quad (4.2.2)$$

Let $I_a(x) = \frac{1}{\Gamma(a)}\int_0^x t^{a-1}e^{-t}dt$ be the incomplete gamma function. Then

$$I_a(bx) = \frac{b^a}{\Gamma(a)}\int_0^x t^{a-1}e^{-bt}dt. \quad (4.2.3)$$

Therefore, it follows from (4.2.2) and (4.2.3) that

$$E\left[m^{i_1}\lambda^{i_2}F(t|\underline{\lambda})^r(1-F(t|\underline{\lambda}))^{n-r}\right]$$

$$=\sum_{l=1}^{L}p_l\int_0^{\infty}m_l^{i_1}\lambda^{i_2}(1-exp(-\lambda t^{m_l}))^r exp(-\lambda(n-r)t^{m_l})\frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)}\lambda^{\alpha_l-1}exp(-\lambda\beta_l)d\lambda$$

$$=\sum_{j=0}^{r}(-1)^j\begin{pmatrix} r \\ j \end{pmatrix}\sum_{l=1}^{L}p_l m_l^{i_1}\frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)}\int_0^{\infty}\lambda^{i_2+\alpha_l-1}exp[-\lambda((n-r+j)t^{m_l}+\beta_l)]d\lambda$$

$$=\sum_{j=0}^{r}(-1)^j\begin{pmatrix} r \\ j \end{pmatrix}\sum_{l=1}^{L}p_l m_l^{i_1}\frac{\Gamma(i_2+\alpha_l)\beta_l^{\alpha_l}}{\Gamma(\alpha_l)[(n-r+j)t^{m_l}+\beta_l]^{i_2+\alpha_l}},\qquad 0\le i_1+i_2\le k. \tag{4.2.4}$$

In particular $E\left[m^{i_1}\lambda^{i_2}\right]=\sum_{l=1}^{L}p_l m_l^{i_1}\frac{\Gamma(i_2+\alpha_l)}{\Gamma(\alpha_l)\beta_l^{i_2}},\quad 0\le i_1+i_2\le k.$

Similarly,

$$E\left[F(t|\underline{\lambda})\right]=1-\sum_{l=1}^{L}p_l\frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)}\int_0^{\infty}\lambda^{\alpha_l-1}e^{-\lambda(t^{m_l}+\beta_l)}d\lambda=1-\sum_{l=1}^{L}p_l\frac{\beta_l^{\alpha_l}}{(t^{m_l}+\beta_l)^{\alpha_l}}. \tag{4.2.5}$$

Consequently, from (4.1.9), (4.2.4) and (4.2.5), the Bayes risk for the sampling plan $(n,t,p)$ is given by

$$R(n,t,p)\ =nC_s-nr_s\sum_{l=1}^{L}p_l\frac{\beta_l^{\alpha_l}}{(t^{m_l}+\beta_l)^{\alpha_l}}+ta_t+C_r+\sum_{r=0}^{[n(1-p)-p]}\sum_{j=0}^{r}$$

$$(-1)^j\begin{pmatrix} n \\ r \end{pmatrix}\begin{pmatrix} r \\ j \end{pmatrix}\sum_{l=1}^{L}p_l\left\{\sum\nolimits_{0\le i_1+i_2\le k}C^*_{i_1i_2}m_l^{i_1}\frac{\Gamma(i_2+\alpha_l)\beta_l^{\alpha_l}}{\Gamma(\alpha_l)[(n-r+j)t^{m_l}+\beta_l]^{i_2+\alpha_l}}\right\}. \tag{4.2.6}$$

From Theorem 4.1, we then have the following result

**Corollary 4.1** *For the case of the Weibull distribution, the optimal sampling size $n_0$ satisfies the following inequality*

$$n_0\le\min\left\{\frac{C_r}{C_s-r_s},\frac{1}{C_s-r_s}\sum_{l=1}^{L}p_l\left[\sum\nolimits_{0\le i_1+i_2\le k}C_{i_1i_2}m_l^{i_1}\frac{\Gamma(i_2+\alpha_l)}{\beta_l^{i_2}\Gamma(\alpha_l)}\right],\frac{R(n,t_n,p_n)}{C_s-r_s}\right\}. \tag{4.2.7}$$

where $R(n, t_n, p_n)$ is the Bayes risk for sampling plan $(n, t_n, p_n)$.

The discretization method can be applied in the present case. To this end, we should determine an upper bound $T_U$. In practice, let $T_U > 1$ be such that $P(0 \leq X \leq T_U) \geq 1 - v$, where $0 < v < 1$. By the result (4.1.13), we have

$$P(0 \leq X \leq T_U) = \int F(T_U|\underline{\lambda})\pi(\underline{\lambda})d\underline{\lambda}$$

$$= \sum_{l=1}^{L} p_l \int_0^{\infty} \int_0^{T_U} \lambda m_l x^{m_l - 1} e^{-\lambda x^{m_l}} \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} \lambda^{\alpha_l - 1} e^{-\beta_l \lambda} dx d\lambda$$

$$= \sum_{l=1}^{L} p_l \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} \int_0^{\infty} (1 - e^{-\lambda T_U^{m_l}}) \lambda^{\alpha_l - 1} e^{-\beta_l \lambda} d\lambda = 1 - \sum_{l=1}^{L} p_l \left(1 + \frac{T_U^{m_l}}{\beta_l}\right)^{-\alpha_l}.$$

Let $m_0 = \min\{m_l\}$, $\beta_0 = \max\{\beta_l\}$, $\alpha_0 = \min\{\alpha_l\}$. It follows that

$$P(0 \leq X \leq T_U) \geq 1 - \left(1 + \frac{T_U^{m_0}}{\beta_0}\right)^{-\alpha_0} = 1 - v.$$

Hence

$$T_U = \left\{\beta_0(v^{-1/\alpha_0} - 1)\right\}^{1/m_0}. \tag{4.2.8}$$

In Section 4.5, we shall study some numerical examples for illustration of the model and the algorithm.

## 4.3 The case of the two-parameter exponential distribution

Suppose that the lifetime $X$ of an item in a batch follows a two-parameter exponential distribution. In this case, the distribution $F(x|\underline{\lambda})$ is a two-parameter exponential distribution with the density function

$$f(x|\underset{\sim}{\lambda}) = \lambda exp[-\lambda(x-\mu)], \qquad x \geq \mu > 0, \tag{4.3.1}$$

where $\underset{\sim}{\lambda} = (\lambda, \mu)$, with unknown $\lambda$ and $\mu$. A conjugate prior distribution for $(\lambda, \mu)$ was suggested by Varde (1969). It is of a four-parameter distribution with the density function as

$$g(\lambda, \mu) = \frac{1}{A}\lambda^{\alpha-1} exp[-\lambda(\beta - \gamma\mu)], \qquad \lambda > 0, 0 < \mu \leq \eta, \tag{4.3.2}$$

where $\alpha > 0$, $\beta > 0$, $\alpha \leq \gamma < \beta/\eta$, $A = \frac{1}{\gamma}[ln\beta - ln(\beta - \gamma\eta)]$, for $\alpha = 1$ and $\frac{\Gamma(\alpha-1)}{\gamma}\left[\frac{1}{(\beta-\gamma\eta)^{\alpha-1}} - \frac{1}{\beta^{\alpha-1}}\right]$ otherwise. Assume that four parameters $\alpha$, $\beta$, $\gamma$ and $\eta$ are all known.

Therefore, it follows from (4.3.1) and (4.3.2) that

$$E\left[\lambda^{i_1}\mu^{i_2}F(t|\underset{\sim}{\lambda})^r(1-F(t|\underset{\sim}{\lambda}))^{n-r}\right]$$

$$= \int_0^{\eta_*}\int_0^{\infty} \tfrac{1}{A}\lambda^{\alpha-1+i_1}\mu^{i_2}e^{-\lambda(\beta-\gamma\mu)}(1-e^{-\lambda(t-\mu)})^r e^{-\lambda(n-r)(t-\mu)}d\lambda d\mu$$

$$= \sum_{j=0}^{r}(-1)^j\binom{r}{j}\tfrac{1}{A}\int_0^{\eta_*}\int_0^{\infty}\lambda^{\alpha+i_1-1}\mu^{i_2}e^{-\lambda[(\beta-\gamma\mu)+(n-r+j)(t-\mu)]}d\lambda d\mu$$

$$= \tfrac{\Gamma(\alpha+i_1)}{A}\sum_{j=0}^{r}(-1)^j\binom{r}{j}\int_0^{\eta_*}\tfrac{\mu^{i_2}}{[\beta+(n-r+j)t-(\gamma+(n-r+j))\mu]^{\alpha+i_1}}du, \tag{4.3.3}$$

$$0 \leq i_1 + i_2 \leq k,$$

where $\eta_* = \min\{\eta, t\}$. In particular, $E[\lambda^{i_1}\mu^{i_2}] = \frac{\Gamma(\alpha+i_1)}{A}\int_0^{\eta_*}\frac{\mu^{i_2}}{(\beta-\gamma\mu)^{\alpha+i_1}}du, 0 \leq i_1 + i_2 \leq k$.

Similarly,

$$E\left[F(t|\underset{\sim}{\lambda})\right] = \tfrac{\Gamma(\alpha)}{A}\left[\int_0^{\eta_*}\tfrac{d\mu}{[\beta-\gamma\mu]^{\alpha}} - \int_0^{\eta_*}\tfrac{d\mu}{[\beta+t-(\gamma+1)\mu]^{\alpha}}\right] = \tfrac{\Gamma(\alpha)}{A}B(\alpha), \tag{4.3.4}$$

where

$$B(\alpha) = \begin{cases} \frac{1}{\alpha-1}\left[\frac{1}{\gamma}\left(\frac{1}{(\beta-\gamma\eta_*)^{\alpha-1}} - \frac{1}{\beta^{\alpha-1}}\right) - \frac{1}{\gamma+1}\left(\frac{1}{(\beta+t-(\gamma+1)\eta_*)^{\alpha-1}} - \frac{1}{(\beta+t)^{\alpha-1}}\right)\right], & \alpha \neq 1, \\ \frac{1}{\gamma}(ln\beta - ln(\beta-\gamma\eta_*)) - \frac{1}{\gamma+1}(ln(\beta+t) - ln(\beta+t-(\gamma+1)\eta_*)), & \alpha = 1. \end{cases} \tag{4.3.5}$$

Consequently, from (4.1.9), (4.3.3),(4.3.4) and (4.3.5), the Bayes risk for the sampling plan $(n, t, p)$ is given by

$$R(n,t,p) = n(C_s - r_s) + n\frac{\Gamma(\alpha)}{A}B(\alpha)r_s + ta_t + C_r + \sum_{r=0}^{[n(1-p)]}\sum_{j=0}^{r}(-1)^j \begin{pmatrix} n \\ r \end{pmatrix}\begin{pmatrix} r \\ j \end{pmatrix}$$

$$\sum_{0\leq i_1+i_2\leq k} C^*_{i_1 i_2}\frac{\Gamma(\alpha+i_1)}{A}\int_0^{\eta_*}\frac{\mu^{i_2}}{[\beta+(n-r+j)t-(\gamma+(n-r+j))\mu]^{\alpha+i_1}}du, \tag{4.3.6}$$

From Theorem 4.1, we can get the following result

**Corollary 4.2** *For the case of the two−parameter exponential distribution, the optimal sampling size $n_0$ satisfies the following inequality*

$$n_0 \leq \min\left\{\frac{C_r}{C_s-r_s}, \frac{1}{C_s-r_s}\sum_{0\leq i_1+i_2\leq k} C_{i_1 i_2}\frac{\Gamma(\alpha+i_1)}{A}\int_0^{\eta_*}\frac{\mu^{i_2}}{(\beta-\gamma\mu)^{\alpha+i_1}}du, \frac{R(n,t_n,p_n)}{C_s-r_s}\right\}. \tag{4.3.7}$$

Then, we can develop a finite algorithm and a discretization method for an optimal sampling plan. However, $\mu$ is clearly the lower bound for the lifetime $X$, and the upper bound $T_U$ for the lifetime $X$ can be chosen so as $P(\mu \leq X \leq T_U) = 1-v$, where $v$ is a preassigned number satisfying $0 < v < 1$.

