Title
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Permalink
https://escholarship.org/uc/item/59b0h927

Journal
Physical Review D, 55(2)

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Publication Date
1996-06-01

LBL-34948
UC-414
Preprint

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June 1996
Submitted to *Physical Review D*

LBL-34948
UCB-PTH-93/37
ITP-SB-95-38
hep-th/9606052
June, 1996

SUPERGRAVITY AT ONE LOOP II: CHIRAL AND YANG-MILLS MATTER*

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Abstract

We present the full calculation of the divergent one-loop contribution to the effective boson Lagrangian for supergravity, including the Yang-Mills sector and the helicity-odd operators that arise from integration over fermion fields. The only restriction is on the Yang-Mills kinetic energy normalization function, which is taken diagonal in gauge indices, as in models obtained from superstrings.

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grants PHY-95-14797, PHY-90-21139 and PHY-93-09888.

1. Introduction

Understanding the structure of the divergences in supergravity is a necessary step in determining the counterterms [1], [2], [3] that are needed to fully restore modular invariance in an effective supergravity theory from superstrings. The determination of these loop corrections may also provide a guide to the construction of an effective theory for a composite chiral multiplet that is a bound state of strongly coupled Yang-Mills superfields, which in turn could shed light on gaugino condensation as a mechanism for supersymmetry breaking.

In a recent paper [4] (hereafter referred to as I), we gave the divergent contributions to the bosonic Lagrangian in a general supergravity theory coupled to chiral matter, in a general bosonic background, averaged over quantum fermion helicities. That work extended and completed the results of several earlier calculations [5]–[8]. In particular, using specific choices of the gauge fixing and of the expansion of the action, we were able to cast the results in an especially simple form in which most of the one-loop corrections can be interpreted in terms of renormalizations. In the present paper we extend these results to incorporate the Yang-Mills sector [9], including helicity-odd operators that arise from integration over quantum fermions. Our results are completely general, except that we assume that the tree-level gauge kinetic energy normalization function $f(z)$ [10], where z represents the complex scalar fields of the theory, is proportional to the unit matrix. This is the case for all known theories derived from superstrings, up to possible multiplicative constants for different factor gauge groups that correspond to higher affine levels [11]. This modification is easily incorporated into our formalism, as explained in Section 5.

The generalization of the results of I to the more general case considered here can be summarized as follows. We define an operator of dimension d as a Kähler invariant operator whose term of lowest dimension is d, where scalar and Yang-Mills fields are assigned the canonical dimension of unity. Then, among the ultra-violet divergent terms generated at one loop, all operators of dimension 6 or less (as well as many operators of dimension 8) that involve neither the Kähler curvature nor derivatives of the gauge kinetic function can be absorbed by field redefinitions, interpreted as renormalizations of the Kähler potential, or take the form $F_{ab}(z,\bar{z})\left(W^aW^b\right)_F+$ h.c., where W^a is a chiral Yang-Mills supermultiplet, the subscript denotes the F-component, and the matrix-valued function $F_{ab}(z,\bar{z})$ is not in general holomorphic. The remaining terms of dimension 8 and higher must be interpreted as arising from higher order spinorial derivatives of superfield operators.

As noted in I, the effective cut-off for effective theories derived from superstrings is field dependent [3], [12], [13]; moreover the field dependence is different for loop corrections arising from different sectors of the theory [3], [13]. As in I we use here a single cut-off and neglect its derivatives; terms involving derivatives of the cut-off have a different dependence on the moduli and must be considered together with terms that are one-loop finite. Our results, some of which are collected in the appendix, are presented in such a way that the contributions from different sectors can be isolated and the corresponding Pauli-Villars contributions can easily be evaluated.

In Section 2 we discuss gauge fixing and the definition of the action expansion and in Section 3 we evaluate the helicity-odd fermion loop contributions. Our result for the one-loop corrected effective action is given in Section 4, and applied to generic models from string theory in Section 5. We summarize our results and discuss applications in Section 6.

In I we included appendices that define our conventions and list the operators that appear in the quantum action as defined by our gauge fixing and expansion prescriptions, as well as the traces of products of these operators that determine the divergent terms in the effective one loop action. Appendix C of this paper extends that compilation to include operators involving the Yang-Mills background field and new operators arising from integration over Yang-Mills quantum fields. Additional conventions and techniques used in the evaluation of helicity-odd fermion traces are included in Appendix A.

In Appendix B we specify our Yang-Mills sign conventions and list relations among the covariant scalar derivatives of the Kähler potential K, the superpotential W and the gauge field normalization function f that follow from gauge invariance of these functions and that are useful in evaluating traces. Corrections to I are included in footnotes to the text.

2. Gauge Fixing and the Expansion of the Action

Our gauge fixing procedure is described in I. Here we generalize the formalism of I to the case $x \neq$ constant, where $x = \mathrm{Re} f(z)$ is the inverse squared gauge coupling. In the general supergravity Lagrangian [10], the function $f_{ab}(z)$, where a, b are gauge indices, that determines the inverse squared gauge coupling constant, is matrix-valued. Throughout this paper we set

$$f_{ab}(z) = \delta_{ab} f(z) \equiv \delta_{ab}\,(x + iy)\,.$$

The Yang-Mills gauge fixing prescription is modified when $x \neq$ constant, and, since we are now including background as well as quantum Yang-Mills fields, gauge-graviton ghost mixing must be included. We discuss only gauge fixing of the bosonic sector in this section. The fermion sector gauge fixing is unchanged[1] from that defined in I, and is summarized in Appendix C.2. Our gauge sign conventions are those of [10] and are defined in Appendix B.

The gauge-fixed Lagrangian is defined by[2]

$$\mathcal{L} \to \mathcal{L} + \mathcal{L}_{gf}, \quad \mathcal{L}_{gf} = -\frac{\sqrt{g}}{2} C_A Z^{AB} C_B, \quad Z = \begin{pmatrix} \delta^{ab} & 0 \\ 0 & -g^{\mu\nu} \end{pmatrix}, \quad C = \begin{pmatrix} C_a \\ C_\mu \end{pmatrix},$$

[1] There are some sign errors in the fermionic part of the Lagrangian and gauge fixing terms given in I that are corrected in Appendix C of this paper; they do not affect the results of I.

[2] There is a factor 2 missing in the last term in (2.6) of I.

$$C^a = \mathcal{D}''^\mu \hat{\mathcal{A}}^a_\mu + \frac{i}{\sqrt{x}} K_{i\bar{m}} \left[(T^a \bar{z})^{\bar{m}} \hat{z}^i - (T^a z)^i \hat{z}^{\bar{m}} \right],$$

$$\sqrt{2} C_\mu = \left(\nabla^\nu h_{\mu\nu} - \frac{1}{2} \nabla_\mu h^\nu_\nu - 2\mathcal{D}_\mu z^I Z_{IJ} \hat{z}^J + 2\mathcal{F}^a_{\mu\nu} \hat{\mathcal{A}}^\nu_a \right), \qquad (2.1)$$

where hatted variables refer to quantum fields and unhatted ones refer to background fields, $h_{\mu\nu}$ is the quantum part of the space-time metric whose classical part is $g_{\mu\nu}$, and $K_{i\bar{m}}$ is the Kähler metric, which here is a function of the background fields. Following [9] we have introduced canonically normalized Yang-Mills fields:

$$\mathcal{A}_\mu = \sqrt{x} A_\mu, \quad \hat{\mathcal{A}}_\mu = \sqrt{x} \hat{A}_\mu, \quad \mathcal{F}_{\mu\nu} = \sqrt{x} F_{\mu\nu}, \quad \sqrt{x} \mathcal{D}_\mu A_\nu = \mathcal{D}'_\mu \mathcal{A}_\mu, \quad (2.2)$$

and we have adopted the shorthand notation

$$\mathcal{D}'_\mu = \mathcal{D}_\mu - \frac{\partial_\mu x}{2x}, \quad \mathcal{D}''_\mu = \mathcal{D}_\mu + \frac{\partial_\mu x}{2x}, \qquad (2.3)$$

where $\mathcal{D}_\mu$ is the gauge and general coordinate invariant derivative. Under a gauge transformation with parameter $\beta = T_a \beta^a$ and fixed background fields we have, neglecting terms of order $\hat{z}, \hat{A}$:

$$\delta \hat{z}^i = -i(\beta z)^i, \quad \delta \hat{z}^{\bar{m}} = +i(\beta \bar{z})^{\bar{m}}, \quad \delta \hat{\mathcal{A}}^a_\mu = \sqrt{x} \mathcal{D}_\mu \beta^a. \qquad (2.4)$$

If we implement the gauge fixing condition in the usual way, the ghost determinant contains a factor $\mathrm{Det}^{\frac{1}{2}} x$ that translates into a quartically divergent term proportional to $\mathrm{Tr} \ln x$ in the effective action. Note however that we have rescaled the quantum Yang-Mills fields [9] [see (2.2) above] and the quantum gaugino fields [5] (see Appendix C.2 below) in order to canonically normalize their kinetic energy. If we rescale the gauge parameter in the same way as the Yang-Mills supermultiplet, and take, instead of β, the gauge parameter

$$\gamma = \sqrt{x} \beta, \quad \sqrt{x} \mathcal{D}_\mu \beta = \mathcal{D}'_\mu \gamma,$$

we get

$$\delta \hat{\mathcal{A}}_\mu = \mathcal{D}'_\mu \gamma, \quad \delta \hat{z}^i = -\frac{i}{\sqrt{x}} (\gamma z)^i, \quad \delta \hat{z}^{\bar{m}} = +\frac{i}{\sqrt{x}} (\gamma \bar{z})^{\bar{m}}, \qquad (2.5)$$

and no $\mathrm{Tr}\ln x$ term is generated in the ghost determinant. We therefore adopt the prescription (2.5).

Under a general coordinate transformation $x \to x' = x + \epsilon$, we have

$$\delta\hat{z}^i = \epsilon^\mu \partial_\mu z^i, \quad \delta\hat{\mathcal{A}}_\nu = \sqrt{x}\left(\epsilon^\sigma \nabla_\sigma A_\nu + A_\sigma \nabla_\nu \epsilon^\sigma\right),$$

which is general coordinate, but not gauge, covariant. To obtain a manifestly gauge covariant result, we add a compensating gauge transformation with parameter $\gamma^a(\epsilon^\mu) = -\epsilon^\mu \mathcal{A}^a_\mu$, giving

$$\delta\hat{z}^i = \epsilon^\mu \mathcal{D}_\mu z^i, \quad \delta\hat{\mathcal{A}}_\nu = \epsilon^\sigma \mathcal{F}_{\sigma\nu}. \tag{2.6}$$

Then, relabelling the gauge parameter as $\epsilon_a \equiv \gamma_a$, the ghost determinant M is obtained in the usual way as

$$M^A_B = \frac{\partial}{\partial \epsilon_A}\delta C_B, \tag{2.7}$$

where the variation δC is determined from

$$\delta\hat{z}^i = -\frac{i}{\sqrt{x}}(T_b z)^i \epsilon^b + \epsilon^\mu \mathcal{D}_\mu z^i, \quad \delta\hat{\bar{z}}^{\bar{m}} = \frac{i}{\sqrt{x}}(T_b \bar{z})^{\bar{m}}\epsilon^b + \epsilon^\mu \mathcal{D}_\mu \bar{z}^{\bar{m}},$$

$$\delta\hat{\mathcal{A}}^a_\mu = \mathcal{D}'_\mu \epsilon^a + \epsilon^\sigma \mathcal{F}^a_{\sigma\mu}, \quad \delta h_{\mu\nu} = \nabla_\nu \epsilon_\mu + \nabla_\mu \epsilon_\nu. \tag{2.8}$$

This gives a contribution to the gauge-fixed Lagrangian:

$$\begin{aligned}
g^{-\frac{1}{2}}\mathcal{L}_{gh} &= \bar{c}^B M^A_B c^0_A \equiv \bar{c}Z\left(\hat{D}^2 + H_{gh}\right)c \\
&= \bar{c}^b\left[(\mathcal{D}''_\mu \mathcal{D}'^\mu)^a_b + q^a_I q^I_b\right]c_a - \bar{c}^\nu\sqrt{2}\left[\mathcal{D}''^\mu \mathcal{F}^a_{\nu\mu} + q^a_I\left(\mathcal{D}_\nu z^I\right)\right]c_a \\
&\quad -\bar{c}^\mu\left[\nabla^2 g_{\mu\nu} - r_{\mu\nu} - 2\left(\mathcal{D}_\mu z^I\right)Z_{IJ}\left(\mathcal{D}_\nu z^J\right) + 2\mathcal{F}^a_{\mu\rho}\mathcal{F}_{a\nu}{}^\rho\right]c^\nu \\
&\quad -\bar{c}^a\sqrt{2}\left[\left(\mathcal{D}_\mu z^I\right)q_{aI} - \mathcal{F}_{a\mu\nu}\mathcal{D}'^\nu\right]c^\mu, \quad c^a_0 = c^a, \quad c^\mu_0 = -\sqrt{2}c^\mu, \\
q^a_i &= \frac{i}{\sqrt{x}}(T^a \bar{z})^{\bar{m}} K_{i\bar{m}}, \quad q^i_a = -\frac{i}{\sqrt{x}}(T_a z)^i.
\end{aligned} \tag{2.9}$$

The rescaling of the graviton ghost in order to canonically normalize the ghost kinetic energy yields a factor $\mathrm{Det}^{-\frac{1}{2}}2$ in the functional integration that cancels a factor $\mathrm{Det}^{\frac{1}{2}}2$ from the gravitino auxiliary field [5], [4]. The matrix elements of H_{gh} and of the covariant derivative $\hat{D}$ are given in (2.11), (C.29) and (C.30).

Finally, as discussed in I, we modify the graviton propagator by adding terms that are proportional to $\mathcal{L}_A = \partial\mathcal{L}/\partial\phi^A$, where ϕ^A is any field. This modification, which is equivalent to a nonlinear redefinition of the quantum variables, does not change the S-matrix and can lead to simplifications as well as enhancing manifest covariance under the symmetries of the theory [14]. We define the graviton propagator by[3] (2.20) and (2.21) of I, and by

$$\Delta^{-1}_{\mu\nu,a\rho} = \mathcal{L}_{\mu\nu,a\rho} - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{a\rho} + \frac{1}{2}g_{\mu\rho}\mathcal{L}_{a\nu} + \frac{1}{2}g_{\nu\rho}\mathcal{L}_{a\mu} = \mathcal{L}_{\mu\nu,a\rho} + 4P_{\mu\nu,\rho\sigma}\mathcal{L}^{\sigma}_{a},$$

$$\mathcal{L}_{\mu\nu,a\rho} = g_{\mu\mu'}g_{\nu\nu'}g_{\rho\rho'}\frac{\partial^2}{\partial g_{\mu'\nu'}\partial A^a_{\rho'}}\mathcal{L}, \quad \mathcal{L}^{\sigma}_{a} = g^{\sigma\rho}\mathcal{L}_{a\rho} = \frac{\partial}{\partial A^a_\sigma}\mathcal{L}. \tag{2.10}$$

It should be emphasized that the propagator modifications that we use have been chosen purely for convenience; they considerably simplify the matrix elements that are listed in Appendix C.1, and are not necessarily derivable from a generalized metric [14]. A natural choice[4] for this metric would be $G_{AB} = \sqrt{g}\,(Z_\Phi)_{AB}$, where A, B run over all bose degrees of freedom and the metric Z_Φ is defined in (2.11) below. Then defining $\Delta^{-1}_{AB} = \mathcal{L}_{AB} - \Gamma^C_{AB}\mathcal{L}_C$, where Γ^C_{AB} is the Christoffel connection derived from the metric G_{AB}, the propagator corrections would be precisely half the ones used here (with additional corrections to scalar propagator Δ^{-1}_{IJ} and the vector propagator $\Delta^{-1}_{a\rho,b\sigma}$ proportional to $\mathcal{L}_{\mu\nu,\rho\sigma}$). It is possible that the use of this generalized metric would reduce the need for field redefinitions as described in Section 4 [see (4.11-13)], but its use would make the intermediate calculations more cumbersome.

