PSEUDOVARIETIES OF FINITE SEMIGROUPS AND APPLICATIONS

by

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Abstract

There are two motivations to study the theory of pseudovarieties of finite semigroups. One is initiated from Universal Algebras. The famous Birkhoff's Theorem [19] asserts that a family of algebras is an equational variety if and only if the family is closed under the operations in passing it to all subalgebras and quotient algebras as well as under arbitrary direct products, which allows us to study the classes of algebras with same properties to get the universal results, instead of one specific algebra. A natural generalization of the theory is to restrict our attention on finite algebras, this is the concept of pseudovariety of finite algebras. While it may lose some properties in generality, but on the other hand, it enriches the theory in structural terms, since besides the algebraic structure some specific combinatorial properties of finite sets play a crucial role. Within the study of pseudovarities, the pseudovarieties of finte semigroups are particularly interesting not only because that semigroups are the most general algebraic systems, but also they have abundant relationships with many combinatorial problems. Hence, the theory gives us a general method in dealing with some combinatorial problems. And also, finite semigroups always play an important role in the study of the semigroup theory.

The other motivation for studying pseudovarieties of finite semigroups is initiated from algebraic automata and formal languages theory. The theory of automata and formal languages was started from an attempt to model mathematically the events in nerve nets done by Kleene [30] in the years of fifties. In the sixties, there are Schützenberger and his school, and in the beginning of the seventies, there are Brzozowski, McNaughton, Simon and Zalcstein, their highly motivated work leaded to a bunch of remarkable and fruitful results. The results they obtained allow us to test the combinatorial properties of rational languages by checking the algebraic properties of the respective syntactic semigroups. Subsequently, the work of Schützenberger and Eilenberg [23,24] showed the importance of the notion of pseudovarieties of finite semigroups and monoids in the study of certain classes of recognizable

languages. And the writings of other researchers, for example, Brzozowski and Simon [17], Pin [43], Straubing [55] , Thérien [62], acknowledge this importance. One particular problem, called the dot-depth problem of star-free languages is most interesting and important with its connections to semigroup theory, mathematical logic and computational complexity. The first two hierarchies have been effectively determined by means of their corresponding pseudovarieties of finite semigroups. The decidability of dot-depth 2 hierarchy is till an open problem.

This dissertation tries to give a lucid and systematic survey to the theory of pseudovarieties of finite semigroups. The dissertation is composed of five chapters. In chapter 1, we introduce briefly the essences of universal algebras, definition of pseudovariety of finite algebras as well as some of their properties and methods which will be used later on are given. In chaper 2, we give some remarkable algebraic theorems of automata and regular languages. In particular, the important theorem of variety, which asserts that there is a one-one correspondence between the varieties of languages and the pseudovarieties of finite monoids are discribed. At the last part of chapter 2, we introduce some classical varieties of languages and their corresponding pseudovarieties. In chapter 3, we adopt the Green's relations and other algebraic structures to characterize some sort of pseudovarieties of finite semigroups and monoids. In chapter 4, the well known dot-depth problem is discussed. We will review the main results and some recent partial results related to this problem. In the last chapter, we study the pseudovarieties generated by some power semigroups which are related to the dot-depth 2 problem, through some of the identities satisfying by them.

Contents

Chapter 1 Pseudovarieties of finite algebras

§1 Elements of universal algebra

Semigroups, monoids, groups, rings, fields, lattices, Boolean algebras and the vector spaces over a given field are examples of algebraic structures, i.e., they are all sets endowed with some operations, gathered in classes according to the type of the operations under consideration. The concepts of homomorphisms, congrueces, sub-structures, direct products etc. are defined in a similar way for all those classes. There are, on the other hand, various results of the same kind that are valid for each of the indicated classes such as, for instance, the homomorphism and the isomorphism theorems. These results suggest that one can adopt a more general perspective in the study of arbitrary algebraic structures, and it is this the objective of the mathematical discipline: Universal Algebra. We consider here only some aspects of this discipline that are relevent in our study of pseudovarieties.

We first start by defining the concept of "algebraic types".

Definition 1.1 An algebraic type is a pair $\tau = (\ \mathcal{O},\ \alpha\ )$, where $\mathcal{O}$ is a set and α is a function from $\mathcal{O}$ to the set of non-negative integers. Each element f in $\mathcal{O}$ is said to be an operation symbol and $\alpha(f)$ is said to be its arity. The set of all operations with arity n is denoted by $\mathcal{O}_n$.

Definition 1.2 An algebra $\mathcal{A}$ of type τ is a nonempty set A, called the universe of $\mathcal{A}$, with functions corresponding to each f in $\mathcal{O}$, f_A: $A^n \to A$ with $n = \alpha(f)$, called the interpretation of f in $\mathcal{A}$. If $\alpha(f) = 0$, then f_A: $\{\emptyset\} \to A$ is a constant in A. We call an algebra trivial if its universe is just a singleton set, and the algebra is finite or infinite depends on whether its universe is finite or infinite respectively. For the sake of simplification, we use the same notation to indicate an algebra and its universe.

Some examples will make it clear of the definition.

1. A semigroup is an algebra of type ($\mathcal{O}$ ={ $\cdot$ }, α = { ($\cdot$, 2) }), in which the associate indentity: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ is valid. i.e., a nonempty set S with a binary operation satisfing the associate law. We denote the semigroup S by the symbol $< S; \cdot >$.

2. A monoid is an algebra of type ($\mathcal{O} = \{\cdot, e\}, \alpha = \{(\cdot, 2), (e, 0)\}$). i.e. a nonempty set M with a binary operation and a constant $1 \in M$ and satisfies $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; $1 \cdot x = x \cdot 1 = x$ for any $x, y, z \in M$. We denote it by $< M; \cdot, 1 >$. In other words, a monoid is a semigroup with the multiplicative identity 1.

3. A semi-ring is an algebra of type ($\mathcal{O} = \{+, \cdot, 0\}, \alpha = \{(+, 2), (\cdot, 2), (0, 0)\}$) such that

   - $< A; +, 0 >$ is a commutative monoid
   - $< A; \cdot >$ is a semigroup
   - for any $x, y, z \in A$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$
   - for any $x \in A$, $x \cdot 0 = 0 \cdot x = 0$

   In other words, a semi-ring is a monoid $< A; +, 0 >$ and a semigroup $< A; \cdot >$ which are conjoint with the distributive laws.

Let A be an algebra of type τ, say B is a subalgebra of A if B is a subset of A which is closed under the restriction of each operation of A on B. we can see immeadiatly that B is also an algebra of type τ itself. B is said to be the subalgebra generated by a subset $X \subseteq A$ if B is the intersection of all the subalgebras containing the subset X of A.

Let A, B be two algebras of the same type τ = ($\mathcal{O}$, α). Then, a homomorphism φ: $A \to B$ is a function which maps from the universe set A into the universe set B such that for any $n \in N$, $f \in \mathcal{O}_n$ and $a_1, \ldots, a_n \in A$, we have

$$\varphi[f_A(a_1, \ldots, a_n)] = f_B(\varphi a_1, \ldots, \varphi a_n).$$

Similarly, we can define the homomorphic images, isomorphisms, endomorphisms and automorphisms between the algebras A and B.

A congruence on the algebra A of type τ is an equivalence relation θ on the universe set A such that for any $f \in \mathcal{O}_n$ and a_i, $b_i \in A$, we have

$$a_i \theta b_i (i = 1, ..., n) \Rightarrow f(a_1, ..., a_n) \theta f(b_1, ..., b_n).$$

Let φ: A $\rightarrow$ A$/\theta$ be a mapping which maps every element of A into its equivalent classes. If θ is also a congruence, then there is a natural reduced algebraic structure of type τ on A$/\theta$ such that φ is a homomorphism. The algebra A/θ is said to be the quotient algebra of A determined by the congruence θ on A.

The kernal of a homomorphism $\varphi : A \rightarrow B$ is the set:

$$ker\varphi = \{(a_1, a_2) \in A \times A : \varphi a_1 = \varphi a_2\}$$

Clearly, $\ker\varphi$ is also a congruence on the algebra A. Same as groups, rings and other classes of algebras, we have the following general Homomorphism theorem:

Proposition 1.3 Let φ: A $\rightarrow$ B be an onto homomorphism. Then there is exactly one homomorphism ψ: A$/ker\varphi \rightarrow$ B such that $\varphi = \psi \circ \nu$, where ν is the natural homomorphism A $\rightarrow$ A$/ker\varphi$. Moreover, ψ is an isomorphism. In other words, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\nu}\downarrow & \nearrow_{\psi} & \\ A/\ker\varphi & & \end{array}$$

We now define the term algebra of a given arbitrary type $\tau = (\mathcal{O}, \alpha)$. Let X be a set, called it the set of variables, such that X$\cup\mathcal{O}_0 \neq \emptyset$. we consider the set $\mathcal{S}$ of all words defined on the alphabet X$\cup\mathcal{O} \cup \{(,)\}$. The set $T(X)$ of all terms of type τ on X is the intersection of all subsets $\mathcal{F}$ of $\mathcal{S}$ such that

1. $X \cup \mathcal{O}_0 \subseteq \mathcal{F}$,

2. if $t_1, ..., t_n \subseteq \mathcal{F}$ and $f \in \mathcal{O}_n$, then $f(t_1, ..., t_n) \in \mathcal{F}$.

On the set $T(X)$, we may easily define a structure $\mathcal{T}(X)$ of algebra of type τ. For $f \in \mathcal{O}_n$, we define its interpretation in $\mathcal{T}(X)$ to be the function

$$T(X)^n \to T(X)$$

$$(t_1, ..., t_n) \mapsto f(t_1, ..., t_n)$$

In this way, we obtain an algebra of terms in X of type τ.

Definition 1.4 Let $\mathcal{K}$ be a class fo algebras of type τ and F an algebra of the same type generated by a set $X \subseteq F$. Then we say F has the universal mapping property for $\mathcal{K}$ over X if for any $A \in \mathcal{K}$ and any function $\varphi : X \to A$, there exists an unique homomorphism $\bar{\varphi} : F \to A$ such that $\bar{\varphi}|_X = \varphi$.

Now, we define the congruence on $T(X)$ associated to the class $\mathcal{K}$ by:

$$\theta_X\mathcal{K} = \bigcap\{ker\ \varphi\ |\varphi : T(X) \to A \text{ is a homomorphism with } A \in \mathcal{K}\}$$

Denote the qoutient algebra $T(X)/\theta_X\mathcal{K}$ by $F_{\bar{X}}\mathcal{K}$, where $\bar{X}$ stands for the set $X/\theta_X\mathcal{K}$ of the equivalence classes of the elements of X. Notice that the natural projection $T(X) \to F_{\bar{X}}\mathcal{K}$ defines a bijection $X \to \bar{X}$, except when $\mathcal{K}$ consists exclusively of trivial algebras. The algebra $F_{\bar{X}}\mathcal{K}$ is called the $\mathcal{K}$-free algebra on X.

The following theorem concerning universal mappings was due to Birkhoff.

Theorem 1.5 (Birkhoff) The algebra $F_{\bar{X}}\mathcal{K}$ has the universal mapping property for $\mathcal{K}$ over $\bar{X}$.

Proof. Let $\pi : T(X) \to T(X)/\theta_X\mathcal{K} = F_{\bar{X}}\mathcal{K}$ be the natural homomorphism. Let $\bar{\pi} = \pi|_X : X \to \bar{X}$ be the restriction of π on X. Then for any $A \in K$ and any function $\varphi : \bar{X} \to A$, we can easily see that $\varphi \circ \bar{\pi}$ is a map: $X \to A$. This map $\varphi \circ \bar{\pi}$ can be extended to a homomorphism $\psi : T(X) \to A$. In fact, we have $\theta_X\mathcal{K} \subseteq \ker\psi$ since $A \in \mathcal{K}$. Hence, we can easily see that ψ induces a homomorphism $\bar{\psi} : F_{\bar{X}}\mathcal{K} \to A$ such that $\bar{\psi} \circ \pi = \psi$ and $\bar{\psi}|_{\bar{X}} = \varphi$. Now suppose that there is another homomorphism $\lambda : F_{\bar{X}}\mathcal{K} \to A$ with $\lambda|_{\bar{X}} = \varphi$,then we have a homomorphism

$$\lambda \circ \pi : T(X) \to F_{\bar{X}}\mathcal{K} \to A,$$

and its restriction on the set X is

$$\lambda \circ \pi|_X = \lambda|_{\bar{X}} \circ \bar{\pi} = \varphi \circ \bar{\pi}.$$

Hence, we obtain that $\lambda \circ \pi = \psi = \bar{\psi}\pi$ since $T(X)$ is the free algebra on the set X. Therefore, they reduce the same homomorphism $\lambda = \bar{\psi}$. i.e. such a homomorphism exists and is unique.

An identity of type τ on X is a pair $(p, q) \in T(X) \times T(X)$, usually indicated by a formal equality $p = q$. We represent it by $Id(X)$; the set of all identities on X. We say an algebra A satisfies an identity $p = q$, writen by $A \models p = q$, if and only if for any homomorphism $\varphi : T(X) \to A$, we have $\varphi p = \varphi q$. The set of all identities on X which is valid in all algebras of a class $\mathcal{K}$ is indicated by $Id_X(\mathcal{K})$. For a set Σ of identities, we write $\mathcal{K} \models \Sigma$ if $A \models p = q$ for any $A \in \mathcal{K}$ and $(p, q) \in \Sigma$. Let Σ_1, Σ_2 be two sets of identities of type τ. Then, we write $\Sigma_1 \vdash \Sigma_2$ if for any algebra of that type A and $A \models \Sigma_1$, it is certainly true that $A \models \Sigma_2$. The sets Σ_1 and Σ_2 are said to be equivalent if $\Sigma_1 \vdash \Sigma_2$ and $\Sigma_2 \vdash \Sigma_1$ hold simultaniously. In particularly, we say that two identities $\varepsilon_1 : p_1 = q_1$ and $\varepsilon_2 : p_2 = q_2$ are equivalent if both $\varepsilon_1 \vdash \varepsilon_2$ and $\varepsilon_2 \vdash \varepsilon_1$ hold. (The above notions are the fundamental concepts in equational logic, the reader is refered to [19] for more details.)

We define the following operations on some classes of algebras. For a class $\mathcal{K}$ of algebras of type τ, let $S\mathcal{K}, H\mathcal{K}, P\mathcal{K}, P_{fin}\mathcal{K}$ be the classes of all algebras that may be obtained as subalgebras, homomorphic images, direct products, finitary direct products from the algebras of $\mathcal{K}$ respectively. Equipped with these operators, we can give the definition of "variety".

Definition 1.6 A variety is a class of algebras of the same type closed under the operators H, S and P.

For a class $\mathcal{K}$, we let $V\mathcal{K}$ be the variety generated by $\mathcal{K}$, i.e., the intersection of all varieties containing $\mathcal{K}$. The following proposition is a fundamental result obtained by Tarski.

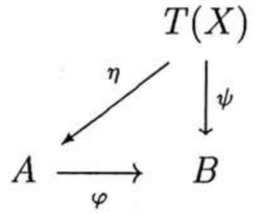
Proposition 1.7 (Tarski). $V = HSP$.

Proof. We sketch the proof as follows. For any operators O_1, O_2 on classes of algebras, write $O_1 \leq O_2$ if $O_1\mathcal{K} \subseteq O_2\mathcal{K}$ for any class of algebras $\mathcal{K}$. This defines a partial order on the class of all unary operators. We now consider the subsemigroup generated by the operators S, H, P, P_{fin}. Firstly, the equations $S^2 = S, H^2 = H, P^2 = P$ and $P_{fin}^2 = P_{fin}$ are obvious. Then we have $SH \leq HS$, as for any $A \in \mathcal{K}$ and $B \in SH(A)$, i.e. there is an epimorphism $\alpha : A \to B'$ and B is a subalgebra of B', then we know that $\alpha^{-1}(B)$ is a subalgebra of A and the restriction of α on that subalgebra $\alpha' : \alpha^{-1}(B) \to B$ is an epimorphism. i.e. $B \in HS\mathcal{K}$. Similarly, we can also obtain the inequalities $OS \leq SO$ and $OH \leq HO$ for any $O \in \{P, P_{fin}\}$. Thus, the equation $V = HSP$ is a consequence of the above relations.

Fix an algebraic type τ, there exists a natural lattice structure in the set of all varieties of algebras of type τ: Let $\mathcal{V}_1, \mathcal{V}_2$ be two varieties of algebras of that type. Then, we can easily check that $\mathcal{V}_1 \cap \mathcal{V}_2$ is still a variety of the same type. Let $\mathcal{V}_1 \vee \mathcal{V}_2$ be the intersection of all varieties containing $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e. $V(\mathcal{V}_1 \cup \mathcal{V}_2)$. Then we know that those varieties form a lattice with the operations $\cap$ and $\vee$, and the 0 element is just the empty variety and the 1 element is just the variety of all algebras of that type.

Lemma 1.8 The classes $\mathcal{K}, S\mathcal{K}, H\mathcal{K}, P\mathcal{K}, P_{fin}\mathcal{K}$ satisfy the same identities.

Proof. We only need to prove the case for the class $H\mathcal{K}$, and the other cases can be proved similarly. Let $A \in \mathcal{K}$ and $A \models p = q$ with $p, q \in T(X)$. Now, let $B \in H\mathcal{K}$, i.e. there is an epimorphism $\varphi : A \to B$. Then for any homomorphism $\psi : T(X) \to B$, there exists some homomorphism (not mecessarily unique) $\eta : T(X) \to A$ such that $\varphi \circ \eta = \psi$.



Hence, since $\eta p = \eta q$, by hypothesis, we certainly have $\psi p = \psi q$. i.e. $B \models p = q$. Conversely, as $\mathcal{K} \subseteq H\mathcal{K}$, any identity satisfied by $H\mathcal{K}$ is obviously satisfied by the class $\mathcal{K}$.

Now we formulate a crucial connection between the $\mathcal{K}$-free algebras and the identities valid in that class.

Proposition 1.9 Let $\mathcal{K}$ be a class of algebras of type τ. Then for terms $p, q \in T(X)$ of type τ, the following conditions are equivalent:

1. $\mathcal{K} \models p = q$,
2. $F_{\bar{X}}\mathcal{K} \models p = q$,
3. $(p, q) \in \theta_X \mathcal{K}$.

Proof.

(1) $\Rightarrow$ (2): As $F_{\bar{X}}\mathcal{K}$ is the quotient algebra of $T(X)$ over $\theta_X\mathcal{K}$ and by the definition of $\theta_X\mathcal{K}$, we know immediately that $F_{\bar{X}}\mathcal{K}$ can be subdirectly embedded in a direct product of subalgebras of algebras of $\mathcal{K}$. i.e. $F_{\bar{X}}\mathcal{K} \in HSPS\mathcal{K}$, and so by lemma 1.8, we know (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3): Consider the natrual homomorphism $\pi : T(X) \to T(X)/\theta_X\mathcal{K} = F_{\bar{X}}\mathcal{K}$. Now, $F_{\bar{X}}\mathcal{K} \models p = q$ gives $\pi p = \pi q$, i.e. $(p, q) \in \theta_X\mathcal{K}$. Thus, (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1): For any $A \in \mathcal{K}$ and homomorphism $\varphi : T(X) \to A$. By definition and hypothesis of (3), we have $(p, q) \in \theta_X\mathcal{K} \subseteq \ker\varphi$, i.e., $\varphi p = \varphi q$. This shows that $\mathcal{K} \models p = q$.

Corollary 1.10 Let $\mathcal{K}$ be a class of algebras of the same type, and suppose $p, q \in T(X)$. Then for any set of variables Y with $|Y| \geq |X|$, we have

$$K \models p = q \Leftrightarrow F_{\bar{Y}}\mathcal{K} \models p = q$$

Proof. The direction "$\Rightarrow$" is obvious since $F_{\bar{Y}}\mathcal{K} \in HSPS\mathcal{K}$. For the converse part, we choose $X_0 \supseteq X$ such that $|X_0| = |Y|$. Then we have

$$F_{\bar{X}_0}\mathcal{K} \cong F_{\bar{Y}}\mathcal{K}$$

and by proposition 1.9, we hence infer that

$$\mathcal{K} \models p = q \Leftrightarrow F_{\bar{X}_0}\mathcal{K} \models p = q$$

Then, it follows that

$$\mathcal{K} \models p = q \Leftrightarrow F_{\bar{Y}}\mathcal{K} \models p = q.$$

Corollary 1.11 Suppose that $\mathcal{K}$ is a class of algebras of the same type and X is an infinite set of variables. Then for any set of variables Y, we have

$$\theta_Y\mathcal{K} = \theta_Y(F_{\bar{X}}\mathcal{K}).$$

Proof. For any $(p,q) \in \theta_Y\mathcal{K}$, as any term involves only a finite number of variables, we can suppose that $p, q \in T(\{y_1, ..., y_n\})$. As $|X|$ is infinite, $|\{y_1, ..., y_n\}| < |X|$, hence by corollary 1.10, we deduce that

$$\mathcal{K} \models p = q \Leftrightarrow F_{\bar{X}}\mathcal{K} \models p = q$$

so the corollary is proved by proposition 1.9.

Given a set Σ of identities of type τ, let $[\Sigma]$ be the class of all algebras of type τ that satisfies all identities of Σ. Note that, by lemma 1.8, $[\Sigma]$ is a variety, which is said to be defined by Σ. A class of algebras of form $[\Sigma]$ is said to be an equational class.

Lemma 1.12 Let $\mathcal{V}$ be a variety and X an infinite set. Then we have $\mathcal{V} = [Id_X\mathcal{V}]$.

Proof. Let $\mathcal{V}' = [Id_X\mathcal{V}]$. Then, it is trivial to see that $\mathcal{V}' \supseteq \mathcal{V}$ since, by definition of $Id_X\mathcal{V}$, we have $\mathcal{V} \models Id_X\mathcal{V}$. Conversely, since $\mathcal{V}' \supseteq \mathcal{V}$, we deduce the inclusion $Id_X\mathcal{V}' \subseteq Id_X\mathcal{V}$. However, by definition of $\mathcal{V}'$, we also have the reverse inclusion $Id_X\mathcal{V}' \supseteq Id_X\mathcal{V}$. Hence, we obtain the equality between these two sets. Consequently, by proposition 1.9, we have $F_{\bar{X}}\mathcal{V}' = F_{\bar{X}}\mathcal{V}$. Now given any set of variables Y, as X is an infinite set, we have, by corollary 1.11,

$$\theta_Y\mathcal{V}' = \theta_Y(F_{\bar{X}}\mathcal{V}') = \theta_Y(F_{\bar{X}}\mathcal{V}) = \theta_Y\mathcal{V}.$$

thereby

$$F_{\bar{Y}}\mathcal{V}' = F_{\bar{Y}}\mathcal{V}.$$

Now for any $A \in \mathcal{V}'$, we can take a sufficiently large set Y and let $\varphi : \bar{Y} \to A$ be a surjective map. Then, by Theorem 1.5, there exists an epimorphism $\psi : F_{\bar{Y}}\mathcal{V}' \to A$, i.e. $A \in H(F_{\bar{Y}}\mathcal{V}')$. Hence, $A \in H(F_{\bar{Y}}\mathcal{V}) \subseteq \mathcal{V}$, and so $\mathcal{V}' \subseteq \mathcal{V}$. This leads to $\mathcal{V}' = \mathcal{V}$.