## 4.4 The case of the gamma distribution

Suppose that the lifetime $X$ of an item in a batch follows a gamma distribution, i.e., $F(x|\underset{\sim}{\lambda})$ is a Gamma distribution $\Gamma(m,\lambda)$ with the density function

$$f(x|\underset{\sim}{\lambda}) = \begin{cases} \frac{\lambda^m}{\Gamma(m)} x^{m-1} e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{4.4.1}$$

where $\underset{\sim}{\lambda} = (m, \lambda)$, with unknown $\lambda$ and $m$. Assume that $m$ has a discrete prior distribution such that $P(m = m_l) = p_l$, $(l = 1, \ldots, L)$, with known $p_l$ and $\sum_{l=1}^{L} p_l = 1$; the conditional prior distribution of $\lambda$ given $m_l$ is a gamma distribution $\Gamma(\alpha_l, \beta_l)$, where $\alpha_l > 0$ and $\beta_l > 0$ are known. Hence, the joint prior distribution $\underset{\sim}{\lambda} = (m, \lambda)$ is given by

$$\pi(\underset{\sim}{\lambda}) = P(m = m_l, \lambda \leq u) = p_l \int_0^u \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} v^{\alpha_l - 1} e^{-\beta_l v} dv, \quad l = 1, \ldots, L. \tag{4.4.2}$$

Consequently, we have

$$\begin{aligned} E\left[F(t|\underset{\sim}{\lambda})\right] &= \sum_{l=1}^{L} p_l \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} \int_0^\infty \lambda^{\alpha_l - 1} e^{-\lambda \beta_l} \left( \int_0^t \frac{\lambda^{m_l}}{\Gamma(m_l)} x^{m_l - 1} e^{-\lambda x} dx \right) d\lambda \\ &= \sum_{l=1}^{L} p_l \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} \int_0^t \frac{x^{m_l - 1}}{\Gamma(m_l)} \left( \int_0^\infty \lambda^{m_l + \alpha_l - 1} e^{-\lambda(\beta_l + x)} d\lambda \right) dx \\ &= \sum_{l=1}^{L} p_l \frac{\beta_l^{\alpha_l} \Gamma(m_l + \alpha_l)}{\Gamma(\alpha_l)\Gamma(m_l)} \int_0^t \frac{x^{m_l - 1}}{(\beta_l + x)^{m_l + \alpha_l}} dx = \sum_{l=1}^{L} p_l I_{s_l}(m_l, \alpha_l), \end{aligned} \tag{4.4.3}$$

where $s_l = \frac{t}{t + \beta_l}$, $I_s(a, b)$ is the incomplete Beta ratio and has been used in Chapter 2.

Similarly,

$$E\left[m^{i_1} \lambda^{i_2} F(t|\underset{\sim}{\lambda})^{r+j}\right] = \sum_{l=1}^{L} p_l m_l^{i_1} \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} \int_0^\infty \lambda^{i_2 + \alpha_l - 1} e^{-\lambda \beta_l} \left( \int_0^t \frac{\lambda^{m_l}}{\Gamma(m_l)} x^{m_l - 1} e^{-\lambda x} dx \right)^{r+j} d\lambda$$

$$= \sum_{l=1}^{L} p_l \frac{m_l^{i_1}\beta_l^{\alpha_l}}{\Gamma(\alpha_l)[\Gamma(m_l)]^{r+j}} \int_0^t \cdots \int_0^t t_1^{m_l-1}\cdots t_{r+j}^{m_l-1}$$

$$\left(\int_0^\infty \lambda^{(r+j)m_l+i_2+\alpha_l-1} e^{-\lambda(\beta_l+t_1+\cdots+t_{r+j})} d\lambda\right) dt_1\cdots dt_{r+j}$$

$$= \sum_{l=1}^{L} p_l \frac{m_l^{i_1}\beta_l^{\alpha_l}\Gamma((r+j)m_l+i_2+\alpha_l)}{\Gamma(\alpha_l)[\Gamma(m_l)]^{r+j}} \int_0^t \cdots \int_0^t \frac{t_1^{m_l-1}\cdots t_{r+j}^{m_l-1}}{(\beta_l+t_1+\cdots+t_{r+j})^{(r+j)m_l+i_2+\alpha_l}} dt_1\cdots dt_{r+j}$$

$$= \sum_{l=1}^{L} p_l \frac{m_l^{i_1}\Gamma(i_2+\alpha_l)}{\beta_l^{i_2}\Gamma(\alpha_l)} \left[\frac{\Gamma((r+j)m_l+i_2+\alpha_l)}{\Gamma(i_2+\alpha_l)[\Gamma(m_l)]^{r+j}} \int_0^{t/\beta_l} \cdots \int_0^{t/\beta_l} \frac{t_1^{m_l-1}\cdots t_{r+j}^{m_l-1}}{(1+t_1+\cdots+t_{r+j})^{(r+j)m_l+i_2+\alpha_l}} dt_1\cdots dt_{r+j}\right]$$

$$= \sum_{l=1}^{L} p_l \frac{m_l^{i_1}\Gamma(i_2+\alpha_l)}{\beta_l^{i_2}\Gamma(\alpha_l)} D_{r+j+1}(t^*,\cdots,t^*;m_l,\cdots,m_l;i_2+\alpha_l), \tag{4.4.4}$$

where $t^* = t/[\beta_l + (r+j)t]$ and $D_n(x_1,\cdots,x_{n-1};a_1,\cdots,a_{n-1};a_n)$ is the Dirichlet distribution. The proof of the last form of (4.4.4) is given in Appendix 2 .

In particular

$$E\left[m^{i_1}\lambda^{i_2}\right] = \sum_{l=1}^{L} p_l \frac{m_l^{i_1}\Gamma(i_2+\alpha_l)}{\beta_l^{i_2}\Gamma(\alpha_l)}. \tag{4.4.5}$$

Hence, from (4.1.9), (4.4.3) and (4.4.4), the Bayes risk for the sampling plan $(n,t,p)$ is given by

$$R(n,t,p) = n(C_s - r_s) + nr_s \sum_{l=1}^{L} p_l I_{s_l}(m_l,\alpha_l) + ta_t + C_r + \sum_{r=0}^{[n(1-p)]} \sum_{j=0}^{n-r} (-1)^j \binom{n}{r}$$

$$\binom{n-r}{j} \sum_{0\le i_1+i_2\le k} C^*_{i_1 i_2} \sum_{l=1}^{L} p_l \frac{m_l^{i_1}\Gamma(i_2+\alpha_l)}{\beta_l^{i_2}\Gamma(\alpha_l)} D_{r+j+1}(t^*,\cdots,t^*;m_l,\cdots,m_l;i_2+\alpha_l). \tag{4.4.6}$$

Using the results of Theorem 4.1 and (4.4.5), we obtain the following Corollary:

**Corollary 4.3** *For the case of the gamma distribution, the optimal* *sampling size $n_0$ satisfies the following inequality*

$$n_0 \leq \min\left\{\frac{C_r}{C_s-r_s}, \frac{1}{C_s-r_s}\sum_{0\leq i_1+i_2\leq k} C_{i_1i_2}\sum_{l=1}^{L} p_l\frac{m_l^{i_1}\Gamma(i_2+\alpha_l)}{\beta_l^{i_2}\Gamma(\alpha_l)}, \frac{R(n,t_n,p_n)}{C_s-r_s}\right\}. \qquad (4.4.7)$$

## 4.5 Numerical examples and sensitivity analysis

To illustrate the model and the algorithm for the determination of an optimal sampling plan and the minimum Bayes risk developed in this chapter, we assume that the degree $k$ of the loss function (4.1.4) is 2 and consider the Weibull distribution case. Then by (4.2.6), the Bayes risk of single sampling plan $(n,t,p)$ is given by

$$R(n,t,p) = nC_s - nr_s\sum_{l=1}^{L} p_l\frac{\beta_l^{\alpha_l}}{(t^{m_l}+\beta_l)^{\alpha_l}} + ta_t + C_r + \sum_{r=0}^{[n(1-p)-p]}\sum_{j=0}^{r}$$

$$(-1)^j\binom{n}{r}\binom{r}{j}\sum_{l=1}^{L} p_l\left\{I_{00}+I_{10}+I_{01}+I_{11}+I_{20}+I_{02}\right\}, \qquad (4.5.1)$$

where $I_{00} = (C_{00}-C_r)\frac{\beta_l^{\alpha_l}}{[(n-r+j)t^{m_l}+\beta_l]^{\alpha_l}}$, $I_{10} = C_{10}m_l\frac{\beta_l^{\alpha_l}}{[(n-r+j)t^{m_l}+\beta_l]^{\alpha_l}}$,

$I_{01} = C_{01}\frac{\alpha_l\beta_l^{\alpha_l}}{[(n-r+j)t^{m_l}+\beta_l]^{1+\alpha_l}}$, $I_{11} = C_{11}m_l\frac{\alpha_l\beta_l^{\alpha_l}}{[(n-r+j)t^{m_l}+\beta_l]^{1+\alpha_l}}$,

$I_{20} = C_{20}m_l^2\frac{\beta_l^{\alpha_l}}{[(n-r+j)t^{m_l}+\beta_l]^{\alpha_l}}$, and $I_{02} = C_{02}\frac{\alpha_l(\alpha_l+1)\beta_l^{\alpha_l}}{[(n-r+j)t^{m_l}+\beta_l]^{2+\alpha_l}}$,

In particular, when the shape parameter $m$ is known, i.e., $L=1, p_1=1, m_1=m, \alpha_1=\alpha$ and $\beta_1=\beta$, then the Bayes risk is given by

$$R(n,t,p) = nC_s - nr_s\frac{\beta^\alpha}{(t^m+\beta)^\alpha} + ta_t + C_r + \sum_{r=0}^{[n(1-p)-p]}\sum_{j=0}^{r}(-1)^j\binom{n}{r}\binom{r}{j}$$

$$\left\{(C_0-C_r)\frac{\beta^\alpha}{[(n-r+j)t^m+\beta]^\alpha} + C_1\frac{\alpha\beta^\alpha}{[(n-r+j)t^m+\beta]^{1+\alpha}} + C_2\frac{\alpha(\alpha+1)\beta^\alpha}{[(n-r+j)t^m+\beta]^{2+\alpha}}\right\}, \qquad (4.5.2)$$

where, $C_0 = C_{00} + C_{10}m + C_{20}m^2$, $C_1 = C_{01} + C_{11}$ and $C_2 = C_{02}$.