[3](2.21) of I should read: $\Delta^{-1}_{\mu\nu,\rho\sigma} \rightarrow \Delta^{-1}_{\mu\nu,\rho\sigma} - 2P_{\mu\nu,\rho\sigma}\mathcal{L}^{\lambda}_{\lambda} - \frac{1}{2}[g_{\mu\nu}\mathcal{L}_{\rho\sigma} + g_{\rho\sigma}\mathcal{L}_{\mu\nu}] + \frac{1}{2}[g_{\mu\rho}\mathcal{L}_{\nu\sigma} + g_{\nu\rho}\mathcal{L}_{\mu\sigma} + g_{\mu\sigma}\mathcal{L}_{\nu\rho} + g_{\nu\sigma}\mathcal{L}_{\mu\rho}]$.

[4]This choice for $G^{\mu\nu,\rho\sigma}$ coincides with that of Fradkin and Tseytlin [14] for the case of supergravity with their parameter $t = 1$, which corresponds to $\lambda = -1/2$ in their pure gravity case.

Once the above prescriptions have been implemented, the quadratic quantum Lagrangian for the bosonic sector takes the general form:

$$\begin{aligned}\mathcal{L}_{\text{bose}} + \mathcal{L}_{gh} &= -\frac{1}{2}\Phi^T \left[Z_\Phi \left(D^2 + M_\Phi^2\right) + \{D_\mu, X_\Phi^\mu\}\right] \Phi \\ &\quad + \frac{1}{2}\bar{c}\left[Z_{gh}\left(\mathcal{D}^2 + M_{gh}^2\right) + \{\mathcal{D}_\mu, X_{gh}^\mu\}\right] c,\end{aligned}$$

where $\Phi = (h_{\mu\nu}, \hat{\mathcal{A}}^a, \hat{z}^i, \hat{z}^{\bar{m}})$, D_μ is covariant under scalar field redefinitions as well as gauge and general coordinate transformations, and the X_μ connect fields of different spin; in addition, there is a vector-vector connection [9] in X_Φ^μ. Following the procedure described in [9], we introduce off-diagonal connections in both the bosonic and ghost sectors, as well as an additional connection for the gauge fields, so as to cast the quantum Lagrangian for the full gauge-fixed bosonic sector in the form

$$\begin{aligned}\mathcal{L}_{\text{bose}} + \mathcal{L}_{gh} &= -\frac{1}{2}\Phi^T Z_\Phi \left(\hat{D}_\Phi^2 + H_\Phi\right)\Phi + \frac{1}{2}\bar{c}Z_{gh}\left(\hat{D}_{gh}^2 + H_{gh}\right) c, \\ \hat{D}_\mu^\Phi &= D_\mu + V_\mu, \quad (V_\mu)_{a\rho,b\sigma} = -\delta_{ab}\epsilon_{\rho\mu\sigma\nu}\frac{\partial^\nu y}{2x}, \\ (ZV_\mu)_{\alpha\beta,a\nu} &= (V_\mu)_{a\nu,\alpha\beta} = \frac{1}{4}\left(\mathcal{F}_{a\beta\mu}g_{\alpha\nu} + \mathcal{F}_{a\alpha\mu}g_{\beta\nu}\right), \\ (V_\mu)_{a\nu,i} &= (V_\mu)_{i,a\nu} = \left[(V_\mu)_{\bar{\imath},a\nu}\right]^* = \frac{1}{4x}f_i\left(\mathcal{F}_{a\mu\nu} - i\tilde{\mathcal{F}}_{a\mu\nu}\right), \\ \hat{D}_\mu^{gh} &= \mathcal{D}_\mu + B_\mu, \quad (B_\mu)_{a\nu} = (B_\mu)_{\nu a} = -\frac{1}{\sqrt{2}}\mathcal{F}_{a\nu\mu}. \end{aligned} \tag{2.11}$$

This introduces corresponding shifts in the background field-dependent "squared mass" matrices:

$$M_\Phi^2 \to H_\Phi = M_\Phi^2 - V_\mu V^\mu, \quad M_{gh}^2 \to H_{gh} = M_{gh}^2 - B_\mu B^\mu. \tag{2.12}$$

The elements of M_Φ^2 were evaluated in [9]; here they are somewhat modified by the different Yang-Mills gauge fixing and action expansion. These modified matrix elements are listed in Appendix C.1 below.

As explained in Section 3 and Appendix A, we evaluate the fermion determinant by first writing it in two-component notation, separating it into helicity-even and -odd contributions, and then recasting these two contributions in Lorentz covariant four-component notation. As discussed in [13], this separation is not uniquely defined. The choice that respects supersymmetry as well as manifest gauge and Kähler covariance allows a consistent Pauli-Villars regulation. We follow that choice here; the corresponding matrix elements are given in the Appendix. The contribution from fermion loops to the effective action is evaluated (see Appendix A) by introducing [5] the 8×8 matrices

$$D_\mu = \begin{pmatrix} D_\mu^+ & 0 \\ 0 & D_\mu^- \end{pmatrix}, \quad M_\Theta = \begin{pmatrix} 0 & M \\ \bar{M} & 0 \end{pmatrix}, \quad \not{D} = \gamma^\mu D_\mu \tag{2.13}$$

that operate on an eight component fermion $f^T = (f_L, \; f_R = f_L^c)$. The helicity averaged contribution of the fermion determinant is then

$$-\frac{i}{4}\mathrm{Tr}\ln(-i\not{D} + M_\Theta)_+ = -\frac{i}{8}\mathrm{Tr}\ln\left(\not{D}^2 + M_\Theta^2 - i[\not{D}, M_\Theta]\right), \tag{2.14}$$

Because the fermion mass matrix and connection contain the terms $\sigma^{\mu\nu}M_{\mu\nu}$ and $iL_\mu\gamma_5$, respectively, they do not commute with γ_μ; thus

$$\begin{aligned}
\not{D}^2 &= D^2 + \frac{1}{4}[\gamma^\mu, \gamma^\nu]G_{\mu\nu} + \frac{1}{2}\{D_\nu, \gamma^\mu[D_\mu, \gamma^\nu]\} - \frac{1}{2}[D_\nu, \gamma^\mu[D_\mu, \gamma^\nu]], \\
[\not{D}, M_\Theta] &= \frac{1}{2}\{\gamma_\mu, D^\mu M_\Theta\} + \frac{1}{2}\{D^\mu, [\gamma_\mu, M_\Theta]\} + \frac{1}{2}[M_\Theta, [D^\mu, \gamma_\mu]], \\
D^\mu M_\Theta &\equiv [D^\mu, M_\Theta].
\end{aligned} \tag{2.15}$$

Therefore, in analogy with the boson case discussed above, we write

$$-\frac{i}{4}\mathrm{Tr}\ln(-i\not{D} + M_\Theta)_+ = -\frac{i}{8}\mathrm{Tr}\ln\left(\hat{D}_\Theta^2 + H_\Theta\right), \tag{2.16}$$

$$\begin{aligned}
H_\Theta &= M_\Theta^2 - \frac{i}{2}\{\gamma^\mu, D_\mu M_\Theta\} + \frac{1}{4}[\gamma^\mu, M_\Theta][\gamma_\mu, M_\Theta] - \frac{i}{2}[M_\Theta, [D^\mu, \gamma_\mu]] + \frac{1}{4}[\gamma^\mu, \gamma^\nu]G_{\mu\nu} \\
&\quad -\frac{1}{4}\gamma^\mu[D_\mu, \gamma^\nu]\gamma^\rho[D_\rho, \gamma_\nu] - \frac{1}{2}[D_\nu, \gamma^\mu[D_\mu, \gamma^\nu]] + \frac{i}{4}\{[\gamma^\mu, M_\Theta], \gamma^\nu[D_\nu, \gamma_\mu]\}, \\
\hat{D}_\mu^\Theta &= D_\mu - \frac{i}{2}[\gamma_\mu, M_\Theta] + \frac{1}{2}\gamma^\nu[D_\nu, \gamma_\mu].
\end{aligned} \tag{2.17}$$

3. Helicity-Odd Fermion Loop Contributions

In this section we determine the helicity-odd operators that arise from integration over fermionic degrees of freedom. They are particularly relevant to the evaluation of anomalies [2], [3], in effective supergravity theories, which is currently of special interest in attempts to extract physics from string theory. We show that these terms are finite, except in the presence of a Yang-Mills sector with a nontrivial kinetic normalization function $f(z)$, in which case there are logarithmically divergent contributions that are invariant under chiral $U(1)_R$ transformations, *i.e.*, under Kähler (or modular) transformations up to a possible dependence of the cut-off on the Kähler potential. We also indicate how the finite contributions to the effective action can be obtained.

A. General formalism

The fermion loop contribution is given by

$$\mathcal{L}_1 = -\frac{i}{2}\mathrm{Tr}\ln\left(-i\not{D} + M_\Theta\right) \equiv -\frac{i}{2}\mathrm{Tr}\ln\mathcal{M}. \tag{3.1}$$

To evaluate the determinant (3.1), we write

$$T = \mathrm{Tr}\ln\mathcal{M} = T_+ + T_-, \quad T_\pm = \frac{1}{2}\left[\mathrm{Tr}\ln\mathcal{M}(\gamma_5) \pm \mathrm{Tr}\ln\mathcal{M}(-\gamma_5)\right]. \tag{3.2}$$

Only T_+ has been calculated previously for supergravity [4]–[8]. Here we will evaluate the additional contribution, T_-:

$$\begin{aligned} T_- &= -\frac{1}{2}\mathrm{Tr}\ln\mathcal{M}(-\gamma_5)\mathcal{M}^{-1}(\gamma_5) = -\frac{1}{2}\mathrm{Tr}\ln\{1 - \mathcal{M}^{-1}[\mathcal{M}(\gamma_5) - \mathcal{M}(-\gamma_5)]\} \\ &= \frac{1}{2}\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}\{\mathcal{M}^{-1}[\mathcal{M}(\gamma_5) - \mathcal{M}(-\gamma_5)]\}^n. \end{aligned} \tag{3.3}$$

Using the techniques described in [15], [5], we can write the trace in (3.3) as (see Appendix A)

$$T_- = \int d^4x T(x), \quad T(x) = \int \frac{d^4p}{(2\pi)^4} T(p,x), \tag{3.4}$$

and then expand $T(p,x)$ as

$$T(p,x) = \mathrm{Tr}\sum_{n=1}^{\infty}\frac{2^n}{2n}\{\sum_{\ell=0}^{\infty}(-\mathcal{R})^{\ell}\mathcal{R}_5\}^n, \tag{3.5}$$

where $\mathcal{R}, \mathcal{R}_5$ are defined in (A.19–20):

$$\begin{aligned}
\mathcal{R} &= \frac{1}{-p^2}\left[p^2 - T^{\mu\nu}\Delta_\mu\Delta_\nu + \hat{h} + X + (p^\nu + G^\nu)\, P_{\mu\nu}\widehat{M}^\mu\right], \\
\mathcal{R}_5 &= \frac{1}{-p^2}\left[(p^\nu + G^\nu)\, P_{\mu\nu}\hat{N}^\mu\right].
\end{aligned} \tag{3.6}$$

The operators appearing in (3.5) are defined in Appendix A as power series of the form $\sum_n c_n(O)(D\cdot\partial/\partial p)^n O$, where $D_\mu = D_\mu^+ R + D_\mu^- L$ is the fully covariant derivative defined in (A.8) of the Appendix, and the operator O is a function of the background bosons. The coefficients $c_n(O)$ are constants with, in particular, $c_0(G) = 0$ in the expansion of $G_\mu^\pm$; more specifically

$$\not{G}^\pm = \gamma^\mu G_\mu^\pm \quad G_\mu^\pm = \frac{1}{2}G_{\nu\mu}^\pm\frac{\partial}{\partial p_\nu} + O\left(\frac{\partial^2}{\partial p\partial p}\right), \quad G_{\mu\nu}^\pm = -G_{\nu\mu}^\pm = [D_\mu^\pm, D_\nu^\pm] \tag{3.7}$$

Thus we have to evaluate the following contribution to the effective one-loop Lagrangian:

$$\mathcal{L}_1 \ni -\frac{i}{2}T_- = -i\int\frac{d^4p}{4(2\pi)^4}\mathrm{Tr}\sum_{n=1}^{\infty}\frac{2^n}{n}\{\sum_{\ell=0}^{\infty}(-\mathcal{R})^{\ell}\mathcal{R}_5\}^n, \tag{3.8}$$

where now the trace is over only Dirac indices and internal quantum numbers (and Lorentz indices for the gravitino).

To keep the integrals finite, the integration should be performed including Pauli-Villars regulator masses μ_0: $-p^{-2} \to (-p^2+\mu_0^2)^{-1}$ in the derivative expansion. However, as shown below, T_-, when suitably defined, contains no quadratically divergent terms. Once the integrals are properly regulated–including the appropriate definitions of $T_\pm$–the coefficients of log divergent terms are independent of the regularization scheme. On the other hand, if one wishes to evaluate finite terms, one has either to expand around an infrared regulator mass μ_0 or, alternatively, to resum the derivative expansion [17] [18]. In particular, the ultra-violet finite terms include the standard chiral anomaly. We explicitly evaluated this term for the vector-vector-axial vertex induced by Dirac fermions with a common mass μ_0, and recovered the large mass limit of the Adler-Rosenberg formula [19]; the complete expression for this formula requires a resummation of the derivative expansion which will be presented elsewhere [18]. We emphasize that, because of the anomaly, Kähler invariance is broken at the quantum level. Classically, this invariance permits a choice [10] of Kähler gauge such that the classical Lagrangian is derivable from only two functions of the scalar fields, the (in general matrix-valued) gauge normalization function $f_{ab}(z)$ and the generalized Kähler potential $\mathcal{G}(z,\bar{z}) = K(z,\bar{z}) + \ln|W(z)|^2$, where K and W are the Kähler potential and the superpotential, respectively. For the purpose of calculating the anomaly [2], [3], one has to undo the Kähler rotation of Cremmer *et al.* [10], by performing a phase transformation [20] on the fermion fields. As in I we work throughout in this Kähler covariant formalism.

As was discussed in [13], the separation (3.2) of T into helicity-odd and -even parts is not uniquely defined because we can interchange terms that are even and odd in γ_5 using $\gamma_5 = (i/24)\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$ and similar identities. In most cases the correct choice is dictated by gauge or Kähler covariance. The remaining ambiguities are resolved by supersymmetry. A fully SUSY-invariant result for the quadratically divergent terms requires the introduction of Pauli-Villars regulator fields [8], [16]; there is a unique definition of the matrix elements that allows a supersymmetric Pauli-Villars regularization [13]. Specifically, this fixes the forms of the fermion mass matrix and connection matrix:

$$\begin{aligned} M &= m + \left(\alpha_a F^a_{\mu\nu} + i\beta_a\gamma_5\tilde{F}^a_{\mu\nu}\right)\sigma^{\mu\nu}, \quad \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \\ D_\mu &= \mathcal{D}_\mu + i\Gamma_\mu\gamma_5 - \frac{1}{24}L_\mu\epsilon^{\lambda\nu\rho\sigma}\gamma_\lambda\gamma_\nu\gamma_\rho\gamma_\sigma, \end{aligned} \tag{3.9}$$

where Γ_μ, L_μ, m, and α, β are proportional to the unit matrix in Dirac space.

