The Radicals of Semigroup Algebras with Chain Conditions

by

AU Yun-Nam

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AU Yun-nam

The Chinese University of Hong Kong
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The Chinese University of Hong Kong
Graduate School

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Dr. Shum Kar Ping
Supervisor

Lam Siu Por

Dr. Lam Siu Por

K. M. Yeung

Dr. Yeung Kit Ming

Prof. M. Tokizawa

Guo Yuqi

Prof. Guo Yuqi
External Examiner

Abstract

Let K be a field and S an arbitrary semigroup. It is known that the theory of semigroup algebras $K[S]$ is closely related both to semigroup theory and ring theory. J. Okninski [Okn1] has recently given a detail survey on semigroup algebras in 1990. In this thesis, the radicals of the semigroup algebras $K[S]$ with chain conditions are particularly investigated. Moreover, the Jacobson radicals and other radicals of graded rings are particularly studied as they are important tools in studying semigroup algebras.

The Jacobson radicals and other radicals of semigroup rings $R[S]$ over commutative semigroups S were firstly investigated by J. Krempa, E. Jespers, J. Okninski, and P. Wauters since 1980. Moreover, the Jacobson radical of $R[S]$ when R satisfies $J(R) = J_1(R)$ was described by E. Jespers. We review here these results in the beginning of chapter 3.

When the algebras over some non-commutative semigroups, the band graded ring theory given by W.D. Munn and A.V. Kelarev provides us another approaches to study the non-commutative semigroup algebras $K[S]$. Furthermore, the semiprimitivity problems of inverse semigroup algebras and PI semigroup algebras are investigated in chapter 3.

The artinian semigroup algebra $K[S]$ was studied by E.I. Zelmanov. He showed that the semigroup S must be finite if $K[S]$ is artinian in 1977. Later on, P. Wauters showed that if semilattice graded ring is semilocal, then the base semilattice must be finite. Recently, some finiteness conditions on groupoid graded rings are solved by A.V. Kelarev in 1995. More finiteness conditions on semigroup algebras are fully investigated in monograph of J. Okninski. All these results will be described and further investigated in chapter 4. We cite some results from J. Okninski [Okn1] and some recent results on artinian semigroup graded rings from M.V. Clase, E. Jespers, A.V. Kelarev and J. Okninski to investigate some finiteness conditions on $K[S]$. Some modifications and simplifications of the relevent results are obtained.

In chapter 5, the relationship between the Gelfand-Kirillov dimension on semigroup algebras $K[S]$ and the growths of the base semigroups is studied. Attemptions have been made to extend the second layer conditions on noetherian semigroup algebras from the well known results on group algebras.

Contents

Introduction

Let R be any ring with unity and S an arbitrary semigroup. Denote the semigroup ring by $R[S]$. It is noticed that the theory of semigroup ring $R[S]$ is closely related to semigroup theory and ring theory. In the case that S is a group G, a full survey on the algebraic structure of group ring was given by D.S. Passman in his monograph [Pas1] in 1977. This monograph is an important text for group algebras. Moreover, the commutative semigroup rings are fully investigated by R. Gilmer [Gil] in 1985. In the case if R is a field K, then the structure of $K[S]$ are fully studied by J. Okninski in his recent monograph [Okn1] in 1990. Okninski has given an intensive survey on cancellative semigroup algebras with finiteness conditions, and also the semigroup PI-algebras. As motivated by these monographs, we focus in this thesis on the topics of the radicals and finiteness conditions on semigroup algebras.

The monograph of Okninski was published in 1990. In this paper, we recognize some of the recent results on semigroup algebras obtained in the literature after 1990 and study these algebras by another approach. We notice that the graded rings are important tools for studying the structure of semigroup algebras. For example, if S is a Clifford semigroup, then S can be decomposed into semilattice of groups. As a result, the corresponding semigroup algebra becomes a semilattice graded ring, i.e.

$$K[S] = \sum_{\alpha \in \Gamma} K[G_\alpha]$$

where Γ is semilattice and $K[G_\alpha]$ is a group algebra. The structure of $K[S]$ is therefore affected by G_α and the order structure of the base semilattice Γ. Throughout this these, we will use these techniques to study the radicals and also some finiteness conditions of semigroup algebras.

In chapter 1, some basic properties and notations of semigroups such as semigroup algebras; group algebras; graded rings; crossed products and smash products are given. We will use these properties and results of the above algebraic structures to study the structure of semigroup algebras.

In chapter 2, the theory of the radicals of graded rings which are frequently used in subsequent works will be established. Let $\mathcal{J}$ be any one of the Jacobson, Brown-McCoy, Prime, Levitzki radicals of graded ring R respectively. Then we will concentrate on the relationships between the graded radicals $\mathcal{J}_{gr}$ and the radical $\mathcal{J}$ on the group graded rings, where G is finite or infinite. After the radicals of group graded ring is described, we study the rings graded by semilattices and also bands. From A.V. Kelarev [Kel1] in 1991, we obtain some descriptions for radicals $\mathcal{J}$ of band graded rings.

The Jacobson radicals of special band-graded rings described by W.D. Munn [Mun7] are given in chapter 2. These recent results will be used in chapter 3 to shorten some of the proofs of some results concerning Jacobson radicals of special band-graded rings in the literature.

Chapter 3 contains two parts. One part concerns the commutative semigroup rings, i.e. the semigroup S involved is a commutative semigroup. The another part is an investigation of the non-commutative semigroup algebras of an arbitrary semigroup.

In the first part of chapter 3, We first examine commutative cancellative semigroup. The radicals of semigroup rings have been investigated by J. Krempa, J. Okninski, E. Jespers and P. Wauters [Kre2, JW1, OW]. On the other hand, semigroup rings involved arbitrary commutative semigroups were investigated by W.D. Munn. In fact, W.D. Munn described the Jacobson radicals of commutative semigroup algebras over a field in 1983 (see [Mun1]) and over commutative rings with unity in 1984 (see [Mun2]). Further, J. Okninski and P. Wauters [OW] **(1986)** generalized the results of Munn to prime and Levitzki radicals with arbitrary coefficents. In addition, E. Jespers gave a complete description of Jacobson radical if R satisfies $J_1(R) = J_\infty(R)$ and S is arbitrary commutative semigroup [Jes1] **(1987)**, i.e.

$$J(R[S]) = J_1(R)[S] + \sum_{p \in \mathrm{P}} I(J_{1,p}(R), S, \xi_p) + J(R, S_P, \Gamma')$$

where $J(R, S_P, \Gamma')$ is defined in section 3.2.

If S is a separative semigroup, then it is known that S is semilattice of commutative cancellative semigroups, say $S = \cup_{\alpha \in \Gamma} S_\alpha$. In this case, S can be embedded into Q, where $Q = \cup_{\alpha \in \Gamma} G_\alpha$ and G_α is group of fractions of S_α. We also investigate the relationship between the semigroup rings $K[S]$ and $K[Q]$.

The another part of chapter 3 is to examine the structure of algebras over an arbitrary semigroup. To simplify our work, we only consider $K[S]$. At first, in section 3.3, S is cancellative semigroup which may not possess a group of (right or left) fractions. We extract some basic properties described by Okninski [Okn1, Chapter 7, Chapter 9] and cite some of the recent results of Okninski [Okn3, Okn4]. A necessary condition for $K[S]$ to be prime or semiprime is given. When the Jacobson radical $J(K[S]) \neq 0$, there is a subsemigroup P which has a group of (right or left) fractions, say the reversive semigroup, also $J(K[P]) \neq 0$. The Jacobson radical of $K[S]$ are described when S is a subsemigroup of a polycyclic-by-finite and nilpotent group.

Furthermore, the algebras of completely 0-simple semigroups will also be studied. For this topics, we adopt the Munn algebras and the graded rings as our approaches. We find that the maximal subgroup of a 0-simple semigroup plays an vital role in sturcture of $K[S]$. The structure and semprimitivity problem of inverse semigroup algebras, like group algebras, are paticularly studied. Domanov (stated in the survey of Munn [Mun3]) showed that if S is inverse semigroup then for any maximal subgroup H of S, $J(K[H]) = 0$ implies that $J(K[S]) = 0$. The converse is not true unless $E(S)$, the set of all idempotents of S, is pseudofinite.

The remaining of chapter 3 provides some results on other semigroups. We shall make use of the results of the band-graded rings and to give descriptions of the radicals of completely 0-simple, inverse, cancellative semigroup algebras. Separative semigroups and completely regular semigroups will be particularly considered. Finally, we also make use of some known results from semigroup PI-algebras $K[S]$ in our discussion. We will show that $J(K[S]) = \mathcal{B}(K[S])$ if $K[S]$ is PI semigroup algebra (not necessarily finitely generated).

In chatper 4, the finiteness conditions of semigroup algebras are examined. First, we study the finiteness condition on graded rings and find the necessary condition of semilattice graded rings to be semilocal. We also show that the base semilattice must be finite. Recently, Kelarev [Kel7] has generalized the cases of finite semigroup graded rings to the groupoid graded rings in 1995. These results solve some problems concerning finite semigroup graded rings. In section 4.2, semiprime rings and Goldie rings will be studied. Following the results of Jespers and Okninski [JO1] in the case of S being nilpotent, we obtain $J(K_0[S]) = \mathcal{B}(K_0[S])$, where $J(K_0[S])$ is the Jacobson radical of $K_0[S]$.

After studyiing the semprimeness of semigroup algebras. We investigate the noetherian semigroup algebras. The objective of this section is to find the necessary condition for S to be right noetherian. It has been conjectured that S is finitely generated if $K[S]$ is right noetherian [Okn1, Part V, Problem 7]. It will be shown here that when $K[S]$ is (right and left) noetherian, then S is finitely generated. Moreover, we also examine the structure of right noetherian inverse semigroup algebras and the right noetherian semigroup PI-algebras.

It is known in the literature that $K[S]$ is semilocal, that is, $K[S]/J(K[S])$ is artinian. The descending chain conditions for $K[S]$ are discussed in section 4.4. We find that there are some recent results concerning about d.c.c on semigroup graded rings given by M.V. Clase, E. Jespers, A.V. Kelarev and J. Okninski [CJKO, JO2, Kel6] in 1995. We apply these results to semilocal (right perfect, semisimple) semigroup algebras and then give another description for $K[S]$. Moreover, we show that there is a close relationship between $K[S]$ and $K[H]$ when H is maximal subgroup of S.

In the last chapter, we study the dimensions and prime ideals of $K[S]$ as applications. In 1993, Okninski [Okn2] wrote a paper on Gelfand-Kirillov dimension on semigroup algebra. We notice in that paper that Gelfand-Kirillov dimensions are connected to the growths and the ranks of semigroups, therefore we study the Krull, classical Krull, and Gelfand-Krillov dimensions on semigroup algebras and apply these to the noetherian cases. We obtain that if $K[S]$ is right noetherian, then the Gelfand-Kirillov dimension $GK(K[S]) < \infty$ iff $GK(K[T]) < \infty$ for every cancellative subsemigroup T of S. That is equivalent to say that T has group of fractions of nilpotent-by-finite.

The prime ideals and prime spectrum are important topics on noetherian algebras. As an application, we examine the links between prime ideals and second layer condition described by [Jat, GW, MR]. By the results of Jategaonkar [Jat], we can state that if G is polycyclic-by-finite, then $K[G]$ satisfies strongly second layer condition. In this respect, the following question arises naturally:

Let S be a complelely 0-simple semigroup with a maximal subgroup polycyclic-by-finite group. Let $K_0[S]$ be a noetherian semigroup algebra:

Is $K[S]$ has second layer condition ?

If S is an inverse semigroup in which every maximal subgroup of S is polycyclic-by-finite, does $K[S]$ has the (strong) second layer condition ?

Chapter 1

Preliminaries

In this chapter, some definitions and elementary results on semigroup algebras, group algebras and their related topics are given. These results will be useful in the subsequent discussion and will be frequently referred.

To investigate the structure of semigroup algebras, we have to know some properties of semigroups and groups. Furthermore, semigroup algebras and group algebras can be treated as graded rings. The description of radicals in graded ring (graded by groups or semigroups) will lead to another approach to characterize the radicals and structure of semigroup algebras. Hence, we recall the properties of graded rings and apply them to study semigroup algebras.

1.1 Some Semigroup Properties

The general definitions and notations of semigroups are taken from [How, Okn1, Pet].

Definition 1.1.1 *Let $S = \mathcal{M}^0(G, I, \Lambda; P)$ be a Rees matrix semigroup with sandwich matrix P. If $i \in I$, then the set $\{(g, i, m) \in S | g \in G^0, m \in \Lambda\}$ is denoted by $S_{(i)}$, and is called the ith* ***row*** *of S. Similarly , for any $m \in \Lambda$, $S^{(m)}$ is called the mth* ***column*** *of S. Clearly, $S_{(i)}^{(m)} = S_{(i)} \cap S^{(m)}$.*

Theorem 1.1.2 *Let $S = \mathcal{M}^0(G, I, \Lambda; P)$ be a semigroup of matrix type with zero θ.*

(i). *For any subsets $J \subseteq I, N \subseteq \Lambda$, $S_{(J)}$ is a right ideal of S and $S^{(N)}$ a left ideal of S.*

(ii). *For any subset* $J \subseteq I, N \subseteq \Lambda$, $S_{(J)}^{(N)}$ *is a semigroup isomorphic to a semigroup of matrix type* $\mathcal{M}^0(G, J, N; P_{NJ})$, *where* $P_{NJ} = (p'_{nj})$ *is the* $N \times J$ *submatrix of* P *defined by* $p'_{nj} = p_{nj}$ *for* $j \in J, n \in N$.

(iii). *The set of nonzero idempotents of* S *is* $\{(g, i, m) \in S | p_{mi} \neq \theta_G, g = p_{mi}^{-1}\}$, *where* θ_G *is the zero of* G^0.

(iv). *Every maximal subgroup of* S *is of the form* $S_{(i)}^{(m)}$ *or* $(S_{(i)}^{(m)} \setminus \theta)$ *for some* $i \in I, m \in \Lambda$.

Let $S = \mathcal{M}^0(G, I, \Lambda; P)$ be a Rees matrix semigroup over a group G and H a normal subgroup of G. Also let $\phi_H((g, i, m)) = (gH, i, m)$. Then

$$\phi_H : \mathcal{M}^0(G, I, \Lambda; P) \to \mathcal{M}^0(G/H, I, \Lambda, P_H)$$

is a homomorphism, where P_H is the matrix $(\bar{a}_{ij})$, where $\bar{a}_{ij}$ is the natural homomorphic image of a_{ij} in the quotient group G/H.

Let $S = \mathcal{M}^0(G, I, \Lambda; P)$. Taking $H = G$, then we have the image $T = \mathcal{M}^0(1^0, I, \Lambda; P')$, where

$$P' = (p'_{mi}) = \begin{cases} p'_{mi} = 1 & \text{if } p_{mi} \neq \theta_G \\ p'_{mi} = \theta & \text{if } p_{mi} = \theta_G \end{cases}$$

Write θ as the zero of 1^0 in order to distinguish it from the zero of G^0. Then, T is said to be **elementary Rees matrix semigroup**. The algebras graded by elementary Rees semigroup will be discussed in section 3.4.

We now consider the larger class of semigroups. A semigroup S is called a weakly periodic semigroup if for every $s \in S$, there exists $n \geq 1$ such that $S^1 s^n S^1$ is an idempotent ideal. The condition is equivalent to saying that a power of every element of S determines a 0-simple principal factor of S. (This is because if $a = s^n$ then $a^2 \neq \theta$ in the principal factor in $J(a)/I(a)$.) S is called π-regular if for every $s \in S$, there exists $n \geq 1$ such that s^n is a regular element of S. The semigroup S is called strongly π-regular if s^n lies in a subgroup of S. (Someone also called this semigroup as "epigroup" or "group-bound semigroup" etc.)

Notice that periodic; locally finite; regular; inverse; semisimple semigroups are all weakly periodic semigroups.

Let $\mathcal{H}, \mathcal{J}, \mathcal{R}, \mathcal{L}$ be the Green relations on the semigroup S [How, Chapter 2]. Also, let M_R, M_L and M_J stand for the minimal conditions on $S/\mathcal{R}, S/\mathcal{L}$ and $S/\mathcal{J}$ respectively. Then we have the following results.

Proposition 1.1.3 *Let J be an ideal of S. Then S satisfies the condition M_R (M_L) iff the semigroups J and S/J both satisfy M_R (M_L).*

Lemma 1.1.4 [Okn1, Lemma 3.1] *Let S be a semigorup satisfying any one of the conditions M_R, M_L, M_J. Then S is weakly periodic. Moreover, if all 0-simple principal factors of S are completely 0-simple, then S must be strongly π-regular.*

Theorem 1.1.5 [Okn1, Th. 3.3] *Let S be a weakly periodic semigroup. Then S has finitely many $\mathfrak{J}$-classes determining 0-simple principal factors iff there exists a chain of ideals $J_1 \subseteq \cdots \subseteq J_n = S$ of S such that J_1 and all $J_i/J_{i-1}, i > 1$ are 0-simple or nil. Moreover, if we let k denote the number of $\mathfrak{J}$-classes determining the 0-simple principal factors of S, then the ideals J_i can be chosen so that $n < 2k$. In addition, if S is a strongly π-regular semigroup of this type, then the non-nil semigorups $J_1, J_i/J_{i-1}$ are completely 0-simple.*

A semigroup S is called locally finite if all its finitely generated (f.g.) subsemigroups are finite. Clearly, all locally finite semigroup is periodic. The following are some properties concerning locally finiteness extracted from the text of [Okn1, Chapter 2].

Proposition 1.1.6 (i). *Let S be a finitely generated semigroup. If T is a subsemigroup of S such that $S \setminus T$ is finite , then T is finitely generated.*

(ii). *Let J be an ideal of a semigroup S. Then S is locally finite iff the semigroups J and S/J are locally finite.*

(iii). *Let S be a completely 0-simple semigroup. Then S is locally finite iff every maximal subgroup of S is locally finite.*

If the semigroup S has zero θ, then S is called nil if for every $s \in S, s^n = \theta$. S is called left T-nilpotent if any $s_1, s_2 \cdots, \in S$, there exists $n \geq 1$ such that $s_1 s_2 \cdots s_n = \theta$. S is left T-nilpotent iff S satisfies M_R. We notice that every nil semigroup is locally finite. Later on, we will discuss the nilpotent semigroup (this is the extension of nilpotent group).

Proposition 1.1.7 [Okn1, Prop. 2.13] *Let S be a nil semigroup. Then the following statements hold:*

(i). *If S has a.c.c. on its right and left annihilator ideals, then S is power nilpotent.*

(ii). *If S is multiplicative subsemigroup of a ring R with finite right Goldie dimension and S satisfies a.c.c. on right annihilator ideals, then S is power nilpotent.*

(iii). *If $S \subseteq M_n(D)$ for a division algebra D, then S is power nilpotent.*

We now consider the natural semigroup arising from general complete matrix ring $M_n(D)$, where D is a division ring.

Let $a \in M_n(D)$. Then define the rank $rk(a)$ of a as the dimension of the subspace $(D^n)a$ of D^n over D. Put $I_j = \{a \in M_n(D) | rk(a) \leq j\}$ for $j = 0, \cdots, n$. Then every I_j is an ideal of the semigroup $M_n(D)$ as semigroup under the matrix multiplication.

Theorem 1.1.8 [Okn1, Th. 1.6] *For any division algebra D and any integer $n \geq 1$,*

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = M_n(D)$$

is an ideal chain of multiplicative semigroup.

Moreover, every Rees factor I_i/I_{i-1}, $j = 1, 2, \cdots, n$, is a completely 0-simple semigroup, the maximal subgroups of $M_n(D)$ are isomorphic to the full skew linear groups of the corresponding algebras $M_j(D)$. In particular, $M_n(D)$ is a completely semisimple semigroup.

Now, we turn to consider another class of semigroups. Call a semigroup S is left (right) cancellative if for any $a, b, x \in S$, $xa = xb$ ($ax = bx$) implies that $a = b$. A weakened version of cancellative semigroup is" separative". The basic properties of separative semigroups are given by M. Petrich [Pet]. It is known that S is separative iff S is a semilattice of cancellative semigroups. A commutative separative semigroup is semilattice of commutative cancellative semigroups which can be embedded in a semilattice of groups (cf. [Pet, II.6.6]).

A semigroup S is archimedean if for any $a, b \in S$ there exists a positive integer n such that $a^n \in SbS$. Moreover, let Γ be a semilattice and $S = \cup_\alpha S_\alpha$ with $S_\alpha S_\beta = S_\beta S_\alpha \subseteq S_{\alpha\beta}$ for any $\alpha, \beta \in \Gamma$ and S_α is of type $\mathfrak{T}$. Then S is said to be semilattice of semigroups with type $\mathfrak{T}$.

Separative and cancellative semigroups both play important roles on the structure of semigroup algebras, especially on finding the radicals of commutative semigroup rings. Some more properties of cancellative semigroups will be discussed in chapter 3.

1.2 General Properties of Semigroup Algebras

We now investigate the structure of semigroup algebras. We first present the general properties of semigroup algebras which will be frequently used in our following discussion. The proofs of theorems are omitted.

Denote the lattices of the right, left, and the two-sided ideals of $K[S]$ by $\mathbf{R}(K[S])$, $\mathbf{L}(K[S])$ and $\mathbf{T}(K[S])$ respectively.

Let ρ be a right congruence on S and $\phi : S \to S/\rho$ the natural mapping onto the set S/ρ (equivalence classes). Then let $I(K, S, \rho)$ be the right ideal of $K[S]$ generated by the set $\{s - t | s, t \in S, (s,t) \in \rho\}$ such that

$$I(K, S, \rho) = \{\sum_{s \in S} r(s-t) \mid r \in K,\ (s,t) \in \rho\}.$$

In the semigroup ring, we always replace K by an arbitrary ring R.

Moreover, $K[S/\rho]$ is a right $K[S]$-module with $\phi_\rho(s) * t = \phi_\rho(st)$. Sometimes, we just denote $I(K, S, \rho)$ by $I(\rho)$ for simplification.

Lemma 1.2.1 [Okn1] *For any right congruence ρ on S , $\bar{\phi}_\rho : K[S] \to K[S/\rho]$ is a homomorphism of right $K[S]$-modules such that*

$$ker(\bar{\phi}_\rho) = I(\rho) = \sum_{s \in S} \omega_s(\rho)$$

where

$$\omega_s(\rho) = \{\sum_{i=1}^{m} \beta_i s_i \in K[S] \; | m \geq 1, \sum_{i=1}^{m} \beta_i = 0, (s, s_i) \in \rho \; \forall i = 1 \cdots m\}$$

If ρ is a congruence on S and $\bar{\phi}_\rho$ is the above homomorphism, then $ker(\bar{\phi}_\rho) = I(\rho)$ and $K[S/\rho] \cong K[S]/I(\rho)$.

Definition 1.2.2 *Let J be an ideal (one sided or two sided)of $K[S]$. Denote the set $\{(s,t) \in S \times S \; | s - t \in J\}$ by $\sim_J$.*
Then "$\sim_J$" is a congruence on S induced by the ideal J of $K[S]$.

Let J be a right ideal of $K[S]$. Then $J \to \sim_J$ is an order-preserving $\wedge$-complete semilattice homomorphism of $\mathbf{R}(K[S])$ onto $\mathbf{R}(S)$ which is lattice of right congruences on S, and $\sim_{I(\rho)} = \rho$ for any $\rho \in \mathbf{R}(S)$. For the right congruences ρ_1 and ρ_2 on S, we have $I(\rho_1 \wedge \rho_2) = I(\rho_1) \cap I(\rho_2)$ and $I(\rho_1 \vee \rho_2) = I(\rho_1) + I(\rho_2)$.

If we consider the universal congruence $\iota = S \times S$ on S, then the ideal

$$I(K, S, \iota) = \{s - t | s, t \in S\}K = \{\sum \alpha_s \, s \in K[S] | \sum \alpha_s = 0\}.$$

The above ideal is denoted by $\omega(K[S])$ and is called the augmentation ideal of $K[S]$, where the corresponding homomorphism $K[S] \to K$ is consequently called the augmentation mapping.

Let S be a semigroup with zero θ and $K_0[S]$ a factor algebra $K[S]/K\theta$. If S has a zero element, then put $K_0[S] = K[S]$. For any $a = \sum \alpha_s s, \alpha \in K$, let $supp_0(a)$ be the set $\{s \in S \setminus \{\theta\} | \alpha \neq 0\}$. Clearly, $supp_0(a) = supp(a) \setminus \{\theta\}$.

Therefore, $K_0[S^0] \cong K[S^0]/K\theta \cong K[S]$. Let I be an ideal of the semigroup S. Then $K_0[S/I] \cong K[S]/K[I]$.

For any algebras , we can adjoin an identity in these algebras. Notice that $K[S]^1 \cong K[S^1]$ if S has no identity.

Let L/F be any field extension with $L[S] \cong L \otimes_F F[S]$. If S, T are semigroups, then $K[S] \otimes_K K[T] \cong K[S \times T]$. Further, if S, T have zero elements θ_S, θ_T, respectively, then $K_0[S] \otimes_K K_0[T] \cong K_0[(S \times T)/I]$, where $I = \{(s,t) \in S \times T | s = \theta_S \text{ or } t = \theta_T\}$.

We say $Z \subseteq S$ a left group-like subset if for any $z \in Z$ and $s \in S$, $s \in Z$ if $zs \in Z$.

Lemma 1.2.3 [Okn1, Lemma 4.15, Coro. 4.16] *Let Z be a subsemigroup of a semigroup S and $J(R)$ the Jacobson radical of ring R. Then the following facts hold:*

(i). *If Z is left group-like in S, then $K[Z]$ is a direct summand of the left $K[S]$-module $K[S]$.*

(ii). *If the elements of S are not zero divisors in $K[S]$, then the converse holds.*

(iii). *For every subalgebra R of $K[S]$, we have $J(R) \cap K[Z] \subseteq J(R \cap K[Z])$.*

Proposition 1.2.4 [Okn1, Prop. 6.8] *Let S be a semilattice of semigroup S_α where $\alpha \in \Gamma$. Then the following statements hold:*

(i). *$K_0[S]$ is subdirect product of all algebras $K_0[S/T_\alpha]$, where $T_\alpha = \cup_{\beta \not\geq \alpha} S_\beta$.*

(ii). *If every $K[S_\alpha]$ has a unity, then $K[S]$ is a subdirect product of all $K[S_\alpha], \alpha \in \Gamma$.*

Later on, in chapter 2, we will make use of the semilattice decomposition of semigroups to study the radicals of algebras.

1.3 Group Algebras

In this section, we answer the following question: let S be a cancellative semigroup which can be embedded into some group G. Does $K[G]$ affect the properties of $K[S]$? Before answering this question, we give some properties of group algebras and we will discuss the cancellative subsemigroups of G in section 3.3.

1.3.1 Some Basic Properties of Groups

Thoughout this section, let $Z(G)$ be the center of the group G; $C_G(a)$ be centralizer of element a in G and G' means the commutator subgroup of G. If N is normal subgroup of G, then we denote it by $N \lhd G$.

In the following chapters, we always consider nilpotent and polycyclic-by-finite groups. We list here some properties of nilpotent groups extracted from [Kar, Pas1, Rob]. It fact, many properties of nilpotent groups and its related subgroups can be found in [Rob, Chapter 5].

Proposition 1.3.1 *Let p be any prime number. Then the following statements hold:*

(i). *If G is nilpotent group and $(e) \neq N \lhd G$, then $N \cap Z(G) \neq (e)$.*

(ii). *If G is nilpotent group and G/G' is finitely generated, then G is noetherian.*

(iii). *A finitely generated nilpotent group has a central series whose factors are cyclic with prime or infinite orders.*

(iv). *If G is a group with $Z(G)$ is torsion-free, then each upper central factor is torsion-free.*

For the other type of groups, we only consider polycyclic-by-finite group. Let $\mathfrak{C}$ be a family of groups, then a group G is poly-$\mathfrak{C}$ if G has a finite subnomal series :

$$G = G_n \supset G_{n-1} \supset \cdots \supset G_0 = \{1\}$$

with each quotient G_{i+1}/G_i belonging to the family $\mathfrak{C}$. If $\mathfrak{C}$ is closed under taking subgroups and homomorphic images, then so is the the class of poly-$\mathfrak{C}$ groups.

A group G is poly-(cyclic, finite) if it admits a subnormal series such that every factor G_i/G_{i-1} is either cyclic or finite. It is known that this type of group has a characteristic subgroup of finite index that is poly-(infinite cyclic) [Pas1, Lemma

10.2.5]. We call G polycyclic-by-finite group and the corresponding algebras of G are important for noetherian algebras. We list here some properties of polycyclic-by-finite groups.

Proposition 1.3.2 *Let G be a polycyclic-by-finite group. Then the following statements hold:*

(i). [Rob, 5.4.17] *G is residually finite.*

(ii). [Rob, 5.4.18] *If the polycylic normal subgroup W of G is not nilpotent, then G must have a finite nonnilpotent image.*

(iii). [Rob, 5.4.15] *An infinite polycyclic group G contains a nontrival torsion-free abelian normal subgroup.*

(iv). [Rob, 15.1.6] *A polycyclic group has a normal subgroup of finite index whose derived subgroup is nilpotent.*

1.3.2 General Properties of Group Algebras

The structure of group algebras was extensively studied by D. Passman. In this section, we refer to his monograph [Pas1] and some results taken from [Kar, Pas2, Row] as well. We denote $K[G]$ be a group algebras. The following theorem is an important theorem concerning finite group algebras, was obtained by Maschke.

Theorem 1.3.3 *(Maschke) Let G be a finite group and K a field. Then the following facts are well known:*

(i). *If $char(K) = 0$, then $K[G]$ is semiprimitive.*

(ii). *If $char(K) = p$, then $K[G]$ is semiprimitve iff G contains no elements of order p. We call this type of group **$\mathbf{p}^0$-groups**.*

Moreover, if $|G| \neq 1$, then $K[G]$ is never simple since it always contains an augmentation ideals.

Let $K[G]$ be a group algebra. Then the following augmentation map is given by

$$\omega : K[G] \to K$$

and the augmentation ideal is $\omega(K[G])$ since it is the kernel of the augmentation map ω.

Proposition 1.3.4 *Let H be a subgroup of an aribitrary group G. Suppose X is the generating set of H. Then*

$$K[G] \cdot \omega(K[H]) = \sum_{x \in X} R[G](x-1) \quad \text{and} \quad \omega(K[H]) \cdot R[G] = \sum_{x \in X} (x-1)K[G]$$

Moreover, if H is infinite then the left annihilator $\mathfrak{l}(\omega(K[H]) = 0$; if H is finite and if $\hat{H}$ is the sum of the elements of H, i.e. $\hat{H} = \sum_{h \in H} h \in K[G]$, then $\mathfrak{l}(\omega(K[H])) = K[G]\hat{H}$ and $\mathfrak{l}(\hat{H}) = K[G]\omega(K[H])$.

Lemma 1.3.5 *Let G be a nontrival group. Then $\omega(K[G])$ is nilpotent iff $char(K) = p$ for some prime p and G is a finite p-group.*

The following is an interesting result concerning the Jacobson radical $J(A)$ of an algebra A over a field K.

Proposition 1.3.6 [Kar] *Let A be an algebra over a field K. Any $x \in J(A)$ is either transcendental over K or nilpotent. In particular, the Jacobson radical of algebraic algebra over K is nil.*

By summarizing the results given in [Kar, Ch. 3], we can state the following proposition.

Proposition 1.3.7 *For any K-algebras A and B we have*

$$(1 \otimes_K B) \cap J(A \otimes_K B) \subseteq J(1 \otimes_K B).$$

Moreover, if either one of following conditions holds, namely:

(i). *A is algebraic over K.*

(ii). *$J(B)$ is nilpotent.*

(iii). *A and B are commutative and $J(B)$ is nil.*

Then, we have $B \cap J(A \otimes_K B) = J(B)$

Proposition 1.3.8 *Let A be a field over K and A, B be K-algebras. The the following statements hold:*

(i). *A/K is separable extension and $J(A \otimes_K B) = A \otimes_K (B \cap J(A \otimes_K B))$. If A is also algebraic over K, then $J(A \otimes_K B) = A \otimes_K J(B)$.*

(ii). *Let A/K be a purely transcendental field extension of the field K. Then $J(A \otimes_K B) = A \otimes_K (B \cap J(A \otimes_K B))$.*

(iii). *If A/K is finite extension of the field K, then*

$$[J(A \otimes_K B)]^n \subseteq A \otimes_K J(B) \subseteq J(A \otimes_K B).$$

Applying the above results to group algebras, the following results on semiprimitive group algebra is now obvious.

Corollary 1.3.9 [Pas3]

(i). *Assume that $char(K) = 0$ with K not algebraic over the rationals $\mathbb{Q}$. If G is any group, then $K[G]$ is semiprimitive.*

(ii). *Assume that $char(K) = p$ with K not algebraic over the Galois field $GF(p)$. If G is p'-group, then $K[G]$ is semiprimitive.*

1.3.3 Δ-Method for Group Algebras

We now describe a special method, namely the Δ-method, to study the Jacobson radical of group algebras. This method was given by D.S. Passman [Pas1]. Consider the following subsets of G:

$$\Delta(G) = \{x \in G| \, |G : C_G(x)| < \infty\}$$

and

$$\Delta^+(G) = \{x \in \Delta(G)|o(x) < \infty\}.$$

Obviously $\Delta(G)$ and $\Delta^+(G)$ are characteristic subgroups of G. If $x \in \Delta(G)$, then x has finite number of conjugates. Moreover, it is clear that $\Delta^+(G)$ is generated by the finite normal subgroups of G and $\Delta(G)/\Delta^+(G)$ is torsion-free abelian. If $\Delta^+(G) = (e)$, then $K[\Delta(G)]$ becomes a domain when K is a field. If $G = \Delta(G)$, then G is said to be a FC-group i.e. finite conjugate group. Notice that in a f.g FC-group, we always have $[G : Z(G)] < \infty$ and G' is finite.

Let $\Delta^p(G) = \langle g \in \Delta(G)|g \text{ has order is a power of } p\rangle$. Then Δ^+/Δ^p is locally finite by knocking out the elements of order p. For the finite normal subgroup N of G, we have:

$$\Delta(G/N) = \Delta(G)/N \; ; \; \Delta^+(G/N) = \Delta^+(G)/N \text{ and } \Delta^p(G/N) = \Delta^p(G)/N.$$

Also, if $[\Delta^p(G) : H] < \infty$, then $[H : \Delta^p(H)] < \infty$.

We say that a subset T of G is large if for all subgroups W of finite index in G, $T \cap W$ cannot be covered by a finite union of cosets of subgroups with infinite index. We say that T is very large if T and all its right translates Tx are large.

Lemma 1.3.10 *Let T be a very large subset of G. If $\sum_i^n \alpha_i x \beta_i = 0$ in $K[G]$ for all $x \in T$, then the identity holds for all $x \in G$.*

We can easily see that the primeness of group algebra $K[G]$ is related to its base group G and its FC-center. The primeness of group ring was given by Connell.

Theorem 1.3.11 [Pas1, Th. 4.2.10] *(Connell) Let K be any field with char$(K) = 0$, the following conditions are equivalent:*

(i). *$K[G]$ is prime.*

(ii). *$Z(K[G])$ is an integral domain.*

(iii). *G contains no finite nontrival normal subgroup.*

(iv). *$\Delta(G)$ is torsion-free abelian.*

(v). *$K[\Delta(G)]$ is an integral domain.*

Theorem 1.3.12 [Pas1, Th. 4.2.13] *Let K be any field with char$(K) = p$. Then the following conditions are equivalent:*

(i). *$K[G]$ is semiprime.*

(ii). *$Z(K[G])$ is semiprime.*

(iii). *$Z(K[G])$ is semisimple.*

(iv). *G contains no finite normal subgroups H with p divides $|H|$, that is $p|\ |H|$.*

(v). *$\Delta(G)$ is a p'-group.*

We define the **nilpotent radical** of the ring R be the sum of all nilpotent ideals of R, denote it by $N(R)$. Note that $N(R)$ may not be nilpotent. The nilpotent radical is not a radical property as pointed out by Divinsky [Div]. Indeed, there exists a finitely generated K-algebra A with $N(A/N(A)) \neq 0$. However, the nilpotent radical defined by Passman acts a crucial role in studying the semiprimitivity problems in group algebras. The details are given in [Pas1, Chapter 8].

We have following theorem concerning the nilpotent radicals of group algebras.

Theorem 1.3.13 [Pas3] *Let $\Delta^+ = \Delta^+(G)$ and $char(K) = p > 0$. Then*

(i). $N(K[G]) = J(K[\Delta^+]) \cdot K[G]$.

(ii). $J(K[\Delta^+]) = \bigcup_{W \in \mathcal{W}(G)} J(K[W])$, *where $\mathcal{W}(G)$ is the set of all finite normal subroups of G.*

(iii). $N(K[G]) \neq 0$ *iff Δ^+ contains an element of order p and iff G has a finite normal subgroup with order divisible by p.*

All other properties related to group ring can be found in [Pas1, Pas2] and his recent survey paper [Pas3].

1.4 Graded Algebras

Every group ring and semigroup ring can be viewed as graded algebras. We recall here to some basic results from graded algebras. We refer to [Jes3, JW1, Kar, NV] for the properties of group graded rings and semigroup graded rings.

Let S be a semigroup. A ring is S-graded if

$$R = \sum_{s \in S} R_s$$

is a direct sum of additive subgroup R_s, indexed by the elements $s \in S$ such that

$$R_s R_t \subseteq R_{st}.$$

A left R-module is S-graded if $M = \oplus_{s \in S} M_s$ is a direct sum of additive groups such that $R_s M_t \subset M_{st}$, for all $s, t \in S$. If S is a group, then R is strongly graded and $1 \in R_e$, where e is the identity of the group S.

The well-known examples of semigroup graded rings are the polynomial rings $R[x_1, x_2, \cdots, x_n]$ with commuting variables. It is graded by free commutative monoids with rank n and the polynomial ring $R\{x_1, x_2, \cdots, x_n\}$ in non-commuting variables is graded by free non-commutative monoids with rank n.

Let A be a S-graded ring. If I is an ideal or subalgebra of A, then I is said to be S-homogeneous (or say S-graded) if

$$I = \sum_{x \in S} (I \cap A_x),$$

i.e. $\sum_{x \in S} r_x \in I$ implies that $r_x \in I$.

Let B be another S-graded algebra. A homomorphism $f : A \to B$ is called graded homomorphism if $f(A_x) \subseteq B_x$ for all $x \in S$. Moreover, R is called nondegenerate if $(rR)_e = (0)$ or $(Rr)_e = (0)$ for some $r \in R$, implies $r = 0$.

In this section, the general properties of semigroup graded algebras will be investigated.

Proposition 1.4.1 [Kar, Prop. 22.5] *Let S be a right (or left) cancellative monoid with identity e. Let R be a S-graded algebra. Then R_e is a subalgebra of R.*

Moreover, if R is a S-graded with S not necessary right (or left) cancellative, then the following statements hold:

If R_e is subalgebra of R (contains $1 \in R$), then

(i). *For $x \in S$, R_x is a (R_e, R_e)-bimodule under the left and right multiplication by the elements of R_e.*

(ii). $R_e \cap U(R) = U(R_e)$ *and* $R_e \cap J(R) \subseteq J(R_e)$.

Let ${}_{R-gr}\mathcal{M}$ be the category of all G-graded left R-modules and the morphisms are the set of graded homomorphisms. If $M, N \in {}_{R-gr}\mathcal{M}$ then $Hom_{R-gr}(M, N)$ is the graded homomorphisms from M to N.

Theorem 1.4.2 [NV, Th. I.3.4] *The following conditions are equivalent in a strongly G-graded ring R.*

(i). *R is a strongly G-graded ring.*

(ii). *Every graded R-module is strongly G-graded.*

(iii). *The functors $R \otimes_{R_e} -$ and $(-)_e$ are equivalent between the Grothendieck categories ${}_{R-gr}\mathcal{M}$ and ${}_{R_e}\mathcal{M}$-mod.*

1.5 Crossed Products and Smash Products

It is known that the radical problems on group-graded rings and even semigroup-graded rings (cf. [Kar, Jes3, Pas1]) can be solved by using the techniques of crossed products. We adopt the notations of crossed products given from [Kar] because the crossed product defined by Karpilovsky not only for groups but also for semigroups. Let k be commutative ring with unit, A be a k-algebra and $\mathrm{Aut}_k(A)$ be the group of all k-algebra automorphisms of A. The unit group of A is denoted by $U(A)$. Then, for the multiplicative monoid M, we consider the following mappings:

$$\sigma : M \to \mathrm{Aut}_k(A)$$

and

$$\alpha : M \times M \to U(A)$$

We call (M, A, σ, α) a crossed system for M over A if for all $x, y, z \in M$ and $a \in A$, the following equalities holds:

$$^{x}(^{y}a) = \alpha(x,y)\ (^{xy}a)\alpha(x,y)^{-1}$$

$$\alpha(x,y)\alpha(xy,z) = \ (^{x}\alpha(y,z))\alpha(x,yz)$$

$$\alpha(x,e) = \alpha(e,x) = e$$

where $^{x}a = \sigma(x)(a)$ for all $a \in A, x \in M$, and e is the identity of M.

Proposition 1.5.1 [Kar, Prop. 23.3] *Let (M, A, σ, α) be a crossed system for M over A and $A * M$ the free A-module freely generated by the elements $\bar{x}, x \in M$, with multiplication defined by*

$$(a_1\bar{x})(a_2\bar{y}) = a_1\ {}^{x}a_2\alpha(x,y)\overline{xy}$$

for all $a_i \in A$ and $x, y \in M$. Then $A * M$ is a strongly M-graded k-algebra with identity element $1 \cdot \bar{e}$, $(A * M)_e = A \cdot \bar{e}$ and with

$$(A * M)_x = A\bar{x} = \bar{x}A \qquad \forall x \in M$$

We say that the k-algebra $A * M$ as the crossed product of M over A. It is mainly composed by two parts:

(a) (Twisting) $\bar{x}\bar{y} = \alpha(x,y)\overline{xy}$, and

(b) (Action) $\bar{x}r = \ ^{x}r\bar{x}$.

Note that if $\sigma(x) = 1$ and $\alpha(x, y) = e$ for all $x, y \in M$, then $A * M$ becomes a monoid ring. If only $\alpha(x, y) = e$, then $A * M$ is a skew monoid ring.

Note that A can be regarded as a subalgebra of $A * M$. However, there is no natural embedding of M into $A * M$ in general.

Let $A * G$ be a crossed product of a group G over an R-algebra A. For a subgroup H of G, we denote

$$A * H = \{\sum_{h \in H} x_h \bar{h} | \bar{h} \in A\}$$

Then $A * H$ is a subalgebra of $A * G$ which is the crossed product of H over A.

Corollary 1.5.2 [Kar, Coro. 23.6] *Let $N \lhd G$ and $A * G$ a crossed product of G over an R-algebra A. Then*

$$A * G = (A * N) * (G/N).$$

For group algebras, we know that if G is a group with a normal subgroup N, then $K[G] = K[N] * (G/N)$.

In case if G is finite, then Cohen and Montgomery [CM1] **(1984)** have related the group-graded algebras to Hopf algebras.

Denote the dual algebra $k[G]^* = Hom_k(k[G], k)$. Let G be a finite group and A is G-graded k-algebra, where k is any commutative ring with unity. We can check that $k[G]^*$ is a bialgebra. The smash product $A\#k[G]^*$ is the free left A-module on the generators set $\{p_g \in k[G]^* | \ g \in G\}$, which is a set of orthogonal idempotents whose sum is 1, with multiplication define by the rule

$$(a\#p_g)(b\#p_h) = ab_{gh^{-1}}\#p_h$$

where $b_{gh^{-1}}$ is the homogeneous element of b in $A_{gh^{-1}}$.