Now, some numerical examples are studied here. For the purpose of comparison, the standard values of the parameters and coefficients are taken as: $m_1 = 2.5$, $m_2 = 2.1$, $\alpha_1 = 2.3$, $\alpha_2 = 2.5$, $\beta_1 = 1$, $\beta_2 = 1.2$ $p_1 = p_2 = 0.5$, $C_{00} = 30$, $C_{10} = 20$, $C_{01} = 20$, $C_{11} = 20$, $C_{20} = 10$, $C_{02} = 10$, $C_r = 350$, $C_s = 0.5$, $r_s = 0.2$, $a_s = 25$. The numerical results are tabulated in Tables 4.1-4.15. In each table, only one parameter or one coefficient can change and the others are fixed. The values of the varying parameters or that of the varying coefficients are given in column 1. The optimal sampling plan $(n_0, t_0, p_0)$ and the corresponding minimum Bayes risks $R^*$ are given by columns 2-5. Note that the sampling plan $(0, 0, 0)$ represents the plan accepting a batch without sampling, and $(0, \infty, 1)$ denotes the plan rejecting a batch without sampling. In these examples, $T_U$ is chosen by (4.2.8) such that $P(0 \leq X \leq T_U) \geq 0.95$. Then, for each $n$, a sequence of the single sampling plan $(n, t, p)$ is studied in the following way: $t = iT_U/100$, $p = j/50$, $i = 0, 1, \ldots, 100$, $j = 0, 1, \ldots, 50$. By comparison, the minimum Bayes risk is obtained and the corresponding optimal single sampling plan is determined accordingly.

Sensitivity analysis is to study the behaviour of the optimal solution due to the changes in the parameter (coefficient) values. Because the parameters (coefficients) are difficult to estimate accurately in practice, we should investigate the effect of using some inaccurate parameters in the model. Similar to Chapter 2, the standard parameters (coefficients ) given above are taken as the true parameters (coefficients), and others are estimated parameters ( coefficients ). In Table 4.1-4.15, the true parameter (coefficient) and the corresponding minimum Bayes risk are also marked by'*'. By using an estimated sampling plan, the true (but not the minimum) Bayes risk $R$ and their efficiencies are given in columns 6-7 in each table respectively. For example, in Table 4.4, if $\beta_1$ and $\beta_2$ are inaccurately estimated as $\beta_1 = 1.5$ and $\beta_2 = 1.3$, the estimated Bayes risk is 284.6903 and the estimated sampling plan is $(16, 0.4208, 0.6400)$. On the other hand, by using the estimated plan, as the true $\beta_1 = 1.0$ and $\beta_2 = 1.2$, the true Bayes risk (but not the minimum) is 313.7863. Because the true minimum Bayes risk is 310.0326, the efficiency of estimated sampling plan is then equal to $R^*/R = 310.0326/313.7863 = 0.9880$. In fact, a number of examples show that efficiencies will not be less than 0.95 in most cases and therefore the model is not sensitive to parameters and coefficients.

Note here, as in Chapter 2, all the results in this chapter can be applied to the case for the exponential distribution with Type I censoring. In Table 4.15, we study the exponential distribution case by taking $m = 1$ and the lifetime $X$ is the exponential distribution $Exp(\lambda) = W(1, \lambda)$.

Table 4.1. *The minimum Bayes risk and optimal sampling plans as p varies*

| $(p_1, p_2)$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| (1.0,0.0) | 322.6250 | 21 | 0.4974 | 0.6800 | 313.7946 | 0.9880 |
| (0.8,0.2) | 318.3698 | 19 | 0.5356 | 0.5800 | 311.0961 | 0.9965 |
| (0.6,0.4) | 314.7640 | 18 | 0.5356 | 0.5600 | 310.2969 | 0.9991 |
| (0.5,0.5)* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| (0.4,0.6) | 306.7913 | 19 | 0.5356 | 0.5400 | 310.0891 | 0.9998 |
| (0.2,0.8) | 299.0952 | 18 | 0.4782 | 0.5600 | 312.0969 | 0.9934 |
| (0.0,1.0) | 289.8369 | 16 | 0.4463 | 0.5800 | 315.3425 | 0.9832 |

Table 4.2. *The minimum Bayes risk and optimal sampling plans as m varies*

| $(m_1, m_2)$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| (3.5,3.0) | 350.0000 | 0 | $\infty$ | 1.0000 | 350.0000 | 0.8858 |
| (3.0,2.5) | 336.7112 | 16 | 0.6312 | 0.5800 | 320.7728 | 0.9665 |
| (2.5,2.3) | 314.3263 | 20 | 0.4787 | 0.6600 | 311.0928 | 0.9966 |
| (2.5,2.1)* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| (2.3,2.1) | 301.4738 | 20 | 0.4591 | 0.6200 | 311.5651 | 0.9950 |
| (2.3,1.8) | 295.3520 | 15 | 0.5541 | 0.4200 | 314.4438 | 0.9859 |
| (2.0,1.0) | 263.9986 | 0 | 0.0000 | 0.0000 | 342.7153 | 0.9046 |

Table 4.3. *The minimum Bayes risk and optimal sampling plans as $\alpha$ varies*

| $(\alpha_1, \alpha_2)$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| (4.0,2.5) | 333.9076 | 17 | 0.6121 | 0.4800 | 311.1618 | 0.9964 |
| (3.5,3.0) | 341.1447 | 16 | 0.5930 | 0.5200 | 311.2568 | 0.9961 |
| (3.0,2.5) | 323.8579 | 18 | 0.5739 | 0.5200 | 310.2984 | 0.9991 |
| (2.3,2.5)* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| (2.3,2.3) | 304.4841 | 17 | 0.5356 | 0.5400 | 310.0326 | 0.0000 |
| (1.5,2.5) | 285.2094 | 16 | 0.4399 | 0.6400 | 311.8190 | 0.9943 |
| (2.0,1.5) | 272.4776 | 16 | 0.4973 | 0.5800 | 316.3689 | 0.9799 |

Table 4.4. *The minimum Bayes risk and optimal sampling plans as $\beta$ varies*

| $(\beta_1, \beta_2)$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| (2.0,2.5) | 217.0375 | 0 | 0.0000 | 0.0000 | 342.7153 | 0.9046 |
| (1.5,1.3) | 284.6903 | 16 | 0.4208 | 0.6400 | 313.7863 | 0.9880 |
| (1.0,1.5) | 298.2915 | 18 | 0.5547 | 0.5200 | 310.0573 | 0.9999 |
| (1.0,1.2)* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| (1.0,1.0) | 318.6458 | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| (0.8,0.5) | 346.3904 | 16 | 0.4782 | 0.7000 | 314.0967 | 0.9871 |
| (0.5,0.8) | 345.9134 | 17 | 0.5356 | 0.6000 | 311.6655 | 0.9947 |

Table 4.5. *The minimum Bayes risk and optimal sampling plans as $C_{00}$ varies*

| $C_{00}$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 0.0 | 290.0937 | 18 | 0.5547 | 0.5000 | 311.8020 | 0.9943 |
| 15.0 | 300.3197 | 18 | 0.5356 | 0.5200 | 310.5468 | 0.9983 |
| 25.0 | 306.8510 | 18 | 0.5547 | 0.5200 | 310.0573 | 0.9999 |
| 30.0* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 35.0 | 313.2008 | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 50.0 | 322.1665 | 16 | 0.5356 | 0.5800 | 310.6179 | 0.9981 |
| 55.0 | 325.0534 | 16 | 0.5258 | 0.6000 | 3100.2981 | 0.9991 |

Table 4.6. *The minimum Bayes risk and optimal sampling plans as $C_{10}$ varies*

| $C_{10}$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| -5.0 | 278.0482 | 16 | 0.4973 | 0.5000 | 319.6594 | 0.9702 |
| 4.5 | 293.8881 | 18 | 0.5165 | 0.5200 | 311.8200 | 0.9942 |
| 14.5 | 304.7700 | 16 | 0.5356 | 0.5200 | 310.4678 | 0.9985 |
| 20.0* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 24.5 | 314.0683 | 18 | 0.5356 | 0.5600 | 310.2969 | 0.9991 |
| 34.5 | 322.0007 | 18 | 0.5547 | 0.5600 | 311.2303 | 0.9961 |
| 44.5 | 328.8021 | 17 | 0.5547 | 0.6000 | 313.3548 | 0.9894 |

Table 4.7. *The minimum Bayes risk and optimal sampling plans as $C_{01}$ varies*

| $C_{01}$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| -5.0 | 277.6198 | 18 | 0.5165 | 0.5200 | 323.4521 | 0.9500 |
| 4.5 | 285.9493 | 17 | 0.5356 | 0.4800 | 311.7397 | 0.9951 |
| 14.5 | 301.8804 | 18 | 0.5356 | 0.5200 | 310.5468 | 0.9983 |
| 20.0* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 24.5 | 316.4940 | 18 | 0.5356 | 0.5600 | 310.2969 | 0.9991 |
| 34.5 | 329.7287 | 17 | 0.5356 | 0.6000 | 311.6655 | 0.9947 |
| 44.5 | 341.3615 | 16 | 0.5547 | 0.6400 | 316.9126 | 0.9783 |

Table 4.8. *The minimum Bayes risk and optimal sampling plans as $C_{11}$ varies*

| $C_{11}$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| -5.0 | 238.3403 | 0 | 0 | 0 | 341.7153 | 0.9046 |
| 9.5 | 282.7569 | 16 | 0.4782 | 0.5200 | 316.2539 | 0.9803 |
| 14.5 | 296.9911 | 16 | 0.5164 | 0.5200 | 311.1626 | 0.9963 |
| 20.0* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 24.5 | 319.0164 | 18 | 0.5547 | 0.5600 | 311.2303 | 1.9961 |
| 34.5 | 334.1999 | 17 | 0.5739 | 0.6000 | 315.5455 | 0.9825 |
| 44.5 | 344.7326 | 17 | 0.5739 | 0.6600 | 322.4833 | 0.9614 |