$\mathcal{D}_\mu$, which contains the spin connection, is the gauge and general coordinate covariant derivative, Γ_μ is the Kähler connection, $F_{\mu\nu}$ is the Yang-Mills field strength, and L_μ is an additional axial connection for gauginos arising from the noncanonical form of the kinetic energy term. $T_\pm$ are defined by (3.2) using the explicit γ_5-dependence in (3.9). Then the operators appearing in the derivative expansion of (3.6) take the form:

$$\begin{aligned}
G^\pm_{\mu\nu} &= \tilde{G}^\pm_{\mu\nu} + i\gamma_5 L^\pm_{\mu\nu} - [L_\mu, L_\nu], \quad \tilde{G}^\pm_{\mu\nu} = [\tilde{D}^\pm_\mu, \tilde{D}^\pm_\nu], \\
\tilde{D}^\pm_\mu &= \mathcal{D}^\pm_\mu \pm i\Gamma_\mu + \Gamma'_\mu, \quad L^\pm_{\mu\nu} = \tilde{D}^\pm_\mu L_\nu - \tilde{D}^\pm_\nu L_\mu, \quad \tilde{D}^\pm_\mu L_\nu \equiv [\tilde{D}^\pm_\mu, L_\nu], \\
\mathcal{J}_\mu &= \frac{i}{2}\left(\tilde{D}^+_\mu - \tilde{D}^-_\mu\right) = \frac{i}{2}\left(\mathcal{D}^+_\mu - \mathcal{D}^-_\mu\right) - \Gamma_\mu, \quad M_I = \frac{1}{2}\left(M - \bar{M}\right), \\
M &= m + M_\sigma = m + M_{\mu\nu}\sigma^{\mu\nu}, \quad \bar{M} = \bar{m} + \bar{M}_\sigma = \bar{m} + \bar{M}_{\mu\nu}\sigma^{\mu\nu}, \\
M_{\mu\nu} &= \alpha F_{\mu\nu} - i\beta\tilde{F}_{\mu\nu}, \quad \bar{M}_{\mu\nu} = \bar{\alpha}F_{\mu\nu} + i\bar{\beta}\tilde{F}_{\mu\nu},
\end{aligned} \tag{3.10}$$

where Γ_μ is the Kähler connection and Γ'_μ is an off-diagonal λ-ψ connection. We consider only the case where the gauge field normalization function $f(z)$ is diagonal in gauge indices; then, since Γ_μ is diagonal, L_μ commutes with $\mathcal{J}_\nu$, and we have

$$\begin{aligned}
L^+_{\mu\nu} &= L^-_{\mu\nu} \equiv \hat{L}_{\mu\nu} = L_{\mu\nu} + [\Gamma'_\mu, L_\nu] - [\Gamma'_\nu, L_\mu], \\
L_{\mu\nu} &= \nabla_\mu L_\nu - \nabla_\nu L_\mu, \quad [L_\mu, L_\nu] = 0.
\end{aligned} \tag{3.11}$$

Note that the spin connection in $\tilde{D}_\mu$ [see eq. (A.12) of I] drops out of the covariant derivatives $\tilde{D}_\mu M$. This is because we have taken the vierbein, and therefore γ_μ, to be covariantly constant [21]: $[\tilde{D}_\mu, \gamma_\nu] = 0$. The spin connection is even in γ_5 and therefore contributes to $\tilde{D}_\mu M$ through the commutator which vanishes [see the definitions (3.27) below].

To identify the ultraviolet divergences, we have to study the large p behavior of the integrand in (3.8) and keep terms up to $O(p^{-4})$. *A priori* $\mathcal{R}, \mathcal{R}_5 \sim p^{-1}$, so the ultraviolet divergent part of (3.7) can occur only in terms with $n \leq 4$, $\ell \leq 4 - n$. Aside from terms involving L_μ, by construction, the integrand is odd in γ_5, and we need at least four γ_μ's to get a nonvanishing trace:

$$T \propto \mathrm{Tr}\left(A_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) = -4i\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}A_{\mu\nu\rho\sigma}, \tag{3.12}$$

so $\mathrm{Tr}\mathcal{R}_5 = 0$. Finally, we note that $G^\pm_\mu$ in (3.7) vanishes except when sandwiched between functions of p, and is of order p^{-1} in power counting. Once all p-differentiations have been performed, surviving terms must have at least three γ_μ's that are not contracted with p^μ because of antisymmetry. After integration over p, the tensor $A_{\mu\nu\rho\sigma}$ in (3.12) can be constructed only from the four-vectors $\mathcal{J}_\mu$ and L_μ, the tensors $M_{\mu\nu}, G^\pm_{\mu\nu}$, the Riemann tensor, and their covariant derivatives $\tilde{D}_\mu$. Each factor of $G^\pm_{\mu\nu}$ and of D_μ reduces the apparent divergence of a given term by one power of p. Furthermore, in the covariant derivative expansions (A.19–20) of the operators O appearing in (3.5) the indices $\mu_i \cdots \mu_n$ in $D_{\mu_i}\cdots D_{\mu_n}O$ are automatically symmetrized, so at most one derivative of each operator can contribute to $A_{\mu\nu\rho\sigma}$ in (3.12).

B. Quadratically divergent contributions

By construction, T_- is antisymmetric under $\gamma_5 \to -\gamma_5$. Therefore we can evaluate, instead of (3.5)

$$T_- \quad\to\quad \frac{1}{2}\left[T_-(\gamma_5) - T_-(-\gamma_5)\right], \tag{3.13}$$

where $T_-(-\gamma_5)$ is obtained from $T_-(\gamma_5)$ by the substitutions

$$(D^+, D^-, M, \bar{M}, \mathcal{J}, M_I) \to (D^-, D^+, \bar{M}, M, -\mathcal{J}, -M_I).$$

The matrices $\mathcal{R}, \mathcal{R}_5$ are defined in (A.19–20). Since $\int d^4p\mathrm{Tr}\mathcal{R}_5 = 0$, the potentially quadratically divergent contribution to T_- is

$$\mathrm{Tr}\left(\mathcal{R}_5^2 - \mathcal{R}\mathcal{R}_5\right) \to \frac{1}{p^4}\mathrm{Tr}\left[(p^\mu N_\mu - p^\mu M_\mu)\,p^\nu N_\nu\right], \tag{3.14}$$

with N_ν, M_ν given in (A.15). Under Lorentz invariant integration, with $M = m + \sigma_{\mu\nu}M^{\mu\nu}$, we have

$$\int d^4p\ \not{p}M\ \not{p}M'(1\pm\gamma_5) \propto \int d^4p\ p^2\gamma_\mu M\gamma^\mu M'(1\pm\gamma_5) = 4\int d^4p\ m\ \not{p}M'(1\pm\gamma_5).$$

It follows that there are no quadratically divergent contribution involving the mass matrix. The averaging procedure (3.13) eliminates a residual spurious quadratic divergence proportional to $\mathrm{Tr}\mathcal{J}_\mu\mathcal{J}^\mu$. This divergence would vanish identically if a Pauli-Villars regularization were used with P-V masses that leave all classical symmetries unbroken. However this is not in general possible for the classical Kähler symmetry.[5] Moreover, in the Pauli-Villars regularization described in [13], there are no P-V fields that can regulate quadratic divergences proportional to $M_{\mu\nu}M^{\mu\nu}$, so the integrals, which are ill-defined unless they are explicitly regulated, must be defined in such a way that these divergences do not appear. Note that no quadratically divergent contribution to T_- arises if (3.3), as defined by (A.6), is expanded without performing the the transformation (A.16) that makes use of partial integration, which is ill-defined if the integrals are not finite. However this transformation renders many terms explicitly covariant and thereby considerably simplifies the derivative expansion.

C. Logarithmically divergent contributions

In the remainder of this section, T_- is understood as the average (3.13). Since we encounter only logarithmic divergences, after symmetric integration we may make the replacements:

$$\begin{aligned} p_\mu p_\nu f(p^2) &\rightarrow \frac{p^2}{4} g_{\mu\nu} f(p^2), \\ p_\mu p_\nu p_\rho p_\sigma f(p^2) &\rightarrow \frac{p^4}{24}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right) f(p^2). \end{aligned} \tag{3.15}$$

To evaluate the terms with p-derivatives, we write

$$\begin{aligned} \frac{1}{-p^2}p^\mu\frac{\partial}{\partial p_\nu} &\rightarrow -\frac{1}{-p^2}A^{\mu\nu}, \quad \frac{1}{-p^2}p^\mu G_{\nu\mu}\frac{\partial}{\partial p_\nu} \rightarrow 0, \quad A^{\mu\nu} = g^{\mu\nu} - \frac{2}{p^2}p^\mu p^\nu \\ \frac{\partial}{\partial p_\nu}\frac{1}{-p^2}p^\mu &\rightarrow \frac{1}{-p^2}A^{\mu\nu}, \quad p^\mu G_{\nu\mu}\frac{\partial}{\partial p_\nu}\frac{1}{-p^2}p^\rho \rightarrow \frac{1}{-p^2}p^\mu G_{\nu\mu}g^{\rho\nu}, \end{aligned} \tag{3.16}$$

[5] A detailed discussion of Pauli-Villars regularization of T_- will be given elsewhere [18].

where the first line is obtained by partial integration over p, and it is understood that operators multiplying the first (second) line on the left (right) are independent of p. Similarly

$$
\begin{aligned}
\frac{\partial^2}{\partial p_\mu \partial p_\nu}\frac{1}{-p^2}p^\rho &\rightarrow \frac{2}{p^4}\left(g^{\rho\mu}p^\nu + g^{\rho\nu}p^\mu + g^{\mu\nu}p^\rho - \frac{4}{p^2}p^\mu p^\nu p^\rho\right), \\
\frac{\partial^2}{\partial p_\mu \partial p_\nu}\frac{1}{-p^2}\,\not{p}p^\rho p^\sigma r_{\mu\rho\sigma\nu} &\rightarrow \frac{1}{-p^2}\left(r\,\not{p} - \frac{2}{p^2}p^\mu p^\nu\,\not{p}r_{\mu\nu} + 2r^{\mu\nu}\gamma_\mu p_\nu\right), \\
\frac{1}{-p^2}\frac{\partial^2}{\partial p_\mu \partial p_\nu}p^\rho p^\sigma r_{\mu\rho\sigma\nu} &\rightarrow -\frac{2}{p^4}p^\mu p^\nu r_{\mu\nu},,
\end{aligned}
\tag{3.17}
$$

where the last line is obtained by partial integration.

It is easy to see that the nonvanishing terms in T_- involve the connection L_μ and/or the off-diagonal mass $M_{\mu\nu}$. In the absence of these contributions, since $\epsilon^{\mu\nu\rho\sigma}r_{\mu\nu\rho\tau} = 0$, the only helicity-odd terms are:

$$
\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[(D^e_\mu\mathcal{J}_\nu)\mathcal{J}_\rho\mathcal{J}_\sigma], \quad \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[G^A_{\mu\nu}\mathcal{J}_\rho\mathcal{J}_\sigma], \quad \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[G^V_{\mu\nu}D^e_\rho\mathcal{J}_\sigma], \tag{3.18}
$$

where

$$
D^e_\mu = \frac{1}{2}\left(D^+_\mu + D^-_\mu\right) \equiv \partial_\mu + \mathcal{J}'_\mu, \quad G^{A(V)}_{\mu\nu} = \frac{1}{2}[G^+_{\mu\nu} - (+)G^-_{\mu\nu}].
$$

The first term in (3.18) can be written

$$
\frac{1}{3}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left[D^e_\mu\left(\mathcal{J}_\nu\mathcal{J}_\rho\mathcal{J}_\sigma\right)\right] = \frac{1}{3}\epsilon^{\mu\nu\rho\sigma}\partial_\mu(\mathrm{Tr}[\mathcal{J}_\nu\mathcal{J}_\rho\mathcal{J}_\sigma]),
$$

where we used cyclic permutations in the trace together with the relation

$$
\mathrm{Tr}[D^e_\mu(\mathcal{J}\mathcal{J}\mathcal{J})] = \mathrm{Tr}\{\partial_\mu(\mathcal{J}\mathcal{J}\mathcal{J}) + i[\mathcal{J}'_\mu, \mathcal{J}\mathcal{J}\mathcal{J}]\} = \partial_\mu\mathrm{Tr}(\mathcal{J}\mathcal{J}\mathcal{J}). \tag{3.19}
$$

Note that if a field-dependent ultraviolet regulator mass Λ is present one cannot drop the total derivative on the right hand side of (3.19), but integrating by parts gives $\partial\ln\Lambda = \partial\Lambda/\Lambda$ which is finite for $\Lambda \rightarrow \infty$. For the second term in (3.18), defining $D^\pm_\mu = \partial_\mu + \Gamma^\pm_\mu$, we have

$$
G^\pm_{\mu\nu} = \partial_\mu\Gamma^\pm_\nu - \partial_\nu\Gamma^\pm_\mu + [\Gamma^\pm_\mu, \Gamma^\pm_\nu] = D_\mu\Gamma^\pm_\nu - D_\nu\Gamma^\pm_\mu - [\Gamma^\pm_\mu, \Gamma^\pm_\nu]. \tag{3.20}
$$

By the above argument the $D\Gamma$ terms give finite contributions, so we are left with

$$\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[(\Gamma^+_\mu\Gamma^+_\nu - \Gamma^-_\mu\Gamma^-_\nu)(\Gamma^+_\rho - \Gamma^-_\rho)(\Gamma^+_\sigma - \Gamma^-_\sigma)] = 0,$$

again using cyclic permutations of the trace. Since $\epsilon^{\mu\nu\rho\sigma}D^e_\mu[D^e_\nu, D^e_\rho]$ vanishes by virtue of the Bianchi identity, the third term in (3.18) reduces (up to a total derivative) to the same form as the first term: $G^V \to [\mathcal{J}, \mathcal{J}]$.

First consider the terms quartic in $\mathcal{R}, \mathcal{R}_5$. To obtain the logarithmically divergent piece, we drop all p-derivatives:

$$\mathcal{R} \to \frac{1}{-p^2}p_\mu M^\mu, \quad \mathcal{R}_5 \to \frac{1}{-p^2}p_\mu N^\mu. \tag{3.21}$$

We note that $F^{\mu\nu}_a F^b_{\nu\rho} F^{c\,\rho}_{\mu}$ and $F^{\mu\nu}_a F^b_{\nu\rho} \tilde{F}^{c\,\rho}_{\mu}$ vanish if any two of the indices a, b, c are equal; there are therefore no terms cubic in M_σ. Then using $\gamma_\mu M \gamma^\mu = 4m$, together with Eqs. (A.23) and (B.12–13) and cyclic permutivity of the trace, we obtain:

$$\begin{aligned}
H(M_1, M_2) &\equiv \mathrm{Tr}\left(\not{p}M_1 \not{p} \not{\mathcal{J}} \not{p}M_2 \not{p} \not{\mathcal{J}}\gamma_5\right) \to \frac{16i}{3}p^4\mathrm{Tr}\left(\widetilde{M}_1^{\mu\nu}\mathcal{J}_\nu M^2_{\mu\rho}\mathcal{J}^\rho - M_1^{\mu\nu}\mathcal{J}_\nu \widetilde{M}^2_{\mu\rho}\mathcal{J}^\rho\right), \\
F(M_1, M_2) &\equiv \mathrm{Tr}\left(\not{p}M_1 \not{p}M_2 \not{p} \not{\mathcal{J}} \not{p} \not{\mathcal{J}}\gamma_5\right) \to 4p^4\mathrm{Tr}\left[\left(\widetilde{M}_1^{\mu\nu}m_2 - m_1\widetilde{M}_2^{\mu\nu}\right)\mathcal{J}_\mu\mathcal{J}_\nu\right] \\
&\quad + \frac{4i}{3}p^4\mathrm{Tr}\left[\left(M^1_{\mu\rho}\widetilde{M}_2^{\mu\nu} - \widetilde{M}_1^{\mu\nu}M^2_{\mu\rho}\right)\{\mathcal{J}^\rho, \mathcal{J}_\nu\}\right],
\end{aligned}$$

$$\begin{aligned}
F'(M_1, M_2, M_3, M_4) &= -F'(M_4, M_1, M_2, M_3) \equiv \mathrm{Tr}\left(\not{p}M_1 \not{p}M_2 \not{p}M_3 \not{p}M_4\gamma_5\right) \\
&\to \frac{16i}{3}p^4\mathrm{Tr}\left(\widetilde{M}_1^{\mu\nu}M_2^{\rho\sigma}M^3_{\mu\nu}M^4_{\rho\sigma} - M_1^{\mu\nu}M_2^{\rho\sigma}M^3_{\mu\nu}\widetilde{M}^4_{\rho\sigma}\right) \\
&\quad + 8ip^4\mathrm{Tr}\left(m_1 M^2_{\mu\nu}m_3\widetilde{M}_4^{\mu\nu} - m_4 M^1_{\mu\nu}m_2\widetilde{M}_3^{\mu\nu}\right),
\end{aligned} \tag{3.22}$$

where $M_i = M, \bar{M}, M_I$, $\widetilde{M}_i^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(M_i)_{\rho\sigma}$, and the traces on the right hand sides are over internal indices only. In evaluating these expressions we used the fact that since $\mathrm{Tr}\left(M^1_\sigma M^2_\sigma M^3_\sigma M^4_\sigma\gamma_5\right) = \mathrm{Tr}\left(M^4_\sigma M^1_\sigma M^2_\sigma M^3_\sigma\gamma_5\right)$, these terms do not contribute to

$$\frac{1}{2}\left[F'(M_1, M_2, M_3, M_4) - F'(M_4, M_1, M_2, M_3)\right] = F'(M_1, M_2, M_3, M_4).$$

