Hierarchical Production Planning for Discrete Event Manufacturing Systems

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ABSTRACT

Most manufacturing systems organize production by using a number of machines in tandem or a flowshop configuration. One of the characterising features of the dynamics of a flowshop is that inventory levels of semi-processed parts in buffers between any two machines, known as internal buffers, must be nonnegative. This feature, together with the fact that machines are usually unreliable, namely, they are subject to the random discrete events of breakdowns and repairs, makes the optimal production planning in a flowshop an extremely difficult problem, both theoretically and computationally.

An effective way to analytically cope with the difficulty is to use the so-called hierarchical control approach. The idea is to average out the uncertainty in machines' capacities and replace the original *stochastic* problem by a *deterministic* (also called *limiting*) problem. One then tries to show that, under certain circumstances, the two problems are in fact very close to each other as the rate of change in machines' states becomes very large. Based on this, one can somehow use a good production policy for the deterministic problem, which is much easier to obtain, to construct a good policy for the original stochastic problem.

In this thesis, we first consider open-loop production planning in manufacturing systems with two tandem unreliable machines and finite buffers. It is

emphasized that two-machine flowshops are basic elements in general manufacturing systems and possess the major difficulties in terms of various constraints. Asymptotic optimal production policies for the original problem are explicitly constructed from near-optimal policies of the limiting problem, and the error estimate for the constructed policies is obtained. Algorithms of constructing these polices are presented.

Note that open-loop controls are of theoretical importantance in justifying the hierarchical control approach. While asymptotically optimal, the constructed open-loop controls are however not expected to perform well unless the rate of change in machine states is unrealistically large. What is required therefore is a construction of asymptotic optimal *feedback* controls. In this thesis, we subsequently consider the feedback controls for the stochastic two-machine flowshops. Once again, the idea is to use the hierarchical approach to replace the stochastic problem by a deterministic problem. Explicit optimal feedback controls for the deterministic problem are obtained. Furthermore, beginning with the solutions of the deterministic problem, a feedback control for the stochastic flowshops is analytically constructed, which is proved to be asymptotically optimal with respect to the rate of change in machine states.

Finally, we numerically compare the performance of our constructed policy, referred to as *Hierarchical Control* (HC) policy with two well known heuristic

policies known as *Kanban Control* (KC) policy and *Two Boundary Control* (TBC) policy. We show that HC performs, while simpler to construct, to understand, and to implement, as well as or better than KC and TBC.

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Notation

This thesis is divided into seven chapters, in which each of the first six chapters is divided into sections and some sections are divided into subsections. In any given chapter, say Chapter 5, sections are numbered consecutively as 5.1, 5.2, 5.3 and so on. The subsections in Section 5.3 are numbered consecutively as 5.3.1, 5.3.2, $\cdots$. Similarly, mathematical expressions in Chapter 5, such as equations, inequalities, and conditions, will be numbered consecutively as (5.1), (5.2), (5.3), $\cdots$. Figures and tables in that chapter are numbered consecutively as Fig. 5.1, Fig. 5.2, $\cdots$ and Table 5.1, Table 5.2, $\cdots$. Also, theorems in Chapter 5 are numbered consecutively as Theorem 5.1, Theorem 5.2, $\cdots$. The same numbering scheme is used for lemmas, corollaries, definitions, remarks, assumptions, algorithms, and examples.

We provide clarification of some frequently-used terms in this thesis. The terms "control", "policy", "planning", and "decision" are used interchangeably. The terms "surplus", "inventory/shortage", and "inventory/backlog" are used interchangeably. The terms "dynamic programming equation", "Hamilton-Jacobi-Bellman equation", and "HJB equation" are used interchangeably.

We make use of the following notation in this thesis:

z^+	:	$=\max\{z,0\}$, for any real number z;
z^-	:	$=\max\{-z,0\}$, for any real number z;
$\|(z_1,\cdots,z_n)\|$	:	$=\|z_1\|+\cdots+\|z_n\|$, for any vector $(z_1,\cdots,z_n)$ with any positive integer n;
A'	:	the transpose of a vector or matrix A;
B^c	:	the complement of a set B;
$B_1 \cap B_2$	:	the intersection of sets B_1 and B_2;
$B_1 \cup B_2$	:	the union of sets B_1 and B_2;
$C^1(A)$	:	set of continuously differentiable functions defined on a set A;
C_0, K_0	:	positive constants required in definition of the cost function;
$C, K, C_i, K_i, i=1,2,\cdots$	:	positive constants required in the analysis;
χ_B	:	the indicator function of a set B;
$E\xi$	:	the expectation of a random variable ξ;
I_n	:	the n-identity matrix ;
$J, J^0, J^\varepsilon, \cdots$	:	cost functions ;

$O(y)$	:	a variable such that $\sup_y \lvert O(y)\rvert / \lvert y\rvert < \infty$;
$P(A)$	:	the probability of any event A;
R^n	:	n-dimensional Euclidean space ;
$\square$	:	indicator of the end of a proof.

Chapter 1

Introduction

1.1 Manufacturing Systems: An Overview

Most manufacturing firms are large, complex systems characterized by several decision subsystems, such as finance, personnel, marketing, and operations. They may have a number of plants and warehouses and produce a large number of different products using a wide variety of machines in tandem or a flowshop configuration. In such manufacturing systems, raw parts are fed into the first machine, get processed sequentially from one machine to the next, and eventually come out as finished parts from the last machine. Moreover, these systems are subjected to various discrete events such as machine failures and repairs, construction of new facilities, purchasing new equipment, hiring and layoff of workers, new product introductions, etc. These events could be deterministic and/or stochastic. Management must recognize and react to these events.

Typically, the managers of a manufacturing firm make production plans for finished products by considering forecasts of demand, sales, raw material availability, inventory levels, and plant (or machines) capacity. Frequently, they use *Materials Requirements Planning* or *MRP* (cf. Orlicky [36]). From the resulting high level plan, the requirements for the components that go into the final products can be determined. The various departments that are responsible for the manufacture of the components schedule their activities so as to meet the requirements dictated by the master production and the materials requirements plans (cf. Halevi [24] and Hitomi [27]). Unfortunately, MRP does not account for the finite (and varying) capacity of a manufacturing system.

The manufacturing systems under consideration in this thesis are the stochastic systems with machines in tandem (also called flowshops). The machines are unreliable, namely, they are subject to breakdowns and repairs (cf. Kimemia and Gershwin [32] and Akella and Kumar [2]). One of the characterizing features of a flowshop is that inventory levels of semi-processed parts in buffers between any two machines, known as internal buffers, must be nonnegative, and the sizes of both internal and external buffers are practically finite. This feature, together with the fact that machines are unreliable, makes the optimal production planning in a flowshop an extremely difficult problem, both theoretically and computationally.

1.2 Previous Research

Beginning with Thompson and Sethi [58], Sethi and Thompson [40] and Kimemia and Gershwin [32], there has been a substantial interest in analyzing production planning problems under uncertainty as continuous-time stochastic optimal control problems with an objective of minimizing costs of inventory/shortages and of production over a finite or infinite horizon. While Sethi and Thompson [40] formulated uncertainty in demand as a diffusion process, Kimemia and Gershwin modelled production capacity as a finite state Markov Process. Since then, a number of authors such as Bensoussan et al. [6], Akella and Kumar [2], Fleming, Sethi and Soner [13], Haurie and van Delft [25], Sethi et al. [39], Ghosh, Arapostathis and Markus [22], and Lou, Sethi and Zhang [34], have extended one or the other or both.

With the exception of Akella and Kumar [2] (see also Bielecki and Kumar [7] and Sharifinia [51]), who explicitly solved the infinite horizon problem of a manufacturing system consisting of a single failure-prone machine with two states: up and down, and with a simple discounted cost structure, this line of research has resulted in existence and partial characterization of optimal production policies.

Indeed, it is by now well known that computation of optimal solutions is extremely difficult except in simple cases.

The recognition of the complexity of the production planning problems in stochastic manufacturing systems has resulted in various attempts to obtain sub-optimal or near-optimal controls. We shall mention some of these efforts. Gershwin, Akella and Choong [20] proposed a heuristic approximation of the value function of the problem in order to obtain near-optimal controls. Caramanis and Sharifnia [11] utilize a capacity set modification, based on the work of Kimemia and Gershwin [32] and Sharifnia [51], in order to design near-optimal controllers. Ho and Cao [28] develop perturbation analysis to obtain consistent gradient estimates based on a single simulation run. Caramanis and Liberopoulos [10] use perturbation analysis [28] to obtain approximate solution of the dynamic programming equation for the value function. Yan, Zhou and Yin [61] use perturbation analysis [28] and stochastic approximation to obtain optimal number of Kanbans in a two-machine flowshop. Van Ryzin, Lou and Gershwin [59], Lou and Van Ryzin [35], and Bai and Gershwin [4] provide an approximation of optimal feedback controls in the case of manufacturing systems consisting of two or three machines in tandem.

Of particular importance is the so-called hierarchical controls approach based on the reduction of a given complex problem into simpler approximate problems or subproblems and to construct a satisfactory solution for the given problem from the solutions of the simpler problems. Moreover, in cases with stochastic systems, in which fluctuation rates or frequencies of some processes are much faster than the frequencies associated with other processes, the hierarchical approach provides us with solutions that are asymptotically optimal as the frequencies of the faster processes tend to become infinitely large. The approach is used by Gershwin [18], Lehoczky et al. [33], Sharifnia, Caramanis and Gershwin [52], Sethi, Zhang and Zhou [47, 45], Soner [55], Sethi and Zhang [42, 43], and Sethi and Zhou [48, 49, 50], to name a few; see also Sethi and Zhang [44] for a recent book on the topic. Furthermore, Gershwin, Caramanis and Murray [21] and Samaratunga, Sethi and Zhou [38] have reported some simulation experience with the hierarchical approach, while Srivatsan, Bai and Gershwin [57] have looked into its application to semiconductor manufacturing.

1.3 Motivation

In the previous research on hierarchical production planning [45, 47, 48] for production planning problems in stochastic manufacturing systems, it is assumed that the sizes of internal and external buffers are infinite, which is a reasonable assumption only when the buffers have very large spaces so that the expected inventory levels will never exceed the sizes of respective buffers.

Unfortunately, this assumption is hardly reasonable in *real* manufacturing systems due to obvious reasons. Recently, Sethi, Zhang, and Zhou [46] augmented their method of lifting and modification (developed in [45]) by another special technique called "squeezing" in order to construct asymptotic optimal open-loop controls for a two-machine flowshop with only the internal buffer being upper bounded. Practically, however, the managers of manufacturing firms must also take the upper bound of the external buffer (such as warehouses) into consideration, especially in the situation of scarce space and/or high rent. Indeed, production planning problems for stochastic manufacturing systems with finite buffers were cited as outstanding open problems by the previous research [46, 48, 44].

One of the main purposes of this thesis is to study hierarchical open-loop production planning for two-machine flowshops with *both* internal and external buffers being of finite size and with general production costs and inventory/backlog costs. It is worth indicating that these two-machine flowshops are relatively simple manufacturing systems and are, at the same time, sufficiently rich for possible applications. This is because the state (inventory) constraint represents a typical complexity present in systems with machines in tandem. A major difficulty in this case is proving the Lipschitz continuity of the value functions for both the original (stochastic) problem and the limiting (deterministic) problem, which plays an essential role in the hierarchical analysis. It should be noted that in [45, 46], the constructive proof of the Lipschitz property needs only to take care of the constraints on internal buffer without worrying about the external buffer. The proof does not go through in our case. To handle the problem, we shall in this thesis introduce and prove some *weak-Lipschitz* property, which is weaker than Lipschitz continuity but sufficient for subsequent analysis.

Another major difficulty is in constructing asymptotic optimal controls and obtaining error estimates. Owing to the upper bound constraints on both buffers, the lifting, the squeezing and modification method in [46] does no longer work. The reason is that lifting and squeezing the internal buffer will violate the upper bound constraint on the external buffer. In this thesis, we develop a "constraint domain approximation" method to overcome the difficulty and to obtain the asymptotic optimal controls. As in [45, 46], we shall give the error estimate of the constructed asymptotic controls.

Note that the open-loop policy does not react to the current inventory/backlog state, whereas the feedback policy does. Moreover, basically only open-loop controls were constructed in [45, 47, 48] which were shown to be asymptotic optimal. Open-loop controls are of theoretical importance in justifying the hierarchical control approach. While asymptotically optimal, the constructed open-loop controls are not expected to perform well unless the rate of change in machine states is unrealistically large. What is required therefore is a construction of asymptotic optimal *feedback* controls. Another main purpose of this thesis is to study hierarchical feedback production planning for the two-machine flowshops with finite buffers. Once again, the idea is to use the hierarchical approach to replace the stochastic problem by a deterministic problem. We first solve explicitly the deterministic problem by virtue of "weak-Lipschitz" and "constraint domain approximation" which we developed. Then, based on the explicit characterization of the optimal controls for deterministic problem, a suitable feedback control for the stochastic flowshops, analytically constructed, is proved to be asymptotically optimal with respect to the rate of change in machine states.

Finally, we shall compare the performance of our constructed policy, referred to as *Hierarchical Control* policy, to a stochastic extension of *Kanban Control* policy developed in Sethi et al. [41] and *Two Boundary Control* policy developed in van Ryzin, Lou and Gershwin [59] and Lou and Van Ryzin [35]. It will be shown that hierarchical controls perform better or no worse than Kanban controls. The costs of hierarchical controls and two-boundary controls are not significantly different, although the former is a much simpler policy than the latter.

1.4 Outline of the Thesis

The plan of this thesis is as follows:

In the next chapter, we formulate the deterministic and stochastic production planning problems for two-machine flowshops respectively. We review the concept of Markov chain and state some lemmas which will be used later on.

In Chapter 3, we consider the open-loop production planning problem for the stochastic flowshops. Since the sizes of both internal and external buffers are practically finite, the problem is one with state constraints. A deterministic limiting problem in which the stochastic machines capacities are replaced by their mean capacities is considered instead. "Weak-Lipschitz" property of the value functions for both original and limiting problems is introduced and proved, and a "constraint domain approximation" approach is developed to show that the value function of the original problem converges to that of the limiting problem as the rate of change in machines' states approaches infinity. Production policies for the original problem are explicitly constructed from near-optimal policies of the limiting problem in a way which guarantees their asymptotic optimality, and the error estimate for the constructed policies is obtained. Algorithms of constructing these polices are presented.

In this thesis, we would like to eventually construct optimal feedback controls for stochastic two-machine flowshops with machines subject to random breakdowns and repairs. As the problem is extremely difficult to solve, it can be approximated by a deterministic problem in which the stochastic machines' capacities are replaced by their average capacities when the rates of machine failures and repairs become large. Therefore, in Chapter 4, we first construct explicitly optimal feedback controls for the deterministic problem with both internal and external buffers being finite.

In Chapter 5, we consider the feedback production planning for the stochastic flowshops. Based on the explicit characterization of optimal controls for the deterministic problem in Chapter 4, a suitable feedback control for the stochastic flowshops is analytically constructed, which is proved to be asymptotically optimal with respect to the rate of change in machine states.

In Chapter 6, we report numerical computations of hierarchical controls and compare the performance of these controls with heuristic methods known as Kanban controls and two-boundary controls. Finally, we conclude this thesis and give some future research directions in Chapter 7.

Chapter 2

Preliminaries

2.1 Problem Formulation: Deterministic Production Planning

In this chapter, we first consider a deterministic dynamic two-machine flowshop or a deterministic manufacturing system consisting of two machines M_1 and M_2 in tandem as shown in Fig. 2.1. We assume that the machines M_1 and M_2 can produce mostly a_1 and a_2 per time unit, respectively. Then the machines M_1 and M_2 have maximum production capacities a_1 and a_2, respectively.

We use $w_1(t)$ and $w_2(t)$, $t \geq 0$, to denote the production rates on the first and the second machine, respectively. We denote the inventory level in the exit buffer of M_1 (i.e., the internal buffer) as $x_1(t) \geq 0$ and the surplus level of the

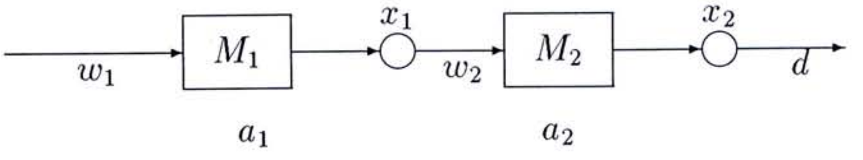

Figure 2.1: A deterministic manufacturing system with two machines in tandem

finished product as $x_2(t)$. A positive surplus means inventory and a negative surplus means shortage. The rate of demand d facing the system is assumed to be a constant.

Let b_1 and b_2 denote the sizes of the internal buffer and the external buffer, respectively. Then, if the buffers are full, we can not put any more in there. Therefore, the state constraint that $0 \leq x_1(t) \leq b_1$ and $x_2(t) \leq b_2$ for all $t \geq 0$ must be satisfied. Let $S = [0, b_1] \times (-\infty, b_2] \subset R^2$ denote the (state) constraint domain. Then the system can be written as follows:

$$\left\{ \begin{array}{ll} \dot{x}_1(t) = w_1(t) - w_2(t), & x_1(0) = x_1 \\ \dot{x}_2(t) = w_2(t) - d, & x_2(0) = x_2 \end{array} \right. , \quad \mathbf{x} = (x_1, x_2) \in S, \tag{2.1}$$

where the input rate to each of the machines is subject to the capacity of the respective machine, namely,

$$0 \le w_i(t) \le a_i, \ t \ge 0, \ i = 1, 2. \tag{2.2}$$

Here and elsewhere we use boldface letters to stand for vectors (e.g., $\mathbf{x} = (x_1, x_2)$, $\mathbf{w} = (w_1, w_2)$, etc.).

Now we can define the set of admissible controls $\mathbf{w}(\cdot) = (w_1(\cdot), w_2(\cdot))$ as follows.

Definition 2.1. A control (policy) $\mathbf{w}(\cdot) = (w_1(\cdot), w_2(\cdot))$ is *admissible* with respect to the initial state value $\mathbf{x} = (x_1, x_2) \in S$ if (i) $\mathbf{w}(t)$ is measurable in t, (ii) $0 \le w_i(t) \le a_i$ for $t \ge 0$ and $i = 1, 2$, and (iii) the corresponding state

$$\mathbf{x}(t) = (x_1(t), x_2(t)) \in S \text{ for all } t \ge 0. \tag{2.3}$$

Definition 2.2. A function $\mathbf{w} = \mathbf{w}(\mathbf{x}) : S \to R^2$ is an *admissible feedback control (policy)* if for any given initial $\mathbf{x} = (x_1, x_2) \in S$, equations

$$\begin{cases} \dot{x}_1(t) = w_1(\mathbf{x}(t)) - w_2(\mathbf{x}(t)), & x_1(0) = x_1 \\ \dot{x}_2(t) = w_2(\mathbf{x}(t)) - d, & x_2(0) = x_2 \end{cases},$$

have a unique solution $\mathbf{x}(\cdot)$, and $\mathbf{w}(\mathbf{x}(\cdot))$ is admissible with respect to $\mathbf{x}$.

Our problem is to find an admissible control $\mathbf{w}(\cdot)$ that minimizes the cost function

$$J(\mathbf{x}, \mathbf{w}(\cdot)) = \int_0^\infty e^{-\rho t} G(\mathbf{x}(t), \mathbf{w}(t))dt, \tag{2.4}$$

where $G(\mathbf{x}, \mathbf{w})$ is the runnning cost of having surplus $\mathbf{x}$ and production rate $\mathbf{w}$ and $\rho > 0$ is the discount rate. To make it more precisely, considering the continuous compounding (discount) interest rate $\rho > 0$, the present-value for unit-cost at time t is $e^{-\rho t}$. Therefore, the right hand side of (2.4) is the total sum of present values of the runnning costs over the long-run time horizon.

For a feedback control $\mathbf{w}$, on the other hand, we shall write the cost $J(\mathbf{x}, \mathbf{w}(\mathbf{x}(\cdot)))$ simply as $J(\mathbf{x}, \mathbf{w})$, where $\mathbf{x}(\cdot)$ is the corresponding trajectory under $\mathbf{w}$ with the initial state $\mathbf{x}$.

We use $\tilde{\mathcal{A}}(\mathbf{x})$ to denote the set of admissible controls with respect to the initial state $\mathbf{x} \in S$, and $v(\mathbf{x})$ to denote the minimal cost, i.e.,

$$v(\mathbf{x}) = \inf_{\mathbf{w}(\cdot) \in \tilde{\mathcal{A}}(\mathbf{x})} J(\mathbf{x}, \mathbf{w}(\cdot)). \tag{2.5}$$

We use $\bar{\mathcal{P}}$ to denote our deterministic control problem, i.e.,

$$\bar{\mathcal{P}}: \begin{cases} \text{minimize} & J(\mathbf{x}, \mathbf{w}(\cdot)) = \int_0^\infty e^{-\rho t} G(\mathbf{x}(t), \mathbf{w}(t)) dt \\ \text{subject to} & \begin{cases} \dot{x}_1(t) = w_1(t) - w_2(t), & x_1(0) = x_1, \\ \dot{x}_2(t) = w_2(t) - d, & x_2(0) = x_2, \\ \mathbf{w}(\cdot) \in \tilde{\mathcal{A}}(\mathbf{x}) \end{cases} \\ \text{value function} & v(\mathbf{x}) = \inf_{\mathbf{w}(\cdot) \in \tilde{\mathcal{A}}(\mathbf{x})} J(\mathbf{x}, \mathbf{w}(\cdot)). \end{cases}$$

In order to formulate the stochastic production planning problems, we have to review the finite state Markov chain.

2.2 Markov Chain

Let $k(\cdot) = \{k(t) : t \geq 0\}$ denote a stochastic process defined on a standard probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathcal{M} = \{k_1, \cdots, k_n\}$. Then $k(\cdot) = \{k(t) : t \geq 0\}$ is a *Markov chain* if

$$P(k(t+s) = k_i | k(r) : r \leq s) = P(k(t+s) = k_i | k(s)), \tag{2.6}$$

for all $s, t \geq 0$ and $k_i \in \mathcal{M}$. We shall also write $k(\cdot)$ as $k(t), t \geq 0$, or simply $k(t)$ if there is no confusion.

Equation (2.6) may be interpreted as stating that, for a Markov chain, the conditional distribution of any future state $k(t+s)$, given the past states $k(r), r < s$ and the present state $k(s)$, is independent of the past states and depends only on the present state. This is called the *Markovian* property. Let us assume that the transition probability $P(k(t+s) = k_j | k(s) = k_i)$ is stationary, i.e., it is independent of s. This allows us to introduce the notation $P_{ij}(t) = P(k(t+s) = k_j | k(s) = k_i)$. The value $P_{ij}(\cdot)$ represents the probability that the process will, when in state k_i, next make a transition into state k_j. Then,

$$\begin{cases} P_{ij}(t) \geq 0, k_i, k_j \in \mathcal{M} \\ \sum_{j=1}^{n} P_{ij}(t) = 1, k_i \in \mathcal{M} \\ P_{ij}(t+s) = \sum_{l=1}^{n} P_{il}(s) P_{lj}(t), t, s \geq 0, k_i, k_j \in \mathcal{M} \\ \text{(The Chapman-Kolmogorov relation).} \end{cases}$$

Let $P(t)$ denote the $n \times n$ matrix $(P_{ij}(t))$ of stationary transition probabilities. We shall refer to $P(t)$ as the transition matrix of Markov chain $k(\cdot)$. We postulate that

$$\lim_{t \to 0} P(t) = I_n,$$

where I_n denotes the $n \times n$ identity matrix.

Let Q denote an $n \times n$ matrix such that $Q = (q_{ij})$ with $q_{ij} \geq 0$ for $j \neq i$ and $q_{ii} = -\sum_{j \neq i} q_{ij}$. Consider a finite state Markov chain $k(\cdot)$ governed by Q (cf.

Eithier and Kurtz [12]), i.e.,

$$Q\phi(\cdot)(i) = \sum_{j \neq i} q_{ij}(\phi(j) - \phi(i)),$$

for any function ϕ on $\mathcal{M}$. The matrix Q is called the *infinitesimal generator* (or simply *generator*) of $k(\cdot)$.

The transition matrix $P(t)$ is determined uniquely by the generator Q according to the following differential equation (cf. Karlin and Taylor [31]):

$$\dot{P}(t) = P(t)Q = QP(t), P(0) = I_n.$$

Thus,

$$q_{ij} = \begin{cases} \lim_{t \to 0^+} \frac{P_{ij}(t)-1}{t}, & \text{if } j = i \\ \lim_{t \to 0^+} \frac{P_{ij}(t)}{t}, & \text{if } j \neq i \end{cases}$$

can be interpreted as the transition rate from state k_i to state k_j when $i \neq j$, and as the (negative of the) transition rate out of state k_i when $j = i$.

An $n \times n$ matrix Q is said to be *(strongly) irreducible*, or simply *irreducible*, if the equations

$$\boldsymbol{\nu} Q = 0 \text{ and } \sum_{i=1}^{n} \nu_i = 1 \tag{2.7}$$

have a unique solution $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_n)$ and $\boldsymbol{\nu} > \mathbf{0}$.

Remark 2.1. An $n \times n$ matrix Q is said to be *weakly irreducible*, if the equations

$$\boldsymbol{\nu} Q = 0 \text{ and } \sum_{i=1}^{n} \nu_i = 1 \tag{2.8}$$

have a unique solution $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_n)$ and $\boldsymbol{\nu} \geq \mathbf{0}$.

The solution $\boldsymbol{\nu}$ to equations (2.7) will be termed an *equilibrium distribution.*

2.3 Problem Formulation: Stochastic Production Planning

In this thesis, we formulate the stochastic production planning for manufacturing system with two machines in tandem as shown in Fig. 2.2. The machines are unreliable, and they may breakdown at random times. When they are down, they will be brought to a repairer for repairing. The repairing time is also random. Each machine has a finite number of states, resulting in a finite state machine capacity process denoted by $\mathbf{k}(\varepsilon, t) = (k_1(\varepsilon, t), k_2(\varepsilon, t))$, defined on a standard probability space $(\Omega, \mathcal{F}, P)$, with values in $\mathcal{M} = \{\mathbf{k}^1, \cdots, \mathbf{k}^p\}$ for some given integer $p \geq 1$, where $\mathbf{k}^j = (k_1^j, k_2^j)$ with k_i^j denoting the capacity of the i-th machine in state j, $j = 1, 2, \cdots, p$ and $i = 1, 2$, and ε is a parameter. The precise meaning of ε will be given later on. The rate of demand d facing the

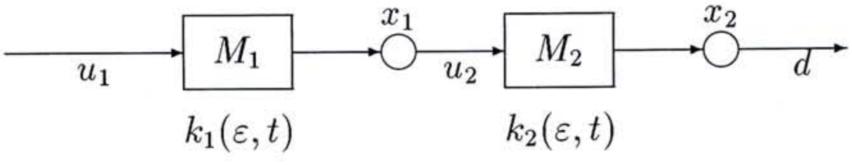

Figure 2.2: A stochastic manufacturing system with two machines in tandem

system is assumed to be a constant. We use $u_1(t)$ and $u_2(t)$ (controls in this problem) to denote the input production rates to the first and the second machine, respectively. As in Section 2.1, we denote the inventory level of the internal buffer as $x_1(t)$ and the surplus level of the external buffer as $x_2(t)$.

We also use b_1 and b_2 to denote the sizes of the internal buffer and the external buffer, respectively, and $S = [0, b_1] \times (-\infty, b_2] \subset R^2$ to denote the (state) constraint domain. Then the system can be written as follows:

$$\left\{ \begin{array}{ll} \dot{x}_1(t) = u_1(t) - u_2(t), & x_1(0) = x_1 \\ \dot{x}_2(t) = u_2(t) - d, & x_2(0) = x_2 \end{array} \right. , \quad \mathbf{x} = (x_1, x_2) \in S, \tag{2.9}$$

where the input rate to each of the machines is subject to the capacity of the respective machine, namely,

$$0 \le u_i(t) \le k_i(\varepsilon, t), \text{ for all } t \ge 0,\ i = 1, 2. \tag{2.10}$$

Remark 2.2. We did not impose a lower bound on the external buffer as it would not be realistic. In fact, if the surplus level reached the lower bound (if we did impose one) and M_2 is brokendown, then the controller could do nothing to prevent the violation of the lower bound constraint.

Remark 2.3. In this thesis, the capacity process $\mathbf{k}(\varepsilon, t) = \mathbf{k}(\frac{t}{\varepsilon}) \in \mathcal{M}$ is assumed to be a finite state Markov chain, where $\mathbf{k}(t)$ is a given Markov chain with an irreducible generator $Q = (q_{ij})$ independent of ε and $\min_{ij}\{|q_{ij}| : q_{ij} \neq 0\} = 1$. Then, $\mathbf{k}(\varepsilon, t)$ has the generator $Q^\varepsilon = \varepsilon^{-1}Q$.

Here, we use the normalized generator $Q = (q_{ij})$ (i.e., $\min_{ij}\{|q_{ij}| : q_{ij} \neq 0\} = 1$) in order to uniquely give ε. In other words, ε is a small parameter representing the reciprocal of the fluctuation rate of the machines' capacities.

We use $\mathbf{x}(t) = (x_1(t), x_2(t))$ and $\mathbf{u}(t) = (u_1(t), u_2(t))$ to denote the state and the control processes of the problem, respectively. We define now the set of admissible controls $\mathbf{u}(\cdot)$.

Definition 2.3. We say that a control $\mathbf{u}(\cdot) = (u_1(\cdot), u_2(\cdot))$ is *admissible* with respect to the initial state $\mathbf{x} = (x_1, x_2) \in S$ if: (i) $\mathbf{u}(\cdot)$ is adapted to $\mathcal{F}_t^\varepsilon =$

$\sigma\{\mathbf{k}(\varepsilon,s) : 0 \le s \le t\}$, the σ-algebra generated by the machine capacity process up to time t, (ii) $0 \le u_i(t) \le k_i(\varepsilon,t)$ for $t \ge 0$ and $i = 1,2$, and (iii) the corresponding state

$$\mathbf{x}(t) = (x_1(t), x_2(t)) \in S \text{ for all } t \ge 0. \tag{2.11}$$

In words, a production plan is admissible if (i) it depends only on the past realizations of the random capacity process, (ii) the input rates are nonnegative and satisfy the production capacity constraints at any time, and (iii) the corresponding inventory level in the internal buffer dose not fall below zero or exceed the buffer size, and the surplus level does not exceed the external buffer size.

Definition 2.4. We say that a function $\mathbf{u} = S \times \mathcal{M} \to R^2$ is an *admissible feedback control* if (i) for any given initial $\mathbf{x} = (x_1, x_2) \in S$, the following equation has a unique solution $\mathbf{x}(\cdot)$:

$$\begin{cases} \dot{x}_1(t) = u_1(\mathbf{x}(t), \mathbf{k}(\varepsilon,t)) - u_2(\mathbf{x}(t), \mathbf{k}(\varepsilon,t)), & x_1(0) = x_1 \\ \dot{x}_2(t) = u_2(\mathbf{x}(t), \mathbf{k}(\varepsilon,t)) - d, & x_2(0) = x_2 \end{cases},$$

and (ii) $\mathbf{u}(\mathbf{x}(\cdot), \mathbf{k}(\varepsilon,\cdot))$ is admissible with respect to $\mathbf{x}$.

The production planning problem is to find an admissible control $\mathbf{u}(\cdot)$ that minimizes the cost function:

$$J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)) = E\int_0^\infty e^{-\rho t}G(\mathbf{x}(t),\mathbf{u}(t))dt, \tag{2.12}$$

where E denotes the expectation, $G(\mathbf{x},\mathbf{u})$ is the runnning cost of having surplus $\mathbf{x}$ and production rate $\mathbf{u}$, $\mathbf{k} = (k_1, k_2)$ is the initial value of $\mathbf{k}(\varepsilon, t)$, and $\rho > 0$ is the given discount rate.

Let $\mathcal{A}^\varepsilon(\mathbf{x},\mathbf{k})$ denote the set of admissible controls with respect to $\mathbf{x}(0) = \mathbf{x} \in S$, $\mathbf{k}(\varepsilon, 0) = \mathbf{k}$, and $v^\varepsilon(\mathbf{x},\mathbf{k})$, the value function, denote the minimal expected cost, i.e.,

$$v^\varepsilon(\mathbf{x},\mathbf{k}) = \inf_{\mathbf{u}(\cdot)\in\mathcal{A}^\varepsilon(\mathbf{x},\mathbf{k})} J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)). \tag{2.13}$$

We use $\mathcal{P}^\varepsilon$ to denote our control problem, i.e.,

$$\mathcal{P}^\varepsilon : \begin{cases} \text{minimize} & J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)) = E\int_0^\infty e^{-\rho t}G(\mathbf{x}(t),\mathbf{u}(t))dt \\ \text{subject to} & \begin{cases} \dot{x}_1(t) = u_1(t) - u_2(t), & x_1(0) = x_1, \\ \dot{x}_2(t) = u_2(t) - d, & x_2(0) = x_2, \\ \mathbf{u}(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x},\mathbf{k}), & \mathbf{x} \in S \end{cases} \\ \text{value function} & v^\varepsilon(\mathbf{x},\mathbf{k}) = \inf_{\mathbf{u}(\cdot)\in\mathcal{A}^\varepsilon(\mathbf{x},\mathbf{k})} J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)). \end{cases} \tag{2.14}$$

Example 2.1. A manufacturing company is contemplating the acquisition of a transfer line to make desktop widgets. The making of widgets is a two-step process. It requires rough drilling and finished reaming. This system is a two-machine flowshop. Machine 1 is a rough drilling machine, and Machine 2 is a final reaming machine. The internal buffer stores the drilled widgets, and the external buffer stores the final reamed widgets. These two machines are subject to breakdown and repair randomly. The drilling machine has a mean time of 40 working hours before failure and a mean time of 2 hours for repairing. The reaming machine has a mean time of 50 working hours and a mean time of 2 hours for repairing. Therefore, this is a stochastic two-machine flowshop.

Example 2.2. A canned food factory has two workshops: the first one is to process the food and the second one is to can the food. This system can also be regarded as a two-machine flowshop. Machine 1 is the first workshop, and Machine 2 is the second workshop. (Note, here the machines are not physical ones; rather, they are workshops performing different kinds of jobs). The internal buffer is a cold storage which is used to store the processed food. The external buffer is a warehouse which is used to store the canned food.

Before solving the problem $\mathcal{P}^\varepsilon$, we have to review some relevant results which will be used later on.

2.4 Some Lemmas

In this section, we consider a Markov chain $\mathbf{k}(\varepsilon, t), t \geq 0$, with generator $Q^{\varepsilon} = \varepsilon^{-1}Q$, where ε is a small parameter, $Q = (q_{ij})$ is a matrix such that $q_{ij} \geq 0$ if $j \neq i$, $q_{ii} = -\sum_{j\neq i} q_{ij}$, and $\min_{ij}\{|q_{ij}| : q_{ij} \neq 0\} = 1$. Moreover, Q is irreducible. Let $\nu = (\nu^1, \cdots, \nu^p) > 0$ denote the equilibrium distribution of Q.

Now we state the following lemmas concerning the asymptotic property of the Markov chain $\mathbf{k}(\varepsilon, t)$ for small ε which will be used later on.

Lemma 2.1. *There exist positive constants C and κ such that for sufficiently small ε,*

$$|P(\mathbf{k}(\varepsilon, t) = \mathbf{k}^j) - \nu^j| \leq C(\varepsilon + e^{-K\varepsilon^{-1}t}), \quad \textit{for all } t \geq 0,\ j = 1, 2, \cdots, p.$$

Proof. See [42] for the proof. □

Lemma 2.2. *There exists a positive constant C such that for sufficiently small ε and for any bounded deterministic measurable process $\beta(\cdot)$,*

$$E\left|\int_0^t (\chi_{\{\mathbf{k}(\varepsilon,s)=\mathbf{k}^j\}} - \nu^j)\beta(s)ds\right|^2 \leq C\varepsilon(1 + t^2),$$

for all $t \geq 0$, $j = 1, 2, \cdots, p$.