Smash product is an important tool for studying group graded rings, in particular, the duality theorems of Cohen and Montgomery [CM1]. We summarize some important properties and duality theorems:

Theorem 1.5.3 [CM1, Th. 3.2, 3.3] **(1984)** *Let $R = A * G$, the skew group ring over commutative ring A. Then*

$$(A * G)\#k[G]^* \cong M_n(A).$$

Let R be G-graded k-algebra and G is finite group with order n, then the skew ring over group G

$$(R\#k[G]^*) * G \cong M_n(R).$$

In order to extend smash product to infinite groups, we use the notations given in [Bea]:

Notation 1.5.4 *Let R be a G-graded ring and let $R\#G^*$ be the free left R-module on the generators $p_g, g \in G$. Denote rp_g by $r\#p_g$ and the multiplication is given by*

$$(rp_g)(sp_h) = rs_{gh^{-1}}p_h$$

and multiplication of such elements is defined by linearity, while $\{p_g\}$ is a set of orthogonal idempotents.

The product $R\#G^*$ is called the **generalized smash product** of R and G. Note that we do not assume that R is a G-graded ring with unity.

Proposition 1.5.5 [BS] **(1991)** *Let $R\#G^*$ be the generalized smash product defined above, R a G-graded ring without unity and R^1, a ring extension of R which R^1 has unity* (1)*. Then*

(i). *$R\#G^*$ is an ideal of $R^1\#G^*$.*

(ii). *For $g \in G$, we define $(R\#G^*)_g = \sum_{h\in G} R_{gh^{-1}}p_h$. Then*

$$R\#G^* = \sum_{g\in G}(R\#G^*)_g$$

and R is isomorphic to $(R\#G^*)_e$, where e is identity of G.

(iii). *Let R be a G-graded ring with unity. Denote R^G as the set of all fixed points under the group action. Then g acts on left by ${}^g(rp_h) = rp_{hg^{-1}}$ and acts on right by $(rp_h)^g = rp_{hg}$. From* [Qui]*, if G is finite then $(R\#G)^G \cong R$. Also, $(R\#G^*)^G = 0$ when G is infinite.*

(iv). [BS, Prop. 2.1] *The caterogies of irreducible left G-graded R-modules and irreducible left $(R\#G^*)$-modules are isomorphic.*

Note that the duality theorems given by Beattie on crossed products has been recently unified by Y.H. Xu and K.P. Shum [XS], by introducing the concept of double crossed products.

(1) R^1 is obtained from R by an adjunction of the unit element by the ring of integers, i.e. $R^1 = R \oplus \mathbb{Z}$, with multiplication $(a, n)(b, m) = (ab + m \cdot a + n \cdot b, nm)$

Chapter 2

Radicals of Graded Rings

In this chapter, we give some notations and terminologies for studying radicals (mainly Jacobson radical) of semigroup algebras and group algebras through graded ring theory. We will make use of these results to investigate the radicals of semigroup algebras.

2.1 Jacobson Radical of Crossed Products

Before studying the Jacobson radical of group algebras or semigroup algebras, we first review the properties of radicals of crossed products and graded rings.

Theorem 2.1.1 [Kar, Th. 23.4] *Let k be commutative ring with unity. Let A be a k-algebra and S a multiplicative monoid. Assume $A*S$ is a crossed product of S over A (constructed in Chapter 1, Section 1.5). Then*

(i). $A \cap J(A * S) \subseteq J(A)$

(ii). *If S is finite of order n, then*

(a) $A \cap J(A * S) = J(A)$.

(b) $J(A * S)^n \subseteq J(A) \cdot (A * S) \subseteq J(A * S)$.

(c) $J(A * S) = J(A)(A * S)$, *provided every $A * S$-module is A-projective.*

(d) $J(A * S)$ *is nilpotent, if* $J(A) = 0$.

The problem describing the radicals of semigroup rings seems to be rather complicated. However, for arbitrary semigroup, if there is a least semilattice congruence

η on S, then S/η is the greatest semilattice decompostion of S into certain type of subsemigroups, say $\cup_{\alpha\in\Gamma}S_\alpha$, where Γ is semilattice. To reduce the problem, we consider $R[S] = \sum_{\alpha\in\Gamma} R[S_\alpha]$ is a Γ-graded ring. Thus, the studying the radicals of graded rings is essential the same as the semigroup algebras.

In the following, we assume all S-graded rings are contracted, i.e. $R_\theta = 0$ where θ is the zero of semigroups. Throughout this section, S is right cancellative semigroup with unity (e.g. cancellative monoid, group) and R is a S-graded algebra. Right cancellative properties ensure that $xy = e$ implies $yx = e$, where e is identity of S. A left R-module M is called graded simple if $RM = M$, and $\{0\}, M$ are the only graded submodules of M.

Definition 2.1.2 *The graded Jacobson radical of R, denoted by $J_{gr}(R)$, is the set of elements of R which annihilates all S-graded simple left R-modules. Let V be graded R-module, then $J_{gr}(V)$ is defined to be the intersection all graded-maximal submodules of V.*

The graded Jacobson radical has the following properties :

Proposition 2.1.3 [Kar] *Let S be right cancellative monoid. Then the following statements hold for the S-graded algebra R:*

(i). *Let V be graded R-module. If V is finitely generated nonzero module, then $J_{gr}(V) \neq V$.*

(ii). *$J_{gr}(R)$ is a homogeneous (graded) ideal of R.*

(iii). *$J_{gr}(R)$ is the largest proper homogeneous (graded) ideal of I of R such that $I \cap R_1 \subseteq J(R_1)$.*

(iv). *$J_{gr}(R)$ contains all homogeneous (graded) nil left ideals of R.*

(v). *$J(R)$ is graded, then $J(R) \subseteq J_{gr}(R)$ with equality if $R/J(R)$ is artinian.*

2.2 Graded Radicals and Reflected Radicals

If S is finite group, then there are some connections between smash products and Hopf algebras [CM1] (see Section 1.6). In the following, we always use the concept of generalized smash product described in 1.5.4.

Theorem 2.2.1 [Bea] *Let G be arbitrary group and R is G-graded ring with unity. Then $J(R\#G^*) = J_{gr}(R)\#G^*$.*

Proof. For $x = \sum_{i=1}^{n} r_i p_{g_i} \in J(R\#G^*)$. We have $r_i p_{g_i} \in J(R\#G^*)$ because p_g is an orthogonal idempotent and the Jacobson radical is a two-sided ideal. Therefore, it suffices to show that $rp_g \in J(R\#G^*)$ implies $r \in J_{gr}(R)$.

Note that $J(R\#G^*)$ is G-stable (invariant under group action). Thus, if $rp_g \in J(R\#G^*)$, then $rp_h \in J(R\#G^*)$ for any other $h \in G$. Let V be an irreducible graded left R-module. Then $V^{\#}$ is also an irreducible left $R\#G^*$-module by the categorial isomorphism (in Prop. 1.5.5 (iii)). For any $v \in V^{\#}$ and all $g \in G$, if rp_g annihilates V such that $0 = rp_g v = rv_g$, then r annihilates V and $r \in J_{gr}(R)$. Thus $J(R\#G^*) \subseteq J_{gr}(R)\#G^*$.

Conversely, let $x = \sum_{i=1}^{n} r_i p_{g_i} \in J_{gr}(R)\#G^*$, where $r_i \in J_{gr}(R)$. Also, let M be an irreducible left $R\#G^*$-module. Then M' is a graded irreducible left R-module by the functor $(\ )'$. Thus, $r_i M' = 0$ implies that $r_i M'_{g_i} = 0 = r_i p_{g_i} M$ and $x = \sum_{g_i} r_i p_{g_i}$. Hence $x \in J(R\#G^*)$. Thus, $J(R\#G^*) = J_{gr}(R)\#G^*$. □

Corollary 2.2.2 $J_{gr}(R) \cap R_e = J(R_e)$ *where e is the identity of G.*

Proof. Since $p_e(R\#G^*)p_e = R_e p_e = R_e$, it follows that

$$\begin{aligned} J(R_e) &= J(p_e(R\#G^*)p_e) \\ &= p_e(J(R\#G^*))p_e \\ &= p_e(J_{gr}(R)\#G^*)p_e \\ &= (J_{gr}(R))_e p_e \\ &= J_{gr}(R) \cap R_e \end{aligned}$$

□

However, in general, it is not true that $J_{gr}(R)$ is always contained in $J(R)$. For instance, consider C as a commutative domain with $J(C) \neq (0)$. Let $R = C\{x\}$ be the group ring of C over $\mathbb{Z}$ as usual. Clearly, $C\{x\}$ contains no nil ideals. Then, we can observe that $J(C\{x\}) = (0)$, but $J_{\mathbb{Z}}(R) = J(C)\{x\}$.

We now consider the case when G is a finite group. From Theorem 2.1.1, we deduce the following theorem:

Theorem 2.2.3 [CM1, CM2] *Let R be ring graded by a finite group G. Then* (i). $J_{gr}(R) \subseteq J(R)$, *in particular,* $J_{gr}(R) = J(R)_{gr}$ *is the largest homogeneous part in* $J(R)$.

(ii). $J(R)^{|G|} \subseteq J_{gr}(R)$.

(iii). $|G|J(R) \subseteq J_{gr}(R)$.

(iv). *If* $|G|$ *is the member of the units in* R, *then* $J_{gr}(R) = J(R)$.

Proof. (i) Consider $R\#G^*$, from Theorem 2.1.1 and 2.2.1, we have $J_{gr}(R) = R \cap J(R\#G^*)$. Thus $x \in J_{gr}(R)$ is quasi-invertible in $R\#G^*$. Since $R\#G^*$ is free over R, any element of R invertible in $R\#G^*$ is already invertible in R. Thus $J_{gr}(R)$ is a quasi-regular ideal in R, so is contained in $J(R)$.

For (ii), by using the crossed product propeties, we obtain from Theorem 2.1.1, that

$$J((R\#G^*) * G)^{|G|} \subseteq J(R\#G^*) * G = (J_{gr}(R)\#G^*) * G.$$

and then for the identity $e \in G$, we have

$$p_e J((R\#G^*) * G)^{|G|} p_e \subseteq (p_e((J_{gr}(R)\#G^*) * G)p_e).$$

By using duality theorem (see Theorem 1.5.3), we have for $n = |G|$ and $p_e \mapsto e_1$, some idempotent $n \times n$ matrix

$$e_1 J(M_n(R))^n e_1 \subseteq e_1 M_n(J_{gr}(R)) e_1$$

$$J(R)^{|G|} \subseteq J_{gr}(R)$$

The results then follow.

(iii) (iv) See [Kar, Th. 30.10 (iii)] for details. □

Recall that the prime radical $\mathcal{B}$ is the intersection of prime ideals of R. Let R be G-graded algebra. A graded ideal I is called graded prime if $JK \subseteq I$, for some J, K graded ideals of R, then $J \subseteq I$ or $K \subseteq I$. It is known that the graded prime radical $\mathcal{B}_{gr}(R)$ is the intersection of all graded prime ideals of R.

Graded prime radical in $R\#G^*$, when G is finite, was studied by Cohen and Montgomery [CM1]. Beattie and Stewart [BS] then considered the generalized smash product $R\#G^*$ and G may be infinite. If J is a graded ideal of graded algebra, then from Proposition 1.5.5, $J\#G^*$ is an ideal of $R\#G^*$.

If I is ideal of $R\#G^*$, then define I_R and $I^\flat$ of R be the sets

$$I_R = \{r : r \in R | rp_g \in I \text{ for all } g \in G\}$$

and $I^\flat = (I_R)_{gr}$. Clearly, $I^\flat$ is the largest graded (homogeneous) ideal in I_R.

If λ is a radical class in the category of associative algebras, then we define the **reflected radical** of λ by

$$\lambda_{ref} = \{R : R \text{ is a } G\text{-graded algebra with } R\#G^* \in \lambda\}.$$

Clearly, λ_{ref} is a radical class of G-graded algebra.

Proposition 2.2.4 [BS, Prop. 1.2] *If λ is a radical in the category of associative rings, then for R a G-graded ring, $\lambda_{ref}(R) = (\lambda(R\#G^*))^\flat$, and thus $\lambda_{ref}(R)\#G^* = \lambda(R\#G^*)$.*

Note that the reflected Jacobson radical coincides the graded Jacobson radical by Theorem 2.2.1.

We now consider the prime radical $\mathcal{B}$ of the graded algebra R.

Lemma 2.2.5 [CM1, Lemma 5.1] *Let R be a graded algebra and G a finite group. If I is a graded ideal of R, then I is graded prime if and only if $I = P_{gr}$, the associated graded ideal of some prime ideal P of R. Consequently, $\mathcal{B}_{gr}(R) = (\mathcal{B}(R))_{gr}$, the largest graded ideal in $\mathcal{B}(R)$.*

Theorem 2.2.6 [BS] *Let R be graded ring over group G. Then the following conditions hold:*

(i). $\mathcal{B}_{gr}(R) \subseteq \mathcal{B}_{ref}(R)$

(ii). *If G is finite, $\mathcal{B}_{gr}(R) = \mathcal{B}_{ref}(R)$.*

(iii). *If G is infinite, then inclusion in (i) may be proper.*

Proof. (i) If P is a prime ideal of $R\#G^*$, then $P^\flat$ is a graded prime of R and thus $\mathcal{B}_{gr}(R)\#G^* \subseteq \mathcal{B}(R\#G^*)$ so that $\mathcal{B}_{gr}(R) \subseteq \mathcal{B}_{ref}(R)$.

(ii) Suppose that the G is finite, and R has an unity. Then, the prime radical of $R\#G^*$ was studied in [CM1]. Since the prime radical is a hereditary radical and R contains no identity so we have

$$\begin{aligned}
\mathcal{B}(R\#G^*) &= \mathcal{B}(R^1\#G^*) \cap R\#G^* \\
&= (\mathcal{B}_{gr}(R^1\#G^*)) \cap R\#G^* (\text{ by [CM1, Th. 5.3]}) \\
&= (\mathcal{B}(R^1)_{gr}\#G^*) \cap R\#G^* \\
&= (\mathcal{B}(R^1)_{gr} \cap R)\#G^* \\
&= (\mathcal{B}(R^1) \cap R)_{gr}\#G^* \\
&= \mathcal{B}(R)_{gr}\#G^* = \mathcal{B}_{gr}(R)\#G^*
\end{aligned}$$

(iii) When G is infinite, we have already known that $\mathcal{B}_{gr}(R) \subsetneq \mathcal{B}_{ref}(R)$. Let K be a field and let $R = K[t]$, the polynomial ring graded by $G = \mathbb{Z}$ in the usual way. Since (0) is graded prime ideal, $\mathcal{B}_{gr}(R) = (0)$. Let I be the principal left ideal $(R\#G^*)(tp_0)$ of $R\#G^*$. Then $I^2 = (0)$. Thus $J = I + I(R\#G^*)$ is a nilpotent two-sided ideal of $R\#G^*$, and therefore, $\mathcal{B}(R\#G^*) = \mathcal{B}_{ref}(R)\#G^*$ is nonzero. □

We quote the following results from Beattie and Stewart [BS] to describe the graded version of Levitzki and Brown-McCoy radical. The details of proof are referred to [BS].

Recall that the Levitzki radical $\mathcal{L}$ is the intersection of the prime ideals P of R such that R/P has no nonzero locally nilpotent ideals, $\mathcal{L}_{gr}(R)$ is hence the intersection of all graded prime ideals P of R such that R/P has no nonzero graded locally nilpotent ideals.

Lemma 2.2.7 [BS, Prop. 3.2] *For any G-graded ring R, we have $\mathcal{L}_{gr}(R) = (\mathcal{L}(R))_{gr}$.*

Theorem 2.2.8 [BS, Th. 3.3] *Suppose that the ring R is graded by a group G. Then*

(i). *For any group G, $\mathcal{L}_{gr}(R) \subseteq \mathcal{L}_{ref}(R)$.*

(ii). *If G is locally finite, $\mathcal{L}_{gr}(R) = \mathcal{L}_{ref}(R)$.*

(iii). *For infinite group G, the inclucsion in (i) may be proper.*

The Brown-McCoy radical is intersection of ideals M whose R/M is a simple ring with unity. Similarly, we define $\mathcal{G}_{gr}(R)$ be the intersection of the graded ideals M of R such that R/M is a graded simple ring with identity.

Proposition 2.2.9 [BS, Prop. 3.5] *For all G-graded rings R, we have $\mathcal{G}(R)_{gr} \subseteq \mathcal{G}_{gr}(R)$ and this inclusion may be proper.*

Theorem 2.2.10 [BS, Th. 3.6] *For all group G, $\mathcal{G}_{gr}(R) \subseteq \mathcal{G}_{ref}(R)$. If G is finite, then $\mathcal{G}_{gr}(R) = \mathcal{G}_{ref}(R)$.*

Moreover, we can extend Theorem 2.2.3 to any radical described above. By using the duality theorems of finite group actions and coactions, we can summarize the results from [BS, CM1, Jes3, JP] and obtain the following theorem.

Theorem 2.2.11 *Let the radicals $\mathcal{H} = \mathcal{B}, \mathcal{L}, \mathcal{G}$ be the prime; Levitzki and Brown-McCoy radicals respectively. Suppose that A is a G-graded ring and G is finite group. Then we have*

$$|G|\mathcal{H}(A) \subseteq \mathcal{H}_{gr}(A) \subseteq \mathcal{H}(A).$$

Proof. If $\mathcal{H} = \mathcal{B}, \mathcal{L}$, from Lemma 2.2.5, and 2.2.7 and using the terminology of prime and graded prime ideal in [CM1], we can see that

$$|G|\mathcal{H}(R) \subseteq \mathcal{H}_{gr}(R) \subseteq \mathcal{H}(R).$$

For $\mathcal{H} = \mathcal{G}$, if R is an algebra strongly graded by a finite group G, then from [JP, Prop. 2], $\mathcal{G}(R) \cap R_e = \mathcal{G}(R_e)$ and

$$|G|\mathcal{G}(R) \subseteq \mathcal{G}(R_e)R = R\mathcal{G}(R_e) = \mathcal{G}_{gr}(R).$$

If R is graded by a finite group G and $|G| = n$, then by duality theorem of group coactions, we have a homomorphism defined by

$$\phi : (R\#G^*) * G \longrightarrow M_n(R)$$

and

$$\phi : (rp_g * h) \mapsto \sum_{f \in G} r_{fg^{-1}} e_{f,gh}$$

where $e_{f,gh}$ is the matrix with 1 in the (f, gh)-entry and zero otherwise. Obviously, ϕ is an isomorphism. If follows that $M_n(\mathcal{G}_{gr}(R)) = \phi((\mathcal{G}_{gr}(R)\#G^*) * G)$. On the other hand, by using Theorem 2.2.10, we have for G is finite, $\mathcal{G}_{gr}(R)\#G^* = \mathcal{G}_{ref}(R)\#G^* = \mathcal{G}(R\#G^*)$.

Also, we have

$$M_n(\mathcal{G}_{gr}(R)) \underset{\cong}{\overset{\phi}{\longleftarrow}} [\mathcal{G}_{gr}(R)\#G^*] * G = [\mathcal{G}_{gr}(R\#G^*)] * G$$

$$\subseteq \mathcal{G}[(R\#G^*) * G] \underset{\cong}{\overset{\phi}{\longrightarrow}} \mathcal{G}(M_n(R)) = M_n(\mathcal{G}(R))$$

Hence, $M_n(\mathcal{G}_{gr}(R)) \subseteq M_n(\mathcal{G}(R))$. Consequently,
$|G|\mathcal{G}((R\#G^*) * G) \subseteq \mathcal{G}_{gr}((R\#G^*) * G)$ and so

$$M_n(|G|\mathcal{G}(R)) = |G|M_n(\mathcal{G}(R)) \subseteq M_n(\mathcal{G}_{gr}(R)) \subseteq M_n(\mathcal{G}(R))$$

Hence, $|G|\mathcal{G}(R) \subseteq \mathcal{G}_{gr}(R) \subseteq \mathcal{G}(R)$ as required. □

2.3 Radicals of Group-graded Rings

We now generalize the results of Karpilovsky (see [Kar, Th. 30.28]) to other radicals described above.

Lemma 2.3.1 [JP] *Let R be a ring graded by a group G which is residually p-finite for two distinct primes p. If the radicals $\mathcal{H} = \mathcal{B}, \mathcal{L}, J$ or $\mathcal{G}$, then $\mathcal{H}(R)$ is homogeneous.*

Proof. First we notice that if G is residually p-finite, then for every finite subset T of G, there exists N, the normal subgroup of G with G/N is p-group, such that $sN \neq tN$ for every $s, t \in T$, $s \neq t$.

Suppose that $r = \sum_{g \in T} r_g \in \mathcal{H}(R)$ and T a finite subset of G. Then by the above result, there exists a normal subgroup N such that G/N is finite p-group satisfying the above situation. Consider R as a G/N-graded ring. Then, by the properties of N, each component r_g, for all $g \in T$, must be a homogeneous component. By Theorem 2.2.11, we have $|G/N|r \in \mathcal{H}_{gr}(R) \subseteq \mathcal{H}(R)$ and hence $|G/N|r_g \in \mathcal{H}(R_g)$ for every $g \in T$. Thus, $p^n r_g \in \mathcal{H}(R_g)$ for some $n > 0$. Since this holds for two distinct prime numbers we obtain that $r_g \in \mathcal{H}(R_g)$. □

Furthermore, we can extend the above result to subdirect product of groups:

Lemma 2.3.2 [Kar, Lemma 30.20] *Let $\{G_i | \ i \in I\}$ be a collection of arbitrary groups, let G be a subdirect product of G_i, and let A be a G-graded algebra. Then, for the radicals $\mathcal{H} = \mathcal{B}, \mathcal{L}, J, \mathcal{G}$, we have :*

(i). *If for each $i \in I$, $\mathcal{H}(A)$ is a G_i-graded ideal of A, then $\mathcal{H}(A)$ is a G-graded ideal of A.*

(ii). *If each G_i is finite, then for any $a = \sum_{g \in G} a_g \in \mathcal{H}(A)$, where $a_g \in A_g$, there exists a positive integer n_a such that $n_a a_g \in J(A)$ for all $g \in G$. Furthermore, n_a divides $|H_1||H_2| \cdots |H_k|$ for some $k = k(a)$ and some $H_t \in \{G_i | \ i \in I\}, 1 \leq t \leq k$.*

The following lemma only serves for the case of Jacobson radicals.

Lemma 2.3.3 [Kar, Lemma 30.27] *Let A be a G-graded algebra. For any finitely generated subgroup H_i of G, let A_{H_i} be the subalgebra graded by H_i in A. If $J(A_{H_i})$ is a graded ideal, then $J(A)$ is a graded ideal of A.*

Proposition 2.3.4 (i). *G is locally free if every finitely generated group is free.*

(ii). *G is residually free if G is a subdirect product of free groups.*

(iii). *G is residually p-finite if G is a subdirect product of finite p-groups.*

(iv). *G is free solvable if* $G \cong F/F^{(n)}$ *for some free groups F.*

We note that free group, finitely generated torsion-free nilpotent group G and free solvable group are all residually finite p-groups for every prime p (ref. [Kar, Section 30]). Now we slightly modify the theorem given in [Kar, Th. 30.28]:

Theorem 2.3.5 *Let R be a ring graded by a group G where G is a group of any one of the following types:*

(a). *G is abelian and the orders of finite subgroup of G are units in R.*

(b). *G is residually free, or free solvable or torsion-free nilpotent.*

(c). *G is locally finite and the orders of finite subgroups of G are units in R.*

(d). *G is locally free.*

If G satisfies one of the following cases:
Case I : $\mathcal{H} = J$ *and G is type (a),(b),(c),(d), or subdirect prodcut of them.*
Case II: $\mathcal{H} = \mathcal{B}, \mathcal{L}, \mathcal{G}$ *and G is type (a),(b), or subdirect product of them.*

Then the following properties hold:

(i). $\mathcal{H}(R)$ *is a graded ideal.*

(ii). $\mathcal{H}(R) \subseteq \mathcal{H}_{gr}(R)$ *if* $\mathcal{H} = J$.

(iii). $\mathcal{H}(R) = (\mathcal{H}(R) \cap R_e)R = R(\mathcal{H}(R) \cap R_e)$, *if R is strongly G-graded.*

Proof. (i) If the group G is of type (a) and (b), then G is residually p-finite. By Theorem 2.2.11, Lemma 2.3.1, Lemma 2.3.2, we can see that $\mathcal{H}(R)$ is graded.

If the group G is type (c) and type (d), then by above and Lemma 2.3.3, we obtain the graded ideal $J(R)$.

(ii) Notice that if $J(R)$ is graded, then $J(R) \cap R_e \subseteq J(R_e)$. And if $J_{gr}(R)$ is the largest graded ideal I of R such that $I \cap R_e \subseteq J(R_e)$ then $J(R) \subseteq J_{gr}(R)$.

(iii) $\mathcal{H}_{gr}(R) \cap R_e = \mathcal{H}(R_e)$ when R is strongly graded. Since $\mathcal{H}(R)$ is graded, then $\mathcal{H}(R) = (\mathcal{H}(R) \cap R_e)R$. □

There is an important corollary, for the case of semigroups. For A is S-graded algebra, S is a submonoid of group G. Then A can be regarded as an G-graded algbera via $A_g = 0$ for $g \notin S$.

Corollary 2.3.6 [Kar, Th. 30.30] *Let S be submonoid of G, where G is one of the above types (a,b,c,d) in above theorem. If A is S-graded algerba, then $J(A)$ is a graded ideal of A and $J(A) \subseteq J_{gr}(A)$.*

Remark: Under the construction of A to G-graded algebra, A must be not strongly G-graded because $1 \notin A_g A_{g^{-1}}$ for some $g \in G$.

For $R * G$, the crossed product of G over R, we have a similar theorem:

Theorem 2.3.7 [Kar, Th. 33.30] *If R be an arbitrary ring and G be a group of one of type of above theorem. Then $J(R * G)$ is a graded ideal of $R * G$ and*

$$J(R * G) = ((J(R * G) \cap R) * G \subseteq J(R) * G.$$

2.4 Algebras Graded by Semilattices

In this section, we introduce some results from semilattice graded rings.

Theorem 2.4.1 [Jes1, OW] *Let $R = \sum_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded algebra, Γ is a semilattice. $\mathcal{H}$ is one of hereditary radical (e.g. Jacobson, Prime, Levitzki and Brown-McCoy radical). If $0 \neq x = \sum_{\alpha \in \Gamma} x_\alpha \in \mathcal{H}(R)$, and β is maximal in $supp_\Gamma(x)$, then $x_\beta \in \mathcal{H}(R_\beta)$. Moreover, if $J(R_\alpha) = R_\alpha$ for all $\alpha \in \Gamma$, then $J(R) = R$.*

Proof. Take $0 \neq x \in \mathcal{H}(R)$ and Γ is semilattice which has partial order in the usual sense. Let $\Gamma_x = \{\alpha \in \Gamma | \alpha \leq \gamma \ , \ \forall \gamma \in supp(x)\}$. We can see that Γ_x is an ideal of Γ. Denote $R' = \sum_{\alpha \in \Gamma_x} R_\alpha$ which is the ideal of R. Then $\mathcal{H}(R) \cap R' = \mathcal{H}(R')$ and $x \in \mathcal{H}(R')$. Let β be the maximal element in $supp_\Gamma(x)$, β is also the maximal in Γ_x. And define $\phi : R' \to R_\beta$ be the projection map to R_β. This is surjective homomorphism because if $\beta = \alpha\delta$ implies $\beta \leq \alpha$ and $\beta \leq \delta$, so $\alpha = \beta = \delta$ by maximality of $\beta \in \Gamma_x$. Therefore

$$\phi(x) = x_\beta \in \mathcal{H}(R_\beta).$$

Moreover, if $J(R_\alpha) = R_\alpha$ and Γ is finite. By induction on $|\Gamma|$, we have $J(R) = R$. For Γ is infinite, for $0 \neq r \in R$, any $y \in R$, $supp(yr)$ is finite and let Γ' be subsemilattice generated by $supp_\Gamma(yr)$, it is a finite semilattice. Then $yr \in R' = \sum_{\alpha \in \Gamma'} R_\alpha = J(R')$. This means that yr has quasi-inverse in R' and hence in R. □

Corollary 2.4.2 [Jes3] *Let $\mathcal{H}$ be a hereditary radical (e.g. Jacobson, Prime, Levitzki radical etc.) and S a semilattice. If R is a S-graded ring and $\mathcal{H}(R_s) = \{0\}$ for each $s \in S$, then $\mathcal{H}(R) = \{0\}$. Moreover, if R is a field or R is a semiprimitive ring, then $J(R[S]) = 0$ if S is semilattice.*

Corollary 2.4.3 [Wau, Coro. 1.3] *Let S be a two-elements semilattice. That is $S = \{e, f\}$ with $e^2 = e, f^2 = f = fe = ef$. Let $\mathcal{H}$ be a supernilpotent radical (hereditary and contains all nilpotent ideals), and R is a S-graded ring. If R_f has an unity 1_f, then*

$$\mathcal{H}(R) = \{r = r_e + r_f \mid r_e \in \mathcal{H}(R_e), 1_f r \in \mathcal{H}(R_f)\}$$

and if $\mathcal{H}(R_f) = 0$, then

$$\mathcal{H}(R) = \{r - 1_f r \mid r \in \mathcal{H}(R_e)\}.$$

2.5 Algebras Graded by Bands

More generally, if all semilattices are bands, then we can extend the above result to the ring graded by bands. Recently, Kelarev and Munn have considered the ring R graded by bands [Kel1, Mun7]. Denoted $R = \sum_B R_b$ where B is a band. Sometimes, we also call R a band-sums. The following results are extracted from [Kel1, Mun7].

Recall that a band is a rectangular band if it is satisfies the identity $xyx = x$. Let B be a band and define a relation σ on B by

$$(x, y) \in \sigma \Leftrightarrow xyx = x \text{ and } yxy = y \text{ for all } x, y \in B.$$

Then B is semilattice of rectangular bands. Let $\pi : B \to \bar{B} = B/\sigma$, for $s_1, s_2 \in B$, we say $s_1 \le s_2$ if $\bar{s_1} \le \bar{s_2}$ where $\bar{s} = \{t \in B | sts = s, tst = t\}$.
Also, we denote

$$\hat{s} = \{t \in B | \bar{s} \le \bar{t}\} = \{t \in B | sts = s\}.$$

2.5.1 Hereditary Radicals of Band-graded Rings

For $r \in R$, write $r_{\hat{s}} = \sum_{t \in \hat{s}} r_t$ and $A_{\hat{s}} = \{a_{\hat{s}} | a \in A\}$. The following formation and characterization are given by Kelarev [Kel1].

Let $\mathcal{H}$ be hereditary radical and the collection of ideals

$$\mathbf{I}(R, B, \mathcal{H}) = \{A \text{ is ideal in } R |\ A_{\hat{b}} \cap R_b \subseteq \mathcal{H}(R_b) \text{ for all } b \in B\}.$$

Let $\mathcal{H}(R, B)$ be the sum of all ideals in $\mathbf{I}(R, B, \mathcal{H})$. Then, we say that $\mathcal{H}$ is **determined by the component of the B-sums** if $\mathcal{H}(R) = \mathcal{H}(R, B)$. Note that $\mathcal{H}(R, B)$ is the largest ideal in $\mathbf{I}(R, B, \mathcal{H})$.

If Γ is a semilattice, $\mathcal{H}$ a radical, and $R = \sum_{s \in \Gamma} R_s$, then $\mathcal{H}(R) \in \mathbf{I}(R, S, \mathcal{H})$. Moreover, if P is a right (left) zero band, $\mathcal{H}$ be left (right) hereditary radical, then $\mathcal{H}(R) \in \mathbf{I}(R, P, \mathcal{H})$.

Moreover, a radical is called **C-local (C_r-local, C_l-local)** if its radical class is closed under unions of ascending chains of subrings (right ideals, left ideals).

Definition 2.5.1 *A radical $\mathcal{H}$ is called* ***countably definable*** *if*

(i). *$\mathcal{H}$ is C-local or there exists a nonradical ring A being a union of an ascending chain of radical subrings $A_1 \subseteq A_2 \subseteq \cdots A_n \subseteq \cdots$, for all $n \in \mathbb{N}$.*

(ii). *$\mathcal{H}$ is C_r-local (C_l-local) or there exists a nonradical ring A being a union of an ascending chain or radical right (left) ideals $A_1 \subseteq A_2 \subseteq \cdots A_n \subseteq \cdots$, for all $n \in \mathbb{N}$.*

The radicals of Jacobson, Levitzki, prime and Brown-McCoy are clearly countably definable.

Recall that a radical is **right summing** iff in every ring the sum of any two $\mathcal{H}$-radical right ideal is $\mathcal{H}$-radical. A radical is **right hereditary** iff its radical class is closed under right ideals.

Lemma 2.5.2 *The radical $\mathcal{H}$ is determined by the components of B-sums while a band B contains semilattice T with infinite descending chain. (or infinite left zero band L). If there is a chain of subrings (or right ideal) $A_1 \subset A_2 \subset \cdots$ with $\mathcal{H}(A_i) = 0$ and $A = \cup_{i=1}^{\infty} A_i$, then $\mathcal{H}(A) = 0$.*

Proof. Let $A[T]$ be semigroup ring and suppose $t_1 \geq t_2 \geq t_3 \geq \cdots$. Let R_{t_i} be the subring $A_i t_i$. Clearly, $R_b = 0$ if $b \in B \backslash T$. Let $R = \sum_{b \in B} R_b$. If $\mathcal{H}$ is determined by the B-sums and $\mathcal{H}(R_{t_i}) \cong \mathcal{H}(A_i t_i) \cong \mathcal{H}(A_i) = 0$, then $\mathcal{H}(R) = 0$. Let $\varphi : A[T] \to A$ be the augmentation map. It is clear that $\varphi(R_{t_i}) = A_i$ and hence $\varphi(R) = A$. Therefore $\mathcal{H}(A) = 0$. The case of left zero band L is similar and routine. □

We now state the main theorem on band-graded rings.

Theorem 2.5.3 [Kell] **(1991)** *Let B be a band which is a semilattice S of rectangular bands Q_s, where Q_s is the direct product of a left zero band L_s and a right zero* *of P_s. A countably definable radical $\mathcal{H}$ is determined by the components of B-sums iff the following conditions hold:*

(i). *S satisfies the descending chain condition or $\mathcal{H}$ is C-local;*

(ii). *Every L_s (P_s) is finite or $\mathcal{H}$ is C_r-local (C_l-local);*

(iii). *Every L_s (P_s) consists of one element or the radical $\mathcal{H}$ is right (left) hereditary, right (left) summing and supernilpotent.*

Proof. We only sketch the proof and the details are found in [Kel1, Th. 1].

Necessity: Suppose $\mathcal{H}$ is determined by the components of B-sums. By Lemma 2.5.2, we know that if S does not satisfy the descending chain condition and B has an infinite semilattice T, then $\mathcal{H}$ is C-local. Hence (i) is proved. Similarly, (ii) follows by Lemma 2.5.2.

(iii) It suffices to show that if L_s is not a singleton, then $\mathcal{H}$ is right summing, right hereditary or supernilpotent. Suppose $L = L_s$ is a two-elements left zero band. We use $L = \{a, b\} \subseteq B$ and take any ring A which is the sum of two $\mathcal{H}$-radical right ideals I and J. Consider $A[L]$ and let $R_a = Ia$ and $R_b = Ib$ and for $t \in B \setminus L$, set $R_t = 0$ and $R = \sum_{s \in B} R_s$. By Lemma 2.5.2, R is B-graded ring. Hence R is $\mathcal{H}$-radical ring. Define $\varphi(r_1 a + r_2 b) = r_1 + r_2$ which maps R onto A. Then A is $\mathcal{H}$-radical ring and hence $\mathcal{H}$ is right summing.

By using similar construction, $\mathcal{H}$ is also right hereditary and supernilpotent.

Sufficiency: The sufficiency can be proved by considering S in the cases of semilattice, left zero band, rectanglar band and band. Then, we have to show that $\mathcal{H}(R)$ is the largest ideal in $\mathbf{I}(R, S, \mathcal{H})$. We omit the details. □

Corollary 2.5.4 *Let J, $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{G}$ be the radicals concerned and R is a ring graded by band B. Then the following facts hold:*

(i). *$J(R) = J(R, B)$ and $\mathcal{L}(R) = \mathcal{L}(R, B)$.*

(ii). *$\mathcal{B}(R) = \mathcal{B}(R, B)$ iff B is a semilattice Γ of rectanglar bands, where Γ satisfies the decending chain condition.*

(iii). *$\mathcal{G}(R) = \mathcal{G}(R, B)$ iff B is a semilattice.*

Proof. We know that $\mathcal{H} = J, \mathcal{L}, \mathcal{B}$ are all right summing, right hereditary, and supernilpoent radicals, however, $\mathcal{B}$ is not C-local and by Theorem 2.5.3(i), Γ must satisfy the decending chain condition. Moreover, $\mathcal{G}$ is not right hereditary and left hereditary, then Theorem 2.5.3(iii), L_s and P_s are singleton for $s \in \Gamma$. Thus $B = \Gamma$ which is a semilattice. $\square$

2.5.2 Special Band-graded Rings

Munn has studied a particular class of band-graded rings in 1992. Let B be band, a ring R is **special** if

(i). For all $\alpha \in B$, R_α is non-zero and has a unity 1_α,

(ii). For all $\alpha, \beta \in B$, $1_\alpha 1_\beta = 1_{\alpha\beta}$.

Denote B/σ by $\bar{B}$ where σ is a semilattice congruence on the band B.

Lemma 2.5.5 *Let B be band and R a speical B-graded ring. let $\alpha, \beta \in B$ with $\bar{\alpha} \geq \bar{\beta}$. Define $\phi_{\alpha,\beta} : R_\alpha \to R_\beta$ by $\phi_{\alpha,\beta}(x) = 1_\beta x 1_\beta$ for all $x \in R_\alpha$. Then*

(i). *$\phi_{\alpha,\beta}$ is homomorphism.*

(ii). *For all $\gamma \in B$ with $\bar{\gamma} \geq \bar{\alpha}$, $\phi_{\alpha,\beta}\phi_{\gamma,\alpha} = \phi_{\gamma,\beta}$.*

(iii). *If $\bar{\alpha} = \bar{\beta}$ then $\phi_{\alpha,\beta}$ is an isomorphism under an inverse isomoprhism $\phi_{\beta,\alpha}$.*

Theorem 2.5.6 [Mun7] **(1992)** *Let R be a special band-graded ring. Then we have*

$$J(R) = \{a \in R \mid \forall \alpha \in B,\ 1_\alpha a_{\hat{\alpha}} 1_\alpha \in J(R_\alpha)\}.$$

Proof. Let T be the set $\{a \in R \mid \forall \alpha \in B,\ 1_\alpha a_{\hat{\alpha}} 1_\alpha \in J(R_\alpha)\}$. Corollary 2.5.4 (i) yields that the Jacobson radical is determined by the components of B, $J(R)_{\hat{\alpha}} \cap R_\alpha \subseteq J(R_\alpha)$. For $a \in J(R)$, we have $1_\alpha a_{\hat{\alpha}} 1_\alpha = (1_\alpha a 1_\alpha)_{\hat{\alpha}} \in J(R)_{\hat{\alpha}}$ and $1_\alpha a_{\hat{\alpha}} 1_\alpha \in R_\alpha$. Thus $1_\alpha a_{\hat{\alpha}} 1_\alpha \in J(R)_{\hat{\alpha}} \cap R_\alpha \subseteq J(R_\alpha)$ for all $\alpha \in B$ and so $a \in T$, i.e. $J(R) \subseteq T$.

Let $\alpha \in B$ and $S_\alpha = \sum_{\beta \in \hat{\alpha}} R_\beta$. Then S_α is subring of R and $\sum_{\beta \in B \setminus \hat{\alpha}} R_\beta$ is an ideal of R. Define $\pi_\alpha : x \mapsto x_{\hat{\alpha}}$. Moreover, there is an epimorphism η_α from S_α to R_α by $x \mapsto 1_\alpha x 1_\alpha$, we have

$$\begin{array}{ccc} R & \xrightarrow{\pi_\alpha} & S_\alpha \\ & {\scriptstyle\psi_\alpha} \searrow & \downarrow {\scriptstyle\eta_\alpha} \\ & & R_\alpha \end{array}$$

Since this holds for all $\alpha \in B$. We have $\psi_\alpha(a) = 1_\alpha a_{\hat{\alpha}} 1_\alpha$ and $\psi_\alpha(J(R)) \subseteq J(R_\alpha)$. Then

$$J(R) \subseteq \bigcap_{\alpha \in B} \psi_\alpha^{-1}(J(R_\alpha)) = T.$$

Moreover $\psi^{-1}(J(R_\alpha))$ is an ideal and so is T.

T is an ideal and by the definition of T, we have $T_{\hat{\alpha}} \cap R_\alpha \subseteq J(R_\alpha)$. By Theorem 2.5.3, $J(R)$ is largest ideal of $\mathbf{I}(R, B, J)$. This shows that $J(R) = T$. □

A special B-graded ring R is called **radically coherent** iff

$$\forall \alpha, \beta \in B \text{ with } \bar{\alpha} \geq \bar{\beta} \qquad \phi_{\alpha,\beta}(J(R_\alpha)) \subseteq J(R_\beta).$$

The following corollaries were due to W.D. Munn [Mun7].

Corollary 2.5.7 [Mun7, Coro. 3.3] *Let R be a special B-graded ring. Then*

(i). $J(R) = \sum_{\bar{\gamma} \in \bar{B}} J(R_{\bar{\gamma}})$ *iff R is radically coherent.*

(ii). $J(R) = \sum_{\gamma \in B} J(R_\gamma)$ *iff B is a semilattice and R is radically coherent.*

Corollary 2.5.8 [Mun7, Coro. 3.4] *Let B be a band and R be a special B-graded ring. If $J(R) = 0$ then B is a semilattice.*

Thus we have the following important corollary for algebras over bands.

Corollary 2.5.9 [Mun7, Coro. 3.5] *Let R be a non-trival ring with unity and B a band. Then $R[B]$ is radically coherent and*

$$J(R[B]) = \{\sum_{b \in B} r_b b \in R[B] \mid \forall \bar{\gamma} \in \bar{B}\ ,\ \sum_{t \in \bar{\gamma}} r_t \in J(R)\}.$$

There is another description on Jacobson radical in band-graded ring. Let A be a finite non-empty subset of a band B and let $\alpha \in \langle A \rangle$, the subsemigroup generated by A. $\phi_{\alpha,\beta} : R_\alpha \to R_\beta$ for $\bar{\alpha} \geq \bar{\beta}$. Define

$$M(A, \alpha) = \bigcap_{\substack{\beta \in B \\ \bar{\alpha} \geq \bar{\beta} \\ \hat{\alpha} \cap A = \hat{\beta} \cap A}} \phi_{\alpha,\beta}^{-1}(J(R_\beta)).$$

Theorem 2.5.10 [Mun7] **(1992)** *Let R be a special band-graded ring and A a finite non-empty subset of B. Let $a \in R$ such that $supp_B(a) \subseteq A$. Then*

$$J(R) = \{a \in R \mid \forall \alpha \in \langle A \rangle, \qquad 1_\alpha a_{\hat{\alpha}} 1_\alpha \in M(A, \alpha)\}$$

Let Γ be semilattice and $\alpha, \beta \in \Gamma$, write $\alpha \succ \beta$ iff $\alpha > \beta$ and subject to no $\gamma \in \Gamma$ such that $\alpha > \gamma > \beta$. Γ is called **pseudofinite** iff it satisfies the following conditions:

(a) For all $\alpha, \beta \in \Gamma$ with $\alpha > \beta$ there exists $\gamma \in \Gamma$ such that $\alpha \succ \gamma \geq \beta$.