Table 4.9. *The minimum Bayes risk and optimal sampling plans as $C_{20}$ varies*

| $C_{20}$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| -0.5 | 270.9667 | 0 | 0.0000 | 0.0000 | 341.7153 | 0.9046 |
| 3.5 | 295.6674 | 15 | 0.4782 | 0.5400 | 313.2685 | 0.9896 |
| 7.5 | 305.3064 | 17 | 0.5164 | 0.5400 | 310.4839 | 0.9985 |
| 10.0* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 15.5 | 318.2312 | 19 | 0.5738 | 0.5400 | 311.1281 | 0.9965 |
| 27.5 | 330.2206 | 19 | 0.5930 | 0.5800 | 317.1108 | 0.9976 |
| 31.5 | 333.1867 | 20 | 0.5930 | 0.6200 | 319.5628 | 0.9702 |

Table 4.10. *The minimum Bayes risk and optimal sampling plans as $C_{02}$ varies*

| $C_{02}$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| -0.50 | 270.8906 | 18 | 0.5156 | 0.4600 | 326.3271 | 0.9501 |
| 3.5 | 286.5717 | 17 | 0.5356 | 0.4800 | 311.7397 | 0.9945 |
| 7.5 | 301.3752 | 18 | 0.5356 | 0.5200 | 310.5468 | 0.9983 |
| 10.0* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 15.5 | 327.6459 | 16 | 0.5547 | 0.5800 | 311.7295 | 0.9946 |
| 23.5 | 348.4094 | 15 | 0.5356 | 0.6800 | 319.2635 | 0.9711 |
| 27.5 | 350.0000 | 0 | $\infty$ | 1.0000 | 350.0000 | 0.8858 |

Table 4.11. *The minimum Bayes risk and optimal sampling plans as $C_s$ varies*

| $C_s$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 0.2 | 303.4961 | 25 | 0.3164 | 0.5800 | 349.6402 | 0.8867 |
| 0.3 | 305.7961 | 23 | 0.5164 | 0.5800 | 310.3963 | 0.9988 |
| 0.4 | 308.1174 | 22 | 0.4974 | 0.6000 | 310.3174 | 0.9991 |
| 0.5* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 0.6 | 311.6329 | 14 | 0.5547 | 0.5200 | 310.2328 | 0.9993 |
| 0.7 | 312.9957 | 13 | 0.5739 | 0.4800 | 310.3956 | 0.9989 |
| 10.0 | 316.2846 | 10 | 0.6121 | 0.4200 | 311.2846 | 0.9959 |

Table 4.12. *The minimum Bayes risk and optimal sampling plans as $r_s$ varies*

| $r_s$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 0.0 | 311.8895 | 15 | 0.5739 | 0.4800 | 310.1483 | 0.9996 |
| 0.1 | 311.0168 | 16 | 0.5547 | 0.5200 | 310.0537 | 0.9998 |
| 0.15 | 310.5352 | 16 | 0.5547 | 0.5200 | 310.0537 | 0.9999 |
| 0.2* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 0.25 | 319.4935 | 18 | 0.5165 | 0.5600 | 310.0746 | 0.9999 |
| 0.35 | 308.1145 | 22 | 0.4974 | 0.6000 | 310.3174 | 0.9990 |
| 0.45 | 306.5767 | 23 | 0.4591 | 0.6600 | 310.6648 | 0.9979 |

Table 4.13. *The minimum Bayes risk and optimal sampling plans as $a_t$ varies*

| $a_t$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 0.0 | 292.7102 | 14 | 0.8608 | 0.1600 | 314.2302 | 0.9866 |
| 10.0 | 300.7285 | 14 | 0.7269 | 0.3000 | 311.6319 | 0.9948 |
| 20.0 | 307.2239 | 17 | 0.5739 | 0.4800 | 310.0933 | 0.9998 |
| 25.0* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0326 | 1.0000 |
| 30.0 | 312.6571 | 18 | 0.5165 | 0.5600 | 310.0623 | 0.9999 |
| 50.0 | 321.5254 | 21 | 0.4017 | 0.7200 | 311.4831 | 0.9953 |
| 70.0 | 328.6426 | 24 | 0.3060 | 0.8400 | 314.8727 | 0.9846 |

Table 4.14. *The minimum Bayes risk and optimal sampling plans as $C_r$ varies*

| $C_r$ | $R(n_0, t_0, p_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| 200.0 | 200.0000 | 0 | $\infty$ | 1.0000 | 350.0000 | 0.8858 |
| 250.0 | 250.0326 | 0 | $\infty$ | 1.0000 | 350.0000 | 0.8858 |
| 300.0 | 288.1618 | 16 | 0.5356 | 0.6400 | 314.5328 | 0.9857 |
| 350.0* | 310.0326* | 17 | 0.5356 | 0.5400 | 310.0325 | 1.0000 |
| 400.0 | 325.7059 | 19 | 0.5356 | 0.4800 | 311.8474 | 0.9942 |
| 450.0 | 336.6851 | 18 | 0.4973 | 0.4600 | 319.5995 | 0.9701 |
| 500.0 | 342.7153 | 0 | 0.0000 | 0.0000 | 342.7153 | 0.9046 |

Table 4.15. *The minimum Bayes risk and optimal sampling plans for the exponential distribution ( m=1) as $\alpha$ and $\beta$ varies*

| $(\alpha, \beta)$ | $R(n_0, r_0, T_0)$ | $n_0$ | $t_0$ | $p_0$ | True risk $R$ | Efficiency of $R$ |
|---|---|---|---|---|---|---|
| (3.5,1.2) | 263.1958 | 22 | 0.1642 | 0.5200 | 228.9742 | 0.9563 |
| (3.0,1.3) | 221.9364 | 17 | 0.1249 | 0.5400 | 228.9325 | 0.9564 |
| (2.5,1.1) | 218.0761 | 19 | 0.1344 | 0.5400 | 228.3340 | 0.9589 |
| (2.3,1.0)* | 218.9572* | 19 | 0.1383 | 0.5400 | 218.9572 | 1.0000 |
| (2.0,0.8) | 228.5112 | 20 | 0.1588 | 0.5200 | 229.8013 | 0.9528 |
| (1.5,0.5) | 243.2496 | 20 | 0.1713 | 0.5200 | 229.8013 | 0.9528 |
| (1.5,0.3) | 293.1258 | 21 | 0.1759 | 0.5400 | 230.7762 | 0.9488 |

# CHAPTER 5.

## Discussion

Up to now, we have mainly studied the Bayesian sampling plans for the Weibull distribution with Type I and Type II censoring. The finite algorithm and discretization method for finding an approximate optimal sampling plan was proposed respectively. From a number of numerical examples, it had been seen that our models and algorithms are efficient and applicable. The sensitivity analyses also show that the efficiencies of the estimated sampling plan in most cases are greater than 0.95, even if the error of an estimated parameter or of a coefficient is over 100%. Therefore, the Bayesian sampling plans are not sensitive to the parameters and the coefficients.

In particular, it should be pointed out that as the exponential distribution is a special Weibull distribution, the models and the methodology developed in this thesis can be applied to the exponential distribution case.

In the final chapter, we shall further discuss the relationship between various sampling plans for the Weibull distribution with the censoring. First of all, the Bayesian sampling plan is compared with the OC curve sampling plan. We explain why our sampling plans can save some resources. Then, the comparison between single and double sampling plan is also made. As noted in Chapter 3, the double sampling plan is a generalization of single sampling plan. If we want to inspect a batch which contains a large number of expensive items so that the Bayes risk is very high, a double sampling plan might be appropriate. Furthermore, we point out that the model with Type I censoring developed in Chapter 4 is a general model and also is a simple model in variable sampling plan. Finally, some methods for estimating the parameters and coefficients are also suggested.

## 5.1 Comparison between the Bayesian variable sampling plans and OC curve sampling plans

In this section, we only make a comparison between the Bayesian single variable sampling plan with Type II censoring and the operating characteristic (OC) curve single sampling plan with Type II censoring. To study the OC curve single sampling plan with Type II censoring, we should start with two points on the OC curve: the producer's risk point $(p_1, 1-a)$ and the consumer's risk point $(p_2, b)$. Given a single sampling plan $(n, r, T)$, the percentage of defectives is given by $p = P(X < T) = 1 - exp(-\lambda T^m)$. Therefore, $p_i = 1 - exp(-\lambda_i T^m)$ and

$$\lambda_i = -T^{-m} ln(1 - p_i), \qquad i = 1, 2. \tag{5.1.1}$$

On the other hand, from (5.1.1), for each $i = 1, 2$, the MLE of $\hat{\theta_{ri}}$ of $\theta_i = 1/\lambda_i$ has a gamma distribution $\Gamma(r, r\lambda_i)$. By (2.1.3), the MLE of $E(X|\lambda_i)$ is equal to $\hat{\theta_{ri}}^{1/m}\Gamma(1 + \frac{1}{m})$. Thus, we have the following equations

$$\begin{cases} P(\hat{\theta_{r1}} \geq [T/\Gamma(1 + \frac{1}{m})]^m) = 1 - a, \\ \\ P(\hat{\theta_{r2}} \geq [T/\Gamma(1 + \frac{1}{m})]^m) = b. \end{cases} \tag{5.1.2}$$

To determine an OC curve single sampling plan $(n_*, r_*, T_*)$, we can substitute (5.1.1) into (5.1.2), and then solve (5.1.2) for $r_*$ and $T_*$. Because the distribution of $\hat{\theta_{ri}}$ is independent of $n$. We cannot determine $n_*$ by this OC curve approach. To overcome this obstacle, suppose a time limit $t_*$ which is specified by the contract or the progress schedule is given. We can then determine $n_*$ so that the expected testing time $E(X_{(r)})$ is not longer than $t_*$. In fact, it follows from (2.1.9) that

$$\frac{\beta^{1/m}\Gamma(\alpha-\frac{1}{m})}{\Gamma(\alpha)}C(n, r_*) \leq t_*. \tag{5.1.3}$$

In practice, the smallest solution of (5.1.3) should be preferred. Otherwise, if there is no time limit or the items are unduly expensive, we may take $n_* = r_*$.

Clearly, the OC curve sampling plan $(n_*, r_*, T_*)$ will have a higher Bayes risk than the optimal sampling plan $(n_0, r_0, T_0)$ has. This means that a higher expenses will normally be involved in the OC curve sampling plan. Furthermore, we can see from Tables 2.1-2.10 that the sizes $n_0$ of the optimal sampling plans $(n_0, r_0, T_0)$ are quite small. This means that using an optimal Bayesian sampling plan will save some resources such as manpower, power, time, etc.

## 5.2 Comparison between single and double sampling plans

The purpose of introducing a double sampling plan is to reduce the Bayes risk and we can then implement the sampling inspection more economically. Therefore, we should compare the optimal double sampling plan with the corresponding optimal single sampling plan.