Proof. The proof can be found in [47]. □

Lemma 2.3. *For any $\delta \in (0, \frac{1}{2})$, there exist positive constants C and κ such that for any bounded deterministic measurable process $\beta(\cdot)$,*

$$P(|\int_0^t (\chi_{\{\mathbf{k}(\varepsilon,s)=\mathbf{k}^j\}} - \nu^j)\beta(s)ds| \geq \varepsilon^{1/2-\delta}) \leq C(e^{-K\varepsilon^{-1}(1+t)} + e^{-K\varepsilon^{-(1-2\delta)}(1+t)^{-3}}),$$

for all $t \geq 0$, $j = 1, 2, \cdots, p$, and sufficiently small ε.

Proof. See [45] for the proof. □

Corollary 2.1. *Let τ be a bounded $\mathcal{F}_t^\varepsilon$−stopping time with $\tau \leq a$, almost surely. Then*

$$P(|\int_0^\tau (\chi_{\{\mathbf{k}(\varepsilon,s)=\mathbf{k}^i\}} - \nu^i)\beta(s)ds| \geq \varepsilon^{\frac{1}{3}}) \leq C(e^{-K\varepsilon^{-1}} + e^{-K\varepsilon^{-1/3}(1+a)^{-3}}),$$

for all $t \geq 0$, $j = 1, 2, \cdots, p$.

Proof. See [50] for the proof. □

Chapter 3

Open-Loop Production Planning in Stochastic Flowshops

3.1 Introduction

This chapter is concerned with open-loop production planning in manufacturing systems with two tandem unreliable machines. Since the sizes of both internal and external buffers are practically finite, the problem is one with state constraints. Using hierarchical control approach, Lehoczky *et al.* [33] studied a production planning problem for a system consisting of parallel identical machines. Sethi, Zhang, and Zhou [47] applied a probabilistic approach to construct a limiting problem which is different in form to that of [33], which enabled them to prove the conjecture made by Gershwin [18] and Lehoczky *et al.* [33] for the special case of separable convex production and inventory costs.

Note that in [33, 47], the control problems in manufacturing systems with tandem machines, in which state constraints are inherent, are not addressed. Furthermore, the method of construction of asymptotic optimal controls developed there may yield inadmissible controls when applied to systems with state constraints. To overcome the difficulty, Sethi, Zhang, and Zhou [45] developed a method of "lifting" and "modification" to construct admissible asymptotic optimal production policies for an N-machine flowshop from a near-optimal policy for the corresponding limiting problem. Nevertheless, they assumed that the sizes of internal and external buffers are infinite. Recently, Sethi, Zhang, and Zhou [46] augmented the method of lifting and modification by another special technique called "squeezing" in order to construct asymptotic optimal controls for a two-machine flowshop with only the internal buffer being upper bounded. Practically, however, the managers of manufacturing firms must also take the upper bound of the external buffer (such as warehouses) into consideration, especially in the situation of scarce space and/or high rent. The purpose of this chapter is to study hierarchical production planning for two-machine flowshops with *both* internal and external buffers being of finite size. One of the major difficulties in this case is proving the Lipschitz continuity of the value functions for both the original problem and the limiting problem, which plays an essential role in the hierarchical analysis. It should be noted that in [45, 46], the constructive proof of the Lipschitz property needs only to take care of the constraints on internal buffer without worrying about the external buffer. The proof does not go through in our case. To handle the problem, we shall in this chapter introduce and prove some *weak-Lipschitz* property, which is weaker than Lipschitz continuity but sufficient for subsequent analysis.

Another major difficulty is in constructing asymptotic optimal controls and obtaining error estimates. Owing to the upper bound constraints on both buffers, the lifting, squeezing and modification method in [46] does no longer work. The reason is that lifting and squeezing the internal buffer will violate the upper bound constraint on the external buffer. In this chapter, a "constraint domain approximation" method is developed to overcome the difficulty. The basic idea behind it is: We approximate the (state) constraint domain by a subset of it whose boundary is distinct from but close enough to that of the original constraint domain. Then we consider two situations: (i) For any initial point in this subset, we construct a control as in [47] from a near-optimal control of the limiting problem. The constructed control may violate the state constraints but the average cumulative duration in which the violation takes place is very small since the initial point is sufficiently away from the boundary of the constraint domain. Therefore, we can *slightly* modify this control to be an admissible one. (ii) For any initial point in the band outside this approximating subset, we can use the weak-Lipschitz property to construct an admissible control. That the final constructed controls in both situations are asymptotic optimal for the original problem can be shown from the facts that the amount of modification is small and the value functions are weak-Lipschitz. As in [45, 46], the order of the error estimate of the constructed asymptotic controls is $\varepsilon^{1/2-\delta}$ for any $\delta > 0$.

The plan of the chapter is as follows. In the next section, we derive the limiting problem from the stochastic production planning problem formulated in Chapter 2 with a separable inventory and production cost. Section 3.3 is devoted to the proof of the weak-Lipschitz continuity of the value functions. In Section 4.4, we introduce a subset approximating the state constraint domain. In Section 3.5 and Section 3.6, we describe the methods of constructing asymptotic optimal open-loop controls and prove the asymptotic optimality for the original problem for cases where the initial state is in and out of the approximating subset, respectively. Finally, Section 3.7 concludes the chapter.

3.2 Limiting Problem

In this chapter, we consider the stochastic production planning problem formulated in Section 2.3 with the running cost function $G(\mathbf{x}, \mathbf{u})$ being a separate cost function, i.e.,

$$G(\mathbf{x}, \mathbf{u}) = h(\mathbf{x}) + c(\mathbf{u}), \tag{3.1}$$

where $h(\mathbf{x})$ is the inventory/backlog cost, $c(\mathbf{u})$ is the production cost.

We make the following assumptions on the functions h and c and the random process $\mathbf{k}(\varepsilon, t)$ throughout this chapter.

Assumption 3.1. h and c are convex functions. For all $\mathbf{x}, \mathbf{x}' \in S$ and $\mathbf{u}, \mathbf{u}'$, there exist constants C_0 and $\kappa_0 \geq 1$ such that

$$\begin{aligned}
&0 \leq h(\mathbf{x}) \leq C_0(1 + |\mathbf{x}|^{\kappa_0}),\\
&|h(\mathbf{x}) - h(\mathbf{x}')| \leq C_0(1 + |\mathbf{x}|^{\kappa_0} + |\mathbf{x}'|^{\kappa_0})|\mathbf{x} - \mathbf{x}'|, \quad \text{and}\\
&|c(\mathbf{u}) - c(\mathbf{u}')| \leq C_0|\mathbf{u} - \mathbf{u}'|.
\end{aligned}$$

Assumption 3.2. Let $\mathcal{M} = \{\mathbf{k}^1, \cdots, \mathbf{k}^p\}$ for some given integer $p \geq 1$, where $\mathbf{k}^j = (k_1^j, k_2^j)$ with k_i^j denoting the capacity of the i-th machine in state j, $j = 1, 2, \cdots, p$ and $i = 1, 2$. For each $\varepsilon > 0$, the capacity process $\mathbf{k}(\varepsilon, t) = \mathbf{k}(\frac{t}{\varepsilon}) \in \mathcal{M}$ is a finite state Markov chain, where $\mathbf{k}(t)$ is a given Markov chain with an irreducible generator $Q = (q_{ij})$ independent of ε and $\min_{ij}\{|q_{ij}| : q_{ij} \neq 0\} = 1$. Then, $\mathbf{k}(\varepsilon, t)$ has the generator $Q^\varepsilon = \varepsilon^{-1}Q$.

Remark 3.1. Assumption 3.2 means that $\mathbf{k}(\varepsilon, t) = \mathbf{k}(\frac{t}{\varepsilon})$ is a fast changing process as ε is sufficiently small.

Intuitively, as the rates of the machine breakdown and repair approach infinity, the problem $\mathcal{P}^\varepsilon$ as formulated in Section 2.3, which is termed the *original problem*, can be approximated by a simpler problem called the *limiting problem*, in which the stochastic machine capacity process $\mathbf{k}(\varepsilon, t)$ is replaced by its average value (cf. [45, 46, 47, 48]).

Let $\nu = (\nu^1, \cdots, \nu^p) > 0$ denote the equilibrium distribution of Q. Note that by Assumption 3.2, ν is the only positive solution of $\nu Q = 0$ and $\sum_{j=1}^p \nu^j = 1$.

To define the limiting problem, we consider the following class of deterministic controls.

Definition 3.1. For $\mathbf{x} \in S$, let $\bar{\mathcal{A}}(\mathbf{x})$ denote the set of the following deterministic measurable controls

$$\mathbf{U}(\cdot) = (\mathbf{u}^1(\cdot), \cdots, \mathbf{u}^p(\cdot)) = ((u_1^1(\cdot), u_2^1(\cdot)), \cdots, (u_1^p(\cdot), u_2^p(\cdot)))$$

such that $0 \le u_i^j(t) \le k_i^j$ for all $t \ge 0$, $i = 1, 2$ and $j = 1, 2, \cdots, p$, and the corresponding solutions $\mathbf{x}(\cdot) = (x_1(\cdot), x_2(\cdot))$ of the following system

$$\begin{cases} \dot{x}_1(t) = \sum_{j=1}^p \nu^j u_1^j(t) - \sum_{j=1}^p \nu^j u_2^j(t), & x_1(0) = x_1, \\ \dot{x}_2(t) = \sum_{j=1}^p \nu^j u_2^j(t) - d, & x_2(0) = x_2 \end{cases} \tag{3.2}$$

satisfy $\mathbf{x}(t) \in S$ for all $t \ge 0$.

Note that the control $\mathbf{U}(\cdot)$ is defined on an enlarged control space. Each of its component vector $\mathbf{u}^j(t) = (u_1^j(t), u_2^j(t))$ represents an admissible control when the machine state is $\mathbf{k}^j = (k_1^j, k_2^j)$.

The objective of this problem is to choose a control $\mathbf{U}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})$, that minimizes

$$J(\mathbf{x}, \mathbf{U}(\cdot)) = \int_0^\infty e^{-\rho t}[h(\mathbf{x}(t)) + \sum_{j=1}^{p} \nu^j c(\mathbf{u}^j(t))]dt. \tag{3.3}$$

We use $\bar{\mathcal{P}}$ to denote the above problem, and will regard this as our limiting problem.

$$\bar{\mathcal{P}}: \begin{cases} \text{minimize} & J(\mathbf{x}, \mathbf{U}(\cdot)) = \int_0^\infty e^{-\rho t}[h(\mathbf{x}(t)) + \sum_{j=1}^{p} \nu^j c(\mathbf{u}^j(t))]dt \\ \text{subject to} & \begin{cases} \dot{x}_1(t) = \sum_{j=1}^{p} \nu^j u_1^j(t) - \sum_{j=1}^{p} \nu^j u_2^j(t), & x_1(0) = x_1, \\ \dot{x}_2(t) = \sum_{j=1}^{p} \nu^j u_2^j(t) - d, & x_2(0) = x_2, \\ \mathbf{U}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x}), & \mathbf{x} \in S \end{cases} \\ \text{value function} & v(\mathbf{x}) = \inf_{\mathbf{U}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})} J(\mathbf{x}, \mathbf{U}(\cdot)). \end{cases}$$

Example 3.1. Consider a 2-machine flowshop in which the original problem is

$$\mathcal{P}^\varepsilon: \begin{cases} \text{minimize} & J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}(\cdot)) = E\int_0^\infty e^{-0.1t}[x_1(t) + 2|x_2(t)|]dt \\ \text{subject to} & \begin{cases} \dot{x}_1(t) = u_1(t) - u_2(t), & x_1(0) = 1, \\ \dot{x}_2(t) = u_2(t) - 0.5, & x_2(0) = 1. \end{cases} \end{cases}$$

We assume that the system has 4 machines states: (1,1), (1,0), (0,1), and (0,0), where the state (1,0) signifies that the first machine is working and the second one is down, etc. We assume further that the breakdown rate and the repair rate are both 50 per year for each machine. This means that both up and down times of each machine have an average duration of $\frac{1}{50}$ year (approximately one week). The associated generator Q of the capacity process is

$$Q^\varepsilon = \begin{pmatrix} -100 & 50 & 50 & 0 \\ 50 & -100 & 0 & 50 \\ 50 & 0 & -100 & 50 \\ 0 & 50 & 50 & -100 \end{pmatrix} = \frac{1}{0.02} \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}.$$

Thus, the small parameter associated with the problem is $\varepsilon = 0.02$. The corresponding equilibrium distribution $\nu = (0.25, 0.25, 0.25, 0.25)$. According to Definition 3.1, the controls for the limiting problem take the form

$$\mathbf{U}(\cdot) = ((u_1^1(\cdot), u_2^1(\cdot)), (u_1^2(\cdot), u_2^2(\cdot)), (u_1^3(\cdot), u_2^3(\cdot)), (u_1^4(\cdot), u_2^4(\cdot))),$$

With

$$0 \le u_1^1(t), u_2^1(t), u_1^2(t), u_2^3(t) \le 1, \qquad u_2^2(t) = u_1^3(t) = u_1^4(t) = u_2^4(t) = 0.$$

Therefore, the limiting problem $\bar{\mathcal{P}}$ is

$$\bar{\mathcal{P}}: \begin{cases} \text{minimize} & J(\mathbf{x}, \mathbf{U}(\cdot)) = \int_0^\infty e^{-0.1t}[x_1(t) + 2|x_2(t)|]dt \\ \text{subject to} & \begin{cases} \dot{x}_1(t) = 0.25(u_1^1(t) + u_1^2(t)) - 0.25(u_2^1(t) + u_2^3(t)), & x_1(0) = 1, \\ \dot{x}_2(t) = 0.25(u_2^1(t) + u_2^3(t)) - 0.5, & x_2(0) = 1. \end{cases} \end{cases}$$

The above $\bar{\mathcal{P}}$ is a deterministic problem.

3.3 Weak-Lipschitz Continuity

The Lipschitz continuity of the value functions of both the original and limiting problems, which played a critical role in proving the asymptotic optimality of hierarchical controls in the preceding research [45, 46, 48], does not follow automatically in tandem machine systems due to the state constraints. It was proved by some rather specific approaches in [45, 46, 48]. These approaches fail, however, for the present case where there are upper bounds on the buffers. On the other hand, some Hölder estimates of the value functions were obtained for deterministic problem with state constraints (cf. [54, 9]), but it is not clear how to adapt the method to stochastic case. To overcome the difficulty, we do not insist on deriving Lipschitz continuity in this chapter; instead, we develop what we call *weak-Lipschitz* property which is weaker than Lipschitz but is sufficient for serving the purpose of deriving asymptotic optimality later on.

Theorem 3.1. *There exists a constant C (independent of ε) such that for all $\mathbf{x} = (x_1, x_2) \in S$, $\mathbf{x}' = (x_1', x_2') \in S$, the following weak-Lipschitz properties hold:*

(i) If $x_2' \leq x_2$, then

$$v^\varepsilon(\mathbf{x}', \mathbf{k}) - v^\varepsilon(\mathbf{x}, \mathbf{k}) \leq C(1 + |\mathbf{x}|^{\mathrm{K}_0} + |\mathbf{x}'|^{\mathrm{K}_0})|\mathbf{x} - \mathbf{x}'|$$

$$\text{and} \quad v(\mathbf{x}') - v(\mathbf{x}) \leq C(1 + |\mathbf{x}|^{\mathrm{K}_0} + |\mathbf{x}'|^{\mathrm{K}_0})|\mathbf{x} - \mathbf{x}'|.$$

(ii) If $x_2' \leq x_2$, $x_1' \geq x_1$ and $|x_1' - x_1| \geq |x_2' - x_2|$, then

$$v^\varepsilon(\mathbf{x}', \mathbf{k}) - v^\varepsilon(\mathbf{x}, \mathbf{k}) \geq -C(1 + |\mathbf{x}|^{\mathrm{K}_0} + |\mathbf{x}'|^{\mathrm{K}_0})|\mathbf{x} - \mathbf{x}'|$$

$$\text{and} \quad v(\mathbf{x}') - v(\mathbf{x}) \geq -C(1 + |\mathbf{x}|^{\mathrm{K}_0} + |\mathbf{x}'|^{\mathrm{K}_0})|\mathbf{x} - \mathbf{x}'|.$$

(iii) If $x_2' \leq x_2$, $x_1' \leq x_1$ and $x_1 \geq |x_1' - x_1| + |x_2' - x_2|$, then

$$v^\varepsilon(\mathbf{x}', \mathbf{k}) - v^\varepsilon(\mathbf{x}, \mathbf{k}) \geq -C(1 + |\mathbf{x}|^{\mathrm{K}_0} + |\mathbf{x}'|^{\mathrm{K}_0})|\mathbf{x} - \mathbf{x}'|$$

$$\text{and} \quad v(\mathbf{x}') - v(\mathbf{x}) \geq -C(1 + |\mathbf{x}|^{\mathrm{K}_0} + |\mathbf{x}'|^{\mathrm{K}_0})|\mathbf{x} - \mathbf{x}'|.$$

Proof. We only show the first inequalities in (i),(ii) and (iii) respectively, the second being similar and, in fact, simpler.

(i) For any $\eta > 0$, let $\mathbf{u}(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}, \mathbf{k})$ be an η-optimal control, i.e.,

$$v^\varepsilon(\mathbf{x}, \mathbf{k}) \geq J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}(\cdot)) - \eta.$$

Note that $\mathbf{u}(\cdot) \notin \mathcal{A}^{\varepsilon}(\mathbf{x}', \mathbf{k})$ in general. We are to construct a control $\mathbf{u}'(\cdot) \in \mathcal{A}^{\varepsilon}(\mathbf{x}', \mathbf{k})$ such that

$$|\mathbf{x}(t) - \mathbf{x}'(t)| \leq 3|\mathbf{x} - \mathbf{x}'|, \tag{3.4}$$

where $\mathbf{x}(\cdot)$ and $\mathbf{x}'(\cdot)$ are the state trajectories with initial states $\mathbf{x}$ and $\mathbf{x}'$ and the controls $\mathbf{u}(\cdot)$ and $\mathbf{u}'(\cdot)$, respectively.

In what follows, we divide into two cases to carry out the analysis according to the positions of the initial x_1 and x_1'.

Case 1. $x_1' \leq x_1$ and $x_2' \leq x_2$. Let $t_0^* > 0$ be the stopping time given by the following:

$$t_0^* := \inf\{t \geq 0 : \int_0^t [(k_1^{\varepsilon}(s) - u_1(s)) + u_2(s)]ds \geq |x_1 - x_1'|\}.$$

We define a new control $\mathbf{u}'(\cdot)$:

$$\mathbf{u}'(t) = (u_1'(t), u_2'(t)) = \begin{cases} (k_1^{\varepsilon}(t), 0), & \text{if } t < t_0^* \\ (u_1(t), u_2(t)), & \text{if } t \geq t_0^* \end{cases}.$$

It is not difficult to show from the definition of t_0^* and $\mathbf{u}'(\cdot)$ that $\mathbf{x}'(t) \in S$ and

(3.4) holds. Thus $\mathbf{u}'(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}', \mathbf{k})$. Moreover,

$$\begin{aligned} v^\varepsilon(\mathbf{x}', \mathbf{k}) - v^\varepsilon(\mathbf{x}, \mathbf{k}) \le & \; J^\varepsilon(\mathbf{x}', \mathbf{k}, \mathbf{u}'(\cdot)) - J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}(\cdot)) + \eta \\ \le & \; E \textstyle\int_0^\infty e^{-\rho t}[|h(\mathbf{x}'(t)) - h(\mathbf{x}(t))| + |c(\mathbf{u}'(t)) - c(\mathbf{u}(t))|]dt + \eta. \end{aligned} \tag{3.5}$$

Note that by Assumption 3.1 along with the fact that

$$|u_1(t) - u_2(t)| \le C_1, \quad |u_2(t) - z| \le C_1,$$

$$|u_1'(t) - u_2'(t)| \le C_1, \quad |u_2'(t) - z| \le C_1,$$

for some constant C_1, we have

$$|h(\mathbf{x}'(t)) - h(\mathbf{x}(t))| \le C_0(1 + |\mathbf{x}'(t)|^{\mathrm{K}_0} + |\mathbf{x}(t)|^{\mathrm{K}_0})|\mathbf{x}'(t) - \mathbf{x}(t)|,$$

$$|\mathbf{x}'(t)| \le C_1(1 + |\mathbf{x}'| + t), \text{ and } |\mathbf{x}(t)| \le C_1(1 + |\mathbf{x}| + t).$$

It follows therefore that

$$E \int_0^\infty e^{-\rho t}|h(\mathbf{x}'(t)) - h(\mathbf{x}(t))|dt \le C_2(1 + |\mathbf{x}|^{\mathrm{K}_0} + |\mathbf{x}'|^{\mathrm{K}_0})|\mathbf{x} - \mathbf{x}'|, \tag{3.6}$$

for some constant C_2.

Furthermore, since $\mathbf{u}'(t) = \mathbf{u}(t)$ for $t \geq t_0^*$, we have

$$\begin{aligned} E\int_0^\infty e^{-\rho t}|c(\mathbf{u}'(t)) - c(\mathbf{u}(t))|dt \leq\ & C_0 E\int_0^{t_0^*} e^{-\rho t}|\mathbf{u}'(t) - \mathbf{u}(t)|dt \\ \leq\ & C_0 E\int_0^{t_0^*}[|k_1^\varepsilon(t) - u_1(t)| + |u_2(t)|]dt \\ =\ & C_0|x_1 - x_1'| \\ \leq\ & C_0|\mathbf{x} - \mathbf{x}'|. \end{aligned} \tag{3.7}$$

Combining (3.5), (3.6), and (3.7), we obtain

$$v^\varepsilon(\mathbf{x}', \mathbf{k}) - v^\varepsilon(\mathbf{x}, \mathbf{k}) \leq C_3(1 + |\mathbf{x}|^{\mathrm{K}_0} + |\mathbf{x}'|^{\mathrm{K}_0})|\mathbf{x} - \mathbf{x}'| + \eta, \tag{3.8}$$

where $C_3 > 0$ is a constant independent of η.

Case 2. $x_1' > x_1$ and $x_2' \leq x_2$. Let $t_1^* > 0$ be the stopping time given by the following:

$$t_1^* := \inf\{t \geq 0 : \int_0^t u_1(s)ds \geq |x_1 - x_1'|\}.$$

We define a new control $\mathbf{u}'(\cdot)$:

$$\mathbf{u}'(t) = (u_1'(t), u_2'(t)) = \begin{cases} (0, u_2(t)), & \text{if } t < t_1^* \\ (u_1(t), u_2(t)), & \text{if } t \geq t_1^* \end{cases}.$$

By an argument similar to that in Case 1, it is not difficult to show that $\mathbf{u}'(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}', \mathbf{k})$ and (3.8) holds.

Since η is arbitrary, we conclude from (3.8) that

$$v^{\varepsilon}(\mathbf{x}',\mathbf{k}) - v^{\varepsilon}(\mathbf{x},\mathbf{k}) \leq C_3(1+|\mathbf{x}|^{\mathrm{K_0}}+|\mathbf{x}'|^{\mathrm{K_0}})|\mathbf{x}-\mathbf{x}'|. \tag{3.9}$$

(ii) For any $\eta > 0$, let $\mathbf{u}'(\cdot) \in \mathcal{A}^{\varepsilon}(\mathbf{x}',\mathbf{k})$ be an η-optimal control. Let $t_2^* > 0$ be the stopping time given by the following:

$$t_2^* := \inf\{t \geq 0 : \int_0^t u_2'(s)ds \geq |x_1 - x_1'|\}.$$

We define a new control $\mathbf{u}(\cdot)$:

$$\mathbf{u}(t) = (u_1(t), u_2(t)) = \begin{cases} (u_1'(t), 0), & \text{if } t < t_2^* \\ (u_1'(t), u_2'(t)), & \text{if } t \geq t_2^* \end{cases}.$$

We want to show that $\mathbf{u}(\cdot) \in \mathcal{A}^{\varepsilon}(\mathbf{x},\mathbf{k})$ and (3.4) holds.

Under the conditions that $x_2' \le x_2$, $x_1' \ge x_1$, and $|x_1' - x_1| \ge |x_2' - x_2|$, we have

$$\begin{aligned}
\text{for } t < t_2^* : \quad & x_1(t) = x_1 + \textstyle\int_0^t u_1'(s)ds \ge 0, \\
& x_1(t) = x_1'(t) + (x_1 - x_1') + \textstyle\int_0^t u_2'(s)ds \le x_1'(t) \le b_1, \\
& x_2(t) = x_2 - \textstyle\int_0^t d ds \le b_2, \\
& x_1 - x_1' \le x_1(t) - x_1'(t) \le 0, \text{ and} \\
& x_1 - x_1' \le x_2(t) - x_2'(t) \le x_2 - x_2'; \\
\text{for } t \ge t_2^* : \quad & x_1(t) = x_1'(t) + (x_1 - x_1') + \textstyle\int_0^{t_2^*} u_2'(s)ds = x_1'(t), \text{ hence} \\
& 0 \le x_1(t) \le b_1, \text{ and} \\
& x_2(t) = x_2'(t) + (x_2 - x_2') - \textstyle\int_0^{t_2^*} u_2'(s)ds \\
& \qquad = x_2'(t) + |x_2 - x_2'| - |x_1 - x_1'| \le x_2'(t) \le b_2.
\end{aligned}$$

Thus $(x_1(t), x_2(t)) \in S$ and (3.4) holds. Moreover, we can easily show as in (i) that

$$v^\varepsilon(\mathbf{x}', \mathbf{k}) - v^\varepsilon(\mathbf{x}, \mathbf{k}) \ge -C(1 + |\mathbf{x}|^{\mathrm{K}_0} + |\mathbf{x}'|^{\mathrm{K}_0})|\mathbf{x} - \mathbf{x}'|. \tag{3.10}$$

(iii) We choose an auxiliary initial state $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$, where $\bar{x}_1 = x_1 - (|x_1' - x_1| + |x_2' - x_2|) = x_1' - (x_2 - x_2')$ and $\bar{x}_2 = x_2$. It is easy to see by the assumptions that $(\bar{x}_1, \bar{x}_2) \in S$.

To proceed, we first consider the difference of the two value functions with the initial states (x_1', x_2') and $(\bar{x}_1, \bar{x}_2)$, respectively. Because $\bar{x}_1 \le x_1'$, $\bar{x}_2 \ge x_2'$, and

$|\bar{x}_1 - x_1'| = |\bar{x}_2 - x_2'|$, we obtain by (ii) that

$$\begin{aligned} v^\varepsilon(\mathbf{x}',\mathbf{k}) - v^\varepsilon(\bar{\mathbf{x}},\mathbf{k}) &\geq -C(1+|\bar{\mathbf{x}}|^{\mathrm{K_0}} + |\mathbf{x}'|^{\mathrm{K_0}})|\bar{\mathbf{x}} - \mathbf{x}'| \\ &\geq -C_4(1+|\mathbf{x}|^{\mathrm{K_0}} + |\mathbf{x}'|^{\mathrm{K_0}})|\mathbf{x} - \mathbf{x}'|. \end{aligned} \tag{3.11}$$

Now we consider the difference of the two value functions with the initial states (x_1, x_2) and $(\bar{x}_1, \bar{x}_2)$, respectively. Since $\bar{x}_2 = x_2$, we have by (i)

$$\begin{aligned} v^\varepsilon(\bar{\mathbf{x}},\mathbf{k}) - v^\varepsilon(\mathbf{x},\mathbf{k}) &\geq -C(1+|\bar{\mathbf{x}}|^{\mathrm{K_0}} + |\mathbf{x}|^{\mathrm{K_0}})|\bar{\mathbf{x}} - \mathbf{x}| \\ &\geq -C_5(1+|\mathbf{x}|^{\mathrm{K_0}} + |\mathbf{x}'|^{\mathrm{K_0}})|\mathbf{x} - \mathbf{x}'|. \end{aligned} \tag{3.12}$$

Thus, by adding (3.11) and (3.12) up, we obtain

$$v^\varepsilon(\mathbf{x}',\mathbf{k}) - v^\varepsilon(\mathbf{x},\mathbf{k}) \geq -C(1+|\mathbf{x}|^{\mathrm{K_0}} + |\mathbf{x}'|^{\mathrm{K_0}})|\mathbf{x} - \mathbf{x}'| \tag{3.13}$$

for some constant C. The proof is now complete. □

3.4 Constraint Domain Approximation

In this section, we shall introduce an appropriate subset S_ε of the state constraint domain S, with the distance between the boundaries of S_ε and S being of order $\varepsilon^{1/2-\delta}$. Here δ is any given constant satisfying $0 < \delta < \frac{1}{2}$. Consequently, S_ε approximates S from inside as ε approaches 0.

Let $b := 2(b_1 + b_2)/\min\{b_1, b_2\}$ and $S_\varepsilon := [b\varepsilon^{1/2-\delta}, b_1 - b\varepsilon^{1/2-\delta}] \times (-\infty, b_2 - b\varepsilon^{1/2-\delta}]$. The following lemma is very important in the sequel, which says that starting from any point in S_ε, there are near-optimal controls with error order $\varepsilon^{1/2-\delta}$ for both original and limiting problems so that the corresponding state trajectories will never reach the boundary of S and, what is more, the difference between the trajectories and the boundary is of order $\varepsilon^{1/2-\delta}$.

For convenience in exposition in the rest of the thesis, we use the convention that the phrase "sufficiently small ε" stands for "$\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$".

Lemma 3.1. *There exists a constant C (independent of ε) such that for any $\mathbf{x} \in S_\varepsilon$, there is $\mathbf{u}(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}, \mathbf{k})$ (resp. $\bar{\mathbf{U}}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})$) satisfying*

(i) $J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}(\cdot)) - v^\varepsilon(\mathbf{x}, \mathbf{k}) \leq C(1 + |\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}$

(resp. $J(\mathbf{x}, \bar{\mathbf{U}}(\cdot)) - v(\mathbf{x}) \leq C(1 + |\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}$*),*

(ii) $\varepsilon^{1/2-\delta} \leq x_1(t) \leq b_1 - \varepsilon^{1/2-\delta}$ *and* $x_2(t) \leq b_2 - 2\varepsilon^{1/2-\delta}$

(resp. $\varepsilon^{1/2-\delta} \leq \bar{x}_1(t) \leq b_1 - \varepsilon^{1/2-\delta}$ *and* $\bar{x}_2(t) \leq b_2 - 2\varepsilon^{1/2-\delta}$*)*

for sufficiently small ε, where $(x_1(\cdot), x_2(\cdot))$ (resp. $(\bar{x}_1(\cdot), \bar{x}_2(\cdot))$) is the state trajectory of the original problem (resp. the limiting problem) with initial state $\mathbf{x}$ and the control $\mathbf{u}(\cdot)$ (resp. $\bar{\mathbf{U}}(\cdot)$).

Proof. We only prove the result in terms of the original problem, the other being similar.

Take $a = 1 - (2\varepsilon^{1/2-\delta})/\min\{b_1, b_2\}$. We have $\frac{1}{2} \leq a < 1$ for sufficiently small

ε. Set $y_1 = x_1/a$, $y_2 = x_2/a$. Since

$$ab_1 = b_1 - (2b_1\varepsilon^{1/2-\delta})/\min\{b_1, b_2\} \geq b_1 - [2\varepsilon^{1/2-\delta}(b_1+b_2)]/\min\{b_1, b_2\} = b_1 - b\varepsilon^{1/2-\delta},$$

we obtain

$$0 \leq y_1 = x_1/a \leq (b_1 - b\varepsilon^{1/2-\delta})/a \leq b_1.$$

By the same reason, $y_2 \leq b_2$, which implies $\mathbf{y} = (y_1, y_2) \in S$. Hence, there exists $\hat{\mathbf{u}}(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{y}, \mathbf{k})$ such that

$$J^\varepsilon(\mathbf{y}, \mathbf{k}, \hat{\mathbf{u}}(\cdot)) - v^\varepsilon(\mathbf{y}, \mathbf{k}) \leq \varepsilon^{1/2-\delta}. \tag{3.14}$$

Let $t_3^* \geq 0$ be defined as follows:

$$t_3^* = \inf\{t \geq 0 : \int_0^t (k_1^\varepsilon(s) - \hat{u}_1(s) + \hat{u}_2(s))ds \geq a^{-1}\varepsilon^{1/2-\delta}\}.$$

By using this t_3^*, we define a control process $\mathbf{u}(t) = (u_1(t), u_2(t))$:

$$\mathbf{u}(t) = (u_1(t), u_2(t)) = \begin{cases} (k_1^\varepsilon(t/a), 0), & \text{if } 0 \leq t < at_3^* \\ (\hat{u}_1(t/a), \hat{u}_2(t/a)), & \text{if } t \geq at_3^* \end{cases}.$$

It is easy to check that

$$\int_0^t |\mathbf{u}(t) - \hat{\mathbf{u}}(t)| dt \leq C_6(1+t)\varepsilon^{1/2-\delta}. \tag{3.15}$$

Let $\mathbf{x}(\cdot) = (x_1(\cdot), x_2(\cdot))$ denote the trajectory of the problem $\mathcal{P}^\varepsilon$ under $\mathbf{u}(\cdot)$ with initial state $\mathbf{x}(0) = \mathbf{x}$, and let $\mathbf{y}(t) = (y_1(\cdot), y_2(\cdot))$ be the trajectory under $\hat{\mathbf{u}}(\cdot)$ with initial state $\mathbf{y}(0) = \mathbf{y}$. Then,

$$\begin{aligned}
\text{for } t < at_3^* : x_1(t) &\geq x_1 \geq \varepsilon^{1/2-\delta}; \\
\text{for } t \geq at_3^* : x_1(t) &= ay_1(t/a) + a \textstyle\int_0^{t_3^*} [(k_1^\varepsilon(s) - \hat{u}_1(s)) + \hat{u}_2(s)] ds \\
&= ay_1(t/a) + \varepsilon^{1/2-\delta} \geq \varepsilon^{1/2-\delta},
\end{aligned}$$

since $y_1(t/a) \geq 0$. So $x_1(t) \geq \varepsilon^{1/2-\delta}$ for all $t \geq 0$. Similarly, for $t \geq 0$,

$$\begin{aligned}
x_1(t) &\leq ay_1(t/a) + a \textstyle\int_0^{t_3^*} [(k_1^\varepsilon(s) - \hat{u}_1(s)) + \hat{u}_2(s)] ds \\
&\leq (1 - 2\varepsilon^{1/2-\delta}/\min\{b_1, b_2\})b_1 + \varepsilon^{1/2-\delta} \leq b_1 - \varepsilon^{1/2-\delta},
\end{aligned}$$

and

$$\begin{aligned}
x_2(t) &\leq ay_2(t/a) + a \textstyle\int_0^{\min\{t/a, t_3^*\}} [-\hat{u}_2(s)] ds \\
&\leq (1 - 2\varepsilon^{1/2-\delta}/\min\{b_1, b_2\})b_2 \leq b_2 - 2\varepsilon^{1/2-\delta},
\end{aligned}$$

since $y_1(t/a) \leq b_1$ and $y_2(t/a) \leq b_2$.