(b) For all $\alpha \in \Gamma$, $|\{\beta \in \Gamma : \alpha \succ \beta\}| < \infty$.

Definition 2.5.11 *Let A be a subsemilattice of B. Suppose A is finite and let $\alpha \in A$ we define*

$$\rho(A, \alpha) = \begin{cases} 1_\alpha & \textit{if } \alpha \textit{ is the least element of } A. \\ \prod\limits_{\substack{\gamma \in A \\ \alpha \succ_A \gamma}} (1_\alpha - 1_\gamma) & \textit{otherwise.} \end{cases}$$

We can see that $\rho(A, \alpha)$ is central idemoptent of R, $\alpha \in A$ which is finite set and $1_\alpha = \sum_{\beta \in \alpha A} \rho(A, \beta) = \sum\limits_{\substack{\beta \in A \\ \alpha \geq \beta}} \rho(A, \beta)$.

By using the above definition, W.D. Munn obtained the following result:

Theorem 2.5.12 [Mun7] **(1992)** *Let R be special Γ-graded ring and let A be a finite subsemilattice of the semilattice Γ. Let $J_A(R) = \{a \in J(R) : supp_\Gamma(a) \subseteq A\}$. Then*

$$J_A(R) = \sum_{\alpha \in A} M(A, \alpha) \rho(A, \alpha).$$

Corollary 2.5.13 [Mun7, Coro. 5.4] *Let R be a special semilattice-graded ring with Γ pseudofinite. Then*

$$J(R) = 0 \quad \textit{iff} \quad \forall \alpha \in \Gamma, \qquad J(R_\alpha) = 0.$$

If Γ is pseudofinite, then $\rho(\alpha) = \rho(B, \alpha)$ because the set $\{\gamma : \alpha \succ \gamma\}$ is finite.

Corollary 2.5.14 [Mun7, Coro. 5.5] *Assume that each principal ideal of Γ is finite. If R is a special Γ-graded ring, then*

$$J(R) = \sum_{\alpha \in \Gamma} J(R_\alpha) \rho(\alpha).$$

Finally, we consider the nilness of the band-graded ring R by using the above techniques. We have the following results.

Theorem 2.5.15 [Mun7, Th. 6.2] *Let R be a special band-graded ring. If $J(R_\alpha)$ is nil for all $\alpha \in B$ then $J(R)$ is nil. If R is radically coherent and $J(R)$ is nil then $J(R_\alpha)$ is nil for all $\alpha \in B$.*

Call a band B **normal** iff all $\alpha, \beta, \gamma, \delta \in B$, $\alpha\beta\gamma\delta = \alpha\gamma\beta\delta$.

Theorem 2.5.16 [Mun7, Th. 6.4] *Let R be a radically coherent special band-graded ring and n a positive integer. Then we have the following statements:*

(i). *If $J(R)^n = 0$ then $J(R_\alpha)^n = 0$ for all $\alpha \in B$.*

(ii). *If B is normal and $J(R_\alpha)^n = 0$ for all $\alpha \in B$, then*

$$RJ(R)^n R = 0.$$

When R is a non-trival ring with unity and B is normal band, we have $J(R[B])$ is nilpotent iff $J(R)$ is nilpotent. If $J(R)^n = 0$ for some positive integer n then $J(R[B])^{n+2} = 0$.

We will apply the above grading technique to study the structure of semigroup algebras and group algebras in the following chapters.

Chapter 3

Radicals of Semigroup Algebras

In this chapter, we investigate the radicals of algebras of an arbitrary semigroup. We consider the polynomial rings and commutative semigroup algebras. After describing them, we study the algebras of non-commutative cancellative semigroups. For other non-commutative cases, we make use of graded ring theory to give some generalizations, especially for algebras of completely 0-simple semigroups. In the last section of this chapter, we also describe the radicals of PI (polynomial identity) semigroup algebras.

3.1 Radicals of Polynomial Rings

The first major result on radical of polynomial rings was obtained by S.A. Amitsur in 1956, [Ami].

Lemma 3.1.1 [JW1, Lemma 4.1] *If S is any semigroup and $\mathcal{J}$ any hereditary radical property then, for any ring R, $\mathcal{J}(R[S])$ is an ideal of $R^1[S]$. Morever, if S has unity element, then*

$$(\mathcal{J}(R[S]) \cap R)[S] \subseteq \mathcal{J}(R[S]).$$

If S has no unity, we also have

$$\mathcal{J}(R[S]) = \mathcal{J}(R^1[S^1]) \cap R[S].$$

Therefore, it is not necessary to divide the case whether R and S contains identities or not. In the following, we assume that all rings will have unity unless specify otherwise.

We now consider the Amitsur's result.

Theorem 3.1.2 [Ami, JW1, Kre2] *Let S be free commutative moniod with rank n, (finite and infinite) and R an arbitrary ring (with or without unity). Suppose that $\mathcal{B}$ is that prime radical, $\mathcal{L}$ is the Levitzki radical, $\mathcal{G}$ is the Brown-McCoy radical and J is the Jacobson radical. Then*

(i). $\mathcal{B}(R[S]) = \mathcal{B}(R)[S]$.

(ii). $\mathcal{L}(R[S]) = \mathcal{L}(R)[S]$.

(iii). $J(R[S]) = J_n(R)[S]$, *where* $J_n(R) = J(R[S]) \cap R$.

(iv). $\mathcal{G}(R[S]) = \mathcal{G}_n(R)[S]$, *where* $\mathcal{G}_n(R) = \mathcal{G}(R[S]) \cap R$.

Moreover,

$$J(R) = J_0(R) \supseteq J_1(R) \supseteq J_2(R) \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} J_n(R) = J_\infty(R)$$

Similarly,

$$\mathcal{G}(R) = \mathcal{G}_0(R) \supseteq \mathcal{G}_1(R) \supseteq \mathcal{G}_2(R) \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} \mathcal{G}_n(R) = \mathcal{G}_\infty(R)$$

Furthermore, $J_1(R)$ is a nil ideal and $J_n(R/J_n(R)) = 0$. In particular, $\mathcal{L}(R) \subseteq J_n(R) \subseteq \mathcal{N}(R)$, where $\mathcal{N}(R)$ is the upper nil radical.

Proof. We just present the case for Jacobson radicals. The cases of other radicals can be proved similarly. In the cases of prime and Levitzki radicals, the reader is referred to [Kre2]. In the case of Brown-McCoy radicals. The reader can find the proof in [JKW, Th. 2.5].

By Theorem 2.1.1, we know $J(R[S]) \cap R)[S] \subseteq J(R[S])$. Moreover, as S is a free commutative monoid, S can be embedded into a commutative free group with rank n. We regard $R[S]$ as G-graded ring and by Theorem 2.3.5 (i), $J(R[S])$ is graded. Thus, $\sum r_s s \in J(R[S])$ implies that $rs \in J(R[S])$. We only need to consider rs. It suffices to show that $rs \in J(R[S])$ implies that $r \in J(R[S])$. When n is finite, S is free of rank n then $s = s_1^{m_1} s_2^{m_2} \cdots s_n^{m_n}$, where $m_i \in \mathbb{N}$. Moreover, as the Jacobson radical is invariant of automorphism of $R[S]$, we have an automorphism $\phi : s_i \mapsto s_i + 1$ for $i \in \{1, \cdots, n\}$. Then $\phi(rs) = r(s_1+1)^{m_1} \cdots (s_n+1)^{m_n}$. As the $J(R[S])$ is S-graded, for all $r \in J(R[S])$, we have, $J(R[S]) = (J(R[S]) \cap R)[S]$ as required.

Let T be a free commutative semigroup of rank $k+1$, with generators $\{t_1, \cdots, t_{k+1}\}$. Let H be a free subsemigroup of T of rank k with generators $\{t_1, \cdots, t_k\}$. Then H is a grouplike subsemigroup of T. By Lemma 1.2.3, we have

$$J(R[T]) \cap R \subseteq J(R[T]) \cap R[H] \subseteq J(R[H]) = J_k(R)[H].$$

Therefore $J_{k+1}(R) \subseteq J_k(R)$, as required. .

For the case that if S is free commutative semigroup of infinite rank, we can take $0 \neq x \in J(R[S])$, $supp(x)$ is finite set with l generator. Let S_l be the free semigroup group generated by $supp(x)$. Then $x \in R[S_l]$ and

$$J(R[S]) \cap R[S_l] \subseteq J(R[S_l]) = J_l(R)[S_l] \subseteq J_l(R)[S] \subseteq J(R)[S]$$

Moreover, if S has infinite rank, $\exists k \geq l$, such that S_k is a rank k free subsemigroup which is a grouplike and $S_l \subset S_k$. This leads to x is in $J_k(R)[S_k] \subseteq J_k(R)[S]$. Hence,

$$x \in \bigcap_{n=1}^{\infty} J_n(R)[S]. \qquad \text{Thus, } J_\infty(R) = J(R[S]) \cap R[S].$$

It remains to prove that $J_1(R)$ is a nil ideal of R. If $r \in J_1(R) = J(R[x]) \cap R$ then $1 + rx$ is invertible in $J(R[x])$. However, as the inverse in power series of $R[[x]]$, $d = 1 - rx + r^2x + \cdots$. Since $d \in R[x] \subseteq R[[x]]$, there exists n such that $r^n = 0$. This shows that $J_1(R)$ is a nil ideal.

Since $J(R[S]) = J_n(R)[S]$, $R[S]/J(R[S]) \cong (R/J_n(R))[S]$. Hence $(R/J_n(R))[S]$ is semiprimitive and so $J_n(R/J_n(R)) = 0$.

It is easy to see that $J_n(R) \subseteq \mathcal{N}(R)$. By [Kar, Coro. 33.13], we have

$$\mathcal{L}(R)[S] \subseteq J(R[S]) \subseteq \mathcal{N}(R)[S],$$

and consequently, $\mathcal{L}(R) \subseteq J_n(R) \subseteq \mathcal{N}(R)$. □

3.2 Radicals of Commutative Semigroup Algebras

We first classify commutative semigroups. Let $\mathbb{P}$ be the set of prime numbers. Let $p \in \mathbb{P}$, then S is p-separative if for any $s, t \in S$, $s^p = t^p$ implies $s = t$. The least separative (p-separative) congruence ξ (respectively ξ_p) is defined by

$$\xi = \{(s,t) \mid \exists n : st^n = t^{n+1}, \text{ and } s^n t = s^{n+1}\},$$

$$(\xi_p = \{(s,t) \mid \exists n : s^{p^n} = t^{p^n}\}, \text{ respectivily })$$

It can be easily seen that $\xi \subseteq \xi_p$ and ξ_p/ξ is the least p-separative congrunce on S/ξ, $p \in \mathbb{P}$.

In chapter 1, we know that the commutative separative semigroup is embeddable in a semilattice of abelian groups, say $S = \cup_{\alpha \in \Gamma} S_\alpha$, where Γ is semilattice and S_α is abelian group or cancellative semigroup without idempotent. (cf. [Mun1]).

In this section, we assume all semigroups S are commutative without zero element and R is an arbitrary ring with unity unless we otherwise stated.

3.2.1 Commutative Cancellative Semigroups

Let S be a cancellative commutative semigroup S satisfying the Ore condition. Then S has group of fractions SS^{-1} and denote it by the group $Q(S)$. We define the torsion-free rank of this semigroup is to be the torsion-free rank of the group $Q(S)$. Moreover, we can define the rank of semigroup. From [Okn1] chapter 23, we denote the rank of the semigroup S by $rk(S)$, that is,

$rk(S) = sup\{n \in \mathbb{N} :$ S has a free commutative subsemigroup on n free generators$\}$.

We observe that $rk(S)$ coincides with the torsion-free rank of S when S is a commutative cancellative semigroup (see [Okn1, Prop. 23.1]).

Moreover, in section 3.1, we denote $J_n(R) = J(R[x_1, x_2, \cdots, x_n]) \cap R$ for the Jacobson radical of the polynomial ring $R[x_1, x_2, \cdots, x_n]$. However, it is not clear to see that $J_n(R)[S] \subseteq J(R[S])$ for any arbitrary semigroup S.

Lemma 3.2.1 [JW1] *Let R be a ring and S a commutative cancellative semigroup with torsion-free rank n. Then $J_n(R)[S] \subseteq J(R[S])$*

Proof. First, we suppose that S is group. Clearly, S has torsion-free rank n. Hence, there exists a free subgroup F with rank n such that S/F is a torsion abelian group. Let T be the free subsemigroup of F such that $F = Q(T)$. It is easy to see that $R[F]$ is a normalizing extension of $R[T]$. Thus, $J(R[F]) \cap R[T] = J(R[T]) = J_n(R)[T]$ by Theorem 3.1.2. Therefore, $J_n(R) \subseteq J(R[F])$.

Now, take $a \in J_n(R)[S]$, $a = \sum r_i s_i$, where $r \in J_n(R)$ and $s \in S$. Take b from $R[S]$, we have a subgroup generated by F, $supp(a)$ and $supp(b)$, say H. H is a finitely generated and H/F is a finite group since $H/F \subseteq S/F$ is a torsion abelian group. Then, $R[H]$ is a normalizing extension of $R[F]$. Therefore,

$$ab \in J(R[F]) = J(R[H]) \cap R[F] \subseteq J(R[H]).$$

ab has a right quasi-inverse in $R[H]$ and also in $R[S]$. As a result, we have $J_n(R)[S] \subseteq J(R[S])$.

When S is semigroup, it suffices to show that for $0 \neq a \in J_n(R)$, a belongs to each maximal ideal of $R[S]$. Let M be a maximal ideal of $R[S]$ and let T be the set $\{x \in S | x \notin M\}$. If T is nonempty set, then T is a semigroup because M is also a prime ideal. Moreover, if $x \in S \setminus T$, then $xy \in S \setminus T$ for all $y \in S$.

Define $\pi : R[S] \to R[T]$ by $\sum_{s \in S} r_s s \mapsto \sum_{s \in T} r_s s$. Clearly, π is a ring epimorphism. $\pi(M)$ is hence a maximal ideal of $R[T]$ and $\pi(M) \cap T = \emptyset$. Also, $R[Q(T)]$

is localization of $R[T]$. As $\pi(M)R[Q(T)]$ is a maximal ideal of $R[Q(T)]$, $\pi(M) = \pi(M)R[Q(T)] \cap R[T]$ by the maximality of $\pi(M)$. Let $rk(Q(T)) = m \leq rk(Q(S)) = n$ and $a \in J_n(R) \subseteq J_m(R)$. Then, we have $a = \pi(a) \in \pi(M)R[Q(T)]$. This implies that $\pi(a) \in \pi(M)R[Q(T)] \cap R[T] = \pi(M)$ and hence $a \in M$. $\square$

Lemma 3.2.2 [JW1] *Let A be an ideal of the ring R and p an odd prime number. Let $A_p = \{r \in A | pr \in A\}$ and S be a semigroup with $(s,t) \in \xi_p$. Then*

$$A_p[S](s-t) = \{\sum_{v \in S} r_v(sv - tv) | r_v \in A_p\}$$

is a nilpotent ideal modulo $A[S]$ in $R[S]$. In particular, $I(A_p, S, \xi_p)$ (defined in Chapter 1) is a sum of nilpotent ideals modulo $A[S]$. Furthermore, $I(R, S, \xi)$ is a sum of nilpotent ideals.

From the survey paper of Jespers and Wauters on Jacobson radical of semigroup rings in [JW1], we obtain the main result about commutative cancellative semigroups.

Theorem 3.2.3 [JKW, JW1] *Let R be a ring and S a cancellative semigroup of torsion-free rank n. Let radical $\mathcal{H} = J, \mathcal{G}$ (the Jacobson and Brown-McCoy radicals respectively). Then*

$$\mathcal{H}(R[S]) = \mathcal{H}_n(R)[S] + \sum_{p \in \mathrm{P}} I(\mathcal{H}_{n,p}(R), S, \xi_p)$$

where $\mathcal{H}_{n,p}(R) = (\mathcal{H}_n(R))_p = \{r \in R | pr \in \mathcal{H}_n(R)\}$. In particular, $\mathcal{H}(R[S]) = \mathcal{H}(R(Q[S]) \cap R[S]$.

If $\mathcal{H} = \mathcal{B}, \mathcal{L}$ (Prime radicals and Levitzki radicals), then

$$\mathcal{H}(R[S]) = \mathcal{H}(R)[S] + \sum_{p \in \mathrm{P}} I(\mathcal{H}(R), S, \xi_p).$$

Proof. We only prove the case for Jacobson radical as the other cases such as the prime, Levitzki and Brown-McCoy radicals are similar.

By Lemma 3.2.1 it yields that $J_n(R)[S] \subseteq J(R[S])$ and $\sum_{p \in \mathrm{P}} I(J_{n,p}(R), S, \xi_p) \subseteq J(R[S])$. It suffices to show that $J(R[S]) \subseteq J_n(R)[S] + \sum_{p \in \mathrm{P}} I(J_{n,p}(R), S, \xi_p)$. We may assume that $J_n(R) = 0$ since $J_n(R)[S] \subseteq J(R[S])$ and $R[S]/J_n(R)[S] \cong (R/J_n(R))[S]$.

Let $d \in J(R[S])$ and let $D = \langle supp(d) \rangle$ be a finitely generated group in $Q(S)$. Then $D \cong G_1 \times F'$, where G_1 is finite group and F' is free subgroup with rank $\leq n$ of $Q(S)$. Add more free generators on F' and make $D < G \cong G_1 \times F$, where F is free subgroup of n free generators. Then we have $d \in R[S \cap G]$. Denote $S \cap G$ by H and so $Q(S)/G$ is torsion, we have $Q(H) = G$. Thus, H is a grouplike subsemigroup of S and so

$$J(R[S]) \cap R[H] \subseteq J(R[H]).$$

Therefore, $d \in J(R[H])$. Denote the subsemigroup of G generated by $H \cup G_1$ by H'. Then $H' = G_1 \times H''$. Write $H'' = F \cap H$. We then have H'' is torsion-free subsemigroup of S with rank n. Therefore, $R[H']$ is a normalizing extension of $R[H'']$ and $R[H'] = (R[H''])[G_1]$ is finite group graded ring. If we assume that $J_n(R) = 0$, then $J(R[H'']) = 0$. This leads to

$$d \in J(R[H']) = J(R(H'')[G_1]) = \{\sum d_i(x_i - y_i) : d_i \in R[H''], x_i, y_i \in G_1,$$

$$x_i^{p_i^k} = y_i^{p_i^k} \text{ for some k} \geq 0\ , p_i \in \mathbb{P}, p_i d_i = 0\}.$$

Let $d_i = \sum_j r_{ij} h_j \in R[H'']$ with $r_{ij} \in R$ and $h_j \in H''$ for all i, j. Then $d = \sum_{i,j} r_{ij}(h_j x_i - h_j y_i)$ and $(h_j x_i)^{p_i^k} = (h_j y_i)^{p_i^k}$ and $p_i r_{ij} = 0$ for all i and j. This completes the proof of the first part.

For the proof of the second part, we let p be any prime p. Since $(s,t) \in \xi_p$ is over $Q(S)$ iff $(s,t) \in \xi_p$ over S, we have $I(J_{n,p}(R), S, \xi_p) = I(J_{n,p}(R), Q(S), \xi_p) \cap R[S]$ and

$$\begin{aligned} J(R[S]) &= J_n(R)[S] + \sum_{p \in \mathrm{P}} I(J_{n,p}(R), S, \xi_p) \\ &= (J_n(R)[Q(S)] + \sum_{p \in \mathrm{P}} I(J_{n,p}(R), Q(S), \xi_p)) \cap R[S] \\ &= J(R[Q(S)]) \cap R[S]. \end{aligned}$$

□

Corollary 3.2.4 *Let R be a field or a domain with $J(R) = 0$. If S is a commutative cancellative semigroup, then $J(R[S]) = I(R, S, \xi_p)$, where $char(R) = p$.*

3.2.2 General Commutative Semigroups

After solving the problem on commuative cancellative semigroup case, we now extend the problem to any commutative semigroups. Since $I(R.S, \rho)$ is the kernel of $\phi : R[S] \to R[S/\rho]$, we have seen in [Mun2] that $J(R[S])/I(R, S, \xi) \cong J(R[S/\xi])$ if ξ is the least separative congruence.

Now, we consider the separative case. Since S is a separative semigroup, S has semilattice decomposition into commtative semigroups, say, $S = \cup_{\alpha \in \Gamma} S_\alpha$. Define

$$I_\alpha = \{x \in J(R[S_\alpha]) | (\forall \beta \in \Gamma)\ R[S_\beta]x \subseteq J(R[S_{\alpha\beta}])\}.$$

Clearly, I_α is an ideal of $R[S_\alpha]$.

Proposition 3.2.5 [OW, Lemma 4.3] *Let S be a separative semigroup, $S_\alpha(\alpha \in \Gamma)$ the archimedean compoments of S. Let R be ring. Suppose for each $\alpha \in \Gamma, I_\alpha = J(R[S_\alpha])$. Then $J(R[S])$ is Γ-graded ring, i.e.*

$$J(R[S]) = \sum_{\alpha\in\Gamma} J(R[S_\alpha]).$$

Lemma 3.2.6 [Jes1, Lemma 3.7] *Let R be arbitrary ring and S a separative semigroup with semilattice Γ of commutative cancellative archimedean components S_α, $S = \cup_{\alpha\in\Gamma} S_\alpha$. If ξ_p is p-congruence on S, then the ideal*

$$\sum_{p\in P} I(J_{n,p}(R), S, \xi_p) = \sum_{\alpha\in\Gamma}\sum_{p\in P} I(J_{n,p}(R), S_\alpha, \xi_p).$$

Theorem 3.2.7 [OW, Lemma 4.5] *If*

(i). *S is periodic semigroup, or;*

(ii). *If S is a commutative semigroup and R a ring such that $J(R) = J_\infty(R)$,*

then

$$J(R[S]) = J(R)[S] + I(R, S, \xi) + \sum_{p\in P} I(J_{0,p}(R), S, \xi_p).$$

Not all rings R satisfy the condition $J(R) = J_\infty(R)$ even if R is commutative. If $J(R)$ is not nil, then $J(R) \neq J_1(R)$. Hence $J(R) \neq J_\infty(R)$ in general. We now consider the situation that $J_1(R) = J_\infty(R)$. Note that $J_1(R)$ is a nil ideal and if R is noetherian or if R satisfies a polynomial identity (e.g. R is commutative), then the condition $J_1(R) = J_\infty(R)$ is often satisfied.

Krempa showed in [Kre1] **(1972)** that this condition is related to the Koethe conjecture, that is, *if a ring R contains a one-sided nil ideal A, can A be contained in a two-sided nil ideal of R?* Krempa has shown that the Koethe conjecture is equivalent to $J_1(R) = \mathcal{N}(R)$, the upper nil radical of R. A stronger conjecture is : *if R is a nil ring, then the polynomial ring $R[x]$ over R is a nil ring.* If this statment holds, then all $J_\infty(R) = \mathcal{N}(R)$.

Jespers [Jes1] **(1987)** has considered the case when R satisfies $J_1(R) = J_\infty(R)$. The description of $J(R[S])$ is very complicated and this complexity depends on the subsemigroup of periodic elements and the order structure of the corresponding semilattice. We now simplify the proof of Jespers, by using the recent results obtained by Kelarev and Munn [Kel2, Mun7] **(1991,1992)**.

Let $S = \cup_{\alpha\in\Gamma} S_\alpha$ be separative semigroup, where every S_α is a commutative cancellative semigroup. Let Q_α be the group of fractions of cancellative semigroup S_α with identity e_α. Set $Q = \cup_{\alpha\in\Gamma} Q_\alpha$ which is semilattice of abelian groups, i.e. $S \hookrightarrow Q$. The identity e_α is clearly central, i.e. $e_\alpha e_\beta = e_{\alpha\beta} = e_{\beta\alpha} = e_\beta e_\alpha$ for all $\alpha, \beta \in \Gamma$. It is known that $R[Q] = \sum_{\alpha\in\Gamma} R[Q_\alpha]$ is special semilattice-graded ring (see Section 2.5).

Using the results given in Section 2.5, we can state the following:

Lemma 3.2.8 [Kel2, Lemma 1] *Let R be an arbitrary ring, $S = \cup_{\alpha\in\Gamma} S_\alpha$ and $\alpha \in \Gamma$, the semilattice decomposition. The radical $J(R[S])$ is the largest ideal among ideals I of $R[S]$ such that $x_\mu \in J(R[S_\mu])$ for any $x \in I$, μ is maximal elements in $supp_\Gamma(x)$.*

Lemma 3.2.9 [Kel2] *Let R and S and Q be rings and semigroups described above. Then we have*

$$J(R[S]) = R[S] \cap J(R[Q])$$

Proof. Let $x \in J(R[Q]) \cap R[S]$. Since μ is the maximal element in $supp_\Gamma(x)$. we have $x_\mu \in J(R[Q_\mu])$. Since Q_μ is group of quotient of S_μ and S_μ is cancellative. so by Theorem 2.4.1, $x_\mu \in J(R[S_\mu])$. By Lemma 3.2.8, $x \in J(R[S])$.

Conversely, if $x \in J(R[S])$, then $z = axb$ for $a, b \in (R[Q])^1$, where $a \in R[Q_\gamma]$ and $b \in R[Q_\lambda]$, for some $\gamma, \lambda \in \Gamma$. Clearly, μ is maximal elements in $supp_\Gamma(z)$. Take $t \in S_\mu$. We can note that $xt \in J(R[S])$. $(xt)_\mu \in J(R[Q_\mu])$. Therefore $z_\mu = (at)_\mu (xt)_\mu (bt)_\mu t^{-3} \in J(R[Q_\mu])$. The above lemma implies that the ideal generated by x is contained in $J(R[Q])$. □

Let S_P be the subsemigroup of all periodic elements of S. Let Γ' be the set of the elements $\alpha \in \Gamma$ such that S_α is not periodic.

Notation 3.2.10 *Let the set*

$$J(R, S_P, \Gamma') = \Big\{x = \sum_{\alpha\in\Gamma} x_\alpha \in J(R[S_P]) \mid x_\alpha \in R[(S_\alpha)_P] \text{ and}$$
$$x \text{ satisfies the condition (*): } \forall\delta \in \Gamma' : \sum_{\alpha\geq\delta} x_\alpha e_\delta \in \sum_{p\in \mathrm{P}} I(J_{1,p}(R), (S_\delta)^1_P, \xi_p)\Big\},$$

where e_δ is the identity element of $Q(S_\delta)$.

Now, we prove the main theorem by using the terminologies and techniques described in Chapter 2.

Lemma 3.2.11 *Let R be a ring such that $J_1(R) = J_\infty(R)$. Then*

$$J_1(R)[S] + J(R, S_P, \Gamma') + \sum_{p\in P} I(J_{1,p}(R), S, \xi_p)$$

is in the Jacobson radical of $R[S]$.

Proof. Let $S = \cup_{\alpha\in\Gamma} S_\alpha \subseteq \cup_{\alpha\in\Gamma} Q_\alpha = Q$, where Q_α is abelian group of fractions of S_α. $R[Q]$ is special semilattice-graded ring. By Theorem 2.5.6, we have

$$J(R[Q]) = \{a \in R[Q] : a_{\hat{\alpha}} e_\alpha \in J(R[Q_\alpha])\}.$$

Also, by Lemma 3.2.9, we have $J(R[S]) = J(R[Q]) \cap R[S]$, we have

$$J(R[S]) = \{a \in R[S] : a_{\hat{\alpha}} e_\alpha \in J(R[Q_\alpha])\}.$$

This means that $J_1(R)[S] = \sum_{\alpha\in\Gamma} J_1(R)[S_\alpha]$ and $J_1(R) = J_n(R)$ for all $n \geq 1$. For $\alpha \in \Gamma$, we have the following cases:

(a.) If $rk(S_\alpha) \geq 1$, then

$$J(R[S_\alpha]) = J_1(R)[S_\alpha] + \sum_{p\in P} I(R, S_\alpha, \xi_p).$$

Hence, $J_1(R)[S_\alpha] \subseteq J(R[S_\alpha])$.

(b.) If $rk(S_\alpha) = 0$, then we have $J_1(R) \subseteq J(R)$ and hence $J_1(R) = J_n(R)$ for $n \geq 1$. This shows that $J_1(R)[S_\alpha] \subseteq J(R[S_\alpha])$.

For all $x \in J_1(R)[S]$, we now have $x_{\hat{\alpha}} e_\alpha \in J_1(R)[Q_\alpha] \subseteq J(R[Q_\alpha])$. This proves that $J_1(R)[S] \subseteq J(R[S])$.

Lemma 3.2.6 yields $T = \sum_{p\in P} I(J_{1,p}(R), S, \xi_p) = \sum_{\alpha\in\Gamma} \sum_{p\in P} I(J_{1,p}(R), S_\alpha, \xi_p)$ and for each $\alpha \in \Gamma$, $\sum_{p\in P} I(J_{1,p}(R), S_\alpha, \xi_p) \subseteq J(R[S_\alpha])$. For $x \in T$, we have $x_{\hat{\alpha}} e_\alpha = \sum_{t\geq\alpha} x_t e_\alpha$ and $x_t e_\alpha$ lie in $\sum_{p\in P} I(J_{1,p}(R), Q_\alpha, \xi_p) \subseteq J(R[Q_\alpha])$. Hence, $T \subseteq J(R[S])$.

Now consider $x \in J(R, S_P, \Gamma')$. Then $x \in J(R[S_P])$ and x satisfies the conditions $(*)$. If $\alpha \notin \Gamma'$, for all α, then e_α is in $S_\alpha \subseteq S$. As S_α is an abelian periodic group, S_α is subgroup of S_P. Hence we have

$$x_{\hat{\alpha}} e_\alpha = (\sum_{t\geq\alpha} x_t) e_\alpha \in J(R[S_P]) \cap R[S_\alpha] \subseteq J(R[S_\alpha]) = J(R[Q_\alpha]) \cap R[S_\alpha].$$

Moreover, by the definition of $J(R, S_P, \Gamma')$, we have $x \in J(R, S_P, \Gamma')$ for $\delta \in \Gamma'$,

$$x_{\hat{\delta}} e_\delta \in \sum_{p \in \mathrm{P}} I(J_{1,p}(R), (S_\delta)^1_P, \xi_p) \subseteq I(J_{1,p}(R), Q_\delta, \xi_p) \cap R[(S_\delta)^1_P].$$

Since Q_δ is not a periodic group, its torsion-free rank ≥ 1, we have, by Theorem 3.2.3,

$$J(R[Q_\delta]) = J_1(R)[Q_\delta] + \sum_{p \in \mathrm{P}} I(J_{1,p}(R), Q_\delta, \xi_p).$$

Hence $x_{\hat{\delta}} e_\delta \in J(R[Q_\delta]) \cap R[S^1_\delta]$. This proves that $J(R, S_P, \Gamma') \subseteq J(R[S])$.

From the above containment, we have

$$J_1(R)[S] + \sum_{p \in \mathrm{P}} I(J_{1,p}(R), S, \xi_p) + J(R, S_P, \Gamma') \subseteq J(R[S]).$$

It remains to prove that the left hand side is an ideal of $R[S]$. Assume that $J_1(R) = 0$. Let $J' = J + K$, where $J = J(R, S_P, \Gamma')$ and $K = \sum_{p \in \mathrm{P}} I(0_p, S, \xi_p)$. We want to prove that J' is an ideal of $R[S]$. It suffices to show that for any $s \in S$ and $a \in J$, $as \in J'$. As $as \in J(R[S])$ for all $a \in J$. By Lemma 3.2.11, we have $(as)_{\hat{\delta}} e_\delta \in J(R[Q_\delta])$ for all $\delta \in \Gamma$.

If $s \in S_\gamma$ for $\gamma \notin \Gamma'$, then s is a periodic element. Hence, $as \in J(R[S_P])$. By above, we have for $\delta \in \Gamma'$, $(as)_{\hat{\delta}} e_\delta \in J(R[S_\delta])$ and $as \in J \subseteq J'$.

If $s \in S_\gamma$ for $\gamma \in \Gamma'$ and $a \in J$, then $a = \sum_{\alpha \geq \gamma} a_\alpha + a_2$, where $supp_\Gamma(a_2) \cap \{\alpha | \alpha \geq \gamma\} = \emptyset$. Hence, $as = (a_{\hat{\gamma}} e_\gamma)s + a_2 s$. This leads to $a_{\hat{\gamma}} s \in \sum_{p \in \mathrm{P}} I(0_p, S^1_\gamma, \xi_p)s \subseteq J'$. By induction hypothesis on $|supp_\Gamma(a)|$, we have $as \in J'$ (cf. [Jes1, Lemma 3.9]). □

We now modify the proof of Jespers [Jes1].

Theorem 3.2.12 *Let R be a ring such that $J_1(R) = J_\infty(R)$ and let S be a separative semigroup. Then*

$$J(R[S]) = J_1(R)[S] + \sum_{p \in \mathrm{P}} I(J_{1,p}(R), S, \xi_p) + J(R, S_P, \Gamma')$$

Proof. Take $x \in J(R[S])$ but $x \notin (J_1(R)[S] + \sum_{p \in \mathrm{P}} I(J_{1,p}(R), S, \xi_p) + J(R, S_P, \Gamma'))$. We may assume that $J_1(R) = 0$. Let J', J, K defined in Lemma 3.2.11. Select x with a minimun $|supp_\Gamma(x)|$. Then , by Theorem 2.4.1, we have $x_\mu \in J(R[S_\mu])$, where μ is maximal element in $supp_\Gamma(x)$. We divide the proof into the following two cases.

(i). If $\mu \in \Gamma'$ and S_μ is not periodic group, then by Theorem 3.2.3 we have

$$x_\mu \in J(R[S_\mu]) = J_n(R)[S_\mu] + \sum_{p \in \mathrm{P}} I(J_{n,p}(R), S_\mu, \xi_p)$$

with torsion free rank $n \geq 1$. Since $J_\infty(R) = J_1(R) = 0$,

$$x_\mu \in \sum_{p \in P} I(0_p, S_\mu, \xi_p) \subseteq K \subseteq J' \subseteq J(R[S]).$$

Obviously, $|supp_\Gamma(x - x_\mu)| \leq |supp_\Gamma(x)|$ and $x - x_\mu \in J(R[S])$. By the minimality of x, we have $x - x_\mu \in J'$. However, since $x_\mu \in J'$, we have $x \in J'$. This contradicts to the choice of x.

(ii). If $\mu \notin \Gamma'$ and S_μ is periodic abelian group. Then, since μ is maximal, we have

$$x_\mu \in J(R[S_\mu]) = J(R[S_P]) \cap R[S_\mu],$$

and $x_\mu e_\delta \in R[Q_{\mu\delta}]$ for all $\delta \in \Gamma$. Now take $\delta \in \Gamma'$. If $\mu > \delta$, then for all $t \in S_\mu$, we have $te_\delta \in S_\delta$. Thus

$$(x_\mu)_{\hat{\delta}} e_\delta = x_\mu e_\delta \in J(R[Q_\delta]) \cap R[(S_\delta)_P].$$

Thus, we have shown that $x_\mu \in J \subseteq J' \subseteq J(R[S])$, by the minimality of x. Hence, we obtain $x - x_\mu \in J'$ and so $x \in J'$. However, this contradicts the choice of x. Therefore,

$$J(R[S]) = J_1(R)[S] + \sum_{p \in P} I(J_{1,p}(R), S, \xi_p) + J(R, S_P, \Gamma')$$

□

The above theorem leads to some corollaries :

Corollary 3.2.13 [Jes1, Coro. 3.11] *Let R be a ring such that $J_1(R) = J_\infty(R)$ for all $n \in \mathbb{N}$ and let S be a semigroup. Then*

$$J_1(R)[S] + \sum_{p \in P} I(J_{1,p}(R), S, \xi_p) \subseteq J(R[S])$$

$$\subseteq I(R, S, \xi) + J_1(R)[S] + \sum_{p \in P} I(J_{1,p}(R), S, \xi_p) + J(R)[S_P] + \sum_{p \in P} I(J_{0,p}(R), S_P, \xi_p).$$

Moreover, if R is a Noetherian ring or satisfies a polynomial identity, then the nilness of $J(R)$ implies the nilness of $J(R[S])$. The converse statment is also true when S is periodic.

Corollary 3.2.14 [Jes1, Coro. 3.13] *Let S be torsion-free semigroup and denote $E(S)$ be set of idempotents. If R is a ring such that $J_1(R) = J_\infty(R)$, then*

$$J(R[S]) = J_1(R)[S] + \{\sum_{\alpha \in \Gamma} a_\alpha e_\alpha \in J(R)[E(S)] \mid \forall \delta \in \Gamma' \ , \ \sum_{\alpha \geq \delta} a_\alpha = 0\}$$

Corollary 3.2.15 [Jes1, Coro. 3.14] *Let S be semigroup such that each archimedean componet of S/ξ has torsion-free rank at least one. If R a ring such that $J_1(R) = J_\infty(R)$, then*

$$J(R[S]) = J_1(R)[S] + I(R, S, \xi) + \sum_{p \in \mathrm{P}} I(J_{1,p}(R), S, \xi_p).$$

3.2.3 The Nilness and Semiprimitivity of Commutative Semigroup Algebras

If R is a commutative ring with unity, then $J_1(R) = J_\infty(R)$ is always true. Now we make use of the description of Jacobson radical for algebras of commutative semigroup in Theorem 3.2.3 and Theorem 3.2.12 to determine the semiprimitvity of commutative semigroup rings.

Lemma 3.2.16 *Let S be a commutative semigroup and F a field with characteristic p (zero or prime). If $\mathcal{B}(F[S]) = 0$ then S is p-separative.*

Proof. Suppose $p \neq 0$, $\mathcal{B}(F[S]) = 0$ and $x, y \in S$ such that $x^p = y^p$. Then, in $F[S]$, $(x-y)^p = 0$. As there is no nilpotent elements, $x = y$. On the other hand, let $p = 0$ and $x, y \in S$. If $x^2 = xy = y^2$, then $(x-y)^2 = 0$. This leads to $x = y$. Therefore S must be separative. □

Theorem 3.2.17 [Mun1] *Let S be a commutative semigroup and K a field. Then*

$$J(K[S]) = \mathcal{B}(K[S]) = I(K, S, \rho)$$

where

$$\rho = \begin{cases} \text{the least separative congruence on } S & \text{if } K \text{ has characteristic } 0 \\ \text{the least } p\text{-separative congruence on } S & \text{if } K \text{ has prime characteristic } p \end{cases}$$

Proof. This is a direct consequence of Corollary 3.2.7. □

Lemma 3.2.18 [Mun2] *Let R be a commutative ring with unity and S a commutative semigroup. Let $Spec(R)$ be the set of all prime ideals of R. If $P \in Spec(R)$, let*

$$\tau_P = \begin{cases} \xi(S) & \text{if } char(R/P) = 0. \\ \xi_p(S) & \text{if } char(R/P) = p. \end{cases}$$

and $\phi_P : R[S] \to (R/P)(S/\tau_P)$ *be the natural morphism. Then*

$$\mathcal{B}(R[S]) = \bigcap_{P \in Spec(R)} ker\phi_P$$

Corollary 3.2.19 *Let* R *be an integral domain and* T *a cancellative commutative semigroup without idempotent. If* T *is p-separative, and if* $char(R) = p \in \mathbb{P}$*. Then* $J(R[T]) = 0$.

Corollary 3.2.20 *Let* R *be arbitrary commutative ring and* S *a commutative semigroup. Let* $F = R/\mathcal{B}(R)$, $T = S/\xi$*. Then the radical* $J(R[S])$ *is nil iff* $J(F[T])$ *is nil.*

Proof. The proof follows from $R[S]/I(R, S, \xi) \cong R[T]$, $R[T]/\mathcal{B}(R)[T] \cong F[T]$. □

From [Kel2], we obtain another approach to describe the Jacobson radical of semigroup ring and we can determine the nilness of $J(R[S])$ without using the fact $J(R) = J_1(R)$.

Notation 3.2.21 *Let* S *be a separative semigroup such that* $S = \cup_{\alpha\in\Gamma} S_\alpha \subseteq \cup_{\alpha\in\Gamma} Q_\alpha = Q$*. Let* $\mu \in \Gamma$ *and* Λ *be finite (or empty) subset of* $\mu\Gamma$*.*

Denote the product $x \prod_{\lambda\in\Lambda}(e_\mu - e_\lambda)$ *by* (μ, x, Λ)*. If* $\Lambda = \emptyset$*, then* $(\mu, x, \Lambda) = x$*.*

We call (μ, x, Λ) *the* ***simplest element*** *if* $xe_\alpha \in J(R[Q_\alpha])$ *for any* $\alpha \in \mu\Gamma \setminus \Lambda\Gamma$*.*

We now provide another proof of [Kel2, Th. 1].

Theorem 3.2.22 *Let* S *be a separative semigroup and* $x \in R[S]$*. Then for any maximal element* μ *in* $supp_\Gamma(x)$ *and the set of maximal elements* Λ *in the finite set*

$$\mu(supp_\Gamma(x)) \setminus \{\mu\} \qquad \text{and} \qquad y = (\mu, x_\mu, \Lambda),$$

we have

(i). $x \in J(R[S])$ *iff both* $y, x - y \in J(R[Q])$*;*

(ii). $y \in J(R[Q])$ *iff* y *is the simplest element of* $R[Q]$*.*

Proof. (i) Let $x \in J(R[S]) \subseteq J(R[Q])$ and $R[Q] = \sum_{\alpha\in\Gamma} R[Q_\alpha]$ where Q_α is the group of fractions of S_α and $S = \cup_{\alpha\in\Gamma} S_\alpha$ is a semilattice decomposition of S. Then, we can regard $R[Q]$ as a special semilattice-graded ring. Let $A = supp_\Gamma(x)$. Then, by Theorem 2.5.12, we have $x \in J_A(R[Q]) = \sum_{\beta \in A} M(A,\beta)\rho(A,\beta)$. By Theorem 2.5.10, we also know that $x_{\hat{\mu}}e_\mu \in M(A,\mu)$. If μ is maximal in $supp_\Gamma(x)$, $x_{\hat{\mu}} = x_\mu$, then $x_\mu e_\mu = x_\mu \in M(A,\mu)$. It suffices to show that $y = x_\mu \rho(A,\mu) = (\mu, x_\mu, \Lambda)$. We can check that, by the definition of Λ, $\prod_{\lambda\in\Lambda}(e_\mu - e_\lambda) = \rho(A,\mu)$ because all $\lambda \in \Lambda \Leftrightarrow \mu \succ_A \lambda$. Thus $y = (\mu, x_\mu, \Lambda) \in J(R[Q])$. The converse of (i) is clear.

(ii) If $y = (\mu, x_\mu, \Lambda)$ is the simplest element of $R[Q]$, then we can observe that $supp_\Gamma(x) \subseteq \mu\Gamma$. Since $\alpha \in \mu\Gamma \setminus \Lambda\Gamma$, $y_{\hat{\alpha}} = x_\mu$ and so we have

$$y_{\hat{\alpha}} e_\alpha = x_\mu e_\alpha \in J(R[Q_\alpha]).$$

As $\beta \in \Lambda\Gamma$, we have $y_{\hat{\beta}} e_\beta = 0$. Thus by Theorem 2.5.6, we have $y \in J(R[Q])$.