By using the preceding lemma, we immediately obtain the following classical result due to G. Birkhoff [13].

Theorem 1.13 (Birkhoff) A class of algebras of the same type is equational if and only if it is a variety.

§2 Pseudovarieties of finite algebras

Pseudovariety is a natural generalization of the concept of variety in Universal Algebra when we restrict our attention on finite algebras. In this session, we introduce in general the theory of pseudovarieties of finite algebras and the discription of pseudovarieties by different means. More specific results when we fix the algebraic type to semigroups will be studied in chapter 3.

Definition 2.1 A pseudovariety is a class of finite algebras of a given type which is closed under the operators H, S and P_{fin}.

For a class $\mathcal{K}$ of finite algebras, the pseudovariety generated by $\mathcal{K}$ is denoted by $\boldsymbol{V}(\mathcal{K})$, i.e., the intersection of all pseudovarieties containing $\mathcal{K}$. In view of Proposition 1.7, we have the following proposition:

Propsition 2.2 $V = HSP_{fin}$.

Given a class $\mathcal{K}$ of algebras, we use $\mathcal{K}^F$ to represent the subclass consisting of the finite algebras of $\mathcal{K}$. It can be easily seen that $\mathcal{V}^F$ is a pseudovariety for any variety $\mathcal{V}$. Say pseudovariety $\boldsymbol{V}$ is an equational pseudovariety if $\boldsymbol{V} = \mathcal{V}^F$ for some variety $\mathcal{V}$.

We remark that not all pseudovarieties are equational. For example, we have the following proposition.

Proposition 2.3 All finite nilpotent semigroups forms a pseudovariety of finite semigroups, denoted it by $\boldsymbol{Nil}$, and it is not equational.

Proof. Let us check the three conditions for a pseudovariety:

1. Let S and T be two finite nilpotent semigroups, with $S^n = 0$ and $T^m = 0$.Then $(S \times T)^{nm} = 0$, hence $S \times T$ is also nilpotent.
2. any subsemigroup of a nilpotent semigroup is obviously nilpotent.
3. if $\varphi : S \to T$ is an epimorphism and $S^n = 0$, then we have $T^n = (\varphi(S))^n = \varphi(S^n) = 0$. Therefore, T is also nilpotent.

Now we prove the second part of the proposition. Suppose that $\boldsymbol{Nil}$ is equational. Take an identity, say $p = q$, where $p, q \in T(X) = X^+$ satisfied by it. Let $r = \max(|p|, |q|)$, where $|x|$ means the length of the word x in X^+. We construct an finite semigroup as follows: Let S be the free semigroup generated by the letters $s_1, ..., s_r$. Then, define an equivalence relation ρ on S by letting all words with length larger than r form an equivalence class, denoted it by $\bar{0}$, and all other words are not equivalent to each other. Then we know easily that ρ is a congruence and $\bar{0}$ is in fact the zero element of the quotient semigruop $\bar{S}$=S/ρ. Then, we can see immediately that $(\bar{S})^{r+1} = \bar{0}$ and clearly this finite semigroup is not satisfied by the identity $p = q$. Therefore, we have proved that the pseudovariety $\boldsymbol{Nil}$ is not equational.

From now on, for the sake of simplicity, we denote the quotient algebras $\theta_{X_n}\mathcal{K}$ and $F_{\bar{X}_n}\mathcal{K}$ by $\theta_n\mathcal{K}$ and $F_n\mathcal{K}$ respectively, where X_n stands for the finite variables set $\{x_1, ..., x_n\}$. We have an useful lemma.

lemma 2.4 Let $\boldsymbol{V}$ be a pseudovariety. Then $F_n\boldsymbol{V} \in \boldsymbol{V}$ if and only if $F_n\boldsymbol{V}$ is finite.

Proof. The direction "$\Rightarrow$" is trivial. For the converse part, we suppose that $|F_n\boldsymbol{V}| = N$. Now, by the definition of $\theta_n\boldsymbol{V}$, we have

$$\theta_n\boldsymbol{V} = \bigcap\{\ker\varphi|\varphi : T(X_n) \to A \text{ for some } A \in \boldsymbol{V}\}$$

Consequently, we can reduce any $\varphi : T(X) \to A$ to a homomorphism $\bar{\varphi} : F_n\boldsymbol{V} \to A$ by the definition of $F_n\boldsymbol{V}$. Let us consider the kernels of all those homomorphisms. In fact, they are the congruences in $F_n\boldsymbol{V}$, so there are only a finite number of distinct kernels since $F_n\boldsymbol{V}$ is finite. Take a complete set of its corresponding homomorphisms, namely, $\{\bar{\varphi}_1, ..., \bar{\varphi}_r\}$. In view of the proof of Theorem 1.5, each homomorphism $\bar{\varphi} : F_n\boldsymbol{V} \to A$ uniquely determines a homomorphism $\varphi : T(X) \to A$. Therefore, $\{\varphi_1, ..., \varphi_r\}$ is a complete set of representives with distinct kernels. Thus, we deduce that

$$\theta_n\boldsymbol{V} = \bigcap_{i=1}^{r} \ker\varphi_i$$

Therefore, $F_n\boldsymbol{V}$ can be subdirectly embedded in a finite direct product of some algebras in $\boldsymbol{V}$, i.e. $F_n\boldsymbol{V} \in HSP_{fin}(\boldsymbol{V}) = \boldsymbol{V}$. The proof is complete.

We are now ready to give a sufficient condition for a pseudovariety to be equational.

Proposition 2.5 For a pseudovariety $\boldsymbol{V}$, if $F_n\boldsymbol{V}$ is finite for any $n \geq 1$, then $\boldsymbol{V}$ is equational, i.e. $V(\boldsymbol{V})^F = \boldsymbol{V}$.

Proof. The inclusion $V(\boldsymbol{V})^F \supseteq \boldsymbol{V}$ is obvious. Conversely, let $A \in V(\boldsymbol{V})$ and A is finite. Suppose that $|A| = n$. Consider a surjective map $\varphi : X_n \to A$. Then there exists an epimorphism $\psi : F_n(V(\boldsymbol{V})) \to A$, in view of Theorem 1.5. We also have the equality $F_{\bar{X}}\mathcal{K} = F_{\bar{X}}V(\mathcal{K})$ for any class of algebras $\mathcal{K}$ since $\mathcal{K}$ and $V(\mathcal{K})$ satisfy the same set of identities on any variables set X, by proposition 1.9. Hence, we deduce that $\psi : F_n\boldsymbol{V} \to A$ is an epimorphism.

i.e. $A \in H(F_n\boldsymbol{V})$, and therefore $A \in \boldsymbol{V}$ since $F_n\boldsymbol{V}$ is finite and by the lemma above.

Corollary 2.6 The relation $\boldsymbol{V}(A) = V(A)^F$ holds for any finite algebra A.

Proof. Since $\boldsymbol{V}(A)$ and A satisfy the same identities on any variables set, we have

$$\theta_n\boldsymbol{V}(A) = \theta_n(A) = \bigcap\{\ker\varphi|\varphi : T(X_n) \to A\}$$

for any $n \geq 1$. Now, as A is finite, and any homomorphism $\varphi : T(X_n) \to A$ is uniquely determined by its restriction on each set X_n, i.e $\varphi|_{X_n} : X_n \to A$, whence there are only a finite number of such homomorphisms. By the same reason in lemma 2.4, we know that there is a subdirectly embeding from $F_n\boldsymbol{V}(A)$ to a finite product of A. Therefore, $F_n\boldsymbol{V}(A)$ is finite since A is finite, so $\boldsymbol{V}(A)$ is equational by proposition 2.5, i.e. $\boldsymbol{V}(A) = V(\boldsymbol{V}(A))^F = V(A)^F$.

Similar to the discussion in section 1, we know that the set of all pseudovarieties of finite algebras of a fixed type forms a lattice under $\cap$ and $\vee$, where $\boldsymbol{V}_1 \vee \boldsymbol{V}_2 = \boldsymbol{V}(\boldsymbol{V}_1 \cup \boldsymbol{V}_2)$ for any two pseudovarieties $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$.

Proposition 2.7 Any pseudovariety is a union of a directed family of equational pseudovarieties. If the algebraic type is finite, then the family may be chosen so as to constitute a chain.

Proof. Let $\boldsymbol{V}$ be a pseudovariety and let $\mathcal{C}$ be a complete set of representatives of the isomorphism classes of the elements of $\boldsymbol{V}$. If $A_1, A_2 \in \mathcal{C}$, then there exists $B \in \mathcal{C}$ such that $A_1 \times A_2 \cong B$, and so it can be easily checked that $\boldsymbol{V}(A_1) \vee \boldsymbol{V}(A_2) = \boldsymbol{V}(B)$. Hence, the set of all pseudovarieties of the form $\boldsymbol{V}(A)$ with A running through the set $\mathcal{C}$ is a directed family of equational pseudovarieties by corollary 2.6, and the union is clearly $\boldsymbol{V}$.

If the algebraic type is finite, then for any positive integer n, there are only a finite number of algebras with cardinality n under isomorphism. Hence, we may assume that $\mathcal{C}$ is countable, say $\mathcal{C} = \{A_1, A_2, ...\}$, and so $\boldsymbol{V}$ is the union of the ascending chain of equational pseudovarieties

$$\boldsymbol{V} = \bigcup_{n\geq 1} \boldsymbol{V}(S_n)$$

where $S_n = A_1 \times ... \times A_n$ for any $n \geq 1$.

In the case if the type of the pseudovariety is finite, we have another form for the preceding result which is a classical result due to Eilenberg and Schützenberger [24].

Theorem 2.8 Let $\boldsymbol{V}$ be a pseudovariety of finite type. Then there exists a sequence $(\varepsilon_n)_{n\geq 1}$ of identities such that $\boldsymbol{V} = \bigcup_{k\geq 1}[\varepsilon_n : n \geq k]^F$.

Proof. We adopt here the proof provided by Ash in [10]. Take $(\boldsymbol{V}_n)_{n\geq 1}$ be an ascending chain of equational pseudovarieties such that $\boldsymbol{V} = \bigcup_{n\geq 1} \boldsymbol{V}_n$ by proposition 2.7 above. Consider the class $\mathcal{D}$ of all finite algebras of the type in question which is a complement of $\boldsymbol{V}$. Let $\{B_1, B_2, ...\}$ be a countable enumeration of a complete set of representatives of the isomorphism classes of $\mathcal{D}$, which can be chosen since the algebraic type is finite.

Now for each $i, j \geq 1$, we have $B_j \notin \boldsymbol{V}_i$. As V_i is an equational pseudovariety, there exists an identity ε_{ij} that is valid in $\boldsymbol{V}_i$ but fails in B_j. Let us consider the set $\Sigma_i = \{\varepsilon_{ij} : j \leq i\}$ of identities for $i \geq 1$. Then, any set Σ_i is finite and so we have a countable sequence of identities

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup ...$$

We assert that this sequence is what we want.

If $A \in \boldsymbol{V}$, suppose that $A \in \boldsymbol{V}_n$, then $A \in \boldsymbol{V}_i$ for every $i \geq n$, and so $A \models \varepsilon_{ij}$ for any $i \geq n$ and any $j \geq 1$, i.e. A satisfies the identities $\Sigma_n \cup \Sigma_{n+1} \cup ....$ Conversely, if $A \notin \boldsymbol{V}$, then $A \cong B_j$ for some j. This means that A fails to satisfy all the identities ε_{ij} for every $i \geq j$ in the above sequence of identities Σ.

We give still another concept to characterize the pseudovarieties which will simplify the description and will be useful later on in chapter 5.

Definition 2.9 A generalized variety is a directed union of varieties.

From proposition 2.7, we can immediately deduce the following result.

Proposition 2.10 A class of finite algebras is a pseudovariety if and only if it is of the form $\mathcal{W}^F$ for some generalized variety $\mathcal{W}$.

§3 Implicit operations and pseudoidentities

Implicit operations were introduced by Banaschewski [11] and Reiterman [47], aiming to estiblish an analogous Birkhoff theorem for pseudovarieties.

Let $\boldsymbol{V}$ be a pseudovariety of finite algebras. An operation defined on the members of $\boldsymbol{V}$ that commutes with all homomorphisms is said to be an implicit operation. Formally, for an integer $n \geq 1$, an n-ary implicit operation π on $\boldsymbol{V}$ is a family $\pi = (\pi_A)$ indexed by the elements $A \in \boldsymbol{V}$ such that

1. for each $A \in \boldsymbol{V}$, $\pi_A : A^n \to A$ is a mapping,
2. for each homomorphism $\varphi : A \to B$ between the elements of $\boldsymbol{V}$, the following diagram is commutative:

$$\begin{array}{ccc} A^n & \xrightarrow{\pi_A} & A \\ {\scriptstyle \varphi^n}\downarrow & & \downarrow{\scriptstyle \varphi} \\ B^n & \xrightarrow[\pi_B]{} & B \end{array}$$

We represent the set of all n-ary implicit operations on $\boldsymbol{V}$ by $\bar{\Omega}_n \boldsymbol{V}$. Suppose the algebraic type of $\boldsymbol{V}$ is $\tau = (\mathcal{O}, \alpha)$. Then, for any $f \in \mathcal{O}_k$ and $\pi_1, ..., \pi_k \in \bar{\Omega}_n \boldsymbol{V}$, we define $f(\pi_1, ..., \pi_k) = \pi$ such that for any $A \in \boldsymbol{V}$,

$$\pi_A = f(\pi_{1A}, ..., \pi_{kA}) : A_n \to A$$

and for any homomorphism $\varphi : A \to B$ between the elements of $\boldsymbol{V}$, we have the following equalities:

$$\begin{aligned}\varphi\pi_A(a_1,...,a_n) &= \varphi f(\pi_{1A}(a_1,...,a_n),...,\pi_{kA}(a_1,...,a_n)) \\ &= f(\varphi\pi_{1A}(a_1,...,a_n),...,\varphi\pi_{kA}(a_1,...,a_n)) \\ &= f(\pi_{1B}\varphi^n(a_1,...,a_n),...,\pi_{kB}\varphi^n(a_1,...,a_n)) \\ &= f(\pi_{1B},...,\pi_{kB})\varphi^n(a_1,...,a_n) \\ &= \pi_B\varphi^n(a_1,...,a_n)\end{aligned}$$

for any $a_1, ..., a_n \in A$.

Thus, the π we defined above is also in $\bar{\Omega}_n\boldsymbol{V}$, and hence we can endow an algebraic structure τ on the set $\bar{\Omega}_n\boldsymbol{V}$.

Now, we consider the following projections $x_1, ..., x_n$, say,

$$(x_i)_A(a_1,...,a_n) = a_i, \forall A \in \boldsymbol{V}, a_i \in A, i = 1,...,n.$$

They are obviously implicit operations on $\boldsymbol{V}$. We can hence use the notation $\Omega_n\boldsymbol{V}$ to represent the subalgebra of $\bar{\Omega}_n\boldsymbol{V}$ generated by those projections and call its elements the n-ary explicit operations on $\boldsymbol{V}$. Compare with the variables set $X_n = x_1, ..., x_n$ and the free algebra $F_{X_n}\boldsymbol{V}$, we easily know that $\Omega_n\boldsymbol{V}$ is isomorphic to $F_{X_n}\boldsymbol{V}$.

We now introduce an important and useful implicit operation on the pseudovarieties of finite semigroups. For any finite semigroup S and $s \in S$, let $\{s, s^2, ..., s^r, ..., s^{r+k-1}\}$ be the subsemigroup generated by s with $s^{r+k} = s^r$, where r is the smallest integer such that $s^r = s^p$ for some $p > r$, called it the index of s and k is called the periodic of s. Also, clearly $\{s^r, ..., s^{r+k-1}\}$ is a subgroup of the semigroup S (the reader is refered to [28] for all these termnologies and properties). Therefore, there is exactly one idempotent of the form s^n with $n \geq 1$ in that subgroup generated by s, denoted it by s^ω. We thus define a new unary operation $x \mapsto x^\omega$ in any finite semigroup. It is easy to verify that this operation commutes with all homomorphisms, i.e., if $\varphi : T \to S$ is a semigroup homomorphism and $s \in S$, then $\varphi(s^\omega) = (\varphi s)^\omega$. Hence, this operation is an unary implict operation on any pseudovariety of finite semigroups. This implicit operation may not explicit, for example, on $\boldsymbol{S}$, the pseudovariety of all finite semigroups, or even on $\boldsymbol{Com}$, the pseudovariety of all finite commutative semigroups. It may also be explicit, for example, on any pseudovariety of finite idempotent semigroups, where it is equivalent to any variable $x \in X$, or on the pseudovariety $\boldsymbol{Z}_p$ which is generated by the cyclic group of integers with order p, where it is equivalent to the explicit operation x^p for any $x \in X$.

We have just seen that the inclusion $\Omega_n \boldsymbol{V} \subseteq \bar{\Omega}_n \boldsymbol{V}$ may be strict.

Now we define a topology on the set $\bar{\Omega}_n \boldsymbol{V}$ by the construction of the following metric: given $\pi, \rho \in \bar{\Omega}_n \boldsymbol{V}$, let

$$r(\pi, \rho) = \min\{|A| : A \in \boldsymbol{V}, \pi_A \neq \rho_A\}$$

where the convention $\min \emptyset = \infty$ is adopted. Define the distance $d(\pi, \rho)$ between π and ρ by

$$d(\pi, \rho) = 2^{-r(\pi,\rho)}$$

It can then be immediately checked the validity of the following properties:

1. $d(\pi, \rho) = 0$ if and only if $\pi = \rho$,
2. $d(\pi, \rho) = d(\rho, \pi)$,
3. $d(\pi, \rho) \leq \max\{d(\pi, \sigma), d(\sigma, \rho)\}$,
4. $d(f(\pi_1, ..., \pi_k), f(\rho_1, ..., \rho_k)) \leq \max\{d(\pi_i, \rho_i) : i = 1, ..., k\}$ for any $f \in \mathcal{O}_k$.

Therefore, this metric d defines a topology on $\bar{\Omega}_n \boldsymbol{V}$. Under this topology, Reiterman [47] showed that the subspace $\Omega_n \boldsymbol{V}$ is dense in $\bar{\Omega}_n \boldsymbol{V}$. i.e., every implicit operation is the limit of a sequence of explicit ones. For instance, it is easy to verify that the unary operation ω is the limit of the sequence of the explicit operations $x^{n!}$.

The algebra $\bar{\Omega}_n \boldsymbol{V}$ plays an analogous role in pseudovarieties as the $\mathcal{V}$-free algebra $F_n\mathcal{V}$ in varieties. We have the following theorem due to Reiterman [47].

Theorem 3.1 Let A be a finite algebra. Then $A \in \boldsymbol{V}$ if and only if there exist a positive integer n and a surjective continuous homomorphism $\bar{\Omega}_n \boldsymbol{V} \to A$.

The second theorem we will present in this section is also due to Reiterman. It is an analogue of Birkhoff's theorem on the definition of varieties by identities. Let $\boldsymbol{V}$ be a pseudovariety. A pseudoidentity on $\boldsymbol{V}$ is a formal identity of implicit operations, $\pi = \rho$, where $\pi, \rho \in \bar{\Omega}_n \boldsymbol{V}$ for some n. We say that an algebra A satisfies the pseudoidentity $\pi = \rho$ if $\pi_A = \rho_A$. If Σ is a set of pseudoidentities on $\boldsymbol{V}$, we write $[[\Sigma]]_{\boldsymbol{V}}$ for the class of all algebras in $\boldsymbol{V}$ that satisfies each pseudoidentities in Σ. It is easy to see that such a class is a pseudovariety, and the converse was also proved by Reiterman [47].

Theorem 3.2 Let $\boldsymbol{V}$ be a pseudovariety of finite algebras and let $\boldsymbol{W}$ be a subclass of $\boldsymbol{V}$. Then $\boldsymbol{W}$ is itself a pseudovariety if and only if $\boldsymbol{W} = [[\Sigma]]_{\boldsymbol{V}}$ for some set Σ of pseudoidentities on $\boldsymbol{V}$.

In section 2, we have seen that $\boldsymbol{Nil}$, the pseudovariety of finite nilpotent semigroups, is not equational. Now, by the notion of pseudoidentities, we have the following characterization.

Proposition 3.3 $\boldsymbol{Nil} = [[x^\omega y = y x^\omega = x^\omega]]_{\boldsymbol{S}}$.

Proof. For any finite nilpotent semigroup S and $x \in S$, as the zero element is the only idempotent in the nilpotent semigroup, we have $x^\omega = 0$, therefore, for any $y \in S$, we deduce that $x^\omega y = y x^\omega = x^\omega = 0$.

Conversely, suppose a finite semigroup S satisfies the pseudoidentity $x^\omega y = y x^\omega = x^\omega$ for any $y \in S$, we then deduce that $x^\omega = 0$ for any $x \in S$, and this implies that $E(S)$, the set of all idempotents of S, consists only the zero element. Let $n = |S|$, the cardinality of S. Then, we have $S^n = SE(S)S$ (see chapter 1, [43]) $= S \cdot 0 \cdot S = 0$, i.e., S is nilpotent.

Let us consider another example.

Let $\boldsymbol{G}$ be the class of all finite groups, we regard it as a subclass of

$\boldsymbol{M}$, the pseudovariety of all finite monoids. Then we can see that $\boldsymbol{G}$ is a pseudovariety, and $\boldsymbol{G} = [[x^\omega y = yx^\omega = y]]_{\boldsymbol{M}}$. For simplification, we call a sub-pseudovariety of $\boldsymbol{M}$ an M-variety. Similarly, we call a sub-pseudovariety of $\boldsymbol{S}$ an S-variety.

In closing this chapter, a natural question on implicit operations arises. Are there examples of implicit operations on the pseudovariety of finite semigroups other than those obtained by just combining words and ω-powers? Some examples were given in [67]. Indeed, $\Omega_n \boldsymbol{V}$ is at most countable by definition for any pseudovariety $\boldsymbol{V}$, but Almeida and Azevedo recently proved that $\bar{\Omega}_n \boldsymbol{S}$ has the power of the continuum [8].