To start with, we note that any single sampling plan $(n, r, T)$ can be regarded as a double sampling plan $(n, r, 0, 0, T, T, T)$. Hence, for an optimal single sampling plan $(n_0, r_0, T_0)$ and an optimal double sampling plan $(n_{10}, r_{10}, n_{20}, r_{20}, T_{00}, T_{10}, T_{20})$, we always have

$$R(n_0, r_0, T_0) \geq R(n_{10}, r_{10}, n_{20}, r_{20}, T_{00}, T_{10}, T_{20}).$$

Consequently, an optimal double sampling plan is always better than an optimal single sampling plan.

The efficiency of a sampling plan is defined as the ratio of the minimum Bayes risk to the Bayes risk of the sampling plan (see Hald (1981)). The efficiency of an optimal single sampling plan can be defined as the ratio of the Bayes risk $R_2^*$ of an optimal double sampling plan to the Bayes risk $R_1^*$ of the optimal single sampling plan. This is a measure for the comparison between single and double sampling plans. In Tables 5.2.1-5.2.10, the minimum Bayes risks $R_1^*$ and $R_2^*$ are tabulated in column 3, and the efficiencies of $R_1^*$ are recorded in column 11 respectively for demonstration. We can see from Tables 5.2.1-5.2.10 that, in most cases, the efficiencies are about 98%. An optimal double sampling plan will reduce the Bayes risk by 2%. In practice, if we want to inspect a batch which contains a large number of expensive items so that the Bayes risk is very high, an optimal double sampling plan should be suggested, otherwise, an optimal single sampling plan might be applied satisfactorily.

## 5.3 Comparison of both the models

Assume that the quality of an item is measured by a random variable with a Weibull distribution. Here, we shall compare the model for the Weibull distribution with Type II censoring developed in Chapter 2 ( the model 2) with the model for Type I proposed in Chapter 4 (the model 1 ). To this end, we assume that the degree $k$ in the loss function is 2 and consider the case that the shape parameter in the Weibull distribution is known. In Tables 5.3.1-5.3.10, we study the optimal sampling plans with the model 1 and the model 2 and corresponding minimum Bayes risks. We can see that the minimum Bayes risks of sampling plans with Type I censoring are almost always smaller than those with Type II censoring . From Tables 5.3.1-5.3.10, we can also see that in most cases, the efficiencies are about 97%. It seems that the model 1 is better than the model 2. It is more important that the model 1 is suitable to any lifetime distribution. In practice, if the decision maker cannot decide to use model 1 or the model 2, such a comparison of minimum Bayes risks may be helpful for making a decision. Especially, when the distribution of the lifetime is not the exponential distribution, the model 1 will be a power tool for the sampling inspection.

## 5.4 Choice of parameters and coefficients

To apply our model to a practical problem, the first thing we need to do is to choose the parameters, including the parameters $\alpha$ and $\beta$ of prior distribution $\Gamma(\alpha, \beta)$ and to determine the coefficients $C_0, \ldots, C_k, C_s, r_s, a_t$ and $C_r$.

For choosing the values $\alpha, \beta$ of the prior distribution $\Gamma(\alpha, \beta)$, suppose that we have already observed I batches. Assume that the distribution of the lifetime $X$ in these batches has the Weibull distribution $W(m, \lambda)$ with common shape parameter $m$ but different scale parameters $\lambda_1, \ldots, \lambda_I$, then under the assumption $\lambda \sim \Gamma(\alpha, \beta)$, we can estimate $\alpha$ and $\beta$ from the data $\lambda_1, \ldots, \lambda_I$.

On the other hand, the values of the coefficients $C_0, \ldots, C_k$ ( or $C_{i_1 \ldots i_m}$ )can be obtained by omitting the remainder of the Taylor expansion in (1.2.4) or (1.2.5) ( or (4.1.6)) but we should make sure that (1.2.3)(or (4.1.5)) holds. However, the degree $k$ should be determined by the preassigned precision. The sampling cost per item $C_s$ includes the normal price of an item and the inspection cost per item, while $r_s$, the salvage value of an unfailed item after testing, is the reduced price of the item. Hence $C_s$ and $r_s$ can be determined easily. With regard to $a_t$, it is the time-consuming cost which contains the labour wages, energy charge, etc. Thus $a_t$ can be evaluated from the practical situation accordingly. Finally, $C_r$ is the cost due to rejecting the batch. It might include the total cost of the batch if the batch is scrapped, or the the sampling cost and the inspection cost of the batch if the rejected batch is then sorted through complete test. It may also include the loss of a deposit as the security and loss of goodwill. However, the loss of goodwill is not easy to estimate precisely.

Table 5.2.1. *Comparisons of the minimum Bayes risk and optimal sampling plans for single(S) and double(D) sampling plan as m varies*

| $m$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | S | 42.74 | 6 | 4 | | | 0.375 | | | 0.988 |
| | D | 42.25 | 4 | 3 | 4 | 3 | 0.599 | 0.150 | 0.346 | |
| 1.0 | S | 42.75 | 6 | 4 | | | 0.430 | | | 0.989 |
| | D | 42.26 | 4 | 3 | 4 | 3 | 0.571 | 0.291 | 0.431 | |
| 2.0 | S | 42.76 | 6 | 4 | | | 0.599 | | | 0.995 |
| | D | 42.27 | 4 | 3 | 6 | 4 | 0.655 | 0.458 | 0.571 | |
| 2.5* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.25* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.655 | |
| 3.0 | S | 42.75 | 6 | 4 | | | 0.683 | | | 0.989 |
| | D | 42.28 | 4 | 3 | 6 | 4 | 0.739 | 0.599 | 0.655 | |
| 3.5 | S | 42.74 | 6 | 4 | | | 0.711 | | | 0.989 |
| | D | 42.28 | 4 | 3 | 4 | 3 | 0.767 | 0.627 | 0.711 | |
| 4.0 | S | 42.76 | 5 | 4 | | | 0.739 | | | 0.989 |
| | D | 42.30 | 4 | 3 | 4 | 3 | 0.766 | 0.655 | 0.739 | |

Table 5.2.2. *Comparisons of the minimum Bayes risk and optimal sampling plans for single and double sampling plan as α varies*

| $\alpha$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.5 | S | 31.56 | 6 | 3 | | | 0.551 | | | 0.977 |
| | D | 30.85 | 6 | 2 | 6 | 2 | 0.551 | 0.373 | 0.551 | |
| 2.0 | S | 37.73 | 6 | 4 | | | 0.588 | | | 0.982 |
| | D | 37.05 | 4 | 2 | 6 | 4 | 0.677 | 0.455 | 0.588 | |
| 2.2 | S | 39.82 | 6 | 4 | | | 0.626 | | | 0.987 |
| | D | 39.30 | 5 | 3 | 6 | 4 | 0.698 | 0.517 | 0.626 | |
| 2.5* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.26* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 2.7 | S | 44.43 | 5 | 4 | | | 0.640 | | | 0.989 |
| | D | 43.96 | 4 | 3 | 4 | 3 | 0.713 | 0.543 | 0.640 | |
| 3.0 | S | 47.68 | 6 | 5 | | | 0.591 | | | 1.000 |
| | D | 47.68 | 6 | 5 | 0 | 0 | 0.591 | 0.591 | 0.591 | |
| 3.5 | S | 50.00 | 0 | 0 | | | $\infty$ | | | 1.000 |
| | D | 50.00 | 0 | 0 | 0 | 0 | $\infty$ | $\infty$ | $\infty$ | |

Table 5.2.3. *Comparisons of the minimum Bayes risk and optimal sampling plans for single and double sampling plan as $\beta$ varies*

| $\beta$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.6 | S | 50.00 | 0 | 0 | | | $\infty$ | | | 1.000 |
| | D | 50.00 | 0 | 0 | 0 | 0 | $\infty$ | $\infty$ | $\infty$ | |
| 0.8 | S | 46.00 | 4 | 3 | | | 0.658 | | | 0.996 |
| | D | 45.84 | 5 | 4 | 4 | 3 | 0.658 | 0.591 | 0.636 | |
| 0.9 | S | 44.43 | 5 | 4 | | | 0.626 | | | 0.989 |
| | D | 43.95 | 4 | 3 | 4 | 3 | 0.715 | 0.564 | 0.640 | |
| 1.0* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.26* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 1.1 | S | 40.99 | 6 | 4 | | | 0.628 | | | 0.988 |
| | D | 40.53 | 4 | 3 | 6 | 4 | 0.689 | 0.504 | 0.628 | |
| 1.4 | S | 35.69 | 5 | 3 | | | 0.564 | | | 0.981 |
| | D | 35.02 | 4 | 2 | 6 | 5 | 0.603 | 0.367 | 0.603 | |
| 2.0 | S | 50.00 | 0 | 0 | | | 0.000 | | | 1.000 |
| | D | 50.00 | 0 | 0 | 0 | 0 | 0.000 | 0.000 | 0.000 | |

Table 5.2.4. *Comparisons of the minimum Bayes risk and optimal sampling plans for single and double sampling plan as $C_0$ varies*

| $C_0$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | S | 40.75 | 5 | 4 | | | 0.627 | | | 0.986 |
| | D | 40.18 | 4 | 3 | 6 | 4 | 0.683 | 0.517 | 0.599 | |
| 2.5 | S | 41.51 | 6 | 4 | | | 0.627 | | | 0.987 |
| | D | 40.99 | 4 | 3 | 6 | 4 | 0.683 | 0.515 | 0.627 | |
| 4.0 | S | 42.26 | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 41.76 | 4 | 3 | 4 | 3 | 0.711 | 0.515 | 0.627 | |
| 5.0* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.26* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 5.5 | S | 43.01 | 6 | 4 | | | 0.655 | | | 0.988 |
| | D | 42.50 | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 7.0 | S | 43.69 | 6 | 4 | | | 0.655 | | | 0.989 |
| | D | 43.24 | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.543 | |
| 10.0 | S | 45.02 | 4 | 3 | | | 0.683 | | | 0.991 |
| | D | 44.60 | 4 | 3 | 5 | 3 | 0.739 | 0.571 | 0.655 | |

Table 5.2.5. *Comparisons of the minimum Bayes risk and optimal sampling plans for single and double sampling plan as $C_1$ varies*

| $C_1$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| -5.0 | S | 32.22 | 5 | 4 | | | 0.514 | | | 0.970 |
| | D | 31.25 | 5 | 3 | 6 | 5 | 0.571 | 0.403 | 0.515 | |
| -1.0 | S | 37.30 | 6 | 4 | | | 0.572 | | | 0.979 |
| | D | 36.54 | 4 | 3 | 6 | 5 | 0.627 | 0.431 | 0.571 | |
| 4.0 | S | 41.98 | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 41.49 | 4 | 3 | 6 | 4 | 0.683 | 0.515 | 0.627 | |
| 5.0* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.26* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 5.5 | S | 43.12 | 6 | 4 | | | 0.655 | | | 0.989 |
| | D | 42.26 | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 7.0 | S | 44.12 | 6 | 4 | | | 0.655 | | | 0.989 |
| | D | 43.65 | 4 | 3 | 5 | 3 | 0.739 | 0.571 | 0.655 | |
| 0.0 | S | 45.75 | 4 | 3 | | | 0.711 | | | 0.991 |
| | D | 45.36 | 4 | 3 | 5 | 3 | 0.767 | 0.627 | 0.683 | |