Conversely, let $y \in J(R[Q])$, so that $y = (\mu, x_\mu, \Lambda)$. For $\alpha \in \mu\Gamma \setminus \Lambda\Gamma$, α is a maximal element of ye_α. By Theorem 2.4.1, $xe_\alpha = (ye_\alpha)_\alpha \in J(R[Q_\alpha])$. Thus $(ye_\alpha)_\alpha = xe_\alpha \in J(R[Q_\alpha])$. If $\alpha \in \Lambda\Gamma$, then $\alpha \leq \mu$. This implies that

$$ye_\alpha = x_\mu(\prod_{\alpha\in\Lambda}(e_\mu - e_\alpha)e_\alpha) = x_\mu(e_\alpha - e_\alpha) = 0.$$

Hence, y is the simplest element of $R[Q]$. □

We now follow [Kel2] by using matrix approach, to represent the commutative semigroups. By applying these results, we study the structure of the Jacobson radical of $R[S]$, where $J(R) \neq \mathcal{B}(R)$ in R.

Let $T = S/\xi$ where ξ is the least congruence on the semigroup T. Let G be a finite subgroup of a semigroup T, e the identity of G. Take a finite set of idemopents E of T. Let I be an ideal generated by E but not containing G. Write down all the subgroups $H_1, H_2, \cdots, H_n$ of G such that $H_i = \{h \in G | ht_i = et_i\}$ for a non-periodic element $t_i \in GT \setminus I$. Let $G = \{g_1, \cdots, g_m\}$. The matrix of the conjugacy relation of G by H_i is the $(m \times m)$-matrix $D_i = [d_{jk}]$ such that

$$d_{jk} = \begin{cases} 1 & \text{when } g_j \in H_i g_k \\ 0 & \text{otherwise} \end{cases}$$

Write $D_I(G) = [D_1 | \cdots | D_n]$

If $n = 0$ such that there is not any non-periodic element in $GT \setminus I$, then $D_I(G) = [0]$.

Definition 3.2.23 *For a ring R, we let $\pi(R)$ be the set of all q such q is prime or zero and $J(R)/B(R)$ has a nonzero element with an additive period q. We say that G is q-complete in T, if q divides $|G|$ or q does not divide the determinant of an $(m \times m)$-submatrix of $D_I(G)$ for any ideal I.*

Theorem 3.2.24 [Kel2, Th. 1] **(1992)** *Let $R[S]$ be a commutative semigroup ring, ξ the least separative congruence on S, and $T = S/\xi$. Then the Jacobson radical $J(R[S])$ is nil iff for any $q \in \pi(R)$ every finite subgroup G of T is q-complete in T.*

The proof of this theorem is rather complicated and we need to use Theorem 3.2.22 to check the matrix representation of subgroup of S. The reader is referred to [Kel2] for details. The proof is omitted here.

Corollary 3.2.25 [Mun2] *If S contains no idempotent elements, then $J(R[S])$ is nil.*

3.3 Radicals of Cancellative Semigroup Algebras

After the properties concerning radicals of algebras in commutative semigroups are investigated, we want to use the similar methods to describe the radicals in algebras of non-commutative semigroups. However, it is not easy to describe the Jacobson radicals in group algebras. Our aim here and section 3.4 is:

(a) to describe the radicals (mainly Jacobson radicals and prime radicals) of some particular types of semigroups;

(b) to find the necessary and sufficent conditions for semiprimitive of algebras of non-commutative semigroups.

We first examine cancellative semigroups. Since some cancellative semigroups may be embedded in some groups. Hence, studying the relationship between groups and cancellative semigroups and also their related algebras are essential. To simplify our works, we only give characterizations for the algebras over the field K with $char(K) = 0$ or p.

3.3.1 Group of Fractions of Cancellative Semigroups

A cancellative semigroup S is called a group of right fractions iff S satisfies the right Ore conditions: For every $s, t \in S$, such that $sS \cap tS \neq \emptyset$.

Definition 3.3.1 *For arbitrary semigroup S, we consider the relation $\rho_l \subseteq S^1 \times S^1$ on S defined by*

$$(s,t) \in \rho_l \qquad \text{if for every } x \in S^1 \qquad sxS \cap txS \neq \emptyset.$$

where $s, t \in S^1$. The relation is clearly a congruence on S and ρ_l is called the left reversive congruence. Let ρ_r be the left-right dual congruence to ρ_l and let $\tau = \rho_l \cap \rho_r$. Of course, ρ_r, ρ_l, τ are all congruences on semigroup S.

Assume S is cancellative semigroup with no noncommutative free subsemigroups. Then S has a two-sided group of fractions. If S is a right Ore set, then S has a group of right fractions. However, if S is just a subsemigroup of group G, we cannot say that S has group of one-sided fractions. We can find an example in [Okn1, Example 10.13] to show that S can be embedded into a group G but cannot have a group of fractions.

First, we consider the cancellative semigroups which are embedded into some groups. The main reference of these cancellative semigroups can be referred to [Okn1, Ch. 7, 9]. We list some useful results:

Lemma 3.3.2 [Okn1, Lemma 7.5] *Let S be a subsemigroup of a group G. If H is subgroup of G and $\forall x \in G$, $\exists n \in \mathbb{N}$ such that $x^n \in H$, then the followings hold:*

(i). *If G is group of right fractions of S, i.e. $G = SS^{-1}$, then $H = (H \cap S)(H \cap S)^{-1}$.*

(ii). *If $S \cap H$ has a group of right fractions, then S has a group of right fractions and $SS^{-1} = S(S \cap H)^{-1}$.*

Definition 3.3.3 *Let T be a subsemigroup of S. Then T has finite index in S if there exists a finite subset F of S such that for every $s \in S$, there exists $f \in F$ with $sf \in T$.*

Proposition 3.3.4 [Okn1, Coro. 7.10]

(i). *Let H be a subgroup of finite index in a group G. If G is generated by semigroup S, then $S \cap H$ is a subsemigroup of finite index in S*

(ii). *Assume that T is a subsemigroup of finite index in S, and let H be a group of right fractions of T. Then S has a group of right fractions $G \supseteq H$ such that $[G : H] < \infty$.*

Proposition 3.3.5 [Okn1, Prop. 7.12] *If S is cancellative semigroup such that either one of the followings holds:*

(i). *S has a.c.c. on right ideals,*

(ii). *$K[S]$ has a finite right Goldie dimension for any field K,*

then S has a group of right fractions.

Summarize the above results in [Okn1, Ch. 7 and Ch. 9], we state following results which are useful for our further discussion.

Lemma 3.3.6 [Okn1] *Let G be the group of right fractions of S. Then*

(i). *For every right (left) ideal T of S, G is right fractions of T.*

(ii). *$\forall s \in S$, G is group of fractions of sSs.*

(iii). *If Z is a right Ore subset of $K[S]$, $\forall a_1, \cdots, a_n \in K[S]Z^{-1}$, then there exists $t \in Z$ such that $a_i t \in K[S]$.*

(iv). *For every right ideal I of $K[S]Z^{-1}$, we have $(I \cap K[S])K[S]Z^{-1} = I$.*

(v). *If Z is right Ore subset of a semigroup S, then Z is also right Ore subset of $K[S]$ and $K[S]Z^{-1} = K[SZ^{-1}]$.*

(vi). *For any right ideals $I_1 \subseteq I_2$ of $K[G]$, we have $I_1 \cap K[S] \subseteq I_2 \cap K[S]$.*

(vii). *If P is prime ideal of $K[G]$ and $K[G]/P$ is Goldie ring, then $P \cap K[S]$ is prime ideal of $K[S]$.*

(viii). *If all prime homomorphic images of $K[G]$ are Goldie rings, then $\mathcal{B}(K[S]) = \mathcal{B}(K[G]) \cap K[S]$.*

(ix). *If S subsemigroup of group G, and S generates a group G, then we have S a very large subset of G. (cf. Chapter 1, Lemma 1.3.10)*

We now examine the conditions for $K[S]$ being prime or semiprime.

Theorem 3.3.7 [Okn1] *G is group generated by S. Then,*

(i). *If $K[G]$ is prime (semiprime), then $K[S]$ is prime (semiprime, respectively).*

(ii). *Assume that G is group of right fractions of S. Then The following conditions are equivalent.*

   (a) *$K[S]$ is prime (semiprime).*

   (b) *$K[G]$ is prime (semiprime).*

(c) *G has no nontrival finite normal subgroups (char(K)=0, or char(K)=p > 0, G has no finite normal subgroups of order divided by p).*

From now on, we consider the subsemigroup which generates a group. Now, we consider all type of cancellative semigroup and make use of the congruences defined in Definition 3.3.1. We then obtain following results.

Definition 3.3.8 *Let x be a nonzero element in $K[S]$ such that $x = \lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_n s_n$. If $x = x_1 + \cdots + x_n$ where $supp_S(x_i)$ lies in different ρ_l-classes of S, then we call S ρ_l-separated iff $supp_S(x_i)S \cap supp_S(x_j)S = \emptyset$ for $i \neq j$.*

Lemma 3.3.9 [Okn3] *Assume that $axc = 0$ for some $0 \neq a, c \in K[S]$ and all $x \in S^1$. We can choose $y \in S^1$ such that if ay, cy are ρ_l-separate. Also, we can also choose $u \in S^1$ when ua, uc are ρ_r-separated. Let $uay = e_1 + e_2 + \cdots + e_q$ and $ucy = d_1 + d_2 + \cdots + d_k$, then $e_i x d_j = 0$ for every i, j and all $x \in S^1$.*

Proof. First, consider $a = \lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_n s_n$. If $(s_1, s_2) \notin \rho_l$, then there exists x_{12} such that $s_1 x_{12} S \cap s_2 x_{12} S = \emptyset$. Then let $a_2 = a x_{12} = \lambda_1 t_1 + \cdots + \lambda_n t_n$ where $t_i = s_i x_{12}$ such that $t_1 S \cap t_2 S = \emptyset$. Repeating the process in finite steps, we have $a' = ay$ for some $y \in S$ such that $a' = a_1 + \cdots + a_m$, where $supp(a_i), supp(a_j)$ are in different ρ_l-classes and

$$supp(a_i)S \cap supp(a_j)S = \emptyset.$$

Moreover, after separate a, we turn to separate cy, then there exists $y' \in S^1$ such that we have $b = af$ and $d = cf$, where let $f = yy'$ and b, d are ρ_l-separated and $bxd = 0$ for all $x \in S$. Similarly, on the right reversive congruence, ua, uc are ρ_r-separated for some $u \in S^1$. Let $uaf = e_1 + \cdots + e_q$ and $ucf = d_1 + \cdots + d_k$ and by τ-separative, we have $e_i S \cap e_j S = \emptyset$ for any i, j. Then $ex = 0$ implies $e_i x = e_j x = 0$. By left-right symmetry, the condition for all i, j, $e_i x d_j = 0$ follows. □

Lemma 3.3.10 [Okn3, Lemma 3] *Assume that T is a cancellative semigroup generated by a subset F such that F lies in a single ρ_l-class in T. Then T has a group H of right fractions.*

Theorem 3.3.11 [Okn3] *Let S be a cancellative semigroup. Then*

(i). *$K[S/\tau]$ is prime. In particular, we have $\mathcal{B}(K[S]) \subseteq I(K, S, \tau)$.*

(ii). *$K[S/\rho_l]$ is prime.*

(iii). *If* $char(K) = 0$, *then* $K[S]$ *is semiprime.*

Proof. (i) We can assume τ is trival. Suppose that $K[S]$ is not prime. Then there exist nonzero $a, c \in K[S]$ then $aS^1c = 0$. For $axc = 0$, then by Lemma 3.3.9, there are $u, y \in S$ such that $uay = e_1 + \cdots + e_n$ and $ucy = d_1 + \cdots + d_m$ where $supp(e_i)$ is in single τ-class and also $supp(d_j)$ does. Since τ is trival, then $e_i, d_j \in S$, hence $e_i x d_j$ leads to contradiction. Hence $K[S]$ is prime.

Since $I(K, S, \tau)$ is the kernel of $K[S] \to K[S/\tau]$, $\mathcal{B}(K[S]) \subseteq I(K, S, \tau)$.

(ii) Since in S/ρ_l, τ_{S/ρ_l} is trival. Thus the assertion follows from (i).

(iii) Now, if $char(K) = 0$ and suppose $\mathcal{B}(K[S]) \neq 0$, then there exists $a \in K[S]$ such that $aS^1a = 0$. Choose the minimal integer n such that the following condition is satisfied:

There exists a cancellative semigroup U and an element $0 \neq b \in K[U]$ such that $bU^1b = 0$ and $|supp(b)| = n$.

Let $T = \langle supp(b) \rangle$ and suppose that $supp(b)$ lies in a single τ_T-class of T. Then T has a group of right fractions H. We have $\mathcal{B}(K[H]) = 0$ since $char(K) = 0$. By Theorem 3.3.7, we have $\mathcal{B}(K[T]) = 0$. Since $bK[T]b = 0$, $b \in \mathcal{B}(K[T])$. This contradiction completes the proof. □

We now establish the Δ-method of semigroup algebras by extending the method of group algebras for studying the structure of cancellative semigroups.

For some $x \in S$, $\exists t \in S$ such that $xs = tx$, and S is cancellative, t is uniquely determined and denoted it by s^x. First, we define $D_S(s) = \{s^x | \, x \in S\}$. Then, let

$$\Delta(S) = \{s \in S \; : \; |D_S(s)| < \infty\}$$

It is known that if S is a group, then $\Delta(S)$ coincides with the FC-center of S (see Passman [Pas1, Ch. 4]). The general properties of $\Delta(S)$ can be found in [Okn1, Ch. 9].

We can see that $\Delta(S)$ is a right and a left Ore subset in S. Furthermore, we have $\Delta(S)^{-1}S = S\Delta(S)^{-1}$ and $\Delta(S)^{-1}\Delta(S) = \Delta(S)\Delta(S)^{-1}$, the latter being a group.

Define $\hat{S} = \Delta(S)^{-1}S$ and $\hat{\Delta} = \Delta(S\Delta(S)^{-1})$, we have the following result:

Proposition 3.3.12 [Okn1, Prop. 9.8] *Let S be a cancellative semigroup with $\Delta(S) \neq \emptyset$. Then*

(i). *$\hat{S}$ is a cancallative semigroup.*

(ii). $D_{\hat{S}}(s) = D_S(s)$ *for all* $s \in \Delta(S)$.

(iii). $\Delta(S)\Delta(S)^{-1}$ *is an FC-group,* $Z(S) \neq \emptyset$, *and* $\Delta(S)\Delta(S)^{-1} = \Delta(S)Z(S)^{-1} \subseteq \hat{\Delta}$.

(iv). $\hat{S} = SZ(S)^{-1}$.

(v). $\hat{\Delta}$ *is an FC-group.*

We have already obtain a characterization for the primeness of group algebra. (see Theorem 1.3.11). For semigroup algebras case, we have a similar theorem.

Theorem 3.3.13 [Okn1] *Let* S *be a cancellative semigroup and* $\Delta(S) \neq \emptyset$. *Let* K *be any field. For the following conditions:*

(i). $K[S]$ *is prime (semiprime).*

(ii). $K[\hat{S}]$ *is prime (semiprime).*

(iii). $Z(K[\hat{S}])$ *is prime (semiprime).*

(iv). $K[\hat{\Delta}]$ *is prime (semiprime).*

(v). $K[S \cap \hat{\Delta}]$ *is prime (semiprime).*

(vi). $K[\Delta(S)]$ *is prime (semiprime).*

(vii). $Z(K[S])$ *is prime (semiprime).*

We have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii). *In addition, if* $\hat{S}$ *is group, then all the above conditions are equivalent.*

Corollary 3.3.14 [Okn1] *The following conditions are equivalent on* S*:*

(i). $\Delta(S) \neq \emptyset$ *and* $\hat{\Delta} = \hat{S}$.

(ii). S *is a subsemigroup of an FC-group.*

(iii). $Z(S) \neq \emptyset$, *and* $SZ(S)^{-1}$ *is an FC-group.*

Moreover, $Z(K[S]) = Z(K[\Delta(S)])$.

Corollary 3.3.15 *Let* S *be a subsemigroup of an FC-group. Then* $K[S]$ *is prime iff* S *is a commutative torsion-free semigroup.*

The semigroup $\Delta(S)$ contributes a nice property that it is left and right Ore subset in S. The primeness of $K[\Delta], K[S \cap \hat{\Delta}]$ and $K[\hat{\Delta}]$ are equivalent to the fact that $\Delta, S \cap \hat{\Delta}, \hat{\Delta}$ are commutative torsion-free semigroup (by previous discussion). Similarly, the semiprimeness of any of these algebras is equivalent to the fact that the group $\hat{\Delta}$ has no p-torsion, where p is $char(K)$. In particular, if $char(K) = 0$, then $K[\Delta]$ is always semiprime.

However, it is rather difficult to describe the Jacobson radical or to determine whether $K[S]$ is semiprimitive or not, even in the case of group algebras. In the next sections, we will consider the semigroup S which can be embedded into u.p. groups, nilpotent groups and polycyclic-by-finite groups.

3.3.2 Jacobson Radical of Cancellative Semigroup Algebras

Recently Okninski [Okn4] **(1994)** obtain some results on Jacobson radical of cancellative semigroup algebras. The following results are taken from his papers.

We use the notations given in Definition 3.3.1.

Lemma 3.3.16 [Okn4, Lemma 1] *For every $t \in S$ the set*

$$S_t = \{s \in S | (t^r s, t^n) \in \rho_l \text{ for some positive integers } r, n \geq 1\}$$

is a left group-like subsemigroup of S. Also, $J(K[S]) \cap K[S_t] \subseteq J(K[S_t])$.

For the natural homomorphism $\phi : S \to S/\rho_l$, we let

$$U = \{s \in S | (sz, 1) \in \rho_l \text{ for some } z \in S\}$$

Then we can see that $U = \phi^{-1}(H)$, where H is group of units of S/ρ_l. Clearly, $S = U$ iff S is left reversive.

Lemma 3.3.17 [Okn4, Lemma 2] *Let V be the set of ρ_l-separated elements of $K[S]$ and $W = V \cap J(K[S \setminus U])$. Then*

(i). *V is a subsemigroup of the multiplicative semigroup of $K[S]$, in particular $VS, SV \subseteq V$;*

(ii). *If $b \in W$, then the ρ_l-components of b generate a finite power nilpotent semigroup, in particular, W is a nil semigroup;*

Lemma 3.3.18 [Okn4, Lemma 3] *Let $t \in S \setminus U$. Assume that $a + b - ab = 0$ for some $a \in J(K[S_t])$ and b is a quasi-inverse of a. Then $b \in K[A]$ for the subsemigroup A generated in S by $supp_S(a)$. Consequently, $J(K[P]) \cap K[T] \subseteq J(K[T])$ for any subsemigroups T, P of S_t.*

From the above lemmas, we obtain the main theorem of this section.

Theorem 3.3.19 [Okn4] *Let S be a cancellative semigroup which may not contain 1 and not left reversive. Assume that $0 \neq c \in J(K[S])$. Then there exists $s \in S$ such that*

(i). *$S^1csS^1 \subseteq W \setminus \{0\}$, where $W = V \cap J(K[S \setminus U])$.*

(ii). *If c_1 is ρ_l-component of cs and $t \in supp(c_1 s)$, then $c_1 \in J(K[S_t])$ and $S^1c_1S^1$ consists of nilpotent elements.*

(iii). *There exists a left reversive subsemigroup T of S and an elements $u \in S$ such that the natural K-linear projection f of csu onto $K[T]$ is a nonzero element of $J(K[T])$ for which T^1fT^1 consisits of nilpotent elements.*

Proof. (i) Since S is not left reversive, $S \neq U$. By Lemma 3.3.9, there exists $z \in S$ such that $cz \in V$. Then $czq \in W$ for any $q \in S \setminus U$. Choose q and $s = zq$ and hence (i) follows.

(ii) Let $cs = c_1 + \cdots c_n$ be ρ_l-components decomposition of cs. Now $t \in supp_S(c_1)$, and take $y \in supp_S(c_i)$. For every $x \in S_t$, $yx \notin S_t$ (otherwise, there exists $y \in S_t$ and that $yS \cap tS \neq \emptyset$ contradicting cs is ρ_l-separable).

Let $\pi : K[S] \to K[S_t]$ be a natural K-linear projection. Let $a \in K[S_t]$ and $cs \in W$ where W is nil semigroup of $K[S]$ by Lemma 3.3.17 (ii). Since $csa \in J(K[S])$, there exists $d \in K[S]$ such that $csa + d = csad$. Then $\pi(csa) = \pi(c_1 a) = c_1 a \in K[S_t]$ and $\pi(csad) = \pi(c_1 ad)$. Since S_t is left group-like subsemigroup of S, from [Okn1, Lemma 4.14], it follows that $\pi(c_1 ad) = c_1 a\pi(d)$. Thus

$$c_1 a + \pi(d) = \pi(csa + d) = \pi(csad) = c_1 a\pi(d).$$

This shows that $c_1 a$ is a quasi-invertible in $K[S_t]$, so $c_1 \in J(K[S_t])$. For every $x, y \in S^1$, xc_1y is a ρ_l-component of $xcsy$. Hence (ii) follows.

(iii) First, we find a subsemigroup Q of S_t where t is selected above. The projection of csu for some $u \in S$ is a nonzero element. If f is a projection of csu, then $f \in J(K[Q])$ and so Q^1fQ^1 only consists of nilpotent elements. From (ii), such that Q

exists. Select n be the minimal integer such that $|supp(f)| = n$. Let $T \subseteq Q \subseteq S_t$ be the semigroup generated by $supp(f)$. Since $t \in S \setminus U$ and by Lemma 3.3.18, $J(K[Q]) \cap K[T] \subseteq J(K[T])$. Then $f \in J(K[T])$. Suppose T is not left reversive. Then $supp(f)$ does not lie in a single ρ_T-class of T. Repeat (i), fw is ρ_T-separated, so $fw = f_1 + \cdots + f_m$, $m \geq 2$ with $supp(f_i)T \cap supp(f_j)T = \emptyset$ for $i \neq j$. Then $T^1 f_1 T^1$ consists of nilpotent elements and $f_1 \in J(K[T])$ for some $supp(f_1)$. Since the choice of f is minimal, $|supp(f_1)| = |supp(f)|$. This implies that $m = 1$, which contradicts to the assumption on T. Hence T is a left reversive semigroup. □

Corollary 3.3.20 [Okn4] *If $J(K[S]) \neq 0$ for a cancellative semigroup S, then there exists a (left and right) reversive subsemigroup P of S such that $J(K[P]) \neq 0$.*

A semigroup S is said to be u.p. (unique product) semigroup if for any nonempty finite subsets X, Y of S with $|X| + |Y| > 2$ (omit the case of which X, Y are both singletons), there exists in the set $XY = \{xy|\ x \in X, y \in Y\}$ that has an unique presention in the form xy, where $x \in X, y \in Y$. S is called t.u.p. (two unquie product) semigroup if there exists at least two elements which have unique presentations in XY.

Note that the unique product semigroup is an extension of the unique product group. The u.p. group algebras are studies in [Pas1, Ch. 13]. We now refer to [Okn1, Ch. 10] in getting the general results on algebras of u.p. semigroups. If S is u.p. semigroup, then $\mathcal{B}(K[S])$ and $\mathcal{L}(K[S])$ are always zero. However, we don't know about the Jacobson radical.

We now give some the properties of u.p. semigroups.

Theorem 3.3.21 [Okn1, Th. 10.4] *Let S be a u.p. semigroup. Then $K[S]$ is a domain (for any field). If S is t.u p. semigroup and $K[S]$ is an algebra with unity, then S is a monoid and $K[S]$ has trival units.*

Corollary 3.3.22 [Okn1, Coro. 10.5] *If S is t.u.p. semigroup, then $J(K[S]) = 0$.*

Theorem 3.3.23 [Okn1, Th. 10.6] *If semigroup S has a group of right fractions G, then the u.p. property and t.u.p. property coincide.*

Corollary 3.3.24 [Kar, Coro. 31.3] *Every submonoid of a u.p. group is a t.u.p. monoid.*

Theorem 3.3.25 [Okn4] **(1994)** *Let S be a u.p. semigroup. Then $J(K[S]) = 0$.*

Proof. If S is left reversive semigroup, then S has a right group of fractions and Corollary 3.3.22 yields S is also t.u.p. semigroup. Then $J(K[S]) = 0$. If S is not a left reversive semigroup an $J(K[S]) \neq 0$, then by the above Corollary 3.3.20, there exists a reversive subsemigroup P such that $J(K[P]) \neq 0$. However, as P is a reversive semigroup which has a group of right fractions, by Theorem 3.3.23, P is t.u.p. semigroup. This implies that $J(K[P]) = 0$, contradicts the assumption of $J(K[S]) \neq 0$. Therefore, $J(K[S]) = 0$. □

Note that the above theorem solves the Problem 23 listed in the monograph of Okninski [Okn1].

3.3.3 Subsemigroups of Polycyclic-by-Finite Groups

The general properties of polycyclic-by-finite group has been stated in chapter 1. Now, we consider the subsemigroups of polycyclic-by-finite groups. The group algebras of polycyclic-by-finite group is an important tool for studying the noetherian algebras. If S is a cancellative semigroup which can be embedded into a polycyclic-by-finite group G, then the question is : what is the relationship between $K[G]$ and $K[S]$?

Consider the subsemigroups of groups with finite index normal subgroups.

Lemma 3.3.26 *Suppose G is generated by its subsemigroup S and G has a normal subgroup N. Then $K[S] \cap gK[N]$ is a $K[S \cap N]$-module and $K[S] = \sum_{\bar{g} \in G/N} K[S] \cap gK[N]$ for some $\bar{g}$ or say, $K[S]$ has nondegenerate G/N-grading. Suppose G/N is finite group. If $J(K[S \cap N]) = 0$, then $J(K[S])^{|G/N|} = 0$, and $J(K[S]) = 0$ if $|G/N|$ is unit in K.*

Proof. We can easily see that $K[S] \cap K[N]$ is a right $K[S \cap N]$-module. Now, let $K[S](\bar{g}) = K[S] \cap gK[N]$ where $\bar{g} \in G/N$ and $K[S] = \sum K[S](\bar{g})$. Since S generates G, then there exists $0 \neq s_g \in S \cap Ng^{-1}$ such that $s_g(S \cap Ng) \subseteq S \cap N$. Therefore, the G/N-grading is non-degenerate. By Theorem 2.2.3, as G/N-graded algebra, $J(K[S])^{|G/N|} \subseteq J_{gr}(K[S]) \subseteq J(K[S \cap N]) \cdot K[S]$. Thus, if $J(K[S \cap N]) = 0$, then we have $J(K[S])^{|G/N|} = 0$. If $|G/N|$ is unit in K, then $J(K[S]) = 0$. Thus, the semiprimitivity of $K[S]$ depends on the structure of group embedded. □

Lemma 3.3.27 [Okn1] *Let H be a normal subgroup of a group G and S a submonoid of G that generates G as a group. Then the following statements hold:*

(i). *If* $K[S]$ *is right noetherian, then every* $g \in G, K[S] \cap gK[H]$ *is a noetherian right* $K[S \cap H]$*-module.*

(ii). $K[S]$ *is right noetherian iff* $K[S \cap H]$ *is right noetherian, and then* $K[S]$ *is a noetherian right* $K[S \cap H]$*-module.*

As we know, every polycyclic-by-finite group has characteristic subgroup which is finite index and poly-(infinite cyclic), (see Section 1.3). Then we have following results.

Theorem 3.3.28 *Let* S *be a submonoid of a polycyclic-by-finite group. Assume that* W *is a poly-(inifinite cyclic) normal subgroup of finite index in the group* H *generated by* S*. Then,* $J(K[S])$ *is nilpotent, and* $J(K[S]) = 0$ *if* $|H/W| \neq 0$ *in* K *or if* H *has no normal finite p-subgroup of order divisible by* p *if* $char(K) = p > 0$*.*

Proof. From [Pas1, Lemma 13.1.6, 13.1.7], W is a u.p. group and so $S \cap W$ is a u.p. semigroup. Therefore, $J(K[S \cap W]) = 0$ by Theorem 3.3.25. Since $[H : W] < \infty$, then by Theorem 3.3.26, $J(K[S])$ is nilpotent and $J(K[S]) = 0$ if $|H/W| \neq 0$ in K. If $char(K) = p$ and H has no normal p-subgroups of order divisible by p, then by Theorem 1.3.12 (cf. [Pas1, Theorem 4.2.10, 4.2.13]), $K[H]$ is semiprime. Theorem 3.3.7 yields that $K[S]$ is semiprime. Since $J(K[S])$ is nilpotent, the property $\mathcal{B}$ coincides with J. Hence, $J(K[S]) = 0$. □

Theorem 3.3.29 [Okn1] *Let* G *be a group such that every finitely generated subgroup of* G *is polycyclic-by-finite, for example, let* G *be locally free or locally finite. Assume that* S *is a subsemigroup of* G *that has no free noncommutative subsemigroups. Then*

$$J(K[S]) = \mathcal{L}(K[S]) = J(K[H]) \cap K[S],$$

where H *is subgroup generated by* S*. In particular,* $J(K[S]) = 0$ *if* $char(K) = 0$ *or* $char(K) = p > 0$ *and* H *has no p-torsion elements.*

Proof. S has no free noncommutative subsemigroups, then S has a group of right fractions. Take $a \in J(K[S])$, let F be a group generated by $supp(a)$ is polycyclic-by-finite. Since $S \cap F$ is group-like subsemigroup in S, $J(K[S]) \cap K[S \cap F] \subseteq J(K[S \cap F])$. Furthermore, F is group of fractions of $S \cap F$ and F is polycyclic-by-finite, F has a normal poly-(infinite cyclic) subgroup W with finite index. By Theorem 3.3.28, we have $J(K[S \cap F]) = J(K[F]) \cap K[S \cap F]$ which is nilpotent. Hence, $J(K[S])$ is locally nilpotent, i.e. $J(K[S]) = \mathcal{L}(K[S])$.

Let H be group generated by S. We have $J(K[H]) \cap K[S] \subseteq J(K[S])$ if we treat $S \cap H$ as group-like subsemigroup in H. We obtain that $J(K[H])$ is a nil ideal of $K[H]$. Now, retake $a \in J(K[S])$, $b \in K[H]$. Then $ab \in K[F']$ for a finitely generated subgroup F' of H. This shows that $ab \in J(K[H]) \cap K[F'] \subseteq J(K[F'])$, As $ab \in J(K[F'])$ and then ab is a nilpotent element for any arbitrary b. Thus, a is quasi-inverse in $J(K[H])$. This proves that $J(K[S]) \subseteq J(K[H])$. Thus, we conclude that $J(K[S]) = 0$ if $char(K) = 0$ or if $char(K) = p$ and H has no p-torsion elements.
□

3.3.4 Nilpotent Semigroups

The nilpotency of semigroup will be defined by the followings relations : For any $s, t, \in S$

$$\text{Let } x_0(s,t) = s; \qquad x_{n+1}(s,t) = x_n(s,t)w_{n+1}y_n(s,t)$$
$$\text{Let } y_0(s,t) = t; \qquad y_{n+1}(s,t) = y_n(s,t)w_{n+1}x_n(s,t)$$
$$\text{Let } X_n = Y_n \text{ if } x_n(s,t) = y_n(s,t)$$

for any $w_1, w_2 \cdots \in S^1$.

The semigroup is then called nilpotent of class n if S satisfies that identity $X_n = Y_n$ for any $x, y \in S$, where n is the least positive integer with this property. If w_i are taken in S only, then S is called weakly nilpotent of class n. However, if S is cancellative, the two constructions coincide. Thus, from [Okn1, Th. 7.3], we know that S is cancellative weakly nilpotent of class n iff S is subsemigroup of nilpotent group of nilpotency class n.

We give here some examples of nilpotent semigroups:

(i). A subsemigroup of a nilpotent group.

(ii). A power nilpotent semigroup, i.e. a semigroup S with zero θ such that $S^m = \theta$ for some $m \geq 1$.

(iii). An inverse semigroup S of matrix type over a nilpotent group G. i.e. an inverse completely 0-simple semigroup:

$$S = \mathcal{M}^0(G, \Lambda, \Lambda, \Delta)$$

where Δ is $\Lambda \times \Lambda$ identity matrix.

(iv). A completely 0-simple semigroup S is nilpotent iff the maximal subgroup G is nilpotent and S is an inverse semigroup. That is the case of (iii).

In order to get some general properties of cancellative weakly nilpotent semigroups, the reader is referred ot Okninski [Okn1, Ch. 7].

Theorem 3.3.30 [Okn1, Th. 7.11] *Let S be cancellative semigroup. Then S has a weakly nilpotent subsemigroup T of finite index iff S has a group of fractions that is nilpotent-by-finite.*

Sometimes, we call a cancellative semigroup S **almost nilpotent** if S has a group of fractions that is nilpotent-by-finite. We consider here the radical properties of nilpotent group algebra.

Lemma 3.3.31 [JO1, Pas1] *Let G be nilpotent group. If $char(K) = p > 0$, then $J(K[G]) = \mathcal{B}(K[G])$ and $J(K[G])$ is the K-subspace spanned by the elements $s - t$, where $s^{p^k} = t^{p^k}$ for some $k \geq 0$. If $char(K) = 0$, then $J(K[G]) = 0$.*

Proof. If G is nilpotent group, then from [Pas1, Lemma 8.4.16], it follows that

$$\mathcal{B}(K[G]) = \omega(K[G_p]) \cdot K[G]$$

where G_p is the unique locally finite normal p-subgroup of G and G_p is minimal respect to $\mathbb{O}_p(\Delta(G/G_p)) = (e)$. For G is nilpotent, then G_p is maximal normal p-subgroup of G (see [Rob, (5.2.7)]). (1)

Assume that G is finitely generated nilpotent group. Then G is a finitely generated solvable group and from [Pas3, Theorem 4.5], $J(K[G]) = N(K[G]) = J(K[\Delta^+(G)]) \cdot K[G]$. Let $\mathcal{W}(G)$ be the set of finite normal subgroup of G. Then, by [Pas1], we know that $J(K[\Delta^+(G)]) = \bigcup_{H\in\mathcal{W}(G)} J(K[H])$. Then $J(K[G]) = N(K[G]) \subseteq \mathcal{B}(K[G])$ and so $J(K[G]) = \mathcal{B}(K[G]) = \omega(K[G_p]) \cdot K[G]$ is spanned by $s - t$ where $s, t \in G_p$ with $s^{p^k} = t^{p^k} = e$, the identity of G. Hence, it suffices to show that if $s^{p^k} = t^{p^k}$ for some k, then $st^{-1} \in G_p$.

Assume $s^{p^k} = t^{p^k}$ for some $k \geq 1$. Also, we assume that G is finitely generated and that $|G_p| < \infty$. Then, $\mathbb{O}_p((G/G_p)) = (e)$ by the choice of G_p. We prove that $s \in G_pt$ by applying induction on $|G_p|$. First, we assume that $|G_p| = 1$. Because $\mathbb{O}_p(Z(G/Z(G))) = (e)$ and since $G/Z(G)$ has smaller nilpotency index c, induction

(1) $\mathbb{O}_p(G)$ denote the maximal normal p-subgroup and G_p is maximal (Sylow) p-subgroup of G. In nilpotent group, these two subgroup coincide.

on c then yields that $sZ(G) = tZ(G)$. So $s = tz$ for some $z \in Z(G)$. It follows that $z^{p^k} = e$. The hypothesis on G_p implies that the maximal p-subgroup of $Z(G)$, $Z(G)_p = (e)$. Hence $z = e$, and thus $s = t$.

Secondly, we assume that $|G_p| > 1$. Thus $|\mathbb{O}_p(Z(G))| \geq 1$ and $G_p \cap Z(G) \neq (e)$ implies that $|\mathbb{O}_p(G/Z(G)_p)| < |G_p|$. We denote the coset of $gZ(G)_p$ by $\bar{g}$, $g \in G$. By induction hypothesis, this implies that $\bar{s} \in \bar{t}\mathbb{O}_p(G/Z(G)_p) = \bar{t}G_p/Z(G)_p$. Hence, it follows that $s \in tG_p$.

Conversely, if $s = tG_p$, then by nilpotency of G, by induction hypothesis on $|G_p|$, we have $s^{p^k} = t^{p^k}$ for some k.

If G is not finitely generated, then it suffices to show that $J(K[G]) \subseteq \mathcal{B}(K[G])$. For $a \in J(K[G])$, the support of a is finite and generated a normal subgroup which is finitely generated nilpotent group H and so

$$a \in J(K[G]) \cap K[H] = J(K[H]) = \mathcal{B}(K[H]) = \mathcal{B}(K[G]) \cap K[H],$$

This shows that $a \in \mathcal{B}(K[G])$.

If $char(K) = 0$ and G is f.g nilpotent group, then we can take $W \in \mathcal{W}(G)$. Hence $J(K[W]) = 0$ and so $J(K[G]) = N(K[G]) = 0$ as required. □

Remark: If $s^{p^k} = t^{p^k}$ for some k, then we may say that $(s,t) \in \xi_p$ but we need to notice that ξ_p may not be a congruence on S.

Theorem 3.3.32 [Okn1] *Let S be a cancellative semigroup that either it is an almost nilpotent subsemigroup or is contained in a finite extension of an FC-group. Then*

$$J(K[S]) = \mathcal{B}(K[S]) = J(K[H]) \cap K[S],$$

where H is the group of fractions of S. Moreover, $J(K[S]) = 0$ if $char(K) = 0$ or if $char(K) = p > 0$ and H has no normal subgroups of order divisible by p.

The semigroup algebras of arbitrary nilpotent semigroup will be investigated in Chapter 4 with some finiteness condition. The results on 2-nilpotent semigroups are given in [JO1] **(1994)**. The following problem is not yet solved.

Problem 3.3.33 *Let S be arbitrary n-nilpotent semigroup, what is structure of $K[S]$ when K is any field with $char(K) = 0$ or $char(K) = p$?*

3.4 Radicals of Algebras of Matrix type

In this section, R is a K-algebra and $\mathcal{M}(R, I, \Lambda; P)$ is denoted by $\hat{R}$. In fact, $\hat{R}$ is an **algebra of matrix type**, where I, Λ are index sets and P is a $\Lambda \times I$ matrix over R, where each row and column contains some non-zero elements. Let $X, Y \in \hat{R}$ be the $I \times \Lambda$ matrices with $X \cdot Y = X \circ P \circ Y$, where $\circ$ is the usual matrix multiplication.

W sometimes call the above type algebra as the **Rees algebra** over R. It is particularly interesting to see the relationship between R and $\hat{R}$. This type of algebra can be generalized to the completely 0-simple semigroups because by Rees Theorem, $S \cong \mathcal{M}^0(G, I, \Lambda; P)$ (see Section 1.1), hence it is easy to see that $K_0[S] \cong \mathcal{M}(K[G], I, \Lambda; P)$ (cf. [Okn1, Lemma 5.1]).

3.4.1 Properties of Rees Algebras

We now refer [Okn1, Ch. 5] to get the results of algebras of matrix type. We now list some of the useful results for studying the radicals of these algebras.

A matrix $(a_{j,k}) \in \hat{R}$ with $a_{j,k} = r$ when $(j,k) = (i, \lambda)$ and $a_{j,k} = 0$ for $(j,k) \neq (i, \lambda)$. This matrix is denoted by (r, i, λ). Let J be a right ideal of R, for $i \in I$ and $J_{(i)} = \{(a, i, \lambda) \in \hat{R} : a \in R, \lambda \in \Lambda\}$ for any fixed i. Then $\hat{J} = \sum_{i \in I} J_{(i)}$ are the right ideals of $\hat{R}$.

Let J be an ideal of R. Define

$$\mathfrak{A}(J) = \{X \in \hat{R} : P \circ X \circ P \text{ lies over } J\}$$

Then $\mathfrak{A}(J)$ a right ideal of $\hat{R}$. Moreover we can view $\mathfrak{A} : \mathbf{R}(R) \to \mathbf{R}(\hat{R})$ as a $\wedge$-complete semilattice homomorphism. In addition, if J is a two-sided ideal of R, then $\mathfrak{A}(J) \supseteq \hat{J} = \mathcal{M}(J, I, \Lambda; P)$ and $\hat{R}\,\mathfrak{A}(J)\hat{R} \subseteq \hat{J}$.

On the other hand, if N is right ideal of $\hat{R}$, fix $(i, m) \in I \times \Lambda$, then the set

$$\tilde{N}_{(i)}^{(m)} = \{r \in R : (r, i, m) \in N\}$$

is also a right ideal of R. Let p_{mi} be the (m, i)-entry in matrix P. If p_{mi}, p_{nj} are units in R, then $\tilde{N}_{(i)}^{(m)} = \tilde{N}_{(j)}^{(n)}$. Hence let $\mathbf{T}(\hat{R})$ be the lattice of ideals of $\hat{R}$. The mapping $\mathfrak{D} : \mathbf{T}(\hat{R}) \to \mathbf{T}(R)$ defined by

$$\mathfrak{D}(N) = \tilde{N}_{(i)}^{(m)} \qquad \text{when } p_{mi} \text{ is unit in } R$$

is a well defined mapping.

In the literature [Okn1], the **Munn algebra** over R is the algebra of matrix type $\mathcal{M}(R, I, \Lambda; P)$, where row and column of P contains a unit of R. Thus, for any $i \in I, m \in \Lambda$, N is a two-sided ideal of $\hat{N}$ such that $\mathfrak{D}(N) = \tilde{N}_{(i)}^{(m)}$.

Theorem 3.4.1 [Okn1, Th. 5.12] *Let $\hat{R} = \mathcal{M}(R, I, \Lambda; P)$ be an algebra of matrix type. Let $\mathbf{T}(R)$ be the lattice of ideals of R. Then we have:*

(i). *$\mathfrak{D}$ is a complete lattice homomorphism of $\mathbf{T}(\hat{R})$ onto $\mathbf{T}(R)$.*

(ii). *$\mathfrak{A}$ is $\wedge$-complete semilattice embedding of $\mathbf{T}(R)$ into $\mathbf{T}(\hat{R})$.*

(iii). *$\mathfrak{D}\mathfrak{A}$ is the identity mapping on $\mathbf{T}(R)$.*

(iv). *$\mathfrak{A}\mathfrak{D}$ is the identity mapping on $\mathfrak{A}(\mathbf{T}(R))$.*

(v). *For any $J \in \mathbf{T}(R)$, $\mathfrak{A}(J)$ is maximal among all ideals N of $\hat{R}$ with the property $\mathfrak{D}(N) = J$.*

Now, we can see that the mappings $\mathfrak{A}, \mathfrak{D}$ established an one-to-one correspondence between the important classes of ideals in R and $\hat{R}$. As a consequence, we can give some descriptions for the prime radicals and Jacobson radicals of $\hat{R}$.

Lemma 3.4.2 [Okn1, Lemma 5.13] *Let N be a semiprime ideal of $\hat{R}$. Then there exists an ideal J of R such that $\mathfrak{A}(J) = N$.*

Proposition 3.4.3 [Okn1, Prop. 5.14] *Let $\hat{R}$ be an algebra of matrix type over R. Then the mappings $\mathfrak{A}, \mathfrak{D}$ establish an one-to-one correspondence between the sets of maximal, prime, and semiprime ideals of $\hat{R}$ and R.*

Corollary 3.4.4 [Okn1, Coro. 5.15]

$$\mathcal{B}(\hat{R}) = \mathfrak{A}(\mathcal{B}(R)) = \{X \in \hat{R} : P \circ X \circ P \text{ lies over } \mathcal{B}(R)\}$$

It should be noticed that the algebra $\mathcal{M}(R, I, \Lambda; P)$ may not have an unity even R has unity. Therefore, even $\mathfrak{D}, \mathfrak{A}$ make an one-to-one correspondence between the maximal ideals of $\hat{R}$ and R, it is not easy to see what $J(\hat{R})$ looks like.