Finally, since the expression (3.6) for $\mathcal{R}_5$ is odd in γ_5: $[\mathcal{R}_5(\gamma_5)]^4 = +[\mathcal{R}_5(-\gamma_5)]^4$, it follows that $\mathrm{Tr}(\mathcal{R}_5)^4$ does not contribute to $T(\gamma_5) = -T(-\gamma_5)$. The logarithmically divergent contributions from the quartic terms in (3.8) are therefore given by:

$$\mathrm{Tr}\Big[-\mathcal{R}^3\mathcal{R}_5 + \mathcal{R}_5\mathcal{R}^2\mathcal{R}_5 + \mathcal{R}^2\mathcal{R}_5^2 + (\mathcal{R}\mathcal{R}_5)^2 \\ -\frac{4}{3}\left(\mathcal{R}_5^2\mathcal{R}\mathcal{R}_5 + \mathcal{R}_5\mathcal{R}\mathcal{R}_5^2 + \mathcal{R}\mathcal{R}_5^3\right)\Big] \to \frac{1}{p^4}\left(T_4 + T_4'\right). \quad (3.23)$$

For the terms quartic in M we obtain

$$\begin{aligned} T_4' = -\frac{1}{4}F'(M,\bar{M},M,\bar{M}) &= -\frac{4i}{3}\mathrm{Tr}\left(\widetilde{M}^{\mu\nu}\bar{M}^{\rho\sigma}M_{\mu\nu}\bar{M}_{\rho\sigma} - M^{\mu\nu}\bar{M}^{\rho\sigma}M_{\mu\nu}\widetilde{\bar{M}}_{\rho\sigma}\right) \\ &\quad -2i\mathrm{Tr}\left(m\bar{M}_{\mu\nu}m\widetilde{\bar{M}}^{\mu\nu} - \bar{m}M_{\mu\nu}\bar{m}\widetilde{M}^{\mu\nu}\right), \quad (3.24) \end{aligned}$$

and for the terms quadratic in M, we find:

$$\begin{aligned} \mathrm{Tr}\mathcal{R}^3\mathcal{R}_5 &\to 0, \quad \mathrm{Tr}(\mathcal{R}\mathcal{R}_5)^2 \to -\frac{1}{p^8}H(M,\bar{M}), \\ \mathrm{Tr}\mathcal{R}^2\mathcal{R}_5^2 &= \mathrm{Tr}\mathcal{R}_5\mathcal{R}^2\mathcal{R}_5 \to \frac{1}{p^8}\frac{1}{2}\left[F(M,\bar{M}) - F(\bar{M},M)\right] = \frac{1}{p^4}\left(T_4'' + T_4'''\right), \\ \mathrm{Tr}\mathcal{R}\mathcal{R}_5^3 &= \mathrm{Tr}\mathcal{R}_5\mathcal{R}\mathcal{R}_5^2 = \mathrm{Tr}\mathcal{R}_5^2\mathcal{R}\mathcal{R}_5 \to \frac{1}{p^8}\frac{1}{2}\Big[H(M,M_I) + H(\bar{M},M_I) \\ &\quad -F(M,M_I) - F(\bar{M},M_I) + F(M_I,\bar{M}) + F(M_I,M)\Big] \\ &= \frac{1}{p^4}\left(T_4'' + T_4'''\right) - \frac{1}{p^8}\frac{1}{2}H(M,\bar{M}) = -\frac{1}{2p^4}T_4, \\ T_4'' &= \mathrm{Tr}\left(\left[\{\bar{m},\widetilde{M}^{\mu\nu}\} - \{m,\widetilde{\bar{M}}^{\mu\nu}\}\right][\mathcal{J}_\mu,\mathcal{J}_\nu]\right) \\ T_4''' &= \frac{2i}{3}\mathrm{Tr}\left[\left(\{M_{\mu\rho},\widetilde{\bar{M}}^{\mu\nu}\} - \{\bar{M}_{\mu\rho},\widetilde{M}^{\mu\nu}\}\right)\{\mathcal{J}^\rho,\mathcal{J}_\nu\}\right]. \quad (3.25) \end{aligned}$$

Then

$$\begin{aligned} T_4 &= -2\left(T_4'' + T_4'''\right) + \frac{1}{p^4}H(M,\bar{M}) \equiv -2T_4'' - t_4 \\ &= -2T_4'' - \frac{8i}{3}\mathrm{Tr}\left(\{\mathcal{J}^\rho,M_{\mu\rho}\}\{\mathcal{J}_\nu,\widetilde{\bar{M}}^{\mu\nu}\} - \{\mathcal{J}^\rho,\bar{M}_{\mu\rho}\}\{\mathcal{J}_\nu,\widetilde{M}^{\mu\nu}\}\right). (3.26) \end{aligned}$$

To evaluate the cubic and quadratic terms, we use a shorthand notation according to which the covariant derivatives imply the matrix products:

$$D^{\pm}_\mu \mathcal{J}_\nu \equiv [D^{\pm}_\mu, \mathcal{J}_\nu], \quad D^{\pm}_\mu M \equiv D^{\pm}_\mu M - M D^{\mp}_\mu, \tag{3.27}$$

where here M is any mass matrix. Using the Dirac traces in (A.23), the first identity in (B.12), and the additional identities

$$\begin{aligned} \mathrm{Tr}\,([A,B]C) &= -\mathrm{Tr}\,(A[B,C]), \quad D_\mu(M\bar{M}) = [d^+_\mu, M\bar{M}], \quad D_\mu(\bar{M}M) = [d^-_\mu, \bar{M}M], \\ [d^+_\mu, MM_I] &= (D_\mu M)M_I + MD^-_\mu M_I, \quad [d^-_\mu, \bar{M}M_I] = (D_\mu \bar{M})M_I + \bar{M}D^+_\mu M_I, \\ \mathrm{Tr}\,(\{A,B\}CD) &= \mathrm{Tr}\,(B\{A,CD\}) = \mathrm{Tr}\,(B\{A,C\}D) - \mathrm{Tr}\,(BC[A,D]), \end{aligned} \tag{3.28}$$

together with the facts [see (A.23)] that $\mathrm{Tr}\,(\sigma \cdot A\gamma_\mu \sigma \cdot B\gamma_\nu)$ and $\mathrm{Tr}\,(\sigma \cdot A\gamma_\mu \sigma \cdot B\gamma_\nu\gamma_5)$ are symmetric in $\{\mu,\nu\}$, and that $[L_\mu, \mathcal{J}_\nu] = 0$, we obtain

$$\begin{aligned} \mathrm{Tr}\mathcal{R}^2\mathcal{R}_5 \;\to\;& \frac{1}{p^4}\mathrm{Tr}\Big\{ -2i\widetilde{X}^{\mu\nu}_-(M,\bar{M})\tilde{D}^+_\mu \mathcal{J}_\nu - 2i\widetilde{X}^{\mu\nu}_-(\bar{M},M)\tilde{D}^-_\mu \mathcal{J}_\nu - L(M,\bar{M}) \\ & + \left[\widetilde{X}^{\mu\nu}_-(M_I,\bar{M}) - \widetilde{X}^{\mu\nu}_-(M,M_I) + \widetilde{X}^{\mu\nu}_+(M,M_I)\right]\tilde{G}^+_{\mu\nu} \\ & + \left[\widetilde{X}^{\mu\nu}_-(M_I,M) - \widetilde{X}^{\mu\nu}_-(\bar{M},M_I) + \widetilde{X}^{\mu\nu}_+(\bar{M},M_I)\right]\tilde{G}^-_{\mu\nu} \\ & + \frac{4}{3}\left[X^+(M,\bar{M}) + X^-(\bar{M},M)\right] + 2\left(\bar{m}M^{\mu\nu} - m\bar{M}^{\mu\nu}\right)\hat{L}_{\mu\nu} \\ & - \hat{L}_{\mu\nu}\left[X^{\mu\nu}_-(M_I,M) + X^{\mu\nu}_-(M_I,\bar{M})\right]\Big\}, \\ \mathrm{Tr}\mathcal{R}\mathcal{R}_5^2 \;+\;& \mathrm{Tr}\mathcal{R}_5\mathcal{R}\mathcal{R}_5 \to \frac{1}{p^4}\mathrm{Tr}\Big\{ -4i\left[\widetilde{X}^{\mu\nu}_-(M_I,\bar{M}) - \widetilde{X}^{\mu\nu}_-(M,M_I)\right]\tilde{D}^+_\mu \mathcal{J}_\nu \\ & + 4i\left[\widetilde{X}^{\mu\nu}_-(M_I,M) - \widetilde{X}^{\mu\nu}_-(\bar{M},M_I)\right]\tilde{D}^-_\mu \mathcal{J}_\nu - 2L(\bar{M},M_I) + 2L(M_I,M) \\ & - \frac{8}{3}\left[X^+(M,M_I) + X^-(M_I,M) - X^+(M_I,\bar{M}) - X^-(\bar{M},M_I)\right] \\ & + \left[\widetilde{X}^{\mu\nu}_+(M_I,M_I) - 2\widetilde{X}^{\mu\nu}_-(M_I,M_I)\right]\left(\tilde{G}^+_{\mu\nu} - \tilde{G}^-_{\mu\nu}\right)\Big\}, \\ \mathrm{Tr}\mathcal{R}_5^3 \;\to\;& \frac{1}{p^4}\mathrm{Tr}\Big\{6i\widetilde{X}^{\mu\nu}_-(M_I,M_I)\left(\tilde{D}^+_\mu \mathcal{J}_\nu + \tilde{D}^-_\mu \mathcal{J}_\nu\right) \\ & - 4\left[X^+(M_I,M_I) + X^-(M_I,M_I)\right] + 3L(M_I,M_I)\Big\}, \end{aligned} \tag{3.29}$$

where

$$
\begin{aligned}
X^{\pm}(M_1, M_2) &= \left(\tilde{D}^{\pm}_{\rho} M_1^{\mu\rho} \widetilde{M}^2_{\mu\nu} - \tilde{D}^{\pm}_{\rho} \widetilde{M}_1^{\mu\rho} M^2_{\mu\nu} + \widetilde{M}^1_{\mu\nu} \tilde{D}^{\mp}_{\rho} M_2^{\mu\rho} - M^1_{\mu\nu} \tilde{D}^{\mp}_{\rho} \widetilde{M}_2^{\mu\rho}\right) \mathcal{J}^{\nu}, \\
X^{\mu\nu}_{\pm}(M_1, M_2) &= M_1^{\mu\nu} m_2 \pm m_1 M_2^{\mu\nu} \qquad \widetilde{X}^{\mu\nu}_{\pm} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \widetilde{X}^{\pm}_{\rho\sigma}, \\
L(M_1, M_2) &= 2\{L_\mu, m_1\}\{\mathcal{J}_\nu, m_2\} + \frac{4}{3}\{L_\mu, M_1^{\mu\nu}\}\{\mathcal{J}_\nu, M^2_{\mu\nu}\} \\
&\quad + \frac{8}{3}\left(\{L_\mu, M_1^{\mu\rho}\}\{\mathcal{J}^\nu, M^2_{\rho\nu}\} + \{L^\nu, M_1^{\mu\rho}\}\{\mathcal{J}_\mu, M^2_{\rho\nu}\}\right). \qquad (3.30)
\end{aligned}
$$

Again, the traces on the right are over internal indices only. Here and throughout the remainder of this section, $\tilde{G}^{\pm}_{\mu\nu}$ is understood as one fourth of the Dirac trace of $[\tilde{D}^{\pm}_{\mu}, \tilde{D}^{\pm}_{\nu}]$, and has no contribution from the spin connection, and the derivative operators $\tilde{D}_\mu$ are understood to operate only on the object to their immediate right. The expressions (3.30) can be simplified further using the relations

$$
\begin{aligned}
X^{\mu\nu}\left(\tilde{D}^{+}_{\mu}\mathcal{J}_\nu + \tilde{D}^{-}_{\mu}\mathcal{J}_\nu\right) &= \frac{i}{2} X^{\mu\nu}\left(\tilde{G}^{+}_{\mu\nu} - \tilde{G}^{-}_{\mu\nu}\right), \\
X^{\mu\nu}\left(\tilde{D}^{+}_{\mu}\mathcal{J}_\nu - \tilde{D}^{-}_{\mu}\mathcal{J}_\nu\right) &= -2i X^{\mu\nu}[\mathcal{J}_\mu, \mathcal{J}_\nu], \\
\{\mathcal{J}_\mu, M\} &= \frac{i}{2}\left(\tilde{D}^{+}_{\mu} M - \tilde{D}^{-}_{\mu} M\right), \qquad (3.31)
\end{aligned}
$$

that follow from the definitions (3.10) and (3.27). Defining

$$
\begin{aligned}
X_1 &= \mathrm{Tr}\left[X^{+}(M, \bar{M}) + X^{-}(\bar{M}, M)\right], \\
X_2 &= \mathrm{Tr}\left[X^{+}(M_I, M_I) + X^{-}(M_I, M_I)\right] = i\mathrm{Tr}\left(\tilde{D}^{+\sigma} \widetilde{M}^I_{\sigma\mu} \tilde{D}^{-}_{\rho} M_I^{\rho\mu} - \tilde{D}^{+\sigma} M^I_{\sigma\mu} \tilde{D}^{-}_{\rho} \widetilde{M}_I^{\rho\mu}\right), \\
X_3 &= i\mathrm{Tr}\left(\tilde{D}^{\sigma} \widetilde{M}_{\sigma\mu} \tilde{D}^{-}_{\rho} M_I^{\rho\mu} - \tilde{D}^{\sigma} M_{\sigma\mu} \tilde{D}^{-}_{\rho} \widetilde{M}_I^{\rho\mu} - \tilde{D}^{+\sigma} \widetilde{M}^I_{\sigma\mu} \tilde{D}_{\rho} \bar{M}^{\rho\mu} + \tilde{D}^{+\sigma} M^I_{\sigma\mu} \tilde{D}_{\rho} \widetilde{\bar{M}}^{\rho\mu}\right), \\
X_4 &= \mathrm{Tr}\left[X^{+}(M, M_I) + X^{-}(M_I, M) - X^{+}(M_I, \bar{M}) - X^{-}(\bar{M}, M_I)\right] \\
&= -X_1 + X_3 - i\mathrm{Tr}\left(\tilde{D}^{\sigma} M_{\sigma\mu} \tilde{D}_{\rho} \widetilde{\bar{M}}^{\rho\mu} - \tilde{D}^{\sigma} \bar{M}_{\sigma\mu} \tilde{D}_{\rho} \widetilde{M}^{\rho\mu}\right), \qquad (3.32)
\end{aligned}
$$

where we dropped total derivatives, we obtain

$$
T_3 = \mathrm{Tr}\left(\mathcal{R}^2\mathcal{R}_5 - \mathcal{R}\mathcal{R}_5^2 - \mathcal{R}_5\mathcal{R}\mathcal{R}_5 + \frac{4}{3}\mathcal{R}_5^3\right) \to \frac{1}{p^4}\left(\frac{4}{3}X_3 - \frac{8}{3}X_2 + t_4 + 2T_4''\right)
$$

$$
\begin{aligned}
&-\frac{4i}{3p^4}\mathrm{Tr}\left(\tilde{D}^\sigma M_{\sigma\mu}\tilde{D}_\rho\widetilde{\bar{M}}^{\rho\mu} - \tilde{D}^\sigma\bar{M}_{\sigma\mu}\tilde{D}_\rho\widetilde{M}^{\rho\mu}\right)\\
&+\frac{1}{p^4}\mathrm{Tr}\Big\{\widetilde{X}_-^{\mu\nu}(M,\bar{M})\tilde{G}^+_{\mu\nu} - \widetilde{X}_-^{\mu\nu}(\bar{M},M)\tilde{G}^-_{\mu\nu} + 2\left(\bar{m}M^{\mu\nu} - m\bar{M}^{\mu\nu}\right)\\
&-\left[X_-^{\mu\nu}(M_I,M) + X_-^{\mu\nu}(M_I,\bar{M})\right]\hat{L}_{\mu\nu} + \widetilde{X}_+^{\mu\nu}(M_I,M_I)\left(\tilde{G}^+_{\mu\nu} - \tilde{G}^-_{\mu\nu}\right)\\
&-\left[\widetilde{X}_+^{\mu\nu}(M,M_I)\tilde{G}^+_{\mu\nu} + \widetilde{X}_+^{\mu\nu}(\bar{M},M_I)\tilde{G}^-_{\mu\nu}\right] - L(M,\bar{M})\Big\}, \qquad (3.33)
\end{aligned}
$$