Table 5.2.6. *Comparisons of the minimum Bayes risk and optimal sampling plans for single and double sampling plan as $C_2$ varies*

| $C_2$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | S | 26.25 | 0 | 0 | | | 0.000 | | | 1.000 |
| | D | 26.25 | 0 | 0 | 0 | 0 | 0.000 | 0.000 | 0.000 | |
| 2.5 | S | 37.15 | 6 | 4 | | | 0.515 | | | 0.983 |
| | D | 36.54 | 3 | 2 | 6 | 5 | 0.571 | 0.346 | 0.548 | |
| 4.5 | S | 41.97 | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 41.49 | 4 | 3 | 6 | 4 | 0.683 | 0.515 | 0.599 | |
| 5.0* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.26* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 5.5 | S | 43.42 | 6 | 4 | | | 0.655 | | | 0.989 |
| | D | 42.97 | 4 | 3 | 4 | 3 | 0.711 | 0.599 | 0.655 | |
| 7.0 | S | 45.02 | 5 | 4 | | | 0.711 | | | 0.989 |
| | D | 44.53 | 4 | 3 | 5 | 3 | 0.767 | 0.599 | 0.683 | |
| 10.0 | S | 46.94 | 4 | 3 | | | 0.795 | | | 0.991 |
| | D | 46.53 | 4 | 3 | 5 | 3 | 0.823 | 0.683 | 0.739 | |

Table 5.2.7. *Comparisons of the minimum Bayes risk and optimal sampling plans for single and double sampling plan as $C_s$ varies*

| $C_s$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.21 | S | 41.09 | 8 | 4 | | | 0.627 | | | 0.976 |
| | D | 40.15 | 6 | 3 | 6 | 4 | 0.719 | 0.515 | 0.627 | |
| 0.3 | S | 41.57 | 7 | 4 | | | 0.627 | | | 0.983 |
| | D | 40.87 | 6 | 4 | 6 | 4 | 0.711 | 0.543 | 0.627 | |
| 0.4 | S | 42.17 | 6 | 4 | | | 0.627 | | | 0.987 |
| | D | 41.62 | 5 | 3 | 6 | 4 | 0.711 | 0.515 | 0.627 | |
| 0.5* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.26* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 0.6 | S | 43.28 | 4 | 3 | | | 0.655 | | | 0.988 |
| | D | 42.78 | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 0.7 | S | 43.68 | 4 | 3 | | | 0.655 | | | 0.990 |
| | D | 43.25 | 2 | 2 | 4 | 3 | 0.739 | 0.515 | 0.627 | |
| 1.0 | S | 44.67 | 2 | 2 | | | 0.655 | | | 0.988 |
| | D | 44.15 | 2 | 2 | 2 | 2 | 0.711 | 0.543 | 0.627 | |

Table 5.2.8. *Comparisons of the minimum Bayes risk and optimal sampling plans for single and double sampling plan as $r_s$ varies*

| $r_s$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | S | 42.98 | 5 | 4 | | | 0.627 | | | 0.989 |
| | D | 42.52 | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 1.0 | S | 42.89 | 5 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.39 | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 1.5 | S | 42.84 | 5 | 4 | | | 0.627 | | | 0.987 |
| | D | 42.32 | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 0.2* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.26* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 0.25 | S | 42.67 | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.18 | 4 | 3 | 5 | 3 | 0.711 | 0.543 | 0.627 | |
| 0.3 | S | 42.57 | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.06 | 5 | 3 | 5 | 3 | 0.711 | 0.543 | 0.627 | |
| 0.4 | S | 42.46 | 6 | 3 | | | 0.653 | | | 0.983 |
| | D | 42.52 | 6 | 3 | 6 | 3 | 0.711 | 0.543 | 0.627 | |

Table 5.2.9. *Comparisons of the minimum Bayes risk and optimal sampling plans for single and double sampling plan as $a_s$ varies*

| $a_s$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | S | 41.14 | 5 | 5 | | | 0.627 | | | 0.986 |
| | D | 40.56 | 3 | 3 | 5 | 5 | 0.711 | 0.487 | 0.627 | |
| 1.0 | S | 42.16 | 6 | 5 | | | 0.628 | | | 0.987 |
| | D | 41.63 | 4 | 3 | 5 | 3 | 0.711 | 0.515 | 0.627 | |
| 1.5 | S | 42.48 | 5 | 4 | | | 0.627 | | | 0.987 |
| | D | 41.95 | 4 | 3 | 4 | 3 | 0.711 | 0.515 | 0.627 | |
| 2.0* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.26* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 2.5 | S | 43.00 | 6 | 4 | | | 0.627 | | | 0.989 |
| | D | 42.53 | 5 | 3 | 5 | 3 | 0.711 | 0.543 | 0.627 | |
| 3.0 | S | 43.23 | 6 | 4 | | | 0.655 | | | 0.989 |
| | D | 42.74 | 5 | 3 | 5 | 3 | 0.711 | 0.543 | 0.627 | |
| 5.0 | S | 43.89 | 6 | 3 | | | 0.655 | | | 0.991 |
| | D | 43.52 | 4 | 2 | 6 | 3 | 0.739 | 0.515 | 0.627 | |

Table 5.2.10. *Comparisons of the minimum Bayes risk and optimal sampling plans for single and double sampling plan as $C_r$ varies*

| $C_r$ | Sampling Plan | $R^*$ | $n_1$ | $r_1$ | $n_2$ | $r_2$ | $T_0$ | $T_1$ | $T_2$ | Eff of $R_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10.0 | S | 10.00 | 0 | 0 | | | $\infty$ | | | 1.000 |
| | D | 10.00 | 0 | 0 | 0 | 0 | $\infty$ | $\infty$ | $\infty$ | |
| 30.0 | S | 29.59 | 3 | 2 | | | 0.823 | | | 0.999 |
| | D | 29.56 | 3 | 2 | 3 | 1 | 0.823 | 0.767 | 0.795 | |
| 40.0 | S | 36.91 | 4 | 3 | | | 0.711 | | | 0.992 |
| | D | 36.61 | 3 | 2 | 5 | 3 | 0.711 | 0.543 | 0.627 | |
| 50.0* | S | 42.77* | 6 | 4 | | | 0.627 | | | 0.988 |
| | D | 42.26* | 4 | 3 | 4 | 3 | 0.711 | 0.543 | 0.627 | |
| 60.0 | S | 47.46 | 6 | 4 | | | 0.599 | | | 0.985 |
| | D | 46.76 | 4 | 3 | 6 | 5 | 0.655 | 0.459 | 0.599 | |
| 70.0 | S | 51.28 | 6 | 5 | | | 0.543 | | | 0.981 |
| | D | 50.33 | 4 | 3 | 6 | 5 | 0.627 | 0.431 | 0.543 | |
| 100.0 | S | 58.86 | 6 | 5 | | | 0.487 | | | 0.974 |
| | D | 57.32 | 6 | 4 | 6 | 6 | 0.543 | 0.374 | 0.487 | |

Table 5.3.1 Comparisons of minimum Bayes risk for Type I and II censoring as $\alpha$ varies ($\beta = 1, m = 2.5, C_0 = C_1 = C_2 = 10, C_r = 100, C_s = 0.5, r_s = 0.2, a_t = 10$)

| $\alpha$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| 1.5 | 1 | 60.5104 | 10 | - | 0.5259 | - | 0.5200 |
| | 2 | 62.3152 | 10 | 4 | - | 0.6268 | - |
| 2.1 | 1 | 77.7094 | 11 | - | 0.5896 | - | 0.4600 |
| | 2 | 79.0724 | 11 | 5 | - | 0.6097 | - |
| 2.5 | 1 | 87.1153 | 11 | - | 0.6057 | - | 0.4600 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | |
| 2.7 | 1 | 91.0878 | 12 | - | 0.5896 | - | 0.5200 |
| | 2 | 92.0575 | 11 | 5 | - | 0.6402 | |
| 3.1 | 1 | 97.6807 | 10 | - | 0.6057 | - | 0.5200 |
| | 2 | 98.3276 | 11 | 5 | - | 0.6538 | - |

Table 5.3.2 Comparisons of minimum Bayes risk for Type I and II censoring as $\beta$ varies ($\alpha = 2.5, m = 2.5, C_0 = C_1 = C_2 = 10, C_r = 100, C_s = 0.5, r_s = 0.2, a_t = 10$)

| $\beta$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 1 | 100 | 0 | - | $\infty$ | - | 1.0000 |
| | 2 | 100 | 0 | 0 | - | $\infty$ | - |
| 0.7 | 1 | 98.6729 | 11 | - | 0.5898 | - | 0.5600 |
| | 2 | 98.1358 | 9 | 4 | - | 0.6742 | - |
| 1.0 | 1 | 87.1153 | 11 | - | 0.6060 | - | 0.4600 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | - |
| 1.2 | 1 | 79.0048 | 12 | - | 0.5579 | - | 0.5200 |
| | 2 | 81.0783 | 12 | 5 | - | 0.6176 | - |
| 1.4 | 1 | 71.1457 | 10 | - | 0.5260 | - | 0.5200 |
| | 2 | 72.4999 | 0 | 0 | - | 0 | - |

Table 5.3.3 Comparisons of minimum Bayes risk for Type I and II censoring as $m$ varies ($\alpha = 2.5, \beta = 1, C_0 = C_1 = C_2 = 10, C_r = 100, C_s = 0.5, r_s = 0.2, a_t = 10$)

| $m$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 1 | 81.8605 | 10 | - | 0.1116 | - | 0.4200 |
| | 2 | 82.9033 | 13 | 6 | - | 0.3465 | - |
| 1.0 | 1 | 83.8560 | 12 | - | 0.2550 | - | 0.5200 |
| | 2 | 85.2981 | 13 | 5 | - | 0.4306 | - |
| 2.0 | 1 | 86.3542 | 12 | - | 0.5101 | - | 0.5200 |
| | 2 | 87.5686 | 11 | 5 | - | 0.6268 | - |
| 2.5 | 1 | 87.1153 | 11 | - | 0.6056 | - | 0.4600 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | - |
| 3.0 | 1 | 87.6375 | 12 | - | 0.6376 | - | 0.5200 |
| | 2 | 88.7508 | 11 | 5 | - | 0.6828 | - |

Table 5.3.4 Comparisons of minimum Bayes risk for Type I and II censoring as $C_0$ varies ($\alpha = 2.5, \beta = 1, m = 2.5, C_1 = C_2 = 10, C_r = 100, C_s = 0.5, r_s = 0.2, a_t = 10$)

| $C_0$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 1 | 81.1404 | 13 | - | 0.5897 | - | 0.4800 |
| | 2 | 82.8237 | 12 | 6 | - | 0.5987 | - |
| 6.0 | 1 | 84.8735 | 12 | - | 0.5738 | - | 0.5200 |
| | 2 | 86.1622 | 12 | 6 | - | 0.6267 | - |
| 10.0 | 1 | 87.1153 | 11 | - | 0.6060 | - | 0.4600 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | - |
| 20.0 | 1 | 92.2916 | 11 | - | 0.5738 | - | 0.5600 |
| | 2 | 93.0508 | 11 | 5 | - | 0.6548 | - |
| 24.0 | 1 | 94.2224 | 11 | - | 0.5898 | - | 0.5600 |
| | 2 | 94.8426 | 10 | 4 | - | 0.6828 | - |