Chapter 2 Algebraic automata and formal languages theory

§1 Semigroup automata theory

Let A be a finite set $\{a_1, a_2, ..., a_n\}$, called it an alphabet set. We use A^* to represent the free monoid generated by A, i.e., all words of letters in A including the empty word, which is the identity element in A^*, denoted by λ. The multiplication in A^* of u, v is just their concatenation. A language over the alphbet A is a subset $L \subseteq A^*$. We denote it by A^+, that is, the free semigroup $A^* \backslash \{\lambda\}$.

Definition 1.1 An Automaton is a triple $\mathcal{A} =< A, Q, \circ >$, where A is an alphabet set $\{a_1, a_2, ..., a_n\}$, Q is a finite set called a set of states, $\circ$ is a binary map: $Q \times A \to Q$.

A language L over the alphbet A is said to be recognized by the automaton $\mathcal{A}$ if there exists $q_0 \in Q$ called the initial state, a subset $T \subseteq Q$ called the set of terminal states and we have that $u \in L \Leftrightarrow q_0 \circ u \in T$, where $q_i \circ u$ is inductively defined by the equations:

$$q_i \circ \lambda = q_i,$$

$$q_i \circ (a_j w) = (q_i \circ a_j) \circ w.$$

The map $\circ : Q \times A \to Q$ can thus be extended to a map from $Q \times A^*$ to Q without indication furthermore.

We now introduce some operations on languages over the same alphabet.

Let L, K be two languages over the alphabet A. Define the following operations on L and K:

1. Boolean operations: $L \cap K, L \cup K, L \setminus K$ are just the usuall Boolean operations on sets.

2. Product: $LK = \{uv \in A^* : u \in L, v \in K\}$.

3. Star: L^* is submonoid of A^* generated by L. i.e., $L^* = L \cup L^2 \cup L^3 \cup \ldots$

4. Left(right) quotient of L by K:

$$K^{-1}L = \{v \in A^* : Kv \cap L \neq \emptyset\} = \{v \in A^* : \exists u \in K, s.t.\ uv \in L\}$$
$$LK^{-1} = \{v \in A^* : vK \cap L \neq \emptyset\} = \{v \in A^* : \exists u \in K, s.t.\ uv \in L\}$$

Definition 1.2 If $\eta : A^* \to M$ is a monoid homomorphism and M is a finite monoid, then η is said to recognize $L \in A^*$ if there exists a subset H in M such that $L = \eta^{-1}(H)$. By extension, we also say that M recognizes L.

Theorem 1.3 A language L is recognized by an automaton if and only if it is recognized by a finite monoid.

Proof. "$\Rightarrow$". Suppose that $\mathcal{A} =< A, Q, \circ >$ recognizes L with an initial state q_0 and a set of terminal states T. Then, for any $u \in A$, we can define a mapping

$$\hat{u} : Q \to Q$$
$$q \mapsto q \circ u$$

Clearly, $\hat{u}$ is an element of $\mathcal{T}(Q)$, the transition monoid of Q, with the following property: $\hat{u_1}\hat{u_2} = (u_1\hat{}u_2)$. So

$$\varphi : A^* \to \mathcal{T}(Q)$$
$$u \mapsto \hat{u}$$

is an monoid embedding, and the image $\varphi(A^*)$, denoted by $M(\mathcal{A})$, called the transition monoid of the automaton $\mathcal{A}$, is a submonoid of $\mathcal{T}(Q)$. As Q is a finite set, we know $\mathcal{T}(Q)$, hence $M(\mathcal{A})$ is also finite. Then, we assert that $\varphi : A^* \to M(\mathcal{A})$ with $\varphi(L) \subseteq M(\mathcal{A})$ recognizes L. To prove this statement, we only need to show that $\varphi^{-1}(\varphi(L)) = L$. For any $u \in \varphi^{-1}(\varphi(L))$, there is, by definition, an element $v \in L$ such that $\varphi u = \varphi v$. In other words, this means that $\hat{u}, \hat{v}$ are the same functions in $\mathcal{T}(Q)$. In particular, we have $q_0 \circ u = \hat{u}(q_0) = \hat{v}(q_0) = q_0 \circ v$ and, since $v \in L$ implies $q_0 \circ v \in T$, hence $q_0 \circ u \in T$ and we therefore deduce from this fact that $u \in L$. The opposite inclusion is obvious and therefore the statement is proved.

"$\Leftarrow$". Suppose $\eta : A^* \to M$ recognizes L. i.e., there exists $H \subseteq M$ such that $\eta^{-1}(H) = L$. We construct an automaton $\mathcal{A} =< A, M, \circ >$, where

$$\begin{aligned} \circ : M \times A^* &\to M \\ (m, \omega) &\mapsto m \circ \omega = m \cdot \eta(\omega) \end{aligned}$$

and take the initial state $1 \in M$, the set of terminal states $H \subseteq M$. Then we have

$$\omega \in L = \eta^{-1}(H) \Leftrightarrow \eta(\omega) \in H \Leftrightarrow 1 \circ \omega = 1 \cdot \eta(\omega) \in H$$

Thus the automaton $\mathcal{A}$ recognizes L. The proof is complete.

The following statements are expressed in terms of monoids. They consist of some classical results of the theory of automata.

Proposition 1.4 The languages A^* and $\emptyset$ are recognized by the trivial monoid $\{1\}$.

Proof. Let $\varphi : A^* \to \{1\}$ be a trivial morphism. Then it is clear that $\varphi^{-1}\{1\} = A^*$ and $\varphi^{-1}\emptyset = \emptyset$.

Proposition 1.5 Let L be a language of A^*. If M recognizes L, then M recognizes $A^* \setminus L$.

Proof. Let $\eta : A^* \to M$ and $P \subseteq M$ be such that $L = \eta^{-1}(P)$. Then $\eta^{-1}(M \setminus P) = A^* \setminus L$.

Proposition 1.6 Let L_1, L_2 be two languages of A^* recognized respectively by the monoids M_1, M_2. Then $L_1 \cap L_2$ and $L_1 \cup L_2$ are recognized by $M_1 \times M_2$.

Proof. Let $\eta_1 : A^* \to M_1, \eta_2 : A^* \to M_2, P_1 \subseteq M_1$, and $P_2 \subseteq M_2$ be such that $L_1 = \eta_1^{-1}(P_1)$ and $L_2 = \eta_2^{-1}(P_2)$. Let $\eta : A^* \to M_1 \times M_2$ be the morphism defined by $\eta u = (\eta_1 u, \eta_2 u)$. We then have the formula

$$\eta^{-1}(P_1 \times P_2) = L_1 \times L_2$$

and

$$\eta^{-1}((P_1 \times M_2) \cup (M_1 \times P_2)) = L_1 \times L_2.$$

Proposition 1.7 Let $\varphi : A^* \to B^*$ be a morphism of free monoids and $L \subseteq B^*$ a language recognized by a monoid M. Then M recognizes $\varphi^{-1}(L)$ also.

Proof. Let $\eta : B^* \to M$ and $P \subseteq M$ be such that $L = \eta^{-1}(P)$. Then $\varphi^{-1}(L) = \varphi^{-1}(\eta^{-1}(P)) = (\eta\varphi)^{-1}(P)$, which proves that $\varphi^{-1}(L)$ is recognized by the morphism $\eta\varphi : A^* \to M$.

Proposition 1.8 Let L be a language of A^* which is recognized by M. Let K be an arbitrary language of A^*. Then M recognizes $K^{-1}L$ and LK^{-1}.

Proof. Let $\eta : A^* \to M$ and $P \subseteq M$ such that $L = \eta^{-1}(P)$. Put

$$Q = \{m \in M : \exists u \in K, (\eta u)m \in P\}$$

It then follows that

$$\begin{aligned}
\eta^{-1}(Q) &= \{v \in A^* : \eta v \in Q\} \\
&= \{v \in A^* : \exists u \in K, (\eta u)(\eta v) \in P\} \\
&= \{v \in A^* : \exists u \in K, \eta(uv) \in P\} \\
&= \{v \in A^* : \exists u \in K, uv \in L\} \\
&= K^{-1}L.
\end{aligned}$$

Thus, $K^{-1}L$ is recognized by M. The proof for LK^{-1} is similar and hence it is omitted.

Definition 1.9 Let L be a language over an alphabet A. Define a congruence $\sim_L$ in A^* by: $u \sim_L v$ if and only if for any $x, y \in A^*$,

$xuy \in L \Leftrightarrow xvy \in L$. The quotient monoid $A^*/\sim_L$ is called the syntactic monoid of L, denote it by $M(L)$.

Theorem 1.10 $M(L)$ is the smallest monoid that recognizes L. That is to say a finite monoid M recognizes L if and only if $M(L) \prec M$, where $M(L) \prec M$ means that $M(L)$ is a homomorphic image of some submonoids of M.

Proof.

1) $M(L)$ recognizes L.

Let $\eta : A^* \to A^*/\sim_L = M(L)$ be the canonical morphism. Take $H = \eta(L) \subseteq M(L)$. We shall show that $L = \eta^{-1}(H)$. The inclusion from left to right is obvious. Conversely, for any $u \in \eta^{-1}(H) = \eta^{-1}(\eta(L))$, there exists $v \in L$ such that $\eta u = \eta v$, i.e. $u \sim_L v$. Since $v \in L$, we also have $u \in L$ by taking $x = y = 1$ in the definition of $\sim_L$. Therefore, $M(L)$ recognizes L.

2) $M(L) \prec M \Rightarrow M$ recognizes L.

By definition, there exists a submonoid N of M and a surjective homomorphism $\beta : N \to M(L)$. Let us associate with each letter $a \in A$ an element $\varphi(a)$ of $\beta^{-1}\eta(a)$. Thus, we can define a function $\varphi : A \to N$, which can be extended to a morphism $\varphi : A^* \to N$ by letting $\eta = \beta\varphi$.

Put $P = \alpha\beta^{-1}\eta(L) \subseteq M$. It then follows that

$$(\alpha\varphi)^{-1}P = \varphi^{-1}\alpha^{-1}\alpha\beta^{-1}\eta(L) = \varphi^{-1}\beta^{-1}\eta(L) = \eta^{-1}\eta(L) = L.$$

This shows that $\alpha\varphi : A^* \to M$ indeed recognizes L.

3) M recognizes $L \Rightarrow M(L) \prec M$.

By our definition, there exists a homomorphism $\varphi : A^* \to M$ and a subset P of M such that $L = \varphi^{-1}(P)$. Put $N = \varphi(A^*)$; N is a submonoid of M. Suppose that $\varphi u = \varphi v$ and $xuy \in L$. Then, $\varphi(xuy) = \varphi(xvy) \in \varphi(L) = P$ and therefore $xvy \in L$ since $\varphi^{-1}(P) = L$. It follows from this fact that $\varphi u = \varphi v$ implies that $u \sim_L v$. Thus, φ induces a surjective morphism $\pi : N \to A^*/\sim_L = M(L)$. i.e. $M(L) \prec M$.

The following proposition discribes the link between the syntactic monoids and the language operations. In fact, it is an immediate consequence of propositions 1.5-1.8.

Proposition 1.11 Let L, L_1, L_2 be recognizable languages over the set of alphabets A. Let K be an arbitrary language over A. Then we have the following statements:

1. $M(A^*\backslash L) = M(L)$.
2. $M(L_1 \cap L_2) \prec M(L_1) \times M(L_2), M(L_1 \cup L_2) \prec M(L_1) \times M(L_2)$.
3. $M(LK^{-1}) \prec M(L), M(K^{-1}L) \prec M(L)$.
4. if $\varphi : B^* \to A^*$ is a morphism of free monoids, then $M(\varphi^{-1}L) \prec M(L)$.

§2 Variety theorem

It is well known that the syntactic monoid is an effective tool in studying the recognizable languages, and because their syntactic monoid is finite, many questions on it can be acturally solved algorithmically. The syntactic monoid is decided when the recognizable language is given (algorithms to determine the syntactic monoid when the language is given by an automaton or a rational form were given in [43]). But, on the other hand, there are usually non-unique languages over an alphabet set which can be recognized by a given finite monoid. So, we need the concept of variety of languages, and we find that there is a one-one corresponding relationship between the varieties of languages and the pseudovarieties of finite monoids.

Definition 2.1 A variety of languages $\mathcal{V}$ is a function which associates with any alphabet set A to a class of languages $A^*\mathcal{V}$ over that alphabet, such that

1. for any A, $A^*\mathcal{V}$ is a Boolean algebra.

2. if $\varphi : A^* \to B^*$ is a free monoid morphism, $L \in B^*\mathcal{V}$ implies $\varphi^{-1}(L) \in A^*\mathcal{V}$.

3. if $L \in A^*\mathcal{V}$ and if $a \in A$, then $a^{-1}L$ and $La^{-1} \in A^*\mathcal{V}$.

For a pseudovariety of finite monoids $\boldsymbol{V}$, we introduce a function $\boldsymbol{V} \to \mathcal{V}$ by: for any alphabet A, let $A^*\mathcal{V}$ be the set of languages of A^* whose syntactic monoid is within $\boldsymbol{V}$.

The following proposition can be regarded as an equivalent definition of definition 2.1.

Proposition 2.2 $A^*\mathcal{V}$ is the set of languages of A^* recognized by a monoid in $\boldsymbol{V}$.

Proof. For $L \in A^*\mathcal{V}$, we have $M(L) \in \boldsymbol{V}$ which recognizes L by definition 2.1. Conversely, if L is a language of A^* and recognized by $M \in \boldsymbol{V}$, we have $M(L) \prec M$ by theorem 1.10. Therefore, we have $M(L) \in \boldsymbol{V}$ by the definition of pseudovariety. Thus, $L \in A^*\mathcal{V}$.

We find that the function $\mathcal{V}$ under the function defined above is acturally a variety of languages.

Proposition 2.3 Let $\boldsymbol{V}$ be a pseudovariety of finite monoids. If $\boldsymbol{V} \to \mathcal{V}$, then $\mathcal{V}$ is a variety of languages.

Proof. Let $L, L_1, L_2 \in A^*\mathcal{V}$ and $a \in A$. Then by definitions, we know that $M(L), M(L_1)$ and $M(L_2)$ are in $\boldsymbol{V}$. From the results of section 1 and that $\boldsymbol{V}$ is a pseudovariety, we know that $M(A^*\backslash L), M(L_1 \cap L_2), M(L_1 \cup L_2), M(a^{-1}L)$ and $M(La^{-1}) \in \boldsymbol{V}$. Hence the corresponding languages are clearly in $A^*\mathcal{V}$. For condition (2), let $\varphi : A^* \to B^*$ be a free monoid morphism, $L \in B^*\mathcal{V}$. Also by proposition 1.11, we have $M(\varphi^{-1}L) \prec M(L) \in \boldsymbol{V}$. Hence, $M(\varphi^{-1}L) \in \boldsymbol{V}$ and $\varphi^{-1}(L) \in A^*\mathcal{V}$.

The next proposition tells us that the function $\boldsymbol{V} \to \mathcal{V}$ is injective.

Proposition 2.4 Let $\boldsymbol{V}, \boldsymbol{W}$ be two pseudovarieties of finite monoids. Suppose that $\boldsymbol{V} \to \mathcal{V}$ and $\boldsymbol{W} \to \mathcal{W}$. Then $\boldsymbol{V} \subseteq \boldsymbol{W}$ if and only if for every set of alphabets A, $A^*\mathcal{V} \subseteq A^*\mathcal{W}$. In particular, $\boldsymbol{V} = \boldsymbol{W}$ if and only if $\mathcal{V} = \mathcal{W}$.

Proof. If $\boldsymbol{V} \subseteq \boldsymbol{W}$, it follows immediately from the definitions that $A^*\mathcal{V} \subseteq A^*\mathcal{W}$ for any alphabet A. The converse is based on the following proposition.

Proposition 2.5 Let $\boldsymbol{V}$ be a pseudovariety of finite monoids and $M \in \boldsymbol{V}$. Then there exists a finite alphabet A and languages $L_1, ..., L_k \in A^*\mathcal{V}$ such that $M \prec M(L_1) \times ... \times M(L_k)$.

Proof. Since M is finite, there exists a finite alphabet set A and a surjective map $\alpha : A \to M$. We extend it to an epimorphism $\varphi : A^* \to M$. Now, for any $m \in M$, the language $L_m = \varphi^{-1}(m)$ is recognized by M and therefore $L_m \in A^*\mathcal{V}$. Let $\sim_{L_m}$ be the syntactic congruence in A^* for each language L_m. We first assert that

$$\ker \varphi = \bigcap_{m \in M} \sim_{L_m}$$

To prove it, for any $(u, v) \in \ker \varphi$, we have, for any $(x, y) \in A^*$,

$$xuy \in L_m \Leftrightarrow \varphi(xuy) = m \Leftrightarrow \varphi(x)\varphi(u)\varphi(y) = m \Leftrightarrow \varphi(x)\varphi(v)\varphi(y) = m$$

$$\Leftrightarrow \varphi(xvy) = m \Leftrightarrow xvy \in L_m$$

Hence $u \sim_{L_m} v$ for any $m \in M$; conversely, if $u \sim_{L_m} v$ for any $m \in M$, then we suppose that $\varphi(u) = m \in M$ since φ is surjective. Thus $u \sim_{L_m} v$ implies that

$$1 \cdot u \cdot 1 = u \in L_m \Leftrightarrow 1 \cdot v \cdot 1 = v \in L_m$$

so $\varphi(u) = m$ and hence $u \in L_m$. As this is equivalent to say that $v \in L_m$ hence $\varphi(v) = m$ as well, i.e. $(u, v) \in \ker \varphi$.

By above, we conclude that there exists a subdirect embedding

$$M \hookrightarrow \prod_{m \in M} A^*/\sim_{L_m} = \prod_{m \in M} M(L_m)$$

and the proposition is proved since M is finite.

Now we complete the proof of proposition 2.4.

Suppose that $A^*\mathcal{V} \subseteq A^*\mathcal{W}$ for any finite alphabet A and let $M \in \boldsymbol{V}$. Then by propositon 2.5 above, we have $M \prec M(L_1) \times ... \times M(L_k)$, where $L_1, ..., L_k \in A^*\mathcal{V}$ for some finite alphabet A. We deduce from this that $L_1, .., L_k \in A^*\mathcal{W}$, i.e. $M(L_1), ..., M(L_k) \in \boldsymbol{W}$. Therefore, $M \in \boldsymbol{W}$.

Now we are ready to prove that the function $\boldsymbol{V} \to \mathcal{V}$ is surjective. The proof follows from J. E. Pin [43].

Proposition 2.6 For every variety of languages $\mathcal{V}$, there exists a pseudovariety of finite monoids $\boldsymbol{V}$ such that $\boldsymbol{V} \to \mathcal{V}$.

Proof. We construct $\boldsymbol{V}$ from $\mathcal{V}$ as following:

$$\boldsymbol{V} = \boldsymbol{V}(\{M(L) : L \in A^*\mathcal{V} \text{ for some alphabet } A\})$$

Suppose that $\boldsymbol{V} \to \mathcal{W}$; we shall in fact show that $\mathcal{V} = \mathcal{W}$. First of all, we let $L \in A^*\mathcal{V}$, then we have $M(L) \in \boldsymbol{V}$ by definition and so $L \in A^*\mathcal{W}$. i.e. for every alphabet A, we have $A^*\mathcal{V} \subseteq A^*\mathcal{W}$.

Conversely, let $L \in A^*\mathcal{W}$, then $M(L) \in \boldsymbol{V}$ and so there exist an integer $n > 0$, alphabets $A_i, i = 1, ..., n$ and languages $L_i \in A_i^*\mathcal{V}, i = 1, ..., n$ such that $M(L) \prec M(L_1) \times ... \times M(L_n) = M$. Let $\pi_i : M \to M(L_i)$ be the ith projection defined by $\pi_i(m_1, .., m_n) = m_i$. Since $M(L)$ divides M, M recognizes L and there exists a morphism $\varphi : A^* \to M$ and a subset P of M such that $L = \varphi^{-1}(P)$. Finally, we put $\varphi_i = \pi_i\varphi$. We let $\eta_i : A_i^* \to M(L_i)$ be the syntatic morphism of $L_i, i = 1, ..., n$. Since $\eta_i : A_i^* \to M(L_i)$ is surjective, A^* is a free monoid, so there exists a morphism $\psi_i : A^* \to A_i^*$ such that $\varphi_i = \psi_i\eta_i$. We summarize these morphisms by the following commutative diagram:

$$\begin{array}{ccc} A^* & \xrightarrow{\psi_i} & A_i^* \\ {\scriptstyle\varphi}\big\downarrow & \overset{\varphi_i}{\searrow} & \big\downarrow{\scriptstyle\eta_i} \\ M & \xrightarrow[\pi_i]{} & M(L_i) \end{array}$$

We still need to prove that $L \in A^*\mathcal{V}$. This can be proved via the following three steps:

1. As $L = \varphi^{-1}(P) = \bigcup_{m \in P} \varphi^{-1}(m)$, and $\mathcal{V}$ is a variety of languages i.e. closed under union, it suffices to show that $\varphi^{-1}(m) \in A^*\mathcal{V}$ for any $m \in M$.

2. For any $m \in M$, suppose that $m = (m_1, ..., m_n)$, where $m_i \in M(L_i)$. We have
$$\varphi^{-1}(m) = \varphi^{-1}((m_1, ..., m_n)) = \bigcap_{i=1}^{n} \varphi_i^{-1}(m_i)$$
The reason for this is
$$u \in \varphi^{-1}(m) \Leftrightarrow \varphi(u) = m = (m_1, ..., m_n) \Leftrightarrow \pi_i\varphi(u) = m_i, \forall i = 1, ..., n$$
$$\Leftrightarrow \varphi_i(u) = m_i, \forall i = 1, ..., n \Leftrightarrow u \in \varphi_i^{-1}(m_i), \forall i = 1, ..., n \Leftrightarrow u \in \bigcap_{i=1}^{n} \varphi_i^{-1}(m_i).$$
Since $A^*\mathcal{V}$ is closed under intersection, it suffices to show that $\varphi_i^{-1}(m_i) \in A^*\mathcal{V}$ for $i = 1, ..., n$.

3. $\varphi_i = \eta_i\psi_i$ implies $\varphi_i^{-1}(m_i) = \psi_i^{-1}(\eta_i^{-1}m_i)$. Since $\psi_i : A^* \to A_i^*$ is a free monoid morphism and a variety of language is closed under the inverse of free monoid morphisms, we need only to prove that $\eta_i^{-1}m_i \in A_i^*\mathcal{V}$ for any $i = 1, ..., n$, which follows from the following lemma.