where t_4, T_4'' are defined in (3.25–26), and

$$
t_4 = \frac{4}{3}X_4 - \frac{8}{3}X_2. \qquad (3.34)
$$

Finally, to obtain the logarithmically divergent parts of $\mathrm{Tr}\mathcal{R}\mathcal{R}_5$ and $\mathrm{Tr}\mathcal{R}_5^2$, we use (3.15–17), giving

$$
\begin{aligned}
\mathrm{Tr}\mathcal{R}_5^2 \;&\to\; \frac{8}{3p^4}X_2 - \frac{2}{p^4}L(M_I,M_I) + \frac{1}{p^4}\widetilde{X}_+^{\mu\nu}(M_I,M_I)\left(\tilde{G}^+_{\mu\nu} - \tilde{G}^-_{\mu\nu}\right),\\
\mathrm{Tr}\mathcal{R}\mathcal{R}_5 \;&\to\; \frac{4}{3p^4}X_3 + \frac{4i}{3p^4}\mathrm{Tr}\left(\{L^\sigma, M_{\sigma\mu}\}\{L_\rho, \widetilde{\bar{M}}^{\rho\mu}\} - \{L^\sigma, \bar{M}_{\sigma\mu}\}\{L_\rho, \widetilde{M}^{\rho\mu}\}\right)\\
&-\frac{1}{p^4}\mathrm{Tr}\left[i\left(\{L^\rho, m\}\tilde{D}_\rho\bar{m} - \tilde{D}^\rho m\{L_\rho, \bar{m}\}\right) + L(M,\bar{M}) + 2L(M_I,M_I)\right]\\
&-\frac{2i}{3p^4}\mathrm{Tr}\left(\{L^\rho, M_{\sigma\mu}\}\tilde{D}_\rho\bar{M}^{\sigma\mu} - \tilde{D}^\rho M_{\sigma\mu}\{L_\rho, \bar{M}^{\sigma\mu}\}\right)\\
&+\frac{8i}{3p^4}\mathrm{Tr}\left(\{L^\sigma, M_{\sigma\mu}\}\tilde{D}_\rho\bar{M}^{\rho\mu} - \tilde{D}^\sigma M_{\sigma\mu}\{L_\rho, \bar{M}^{\rho\mu}\}\right)\\
&-\frac{4i}{3p^4}\mathrm{Tr}\left(\hat{L}_\sigma{}^\rho\{M^{\sigma\mu}, \bar{M}_{\rho\mu}\}\right) - \frac{1}{p^4}\mathrm{Tr}\left(\hat{L}_{\mu\nu}\left[X_-^{\mu\nu}(M,M_I) + X_-^{\mu\nu}(\bar{M},M_I)\right]\right)\\
&+\frac{1}{p^4}\mathrm{Tr}\left[\widetilde{X}_+^{\mu\nu}(M,M_I)\tilde{G}^+_{\mu\nu} + \widetilde{X}_+^{\mu\nu}(\bar{M},M_I)\tilde{G}^-_{\mu\nu}\right]\\
&+\frac{i}{p^4}r^\mu_\nu\mathrm{Tr}\left(\widetilde{M}^{\nu\rho}\bar{M}_{\mu\rho} - M^{\nu\rho}\widetilde{\bar{M}}_{\mu\rho}\right) + \text{total derivative.} \qquad (3.35)
\end{aligned}
$$

Inserting these results in (3.7) gives

$$
-\frac{i}{2}T_- \;=\; g^{\frac{1}{2}}\frac{\ln\Lambda^2}{32\pi^2}\left(T_4' + T_4 + T_3 - \mathrm{Tr}\mathcal{R}\mathcal{R}_5 + \mathrm{Tr}\mathcal{R}_5^2\right)
$$

$$
\begin{aligned}
= \quad & g^{\frac{1}{2}}\frac{\ln\Lambda^2}{32\pi^2}\mathrm{Tr}\Big\{T_4' + \left[\widetilde{X}_-^{\mu\nu}(M,\bar{M})\tilde{G}^+_{\mu\nu} - \widetilde{X}_-^{\mu\nu}(\bar{M},M)\tilde{G}^-_{\mu\nu}\right] \\
& -\frac{4i}{3}\left(\tilde{D}^\sigma M_{\sigma\mu}\tilde{D}_\rho\widetilde{\bar{M}}^{\rho\mu} - \tilde{D}^\sigma\bar{M}_{\sigma\mu}\tilde{D}_\rho\widetilde{M}^{\rho\mu}\right) - ir^\mu_\nu\left(\widetilde{M}^{\nu\rho}\bar{M}_{\mu\rho} - M^{\nu\rho}\widetilde{\bar{M}}_{\mu\rho}\right) \\
& +[\hat{L}_{\mu\nu},\bar{m}]M^{\mu\nu} - [\hat{L}_{\mu\nu},m]\bar{M}^{\mu\nu} + i\left(\{L^\rho,m\}\tilde{D}_\rho\bar{m} - \tilde{D}^\rho m\{L_\rho,\bar{m}\}\right) \\
& -\frac{4i}{3}\left(\{L^\sigma,M_{\sigma\mu}\}\{L_\rho,\widetilde{\bar{M}}^{\rho\mu}\} - \{L^\sigma,\bar{M}_{\sigma\mu}\}\{L_\rho,\widetilde{M}^{\rho\mu}\}\right) \\
& -\frac{8i}{3}\left(\{L^\sigma,M_{\sigma\mu}\}\tilde{D}_\rho\bar{M}^{\rho\mu} - \tilde{D}^\sigma M_{\sigma\mu}\{L_\rho,\bar{M}^{\rho\mu}\}\right) \\
& +\frac{2i}{3}\left[L^\rho\left(\{M_{\sigma\mu},\tilde{D}_\rho\bar{M}^{\sigma\mu}\} - \{\tilde{D}_\rho M^{\sigma\mu},\bar{M}_{\sigma\mu}\}\right) + 2\hat{L}_{\mu\nu}\{M^{\mu\rho},\bar{M}^\nu{}_\rho\}\right]\Big\}. \qquad (3.36)
\end{aligned}
$$

To evaluate (3.36), we note that the connection is block diagonal in the χ-λ-α sector, and the axial part is diagonal in the λ and α sectors, with $\mathcal{J}_{\lambda\lambda} = -\mathcal{J}_{\alpha\alpha}$. Using the reality and symmetry properties of the off-diagonal λ-α masses:

$$
m_{\lambda\alpha} = -\bar{m}_{\lambda\alpha} = m^T_{\lambda\alpha}, \quad M^{\mu\nu}_{\lambda\alpha} = \bar{M}^{\mu\nu}_{\lambda\alpha} = -\left(M^{\mu\nu}_{\lambda\alpha}\right)^T, \qquad (3.37)
$$

it is easy to see that there is no contribution that involves only these masses. For the off-diagonal λ-χ masses:

$$
\begin{aligned}
& m_{\lambda\chi} = m^T_{\lambda\chi}, \quad \widetilde{M}^{\mu\nu}_{\lambda\chi} = iM^{\mu\nu}_{\lambda\chi}, \quad \widetilde{\bar{M}}^{\mu\nu}_{\lambda\chi} = -i\bar{M}^{\mu\nu}_{\lambda\chi}, \quad M^{\mu\nu}_{\lambda\chi} = -\left(M^{\mu\nu}_{\lambda\chi}\right)^T, \\
& M^{\mu\nu}_{\lambda\chi}\bar{M}^{\lambda\chi}_{\mu\nu} = \widetilde{M}^{\mu\nu}_{\lambda\chi}\bar{M}^{\lambda\chi}_{\mu\nu} = 0, \quad \left(M^{\mu\nu}\bar{M}_{\rho\nu}\right)^a_a = \left(M_{\rho\nu}\bar{M}^{\mu\nu}\right)^a_a. \qquad (3.38)
\end{aligned}
$$

It follows from these relations that the last line in (3.36) vanishes.

Using the fermion matrix elements given in Appendix C.2, we obtain the nonvanishing contributions to T_- listed in Appendices C.3–8. Note that these expressions are fully covariant, although the expansion (3.7) of T_- is not. This noncovariance is necessarily the case since T_- contains the chiral anomaly that breaks classical Kähler invariance. However, the logarithmically divergent contributions are Kähler invariant, up to a possible dependence of the effective cut-off on the Kähler potential [12], [3], [13].

The ghostino determinant also contains helicity-odd contributions, but since it has the same form [4] as that of a four-component scalar, its evaluation is straightforward; the result is given in Appendix C.7.

4. The One-Loop Effective Action

The quantum action obtained by the prescriptions defined in I (see section 2 of that paper) and in Section 2 above takes the form

$$\mathcal{L}_q = -\frac{1}{2}\Phi^T Z_\Phi \left(\hat{D}^2 + H_\Phi\right)\Phi + \frac{1}{2}\bar{\Theta} Z_\Theta \left(i \not{D} - M_\Theta\right)\Theta + \mathcal{L}_{gh} + \mathcal{L}_{Gh}. \quad (4.1)$$

The last two terms are the ghost and ghostino terms, respectively, $\Phi = (h_{\mu\nu}, \hat{A}^a, \hat{z}^i, \hat{z}^{\bar{m}})$ is a $2N + 4N_G + 10$ component scalar, $\Theta = (\psi_\mu, \lambda^a, \chi^I = L\chi^i + R\chi^{\bar{\imath}}, \alpha)$ is an $N + N_G + 5$ component Majorana fermion, where N is the number of chiral multiplets, N_G is the number of gauge multiplets, and the matrix valued metrics Z_Φ and Z_Θ are defined in Appendix B of I and in Appendices C.1 and C.2 below. As in I we set background fermion fields to zero, so $\psi_\mu, \lambda^a, \chi^I$ are the quantum gravitino, gaugino and chiral fermions, respectively, and α is the auxiliary field introduced to implement the gravitino gauge fixing condition [4]. The matrix-valued covariant derivative D_μ is defined as in Appendix A of I, and $\hat{D}_\mu$ includes additional terms in the connections that are given in (2.11,17) above.

The one-loop contribution to the effective action is

$$\begin{aligned}\mathcal{L}_1 &= \frac{i}{2}\mathrm{Tr}\ln(\hat{D}^2 + H_\Phi) - \frac{i}{2}\mathrm{Tr}\ln(-i\not{D} + M_\Theta) \\ &\quad + i\mathrm{Tr}\ln(D^2 + H_{Gh}) - i\mathrm{Tr}\ln(\hat{D}^2 + H_{gh}). \end{aligned} \quad (4.2)$$

The general results obtained in [15], [8], [5], [22] give for the bosonic determinant:

$$\frac{i}{2}\mathrm{Tr}\ln(\hat{D}^2 + H_\Phi) = \sqrt{g}\left\{\frac{\Lambda^2}{32\pi^2}\mathrm{Tr}\left(\frac{1}{6}r - H_\Phi\right)\right.$$

$$+\frac{\ln\Lambda^2}{32\pi^2}\mathrm{Tr}\left(\frac{1}{2}H_\Phi^2-\frac{1}{6}rH_\Phi+\frac{1}{12}\hat{G}^\Phi_{\mu\nu}\hat{G}^{\mu\nu}_\Phi+\frac{1}{120}\left[r^2+2r^{\mu\nu}r_{\mu\nu}\right]\right)\Bigg\},\tag{4.3}$$

and for the fermionic determinant we have

$$-\frac{i}{2}\mathrm{Tr}\ln(-i\not{D}+M_\Theta)=-\frac{i}{2}(T_++T_-)=-\frac{i}{8}\mathrm{Tr}\ln[\hat{D}^2+H_\Theta]-\frac{i}{2}T_-,\tag{4.4}$$

where in (4.4) $\hat{D}_\mu$ and H_Θ are the 8×8 matrices defined in (2.14–17). The helicity-averaged part, T_+, of the fermion trace is $-\frac{1}{4}$ times (4.3) with the substitutions $H_\Phi\to H_\Theta$, $\hat{G}^\Phi_{\mu\nu}\to\hat{G}^\Theta_{\mu\nu}$ and the trace includes a trace over Dirac indices, so

$$\frac{1}{4}(\mathrm{Tr}\,\mathbf{1})_\Theta=(\mathrm{Tr}\,\mathbf{1})_\Phi-2N_G=2N+2N_G+10.$$

Similarly, the ghost and ghostino contributions are equivalent to, respectively, -2 times the contribution of a $(4+N_G)$-component scalar and $+2$ times the contribution of a four-component scalar. For bosons, H_Φ and $\hat{D}_\mu$ are defined in Section 2; the matrix elements of H and of

$$\hat{G}_{\mu\nu}=[\hat{D}_\mu,\hat{D}_\nu],\tag{4.5}$$

are given in Appendix C, and the helicity-odd contribution, T_-, of the fermion determinant that was evaluated in Section 3, Eq. (3.6) is given in (C.36). The traces in (4.3–4.4) are given explicitly in Appendix C below and in Appendix B of I. Here we list only the contributions involving background Yang-Mills fields and/or integration over the quantum Yang-Mills supermultiplet that were omitted in I.

If $\mathcal{L}(g,K)$ is the standard Lagrangian [10], [20] for $N=1$ supergravity coupled to matter with space-time metric $g_{\mu\nu}$, Kähler potential K, and gauge kinetic normalization function $f_{ab}=\delta_{ab}(x+iy)$, then the logarithmically divergent part of the one-loop corrected Lagrangian is

$$\mathcal{L}_{eff}=\mathcal{L}(g_R,K_R)+\mathcal{L}_0+\frac{\ln\Lambda^2}{32\pi^2}\left(X^{AB}\mathcal{L}_A\mathcal{L}_B+X^A\mathcal{L}_A\right)+\sqrt{g}\frac{\ln\Lambda^2}{32\pi^2}(L+N_GL_g),\tag{4.6}$$

where the classical Lagrangian $\mathcal{L}(g,K)$ is given in Appendix C below (see footnote 1), $\mathcal{L}_0$ is the one loop correction found[6] in I after renormalization of g, K [see Eq.(3.6) of I], and

$$
\begin{aligned}
L \;=\; & \left[\mathcal{W}^{ab}\left(3C_G\delta_{ab} - D_i(T_bz)^jD_j(T_az)^i\right) + \text{h.c.}\right] - 24e^{-K}a\bar{a}\mathcal{D} \\
& + \frac{N+5}{6}\left[\left(\mathcal{W}^{ab}+\overline{\mathcal{W}}^{ab}\right)\mathcal{D}_a\mathcal{D}_b - x\left(F^a_{\rho\mu} - i\tilde{F}^a_{\rho\mu}\right)\left(F_a^{\rho\nu} + i\tilde{F}_a^{\rho\nu}\right)\mathcal{D}_\nu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\right] \\
& + \frac{N+5}{3}\left[x^2\mathcal{W}_{ab}\overline{\mathcal{W}}^{ab} + 2\mathcal{D}^2 - \mathcal{D}\left(K_{i\bar{m}}\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^{\bar{m}} + 2\hat{V} + 4\mathcal{D}M^2_\psi\right)\right] \\
& + 14x^2\mathcal{W}_{ab}\overline{\mathcal{W}}^{ab} + 12\left(\mathcal{W}^{ab}+\overline{\mathcal{W}}^{ab}\right)\mathcal{D}_a\mathcal{D}_b + 22\mathcal{D}^2 + 2\mathcal{D}\left(11\hat{V} + 8K_{i\bar{m}}\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^{\bar{m}}\right) \\
& + x\left(\mathcal{W}+\overline{\mathcal{W}}\right)\left(K_{i\bar{m}}\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^{\bar{m}} - 2M^2_\lambda - 2V\right) + 4\mathcal{D}\left(27M^2_\psi + 7M^2_\lambda\right) \\
& - 26i\mathcal{D}_\mu z^j\mathcal{D}_\nu\bar{z}^{\bar{m}}K_{i\bar{m}}\mathcal{D}^aF_a^{\mu\nu} + \frac{2}{x}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}R_{\bar{n}i\bar{m}j}\mathcal{D}_aD^j(T^a\bar{z})^{\bar{n}} \\
& + \frac{2}{x}\mathcal{D}_ae^{-K}R^{k\,j}_{\;n\,i}A_k\bar{A}^nD_j(T^az)^i + \frac{e^{-K}}{x}\mathcal{D}_a\left[(T^az)^iR_i{}^j{}_\ell{}^k\bar{A}^\ell A_{jk} + \text{h.c.}\right] \\
& + 2iF^a_{\mu\nu}D_j(T_az)^iR^j_{i\bar{m}k}\mathcal{D}^\mu z^k\mathcal{D}^\nu\bar{z}^{\bar{m}} + \frac{4}{3}\mathcal{D}e^{-K}R^i_jA_i\bar{A}^j + \frac{4}{3}\mathcal{D}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}R_{i\bar{m}} \\
& + \frac{D_i(T_az)^i}{6x}\left[4\mathcal{D}_a\left(\mathcal{D}_\mu z^j\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{j\bar{m}} + \hat{V} + 3M^2_\psi - 2\mathcal{D}\right) + 13iF^a_{\mu\nu}K_{\bar{m}j}\mathcal{D}^\mu z^j\mathcal{D}^\nu\bar{z}^{\bar{m}}\right] \\
& + \frac{1}{2}\left(\frac{19}{x^2} + \rho^i\rho_i\right)\mathcal{D}\left(\partial_\mu x\partial^\mu x + \partial_\mu y\partial^\mu y\right) - \left(\frac{11}{x^2} + 3\rho^i\rho_i\right)\partial_\mu y\partial_\nu x\mathcal{D}^aF_a^{\mu\nu} \\
& - \frac{1}{4x}\left(1 + 3x^2\rho^i\rho_i\right)\left(\partial_\rho x\partial^\nu x + \partial_\rho y\partial^\nu y\right)\left(F^a_{\mu\nu} + i\tilde{F}^a_{\mu\nu}\right)\left(F_a^{\mu\rho} - i\tilde{F}_a^{\mu\rho}\right)
\end{aligned}
$$