Consequently, (ii) of the theorem is proved. To show (i), note that for $t \geq 0$

and sufficiently small ε, we have

$$\begin{aligned}
|x_1(t) - y_1(t)| \leq\ & |x_1 - y_1| + |a \textstyle\int_0^{t/a}[u_1(as) - u_2(as)]ds - \int_0^t[\hat{u}_1(s) - \hat{u}_2(s)]ds| \\
\leq\ & x_1(1-a)/a + a \textstyle\int_0^{t_3^*}[(k_1^\varepsilon(s) - \hat{u}_1(s)) + \hat{u}_2(s)]ds \\
& +(1-a) \textstyle\int_0^t |\hat{u}_1(s) - \hat{u}_2(s)|ds + a \int_t^{t/a} |\hat{u}_1(s) - \hat{u}_2(s)|ds \\
\leq\ & x_1(1-a)/a + aa^{-1}\varepsilon^{1/2-\delta} + C_7(1-a)t + C_7 a(t/a - t) \\
\leq\ & C_8(|\mathbf{x}| + t)\varepsilon^{1/2-\delta}.
\end{aligned}$$

Similarly, $|x_2(t) - y_2(t)| \leq C_9(|\mathbf{x}| + t)\varepsilon^{1/2-\delta}$. Thus,

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leq C_{10}(|\mathbf{x}| + t)\varepsilon^{1/2-\delta}. \tag{3.16}$$

By virtue of (3.15) and (3.16), we have

$$J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}(\cdot)) - J^\varepsilon(\mathbf{y}, \mathbf{k}, \hat{\mathbf{u}}(\cdot)) \leq C_{11}(1 + |\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}. \tag{3.17}$$

On the other hand, since $x_2 < y_2$, $x_1 < y_1$, and

$$\begin{aligned}
|x_1 - y_1| + |x_2 - y_2| &= (x_1 + x_2)(1-a)/a \leq (b_1 + b_2)(1-a)/a \\
&= b\varepsilon^{1/2-\delta}/a \leq x_1/a = y_1,
\end{aligned}$$

we can apply Theorem 3.1 (iii) to get

$$v^\varepsilon(\mathbf{y},\mathbf{k}) - v^\varepsilon(\mathbf{x},\mathbf{k}) \le C(1+|\mathbf{x}|^{\mathrm{K}_0}+|\mathbf{y}|^{\mathrm{K}_0})|\mathbf{x}-\mathbf{y}| = C_{12}(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}. \quad (3.18)$$

Therefore, by (3.14), (3.17) and (3.18), we obtain

$$\begin{aligned} J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)) - v^\varepsilon(\mathbf{x},\mathbf{k}) \le\ & [J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)) - J^\varepsilon(\mathbf{y},\mathbf{k},\hat{\mathbf{u}}(\cdot))] + [J^\varepsilon(\mathbf{y},\mathbf{k},\hat{\mathbf{u}}(\cdot)) \\ & -v^\varepsilon(\mathbf{y},\mathbf{k})] + [v^\varepsilon(\mathbf{y},\mathbf{k}) - v^\varepsilon(\mathbf{x},\mathbf{k})] \\ \le\ & C_{11}(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta} + \varepsilon^{1/2-\delta} \\ & +C_{12}(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta} \\ =\ & C(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}. \quad \square \end{aligned}$$

Based on the above proof, an algorithm can be given for computing a control $\bar{\mathbf{U}}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})$ for the limiting problem satisfying (i) and (ii) of Lemma 3.1 with respect to initial $\mathbf{x} \in S_\varepsilon$.

Algorithm 3.1.

Input $\{\mathbf{x},\ \varepsilon,\ \delta,\ d, b_1,\ b_2\}$.

Set $a := 1 - 2\varepsilon^{1/2-\delta}/\min\{b_1, b_2\}, \qquad \mathbf{y} := \mathbf{x}/a.$

Let $\hat{\mathbf{U}}(\cdot) = (\hat{\mathbf{u}}^1(\cdot), \cdots, \hat{\mathbf{u}}^p(\cdot)) = ((\hat{u}_1^1(\cdot), \hat{u}_2^1(\cdot)), \cdots, (\hat{u}_1^p(\cdot), \hat{u}_2^p(\cdot))) \in \bar{\mathcal{A}}(\mathbf{y})$ be any near-optimal control for $\bar{\mathcal{P}}$.

Set

$$t_4^* := \inf\{t \geq 0 : \int_0^t (\sum_{j=1}^p \nu^j k_1^j - \sum_{j=1}^p \nu^j \hat{u}_1^j(t) + \sum_{j=1}^p \nu^j \hat{u}_2^j(t)) ds \geq a^{-1}\varepsilon^{1/2-\delta}\}.$$

$$\bar{\mathbf{u}}^j(t) = (\bar{u}_1^j(t), \bar{u}_2^j(t)) := \begin{cases} (k_1^j, 0), & \text{if } 0 \leq t < at_4^* \\ (\hat{u}_1^j(t/a), \hat{u}_2^j(t/a)), & \text{if } t \geq at_4^* \end{cases}, \text{ for } j = 1, \cdots, p.$$

Output $\bar{\mathbf{U}}(\cdot) = (\bar{\mathbf{u}}^1(\cdot), \cdots, \bar{\mathbf{u}}^p(\cdot))$.

Remark 3.2. Compared with the original stochastic problem, it is much easier to get the near-optimal control $\hat{\mathbf{U}}(\cdot)$ in Algorithm 3.1 for the deterministic problem $\bar{\mathcal{P}}$. As shown in the next chapter, in some cases, even the optimal control for $\bar{\mathcal{P}}$ can be explicitly obtained. Here we do not give algorithms for computing $\hat{\mathbf{U}}(\cdot)$ as they may differ from cases to cases depending on the specific forms of cost functions. It should be emphasized here that the idea of hierarchical controls is to *reduce* the extremely difficult problem into relatively easier problems. Therefore, the focus of this chapter will be on how to realize such reductions.

3.5 Asymptotic Analysis: Initial States in S_ε

In this section, we will state and prove the results regarding the asymptotic behavior of the problem $\mathcal{P}^\varepsilon$ with any initial state $\mathbf{x} \in S_\varepsilon = [b\varepsilon^{1/2-\delta}, b_1 - b\varepsilon^{1/2-\delta}] \times (-\infty, b_2 - b\varepsilon^{1/2-\delta}] \subset S$ for any given $0 < \delta < \frac{1}{2}$.

The following theorem, whose proof will be deferred to the end of this section, says that the value function v^ε of $\mathcal{P}^\varepsilon$ converges to the value function v of $\bar{\mathcal{P}}$ with the convergence rate $\varepsilon^{1/2-\delta}$ for any given $0 < \delta < \frac{1}{2}$.

Theorem 3.2. *For any $0 < \delta < \frac{1}{2}$, there exists a positive constant C such that for all $\mathbf{x} \in S_\varepsilon$ and sufficiently small ε, we have*

$$|v^\varepsilon(\mathbf{x}, \mathbf{k}) - v(\mathbf{x})| \leq C(1 + |\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}. \tag{3.19}$$

This theorem implies, in particular, that $\lim_{\varepsilon\to 0} v^\varepsilon(\mathbf{x}, \mathbf{k}) = v(\mathbf{x})$ for all $\mathbf{k}$ and for each $\mathbf{x} \in S_\varepsilon$.

Next, for a given $\mathbf{x} \in S_\varepsilon$ and $\mathbf{k}(\varepsilon, 0) = \mathbf{k}$, we describe the flow of constructing an asymptotic optimal control $\mathbf{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}, \mathbf{k})$ of the original problem $\mathcal{P}^\varepsilon$ beginning with the near-optimal control $\bar{\mathbf{U}}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})$ given by Algorithm 3.1 for the limiting problem $\bar{\mathcal{P}}$.

Construction of Asymptotic Optimal Controls for Initial $\mathbf{x} \in S_\varepsilon$

For any $0 < \delta < \frac{1}{2}$, let $\mathbf{x} \in S_\varepsilon$ be a fixed initial state and let $\bar{\mathbf{U}}(\cdot) = (\bar{\mathbf{u}}^1(\cdot), \cdots, \bar{\mathbf{u}}^p(\cdot)) \in \bar{\mathcal{A}}(\mathbf{x})$ where $\bar{\mathbf{u}}^j(t) = (\bar{u}_1^j(t), \bar{u}_2^j(t))$ be the near-optimal control for $\bar{\mathcal{P}}$ constructed by Algorithm 3.1. We use $\bar{\mathbf{x}}(t) = (\bar{x}_1(t), \bar{x}_2(t))$ to denote the state of (3.2) under the control $\bar{\mathbf{U}}(\cdot)$ and the initial state $\bar{\mathbf{x}}(0) = \mathbf{x} = (x_1, x_2)$. On account of Lemma 3.1, it holds that $\varepsilon^{1/2-\delta} \leq \bar{x}_1(t) \leq b_1 - \varepsilon^{1/2-\delta}$ and $\bar{x}_2(t) \leq b_2 - 2\varepsilon^{1/2-\delta}$. We will use this control $\bar{\mathbf{U}}(\cdot)$ to construct a near-optimal control for the original problem.

Let

$$\begin{aligned} \tilde{\mathbf{u}}(t) = (\tilde{u}_1(t), \tilde{u}_2(t)) = & \sum_{j=1}^{p} \chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^j\}}(\bar{u}_1^j(t), \bar{u}_2^j(t)) \\ = & \sum_{j=1}^{p} \chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^j\}} \bar{\mathbf{u}}^j(t), \end{aligned} \tag{3.20}$$

and let $\mathbf{y}(\cdot) = (y_1(\cdot), y_2(\cdot))$ be the corresponding trajectory defined as

$$\begin{cases} y_1(t) = x_1 + \int_0^t (\tilde{u}_1(s) - \tilde{u}_2(s)) ds \\ y_2(t) = x_2 + \int_0^t (\tilde{u}_2(s) - d) ds. \end{cases}$$

It can be seen easily that $\tilde{\mathbf{u}}(\cdot)$ satisfies control constraints (2.10). However, $\mathbf{y}(t)$ does not necessarily satisfy (2.11), i.e., $\mathbf{y}(t)$ may not be in S for some $t \geq 0$. To obtain an admissible control for $\mathcal{P}^\varepsilon$, we need to modify $\tilde{\mathbf{u}}(\cdot)$ so that the state trajectory stays in S. This is done as follows.

We set

$$B := \{t \in [0,\infty) | y_1(t) \leq 0\} \cup \{t \in [0,\infty) | y_1(t) \geq b_1\} \cup \{t \in [0,\infty) | y_2(t) > b_2\}. \tag{3.21}$$

Define

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) := \begin{cases} (\tilde{u}_1(t), \tilde{u}_2(t)), & \text{if } t \in B^c \\ (0,0), & \text{if } t \in B \end{cases}, \tag{3.22}$$

which clearly satisfies the control constraints (2.10). Let $\mathbf{x}^\varepsilon(\cdot)$ denote the trajectory under such $\mathbf{u}^\varepsilon(\cdot)$ with initial value $\mathbf{x}^\varepsilon(0) = \mathbf{x}$. Then, it will be shown later that $\mathbf{x}^\varepsilon(t) \in S$ for all $t \geq 0$, which implies that the constructed control $\mathbf{u}^\varepsilon(\cdot)$ is admissible.

The following theorem stipulates that $\mathbf{u}^\varepsilon(\cdot)$ is asymptotically optimal for the original problem with the rate of error estimate to be of order $\varepsilon^{1/2-\delta}$.

Theorem 3.3. *For any $0 < \delta < \frac{1}{2}$ and $\mathbf{x} \in S_\varepsilon$, let $\bar{\mathbf{U}}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})$ be the output of Algorithm 3.1. Then, for the control $\mathbf{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}, \mathbf{k})$ constructed in (3.20)-(3.22) above, we have*

$$|J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}^\varepsilon(\cdot)) - v^\varepsilon(\mathbf{x}, \mathbf{k})| \leq C(1 + |\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}, \tag{3.23}$$

for some constants C and sufficiently small ε.

To summarize, we can now give an algorithm for constructing one such asymptotic optimal policy $\mathbf{u}^\varepsilon(\cdot)$ for initial state $\mathbf{x} \in S_\varepsilon$.

Algorithm 3.2.

Input $\{\mathbf{x}, \varepsilon, \delta, d, b_1, b_2\}$.

Get $\bar{\mathbf{U}}(\cdot) = (\bar{\mathbf{u}}^1(\cdot), \cdots, \bar{\mathbf{u}}^p(\cdot))$ where $\bar{\mathbf{u}}^j(t) = (\bar{u}_1^j(t), \bar{u}_2^j(t))$ by performing Algorithm 3.1 for initial $\mathbf{x}$.

Set

$$\begin{aligned}\tilde{\mathbf{u}}(t) = (\tilde{u}_1(t), \tilde{u}_2(t)) := & \sum_{j=1}^{p} \chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^j\}}(\bar{u}_1^j(t), \bar{u}_2^j(t)) \\ = & \sum_{j=1}^{p} \chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^j\}}\bar{\mathbf{u}}^j(t).\end{aligned}$$

$$y_1(t) := x_1 + \int_0^t (\tilde{u}_1(s) - \tilde{u}_2(s))ds,$$

$$y_2(t) := x_2 + \int_0^t (\tilde{u}_2(s) - d)ds.$$

$$B_1 := \{t \in [0,\infty) | y_1(t) \leq 0\},$$

$$B_2 := \{t \in [0,\infty) | y_1(t) \geq b_1\},$$

$$B_3 := \{t \in [0,\infty) | y_2(t) > b_2\},$$

$$B := B_1 \cup B_2 \cup B_3.$$

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) := \begin{cases} (\tilde{u}_1(t), \tilde{u}_2(t)), & \text{if } t \in B^c \\ (0,0), & \text{if } t \in B \end{cases}.$$

Output $\mathbf{u}^\varepsilon(\cdot)$.

In the following, we shall prove Theorems 3.2 and 3.3 along with required auxiliary lemma.

Lemma 3.2. *For any given* $\delta > 0$ *and any given* $\mathbf{x} \in S_\varepsilon$, *let* $\bar{\mathbf{U}}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})$ *be the output of Algorithm 3.1. Then for the control* $\mathbf{u}^\varepsilon(\cdot)$ *constructed in (3.20)-(3.22), it holds that* $\mathbf{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}, \mathbf{k})$, *and*

$$J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}^\varepsilon(\cdot)) - v(\mathbf{x}) \leq C(1 + |\mathbf{x}|^{K_0+1})\varepsilon^{1/2-\delta}, \tag{3.24}$$

for some positive constant C and sufficiently small ε.

Proof. With $\bar{\mathbf{U}}(\cdot)$ specified in the lemma, we define $\bar{\mathbf{x}}(\cdot)$, $(\mathbf{y}(\cdot), \tilde{\mathbf{u}}(\cdot))$, $(\mathbf{x}^\varepsilon(\cdot), \mathbf{u}^\varepsilon(\cdot))$, and B_1, B_2, B_3, B as before and $B_4 := B_1 \cup B_2 = \{t \geq 0 | y_1(t) \leq 0 \text{ or } y_1(t) \geq b_1\}$, $B_5 := B_4^c \cap B_3 = \{t \geq 0 | 0 < y_1(t) < b_1 \text{ and } y_2(t) > b_2\}$. First, we are to show that $\mathbf{x}^\varepsilon(t) \in S$ for all $t \geq 0$. To this end, we define two auxiliary trajectories $(x_1^1(\cdot), x_2(\cdot))$ and $(x_1^2(\cdot), x_2(\cdot))$ as follows:

$$x_1^1(t) := \begin{cases} 0, & \text{if } t \in B_1 \\ b_1, & \text{if } t \in B_2 \\ y_1(t), & \text{if } t \in B^c \\ 0, & \text{if } t \in B_5 \end{cases}, \quad x_1^2(t) := \begin{cases} 0, & \text{if } t \in B_1 \\ b_1, & \text{if } t \in B_2 \\ y_1(t), & \text{if } t \in B^c \\ b_1, & \text{if } t \in B_5 \end{cases},$$

$$x_2(t) := \begin{cases} b_2, & \text{if } t \in B_3 \\ y_2(t), & \text{if } t \in B_3^c \end{cases}.$$

It should be noted that these trajectories may not correspond to any control policy. However, it is clear that $0 \leq x_1^1(t) \leq b_1, 0 \leq x_1^2(t) \leq b_1$ and $x_2(t) \leq b_2$ for all $t \geq 0$. We now want to show that $x_1^1(t) \leq x_1^\varepsilon(t) \leq x_1^2(t)$ and $x_2^\varepsilon(t) \leq x_2(t)$ for all $t \geq 0$.

For the open set B_5, there exist countable open intervals $(a_i', b_i'), i = 1, 2, \cdots,$

such that $B_5 = \bigcup_i (a'_i, b'_i)$. For a.e.$t \in (0, a'_1)$ and $j = 1, 2$, we have

$$\dot{x}_1^j(t) = (\tilde{u}_1(t) - \tilde{u}_2(t))\chi_{B_4^c} = (\tilde{u}_1(t) - \tilde{u}_2(t))\chi_{B^c} = u_1^\varepsilon(t) - u_2^\varepsilon(t) = \dot{x}_1^\varepsilon(t), \quad (3.25)$$

thus $x_1^j(t) = x_1^\varepsilon(t)$ for $t \in [0, a'_1]$. For $t \in (a'_1, b'_1)$, we have

$$x_1^1(t) = 0 \leq x_1^\varepsilon(t) = x_1^\varepsilon(a'_1) \leq b_1 = x_1^2(t). \quad (3.26)$$

For a.e.$t \in (b'_1, a'_2)$, once again we have $\dot{x}_1^j(t) = \dot{x}_1^\varepsilon(t)$ for $j = 1, 2$ as with (3.25). Therefore it follows from (3.26) that

$$x_1^1(t) \leq x_1^\varepsilon(t) \leq x_1^2(t), \text{ for } t \in [b'_1, a'_2]. \quad (3.27)$$

By induction, it is easy to see that $x_1^1(t) \leq x_1^\varepsilon(t) \leq x_1^2(t)$ for all $t \geq 0$. Similarly, for a.e.$t \in [0, \infty)$,

$$\dot{x}_2(t) \geq \tilde{u}_2(t)\chi_{B_3^c} - d \geq \tilde{u}_2(t)\chi_{B^c} - d = u_2^\varepsilon(t) - d = \dot{x}_2^\varepsilon(t),$$

thus $x_2(t) \geq x_2^\varepsilon(t)$ for all $t \geq 0$. We have now proved that $\mathbf{x}^\varepsilon(t) = (x_1^\varepsilon(t), x_2^\varepsilon(t)) \in S$ for all $t \geq 0$ and, therefore, $\mathbf{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}, \mathbf{k})$.

Next, we will show that the discounted cost of $(\bar{\mathbf{x}}(\cdot), \bar{\mathbf{U}}(\cdot))$, $(\mathbf{y}(\cdot), \tilde{\mathbf{u}}(\cdot))$, and

$(\mathbf{x}^\varepsilon(\cdot), \mathbf{u}^\varepsilon(\cdot))$ over the infinite horizon are close to each other. First of all, it is easy to verify that

$$\begin{aligned} E|\int_0^t(\bar{\mathbf{U}}(s)-\tilde{\mathbf{u}}(s))ds| &\le C_{13}\sum_{j=1}^p E|\int_0^t(\chi_{\{\mathbf{k}(\varepsilon,s)=\mathbf{k}^j\}}-\nu^j)ds| \\ &\le C_{14}\sqrt{1+t^2}\varepsilon^{\frac{1}{2}}, \text{ (by Lemma 2.2)}, \end{aligned}$$

therefore

$$E|\bar{\mathbf{x}}(t)-\mathbf{y}(t)| \le C_{15}\sqrt{1+t^2}\varepsilon^{\frac{1}{2}}. \tag{3.28}$$

We next estimate $E\int_0^\infty e^{-\rho t}|\mathbf{u}^\varepsilon(t)-\tilde{\mathbf{u}}(t)|dt$. Since $\mathbf{u}^\varepsilon(t)=\tilde{\mathbf{u}}(t)$ on $B^c=\{t\in[0,\infty)|0<y_1(t)<b_1 \text{ and } y_2(t)\le b_2\}$, we have

$$\begin{aligned} E\int_0^\infty e^{-\rho t}|\mathbf{u}^\varepsilon(t)-\tilde{\mathbf{u}}(t)|dt = &\ E\int_0^\infty e^{-\rho t}\chi_{\{y_1(t)\le 0\}\cup\{y_1(t)\ge b_1\}\cup\{y_2(t)>b_2\}}|\mathbf{u}^\varepsilon(t)-\tilde{\mathbf{u}}(t)|dt \\ \le &\ E\int_0^\infty e^{-\rho t}\chi_{\{y_1(t)\le 0\}}|\mathbf{u}^\varepsilon(t)-\tilde{\mathbf{u}}(t)|dt \\ &+E\int_0^\infty e^{-\rho t}\chi_{\{y_1(t)\ge b_1\}}|\mathbf{u}^\varepsilon(t)-\tilde{\mathbf{u}}(t)|dt \\ &+E\int_0^\infty e^{-\rho t}\chi_{\{y_2(t)>b_2\}}|\mathbf{u}^\varepsilon(t)-\tilde{\mathbf{u}}(t)|dt \\ \le &\ C_{16}[\int_0^\infty e^{-\rho t}P(y_1(t)\le 0)dt+\int_0^\infty e^{-\rho t}P(y_1(t)\ge b_1)dt \\ &+\int_0^\infty e^{-\rho t}P(y_2(t)>b_2)dt]. \end{aligned}$$

Noting that $\varepsilon^{1/2-\delta}\le \bar{x}_1(t)\le b_1-\varepsilon^{1/2-\delta}$ and $\bar{x}_2(t)\le b_2-2\varepsilon^{1/2-\delta}$ for $t\ge 0$, we have

$$P(y_1(t) \le 0) \le P(\bar{x}_1(t) - y_1(t) \ge \varepsilon^{1/2-\delta}) \le P(|\bar{x}_1(t) - y_1(t)| \ge \varepsilon^{1/2-\delta}),$$

$$P(y_1(t) \ge b_1) \le P(y_1(t) - \bar{x}_1(t) \ge \varepsilon^{1/2-\delta}) \le P(|\bar{x}_1(t) - y_1(t)| \ge \varepsilon^{1/2-\delta}),$$

and

$$P(y_2(t) > b_2) \le P(y_2(t) - \bar{x}_2(t) \ge 2\varepsilon^{1/2-\delta}) \le P(|\bar{x}_2(t) - y_2(t)| \ge 2\varepsilon^{1/2-\delta}).$$

According to the definitions of $(\bar{x}_1(t), \bar{x}_2(t))$ and $(y_1(t), y_2(t))$, we have

$$P(|\bar{x}_1(t) - y_1(t)| \ge \varepsilon^{1/2-\delta}) = P(|\int_0^t \sum_{j=1}^{p} (\chi_{\{\mathbf{k}(\varepsilon,s)=\mathbf{k}^j\}} - \nu^j)(-\bar{u}_1^j(s) + \bar{u}_2^j(s))ds| \ge \varepsilon^{1/2-\delta}),$$

and

$$P(|\bar{x}_2(t) - y_2(t)| \ge 2\varepsilon^{1/2-\delta}) = P(|\int_0^t \sum_{j=1}^{p} (\chi_{\{\mathbf{k}(\varepsilon,s)=\mathbf{k}^j\}} - \nu^j)(-\bar{u}_2^j(s))ds| \ge 2\varepsilon^{1/2-\delta}).$$

Therefore, we can apply Lemma 2.3 with $\beta(t) = -\bar{u}_1^j(t) + \bar{u}_2^j(t)$ and $\beta(t) = -\bar{u}_2^j(t)$, respectively, to conclude that

$$E\int_0^\infty e^{-\rho t}|\mathbf{u}^\varepsilon(t) - \tilde{\mathbf{u}}(t)|dt \le C_{17}\varepsilon^{1/2-\delta}. \tag{3.29}$$

Observe also that

$$|\mathbf{x}^{\varepsilon}(t) - \mathbf{y}(t)| \le C_{18} \int_0^t |\mathbf{u}^{\varepsilon}(s) - \tilde{\mathbf{u}}(s)| ds.$$

Thus,

$$\begin{aligned} E\int_0^\infty e^{-\rho t}|\mathbf{x}^{\varepsilon}(t) - \mathbf{y}(t)|dt \le\ & C_{18}E\int_0^\infty e^{-\rho t}\int_0^t |\mathbf{u}^{\varepsilon}(s) - \tilde{\mathbf{u}}(s)|dsdt \\ =\ & -C_{18}E\rho^{-1}e^{-\rho t}\int_0^t |\mathbf{u}^{\varepsilon}(s) - \tilde{\mathbf{u}}(s)|ds|_{t=0}^{t=\infty} \\ & +C_{18}\rho^{-1}E\int_0^\infty e^{-\rho t}|\mathbf{u}^{\varepsilon}(t) - \tilde{\mathbf{u}}(t)|dt \qquad (3.30) \\ =\ & C_{18}\rho^{-1}E\int_0^\infty e^{-\rho t}|\mathbf{u}^{\varepsilon}(t) - \tilde{\mathbf{u}}(t)|dt \\ \le\ & C_{17}C_{18}\rho^{-1}\varepsilon^{1/2-\delta}. \end{aligned}$$

On the other hand, we can write

$$\begin{aligned} J^{\varepsilon}(\mathbf{x},\mathbf{k},\mathbf{u}^{\varepsilon}(\cdot)) - v(\mathbf{x}) \le\ & J^{\varepsilon}(\mathbf{x},\mathbf{k},\mathbf{u}^{\varepsilon}(\cdot)) - J(\mathbf{x},\bar{\mathbf{U}}(\cdot)) + C(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta} \\ \le\ & E\int_0^\infty e^{-\rho t}[h(\mathbf{x}^{\varepsilon}(t)) - h(\bar{\mathbf{x}}(t))]dt \\ & +E\int_0^\infty e^{-\rho t}[c(\mathbf{u}^{\varepsilon}(t)) - \textstyle\sum_{j=1}^p \nu^j c(\bar{\mathbf{u}}^j(t))]dt \\ & +C(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}, \end{aligned}$$

where $\bar{\mathbf{U}}(t) = (\bar{\mathbf{u}}^1(t), \cdots, \bar{\mathbf{u}}^p(t))$. It follows by (3.28) that

$$\begin{aligned}
& E \int_0^\infty e^{-\rho t}[h(\mathbf{x}^\varepsilon(t)) - h(\bar{\mathbf{x}}(t))]dt \\
\leq\ & C_{19}(1 + |\mathbf{x}|^{\mathrm{K}_0}) E \int_0^\infty e^{-\rho t}(1 + t^{\mathrm{K}_0})|\mathbf{x}^\varepsilon(t) - \bar{\mathbf{x}}(t)|dt \\
\leq\ & C_{19}(1 + |\mathbf{x}|^{\mathrm{K}_0}) \int_0^\infty e^{-\rho t}(1 + t^{\mathrm{K}_0})[E|\mathbf{x}^\varepsilon(t) - \mathbf{y}(t)| \\
& + E|\mathbf{y}(t) - \bar{\mathbf{x}}(t)|]dt \\
\leq\ & C_{20}(1 + |\mathbf{x}|^{\mathrm{K}_0}) \int_0^\infty e^{-\rho t}(1 + t^{\mathrm{K}_0})[E|\mathbf{x}^\varepsilon(t) - \mathbf{y}(t)| \\
& + \sqrt{1 + t^2}\varepsilon^{\frac{1}{2}}]dt \\
\leq\ & C_{21}(1 + |\mathbf{x}|^{\mathrm{K}_0}) \int_0^\infty e^{-\rho t}(1 + t^{\mathrm{K}_0+1})[E|\mathbf{x}^\varepsilon(t) - \mathbf{y}(t)| + \varepsilon^{1/2-\delta}]dt \\
\leq\ & C_{22}(1 + |\mathbf{x}|^{\mathrm{K}_0}) \int_0^\infty e^{-\rho t/2}[E|\mathbf{x}^\varepsilon(t) - \mathbf{y}(t)| + \varepsilon^{1/2-\delta}]dt \\
\leq\ & C_{23}(1 + |\mathbf{x}|^{\mathrm{K}_0})\varepsilon^{1/2-\delta},
\end{aligned}$$

where the last inequality is obtained in a way similar to (3.30).

To complete the proof of the lemma, it remains to show that

$$E \int_0^\infty e^{-\rho t}[c(\mathbf{u}^\varepsilon(t)) - \sum_{j=1}^p \nu^j c(\bar{\mathbf{u}}^j(t))]dt \leq C_{24}\varepsilon^{1/2-\delta}.$$

By recalling (3.29), the convexity of c as well as Lemma 2.2, we have

$$\begin{aligned}
& E\int_0^\infty e^{-\rho t}[c(\mathbf{u}^\varepsilon(t)) - \textstyle\sum_{j=1}^p \nu^j c(\bar{\mathbf{u}}^j(t))]dt \\
= \ & E\int_0^\infty e^{-\rho t}[(c(\mathbf{u}^\varepsilon(t)) - c(\tilde{\mathbf{u}}(t))) + (c(\tilde{\mathbf{u}}(t)) - \textstyle\sum_{j=1}^p \nu^j c(\bar{\mathbf{u}}^j(t))]dt \\
\leq \ & C_{25}(E\int_0^\infty e^{-\rho t}|\mathbf{u}^\varepsilon(t) - \tilde{\mathbf{u}}(t)|dt + \varepsilon^{1/2-\delta}) \\
\leq \ & C_{24}\varepsilon^{1/2-\delta}. \ \square
\end{aligned}$$

Proof of Theorem 3.2. Lemma 3.2 implies that $v^\varepsilon(\mathbf{x},\mathbf{k}) - v(\mathbf{x}) \leq C(1 + |\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}$. To prove the theorem, it suffices to show the opposite inequality

$$v^\varepsilon(\mathbf{x},\mathbf{k}) - v(\mathbf{x}) \geq -C(1 + |\mathbf{x}|^{\mathrm{K}_0})\varepsilon^{1/2-\delta} \tag{3.31}$$

holds for all $\mathbf{x} \in S_\varepsilon$.

To this end, it suffices to prove that for all $\mathbf{x} \in S_\varepsilon$ and $\mathbf{u}(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x},\mathbf{k})$ obtained by Lemma 3.1, there exists $\bar{\mathbf{U}}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})$ such that

$$J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)) - J(\mathbf{x},\bar{\mathbf{U}}(\cdot)) \geq -C(1 + |\mathbf{x}|^{\mathrm{K}_0})\varepsilon^{1/2-\delta}. \tag{3.32}$$

Let $\mathbf{x}(t) = (x_1(t), x_2(t))$ denote the state under $\mathbf{u}(\cdot)$ with initial state $\mathbf{x}(0) = \mathbf{x}$. By Lemma 3.1, we know that

$$\varepsilon^{1/2-\delta} \leq x_1(t) \leq b_1 - \varepsilon^{1/2-\delta} \text{ and } x_2(t) \leq b_2 - 2\varepsilon^{1/2-\delta}, \quad \text{for all } t \geq 0.$$

For each $j = 1, 2, \cdots, p$, let

$$\bar{\mathbf{u}}^j(t) = E[\mathbf{u}(t)|\mathbf{k}(\varepsilon, t) = \mathbf{k}^j] := (\bar{u}_1^j, \bar{u}_2^j).$$

Define $\bar{\mathbf{U}}(t) = (\bar{\mathbf{u}}^1(t), \cdots, \bar{\mathbf{u}}^p(t))$ and let $\bar{\mathbf{x}}(t)$ denote the state trajectory under $\bar{\mathbf{U}}(\cdot)$ of $\bar{\mathcal{P}}$. We need to prove that $\bar{\mathbf{U}}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})$. Actually, it suffices to show that $0 \le \bar{x}_1(t) \le b_1$ and $\bar{x}_2(t) \le b_2$ for sufficiently small ε. For this purpose, we first observe that

$$\begin{aligned} E\mathbf{u}(t) = &\ \textstyle\sum_{j=1}^p E\chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^j\}}\bar{\mathbf{u}}^j(t) \\ = &\ \textstyle\sum_{j=1}^p \nu^j\bar{\mathbf{u}}^j(t) + \sum_{j=1}^p \nu^j(P(\mathbf{k}(\varepsilon,t) = \mathbf{k}^j) - \nu^j) \\ = &\ \textstyle\sum_{j=1}^p \nu^j\bar{\mathbf{u}}^j(t) + O(\varepsilon + e^{-K\varepsilon^{-1}t}), \end{aligned}$$

where the last equality is due to Lemma 2.1. Therefore,

$$|\bar{x}_1(t) - Ex_1(t)| = |\int_0^t [\sum_{j=1}^p \nu^j(\bar{u}_1^j(s) - \bar{u}_2^j(s)) - E(u_1(s) - u_2(s))]ds| \le C_{26}\varepsilon,$$

and

$$|\bar{x}_2(t) - Ex_2(t)| = |\int_0^t [\sum_{j=1}^p \nu^j\bar{u}_2^j(s)) - Eu_2(s)]ds| \le C_{27}\varepsilon.$$

It follows that for sufficiently small ε,

$$\bar{x}_1(t) \geq Ex_1(t) - C_{26}\varepsilon \geq \varepsilon^{1/2-\delta} - C_{26}\varepsilon \geq 0,$$

$$\bar{x}_1(t) \leq Ex_1(t) + C_{26}\varepsilon \leq b_1 - \varepsilon^{1/2-\delta} + C_{26}\varepsilon \leq b_1, \text{ and}$$

$$\bar{x}_2(t) \leq Ex_2(t) + C_{27}\varepsilon \leq b_2 - 2\varepsilon^{1/2-\delta} + C_{27}\varepsilon \leq b_2.$$

Thus, $\bar{\mathbf{U}}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})$ for sufficiently small ε. Moreover,

$$|\bar{\mathbf{x}}(t) - E\mathbf{x}(t)| \leq C_{28}\varepsilon.$$

By the convexity of h and c, we have

$$\begin{aligned} Eh(\mathbf{x}(t)) \geq\ & h(E\mathbf{x}(t)) \\ =\ & h(\bar{\mathbf{x}}(t)) + [h(E\mathbf{x}(t)) - h(\bar{\mathbf{x}}(t))] \\ \geq\ & h(\bar{\mathbf{x}}(t)) - C_{29}(1 + |\mathbf{x}|^{\mathrm{K_0}})(1 + t^{\mathrm{K_0}})|E\mathbf{x}(t) - \bar{\mathbf{x}}(t)| \\ \geq\ & h(\bar{\mathbf{x}}(t)) - C_{28}C_{29}(1 + |\mathbf{x}|^{\mathrm{K_0}})(1 + t^{\mathrm{K_0}})\varepsilon, \end{aligned}$$

and

$$\begin{aligned} Ec(\mathbf{u}(t)) =\ & E\textstyle\sum_{j=1}^{p} \chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^j\}} E[c(\mathbf{u}(t))|\mathbf{k}(\varepsilon,t) = \mathbf{k}^j] \\ =\ & \textstyle\sum_{j=1}^{p} P(\mathbf{k}(\varepsilon,t) = \mathbf{k}^j) E[c(\mathbf{u}(t))|\mathbf{k}(\varepsilon,t) = \mathbf{k}^j] \\ \geq\ & \textstyle\sum_{j=1}^{p} P(\mathbf{k}(\varepsilon,t) = \mathbf{k}^j) c(\bar{\mathbf{u}}^j(t)) \\ \geq\ & \textstyle\sum_{j=1}^{p} \nu^j c(\bar{\mathbf{u}}^j(t)) - C_{30}(\varepsilon + e^{-K\varepsilon^{-1}t}). \end{aligned}$$

Thus,

$$\begin{aligned} J^{\varepsilon}(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)) \geq &\int_0^\infty e^{-\rho t}[h(\bar{\mathbf{x}}(t)) + \textstyle\sum_{j=1}^p \nu^j c(\bar{\mathbf{u}}^j(t))]dt - C_{31}(1+|\mathbf{x}|^{\mathrm{K}_0})\varepsilon \\ = &\; J(\mathbf{x},\bar{\mathbf{U}}(\cdot)) - C_{31}(1+|\mathbf{x}|^{\mathrm{K}_0})\varepsilon. \end{aligned} \tag{3.33}$$

Therefore, (3.31) follows and the proof is concluded . □

Proof of Theorem 3.3. By Theorem 3.2 and Lemma 3.2, we obtain

$$\begin{aligned} 0 \leq &\; J^{\varepsilon}(\mathbf{x},\mathbf{k},\mathbf{u}^{\varepsilon}(\cdot)) - v^{\varepsilon}(\mathbf{x},\mathbf{k}) \\ \leq &\; (J^{\varepsilon}(\mathbf{x},\mathbf{k},\mathbf{u}^{\varepsilon}(\cdot)) - v(\mathbf{x})) + (v(\mathbf{x}) - v^{\varepsilon}(\mathbf{x},\mathbf{k})) \\ \leq &\; C(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}. \;\square \end{aligned}$$

3.6 Asymptotic Analysis: Initial States in $S \setminus S_\varepsilon$

In this section, we continue our asymptotic analysis for the original problem $\mathcal{P}^{\varepsilon}$ with any initial state $\mathbf{x} \in S \setminus S_\varepsilon$. We shall show that Theorem 3.2 also holds for all $\mathbf{x} \in S \setminus S_\varepsilon$. First we state and prove the following lemma.