Lemma 2.7 For a variety of languages $\mathcal{V}$ and an alphabet set A, let $L \in A^*\mathcal{V}$ and $\varphi : A^* \to M(L)$ be its syntactic morphism. Then for any $m \in M(L), \varphi^{-1}(m) \in A^*\mathcal{V}$.

Proof. For $w \in A^*$, let $C(w)$ be the set of all contexts of w in L, that is,
$$C(w) = \{(u, v) \in A^* \times A^* : uwv \in L\} = \{(u, v) \in A^* \times A^* : w \in u^{-1}Lv^{-1}\}.$$

We now see immediately that $w \sim_L w'$ if and only if $C(w) = C(w')$. Therefore, the syntactic equivalent class of w is the following set:

$$\begin{aligned} \varphi^{-1}(\varphi(w)) &= \{w' : C(w') = C(w)\} \\ &= \bigcap_{(u,v)\in C(w)} u^{-1}Lv^{-1} \setminus \bigcup_{(u,v)\notin C(w)} u^{-1}Lv^{-1} \end{aligned}$$

Consider the languages of the form $u^{-1}Lv^{-1}$ for $u, v \in A^*$. Firstly, by the definition of the variety of languages, they are also in $A^*\mathcal{V}$. Secondly, as $\varphi : A^* \to M(L)$ recognizes L, in view of the proof given in proposition 1.8, we know that $u^{-1}Lv^{-1}$ is also recognized by φ. Hence, there are only finite number of languages of the form $u^{-1}Lv^{-1}$ since $M(L)$ is finite. Therefore, $\varphi^{-1}(\varphi(w)) \in A^*\mathcal{V}$ for any $w \in A^*$ as $A^*\mathcal{V}$ is closed under finite Boolean operations and by the above formula. Now, we have $\varphi^{-1}(m) \in A^*\mathcal{V}$ for any $m \in M(L)$ by taking $w \in A^*$ such that $\varphi(w) = m$. This concludes the proof of the lemma.

In conclusion, we can state the following theorem of variety due to Eilenberg [23].

Theorem 2.8 (Eilenberg) The function $\boldsymbol{V} \to \mathcal{V}$ defines a bijection between the pseudovarieties of finite monoids and the varieties of languages.

In the last of this section, we give a "semigroup" version of language theory and of this theorem. We define languages as subsets of the free semigroup A^+ over the alphabet set A. The syntactic semigroup of the language L is the quotient of A^+ by the syntactic congruence $\sim_L$ (defined as same as which in monoid case). The other definitions and propositions can be adopted for this case word by word by replacing the symbol $*$ by $+$ and the word monoid by semigroup. To distinguish the two types of varieties of languages, we speak of $*$-variety or $+$-variety for variety of languages on monoids or variety of languages on semigroups respectively. The most important difference between $*$-varieties and $+$-varieties is the following: a $*$-variety is closed under inverse morphism between free monoids, whereas a $+$-variety is closed under inverse morphism between free semigroups. In particular, we can use morphisms which erase, i.e., morphisms sending certain letters to the empty word. This is impossible in the second case.

§3 Varieties of languages and corresponding pseudovarieties

We now introduce some varieties of languages and their corresponding pseudovarieties, some of them are quite classical and remarkable in the literature.

Definition 3.1 Let A be a finite alphabet. Then the set of rational languages of A^*, denoted it by $A^*\mathcal{R}at$, is the smallest set of languages of A^* such that

1. for any word $u \in A^*$, $\{u\} \in A^*\mathcal{R}at$,

2. $A^*\mathcal{R}at$ is closed under the finite union, product and star operations.

The fundamental theorem regarding rational and recognizable languages over A is due to Kleene [30].

Theorem 3.2 (Kleene). A language L over the alphabet A is a rational language of A^* if and only if L is recognized by a finite monoid.

From the preceding theorem and the fact that the class of all finite monoids forms a pseudovariety, we know immediately that the function $\mathcal{R}at$ defined above is a $*$-variety of languages, and the corresponding pseudovariety is $\boldsymbol{M}$, which is the pseudovariety of all finite monoids.

Definition 3.3 We define $\boldsymbol{I}$ be the trivial M-variety, i.e. the pseudovariety of all trivial monoids.

In view of proposition 1.4, the corresponding $*$-variety of languages is $\mathcal{I}$: for any finite alphabet A, $A^*\mathcal{I} = \{\emptyset, A^*\}$.

In chapter 1, we have seen that $\boldsymbol{Nil}$, the class of all finite nilpotent semigroups, is an S-variety. We now find the corresponding $+$-variety of languages

$\mathcal{N}il$.

Theorem 3.4 For any alphabet A, $A^+\mathcal{N}il$ is the set of all finite or cofinite languages of A^+, where a language $L \in A^+$ is called cofinite if and only if $A^+ \setminus L$ is finite.

Proof. Let L be a finite language and $\eta : A^+ \to S = S(L)$ be its syntactic morphism. Let n be the maximum length of the words in L. Then, all words u of length greater than n are obviously syntactically equivalent (since for any $x, y \in A^*$, $|xuy| > n$ implies $xuy \notin L$). We use 0 to denote the common syntactic image of all these words. 0 is then the zero element of S since for any $x \in S$, we can take any $u \in A^+$ such that $\eta(u) = x$ and take any $v \in A^+$ with $|v| > n$, then we have $0 \cdot x = \eta(v)\eta(u) = \eta(vu) = 0$, since $|vu| > n$. Similarly, we also have $x \cdot 0 = 0$. Moreover, for any $u_1, ..., u_{n+1} \in A^+$, we have $|u_1 \cdots u_{n+1}| > n$ and therefore $\eta(u_1 \cdots u_{n+1}) = \eta(u_1) \cdots \eta(u_{n+1}) = 0$. Since η is surjective, $S^{n+1} = 0$ and S is nilpotent.

If L is cofinite, then $A^+ \setminus L$ is finite and consequently $S(L) = S(A^+ \setminus L)$ is nilpotent.

Conversely, let L be a language recognized by a finite nilpotent semigroup S. Then there exists a morphism $\eta : A^+ \to S$ and a subset P of S such that $L = \eta^{-1}(P)$. Suppose $S^n = 0$ for an integer n as S is nilpotent. Now, if $0 \notin P$, we conclude that for any $u \in A^+$ with $|u| \geq n$ and u is not in L. Since we can write $u = u_1 \cdots u_n$ for $u_i \in A^+$ and therefore $\eta(u) = \eta(u_1) \cdots \eta(u_n) \in S^n = 0$. This statement then enforces that L is finite. If $0 \in P$, we have $0 \notin S \setminus P$ and therefore $A^+ \setminus L = \eta^{-1}(S \setminus P)$ is finite by the same argument. Therefore L is finite or cofinite if $L \in A^+\mathcal{N}il$.

If a M-variety (resp. S-variety) $\boldsymbol{V}$ is generated by a single finite monoid (resp. semigroup), then we can give a direct description for the corresponding variety of languages.

Proposition 3.5 Let $\boldsymbol{V} = \boldsymbol{V}(M)$ be the M-variety (resp. S-variety) generated by the finite monoid (resp. semigroup) M and let $\mathcal{V}$ be its corresponding $*$-variety (resp. $+$-variety) of languages. Then for every alphabet set A, $A^*\mathcal{V}$ (resp. $A^+\mathcal{V}$) is the Boolean algebra generated by the languages of the form $\varphi^{-1}(m)$ where φ is an arbitrary morphism from A^* (resp. A^+) to M and $m \in M$.

Proof. One side inclusion is clear since $\varphi^{-1}(m) \in A^*\mathcal{V}$. For the converse inclusion, we let $L \in A^*\mathcal{V}$. Then, we have $M(L) \in HSP_{fin}(M)$ and so $M(L) \prec M^{(n)}$ for some integer n, (where $M^{(n)}$ stands for the direct product of n copies of M). Hence, $M^{(n)}$ recognizes L and so there exists a morphism $\eta : A^* \to M^{(n)}$ and a subset $P \in M^{(n)}$ such that $L = \eta^{-1}(P)$. Since

$$L = \eta^{-1}(P) = \bigcup_{m \in P} \eta^{-1}(m)$$

It suffices to consider the languages $\eta^{-1}(m)$ for all $m \in M^{(n)}$. Suppose $m = (m_1, ..., m_n)$, where $m_i \in M$. Let $\pi_i : M^{(n)} \to M$ be its i-th projection. Then, we have

$$m = \bigcap_{i=1}^{n} \pi_i^{-1}(m_i)$$

Therefore

$$\eta^{-1}(m) = \bigcap_{i=1}^{n} (\pi_i \eta)^{-1}(m_i)$$

Now since $\pi_i \eta : A^* \to M$ is a morphism and $m_i \in M$, thus, the converse inclusion holds. Therefore, $A^*\mathcal{V}$ is the Boolean algebra generated by the languages of the form $\varphi^{-1}(m)$.

By the preceeding proposition, we are now able to describe the varieties of languages corresponding to some M-varieties or S-varieties. For example, if we let $\boldsymbol{J_1}$ be the class of all finite commutative monoids of idempotents (or equivalently, all finite semilattice monoids). Then, by the definition, $\boldsymbol{J_1} = [[xy = yx, x^2 = x]]_M$ and therefore it is an M-variety. Let Sl_2 be the semilattice consists of two elements $\{0, 1\}$ (i.e., $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$). Then we know that $\boldsymbol{J_1}$ is generated by Sl_2.

Proposition 3.6 $\boldsymbol{J_1} = \boldsymbol{V}(Sl_2)$.

Proof. $\boldsymbol{V}(Sl_2) \subseteq \boldsymbol{J}_1$ is obvious since $Sl_2 \in \boldsymbol{J}_1$. On the other hand, suppose that the equation does not hold. Since the pseudovariety $\boldsymbol{V}(Sl_2)$ is equational (by corollary 2.6, chapter 1), there must exists an identity (explicit) which is satisfied by Sl_2 but can not be deduced from the identities:

$$xy = yx, x^2 = x$$

Take such an identity $u = v$ with $|u| + |v|$ minimum. Firstly, it can be easily seen that no variable can occur more than one time in each side of the identity. If not, for example, let $u = v_1xv_2xv_3$, say, then, we have $u = v_1xxv_2v_3 = v_1xv_2v_3$. Clearly, this new identity is equivalent to $u = v$ but it is obviously shorter. Secondly, if a variable x occurs in u, then, by taking any $y \neq x$ be 1 in the equation $u = v$, we get the identity $x = x$ or $x = 1$. However, the later identity is obviously not satisfied by Sl_2. Hence, we deduce that u and v contain exactly the same variables. It then follows that the identity $u = v$ can be reduced from $xy = yx$, which clearly contradicts to our hypothesis. Therefore, $\boldsymbol{J}_1 = \boldsymbol{V}(Sl_2)$.

Let $\mathcal{J}_1$ be the corresponding $*$-variety of $\boldsymbol{J}_1$. Then we have

Proposition 3.7 For any alphabet A, $A^*\mathcal{J}_1$ is the Boolean algebra generated by the languages of the form A^*aA^*, where a is a letter in A. Equivalently, $A^*\mathcal{J}_1$ is the Boolean algebra generated by the languages of the form B^*, where B is a subset of A.

Proof. Since $\boldsymbol{J}_1 = \boldsymbol{V}(Sl_2)$, we can use proposition 3.5 to describe $A^*\mathcal{J}_1$. For any subset B of A, let $\eta : A \to Sl_2$ is a map such that

$$\eta(a) = \begin{cases} 1 & if\ a \in B \\ 0 & if\ a \in A \setminus B \end{cases}$$

Then, this map η can be extended to a morphism $\bar{\eta} : A^* \to Sl_2$ and it can be easily seen that $B^* = \eta^{-1}(1)$. Conversely, for any morphism $\eta : A^* \to Sl_2$, $\eta^{-1}(1)$ must have the form B^* and $\eta^{-1}(0)$ has the form $A^* \setminus B^*$ for some subset B of A. Therefore $A^*\mathcal{J}_1$ is the Boolean algebra generated by B^*, where B is a subset of A. The remaining part of the proposition follows from the equations:

$$B^* = A^* \setminus \bigcup_{a \in A \setminus B} A^*aA^*$$

$$A^*aA^* = A^* \setminus (A \setminus a)^*.$$

Finally, we introduce an important variety of languages in the formal language theory, namely, the star-free languages.

Definition 3.8 For any finite alphabet set A, the set of star-free languages of A^* is the smallest set, denoted it by $A^*\mathcal{S}$, of languages of A^* such that

1. for any $u \in A^*$, $\{u\} \in A^*\mathcal{S}$,
2. $A^*\mathcal{S}$ is closed under finite Boolean operators and product.

A characterization of the corresponding syntactic monoids of star-free languages was given by the late M. P. Schützenberger in [49]

Theorem 3.9 (Schutzenberger). A language L over a finite alphabet is a star-free language if and only if its syntactic monoid $M(L)$ is aperiodic (i.e. for any $s \in M(L)$, there exists an integer n such that $s^{n+1} = s^n$).

It can be easily checked that all finite aperiodic monoids forms a pseudovariety, we represent it by $\boldsymbol{A}$. We now know immediately from Schutzenberger's theorem that the function $\mathcal{S}$ is a $*$-variety of languages, and the corresponding M-variety is $\boldsymbol{A}$.

We can define similarly the variety of star-free languages on semigroups as we have done at the end in the last section. The corresponding S-variety is the pseudovariety of finite aperiodic semigroups $\boldsymbol{A}_S$, where we denote it by $\boldsymbol{V}_S$, the S-variety generated by $\boldsymbol{V}$ for an M-variety $\boldsymbol{V}$.

There are other characterizations of rational languages and star-free languages, which apeared in logic. In particular, Büchi [18] has shown that a language is rational if and only if it is a set of words satisfying some sentence in the weak monadic second order theory of succesors, and McNaughton [37] further showed that a language is star-free if and only if it is a set of words satisfying some first-order sentences.

Chapter 3 M-varieties and S-varieties

In the last chapter, we have provided some examples of pseudovarieties of finite semigroups and monoids. In this chapter, we will study the algebraic structures that characterize them.

We first give some fundamental definitions and properties of Green's relations on a semigroup S.

Definition 1 On a semigroup S, the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{D}, \mathcal{H}$ are defined by:

1. $a\mathcal{R}b \Leftrightarrow aS^1 = bS^1 \Leftrightarrow \exists x, y \in S, ax = b, by = a.$

2. $a\mathcal{L}b \Leftrightarrow S^1a = S^1b \Leftrightarrow \exists x, y \in S, xa = b, yb = a.$

3. $a\mathcal{J}b \Leftrightarrow S^1aS^1 = S^1bS^1,$

4. $\mathcal{D} = \mathcal{R} \vee \mathcal{L}, \mathcal{H} = \mathcal{R} \cap \mathcal{L}.$

for any $a, b \in S$, where S^1 represents S if there is an identity element in S, otherwise, just add "1" in S as its extra identity element.

Proposition 2 In a finite semigroup S, $\mathcal{D} = \mathcal{J}$.

Proof. $\mathcal{D} \subseteq \mathcal{J}$ is obvious since $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$. For the converse inclusion, suppose that $a\mathcal{J}b$ for $a, b \in S$. Then there exist elements $x, y, u, v \in S^1$ such that $xay = b$ and $ubv = a$. Hence we can deduce that $a = ubv = (ux)a(yv)$, and therefor $(ux)^k a(yv)^k = a$ for any positive integer k. Since S is a finite semigroup, we can take the positive integers, say, m and n such that $(ux)^m = e$ and $(yv)^n = f$ are both idempotents. Now, we have $(ux)^{mn}a(yv)^{mn} = a = eaf$, whence $(ux)^m a = ea = e(eaf) = eaf = a$. This implies that $xa\mathcal{L}a$. Similarly, we can deduce that $ay\mathcal{R}a$, and this relation can imply further that $xay\mathcal{R}xa$. Hence we have $xay\mathcal{R}xa\mathcal{L}a$, and therefore $xay\mathcal{D}a$, i.e. $b\mathcal{D}a$. Thus we proved the inclusion $\mathcal{J} \subseteq \mathcal{D}$.

Definition 3 An element a in a semigroup S is called regular if there exists $x \in S$ such that $axa = a$. The semigroup S is called regular if all the elements of S are regular.

The following proposition can be found in Howie [28]. In fact, this is a well known result in the theory of semigroups.

Proposition 4 A $\mathcal{D}$-class D of a semigroup S contains a regular element if and only if all its elements are regular. If D is regular, then every $\mathcal{R}$-class and every $\mathcal{L}$-class of S contained in D contains at least one idempotent.

The following properties are also well known in semigroups:

Proposition 5 Let H be an $\mathcal{H}$-class of a semigroup S. Then the following conditions are equivalent:

1. H contains an idempotent.

2. There exists elements $a, b \in H$ such that $ab \in H$.

3. H is a maximal subgroup of the semigroup S.

As we are concerned mostly on the finite cases of semigroups (monoids), in the remaining part of this chapter and throughout chapter 4, we regard a semigroup (monoid) is always finite except the free objects.

We say that a semigroup S is $\mathcal{K}$-trivial if for any two elements $x, y \in S$, $x\mathcal{K}y$ implies $x = y$.

We have the following lemma.

Lemma 6 Let $\mathcal{K}$ be one of the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{D}$ or $\mathcal{H}$ and let S be a semigroup. If the restriction of $\mathcal{K}$ on the regular $\mathcal{K}$-classes of S is trivial, then S must be $\mathcal{K}$-trivial.

Proof. We divided the proof into four cases:

1. $\mathcal{K} = \mathcal{R}$. Suppose $a\mathcal{R}b$. Then there exists elements $c, d \in S^1$ such that $ac = b, bd = a$ and whence $acd = a$. Let n be an integer such that $(cd)^n$ is an idempotent. We then have $(cd)^n\mathcal{R}(cd)^n c$, whence $(cd)^n = (cd)^n c$

since the restriction of $\mathcal{R}$ on the regular $\mathcal{R}$-classes is trivial. Hence, we can deduce $a = a(cd)^n = a(cd)^n c = ac = b$ and therefore S is $\mathcal{R}$-trivial.

2. $\mathcal{K} = \mathcal{L}$. By the same argument as 1.

3. $\mathcal{K} = \mathcal{D}$. Suppose $a\mathcal{D}b$. Then there exists element $c \in S$ such that $a\mathcal{R}c$ and $c\mathcal{L}b$. Since the restriction of $\mathcal{D}$ on the regular $\mathcal{D}$-classes is trivial, restrivtion of $\mathcal{R}$ or $\mathcal{L}$ to the regular $\mathcal{R}$-classes or $\mathcal{L}$-classes respectively is consequently trivial. From (a) and (b), it follows that $a = c$ and $c = b$, whence finally we have $a = b$. Therefore S is $\mathcal{D}$-trivial.

4. $\mathcal{K} = \mathcal{H}$. Suppose $x\mathcal{H}y$. Then there exists elements $a, b, c, d \in S^1$ such that $ax = y, by = x, xc = y, yd = x$, whence $x = axd$ and therefore $a^n x d^n = x$ for every n. Choose a non-negative integer n such that $a^n = a^{n+1}$ (which is possible since the restriction of $\mathcal{H}$ on the regular $\mathcal{H}$-classes is trivial). It then follows from this that $a(a^n x d^n) = ax = a^{n+1} x d^n = a^n x d^n = x$, whence $ax = y = x$. Therefore S is $\mathcal{H}$-trivial.

Proposition 7 All $\mathcal{R}$-trivial (respectively $\mathcal{L}$-trivial, $\mathcal{J}$-trivial) monoids form an M-variety, we represent it by $\boldsymbol{R}$ (respectively $\boldsymbol{L}, \boldsymbol{J}$) and we have the following characterizations.

1. $\boldsymbol{R} = \bigcup_{n\geq 1}[[(xy)^n x = (xy)^n]]_M = [[(xy)^\omega x = (xy)^\omega]]_M$

2. $\boldsymbol{L} = \bigcup_{n\geq 1}[[y(xy)^n = (xy)^n]]_M = [[y(xy)^\omega = (xy)^\omega]]_M$

3. $\boldsymbol{J} = \bigcup_{n\geq 1}[[(xy)^n x = (xy)^n = y(xy)^n]]_M = \bigcup_{n\geq 1}[[(xy)^n = (yx)^n, x^n = x^{n+1}]]_M = [[(xy)^\omega = (yx)^\omega, x^\omega = x^{\omega+1}]]_M$

Proof.

1. Let M be an $\mathcal{R}$-trivial (and therefore aperiodic) monoid and let n be an non-negative integer such that $u^n = u^{n+1}$ for every $x, y \in M$. Then, we have $(xy)^n x \mathcal{R} (xy)^n$ since $((xy)^n x)y = (xy)^{n+1} = (xy)^n$. Since M is $\mathcal{R}$-trivial, it follows that $(xy)^n x = (xy)^n$. Conversely, if M satisfies $(xy)^n x = (xy)^n$ for some n, then, by lemma 6, it suffices to show that all regular $\mathcal{R}$-classes of M are trivial. For this purpose, let $e \in E(M)$ and $x \in M$ be such that $e\mathcal{R}x$. Then $ex = x$ and there exists y such

that $xy = e$. Therefore, $x = ex = (xy)^n x = (xy)^n = e$. This shows that M is $\mathcal{R}$-trivial. Note that $(xy)^n = (xy)^\omega$ in the discussion above, we can deduce immediately that M is $\mathcal{R}$-trivial if and only if M satisfies the identity $(xy)^\omega x = (xy)^\omega$.

2. The proof of this part is similar to part 1.

3. Since $\boldsymbol{J} = \boldsymbol{R} \cap \boldsymbol{L}$, it follows from (1) and (2) that any $M \in \boldsymbol{J}$ satisfies the equation $(xy)^n x = (xy)^n = y(xy)^n$ for some non-negative integer n. Taking $y = 1$ in the above equation, we obtain $x^n = x^{n+1}$ and also $(xy)^n = y(xy)^n = (yx)^n y = (yx)^n$. Similarly, we can show that M satisfies $(xy)^\omega x = (xy)^\omega = y(xy)^\omega, (xy)^\omega = (yx)^\omega, x^\omega = x^{\omega+1}$.
   Conversely, suppose that a monoid M satisfies the equations $x^n = x^{n+1}$ and $(xy)^n = (yx)^n$ for some non-negative integer n. Then, we have $(xy)^n = (xy)^{n+1} = (yx)^{n+1} = y(xy)^n x$, whence $(xy)^n = y^n(xy)^n x^n = y^{n+1}(xy)^n x^n = y(xy)^n$ and likewise, $(xy)^n = (xy)^n x$. Similarly, if M satisfies $(xy)^\omega = (yx)^\omega$ and $x^\omega = x^{\omega+1}$, then, they imply that the identity $(xy)^\omega x = (xy)^\omega = y(xy)^\omega$ holds. The proof is complete.