[6]The last five lines of (3.6) in I should read:

$$
\begin{aligned}
- \;& 4\left(\mathcal{D}_\mu\bar{z}^{\bar{m}}\mathcal{D}^\mu z^iK_{i\bar{m}}\right)^2 + \left(\frac{N}{3} - 7\right)\mathcal{D}_\mu z^j\mathcal{D}^\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{m}}\mathcal{D}^\nu\bar{z}^{\bar{n}}K_{i\bar{n}}K_{j\bar{m}} \\
+ \;& \frac{2}{3}\mathcal{D}_\mu\bar{z}^{\bar{m}}\mathcal{D}^\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{n}}\mathcal{D}^\nu z^jK_{i\bar{n}}K_{j\bar{m}} - \frac{2}{3}\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^{\bar{m}}K_{i\bar{m}}\mathcal{D}^\mu z^j\mathcal{D}_\mu\bar{z}^{\bar{n}}R_{j\bar{n}} \\
+ \;& \mathcal{D}_\mu z^j\mathcal{D}^\mu\bar{z}^{\bar{m}}R^k_{j\bar{m}i}\mathcal{D}_\nu z^\ell\mathcal{D}^\nu\bar{z}^{\bar{n}}R^i_{\ell\bar{n}k} + \mathcal{D}_\mu z^j\mathcal{D}^\mu z^iR^k{}_j{}^\ell{}_i\mathcal{D}_\nu\bar{z}^{\bar{n}}\mathcal{D}^\nu\bar{z}^{\bar{m}}R_{\bar{n}k\bar{m}\ell} \\
+ \;& \frac{1}{3}\mathcal{D}_\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{m}}K_{i\bar{m}}R_{j\bar{n}}\left(\mathcal{D}^\mu z^j\mathcal{D}^\nu\bar{z}^{\bar{n}} - \mathcal{D}^\nu z^j\mathcal{D}^\mu\bar{z}^{\bar{n}}\right) \\
+ \;& \mathcal{D}_\mu z^j\mathcal{D}_\nu\bar{z}^{\bar{m}}R^k_{i\bar{m}j}\mathcal{D}^\mu z^\ell\mathcal{D}^\nu\bar{z}^{\bar{n}}R^i_{k\bar{n}\ell} - \mathcal{D}_\mu z^j\mathcal{D}_\nu\bar{z}^{\bar{m}}R^k_{i\bar{m}j}\mathcal{D}^\nu z^\ell\mathcal{D}^\mu\bar{z}^{\bar{n}}R^i_{k\bar{n}\ell} + 4\left(\mathcal{L}_i\bar{A}^iAe^{-K} + \text{h.c.}\right).
\end{aligned}
$$

$$
\begin{aligned}
&-5\left\{\left[\frac{i}{x}\left(F_a^{\nu\mu}-i\tilde{F}_a^{\nu\mu}\right)+\frac{g_{\nu\mu}}{x^2}\mathcal{D}^a\right](\partial_\nu x+i\partial_\nu y)\,K_{i\bar{m}}(T^a z)^i\mathcal{D}_\mu\bar{z}^{\bar{m}}+\text{h.c.}\right\}\\
&-\rho^i\rho_i\left\{\left[ix\left(F_a^{\nu\mu}-i\tilde{F}_a^{\nu\mu}\right)+g_{\nu\mu}\mathcal{D}^a\right](\partial_\nu x+i\partial_\nu y)\,K_{i\bar{m}}(T^a z)^i\mathcal{D}_\mu\bar{z}^{\bar{m}}+\text{h.c.}\right\}\\
&+2ix^2\rho^i\rho_i\mathcal{D}_\mu z^j\mathcal{D}_\nu\bar{z}^{\bar{m}}K_{i\bar{m}}\mathcal{D}^a F_a^{\mu\nu}+2x^2\rho^i\rho_i\mathcal{D}\left[8M_\psi^2+M_\lambda^2+2\hat{V}-2e^{-K}a\bar{a}\right]\\
&-x^2\rho^i\rho_i\left[2x^2\mathcal{W}_{ab}\overline{\mathcal{W}}^{ab}\left(1-x^2\rho^i\rho_i\right)-4x\left(\mathcal{W}+\overline{\mathcal{W}}\right)\mathcal{D}+\left(\mathcal{W}^{ab}+\overline{\mathcal{W}}^{ab}\right)\mathcal{D}_a\mathcal{D}_b+2\mathcal{D}^2\right]\\
&+\frac{x^3}{2}\rho_i\rho^i\left(F^a_{\rho\mu}-i\tilde{F}^a_{\rho\mu}\right)\left(F_a^{\rho\nu}+i\tilde{F}_a^{\rho\nu}\right)\mathcal{D}_\nu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}+2x^2\rho_i\rho^i\mathcal{D}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\\
&+2x\left[4\rho_{ij}(T^a z)^i(T^b z)^j\mathcal{W}_{ab}+i\mathcal{D}_\nu\bar{z}^{\bar{m}}(T^a z)^i\rho_{\bar{m}ij}\left(F_a^{\mu\nu}-i\tilde{F}_a^{\mu\nu}\right)\mathcal{D}_\mu z^j+\text{h.c.}\right]\\
&+\left\{\rho_{ij}\mathcal{D}^\mu z^j\left[\frac{2}{x}(\partial_\mu x-i\partial_\mu y)(T^a z)^i\mathcal{D}_a-\frac{\bar{f}^i}{2}\mathcal{W}(\partial_\mu x+i\partial_\mu y)\right]+\text{h.c.}\right\}\\
&+\left\{\overline{\mathcal{W}}\left[2x^3\rho^i\rho_i M_\lambda^2+\bar{f}^i a_i(\bar{a}-\bar{A})e^{-K}-x^2\rho^{ij}\left(A_{jik}\bar{A}^k-A_{ij}\bar{A}\right)e^{-K}\right]+\text{h.c.}\right\}\\
&+\left\{\left(e^{-K}\bar{A}^j A^{\bar{m}}+\mathcal{D}_\mu z^j\mathcal{D}^\mu\bar{z}^{\bar{m}}\right)\left[4\rho_{\bar{m}ij}(T^a z)^i\mathcal{D}_a-\left(\rho_{\bar{m}ij}+\frac{\bar{f}_{\bar{m}}}{x}\rho_{ij}\right)\bar{f}^i\mathcal{D}\right]+\text{h.c.}\right\}\\
&-\frac{i}{2}K_{i\bar{m}}\left[\mathcal{D}^\nu\bar{z}^{\bar{m}}(T_a z)^i-\mathcal{D}^\nu z^i(T_a\bar{z})^{\bar{m}}\right]\left[\bar{f}^i\rho_{ij}\mathcal{D}^\rho z^j\left(F^a_{\rho\nu}-i\tilde{F}^a_{\rho\nu}\right)+\text{h.c.}\right]\\
&+\frac{\mathcal{D}_a}{2x}\left[K_{k\bar{m}}(T^a z)^k\mathcal{D}^\mu\bar{z}^{\bar{m}}+\frac{i}{2}(\partial_\nu x+i\partial_\nu y)\left(F^{\nu\mu}-i\tilde{F}^{\nu\mu}\right)+\text{h.c.}\right]\left(\rho_{ij}\mathcal{D}_\mu z^i\bar{f}^j+\text{h.c.}\right)\\
&-\left[\mathcal{W}^{ab}\rho_{ij}f^i(T_a z)^j\mathcal{D}_b+x^2\mathcal{D}_\rho z^i\mathcal{D}^\rho z^j\left(2\rho_{ij}\mathcal{W}-R_{\bar{n}i\bar{m}j}\rho^{\bar{m}\bar{n}}\overline{\mathcal{W}}\right)+\text{h.c.}\right]\\
&+2x^2\rho_{ij}\rho^j_{\bar{m}}\mathcal{D}\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^{\bar{m}}+x^4\rho_{ij}\rho^{ij}\mathcal{W}\overline{\mathcal{W}},
\end{aligned}\tag{4.7}
$$

$$
\begin{aligned}
L_g \;=\;& x^6\left(\rho^i\rho_i\right)^2\mathcal{W}\overline{\mathcal{W}}-2M_\lambda^4+3M_\psi^4-2M_\psi^2M_\lambda^2+\hat{V}^2+\mathcal{D}^2+6e^{-K}a\bar{a}M_\psi^2\\
&+2\hat{V}\left(2M_\psi^2-M_\lambda^2+e^{-K}a\bar{a}\right)-e^{-K}\left(\bar{a}^iA_i+\text{h.c.}\right)\left(\hat{V}+M_\psi^2\right)\\
&+e^{-2K}a_i\bar{A}^i\bar{a}^jA_j-2e^{-2K}\left(\bar{a}^iA_ia\bar{A}+\text{h.c.}\right)+x^2\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\rho_{\bar{n}\bar{m}}\mathcal{D}_\nu\bar{z}^{\bar{m}}\mathcal{D}^\nu\bar{z}^n\\
&+e^{-K}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\left[(a_i-A_i)\left(\bar{a}_{\bar{m}}-\bar{A}_{\bar{m}}\right)+x^2\rho_{ik}\bar{A}^k\rho^j_{\bar{m}}A_j+\frac{\bar{f}_{\bar{m}}f_i}{4x^2}a\bar{a}\right]\\
&+e^{-K}\left\{\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\left[(a_i-A_i)\left(\frac{f_j}{2x}\bar{a}-x\rho_{jn}\bar{A}^n\right)-\frac{f_j}{2x}a_i\bar{A}-f_i(a-A)\rho_{jk}\bar{A}^k\right]+\text{h.c.}\right\}\\
&+\frac{e^{-K}}{2x}\left\{\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\bar{f}_{\bar{m}}\left[2\bar{a}a_i-x\rho_{ik}(a-A)\bar{A}^k\right]+\frac{f_if_j}{2x}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\bar{a}(2a-A)+\text{h.c.}\right\}\\
&+x\left(\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j+\text{h.c.}\right)\left(M_\psi^2-\hat{V}\right)+e^{-K}\left[x\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\left(a_k\bar{A}^k-2\bar{A}a\right)+\text{h.c.}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{16x^4}\left|(\partial_\mu x + i\partial_\mu y)(\partial^\mu x + i\partial^\mu y)\right|^2 - x^3\rho^i\rho_i\left(\mathcal{W}+\overline{\mathcal{W}}\right)\left(M_\psi^2+\hat{V}\right)\\
&+x^3\rho^k\rho_k\left[\mathcal{W}\left(x\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j + e^{-K}A_i\bar{a}^i - 2e^{-K}\bar{a}A\right)+\text{h.c.}\right]\\
&+\frac{1}{6}K_{i\bar{m}}K_{j\bar{n}}\left(4\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\mathcal{D}_\nu\bar{z}^{\bar{m}}\mathcal{D}^\nu\bar{z}^{\bar{n}} + \mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{n}}\mathcal{D}_\nu\bar{z}^{\bar{m}}\mathcal{D}^\nu z^j\right)\\
&-\frac{1}{3}\left(\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\right)^2 + x^2\mathcal{W}_{ab}\overline{\mathcal{W}}^{ab} + \frac{1}{2}\left(\mathcal{W}_{ab}+\overline{\mathcal{W}}_{ab}\right)\mathcal{D}^a\mathcal{D}^b\\
&-\frac{1}{3}V^2 + \frac{1}{3}M_\lambda^2\left(\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}} - 2V\right) - \left(\frac{\partial_\mu x\partial^\nu x + \partial_\mu y\partial^\nu y}{x^2}\right)K_{i\bar{m}}\mathcal{D}_\nu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\\
&+\frac{1}{3}V\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}} + \left(\frac{\partial_\nu x\partial^\nu x}{6x^2}+\frac{\partial_\nu y\partial^\nu y}{6x^2}\right)\left(2\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}} - V\right)\\
&+\left(F^a_{\rho\mu}+i\tilde{F}^a_{\rho\mu}\right)\left(F_a^{\rho\nu}-i\tilde{F}_a^{\rho\nu}\right)\left(\frac{\partial^\mu x\partial_\nu x + \partial^\mu y\partial_\nu y}{4x} - \frac{x}{2}K_{i\bar{m}}\mathcal{D}_\nu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\right). \qquad (4.8)
\end{aligned}
$$

Our notation is defined in Appendix B below. Here $\mathcal{W} = \mathcal{W}^a_a$, where

$$
\mathcal{W}^a_b = \frac{1}{4}\left(F^a_{\mu\nu}F_b^{\mu\nu} - iF^a_{\mu\nu}\tilde{F}_b^{\mu\nu}\right) - \frac{1}{2x^2}\mathcal{D}^a\mathcal{D}_b \qquad (4.9)
$$

is the bosonic part of the F-component of the composite chiral supermultiplet constructed from the Yang-Mills chiral superfield $W^a(\theta) = \lambda^a_L + O(\theta)$. The renormalized Kähler potential is

$$
K_R = K + \frac{\ln\Lambda^2}{32\pi^2}\left[e^{-K}\left(A_{ij}\bar{A}^{ij} - 2A_i\bar{A}^i - 4A\bar{A}\right) - 4\mathcal{K}^a_a - \left(12+4x^2\rho_i\rho^i\right)\mathcal{D}\right], \qquad (4.10)
$$

and the renormalized space-time metric is given by

$$
\begin{aligned}
g_{\mu\nu} &= (1-\epsilon)g^R_{\mu\nu} + \epsilon_{\mu\nu},\\
\epsilon &= \epsilon^0 - \frac{\ln\Lambda^2}{32\pi^2}\left[\frac{N_G}{6}(r+V) + \frac{55-N}{6}\mathcal{D} + 2x^2\rho_i\rho^i\mathcal{D} + \frac{2}{3x}\mathcal{D}_a D_i(T^a z)^i + \frac{N_G}{3}M_\lambda^2\right],\\
\epsilon_{\mu\nu} &= \epsilon^0_{\mu\nu} + \frac{\ln\Lambda^2}{32\pi^2}\frac{N_G}{2}\left(r_{\mu\nu} - \frac{1}{2}rg_{\mu\nu}\right) - g_{\mu\nu}\frac{N_G}{6x^2}\left(\partial_\rho x\partial^\rho x + \partial_\rho y\partial^\rho y - \mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\right)\\
&\quad + N_G\left[2\frac{\nabla_\mu\partial_\nu x}{x} - \frac{\partial_\mu x\partial_\nu x}{x^2} + \frac{\partial_\mu y\partial_\nu y}{x^2} - \frac{1}{2}\left(\mathcal{D}_\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{m}} + \mathcal{D}_\nu z^i\mathcal{D}_\mu\bar{z}^{\bar{m}}\right)K_{i\bar{m}}\right]\\
&\quad - g_{\mu\nu}xF^a_{\rho\sigma}F_a^{\rho\sigma}\left(\frac{N+17}{24} + \frac{N_G}{8} - \frac{x^2\rho_i\rho^i}{4}\right)
\end{aligned}
$$

$$+xF^a_{\mu\rho}F_{a\nu}{}^{\rho}\left(\frac{N+29}{6}+\frac{N_G}{2}-x^2\rho_i\rho^i\right) \tag{4.11}$$