Now, consider the Jacobson radical case, let V be a right R-module. Then V^{Λ}, the direct sum of Λ copies of V, may be regarded as $M_{|\Lambda|}(R)$-module structure. Moreover, there is an homomorphsim $\phi : \hat{R} \to M_{|\Lambda|}(R)$ defined by $\phi(X) = P \circ X$ between the algebras $\hat{R}$ and $M_{|\Lambda|}(R)$. This module can be regarded as a $\hat{R}$-module and is denoted by $V^{\Lambda}(P)$. If $v \in V^{\Lambda}(P)$, then we have $v \cdot X = v \circ P \circ X$ for $X \in \hat{R}$.

Let $V_0^{\Lambda}(P) = \{v \in V^{\Lambda}(P) : v \cdot \hat{R} = 0\}$. Then we can check that

$$ann_{\hat{R}}(V^{\Lambda}(P)/V_0^{\Lambda}(P)) = \mathfrak{A}(ann_R(V)) = \{X \in \hat{R} | P \circ X \circ P \text{ lies over } ann_R(V)\}.$$

Recall that if T is a ring with nonzero idempotents e and if V is a right faithful (irreducible) T-module, then Ve is also a faithful (irreducible) eTe-module.

Theorem 3.4.5 *Let N be an ideal of an algebra of matrix type $\hat{R}$. Then N is right primitive iff $N = \mathfrak{A}(J)$ for a right primitive ideal J of R. Moreover, in this case, the division rings associated with the right primitive rings of $\hat{R}/N$ and R/J are isomorphic. Moreover, $\mathfrak{A}(J(R)) = J(\hat{R})$.*

Proof. By Lemma 3.4.2, we know that any right primitive ideal of $\hat{R}$ must be of the form $\mathfrak{A}(J)$ for some ideal J of R. Define a mapping $\varphi : R \to R/J$ which extends an homomorphism:

$$\hat{\varphi} : \hat{R} \to \hat{R}' = \mathcal{M}(R/J, I, \Lambda; \varphi(P)),$$

where $\varphi(P)$ is matrix $(\varphi(p_{\lambda,i}))$.

Then, it can be seen that the kernel of $\hat{\varphi}$ lies in $\mathfrak{A}(J)$. Hence, without loss of generality, we may assume that $J = 0$. Then we can see that R is right primitive iff $\hat{R}/\mathfrak{A}(0)$ is right primitive. Furthermore, for the case $\mathfrak{A}(0)$, we can show that $\mathfrak{A}(0)^3 = 0$, then hence $\mathfrak{A}(0) \cap E\hat{R}E = 0$, where E is choosen idempotent for VE. This shows that VE is an irreducible module. Thus, $R \cong E\hat{R}E/(E\hat{R}E \cap \mathfrak{A}(0)) \cong E'(\hat{R}'/\mathfrak{A}(0))E'$ for an idempotent E' in $\hat{R}/\mathfrak{A}(0)$. Thus,

$$\begin{aligned} J(\hat{R}) &= \bigcap\{N | N \text{ right primitive ideal of } \hat{R}\} \\ &= \bigcap\{\mathfrak{A}(J) | J \text{ right primitive ideal of } R\} \end{aligned}$$

Hence, we have shown that $J(\hat{R}) = \mathfrak{A}(J(R))$ by the property of $\mathfrak{A}$. □

In particular, if S is completely 0-simple and $S = \mathcal{M}^0(G, I, \Lambda; P)$, then there exists a linkage between $K_0[S]$ and $K[G]$, where G is a maximal subgroup of S. Hence, we can give a description on the radicals of $K_0[S]$, where S is completely 0-simple semigroup.

Corollary 3.4.6 *Let $S = \mathcal{M}^0(G, I, \Lambda; P)$ be a completely 0-simple semigroup and K any field. If $J(K[G]) = 0$, then $J(K_0[S]) = \mathfrak{A}(0)$. Moreover, $J(K_0[S])$ is nilpotent.*

Proof. Obviously, $K_0[S] \cong \mathcal{M}(K[G], I, \Lambda; P) = \widehat{K[G]}$, by Theorem 3.4.5, hence,

$$J(K_0[S]) = \mathfrak{A}(0) = \{X \in \hat{R} : P \circ X \circ P = 0\}.$$

Thus, $J(K_0[S])^3 = 0$, and so $J(K_0[S])$ is nilpotent. □

Corollary 3.4.7 *Let K be a field with $char(K) = p$ and S a completely 0-simple semigroup isomorphic to $\mathcal{M}^0(G^0, I, \Lambda; P)$. If $\mathcal{B}(K[G]) = J(K[G])$ (e.g G is niloptent-by-finite or finite extension of FC group), then $J(K_0[S]) = \mathcal{B}(K_0[S])$.*

Proof. In Lemma 3.3.31 and Theorem 3.3.32, we know that if G is niloptent-by-finite or FC-by-finite, then $J(K[G]) = \mathcal{B}(K[G])$. Since $K_0[S] \cong \mathcal{M}(K[G], I, \Lambda; P) = \widehat{K[G]}$, we have

$$J(K_0[S]) = \mathfrak{A}(J(K[G]) = \mathfrak{A}(\mathcal{B}(K[G]) = \mathcal{B}(K_0[S]).$$

□

3.4.2 Algebras Graded by Elementary Rees Matrix Semigroups

Let $S = \mathcal{M}^0(1^0, I, \Lambda; P)$ be an elememtary Rees matrix semigroup. In this section, our aim is to show that every Rees matrix algebras $\hat{R}$ have a S-grading. This provides another approach to study the structure of Munn algebras. Also, the method of S-grading can be applied to other algebras.

Let S be an 0-rectangular band. For any $(1, i, m) \in S$, define

$$\begin{cases} (1,i,m)(1,j,n) = (1,i,n) & \text{if } p_{m,j} = 1 \\ (1,i,m)(1,j,n) = \theta & \text{if } p_{m,j} = 0 \end{cases}$$

If we put $\hat{R}_{im} = \hat{R}_{(i)}^{(m)}$ and $\hat{R}_\theta = 0$, then $\hat{R} = \sum_{(1,i,m)\in S} \hat{R}_{im}$. This means that $\hat{R}$ is a contracted S-graded algebra.

For elementary Rees matrix semigroups, we have

Theorem 3.4.8 [CJ] **(1994)** *Let $S = \mathcal{M}^0(1^0, I, \Lambda; P)$ be an elementary Rees matrix semigroup. Let R be a contracted S-graded ring. Let $\mathcal{H}$ be Jacobson, prime, Levitzki radicals respectively. Let*

$$T = \{x \in R : RxR \subseteq \sum_{i,\lambda} \mathcal{H}(R_{i\lambda})\}.$$

Then $\mathcal{H}(R) = T$.

Proof. By Corollary 2.5.4, we know that the Jacobson and Levitzki radical can be determined by the components of band-sum. Moreover, since S is a rectangular band only, $\mathcal{B}$ can be also determined by this sum. Hence, $\mathcal{H}(R)_{\hat{b}} \cap R_b \subseteq \mathcal{H}(R_b)$. Since S is an 0-rectangular band, we have $\mathcal{H}(R)_{\hat{b}} = \mathcal{H}(R)_b$.

In case, if T is an ideal, then we can see that for any $x \in T \cap R_b$, $RxR \subseteq \sum_{b\in S}\mathcal{H}(R_b)$, so $R_bxR_b \subseteq \mathcal{H}(R_b)$. With $\mathcal{H}(R,S) = \mathcal{H}(R)$, we have $T \subseteq \mathcal{H}(R)$.

Conversely, for any $x \in \mathcal{H}(R)$, we have $RxR \subseteq \mathcal{H}(R)$. Hence, for each $b \in S$, $(RxR)_b \subseteq \mathcal{H}(R_b)$, This implies that $RxR \subseteq \sum \mathcal{H}(R_b)$ and hence $\mathcal{H}(R) \subseteq T$. □

Here is another description of the Jacobson radical of algebra of complete 0-simple semigroup similar to Corollary 3.4.7.

Corollary 3.4.9 *Let $S = \mathcal{M}^0(G, I, \Lambda; P)$ be a completely 0-simple semigroup. If $J(K[G]) = \mathcal{L}(K[G])$ (or $J(K[G] = \mathcal{B}(K[G]))$, then*

$$J(K_0[S]) = \mathcal{L}(K_0[S]) \qquad (\text{ or } J(K_0[S]) = \mathcal{B}(K_0[S])).$$

Proof. From the above arguments, we have $K_0[S] = \mathcal{M}(K[G], I, \Lambda; P)$. Let $T = \mathcal{M}^0(1^0, I, \Lambda; P')$ be an elementary Rees matrix semigroup. Then, we can see that $R = K_0[S]$ is a T-graded algebra. Let $(1, i, \lambda) \in T$ and write $R_{i\lambda} = (K[G], i, \lambda)$ (set of (a, i, λ), where $a \in K[G]$). Then we have

$$J(R_{i\lambda}) \cong J(K[G]) = \mathcal{L}(K[G]) \cong \mathcal{L}(R_{i\lambda}).$$

□

Corollary 3.4.10 *Let T be a subsemigroup of a completely 0-simple semigroup S. Then*

$$J(K_0[T])^3 \subseteq \sum_{\substack{i\in I\\ \lambda\in\Lambda}} J(T_{i\lambda}),$$

where $T_{i\lambda}$ is the row and column of T in $S \cong \mathcal{M}^0(G^0, I, \Lambda; P)$.

Corollary 3.4.11 *Consider $S \cong \mathcal{M}^0(G, I, \Lambda; P)$ and let S' be its corresponding elementary Rees matrix semigroup. Let R be a S-graded ring. Denote $R'_{i\lambda} = \sum_{g\in G} R_{(g,i,\lambda)}$ for $i, \in I, \lambda \in \Lambda$. If G is a group satisfying any condition of Theorem 2.3.5, then*

$$J(R)^3 \subseteq \sum_{(1,i,\lambda)\in S'_1} J_{gr}(R'_{i\lambda}) + \sum_{(1,i,\lambda)\in S'_2} R'_{i\lambda}$$

where $S'_1 = \{(1, i, \lambda) : p'_{\lambda,i} = 1\}$ and $S'_2 = \{(1, j, \mu) : p'_{\mu,j} = 0\}$. Hence, $J(R)^3$ is S'-homogeneous.

Proof. Since R can be regarded as a S'-graded ring with each component $R'_{i\lambda}$ a G-graded algebra. Then in Theorem 2.3.5 $J(R'_{i\lambda}) = J_{gr}(R'_{i\lambda})$ for $(1, i, \lambda) \in S'_1$. If $(1, j, \mu) \in S'_2$, then $(R'_{j\mu})^2 = 0$. Thus, $J(R'_{j\mu}) = R'_{j\mu}$. By Theorem 3.4.8, we also have

$$J(R)^3 \subseteq \sum_{(1,i,\lambda)} J(R_{i\lambda}) = \sum_{(1,i,\lambda)\in S'_1} J_{gr}(R'_{i\lambda}) + \sum_{(1,i,\lambda)\in S'_2} R'_{i\lambda}.$$

Hence, $J(R)^3$ is a S'-homogeneous ideal. □

Referring to the properties of graded rings, we have the following lemma:

Lemma 3.4.12 [CJ, Lemma 18] *Let I be an ideal of a ring R such that $J(I)$ and $J(R/I)$ are both locally nilpotent. Then $J(R)$ is locally nilpotent.*

We prove the main theorem of this section.

Theorem 3.4.13 [CJ] **(1994)** *Let S be a locally finite semigroup and R a contracted S-graded ring. If $J(R_e)$ is locally nilpotent, then $J(R)$ is also locally nilpotent.*

Proof. If S does not contain a zero element, then we may adjoin it with the zero θ and put $R_\theta = 0$. Otherwise, θ is an idempotent of S and obviously $J(R_\theta) = 0$, $J(R_\theta)$ is locally nilpotent. Now, R becomes a contracted S^0-graded algebra. Hence, $J(R)$ is locally nilpotent iff $J(R) \cong J(R/R_\theta)$ is locally nilpotent. Therefore, we may assume that R itself is a contracted S-graded ring with S containing zero.

Let $X = \{a_1, \cdots, a_n\}$ be a finite subset of $J(R)$ and A a subring generated by X. Let $T = \cup_{i=1}^{n} supp_S(a_i)$ and let B be the subsemigroup of S generated by T. Since S is locally finite, B is finite, so A is subring with finite support.

Now, we show that the order of the subsemigroup generated by the support of A^k for all k is strictly smaller that $|B|$.

Suppose that I is an the ideal generated by T. Then, we have $B \subseteq I$. Since I is finitely generated, the set of ideals is strictly contained in I, thus it contains some maximal elements. Let M be the maximal ideal contained in I. Then I/M is either 0-simple or null.

By our construction, we can see that $A \subseteq R_T \subseteq R_I$ and $A \subseteq J(R)$, so $A \subseteq R_I \cap J(R) = J(R_I)$ since R_I is ideal. This shows that R_I/R_M is a contracted I/M-graded ring. Since $A \subseteq J(R_I)$, the image of A, namely $\bar{A}$, is a finitely generated subring of $J(R_I/R_M)$.

If I/M is a null semigroup then $(R_I/R_M)^2 = 0$. This implies that $J(R_I/R_M)$ is nilpotent.

If I/M is a 0-simple semigroup, then by the locally finiteness of S, we have I/M is completely 0-simple semigroup and is isomorphic to $\mathcal{M}^0(G, I, \Lambda; P)$ for some group G which is locally finite (see Proposition 1.1.6).

Recall that G is locally finite, then by Theorem 2.2.8 (ii), we know that if $\tilde{R}$ is a G-graded ring, then $\mathcal{L}_{gr}(\tilde{R}) = \mathcal{L}_{ref}(\tilde{R})$. Moreover, if $J(\tilde{R}_e)$ is locally nilpotent, then $J(\tilde{R}_e) = \mathcal{L}(\tilde{R}_e)$ and $J_{gr}(\tilde{R}) \cap \tilde{R}_e = J(\tilde{R}_e) = L(\tilde{R}_e) = \mathcal{L}_{gr}(\tilde{R}) \cap \tilde{R}_e$. These facts lead to

$$(\mathcal{L}(\tilde{R}))_{gr} = \mathcal{L}_{gr}(\tilde{R}) = \mathcal{L}_{ref}(\tilde{R}) = J_{ref}(\tilde{R}) = J_{gr}(\tilde{R}).$$

Hence $J_{gr}(\tilde{R})$ is locally nilpotent as well under the above conditions. Let I be a finitely generated subring of $J(\tilde{R})$ with generator set X. Let H be a subgroup generated by X. Since G is locally finite, H is finite group. Thus, $J(R_H)^{|n|} \subseteq J_{gr}(\tilde{R}_H)$, where $n = |H|$. Hence, $I \subseteq J(\tilde{R}_H)$ is nilpotent. Therefore, $J(\tilde{R})$ is locally nilpotent.

Now, let $\bar{R}$ be a contracted algebra graded by $\mathcal{M}^0(G, I, \Lambda; P)$ with G is locally finite and $\bar{R}_{ij}$ is G-graded so that $J(\bar{R}_{ij}) = \mathcal{L}(\bar{R}_{ij})$ as above. This means that $\bar{R}$ can be now graded by an elementary Rees matirix semigroup and each of its component is locally nilpotent. Hence, $J(\bar{R})$ is locally nilpotent due to Theorem 3.4.8. Thus, $J(\bar{R}) = \mathcal{L}(\bar{R})$. Hence, $\bar{R} = R_I/R_M$ and $J(R_I/R_M)$ is locally nilpotent.

Moreover, if $\bar{A}$ is finitely generated subring in $J(R_I/R_M)$, then there exists an integer $k \geq$ such that $\bar{A}^k = 0$ or $A^k \subseteq R_M$. Hence $A^k \subseteq R_{M \cap B}$. However, we cannot obtain $B \not\subseteq M$ and in fact $B \cap M$ contains the support of A^k. Thus, the subsemigroup generated by the support of A^k is strictly smaller than the support of B. By induction hypothesis, we know that A nilpotent, and consequently $J(R)$ is locally nilpotent. □

We now extened the above case to locally finite semigroup algebras.

Corollary 3.4.14 *Let S be any locally finite semigroup. Then $J(K[S]) = \mathcal{L}(K[S])$.*

3.5 Radicals of Inverse Semigroup Algebras

If S is an inverse semigroup, then the maximal subgroups are the subgroups

$$H_e = \{s \in S | ss^{-1} = s^{-1}s = e\},$$

where $e \in E(S)$ which is a semilattice of idempotents of S. The semigroup S is called **combinatorial** iff each of its maximal subgroup is trival. A inverse semigroup is called **Clifford semigroup** if every idempotent of S is central. An inverse semigroup

S is completely 0-simple and is isomorphic to a **Brandt semigroup**, i.e. $S \cong \mathcal{M}(G, \Lambda, \Lambda; \Delta)$, where Δ is the identity matrix. Moreover, each principal factor of an inverse semigroup is of course 0-simple.

3.5.1 Properties of Inverse Semigroup Algebras

In this part, we assume that all semigroups S are inverse semigroups. From the previous sections, we immedicately have the following result.

Lemma 3.5.1 *Let K be any field, and S an inverse completely 0-simple semigroup. If S contains finitely many idemoptents, then I is finite and $K_0[S] \cong M_{|I|}(K[G])$.*

Now we revise some results of Munn [Mun3, Mun4, Mun6] **(1986,1987,1992)**, concerning the right nil ideals of inverse semigroup algebras.

Let S be an inverse semigroup and $e \in E(S)$. Let H_e be group of units of eSe. Notice that H_e also the $\mathcal{H}$-class of e in S. Denote the right units subsemigroup of eSe by P_e, that is

$$P_e = \{x \in eSe \mid xy = e \text{ for some } y \in eSe\}.$$

Note that $P_e \supseteq H_e$ and $P_e = R_e \cap eSe$, where R_e is $\mathcal{R}$-class containing e in S. For a prime number p, $x \in H_e$, is called a p-element iff x has order p^r for some $r \in \mathbb{N}$.

Lemma 3.5.2 [Mun4, Lemma 1] *Let S be a semigroup, K a field with $char(K) = p$ and $q = p^r$ for some $r \in \mathbb{N}$. Then $x \in K[S]$, also for any elements $x_1, x_2, \cdots, x_n \in S$ and $\alpha_1, \alpha_2, \cdots \alpha_n \in K$, we have*

$$\left(\sum_{i=1}^{n} \alpha_i x_i\right)^q = \sum_{i=1}^{n} \alpha_i^q x_i^q + c,$$

where c is a linear combination of the elements of the form $uv - vu$, with $u, v \in \langle x_1, \cdots, x_n\rangle$.

We need some technical lemmas for studying the inverse semigroup algebras.

Lemma 3.5.3 [Mun3, Lemma 5.1] *Let S be an inverse semigroup and T a nonempty finite subset of S. Let e be maximal in $\{xx^{-1} : x \in T\}$. Then for all $x, y \in T$*

$$xy^{-1} = e \qquad \Longrightarrow \qquad x = y.$$

Lemma 3.5.4 [Mun4, Lemma 2] *Let S be an inverse semigroup containing an idempotent e and K a field with char$(K) = p$. Let a be a nilpotent element of $K[S]$ such that $e \in supp(a) \subseteq eSe$. Then either (i) $supp(a) \cap H_e$ contains a p-element or (ii) $supp(a) \cap (P_e \setminus H_e) \neq \emptyset$.*

Lemma 3.5.5 [Mun6] *Let S be an inverse semigroup, K a field and A a nonzero right ideal. Then there exist $e \in E(S)$ and $a \in A$ such that*

$$e \in supp(ea) \subseteq H_e \cup (eSe \setminus P_e).$$

Proof. Choose $b \in A \setminus 0$ and let $x_1, \cdots x_n$ be the elements of $supp(b)$. Let e be maximal element in $T = \{x_1x_1^{-1}, \cdots, x_nx_n^{-1}\}$. Then $e = xx^{-1}$ for some $x \in supp(b)$. Without loss of generality, let

$$e = x_jx_j^{-1} \qquad \text{for } 1 \leq j \leq k \qquad \text{and for } k \leq n.$$

Then let $T' = \{x_1^{-1}x_1, x_2^{-1}x_2, \cdots, x_k^{-1}x_k\}$. Denote the minimal element in T' by f. Without loss of generality, let $f = x_1^{-1}x_1$. Take $a = bx_1^{-1}$. Since A is a right ideal of $K[S]$, $a \in A$. We now check that ea has the corresponding properties.

Consider the element ea. Let $y_i = ex_ix_1^{-1}$ for $i = 1, \cdots n$. Then $supp(ea) \subseteq \{y_i\}_{i=1,\cdots,n}$. First of all $y_1 = e$ and $y_i = ey_i = ex_ix_1^{-1}x_1x_1^{-1} = ey_ie$, hence $supp(a) \subseteq eSe$.

Suppose $y_k = ex_kx_1^{-1} = e$ for $k > 1$. Since $y_k = (ex_k)(ex_1)^{-1}$ and e is also the maximal element of the set eT. By Lemma 3.5.3, we have $ex_k = ex_1 = x_1$. Hence $ex_kx_k^{-1} = (ex_k)(ex_k)^{-1} = x_1x_1^{-1} = e$, that is $e \leq x_kx_k^{-1}$. This shows that $e = x_kx_k^{-1}$ and $x_k = ex_k = x_1$. Hence if $y_k = e$ then $k = 1$. Therefore,

$$e \in supp(ea) \subseteq eSe$$

To prove the second inclusion, it suffices to show that for $y \in supp(ea)$, and $yy^{-1} = e$ iff $y^{-1}y = e$. We know that $x_jx_j^{-1} = e$ for $j = 1, \cdots k$. For $j = 1, \cdots k$, if $y_jy_j^{-1} = ex_jx_1^{-1}x_1x_j^{-1}e = e$. Similar to above, we have

$$y_j^{-1}y_j = x_jx_1^{-1}x_1ex_j^{-1} = e.$$

By some rearrangement, we have $x_1^{-1}x_1 \geq x_j^{-1}x_j$. By the choice of x_1 (minimal in $x_i^{-1}x_i$.), we have $x_j^{-1}x_j = x_1^{-1}x_1^{-1}$. Then $y_jy_j^{-1} = e$ and $y_j^{-1}y_j = x_1x_j^{-1}x_jx_1^{-1} = x_1x_1^{-1} = e$. Then $y_j \in H_e$.

For $j > k$, $y_j = ex_jx_1^{-1}$. If $y_jy_j^{-1} = e$, then we can also show that $(ex_je)(ex_je)^{-1} = e$, whence $e \leq x_jx_j^{-1}$. This contradicts to $e > x_jx_j^{-1}$. Hence $y_jy_j^{-1} \neq e$ for $j > k$.

Thus, $e \subseteq \{y_i\}_{i=1,\cdots,n} \subseteq H_e \cup ((eSe) \setminus P_e)$. $\square$

By the above lemma, we obtain the following modified results of Munn [Mun4].

Lemma 3.5.6 *Let S be an inverse semigroup. Let K be a field and A a nonzero ideal of $K[S]$. Then there exists $e \in E(S)$ and $a \in A \setminus 0$ such that $e \in supp(a) \subseteq eSe$ and $supp(a) \cap (P_e \setminus H_e) = \emptyset$.*

Consider the algebra over field K with $char(K) = 0$. We have the following lemma.

Lemma 3.5.7 [Mun6] *Let S be an inverse semigroup and K a field of characteristic 0. Let b be a nonzero nilpotent element of $K[S]$. Then there is an infinite set $\mathcal{P}_b$ of prime numbers such that for each $p \in \mathcal{P}_b$, there exists a field F_p of characteristic p and a nonzero nilpotent element c of $F_p[S]$ such that $supp(c) = supp(b)$.*

Hence, we obtain the following Munn's theorem for $K[S]$ which is analogous to the group case.

Theorem 3.5.8 [Mun6] *Let S be an inverse semigroup and K be a field of characteristics 0 or prime that is not the order of an element in a subgroup of S. Then $K[S]$ has no nonzero nil right ideals.*

Proof. Suppose that $K[S]$ has a nonzero nil right ideal A. Then, by Lemma 3.5.5, there exist $e = e^2$ and $a \in A$ such that

$$e \in supp(ea) \subseteq H_e \cup (eSe \setminus P_e).$$

Since A is nil, there exists a positive integer k such $a^k = 0$. Now, $supp(ea) \subseteq eSe$ and

$$supp((ea)^r) \subseteq [supp(ea)]^r \subseteq eSe.$$

Hence, by induction, $(ea)^r = ea^r$ for all positive intergers r and so, in particular, $(ea)^k = ea^k = 0$. Write $b = ea$, so b is nilpotent. Thus, we have

$$e \in supp(b) \subseteq eSe, \qquad supp(b) \cap (P_e \setminus H_e) = \emptyset.$$

First, if $char(K) = p$. Then, since b is nilpotent, by Lemma 3.5.4 $supp(b) \cap H_e$ contains p element. Thus H_e contains an element of order p, contrary to our hypothesis. The result therefore holds in the prime characteristics case.

On the other hand, if $char(K) = 0$ and $\mathcal{P}_b$ is defined in Lemma 3.5.7. Let $p \in \mathcal{P}_b$. Then there exists a field F_p such that $F_p[S]$ contains some nonzero nilpotent elements c and $supp(c) = supp(b)$. Applying relation in Lemma 3.5.5, we know that $supp(c) \cap H_e$ contains a p-element. Thus $supp(b) \cap H_e$ contains p-element for all $p \in \mathcal{P}_b$. However, $\mathcal{P}_b$ is an infinte set, which contracdicts to the finiteness of $supp(b)$. Hence, $b = 0$ and $a = 0$. □

3.5.2 Radical of Algebras of Clifford Semigroups

It is well known that a Clifford semigroup is a semilattice of groups. Let S be such type of semigroup, i.e. $S = \cup_{\alpha \in \Gamma} G_\alpha$, where Γ is a semilattice and all G_α's are groups. Also, we have the structure homomorphism $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ with $\alpha \geq \beta$. Consider $K[S]$, by additivity, $K[S] = \sum_{\alpha \in \Gamma} K[G_\alpha]$. Then $K[S]$ is a Γ-graded algebra, with $R = K[S]$ and $R_\alpha = K[G_\alpha]$. Moreover, in each $\alpha \in \Gamma$ and R_α is nonzero and has a unity $1_\alpha = 1 \cdot e_\alpha$, where e_α is identity of G_α. We can easily see that $1_\alpha 1_\beta = 1_{\alpha\beta}$. By Section 2.5, $K[S]$ is known as a special Γ-graded algebra. We now use the results of Section 2.2.5 to describe the radicals of Clifford semigroups.

Theorem 3.5.9 *Let $S = \cup_{\alpha \in \Gamma} G_\alpha$ be a Clifford semigroup and $E(S)$ the set of all idempotents of S is pseudofinite. Then $J(K[S]) = 0$ iff for every $\alpha \in \Gamma$, $J(K[G_\alpha]) = 0$.*

Proof. This theorem is a direct consequence of Corollary 2.5.13. □

Theorem 3.5.10 *Let $K[S]$ be semigroup algebra of Clifford semigroup. If each principal ideal of Γ is finite, then*

$$J(K[S]) = \sum_{\alpha \in \Gamma} J(K[G_\alpha])\rho(\alpha)$$

Proof. This is a direct consequence of Corollary 2.5.14. □

Note: When $E(S)$ is pseudofinite, $\rho(\alpha) = \prod_{\substack{\mu \in E(S) \\ \alpha \succ \mu}} (e_\alpha - e_\mu)$ is defined in Section 2.2.5.

Therefore, the semiprimitivity of algebras of this type of semigroup depends on the semiprimitivity of group algebras. Recently, Passman has given a survey on the semiprimitivity of group algebras and make his focus on locally finite groups. Hence, we can refer to his results on group algebras and apply them to locally finite Clifford semigroups.

3.5.3 Semiprimitivity Problems of Inverse Semigroup Algebras

In this section, we introduce an important theorem for inverse semigroup algebras. These materials are mainly taken from Domanov (1976) (stated in [Mun3]). We first give the main results of this section.

Theorem 3.5.11 [Mun3] *Let S be an inverse semigroup and K a field. If $J(K[H_e]) = 0$ for each maximal subgroup H_e of S, where $e \in E(S)$, then $J(K[S]) = 0$.*

In order to prove this theorem, we need to examine the structure of the inverse semigroup S and its subgroup H_e. Let $e \in E(S)$ and D be the $\mathcal{D}$-class of S containing e. Consider a right $K[H_e]$-module W. For each $f \in E(D)$, let V_f be an isomorphic copy of W. Since there exists an isomorphism $\phi_{e,f} : H_e \to H_f$, we can define $\theta_f : W \to V_f$ as a module isomorphism. In this way, V_f can then be regarded as a $K[H_f]$-module.

Let $V = \bigoplus_{f \in E(D)} V_f$. Then we perform the construction as followings:

Suppose V is constructed above and let $x \in S$. Define for each $f \in E(D)$ and $v \in V_f$,

$$v \cdot x = \begin{cases} v\theta_f^{-1}(r_f x r_g^{-1})\theta_g & \text{if } f \leq xx^{-1} \text{ and } g = x^{-1}fx, \\ 0 & \text{if } f \not\leq xx^{-1} \end{cases}$$

Then, it is not difficult to check that V is a well defined right $K[S]$-module. As this module V depends on e and W, we just denote it by $V(e, W)$.

From [Mun3, Lemma 4.6], it is shown that if W is an irreducible $K[H_e]$-module then $V(e, W)$ is irreducible $K[S]$-module.

Let $\mathcal{M}_e$ be the family of $K[H_e]$-modules. Then $\mathcal{M}_e$ is faithful iff the intersection of the annihilators of all the modules in the family $\mathcal{M}_e$ is zero. Now define

$$\mathcal{M} = \{V(e, W)\}_{W \in \mathcal{M}_e, e \in T}$$

where T is a subset of $E(S)$ with exactly one element from each $\mathcal{D}$-class. Thus, by a result in [Mun3, Lemma 4.7], it yields that $\mathcal{M}$ is a faithful family of right $K[S]$-modules.

We now turn to the proof of Theorem 3.5.11.

Suppose that $K[H_e]$ is semiprimitive for all $e \in E(S)$ and T is constructed above. Then there exists a faithful family $\mathcal{M}_e$ of irreducible $K[H_e]$ modules. For each $W \in \mathcal{M}_e$, form a $K[S]$-module $V(e, W)$. From the above construction, it can be easily seen $V(e, W)$ is irreducible and $\mathcal{M}$ is faithful. Hence $K[S]$ is semiprimitive. The proof is completed. □

Corollary 3.5.12 *Let S be an inverse semigroup and K a field that is not algebraic over its prime subfield. Suppose, if $char(K) = p > 0$ and no subgroup of S has an element of order p, then $J(K[S]) = 0$.*

Corollary 3.5.13 *Let S be a combinatorial inverse semigroup and let K be a field. Then $J(K[S]) = 0$.*

However, the converse of Theorem 3.5.11 does not hold. We can find an example in [Mun3, Example 4.10] which shows that $J([K[S]) = 0$ but $J(K[G]) \neq 0$ for some maximal subgroup of S. On the other hand, we give here some conditions for $E(S)$ which make the converse of Theorem 3.5.11 true. In the following, we use an example given by Ponizovskii to verify it in a different way.

We describe the example obtained by Ponizovskii as follows:

Theorem 3.5.14 [Pon1] **(1990)** *Let S be an inverse semigroup. The following conditions are equivalent.*

(i). *$E(S)$ is a pseudofinite semilattice of idempotents*

(ii). *If $J(K[S]) = 0$, then for each maximal subgroup G of S, $J(K[G]) = 0$.*

Proof. Munn [Mun5] showed that (i) implies (ii). Hence we have to prove that (ii) implies (i). we construct an example to show that $E = E(S)$ is a non-pseudofinite semilattice and $J(K[S]) = 0$ but there are subgroup G such that $J(K[G]) \neq 0$.

Let K be field with $char(K) = p$. Construct S as a semilattice of groups $(E, G_\alpha, \phi_{\alpha,\beta})$. First, without loss of generality, take $\alpha \in E$ which is maximal. Let G_α be a proper subgroup of G with $J(K[G_\alpha]) \neq 0$ but $J(K[G]) = 0$. For example, $G = \mathcal{S}_\infty$, an infinite locally finite symmetric group and $\mathcal{S}_\infty$ has subgroup G_α with order p when $char(K) = p$. If $\beta < \alpha$, let $G_\beta = G$ and $\phi_{\alpha,\beta}$ is an inclusion map. If $\beta \not\leq \alpha$, then $G_\beta = \{e\}$. Moreover, define

$$\begin{cases} \gamma < \beta < \alpha & \phi_{\beta,\gamma} = \iota \text{ identity mapping} \\ \gamma < \beta \not\leq \alpha & \phi_{\beta,\gamma} : \{e\} \to G_\gamma \text{ is the trival inclusion .} \end{cases}$$

Then in this semigroup, $K[G_\alpha]$ is not semiprimitive, and if $\beta \not\leq \alpha$, then $K[G_\beta]$ is semiprimitive. If $\beta < \alpha$, then $x \in J(K[G_\alpha])$ is nonzero, we have $\phi_{\alpha,\beta}(x) \notin J(K[G_\beta])$.

Refer to Section 2.5, $K[S]$ now becomes a special semilattice-graded ring when α is maximal. If there is a nonzero element $a \in J(K[S])$, then $a_\alpha e_\alpha \in J(K[G_\alpha])$ by Theorem 2.5.6. Assume that $b = ae_\alpha \neq 0$. Let $D = supp(b) \setminus \{\alpha\}$. Since E is non-pseudofinite, there exists an infinite β such that $\alpha \succ \beta$. Moreover, suppose that there exists $\gamma \prec \alpha$ and $\gamma \in \alpha E \setminus D$ such that $\gamma d = \gamma$, then $d \leq \gamma$ for some $d \in D$. Thus $\gamma = d$. This contradiction leads that for any $\gamma \in \alpha E \setminus D$,

$$a_\alpha = a_{\hat{\gamma}}$$

$$\text{but} \qquad a_{\hat{\gamma}} e_\gamma = \phi_{\alpha,\gamma}(a_\alpha e_\alpha) \notin J(K[G_\gamma]).$$

However, this result contradicts the description of $J(R)$ in Theorem 2.5.6 and the choice of a.

Thus $a_\alpha = 0$ and $\alpha \notin supp(a)$. Hence, $a \in J(K[S]) \cap R' = J(R')$, $R' = K[S] \setminus K[G_\alpha]$ which is an ideal of $K[S]$. We can check that $J(R') = 0$ since for every maximal subgroup H, $J(K[H]) = 0$. Thus, $a = 0$ and $J(K[S]) = 0$. □

The above construction shows the importance of pseudofiniteness of $E(S)$ in an inverse semigroup S.

Finally, we describe the structure of $K[S]$ over a completely semisimple inverse semigroup S.

Theorem 3.5.15 *If S is completely semisimple inverse semigroup with finite $E(S)$ and K is any fields, then $K[S]$ has an identity and*

$$K[S] \cong M_{n_1}(K[G_1]) \oplus M_{n_2}(K[G_2]) \oplus \cdots \oplus M_{n_k}(K[G_k]).$$

Proof. Since S is a completely semisimple inverse semigroup with $E(S)$ is finite, S has a prinipical series

$$S = S_n \supseteq S_{n-1} \supseteq \cdots \supseteq S_0,$$

where S_i/S_{i-1} is completely 0-simple and S_0 is completely simple.

As $T = S_0$ is minimal nonzero ideal of S and T is completely 0-simple inverse semigroup, $T \cong \mathcal{M}(G^0, I, I, \Delta)$ with finite index set I. By Lemma 3.5.1, we have $K_0[T] \cong M_{|I|}(K[G_0])$, where G_0 is maximal subgroup contained in T. Thus, $M_{|I|}(K[G_0])$ has an identity. Then

$$K_0[S] \cong K_0[T] \oplus K_0[S/T]$$

(see Prop. 1.2.4). We can complete the proof by using induction hypothesis. Note that each G_i can be selected from S_i/S_{i-1}. □

Finally, if $char(K) = 0$ and if the maximal subgroup G is finite, then it is easy to see that $K[S]$ is semisimple.

3.6 Other Semigroup Algebras

3.6.1 Completely Regular Semigroup Algebras

By a completely regular semigroup S, we mean S is the unions of groups. Moreover, every completely regular semigroup is also a semilattices of completely simple semigroups, i.e. S/η is semilattice of completely simple semigroups, where the least semilattice congruence $\eta = \mathcal{D} = \mathcal{J}$. Since S is a completely regular semigroups. S is a semilattice of completely simple semigroups. Using the theory of semilattice graded rings and the Munn algebras, the structure and radicals of this semigroup algebras $K[S]$ can be found.

In this section, we consider the class of the semigroups having band decompositions.

Proposition 3.6.1 [Pet, Prop IV.1.7] *Let S be a completely regular semigroup. Then the following conditions are equivalent:*

(i). *S is a band of groups.*

(ii). *$\mathcal{H}$ is a congruence on S.*

(iii). *$a^2bS = abS$, $Sab^2 = Sab$ for all $a, b \in S$.*

Using the above characterization for band graded rings, we obtain the following results.

Theorem 3.6.2 *If S is band of groups, then for any field K, $K[S]$ is special band-graded algebra with band B and*

$$J(K[S]) = \{a \in J(K[S]) \mid \forall \alpha \in B; \qquad e_\alpha a_{\hat{\alpha}} e_\alpha \in J(K[G_\alpha])\}.$$

Proof. This is a direct consequence of Theorem 2.5.6. □

Theorem 3.6.3 *Let S be a strong semilattice of completely simple semigroups, and K any field with $char(K) = p$. Then, $K[S] = \sum_{\beta \in B} K[G_\beta]$ where B is a normal band. Moreover, $J(K[G_\beta])$ is nil (e.g $J(K[G_\beta]) = \mathcal{B}(K[G_\beta])$) iff $J(K[S])$ is a nil ideal.*

Proof. By [Pet, IV.4.3], S is a strong semilattice of completely simple semigroups, i.e. $S = \cup_{\gamma\in\Gamma} S_\gamma$ iff $S_\gamma/\mathcal{H}$ is rectangular band. This means that S_γ is union of groups. Let

$$B \cong \bigcup_{\gamma\in\Gamma} S_\gamma/\mathcal{H}.$$

Then B becomes a normal band and the first part is therefore proved. In particular, we know that $K[S]$ is a special band-graded ring. By Theorem 2.5.15, it suffices to show that $K[S]$ is radically coherent. In fact, B is the band induced by the semilattice decomposition of S, hence B is a strong semilattice of rectangular bands. If $\bar{\alpha} \geq \bar{\beta}$, then we can define $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ such that $\phi_{\alpha,\beta}(x) = e_\beta x e_\beta$, for all $x \in G_\alpha$, where $G_\alpha \subseteq S_{\bar{\alpha}}$ and $G_\beta \subseteq S_{\bar{\beta}}$. Thus, $\phi_{\alpha,\beta}$ induces an homomorphism

$$\widehat{\phi_{\alpha,\beta}} : K[G_\alpha] \to K[G_\beta].$$

It is not difficult to see that $\widehat{\phi_{\alpha,\beta}}(J(K[G_\alpha]) \subseteq J(K[G_\beta])$. Hence, $K[S]$ is radical coherent, and the second part is proved. □

3.6.2 Separative Semigroup Algebras

The radicals of commutative separative semigroup algebras have been completely solved in section 3.3. However, for non-commutative separative case, it is so difficult to describe the radicals because even S has a semilattice decomposition of cancellative semigroups, not all cancellative semigroup can be embedded into groups. However, if S is separative, then there exists a greatest semilattice decomposition of cancellative semigroups, say $S = \cup_{\alpha\in\Gamma} S_\alpha$. Assume that each cancellative semigroup generates a particular group, then we can still describe the radicals for certain separative semigroups.

Lemma 3.6.4 [JO1] **(1994)** *Let S be a separative semigroup with a semilattice decomposition $\cup_{\alpha\in\Gamma} S_\alpha$, where each S_α is a cancellative semigroup. If S_α is also n_α-nilpotent, then S can be embedded into a semilattice $Q = \cup_{\alpha\in\Gamma} G_\alpha$, where each G_α is a two-sided fractions group of S_α. Moreover, S is nilpotent iff $n = sup\{n_\alpha | \alpha \in \Gamma\} < \infty$. In this case, Q is also nilpotent semigroup.*

Proof. By Theorem 3.3.30, S_α has group of fractions G_α. Let $G_\alpha = \{s_\alpha t_\alpha^{-1} | s_\alpha, t_\alpha \in S_\alpha\}$. It suffices to show that the multiplication on $Q = \cup_{\alpha\in\Gamma} G_\alpha$ is well defined.

For arbitrary $\alpha, \beta \in \Gamma$, any $s_\alpha \in S_\alpha$ and $s_\beta \in S_\beta$. We can write $x = s_\alpha s_\beta$, $y = s_\beta s_\alpha$, it is obvious that $x, y \in S_{\alpha\beta}$ since $S = \cup_{\alpha\in\Gamma} S_\alpha$. By the nilpotency of $S_{\alpha\beta}$, for any sequences, $X_k(x,y) = Y_k(x,y)$ for $k \geq n_{\alpha\beta}$, which is the nilpotent class of $S_{\alpha\beta}$. Then there exists $a_{\alpha\beta}, b_{\alpha\beta}$ in $S_{\alpha\beta}$ such that $s_\alpha a_{\alpha\beta} = s_\beta b_{\alpha\beta}$.

The multiplication of Q is defined by

$$(a_\alpha b_\alpha^{-1})(c_\beta d_\beta^{-1}) = (a_\alpha C_{\alpha\beta})(d_\beta B_{\alpha\beta})^{-1},$$

where $b_\alpha C_{\alpha\beta} = c_\beta B_{\alpha\beta}$, $C_{\alpha\beta}, B_{\alpha\beta} \in S_{\alpha\beta}$. Then this multiplication of Q is well defined. Hence, it follows that Q is a semigroup.

If $sup\{n_\alpha\}$ exists, then S is clearly nilpotent. Moreover, the multiplication of Q induced by S also leads to nilpotency. Under this case, Q is also nilpotent. □

We have already known that the semilattice Γ is locally finite. If S is a separative nilpotent semigrop, then we have the following theorem.

Theorem 3.6.5 *Let S be a separative nilpotent semigroup in which each semilattice component S_α of S is weakly n_α-nilpotent that is $S = \cup_{\alpha\in\Gamma} S_\alpha$ and each S_α generates a group G_α. Let $Q = \cup_{\alpha\in\Gamma} G_\alpha$ where G_α is group generated by S_α. For any field K, we have*

(i). *If Γ is finite, then $J(K[S]) = \mathcal{B}(K[S])$,*

(ii). *If Γ is infinite, then $J(K[S]) = \mathcal{L}(K[S])$.*

Proof. The proof follows from Theorem 3.3.32, as for each $\alpha \in \Gamma$, we have

$$J(K[S_\alpha]) = \mathcal{B}(R[S_\alpha]) = \mathcal{L}(K[S_\alpha]) = J(K[S_\alpha]) \cap K[S_\alpha].$$

This shows that $J(K[S_\alpha])$ is locally nilpotent. As a direct consequence of Theorem 3.4.13, we know that $J(R[S])$ is locally nilpotent and equal to $\mathcal{L}(K[S])$. Moreover, if Γ is finite, then $J(K[S]) = \mathcal{B}(K[S])$ by induction hypothesis. □

Corollary 3.6.6 *Let S be a separative semigroup with the greatest semilattice decomposition of cancellative semigroups $\cup_{\alpha\in\Gamma} S_\alpha$, where each S_α generates a group G_α. Let $Q = \cup_{\alpha\in\Gamma} G_\alpha$ where G_α is group generated by S_α. Moreover, if each G_α is nilpotent-by-finite or FC-by-finite, then $\mathcal{B}(K[S_\alpha]) = J(K[S_\alpha])$. If Γ is finite, then $\mathcal{B}(K[S]) = J(K[S])$.*

For the commutative case, the relation $\xi_p = \{(s,t) | s^{p^k} = t^{p^k} \text{for some } k\}$ is p-separative congruence (cf. Section 3.2) but ξ_p may not be a congruence in noncommutative semigroup. Hence $I(K, S, \xi_p)$ may not be an ideal of $K[S]$. Recently, [JO1] has shown that when S is 2-nilpotent semigroup and K is a field of characteristic $p > 0$, then ξ_p is a congruence on S, and $J(K[S]) = \mathcal{B}(K[S]) = I(K, S, \xi_p)$. We now extend this theorem to a more general situation.