Lemma 3.3. *For any $\mathbf{x} \in S \setminus S_\varepsilon$, there exists an $\mathbf{x}' \in S_\varepsilon$ such that*

$$|v^{\varepsilon}(\mathbf{x},\mathbf{k}) - v^{\varepsilon}(\mathbf{x}',\mathbf{k})| \leq C(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta} \tag{3.34}$$

$$|v(\mathbf{x}) - v(\mathbf{x}')| \leq C(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta} \tag{3.35}$$

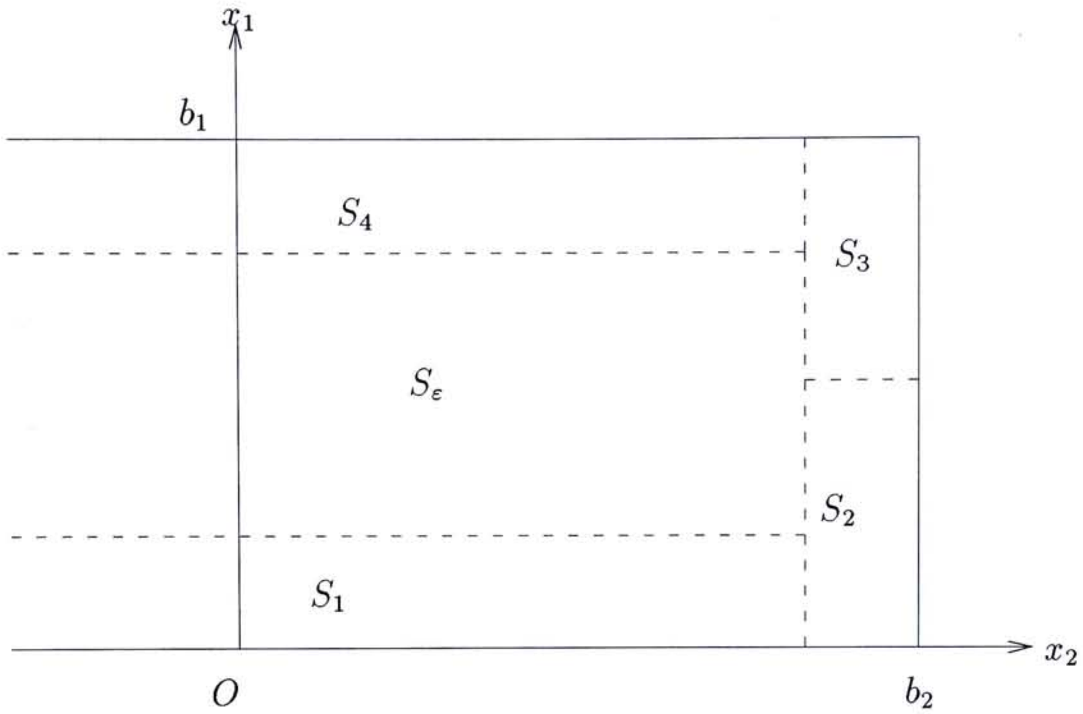

Figure 3.1: The partition of the state constraint domain

for some positive constant C (independent of ε) and sufficiently small ε.

Proof. We only show the first inequality, the second being similar. We divide the band $S \setminus S_\varepsilon$ into four parts (as shown in Fig. 3.1):

$$\begin{aligned} S \setminus S_\varepsilon = \; & \{[0, b\varepsilon^{1/2-\delta}) \times (-\infty, b_2 - b\varepsilon^{1/2-\delta}]\} \\ & \cup\{[0, b_1 - 3b\varepsilon^{1/2-\delta}) \times (b_2 - b\varepsilon^{1/2-\delta}, b_2]\} \\ & \cup\{[b_1 - 3b\varepsilon^{1/2-\delta}, b_1) \times (b_2 - b\varepsilon^{1/2-\delta}, b_2]\} \\ & \cup\{(b_1 - b\varepsilon^{1/2-\delta}, b_1] \times (-\infty, b_2 - b\varepsilon^{1/2-\delta}]\} \\ := \; & S_1 \cup S_2 \cup S_3 \cup S_4, \end{aligned}$$

and then discuss four cases.

Case 1. $\mathbf{x} = (x_1, x_2) \in S_1$.

We choose $\mathbf{x}' = (x_1', x_2') \in S_\varepsilon$, where $x_1' = b\varepsilon^{1/2-\delta}$, $x_2' = x_2$. By Theorem 3.1 (i), the above (3.34) holds.

Case 2. $\mathbf{x} = (x_1, x_2) \in S_2$.

We choose $\mathbf{x}' = (x_1', x_2') \in S_\varepsilon$, where $x_1' = x_1 + |x_2 - x_2'| + b\varepsilon^{1/2-\delta}$, $x_2' = b_2 - b\varepsilon^{1/2-\delta}$. By Theorem 3.1 (i) and (ii), the above (3.34) holds.

Case 3. $\mathbf{x} = (x_1, x_2) \in S_3$.

We choose $\mathbf{x}' = (x_1', x_2') \in S_\varepsilon$, where $x_1' = b_1 - 3b\varepsilon^{1/2-\delta}$, $x_2' = b_2 - b\varepsilon^{1/2-\delta}$. By Theorem 3.1 (i) and (iii), the above (3.34) holds.

Case 4. $\mathbf{x} = (x_1, x_2) \in S_4$.

We choose $\mathbf{x}' = (x_1', x_2') \in S_\varepsilon$, where $x_1' = b_1 - b\varepsilon^{1/2-\delta}$, $x_2' = x_2$. By Theorem 3.1 (i), the above (3.34) holds. □

Theorem 3.4. *For any $0 < \delta < \frac{1}{2}$, there exists positive constant C such that for all $\mathbf{x} \in S$ and sufficiently small ε, we have*

$$|v^\varepsilon(\mathbf{x}, \mathbf{k}) - v(\mathbf{x})| \le C(1 + |\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}. \tag{3.36}$$

Proof. We only need to show (3.36) for $\mathbf{x} \in S \setminus S_\varepsilon$. By Lemma 3.3, there exists an $\mathbf{x}' \in S_\varepsilon$, such that (3.34) and (3.35) hold. Applying Theorem 3.2 to $\mathbf{x}' \in S_\varepsilon$, we get

$$\begin{aligned} |v^\varepsilon(\mathbf{x},\mathbf{k}) - v(\mathbf{x})| \leq \; & |v^\varepsilon(\mathbf{x},\mathbf{k}) - v^\varepsilon(\mathbf{x}',\mathbf{k})| + |v^\varepsilon(\mathbf{x}',\mathbf{k}) - v(\mathbf{x}')| + |v(\mathbf{x}') - v(\mathbf{x})| \\ \leq \; & C(1 + |\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}. \quad \square \end{aligned} \tag{3.37}$$

Construction of Asymptotic Optimal Controls for Initial $\mathbf{x} \in S \setminus S_\varepsilon$

For any given initial $\mathbf{x} \in S \setminus S_\varepsilon$, by Lemma 3.3, there exists an $\mathbf{x}' \in S_\varepsilon$ such that (3.34) holds. For such an $\mathbf{x}' \in S_\varepsilon$, let the control $\mathbf{u}'(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}',\mathbf{k})$ be constructed in a fashion similar to (3.20)-(3.22), with $\mathbf{x}$ replaced by $\mathbf{x}'$. Then by Theorem 3.3, we have

$$|J^\varepsilon(\mathbf{x}',\mathbf{k},\mathbf{u}'(\cdot)) - v^\varepsilon(\mathbf{x}',\mathbf{k})| \leq C(1 + |\mathbf{x}'|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}, \tag{3.38}$$

for some constants C and ε small enough.

In the following, we will first construct such $\mathbf{x}' \in S_\varepsilon$ according to the positions of $\mathbf{x} \in S \setminus S_\varepsilon$ as in the proof of Lemma 3.3. Then we will use the control $\mathbf{u}'(\cdot) = (u_1'(\cdot), u_2'(\cdot)) \in \mathcal{A}^\varepsilon(\mathbf{x}',\mathbf{k})$ to construct an asymptotic optimal control $\mathbf{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x},\mathbf{k})$ for the original problem as in the proof of Theorem 3.1.

Case 1. $\mathbf{x} = (x_1, x_2) \in S_1$.

Let $\mathbf{x}' = (x_1', x_2') = (b\varepsilon^{1/2-\delta}, x_2) \in S_\varepsilon$. Define $\mathbf{u}^\varepsilon(t)$:

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) = \begin{cases} (k_1^\varepsilon(t), 0), & \text{if } t < t_5^* \\ (u_1'(t), u_2'(t)), & \text{if } t \geq t_5^* \end{cases},$$

where $t_5^* > 0$ is the stopping time given by

$$t_5^* := \inf\{t \geq 0 : \int_0^t [(k_1^\varepsilon(s) - u_1'(s)) + u_2'(s)]ds \geq |x_1 - x_1'|\}.$$

Case 2. $\mathbf{x} = (x_1, x_2) \in S_2$.

Let $\mathbf{x}' = (x_1', x_2') = (x_1 + |x_2 - (b_2 - b\varepsilon^{1/2-\delta})| + b\varepsilon^{1/2-\delta}, b_2 - b\varepsilon^{1/2-\delta}) \in S_\varepsilon$. Define $\mathbf{u}^\varepsilon(t)$:

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) = \begin{cases} (u_1'(t), 0), & \text{if } t < t_6^* \\ (u_1'(t), u_2'(t)), & \text{if } t \geq t_6^* \end{cases},$$

where $t_6^* > 0$ is the stopping time given by

$$t_6^* := \inf\{t \geq 0 : \int_0^t u_2'(s)ds \geq |x_1 - x_1'|\}.$$

Case 3. $\mathbf{x} = (x_1, x_2) \in S_3$.

Let $\mathbf{x}' = (x_1', x_2') = (b_1 - 3b\varepsilon^{1/2-\delta}, b_2 - b\varepsilon^{1/2-\delta}) \in S_\varepsilon$. Further set $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2) \in$

S, where $\bar{x}_1 = x_1' - (x_2 - x_2') \geq 0$, $\bar{x}_2 = x_2$. Then define $\bar{\mathbf{u}}(t)$ as follows

$$\bar{\mathbf{u}}(t) = (\bar{u}_1(t), \bar{u}_2(t)) = \begin{cases} (u_1'(t), 0), & \text{if } t < t_{71}^* \\ (u_1'(t), u_2'(t)), & \text{if } t \geq t_{71}^* \end{cases},$$

where $t_{71}^* > 0$ is the stopping time given by

$$t_{71}^* := \inf\{t \geq 0 : \int_0^t u_2'(s)ds \geq |\bar{x}_1 - x_1'|\}.$$

Further, we define $\mathbf{u}^\varepsilon(t)$:

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) = \begin{cases} (0, \bar{u}_2(t)), & \text{if } t < t_{72}^* \\ (\bar{u}_1(t), \bar{u}_2(t)), & \text{if } t \geq t_{72}^* \end{cases},$$

where $t_{72}^* > 0$ is the stopping time given by

$$t_{72}^* := \inf\{t \geq 0 : \int_0^t \bar{u}_1(s)ds \geq |x_1 - \bar{x}_1|\}.$$

Case 4. $\mathbf{x} = (x_1, x_2) \in S_4$.

Let $\mathbf{x}' = (x_1', x_2') = (b_1 - b\varepsilon^{1/2-\delta}, x_2) \in S_\varepsilon$. Define $\mathbf{u}^\varepsilon(t)$:

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) = \begin{cases} (0, u_2'(t)), & \text{if } t < t_8^* \\ (u_1'(t), u_2'(t)), & \text{if } t \geq t_8^* \end{cases},$$

where $t_8^* > 0$ be the stopping time given by the following:

$$t_8^* := \inf\{t \geq 0 : \int_0^t u_1'(s)ds \geq |x_1 - x_1'|\}.$$

Let $\mathbf{x}^\varepsilon(\cdot)$ be the trajectory under $\mathbf{u}^\varepsilon(\cdot)$ constructed above with initial $\mathbf{x}$. Then by the proof of Theorem 3.1, $\mathbf{u}^\varepsilon(\cdot)$ is admissible, and

$$J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}^\varepsilon(\cdot)) - J^\varepsilon(\mathbf{x}', \mathbf{k}, \mathbf{u}'(\cdot)) \leq C(1 + |\mathbf{x}|^{\mathrm{K}_0})\varepsilon^{1/2-\delta}. \tag{3.39}$$

The following theorem stipulates that $\mathbf{u}^\varepsilon(\cdot)$ is asymptotically optimal for the original problem with the error estimate being of order $\varepsilon^{1/2-\delta}$.

Theorem 3.5. *For any $0 < \delta < \frac{1}{2}$ and $\mathbf{x} \in S \setminus S_\varepsilon$, let the control $\mathbf{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}, \mathbf{k})$ constructed as above. Then we have*

$$|J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}^\varepsilon(\cdot)) - v^\varepsilon(\mathbf{x}, \mathbf{k})| \leq C(1 + |\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}, \tag{3.40}$$

for some constants C and ε small enough.

Proof. By the above (3.34), (3.38) and (3.39), we obtain

$$\begin{aligned} 0 \le J^{\varepsilon}(\mathbf{x},\mathbf{k},\mathbf{u}^{\varepsilon}(\cdot)) - v^{\varepsilon}(\mathbf{x},\mathbf{k}) \le\ & [J^{\varepsilon}(\mathbf{x},\mathbf{k},\mathbf{u}^{\varepsilon}(\cdot)) - J^{\varepsilon}(\mathbf{x}',\mathbf{k},\mathbf{u}'(\cdot))] \\ & +[J^{\varepsilon}(\mathbf{x}',\mathbf{k},\mathbf{u}'(\cdot)) - v^{\varepsilon}(\mathbf{x}',\mathbf{k})] \\ & +[v^{\varepsilon}(\mathbf{x}',\mathbf{k}) - v^{\varepsilon}(\mathbf{x},\mathbf{k})] \\ \le\ & C(1+|\mathbf{x}|^{\mathrm{K}_0+1})\varepsilon^{1/2-\delta}. \quad \square \end{aligned} \tag{3.41}$$

The following is an algorithm for constructing asymptotic optimal policy for initial states in $S \setminus S_{\varepsilon}$.

Algorithm 3.3

Input $\{\mathbf{x}, \varepsilon, \delta, d, b_1, b_2, b\}$.

While

$\mathbf{x} \in S_1$ *Do* **Procedure 1**

$\mathbf{x} \in S_2$ *Do* **Procedure 2**

$\mathbf{x} \in S_3$ *Do* **Procedure 3**

$\mathbf{x} \in S_4$ *Do* **Procedure 4**

End.

Output $\mathbf{u}^{\varepsilon}(\cdot)$.

Procedure 1

Set $\mathbf{x}' = (x_1', x_2') := (b\varepsilon^{1/2-\delta}, x_2)$.

Get $\mathbf{u}'(\cdot) = (u_1'(\cdot), u_2'(\cdot))$ by performing Algorithm 3.2 for initial $\mathbf{x}'$.

$$t_5^* := \inf\{t \geq 0 : \int_0^t [(k_1^\varepsilon(s) - u_1'(s)) + u_2'(s)]ds \geq |x_1 - x_1'|\}.$$

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) := \begin{cases} (k_1^\varepsilon(t), 0), & \text{if } t < t_5^* \\ (u_1'(t), u_2'(t)), & \text{if } t \geq t_5^* \end{cases}.$$

Procedure 2

Set $\mathbf{x}' = (x_1', x_2') := (x_1 + |x_2 - (b_2 - b\varepsilon^{1/2-\delta})| + b\varepsilon^{1/2-\delta}, b_2 - b\varepsilon^{1/2-\delta})$.

Get $\mathbf{u}'(\cdot) = (u_1'(\cdot), u_2'(\cdot))$ by performing Algorithm 3.2 for initial $\mathbf{x}'$.

$$t_6^* := \inf\{t \geq 0 : \int_0^t u_2'(s)ds \geq |x_1 - x_1'|\}.$$

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) := \begin{cases} (u_1'(t), 0), & \text{if } t < t_6^* \\ (u_1'(t), u_2'(t)), & \text{if } t \geq t_6^* \end{cases}.$$

Procedure 3

Set $\mathbf{x}' = (x_1', x_2') := (b_1 - 3b\varepsilon^{1/2-\delta}, b_2 - b\varepsilon^{1/2-\delta})$.

$\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2) := (x_1' - (x_2 - x_2'), x_2)$.

Get $\mathbf{u}'(\cdot) = (u_1'(\cdot), u_2'(\cdot))$ by performing Algorithm 3.2 for initial $\mathbf{x}'$.

$$t_{71}^* := \inf\{t \geq 0 : \int_0^t u_2'(s)ds \geq |\bar{x}_1 - x_1'|\}.$$

$$\bar{\mathbf{u}}(t) = (\bar{u}_1(t), \bar{u}_2(t)) := \begin{cases} (u_1'(t), 0), & \text{if } t < t_{71}^* \\ (u_1'(t), u_2'(t)), & \text{if } t \geq t_{71}^* \end{cases}.$$

$$t_{72}^* := \inf\{t \geq 0 : \int_0^t \bar{u}_1(s)ds \geq |x_1 - \bar{x}_1|\}.$$

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) := \begin{cases} (0, \bar{u}_2(t)), & \text{if } t < t_{72}^* \\ (\bar{u}_1(t), \bar{u}_2(t)), & \text{if } t \geq t_{72}^* \end{cases}.$$

Procedure 4

Set $\mathbf{x}' = (x_1', x_2') := (b_1 - b\varepsilon^{1/2-\delta}, x_2)$.

Get $\mathbf{u}'(\cdot) = (u_1'(\cdot), u_2'(\cdot))$ by performing Algorithm 3.2 for initial $\mathbf{x}'$.

$$t_8^* := \inf\{t \geq 0 : \int_0^t u_1'(s)ds \geq |x_1 - x_1'|\}.$$

$$\mathbf{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) := \begin{cases} (0, u_2'(t)), & \text{if } t < t_8^* \\ (u_1'(t), u_2'(t)), & \text{if } t \geq t_8^* \end{cases}.$$

3.7 Concluding Remarks

In this chapter, we have studied hierarchical open-loop planning for the stochastic two-machine flowshops with finite buffers. The methodology is based on the state constraint domain approximation and weak-Lipschitz property developed in this chapter.

The controls, which may be more precisely called partially open-loop controls, are of theoretical importance in deriving one of the main results, namely Theorem 3.4, which states how close to one another the original and the limiting problems are. While the asymptotic result derived in Theorem 3.4 is independent of the type of controls under consideration, the construction method developed in this chapter does not extend to feedback controls. The difficulty lies in the fact that one cannot in general expect to have a near-optimal Lipschitz feedback control for the limiting problem. Thus it is not clear how to estimate the difference between the trajectories of the original and the limiting problems.

Recently, Sethi and Zhou [50] have obtained near-optimal feedback controls for a two-machine under the assumption that both internal and external buffers are infinite. They use a probabilistic approach in estimating the error bound associated with these controls. Moreover, Samaratunga, Sethi, and Zhou [38] evaluate these policies in an extensive simulation study by comparing them to some existing heuristic policies such as Kanban policy and Two Boundary policy (see [59]). In Chapter 5, we will extend the method in [50] to the case of the finite internal and external buffers treated in this chapter. We should once again emphasize that Theorem 3.4 proved in this chapter is an important link in the process of carrying out the extension as stated in Chapter 5.

Finally, we would like to mention that the material presented in this chapter are published in [14, 15].

Chapter 4

Feedback Production Planning in Deterministic Flowshops

4.1 Introduction

There has been a considerable amount of interest in the study of hierarchical controls in stochastic manufacturing systems, since Gershwin[18] proposed a frequency-based hierarchical framework similar to that of singular perturbations[37, 5], and then developed on a rigorous basis by Lehoczky *et al.* [33], Sethi, Zhang, and Zhou [45, 47, 48], and Fong and Zhou [14, 15].

Note that basically only open-loop controls were constructed in [14, 15, 45, 47, 48] which were shown to be asymptotic optimal. While asymptotically optimal, the constructed open-loop controls are not expected to perform well unless the rate of change in machine states is unrealistically large. What is required therefore is a construction of asymptotic optimal *feedback* controls. Such constructions had been available for single or parallel machine systems (cf. [33, 47]), in which no state constraints are present. In such systems, either the Lipschitz property of optimal control for the corresponding deterministic systems or the monotonicity property of optimal control with respect to the state variables makes the proof of asymptotic optimality go through. Unfortunately, these properties are no longer available in the case of flowshops. To overcome the difficulty, Sethi and Zhou [49, 50] constructed asymptotic optimal feedback controls for stochastic two-machine flowshops based on the explicit characterization of optimal controls for the corresponding deterministic problem. Moreover, Samaratunga, Sethi and Zhou [38] made a computational investigation of such constructed controls and found them to perform very well in comparison to other existing methods in the literature (cf. [35, 41, 56, 59]). Nevertheless, they assumed that the size of the external buffer is infinite. Unfortunately, this assumption is hardly reasonable in *real* situations since external buffers usually represent warehouses or storages, whose limited capacities must be taken into consideration especially in the situation of scarce space and/or high rent. We will tackle the problem with finite external buffer. To do this, we have to first solve the limiting deterministic problem in this chapter since the hierarchical planning is based on the solutions to the deterministic problem. In order to obtain an explicit optimal feedback control for the deterministic problem, we first construct one by intuition, then prove that the corresponding cost of the constructed control satisfies the Hamilton-Jacobi-Bellman (HJB) equation. The optimality would then follow if the HJB equation could admit a unique solution. However, since the HJB equation involves complicated boundary conditions due to the presence of the state constraints, the uniqueness of its solutions is not at all clear. Soner [54] and Cannarsa, Gozzi and Soner [9] studied the problem with general state constraints, but it is also not clear how to apply their results to the present case. To overcome this major difficulty, we do not insist on the uniqueness. Rather, we employ the idea of "constraint domain approximation" developed in the last chapter, to show that any cost function that satisfies the HJB only in the *interior* of the state constraint domain must be bounded above by the value function. This way we can obtain explicit optimal feedback controls rigourously for the corresponding deterministic problem. Based on it, we are able to construct suitable feedback controls for the stochastic problem and prove their asymptotic optimality in the next chapter.

The plan of this chapter is as follows. In the next section we make some assumptions for the deterministic problem as formulated in Chapter 2. In Sections 4.3, we solve explicitly the deterministic problem. Section 4.4 concludes the chapter.

4.2 Assumptions

In order to study the optimal feedback production planning for the deterministic manufacturing system formulated in Section 2.1, we need to make the following assumptions on the function $G(\mathbf{x}, \mathbf{w})$ and the maximum production capacities a_1 and a_2 throughout this chapter.

Assumption 4.1. $G(\mathbf{x}, \mathbf{w}) = c_1 x_1 + c_2^+ x_2^+ + c_2^- x_2^-$.

where c_1, c_2^+ and c_2^- are given *nonnegative* constants, $x^+ = \max\{x, 0\}$, and $x^- = \max\{-x, 0\}$.

Assumption 4.2. $a_1 \geq a_2 \geq d$.

In Assumption 4.1, c_1, c_2^+, c_2^- are the unit holding costs of the inventory/shortage levels of the two buffers, respectively.

Assumption 4.2 implies that the system has enough capacity to meet the demand, and the second (downstream) machine is the bottleneck.

We should note that the problem can also be solved without Assumption 4.2. However, the assumption is a sensible one in practice, since otherwise the shortage will increase to infinity in the long run.

4.3 Optimal Feedback Controls

Let S_0 be the interior of the state space S, i.e., $S_0 = (0, b_1) \times (-\infty, b_2)$. Let ∂S be the boundary of S. As is well-known, if the value function v of the deterministic problem $\bar{\mathcal{P}}$ (see (2.5)) is in $C^1(S_0)$, then it solves the following HJB equation in S_0:

$$\begin{aligned}\rho v(\mathbf{x}) &= \inf_{0\le w_1\le a_1, 0\le w_2\le a_2}[(w_1 - w_2)v_{x_1}(\mathbf{x}) + (w_2 - d)v_{x_2}(\mathbf{x})] + h(\mathbf{x}) \\ &= \inf_{0\le w_1\le a_1, 0\le w_2\le a_2}[w_1 v_{x_1}(\mathbf{x}) + w_2(v_{x_2}(\mathbf{x}) - v_{x_1}(\mathbf{x}))] - d v_{x_2}(\mathbf{x}) + h(\mathbf{x}), \\ &\text{for all } \mathbf{x} \in S_0.\end{aligned} \tag{4.1}$$

The uniqueness of solutions to (4.1) relies on some appropriate boundary conditions, which are nevertheless very hard to obtain due to the state constraints present. In this chapter, we do not insist on the uniqueness. Rather, we prove the following result which will be enough for serving our purpose later on.

Theorem 4.1. *If a given function J^* defined on S satisfies*

(i) $J^* \in C^1(S_0)$,

(ii) J^* is continuous and is of polynomial growth on S, and

(iii) J^* satisfies the HJB equation (4.1) in S_0,

then $J^*(\mathbf{x}) \le v(\mathbf{x})$ for all $\mathbf{x} \in S$.

To prove this theorem, we have to introduce the following lemma which says that starting from any point in S_0, there are near-optimal controls for $\bar{\mathcal{P}}$ so that the corresponding state trajectories will never reach the boundary of S.

Lemma 4.1. *For any* $\mathbf{x} \in S_0$ *and any* η *satisfying* $0 < \eta < \eta_0 = \inf_{\mathbf{x}' \in \partial S} |\mathbf{x} - \mathbf{x}'|$, *there exist a constant* C *(independent of* η*) and a control* $\mathbf{w}(\cdot) = (w_1(\cdot), w_2(\cdot)) \in \tilde{\mathcal{A}}(\mathbf{x})$ *such that*

(i) $\eta \le x_1(t) \le b_1 - \eta$ *and* $x_2(t) \le b_2 - 2\eta$,

(ii) $J(\mathbf{x}, \mathbf{w}(\cdot)) - v(\mathbf{x}) \le C(1 + |\mathbf{x}|)\eta$,

where $\mathbf{x}(\cdot) = (x_1(\cdot), x_2(\cdot))$ *is the state trajectory of the deterministic problem* $\bar{\mathcal{P}}$ *under* $\mathbf{w}(\cdot)$ *with initial state* $\mathbf{x}$.

Proof. This proof is similar to that of Lemma 3.1, with $\varepsilon^{1/2-\delta}$ replaced by η. □

Proof of Theorem 4.1. First we will show that $J^*(\mathbf{x}) \le v(\mathbf{x})$, for all $\mathbf{x} \in S_0$. Indeed, for sufficiently small $\eta > 0$, there is a control $\mathbf{w}(\cdot)$ as specified in Lemma 4.1. Since the corresponding state trajectory $\mathbf{x}(\cdot)$ under $\mathbf{w}(\cdot)$ lies in the interior S_0 and J^* satisfies the HJB equation in S_0, we have by using Dynkin's formula

$$\begin{aligned} e^{-\rho T} J^*(\mathbf{x}(T)) - J^*(\mathbf{x}) = & \int_0^T e^{-\rho t}[-\rho J^*(\mathbf{x}(t)) + (w_1 - w_2) J^*_{x_1}(\mathbf{x}(t)) \\ & + (w_2 - d) J^*_{x_2}(\mathbf{x}(t))]dt \qquad (4.2) \\ \ge & -\int_0^T e^{-\rho t} h(\mathbf{x}(t))dt \end{aligned}$$

Letting $T \to \infty$ and noting that J^* is of polynomial growth, we have

$$J^*(\mathbf{x}) \leq \int_0^\infty e^{-\rho t} h(\mathbf{x}(t))dt = J(\mathbf{x}, \mathbf{w}(\cdot)).$$

Hence, by Lemma 4.1 (ii), we have

$$J^*(\mathbf{x}) \leq v(\mathbf{x}) + C(1 + |\mathbf{x}|)\eta.$$

Letting $\eta \to 0$, we obtain $J^*(\mathbf{x}) \leq v(\mathbf{x})$.

Secondly, for any initial point $\mathbf{x} \in \partial S$, there is a sequence $\{\mathbf{x}_n\}_0^\infty$ in S_0, such that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$. For $\mathbf{x}_n \in S_0$, we have had $J^*(\mathbf{x}_n) \leq v(\mathbf{x}_n)$. Since J^* is continuous on S, we obtain $J^*(\mathbf{x}) \leq v(\mathbf{x})$ by letting $n \to \infty$. The theorem is proved. □

Next we provide an explicit optimal solution of $\bar{\mathcal{P}}$. It is obtained in two subcases depending on the cost parameters. We show that in each case the cost function associated with the control satisfies the conditions of Theorem 4.1. This will establish the optimality of the controls.

4.3.1 The Case $c_1 < c_2^+$

We conjecture that the following feedback control is optimal for $\bar{\mathcal{P}}$.

$$\mathbf{w}^*(\mathbf{x}) = \begin{cases} (d,d), & x_1 = x_2 = 0, \\ (0,0), & 0 \le x_1 \le b_1, 0 < x_2 \le b_2, \\ (0,d), & 0 < x_1 \le b_1, x_2 = 0, \\ (0,a_2), & 0 < x_1 \le b_1, x_2 < 0, \\ (a_2,a_2), & x_1 = 0, x_2 < 0. \end{cases} \tag{4.3}$$

This control is shown in Fig. 4.1 along with the corresponding movements of the state trajectories. For example, the control in the left quadrant interior is $(0, a_2)$ and the trajectory ABO is obtained by using control $(0, a_2)$ along AB and control (a_2, a_2) along BO.

The intuition behind our conjecture is simply that $(x_1, x_2) = (0, 0)$ is the most desirable state of the system and the control policy should be such that the state trajectory will reach this state as quickly as possible and then stay there forever. More specifically, when $(x_1, x_2) = (0, 0)$, staying there with the use of control $\mathbf{w}^* = (d, d)$ results in the minimum possible value of zero to the objective function of $\bar{\mathcal{P}}$. When $0 \le x_1 \le b_1, 0 < x_2 \le b_2$, there is no need to produce anything since both x_1 and x_2 are in surplus situation. The control $\mathbf{w}^* = (0, 0)$ decreases x_2 and keeps x_1 unchanged, which is desirable if $c_1 < c_2^+$. When $0 < x_1 \le b_1, x_2 = 0$, it is clear that in view of $c_1 < c_2^+$, the cheapest way to get to (0,0) is $\mathbf{w}^* = (0, d)$. When $0 < x_1 \le b_1, x_2 < 0$, the control $\mathbf{w}^* = (0, a_2)$ is easily seen to be the fastest way to decrease x_1 and increase x_2 at the same time. Finally, when $x_1 = 0, x_2 < 0$, $\mathbf{w}^* = (a_2, a_2)$ is the fastest way to reduce shortage while maintaining the zero inventory in the exit buffer of M_1.

Since $\mathbf{w}^*$ is explicitly given, it is easy to compute the corresponding cost function $J^*(\mathbf{x}) = J(\mathbf{x}, \mathbf{w}^*)$ given below:

$$J^*(x_1, x_2) = \begin{cases} \frac{c_1 d}{\rho^2}(e^{-\frac{\rho(x_1+x_2)}{d}} - 1) + \frac{(c_2^+ - c_1)d}{\rho^2}(e^{-\frac{\rho x_2}{d}} - 1) + \frac{c_1 x_1 + c_2^+ x_2}{\rho}, \\ \text{if } 0 \le x_1 \le b_1, 0 \le x_2 \le b_2; \\ \\ \frac{c_1 a_2}{\rho^2}(e^{-\frac{\rho x_1}{a_2}} - 1) + \frac{c_2^-(a_2-d)}{\rho^2}(e^{\frac{\rho x_2}{a_2-d}} - 1) + \frac{c_1 x_1 - c_2^- x_2}{\rho}, \\ \text{if } 0 \le x_1 \le b_1, x_2 < 0, \frac{x_1}{a_2} \le \frac{-x_2}{a_2-d}; \\ \\ \frac{c_1 d}{\rho^2}(e^{-\frac{\rho(x_1+x_2)}{d}} - 1) + \frac{(c_1+c_2^-)(a_2-d)}{\rho^2}(e^{\frac{\rho x_2}{a_2-d}} - 1) + \frac{c_1 x_1 - c_2^- x_2}{\rho}, \\ \text{if } 0 \le x_1 \le b_1, x_2 < 0, \frac{x_1}{a_2} > \frac{-x_2}{a_2-d}. \end{cases} \tag{4.4}$$

One can easily check that $J^* \in C^1(S_0)$ and J^* is continuous on S. Moreover,

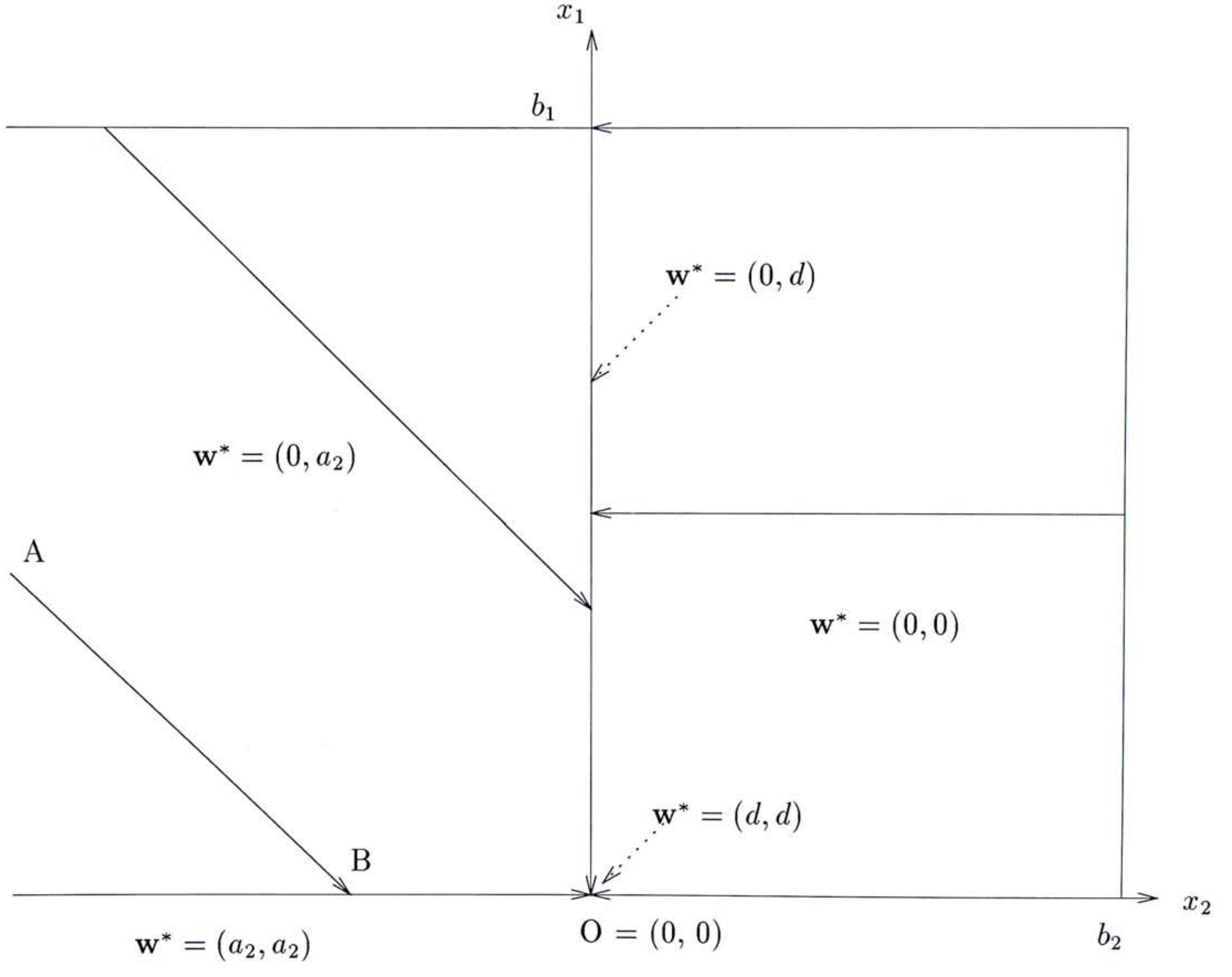

Figure 4.1: Optimal control and state movements when $c_1 < c_2^+$

we can calculate that when $0 < x_1 < b_1, 0 \leq x_2 < b_2$,

$$\begin{aligned} J^*_{x_1}(x_1, x_2) &= \tfrac{c_1}{\rho}(1 - e^{-\frac{\rho(x_1+x_2)}{d}}) \geq 0, \\ J^*_{x_2}(x_1, x_2) &= \tfrac{c_1}{\rho} e^{-\frac{\rho x_2}{d}}(1 - e^{-\frac{\rho x_1}{d}}) + \tfrac{c_2^+}{\rho}(1 - e^{-\frac{\rho x_2}{d}}), \\ J^*_{x_2}(x_1, x_2) - J^*_{x_1}(x_1, x_2) &= \tfrac{c_2^+ - c_1}{\rho}(1 - e^{-\frac{\rho x_2}{d}}) \geq 0. \end{aligned}$$

Hence,

$$\begin{aligned} & \inf_{0 \leq w_1 \leq a_1, 0 \leq w_2 \leq a_2} [(w_1 - w_2) J^*_{x_1}(\mathbf{x}) + (w_2 - d) J^*_{x_2}(\mathbf{x})] + h(\mathbf{x}) \\ = \; & \inf_{0 \leq w_1 \leq a_1, 0 \leq w_2 \leq a_2} [w_1 J^*_{x_1}(\mathbf{x}) + w_2 (J^*_{x_2}(\mathbf{x}) - J^*_{x_1}(\mathbf{x}))] - d J^*_{x_2}(\mathbf{x}) + h(\mathbf{x}) \\ = \; & \tfrac{c_1 d}{\rho}(e^{-\frac{\rho(x_1+x_2)}{d}} - 1) + \tfrac{(c_2^+ - c_1)d}{\rho}(e^{-\frac{\rho x_2}{d}} - 1) + c_1 x_1 + c_2^+ x_2 \\ = \; & \rho J^*(\mathbf{x}), \text{ when } 0 \leq x_1 \leq b_1, 0 \leq x_2 \leq b_2. \end{aligned}$$

The same equality can be derived also in the region $0 < x_1 < b_1, x_2 < 0$. This means that J^* satisfies the HJB equation (4.1) in S_0. By Theorem 4.1, we have $J^* \leq v$. This implies $J^* = v$ since v is the minimum cost. Therefore, we have proved the following result.