Table 5.3.5 Comparisons of minimum Bayes risk for Type I and II censoring as $C_1$ varies ($\alpha = 2.5, \beta = 1, m = 2.5$, $C_0 = C_2 = 10, C_r = 100, C_s = 0.5, r_s = 0.2, a_t = 10$)

| $C_1$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| -5.0 | 1 | 70.7728 | 12 | | 0.5579 | - | 0.4200 |
| | 2 | 72.9880 | 12 | 6 | - | 0.5427 | - |
| 5.0 | 1 | 82.4599 | 12 | - | 0.5579 | - | 0.5200 |
| | 2 | 83.9278 | 12 | 6 | - | 0.5988 | - |
| 10.0 | 1 | 87.1153 | 11 | - | 0.6060 | - | 0.4600 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | - |
| 15.0 | 1 | 91.0396 | 12 | - | 0.6057 | - | 0.5200 |
| | 2 | 91.7666 | 11 | 5 | - | 0.6548 | - |
| 23.0 | 1 | 96.0638 | 11 | - | 0.6216 | - | 0.5600 |
| | 2 | 96.2107 | 10 | 4 | - | 0.7109 | - |

Table 5.3.6 Comparisons of minimum Bayes risk for Type I and II censoring as $C_2$ varies ($\alpha = 2.5, \beta = 1, m = 2.5$, $C_0 = C_1 = 10, C_r = 100, C_s = 0.5, r_s = 0.2, a_t = 10$)

| $C_2$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 1 | 35 | 0 | - | 0 | - | 0.0000 |
| | 2 | 35 | 0 | 0 | - | 0 | - |
| 6.0 | 1 | 77.8819 | 11 | - | 0.5419 | - | 0.4600 |
| | 2 | 79.9180 | 11 | 5 | - | 0.5427 | - |
| 10.0 | 1 | 87.7849 | 12 | - | 0.5615 | - | 0.5200 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | - |
| 14.0 | 1 | 92.6073 | 13 | - | 0.6057 | - | 0.5400 |
| | 2 | 93.0291 | 11 | 5 | - | 0.6828 | - |
| 24.0 | 1 | 99.9051 | 11 | - | 0.6854 | - | 0.5600 |
| | 2 | 99.2163 | 10 | 4 | - | 0.7949 | - |

Table 5.3.7 Comparisons of minimum Bayes risk for Type I and II censoring as $C_r$ varies ($\alpha = 2.5, \beta = 1, m = 2.5$, $C_0 = C_1 = C_2 = 10, C_s = 0.5, r_s = 0.2, a_t = 10$.)

| $C_r$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| 50.0 | 1 | 50 | 0 | - | $\infty$ | - | 1.0000 |
| | 2 | 50 | 0 | 0 | - | $\infty$ | - |
| 80.0 | 1 | 76.9369 | 9 | - | 0.5898 | - | 0.5600 |
| | 2 | 77.2591 | 10 | 4 | - | 0.6828 | - |
| 100.0 | 1 | 87.1153 | 11 | - | 0.6060 | - | 0.4600 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | - |
| 140.0 | 1 | 101.8998 | 13 | - | 0.5896 | - | 0.4000 |
| | 2 | 104.1377 | 13 | 7 | - | 0.5427 | - |
| 180.0 | 1 | 111.7017 | 14 | - | 0.5579 | - | 0.3600 |
| | 2 | 114.5916 | 14 | 8 | - | 0.5147 | - |

Table 5.3.8 Comparisons of minimum Bayes risk for Type I and II censoring as $C_s$ varies ($\alpha = 2.5, \beta = 1, m = 2.5$, $C_0 = C_1 = C_2 = 10, C_r = 100, r_s = 0.2, a_t = 10$)

| $C_s$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 1 | 82.5650 | 16 | - | 0.5419 | - | 0.5800 |
| | 2 | 83.6685 | 16 | 6 | - | 0.6268 | - |
| 0.4 | 1 | 85.7650 | 16 | - | 0.5419 | - | 0.5800 |
| | 2 | 86.8649 | 15 | 6 | - | 0.6268 | - |
| 0.5 | 1 | 87.1153 | 11 | - | 0.6060 | - | 0.4600 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | - |
| 0.8 | 1 | 89.0763 | 8 | - | 0.6535 | - | 0.3800 |
| | 2 | 90.8613 | 7 | 4 | - | 0.6268 | - |
| 1.0 | 1 | 91.1518 | 6 | - | 0.6854 | - | 0.3400 |
| | 2 | 92.1487 | 6 | 4 | - | 0.6376 | - |

Table 5.3.9 Comparisons of minimum Bayes risk for Type I and II censoring as $r_s$ varies ($\alpha = 2.5, \beta = 1, m = 2.5$, $C_0 = C_1 = C_2 = 10, C_r = 100, C_s = 0.5, a_t = 10$)

| $r_s$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 1 | 88.1789 | 10 | - | 0.6535 | - | 0.4200 |
| | 2 | 89.1589 | 9 | 5 | - | 0.6268 | - |
| 0.1 | 1 | 87.7024 | 11 | - | 0.6057 | - | 0.4600 |
| | 2 | 88.7471 | 10 | 5 | - | 0.6268 | - |
| 0.2 | 1 | 87.1153 | 11 | - | 0.6060 | - | 0.4600 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | - |
| 0.3 | 1 | 86.3573 | 13 | - | 0.5579 | - | 0.5400 |
| | 2 | 87.4621 | 14 | 5 | - | 0.6268 | - |
| 0.5 | 1 | 84.2802 | 16 | - | 0.4941 | - | 0.6400 |
| | 2 | 85.3021 | 16 | 5 | - | 0.6268 | - |

Table 5.3.10 Comparisons of minimum Bayes risk for Type I and II censoring as $a_t$ varies ($\alpha = 2.5, \beta = 1, m = 2.5$, $C_0 = C_1 = C_2 = 10, C_r = 100, C_s = 0.5, r_s = 0.2$)

| $a_t$ | Models* | Bayes Risk | $n_0$ | $r_0$ | $t_0$ | $T_0$ | $p_0$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 1 | 80.1472 | 10 | - | 0.8129 | - | 0.2200 |
| | 2 | 79.1930 | 8 | 8 | - | 0.6268 | - |
| 4.0 | 1 | 83.1963 | 10 | - | 0.7173 | - | 0.3200 |
| | 2 | 83.5678 | 9 | 7 | - | 0.6268 | - |
| 10.0 | 1 | 87.1153 | 11 | - | 0.6060 | - | 0.4600 |
| | 2 | 88.2085 | 11 | 5 | - | 0.6268 | - |
| 16.0 | 1 | 90.3749 | 13 | - | 0.5101 | - | 0.6200 |
| | 2 | 91.8637 | 13 | 4 | - | 0.6268 | - |
| 20.0 | 1 | 92.3441 | 16 | - | 0.4622 | - | 0.7000 |
| | 2 | 92.9244 | 14 | 4 | - | 0.6268 | - |

# Appendix 1 Some Lemmas

**Lemma A1.**

$$E\left[\lambda^l P(\hat{\theta}_r \geq T_m)\right] = E\left[\lambda^l P(\hat{\theta}_r^{\,1/m}\Gamma(1+\frac{1}{m}) \geq T)\right] = \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(1 - I_s(r, \alpha+l)),$$

*where* $T_m = T^m\Gamma(1+\frac{1}{m})$, $s = rT_m/(rT_m+\beta)$, *and* $\alpha + l > 0$.

**Proof .** For any fixed $l$ such that $\alpha + l > 0$, we have

$$\begin{aligned}
E\left[\lambda^l P(\hat{\theta}_r \geq T_m)\right] &= \int_0^\infty\int_{T_m}^\infty \frac{r^r\beta^\alpha}{\Gamma(r)\Gamma(\alpha)}\lambda^{r+\alpha+l-1}u^{r-1}e^{-(ru+\beta)\lambda}du d\lambda \\
&= \frac{r^r\beta^\alpha}{\Gamma(r)\Gamma(\alpha)}\int_{T_m}^\infty \Gamma(r+\alpha+l)\frac{u^{r-1}}{(ru+\beta)^{(r+\alpha+l)}}du \\
&= \frac{r^r\beta^\alpha\Gamma(r+\alpha+l)}{\Gamma(r)\Gamma(\alpha)}\int_{s/(1-s)}^1 \frac{y^{r-1}}{r^r\beta^{\alpha+l}(1+y)^{(r+\alpha+l)}}dy \\
&= \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(1 - I_s(r,\alpha+l)),
\end{aligned}$$

Lemma A2 follows from Lemma A1 directly.

**Lemma A2.**

$$E\left[\lambda^l P(T_{1m} \leq \hat{\theta}_1 < T_{0m})\right] = \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(I_{s_0}(r_1,\alpha+l) - I_{s_1}(r_1,\alpha+l))$$

*where* $T_{im} = T_i^m\Gamma(1+\frac{1}{m})$, $s_i = (r_1T_{im})/(r_1T_{im}+\beta)$, $i = 0, 1$, *and* $\alpha + l > 0$.

**Lemma A3.**

$$P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} < T_{2m})$$

$$= \frac{r_1^{r_1} r_2^{r_2}}{\Gamma(r_1)\Gamma(r_2)} \int_{T_{1m}}^{T_{*m}} \int_{0}^{T_m(u)} \lambda^{r_1+r_2} u^{r_1-1} v^{r_2-1} e^{-(r_1 u + r_2 v)\lambda} dv du,$$

*where* $T_m(u) = \frac{r}{r_2} T_m - \frac{r_1}{r_2} u;\ \ T_{*m} = \min\{T_{0m}, \frac{r}{r_1} T_{2m}\}.$

**proof.**

$$\begin{aligned}
P & \ (T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} < T_{2m}) \\
&= \int_{T_{1m}}^{T_{0m}} P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} < T_{2m} | \hat{\theta}_1 = u) \frac{(r_1\lambda)^{r_1}}{\Gamma(r_1)} u^{r_1-1} e^{-r_1 \lambda u} du \\
&= \frac{(r_1\lambda)^{r_1}}{\Gamma(r_1)} \int_{T_{1m}}^{T_{0m}} P(\hat{\theta}_2 < \frac{r}{r_2} T_{2m} - \frac{r_1}{r_2} u) u^{r_1-1} e^{-r_1\lambda u} du \\
&= \frac{(r_1\lambda)^{r_1}}{\Gamma(r_1)} \int_{T_{1m}}^{T_{*m}} \left( \int_{0}^{T_m(u)} \frac{(r_2\lambda)^{r_2}}{\Gamma(r_2)} v^{r_2-1} e^{-r_2\lambda v} dv \right) u^{r_1-1} e^{-r_1\lambda u} du \\
&= \frac{r_1^{r_1} r_2^{r_2}}{\Gamma(r_1)\Gamma(r_2)} \int_{T_{1m}}^{T_{*m}} \int_{0}^{T_m(u)} \lambda^{r_1+r_2} u^{r_1-1} v^{r_2-1} e^{-(r_1 u + r_2 v)\lambda} dv du.
\end{aligned}$$

**Lemma A4.**

$$E\left[\lambda^l P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} \geq T_{2m})\right] = \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l} (I_{s_0}(r_1, \alpha+l) - I_{s_1}(r_1, \alpha+l))$$

$$- \frac{r_1^{r_1} \beta^\alpha \Gamma(r_1+\alpha+l)}{\Gamma(r_1)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \frac{u^{r_1-1} I_{s(u)}(r_2, r_1+\alpha+l)}{(r_1 u + \beta)^{r_1+\alpha+l}} du$$

*where* $s(u) = (rT_{2m} - r_1 u)/(rT_{2m} + \beta)$; $s_i = (r_1 T_{im})/(r_1 T_{im} + \beta)$, $i = 0, 1$, *and* $\alpha + l > 0$.