We recall that a semigroup S is aperiodic if, for every $x \in S$, there exists a non-negative integer n such that $x^n = x^{n+1}$. i.e., the period of every element in S is 1. The following proposition gives various characterizations of aperiodic semigroups.

Proposition 8 [43] Let S be a semigroup. Then the following conditions are equivalent.

1. S is aperiodic.

2. There exists an integer m such that, for every $x \in S$, $x^m = x^{m+1}$.

3. S is $\mathcal{H}$-trivial.

4. There is no non-trivial subgroups in S.

5. $x^\omega = x^{\omega+1}$ for any $x \in S$.

Proof.

(1) $\Rightarrow$ (2): For each $x \in S$, let us take the smallest integer n_x such that $x^{n_x} = x^{n_x+1}$ and let $m = \max_{x \in S} n_x$. Then $x^m = x^{m+1}$ for every $x \in S$.

(2) $\Rightarrow$ (4): Let G be a subgroup in S. Let $x \in G$ and $x^n = x^{n+1}$ for some n. As $x \in G$, we also have $x^n \in G$ and $x^n = x^n x = ... = x^{2n}$, so x^n is an idempotent in G and hence is an identity e in G. We can deduce that $x = xe = xx^n = x^{n+1} = x^n = e$, i.e. G is trivial.

(4) $\Rightarrow$ (3): By lemma 6, it suffices to verify that all regular $\mathcal{H}$-classes are trivial. As we know that any regular $\mathcal{H}$-class is in bijection with a subgroup in S, S is of course $\mathcal{H}$-trivial.

(3) $\Rightarrow$ (1): Let $x \in S$ and r be the index of x. If p is the period of x with $x^{r+p} = x^r$, then we know that $\{x^r, ..., x^{r+p-1}\}$ is a subgroup in S. As any subgroup of S is contained in some $\mathcal{H}$-classes, it is therefore trivial by hypothesis, i.e. $x^r = x^{r+1}$.

(1) $\Rightarrow$ (5): For any $x \in S$, there exists an integer n such that $x^n = x^{n+1}$ by hypothesis. Then, we have $x^n = x^{n+1} = ... = x^{2n}$. Thus, x^n is an idempotent, and so $x^n = x^\omega$. Hence, we deduce that $x^\omega = x^{\omega+1}$.

(5) $\Rightarrow$ (1): This part is obvious since $x^\omega = x^n$ for some non-negative integer n.

Similarly, the equivalent conditions in proposition 8 hold for aperiodic monoids if we replace the semigroups by monoids. Recall that $\boldsymbol{A}$ is the M-variety of aperiodic monoids, we therefore obtain the following result.

Proposition 9 $\boldsymbol{A} = \bigcup_{n \geq 1} [[x^n = x^{n+1}]]_M = [[x^\omega = x^{\omega+1}]]_M$.

Recall that $\boldsymbol{A_S}$ is the S-variety of aperiodic semigroups, by proposition 8, we also have

Proposition 10 $\boldsymbol{A_S} = \bigcup_{n \geq 1} [[x^n = x^{n+1}]]_S = [[x^\omega = x^{\omega+1}]]_S$.

We call a semigroup S locally trivial if for every $s \in S$ and $e \in E(S)$, $ese = e$ holds. As for an idempotent e in a semigroup S, the subsemigroup eSe is the largest submonoid with e as its identity contained in S. We call it the local semigroup associated with the element e. The condition above indicates that all the local semigroups of S are trivial.

Proposition 11 [43] Let S be a semigroup. Then, the following conditions are equivalent:

1. S is locally trivial.

2. $E(S)$ is the minimal ideal of S.

3. $esf = ef$ holds for every $s \in S$ and $e, f \in E(S)$.

Proof.

(1) $\Rightarrow$ (2): Let I be a minimal ideal of S and let $x \in I$. Then for every $e \in E(S)$, we have $e = exe \in I$ since I is an ideal. Therefore $E(S) \subseteq I$. On the other hand, for any $e \in E(S)$ and $s \in S$, we have $(es)^2 = eses = es$ and $(se)^2 = sese = se$. This shows that $E(S)$ is an ideal (non-empty as S is finite) of S contained in I and hence $E(S) = I$.

(2) $\Rightarrow$ (3): For any $e, f \in E(S)$ and $s \in S$, we have $ef, esf \in E(S)$ since $E(S)$ is an ideal of S. Now, $E(S)$ is a simple idempotent semigroup, so it is a rectangular band (see chapter 3.3, [43]). Thus, we have $esf = e(esf)f = ef$.

(3) $\Rightarrow$ (1): Just take $e = f$ in condition (3), and the proof follows.

Lemma 12 Let S be a semigroup with n elements. Then, for any $s_1, ..., s_n \in S$, there exist $t_1, t_2 \in S$ and $e \in E(S)$ such that $s_1 s_2 \cdots s_n = t_1 e t_2$.

Proof. Let $p_k = s_1 s_2 \cdots s_k$ $(k = 1, ..., n)$.We then have the n products $p_1, ..., p_n$ in S. If they are all distinct, then at least one of them, say p_k, is an idempotent, and so $s_1 \cdots s_n = p_k \cdot p_k \cdot s_{k+1} \cdots s_n$ has the desired form. If $p_i = p_j$ with $i < j$, then

$$p_i = p_j = p_i s_{i+1} \cdots s_j = p_i(s_{i+1} \cdots s_j)^2 = ... = p_i(s_{i+1} \cdots s_j)^\omega$$

This forces $s_1 \cdots s_n = p_i \cdot e \cdot s_{j+1} \cdots s_n$, where $e = (s_{i+1} \cdots s_j)^\omega$.

Let $\boldsymbol{LI}$ be the class of all locally trivial semigroups. Then we obtain the following characterizations of $\boldsymbol{LI}$.

Proposition 13 $\boldsymbol{LI}$ is an S-variety and $\boldsymbol{LI} = [[x^\omega y x^\omega = x^\omega]]_{\boldsymbol{S}} = \bigcup_{n\geq 1}[[x^n y x^n = x^n]]_{\boldsymbol{S}} = \bigcup_{n\geq 1}[[x_1 \cdots x_n y z_1 \cdots z_n = x_1 \cdots x_n z_1 \cdots z_n]]_{\boldsymbol{S}}$

Proof. Let S be a locally trivial semigroup, i.e. for any $e \in E(S), s \in S$ we have $ese = e$. Then, for any $x^\omega \in E(S), y \in S$, we also have $x^\omega y x^\omega = x^\omega$.

Suppose a semigroup S satisfies the identity $x^\omega y x^\omega = x^\omega$. For any $x \in S$, let n_x be the least integer such that $x^{n_x} = x^\omega$. Let N be the product of all integers n_x for $x \in S$ (note that S is finite). Then, we have $x^N = x^\omega$ for any x. Therefore, the equation $x^N y x^N = x^N$ is satisfied by the semigroup S.

Conversely, if a semigroup S satisfies the equation $x^n y x^n = x^n$ for some integer n, then for any $e \in E(S), s \in S$, we have $ese = e^n s e^n = e^n = e$, whence S is locally finite.

We now prove the remaining part of proposition 13. Suppose again that S is a locally trivial semigroup with n elements. For any $x_1, ..., x_n \in S$, by lemma 12, there exist $t_1, t_2 \in S, e \in E(S)$ such that $x_1 \cdots x_n = t_1 e t_2$. Similarly, for any $z_1, ..., z_n \in S$, there exist $s_1, s_2 \in S, f \in E(S)$ and $z_1 \cdots z_n = s_1 f s_2$. So we can deduce that for any $y \in S$, $x_1 \cdots x_n y z_1 \cdots z_n = t_1 e t_2 y s_1 f s_2 = t_1 e f s_2 = t_1 e t_2 s_1 f s_2 = x_1 \cdots x_n z_1 \cdots z_n$ by proposition 11 and this proves that S is loclly trivial.

Conversely, suppose a semigroup S satisfies the equation $x_1 \cdots x_n y z_1 \cdots z_n = x_1 \cdots x_n z_1 \cdots z_n$. Then, for any $e \in E(S), s \in S$, $ese = e^n s e^n = e^n e^n = e$, i.e. S is locally trivial.

Let $\boldsymbol{K}$ ($\boldsymbol{K}^r$) be the class of all semigroups S such that $e = es$ ($e = se$) for all $s \in S$ and $e \in E(S)$. Then we have the following results:

Proposition 14 $\boldsymbol{K}$ ($\boldsymbol{K}^r$) is an S-variety and we have

1. $\boldsymbol{K} = [[x^\omega y = x^\omega]]_{\boldsymbol{S}}$,
2. $\boldsymbol{K}^r = [[y x^\omega = x^\omega]]_{\boldsymbol{S}}$.

Proof.

1. Let S be a semigroup in the class $\boldsymbol{K}$. For any $x, y \in S$, as x^ω is an identity in S by definition, hence $x^\omega y = x^\omega$. Conversely, suppose a

semigroup S satisfies the pseudoidentity $x^\omega y = x^\omega$ then obviously, for any $e \in E(S)$ and $s \in S$, we have $es = e^\omega s = e^\omega = e$.

2. The proof is similar to part 1.

By the definition of the S-variety $\boldsymbol{Nil}$, we can immediately obtain that $\boldsymbol{Nil} = \boldsymbol{K} \cap \boldsymbol{K}^r$. We also have the following proposition given in [43].

Proposition 15 $\boldsymbol{LI} = \boldsymbol{K} \vee \boldsymbol{K}^r$.

Let $\boldsymbol{V}$ be an M-variety. Let $\boldsymbol{LV}$ be the class of all semigroups which are locally in $\boldsymbol{V}$, i.e., the class of semigroups S with $eSe \in \boldsymbol{V}$ for every $e \in E(S)$. Then we have the following proposition.

Proposition 16 $\boldsymbol{LV}$ is an S-variety and $\boldsymbol{L}(\boldsymbol{LV}) = \boldsymbol{LV}$ for any M-variety $\boldsymbol{V}$.

Proof. Suppose S_1 and S_2 are semigroups in $\boldsymbol{LV}$. Consider the semigroup $S_1 \times S_2$, note that any idempotent e of $S_1 \times S_2$ should has the form (e_1, e_2) with $e_i \in E(S_i)$. Hence, $eS_1 \times S_2 e = (e_1, e_2)S_1 \times S_2(e_1, e_2) = e_1S_1e_1 \times e_2S_2e_2 \in \boldsymbol{V}$ since $e_iS_ie_i \in \boldsymbol{V}$. Therefore $S_1 \times S_2 \in \boldsymbol{LV}$.

For a subsemigroup R of S and any idempotent e of R, we have eRe is a submonoid of eSe, and therefore $eRe \in \boldsymbol{V}$. Thus $R \in \boldsymbol{LV}$

Finally, let $\varphi : S \to T$ be an epimorphism of semigroups. Then for any idempotent $e \in T$, there is an idempotent f of S such that $\varphi f = e$, so $eTe = \varphi(fSf)$ and hence $eTe \in \boldsymbol{V}$. This leads to $T \in \boldsymbol{LV}$. Thus, $\boldsymbol{LV}$ is an S-variety.

For the remaining part of the proposition, we first notice that $\boldsymbol{V} \subseteq \boldsymbol{LV}$. Hence we immediately have $\boldsymbol{LV} \subseteq \boldsymbol{L}(\boldsymbol{LV})$. For the inverse inclusion, suppose $S \in \boldsymbol{L}(\boldsymbol{LV})$. Then, by definition, we have $eSe \in \boldsymbol{LV}$ for any $e \in E(S)$. By using the definition again, for the idempotent e of the semigroup eSe, we have $e(eSe)e \in \boldsymbol{V}$, i.e. $eSe \in \boldsymbol{V}$. Therefore $S \in \boldsymbol{LV}$ and so we obtain the inclusion $\boldsymbol{L}(\boldsymbol{LV}) \subseteq \boldsymbol{LV}$.

Observe that, in the preceding proposition, we have proved again that all locally trivial semigroups form an S-variety $\boldsymbol{LI}$, where $\boldsymbol{I}$ is just the trivial M-variety $\boldsymbol{I}$.

Recall that $\boldsymbol{G}$ is the M-variety of all groups. We have the following proposition.

Proposition 17 $\boldsymbol{LI} = \boldsymbol{LG} \cap \boldsymbol{A_S}$.

Proof. $\boldsymbol{LI} \subseteq \boldsymbol{LG}$ and $\boldsymbol{LI} \subseteq \boldsymbol{A_S}$ are clear. Conversely, for a semigroup which is locally groups and aperiodic, then it is locally trivial. Thus $\boldsymbol{LI} = \boldsymbol{LG} \cap \boldsymbol{A_S}$.

The concept of relational morphism will be useful in the following discussions. Roughly speaking, relational morphisms allow us to "inverse" the arbitrary surjective (not necessarily injective) morphisms.

Definition 18 A relational morphism $\tau : S \to T$ between two semigroups S and T is a mapping from S into the set of all subsets of T, such that

1. $\tau(s) \neq \emptyset$ for any $s \in S$,
2. $\tau(s_1)\tau(s_2) \subseteq \tau(s_1 s_2)$.

A relational morphism τ is called injective if the condition $\tau(s_1) \cap \tau(s_2) \neq \emptyset$ implies $s_1 = s_2$. τ is called surjective if, for every $t \in T$, there exists $s \in S$ such that $t \in \tau(s)$, or equivalently, $\cup_{s \in S} \tau(s) = T$. If a relational morphism $\tau : S \to T$ is surjective, then we can define $\tau^{-1} : T \to S$ by letting $\tau^{-1}(t) = \{s \in S | t \in \tau(s)\}$ for any $t \in T$. Thus, τ^{-1} is also a relational morphism. Furthermore, the composition of two relational morphisms $\tau : S \to T$ and $\sigma : T \to R$ defined by $\sigma\tau(s) = \sigma(\tau(s))$ is again a relational morphism.

Let $\tau : S \to T$ be a relational morphism and let R be its map graph, that is,

$$R = \{(s,t) \in S \times T | t \in \tau(s)\}.$$

Then, it is clear that R is a subsemigroup of $S \times T$. Let $\alpha : R \to S$ and $\beta : R \to T$ be the morphisms induced by the projections of $S \times T$ to S and T

respectively. Then α is surjective and we have $\tau = \beta\alpha^{-1}$. This factorization of τ is called the canonical factorization of the relational morphism τ.

The following lemmas can be easily seen. We omit the proofs.

Lemma 19 Let $\tau : S \to T$ be a relational morphism and $\tau = \beta\alpha^{-1}$ its canonical factorization. Then τ is injective if and only if β is injective.

Lemma 20 Let $\tau : S \to T$ be a relational morphism. Then for any subsemigroup S' of S, $\tau(S')$ is a subsemigroup of T. And for any subsemigroup T' of T, $\tau^{-1}(T')$ is a subsemigroup of S.

Let $\boldsymbol{V}$ be an S-variety. A relational morphism $\tau : S \to T$ is called a (relational) $\boldsymbol{V}$-morphism if, for any subsemigroup T' of T which is also an element of $\boldsymbol{V}$, we have $\tau^{-1}(T') \in \boldsymbol{V}$.

By routine checking, we can prove the following lemmas:

Lemma 21 Let $\boldsymbol{V}$ be an S-variety. Let $\tau : S \to T$ be a relational morphism and $\tau = \beta\alpha^{-1}$ its canonical factorization. Then τ is a $\boldsymbol{V}$-morphism if and only if β is a $\boldsymbol{V}$-morphism.

Lemma 22 If $\tau : S \to T$ and $\sigma : T \to R$ are $\boldsymbol{V}$-morphisms, then $\sigma\tau : S \to R$ is also a $\boldsymbol{V}$-morphism.

Let $\boldsymbol{V}$ be an S-variety and $\boldsymbol{W}$ an S- (resp. M-) variety. We define $\boldsymbol{V}^{-1}\boldsymbol{W}$ to be the class of all semigroups (resp. monoids) S such that there exists a $\boldsymbol{V}$-morphism $\tau : S \to T$ with $T \in \boldsymbol{W}$.

Proposition 23 If $\boldsymbol{V}$ is an S-variety and $\boldsymbol{W}$ is an S- (resp. M-) variety, then $\boldsymbol{V}^{-1}\boldsymbol{W}$ is an S- (resp. M-) variety.

Proof. Suppose $S_1, S_2 \in \boldsymbol{V}^{-1}\boldsymbol{W}$ and $\tau_1 : S_1 \to T_1, \tau_2 : S_2 \to T_2$ are $\boldsymbol{V}$-morphisms with $T_1, T_2 \in \boldsymbol{W}$. It can be easily checked that $\tau_1 \times \tau_2 : S_1 \times S_2 \to T_1 \times T_2$ is a relational morphism and therefore $S_1 \times S_2 \in \boldsymbol{V}^{-1}\boldsymbol{W}$ since $T_1 \times T_2 \in \boldsymbol{W}$.

Suppose R is a subsemigroup of $S \in \boldsymbol{V}^{-1}\boldsymbol{W}$ and $\tau : S \to T$ is a $\boldsymbol{V}$-morphism with $T \in \boldsymbol{W}$. Let $\eta : R \to S$ be the inclusion morphism. Then, by definition, η is clearly a $\boldsymbol{V}$-morphism. Hence $\tau\eta : R \to T$ is a $\boldsymbol{V}$-morphism by lemma 22, and therefore $R \in \boldsymbol{V}^{-1}\boldsymbol{W}$.

Finally, suppose $\varphi : S \to R$ is an epimorphism and $S \in \boldsymbol{V}^{-1}\boldsymbol{W}$ with a $\boldsymbol{V}$-morphism $\tau : S \to T$ and $T \in \boldsymbol{W}$. Since φ is an epimorphism, φ^{-1} is an injective relational morphism. Let $\varphi^{-1} = \beta\alpha^{-1}$ be its canonical factorization. Then β is an injective semigroup morphism by Lemma 19, and by our definition, it is clear that β is a $\boldsymbol{V}$-morphism for any S-variety $\boldsymbol{V}$. Hence φ^{-1} is also a $\boldsymbol{V}$-morphism by lemma 21. Therefore, $\tau\varphi^{-1} : R \to T$ is a $\boldsymbol{V}$-morphism by lemma 22, and so $R \in \boldsymbol{V}^{-1}\boldsymbol{W}$. The proof is complete.

When $\boldsymbol{V} = \boldsymbol{A_S}$, the $\boldsymbol{A_S}$-morphisms are called the aperiodic relational morphisms. We will use the concept of aperiodic relational morphisms in the next chapter.

Define $\boldsymbol{DA}$ be the class of all monoids of which each regular $\mathcal{D}$-class is an idempotent subsemigroup.

Proposition 24 $\boldsymbol{DA}$ is an M-variety.

Proof. Suppose $M \in \boldsymbol{DA}$. Let N be a submonoid of M and let D be a regular $\mathcal{D}$-class of N. Then for any $a \in D$, there exists an idempotent $e \in N$ such that $a\mathcal{R}_N e$ and therefore $a\mathcal{R}_M e$. However, the $\mathcal{D}$-class of e in M only contains idempotents and therefore $a \in E(N)$. Hence, $N \in \boldsymbol{DA}$.

Suppose that $\varphi : M \to N$ is an epimorphism. Let D be a regular $\mathcal{D}$-class of N. Then there exists a regular $\mathcal{D}$-class D' of M such that $D = \varphi(D')$. Since D' contains only idempotents, D is also an idempotent subsemigroup and therefore $N \in \boldsymbol{DA}$.

Finally, let $M_1, M_2 \in \boldsymbol{DA}$. We can see easily that the $\mathcal{D}$-classes of $M_1 \times M_2$ are of the form $D_1 \times D_2$, where D_i is a $\mathcal{D}$-class of M_i for $i = 1, 2$. Therefore, if $D_i \subseteq E(M_i)$, we have $D_1 \times D_2 \subseteq E(M_1 \times M_2)$ and so $M_1 \times M_2 \in \boldsymbol{DA}$. Hence, the proposition is proved.

Another direction in the study of pseudovarieties is the structures of the lattice of all pseudovarieties and its sublattices. For example, the existence of maximal or minimal elements, finite or infinite chains and the isomorphisms between two sublattices. Margolis proved that there is no maximal M-variety (resp. S-variety) in the lattice fo all M-varieties (resp. S-varieties) and Pin in [43] summarized all minimal M-varieties and S-varieties. The reader is refered to the papers [36], [43] and [46] if necessary.

Chapter 4 The dot-depth hierarchy

In chapter 2, we have already seen that the variety of star-free languages (on monoids) is corresponding to the M-variety $\boldsymbol{A}$ and its relation with formal logic theory. In this chapter, we introduce hierarchies of varieties of languages which is a decomposition theory for the variety of star-free languages. This theory has been studied by numerous authors in the literature since its first introduction by Brzozowski and Cohen [15] in 1971. Further investigations showed its connection with formal logic [40] [63] and the complexity of Boolean circuits [12] [26]. We will review the main results about these hierarchies and discuss some recent partial results.

§1 The dot-depth problem

As we can see, the variety of rational languages is the variety of letters of the given alphabet and is closed under the Boolean operations, star and product; Restricting ourselves to the Boolean operations and product only, we then get the variety of star-free languages. Thus, a natrual question arises: For a given a star-free language L, what is the minimal number of times that one should use the product operation only to describe L? This problem motivates the definitions of hierarchies of star-free languages.

The following is a precise definition of the dot-depth hierarchy. Due to some technical reasons, the hierarchy is usually defined for languages on semigroups only.

Definition 1.1 The dot-depth hierarchy is a hierarchy of +-varieties of languages $\mathcal{B}_i$: For any finite alphabet set A, we let $A^+\mathcal{B}_i$ as follows:

1. $A^+\mathcal{B}_0 =$ Boolean algebra generated by $\{\{u\} : u \in A^+\}$,
2. $A^+\mathcal{B}_{k+1} =$ Boolean algebra generated by $\{L_1L_2\cdots L_j : L_i \in A^+\mathcal{B}_k, j \geq 1\}$.

The dot-depth of a star-free language $L \subseteq A^+$ is the least integer k such that $L \in A^+\mathcal{B}_k$. Similarly, if S is an aperiodic semigroup, we define the dot-depth of S to be the non-negative integer k such that $S \in \boldsymbol{B}_k \setminus \boldsymbol{B}_{k-1}$.

An outstanding open problem is whether there is an effective means to determine the dot-depth for a given star-free language? We see immediately that this problem is equivalent to whether there is an effective algorithm to determine the dot-depth for a given aperiodic semigroup?