Theorem 4.2. $\mathbf{w}^*$ *constructed in (4.3) is an optimal feedback control for the deterministic problem* $\bar{\mathcal{P}}$ *with* $c_1 < c_2^+$.

4.3.2 The Case $c_1 \geq c_2^+$

In this case, we conjecture the optimal control to be identical to that in the case $c_1 < c_2^+$ except in the region $x_1 > 0, x_2 \geq 0$, where it is more desirable to reduce x_1 as quickly as possible. When $0 < x_1 \leq b_1, x_2 = b_2$, the control $\mathbf{w}^* = (0, d)$ is the fastest way to reduce surplus, i.e., decreases x_1 and keeps $x_2 = b_2$ unchanged, if $c_1 \geq c_2^+$. Thus,

$$\mathbf{w}^*(\mathbf{x}) = \begin{cases} (d, d), & x_1 = x_2 = 0, \\ (0, 0), & x_1 = 0, 0 < x_2 \leq b_2, \\ (a_2, a_2), & x_1 = 0, x_2 < 0, \\ (0, a_2), & 0 < x_1 \leq b_1, x_2 < b_2, \\ (0, d), & 0 < x_1 \leq b_1, x_2 = b_2; \end{cases} \tag{4.5}$$

see Fig. 4.2.

The corresponding cost function $J^*(\mathbf{x}) = J(\mathbf{x}, \mathbf{w}^*)$ is as follows:

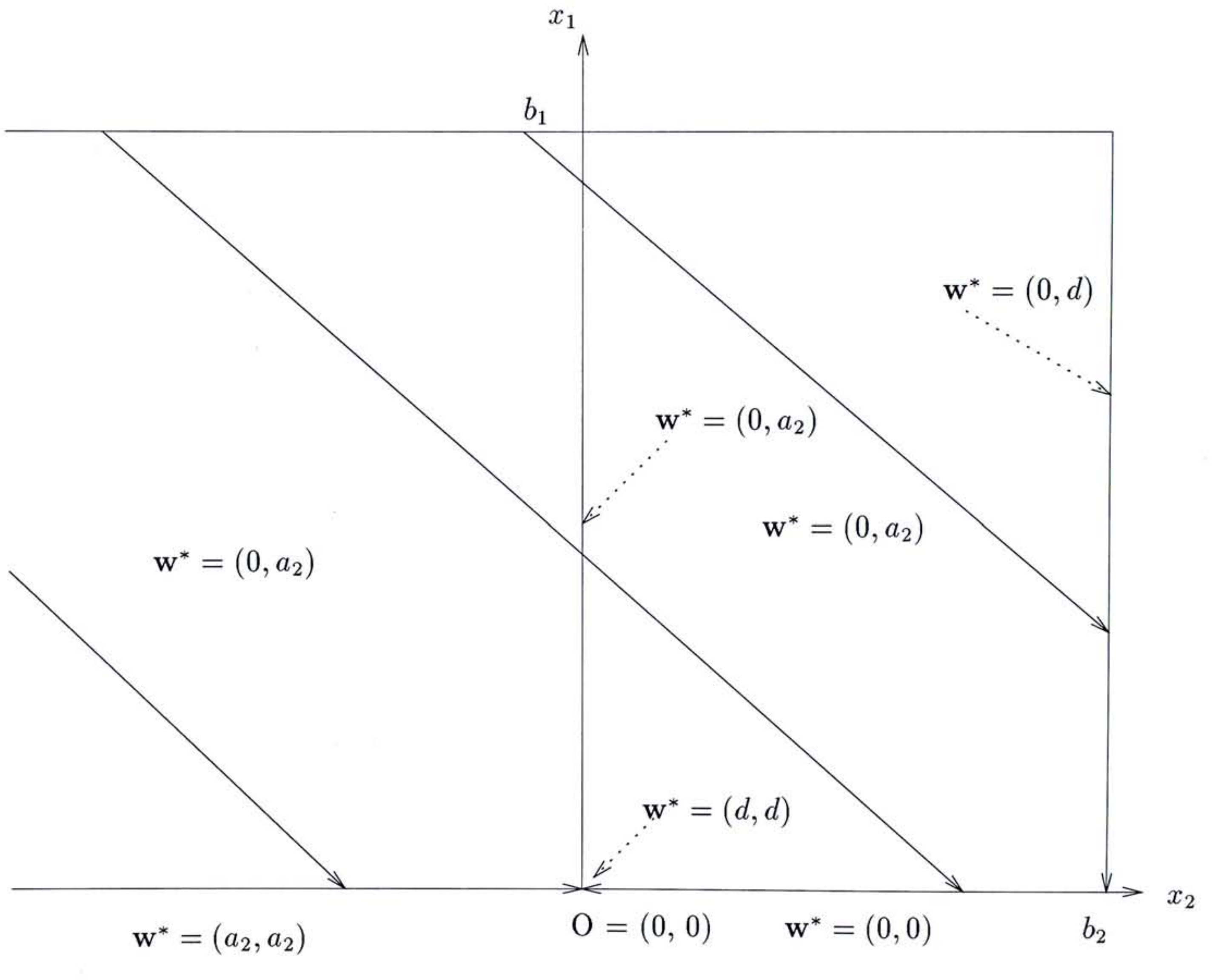

Figure 4.2: Optimal control and state movements when $c_1 \geq c_2^+$

$$
J^*(x_1,x_2) = \begin{cases}
J_1^*(x_1,x_2) = \frac{(c_1-c_2^+)a_2}{\rho^2}(e^{-\frac{\rho x_1}{a_2}}-1) + \frac{c_2^+ d}{\rho^2}(e^{-\frac{\rho(x_1+x_2)}{d}}-1) + \frac{c_1x_1+c_2^+x_2}{\rho}, \\
\text{if } 0 \le x_1 \le b_1, 0 \le x_2 \le b_2, x_2 + \frac{a_2-d}{a_2}x_1 \le b_2; \\
\\
J_2^*(x_1,x_2) = \frac{(c_1-c_2^+)(a_2-d)}{\rho^2}(e^{-\frac{\rho(b_2-x_2)}{a_2-d}}-1) + \frac{(c_1-c_2^+)d}{\rho^2}(e^{-\frac{\rho(x_1+x_2-b_2)}{d}}-1) \\
+\frac{c_2^+ d}{\rho^2}(e^{-\frac{\rho(x_1+x_2)}{d}}-1) + \frac{c_1x_1+c_2^+x_2}{\rho}, \\
\text{if } 0 \le x_1 \le b_1, 0 \le x_2 \le b_2, x_2 + \frac{a_2-d}{a_2}x_1 > b_2; \\
\\
J_3^*(x_1,x_2) = \frac{c_1a_2}{\rho^2}(e^{-\frac{\rho x_1}{a_2}}-1) + \frac{c_2^-(a_2-d)}{\rho^2}(e^{\frac{\rho x_2}{a_2-d}}-1) + \frac{c_1x_1-c_2^-x_2}{\rho}, \\
\text{if } 0 \le x_1 \le b_1, x_2 < 0, x_2 + \frac{a_2-d}{a_2}x_1 \le 0; \\
\\
J_4^*(x_1,x_2) = \frac{c_2^+ d}{\rho^2}(e^{-\frac{\rho(x_1+x_2)}{d}}-1) + \frac{(c_2^++c_2^-)(a_2-d)}{\rho^2}(e^{\frac{\rho x_2}{a_2-d}}-1) \\
+\frac{(c_1-c_2^+)a_2}{\rho^2}(e^{-\frac{\rho x_1}{a_2}}-1) + \frac{c_1x_1-c_2^-x_2}{\rho}, \\
\text{if } 0 \le x_1 \le b_1, x_2 < 0, 0 < x_2 + \frac{a_2-d}{a_2}x_1 \le b_2; \\
\\
J_5^*(x_1,x_2) = \frac{(c_2^++c_2^-)(a_2-d)}{\rho^2}(e^{\frac{\rho x_2}{a_2-d}}-1) + \frac{(c_1-c_2^+)(a_2-d)}{\rho^2}(e^{-\frac{\rho(b_2-x_2)}{a_2-d}}-1) \\
+\frac{(c_1-c_2^+)d}{\rho^2}(e^{-\frac{\rho(x_1+x_2-b_2)}{d}}-1) + \frac{c_2^+ d}{\rho^2}(e^{-\frac{\rho(x_1+x_2)}{d}}-1) + \frac{c_1x_1-c_2^-x_2}{\rho}, \\
\text{if } 0 \le x_1 \le b_1, x_2 < 0, x_2 + \frac{a_2-d}{a_2}x_1 > b_2.
\end{cases}
\tag{4.6}
$$

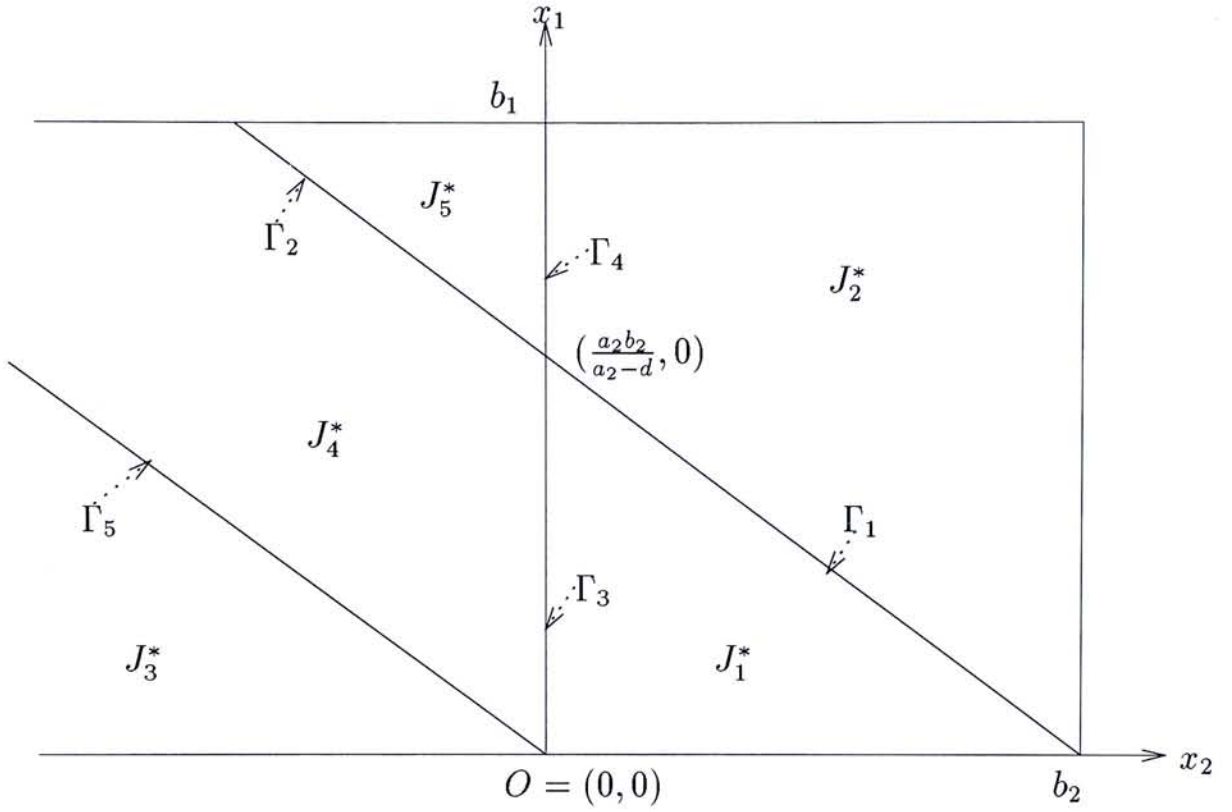

Figure 4.3: Switching manifolds for cost functions with different initial state

Let

$$\Gamma_1 = \{(x_1, x_2) | 0 \le x_1 \le b_1, 0 \le x_2 \le b_2, x_2 + \tfrac{a_2-d}{a_2} x_1 = b_2\},$$

$$\Gamma_2 = \{(x_1, x_2) | 0 \le x_1 \le b_1, x_2 < 0, x_2 + \tfrac{a_2-d}{a_2} x_1 = b_2\},$$

$$\Gamma_3 = \{(x_1, x_2) | 0 < x_1 \le \tfrac{a_2 b_2}{a_2-d}, x_2 = 0\},$$

$$\Gamma_4 = \{(x_1, x_2) | \tfrac{a_2 b_2}{a_2-d} < x_1 \le b_1, x_2 = 0\},$$

$$\Gamma_5 = \{(x_1, x_2) | 0 \le x_1 \le b_1, x_2 < 0, x_2 + \tfrac{a_2-d}{a_2} x_1 = 0\},$$

as shown in Fig. 4.3.

Remark 4.1. If $\frac{b_1}{a_2} < \frac{b_2}{a_2-d}$, then the set $\{(x_1,x_2)|0 \le x_1 \le b_1, x_2 < 0, x_2 + \frac{a_2-d}{a_2}x_1 > b_2\}$ is empty, in which case the switching line Γ_2 and the function J_5^* do not exist.

It is easy to see that J^* is continuous on S. In fact, J^* is also continuously differentiable in S_0, although not as easily seen as in the previous cases. Moreover, the continuous differentiability of J^* along the switching curve $\Gamma_i, i = 1,2,3,4,5,$ is not at all obvious. Let us now show this for the switching curve Γ_1 for example.

We define the following regions:

$$\text{I}: \quad 0 \le x_1 \le b_1, 0 \le x_2 \le b_2, x_2 + \tfrac{a_2-d}{a_2}x_1 \le b_2,$$

$$\text{II}: \quad 0 \le x_1 \le b_1, 0 \le x_2 \le b_2, x_2 + \tfrac{a_2-d}{a_2}x_1 > b_2.$$

Then, for any $\mathbf{x}^0 = (x_1^0, x_2^0) \in \Gamma_1$,

$$\lim_{\mathbf{x}\in\text{I},\mathbf{x}\to\mathbf{x}^0} \frac{\partial J_1^*(\mathbf{x})}{\partial x_1} = -\frac{1}{\rho}[(c_1 - c_2^-)e^{-\frac{\rho x_1^0}{a_2}} + c_2^+ e^{\frac{\rho(x_1^0+x_2^0)}{d}} - c_1].$$

On the other hand,

$$\begin{aligned}\lim_{\mathbf{x}\in\text{II},\mathbf{x}\to\mathbf{x}^0} \tfrac{\partial J_2^*(\mathbf{x})}{\partial x_1} &= -\tfrac{1}{\rho}[(c_1 - c_2^-)e^{-\frac{\rho x_1^0}{a_2}} + c_2^+ e^{\frac{\rho(x_1^0+x_2^0)}{d}} - c_1] \\ &= \lim_{\mathbf{x}\in\text{I},\mathbf{x}\to\mathbf{x}^0} \tfrac{\partial J_1^*(\mathbf{x})}{\partial x_1}.\end{aligned}$$

Similarly, we have

$$\lim_{\mathbf{x}\in \mathrm{I}, \mathbf{x}\to\mathbf{x}^0} \frac{\partial J_1^*(\mathbf{x})}{\partial x_2} = -\frac{1}{\rho}[c_2^+ e^{\frac{\rho(x_1^0+x_2^0)}{d}} - c_2^+],$$

and

$$\begin{aligned}
\lim_{\mathbf{x}\in \mathrm{II}, \mathbf{x}\to\mathbf{x}^0} \tfrac{\partial J_2^*(\mathbf{x})}{\partial x_2} &= -\tfrac{1}{\rho}[(c_1 - c_2^-)e^{-\frac{\rho x_1^0}{a_2}} + c_2^+ e^{\frac{\rho(x_1^0+x_2^0)}{d}} - (c_1 - c_2^-)e^{-\frac{\rho x_1^0}{a_2}} - c_2^+] \\
&= -\tfrac{1}{\rho}[c_2^+ e^{\frac{\rho(x_1^0+x_2^0)}{d}} - c_2^+] \\
&= \lim_{\mathbf{x}\in \mathrm{I}, \mathbf{x}\to\mathbf{x}^0} \tfrac{\partial J_1^*(\mathbf{x})}{\partial x_2}.
\end{aligned}$$

So J^* is continuously differentiable along Γ_1. Similarly, we can show that J^* is continuously differentiable elsewhere in S_0. With similar calculations, we can prove that J^* satisfies the HJB equation and, therefore, we have also proved the following theorem in view of Theorem 4.1.

Theorem 4.3. $\mathbf{w}^*$ *constructed in (4.5) is an optimal feedback control for the deterministic problem* $\bar{\mathcal{P}}$ *with* $c_1 \geq c_2^+$.

4.4 Concluding Remarks

In this chapter, we have obtained explicitly optimal feedback production policies for a dynamic deterministic two-machine flowshop that minimize discounted costs of inventories and shortages over an infinite horizon. The optimal control problem involves a state constraint as the inventory in the buffer between the two machines must remain nonnegative and both buffer sizes are finite. Optimal policies involve both bang-bang and singular controls.

In the next chapter, we shall use the results obtained in this chapter to construct feedback controls that are asymptotically optimal for stochastic two-machine flowshops with unreliable machines.

Chapter 5

Feedback Production Planning in Stochastic Flowshops

5.1 Introduction

In the previous chapter, we explicitly obtained the optimal feedback control for dynamic deterministic two-machine flowshops. In this chapter, we investigate the feedback controls for stochastic two-machine flowshops with machines subject to random breakdowns and repairs. Since the sizes of both internal and external buffers are practically finite, the problem is one with state constraints. As the problem is extremely difficult to solve, it can be approximated by a deterministic problem in which the stochastic machines' capacities are replaced by their average capacities when the rates of machine failures and repairs become large. Furthermore, based on the explicit characterization of optimal controls for deterministic problem in Chapter 4, suitable feedback controls for the stochastic flowshops are analytically constructed, which are proved to be asymptotically optimal with respect to the rate of change in machine states. It should be pointed out that we will make a computational investigation of such constructed controls in the next chapter in comparison to other existing methods in the literature [59, 41].

The plan of this chapter is as follows. In the next section we derive the corresponding deterministic (limiting) problem from the stochastic problem of the two-machine flowshops formulated in Chapter 2 with linear inventory/backlog cost. In Sections 5.3, we construct asymptotic optimal feedback controls for the original stochastic problem. Section 5.4 concludes the chapter.

5.2 Original and Limiting Problems

In this chapter, we consider a dynamic stochastic flowshop consisting of two unreliable machines, M_1 and M_2, in which each machine is assumed to have two states: up and down. We assume that the first and the second machines have maximum production capacities m_1 and m_2, respectively. Therefore, the system has four machines capacity states: $\mathbf{k}^1 = (m_1, m_2)$ corresponds to both machines up, $\mathbf{k}^2 = (m_1, 0)$ to M_1 up and M_2 down, $\mathbf{k}^3 = (0, m_2)$ to M_1 down and M_2 up, and $\mathbf{k}^4 = (0,0)$ to both machines down. Let $\mathcal{M} = \{\mathbf{k}^1, \mathbf{k}^2, \mathbf{k}^3, \mathbf{k}^4\}$.

Here, we consider the running cost function $G(\mathbf{x}, \mathbf{u})$ as the one in the last chapter, i.e.,

$$G(\mathbf{x}, \mathbf{u}) = c_1 x_1 + c_2^+ x_2^+ + c_2^- x_2^-. \tag{5.1}$$

For a feedback control $\mathbf{u}$, we shall write the cost $J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}(\mathbf{x}(\cdot), \mathbf{k}(\varepsilon, \cdot)))$ as simply $J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u})$, where $\mathbf{x}(\cdot)$ is the corresponding trajectory under $\mathbf{u}$ with the initial state $\mathbf{x}$ and the initial machine state $\mathbf{k}$.

We use $\mathcal{A}^\varepsilon(\mathbf{x})$ to denote the set of admissible controls with respect to the initial state $\mathbf{x} \in S$ and the initial machine state $\mathbf{k}$, and $v^\varepsilon(\mathbf{x}, \mathbf{k})$ to denote the value function, i.e.,

$$v^\varepsilon(\mathbf{x}, \mathbf{k}) = \inf_{\mathbf{u}(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x})} J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}(\cdot)). \tag{5.2}$$

We make the following assumptions throughout this chapter:

Assumption 5.1. The capacity process $\mathbf{k}(\varepsilon, t) \in \mathcal{M}$ is a finite state Markov chain with generator $Q^\varepsilon = \varepsilon^{-1} Q$, where $Q = (q_{ij})$ is a 4×4 matrix with $q_{ij} \geq 0$ if $j \neq i$ and $q_{ii} = -\sum_{j \neq i} q_{ij}$. Moreover, Q is irreducible and is taken to be the one that satisfies

$$\min_{ij} \{|q_{ij}| : q_{ij} \neq 0\} = 1.$$

Let $\boldsymbol{\nu} = (\nu^1, \nu^2, \nu^3, \nu^4) > 0$, the only positive solution of

$$\boldsymbol{\nu} Q = 0 \text{ and } \sum_{j=1}^{4} \nu^j = 1,$$

denote the equilibrium distribution of Q.

Assumption 5.2. $a_1 \geq a_2 > d$, where $\mathbf{a} = (a_1, a_2)$ denotes the average maximum capacities of M_1 and M_2 respectively, i.e., $a_1 = (\nu^1 + \nu^2) m_1$ and $a_2 = (\nu^1 + \nu^3) m_2$.

Remark 5.1. Assumption 5.1 means that $\mathbf{k}(\varepsilon, t) = \mathbf{k}(\frac{t}{\varepsilon})$ is a fast changing process as ε is sufficiently small.

Remark 5.2. It is usual in the literature to assume, as in Assumption 5.2, that on average the first machine has at least the same capacity as the second. This is also the case that is encountered frequently in practice so as to avoid excessive starvation of the second machine.

Similar to Chapter 3, we employ the hierarchical production planning to solve this feedback control problem. To define the limiting problem, we consider the following class of deterministic controls.

Definition 5.1. For $\mathbf{x} \in S$, let $\bar{\mathcal{A}}(\mathbf{x})$ denote the set of the deterministic measurable controls

$$\mathbf{U}(\cdot) = (\mathbf{u}^1(\cdot), \cdots, \mathbf{u}^4(\cdot)) = ((u_1^1(\cdot), u_2^1(\cdot)), \cdots, (u_1^4(\cdot), u_2^4(\cdot)))$$

such that $0 \le u_i^j(t) \le k_i^j$ for $t \ge 0$, $i = 1,2$ and $j = 1,2,3,4$, and that the corresponding solutions $\mathbf{x}(\cdot)$ of the following system

$$\begin{cases} \dot{x}_1(t) &= \sum_{j=1}^4 \nu^j u_1^j(t) - \sum_{j=1}^4 \nu^j u_2^j(t), \quad x_1(0) = x_1, \\ \dot{x}_2(t) &= \sum_{j=1}^4 \nu^j u_2^j(t) - d, \quad x_2(0) = x_2. \end{cases} \tag{5.3}$$

satisfy $\mathbf{x}(t) \in S$ for all $t \ge 0$.

We can now specify the limiting problem as

$$\bar{\mathcal{P}} : \begin{cases} \text{minimize} & J(\mathbf{x}, \mathbf{U}(\cdot)) = \int_0^\infty e^{-\rho t} h(\mathbf{x}(t))dt \\ \text{subject to} & \begin{cases} \dot{x}_1(t) = \sum_{j=1}^4 \nu^j u_1^j(t) - \sum_{j=1}^4 \nu^j u_2^j(t), & x_1(0) = x_1, \\ \dot{x}_2(t) = \sum_{j=1}^4 \nu^j u_2^j(t) - d, & x_2(0) = x_2, \\ \mathbf{U}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x}) \end{cases} \\ \text{value function} & v(\mathbf{x}) = \inf_{\mathbf{U}(\cdot) \in \bar{\mathcal{A}}(\mathbf{x})} J(\mathbf{x}, \mathbf{U}(\cdot)). \end{cases}$$

Let

$$w_1(t) = \sum_{j=1}^4 \nu^j u_1^j(t) \text{ and } w_2(t) = \sum_{j=1}^4 \nu^j u_2^j(t). \tag{5.4}$$

Then the limiting problem $\bar{\mathcal{P}}$ can be rewritten as

$$\bar{\mathcal{P}}:\begin{cases}\text{minimize} & J(\mathbf{x},\mathbf{w}(\cdot))=\int_0^\infty e^{-\rho t}h(\mathbf{x}(t))dt\\ \text{subject to} & \begin{cases}\dot{x}_1(t)=w_1(t)-w_2(t), & x_1(0)=x_1,\\ \dot{x}_2(t)=w_2(t)-d, & x_2(0)=x_2,\\ \mathbf{w}(\cdot)\in\tilde{\mathcal{A}}(\mathbf{x}) \end{cases}\\ \text{value function} & v(\mathbf{x})=\inf_{\mathbf{w}(\cdot)\in\tilde{\mathcal{A}}(\mathbf{x})}J(\mathbf{x},\mathbf{w}(\cdot)),\end{cases}$$

where $\tilde{\mathcal{A}}(\mathbf{x})$, $\mathbf{x}\in S$, can also be considered, with an abuse of notation,to be the set of the deterministic measurable controls $\mathbf{w}(\cdot)=(w_1(\cdot),w_2(\cdot))$ with

$$0\le w_1(t)\le a_1,\ \ 0\le w_2(t)\le a_2,$$

and with the corresponding solution $\mathbf{x}(\cdot)$ of the state equation appearing in $\bar{\mathcal{P}}$ satisfying $\mathbf{x}(t)\in S$ for all $t\ge 0$.

It is standard to show that the value functions v^ε and v are continuous and convex in $\mathbf{x}$.

Theorem 3.4 says that the problem $\bar{\mathcal{P}}$ is indeed a limiting problem in the sense that the value function v^ε of $\mathcal{P}^\varepsilon$ converges to the value function v of $\bar{\mathcal{P}}$. Moreover, it gives the corresponding convergence rate. Note that the proof of it does not involve using any feedback controls.

Next, we define what are asymptotic optimal feedback controls and give some estimates related to the behaviour of the process $\mathbf{k}(\varepsilon, t)$ for small ε.

Definition 5.2. An admissible feedback control $\mathbf{u}^\varepsilon$ is *asymptotic optimal* if

$$\lim_{\varepsilon \to 0} |J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}^\varepsilon) - v^\varepsilon(\mathbf{x}, \mathbf{k})| = 0$$

for all $(\mathbf{x}, \mathbf{k})$. Moreover, if there exist positive constants $C(\mathbf{x})$ and γ such that

$$|J^\varepsilon(\mathbf{x}, \mathbf{k}, \mathbf{u}^\varepsilon) - v^\varepsilon(\mathbf{x}, \mathbf{k})| \le C(\mathbf{x})\varepsilon^\gamma,$$

then $C(\mathbf{x})\varepsilon^\gamma$ is called an *asymptotic error bound* and γ is called the *rate of convergence.*

The purpose of this chapter is to construct asymptotic optimal controls for our original problem $\mathcal{P}^\varepsilon$. The construction will begin with the explicit optimal controls for the limiting deterministic problem $\bar{\mathcal{P}}$, which had been solved in Chapter 4.

5.3 Asymptotic Optimal Feedback Controls for $\mathcal{P}^\varepsilon$

In this section, we construct a feedback control for $\mathcal{P}^\varepsilon$ along with some discussions for its asymptotic optimality. The main idea is to use a control for the original stochastic problem $\mathcal{P}^\varepsilon$ that *has the same form* as an optimal feedback control for the limiting deterministic problem $\bar{\mathcal{P}}$ which is obtained in Chapter 4, and then to show that the two trajectories of $\mathcal{P}^\varepsilon$ and $\bar{\mathcal{P}}$ under their respective controls are very close to each other *on average.*

5.3.1 The Case $c_1 < c_2^+$

The optimal control $\mathbf{w}^*$ for the limiting problem $\bar{\mathcal{P}}$ when $c_1 < c_2^+$ is presented in (4.3). Clearly, this control is not feasible for the stochastic problem $\mathcal{P}^\varepsilon$ when one or both of the machines are broken down. Since our purpose here is to construct an asymptotic optimal control for $\mathcal{P}^\varepsilon$ beginning with $\mathbf{w}^*$, we first rewrite $\mathbf{w}^*(\mathbf{x})$ as $\mathbf{w}^*(\mathbf{x}) = \mathbf{r}^*(\mathbf{x}, \mathbf{a})$, where

$$\mathbf{r}^*(\mathbf{x},\mathbf{k}) = \begin{cases} (\min\{k_1,k_2,d\},\min\{k_1,k_2,d\}), & x_1 = x_2 = 0, \\ (0,0), & 0 \le x_1 \le b_1, 0 < x_2 \le b_2, \\ (0,\min\{k_2,d\}), & 0 < x_1 \le b_1, x_2 = 0, \\ (0,k_2), & 0 < x_1 \le b_1, x_2 < 0, \\ (\min\{k_1,k_2\},\min\{k_1,k_2\}), & x_1 = 0, x_2 < 0. \end{cases} \tag{5.5}$$

However, one can easily see that $\mathbf{u}^*(\mathbf{x},\mathbf{k}) = \mathbf{r}^*(\mathbf{x},\mathbf{k})$ is *not* asymptotic optimal for the original problem $\mathcal{P}^\varepsilon$, when the unit shortage cost c_2^- is strictly positive. This is because when the trajectory under this control reaches $(0,x_2), x_2 < 0$, it moves along the line $x_1 = 0$ at a different average rate than the rate in $\bar{\mathcal{P}}$ with $\mathbf{w}^*(\mathbf{x})$. In other words, there is a significant loss of capacity on behalf of M_2 during the movement along $x_1 = 0, x_2 < 0$, whenever M_1 is under repair. This is a consequence of the requirement that $x_1(t)$ is not to become negative. To mitigate the effect of the capacity loss phenomenon, we try to reduce the time spent on the boundary $x_1 = 0$.

To this end, we introduce a small region $\{\mathbf{x}|0 \le x_1 < \varepsilon^{\frac{1}{3}}, x_2 < 0\}$ as a neighbourhood of $x_1 = 0, x_2 < 0$, where the policy is such that there is a tendency for the state trajectory to go away from $x_1 = 0$ and toward $x_1 = \varepsilon^{\frac{1}{3}}$, while still staying in the neighbourhood. This tendency is in a marked contrast from the feedback policy $\mathbf{r}^*(\mathbf{x},\mathbf{k})$, which brings the trajectory down to $x_1 = 0$ as quickly as possible.

Let us therefore introduce the following function:

$$\mathbf{r}^\varepsilon(\mathbf{x},\mathbf{k}) = \begin{cases} (0, k_2), & \text{if } \mathbf{x} \in \Gamma'_1, \\ (\min\{k_1,k_2\}, k_2), & \text{if } \mathbf{x} \in \Gamma'_2, \\ (k_1, k_2), & \text{if } \mathbf{x} \in \Gamma'_3, \\ (k_1, \min\{k_1,k_2\}), & \text{if } \mathbf{x} \in \Gamma'_4, \\ (k_1, \min\{k_1,k_2,d\}), & \text{if } \mathbf{x} \in \Gamma'_5, \\ (k_1, \min\{k_2,d\}), & \text{if } \mathbf{x} \in \Gamma'_6, \\ (\min\{k_1,k_2,d\}, \min\{k_2,d\}), & \text{if } \mathbf{x} \in \Gamma'_7, \\ (0, \min\{k_2,d\}), & \text{if } \mathbf{x} \in \Gamma'_8, \\ (0,0), & \text{if } \mathbf{x} \in \Gamma'_9. \end{cases} \tag{5.6}$$

where, as shown in Fig. 5.1,

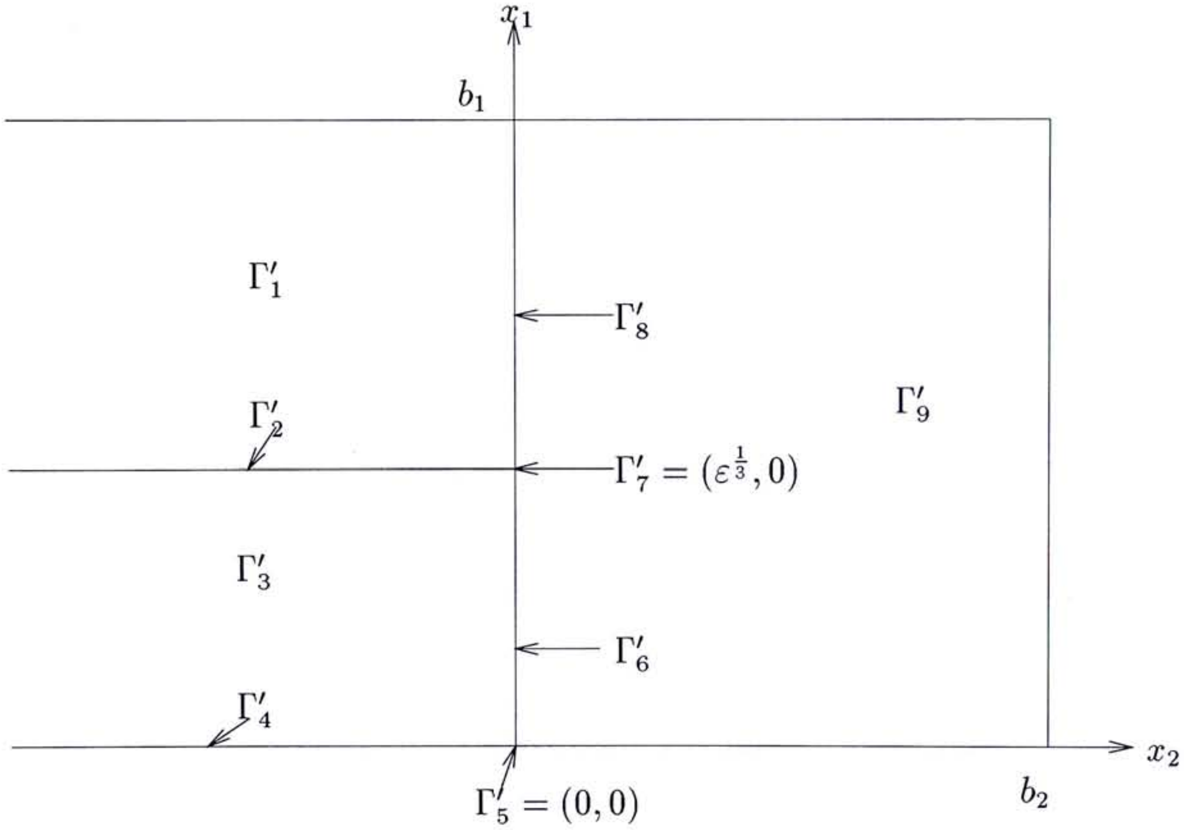

Figure 5.1: Switching manifolds for hierarchical control when $c_1 < c_2^+$

$$\Gamma_1' = \{(x_1, x_2) | \varepsilon^{\frac{1}{3}} < x_1 \le b_1, x_2 < 0\},$$

$$\Gamma_2' = \{(x_1, x_2) | x_1 = \varepsilon^{\frac{1}{3}}, x_2 < 0\},$$

$$\Gamma_3' = \{(x_1, x_2) | 0 < x_1 < \varepsilon^{\frac{1}{3}}, x_2 < 0\},$$

$$\Gamma_4' = \{(x_1, x_2) | x_1 = 0, x_2 < 0\},$$

$$\Gamma_5' = \{(x_1, x_2) | x_1 = 0, x_2 = 0\},$$

$$\Gamma_6' = \{(x_1, x_2) | 0 < x_1 < \varepsilon^{\frac{1}{3}}, x_2 = 0\},$$

$$\Gamma_7' = \{(x_1, x_2) | x_1 = \varepsilon^{\frac{1}{3}}, x_2 = 0\},$$

$$\Gamma_8' = \{(x_1, x_2) | \varepsilon^{\frac{1}{3}} < x_1 \le b_1, x_2 = 0\},$$

$$\Gamma_9' = \{(x_1, x_2) | 0 \le x_1 \le b_1, 0 < x_2 \le b_2\}.$$

In connection with these regions, $\Gamma_i', i = 1, 3, 9$, are called *interiors*, and the remaining ones are called *boundaries*.