Theorem 3.6.7 *Let S be separative nilpotent semigroup, If $(s,t) \in \xi_p$ then $s - t \in \mathcal{B}(K[S])$. The relation ξ_p is a congruence on S. Moreover,*

$$J(K[S]) = \mathcal{L}(K[S]) \supseteq \sum_{p \in \mathrm{P}} I(K, S, \xi_p).$$

Proof. Since S is a separative nilpotent semigroup and has a semilattice decomposition of cancellative semigroups, from Lemma 3.6.4, S can be embedded into $Q = \cup_{\alpha\in\Gamma} G_\alpha$. where G_α is group of fractions of S_α. If $a \in J(K[S_\alpha]) = J(K[G_\alpha]) \cap K[S_\alpha]$, then $a \in J(K[G_\alpha])$, where G_α is a nilpotent group. Lemma 3.3.31 then yields $a = \sum_i k_i(s_i - t_i)$ with $(s_i, t_i) \in \xi_p$. Note that if $(s_i, t_i) \in \xi_p$, then $s_i, t_i \in S_\alpha$ for some $\alpha \in \Gamma$. For $e_\alpha \in Q$, e_α is the identity of G_α and so e_α is central. Assume that the semilattice Γ is finite and let α be the maximal element in Γ. Denote $I = S \setminus S_\alpha$. Then we have

$$\begin{aligned} aK[I] &\subseteq \sum_i (s_i - t_i)K[I] \\ &\subseteq \sum_i \sum_{\substack{\beta\in\Gamma \\ \beta\neq\alpha}} (s_i - t_i)K[S_\beta] \\ &\subseteq \sum_i \sum_{\substack{\beta\in\Gamma \\ \beta\neq\alpha}} (s_i e_{\alpha\beta} - t_i e_{\alpha\beta})K[e_{\alpha\beta} S_\beta] \end{aligned}$$

Since all $e_{\alpha\beta}$ are central so that $s_i e_{\alpha\beta}$ and $t_i e_{\alpha\beta}$ are ξ_p-related. Hence, $s_i e_{\alpha\beta} - t_i e_{\alpha\beta} \in J(K[G_{\alpha\beta}])$. Consequently, for $\beta \neq \alpha$, we have

$$\sum_i (s_i e_{\alpha\beta} - t_i e_{\alpha\beta})K[e_\alpha S_\beta] \subseteq K[S_{\alpha\beta}] \cap J(K[G_{\alpha\beta}]) = J(K[S_{\alpha\beta}]).$$

By using induction hypothesis on the order of Γ, we obtain $aK[I] \subseteq J(K[S])$. Moreover, since $a \in J(K[S_\alpha]) = J(K[G_\alpha]) \cap K[S_\alpha]$, $aK[S] \subseteq J(K[S_\alpha] + J(K[I])$ and $a \in J(K[S])$. Thus, $J(K[S_\alpha]) \subseteq J(K[S])$ for some maximal $\alpha \in \Gamma$. By using induction again, we have $\sum_{\alpha\in\Gamma} J(K[S_\alpha]) \subseteq J(K[S])$.

On the other hand, assume that $s, t \in S$ such that $s - t$ nilpotent. We may assume that $s \in S_\alpha, t \in S_\beta$. Hence $(s-t)^k = s^k - st^{k-1} + \cdots + t^k = 0$ iff $\alpha\beta = \alpha = \beta$ iff $s, t \in S_\alpha$. Therefore, $s - t \in \mathcal{B}(K[S])$ iff $s - t \in \mathcal{B}(K[S_\alpha])$ for some $\alpha \in \Gamma$. Hence ξ_p is a congruence on S and hence

$$I(K, S, \xi_p) = \sum_{\alpha\in\Gamma} I(K, S_\alpha, \xi_p) \subseteq \mathcal{B}(K[S]).$$

Using Theorem 3.6.5, we then obtain $J(K[S]) = \mathcal{L}(K[S]) \supseteq \mathcal{B}(K[S]) \supseteq I(K, S, \xi_p)$.
□

3.7 Radicals of PI-semigroup Algebras

3.7.1 PI-Algebras

The algebras or rings that satisfy an polynomial identity are rather useful in the aspect of geometry, in particular, the Azumaya algebras and division algebras. Now, we study semigroup algebras satisfying a polynomial identity and obtain some results on semigroup algebras. The reader is referred to [Pas1, Row] for more properties of PI-algebras.

Definition 3.7.1 *Let R be a k-algebra. Then R is called a PI-algebra if R satisfies a polynomial identity f over k.*

We state the following important result of PI-algebras.

Theorem 3.7.2 [Okn1, Pas1, Row] *Let R be a PI-algebra satisfying a polynomial identity of degree n.*

(i). *For every prime ideal P, the localization of R/P with respect to its center is isomorphic to the matrix algebra $M_r(D)$ over a division K-algebra D such that $dim_{Z(D)} M_r(D) \leq (n/2)^2$. Moreover, R/P can be embedded into $M_N(L)$ for a field $L \supseteq K$, and an integer $N \leq n/2$*

(ii). *If P is a right primitive ideal of R, then $R/P \cong M_r(D)$ with D, r as above.*

(iii). *For any nil ideal of R, $J(I) = \mathcal{B}(I)$ and if R is a finitely generated K-algebra, then $J(R)$ is a nilpotent ideal of R.*

3.7.2 Permutational Property and Algebras of Permutative Semigroups

In order to study the PI semigroup algebras, we first have to examine some of the properties of the given semigroup. A semigroup is said to have the property $\mathfrak{P}_n$ if for any elements $s_1, \cdots, s_n \in S$, there exists a nontrival permutation σ in the symmetric group $\mathcal{S}_n$ such that $s_1 \cdots s_n = s_{\sigma(1)} \cdots s_{\sigma(n)}$.

Proposition 3.7.3 [Okn1, Prop.19.1] *Assume that $K[S]$ satisfies a polynomial identity of degree n. Then S has the property by $\mathfrak{P}_n$.*

Hence, the studying permuntational property of semigroup is a must. However, if S satisfies a permutational identity, it is still not sufficient to show that $K[S]$ is a PI-algebra.

Summerize the results in [Okn1], we obtain some properties of semigroups satisfying permutational property. The proofs can be found in [Okn1, Chapter 19].

Theorem 3.7.4 [Okn1] *Denote the permutational property $\mathfrak{P}$.*

(i). *If S is periodic with $\mathfrak{P}$, then S is locally finite.*

(ii). *If S is cancellative, then S has a two-sided group of fractions G which is finite-by-abelian-by-finite iff S has $\mathfrak{P}$.*

(iii). *If S is a finitely generated cancellative semigroup, then S has an abelian-by-finite group of fractions iff S has $\mathfrak{P}$.*

(iv). *(Domanov) Assume that S is a 0-simple semigroup with $\mathfrak{P}$. Then S is completely 0-simple.*

The semigroup S is called a **permutative semigroup** if there exists an integer $n \geq 2$ and a nontrival permutation σ, taken from the symmetric group $\mathcal{S}_n$, such that

$$x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

for every $x_1, \cdots, x_n \in S$. Clearly this property is stronger than above property described. The above formula is a multilinear identity. Assume that all algebras are over field K (or commutative ring R). It is then obvious to see that $K[S]$ is a PI-algebra. This is a special kind of PI-algebras. we will see that the Jacobson radical is determined by the congruence on the semigroup S.

Proposition 3.7.5 *If S is a permutative semigroup, then $K[S]^m(xy-yx)K[S]^m = 0$. for some $m \geq 1$ and any $x, y \in S$. Consequently, the commutator ideal of $K[S]$ is nilpotent.*

Theorem 3.7.6 [Okn1] *Let S be a permutative semigroup. Then $J(K[S]) = I(K, S, \rho)$, where ρ is the congruence on S such that*

$$S/\rho = \begin{cases} (S/\mu)/\xi & if\ char(K) = 0 \\ (S/\mu)/\xi_p & if\ char(K) = p > 0 \end{cases}$$

where μ is the least commutative congruence on S and ξ, ξ_p are the least separative and p-separative congruence on S/μ respectively. Moreover, $J(K[S])$ is a sum of nilpotent ideals of $K[S]$ and coincides with the set of nilpotent elements of $K[S]$

Proof. The congruence determined by the commutator ideal of $K[S]$ is a commutative congruence on S, so $\mu \subseteq \sim_{[K[S],K[S]]}$. Since S/μ is commutative, $J(K[S/\mu]) = I(K, S/\mu, \xi)$ if $char(K) = 0$ (or $J(K[S/\mu]) = I(K, S/\mu, \xi_p)$ if $char(K) = p$). Hence $J(K[S/\mu])$ is sum of nilpotent ideals of $K[S/\mu]$. This leads to $I(\mu)$ is nilpotent and consequently, $J(K[S]) = I(\rho)$. □

3.7.3 Radicals of PI-algebras

Polycyclic-by-finite groups are related to noetherian algebras. Moreover, the properties of PI-algebras also give more generalization on algebras of cancellative semigroups and its groups of fractions. We now point out when the group algebras would be PI-algebras.

Lemma 3.7.7 [Pas1, Th. 5.2.14] *Suppose that $K[G]$ satisfies a polynomial identity of degree n. Then $[G : \Delta(G)] < n/2$ and $|\Delta(G)'| < \infty$.*

Theorem 3.7.8 [Okn1] *Let G be a group. Then the following statements hold:*

(i). *$K[G]$ is a PI-algebra iff the following conditions hold*

(a) *G is abelian-by-finite, that is, G has an abelian normal subgroup A of finite index, if $char(K) = 0$.*

(b) *G is p-abelian-by-finite, that is, G has a normal subgroup A of finite index such that the commutator subgroup A' is a p-group, if $char(K) = p > 0$.*

(ii). *If $K[G]$ is a PI-algebra, then $J(K[G]) = \mathcal{B}(K[G])$ and $J(K[G]) = 0$ if $char(K) = 0$ or $char(K) = p > 0$ and G has no normal subgroup of order divisible by p. If $char(K) = p > 0$ and the maximal normal p-subgroup $\mathbb{O}_p$ of G is finite, then $J(K[G])$ is a nilpotent ideal.*

(iii). *For all elements $g, h \in G$, we have $(g, h) \in \sim_{J(K[G])}$ iff g, h are in the same coset of the maximal normal p-subgroup $\mathbb{O}_p$ of G, where $p = char(K) = p > 0$.*

Corollary 3.7.9 [Okn1] *Let S be a cancellative semigroup and K a field with $char(K) = p$. Assume that $K[S]$ is a PI-algebra. Then S has a group of fractions G such that $K[G]$ is a PI-algebra. Moreover,* (i). $\mathcal{B}(K[S]) = J(K[S]) = J(K[G]) \cap K[S]$, *and* $J(K[S]) = 0$ *if* $p = 0$, *or if* $p > 0$ *and* G *has no normal subgroup whose order is divisible by* p.

(ii). *For any* s, t, $(s,t) \in \sim_{J(K[S])}$ *iff* s *and* t *are in the same coset of some normal* p*-subgroup of* G.

Theorem 3.7.10 [Okn1] *Suppose that* $K[S]$ *satisfies a polynomial identity. Then* $J(K[S]) = \mathcal{B}(K[S])$.

Proof. As the algebra $K[S]/\mathcal{B}(K[S])$ is a subdirect product of prime algebras A_i, $i \in I$. From Theorem 3.7.2 (ii), A_i is prime PI-algebra and can be embedded into a matrix algebra $M_n(D_i)$ and a semiprime PI-algebra has no nonzero nil ideal. Consider the following homomorphism:

$$\phi_{\mathcal{B}} : K[S] \to K[S]/\mathcal{B}(K[S]) \hookrightarrow \prod_{i \in I} A_i$$

where $\phi_{\mathcal{B}}(a) = (a_i)_{i \in I}$, $a_i \in A_i$. Let M_i be the kernel of the homomorphism $\phi_i : K[S] \to A_i$. Then we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_i & \longrightarrow & K[S] & \xrightarrow{\phi_i} & A_i & \longrightarrow & 0 \\ & & & & \lambda_{M_i}\big\downarrow & \nearrow \sigma_i & & & \\ & & & & K[S/\sim_{M_i}] & & & & \end{array}$$

where $\sigma_i \lambda_{M_i} = \phi_i$ and $\phi_{\mathcal{B}}(a) = (\phi_i(a))_{i \in I}$.

Let $a \in J(K[S])$ and if $a_i^n = 0$ for all $i \in I$, then $\phi_{\mathcal{B}}(a)^n = 0$. This implies that $a^n \in \mathcal{B}([S])$. Thus, $J(K[S])$ is nil ideal and hence $J(K[S]) = \mathcal{B}(K[S])$.

Since $S/\sim_{M_i}$ can be embedded into $A_i \hookrightarrow M_n(D)$ for some division rings, we may denote S by $S/\sim_{M_i} \subseteq M_n(D)$. In chaper 1, we have already stated that $M_n(D)$ is a completely semisimple semigroup with I_i/I_{i-1} a completely 0-simple semigroup. Now, if $S \hookrightarrow I_i/I_{i-1}$, then by Coro. 3.4.10, $J(K_0[S])^3$ is a sum of nil ideals $J(K_0[T_{i,j}])$, where $K_0[T_{i,j}]$ are PI-algebras of the cancellative subsemigroup $T_{i,j}$ of S which intersects the $\mathcal{H}$-classes of I_i/I_{i-1}. This shows that $J(K_0[S])$ is nil ideal and hence $J(K_0[S]) = \mathcal{B}(K_0[S])$.

If $S \subseteq M_n(D)$ but not in I_i/I_{i-1}, then we refine S by $S_j = I_j \cap S$, where $j = 0, 1, \cdots, n$. Now, for the least number k, write $S = S_k$ and $S_k/S_{k-1} \subseteq I_k/I_{k-1}$. For any $a \in J(K[S])$, we have

$$\phi_{K[S_{k-1}]}(a) \in J(K_0[S_k/S_{k-1}]) \subseteq K_0[I_k/I_{k-1}] \cong \mathcal{M}^0(K[G], I, \Lambda; P)$$

Hence, $\phi_{K[S_{k-1}]}(a^r) = \phi_{K[S_{k-1}]}(a)^r = 0$ for some $r \geq 1$. This means that

$$a^r \in K[S_{k-1}] \cap J(K[S]) \subseteq J(K[S_{k-1}]).$$

If $S_{k-1} = \theta$, then $a^r = 0$. Otherwise, we continue the same process repeatedly, we eventually get $J(K[S])$ a nil ideal. This completes the proof. $\square$

Chapter 4

Finiteness Conditions on Semigroup Algebras

4.1 Introduction

In this chapter, finiteness conditions on semigroup algebras will be investigated. The main object of this chapter is to find some necessary and sufficent conditions for $K[S]$ which are noetherian, artinian, semisimple and also the other related topics. We also consider the graded rings by groups, semigroups, groupiods, and apply these cases to semigroup algebras over any field.

We first recall some basic properties of ring theory and graded ring theory. The references of graded ring theory can be obtained in [JW2, Kar, NV, Wau].

4.1.1 Preliminaries

Call a class $\mathcal{K}$ of rings closed under right ideals (left ideals, homomorphic images) if for every ring $R \in \mathcal{K}$, the class $\mathcal{K}$ contains all right ideals (left ideals; homomorphic images) of R. Say that $\mathcal{K}$ is closed under (finite) sums of one-sided ideals if $\mathcal{K}$ contains every ring which is a (finite) sum of its right ideals or a (finite) sum of its left ideals belonging to $\mathcal{K}$. A class $\mathcal{K}$ is called closed under ideal extensions if $\mathcal{K}$ contains every ring R such that $R/I \in \mathcal{K}$ for an ideal I of R.

Lemma 4.1.1 *The classes of semilocal, semiprimary, right perfect, left perfect, nilpotent, right T-nilpotent, left T-nilpotent rings are closed under ideal extensions, right and left ideals, homomorphic images and finite sums of one-sided ideals.*

It is noted that the class of right artinian or noetherian rings are not closed under finite sums of left ideals. For example, the semigroup algebra $\mathbb{Q}[B]$ over the two-element right zero band $B = \{a, b\}$ is neither right artinian nor right noetherian, althought it is the sum of two left ideals $\mathbb{Q}a$ and $\mathbb{Q}b$ which are isomorphic to $\mathbb{Q}$.

4.1.2 Semilattice Graded Rings

Let Γ be a semilattice and R a ring graded by Γ.

Lemma 4.1.2 [JW2, Lemma 1.1] *Let Γ be a semilattice. Suppose that*

(i). *Γ satisfies the d.c.c.;*

(ii). *Γ satisfies the a.c.c.;*

(iii). *Γ does not contain an infinite subset of incomparable elements.*

Then Γ is finite.

Theorem 4.1.3 [Wau, JW2] **(1986)** *Let R be a ring graded by a semilattice Γ such that $J(R_\alpha) \neq R_\alpha$ for all $\alpha \in \Gamma$. Then the following statements hold:*

(i). *R is semilocal iff Γ is finite and each R_α is semilocal for all $\alpha \in \Gamma$.*

(ii). *If R_α has unity e_α then R is semiperfect iff Γ is finite and R_α is semiperfect for all $\alpha \in \Gamma$.*

(iii). *R is left perfect ring iff Γ is finite and R_α is left perfect for all $\alpha \in \Gamma$.*

(iv). *R is semiprimary ring iff Γ is finite and R_α is semiprimary for all $\alpha \in \Gamma$.*

Proof. (i) If $R = \sum_{\alpha\in\Gamma} R_\alpha$ and R is semilocal, then $R/J(R)$ is artinian. Let $R_{\breve{\alpha}} = \sum_{\alpha\geq\beta} R_\beta$ be an ideal of R. From the proof of Theorem 2.4.1, $\pi_\alpha : R_{\breve{\alpha}} \to R_\alpha$ is a projection homomorphism. Suppose that Γ has infinite chain $\alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_n > \cdots$ and $R/J(R)$ is artinian. Then $R_{\breve{\alpha_k}} + J(R) = R_{\alpha_{\breve{k+1}}} + J(R)$ for some k. This implies that $J(R_{\alpha_k}) = R_{\alpha_k}$, which contradicts the assumption. Therefore, Γ satisfies d.c.c. If $R/J(R)$ is noetherian, then Γ has a.c.c. Let $\{\beta_1, \beta_2, \cdots, \beta_k, \cdots\}$ be an infinite subset of Γ of incomparable elements. Then for some $m > n$, we have

$$\sum_{i=1}^{n} R_{\breve{\beta}_i} = \sum_{i=1}^{m} R_{\breve{\beta}_i}.$$

If $r \in R_{\beta_m}$, then there exists an element $x \in \sum_{i=1}^{n} R_{\breve{\beta}_i}$ such that $r - x \in J(R)$. As $supp(x) \subseteq \Gamma\beta_1 \cup \Gamma\beta_2 \cdots \Gamma\beta_n$ for all $s \in R_{\beta_m}$, $(r-x)s = rs - xs \in J(R)$. Let $\alpha \in supp(xs)$ and $\alpha = \gamma\beta_m$, where $\gamma \leq \beta_i$ for any $i = 1, \cdots, n$. Then $\alpha \leq \beta_m$. If $\alpha = \beta_m$ and $\beta_m \leq \gamma \leq \beta_i$, then β_i is comparable with β_m. This leads to a contradiction. Hence $\alpha < \beta_m$. Therefore, $rs - xs \in J(R) \cap R_{\breve{\beta_m}} = J(R_{\breve{\beta_m}})$ and $rs = \pi_{\beta_m}(rs - xs) \in \pi_{\beta_m}(J(R_{\breve{\beta}})) \subseteq J(R_{\beta_m})$. So $rR_{\beta_m} \subseteq J(R_{\beta_m})$ whence $r \in J(R_{\beta_m})$, as the case holds for any $r \in R_{\beta_m}$, which contradicts the hypothesis $J(R_\beta) \neq R_\beta$. Thus, Γ contains no infinite subset of incomparable elements. By Lemma 4.1.2, Γ is finite.

Conversely, if R_α is semilocal for all $\alpha \in \Gamma$ and $|\Gamma| = n$, then we can show that the theorem hold by induction on $|\Gamma|$. First, let $|\Gamma| = 2$, say $\Gamma = \{\alpha, \beta\}$ with $\alpha > \beta$. Then R_β is an ideal of R and $R/J(R_\beta) = R_\alpha \oplus (R_\beta/J(R_\beta))$. In particular, we may assume that $J(R_\beta) = 0$. As R_β is semisimple artinian and so there is an identity $e_\beta \in R_\beta$. From Theorem 2.4.2, $J(R) = \{r - e_\beta r | r \in J(R_\alpha)\}$. Using $\pi_\alpha(J(R)) = J(R_\alpha)$ and that R_α is semilocal, it follows that $R/J(R)$ is artinian.

Now, by induction hypothesis, we assume that $|\Gamma| = n+1$. Select a maximal element in Γ, say α. Then $R = R_\alpha \oplus R'$. By hypothesis, $R'/J(R')$ is artinian and R' is an ideal of R, hence $J(R) \cap R' = J(R')$. Moreover, $R_\alpha/J(R_\alpha)$ is also artinian. We can reduce this case to $\Gamma' = \{\alpha, \beta\}$, where $R_\beta = R'$. By using the above arguments, we can prove that R is semilocal.

(ii) In case of R is semiperfect, then R is semilocal. Hence R_α is semilocal and Γ is finite. It suffices to show that $J(R_\alpha)$ lifts the idempotents iff $J(R)$ lifts the idempotents in R. Since Γ is finite, there exists α which is maximal and let $\Gamma' = \Gamma \setminus \{\alpha\}$. As $R_{\Gamma'}$ is an ideal, $R/R_{\Gamma'} \cong R_\alpha$. If R is semiperfect, then R_α is clearly semiperfect. It remains to show that $R_{\Gamma'}$ is semiperfect.

As by (i), we know that $R_{\Gamma'}$ is semilocal. Let $e \in R_{\Gamma'}$, such that $(e - e^2) \in J(R_{\Gamma'})$. Then $(e - e^2) \in J(R)$ and by the semiperfectness of R, we can find an idempotent $f \in R$ such that $f + J(R) = e + J(R)$. (since the idempotent is lifted by $J(R)$ and $(f - e) \in J(R)$). Write $f = f_\alpha + f'$, $f' \in R_{\Gamma'}$. Then $f_\alpha = f_\alpha^2$, $f - e = f_\alpha + (f' - e) \in J(R)$ and so $f_\alpha \in J(R_\alpha)$. Therefore, $f_\alpha = 0$ and $f \in R_{\Gamma'}$.

Conversely, suppose that Γ' is finite and R_α is semiperfect for all $\alpha \in \Gamma$. By using similar method in (i), we can reduce the theorem to the case $|\Gamma| = 2$. In this cases, R_α and R_β are both semiperfect with unity e_α, e_β repectively and $\alpha > \beta$. Let $x = x_\alpha + x_\beta$ such that $(x - x^2) \in J(R)$. Then $x_\alpha - x_\alpha^2 \in J(R_\alpha)$ and so $y_\alpha - x_\alpha \in J(R_\alpha)$ for some $y_\alpha = y_\alpha^2 \in R_\alpha$, by the semiperfectness of R_α. On the other hand, $(e_\alpha x) - (e_\alpha x)^2 = e_\alpha(x - x^2) \in J(R) \cap R_\beta = J(R_\beta)$. Since R_β is semiperfect, there exists an element $y_\beta - y_\beta^2 \in R_\beta$ with $y_\beta - e_\beta x \in J(R_\beta)$. Put $y = y_\alpha + y_\beta$. It is then easy to check that $y = y^2$ and $y - x \in J(R)$.

The proofs of (iii) and (iv) are similar to (i) and (ii) and hence are omitted. (cf. [Wau, Prop. 1.11, Prop. 1.13]). □

4.1.3 Group Graded Rings

The Jacobson radical of the group graded ring has been discussed in Section 2.2. We review here some of the finiteness condition of group graded rings. The main references are [Kar, Kel5, NV], etc.

Let G be any group and R a G-graded ring. If M is a graded left R-module such that $M \in_{\mathrm{R-gr}} \mathcal{M}$, then from Theorem 1.4.2, we know that ${}_{\mathrm{R-gr}}\mathcal{M}$ and ${}_{\mathrm{R_e}}\mathcal{M}$ are categorical equivalent.

Lemma 4.1.4 [NV, Lemma II.3.2] *Let R be a ring graded by a group G (not necessarily strongly graded). If $M \in_{\mathrm{R-gr}} \mathcal{M}$ is a left gr-noetherian (gr-artinian) then M_σ is left noetherian (artinian) in ${}_{R_e}\mathcal{M}$, for all $\sigma \in G$.*

Corollary 4.1.5 [NV, Coro. II.3.3] *Let R be a ring graded by a finite group G. If $M \in_{\mathrm{R-gr}} \mathcal{M}$ is left gr-noetherian (artinian), then M is left noetherian (artinian) in ${}_R\mathcal{M}$.*

Theorem 4.1.6 [NV, Th. II.3.5] *Let R be a $\mathbb{Z}$-graded ring and $M \in {}_{\mathrm{R-gr}}\mathcal{M}$. Then M is left gr-noetherian iff M is a left noetherian R-module.*

We have some theorems for some classes of infinite groups, for instance, we have the following theorem:

Theorem 4.1.7 [NV, Th. II.3.7] *Let R be a strongly G-graded ring, where G is a polycyclic-by-finite group. If R_e is left noetherian ring then R is left noetherian ring.*

We can also consider the homological properties of group graded rings. Suppose that G is finite and R is a G-graded ring with unity. For the smash product $R\#G^*$, we have following results from [JJ].

Proposition 4.1.8 [JJ, Prop. 2.1] *Let M be a right $R\#G^*$-module and N a right R-module. Then there is a natural isomorphism between*

$$\begin{aligned} \tilde{}: Hom_R(M, N) &\rightarrow Hom_{R\#G^*}(M, N \otimes_R (A\#G^*)) \\ f &\mapsto \tilde{f} \end{aligned}$$

where $\tilde{f}(m) = \sum_{g\in G} f(m)p_g \otimes p_g$.

Theorem 4.1.9 *Suppose that the ring* R *is a* G*-graded ring. Let* V *be a right* $R\#G^*$*-module. Then the following statements hold:*

(i). V *is projective iff* V_R *is projective* R*-module.*

(ii). V *is injective iff* V_R *is injective* R*-module.*

(iii). V *is flat iff* V_R *is flat* R*-module.*

4.1.4 Groupoid Graded Rings

Groupoid graded rings have been recently generalized by Kelarev in [Kel7] **(1995)**. A graded ring R is said to have a finite support if only a finite number of the homogeneous components of R are nonzero. If R is graded by any set S and for all $s, t \in S$, then there exists $u \in S$ such that $R_sR_t \subseteq R_u$, in fact, this multiplication makes R a groupoid graded ring. In this groupoid graded rings, R is graded by finite groupoid iff R is graded ring with finite support. Hence, if we have results on finite groupoid graded ring, then we can transfer these results to group or semigroup graded rings with finite supports.

Theorem 4.1.10 [Kel7] *Let* $\mathcal{K}$ *be a class of rings containing all rings with zero multiplication (i.e.* $R^2 = 0$*). Suppose* $\mathcal{K}$ *is closed under homomorphic images, right and left ideals, ring extensions and also closed under finite sums of one sided ideals. Then the followings are equivalent:*

(i). *For each finite groupoid* S *and the* S*-graded ring, we have* $R = \sum_{s\in S} R_s \in \mathcal{K}$ *iff* $R_e \in \mathcal{K}$ *for every* $e \in E(S)$.

(ii). *For each finite semigroup* S *and the* S*-graded ring* R, $R = \sum_{s\in S} R_s \in \mathcal{K}$ *iff every* $e \in E(S)$.

(iii). *For every finite group with identity* e*, the* G*-graded ring* $R = \sum_{s\in S} R_s$ *is in* $\mathcal{K}$ *iff* $R_e \in \mathcal{K}$.

Proof. The implications of (i) to (ii) and (ii) to (iii) are trival. The key step is prove (iii) implies (i).

Let S be any finite groupoid. Let $R = \sum_{s \in S} R_s$ be a S-graded ring. Suppose (iii) holds but (i) does not hold, then define a class $\mathfrak{K}$ by the collection of all counter examples of (i). Then $\mathfrak{K}$ must be the following collection:

$$\{R \in \mathcal{K} : R \text{ is } S\text{-graded ring but } \exists e \in E(S)\ R_e \notin \mathcal{K}\} \text{ or}$$

$$\{R \notin \mathcal{K} : R_e \in \mathcal{K} \text{ for all } e \in E(S)\}$$

Let I be a homogeneous two-sided ideal of R. Then, we can see that $R \in \mathfrak{K}$ iff either I or $R/I \in \mathfrak{K}$. We now proceed to find a homogeneous two-sided ideal I which gives a contradiction.

Step 1: As S is finite, we can assume R is S-graded ring and choose $R \in \mathfrak{K}$ with minimal $|S|$. Take an additive subgroup A of R_s for some s such that $supp(AR) \neq S$.

Step 2: If $AR = 0$, then take $I = R^1 A$. Thus, $I^2 = 0$, and so $I \in \mathcal{K}$ and $I_t \in \mathcal{K}$ for every $t \in S$.

Step 3: If $AR = P \neq 0$, then by the minimality of $|S|$, we know that P satisfies (i). The following cases then arises:

Case I : If $R \in \mathcal{K}$ but $R \in \mathfrak{K}$, then P is a right ideal of R, and so $P \in \mathcal{K}$. Thus, for all $e \in E(S)$, $P_e \in \mathcal{K}$.

Case II: If $R \notin \mathcal{K}$ but all $e \in E(S)$, then $R_e \in \mathcal{K}$ since P_e is right ideal of R_e. This implies that $P_e \in \mathcal{K}$ and by (i) $P \notin \mathfrak{K}$, $P \in \mathcal{K}$.

In the above two cases, P is a right ideals satisfying (i), hence, $P \in \mathcal{K}$ iff $P_e \in \mathcal{K}$ for all $e \in E(S)$.

However since $supp(R_x P) \neq S$ for every $x \in S$, we know that $R_x A$ is contained in R_{xs}. This means that $(R_x A)R$ and $((R_x A)R)_e$ are in $\mathcal{K}$, for every $e \in E(S)$. Since $\mathcal{K}$ is closed under finite sum of ideals, so $I = R^1 P = P + \sum_{x \in S} R_x$ is a homogeneous two side-ideal of R. Hence $A \subseteq I$ and for all $e \in E(S)$, $I_e \in \mathcal{K}$. These lead to $I \in \mathcal{K}$ and I is therefore not a counter example of (i). Hence R/I is the other counter example of (i).

Step 4: If $s \in S$ and $sS \neq S$, then $supp(R_s R) \subseteq sS \neq S$. Put $A = R_s$ above. Then AR can be graded by the groupoid $T = S \setminus \{s\}$, where the sth homogeneous component is zero and AR is in $\mathfrak{K}$ which contradicts to the minimality of $|S|$. Therefore $supp(R_s R) = S$ and $sS = S$. Similarity, we have $Ss = S$. Thus, S is a left and right simple groupoid.

Step 5: If $(st)x \neq s(tx)$ then $R_sR_tR_x \subseteq R_{(st)x} \cap R_{s(tx)}$. This implies that $R_sR_tR_x = 0$. As a result, we have $supp(R_sR_tR) \neq S$. Since S is finite, we obtain $supp(R_sR) \neq S$, contradicting wth the minimality of $|S|$. Therefore, S is a semigroup and S is left and right simple so that S is group.

Step 6: By assumption of (iii), R is a group graded ring. Moreover, $R \in \mathcal{K}$ iff $R_e \in \mathcal{K}$. This contradicts that R is a counter example of (i). Therefore, (i) is proved. □

As all semilocal rings have properties like $\mathcal{K}$, we have the following corollary.

Corollary 4.1.11 [Kel7] *Let $\mathcal{K}$ be the class of all semilocal (right perfect, left perfect, semiprimary, nilpotent, locally nilpotent, T-nilpotent, prime radical, quasiregular, PI) rings, S is a semigroup and $R = \sum_{s \in S} R_s$ an S-graded ring with finite support. Then $R \in \mathcal{K}$ iff $R_e \in \mathcal{K}$ for every $e \in E(S)$.*

Applying the above results to semigroup algebras, we get the following corollary.

Corollary 4.1.12 *Let K be field with $char(K) = p$ and S a Rees matrix semigroup $\mathcal{M}^0(G^0, I, \Lambda; P)$, where I and Λ are finite. Then $K_0[S]$ is semilocal (or, in $\mathcal{K}$) iff $K[G]$ is semilocal (or, in $\mathcal{K}$).*

4.1.5 Semigroup Graded PI-Algebras

A class $\mathcal{A}$ of algebras is called S-closed if $\mathcal{A}$ contains $R = \oplus_{s \in S} R_s$ where $R_s \in \mathcal{A}$ for all $s \in S$. From the above equality, we say that a class $\mathcal{K}$ of semilocal rings S-closed if S is a finite semilattice. Recently, Kelarev described the conditions that make the class of PI-algebras S-closed.

Theorem 4.1.13 [Kel3] *The class PI-algebras is S-closed iff S has a finite ideal chain*

$$\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S$$

such that each S_{i+1}/S_i is finite or nilpotent. (If S has zero, let $S_0 = \{0\}$.)

To prove this theorem, we need some technical lemmas on PI-algebra. From [Row], it is known that the class of PI-algebras is closed under one-sided ideal. Moreover, if R is a PI-algebra, then the matrix ring $M_t(R)$ is also PI-algebra for any integer t. Then by applying the smash product of group graded rings and by using duality theorem, we obtain the following lemma.

Lemma 4.1.14 [Kel3] *The class of PI-algebras is closed under finite groups.*

Proof. By Theorem 4.1.10, R_e is a PI algebra iff R is. If R_e is PI algebras, then we know that R is embeddable in $R\#G^*$ and so by [CM1], $A_{p,h} = p_g(R\#G^*)p_h = I_{gh^{-1}}p_h$, also $R\#G^* = A = \sum A_{g,h}$, and each $A_{g,h} \cong A_{e,e} \cong R_e$. Thus, A is a PI-algeba and R is PI algebra. □

Lemma 4.1.15 [Kel3, Lemma 7,8] *If the class of PI-algebras is S-closed, then S is a periodic semigroup and the number of idempotents in S is finite.*

Moreover, if S contains a subgroup G, then G is finite. If S is nil, then S is nilpotent.

We now sketch the proof of Theorem 4.1.13.
The necessary part follows from Lemma 4.1.15. Now we prove the sufficiency. Assume that S has a chain with the properties above, $R = \sum_{s\in S} R_s$. Let $A_i = \sum_{s\in S_i} R_s$. Then, the quotient algebra $T = A_i/A_{i+1}$ is a Q_i-graded, where $Q_i = S_i/S_{i-1}$. For $q \in Q_i$. T_q is PI. Then T is a PI-algebra since Q_i is a finite semigroup or is nilpotent. Thus, every factor T is a PI-algebra. Thus, we can show that R is a PI-algabra by induction hypothesis. □

4.1.6 Application to Semigroup Algebras

We now apply the results in section 4.1.2 and 4.1.4 to semigroup algebras.

Theorem 4.1.16 *If S is finite semigroup (group) and K field with $char(K) = p$, then $K[S]$ is member of class $\mathcal{K}$, the class is same as Coro. 4.1.11.*

Theorem 4.1.17 *If S is a semilattice (band) of groups, $S = \cup_{\alpha\in\Gamma} G_\alpha$, then $K[S] \in \mathcal{K}$ iff $K[G_\alpha] \in \mathcal{K}$ for any $\alpha \in \Gamma$, where Γ is finite semilattice (band).*

Proof. $R = K[S]$ is graded by finite semilattice or band. By Corollary 4.1.12, $R_e \in \mathcal{K}$ iff $R \in \mathcal{K}$. Hence, $K[G_\alpha] \in \mathcal{K}$ for all $\alpha \in \Gamma$. □

4.2 Semiprime and Goldie Rings

In this section, we are interested in the following question: When will a semigroup algebra be semiprime right Goldie or right noetherian? However, for non-cancellative semigroups, the problems seem to be rather complicated. In [Jes2], a solution is given for semigroup ring $R[S]$ which is an inverse semigroup. Finally, we examine the case when S being nilpotent. There are some nice results for noetherian algebras.

Now, let S be a submonoid of polycyclic-by-finite group. Note that if G is polycyclic-by-finite group, then $K[G]$ is noetherian ring and satisfies the a.c.c condition on right (left) ideals. Using the results in section 3.3, we have following result.

Theorem 4.2.1 *Let S be a submonoid of a polycyclic-by-finite group and W a poly-(infinite cyclic) normal subgroup of finite index in the group H generated by S.*

Moreover, the following conditions are equivalent:

(i). *S has a group of right fractions.*

(ii). *$K[S]$ is a right Goldie ring.*

(iii). *$K[S \cap W]$ is a semiprime right Goldie ring.*

Furthermoer, $J(K[S]) = J(K[H]) \cap K[S]$, and $K[S]$ is an Ore domain if H is a torsion-free group.

Consider the inverse semigroup, by using the characterization on inverse semigroup algebras (see section 3.5), we have $R[S] \cong \bigoplus_{k=1}^{n} M_{n_k}(R[G_k])$, and hence we obtain the following theorem in [Jes2], **(1988)**.

Theorem 4.2.2 *Let R be a ring with unity and S an inverse semigroup. Then $R[S]$ is semiprime right Goldie iff $E(S)$ is finite and for every maximal subgroup G of S, the group ring $R[G]$ is semiprime right Goldie. If $R[S]$ is prime right Goldie, then S is a group.*

For the semigroup algebra $K[S]$ with ascending chain condition on right annihilators, the following technical lemmas are useful for the investigation of the Goldie and noetherian semigroup algebras.

Lemma 4.2.3 [Okn2, Lemma 1] *Assume $K[S]$ has a.c.c. on its right annihilator ideals. Then for every subsemigroup T of S, there exists $u \in T$ such that uT is a left cancellative semigroup.*

Lemma 4.2.4 [Okn2, Lemma 3] *Let S be a semigroup with no free noncommutative subsemigroups. Assume further that $K[S]$ has a.c.c. on its right annihilator ideals* *and J is a nilpotent ideal of $K[S]$. Let $U' \subseteq S'$ be a cancellative subsemigroup of the image S' of S under the natural homomorphism $K[S] \to K[S]/J$. Then there exists an element y in the inverse image U of U' in S such that yUy is a cancellative semigroup.*

Lemma 4.2.5 *Let S be a nilpotent cancellative semigroup with quotient group G, and P be prime ideal of $K[S]$. If $P \cap S = \emptyset$, then*

(i). *$PK[G] = K[G]P$ is a two-sided ideal of $K[G]$*

(ii). *$Q = PK[G]$ is a prime ideal of $K[G]$, $Q \cap K[S] = P$, and $K[G]/Q$ is a localization with respect to an Ore subset of $K[S]/P$.*

(iii). *If all prime ideals Q of $K[G]$ satisfying $Q \cap G = \emptyset$, and $K[G]/Q$ is Goldie ring. Then $P = Q \cap K[S]$ is also prime and satisfying $P \cap S = \emptyset$.*

Proof. First of all, since S is a nilpotent cancellative semigroup, we have shown that S has a group of fractions G which is a nilpotent group (see section 3.3.4.).

Now, take $\alpha \in PK[G] \cap K[S]$. Then there exists $s \in S$ such that $\alpha s \in P$. For any $t, w_1, \cdots, w_n \in S$ and n is a nilpotency class of S, we obtain

$$\alpha y_n(s,t) = \alpha x_n(s,t) \in P.$$

by using the notation in section 3.3.4. Hence $\alpha y_{n-1} w_n x_{n-1} \in P$ for all $w_n \in S$,

$$\alpha y_{n-1}(s,t) K[S] x_{n-1}(s,t) \subseteq P.$$

Since $P \cap S = \emptyset$, we have $x_n(s,t) \notin P$. By the primeness of P, we also have $\alpha y_n(s,t) \in P$. Repeating the process in this way, we can show that $\alpha y_{n-1}(s,t) \in P$ and so on. Eventually, we obtain that $\alpha y_1(s,t) = \alpha t w_1 s \in P$ for all $t, w_1 \in S$. This implies that $\alpha \in P$.

Now we want to show that $K[G]P = PK[G]$. Let $p \in P$ and $t \in S$. Since *supp(p)* is finite, there exists $u \in S$ such that

$$upt^{-1} \in K[S] \cap PK[G] = P \subseteq K[G]P.$$

Hence, $pt^{-1} = u^{-1}upt^{-1} \in K[G]P$ and so $PK[G]$ is generated by pt^{-1} for some arbitrary p, t. This proves the inclusion containment. By symmetry, we have $K[G]P = PK[G]$.

Now, let Q be a maximal ideal of $K[G]$ with respect to the condition $Q \cap K[S] = P$. Then, by noting that $Q = (Q \cap K[S])K[G] = PK[G]$ and Q is prime due to the fact that P is prime, we know that $K[G]/Q$ is a localization of $K[S]/P$ with respect to the image of S in $K[S]/P$. That is, $(K[S]/P)\bar{S}^{-1} \cong K[G]/PK[G]$, where $\bar{S}$ is an image of S in $K[S]/P$.

(iii) Since $SS^{-1} = G$ and by Lemma 3.3.6(iv), we know that $Q \cap K[S]$ is a prime ideal. Since $Q \cap G = \emptyset$, $P \cap S \subseteq Q \cap G = \emptyset$. Hence, $P \cap S = \emptyset$. This completes the proof. In particular, if G is finitely generated nilpotent, then $K[G]$ is noetherian and so it satisfies the condition that $K[G]/P$ is Goldie. □

Proposition 4.2.6 *Let S be a nilpotent semigroup, and P a prime ideal of $K[S]$ such that $S\backslash(S\cap P)$ is a subsemigroup of S. Then $S/\sim_P$ is a 0-cancellative semigroup and there exists a prime ideal Q of $K[G]$ such that $K[G]/Q$ is isomorphic to a localization of $K[S]/P$, where G is quotient group of $(S/\sim_P)\setminus\{\theta\}$, and θ is zero of $S/\sim_P$.*

Proof. Let $T = (S/\sim_P)\setminus\{\theta\}$ be a semigroup. We first prove that T is right cancellative. Assume that $a, b, x \in T$ such that $ax = bx$. Then for any $w_1, w_2, \cdots w_n \in T$, $x_n(a,b) = y_n(a,b)$. where n is the nilpotency class of S. Replacing w_i by $xv_i \in T$ for all i, then we obtain

$$x_{n-1}(a,b)xv_ny_{n-1}(a,b) = y_{n-1}(a,b)xv_nx_{n-1}(a,b),$$

and $x_{n-1}(a,b)x = y_{n-1}(a,b)x \neq \theta$ by $ax = bx$. Hence,

$$(x_{n-1}(a,b)x)K[S](y_{n-1}(a,b) - x_{n-1}(a,b)) = 0$$

The primeness of $K[S]/P$ then implies that $y_{n-1}(a,b) = x_{n-1}(a,b)$.