**Proof.** First of all, we have

$$E\left[\lambda^l P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} < T_{2m})\right]$$

$$= \frac{r_1^{r_1} r_2^{r_2} \beta^\alpha}{\Gamma(r_1)\Gamma(r_2)\Gamma(\alpha)} \int_0^\infty \int_{T_{1m}}^{T_{*m}} \int_0^{T_m(u)} \lambda^{r_1+r_2+\alpha+l-1} u^{r_1-1} v^{r_2-1} e^{-(r_1 u + r_2 v + \beta)\lambda} du dv d\lambda$$

$$= \frac{r_1^{r_1} r_2^{r_2} \beta^\alpha \Gamma(r_1+r_2+\alpha+l)}{\Gamma(r_1)\Gamma(r_2)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \left( \int_0^{T_m(u)} \frac{v^{r_2-1}}{(r_1 u + r_2 v + \beta)^{r_1+r_2+\alpha+l}} dv \right) u^{r_1-1} du$$

$$= \frac{r_1^{r_1} \beta^\alpha \Gamma(r_1+r_2+\alpha+l)}{\Gamma(r_1)\Gamma(r_2)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \frac{u^{r_1-1}}{(r_1 u + \beta)^{r_1+\alpha+l}} B(r_2, r_1 + \alpha + l) I_{s(u)}(r_2, r_1 + \alpha + l) du$$

$$= \frac{r_1^{r_1} \beta^\alpha \Gamma(r_1+\alpha+l)}{\Gamma(r_1)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \frac{u^{r_1-1}}{(r_1 u + \beta)^{r_1+\alpha+l}} I_{s(u)}(r_2, r_1 + \alpha + l) du$$

Then, it follows from Lemma A2 that

$$E\left[\lambda^l P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} \geq T_{2m})\right]$$

$$= E\left[\lambda^l P(T_{1m} \leq \hat{\theta}_1 < T_{0m})\right] - E\left[\lambda^l P(T_{1m} \leq \hat{\theta}_1 < T_{0m}, \hat{\theta} < T_m)\right]$$

$$= \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(1 - I_{s_1}(r_1, \alpha + l)) - \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(1 - I_{s_0}(r_1, \alpha + l))$$

$$- \frac{r_1^{r_1} \beta^\alpha \Gamma(r_1+\alpha+l)}{\Gamma(r_1)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \frac{u^{r_1-1}}{(r_1 u + \beta)^{r_1+\alpha+l}} I_{s(u)}(r_2, r_1 + \alpha + l) du$$

$$= \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}(I_{s_0}(r_1, \alpha + l) - I_{s_1}(r_1, \alpha + l)) - \frac{r_1^{r_1} \beta^\alpha \Gamma(r_1+\alpha+l)}{\Gamma(r_1)\Gamma(\alpha)} \int_{T_{1m}}^{T_{*m}} \frac{u^{r_1-1} I_{s(u)}(r_2, r_1+\alpha+l)}{(r_1 u + \beta)^{r_1+\alpha+l}} du.$$

# Appendix 2.

**Theorem A1.**

$$\frac{\Gamma(\sum_{i=1}^{n} a_i)}{\prod_{i=1}^{n}\Gamma(a_i)} \int_0^{x_1}\cdots\int_0^{x_{n-1}} \frac{t_1^{a_1-1}\cdots t_{n-1}^{a_{n-1}-1}}{(1+t_1+\cdots+t_{n-1})^{a_1+\cdots+a_{n-1}+a_n}} dt_1\cdots dt_{n-1}$$

$$= D_n(x_1^*,\cdots,x_{n-1}^*;a_1,\cdots,a_{n-1};a_n),$$

*where* $x_i^* = \frac{x_i}{1+x_1+\cdots+x_{n-1}}$, $i = 1,\ldots,n-1$, *and* $D_n(x_1^*,\cdots,x_{n-1}^*;a_1,\cdots,a_{n-1};a_n)$ *is a the Dirichlet distribution with the density function*

$$p(t_1,\cdots,t_{n-1}) = \begin{cases} \frac{\Gamma(\sum_{i=1}^{n} a_i)}{\prod_{i=1}^{n}\Gamma(a_i)} \prod_{i=1}^{n-1} t_i^{a_i-1}\left(1-\sum_{i=1}^{n-1} t_i\right)^{a_n-1}, & t_i \geq 0,\ \sum_{i=1}^{n-1} t_i < 1, \\ 0, & \textit{otherwise}. \end{cases}$$

**Proof.** Let $t_i^* = \frac{t_i}{1+t_1+\cdots+t_{n-1}}$, $(i = 1,\ldots,n-1)$. then, we have following results

$$(I) \qquad t_i = t_i^*(1+t_1+\cdots+t_{n-1}) = \frac{t_i^*}{1-t_1^*-\cdots-t_{n-1}^*}, \quad i = 1,\ldots,n.$$

$$(II) \qquad 1-t_1^*-\cdots-t_{n-1}^* = \frac{1}{1+t_1+\cdots+t_{n-1}}.$$

$$(III) \qquad |J| = \begin{vmatrix} \frac{\partial t_1}{\partial t_1^*} & \frac{\partial t_1}{\partial t_2^*} & \cdots & \frac{\partial t_1}{\partial t_{n-1}^*} \\ \frac{\partial t_2}{\partial t_1^*} & \frac{\partial t_2}{\partial t_2^*} & \cdots & \frac{\partial t_2}{\partial t_{n-1}^*} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial t_{n-1}}{\partial t_1^*} & \frac{\partial t_{n-1}}{\partial t_2^*} & \cdots & \frac{\partial t_{n-1}}{\partial t_{n-1}^*} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1-t_2^*-\cdots-t_{n-1}^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \frac{t_1^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \cdots & \frac{t_1^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} \\ \frac{t_2^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \frac{1-t_1^*-t_3^*-\cdots-t_{n-1}^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \cdots & \frac{t_2^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{t_{n-1}^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \frac{t_{n-1}^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \cdots & \frac{1-t_1^*-\cdots-t_{n-2}^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} \end{vmatrix}$$

$$= \frac{1}{\delta^2} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{t_2^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \frac{1-t_1^*-t_3^*-\cdots-t_{n-1}^*}{1-t_1^*-\cdots-t_{n-1}^*}^2 & \cdots & \frac{t_2^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{t_{n-1}^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \frac{t_{n-1}^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \cdots & \frac{1-t_1^*-\cdots-t_{n-2}^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} \end{vmatrix}$$

$$= \frac{1}{\delta^2} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ \frac{t_2^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & \frac{1}{1-t_1^*-\cdots-t_{n-1}^*} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{t_{n-1}^*}{(1-t_1^*-\cdots-t_{n-1}^*)^2} & 0 & \cdots & \frac{1}{1-t_1^*-\cdots-t_{n-1}^*} \end{vmatrix}$$

$$= \frac{1}{(1-t_1^*-\cdots-t_{n-1}^*)^n}.$$

where $\delta = 1 - t_1^* - \cdots - t_{n-1}^*$.

Therefore, from $(I)$, $(II)$ and $(III)$, we have

$$I_n = \int_0^{x_1} \cdots \int_0^{x_{n-1}} \frac{t_1^{a_1-1} \cdots t_{n-1}^{a_{n-1}-1}}{(1+t_1+\cdots+t_{n-1})^{a_1+\cdots+a_{n-1}+a_n}} dt_1 \cdots dt_{n-1}$$

$$= \int_0^{x_1^*} \cdots \int_0^{x_{n-1}^*} t_1^{*(a_1-1)} \cdots t_{n-1}^{*(a_{n-1}-1)} (1 - t_1^* - \cdots - t_{n-1}^*)^{(a_n+n-1)} \, |J| \, dt_1^* \cdots dt_{n-1}^*$$

$$= \int_0^{x_1^*} \cdots \int_0^{x_{n-1}^*} t_1^{*(a_1-1)} \cdots t_{n-1}^{*(a_{n-1}-1)} (1 - t_1^* - \cdots - t_{n-1}^*)^{a_n-1} dt_1^* \cdots dt_{n-1}^*$$

where $x_i^* = \frac{x_i}{1+x_1+\cdots+x_{n-1}}$ $(i = 1, \cdots, n-1)$. It follows that

$$\frac{\Gamma(a_1+\cdots+a_{n-1}+a_n)}{\prod_{i=1}^{n}\Gamma(a_i)} \int_0^{x_1} \cdots \int_0^{x_{n-1}} \frac{t_1^{a_1-1} \cdots t_{n-1}^{a_{n-1}-1}}{(1+t_1+\cdots+t_{n-1})^{a_1+\cdots+a_{n-1}+a_n}} dt_1 \cdots dt_{n-1}$$

$$= \int_0^{x_1^*} \cdots \int_0^{x_{n-1}^*} p(t_1, \cdots, t_{n-1}) dt_1 \cdots dt_{n-1}$$

$$= D_n(x_1^*, \cdots, x_{n-1}^*; a_1, \cdots, a_{n-1}; a_n),$$

The following Corollary A1 is the modification of the Theorem A1.

**Corollary A1.**

$$\frac{\Gamma((n-1)a+a_n)}{[\Gamma(a)]^{n-1}\Gamma(a_n)} \int_0^{x} \cdots \int_0^{x} \frac{t_1^{a-1} \cdots t_{n-1}^{a-1}}{(1+t_1+\cdots+t_{n-1})^{(n-1)a+a_n}} dt_1 \cdots dt_{n-1}$$

$$= D_n(x^*, \cdots, x^*; a, \cdots, a; a_n),$$

*where* $x^* = \frac{x}{1+(n-1)x}$.

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