It is interesting to point out that the line $x_1 = \varepsilon^{\frac{1}{3}}, x_2 < 0$ in defining (5.6) is a manifold introduced to reduce capacity losses on M_2 along $x_1 = 0, x_2 < 0$. Moreover, setting a policy $\mathbf{u}(\mathbf{x}, \mathbf{k}) = \mathbf{r}^\varepsilon(\mathbf{x}, \mathbf{k})$ implies that the full machine M_1 capacity be used below the manifold $x_1 = \varepsilon^{\frac{1}{3}}, x_2 < 0$.

We have now constructed the function which we will use after. As we shall see shortly, setting $\mathbf{k} = \mathbf{a}$ in $\mathbf{r}^\varepsilon(\mathbf{x}, \mathbf{k})$ would provide us with a near-optimal control for $\bar{\mathcal{P}}$. Furthermore, $\mathbf{u}^\varepsilon(\mathbf{x}, \mathbf{k}) = \mathbf{r}^\varepsilon(\mathbf{x}, \mathbf{k})$ would yield an asymptotic optimal feedback control for $\mathcal{P}^\varepsilon$, since its cost is understandably, and also provably, not too far from the cost of the near-optimal control $\mathbf{r}^\varepsilon(\mathbf{x}, \mathbf{a})$ in $\bar{\mathcal{P}}$. We begin with defining the control

$$\mathbf{w}^\varepsilon(\mathbf{x}) = (w_1^\varepsilon(\mathbf{x}), w_2^\varepsilon(\mathbf{x})) = \mathbf{r}^\varepsilon(\mathbf{x}, \mathbf{a}) \tag{5.7}$$

for $\bar{\mathcal{P}}$; see Fig. 5.2. Note that $w_2^\varepsilon(\mathbf{x}) = w_2^*(\mathbf{x})$ in the entire state space. Moreover, $w_1^\varepsilon(\mathbf{x}) = w_1^*(\mathbf{x})$, except when $0 \le x_1 \le \varepsilon^{\frac{1}{3}}, x_2 \le 0$. In this region of exception, the control policy has the tendency to go toward $(\varepsilon^{\frac{1}{3}}, 0)$. This policy, while not optimal, is nearly optimal for $\bar{\mathcal{P}}$ for sufficiently small ε, since $|x_1^\varepsilon - x_1^*| \le \varepsilon^{\frac{1}{3}}$. More specifically, we have the following lemma which is clear by the above observations.

Lemma 5.1. *There is a positive constant C such that*

$$|J(\mathbf{x}, \mathbf{w}^\varepsilon) - v(\mathbf{x})| = |J(\mathbf{x}, \mathbf{w}^\varepsilon) - J(\mathbf{x}, \mathbf{w}^*)| \le C\varepsilon^{\frac{1}{3}}.$$

Note that in Definition 5.1 and (5.4) there are two alternative ways to define the class of admissible controls for the limiting problem. Control $\mathbf{w}^\varepsilon(\mathbf{x})$ constructed above can be equivalently written as

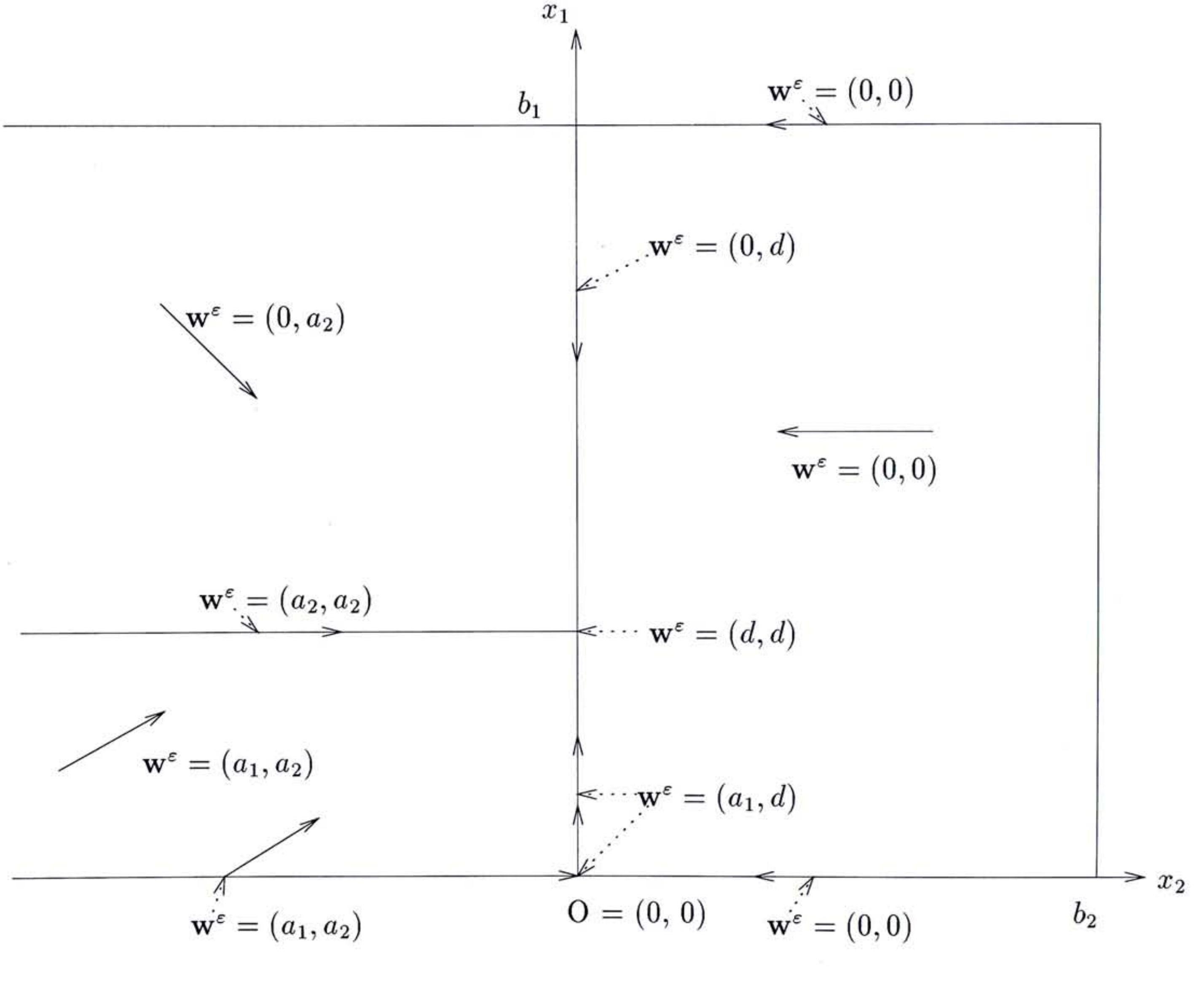

Figure 5.2: Near-optimal control $\mathbf{w}^\varepsilon$ for $\bar{\mathcal{P}}$ and trajectory movements when $c_1 < c_2^+$

$$\mathbf{U}^{\varepsilon}(\mathbf{x}) = ((u_1^1(\mathbf{x}), u_2^1(\mathbf{x})), (u_1^2(\mathbf{x}), u_2^2(\mathbf{x})), (u_1^3(\mathbf{x}), u_2^3(\mathbf{x})), (u_1^4(\mathbf{x}), u_2^4(\mathbf{x})))$$
$$= \begin{cases} ((0, m_2), (0,0), (0, m_2), (0,0)), & \text{if } \mathbf{x} \in \Gamma_1', \\ ((\frac{\nu^1+\nu^3}{\nu^1+\nu^2} m_2, m_2), (\frac{\nu^1+\nu^3}{\nu^1+\nu^2} m_2, 0), (0, m_2), (0,0)), & \text{if } \mathbf{x} \in \Gamma_2', \\ ((m_1, m_2), (m_1, 0), (0, m_2), (0,0)), & \text{if } \mathbf{x} \in \Gamma_3', \\ ((m_1, m_2), (m_1, 0), (0, m_2), (0,0)), & \text{if } \mathbf{x} \in \Gamma_4', \\ ((m_1, \frac{d}{\nu^1+\nu^3}), (m_1, 0), (0, \frac{d}{\nu^1+\nu^3}), (0,0)), & \text{if } \mathbf{x} \in \Gamma_5', \\ ((m_1, \frac{d}{\nu^1+\nu^3}), (m_1, 0), (0, \frac{d}{\nu^1+\nu^3}), (0,0)), & \text{if } \mathbf{x} \in \Gamma_6', \\ ((\frac{d}{\nu^1+\nu^2}, \frac{d}{\nu^1+\nu^3}), (\frac{d}{\nu^1+\nu^2}, 0), (0, \frac{d}{\nu^1+\nu^3}), (0,0)), & \text{if } \mathbf{x} \in \Gamma_7', \\ ((0, \frac{d}{\nu^1+\nu^3}), (0,0), (0, \frac{d}{\nu^1+\nu^3}), (0,0)), & \text{if } \mathbf{x} \in \Gamma_8', \\ ((0,0), (0,0), (0,0), (0,0)), & \text{if } \mathbf{x} \in \Gamma_9'. \end{cases} \tag{5.8}$$

According to Lemma 5.1, the policy $\mathbf{U}^{\varepsilon}(\mathbf{x})$ is asymptotic optimal for $\bar{\mathcal{P}}$ as ε goes to zero.

In view of Lemma 5.1 and Theorem 3.4, a control for $\mathcal{P}^{\varepsilon}$, whose associated expected cost is close to $J(\mathbf{x}, \mathbf{U}^{\varepsilon}) = J(\mathbf{x}, \mathbf{w}^{\varepsilon})$ would be asymptotic optimal. An obvious candidate is

$$\mathbf{u}^{\varepsilon}(\mathbf{x}, \mathbf{k}) = (u_1^{\varepsilon}(\mathbf{x}, \mathbf{k}), u_2^{\varepsilon}(\mathbf{x}, \mathbf{k})) = \mathbf{r}^{\varepsilon}(\mathbf{x}, \mathbf{k}). \tag{5.9}$$

$\mathbf{u}^\varepsilon(\mathbf{x},\mathbf{k})$	$\mathbf{k}=\mathbf{k}^1$	$\mathbf{k}=\mathbf{k}^2$	$\mathbf{k}=\mathbf{k}^3$	$\mathbf{k}=\mathbf{k}^4$
$\mathbf{x}\in\Gamma'_1$	$(0,m_2)$	$(0,0)$	$(0,m_2)$	$(0,0)$
$\mathbf{x}\in\Gamma'_2$	$(\min\{m_1,m_2\},m_2)$	$(0,0)$	$(0,m_2)$	$(0,0)$
$\mathbf{x}\in\Gamma'_3$	(m_1,m_2)	$(m_1,0)$	$(0,m_2)$	$(0,0)$
$\mathbf{x}\in\Gamma'_4$	$(m_1,\min\{m_1,m_2\})$	$(m_1,0)$	$(0,0)$	$(0,0)$
$\mathbf{x}\in\Gamma'_5$	(m_1,d)	$(m_1,0)$	$(0,0)$	$(0,0)$
$\mathbf{x}\in\Gamma'_6$	(m_1,d)	$(m_1,0)$	$(0,d)$	$(0,0)$
$\mathbf{x}\in\Gamma'_7$	(d,d)	$(0,0)$	$(0,d)$	$(0,0)$
$\mathbf{x}\in\Gamma'_8$	$(0,d)$	$(0,0)$	$(0,d)$	$(0,0)$
$\mathbf{x}\in\Gamma'_9$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

Table 5.1: Tabular representation of $\mathbf{u}^\varepsilon(\mathbf{x},\mathbf{k})$ when $c_1 < c_2^+$

The candidate control is represented in a tabular form in Table 5.1 for convenience in exposition. Each component of $\mathbf{u}^\varepsilon(\mathbf{x},\mathbf{k})$ in the table represents the control policy under the corresponding machine state and initial state.

We can also depict the state trajectory movements in Fig. 5.3 under each of the four machine states. In view of (5.8), (5.9) and Theorem 3.4, it is easy to see that the average movement of the state trajectory in Fig. 5.3 at each point of the state space coincides with the movement of the state trajectory of $\bar{\mathcal{P}}$ shown in Fig. 5.2 under the policy (5.7) or (5.8).

With this observation, we can expect the following main result of the chapter to hold.

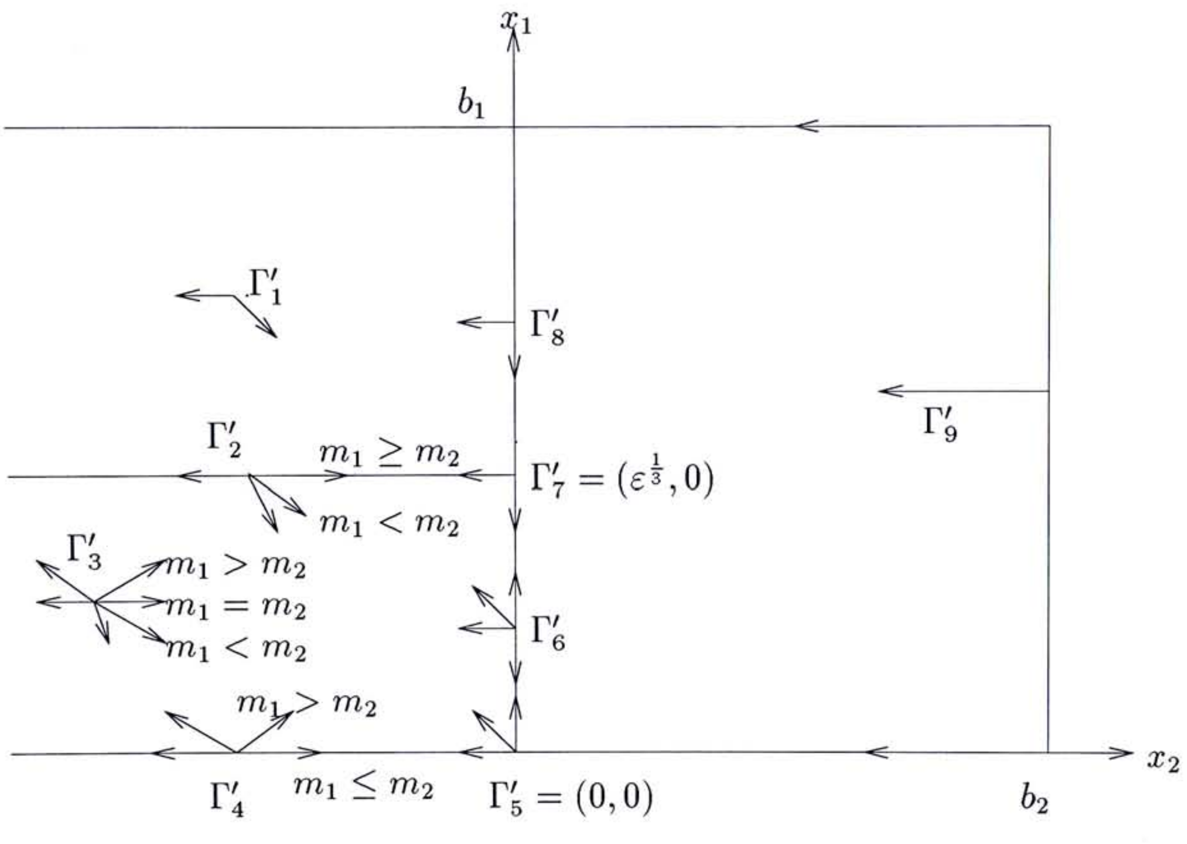

Figure 5.3: Directions of trajectory movements under different machine states when $c_1 < c_2^+$

Theorem 5.1. *There exists a positive constant C such that*

$$|J^{\varepsilon}(\mathbf{x},\mathbf{k},\mathbf{u}^{\varepsilon}(\mathbf{x},\mathbf{k})) - J(\mathbf{x},\mathbf{U}^{\varepsilon}(\mathbf{x}))| < C\varepsilon^{\frac{1}{6}}, \tag{5.10}$$

for sufficiently small ε.

The main idea of the proof of the theorem is to show that the trajectory $\mathbf{x}(\cdot)$ of the original problem and the trajectory $\mathbf{y}(\cdot)$ of the limiting problem (under $\mathbf{u}^{\varepsilon}(\mathbf{x},\mathbf{k})$ and $\mathbf{U}^{\varepsilon}(\mathbf{x})$, respectively) are very *close* to each other in the average sense, if they start from the same initial state. It is very important to note that $\mathbf{u}^{\varepsilon}(\mathbf{x},\mathbf{k}^{i})$ coincides with $\mathbf{u}^{i}(\mathbf{y})$, the i-th component of $\mathbf{U}^{\varepsilon}(\mathbf{y})$, if $\mathbf{x}$ and $\mathbf{y}$ are in one of the same interiors of Γ'_1, Γ'_3, or Γ'_9.

Before we prove the theorem, we need to prove some preliminary results. First of all, we obtain a bound on the rate at which the first components of $\mathbf{x}(\cdot)$ and $\mathbf{y}(\cdot)$ diverge over time. Specifically, we consider

$$\begin{aligned}
&\tfrac{d}{dt}(x_1(t)-y_1(t))^2 \\
=\ & 2(x_1(t)-y_1(t))(\dot{x}_1(t)-\dot{y}_1(t)) \\
=\ & 2(x_1(t)-y_1(t))[(u_1^\varepsilon(\mathbf{x}(t),\mathbf{k}(\varepsilon,t))-u_2^\varepsilon(\mathbf{x}(t),\mathbf{k}(\varepsilon,t))) \\
& -\textstyle\sum_{i=1}^4 \nu^i(u_1^i(\mathbf{y}(t))-u_2^i(\mathbf{y}(t)))] \\
=\ & 2(x_1(t)-y_1(t))\{\textstyle\sum_{i=1}^4 \chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^i\}}[(u_1^\varepsilon(\mathbf{x}(t),\mathbf{k}^i)-u_1^i(\mathbf{y}(t))) \\
& -(u_2^\varepsilon(\mathbf{x}(t),\mathbf{k}^i)-u_2^i(\mathbf{y}(t)))]+\textstyle\sum_{i=1}^4(\chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^i\}}-\nu^i)(u_1^i(\mathbf{y}(t))-u_2^i(\mathbf{y}(t)))\}.
\end{aligned} \tag{5.11}$$

Define

$$\lambda^i(\mathbf{x},\mathbf{y})=(x_1-y_1)[(u_1^\varepsilon(\mathbf{x},\mathbf{k}^i)-u_1^i(\mathbf{y}))-(u_2^\varepsilon(\mathbf{x},\mathbf{k}^i)-u_2^i(\mathbf{y}))], i=1,2,3,4.$$

In the following, we want to show that

$$E\int_0^t \lambda^i(\mathbf{x}(s),\mathbf{y}(s)) \le K\varepsilon^{\frac{1}{3}}(1+t), i=1,2,3,4, \tag{5.12}$$

for some constant K; see Theorem 4.2 and Remark 4.2 in Sethi and Zhang [42]. To this end, let us first introduce several useful lemmas.

Lemma 5.2. *For $i=1,2,3,4$, we have*

(a) $\lambda^i(\mathbf{x},\mathbf{y})=0$, *for* $(\mathbf{x},\mathbf{y})\in\Gamma_1'\times\Gamma_1'$;

(b) $\lambda^i(\mathbf{x},\mathbf{y}) \le 0$, *for* $(\mathbf{x},\mathbf{y}) \in [\Gamma_1' \times (\Gamma_2' \cup \Gamma_7')] \cup [(\Gamma_2' \cup \Gamma_7') \times \Gamma_1']$;

(c) $\lambda^i(\mathbf{x},\mathbf{y}) \le 0$, *for* $\mathbf{x} \in \{(x_1,x_2)|x_1 \ge 0, x_2 < 0\}, \mathbf{y} \in \Gamma_7'$;

(d) $\lambda^i(\mathbf{x},\mathbf{y}) \le K\varepsilon^{\frac{1}{3}}$, *for* $\mathbf{x},\mathbf{y} \in \{(x_1,x_2)|0 \le x_1 \le \varepsilon^{\frac{1}{3}}\}$.

Proof. (a) Since Γ_1' is an interior in which $u_j^\varepsilon(\mathbf{x},\mathbf{k}^i) = u_j^i(\mathbf{y})$ for $i = 1,2,3,4$ and $j = 1,2$, we have $\lambda^i(\mathbf{x},\mathbf{y}) = 0$ on $\Gamma_1' \times \Gamma_1'$ by definition.

(b) This can be proved by direct calculation in each case. For example, when $\mathbf{x} \in \Gamma_1', \mathbf{y} \in \Gamma_2'$, we have

$$x_1 - y_1 > 0, \text{ and}$$
$$(u_1^\varepsilon(\mathbf{x},\mathbf{k}^1) - u_1^1(\mathbf{y})) - (u_2^\varepsilon(\mathbf{x},\mathbf{k}^1) - u_2^1(\mathbf{y})) = (0 - \tfrac{\nu^1+\nu^3}{\nu^1+\nu^2}m_2) - (m_2 - m_2) \le 0,$$

which gives $\lambda^1(\mathbf{x},\mathbf{y}) \le 0$.

(c) This can also be shown by direct calculation.

(d) If both $\mathbf{x}$ and $\mathbf{y}$ are in the ε-strip $\{(x_1,x_2)|0 \le x_1 \le \varepsilon^{\frac{1}{3}}\}$, then $|x_1 - y_1| \le \varepsilon^{\frac{1}{3}}$. The desired result is then easily seen, since all the controls constructed are bounded (independent of ε). □

Lemma 5.3. *Let* $\tilde{\mathbf{x}}(\cdot)$ *be the trajectory of* $\mathcal{P}^\varepsilon$ *under* $\mathbf{u}^\varepsilon(\mathbf{x},\mathbf{k})$, *and* $\tilde{\mathbf{y}}(\cdot)$ *be the trajectory of* $\bar{\mathcal{P}}$ *under* $\mathbf{U}^\varepsilon(\mathbf{x})$, *both starting from the same initial* $\tilde{\mathbf{x}} \in \Gamma_8'$. *Then*

$$\tilde{x}_1(t) \ge \tilde{y}_1(t), \quad \textit{for } t \in [0, \frac{\tilde{x}_1 - \varepsilon^{\frac{1}{3}}}{d}]. \tag{5.13}$$

Proof. Let us pick up a sample point ω. Suppose during the time interval $[0,\tau)$, the machine state is in $\{\mathbf{k}^2,\mathbf{k}^4\}$, and then switch to be in $\{\mathbf{k}^1,\mathbf{k}^3\}$ (τ could be zero) at τ. Then obviously (5.13) holds for $t\in[0,\tau]$. Note that for any $t\in[\tau,\frac{\tau m_2}{m_2-d}]$, we have

$$\tilde{x}_1(t)=\tilde{x}_1-(t-\tau)m_2\geq\tilde{x}_1-td=\tilde{y}_1(t),$$

and the equality holds only at $t=t_0\equiv\frac{\tau m_2}{m_2-d}$.

Here t_0 is nothing but the time of $\tilde{\mathbf{x}}(\cdot)$ hitting Γ_8' under the assumption that the machine state does not switch to $\{\mathbf{k}^2,\mathbf{k}^4\}$ in $[\tau,t_0)$. Now let us assume that the machine state changes to $\{\mathbf{k}^2,\mathbf{k}^3\}$ at time $\theta<t_0$ and stays there until time τ'. By the same argument as above, it is easy to check that $\tilde{x}_1(t)\geq\tilde{y}_1(t)$ for $t\geq\theta$. This situation continues until $t=\frac{\tilde{x}_1-\varepsilon^{\frac{1}{3}}}{d}$, the time of $\tilde{\mathbf{y}}(\cdot)$ reaching the threshold Γ_7'. □

Lemma 5.4. *Let $\tilde{\mathbf{x}}(\cdot)$ be the trajectory of $\mathcal{P}^\varepsilon$ under $\mathbf{u}^\varepsilon(\mathbf{x},\mathbf{k})$ with initial $\tilde{\mathbf{x}}\in\Gamma_1'\cup\Gamma_8'$, and $\tilde{\mathbf{y}}(\cdot)$ be the trajectory of $\bar{\mathcal{P}}$ under $\mathbf{U}^\varepsilon(\mathbf{x})$ with initial $\tilde{\mathbf{y}}\in\Gamma_8'$. If $|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|\leq C\varepsilon^{\frac{1}{3}}$, then*

$$\tilde{x}_1(t)\geq\tilde{y}_1(t)-C\frac{m_2}{m_2-d}\varepsilon^{\frac{1}{3}},\ \text{for } t\in[0,\frac{\tilde{y}_1-\varepsilon^{\frac{1}{3}}}{d}]. \tag{5.14}$$

Proof. Define

$$\begin{aligned}
\mathbf{x}'(t) = (x_1'(t), x_2'(t)) &= \begin{cases} (\tilde{x}_1, \tilde{y}_2 - (m_2 - d)t), & 0 \le t < \frac{\tilde{y}_2 - \tilde{x}_2}{m_2 - d} \\ (\tilde{x}_1(t - \frac{\tilde{y}_2 - \tilde{x}_2}{m_2 - d}), \tilde{x}_2(t - \frac{\tilde{y}_2 - \tilde{x}_2}{m_2 - d})), & t \ge \frac{\tilde{y}_2 - \tilde{x}_2}{m_2 - d}, \end{cases} \\
\mathbf{y}'(t) = (y_1'(t), y_2'(t)) &= (\tilde{y}_1(t) - \tilde{y}_1 + \tilde{x}_1, \tilde{y}_2(t)).
\end{aligned}$$

Then $\mathbf{x}'(0) = \mathbf{y}'(0) = (\tilde{x}_1, \tilde{y}_2) \in \Gamma_8'$. We can therefore apply Lemma 5.3 to obtain

$$\begin{aligned}
\tilde{x}_1(t) = &\ x_1'(t + \tfrac{\tilde{y}_2 - \tilde{x}_2}{m_2 - d}) \ge y_1'(t + \tfrac{\tilde{y}_2 - \tilde{x}_2}{m_2 - d}) \\
= &\ \tilde{y}_1(t + \tfrac{\tilde{y}_2 - \tilde{x}_2}{m_2 - d}) - \tilde{y}_1 + \tilde{x}_1 = \tilde{y}_1(t) - d\tfrac{\tilde{y}_2 - \tilde{x}_2}{m_2 - d} - \tilde{y}_1 + \tilde{x}_1 \\
\ge &\ \tilde{y}_1(t) - C\tfrac{m_2}{m_2 - d}\varepsilon^{\frac{1}{3}},
\end{aligned}$$

for $t \in [0, \frac{\tilde{y}_1 - \varepsilon^{\frac{1}{3}}}{d}]$. This proves the lemma. □

Lemma 5.5. *For $i = 1, 2, 3, 4$, there exists a positive constant K such that the inequality (5.12) holds.*

Proof. We have to analyze various different cases depending on the position of the initial state $\mathbf{x}$.

Case I. The initial state $\mathbf{x} \in \Gamma_1'$.

In this case, both trajectories have the following important features:

(i) The deterministic trajectory $\mathbf{y}(\cdot)$ will either first hit Γ_7' and then stay there forever, or first hit either Γ_2' or Γ_8' and then go along one of these boundaries towards Γ_7'.

(ii) The stochastic trajectory $\mathbf{x}(\cdot)$ will never go up inside Γ_1', and once having entered the "box" $\{(x_1, x_2)|0 \le x_1 \le \varepsilon, x_2 \le 0\}$, it will never go out of it.

Define by τ_x and τ_y, respectively, the stopping times of $\mathbf{x}(\cdot)$ and $\mathbf{y}(\cdot)$ hitting the boundary $\Gamma_2' \cup \Gamma_7' \cup \Gamma_8'$. Note that τ_y is deterministic. We now want to prove the following estimate:

$$P(\tau_y - \tau_x \ge \varepsilon^{\frac{1}{3}}) \le C\varepsilon, \tag{5.15}$$

for some positive constant C.

To show this, we only consider the case when $\mathbf{y}(\cdot)$ hits Γ_8'. The analyses for the other cases are the same. By Corollary 2.1,

$$\begin{aligned}
& P(|\mathbf{x}(\tau_x) - \mathbf{y}(\tau_x)| \ge (a_2 - d)\varepsilon^{\frac{1}{3}}; \tau_x \le \tau_y) \\
\le\ & P(|\textstyle\int_0^{\tau_x} \sum_{i=0}^4 (\chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^i\}} - \nu^i)(u_1^i(\mathbf{y}) - u_2^i(\mathbf{y}))dt| > \frac{1}{2}(a_2 - d)\varepsilon^{\frac{1}{3}}; \tau_x \le \tau_y) \\
& +P(|\textstyle\int_0^{\tau_x} \sum_{i=0}^4 (\chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^i\}} - \nu^i)(u_2^i(\mathbf{y}) - d)dt| > \frac{1}{2}(a_2 - d)\varepsilon^{\frac{1}{3}}; \tau_x \le \tau_y) \\
\le\ & C(e^{-K\varepsilon^{-1}} + e^{-K\varepsilon^{-1/3}(1+\tau_y)^{-3}}) \le C\varepsilon.
\end{aligned}$$

Hence,

$$
\begin{aligned}
& P(\tau_y - \tau_x \geq \varepsilon^{\frac{1}{3}}) \\
\leq\ & P(|\mathbf{x}(\tau_x) - \mathbf{y}(\tau_x)| \geq (a_2 - d)\varepsilon^{\frac{1}{3}}; \tau_x \leq \tau_y; \mathbf{x}(\tau_x) \in \Gamma_8' \cup \Gamma_7') \\
& + P(|\mathbf{x}(\tau_x) - \mathbf{y}(\tau_x)| \geq y_1(\tau_y); \tau_x \leq \tau_y; \mathbf{x}(\tau_x) \in \Gamma_2') \\
\leq\ & C\varepsilon.
\end{aligned}
$$

This proves (5.15).

To complete the proof of the theorem in Case I, we have to study three subcases:

Case I.1. $\mathbf{y}(\tau_y) \in \Gamma_2'$.

In this case, we note the following facts:

(i) $\lambda^i(\mathbf{x}(t), \mathbf{y}(t)) = 0$, $t < \min\{\tau_x, \tau_y\}$, by virtue of Lemma 5.2 (a).

(ii) If $\tau_y \leq \tau_x$, then for $t \geq \tau_y$, we have $\mathbf{y}(t) \in \Gamma_2' \cup \Gamma_7'$ and $\mathbf{x}(t) \in \cup_{i=1}^7 \Gamma_i'$. Hence, by Lemma 5.2 (b) and (c), we have

$$\lambda^i(\mathbf{x}(t), \mathbf{y}(t)) \leq K\varepsilon^{\frac{1}{3}}, i = 1, 2, 3, 4.$$

(iii) If $\tau_y > \tau_x$, then as $t \geq \tau_x$, $\mathbf{y}(t) \in \Gamma_1' \cup \Gamma_2' \cup \Gamma_7'$ and $\mathbf{x}(t) \in \cup_{i=2}^7 \Gamma_i'$. Again by Lemma 5.2 (b) and (c), we have

$$\lambda^i(\mathbf{x}(t), \mathbf{y}(t)) \le K\varepsilon^{\frac{1}{3}}, i = 1, 2, 3, 4.$$

For any $0 \le t \le \tau_y$, we can now estimate

$$\begin{aligned}
& E\int_0^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds \\
= \;& E(\textstyle\int_0^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds; \tau_y \le \tau_x) + E(\int_0^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds; \tau_y > \tau_x) \\
= \;& E(\textstyle\int_0^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds; \tau_y > \tau_x) \\
\le \;& E(\textstyle\int_0^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds; \tau_x < \tau_y < \tau_x + \varepsilon^{\frac{1}{3}}) + KP(\tau_y - \tau_x \ge \varepsilon^{\frac{1}{3}}) \\
= \;& E(\textstyle\int_{\min\{t,\tau_x\}}^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds; \tau_x < \tau_y < \tau_x + \varepsilon^{\frac{1}{3}}) + KP(\tau_y - \tau_x \ge \varepsilon^{\frac{1}{3}}) \\
\le \;& K\varepsilon^{\frac{1}{3}} + C\varepsilon.
\end{aligned}$$

For any $t > \tau_y$, by (ii) and (iii) above,

$$\begin{aligned}
& E(\textstyle\int_{\tau_y}^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds) \\
= \;& E(\textstyle\int_{\tau_y}^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds; \tau_y \le \tau_x) + E(\int_{\tau_y}^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds; \tau_y > \tau_x) \\
\le \;& K\varepsilon^{\frac{1}{3}}t.
\end{aligned}$$

Therefore, (5.12) follows in Case I.1.

Case I.2. $\mathbf{y}(\tau_y) \in \Gamma_8'$.