where the superscript 0 refers to the result of I. The terms in (4.6) proportional to $\mathcal{L}_A$ can be removed by field redefinitions:

$$\phi^A \to \phi^A_R = \phi^A - \frac{\ln\Lambda^2}{32\pi^2}\left(X^A + \frac{1}{2}X^{AB}\mathcal{L}_B\right), \tag{4.12}$$

with

$$\begin{aligned}
X_{i\bar m} &= \frac{N_G}{4x^2\sqrt{g}}f_i f_{\bar m}, \quad X_{a\mu,b\nu} = -\frac{1}{x\sqrt{g}}\left(7+x^2\rho_i\rho^i\right)\delta_{ab}g_{\mu\nu},\\
X^i &= (X^{\bar\imath})^* = 4e^{-K}\bar A^i A + \frac{2}{x}\left(2+x^2\rho^j\rho_j\right)\mathcal{D}_a(T^a z)^i\\
&\quad -4x\mathcal{D}\rho^i - 2\rho^i_{\bar m}(T_a\bar z)^{\bar m}\mathcal{D}_a - N_G\frac{\partial_\mu x}{x}\mathcal{D}^\mu z^i\\
&\quad +N_G\frac{\bar f^i}{2x}\left[x^3\rho^j\rho_j\mathcal{W} + x\rho_{jk}\mathcal{D}_\mu z^j\mathcal{D}^\mu z^k + e^{-K}\left(\bar a^j A_j - 2\bar a A\right) - \hat V - M_\psi^2\right],\\
X_{\mu a} &= \frac{i}{x}\left(16+2x^2\rho^i\rho_i\right)K_{i\bar m}\left[(T_a z)^i\mathcal{D}_\mu\bar z^{\bar m} - (T_a\bar z)^{\bar m}\mathcal{D}_\mu z^i\right]\\
&\quad +x\rho_i\rho^i\left(\partial^\rho x F_{a\rho\mu} + \partial^\rho y\tilde F_{a\rho\mu}\right) + 3\frac{\partial^\rho y}{x}\tilde F_{a\rho\mu} + \frac{\partial^\rho x}{x}(7-N_G)F_{a\rho\mu}\\
&\quad +\frac{1}{2}\left[\left(F_{a\rho\mu} - i\tilde F_{a\rho\mu}\right)\mathcal{D}^\rho z^i\rho_{ij}\bar f^j + \text{h.c.}\right] - \left(5+x^2\rho^i\rho_i\right)\frac{\partial_\mu y}{x^2}\mathcal{D}_a.
\end{aligned} \tag{4.13}$$

The terms in (4.7–8) of the form $g(z,\bar z)\mathcal{W}\overline{\mathcal{W}}$ are the bosonic part of the effective Lagrangian (in the notation of [20])

$$\mathcal{L}_{|W|^4} = \int d^4\theta E g(Z,\bar Z)|WW|^2. \tag{4.14}$$

It should be possible to write the remaining terms in superfield form[7] [up to total derivatives and field redefinitions of the form (4.11-13)], and thus

[7]Note that $F^i = -e^{-K/2}\bar A^i$ and $M = -3e^{-K/2}A$ are the bosonic parts of auxiliary fields of the chiral superfield Z^i and the gravity superfield, respectively. It is easy to show that calculating the one loop corrections before or after elimination of the auxiliary fields in terms of their classical solutions gives the same result to the loop order considered. Our results are expressed in terms of these auxiliary fields in [30].

to extract the fermionic part of the Lagrangian for these higher dimension operators. However, there may be additional fermionic terms, *e.g,* those of the form [23]

$$\mathcal{L}_{W^{2n}} = \int d^4\theta E g(Z,\bar{Z})(WW)^{n>1} + \text{h.c.}, \tag{4.15}$$

that cannot be obtained in this way, as they have no purely bosonic components. The determination of such terms requires retaining fermionic background fields [24], [8], [16].

Notice that the coefficient of $\ln\Lambda^2 F^{\mu\nu}F_{\mu\nu}$ is not a holomorphic function, except in the limits of a flat Kähler metric ($D_i \rightarrow \partial_i$) and flat space-time ($M_{Pl} \rightarrow \infty$, in which case operators of dimension greater than four are suppressed). This nonholomorphicity is distinct from from the holomorphic anomaly [1, 25] that arises from the field-dependence of the infrared regulator masses. In other words, when the Kähler and/or space-time metric is not flat, there are corrections that correspond to D-terms as well as the usual F-terms.

The quadratically divergent contributions to the one-loop Lagrangian are given by (C.33–C.35). The Pauli-Villars regularization of these terms was given in [13]; they contribute additional renormalizations of the metric and the Kähler potential that are determined by the field-dependent squared masses of the Pauli-Villars regulator fields that play the role of effective cut-offs. The field dependence of the effective cut-offs in the logarithmically divergent contribution to the renormalized Kähler potential will generate additional terms in the effective Lagrangian proportional to

$$D_I \ln\Lambda^2 = 2\frac{D_I\Lambda}{\Lambda}, \quad I = i, \bar{\imath},$$

that do not grow with the cut-off, and therefore have to be considered together with the finite terms that we have not evaluated here.

5. The String Dilaton

In effective supergravity from superstring theory, the classical Kähler potential $K(z,\bar{z})$, superpotential $W(z)$ and Yang-Mills normalization function $f_{ab}(z)$ take the forms

$$\begin{aligned} K(z,\bar{z}) &= -\ln(s+\bar{s}) + G(y^i,\bar{y}^{\bar{m}}), \quad W(z) = W(y^i), \\ f_{ab}(z) &= \delta_{ab}k_a s, \quad y^i,\bar{y}^{\bar{m}} \neq s. \end{aligned} \tag{5.1}$$

Although we have restricted our analysis to the case $f_{ab} = \delta_{ab}f$, it is equally applicable to the case $f_{ab} = \delta_{ab}k_a f$, k_a = constant, provided we make the substitutions $F^a_{\mu\nu} \to k_a^{\frac{1}{2}} F^a_{\mu\nu}$, $A^a_\mu \to k_a^{\frac{1}{2}} A^a_\mu$, $T^a \to k_a^{-\frac{1}{2}} T^a$, $c_{abc} \to k_a^{-\frac{1}{2}} c_{abc}$, ($c_{abc} \neq 0$ only if $k_a = k_b = k_c$) in all the relevant equations. Our results are therefore applicable to all known effective tree Lagrangians from superstrings, including those where the integers $k_a \geq 1$ correspond to higher affine levels [11]. In this case the operators $a, \rho_{ij}, 1 - x^2\rho_i\rho^i$, and their covariant derivatives vanish identically. In particular $M^2_\lambda = M^2_\psi \equiv M^2$, and (4.6) reduces to

$$\begin{aligned} \mathcal{L}_{eff} &= \mathcal{L}(g_R, K_R) + \mathcal{L}_0 + \frac{\ln\Lambda^2}{32\pi^2}\left(X^{AB}\mathcal{L}_A\mathcal{L}_B + X^A\mathcal{L}_A\right) + \sqrt{g}\frac{\ln\Lambda^2}{32\pi^2}\left(L + N_G L_g\right), \\ L &= \left(\mathcal{W}^{ab} + \overline{\mathcal{W}}^{ab}\right)\left(3C_G\delta_{ab} - D_i(T_b z)^j D_j(T_a z)^i\right) + 2\mathcal{D}\left(13\hat{V} + 9K_{i\bar{m}}\mathcal{D}_\mu z^i \mathcal{D}^\mu \bar{z}^{\bar{m}}\right) \\ &+ \frac{N+5}{12}\left[(s+\bar{s})^2\mathcal{W}_{ab}\overline{\mathcal{W}}^{ab} + 2\left(\mathcal{W}^{ab} + \overline{\mathcal{W}}^{ab}\right)\mathcal{D}_a\mathcal{D}_b + 8\mathcal{D}^2 - 8\left(\hat{V} + 2M^2\right)\mathcal{D}\right] \\ &- \frac{N+5}{12}\left[(s+\bar{s})\left(F^a_{\rho\mu} - i\tilde{F}^a_{\rho\mu}\right)\left(F_a^{\rho\nu} + i\tilde{F}_a^{\rho\nu}\right) + 4g^\nu_\mu\mathcal{D}\right]\mathcal{D}_\nu z^i \mathcal{D}^\mu \bar{z}^{\bar{m}} K_{i\bar{m}} \\ &+ \frac{7}{2}(s+\bar{s})^2\mathcal{W}_{ab}\overline{\mathcal{W}}^{ab} + 11\left(\mathcal{W}^{ab} + \overline{\mathcal{W}}^{ab}\right)\mathcal{D}_a\mathcal{D}_b + 20\mathcal{D}^2 + 154M^2\mathcal{D} \\ &+ \frac{(s+\bar{s})}{2}\left(\mathcal{W} + \overline{\mathcal{W}}\right)\left(K_{i\bar{m}}\mathcal{D}_\rho z^i \mathcal{D}^\rho \bar{z}^{\bar{m}} - 2\hat{V} + 2\mathcal{D}\right) - 24i\mathcal{D}_\mu z^i \mathcal{D}_\nu \bar{z}^{\bar{m}} K_{i\bar{m}}\mathcal{D}^a F_a^{\mu\nu} \\ &+ \frac{(s+\bar{s})}{4}\left(F^a_{\rho\mu} - i\tilde{F}^a_{\rho\mu}\right)\left(F_a^{\rho\nu} + i\tilde{F}_a^{\rho\nu}\right)\mathcal{D}_\nu z^i \mathcal{D}^\mu \bar{z}^{\bar{m}} K_{i\bar{m}} \\ &+ \frac{D_i(T_a z)^i}{3(s+\bar{s})}\left[4\mathcal{D}^a\left(\hat{V} - 2\mathcal{D} + 3M^2 + K_{j\bar{m}}\mathcal{D}_\mu z^j \mathcal{D}^\mu \bar{z}^{\bar{m}}\right) + 13iF^a_{\mu\nu}K_{\bar{m}j}\mathcal{D}^\mu z^j \mathcal{D}^\nu \bar{z}^{\bar{m}}\right] \\ &- \left(\frac{4}{3}\mathcal{D}R_{i\bar{m}} + \frac{\mathcal{D}_a}{(s+\bar{s})}R_{\bar{n}i\bar{m}j}D^j(T^a\bar{z})^{\bar{n}}\right)\left(e^{-K}\bar{A}^i A^{\bar{m}} + \mathcal{D}_\mu z^i \mathcal{D}^\mu \bar{z}^{\bar{m}}\right) \end{aligned}$$

$$
\begin{aligned}
&+2iF^a_{\mu\nu}D_j(T_az)^iR^j_{i\bar m k}\mathcal{D}^\mu z^k\mathcal{D}^\nu\bar z^{\bar m}+\frac{2e^{-K}}{(s+\bar s)}\mathcal{D}_a\left[(T^az)^iR_i{}^j{}_\ell{}^k\bar A^\ell A_{jk}+\mathrm{h.c.}\right]\\
&-\frac{12}{s+\bar s}\left\{\left[i\partial_\nu s\left(F^{\nu\mu}_a-i\tilde F^{\nu\mu}_a\right)+\frac{2\partial^\mu s}{s+\bar s}\mathcal{D}_a\right]\mathcal{D}_\mu\bar z^{\bar m}K_{i\bar m}(T^az)^i+\mathrm{h.c.}\right\}\\
&-2\frac{\partial_\rho s\partial^\nu\bar s}{(s+\bar s)}\left(F^a_{\mu\nu}+i\tilde F^a_{\mu\nu}\right)\left(F^{\mu\rho}_a-i\tilde F^{\mu\rho}_a\right)+40\frac{\partial_\mu s\partial^\mu\bar s}{(s+\bar s)^2}\mathcal{D}+28i\frac{\partial_\mu s\partial_\nu\bar s}{(s+\bar s)^2}\mathcal{D}^aF^{\mu\nu}_a,\\
L_g\ =\ &\frac{(s+\bar s)^2}{4}\left(\mathcal{W}_{ab}\overline{\mathcal{W}}^{ab}+\mathcal{W}\overline{\mathcal{W}}\right)-\frac{s+\bar s}{2}\left(\mathcal{W}+\overline{\mathcal{W}}\right)\left(M^2+\hat V\right)+\mathcal{D}^2\\
&+\frac{1}{2}\left(\mathcal{W}_{ab}+\overline{\mathcal{W}}_{ab}\right)\mathcal{D}^a\mathcal{D}^b-2\left(\mathcal{D}+\frac{1}{3}V\right)M^2+\frac{1}{3}\left(M^2+V\right)\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar z^{\bar m}K_{i\bar m}\\
&-\frac{1}{3}V^2+\frac{1}{6}K_{i\bar m}K_{j\bar n}\left(4\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\mathcal{D}_\nu\bar z^{\bar m}\mathcal{D}^\nu\bar z^{\bar n}+\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar z^{\bar n}\mathcal{D}_\nu\bar z^{\bar m}\mathcal{D}^\nu z^j\right)\\
&-\frac{1}{3}K_{i\bar m}K_{j\bar n}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar z^{\bar m}\mathcal{D}_\nu z^j\mathcal{D}^\nu\bar z^{\bar n}+\frac{2}{3}\left(2K_{i\bar m}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar z^{\bar m}-V\right)\frac{\partial_\nu s\partial^\nu\bar s}{(s+\bar s)^2}\\
&+\frac{\partial_\mu s\partial^\mu s\partial_\nu\bar s\partial^\nu\bar s}{(s+\bar s)^4}-\frac{2\partial_\mu s\partial_\nu\bar s}{(s+\bar s)^2}K_{i\bar m}\left(\mathcal{D}^\mu z^i\mathcal{D}^\nu\bar z^{\bar m}+\mathcal{D}^\mu\bar z^{\bar m}\mathcal{D}^\nu z^i\right)\\
&+\left(F^a_{\rho\mu}+i\tilde F^a_{\rho\mu}\right)\left(F^{\rho\nu}_a-i\tilde F^{\rho\nu}_a\right)\left(\frac{\partial_\nu s\partial^\mu\bar s}{2(s+\bar s)}-\frac{s+\bar s}{4}K_{i\bar m}\mathcal{D}_\nu z^i\mathcal{D}^\mu\bar z^{\bar m}\right),
\end{aligned}
\tag{5.2}
$$

with, instead of (4.10),

$$
K_R=K+\frac{\ln\Lambda^2}{32\pi^2}\left(e^{-K}\left[A_{ij}\bar A^{ij}-2A_i\bar A^i+(N_G-4)A\bar A\right]-4\mathcal{K}^a_a-16\mathcal{D}\right). \tag{5.3}
$$

Here we have considered only the standard chiral multiplet formulation of supergravity. Their is reason to believe [2], [3], [26] that the dilaton in the effective field theory from superstrings should be described, in fact, by a linear multiplet, which is dual to the chiral multiple used here. It has been shown [27] that a variety of classically dual theories remain equivalent at the quantum level. In [13] it was observed that once the ambiguous matrix elements (3.9) have been fixed in a supersymmetric way that admits Pauli-Villars regularization, the axion y of the dilaton supermultiplet appears only through its dual $h^{\nu\rho\sigma}=\epsilon^{\nu\rho\sigma\mu}\partial_\mu y/4x^2$. This suggests that the properly regulated chiral supergravity theory also remains equivalent to the linear multiplet version for the dilaton at the quantum level. Some loop corrections using the linear multiple formulation have been carried out in [28].

As shown in I, further simplifications occur[8] in specific models, such as the untwisted sectors from orbifold compactifications where the scalar Riemann tensor is covariantly constant and the Ricci tensor is proportional to the Kähler metric for each untwisted sector.

6. Conclusions

In this paper we have completed the results of I by including the gauge sector. The complete divergent part of the one-loop Lagrangian, obtained from the results of this paper and from I, will be presented elsewhere in a short communication [30].