By the definition, we can see that the dot-depth one languages are just the finite and co-finite languages on every alphabet, i.e., $\mathcal{B}_0 = \mathcal{N}il$ and so the corresponding S-variety is $\boldsymbol{B}_0 = \boldsymbol{Nil}$. As the union of all dot-depth hierarchies is generated by letters of the given alphabet and closed under the Boolean operations and products, therefore it is just the +-variety of star-free languages, i.e. $\bigcup_{k\geq 0} \mathcal{B}_k = \mathcal{A}$ and $\bigcup_{k\geq 0} \boldsymbol{B}_k = \boldsymbol{A}_S$.

As the description of the syntactic monoid of the product of some given languages is essential in the study of hierarchies of star-free languages, we shall give the definition of Schützenberger product which was first introduced by Schützenberger [49] and generalized by Straubing [57].

For a monoid M, we consider the power set $\mathcal{P}(M)$ of M, i.e., the set of all subsets of M. Firstly, we know that it is a monoid with the product

$$RS = \{rs | r \in R, s \in S\}$$

for any two subsets R and S of M and the identity element $1 = \{1\}$. Next, it is also a commutative monoid for union of subsets R and S, we denote it by $R + S$, with the empty subset as its identity, we denote it by 0. Then for any $R, R_1, R_2 \subseteq M$, we can easily verify the following formulae:

1. $R(R_1 + R_2) = RR_1 + RR_2$

2. $(R_1 + R_2)R = R_1R + R_2R$

3. $0 \cdot R = R \cdot 0 = 0$

Hence, $\mathcal{P}(M)$ becomes a semiring (with identity) under the union and product operations. We therefore can, for any positive integer n, define the monoid $\mathcal{P}(M)^{n\times n}$ of matrices of size $n \times n$ with entries in $\mathcal{P}(M)$, with the usual matrix product.

Let $M_1, ..., M_n$ be monoids and let $M = M_1 \times \cdots \times M_n$. Then the Schützenberger product of $M_1, ..., M_n$, denoted it by $\Diamond_n(M_1, ..., M_n)$, is the submonoid of $\mathcal{P}(M)^{n\times n}$ composed of all the upper triangular matrices T satisfying the following conditions

1. $T_{ij} = 0$ if $i > j$
2. $T_{ii} = \{(1, ..., 1, m_i, 1, ..., 1)\}$ for a certain $m_i \in M_i$ which appears in the i-th component.
3. $T_{ij} \subseteq 1 \times \cdots \times 1 \times M_i \times M_{i+1} \times \cdots \times M_j \times 1 \times \cdots \times 1$ for $i < j$

Similarly, we have a parallel definition for semigroups. Let $S_1, ..., S_n$ be semigroups. We define the Schützenberger product $\Diamond_n(S_1, ..., S_n)$ of semigroups $S_1, ..., S_n$ be all matrices in $\mathcal{P}(S_1^1 \times \cdots S_n^1)^{n\times n}$ satisfying almost the same conditions in the monoid case with the only difference that the diagonal entries are elements of S_i but not from S_i^1. It can be easily checked that the product $\Diamond_n(S_1, ..., S_n)$ is a semigroup.

The recognizing power of the Schützenberger product is established by the following theorems. The first one is due to Schützenberger [49] in the case of $n = 2$ and to Straubing [57] in general case. The second one is due to Reutenauer [48] for $n = 2$ and to Pin [42] in general.

Theorem 1.2 [49] [57] Let $L_1, ..., L_n \in A^*$ be languages recognized by the monoids $M_1, ..., M_n$ and $a_1, ..., a_{n-1}$ be letters of A. Then the languages $L_1 \cdots L_n$ and $L_1 a_1 \cdots a_{n-1} L_n$ are recognized by the monoid $\Diamond_n(M_1, ..., M_n)$.

Theorem 1.3 [48] [42] Let $M_1, ..., M_n$ be monoids. If $L \in A^*$ is recognized by $\Diamond_n(M_1, ..., M_n)$. Then L is in the Boolean algebra generated by the languages of the form $L_{i_1} a_1 L_{i_2} \cdots a_{r-1} L_{i_r}$ where $1 \leq i_1 \leq ... \leq i_r \leq n$, L_{i_k} is a language recognized by M_{i_k} for $k = 1, ..., r$ and $a_1, ..., a_{r-1}$ are letters of A.

Using the preceding theorems, we can establish an one to one correspondence between the pseudovarieties generated by the Schützenberger products and the varieties of languages generated by the concatenation products (for two languages $L_1, L_2 \in A^*$ (A^+) and $a \in A$, the concatenation product of them is the language $L_1 a L_2$).

Let $\boldsymbol{V}_1, ..., \boldsymbol{V}_n$ be M- (resp. S-) varieties. We denote by $\Diamond_n(\boldsymbol{V}_1, ..., \boldsymbol{V}_n)$ the M- (resp. S-) variety generated by all monoids of the form $\Diamond_n(M_1, ..., M_n)$ where $M_i \in \boldsymbol{V}_i$ for $i = 1, ..., n$. If $\boldsymbol{V}$ is an M- (resp. S-) variety, we define $\Diamond \boldsymbol{V}$ be the union of $\Diamond_n(\boldsymbol{V}, ..., \boldsymbol{V})$ for all $n \geq 1$. Then $\Diamond \boldsymbol{V}$ is also an M- (resp. S-) variety.

The following characterization for the variety of languages corresponding to $\Diamond \boldsymbol{V}$ is an immediate consequence of theorems 1.2 and 1.3 above.

Corollary 1.4 Let $\boldsymbol{V}$ be an M- (resp. S-) variety and let $\mathcal{V}$ be the corresponding $*$- (resp. $+$-) variety. Then a language $L \in A^*$ (resp. A^+) is recognized by a monoid (resp. semigroup) in $\Diamond \boldsymbol{V}$ if and only if it is in the Boolean algebra generated by the languages of the form $L_1 a_1 L_2 \cdots a_{n-1} L_n$ with $n \geq 1$, $L_i \in A^*\mathcal{V}$ (resp. $A^+\mathcal{V}$) for $i = 1, ..., n$ and $a_1, ..., a_{n-1} \in A$.

There is another characterization of the varieties of languages which is closed under concatenation products by using aperiodic relational morphisms. In fact, Straubing [55] proved the following important theorem.

Theorem 1.5 [55] Let $\mathcal{V}$ be a $*$-variety (resp. $+$-variety) and $\boldsymbol{V}$ the corresponding M-variety (resp. S-variety). Then

1. The least $*$-variety (resp. $+$-variety) containing $\mathcal{V}$ and closed under the concatenation product is the $*$-variety (resp. $+$-variety) which is associated to the M-variety (resp. S-variety) $\boldsymbol{A}_{\boldsymbol{S}}^{-1}\boldsymbol{V}$.
2. $\mathcal{V}$ is closed under the concatenation product if and only if $\boldsymbol{V} = \boldsymbol{A}_{\boldsymbol{S}}^{-1}\boldsymbol{V}$.

From the preceding theorem, we know immediately that $\Diamond \boldsymbol{V} = \boldsymbol{A}_{\boldsymbol{S}}^{-1}\boldsymbol{V}$ for any M-variety (resp. S-variety) $\boldsymbol{V}$.

Now, we introduce another hierarchy of varieties of languages related to the concatenation product, this one in A^*, closely related to the dot-depth hierarchy, which is introduced by H. Straubing [57].

For any finite alphabet A, we set

1. $A^*\mathcal{V}_0 = \{\emptyset, A^*\}$.
2. $A^*\mathcal{V}_k =$ Boolean algebra generated by $\{L_0a_1L_1\cdots a_jL_j : L_i \in A^*\mathcal{V}_k, a_i \in A, i = 0,1,...,j.\}$.

We know from chapter 2 that $\mathcal{V}_0$ is just the $*$-variety $\mathcal{I}$ and hence the corresponding M-variety $\boldsymbol{V}_0 = \boldsymbol{I}$. By theorem 1.4 above, we see immediately that every $\mathcal{V}_k$ ($k \geq 0$) is an $*$-variety and we have an algebraic description of their corresponding M-varieties: $\boldsymbol{V}_{k+1} = \Diamond\boldsymbol{V}_k$ for each $k \geq 0$.

Level 1 of the two hierarchies were also characterized algebraically, by Simon [51] [52] for Straubing's hierarchy and by Knast [31] for the dot-depth hierarchy. There are still very rare results concerning the higher levels of these hierarchies in the literature and are worthwhile for exploration.

We now introduce the power pseudovariety of a given pseudovariety, which is related to the dot-depth 2. For a semigroup (resp. monoid) S, if $\mathcal{P}(S)$ is the set of all subsets of S, then $\mathcal{P}(S)$ is a semigroup (resp. monoid) under the product given by

$$AB = \{ab | a \in A, b \in B\}$$

for any two subsets $A, B \subseteq S$. $\mathcal{P}(S)$ is called the power semigroup (resp. monoid) of S. Let $\boldsymbol{V}$ be an S- (resp. M-) variety. We define $\boldsymbol{PV}$ be the S- (resp. M-) variety generated by the semigroups (resp. monoids) of the form $\mathcal{P}(S)$ with $S \in \boldsymbol{V}$.

The operation $\boldsymbol{V} \mapsto \boldsymbol{PV}$ on pseudovarieties corresponds to two important operations on the varieties of languages. Let A, B be alphabets. We call a morphism $\theta : A^* \to B^*$ of free monoids literal if $\theta(A) \subseteq B$, i.e. θ maps the letters to letters. A morphism $\sigma : A^* \to \mathcal{P}(B^*)$ is called a substitution, i.e. σ maps the letters of A to the languages over B. Let $\sigma : A^* \to \mathcal{P}(B^*)$ be a substitution. For any $L \in \mathcal{P}(B^*)$, we let $\sigma^{-1}(L) = \{u \in A^* | \sigma u \cap L \neq \emptyset\}$. Thus we can treat σ as a relation in $A^* \times B^*$ and take inverse images with respect to this relation.

Let $\mathcal{V}$ be a $*$-variety of languages. For any alphabet A, we define $A^*\Lambda\mathcal{V}$ be the Boolean algebra generated by the languages of the form $\theta(L)$, where

$L \in B^*\mathcal{V}$ for some alphabet B and $\theta : B^* \to A^*$ is a literal morphism. For any alphabet A, we let $A^*\Sigma\mathcal{V}$ be the Boolean algebra generated by the languages of the form $\sigma^{-1}(L)$, where $L \in B^*\mathcal{V}$ and $\sigma : A^* \to \mathcal{P}(B^*)$ is a substitution.

The following theorem then characterizes the recognizability power of the pseudovariety $\boldsymbol{PV}$, which summarizes the works of Pin [41], Reutenauer [48] and Straubing [55].

Proposition 1.6 Let $\boldsymbol{V}$ be an M-variety and let $\mathcal{V}$ be the corresponding $*$-variety. Then $\Lambda\mathcal{V} = \Sigma\mathcal{V}$ are $*$-varieties and which correspond to the M-variety $\boldsymbol{PV}$.

Pin and Straubing [45] characterized the M-variety $\boldsymbol{V}_2$ which is related to dot-depth 2.

Theorem 1.7 [45] $\boldsymbol{V}_2 = \Diamond\boldsymbol{J} = \Diamond\boldsymbol{Sl} = \Diamond\boldsymbol{R} = \Diamond\boldsymbol{L} = \Diamond\boldsymbol{DA} = \boldsymbol{PJ}$.

In conclusion, we summerize the important facts and the main results on the two hierarchies mentioned in the literature.

Within the dot-depth hierarchy, we have the following results:

1. Each $\mathcal{B}_i$ is a variety of languages for $i \geq 0$. (see [23], chapter IX)

2. $\bigcup_{k=0}^{\infty} A^+\mathcal{B}_k = A^+\mathcal{S}$.

3. The hierarchy is strict, i.e. for any $k \geq 0$ and if $|A| > 1$, $A^+\mathcal{B}_k$ is contained strictly in $A^+\mathcal{B}_{k+1}$. [16]

4. For the corresponding pseudovarieties $\boldsymbol{B}_k$, $k \geq 0$. $\boldsymbol{B}_k$ is strictly contained in $\boldsymbol{B}_{k+1}$, and $\bigcup_{k=0}^{\infty} \boldsymbol{B}_k = \boldsymbol{A_S}$. This is an immediate consequence of the items above.

5. $\boldsymbol{B}_0 = \boldsymbol{Nil}$.

6. $\boldsymbol{B}_1$ consists of all semigroups S that satisfy the following condition: There exists $n > 0$ such that for all $s, t, u, v \in S$, $e, f \in E(S)$,

$$(esft)^n esfve(ufve)^n = (esft)^n e(ufve)^n$$

It follows from the above statements, one can effectively determine whether a given semigroup belongs to $\boldsymbol{B}_1$ or not, and consequently, we know whether a given recognizable language in A^+ belongs to $A^+\mathcal{B}_1$ or not. [31]

Within the Straubing's hierarchy, we also have the following facts: [58] [62]

1. $\mathcal{V}_k$ is a variety of languages for each $k \geq 0$.

2. $\bigcup_{k=0}^{\infty} A^*\mathcal{V}_k = A^*\mathcal{S}$ for any alphabet A.

3. $A^*\mathcal{V}_k$ is strictly contained in $A^*\mathcal{V}_{k+1}$ for any $k \geq 0$ and any $|A| > 1$.

4. Let $\boldsymbol{V}_k, k \geq 1$ be the corresponding pseudovarieties, then

   (a) $\boldsymbol{V}_k$ is strictly contained in $\boldsymbol{V}_{k+1}$.

   (b) $\bigcup_{k=0}^{\infty} \boldsymbol{V}_k = \boldsymbol{A}$

5. $\boldsymbol{V}_0 = \boldsymbol{I}$, $\boldsymbol{V}_1 = \boldsymbol{J}$ (Simon), $\boldsymbol{V}_2 = \boldsymbol{PJ}$ (Straubing and Pin).

6. $\boldsymbol{B}_k = \boldsymbol{V}_k * \boldsymbol{LI}$ for $k \geq 1$ and $\boldsymbol{V}_k = \boldsymbol{B}_k \cap \boldsymbol{M}$ for $k \geq 0$. It follows from this that $\boldsymbol{V}_k$ consists of all monoids in $\boldsymbol{B}_k$, and that if there is an algorithm for determining membership in $\boldsymbol{V}_k$, with $k \geq 1$, then there is such an algorithm for $\boldsymbol{B}_k$. (Margolis and Straubing)

§2 Lower bounds, upper bounds and partial results

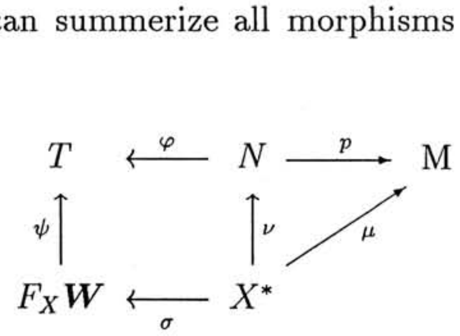
As the exact dot-depth of an aperiodic semigroup is very difficult to be effectively determined, we transfer our attention on its lower bounds and upper bounds, and in some special cases, these bounds may yield the exact dot-depth. Following the discussion on a lower bound for the dot-depth, P. Weil [68] obtained some results.

Let $\boldsymbol{V}$ be an S-variety and $\boldsymbol{W}$ an M- (resp. S-) variety. Then define the Mal'cev product $\boldsymbol{V} \odot \boldsymbol{W}$ as the class of all the quotients of the monoids (resp. semigroups) M such that there exists a morphism $\varphi : M \to N$ with $N \in \boldsymbol{W}$ and $\varphi^{-1}(e) \in \boldsymbol{V}$ for each idempotent e of N. Using the notion of the canonical factorization of a relational morphism, it is easy to verify that $\boldsymbol{V} \odot \boldsymbol{W}$ defines equivalently as the class of all monoids (resp. semigroups) M such that there exists a relational morphism $\tau : M \to N$ with $N \in \boldsymbol{W}$ and $\tau^{-1}(e) \in \boldsymbol{V}$ for each idempotent e of N. It can be verified easily that $\boldsymbol{V} \odot \boldsymbol{W}$ is an M- (resp. S-) variety.

Proposition 2.1 [68] Let $\boldsymbol{V}$ be an S-variety and $\boldsymbol{W}$ a locally finite M-variety. Let M be a finite monoid and hence there exists an finite variables set X and an epimorphism $\mu : X^* \to M$. Let $\sigma : X^* \to F_X\boldsymbol{W}$ be the canonical morphism of the $\boldsymbol{W}$-free monoid on X. Then $M \in \boldsymbol{V} \odot \boldsymbol{W}$ if and only if $\mu\sigma^{-1}(e) \in \boldsymbol{V}$ for each idempotent e of $F_X\boldsymbol{W}$.

Proof. Let $\tau = \sigma\mu^{-1}$. Then $\tau : M \to F_X\boldsymbol{W}$ is a surjective relational morphism and $\tau^{-1} = \mu\sigma^{-1}$. Thus, if $\mu\sigma^{-1}(e) = \tau^{-1}(e) \in \boldsymbol{W}$ for each idempotent $e \in E(F_X\boldsymbol{W})$, then $M \in \boldsymbol{V} \odot \boldsymbol{W}$.

Conversely, suppose that $M \in \boldsymbol{V} \odot \boldsymbol{W}$. Then there exist epimorphisms $p : N \to M$ and $\varphi : N \to T$ such that $N \in \boldsymbol{V} \odot \boldsymbol{W}, T \in \boldsymbol{W}$ and $\varphi^{-1}(e) \in \boldsymbol{V}$ for any idempotent e in T. By the universal property of the free monoid X^* there exists a morphism $\nu : X^* \to N$ such that $\mu = p\nu$. Replacing N by $\nu(N)$ and T by $\varphi\nu(N)$. As $\nu(N)$ and $\varphi\nu(N)$ are submonoids of N and T respectively, they are still in the varieties $\boldsymbol{V} \odot \boldsymbol{W}$ and $\boldsymbol{W}$ respectively. We may assume all morphisms are epimorphisms. Now, as $T \in \boldsymbol{W}$ and $\varphi\nu : X^* \to T$ is an epimorphism, it then induces an epimorphism $\psi : F_X\boldsymbol{W} \to T$ such that $\psi\sigma = \varphi\nu$. We can summerize all morphisms by the following commutative diagram:



Let e be an idempotent of $F_X\boldsymbol{W}$. Note that $\sigma^{-1}(e) \subseteq \sigma^{-1}\psi^{-1}\psi(e)$ and since $\psi\sigma = \varphi\nu$ and $p\nu = \mu$, we have

$$\mu\sigma^{-1}(e) \subseteq \mu\sigma^{-1}\psi^{-1}\psi(e) = p\nu\nu^{-1}\varphi^{-1}\psi(e) = p\varphi^{-1}\psi(e).$$

Now, $e \in E(F_X \boldsymbol{W})$ implies that $\psi(e) \in E(T)$. Then, we can deduce that $\varphi^{-1}\psi(e)$ is in the S-variety $\boldsymbol{V}$. Finally, as p is an epimorphism, this implies that $p\varphi^{-1}\psi(e)$ is also in $\boldsymbol{V}$. Hence, $\mu\sigma^{-1}(e) \in \boldsymbol{V}$. The proof is complete.

We have the following useful lemma.

Lemma 2.2 Let $\boldsymbol{V}_1, \boldsymbol{V}_2$ be S-varieties and $\boldsymbol{V}_3$ a pseudovariety. Then

$$\boldsymbol{V}_1 \odot (\boldsymbol{V}_2 \odot \boldsymbol{V}_3) \subseteq (\boldsymbol{V}_1 \odot \boldsymbol{V}_2) \odot \boldsymbol{V}_3.$$

Proof. For any M in $\boldsymbol{V}_1 \odot (\boldsymbol{V}_2 \odot \boldsymbol{V}_3)$, by the definition, there exists a relational morphism $\tau : M \to N$ with $N \in \boldsymbol{V}_2 \odot \boldsymbol{V}_3$ and $\tau^{-1}(e) \in \boldsymbol{V}_1$ for each idempotent e of N. By the definition again, there exists a relational morphism $\rho : N \to P$ with $P \in \boldsymbol{V}_3$ and $\rho^{-1}(f) \in \boldsymbol{V}_2$ for each idempotent f of P. Now, let $\sigma : M \to P$ be the composition $\sigma = \rho\tau$. Then for each idempotent $f \in E(P)$, we have $\sigma^{-1}(f) = \tau^{-1}(\rho^{-1}(f))$. Observe that $\rho^{-1}(f) \in \boldsymbol{V}_2$, we can deduce that $\sigma^{-1}(f) \in \boldsymbol{V}_1 \odot \boldsymbol{V}_2$. By the definition, we have $M \in (\boldsymbol{V}_1 \odot \boldsymbol{V}_2) \odot \boldsymbol{V}_3$.

As for a relation between the Schützenberger product and the Mal'cev product, Pin [44] proved that $\Diamond\boldsymbol{V} \subseteq \boldsymbol{B}_1 \odot \boldsymbol{V}$ for any pseudovariety $\boldsymbol{V}$. This implies immediately that $\boldsymbol{V}_{n+1} \subseteq \boldsymbol{B}_1 \odot \boldsymbol{V}_n$ for $n \geq 0$ in the Straubing's hierarchy.

Let M be a monoid and let $e \in E(M)$. We define M_e be the subsemigroup of M generated by all elements of M which is greater than e under the partial order $\leq_{\mathcal{J}}$, that is, generated by the set

$$P_e = \{m \in M | MeM \subseteq MmM\}.$$

Let us consider the submonoids of M of the form eM_ee for any $e \in E(M)$. Note that if e is the identity of M, then eM_ee is the group of invertible elements of M. Also it is easy to verify that eM_ee is strictly contained in M if and only if M is not a group.

Let $\boldsymbol{V}$ be an M-variety. We define $\tilde{\boldsymbol{V}}$ be the class of all monoids M such that $eM_e e \in \boldsymbol{V}$ for all $e \in E(M)$. Firstly, we prove that $\tilde{\boldsymbol{V}}$ is also an M-variety.

Lemma 2.3 [68] Let $\varphi : M \to N$ be a monoid morphism. Then we have:

1. If $e \in E(M)$ and $f = \varphi e$, then $\varphi(M_e) = N_f$.
2. If φ is an epimorphism and $f \in E(N)$, then there exists $e \in E(M)$ such that $\varphi e = f$ and $\varphi(M_e) = N_f$.

Proof.