In this case, let us first estimate

$$\begin{aligned}
& P(|\mathbf{x}(\tau_y) - \mathbf{y}(\tau_y)| > \varepsilon^{\frac{1}{3}}; \tau_y \leq \tau_x) \\
\leq\ & P(|\textstyle\int_0^{\tau_y} \sum_{i=0}^4 (\chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^i\}} - \nu^i)(u_1^i(\mathbf{y}) - u_2^i(\mathbf{y}))dt| > \frac{1}{2}\varepsilon^{\frac{1}{3}}) \\
& + P(|\textstyle\int_0^{\tau_y} \sum_{i=0}^4 (\chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^i\}} - \nu^i)(u_2^i(\mathbf{y}) - d)dt| > \frac{1}{2}\varepsilon^{\frac{1}{3}}) \\
\leq\ & C\varepsilon.
\end{aligned}$$

Furthermore, with $\gamma = 2m_2 - d$ and $\bar{\gamma} = 2a_2 - d$, we have

$$\begin{aligned}
& P(|\mathbf{x}(\tau_y) - \mathbf{y}(\tau_y)| > (1+\gamma+\bar{\gamma})\varepsilon^{\frac{1}{3}}; \tau_y > \tau_x) \\
=\ & P(|\mathbf{x}(\tau_y) - \mathbf{y}(\tau_y)| > (1+\gamma+\bar{\gamma})\varepsilon^{\frac{1}{3}}; \tau_x < \tau_y < \tau_x + \varepsilon^{\frac{1}{3}}) + P(\tau_y \geq \tau_x + \varepsilon^{\frac{1}{3}}) \\
\leq\ & P(|\mathbf{x}(\tau_y) - \mathbf{x}(\tau_y)| > \gamma\varepsilon^{\frac{1}{3}}; \tau_x < \tau_y < \tau_x + \varepsilon^{\frac{1}{3}}) \\
& + P(|\mathbf{x}(\tau_x) - \mathbf{y}(\tau_x)| > \varepsilon^{\frac{1}{3}}; \tau_x < \tau_y < \tau_x + \varepsilon^{\frac{1}{3}}) \\
& + P(|\mathbf{y}(\tau_x) - \mathbf{y}(\tau_y)| > \bar{\gamma}\varepsilon^{\frac{1}{3}}; \tau_x < \tau_y < \tau_x + \varepsilon^{\frac{1}{3}}) + C\varepsilon \\
=\ & P(|\mathbf{x}(\tau_x) - \mathbf{y}(\tau_x)| > \varepsilon^{\frac{1}{3}}; \tau_x < \tau_y < \tau_x + \varepsilon^{\frac{1}{3}}) + C\varepsilon \\
\leq\ & C'\varepsilon.
\end{aligned}$$

Combining the above two estimates, we conclude that

$$P(|\mathbf{x}(\tau_y) - \mathbf{y}(\tau_y)| > (1+\gamma+\bar{\gamma})\varepsilon^{\frac{1}{3}}) \leq C\varepsilon.$$

Observe the following facts:

(i) If for a sample point ω, $|\mathbf{x}(\tau_y) - \mathbf{y}(\tau_y)| \le (1+\gamma+\bar{\gamma})\varepsilon^{\frac{1}{3}}$, then by Lemma 5.4,

$$x_1(t) \ge y_1(t) - \frac{(1+\gamma+\bar{\gamma})k_2}{k_2-d}\varepsilon^{\frac{1}{3}},\ t \in [\tau_y, \tau_y + \frac{y_1(\tau_y) - \varepsilon^{\frac{1}{3}}}{d}].$$

Hence,

$$\lambda^i(\mathbf{x}(t), \mathbf{y}(t)) \le K\varepsilon^{\frac{1}{3}},\ t \in [\tau_y, \tau_y + \frac{y_1(\tau_y) - \varepsilon^{\frac{1}{3}}}{d}].$$

(ii) When $t \ge \frac{y_1(\tau_y)-\varepsilon^{\frac{1}{3}}}{d}$, $y(t) \in \Gamma_7'$. In this case, we have by Lemma 5.2 (c) that

$$\lambda^i(\mathbf{x}(t), \mathbf{y}(t)) \le 0.$$

For any $0 \le t \le \tau_y$, we have the following, similar to Case I.1:

$$E\int_0^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds \le K\varepsilon^{\frac{1}{3}} + C\varepsilon.$$

On the other hand, for any $t \ge \tau_y$, we obtain

$$\begin{aligned}
& E\textstyle\int_{\tau_y}^t \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds \\
\le\ & E\textstyle\int_{\tau_y}^{\frac{y_1(\tau_y)-\varepsilon^{\frac{1}{3}}}{d}} \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds \\
=\ & E(\textstyle\int_{\tau_y}^{\frac{y_1(\tau_y)-\varepsilon^{\frac{1}{3}}}{d}} \lambda^i(\mathbf{x}(s), \mathbf{y}(s))ds; |\mathbf{x}(\tau_y) - \mathbf{y}(\tau_y)| \le (1+\gamma+\bar{\gamma})\varepsilon^{\frac{1}{3}}) \\
& +KP(|\mathbf{x}(\tau_y) - \mathbf{y}(\tau_y)| > (1+\gamma+\bar{\gamma})\varepsilon^{\frac{1}{3}}) \\
\le\ & K\varepsilon^{\frac{1}{3}} + K\varepsilon.
\end{aligned}$$

This proves (5.12) in Case I.2.

Case I.3. $\mathbf{y}(\tau_y) \in \Gamma'_7$.

In this case, (5.12) follows easily from Lemma 5.2 (c).

Case II. The initial state $\mathbf{x} \in \cup_{i=2}^{7} \Gamma'_i$.

In this case, both $\mathbf{x}(\cdot)$ and $\mathbf{y}(\cdot)$ will never go out of the ε–strip $\{(x_1, x_2) | 0 \leq x_1 \leq \varepsilon^{\frac{1}{3}}\}$. Therefore, (5.12) holds by Lemma 5.2 (d).

Case III. The initial state $\mathbf{x} \in \Gamma'_9 \cup \Gamma'_8$.

If $\mathbf{x} \in \Gamma'_9$, then initially both $\mathbf{x}(\cdot)$ and $\mathbf{y}(\cdot)$ will coincide until they hit $\cup_{i=5}^{8} \Gamma'_i$. So we may assume that $\mathbf{x} \in \Gamma'_8$.

By Lemma 5.3, if $t \leq \frac{x_1 - \varepsilon^{\frac{1}{3}}}{d}$, it holds that $x_1(t) \geq y_1(t)$, and therefore $\lambda^i(\mathbf{x}(t), \mathbf{y}(t)) \leq 0$. When $t \geq \frac{x_1 - \varepsilon^{\frac{1}{3}}}{d}$, $y(t) \in \Gamma'_7$. In this case, $\lambda^i(\mathbf{x}(t), \mathbf{y}(t)) \leq 0$ as given by Lemma 5.2 (c). So (5.12) holds. □

Let us now turn to the proof of the main result.

Proof of Theorem 5.1. By (5.11), (5.12), and Lemma 2.2, we obtain

$$\begin{aligned} E(x_1(t) - y_1(t))^2 \leq \; & 8K(1+t)\varepsilon^{\frac{1}{3}} + \tfrac{1}{2}E(x_1(t) - y_1(t))^2 \\ & + \tfrac{1}{2}E\left| \int_0^t \sum_{i=1}^{4} (\chi_{\{\mathbf{k}(\varepsilon,t)=\mathbf{k}^i\}} - \nu^i)(u_1^i(\mathbf{y}(t)) - u_2^i(\mathbf{y}(t)))dt \right|^2 \\ \leq \; & 8K(1+t)\varepsilon^{\frac{1}{3}} + \tfrac{1}{2}E(x_1(t) - y_1(t))^2 + C(1+t^2)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}} \\ \leq \; & C(1+t)\varepsilon^{\frac{1}{3}} + \tfrac{1}{2}E(x_1(t) - y_1(t))^2. \end{aligned}$$

Thus,

$$E|x_1(t) - y_1(t)| \leq C(1+t)^{\frac{1}{2}}\varepsilon^{\frac{1}{6}}.$$

By a similar argument, we can get

$$E|x_2(t) - y_2(t)| \leq C(1+t)^{\frac{1}{2}}\varepsilon^{\frac{1}{6}}.$$

This concludes the proof. □

Theorem 5.2. *There exists a positive constant C such that*

$$|J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}^\varepsilon(\mathbf{x},\mathbf{k})) - v^\varepsilon(\mathbf{x},\mathbf{k})| < C\varepsilon^{\frac{1}{6}}. \tag{5.16}$$

Proof. This follows easily from Theorems 3.4, 5.1 and Lemma 5.1. □

5.3.2 The Case $c_1 \geq c_2^+$

We write the optimal control obtained in (4.5) as $\mathbf{w}^*(\mathbf{x}) = \mathbf{r}^*(\mathbf{x},\mathbf{a})$, where

$$\mathbf{r}^*(\mathbf{x},\mathbf{k}) = \begin{cases} (\min\{k_1,k_2,d\},\min\{k_1,k_2,d\}), & x_1 = x_2 = 0, \\ (0,0), & x_1 = 0, 0 < x_2 \le b_2, \\ (0,k_2), & 0 < x_1 \le b_1, x_2 < b_2, \\ (0,\min\{k_2,d\}), & 0 < x_1 \le b_1, x_2 = b_2, \\ (\min\{k_1,k_2\},\min\{k_1,k_2\}), & x_1 = 0, x_2 < 0. \end{cases} \tag{5.17}$$

Again we introduce a switching manifold slightly above the boundary $x_1 = 0, x_2 < 0$ as in Section 5.3.1, and then define the following function:

$$\mathbf{r}^\varepsilon(\mathbf{x},\mathbf{k}) = \begin{cases} (0,k_2), & \text{if } \mathbf{x} \in \Gamma_1'', \\ (\min\{k_1,k_2\},k_2), & \text{if } \mathbf{x} \in \Gamma_2'', \\ (k_1,k_2), & \text{if } \mathbf{x} \in \Gamma_3'', \\ (k_1,\min\{k_1,k_2\}), & \text{if } \mathbf{x} \in \Gamma_4'', \\ (k_1,\min\{k_1,k_2,d\}), & \text{if } \mathbf{x} \in \Gamma_5'', \\ (k_1,\min\{k_2,d\}), & \text{if } \mathbf{x} \in \Gamma_6'', \\ (\min\{k_1,k_2,d\},\min\{k_2,d\}), & \text{if } \mathbf{x} \in \Gamma_7'', \\ (0,\min\{k_2,d\}), & \text{if } \mathbf{x} \in \Gamma_8'', \\ (0,0), & \text{if } \mathbf{x} \in \Gamma_9''. \end{cases} \tag{5.18}$$

where, as shown in Fig. 5.4,

$$\Gamma_1'' = \{(x_1,x_2)|\varepsilon^{\frac{1}{3}} < x_1 \le b_1, x_2 < b_2\} \cup \{(x_1,x_2)|0 < x_1 \le \varepsilon^{\frac{1}{3}}, 0 < x_2 < b_2\},$$

$$\Gamma_2'' = \{(x_1,x_2)|x_1 = \varepsilon^{\frac{1}{3}}, x_2 < 0\},$$

$$\Gamma_3'' = \{(x_1,x_2)|0 < x_1 < \varepsilon^{\frac{1}{3}}, x_2 < 0\},$$

$$\Gamma_4'' = \{(x_1,x_2)|x_1 = 0, x_2 < 0\},$$

$$\Gamma_5'' = \{(x_1,x_2)|x_1 = 0, x_2 = 0\},$$

$$\Gamma_6'' = \{(x_1,x_2)|0 < x_1 < \varepsilon^{\frac{1}{3}}, x_2 = 0\},$$

$$\Gamma_7'' = \{(x_1,x_2)|x_1 = \varepsilon^{\frac{1}{3}}, x_2 = 0\},$$

$$\Gamma_8'' = \{(x_1,x_2)|0 < x_1 \le b_1, x_2 = b_2\},$$

$$\Gamma_9'' = \{(x_1,x_2)|x_1 = 0, 0 < x_2 \le b_2\}.$$

Now let us construct a control

$$\mathbf{w}^\varepsilon(\mathbf{x}) = \mathbf{r}^\varepsilon(\mathbf{x},\mathbf{a}) \tag{5.19}$$

for $\bar{\mathcal{P}}$; see Fig. 5.5. Note that $w^\varepsilon(\mathbf{x}) = w^*(\mathbf{x})$ when $\mathbf{x} \in \Gamma_1''$. In other regions except Γ_1'', the control policy has the tendency to go toward $(\varepsilon^{\frac{1}{3}}, 0)$. This policy, while not optimal, is nearly optimal for $\bar{P}$ for sufficiently small ε, since $|\mathbf{x}^\varepsilon - \mathbf{x}^*| \le \varepsilon^{\frac{1}{3}}$. Similar as before, it is easy to show that Lemma 5.1 holds for this case.

Using Definition 5.1 and (5.4), we can also equivalently write the above control $\mathbf{w}^\varepsilon(\mathbf{x})$ as follows:

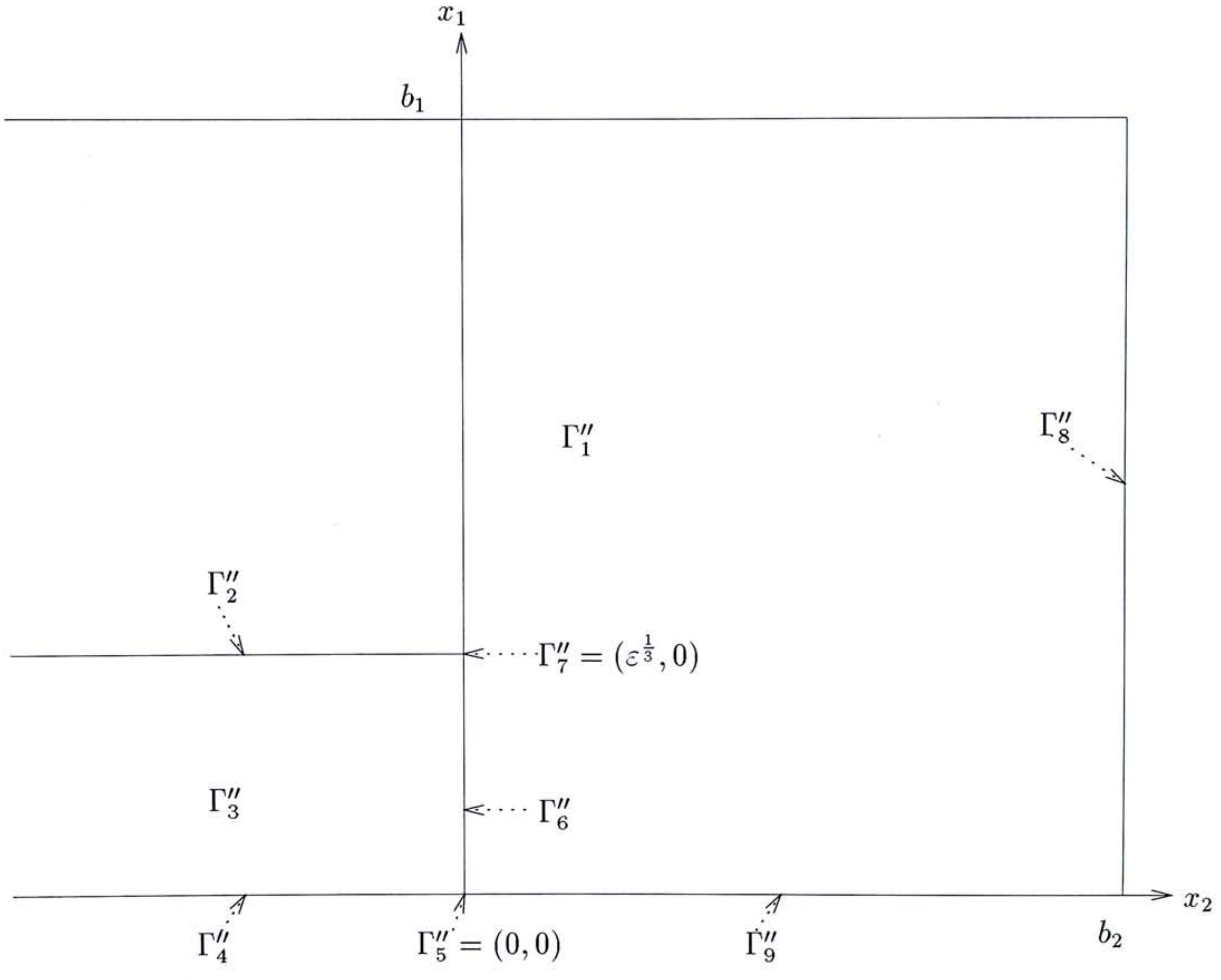

Figure 5.4: Switching manifolds for hierarchical control when $c_1 \geq c_2^+$

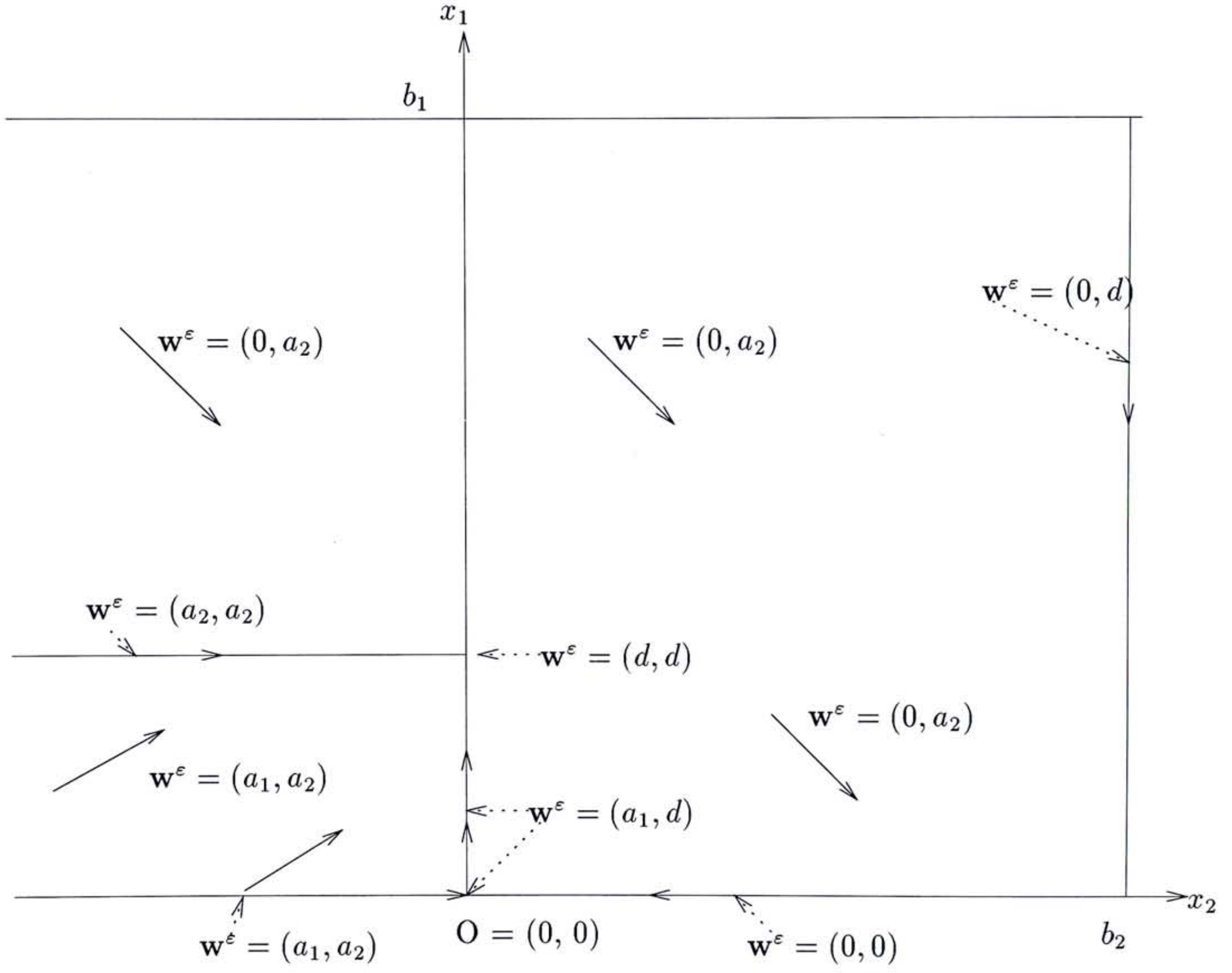

Figure 5.5: Near-optimal control $\mathbf{w}^\varepsilon$ for $\bar{\mathcal{P}}$ and trajectory movements when $c_1 \geq c_2^+$

$$\mathbf{U}^{\varepsilon}(\mathbf{x}) = ((u_1^1(\mathbf{x}), u_2^1(\mathbf{x})), (u_1^2(\mathbf{x}), u_2^2(\mathbf{x})), (u_1^3(\mathbf{x}), u_2^3(\mathbf{x})), (u_1^4(\mathbf{x}), u_2^4(\mathbf{x})))$$

$$= \begin{cases} ((0, m_2), (0,0), (0, m_2), (0,0)), & \text{if } \mathbf{x} \in \Gamma_1'', \\ ((\frac{\nu^1+\nu^3}{\nu^1+\nu^2} m_2, m_2), (\frac{\nu^1+\nu^3}{\nu^1+\nu^2} m_2, 0), (0, m_2), (0,0)), & \text{if } \mathbf{x} \in \Gamma_2'', \\ ((m_1, m_2), (m_1, 0), (0, m_2), (0,0)), & \text{if } \mathbf{x} \in \Gamma_3'', \\ ((m_1, m_2), (m_1, 0), (0, m_2), (0,0)), & \text{if } \mathbf{x} \in \Gamma_4'', \\ ((m_1, \frac{d}{\nu^1+\nu^3}), (m_1, 0), (0, \frac{d}{\nu^1+\nu^3}), (0,0)), & \text{if } \mathbf{x} \in \Gamma_5'', \\ ((m_1, \frac{d}{\nu^1+\nu^3}), (m_1, 0), (0, \frac{d}{\nu^1+\nu^3}), (0,0)), & \text{if } \mathbf{x} \in \Gamma_6'', \\ ((\frac{d}{\nu^1+\nu^2}, \frac{d}{\nu^1+\nu^3}), (\frac{d}{\nu^1+\nu^2}, 0), (0, \frac{d}{\nu^1+\nu^3}), (0,0)), & \text{if } \mathbf{x} \in \Gamma_7'', \\ ((0, \frac{d}{\nu^1+\nu^3}), (0,0), (0, \frac{d}{\nu^1+\nu^3}), (0,0)), & \text{if } \mathbf{x} \in \Gamma_8'', \\ ((0,0), (0,0), (0,0), (0,0)), & \text{if } \mathbf{x} \in \Gamma_9''. \end{cases} \tag{5.20}$$

Finally, we construct a control

$$\mathbf{u}^{\varepsilon}(\mathbf{x}, \mathbf{k}) = \mathbf{r}^{\varepsilon}(\mathbf{x}, \mathbf{k}), \tag{5.21}$$

as specified in Table 5.2.

Similarly, the $\mathbf{u}^{\varepsilon}(\mathbf{x}, \mathbf{k})$ constructed above can be shown to be an asymptotic optimal feedback control for $\mathcal{P}^{\varepsilon}$.

$\mathbf{u}^\varepsilon(\mathbf{x},\mathbf{k})$	$\mathbf{k}=\mathbf{k}^1$	$\mathbf{k}=\mathbf{k}^2$	$\mathbf{k}=\mathbf{k}^3$	$\mathbf{k}=\mathbf{k}^4$
$\mathbf{x}\in\Gamma_1''$	$(0,m_2)$	$(0,0)$	$(0,m_2)$	$(0,0)$
$\mathbf{x}\in\Gamma_2''$	$(\min\{m_1,m_2\},m_2)$	$(0,0)$	$(0,m_2)$	$(0,0)$
$\mathbf{x}\in\Gamma_3''$	(m_1,m_2)	$(m_1,0)$	$(0,m_2)$	$(0,0)$
$\mathbf{x}\in\Gamma_4''$	$(m_1,\min\{m_1,m_2\})$	$(m_1,0)$	$(0,0)$	$(0,0)$
$\mathbf{x}\in\Gamma_5''$	(m_1,d)	$(m_1,0)$	$(0,0)$	$(0,0)$
$\mathbf{x}\in\Gamma_6''$	(m_1,d)	$(m_1,0)$	$(0,d)$	$(0,0)$
$\mathbf{x}\in\Gamma_7''$	(d,d)	$(0,0)$	$(0,d)$	$(0,0)$
$\mathbf{x}\in\Gamma_8''$	$(0,d)$	$(0,0)$	$(0,d)$	$(0,0)$
$\mathbf{x}\in\Gamma_9''$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

Table 5.2: Tabular representation of $\mathbf{u}^\varepsilon(\mathbf{x},\mathbf{k})$ when $c_1 \geq c_2^+$

5.4 Concluding Remarks

In this chapter we have studied stochastic two-machine flowshops with both internal and external buffers of finite sizes. We approximated the original stochastic problem by a much easier deterministic problem, and then constructed asymptotic optimal feedback controls for the original problem based on the explicit optimal feedback control for the deterministic problem obtained in Chapter 4. We introduce a switching manifold (in the case when the average capacity of the first machine exceeds that of the second), whose purpose is to increase the inventory in the internal buffer when it is too close to being empty. This reduced the amount of capacity loss.

One of the main assumptions in this chapter is Assumption 5.2, i.e., the downstream machine is no faster than the upstream one. The opposite case is also very interesting in both theory and practice. However, this case will cause great complication even in deterministic situation [49].

Numerical evaluation of the constructed feedback controls will be conducted in next chapter to evaluate their performance.

Finally, we mention that the results in Chapters 4 and 5 are presented in [16, 17].

Chapter 6

Computational Evaluation of Hierarchical Controls

6.1 Introduction

The purpose of this chapter is to make a computational evaluation of the hierarchical controls by comparing them to the solutions obtained by some other heuristic approaches published in the literature. For this purpose, we select manufacturing systems with two failure-prone machines in tandem with an objective of minimizing a convex, piecewise-linear cost of inventories/shortages discounted over an infinite horizon studied in the last chapter. This system is relatively simple for computational evaluation and is, at the same time, sufficiently rich for possible applications. This is because, such a system has an internal buffer which must contain nonnegative inventories and the sizes of both internal and external buffers must be finite. The state constraint represents a typical complexity present in systems with machines in tandem.

We shall compare the performance of our constructed control in the last chapter, denoted as *Hierarchical Control* (HC) to a stochastic extension of *Kanban Control* (KC) developed in Sethi et al. [41] and *Two Boundary Control* (TBC) developed in van Ryzin, Lou and Gershwin [59] and Lou and Van Ryzin [35]. It turns out that TBC and KC can be shown also to be asymptotically optimal, under the conditions assumed in this chapter.

All of these policies are specified in terms of a number of parameters. KC requires two parameters, which can be termed *thresholds* in the sense of Kimemia and Gershwin [32]. HC and TBC are defined in terms of two and three parameters, respectively; strictly speaking, these cannot be called thresholds. Rather they are simplified turnpike policies, where a *turnpike* is an attractor for the optimal trajectories emanating from different initial states.

We will show by an extensive numerically study that HC performs no worse than KC and TBC, although HC is the simplest policy among the three to construct, understand, and implement.

The plan of this chapter is as follows. In Section 6.2, we state the optimal control problem for a two-machine flowshop and construct asymptotically optimal feedback controls obtained in the previous chapter, and define Kanban and Two Boundary Control Policies. Section 6.3 carries out the computational experiments for these control policies. In Section 6.4, we compare HC with KC and TBC. Finally, Section 6.5 concludes the chapter.

6.2 The Problem and Control Policies under Consideration

6.2.1 The Problem

In this chapter, we consider the following original problem $\mathcal{P}^\varepsilon$ and limiting problem $\bar{\mathcal{P}}$ studied in the last chapter:

$$\mathcal{P}^\varepsilon : \begin{cases} \text{minimize} & J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)) = E\int_0^\infty e^{-\rho t}(c_1 x_1(t) + c_2^+ x_2^+(t) + c_2^- x_2^-(t))dt \\ \text{subject to} & \begin{cases} \dot{x}_1(t) = u_1(t) - u_2(t), & x_1(0) = x_1, \\ \dot{x}_2(t) = u_2(t) - d, & x_2(0) = x_2, \\ \mathbf{u}(\cdot) \in \mathcal{A}^\varepsilon(\mathbf{x}), \end{cases} \\ \text{value function} & v^\varepsilon(\mathbf{x},\mathbf{k}) = \inf_{\mathbf{u}(\cdot)\in\mathcal{A}^\varepsilon(\mathbf{x})} J^\varepsilon(\mathbf{x},\mathbf{k},\mathbf{u}(\cdot)), \end{cases}$$

$$\bar{\mathcal{P}}: \begin{cases} \text{minimize} & J(\mathbf{x}, \mathbf{w}(\cdot)) = \int_0^\infty e^{-\rho t}(c_1 x_1(t) + c_2^+ x_2^+(t) + c_2^- x_2^-(t))dt \\ \text{subject to} & \begin{cases} \dot{x}_1(t) = w_1(t) - w_2(t), & x_1(0) = x_1, \\ \dot{x}_2(t) = w_2(t) - d, & x_2(0) = x_2, \\ \mathbf{w}(\cdot) \in \tilde{\mathcal{A}}(\mathbf{x}), \end{cases} \\ \text{value function} & v(\mathbf{x}) = \inf_{\mathbf{w}(\cdot)\in\tilde{\mathcal{A}}(\mathbf{x})} J(\mathbf{x}, \mathbf{w}(\cdot)). \end{cases}$$

For convenience in exposition, we limit the cost coefficients in the objective functional and the random process $\mathbf{k}(\varepsilon, t)$ to satisfy the following:

Assumption 6.1. $c_1 \leq c_2^+$ and $c_2^- > 0$.

Assumption 6.2. The capacity process $\mathbf{k}(\varepsilon, \cdot)$ is a Markov chain over four states $\{(m_1, m_2), (m_1, 0), (0, m_2), (0, 0)\}$ represented by the generator

$$Q^\varepsilon = \varepsilon^{-1} Q = \varepsilon^{-1} \begin{pmatrix} -2\lambda & \lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & 0 & \lambda \\ \mu & 0 & -(\lambda+\mu) & \lambda \\ 0 & \mu & \mu & -2\mu \end{pmatrix},$$

with m_1 and m_2 denoting machines' capacities of machines M_1 and M_2, respectively, and with the repair rate $\varepsilon^{-1}\mu > 0$ and the failure rate $\varepsilon^{-1}\lambda > 0$ for each machine. Moreover, Q is irreducible and satisfies

$$\min\{\mu, \lambda\} = 1.$$

Assumption 6.3. $a_1 \geq a_2 \geq d$, where $a_1 = \frac{m_1\mu}{\lambda+\mu}$ and $a_2 = \frac{m_2\mu}{\lambda+\mu}$ are the average capacities of machines M_1 and M_2, respectively.

Remark 6.1. The equilibrium distribution ν of the above Q is

$$\nu = (\frac{\mu^2}{(\lambda+\mu)^2}, \frac{\lambda\mu}{(\lambda+\mu)^2}, \frac{\lambda\mu}{(\lambda+\mu)^2}, \frac{\lambda^2}{(\lambda+\mu)^2}).$$

Let us now define various control policies that we shall use for our computational experiments.

6.2.2 Hierarchical Control (HC)

This is the control of type given in equations (5.6) and (5.9) with $\{\varepsilon^{\frac{1}{3}}, 0\}$ replaced by some $\{\theta_1(\varepsilon), \theta_2(\varepsilon)\}$ with $\theta_1(\varepsilon) \geq 0$ and $\theta_2(\varepsilon) \geq 0$, and defined as follows:

$$\mathbf{u}_H(\mathbf{x}, \mathbf{k}) = \begin{cases} (0, k_2), & \text{if } \mathbf{x} \in \Gamma_1, \\ (\min\{k_1, k_2\}, k_2), & \text{if } \mathbf{x} \in \Gamma_2, \\ (k_1, k_2), & \text{if } \mathbf{x} \in \Gamma_3, \\ (k_1, \min\{k_1, k_2\}), & \text{if } \mathbf{x} \in \Gamma_4, \\ (k_1, \min\{k_1, k_2, d\}), & \text{if } \mathbf{x} \in \Gamma_5, \\ (k_1, \min\{k_2, d\}), & \text{if } \mathbf{x} \in \Gamma_6, \\ (\min\{k_1, k_2, d\}, \min\{k_2, d\}), & \text{if } \mathbf{x} \in \Gamma_7, \\ (0, \min\{k_2, d\}), & \text{if } \mathbf{x} \in \Gamma_8, \\ (0, 0), & \text{if } \mathbf{x} \in \Gamma_9, \end{cases} \tag{6.1}$$

where, as shown in Fig. 6.1,

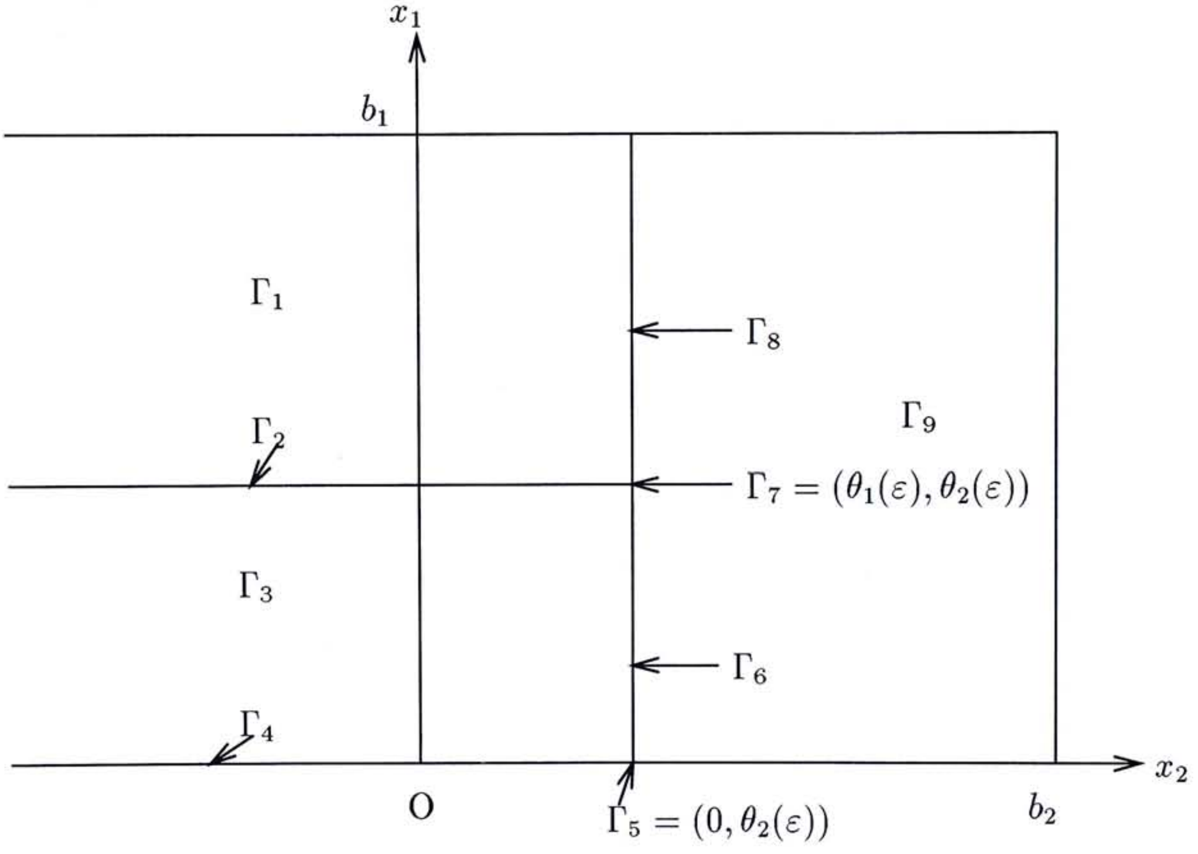

Figure 6.1: Switching manifolds and turnpike point for HC

$$\Gamma_1 = \{(x_1, x_2)|\theta_1(\varepsilon) < x_1 \leq b_1, x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_2 = \{(x_1, x_2)|x_1 = \theta_1(\varepsilon), x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_3 = \{(x_1, x_2)|0 < x_1 < \theta_1(\varepsilon), x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_4 = \{(x_1, x_2)|x_1 = 0, x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_5 = \{(x_1, x_2)|x_1 = 0, x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_6 = \{(x_1, x_2)|0 < x_1 < \theta_1(\varepsilon), x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_7 = \{(x_1, x_2)|x_1 = \theta_1(\varepsilon), x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_8 = \{(x_1, x_2)|\theta_1(\varepsilon) < x_1 \leq b_1, x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_9 = \{(x_1, x_2)|0 \leq x_1 \leq b_1, \theta_2(\varepsilon) < x_2 \leq b_2\},$$

with $(\theta_1(\varepsilon), \theta_2(\varepsilon)) \to (0, 0)$ as $\varepsilon \to 0$. We have shown in the last chapter that this control policy is asymptotically optimal.