Processing in the same way, we obtain that $a = x_0 = y_0 = b$. Thus, T is right cancellative. By symmetry, $S/\sim_P$ is an 0-cancellative semigroup.

Consider the following natural homomorphisms:

$$K[S] \to K[S/\sim_P] \to K_0[S/\sim_P] = K[T] \xrightarrow{\phi} K[S]/P$$

such that $S/\sim_P$ can be embedded in $K[S]/P$. Let $P' = ker\phi$. Then we have

$$K[T]/P' \cong K[S]/P \qquad \text{and} \qquad P' \cap T = \emptyset$$

Since $G = TT^{-1}$, $K[G]/Q$ is a localization of $K[S]/P$ with resepect to the Ore set induced by the prime ideal P. The proof is completed. □

Let T be a subsemigroup of the semigroup S. Denote the image of T in $K[S]/J$ by T' and let $K\{T\}$ be the image of $K[T]$ in $K[S]/J$. Then we can form the following definition.

Definition 4.2.7 *A semigroup S is called uniform if it can be embedded into a completely 0-simple semigroup T such that S intersects non-trivally with all $\mathcal{H}$-classes of T.*

Note that every subsemigroup of a group is uniform. In fact, uniform semigroups occur in the following cases.

Lemma 4.2.8 [Okn2, Lemma 8] *Let U be a completely semisimple semigroup with finitely many $\mathcal{J}$-classes. Assume that S is a subsemigroup of U that intersects only k nonzero $\mathcal{R}$-classes of U. Then there exists a chain of ideals of S, $S = S_n \supseteq S_{n-1} \supseteq \cdots \supseteq S_0$, $n \leq 2k$, such that S_n and every Rees factor S_i/S_{i-1} is a uniform semigroup or a power nilpotent semigroup of nilpotent index less than or equal to $k+1$.*

Lemma 4.2.9 [Okn2, Lemma 9] *Assume that S is a semigroup with a uniform ideal T that has no free noncommutative subsemigroups. Let J be a completely 0-simple closure of T. Then $S \cup J$ has a natural semigroup structure extending that of S. Moreover, if $U \supseteq S$ is a semigroup with a completely 0-simple ideal Z containing T, then J can be chosen so that $J \supseteq Z$ and every $\mathcal{H}$-class of J is contained in an $\mathcal{H}$-class of Z.*

Theorem 4.2.10 [JO1] **(1994)** *Let S be a nilpotent semigroup and P a prime ideal of $K[S]$ such that $K[S]/P$ is right Goldie with the classical ring of quotients $M_n(D)$, D is a division ring. Then the semigroup $S/\sim_P$ has a chain of ideals.*

(Eq:4–2–1) $$(S/\sim_P) = I_r \supseteq I_{r-1} \supseteq \cdots \supseteq I_1 = I \supseteq I_0 = \{\theta\}$$

where θ is zero element if S has a zero element, otherwise $I_0 = \emptyset$. Moreover, the ideal chain (Eq:4–2–1) has following properties,

(i). *Each Rees factor I_i/I_{i-1}, for $1 \leq i \leq r$, is either a power nilpotent semigroup or a uniform semigroup.*

(ii). *I is uniform in a completely 0-simple inverse subsemigroup $\hat{I}$ of $M_n(D)$ with finitely many non-zero idempotents, i.e. $|E(\hat{I})| = q$. Let $\hat{S} = (S/\sim_P) \cup \hat{I}$. We have $\hat{S}$ a nilpotent subsemigroup of $M_n(D)$.*

(iii). *Let $K\{\hat{I}\}$ be the subalgebra of $M_n(D)$ generated by $\hat{I}$. Then $K\{I\} \subseteq K[S]/P \subseteq K\{\hat{I}\}$. Furthermore, the matrix ring over D $M_n(D)$ is the common classical ring of quotients of these three classes algebras, and $K\{\hat{I}\}$ is a localization of $K\{I\}$ with respect to an Ore set.*

(iv). *Denote the maximal subgroup of* $\hat{I}$ *by* G . *Then, there exists a prime ideal* Q *of* $K[G]$ *such that* $K[G]/Q$ *is a Goldie ring and*

$$M_q(K[G]/Q) \cong K\{\hat{I}\}.$$

where q *is the number of nonzero idempotents of* $\hat{I}$.

Proof. (i) Let $\bar{S} = S/\sim_P$, which can be embedded into $K[S]/P$. By this way, we can identify $\bar{S}$ as a subsemigroup of $M_n(D)$. In chapter 1, we have already known that $M_n(D)$ is completely semisimple, so there are ideals $T_i = \{X \in M_n(D) : rk(X) < i\}$ for $i = 0, \cdots, n$ and T_i/T_{i-1} is completely 0-simple semigroup. Let $\overline{S_i} = \bar{S} \cap T_i$, that is $\overline{S_i}$ contains all matrices of S with rank i. Refining $\overline{S_i}$ and by Lemma 4.2.8, it yields an ideal chain, namely,

$$\bar{S} = I_r \supseteq I_{r-1} \supseteq \cdots \supseteq \cdots I_1 = I \supseteq I_0$$

with each I_i/I_{i-1} is either a uniform or a power nilpotent semigroup.

Consider the last nontrivial ideal $I \subseteq \bar{S}$. If I is power nilpotent, then $I^k = \theta$ for some positive integer k and $K\{I\}$ is also nilpotent in the prime algebra $K[S]/P$. This leads to $K\{I\} = \theta$, which contradicts I is nontrival. Therefore I is the smallest uniform ideal. Let $\hat{I}$ be the smallest completely 0-simple subsemigroup of $M_n(D)$ containing I. Since the maximal subgroup G of $\hat{I}$ is generated by a subsemigroup of I, G is a nilpotent group. Moreover, if I is both uniform and nilpotent in $\hat{I}$ then $\hat{I}$ is an inverse semigroup. Hence, $\hat{I}$ is nilpotent semigroup.

Now, we define $\hat{S} = \bar{S} \cup \hat{I} \subseteq M_n(D)$. Then, by Lemma 4.2.9, $\hat{S}$ is a natural extension of $\bar{S}$ and $\hat{I}$. We note here that $\hat{S}\backslash\hat{I} = \bar{S}\backslash I$ and hence $\hat{S}/\hat{I} = \bar{S}/I$ is nilpotent. Choose a positive integer m which is larger than the nilpotency of both $\hat{S}$ and $\hat{I}$. Since $\bar{S}/I = \hat{S}/\hat{I}$, it suffices to show that $x_{m+1}(x, y, w_1, \cdots, w_m) = y_{m+1}(x, y, w_1, \cdots, w_m)$ for $x, y \in \hat{I}$, $w_1, \cdots, w_m \in \hat{S}$.

Assume that both elements x_m and y_m are nonzero. Then, since I is uniform in $\hat{I}$, for each $w_i \in \hat{I}$, we can find $x', y', w_i' \in I$ such that $w_i \mathcal{H} w_i'$, $x \mathcal{H} x'$ and $y \mathcal{H} y'$. Since $\hat{I}$ itself is nilpotent, $x_m(x', y') = y_m(x', y') \neq \theta$. Then $(x', y') \in \mathcal{H}$ implies that $(x, y) \in \mathcal{H}$. As $\hat{I}$ is an inverse semigroup, there exists idempotents g, h with $xg = x, yg = y$ and $hx = x, hy = y$. Hence, $xw_iy = xgw_ihy$. So by the nilpotency of $\hat{I}$, we have

$$x_{m+1}(x, y) = y_{m+1}(x, y).$$

This proves that $\hat{S}$ is nilpotent.

Now, $\hat{I} \cong \mathcal{M}^0(G, q, q; \Delta)$, where Δ is a $q \times q$ identity matrix and $I_{ij} = I \cap \hat{I}_{(i)}^{(j)}$ is a nontrival cancellative subsemigroup of some maximal subgroup of $\hat{I}$. Then, we have the following chain:

$$\begin{array}{c} \bar{S} \\ \cup \\ I_r \\ \cup \\ \vdots \\ \cup \end{array}$$

$$I \cong \begin{pmatrix} I_{11} & \cdots & I_{1q} \\ \vdots & \ddots & \vdots \\ I_{q1} & \cdots & I_{qq} \end{pmatrix} \subseteq \mathcal{M}^0(G, q, q; \Delta)$$

By noting that $\hat{I} \subseteq M_n(D)$, $|E(\hat{I})| = q < \infty$ and the fact $E(\hat{I}) = \{e_1, \cdots, e_q\}$ is the set of non-zero idempotents of $\hat{I}$, we have $K_0[\hat{I}] \cong M_q(K[G])$ and satisfies the identity $e = e_1 + e_2 + \cdots + e_q$. We hence know that $K\{I\}$ is a homomorphic image $K_0[\hat{I}]$, that is $K\{I\} \cong K_0[\hat{I}]/A \cong M_q(K[G]/Q)$ for some ideal Q of $K[G]$ and A is ideal of $K_0[\hat{I}]$. Since $K\{I\}$ is prime, Q is a prime ideal of $K[G]$. Since $\hat{I}$ is ideal of $\hat{S}$ and $e \in K\{\hat{I}\}$, we obtain

$$K\{I\} \subseteq K[S]/P = K\{\bar{S}\} \subseteq K\{\hat{S}\} = K\{\hat{I}\} \subseteq M_n(D).$$

Since $K\{\bar{S}\}$ has a classical ring of quotients $M_n(D)$, $K\{I\}$ and $K\{\hat{I}\}$ have the same classical ring of quotients $M_n(D)$.

As $K_0[\hat{I}] \cong M_q(K[G])$, we may assume that $E(\hat{I}) = \{e_1, e_2, \cdots, e_q\}$, where e_i's are the diagonal orthogonal matrices. We now show that $K_0[\hat{I}]$ is localization of $K_0[I]$ with respect to the Ore set

$$C = \left\{ \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_q \end{pmatrix} : \ 0 \neq c_j \in I_{jj} \text{ for all } j \right\}.$$

Identifying $\hat{I}$ with the subsemigroup of $M_q(K[G])$. Then, obviously, every non-zero element of $\hat{I}$ is a matrix of the form $(g)_{i,j}$, that is, the matrix with $g \in G$ at the (i, j)th position and zero elsewhere. Hence, $I_{jj}I_{jj}^{-1} = G$. Let $(h)_{i,j} \in \hat{I}$. Then we have $h = xy^{-1}$ for some $x, y \in I_{ij}$. Since $y^{-1} \in I_{ij}^{-1} \subseteq G = I_{jj}I_{jj}^{-1}$, there exists $s \in I_{jj}$ such that $y^{-1}s \in I_{jj}$ and so

$$(h)_{i,j}(s)_{j,j} = (hs)_{i,j} = (x)_{i,j}(y^{-1}s)_{j,j} \in I.$$

Let $c \in C$ as defined above. Then, by $(h)_{i,j}c = (h)_{i,j}(s)_{j,j}$, we have $\hat{I} \subseteq IC^{-1}$. Consequently, $\hat{I} = IC^{-1}$. This yields that $K_0[\hat{I}] = K_0[I]C^{-1}$. Hence, $K\{\hat{I}\}$ is a localization of $K\{I\}$. □

Now, we apply the above theorem to the radicals of the algebras over nilpotent semigroup, we hence obtain the extended results in section 3.3.4.

Lemma 4.2.11 [JO1] **(1994)** *Let I be a uniform subsemigroup in a completely 0-simple inverse subsemigroup $\hat{I}$ with finitely many non-zero idempotents, say q, and over a nilpotent group G. Let G_p be the p-subgroup of G if $char(K) = p$. Then*

$$J(K_0[I]) = \mathcal{B}(K_0[I]) = \begin{cases} \{0\} & \text{if } char(K) = 0; \\ M_q(\omega(K[G_p])K[G]) \cap K_0[I] & \text{if } char(K) = p > 0 \end{cases}$$

Theorem 4.2.12 [JO1] **(1994)** *Let S be a nilpotent semigroup such that for every prime ideal P of $K[S]$ either $K[S]/P$ is a right Goldie algebra or $S \setminus (S \cap P)$ is a subsemigroup of S. Then*

$$J(K[S]) = \mathcal{B}(K[S]).$$

Proof. If $S \setminus (S \cap P)$ is a semigroup, then the results follow from Prop. 4.2.6 and Theorem 3.3.32.

Let $K[S]/P$ be right Goldie. Then it suffices to show that for any prime ideal P of S,

$$J(K[S/\sim_P]) \cong \mathcal{B}(K[S/\sim_P]).$$

By Theorem 4.2.10, The semigroup S is decomposed by $S/\sim_P$ which has finitely many ideal factors $T = I_i/I_{i-1}$. This factors are uniform or power nilpotent. By Lemma 4.2.11, we have $J(K_0[T]) = \mathcal{B}(K_0[T])$. Since T is uniform and is nilpotent, we have $J(K_0[T]) = \mathcal{B}(K_0[T])$. Combining the above results and by induction on its chain decomposition of $S/\sim_P$, we have $J(K_0[S]) = \mathcal{B}(K_0[S])$. □

4.3 Noetherian Semigroup Algebras

It is known that if $K[G]$ is right noetherian for any group G iff it is left noetherian by the involution $g \mapsto g^{-1}$. If S is a commutative semigroup, then the ascending chain condition on the congruences on S implies that S is finitely generated (cf. [Gil, Th. 5.10]). In other words, all the noetherian commutative semigroup algebras are finitely generated. In this section, we study the noetherian noncommutative semigroup algebras and find the necessary conditions for the semigorup algebras to be left (or right) noetherian. Recall that there are a.c.c. on the right congruences on S if $K[S]$ is right noetherian.

In the case of cancellative noncommutative semigroups. If S is monoid and has a group of one-sided fractions G. Then by section 3.3, $K[S]$ is prime (semiprime) iff $K[G]$ is prime (semiprime). Moreover, for the categories of modules of $K[S]$ and $K[G]$, we have the following result:

Theorem 4.3.1 [Squ, Th. 2.3, Th. 2.4] *Let S be a moniod having a group of fractions G and R any commutative ring with 1. Let ${}_{K[S]}\mathcal{M}$ be the category of left $R[S]$-modules and ${}_{K[G]}\mathcal{M}$ the category of left $R[G]$-modules, Then the functor*

$$F : (-) \mapsto R[G] \otimes_{R[S]} (-)$$

is exact. Conversely, if S generates a group G but G is not group of fractions of S, then $R[G] \otimes_{R[S]} (-)$ is not exact.

We can see that if S has a group of fractions G, then $K[G]$ is a flat right $K[S]$-module. If $K[S]$ is right noetherian, then $K[G]$ is right noetherian and flat right $K[S]$-module.

Since $K[S]$ is a subalgebra of $K[G]$ and $K[G]$ is a flat right $K[S]$-module, we can induce the left $K[S]$-module to $K[G]$-module. For example, if V is semisimple $K[S]$-module, then we can induce V (denote $V^{|G}$) as a semisimple $K[G]$-module.

The following characterization theorem for $K[S]$ to be right noetherian was due to Okninski.

Proposition 4.3.2 [Okn1] *Let G be a polycyclic-by-finite group, K any field and S is a submonoid of G. Then $K[S]$ is right noetherian iff S satisfies a.c.c. on right ideals. Moreover, in this case, S is finitely generated.*

Notice that even if G is a finitely generated group and $K[G]$ is noetherian, it is still **not** sufficient to imply that $K[S]$ is right noetherian, where S is subsemigroup of G with $SS^{-1} = G$. We give here an example.

Example 4.3.3 *Let $G = \langle x, y \rangle$, free abelian group and $S = \langle 1, xy, xy^2, xy^3, \cdots \rangle$. Then we can see that $G = SS^{-1}$ and $K[G] = K\{x, y\}$ is noetherian but S does not has the a.c.c. on ideals, Hence, $K[S]$ is not noetherian.*

Now, we consider the case of noetherian algebras of general semigroups.

Lemma 4.3.4 *Assume that $K_0[S]$ is a right noetherian ring and S is a completely 0-simple. If $S \cong \mathcal{M}^0(G, I, \Lambda; P)$, then $K[G]$ is also right noetherian ring. Moreover, I is finite when $K_0[S]$ is right noetherian and Λ is finite when $K_0[S]$ is left noetherian.*

Proof. Assume that $K_0[S]$ is right noetherian and $S \cong \mathcal{M}^0(G, I, \Lambda; P)$ is completely 0-simple. Then

$$K_0[S] \cong \mathcal{M}^0(K[G], I, \Lambda; P) = \widehat{K[G]}.$$

The reader is referred to Section 3.4 for the properties of the Munn algebras.

Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be a chain of right ideals of $K[G]$. We then have $\hat{I}_1 \subseteq \hat{I}_2 \subseteq \cdots$ and if $K_0[S]$ is right noetherian, $\hat{I}_i = \hat{I}_{i+1}$ for some i. Hence $I_i = I_{i+1}$. This shows that $K[G]$ is also right noetherian.

The second part is obvious because for any subset $J \subseteq I$, $S_{(J)} = \bigcup_{i \in J} S_{(i)}$ is a right ideal of S. Since S satisfies a.c.c. on its right ideals, whence I is finite. □

Combining Lemma 4.2.3, Lemma 4.2.4 and [Okn1, Ch. 12], we then obtain the following theorem.

Theorem 4.3.5 *Assume that $K[S]$ is right noetherian. Then,*

(i). *S has finitely many right ideals of the form eS, where e is an idempotent in S.*

(ii). *Any 0-simple principal factor of S is completely 0-simple.*

(iii). *S has finitely many completely 0-simple principal factor.*

(iv). *If S is weakly periodic semigroup and S has finitely many $\mathfrak{J}$-classes, then S is strongly π-periodic.*

(v). [Okn2] **(1993)** *If S has no free noncommutative subsemigroup, then every cancallative subsemigroup T of S has a finitely generated group G of two-sided fraction.*

(vi). [Okn2] **(1993)** *The set of isomorphism classes of groups of fractions of the maximal cancellative subsemigroups of $\bar{S}$ is finite, where $\bar{S}$ is image of S in $K[S]/\mathcal{B}(K[S])$.*

Proof. (i) Consider the natural order defined on the set $E(S)$, that is $e \geq f$ iff $ef = fe = f$. (Recall that $E(S)$ may not be semilattice). Then S has no infinite chains of orthogonal idempotents for if otherwise, it will contradict the finiteness of the right Goldie dimension of $K[S]$. Clearly, if $e_1 S \supseteq e_2 S$, then $e_2 e_1 \leq e_1$. This shows that S satisfies the descending chain condition on right principal ideals in $\mathfrak{C} = \{eS | e \in E(S)\}$.

Now, fix $f \in E(S)$ and consider the set:

$$\mathfrak{C}_f = \{eS \subseteq fS | e \in E(S) \text{ and } eS \text{ is maximal in } fS\}.$$

By the a.c.c. on right ideals of $K[S]$, we know that $\mathfrak{C}_f$ is finite. By the fact that there are no infinite descending chains of ideals eS, we know that $\mathfrak{C}$ is finite.

(ii) Assume that the principal factor S_t of S determined by an element $t \in S$ is 0-simple. It suffices to show that if $S_t = StS$ is 0-simple, then StS has primitive idempotent. As $\bar{S} = S/\sim_{\mathcal{B}(K[S])}$ can be embedded into $M_n(D)$ for some $n \geq 1$, so $T = StS/\sim_{\mathcal{B}(K[S])}$, which is 0-simple ideal of $\bar{S}$. This means that T can also be embedded into some completely 0-simple factor I_i/I_{i-1} of $M_n(D)$. (see Section 1.1).

Let $\bar{t}$ be the image of t in $\bar{S}$. By the 0-simplicity of T, we have $u_1\bar{t}v_1 = \bar{t}$. for $u_1, v_1 \in \bar{S}$. Then $u_1 = u_2u_1v_2$. Hence, $u_n\bar{S}^1 \subseteq u_{n+1}\bar{S}^1$. By the noetherian properties on right ideals, we have $u_{m+1} = u_m z$. Then, we can find an idempotent $u_m v_m z$. Moroever, the inverse image $x \in S$ satisfies $x - x^2 \in \mathcal{B}(K[S])$. Thus, StS has a nonzero idempotent and by (i), we know that S has no infinite chains of idempotents. This implies that StS has a primitive element.

(iii) This part is a direct consequence of assertions (i) and (ii), and the proof is hence omitted.

(iv) Since every 0-simple principal factor is completely 0-simple factor, Lemma 1.1.5 and 1.1.6 yield that S is strongly π-regular.

(v) The group of fractions of T exists because of the non-existence of free non-commutative semigroups. Since S has a.c.c. on right congruences, G has a.c.c. on its subgroups and so G is finitely generated.

(vi) First, we observe that $\bar{S} \subseteq M_n(D)$ for some division algebra D. Moreover, as $K[\bar{S}]$ is right noetherian, $\bar{S}$ intersects finitely many $\mathcal{J}$-classes. Let T be a maximal cancellative subsemigroup of $\bar{S}$. Then, the group of fractions of T is contained in the $\mathcal{H}$-class H' of $M_n(D)$. The maximal subgroups of the principal factor, say I_i/I_{i-1} for all i, are mutually isomorphic to each other. Clearly, the isomorphic classes of maximal subgroups are finite since $\bar{S}$ intersects finitely many $\mathcal{J}$-classes of $M_n(D)$. □

By using the above theorem, we obtain the following corollaries.

Corollary 4.3.6 *Let $S \cong \mathcal{M}^0(G^0, I, \Lambda; P)$ be a completely 0-simple semigroup. Then we have* (i). *If T is subsemigroup of S and $K[T]$ is right noetherian. Then $T \cap S_{(i)}^{(m)} \neq \emptyset, \theta$ for only finitely many subsemigroups of $S_{(i)}^{(m)}$, $i \in I$ and $m \in \Lambda$.*

(ii). *If $K_0[S]$ is right noetherian, then I and Λ are finite index sets.*

(iii). *Let U be uniform and can be embedded into S. Then if $K[U]$ is right noetherian, then S has finite rows and columns.*

(iv). *Assume that $J(K[G]) = 0$ and $K_0[S]/J(K_0[S])$ is a finitely generated K-algebra. Then $K_0[S]$ is a finitely generated and hence S is finitely generated.*

Lemma 4.3.7 *Let S be a minimal 0-simple ideal M and also assume S has a.c.c on its left principal ideals. If $K[S]$ is right noetherian, then S is finitely generated semigroup.*

Proof. Suppose $K[S]$ is right noetherian but S is not finitely generated. Then S has a.c.c. on its congruences. Thus, there exists a maximal congruence ρ on S with respect to the property that S/ρ is not finitely generated. Let $T = S/\rho$. Then we have

(a) Any nontrival homomorphic image of T is finitely generated.

(b) For any nil ideal I of $K_0[T]$, the congruence $\sim_I$ is a trival congruence on S. That is $(s,t) \in \sim_I$ implies $s = t$ on S.

Hence, by replacing $\bar{S}$ by S, S may assume not finitely generated and there is no nontrival congruence ρ on S that S/ρ is not finitely generated.

If M is not nilpotent, then $M \cong \mathcal{M}^0(G, I, \Lambda; P)$ is a completely 0-simple semigroup. Also, $K_0[S]$ is right noetherian and if G is finite, then M is finite. Thus, S/M is finitely generated and hence S is finitely generated. This contradicts our assumption. On the other hand, if G is infinite then $K[G]$ is right noetherian since $K_0[M]$ is right noetherian and so G is finitely generated. Consequently, $S \setminus M$ is finitely generated. It suffices to show that M is finitely generated. For this purpose, take $u \in M$ such that $u = (g, i, \lambda)$, where $i \in I, \lambda \in \Lambda$. By some construction (see [Okn1, Th. 12.6]), we can check that M is finitely generated.

In [Okn1], there is an important theorem on the condition for right noetherian algebra $K[S]$ which leads to S is finitely generated. The proof of this theorem is rather constructive and have to make use of the above lemmas. We only give a brief sketch of the proof. The reader is referred to [Okn1, Th. 12.6] for more details.

Theorem 4.3.8 [Okn1] *Assume that $K[S]$ is right noetherian. Then S is a finitely generated semigroup if either of the following holds:*

(i). *S is a weakly periodic semigroup.*

(ii). *S has a.c.c. on principal left ideals.*

(iii). *Every cancellative subsemigroup of any homomorphic image of S has a finite-by-abelian-by-finite group of fractions.*

Proof. (Sketch) We first assume that S is not finitely generated and so there is no nontrival congruence such that S/ρ is not finitely generated. Then S satisfies (a) and (b) in Lemma 4.3.7.

If S is weakly periodic, then there exists a minimal ideal M which is either nil or 0-simple. (See Theorem 1.1.5). If M is nil, then it contradicts (*b*). Therefore M is 0-simple ideal and $K_0[S]$ is right noetherian. Consequently, M is completely 0-simple and $M \cong \mathcal{M}^0(G, I, \Lambda; P)$. The trivalness of G is due to the minimality of M. Moreover, if $K_0[M]$ is right noetherian, then I, Λ are finite and therefore $|M|$ is finite. By using the hypothesis on S, we have S/M is finitely generated and M is finite. Then S is finitely generated (see the proof of Lemma 4.3.7),

If (ii) is satisfied, then S has some minimial nonzero ideals. By the hypothesis on S, we know that M is 0-simple. This is exactly the case of Lemma 4.3.7. Then S is finitely generated.

Finally, we can only deal with the case that S has no minimal ideals. Since $\sim_{\mathcal{B}(K[S])}$ is trival, We may take $S \subseteq M_n(D)$ for some n and a division algebra D. Let T be an ideal of S containing all the least nonzero rank matrices. Then $T \hookrightarrow I_r/I_{r-1}$ for some r. Let $w \in T$. Then, $S^1wS^1 \subseteq T$ and I_r/I_{r-1} is completely 0-simple semigroup. Now, select S^1wS^1 be the minimal ideals among the ideals S^1xS^1 for all $x \neq \theta$, determines the principal factors containing idempotents. Let J be the nongenerators of S^1wS^1. Then it suffices to show that $S \subseteq \langle S \setminus J^2 \rangle$. The details are given in [Okn1, Th. 12.6]. □

Corollary 4.3.9 *If $K[S]$ is noetherian, then S is finitely generated.*

Corollary 4.3.10 *Let S be weakly periodic semigroup. If $K[S]$ is right noetherian and all subgroups of S are locally finite, then S is finite.*

Corollary 4.3.11 *Let S be inverse semigroup. Then the following conditions are equivalent.*

(i). *$K[S]$ is right noetherian*

(ii). *S has finitely many idempotents, and all group algebras $K[G]$ are noetherian, where G is a subgroup of S.*

(iii). *$K[S]$ is noetherian.*

It is conjectured that the finitely generated property of S depends on whether $K[S]$ is right noetherian or not ([Okn1, Problem 7]). We now consider some typical semigroups and add some extra conditions to them in order to make the answer positive.

Theorem 4.3.12 *Assume that S is a semigroup such that $K[S]$ is a right noetherian PI-algebra. Then S is finitely generated.*

Proof. Since $K[S]$ is a PI-algebra, S has a permutation property $\mathfrak{P}$. This means that every cancellative subsemigroup of a homomorphic image of S has $\mathfrak{P}$, hence it contains a finite-by-abelian-by-finite group. By using Theorem 4.3.8, we know that S is finitely generated. □

In section 3.7, we have discussed the criteria for group algebras to be PI-algebras and also consider its subsemigroups. We now consider the (one-sided) noetherian PI-algebras and obtain some results such that its corresponding semigroups are also finitely generated.

Theorem 4.3.13 [Okn1] *Let S be a cancellative monoid. Then the following conditions are equivalent:*

(i). *$K[S]$ is a right noetherian PI-algebra.*

(ii). *$K[S]$ is a right and left noetherian PI-algebra.*

(iii). *S is a finitely generated semigroup with a.c.c. on its right ideals and satisfies the permutation property.*

(iv). *S is a finitely generated subsemigroup of an abelian-by-finite group, and S has a.c.c. on its right ideals.*

Theorem 4.3.14 [Okn1] *Let S be an inverse semigroup. Then the following conditions are equivalent:*

(i). *S is a finitely generated semigroup and satsifies the permutational properties.*

(ii). $K[S]$ *is a right and left noetherian PI-algebras*

It is known that a submonoid S of a finitely generated nilpotent group yields a right noetherian semigroup algebra $K[S]$ iff S has the a.c.c. on right ideals. We now describe the radicals of right noetherian algebras $K[S]$, where S is nilpotent semigroup (in sense of Malcev).

Proposition 4.3.15 [JO1, Prop. 3.9] **(1994)** *Let S be a nilpotent semigroup and $K[S]$ is right noetherian ring. Then for $s, t \in S$, the following conditions are equivalent:*

(i). $s - t \in \mathcal{B}(K[S]) = J(K[S])$.

(ii). *For every $u \in S$, there exists $n \geq 1$ and $v \in T = \langle su, tu\rangle$ such that*

(a) $(su)^n \in T(tu)T$, *and* $(tu)^n \in T(su)T$,

(b) *if* $char(K) = p > 0$, *then* $(v(su)v)^{p^k} = (v(tu)v)^{p^k}$ *for some* $k \geq 0$.

(c) *If* $char(K) = 0$, *then* $v(su)v = v(tu)v$.

Proof. Since S is nilpotent, S has no noncommutative free subsemigroups and $J(K[S]) = \mathcal{B}(K[S])$ (see Theorem 4.2.12). Assume $s - t \in \mathcal{B}(K[S])$. So for every $u \in S$, $(su - tu)^n = 0$ in $K[S]$. Hence condition (a) is proved. Since $K[S]$ is a right noetherian ring, by Lemma 4.2.4, for $T = \langle su, tu\rangle$, there exists $v \in T$ such that vTv is cancellative. Hence,

$$v(su)v - v(tu)v \in K[vTv] \cap \mathcal{B}(K[S]) \subseteq \mathcal{B}(K[vTv]).$$

By Lemma 3.3.37 and Theorem 3.3.8, we have $(v(su)v)^{p^k} = (v(tu)v)^{p^k}$ for some $k \geq 0$ if $char(K) = p$, otherwise $v(su)v = v(tu)v$.

Conversely, let P be a prime ideal of S such that $S/\sim_P \hookrightarrow K[S]/P$. Take any s, t and suppose that the condition (ii) in the statement of theorem is satisfied. By Theorem 4.2.10, it yields that $\bar{S} = S/\sim_P$ has a chain of ideals (Eq:4–2–1) that contains an uniform ideal which can be embedded into a completely 0-simple inverse semigroup. Let $\bar{s}, \bar{t}$ be the images of s, t in $K[S]/P$ and and for any $u \in I$, we have $\bar{s}u, \bar{t}u \in I$. Assume $\bar{s}u \neq \theta$ and $\hat{I}$ is a completely 0-simple inverse semigroup. Then there exists $r \in I$ such that $\bar{s}ur$ is in the maximal subgroup H of $\hat{I}$. Let $T = \langle \bar{s}ur, \bar{t}ur\rangle$ and $(\bar{s}ur)^n \in T(\bar{t}ur)T$. Then $\bar{t}ur \neq \theta$ and so $\bar{s}ur$ and $\bar{t}ur$ are $\mathcal{H}$-related.

By hypothesis, there exists $v \in \langle \bar{s}ur, \bar{t}ur \rangle$ such that if $char(K) = 0$, $v(\bar{s}ur)v = v(\bar{t}ur)v$. Moreover, $(v(\bar{s}ur)v)^{p^k} = (v(\bar{t}ur)v)^{p^k}$ if $char(K) = p$. Since $v, \bar{s}ur, \bar{t}ur \in H$, we know that if $char(K) = 0$, then $\bar{s}ur = \bar{t}ur$. If $char(K) = p$, then by Theorem 4.2.10, it yields that

$$v(\bar{s}ur)v - v(\bar{t}ur)v \in \mathcal{B}(K[H])$$

and $K_0[\hat{I}] \cong M_q(K[H])$ for some $q \geq 1$. It follows that

$$v(\bar{s}ur)v - v(\bar{t}ur)v \in \mathcal{B}(K_0[\bar{I}]) \cap K_0[S/\sim_P] \subseteq \mathcal{B}(K_0[S/\sim_P]).$$

Now, as $K[S]/P$ is the epimorphic image of $K_0[S/\sim_P]$, so we can see that $v(\bar{s}ur)v = v(\bar{t}ur)v$. Due to the belonging in the same subgroup H, $\bar{s}ur = \bar{t}ur$ for every $u \in I$. This implies that $(\bar{s} - \bar{t})I = \{0\}$. Since $K[S]/P$ is prime, $\bar{s} = \bar{t} \in S/\sim_P$. This completes the proof. □

4.4 Descending Chain Conditions

After the discussion of the ascending chain condition on semigroup algebras, in this section we now consider the semigroup algebras with descending chain condition. Properties of semilocal, local and perfect semigroup algebras will be investigated. First of all, we recall some important theorems about group and semigroup algebra.

Theorem 4.4.1 [Pas1, Th. 10.1.1] **(1963)** *(Connell) Let R be ring with unity and G is an arbitrary group. Then $R[G]$ is artinian iff R is artinian and G is a finite group.*

E.I. Zelmanov extended the above result to the class of semigroup algebras.

Theorem 4.4.2 [Zel, Th. 3] **(1977)** *Let R be ring with unity and S any semigroup. Then $R[S]$ is artinian implies that R is artinian and S is fintie. The converse statement holds if S is a monoid.*

4.4.1 Artinian Semigroup Graded Rings

Recently, there are several results on artinian semigroup graded ring [CJKO, Kel6, JO2] **(1995)**. We state these results to investigate the descending chain condition of semigroup algebras. These characterization theorem of semigroup graded rings provide another method in studying the semigroup algebra which is different from Okninski [Okn1].

Throughout this section, we assume R is a S-graded ring but may not have an unity. We also assume that S contains no zero unless other mentioned.

Lemma 4.4.3 [CJKO, Lemma 1] *Let S be a semigroup and R a S-graded ring. Suppose that $\mathcal{F}$ is a family of right ideals of R satisfying the following conditions:*

(i). *there is a natural number k such that $|supp(I)| \leq k$ for all $I \in \mathcal{F}$;*

(ii). *$\cup_{I\in\mathcal{F}} supp(I)$ is infinite.*

Then R is not right artinian.

Lemma 4.4.4 [CJKO, Lemma 2] *Let S be semigroup and R a right artinian S-graded ring. Let I be nilpotent homogeneous ideal of R such that there are only finitely many $s \in S$ with $R_s \not\subseteq I$. Then $supp(I)$ and $supp(R)$ are both finite.*

Proposition 4.4.5 [CJKO, Lemma 4] *Let S be a semigroup with no infinite subgroups and let R be a right artinian S-graded ring. Then there exist finitely many elements $x_1, \cdots, x_n \in S$ such that*

$$R = J(R) + R_{x_1} + R_{x_2} + \cdots + R_{x_n}.$$

Hence, we obtain the following theorem on semigroup graded ring which is similar to Theorem 4.4.2.

Theorem 4.4.6 [CJKO] *Let S be a semigroup with no infinite subgroups and let R be right artinian S-graded ring. Then $supp(R)$ is finite.*

Proof. Clearly, the ring R^2 is both right artinian and right noetherian. Since R/R^2 is a nilpotent right artinian S-graded ring, it follows from Lemma 4.4.4 that $supp(R/R^2)$ is finite. It suffices to show that R^2 has a finite supports.

Consider that $R = R^2$ which is right artinian and right noetherian. Then by Prop. 4.4.5, there is a finite subset $X_0 \subseteq S$ such that $R = J(R) + R_{X_0}$.

Since R is right artinian, $J(R)$ is nilpotent. Let m be its index of nilpotency. Fix k with $1 \leq k < m$. Thus R is right noetherian and so the quotient $J(R)^k/J(R)^{k+1}$ is finitely generated as right R-module. Then there exists $b_1, \cdots b_n \in J(R)^k$ such that

$$J(R)^k = J(R)^{k+1} + \sum_{i=1}^{n}(b_1\mathbb{Z} + b_iR).$$

Note that $b_iR = b_iJ(R) + b_iR_{X_0}$. Since $b_iJ(R) \subseteq J(R)^{k+1}$ and the support of each b_i is finite, $supp(b_iR) \subseteq supp(J(R)^{k+1}) \cup X_{k,i}$, where $X_{k,i} = supp(b_i)X_0$. Therefore

$$J(R)^k \subseteq R_{X_k} + J(R)^{k+1}$$

where $X_k = \cup_{i=1}^{n}(X_{k,i} \cup supp(b_i))$ is finite.

Since $J(R)^m = 0$, $J(R) \subseteq R_{X_1} + R_{X_2} + \cdots + R_{X_{m-1}}$, and therefore the support of R is contained in the finite set $X_0 \cup X_1 \cup \cdots \cup X_{m-1}$. □

Since every band has a trival subgroup, we obtain the following corollary.

Corollary 4.4.7 [Kel6] *Let B be a band and let R be a right artinian B-graded ring. Then the support of R is finite.*

Using the above theorem, we are able to give a short proof of Theorem 4.4.2 which is independent of Zelmanov. We only need to check if $K[S]$ is right artinian, then S contains no infinite subgroup. If $G \subseteq S$, then $G \subseteq eSe$ for $e \in G$. Thus, $K[G]$ is a homomorphic image of $K[eSe]$, so $K[G]$ is right artinian. This implies that G is finite.

4.4.2 Semilocal Semigroup Algebras

In this section, we discuss the structure of semilocal algebra by using the techniques given in [Okn1] as every artinian, semiperfect, semiprimary rings are semilocal. Recall that in the group algebra cases, we have following results:

Theorem 4.4.8 [Pas1, Th. 2.3.11] *If G is a locally finite group, then the group ring $K[G]$ is algebraic. Conversely, if K has characteristic 0 and if $K[G]$ is algebraic, then G is locally finite.*

Theorem 4.4.9 [Pas1, Th. 10.1.6] *Let K be a field and G a locally finite group. Then $K[G]$ is semilocal iff* (i). G *is finite when* $char(K) = 0$.

(ii). G *contains a normal* p*-subgroup* N *of finite index with* $\omega(K[N]) = J(K[N]) \subseteq J(K[G])$ *when* $char(K) = p > 0$.

In both cases, $K[G]/J(K[G])$ *is finite dimensional* K*-algebra.*

Theorem 4.4.10 [Pas1, Th. 10.1.3] *The group ring* $K[G]$ *is perfect iff* G *is finite.*

In order to investigate the structure of semilocal semigroup algebras, we first cite some lemmas on semilocal algebras taken from [Kar, Okn1]. These results are useful in the sequel.

Lemma 4.4.11 *Let* A, B *be algebras over field* K.

(i). *If* B *is an ideal of* A*, then* A *is semilocal iff the algebras* $B, A/B$ *are semilocal.*

(ii). *If* $A \otimes_K B$ *is semilocal and* A, B *are algebras with unities, then,*

(a) A, B *are semilocal algebras.*

(b) *If* B *is a separable field extension of* K*, then* $J(A \otimes_K B) = J(A) \otimes_K B$.

Furthermore, from the graded ring theory, we have the following proposition.

Proposition 4.4.12 [JO2] **(1995)** *Let* S *be a semigroup and* R *an* S*-graded ring. If* R *is semilocal and* $a \in R_s$ *is not nilpotent in* R*, then* s *is periodic in* S.

By using the results given in [Okn1, Ch. 14], we obtain the following propositions.

Proposition 4.4.13 *Assume* $K[S]$ *is semilocal. Then*

(i). S *is a periodic semigroup.*

(ii). S *is locally finite if* $char(K) = 0$.

(iii). $K[G]$ *is semilocal for every subgroup* G *of* S.

Proof. (i) This part follows from Proposition 4.4.12 since $R = K[S]$ can be regarded as a S-graded ring and $R_s = Ks$. We notice that all $s \in R_s$ is not nilpotent. Thus s is periodic and so S is a periodic semigroup.

(ii) Let K be a field with $char(K) = 0$. Then its prime field must be $\mathbb{Q}$. If $K[S]$ is semilocal, then $\mathbb{Q}[S]$ is semilocal and also $\mathbb{Q}[S^1]$. Let $a \in \sum_1^n \lambda_i s_i \in \mathbb{Q}[S^1]$, where $\sum \lambda_i < 1$. Then we naturally define the norm of a by $\| a \| = \sum_1^n \lambda_i^2$. Considering the completion of $\bar{R}$ in the $\mathbb{Q}$-algebra $\mathbb{Q}[S^1]$. Then by hypothesis, $\| a \| < 1$, $(1-a)$ is an invertible element in $\bar{R}$. If $(1-a)$ is von Neumann regular in $\mathbb{Q}[S^1]$, then there exists $b \in \mathbb{Q}[S^1]$ such that $(1-a)b(1-a) = 1-a$. This implies that $(1-a)b = b(1-a) = 1$. Take $b_m = 1+a+a^2+\cdots+a^m$ so that $b-b_m = b(1-a)(b-b_m) = ba^{m+1}$. Consequently, we have

$$||b - b_m|| \leq ||b||\ ||a||^{m+1} \xrightarrow[m\to\infty]{} 0 \text{ because } ||a|| < 1.$$

We can now check that $t \in \langle s_1, \cdots, s_n \rangle$ in S. Since $\lambda_i > 0$, there exists $r \geq 1$ such that $t \in supp(a^r)$. Moreover, $t \in supp(b_m)$ for some $m \geq r$. The coefficient of t in the element a^r is greater than or equal to λ^r, where $\lambda = \min \lambda_i$ it $t \notin supp(b)$. Thus it follows that $||b - b_m|| \geq \lambda^r$ for every $m \geq r$. Hence, $b_m \not\to b$. This means that $t \in supp(b)$, which shows that $\langle s_1, \cdots, s_n \rangle \subseteq supp(b)$ is a finite semigroup.

Moreover, if $\mathbb{Q}[S]$ is semilocal, then for all $a \in \mathbb{Q}[S]$, it can easily prove that there exists $\lambda \in \mathbb{Q}$ such that $\lambda - a$ is invertible. Hence, for any finite subset T of S, we can construct $a \in \mathbb{Q}[T^1] \subseteq \mathbb{Q}[S^1]$ such that $\langle T \rangle$ is finite. Thus S is finite.

To prove (iii), we consider the identity of G which is also the idempotent of S. It is easy to see that $G \subseteq eSe$ and $eJ(K[S])e = J(eK[S]e)$. Thus, $K[G]$ is a homomorphic image of $eK[S]e$. Hence, $K[G]$ is semilocal. □

Proposition 4.4.14 *For $K[S]$ is semilocal and K is separable over its prime field F, we have the followings about $S/\sim_{J(K[S])}$.*

(i). $\sim_{J(K[S])}$ *coincides with* $\sim_{J(F[S])}$.

(ii). *If $K = \mathbb{Q}$, then we have*

(a) *If $(s,t) \in \sim_{J(K[S])}$, then $(s,t) \in \sim_{J(\mathrm{F}_{p_i}[S])}$ for every $,t \in S$ and every prime number p.*

(b) *There exist primes $p_1, \cdots, p_n$ such that*

$$\sim_{J(K[S])} = \bigcap_{i=1}^{n} \sim_{J(\mathrm{F}_{p_i}[S])}$$

Proof. By Prop. 4.4.11 (ii) (b), $J(K[S]) = K \otimes_F J(F[S])$, where F is a prime field of K. Thus, (i) follows.

(ii) As $\sim_{J(K[S])}=\sim_{J(\mathbb{Q}[S])}$, we only need to consider $\mathbb{Q}[S]$. Since S is locally finite when $char(K) = 0$. Then, $\mathbb{Q}[S]$ can be considered as a linear $\mathbb{Q}$-space with basis selected from $\mathbb{Z}[S]$, say $a_1, a_2, \cdots, a_n$. By the natural homomorphism $\mathbb{Z}[S] \to \mathbb{F}_p[S]$ for all prime p, we have

$$J(\mathbb{Q}[S]) \cap \mathbb{Z}[S] \subseteq J(\mathbb{Z}[S])$$

Denote $\sim_p=\sim_{J(\mathbb{F}_p[S])}$. Then it is not difficult to see that $\sim_{J(\mathbb{Z}[S])}\subseteq\sim_p$. Hence (a) is proved.