1. By definition, $\varphi(M_e)$ is generated by the set $\varphi(P_e) = \{\varphi s | e \leq_{\mathcal{J}} s\}$. Since $e \leq_{\mathcal{J}} s$ implies $f \leq_{\mathcal{J}} \varphi s$, we have $\varphi(P_e) \subseteq P_f$ and hence $\varphi(M_e) \subseteq N_f$.
2. Let e be an idempotent in the minimal ideal of the subsemigroup $\varphi^{-1}(f)$ of M. In particular, $\varphi e = f$, so $\varphi(M_e) \subseteq N_f$ by (1). Let now $n \in N$ such that $f \leq_{\mathcal{J}} n$. Then $f = anb$ for some $a, b \in N$. Let x, m and y be elements of $\varphi^{-1}(a), \varphi^{-1}(n)$ and $\varphi^{-1}(b)$ respectively. Then we have $\varphi(xmy) = f$ and so $e \leq_{\mathcal{J}} xmy \leq_{\mathcal{J}} m$. Therefore $m \in M_e$ and hence $n = \varphi m \in \varphi(M_e)$. Thus $P_f \subseteq \varphi(M_e)$ and so $N_f \subseteq \varphi(M_e)$.

Proposition 2.4 [68] Let $\boldsymbol{V}$ be an M-variety. Then $\tilde{\boldsymbol{V}}$ is an M-variety.

Proof. Let M and N be monoids and let $(e, f) \in E(M \times N) = E(M) \times E(N)$. It is easy to verify that $P_{(e,f)} = P_e \times P_f$, so that $(e, f)M \times N_{(e,f)}(e, f) = eM_e e \times fN_f f$. Thus $\tilde{\boldsymbol{V}}$ is closed under direct product.

Next, suppose that N is a submonoid of M and e is an idempotent of N. Then e is also an idempotent of M and $eN_e e \subseteq eM_e e$. So $N \in \tilde{\boldsymbol{V}}$.

Finally, suppose that $\varphi : M \to N$ is an epimorphism and e is an idempotent of N. Then $eN_e = \varphi(fM_f f)$ for some $f \in E(M)$ according to Lemma 2.3. Hence N is also in the class $\tilde{\boldsymbol{V}}$. Thus, we have proved that $\tilde{\boldsymbol{V}}$ is indeed an M-variety.

We now give another description for the operation $\boldsymbol{V} \mapsto \tilde{\boldsymbol{V}}$.

Theorem 2.5 [68] Let $\boldsymbol{V}$ be an M-variety. Then we have $\tilde{\boldsymbol{V}} = \boldsymbol{LV} \odot \boldsymbol{J}_1 = \boldsymbol{LV} \odot \boldsymbol{DA}$.

Proof. For any $M \in \tilde{\boldsymbol{V}}$, we take a finite variables set X such that there is an epimorphism $\mu : X^* \to M$. Let $\sigma : X^* \to 2^X$ be a monoid morphism from X^* to the monoid of subsets of M with union. Then, we know that $2^X = F_X \boldsymbol{J}_1$ and σ is its natural morphism. Now for any idempotent $X_r \subseteq X$ of 2^X, let $e \in E(\mu\sigma^{-1}(X_r))$. Then $e = \mu(x_1 \cdots x_r)$ for $\{x_1, ..., x_r\} = X_r$. In particular, for each $x_i \in X_r$, $e \leq_{\mathcal{J}} \mu(x_i)$ and so $\mu(x_i) \in M_e$. Therefore, $\mu\sigma^{-1}(X_r) \leq M_e$. So, we have $e\mu\sigma^{-1}(X_r)e \leq eM_e e \in \boldsymbol{V}$ and hence $\mu\sigma^{-1}(X_r) \in \boldsymbol{V}$. It is clear that any semilattice is locally finite, i.e., $\boldsymbol{J}_1$ is locally finite. Then we have proved that $M \in \boldsymbol{LV} \odot \boldsymbol{J}_1$, by using proposition 2.1.

Conversely, let us assume that $\mu\sigma^{-1}(X_r) \in \boldsymbol{LV}$ for each idempotent $X_r \subseteq X$. For each $e \in E(M)$ and $x \in X$ such that $e \leq_{\mathcal{J}} \mu x$, there exist $u_x, v_x \in X^*$ such that $e = \mu(u_x x v_x)$. In particular, if $X_r = \mu^{-1}(P_e) \cap X = \{x_1, ... x_r\}$, then $e = \mu(u_{x_1} x_1 v_{x_1} \cdots u_{x_r} x_r v_{x_r})$ and $\sigma(u_{x_1} x_1 v_{x_1} \cdots u_{x_r} x_r v_{x_r}) = X_r$, so $e \in \mu\sigma^{-1}(X_r)$ and hence $eM_e e \in \boldsymbol{LV}$. But since $eM_e e$ is a monoid, $eM_e e \in \boldsymbol{V}$ and $M \in \tilde{\boldsymbol{V}}$. Thus we have proved that $\tilde{\boldsymbol{V}} = \boldsymbol{LV} \odot \boldsymbol{J}_1$.

Now we prove the left part of the proposition. Since $\boldsymbol{J}_1 \subseteq \boldsymbol{DA}$, we have $\tilde{\boldsymbol{V}} \subseteq \boldsymbol{LV} \odot \boldsymbol{DA}$. Conversely, Lemma 2.2 implies that

$$\boldsymbol{LV} \odot \boldsymbol{DA} = \boldsymbol{LV} \odot (\boldsymbol{LI} \odot \boldsymbol{J}_1) \subseteq (\boldsymbol{LV} \odot \boldsymbol{LI}) \odot \boldsymbol{J}_1 = \boldsymbol{LV} \odot \boldsymbol{J}_1$$

since it is clear that $\boldsymbol{LV} \odot \boldsymbol{LI} = \boldsymbol{LV}$. The proof is complete.

We now define an increasing sequence $(\boldsymbol{W}_n)_{n \geq 0}$ of M-varieties by letting

1. $\boldsymbol{W}_0 = \boldsymbol{I}$

2. $\boldsymbol{W}_{n+1} = \tilde{\boldsymbol{W}}_n$ for any $n \geq 0$.

Clearly, we have $\boldsymbol{W}_1 = \tilde{\boldsymbol{I}} = \boldsymbol{LI} \odot \boldsymbol{DA} = \boldsymbol{DA}$. It is also clear that $\boldsymbol{W}_n$ is strictly contained in $\boldsymbol{W}_{n+1}$ for any $n \geq 0$ and that $\cup_{n \geq 0} \boldsymbol{W}_n = \boldsymbol{A}$.

Proposition 2.6 [68] For all $n \geq 0$, we have $\boldsymbol{W}_n = \boldsymbol{LI} \odot \boldsymbol{W}_n$ and $\boldsymbol{LDA} \odot \boldsymbol{W}_n \subseteq \boldsymbol{W}_{n+1}$.

Proof. We have $\boldsymbol{W}_0 = \boldsymbol{LI} \odot \boldsymbol{W}_0$ and $\boldsymbol{W}_1 = \boldsymbol{DA} = \boldsymbol{LDA} \odot \boldsymbol{W}_0$ since that $\boldsymbol{LI} \odot \boldsymbol{I}$ and $\boldsymbol{LDA} \odot \boldsymbol{I}$ are classes of monoids. Let us assume that the formulae $\boldsymbol{LI} \odot \boldsymbol{W}_k = \boldsymbol{W}_k$ and $\boldsymbol{LDA} \odot \boldsymbol{W}_k \subseteq \boldsymbol{W}_{k+1}$ stands for all $k \leq n$. Let $\varphi : M \to N$ be an epimorphism with $N \in \boldsymbol{W}_{n+1}$ and $\varphi^{-1}(e) \in \boldsymbol{LI}$ (resp. $\boldsymbol{LDA}$) for each $e \in E(N)$. Now for any $f \in E(M)$, we have $\varphi(fM_f f) \subseteq (\varphi f)N_{(\varphi f)}(\varphi f)$ by lemma 2.3. But $N \in \boldsymbol{W}_{n+1}$, so $(\varphi f)N_{(\varphi f)}(\varphi f) \in \boldsymbol{W}_n$, and hence we have $fM_f f \in \boldsymbol{LI} \odot \boldsymbol{W}_n$ (resp. $\boldsymbol{LDA} \odot \boldsymbol{W}_n$). Thus $fM_f f \in \boldsymbol{W}_n$ (resp. $\boldsymbol{W}_{n+1}$) for any $f \in E(M)$ and therefore, $M \in \boldsymbol{W}_{n+1}$ (resp. $\boldsymbol{W}_{n+2}$).

Corollary 2.7 $\boldsymbol{V}_n \subseteq \boldsymbol{W}_n$ for any $n \geq 0$.

Proof. For $n = 0$, we have by definition $\boldsymbol{V}_0 = \boldsymbol{W}_0 = \boldsymbol{I}$.

Let us assume that $\boldsymbol{V}_n \subseteq \boldsymbol{W}_n$ for some $n \geq 0$. Then $\boldsymbol{V}_{n+1} \subseteq \boldsymbol{B}_1 \odot \boldsymbol{V}_n \subseteq \boldsymbol{B}_1 \odot \boldsymbol{W}_n$. But also we have $\boldsymbol{B}_1 \subseteq \boldsymbol{LJ} \subseteq \boldsymbol{LDA}$ and so $\boldsymbol{V}_{n+1} \subseteq \boldsymbol{LDA} \odot \boldsymbol{W}_n \subseteq \boldsymbol{W}_{n+1}$. Thus the corollary follows.

As a consequence, we also obtain the following corollary:

Corollary 2.8 Let M be an aperiodic monoid. If $M \notin \boldsymbol{W}_n$ then $M \notin \boldsymbol{V}_n$. In particular, the dot-depth of M is greater than or equal to $n+1$.

Now, observe that the M-varieties $\boldsymbol{W}_n$ are decidable, so that the above criterion gives us an effective lower bound for the dot-depth of a given aperiodic monoid. In addition, we also have the following theorem due to Weil [64].

Theorem 2.9 [64] Let M be an aperiodic monoid and n the maximal number such that there exists a chain $M = M_n \supset \cdots \supset M_1 \supset M_0 = \{1\}$ with $M_{i-1} = e(M_i)_e e$ for some idempotent e of M_i for $i = 1, ..., n$. Then M has dot-depth at least n.

J. E. Pin et al [46] investigated the relationships between locally trivial categories and the unambiguous concatenation products of languages. They have also obtained an effective upper bound of the dot-depth of aperiodic monoids. The following theorem is obtained by them:

Theorem 2.10 [46]

1. Let M be an aperiodic monoid of dot-depth k. Then there exist surjective morphisms

$$M = M_k \to M_{k-1} \to .... \to M_1 \to M_0 = \{1\}$$

such that the dot-depth of M_i is i for $0 \leq i \leq k$.

2. Let M be an aperiodic monoid. Consider a factorization $\pi = \pi_1\pi_2 \cdots \pi_k$ of the morphism $\pi : M \to \{1\}$, where each factor $\sigma : N \to R$ satisfies one of the following conditions:

   (a) There is a $\mathcal{J}$-class J of N such that σ is injective on $N \setminus J$, and $\sigma n = \sigma n'$ implies that n and n' are $\mathcal{H}$-equivalent.

   (b) There is a $\mathcal{J}$-class J of N such that σ is injective on $N \setminus J$, $\sigma n = \sigma n'$ implies that n and n' are $\mathcal{J}$-equivalent, and σ is an injective mapping on the groups contained in S.

   (c) There are two $\mathcal{J}$-class $J_1 \leq J_2$ such that σ is injective on $S \setminus J_2$, $\sigma n = \sigma n'$ implies $n = n'$ or $n, n' \in J_1 \cup J_2$,and σ is injective on the groups contained in N.

   (d) There are two incomparable $\mathcal{J}$-classes J_1 and J_2 such that σ is injective on $S \setminus J_1$ and on $S \setminus J_2$, $\sigma n = \sigma n'$ implies $n = n'$ or $n, n' \in J_1 \cup J_2$, and σ is injective on the groups contained in S.

   Let k be the number of these factors that satisfy condition (c) with J_2 regular. Then the dot-depth of M is less than or equal to k.

Straubing [59] gave an apparently different effective necessary condition for an aperiodic monoid with dot-depth at most two, and went on to prove that this condition is also sufficient for 2-generated monoids, thus proving the decidability of dot-depth two for these monoids. He also conjectured that the condition he gave is sufficient for general case.

A new approach to the dot-depth problem is restricting the general problem to inverse monoids [64] [66], A class of finite inverse monoids (resp. semigroups) is called an $\boldsymbol{IM}$-variety (resp. $\boldsymbol{IS}$-variety) if it is closed under finite direct product, homomorphism image and taking of inverse submonoids (resp. subsemigroups). Thus, the $\boldsymbol{IM}$-varieties (resp. $\boldsymbol{IS}$-varieties) can be dealt with like M-varieties (resp. S-varieties), and we can hence study the counterparts for $\boldsymbol{IM}$- (resp. $\boldsymbol{IS}$-) varieties of the general dot-depth hierarchies [60] [21]. We omit the details.

Chapter 5 Operators $\boldsymbol{P}$ and $\boldsymbol{P}'$

The discussion in chapter 4 showed the importance of the characterization of the pseudovariety $\boldsymbol{PJ}$. We now study the operator $\boldsymbol{P}$ over any pseudovariety $\boldsymbol{V}$. Recall that for a semigroup S, $\mathcal{P}(S)$ is the semigroup of the subsets of S under the product induced by the product in S. We denote it by $\mathcal{P}'(S)$ which is the set of all non-empty subsets of S. Then it is clear that it is a subsemigroup of $\mathcal{P}(S)$. We have the following definitions for the varieties $\mathcal{PV}$, $\mathcal{P}'\mathcal{V}$ and pseudovarieties $\boldsymbol{PV}$, $\boldsymbol{P'V}$ of semigroups. Similar definitions hold if we replace semigroups by monoids.

Definition 1 Let $\mathcal{V}$ be a variety of semigroups. Define $\mathcal{PV}$ (respectively $\mathcal{P}'\mathcal{V}$) be the variety generated by the semigroups of the form $\mathcal{P}(S)$ (respectively $\mathcal{P}'(S)$) with $S \in \mathcal{V}$. Similarly, for a pseudovariety of finite semigroups $\boldsymbol{V}$, define $\boldsymbol{PV}$ (respectively $\boldsymbol{P'V}$) be the pseudovariety generated by the semigroups of the form $\mathcal{P}(S)$ (respectively $\mathcal{P}'(S)$) with $S \in \boldsymbol{V}$.

Note that the function

$$\mathcal{P}'(S) \times P'(T) \to \mathcal{P}'(S \times T)$$

$$(A, B) \mapsto A \times B$$

is an embedding. Thus, we conclude that

$$\mathcal{P}'\mathcal{V} = HSP\{\mathcal{P}'(S) : S \in \mathcal{V}\} = HS\{\mathcal{P}'(S) : S \in \mathcal{V}\},$$

and similarly

$$\boldsymbol{P'V} = HSP_{fin}\{\mathcal{P}'(S) : S \in \boldsymbol{V}\} = HS\{\mathcal{P}'(S) : S \in \boldsymbol{V}\}.$$

Recall that Sl_2 is the semilattice with two elements $\{0,1\}$. Let $\boldsymbol{Sl} = \boldsymbol{V}(Sl_2)$ and $\mathcal{S}l = V(Sl_2)$. Then clearly $\boldsymbol{Sl}$ is thus the pseudovariety of finite semilattices and $\mathcal{S}l$ is thus the variety of semilattices (recall the proof in proposition 3.6, chapter 2 and compare with $\boldsymbol{J}_1$, which is the M-variety of finite semilattice monoids) and we have $\mathcal{S}l^F = \boldsymbol{Sl}$.

The following proposition tells us the relationships between $\mathcal{P}'\mathcal{V}$, $\boldsymbol{P'V}$ and $\mathcal{P}\mathcal{V}$, $\boldsymbol{PV}$.

Proposition 2 [6] Let $\mathcal{V}$ be a variety of semigroups and $\boldsymbol{V}$ a pseudovariety of finite semigroups. Then, we have $\mathcal{P}\mathcal{V} = \mathcal{P}'\mathcal{V} \vee \mathcal{S}l$ and $\boldsymbol{PV} = \boldsymbol{P'V} \vee \boldsymbol{Sl}$.

Proof. First, we have an onto homomorphism

$$\begin{aligned} \mathcal{P}'(S) \times Sl_2 &\rightarrow \mathcal{P}(S) \\ (A, 1) &\mapsto A \\ (A, 0) &\mapsto \emptyset \end{aligned}$$

whence $\mathcal{P}(S) \in H(\mathcal{P}'(S) \times Sl_2)$ for any $S \in \mathcal{V}$. This implies the inclusion $\mathcal{P}\mathcal{V} \subseteq \mathcal{P}'\mathcal{V} \vee \mathcal{S}l$. Conversely, the semigroup Sl_2 is embeddable in every semigroup of the form $\mathcal{P}(S)$ (just take the subsemigroup $\{\emptyset, S\}$), therefore $\mathcal{S}l \subseteq \mathcal{P}\mathcal{V}$ and we then deduce the reverse inclusion $\mathcal{P}'\mathcal{V} \vee \mathcal{S}l \subseteq \mathcal{P}\mathcal{V}$ since $\mathcal{P}'\mathcal{V} \subseteq \mathcal{P}\mathcal{V}$ is obvious.

Similarly, we have $\boldsymbol{PV} = \boldsymbol{P'V} \vee \boldsymbol{Sl}$.

In chapter 3, we have already found many pseudovarieties of finite semigroups which can be represented by a set of pseudoidentities combined by words and ω-powers, hence, the membership problem of them can be effectively solved by checking whether a finite semigroup satisfies those pseudoidentities or not. In this chapter, we are going to describe the pseudovariety $\boldsymbol{PV}$ through the identities satisfied by it. The following discussion on the identities of power semigroups was due to J. Almeida [6].

To determine the identities satisfied by power semigroups, it is convenient to generalize the notion of identity. Let $X = \{x_1, x_2, ...\}$ be a countable set of variables. For $t \in X^+, T \subseteq X^+$ and a semigroup S, we write $S \models t \in T$ if, for any homomorphism $\varphi : X^+ \rightarrow S$, $\varphi t \in \varphi T$.

Let $\alpha : N_p \rightarrow N$ and $\beta : N_q \rightarrow N$ be arbitrary functions corresponding to the arbitrary words $x_{\alpha 1} \cdots x_{\alpha p}, x_{\beta 1} \cdots x_{\beta q} \in X^+$. For the natural number $p' = p + |Im\beta \setminus Im\alpha|$, we fix an ordering $j_{p+1}, j_{p+2}, ..., j_{p'}$ of $Im\beta \setminus Im\alpha$. Extend α to a function $\alpha' : N_{p'} \rightarrow N$ by letting $\alpha' i = \alpha i$ for $i = 1, ..., p$ and $\alpha' i = j_i$ for $i = p + 1, ...p'$. For this choice of α', we obtain a finite nonempty set of functions

$$\beta/\alpha = \{\gamma : N_q \to N_{p'} : \alpha'\gamma i = \beta i, \forall i \in N_q\}$$

and a corresponding set of words

$$F(\alpha, \beta) = \{x_{\gamma 1} \cdots x_{\gamma q} : \gamma \in \beta/\alpha\}.$$

Note that, although the set of words $F(\alpha, \beta)$ depends on the choice of α', the condition $S \models x_1 \cdots x_p \in F(\alpha, \beta)$ is independent of that choice. Similarly, we define a set $F(\beta, \alpha)$.

Lemma 3 [6] Let S be a semigroup. Then the inclusion relation $A_{\alpha 1} \cdots A_{\alpha p} \subseteq A_{\beta 1} \cdots A_{\beta q}$ holds for any $A_1, ..., A_r \in \mathcal{P}'(S)$ if and only if $S \models x_1 \cdots x_p \in F(\alpha, \beta)$.

Proof. "$\Rightarrow$" Let $\varphi : X \to S$ and let $a_i = \varphi x_i, i = 1, 2, ...$ Consider the sets $D_i = \{a_j : \alpha' j = i\}$, As $a_j \in D_{\alpha' j}$, for each $i \in Im\alpha \cup Im\beta$, D_i is clearly a nonempty set. By hypothesis, we have

$$a_1 \cdots a_p \in D_{\alpha 1} \cdots D_{\alpha p} \subseteq D_{\beta 1} \cdots D_{\beta q}.$$

Hence, there exist $z_j \in D_{\beta j} (j \in N_q)$ such that $a_1 \cdots a_p = z_1 \cdots z_q$. Let $\delta \in \beta/\alpha$. Then, for each $j \in N_q$, we have $z_j \in D_{\beta j} = D_{\alpha' \delta j}$. For j such that $\delta j \leq p$, we choose $\gamma j \in N_p$ such that $z_j = a_{\gamma j}$ and $\alpha' \gamma j = \alpha' \delta j$. We can use this information to define a function $\gamma : N_q \to N_{p'}$ by taking $\gamma j = \delta j$ if $\delta j > p$. Thus, $\gamma \in \beta/\alpha$ and

$$\begin{aligned} \varphi(x_1 \cdots x_p) &= a_1 \cdots a_p = z_1 \cdots z_q \\ &= a_{\gamma 1} \cdots a_{\gamma q} = \varphi(x_{\gamma 1} \cdots x_{\gamma q}). \end{aligned}$$

This shows that $S \models x_1 \cdots x_p \in F(\alpha, \beta)$.

"$\Leftarrow$" Let $A_1, ..., A_r \in \mathcal{P}'(S)$ with $Im\alpha \cup Im\beta \subseteq N_r$ and let $z_i \in A_{\alpha' i}, i \in N_{p'}$. For any function $\varphi : X \to S$ such that $\varphi x_i = z_i, i \in N_{p'}$, we have, by hypothesis, $\varphi(x_1 \cdots x_p) \in \varphi F(\alpha, \beta)$, and so there exists $\gamma \in \beta/\alpha$ such that $\varphi(x_1 \cdots x_p) = \varphi(x_{\gamma 1} \cdots x_{\gamma q})$. Hence

$$\begin{aligned} z_1 \cdots z_p &= \varphi(x_1 \cdots x_p) = \varphi(x_{\gamma 1} \cdots x_{\gamma q}) \\ &= z_{\gamma 1} \cdots z_{\gamma q} \in A_{\alpha' \gamma 1} \cdots A_{\alpha' \gamma q} = A_{\beta 1} \cdots A_{\beta q}, \end{aligned}$$

whence $A_{\alpha 1} \cdots A_{\alpha p} \subseteq A_{\beta 1} \cdots A_{\beta q}$.

In terms of the identities satisfied by $\mathcal{P}'(S)$, the preceding lemma has the following formulation.

Theorem 4 [6] Let S be a semigroup and let $u = x_{\alpha 1} \cdots x_{\alpha p}$ and $v = x_{\beta 1} \cdots x_{\beta q}$ be words of X^+. Then $\mathcal{P}'(S) \models u = v$ if and only if $S \models x_1 \cdots x_p \in F(\alpha, \beta)$ and $S \models x_1 \cdots x_q \in F(\beta, \alpha)$.