Next we define two other types of control policies for comparison purposes.

6.2.3 Kanban Control (KC)

Kanban control policy is a threshold type policy. It is defined as follows for some $\{\theta_1(\varepsilon), \theta_2(\varepsilon)\}$ with $\theta_1(\varepsilon) \geq 0$ and $\theta_2(\varepsilon) \geq 0$.

$$\mathbf{u}_K(\mathbf{x},\mathbf{k}) = \begin{cases} (0, k_2), & \text{if } \mathbf{x} \in \Gamma'_1, \\ (\min\{k_1, k_2\}, k_2), & \text{if } \mathbf{x} \in \Gamma'_2, \\ (k_1, k_2), & \text{if } \mathbf{x} \in \Gamma'_3, \\ (k_1, \min\{k_1, k_2\}), & \text{if } \mathbf{x} \in \Gamma'_4, \\ (k_1, \min\{k_1, k_2, d\}), & \text{if } \mathbf{x} \in \Gamma'_5, \\ (k_1, \min\{k_2, d\}), & \text{if } \mathbf{x} \in \Gamma'_6, \\ (\min\{k_1, k_2, d\}, \min\{k_2, d\}), & \text{if } \mathbf{x} \in \Gamma'_7, \\ (0, \min\{k_2, d\}), & \text{if } \mathbf{x} \in \Gamma'_8, \\ (0, 0), & \text{if } \mathbf{x} \in \Gamma'_9, \\ (k_1, 0), & \text{if } \mathbf{x} \in \Gamma'_{10}, \end{cases} \tag{6.2}$$

where, as shown in Fig. 6.2,

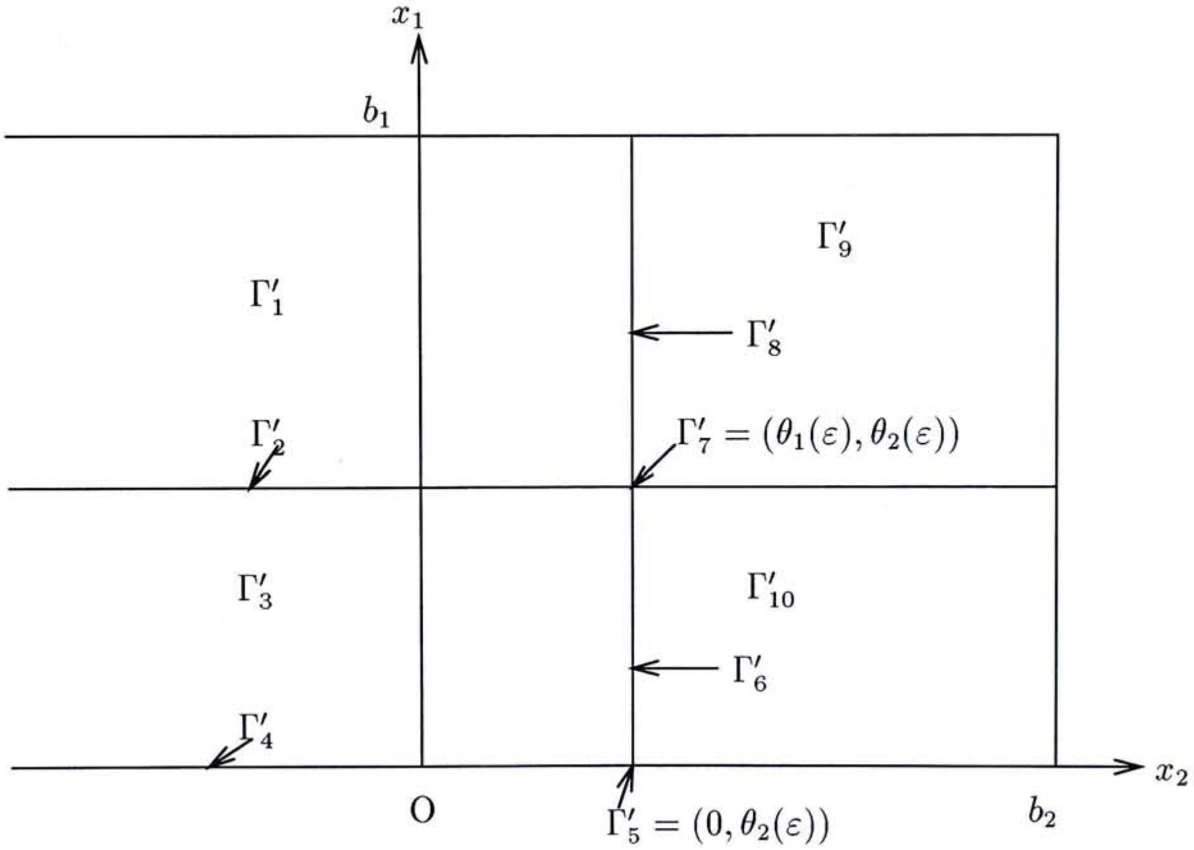

Figure 6.2: Switching manifolds and threshold values for KC

$$\Gamma_1' = \{(x_1, x_2)|\theta_1(\varepsilon) < x_1 \le b_1, x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_2' = \{(x_1, x_2)|x_1 = \theta_1(\varepsilon), x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_3' = \{(x_1, x_2)|0 < x_1 < \theta_1(\varepsilon), x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_4' = \{(x_1, x_2)|x_1 = 0, x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_5' = \{(x_1, x_2)|x_1 = 0, x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_6' = \{(x_1, x_2)|0 < x_1 < \theta_1(\varepsilon), x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_7' = \{(x_1, x_2)|x_1 = \theta_1(\varepsilon), x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_8' = \{(x_1, x_2)|\theta_1(\varepsilon) < x_1 \le b_1, x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_9' = \{(x_1, x_2)|\theta_1(\varepsilon) \le x_1 \le b_1, \theta_2(\varepsilon) < x_2 \le b_2\},$$

$$\Gamma_{10}' = \{(x_1, x_2)|0 \le x_1 < \theta_1(\varepsilon), \theta_2(\varepsilon) < x_2 \le b_2\}.$$

The Kanban control policy is an adaptation of Just-In-Time (JIT) method to our stochastic problem. Indeed Kanban policy reduces to conventional JIT when $\theta_1(\varepsilon) = \theta_2(\varepsilon) = 0$. But given unreliable machines, one can lower the cost by selecting nonnegative values for the threshold inventory levels $\theta_1(\varepsilon)$ and $\theta_2(\varepsilon)$. This is because positive inventory levels hedge against machine breakdowns. While the general idea seems to have been around, the formula (6.2) appears in Sethi et al. [41] for the first time. It should be noted that under Assumptions 6.1 - 6.3, KC is also asymptotically optimal.

We can state the following results.

Theorem 6.1. *Kanban control is asymptotically optimal for $a_1 \geq a_2$ and $c_1 \leq c_2^+$, provided $\theta_1(\varepsilon) = C_1\varepsilon^{\frac{1}{3}}$ and $\theta_2(\varepsilon) = C_2\varepsilon^{\frac{1}{3}}$ for some constants $C_1 > 0$* and $C_2 \geq 0$.

Proof. see [50] for the proof.

Remark 6.2. Sethi and Zhou [50] also proved that Kanban control is not asymptotically optimal for $a_1 > a_2$ and $c_1 > c_2^+$ even if $\theta_1(\varepsilon) = C_1\varepsilon^{\frac{1}{3}}$ and $\theta_2(\varepsilon) = C_2\varepsilon^{\frac{1}{3}}$ for some $C_1 > 0$ and $C_2 \geq 0$.

6.2.4 Two-Boundary Control (TBC)

The Two Boundary Control policy was proposed by van Ryzin, Lou and Gershwin [59] and Lou and van Ryzin [35] as a heuristic approximation of the structure of the optimal switching manifolds. It is defined as follows for some $\{\theta_1(\varepsilon), \theta_2(\varepsilon), \theta_3(\varepsilon)\}$, with $0 \leq \theta_1(\varepsilon) \leq \theta_3(\varepsilon)$ and $\theta_2(\varepsilon) \geq 0$, under an additional assumption that $m_2 \geq 2d$:

$$\mathbf{u}_T(\mathbf{x},\mathbf{k}) = \begin{cases} (0,k_2), & \text{if } \mathbf{x} \in \Gamma_1'' \cup \Gamma_2'', \\ (k_1,k_2), & \text{if } \mathbf{x} \in \Gamma_3'', \\ (k_1,\min\{k_1,k_2\}), & \text{if } \mathbf{x} \in \Gamma_4'', \\ (k_1,\min\{k_1,k_2,d\}), & \text{if } \mathbf{x} \in \Gamma_5'', \\ (k_1,\min\{k_2,d\}), & \text{if } \mathbf{x} \in \Gamma_6'', \\ (\min\{k_1,k_2,d\},\min\{k_2,d\}), & \text{if } \mathbf{x} \in \Gamma_7'', \\ (0,\min\{k_2,d\}), & \text{if } \mathbf{x} \in \Gamma_8'', \\ (0,0), & \text{if } \mathbf{x} \in \Gamma_9'', \\ (k_1,0), & \text{if } \mathbf{x} \in \Gamma_{10}'', \\ (\min\{k_1,d\},\min\{k_2,k_2+(2d-k_2)\text{sgn}(k_1)\}), & \text{if } \mathbf{x} \in \Gamma_{11}'', \\ (\min\{k_1,d\},0), & \text{if } \mathbf{x} \in \Gamma_{12}'', \\ (\min\{k_1,k_2\},k_2), & \text{if } \mathbf{x} \in \Gamma_{13}'', \end{cases} \tag{6.3}$$

where $\text{sgn}(k_1) = 1$ if $k_1 > 0$, $\text{sgn}(k_1) = 0$ if $k_1 = 0$, and as shown in Fig. 6.3,

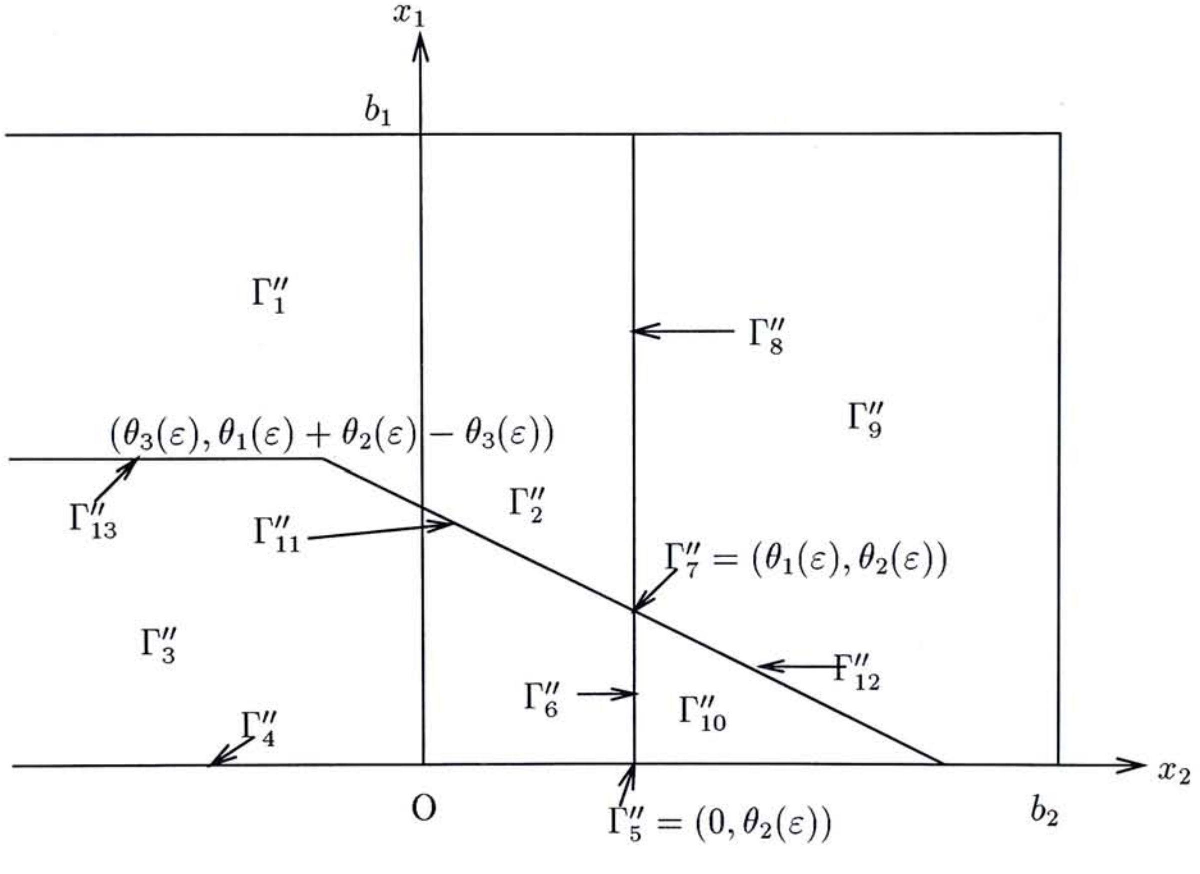

Figure 6.3: Switching manifolds and defining parameters for TBC

$$\Gamma_1'' = \{(x_1, x_2) | \theta_3(\varepsilon) < x_1 \le b_1, x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_2'' = \{(x_1, x_2) | \theta_1(\varepsilon) < x_1 \le \theta_3(\varepsilon), x_2 < \theta_2(\varepsilon), x_1 + x_2 > \theta_1(\varepsilon) + \theta_2(\varepsilon)\},$$

$$\Gamma_3'' = \{(x_1, x_2) | 0 < x_1 < \theta_3(\varepsilon), x_2 < \theta_2(\varepsilon), x_1 + x_2 < \theta_1(\varepsilon) + \theta_2(\varepsilon)\},$$

$$\Gamma_4'' = \{(x_1, x_2) | x_1 = 0, x_2 < \theta_2(\varepsilon)\},$$

$$\Gamma_5'' = \{(x_1, x_2) | x_1 = 0, x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_6'' = \{(x_1, x_2) | 0 < x_1 < \theta_1(\varepsilon), x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_7'' = \{(x_1, x_2) | x_1 = \theta_1(\varepsilon), x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_8'' = \{(x_1, x_2) | \theta_1(\varepsilon) < x_1 \le b_1, x_2 = \theta_2(\varepsilon)\},$$

$$\Gamma_9'' = \{(x_1, x_2) | 0 \le x_1 \le b_1, \theta_2(\varepsilon) < x_2 \le b_2, x_1 + x_2 > \theta_1(\varepsilon) + \theta_2(\varepsilon)\},$$

$$\Gamma_{10}'' = \{(x_1, x_2) x_1 \ge 0, \theta_2(\varepsilon) < x_2 \le b_2, x_1 + x_2 < \theta_1(\varepsilon) + \theta_2(\varepsilon)\},$$

$$\Gamma_{11}'' = \{(x_1, x_2) | \theta_1(\varepsilon) < x_1 \le \theta_3(\varepsilon), \theta_1(\varepsilon) + \theta_2(\varepsilon) - \theta_3(\varepsilon) \le x_2 < \theta_2(\varepsilon),$$
$$x_1 + x_2 = \theta_1(\varepsilon) + \theta_2(\varepsilon)\},$$

$$\Gamma_{12}'' = \{(x_1, x_2) | x_1 \ge 0, x_2 < \theta_2(\varepsilon), x_1 + x_2 = \theta_1(\varepsilon) + \theta_2(\varepsilon)\},$$

$$\Gamma_{13}'' = \{(x_1, x_2) | x_1 = \theta_3(\varepsilon), x_2 < \theta_1(\varepsilon) + \theta_2(\varepsilon) - \theta_3(\varepsilon)\}.$$

Remark 6.3. Samaratunga, Sethi and Zhou [38] merge the switching manifolds Γ_4'' and Γ_{13}'' into Γ_3'', and use $\mathbf{u}_T = (k_1, k_2)$ as the control policy. It should be noted that this control will not be admissible when machine M_1 is down, machine M_2 is up, and $x_1 = 0$.

If $\theta_1(\varepsilon), \theta_2(\varepsilon), \theta_3(\varepsilon) \to 0$ as $\varepsilon \to 0$, then TBC turns out to be asymptotically optimal under Assumptions 6.1 - 6.3 and $m_2 \geq 2d$ [59, 35]. Note also that TBC is defined in Lou and van Ryzin [35] only under these assumptions.

6.2.5 Similarities and Differences between HC, KC, and TBC

Before commencing our computational experiments, let us pause for a moment to briefly examine differences and similarities for the policies under consideration. In each of these policies, the idea is to get fairly quickly to a desirable point in the state space and then stay close to it thereafter. This point can be called a *turnpike* or a *hedging point*; see Haurie and van Delft [26] for further details. How we choose this point in each of the three policies will be described shortly. The structural difference between the three polices is the ways to get to the turnpike. The difference between HC and KC is that the switching manifold $x_1 = \theta_1(\varepsilon)$ in KC is applicable in HC only in the region $x_2 \leq \theta_2(\varepsilon)$. Note that the manifold in HC arises from the consideration of avoiding the capacity loss incurred wherever $x_1 = 0$. The presence of the manifold decreases the occupancy measure of $x_1 = 0$. The manifold $x_1 = \theta_1(\varepsilon)$ in KC, on the other hand, arises from the very local nature of a *threshold type policy*, which it is by definition. That is, whenever $x_i(t)$ is below (above) the threshold $\theta_i(\varepsilon)$, $i = 1$ or 2, the system must behave in a way to increase (decrease) it. Because of this reason, even when $x_2(t)$ is large, machine M_1 will produce when $x_1(t) < \theta_1(t)$. Clearly, a hedge is not needed when we have a large positive surplus. Machine M_1 in HC, on the other hand, will produce only when $x_1(t) < \theta_1(\varepsilon)$, provided $x_2(t) \leq \theta_2(\varepsilon)$. It is clear that HC is *not* a threshold type policy; it is simply a two-parameter policy dictated from capacity loss considerations. TBC is also not of threshold type. It is a three parameter policy, whose first two parameters define the hedging points. An additional degree of freedom is needed in defining TBC to more closely approximate the optimal switching manifolds when both machines are up. It should be noted that even with the additional parameter $\theta_3(\varepsilon)$, TBC is not a generalization of HC. With $\theta_3(\varepsilon) = \theta_1(\varepsilon)$, it does, however, reduce to HC in all but the triangular region $\{x_2 > \theta_2(\varepsilon)\} \cap \{0 \leq x_1\} \cap \{x_1 + x_2 \leq \theta_1(\varepsilon) + \theta_2(\varepsilon)\}$.

6.3 Computational Results

Now, we use problem series P of van Ryzin, Lou and Gershwin [59] as our test data. Data representing the problem series P is given in Table 6.1. The sizes of both internal and external buffers are 50 for all problems P1-P4. We also assume that $\varepsilon = 10.0$, in order to be compatible with van Ryzin et al. [59]. This problem series uses a discount rate of 10%, i.e., $\rho = 0.1$.

Since HC, KC, or TBC are not optimal, one would like to imagine each of them

Problem Name	λ	μ	m_1	m_2	d	c_1^+	c_2^+	c_2^-
P1	1.0	5.0	2.0	2.0	1.0	1.0	2.0	10.0
P2	1.0	2.0	2.0	2.0	1.0	1.0	2.0	10.0
P3	1.0	1.5	2.0	2.0	1.0	1.0	2.0	10.0
P4	1.0	1.2	2.0	2.0	1.0	1.0	2.0	10.0

Note: Both machines have identical parameters λ, μ and m.

Table 6.1: Problem series P for the two-machine case

to be optimal within a certain class of feedback policies. This is unfortunately not possible in the two-machine case with the *discounted cost criterion*, since we cannot obtain parameters of these policies in a way so that they are optimal for every given initial state.

Recognizing the complication, in our empirical studies, we therefore apply two different criteria developed by Samaratunga, Sethi and Zhou [38] to obtain parameter values for HC, KC, and TBC for comparison purposes. The first criterion is to choose an initial state and find the minimum cost for each of the three policies beginning with that state. The parameter values that accomplish this will be known as the *best such for the given initial state.* The second criterion is to compute the costs of various policies defined by the best parameters for the initial state (0,0).

We perform our simulations on a SUN-Station SPARCclassic with an event-driven simulator designed by us specifically for flowshop production. Simulation runs are all started where the machines are up and idle. 100 replications with different random seeds are used to estimate the sample means and 95% confidence intervals.

We now describe the procedures used to compute $\theta_i(\varepsilon)$ for HC, KC, and TBC under the first criterion. For the HC policy as well as the KC policy, we carry out a two dimensional downhill simplex search to obtain the best values for $(\theta_1(\varepsilon), \theta_2(\varepsilon))$. Refer to Simmons [53] for an introduction to such numerical solution approaches. For TBC, we use a three dimensional downhill simplex method to search for the three parameters $\theta_1(\varepsilon), \theta_2(\varepsilon)$, and $\theta_3(\varepsilon)$ starting with $\theta_1(\varepsilon)$ and $\theta_2(\varepsilon)$ as those obtained for HC and $\theta_3(\varepsilon) = \theta_1(\varepsilon) + \theta_2(\varepsilon)$.

We compare HC with the other two policies, namely, KC, and TBC, as well as study their asymptotic behavior as ε decreases. For this purpose we select Problem P1. Recall that the qualitative similarities and differences in these policies have already been discussed in Section 6.2.

In Table 6.2, we select different initial states and use the first criterion to find the best parameter values for different initial states for each of HC, KC, and TBC; Table 6.3 provides the ratios of costs reported in Table 6.2. Table 6.4 uses the second criterion so that the parameter values used for all different initial states are the ones that appear in Table 6.2 in the row with the initial state (0,0).

In Table 6.5, we obtain costs of HC for Problems P1-P4 with $\varepsilon = 10, 1$, and

Initial State (x_1, x_2)	Control Policy					
	HC		KC		TBC	
	Cost	Parameters	Cost	Parameters	Cost	Parameters
(0,50)	801.34	(0.00,1.00)	801.34	(0.00,1.00)	801.34	(0.00,1.00,0.00)
(0,20)	228.38	(2.51,1.52)	235.51	(0.00,3.00)	226.87	(2.21,2.01,3.71)
(0,10)	74.20	(1.00,0.50)	78.83	(0.00,3.22)	73.86	(2.36,1.76,4.38)
(0,5)	23.53	(2.20,2.01)	23.90	(2.29,1.81)	22.41	(2.58,1.55,4.72)
(0,0)	11.80	(2.75,1.58)	11.80	(2.75,1.58)	11.77	(2.65,1.65,4.20)
(0,-5)	193.24	(2.10,1.00)	193.24	(2.10,1.00)	192.68	(2.50,1.81,4.87)
(0,-10)	540.66	(3.25,1.00)	540.66	(3.25,1.00)	539.08	(3.29,2.02,4.89)
(0,-20)	1446.03	(2.50,1.50)	1446.03	(2.50,1.50)	1445.43	(5.77,1.12,6.19)
(20,20)	416.37	(1.00,1.00)	416.37	(0.75,2.50)	415.56	(2.48,2.15,3.19)
(10,10)	150.05	(3.22,1.61)	150.05	(3.22,1.60)	149.88	(2.23,2.22,2.24)
(5,5)	43.95	(2.70,1.64)	43.95	(2.49,1.79)	43.36	(2.57,1.82,2.98)
(5,-5)	177.40	(2.25,1.00)	177.40	(2.25,1.00)	176.45	(2.98,2.04,5.00)
(10,-10)	526.92	(2.25,0.54)	526.92	(2.25,0.54)	526.26	(2.00,1.00,2.00)
(20,-20)	1470.13	(2.00,1.00)	1470.13	(2.00,1.00)	1470.13	(2.00,0.00,2.00)

Note: Simulation relative error $\leq \pm 2\%$, Confidence level = 95%. Comparison is carried out for the same machine failure breakdown sample paths for all policies.

Table 6.2: Comparison of control policies with Criteria I

0.1.

6.4 Comparison of HC with Other Polices

In this section, we analyze these computational results and compare HC with KC and TBC.

HC vs. TBC

From Tables 6.2, 6.3 and 6.4, we see that the costs of HC and TBC are quite

Initial State	Cost Ratio	
(x_1, x_2)	$\frac{KC}{HC}$	$\frac{TBC}{HC}$
(0,50)	1.0000	1.0000
(0,20)	1.0312	0.9934
(0,10)	1.0624	0.9954
(0,5)	1.0157	0.9524
(0,0)	1.0000	0.9975
(0,-5)	1.0000	0.9971
(0,-10)	1.0000	0.9971
(0,-20)	1.0000	0.9996
(20,20)	1.0000	0.9981
(10,10)	1.0000	0.9989
(5,5)	1.0000	0.9866
(5,-5)	1.0000	0.9946
(10,-10)	1.0000	0.9987
(20,-20)	1.0000	1.0000

Table 6.3: Cost ratios corresponding to Table 6.2

Initial State (x_1, x_2)	Control Policy				
	Cost			Cost Ratio	
	HC	KC	TBC	$\frac{KC}{HC}$	$\frac{TBC}{HC}$
(0,50)	801.44	826.51	801.44	1.0313	1.0000
(0,20)	228.73	247.73	227.31	1.0631	0.9938
(0,10)	77.72	86.11	75.58	1.1080	0.9725
(0,5)	25.20	25.49	24.01	1.0115	0.9528
(0,0)	11.80	11.80	11.77	1.0000	0.9975
(0,-5)	195.50	195.50	195.67	1.0000	1.0009
(0,-10)	541.06	541.06	541.14	1.0000	1.0001
(0,-20)	1446.50	1446.50	1446.94	1.0000	1.0003
(20,20)	416.85	416.85	416.93	1.0000	1.0002
(10,10)	151.53	151.53	151.60	1.0000	1.0005
(5,5)	44.21	44.21	44.07	1.0000	0.9968
(5,-5)	181.47	181.47	185.43	1.0000	1.0218
(10,-10)	530.93	530.93	531.17	1.0000	1.0005
(20,-20)	1472.83	1472.83	1477.32	1.0000	1.0030

Note: Simulation relative error $\leq \pm 2\%$, Confidence level = 95%. Comparison is carried out for the same machine failure breakdown sample paths. Therefore, the relative comparison is free of statistical uncertainty. The constant thresholds are obtained from the (0,0) initial inventory row of Table 6.2.

Table 6.4: Comparison of control policies with Criterion II

Problem Name	Control Policy	ε			lim
		10.0	1.0	0.1	$\varepsilon \to 0$
P1	HC	61.62	14.18	4.11	0
P2	HC	294.68	27.31	19.89	0
P3	HC	527.51	62.12	21.45	0
P4	HC	725.23	85.28	30.25	0

Note: Simulation relative error $\leq \pm 2\%$, Confidence level = 95%.

Table 6.5: Asymptotic behavior with respect to ε of costs of HC with initial $\mathbf{x} = (0,0)$ and $\mathbf{k} = (0,2)$.

close to one another. A more detailed comparison reveals that sometimes HC is slightly better and sometimes it is the other way around. Both of these situations are theoretically possible. Of course, if initial $x_2 \leq \theta_2$ for HC (which covers the situation of $x_2 \leq 0$), the trajectory under HC will stay in the region $x_2 \leq \theta_2$. In this case, as indicated in Section 6.2, TBC can duplicate the performance of HC by setting its θ_1 and θ_2 as those of HC and its $\theta_3 = \theta_1$ of HC. However, with three policy parameters to choose, provided the parameter search procedure is accurate, the cost of TBC cannot be larger than that of HC. One can see this in Table 6.2 for $(x_1, x_2) = (0,0), (0,-5), (0,-10), (0,-20), (5,-5), (10,-10)$ and (20,-20). It is also important to point out that the best value of θ_2 for HC obtained in Table 6.2 depends on the initial state (x_1, x_2). Therefore, the relative performance of HC and TBC cannot be decided *a priori* when $x_2 > 0$.

While costs of HC and TBC are not significantly different, it should be emphasized that HC is a much simpler policy than TBC is, with regard to the computation of policy parameters as well as to their implementation. When it comes to implementation, both HC and TBC are technically not difficult to implement. However, TBC given in (6.3) is quite complicated to understand especially along the 45^o manifold in Fig. 6.3, whereas HC does not have this complication. Moreover, it should be noted as in Buzacott and Shantikumar [8] that the lack of understanding of a control policy by the operator may at times outweigh the benefits that could be obtained by implementing a more optimal complicated policy over a less optimal simpler policy.

Finally, the construction of TBC in Section 6.2 requires an additional assumption, $m_2 \geq 2d$, not needed for HC. Moreover, HC can be easily defined in cases when some of the assumptions made in this chapter do not hold , see Chapter 5.

HC vs. KC

Let us now compare HC and KC in detail. Of course, if the initial state is in a shortage situation ($x_2 \leq 0$), then HC and KC must have identical costs. This can be easily seen in Table 6.2 or Table 6.4 when initial (x_1, x_2) = (0, -5), (0, -10), (0, -20), (5, -5), (10, -10) and (20, -20).

On the other hand, if the initial surplus is positive, cost of HC is either the same as or slightly smaller than the cost of KC, as should be expected. This is because, KC being a threshold type policy, the system approaches θ_1 even when there is large positive surplus, implying higher inventory costs. In Tables 6.2, 6.3 and 6.4, we can see this in rows with initial $(x_1, x_2) =$ (0, 5), (0, 10), (0, 20), and (20, 20). Moreover, by the same argument the values of θ_1 for KC must not be larger than those for HC in Table 6.2. Indeed, in cases with large positive surplus, the value of θ_1 for KC must be smaller than that for HC. Furthermore, in these cases with positive surplus, the cost differences in Table 6.4 must be larger than those in Table 6.2, since Table 6.4 uses hedging point parameters that are best for initial $(x_1, x_2) = (0,0)$. These parameters are the same for HC and KC. Thus, the system with an initial surplus has higher inventories in the internal buffer with KC than with HC.

We also note that if the surplus is very large, then KC in order to achieve lower inventory costs sets $\theta_1 = 0$, with the consequence that its cost is the same as that for HC. For example, this happens when the initial $(x_1, x_2) = (0,50)$; see Table 6.2. As should be expected, the difference in cost for initial $(x_1, x_2) = (0,50)$ in Table 6.4 is quite large compared to the corresponding difference in Table 6.2.

Asymptotic Behavior of HC

Before summarizing the chapter in the next section, let us make some important remarks regarding asymptotic optimality. The main intuition behind the asymptotic optimality of HC is that these policies *try* to keep the system away from $x_1 = 0$ boundary, when $x_2 < 0$, without letting x_1 get too big. This is so because with $a_1 \geq a_2$ assumed in the chapter, the optimal solution of the limiting problem $\bar{\mathcal{P}}$ stays on $x_1 = 0$, once it gets there. This is not the case, however, when $a_1 < a_2$ and we do not therefore know how to construct HC as summarized in Chapter 5.

It is also clear from the above that asymptotic optimal controls are not unique.

Moreover, the theory provides only the order of the error bounds. Generally speaking, the higher the order,the faster the convergence, and the better the policy. The main benefit of the asymptotic analysis lies in the identification of the essential structural requirement for asymptotic optimality, as has been done in Section 6.2 for the problem under consideration. Beyond this, we must resort to computational experiments for further evaluation of different asymptotic optimal policies.

In Table 6.5, we have carried out a computational asymptotic analysis as ε decreases. Since the initial condition is assumed to be (0,0), the deterministic value function is zero, to which costs of HC will converge as $\varepsilon \to 0$. The results provide some idea about the rate at which HC are converging to the value 0 as $\varepsilon \to 0$.

6.5 Concluding Remarks

In this chapter, we have compared the performance of hierarchical control policies to some other existing control policies in the literature in the context of two-machine flowshops with unreliable machines.

We have shown that the hierarchical controls perform as well as or better than Kanban controls. There does not appear to be a significant difference in the costs of hierarchical controls and two-boundary controls. Moreover, hierarchical controls are simpler to construct, to understand, and perhaps to implement than are two boundary controls. More importantly, however, the additional degree of freedom in defining TBC does not provide much of an advantage, and that two parameters defining HC are in most cases sufficient to construct effective policies in practice.

Finally, we should emphasize that it is not difficult to construct hierarchical controls for larger systems. We have chosen to deal with a two-machine system because of simplicity in exposition.

Chapter 7

Conclusions and Future Research

In this thesis, we have considered the cost-minimizing manufacturing systems with stochastic discrete events. In particular, the rates at which production machines break down and get repaired are much higher than the rate of discounting costs. The sizes of both internal and external buffers are finite. We have used the hierarchical control approach to study the open-loop and feedback production planning for the manufacturing systems.

In what follows we briefly review the main results that have been obtained. In Chapter 3, we have dealt with the open-loop production planning for stochastic two-machine flowshops with finite buffers. The methodology is based on the state constraint domain approximation and weak-Lipschitz property developed in this chapter. A deterministic limiting problem in which the stochastic machines capacities are replaced by their mean capacities is introduced. We have then shown that the value function of the original problem is close enough to that of the limiting problem as the rate of change in machines' states approaches is very large. Moreover, open-loop production policies for the original problem have been explicitly constructed from optimal or near-optimal policies of the limiting problem in a way which guarantees their asymptotic optimality, and the error estimate for the constructed policies has been obtained. Algorithms of constructing these polices have been presented.

The controls constructed in Chapter 3 are not feedback as they respond neither to the surplus nor to the inventory level except when the internal buffer is full or empty or the external buffer is full. However, this kind of controls, which may be called partially open-loop controls, are of theoretical importance in deriving one of the main results, namely Theorem 3.4, which states how close to one another the original and the limiting problems are. With regards to practical implementation, however, both intuition and simulation suggest that feedback controls would perform better than our partially open-loop controls, especially when there is uncertainty present in the system. So in the rest of this thesis, we studied the feedback controls for the case of two-machine flowshop with linear inventory/shortage cost.

As mentioned earlier, the hierarchical control approach approximates the original stochastic problem by a much easier deterministic problem. In Chapter 4, we have first obtained explicit optimal feedback controls for the deterministic problem on a rigourous basis. Based on it, we have then analytically constructed suitable feedback controls for the original stochastic problem and proved their asymptotic optimality in Chapter 5.

The hierarchical planning approach is further evaluated computationally in Chapter 6, where the constructed policy is compared with other heuristic policies, Kanban control policy and two-boundary control policy, existing in the literature. We have shown that the hierarchical controls perform as well or better than Kanban controls, and hierarchical controls are simpler to construct, to understand, and perhaps to implement than are two boundary controls.

The remainder of this chapter is devoted to indicating some important open problems for my future research.

It is first worth pointing out that unlike the previous methods that work specifically for some special problems [45, 47, 48], the constraint domain approximation method is more general and may adapt to N-machine tandem systems and even job shops as in Sethi and Zhou [42] with finite buffers to obtain near-optimal open-loop controls.

Another interesting case is that the machines capacity process depends on the production rates. Soner [55] and Sethi and Zhang [43] studied this case for unconstrained system, and obtained asymptotic results. It remains an open problem to handle the control-dependent capacity process with constraints on inventory processes.

Note that in Chapters 4, 5, and 6, we considered the two-machine flowshop with linear inventory/backlog cost. It remains an outstanding open problem to investigate the production planning for the two-machine flowshop, or even general N-machine jobshop, with general inventory/backlog cost.

It should also be noted that construction of asymptotic optimal feedback controls for general dynamic stochastic flowshops and jobshops remains a wide open research area. We are able to treat simple two-machine flowshops, since we were able to explicitly solve the limiting problem. What one would like to have is a general methodology that can construct provably asymptotic optimal feedback controls, without requiring a detailed characterization of the optimal feedback controls of the limiting problems. It should be indicated that such a methodology is available for open loop controls (see [45]). This methodology, which is based on the Lipschitz property of open loop controls, cannot be extended, however, to feedback controls that are not likely to be Lipschitz.

Finally, we would like to mention that Zhou and Sethi [62] consider the aggregate systems, which is a stochastic extension of the classical HMMS model [29], with nonlinear dynamics and non-separable cost. They use a maximum principle approach in order to construct asymptotic optimal open-loop controls. It would be interesting to consider non-separable cost of surplus and production and obtain asymptotic optimal feedback controls.

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