From (a) and the above, we can see that

$$\sim_{J(\mathbb{Q}[S])}= \bigcap_{p\in \mathrm{P}} \sim_{J(\mathbb{F}_p[S])}$$

and by the commutative diagram

$$\begin{array}{ccc}
\mathbb{Q}[S] & \xrightarrow[\phi_p]{} & \mathbb{Q}[S/\sim_p] \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_p} \\
\mathbb{Q}[S]/J(\mathbb{Q}[S]) & \xrightarrow[\psi_p]{} & \mathbb{Q}[S/\sim_p]/J(\mathbb{Q}[S/\sim_p])
\end{array}$$

We can see that, by the semilocalness of $\mathbb{Q}[S]$, there are finitely many distinct non-trival congruences of the form $\sim_p$ on S. Hence (b) follows. (see [Okn1, Ch. 14] for details.)

Lemma 4.4.15 [Okn1, Lemma 14.5, Prop. 14.9, Th. 14.10] *We know that $K[S]$ is semilocal and then $K[S]/J(K[S])$ is artinian. Moreover, $S/\sim_{J(K[S])}\hookrightarrow K[S]/J(K[S])$. Also, we have following properties.*

(i). *$K[S]/J(K[S])$ is an artinian and is an algebraic K-algebra for any field K. Also, if $char(K) > 0$, then $S/\sim_{J(K[S])}$ is a locally finite semigroup.*

(ii). *If all subgroup G of S such that $K[G]/J(K[G])$ is finite dimensional, then $K[S]/J(K[S])$ is finite dimensional K-algbera and $S/\sim_{J(K[S]}$ is finite.*

Theorem 4.4.16 *Suppose $char(K) = p$. Then $S/\sim_{J(K[S])}$ is finite if either one of the following conditions holds:*

(i). *K is not algebraic over its prime subfield.*

(ii). *S is locally finite.*

(iii). *S has no infinite subgroups.*

Proof. By Proposition 4.4.13, for every subgroup G of S. $K[G]$ is semilocal.

(i) If K is not algebraic over its prime subfield, then by [Pas1, Th. 10.1.6], $K[G]/J(K[G])$ is a homomorphic image of the finite dimensional algebras $K[G/G_p]$, where G_p is a normal p-subgroup and $[G : G_p] < \infty$. Thus, $K[G]/J(K[G])$ is finite dimensional algebra.

(ii) if S is locally finite, then every subgroup is locally finite. By Theorem 4.4.9, $K[G]/J(K[S])$ hence is finite dimensional.

(iii) As all subgroups of S is finite, by Lemma 4.4.15, $S/\sim_{J(K[S])}$ is finite. □

From section 1.1 (also see [Okn1, Ch. 2,3]), we now know that the structure of weakly periodic semigroups. In particular, when S is a periodic semigroup, we have the following equivalent conditions:

(i). S is locally finite and every subgroup of S has a normal p-subgroup of finite index.

(ii). S has a chain of ideals $S = S_n \supseteq S_{n-1} \cdots \supseteq S_1$ such that any one of S_1 and $S_i/S_{i-1}, i > 1$ are locally nilpotent or complete 0-simple semigroup. Moreover, if $\mathcal{M}^0(G, I, \Lambda; P)$ is a Rees matrix presentation of some completely 0-simple semigroup which is S_i/S_{i-1} or S_1, then G is locally finite and has a normal p-subgroup of finite index.

The relationship between locally finite and semilocal semigroup algebras can be found by applying the above conditions.

Definition 4.4.17 *Call $Z \subseteq E(S)$ is a left p-subset if the following condition is satisfied for all $e, f \in Z$ and $s \in S$:*
The element $ese \in U(eSe)$ iff $efse \in U(eSe)$ and in this case, $eseN = efseN$ for a normal p-subgroup N of $U(eSe)$, where $U(eSe)$ is a group of units of monoid eSe.

Lemma 4.4.18 [Okn1, Lemma 14.12] *Assume that $Z \subseteq E(S)$ is nonempty set contained in an equivalence class of the congruence $\sim_{J(K[S])}$. Then Z is a left p-subset of $E(S)$ if p=char(K). Moreover, if $S/\sim_{J(K[S])}$ is finite, then $E(S)$ is a union of finitely many left (right, respectively) p-subsets.*

Lemma 4.4.19 *If G is locally finite and any subgroup of G has a normal p-subgroup N with finite index. Consider $S = \mathcal{M}^0(G, I, \Lambda; P)$ and let $E(S)$ be the union of finitely* *many left (right, respectively) p-subsets. Then P has finitely many p-equivalent classes of rows (columns, respectively).*

Proof. Let Z be a left p-subset of $E(S)$. Let $e, f \in S$ that is $e = (g, i, m)$ and $f = (h, j, n)$ for some $i, j \in I$ and $m, n \in \Lambda$. Then by the definition of p-subsets, we know that e and efe lie in the same coset of the maximal normal p-subgroup N of $eSe \setminus \{\theta\}$, which is isomorphic to G, where θ is the zero of S.

For $s = (1, l, t)$ where $p_{tl} \neq \theta$ and consider the elements $ese = (gp_{mi}p_{tl}g, i, m)$ and $efse = (gp_{mj}hp_{nl}p_{ti}g, i, m)$. If $p_{ml}, p_{nl} \neq \theta$, then $p_{ml}N = p_{mj}hp_{nl}N$. Since $l \in I$ is arbitrary, we can show that the mth row of P is a multiple of the nth row of the P-module N. However, if G/N is finite, then it follows that there are finiely many p-equivalent classes of rows of P corresponding to the respective columns of $\mathcal{M}^0(G, I, \Lambda; P)$ containing idempotents from Z. Because any column of S contains an idempotent, by the fact that $E(S)$ is covered by finitely many left p-subsets, we know that there are finitely many p-equivalent classes of rows of P. □

Since all right (left) perfect and semisimple ring are semilocal, we obtain the following characterization for these semigroup algebras.

Theorem 4.4.20 *Let S be periodic semigroup and K a field with $char(K) = p > 0$.*

(i). *$K[S]$ is semilocal iff S has a chain of ideals*

$$S = S_n \supseteq S_{n-1} \cdots \supseteq S_1$$

such that any one of S_1 and $S_i/S_{i-1}, i > 1$ is either locally nilpotent or a complete 0-simple semigroup. Moreover, if $\mathcal{M}^0(G, I, \Lambda; P)$ is a Rees matrix presentation of completely 0-simple semigroup of S_i/S_{i-1} or S_1, then G is locally finite and has a normal p-subgroup of finite index. Furthermore, there are finitely many p-equivalent classes of rows (columns, respectively) of P.

(ii). *$K[S]$ is right perfect iff S has chain of ideals in (i) and each nil principal factor is right T-nilpotent and each completely 0-simple factor has a maximal subgroup which is finite.*

(iii). *$K[S]$ is semisimple aritinian iff S has a chain of ideals in (i) such that every S_i/S_{i-1} and S_1 is a completely 0-simple with Rees presentation $\mathcal{M}^0(G, m, m; P)$, for some $m > 1$ and there is an invertiable matrix P in the matrix ring $M_m(K[G])$, where G is a finite group with order not divisible by $char(K) = p > 0$ iff S is a finite strongly p-semigroup semigroup such that there is no subgroup of S with order divisible by $p > 0$.*

Proof. (i) Since S is locally finite, $S/\sim_{J(K[S])}$ is finite. Hence, $E(S)$ is a union of finitely many left p-subsets and S has finitely many $\mathcal{J}$-classes containing some idempotents. The existence condition of (i) and the locally nilpotency of the principal factors are due to the locally finiteness of S. We hence know that the 0-simple principal factor is completely 0-simple.

By Lemma 4.4.18 and Lemma 4.4.19, we have that G is locally finite and has a normal p-subgroup with finite index. Also there are finitely many p-equivalent classes of rows.

(ii) Assume $K[S]$ is right perfect. Then S is periodic and $K[G]$ is semilocal. As $J(K[S])$ is a homomorphic image of $J(K[eSe]) = J(K[S]) \cap K[eSe]$, $K[G]$ is right perfect. Thus G is finite. Moreover, since S has d.c.c. on principal left ideals, S is locally finite as well.

Hence, we obtain a desired chain in S such that all its factors S_i/S_{i-1} is either nil or completely 0-simple. Hence, S is locally finite by the finiteness condition on S and the fact that any nil ideal is right T-nilpotent.

If the converse statement holds, then $K[S]$ is clearly semilocal. Let P be a right T-nilpotent principal factor. Then, $K_0[P]$ is also right T-nilpotent. In fact, if P is a completely 0-simple factor, then the contracted semigroup algebra $J(K_0[P])$ is nilpotent since G is finite (see the proof in Lemma 4.4.18 or treat as graded ring as Section 3.4). This means that $K_0[P]$ is right perfect. Therefore, $K_0[S]$ is right perfect.

(iii) If $K[S]$ is artinian, then S is finite. Hence, $T = S_i/S_{i-1}$ or S_1 is nilpotent or completely 0-simple because S is periodic and finite. Now, every $K_0[T]$ and $K[S_1]$ are semisimple and so all $K_0[T]$ and $K[S_1]$ all have an identity. Thus T or S_1 is isomorphic to some $\mathcal{M}^0(G, m, m; P)$, where P is an invertible matrix in $M_m(K[G])$. Thus, each S is a strongly finite p-semisimple semigroup.

Now suppose that S is a strongly finite p-semisimple semigroup. Then every principal factor of S is a completely 0-simple and has an identity element. This implies that P is invertible and the completely 0-simple factor is isomorphic to $\mathcal{M}(G, m, m; P)$. Hence

$$K[S] \cong M_{n_1}(K[G_1]) \oplus \cdots \oplus M_{n_k}(K[G_k])$$

This shows that $K[S]$ is artinian. The proof is completed. □

By a local algebra, we mean an algebra that its Jacobson radical is a maximal ideal. In particular, if $K[S]/J(K[S]) \cong K$, then $\omega(K[S]) = J(K[S])$.

Theorem 4.4.21 [Okn1, Th. 14.18] *Assume that $char(K) = p$ or S is a locally finite semigroup. Then*

(i). *$K[S]$ is a local algebra.*

(ii). *S is locally finite and eSe is a p-group, where $char(K) = p$ and e is an idempotent of S.*

(iii). *S has a completely simple ideal $T \cong \mathcal{M}(G, I, \Lambda; P)$, where G is a locally finite p-group and S/T is a locally nilpotent semigroup.*

Corollary 4.4.22 [Okn1, Th. 14.17] *Let S be commutative semigroup and K be field with characteristics p. Then the following conditions are equivalent:*

(i). *$K[S]$ is semilocal.*

(ii). *S is periodic and $E(S)$ is finite and every subgroup of S has a p-group of finite index.*

(iii). *S/ξ is a finite semigroup, where ξ is the least p-separative congruence on S.*

From Theorem 4.4.20 and the characterization for semilocal semigroups, we deduce the following corollary.

Corollary 4.4.23 [Okn1, Coro. 14.22]

(i). *If $K[S]$ is right perfect, then S is locally finite.*

(ii). *If S is a completely 0-simple semigroup with no infinite subgroup, and $K[S]$ is semilocal, then it is semiprimary.*

(iii). *If S has no infinite subgroups and has d.c.c. on principal left ideals, and if $K[S]$ is semilocal, then $K[S]$ is right perfect.*

If S is an inverse semigroup, then clearly each principal factor of S is a completely 0-simple inverse semigroup. As $K[S] \cong \bigoplus M_{n_i}(K[G_i])$, $K[S]$ is semisimple artinian iff S is a finite inverse semigroup and each maximal subgroup is not divisible by p. On the other hand, we can use graded ring theory to examine the descending chain condition on $K[S]$. The following theorem is recently obtained by Jespers and Okninski [JO2].

Theorem 4.4.24 [JO2] **(1995)** *Let S be a semigroup and R an S-graded ring with $J(R)$ nil. If R is semilocal, then there exists finitely many subgroup $G_1, \cdots, G_n$ of S, with identity $e_1, \cdots, e_n$ respectively, and there exist homogeneous elements $f_i \in R_{e_i}$ such that*

$$R = J(R) + \sum_{i=1}^{n}\sum_{j=1}^{n(i)} a_{i,j} f_i R_{G_i} f_i b_{i,j},$$

for some finitely many homogeneous elements $a_{i,j}, b_{i,j} \in R$. Furthermore, each $f_i R_{G_i} f_i$ is a semilocal G_i-graded ring. If, moreover, R is left perfect (respectively semiprimary, left Artinian), then each f_i can be chosen to be an idempotent and $f_i R_{G_i} f_i$ is left perfect (respectively semiprimary, left artinian) with an identity.

By applying this theorem to semigroup algebras, we further obtain the following theorem.

Theorem 4.4.25 *Let $K[S]$ be a semilocal semigroup algebra. If $J(K[S])$ is nil and there exists finitely many subgroups $G_1, \cdots, G_n$ of S, then*

$$K[S] = J(K[S]) + \sum_{i=1}^{n}\sum_{j=1}^{n(i)} a_{i,j} K[G_i] b_{i,j}$$

where $a_{i,j}, b_{i,j} \in S$ and $K[G_i]$ is a semilocal group algebra.

Proof. If $K[S]$ is semilocal, then clearly S is periodic. Moreover, we may assume that $J(K[S])$ is nil. Let $R = K[S]$ and suppose that R is graded by S. Then, we have $R_{G_i} = K[G_i]$, $R_{e_i} = Ke_i$. As by above theorem, $a_{i,j}, b_{i,j}$ are homogeneous elements. This leads to $a_{i,j} = k_{i,j}s_{i,j}$, where $s_{i,j} \in S$ for all i, j. Similar to $b_{i,j}$ and note that $k_{i,j}$ is unit in the field K, then by the above theorem, we can prove the desired result. □

For example, if S is locally finite, $J(K[S]) = \mathcal{L}(K[S])$ (see Corollary 3.4.14), then $J(K[S])$ is nil. This theorem leads to a deeper description to $K[S]$ when $K[S]$ is semilocal or left perfect.

Corollary 4.4.26 *Let S be semigroup and K is any field. If $K[S]$ is right perfect, then there exist finitely many $s_1, \cdots, s_n \in S$ such that*

$$K[S] = J(K[S]) + Ks_1 + Ks_2 + \cdots Ks_n.$$

Proof. Since $J(K[S])$ is right T-nilpotent, $J(K[S])$ is nil ideal. By using the above theorem, we have finitely many j and $J(K[G_j])$ is right perfect. Thus, we obtain that

G_j is a finite group. Thus, there are finitely many s_i such that,

$$K[S] = J(K[S]) + \sum_i K s_i.$$

$\square$

Notice that we can also obtained the above corollary by using Theorem 4.4.16 since S contains no proper infinite subgroups. Thus $S/\sim_{J(K[S])}$ is finite and so $K[S]/J(K[S])$ is finite dimensional as well.

Chapter 5

Dimensions and Second Layer Condition on Semigroup Algebras

In this chapter, the dimensions of semigroup algebras (e.g. Gelfand-Kirillov dimension, classical Krull dimension, Krull dimension) are studied and the relationship between semigroups and these dimensions are established. Moreover, we also relate the prime ideals of Goldie algebras and the noetherian semigroup algebras with some of these diemensions.

5.1 Dimensions

5.1.1 Gelfand-Kirillov Dimension

The definition of Gelfand-Kirillov dimension is given in [KL]. The Gelfand-Kirillov dimension is a tool to measure the growth of algebras. Let V be a finite dimensional subspace of a K-algebras A. The Gelfand-Kirillov dimension (GK-dimension) of A is defined by

$$d_V(n) = dim(K + V + V^2 + \cdots + V^n)$$

$$GK(A) = \sup_V(\limsup_{n\to\infty} \ \log_n d_V(n)).$$

Similarly, for any A-module M, we can define the GK-dimension of module of M by

$$GK(M) = \sup_{V,F}(\limsup_{n\to\infty} \ \log_n d_{V,F}(n)).$$

where $d_{V,F}(n) = dim_K(FV^n)$, V is a finite dimensional subspace of A containing 1 and F is a finite dimensional subspace which generates M as an A-module.

The Gelfand-Kirillov dimension is said to be exact if for each short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

we have $GK(M) = \max\{GK(L), GK(N)\}$.

The general properties of GK-dimension can be found in [KL, MR].

Proposition 5.1.1 *Let A, B be K-algebras. Then*

(i). $GK(A \oplus B) = \max\{GK(A), GK(B)\}$.

(ii). *If B is a subalgebra or homomorphic image of A, then $GK(B) \leq GK(A)$.*

(iii). $\max\{GK(A), GK(B)\} \leq GK(A \otimes_K B) \leq GK(A) + GK(B)$.

An algebra A is affine algebra if A is generated by finite subset, i.e $A = K\{a_1, \cdots a_n\}$. Define GK-dimension of A by

$$GK(A) = \sup\{GK(R) | R \text{ an affine } K\text{-subalgebra of } A\}.$$

Then we have:

(iv). $GK(M_n(A)) = GK(A)$.

(v). $GK(A[x]) = GK(A) + 1$.

(vi). *Let $A * G$ be the crossed product of A over a finite group G. Then $GK(A * G) = GK(A)$.*

(vii). *Let P be prime ideal of A such that A/P is right Goldie. If $ht(P)$ is the height of the prime ideal P, then*

$$GK(A) \geq GK(A/P) + ht(P).$$

(viii). *Let N be a nilpotent ideal of A with nilpotent index k, i.e. $N^k = 0$. Then*

$$GK(A) \leq k \cdot GK(A/N).$$

(ix). *Let Ω be a multiplicative closed subset of regular central elements, then*

$$GK(A) = GK(A\Omega^{-1}).$$

For the right noetherian noncommutative algebras, we have the following theorem.

Theorem 5.1.2 [MR, Coro. 8.3.6] *Let R be a right noetherian K-algebra with $GK(R) < \infty$. Then the following statements hold:*

(i). *If R is prime and E is an essential right ideal then $GK(R/E) \leq GK(R) - 1$.*

(ii). *M_R is finitely generated and $MP = 0$ for some prime ideal of R. Also, $M_{R/P}$ is torsion and $GK(M) \leq GK(R/P) - 1$.*

(iii). *Let $P_0 \supseteq P_1 \supseteq \cdots \supseteq P_m$ be a chain of distinct prime ideals of R. Then*

$$GK(R) \geq GK(R/P_0) \geq GK(R/P_m) + m.$$

For group algebras $K[G]$, there are some important theorems. The following is an interesting one.

Theorem 5.1.3 [KL, Th. 11.1] *If G is a finitley generated group and $GK(K[G]) \leq \infty$ iff there is nilpotent normal subgroup of finite index in G.*

The case that G is finitely generated solvable group was also discussed in [KL, Th. 11.2].

5.1.2 Classical Krull and Krull Dimensions

In this section, we review some properties of the classical Krull (cl.Kdim) and Krull ($\mathcal{K}$) dimensions. The reader is referred to [GW, MR] for the definitions and general properties of the classical Krull and Krull dimensions. We list here some important properties of these dimensions.

Theorem 5.1.4 *Let R be ring with $\mathcal{K}(R_R) < \infty$. Then*

(i). *A semiprime ring R with right Krull dimension is a right Goldie ring.*

(ii). *The prime radical $\mathcal{B}$ of R is nilpotent.*

(iii). *$\mathcal{B}(R)$ is a finite intersection of minimal prime ideals, $P_1, \cdots, P_m$.*

(iv). *$\mathcal{K}(R_R) = \sup\{\mathcal{K}(R/P) | P \in Spec(R)\} = \mathcal{K}(R/\mathcal{B}(R))$.*

The dimension of a ring R can be defined in terms of posets and the prime spectrum $Spec(R)$, which is the collection of all prime ideals of R. The classical Krull dimension $cl.Kdim(R)$ is the supremum of the length of chains of prime ideals of R. A prime ring is called **right bounded** if every essential right ideal contains a nonzero (two-sided) ideal. A ring R is called right **fully bounded** if R/P is a right bounded ring for each prime ideal P. If R is right noetherian and right fully bounded, then we call R right **FBN** ring.

Theorem 5.1.5 *Let R be right noetherian ring with $\mathcal{K}(R_R) < \infty$. Then*

(i). $cl.Kdim(R) \leq \mathcal{K}(R_R)$.

(ii). *If R is a right FBN ring, then $cl.Kdim(R) = \mathcal{K}(R_R)$.*

(iii). *Let R be right noetherian ring and G is polycyclic-by-finite and $R * G$ a crossed prodcut of R by G. Then*

$$\mathcal{K}(R_R) \leq \mathcal{K}(R * G) \leq \mathcal{K}(R) + h(G)$$

where $h(G)$ is Hirsch number of group G.

In the case of PI-algebras, we can establish some connections between their dimensions.

Theorem 5.1.6 *Let A be PI K-algebra. Then*

(i). [KL, Th. 10.5] *If A is prime, then $GK(A) = tr.deg_K(A)$.* (1)

(ii). [KL, Th. 10.10] *If A is finitely generated prime, then $GK(A)$ is a nonnegative integer and*

$$GK(A) = cl.Kdim(A) = tr.deg_K(A).$$

(iii). [KL, Th. 10.15] *If A is noetherian PI-algebra with nilpotent radical N, then*

$$GK(A) = GK(A/N) = \max_P\{GK(A/P)\}$$

where P runs through the set of minimal prime ideals of A.

(iv). [KL, Coro. 10.16] *If R is noetherian PI-algebra with finite GK-dim, then GK-dim is exact for finitely generated R-modules and $GK(R)$ is nonnegative integer. If R is finitely generated, then $GK(R) = cl.Kdim(R)$.*

(1) Since A is a prime PI-algebra, so it must be a Goldie algebra. Let Q be the ring of quotients of A and let Z be its center of Q containing K. Then the transcendence degree of A over K is defined by $tr.deg_K(A) = tr.deg_K(Z)$.

5.2 The Growth and the Rank of Semigroups

Let S be a finitely generated semigroup with a system of generators $\{\alpha_1, \cdots, \alpha_m\}$. Let $\gamma(n)$ be the number of elements of S that can be presented as a product of at most n generators s_i. Then the semigroup S is said to have polynomial growth if there exist positive numbers C and d such that

$$\gamma(n) \leq Cn^d \qquad \text{for all } n \geq 1.$$

Also, $GK(S) = \limsup_{n\to\infty} \log_n \gamma(n)$ is called the exponent, or the degree of growth of the semigroup S.

In [KL, Ch. 11], we know that the group G has a polynomial growth iff G is nilpotent-by-finite. Note that the degree of the growth of group G and the growth of its subgroup N with finite index in G coincide. We also see that $GK(S) = 0$ iff S is finite. We now connect the GK-dimension of group algebras with the growths of its corresponding groups.

Theorem 5.2.1 [KL] *Let N be a finitely generated nilpotent group with a lower central series*

$$N = N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n = (e)$$

and let $d(i)$ be the torsion-free rank of the i-quotient N_i/N_{i+1}. Then we have

$$GK(K[N]) = d = \sum_{i=1}^{n} i \cdot d(i) = GK(N).$$

Proof. By referring to the results in [KL, Lemma 11.11, Lemma 11.12, Th. 11.14], we obtain this result. □

From the remark of [Okn1, Ch. 8], we know that $GK(K[S]) = GK(S)$. If $GK(S) < t$ and $d_S(m) < Cm^t$ for some C for almost all m, then S has a polynomial growth. Let V be an m-dimensional algebra, e.g. $V = K[s_1, \cdots, s_m]$ for some arbitrary $s_1, \cdots s_m$. Then $d_V(m) < Cm^t$ is determined by the presentation of $\langle s_1, \cdots, s_m\rangle$ and so the GK-dimension of the corresponding semigroup algebras is less than t. This shows that the GK-dimension of $K[S]$ is independent on the coefficient field and so it coincides with the growth of S. On the other hand, if $GK(K[S])$ is obtained by other means, then we can find the growth of S.

For the properties of polynomial growth of cancellative semigroups, the reader is always referred to [Okn1, Ch. 8].

Theorem 5.2.2 [Okn1, Th. 8.3] *Let S be a finitely generated cancellative semigroup. Then the following conditions are equivalent.*

(i). *S has a polynomial growth.*

(ii). *S has a group of fractions that is nilpotent-by-finite.*

(iii). *S has a weakly nilpotent subsemigroup of finite index.*

Moreover, in this case, the degree of growth $GK(S) = GK(G)$.

In section 3.2.2, we have introduced the rank of semigroup S which is the supremum of n, where S containing the n-generated free commutative subsemigroup, denote by $rk(S)$. Note that if S is a cancellative semigroup with group of fractions G, then $rk(S) = rk(G)$.

In [Okn1, Ch. 23], $Rk(S)$ is defined by $\sup_\rho\{rk(S/\rho)\}$, where ρ runs over the set of all congruences of S. The following result on $Rk(S)$ is obtained by Okninski.

Proposition 5.2.3 [Okn1, Lemma 23.6] *For every semigroup S,*

$$rk(S) \leq Rk(S) = \sup_P\{rk(S/\sim_P)\},$$

where P runs over the set of prime ideals of $K[S]$ for any coefficient field K.

5.3 Dimensions on Semigroup Algebras

Consider the noetherian group algebras $K[G]$. If G is a polycyclic-by-finite group, then it is know that $K[G]$ is noetherian. If G has no free noncommuatative subsemigroups, then by [Okn1, Th. 11.7], G must be nilpotent-by-finite. Thus, we have the following results concerning the dimensions of semigroup algebras.

Theorem 5.3.1 *Let G be a polycyclic-by-finite group and has no free noncommutative subsemigroup. Then $GK(K[G]) < \infty$, where $GK(K[G])$ is an integer and $GK(K[G]) \geq \mathcal{K}(K[G])$.*

Proof. Since G is nilpotent-by-finite and is finitely generated. $GK(K[G])$ is nonnegative integer d by Theorem 5.2.1. Moreover $\mathcal{K}(K[G]) = h(G)$, which is the Hirsch number of G. By the formula of d, it is obviously to see that $d \geq \sum d(i) = h(G)$. then $GK(K[G]) \geq \mathcal{K}(K[G])$. □

Lemma 5.3.2 [Okn1, Lemma 23.3] *Let G be a group with an abelian normal subgroup H of finite index. Then, we have $cl.Kdim(K[G]) \geq cl.Kdim(K[H])$.*

By using the properties of GK dimensions and the necessary conditions for $K[S]$ being PI-algebras (see Section 3.7 or [Okn1, Ch. 20]), we obtain the following result.

Theorem 5.3.3 [Okn1] *Let S be a cancellative monoid such that $K[S]$ is PI-algebra. Then*

$$cl.Kdim(K[S]) = GK(K[S]) = rk(S).$$

Proof. By Prop. 3.7.3, we know that S satisfies the permutational properties $\mathfrak{P}$, so S is a cancallative semigroup with $\mathfrak{P}$. Moreover, S has group of right fractions G such that $K[G]$ is PI-algebra. Thus, G has a normal subgroup H of finite index and H' is a finite p-group of G, where $char(K) = p$. Let T be a finitely generated subsemigroup of S. Then $TT^{-1} = F$ has an abelian normal subgroup Z of finite index. By Theorems 5.2.2 and 5.2.1, the results follow from

$$GK(K[T]) = GK(K[F]) = GK(K[Z]) = rk(Z) = rk(F) = rk(T).$$

Moreover, from Theorem 5.1.1(iv), we have

$$GK(K[S]) = \sup_T\{GK(K[T])\} = \sup_T\{rk(T)\} = rk(S).$$

Since prime PI-algebras are all Goldie rings, by Theorem 5.1.1(viii), we have $GK(K[S]) \geq cl.Kdim(K[S])$. Moreover, it is known that S has a group of fractions G which is abelian-by-finite. Thus, if we let H be this abelian normal subgroup, then from Lemma 5.3.2 and Section 3.3, we have

$$cl.Kdim(K[S]) \geq cl.Kdim(K[G]) \geq cl.Kdim(K[H]) = rk(H) = rk(S).$$

This shows that $cl.Kdim(K[S]) = rk(S) = GK(K[S])$ and $GK(K[S])$ is a nonnegative integer. □

Finally, we find the following generalization in the monograph of Okninski [Okn1].

Theorem 5.3.4 [Okn1, Prop. 23.11, Th. 23.12] *Let S be a monoid such that $K[S]$ is a PI-algebra. If P is a prime ideal of $K[S]$, then*

$$cl.Kdim(K[S]/P) \leq GK(K[S]/P) \leq rk(S/\sim_P) \leq cl.Kdim(K[S/\sim_P]) \leq GK(K[S/\sim_P])$$

Moreover, $rk(S) \leq cl.Kdim(K[S]) = \sup_P\{GK(K[S]/P)\} = Rk(S)$ where the supremum is taken over all the prime ideals of $K[S]$. Consequently, $cl.Kdim(K[S]) = 0$ iff $Rk(S) = 0$ iff S is periodic semigroup.

Theorem 5.3.5 *Let T be a finitely generated cancallative subsemigroup of S such that $GK(T) < \infty$. Let T' be the image of T in $K[S]/\mathcal{B}(K[S])$. Then*

$$GK(T) = GK(K\{T'\}) = GK(T')$$

Proof. Let $H = TT^{-1}$. Then by above theorem, H is nilpotent-by-finite. Moreover, $GK(T) = GK(H) = GK(N) = GK(T/\sim_N)$ where N is a maximal finite normal subgroup of H such that H/N has no nontrival finite normal subgroups. Thus, $K[H/N]$ is prime for any field K and H/N is a group of fractions of $T/\sim_N$. Hence, $K_0[T/\sim_N]$ is also prime and is the homomorphic image of $K[T]/\mathcal{B}(K[T])$. Since $GK(T) = GK(H) = GK(H/N) = GK(T/\sim_N)$, $GK(T) = GK(K[T]/\mathcal{B}(K[T]))$. Let $T', K\{T'\}$ be the images of T and $K[T]$ in $K[S]/\mathcal{B}(K[S])$ respectively. Then $K[T] \cap \mathcal{B}(K[S]) \subseteq \mathcal{B}(K[T])$ and so

$$GK(T) = GK(T') = GK(K\{T'\}).$$

□

On the other hand, if $K[S]$ satisfies the a.c.c. on its right annihilators, then we have a further generalizations, from section 4.2.

Theorem 5.3.6 [Okn2] **(1993)** *Assume that $K[S]$ is right noetherian and S has no free noncommutative subsemigroups. Then*

$$\sup\{GK(T)|T \textit{ is a cancellative subsemigroup of } S\}$$

$$= \sup\{GK(T)|T \textit{ is a cancellative subsemigroup of } \bar{S}\}$$

where $\bar{S}$ is image of S in $K[S]/\mathcal{B}(K[S])$.

Concerning the extension to right Goldie rings, we obtain the following characterization.

Theorem 5.3.7 [Okn2] **(1993)** *Assume that $K[\bar{S}]$ has finite right Goldie dimension and $K[S]/\mathcal{B}(K[S])$ is a right Goldie ring. Assume $GK(T) < \infty$ for every cancellative subsemigroup T of $\bar{S}$. Then $GK(K[S]/\mathcal{B}(K[S])) = GK\{T\}$ for a cancellative semigroup T of $\bar{S}$.*

The following theorem provides the conditions to ensure that $GK(S) < \infty$.

Theorem 5.3.8 [Okn2]**(1993)** *Let $K[S]$ be right noetherian. Then the following conditions are equivalent.*

(i). $GK(K[S]) = GK(S) < \infty$,

(ii). $GK(K[T]) = GK(T) < \infty$ *for every cancellative subsemigroup* T *of* S.

(iii). *every cancellative subsemigroup of* S *has a finitely generated nilpotent-by-finite group of fractions.*

Moreover, in this case $GK(S) \leq rq$ *where* r *denotes the nilpotency index of* $\mathcal{B}(K[S])$ *and* q *is the maximum of the GK-dimensions of the cancellative subsemigroups of* S.

Proof. Obviously (i) implies (ii) since $K[T]$ is a subalgebra of $K[S]$.

Since $K[S]/\mathcal{B}(K[S])$ is semiprime right Goldie ring and $GK(T) < \infty$, for every cancellative subsemigroup of S, by the above Lemma 5.3.7, we have

$$GK(T) = GK(K\{T\}) = GK(K[S]/\mathcal{B}(K[S])$$

for a cancellative subsemigroup T of $\bar{S}$. Since $\mathcal{B}(K[S])$ is nilpotent, by Theorem 5.1.1(viii), we have

$$GK(K[S]) \leq r \cdot GK(K[S]/\mathcal{B}(K[S])) = r \cdot GK(T)$$

for some $T \subseteq \bar{S}$. Therefore, $GK(S) \leq rq$ where q is the maximum of the GK dimensions of the cancellative subsemigroups.

Therefore, if we assume (ii), then $GK(T) < \infty$ for every cancellative subsemigroup T of S. Thus Theorem 5.3.6 yields that

$$\sup_{T \subseteq S}\{GK(T)\} = \sup_{T \subseteq \bar{S}}\{GK(T)\}.$$

Since $K[S]$ is right noetherian, by Theoerem 4.3.5(vi) there are finitely many isomorphism classes of group of fractions of their maximal subgroup. Thus, the supremum exists and $GK(S) \leq rq < \infty$. This shows that (ii) implies (i). Thus, in particular, we have every cancellative subsemigroup T of S satisfies $GK(T) < \infty$ iff T has a finitely generated nilpotent group of fractions G and $GK(T) = GK(G)$ (see Theorem 5.2.1). □

Corollary 5.3.9 *If* S *is nilpotent semigroup and* $K[S]$ *is right noetherian. Then* $GK(S) < \infty$.

5.4 Second Layer Condition

The localizations of noncommutative noetherian rings have been fully studied by Jateganokar [Jat]. This topic, in fact, links the prime ideals of the ring R and a well-known condition, namely the second layer condition in (right) noetherian algebras. In this section, we will not go through the details on such links between prime ideals and the second layer condition, the details can be found in [Jat, GW, MR]. Examples of noetherian ring satisfying second layer condition are enveloping algebras of finite dimensional solvable Lie algebras; noetherian PI algebras and the group algebras of polycyclic-by-finite group, etc.

In this section, we study the strong second layer condition. A prime ideal P in a right noetherian ring R is said to satisfy the **right strong second layer condition** if, for every prime ideal $Q < P$, every finitely generated (P/Q)-primary right (R/Q)-module is unfaithful over R/Q.

Proposition 5.4.1 [Jat, Prop. 8.1.5] *The (right) (strong) second layer condition is a Morita invariant.*

If G is polycyclic-by-finite group, then $K[G]$ has strong second layer condition. There is a natural question for the subsemigroup of G, or completely 0-simple with maximal subgroup G: Do the corresponding algebras satisfy the strong second layer condition ?

First, we recall the following diagram given in the monograph of Jateganokar [Jat].

G :	polycyclic-by-finite	$\Leftarrow$	orbitally sound polycyclic	$\Leftarrow$	f.g. nilpotent
	$\Downarrow$		$\Downarrow$		$\Downarrow$
$K[G]$:	strong second layer condition	$\Leftarrow$	AR-separated	$\Leftarrow$	polycentral

In the above diagram, the group G is called **orbitally sound polycyclic** if for all subgroup H of G, the normal closure H^G and the $core_G(H)$ [(2)] satisfies $[H^G : core_G(H)] < \infty$ whenever $[G : N_G(H)] < \infty$.

Proposition 5.4.2 [Jat, Th. A.4.2] *Every polycyclic-by-finite group contains a characteristic orbitally sound polycyclic subgroup of finite index.*

Definition 5.4.3 *An ideal I in a ring R has the right AR-property if for every right ideal J of R, there is a positive integer n such that $J \cap I^n \subseteq JI$.*

[(2)] The core of subgroup, $core_G(H) = \bigcap_{g \in G} g^{-1}Hg$

A ring R is called right AR-separated if for every pair of prime ideals P and Q in R such that $Q \subseteq P$, there is an ideal I such that $Q \subset I \subseteq P$ and I/Q satisfies the right AR property in R/Q. Left AR separated is analogously defined.

Proposition 5.4.4 [Jat, Prop. 8.1.7] *Any (right) AR separated ring satisfies the (right) strong second layer condition.*

Now, Let S be a submonoid of G, where G is a polycyclic-by-finite group. Then, $K[G]$ is right noetherian and $K[S]$ is right noetherian iff S has a.c.c. on its right ideals (see Prop. 4.3.2). Moreover, S is finitely generated. We are now going to make use of Theorem 4.3.1, Lemma 4.3.2 and Lemma 3.3.6 (vii) in our theory. Without loss of generality, we may assume that S has a group of fractions G. It can be seen that if Q is prime ideal of $K[G]$, then $Q \cap K[S]$ is prime ideal of $K[S]$. If P is prime ideal of $K[S]$, then $Q = K[G] \otimes_{K[S]} P \cong K[G]P$ is also prime ideal if $P \cap S = \emptyset$. Moreover, $Q \cap K[S] = P$ (cf. the case of nilpotent cancellative semigroup in Lemma 4.2.5), so for every prime ideal P of $K[S]$ with $P \cap S = \emptyset$, we have Q lies over P.

Moreover, by Theorem 4.3.2, we know that S is finitely generated. Hence, $K[G]$ can be viewed as a finitely generated $K[S]$-module. Thus $K[S] \hookrightarrow K[G]$ is a ring extension. We stated here two extension theorems from [Let] **(1990)**.

Theorem 5.4.5 [Let, Th. 4.2] *Let $R \hookrightarrow T$ be an extension of noetherian ring such that T is finitely generated as a left and right R-module. If R satisfies the second layer condition then so does T. Moreover, if R satisfies the strong second layer condition then so does T.*

Theorem 5.4.6 [Let, Th. 5.3] *Let $R \hookrightarrow T$ and T be noetherian rings satisfying the second layer condition such that T is a finitely generated right R-module. Let $Q_\alpha \rightsquigarrow Q_\beta$. Then the following statements hold:*

(i). *There exists prime ideals P_α and P_β lying over Q_α and Q_β respecitvity. Then either*

  (a) *$P_\alpha = P_\beta$ or ;*

  (b) *there exists a sequence of prime ideals $P_\alpha = P_1, \cdots, P_t = P_\beta$ with $t \geq 2$, such that*

$$P_1 \rightsquigarrow \cdots \rightsquigarrow P_i \rightsquigarrow P_{i+1} \rightsquigarrow \cdots \rightsquigarrow P_t.$$

(ii). *If $P \cap R$ is semiprime for every prime ideal P of S, then we may choose P_α and P_β in (i) such that (b) occurs.*

By the above theorem, we have the following theorem for polycyclic-by-finite groups.

Theorem 5.4.7 *Let G be polycyclic-by-finite group and S a submonoid of G. Then $K[S]$ satisfies the right second order condition if S has a.c.c. on right ideals.*

For other semigroups S, for example, nilpotent semigroups, we consider the case when $K[S]$ is a prime noetherian ring.

Lemma 5.4.8 [JW3] **(1995)** *Let S be a nilpotent semigroup. If $K_0[S]$ is a prime noetherian algebra, then, by the notation of Theorem 4.2.10, $K\{I\} = K_0[\hat{I}]$, G is poly-(infinite cyclic) and $q = n$.*

Proof. In the proof of Theorem 4.2.10, we have seen that $K_0[\hat{I}]$ is a localization of $K_0[I]$ with respect to the Ore set C. Let φ be the natural homomorphism of $K_0[\hat{I}]$ to $K\{\hat{I}\}$ and $\alpha \in K_0[\hat{I}]$ such that $\varphi(\alpha) = 0$. Let $c \in C$ such that $c\alpha \in K_0[I]$. Since $K_0[I] = K\{I\}$, we have $\varphi(c\alpha) = c\alpha$. Thus $c\alpha = 0$ and so c is invertible. Hence, we conclude that $\alpha = 0$. This leads to $\ker\varphi = 0$ and hence $Q = 0$.

As $K_0[S] \subseteq M_q(K[G])$ and and $K[G]$ has a.c.c. on right ideals, we know that G is nilpotent and also $K[G]$ is a prime ring. Thus we know $Z(G)$ is torsion-free and by [Pas1, Lemma 11.1.3], the upper central series of G, namely $Z_i(G)/Z_{i-1}(G)$ is also torsion-free. Hence, G is a poly-(infinite cyclic) group. □

From Theorem 4.2.10, we know S is arbitrary nilpotent semigroup (not necessarily cancellative), and $K[S]$ is right noetherian, for any prime ideal P of $K[S]$. Thus, $K[S]/P$ can be embedded into $M_q(K[G]/Q)$ for some group G and is a prime ideal of $K[G]$. Note that the group G is a finitely generated nilpotent group. This forces $K[G]/Q$ must be a prime Goldie ring and $M_q(K[G]/Q)$ is prime noetherian ring, hence it is a finite extension of $K[G]/P$. Moreover, it is clear that $M_q(K[G]/Q)$ satisfies the strong second layer condition, by Proposition 5.4.1.

For the class of completely 0-simple nilpotent semigroups, we have the following theorem.

Theorem 5.4.9 *Let $K[S]$ be an algebra of finitely generated completely 0-simple nilpotent semigroup with $|E(S)| < \infty$. Then $K_0[S]$ is noetherian and satisfies the strong second layer condition.*

Proof. From section 3.3.4, we know that the completely 0-simple nilpotent semigroup is inverse and its maximal subgroup G is nilpotent. Hence, G is finitely generated nilpotent group so that $K[G]$ is noetherian and $E(S)$ is finite set. By Theorem

4.3.11, we know that $K_0[S]$ is noetherian and $K_0[S] \cong M_n(K[G])$. This shows that $K_0[S]$ satisfies the strong second layer condition by Theorem 5.4.5 or Proposition 5.4.1. $\square$

In closing the thesis, we wish to point out that the dimensions and prime ideals of noetherian rings are important topic for investigation and the above results can be applied in studying this area. We also post out here an open problem concerning the relation $\sim_P$, where P is a prime ideal of $K[S]$, for solution.

If $P \rightsquigarrow Q$, what is the relationship between $\sim_P$ and $\sim_Q$?

Notations

$char(K)$	Characteristics of field K
$E(S)$	The set of all idempotents of the semigroup S
$\mathcal{H}, \mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{J}$	Green relations
M_R, M_L, M_J	Minimal condition on equivalence classes on $S/\mathcal{R}$, $S/\mathcal{L}$ and $S/\mathcal{J}$ respectively.
$J(a)/I(a)$	Principal factor in semigroup containing a
$\mathbf{R}(R), \mathbf{L}(R), \mathbf{T}(R)$	Lattice of right ideals, left ideals and two-sided ideals respectively
$\mathcal{M}^0(G, I, \Lambda; P)$	Rees matrix semigroup with maximal subgroup G
$K[X]$	Polynomial ring over set X
$K\{X\}$	K-algebra generated by set X
$J(R)$	Jacobson radical
$\mathcal{L}(R)$	Levitzki radical
$\mathcal{B}(R)$	Prime (Baer) radical
$\mathcal{G}(R)$	Brown McCoy radical
$\mathcal{N}(R)$	Upper nil radcial
$N(K[G])$	Nilpotent radical of group algebra
$U(S)$	Units group of monoid S
$\mathfrak{h}(R)$	Set of homogeneous elements of graded ring R

Abbreviations

cf.	(*Latin: confer*) compare
i.e.	that is
iff	if and only if
f.g.	finitely generated
a.c.c.	ascending chain condition
d.c.c.	descending chain condition
e.g.	for example
Ch.	Chapter
Th.	Theorem
Prop.	Proposition
Coro.	Corollary
□	end of proof

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