Lemma 5 If the sides of an identity $u = v$ are both products of distinct variables and $S \models u = v$, then $\mathcal{P}'(S) \models u = v$.

Proof. Let the identity be $x_{\alpha 1} \cdots x_{\alpha p} = x_{\beta 1} \cdots x_{\beta q}$. For any $A_1, ..., A_r \in \mathcal{P}'(S)$, we need to show that $A_{\alpha 1} \cdots A_{\alpha p} = A_{\beta 1} \cdots A_{\beta q}$. By hypothesis, we know that $x_{\alpha i}, i = 1, ..., p$ are distinct variables and $x_{\beta j}, j = 1, ..., q$ are also distinct variables. Now if all the variables in these two sets of variables are distinct, then the equation $A_{\alpha 1} \cdots A_{\alpha p} = A_{\beta 1} \cdots A_{\beta q}$ is obviously satisfied. Let us consider the case that there is one same variable in these two variable sets, say $x_{\alpha h} = x_{\beta k}$. Then, for any $a \in A_{\alpha h} = A_{\beta k}$, we have the following equation

$$A_{\alpha 1} \cdots A_{\alpha(h-1} a A_{\alpha(h+1)} \cdots A_{\alpha p} = A_{\beta 1} \cdots A_{\beta(k-1)} a A_{\beta(k+1)} \cdots A_{\beta q}$$

Therefore,

$$\begin{aligned} A_{\alpha 1} \cdots A_{\alpha p} &= \bigcup_{a \in A_{\alpha h}} A_{\alpha 1} \cdots A_{\alpha(h-1} a A_{\alpha(h+1)} \cdots A_{\alpha p} \\ &= \bigcup_{a \in A_{\beta k}} A_{\beta 1} \cdots A_{\beta(k-1)} a A_{\beta(k+1)} \cdots A_{\beta q} \\ &= A_{\beta 1} \cdots A_{\beta q} \end{aligned}$$

Similarly, if there are more than one (finite number) same variables in those two variable sets, we can take the union, one by one, similar to the discussion above and actually concludes the proof of the equation $A_{\alpha 1} \cdots A_{\alpha p} = A_{\beta 1} \cdots A_{\beta q}$.

The lemma above shows the importance of those identities whose both sides are products of distinct variables in the study of the identities satisfied by power semigroups. We give a name for these kind of identities: linear identities. For an arbitrary set of identities Σ, we use $\mathcal{L}in\Sigma$ to represent the subset of all linear identities of Σ. For a variety of semigroups $\mathcal{V}$, we define $\mathcal{L}in\mathcal{V} = [\mathcal{L}in(Id_X\mathcal{V})]$ for an infinity set of variables X, in view of lemma 1.12, chapter 1, this variety is well defined not depending on the choice of X.

Given a function $\alpha : N_r \to N$, we represent the identity $x_1 \cdots x_r = x_{\alpha 1} \cdots x_{\alpha r}$ by ε_α. Most of what follows depends on the detailed analysis of the consequences of such an identity, in which the following lemma is quite useful.

Lemma 6 [6] Let $\rho : N_r \to N$ be a function. Then the identity ε_ρ is equivalent to the linear identity ε_σ, where

$$\sigma i = \begin{cases} \rho i, & i \in L = \cap_{t=1}^{\infty} Im\rho^t \\ r+i, & i \in N_r \setminus L \end{cases}$$

Proof. Since the domain of ρ is a finite set, there exist some positive integers n and k such that $\rho^{n+k} = \rho^n$. Hence, ρ defines a permutation of the set $L = Im\rho^n$ and, therefore, ε_σ is a linear identity.

The substitution of x_{r+i} by $x_{\rho i}, i \in N_r$ then yields ε_ρ as a consequence of ε_σ. For the converse, note that, given a function $\alpha : N_r \to N$, we have $\varepsilon_\alpha \vdash \varepsilon_{\bar{\alpha}}$, where

$$\bar{\alpha} i = \begin{cases} \alpha i, & i \in Im\alpha \cap N_r \\ r+i, & i \in N_r \setminus Im\alpha \end{cases}$$

Let us consider the functions $\rho^{(l)}$ defined by

$$\rho^{(l+1)} i = \begin{cases} \rho i, & i \in N_r \cap Im\rho^l \\ r+i, & i \in N_r \setminus Im\rho^l \end{cases}$$

where ρ^0 denotes the inclusion $N_r \hookrightarrow N$. As

$$\begin{aligned} Im(\rho^{(l+1)}) \cap N_r &= \{j \in N_r : j = \rho i, \exists i \in N_r \cap Im\rho^l\} \\ &= \rho(N_r \cap Im\rho^l) \cap N_r \\ &= Im(\rho^{l+1}) \cap N_r, \end{aligned}$$

from the preceding observation, we deduce that $\varepsilon_{\rho^{(l)}} \vdash \varepsilon_{\rho^{(l+1)}}$. Since $\rho^{(1)} = \rho$ and $\rho^{(l+1)} = \sigma$, we conclude that $\varepsilon_\rho \vdash \varepsilon_\sigma$.

The identities of the form $x_1 \cdots x_p = x_{\gamma 1} \cdots x_{\gamma q}$ are naturally associated with formal inclusions of the form $x_1 \cdots x_p \in F(\alpha, \beta)$, and so the following result will be of importance to continue the analysis of the identities which are valid in power semigroups.

Proposition 7 [6] Let $\gamma : N_q \to N$ and $\delta : N_p \to N$ be arbitrary functions. Then the set of identities

$$\Sigma = \{ \begin{array}{l} \mu_1 : x_1 \cdots x_p = x_{\gamma 1} \cdots x_{\gamma q}, \\ \mu_2 : x_1 \cdots x_q = x_{\delta 1} \cdots x_{\delta p} \} \end{array}$$

is equivalent to a set of two linear identities with the same left sides as μ_1 and μ_2 respectively.

Proof. We start by extending γ and δ to the functions $\bar{\gamma}, \bar{\delta} : N \to N$ respectively, say by letting them act as the identity function outside the sets N_q and N_p. We then have

$$\Sigma \vdash x_1 \cdots x_p = x_{\gamma 1} \cdots x_{\gamma q} = x_{\bar{\gamma}\delta 1} \cdots x_{\bar{\gamma}\delta p}.$$

Let $\rho = \bar{\gamma}\delta$ and $r = p$. Then by Lemma 6, taking k such that ρ^k is the identity function on the set $L = \cap_{t=1}^{\infty} Im(\bar{\gamma}\delta)^t$ and iterating k times the identity ε_σ given by the lemma, we obtain the identity ε_{σ^k} and therefore,

$$\Sigma \vdash x_1 \cdots x_p = x_{\eta 1} \cdots x_{\eta p},$$

where

$$\eta i = \begin{cases} i, & i \in L \\ p + i, & i \in N_p \setminus L \end{cases}$$

Combining this last identity with μ_2, we obtain the following equation:

$$\Sigma \vdash \lambda_2 : x_1 \cdots x_q = x_{\delta' 1} \cdots x_{\delta' p},$$

where $\delta' j = \delta j$ for $j \in L$ and $\delta' j = s + j$ for $j \in N_p \setminus L$ with $s = \max\{p, q\}$. As $\rho = \bar{\gamma}\delta$ permutes the elements of L, δ is injective on L; hence so is the function δ' and λ_2 is a linear identity. The substitution of $x_(s+j)$ by $x_{\delta j}$ shows that $\delta_2 \vdash \mu_2$. Similarly, Σ allows us to deduce the linear identity

$$\lambda_1 : x_1 \cdots x_p = x_{\gamma' 1} \cdots x_{\gamma' q},$$

where $\gamma' i = \gamma i$ for $i \in M = \cap_{t=1}^{\infty} Im(\bar{\delta}\gamma)^t$ and $\gamma' i = s + i$ for $i \in N_q \setminus M$, μ_1 being obtained from λ_1 by substitution. Hence, the set $\{\lambda_1, \lambda_2\}$ satisfies the required conditions.

The following result in a certain way solves the characterization problem of $\mathcal{P}'\mathcal{V}$ and $\mathcal{P}\mathcal{V}$ through the identities satisfied by them. This result was obtained by J. Almeida [6].

Theorem 8 Let $\mathcal{V}$ be a variety of semigroups. Then the following properties hold:

1. $\mathcal{P}'\mathcal{V} = \mathcal{L}in\mathcal{V}$
2. $\mathcal{P}\mathcal{V} = \mathcal{L}in\mathcal{V} \vee \mathcal{S}l$
3. if $Sl_2 \in \mathcal{V}$, then $\mathcal{P}\mathcal{V} = \mathcal{P}'\mathcal{V}$.

Proof.

1. By Lemma 5, we have $\mathcal{P}'\mathcal{V} \subseteq \mathcal{L}in\mathcal{V}$. Conversely, if $\mathcal{P}'\mathcal{V} \models \varepsilon : x_{\alpha 1} \cdots x_{\alpha p} = x_{\beta 1} \cdots x_{\beta q}$, then by Theorem 4, $\mathcal{V}$ satisfies the formal inclusions $x_1 \cdots x_p \in F(\alpha, \beta)$ and $x_1 \cdots x_q \in F(\beta, \alpha)$. Then, $x_1 \cdots x_p = x_{\gamma 1} \cdots x_{\gamma q}$ for some $\gamma \in \beta/\alpha$, and so

$$\mathcal{V} \models \varepsilon_1 : x_1 \cdots x_p = x_{\gamma 1} \cdots x_{\gamma q}.$$

Similarly, there exists $\delta \in \alpha/\beta$ such that

$$\mathcal{V} \models \varepsilon_2 : x_1 \cdots x_q = x_{\delta 1} \cdots x_{\delta p}.$$

Let $\Sigma = \{\varepsilon_1, \varepsilon_2\}$ and note that each ε_i allows us to deduce the identity ε, for example, substituting x_i by $x_{\alpha' i}$ in ε_1, we obtain ε. From Proposition 7, we know that Σ is equivalent to a set Σ' of linear identities. Hence $\mathcal{V} \models \Sigma'$ and therefore, $\mathcal{L}in\mathcal{V} \models \Sigma'$, and so $\mathcal{L}in\mathcal{V} \models \Sigma$. Hence, the variety $\mathcal{L}in\mathcal{V}$ satisfies the identity ε. Consequently, $\mathcal{L}in\mathcal{V}$ is contained in $\mathcal{P}'\mathcal{V}$.

2. This part follows from 1) and proposition 2.

3. As $Sl_2 \in \mathcal{V}$ by hypothesis, we have $\mathcal{S}l \subseteq \mathcal{V} \subseteq \mathcal{L}in\mathcal{V}$ (the last inclusion is true since $Id_X\mathcal{V} \supseteq Lin(Id_X\mathcal{V})$). Hence $\mathcal{P}\mathcal{V} = \mathcal{L}in\mathcal{V} \vee \mathcal{S}l = \mathcal{L}in\mathcal{V} = \mathcal{P}'\mathcal{V}$ by 1) and 2).

Corollary 9 The semigroup of operators on varieties of semigroups which is generated by $\{\mathcal{P}, \mathcal{P}'\}$ has three elements, namely $\mathcal{P}, \mathcal{P}'$ and $\mathcal{P}^2$, and it is defined by the relations $\mathcal{P}' = \mathcal{P}'^2, \mathcal{P}'\mathcal{P} = \mathcal{P}^2$ and $\mathcal{P}\mathcal{P}' = \mathcal{P}$.

Having solved the problem of discribing $\mathcal{P}\mathcal{V}$ and $\mathcal{P}'\mathcal{V}$ by the identities satisfied by the power varieties, it is natural to study the pseudovarieties $\boldsymbol{PV}$ and $\boldsymbol{P'V}$ following the same idea. From chapter 1, we know that any pseudovariety $\boldsymbol{V}$ is of the form $\mathcal{W}^F$ for some generalized variety $\mathcal{W}$. Generally, we can extend the operator $\mathcal{L}in$ to a generalized variety $\mathcal{W} = \cup_{i \in I}\mathcal{V}_i$ such as

$$\mathcal{L}in\mathcal{W} = \mathcal{L}in(\cup_{i \in I}\mathcal{V}_i) = \cup_{i \in I}\mathcal{L}in\mathcal{V}_i$$

and consequently we can define $\boldsymbol{LinV} = (\mathcal{L}in\mathcal{W})^F$. We may also expect that $\boldsymbol{P'V} = \boldsymbol{LinV}$, but unforturnately, this is generally not true. At the following context, we give a result that if $\boldsymbol{V}$ is a subpseudovariety of the pseudovariety of locally trivial semigroups, then the equation $\boldsymbol{P'V} = \boldsymbol{LinV}$ holds, which was given by J. Almeida [6]. Here we give a detailed proof.

Lemma 10 [6] If $\boldsymbol{V}$ is a pseudovariety of finite semigroups, then $V(\boldsymbol{P'V}) = \mathcal{P}'V(\boldsymbol{V})$.

Proof. First, we have

$$\begin{aligned} V(\boldsymbol{P'V}) &= HSP(HS\{\mathcal{P}'(S) : S \in \boldsymbol{V}\}) \\ &= HSP\{\mathcal{P}'(S) : S \in \boldsymbol{V}\} \\ &\subseteq HS\{\mathcal{P}'(S) : S \in V(\boldsymbol{V})\} \\ &= \mathcal{P}'V(\boldsymbol{V}). \end{aligned}$$

Conversely, suppose that

$$\boldsymbol{P'V} \models \varepsilon : x_{\alpha 1} \cdots x_{\alpha p} = x_{\beta 1} \cdots x_{\beta q}.$$

Then, from theorem 4, we have for any $S \in \boldsymbol{V}$,

$$S \models x_1 \cdots x_p \in F(\alpha, \beta)$$

and

$$S \models x_1 \cdots x_q \in F(\beta, \alpha).$$

Let $F(\alpha, \beta) = \{w_1, ..., w_r\}$. If for any $w_i \in F(\alpha, \beta)$, there exist $S_i \in \boldsymbol{V}$ such that $S_i \not\models x_1 \cdots x_p = w_i$, then $S_1 \times \cdots \times S_r \not\models x_1 \cdots x_p \in F(\alpha, \beta)$. Hence there exist $\gamma \in F(\alpha, \beta)$ and $\delta \in F(\beta, \alpha)$ such that $\boldsymbol{V}$ satisfies the identities

$$\varepsilon_1 : x_1 \cdots x_p = x_{\gamma 1} \cdots x_{\gamma q},$$

$$\varepsilon_2 : x_1 \cdots x_q = x_{\delta 1} \cdots x_{\delta p}.$$

Hence, the identities $\varepsilon_1, \varepsilon_2$ are all valid in the variety $V(\boldsymbol{V})$, and therefore $\mathcal{P}'V(\boldsymbol{V}) \models \varepsilon$ by theorem 8. Hence, $\mathcal{P}'V(\boldsymbol{V}) \subseteq V(\boldsymbol{P}'\boldsymbol{V})$.

An algebra A is called locally finite if all its finitely generated subalgebras are finite. A class $\mathcal{K}$ of algebras is said to be locally finite if all its elements are locally finite.

Lemma 11 Let $\boldsymbol{V}$ be a pseudovariety. Then the variety $V(\boldsymbol{V})$ is locally finite if and only if $F_n\boldsymbol{V}$ is finite for any $n \geq 1$.

Proof.

"$\Rightarrow$" For any $n \geq 1$, $F_n\boldsymbol{V} \in V(\boldsymbol{V})$ is clear, also $F_n\boldsymbol{V}$ is finitely generated by the set $\{\bar{x_1}, ..., \bar{x_n}\}$. Hence $F_n\boldsymbol{V}$ is finite by hypothesis.

"$\Leftarrow$" For any algebra $A \in V(\boldsymbol{V})$ and any finitely generated subalgebra B of A, B is also in the variety $V(\boldsymbol{V})$. Suppose B is generated by n elements, then we can take a surjective map

$$\varphi : X_n \to B,$$

and then we can extend φ to an epimorphism

$$\bar{\varphi} : F_n\boldsymbol{V} \to B.$$

As $F_n\boldsymbol{V}$ is finite by hypothesis and $\bar{\varphi}$ is surjective, B is forced to be finite. Hence the proof is concluded.

From the above lemma and proposition 2.5, chapter 1, we immediately deduce the following corollary.

Corollary 12 Let $\boldsymbol{V}$ be a pseudovariety of finite algebras. If the variety $V(\boldsymbol{V})$ is locally finite, then $V(\boldsymbol{V})^F = \boldsymbol{V}$.

Theorem 13 [6] If a pseudovariety of finite semigroups $\boldsymbol{V}$ is contained in $\boldsymbol{LI}$, then $\boldsymbol{P'V} = \boldsymbol{Lin V}$.

Proof. Suppose that $\boldsymbol{V} = \mathcal{W}^F = \cap_{i\in I}\mathcal{V}_i^F$. Then, we have

$$\begin{aligned}\boldsymbol{P'V} &= HS\{\mathcal{P}'(S) : S \in \cup_{i\in I}\mathcal{V}_i^F)\} \\ &= \cup_{i\in I}HS\{\mathcal{P}'(S) : S \in \mathcal{V}_i^F\} \\ &= \cup_{i\in I}\boldsymbol{P'}\mathcal{V}_i^F\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\boldsymbol{Lin V} &= (\mathcal{L}in\mathcal{W})^F = (\cup_{i\in I}\mathcal{L}in\mathcal{V}_i)^F \\ &= \cup_{i\in I}(\mathcal{L}in\mathcal{V}_i)^F = \cup_{i\in I}(\mathcal{P}'\mathcal{V}_i)^F.\end{aligned}$$

Hence it suffices to show that $\boldsymbol{P'}\mathcal{V}^F = (\mathcal{P}'\mathcal{V})^F$ for any variety $\mathcal{V}$ which is generated by a finite semigroup (recall the proof of proposition 2.7, chapter 1) and such that $\mathcal{V}^F \subseteq \boldsymbol{LI}$ by hypothesis. Suppose now $\mathcal{V} = V(A)$ with a finite semigroup A. As

$$A \in V(A)^F \subseteq \boldsymbol{LI} = \cup_{k\geq 1}[[x_1\cdots x_k y z_1\cdots z_k = x_1\cdots x_k z_1\cdots z_k]],$$

so $A \models x_1\cdots x_k y z_1\cdots z_k = x_1\cdots x_k z_1\cdots z_k$ for some k and therefore this identity is satisfied by $V(A)$. From lemma 5, this identity is also satisfied by $\mathcal{P}'\mathcal{V}$ since it is linear. Now, for any finitely generated semigroup S in the variety $\mathcal{P}'\mathcal{V}$, consider all the words on its finite generators $\{s_1, ..., s_n\}$, any words with length $l \geq 2k+1$ can be equivalent to a word with length $l-1$ by a substitution of identity $x_1\cdots x_k y z_1\cdots z_k = x_1\cdots x_k z_1\cdots z_k$. This shows that there are at most finite words with length less than or equal to $2k$ which are not equal to each other in S, i.e. S is finite. Hence, we proved that $\mathcal{P}'\mathcal{V} = V(\boldsymbol{P}'\mathcal{V}^F)$ (by lemma 10) is locally finite, and so it is equational by corollary 12. i.e. $\boldsymbol{P}'\mathcal{V}^F = (\mathcal{P}'\mathcal{V})^F$, which concludes the proof.

From the proof of the preceding theorem, we notice that the restriction of $V(A)^F \subseteq \boldsymbol{LI}$ is essential to enforce that $\mathcal{P}'\mathcal{V}$ is locally finite, and hence equational.

Another problem on the operator $\boldsymbol{P}$ was asked by Straubing in [55]. For a pseudovariety $\boldsymbol{V}$, let $\boldsymbol{P}^{n+1} = \boldsymbol{P}(\boldsymbol{P}^n\boldsymbol{V})$, then we obtain an ascending chain of pseudovarieties $\boldsymbol{V} \subseteq \boldsymbol{PV} \subseteq \boldsymbol{P}^2\boldsymbol{V} \subseteq .....$ Straubing [55] asked whether this chain is infinite for some pseudovariety $\boldsymbol{V}$? He conjectured that $\boldsymbol{P}^2\boldsymbol{V} = \boldsymbol{P}^3\boldsymbol{V}$ for any M-variety $\boldsymbol{V}$ and proved the result for commutative M-variety, obtained independently by Perrot in [39] as well.

Proposition 14 [55] [39] Let $\boldsymbol{V}$ be a commutative M-variety. Then $\boldsymbol{P}^2\boldsymbol{V} = \boldsymbol{P}^3\boldsymbol{V}$. Furthermore, if $\boldsymbol{V} \neq \boldsymbol{I}$, then $\boldsymbol{PV} = \boldsymbol{P}^2\boldsymbol{V}$ is the M-variety of commutative monoids all of whose subgroups are in $\boldsymbol{V}$.

Margolis and Pin [36] proved the following result:

Proposition 15 For any noncommutative M-variety $\boldsymbol{V}$, $\boldsymbol{P}^3\boldsymbol{V} = \boldsymbol{M}$.

Corollary 16 $\boldsymbol{P}^3\boldsymbol{V} = \boldsymbol{P}^4\boldsymbol{V}$ for any M-variety $\boldsymbol{V}$.

Remark: Pin [41] also gave an example of M-variety, which illustrates that the result gaven in Corollary 16 is the best possible.

Example 17 [41] Let $\boldsymbol{R}_1$ be the M-variety of all $\mathcal{R}$-trivial idempotent monoids. Then the M-varieties $\boldsymbol{R}_1, \boldsymbol{PR}_1, \boldsymbol{P}^2\boldsymbol{R}_1$ and $\boldsymbol{P}^3\boldsymbol{R}_1$ are all distinct.

In closing, we remark that a complete description for the operators generated by $\boldsymbol{P}$ and $\boldsymbol{P}'$ on S-varieties was given by Almeida in [6].

Theorem 18 [6] The semigroup of operators on S-varieties generated by $\{\boldsymbol{P}, \boldsymbol{P}'\}$ has eight elements, namely, $\boldsymbol{P}', \boldsymbol{P}'^2, \boldsymbol{P}'^3, \boldsymbol{P}, \boldsymbol{P}^2, \boldsymbol{P}^3, \boldsymbol{P}\boldsymbol{P}'$, and $\boldsymbol{P}\boldsymbol{P}'^2$, and it is defined by the relations

$$\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{P}^2, \boldsymbol{P}^2\boldsymbol{P}' = \boldsymbol{P}^3, \boldsymbol{P}'^4 = \boldsymbol{P}'^3, \boldsymbol{P}'^3\boldsymbol{P} = \boldsymbol{P}'^2\boldsymbol{P}, \boldsymbol{P}\boldsymbol{P}'^3 = \boldsymbol{P}\boldsymbol{P}'^2.$$

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