Some comments on the implications and applications of our results are in order. It has already been shown [13] that, using the gauge fixing and expansion procedures defined here, the one-loop quadratic divergences, as well as the logarithmic divergences in the flat space limit and in the absence of a dilaton, can be regulated à la Pauli-Villars. Regularization of the full supergravity divergences without a dilaton are under study [18]. An

[8]The four-derivative terms of (4.4) of I should read:

$$\begin{aligned}
&-4\left(\mathcal{D}_\mu\bar{z}^{\bar{m}}\mathcal{D}^\mu z^i K_{i\bar{m}}\right)^2 + \left(\frac{N}{3}-7\right)\mathcal{D}_\mu z^j\mathcal{D}^\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{m}}\mathcal{D}^\nu\bar{z}^{\bar{n}}K_{i\bar{n}}K_{j\bar{m}} \\
&+\frac{2}{3}\mathcal{D}_\mu\bar{z}^{\bar{m}}\mathcal{D}^\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{n}}\mathcal{D}^\nu z^j K_{i\bar{n}}K_{j\bar{m}} - \frac{2}{3}\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^{\bar{m}}K_{i\bar{m}}\sum_\alpha\left(N_\alpha+1\right)\mathcal{D}^\mu z^j\mathcal{D}_\mu\bar{z}^{\bar{n}}K^\alpha_{j\bar{n}} \\
&+\frac{1}{3}\mathcal{D}_\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{m}}K_{i\bar{m}}\sum_\alpha\left(N_\alpha+1\right)K^\alpha_{j\bar{n}}\left(\mathcal{D}^\mu z^j\mathcal{D}^\nu\bar{z}^{\bar{n}} - \mathcal{D}^\nu z^j\mathcal{D}^\mu\bar{z}^{\bar{n}}\right) \\
&+\sum_\alpha\left[\left(N_\alpha+1\right)\left(\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K^\alpha_{i\bar{m}}\right)^2 + \left(N_\alpha+7\right)\mathcal{D}_\mu z^j\mathcal{D}^\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{n}}\mathcal{D}^\nu\bar{z}^{\bar{m}}K^\alpha_{i\bar{m}}K^\alpha_{j\bar{n}}\right. \\
&\left.-\left(N_\alpha+1\right)\mathcal{D}_\mu\bar{z}^{\bar{m}}\mathcal{D}^\mu z^i\mathcal{D}_\nu z^j\mathcal{D}^\nu\bar{z}^{\bar{n}}K^\alpha_{j\bar{m}}K^\alpha_{i\bar{n}}\right].
\end{aligned}$$

objective of this study is to determine the extent to which, in the string theory context, a modular invariant regularization procedure can be achieved that preserves the continuous $SL(2,R)$ symmetry of the classical effective Lagrangian. To obtain the full one-loop Lagrangian, including all finite contributions, requires a resummation of the derivative expansion. A procedure for resummation will be described elsewhere [18].

We have presented our results for one-loop corrections to the classical general supergravity Lagrangian [10, 20] with at most two-derivative terms. As seen in Section 5, the result simplifies considerably for the classical effective Lagrangian derived from string theory, due to the the absence of a potential for the dilaton and the special form of its Kähler potential. These features are modified when the effective Lagrangian includes a nonperturbatively induced [31] superpotential for the dilaton and/or the Green-Schwarz counterterm [2] that is necessary to restore modular invariance. The latter term destroys the no-scale nature of Lagrangians from torus compactification and the untwisted sector of orbifold compactification, and generally destabilizes the effective scalar potential. However this term is of one-loop order and therefore should be considered together with the full one-loop corrections. An interesting question, that will be addressed elsewhere, is whether these corrections can restabilize the potential.

An important unresolved issue in the construction of effective supergravity Lagrangians for gaugino condensation is the correct form of the kinetic term for the composite chiral multiplet that represents the lightest bound state of the confined Yang-Mills sector. It has recently been shown [32], in the context of both the linear and chiral multiplet formulations for the dilaton, that such terms can be generated by higher dimension operators. The contribution (4.14) to the effective Lagrangian determines the leading one-loop contribution to these operators; similar terms occur in string theory [33]. This is one example of how the determination of loop corrections can serve as guide to the construction of such an effective theory.

Acknowledgements. We thank Josh Burton for his collaboration at the initial stage of this work. This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grants PHY-90-21139 and PHY–93–09888.

Appendix

A. Dirac algebra

We work in the Weyl representation for the Dirac matrices; for a flat metric:

$$\begin{aligned}\gamma_0 &= \gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^i = -\gamma_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\ \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \end{aligned} \tag{A.1}$$

To evaluate the fermion determinant, we note that an arbitrary 4×4 Dirac matrix $\mathcal{M}_4$ can be written as

$$\mathcal{M}_4 = RAR + LBL + RCL + LDR, \tag{A.2}$$

where A, B contain an even number of Dirac matrices γ_ν, C, D contain an odd number, A, B, C, D have no explicit γ_5-dependence, and $L = \frac{1}{2}(1 - \gamma_5)$ and $R = \frac{1}{2}(1 + \gamma_5)$ are the helicity projection operators. Then $\mathrm{Tr}\mathcal{M}_4 = \mathrm{Tr}RA + \mathrm{Tr}LB = \mathrm{Tr}\mathcal{M}_8$, where $\mathcal{M}_8$ is the 8×8 matrix

$$\mathcal{M}_8 = \begin{pmatrix} RAR & RCL \\ LDR & LBL \end{pmatrix}, \tag{A.3}$$

and $\mathrm{Tr}f(\mathcal{M}_4) = \mathrm{Tr}f(\mathcal{M}_8)$, where f is any function that can be expanded in a Taylor series. Writing $\mathcal{M}_4 \equiv \mathcal{M}_4(\gamma_5)$, we have

$$\mathcal{M}_4(-\gamma_5) \;=\; RBR + LAL + RDL + LCR,$$

$$\frac{1}{2}\left[\mathrm{Tr}\mathcal{M}_4(\gamma_5)+\mathrm{Tr}\mathcal{M}_4(-\gamma_5)\right] = \frac{1}{2}(\mathrm{Tr}A+\mathrm{Tr}B)=\frac{1}{2}\mathrm{Tr}\begin{pmatrix} A & C \\ D & B\end{pmatrix}. \quad (A.4)$$

Similarly, if f is an arbitrary function of $\mathcal{M}_4$,

$$\frac{1}{2}\left\{\mathrm{Tr}f\left[\mathcal{M}_4(\gamma_5)\right]+\mathrm{Tr}f\left[\mathcal{M}_4(-\gamma_5)\right]\right\}=\frac{1}{2}\mathrm{Tr}f(P), \quad P=\begin{pmatrix} A & C \\ D & B\end{pmatrix}. \quad (A.5)$$

Setting $\mathcal{M}_4=-i\not{D}+M_\Theta$, $f(\mathcal{M}_4)=\ln\mathcal{M}_4$, (A.5) gives the trace T_+ that has been evaluated previously[9] [4]–[8]. To evaluate the determinant T_- we define

$$\mathcal{M}_4=\gamma_0\left(-i\not{D}+M_\Theta\right), \quad (A.6)$$

which is a 4×4 matrix in Dirac space that we write [5] in terms of the 2×2 Pauli σ-matrices as

$$\begin{aligned}
\mathcal{M}_4 &= -\begin{pmatrix} i\tilde{\not{A}} & C \\ \tilde{D} & i\not{B}\end{pmatrix}, \quad \sigma^\mu_\pm=(1,\pm\vec{\sigma}), \quad \sigma^{\mu\nu}_\pm=\frac{i}{2}\left(\sigma^\mu_\pm\sigma^\nu_\mp-\sigma^\nu_\pm\sigma^\mu_\mp\right), \\
\tilde{\not{A}} &= \sigma^\mu_+ d^+_\mu=\sigma^\mu_+\left[\tilde{D}^+_\mu-\tilde{L}_\mu(\sigma_-,\sigma_+)\right], \\
\not{B} &= \sigma^\mu_- d^-_\mu=\sigma^\mu_-\left[\tilde{D}^-_\mu-\tilde{L}_\mu(\sigma_+,\sigma_-)\right], \\
C &= m+M_{\mu\nu}\sigma^{\mu\nu}_+\equiv M(\sigma^{\mu\nu}_+), \quad \tilde{D}=\bar{m}+\bar{M}_{\mu\nu}\sigma^{\mu\nu}_-\equiv\bar{M}(\sigma^{\mu\nu}_-)
\end{aligned}$$

$$\tilde{L}_\mu(\sigma_-,\sigma_+)=\frac{L_\mu}{24}\epsilon_{\lambda\nu\rho\sigma}\sigma^\lambda_-\sigma^\nu_+\sigma^\rho_-\sigma^\sigma_+. \quad (A.7)$$

The matrix elements in $\mathcal{M}_4$ are defined, up to the γ_5 ambiguity noted in [13], in terms of those appearing in the fermionic part of the action (4.1) by:

$$D_\mu=\tilde{D}_\mu+i\gamma_5L_\mu=iD^+_\mu R+iD^-_\mu L, \quad M_\Theta=\bar{M}(\sigma^{\mu\nu})R+M(\sigma^{\mu\nu})L. \quad (A.8)$$

The matrix-valued derivative operator $\tilde{D}_\mu$ is defined in (A.12) of I, the additional gaugino connection L_μ is given in (C.19) below, and the elements of the mass matrix $M_\Theta=\bar{M}R+ML$ are given in (2.16), (2.17), (A.11) and

[9] The contributions from the terms $M_{\mu\nu}\sigma^{\mu\nu}$ were not fully included in [5].

(B.10) of I, together with (C.15) below. The tilde operation on $\not{A}, \not{B}, C, D$ amounts to the interchange $\sigma_+ \leftrightarrow \sigma_-$. Thus

$$
\begin{aligned}
\not{A}L &= \begin{pmatrix} 0 & -\not{A} \\ 0 & 0 \end{pmatrix}, \quad \not{A}R = \begin{pmatrix} 0 & 0 \\ -\tilde{\not{A}} & 0 \end{pmatrix}, \\
\not{A}\tilde{\not{A}} &= R\left(\tilde{\not{D}}^+ - \frac{\not{V}}{24}\epsilon_{\lambda\nu\rho\sigma}\gamma^\lambda\gamma^\nu\gamma^\rho\gamma^\sigma\right)^2 R = R\,\not{D}^2 R, \\
\tilde{\not{B}}\not{B} &= L\left(\tilde{\not{D}}^- - \frac{\not{V}}{24}\epsilon_{\lambda\nu\rho\sigma}\gamma^\lambda\gamma^\nu\gamma^\rho\gamma^\sigma\right)^2 L = L\,\not{D}^2 L, \qquad \text{(A.9)}
\end{aligned}
$$

where the appropriate zero's in the transition from 2×2 to 4×4 matrices is implicit in the last two lines. More, generally, products of $\sigma_\pm^\mu$ can be converted into products of γ^μ by

$$
\begin{aligned}
(\sigma_+\sigma_-)^n\,\sigma_+ &\to -L\gamma^{2n+1}R, \quad (\sigma_-\sigma_+)^n\,\sigma_- \to -R\gamma^{2n+1}L, \\
(\sigma_+\sigma_-)^n &\to L\gamma^{2n}L, \quad (\sigma_-\sigma_+)^n \to R\gamma^{2n}R. \qquad \text{(A.10)}
\end{aligned}
$$

Then defining

$$
\begin{aligned}
S_\pm &= \frac{1}{2}\left[\mathrm{Tr}\ln\mathcal{M}_4(M,\vec{\sigma}) \pm \mathrm{Tr}\ln\mathcal{M}_4(-M,-\vec{\sigma})\right], \\
\mathcal{M}_4(-M,-\vec{\sigma}) &= -\begin{pmatrix} i\not{A} & -\tilde{C} \\ -D & i\tilde{\not{B}} \end{pmatrix} = \mathcal{M}_4(-M,-\gamma_5)\gamma_0, \qquad \text{(A.11)}
\end{aligned}
$$

(A.9–10) immediately gives:

$$
\begin{aligned}
S_+ &= \frac{1}{2}\mathrm{Tr}\ln\left[\mathcal{M}_4(-M,-\vec{\sigma})\mathcal{M}_4(M,\vec{\sigma})\right] \\
&= \frac{1}{2}\mathrm{Tr}\ln\begin{pmatrix} -R[\not{D}_+^2 + M\bar{M}]R & -R[i\,\not{D}^+M - Mi\,\not{D}^-]L \\ -L[i\,\not{D}^-\bar{M} - \bar{M}i\,\not{D}^+]R & -L[\not{D}_-^2 + \bar{M}M]L \end{pmatrix} \\
&= \frac{1}{2}\mathrm{Tr}\ln\left(-\not{D}^2 - M_\Theta^2 + i[\not{D}, M_\Theta]\right) = \frac{1}{2}\mathrm{Tr}\ln\left(-\hat{D}^2 - H_\Theta^2\right). \qquad \text{(A.12)}
\end{aligned}
$$

where $\hat{D} = \hat{D}_\Theta$ and H_Θ are defined in (2.17) Although the matrix in (A.12) is 8×8, the helicity projection operators L, R project out half the elements, so the counting of states is unchanged when we take the Dirac trace. Since

$\mathrm{Tr}\ln \mathcal{M}(M) = \mathrm{Tr}\ln \mathcal{M}(-M)$, we have[10] $S_\pm = T_\pm$, and (A.12) is equivalent to (A.5), up to the ambiguity described in [13]: terms even and odd in γ_5 can be interchanged using $\gamma_5 = (i/24)\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$.

The next step is to cast $S_- = T_-$ in the form of (3.3) and to take its Fourier transform to obtain an expression of the form (3.4), but before performing the p-integration we write

$$
\begin{aligned}
\mathcal{M}^{-1}[\mathcal{M}(\gamma_5) - \mathcal{M}(-\gamma_5)] &= \mathcal{M}^{-1}\mathcal{M}_0^{-1}\mathcal{M}_0[\mathcal{M}(\gamma_5) - \mathcal{M}(-\gamma_5)] \\
&= 2\left(D^2 - \frac{i}{2}\sigma_{\mu\nu}G^{\mu\nu} + iD_\mu M^\mu\right)^{-1} iD_\nu N^\nu, \qquad \text{(A.13)}
\end{aligned}
$$

where $\mathcal{M}_0$ is[11] the matrix (A.11) with

$$
C = D = 0, \quad A_\mu = \tilde{D}^+_\mu + \tilde{L}_\mu(\sigma_+, \sigma_-), \quad B_\mu = \tilde{D}^-_\mu + \tilde{L}_\mu(\sigma_-, \sigma_+),
$$

$$
\begin{aligned}
\frac{1}{2}[\mathcal{M}(\gamma_5) - \mathcal{M}(-\gamma_5)] &= -\begin{pmatrix} \tilde{\not{\mathcal{J}}} & M_I(\sigma_+^{\mu\nu}) \\ -M_I(\sigma_-^{\mu\nu}) & -\not{\mathcal{J}} \end{pmatrix}, \\
\mathcal{J}_\mu = \frac{i}{2}(D^+_\mu - D^-_\mu), &\qquad M_I = \frac{1}{2}(M - \bar{M}), \qquad \text{(A.14)}
\end{aligned}
$$

and

$$
N_\mu = \begin{pmatrix} -R\gamma_\mu \not{\mathcal{J}} R & R\gamma_\mu M_I L \\ -L\gamma_\mu M_I R & L\gamma_\mu \not{\mathcal{J}} L \end{pmatrix}, \quad M_\mu = \begin{pmatrix} 0 & R\gamma_\mu M L \\ L\gamma_\mu \bar{M} R & 0 \end{pmatrix}. \qquad \text{(A.15)}
$$

We then redefine the integrand by [15]

$$
T(p,x) \to UT(p,x)U^{-1}, \quad U = \exp\left(-id\cdot\frac{\partial}{\partial p}\right)\exp\left(i\partial\cdot\frac{\partial}{\partial p}\right), \qquad \text{(A.16)}
$$

[10]In [5] it was incorrectly stated that $S_- = 0$.

[11]It might seem more efficient to take instead $\mathcal{M}_0 = \mathcal{M}_4(-M, -\vec{\sigma})$ but this form turns out to introduce a spurious quadratic divergent term involving $M_{\mu\nu}$. To explicitly regulate ultraviolet (or infrared) divergences, one should introduce a regulator mass matrix μ_0 and set $\mathcal{M}_0 \to \mathcal{M}_0 + \mu_0$; see the discussion in Section 3.

which leaves the (properly regulated) integral unchanged. In the absence of background space-time curvature, the 8×8 matrix valued operator d_μ is simply

$$d_\mu = D_\mu = \frac{\partial}{\partial x^\mu} + a_\mu(x). \tag{A.17}$$

In the presence of space-time curvature, one has to expand [8] the action at $x' = x + y$ in terms of normal coordinates, $\xi^\mu = y^\mu + \frac{1}{2}\gamma^\mu_{\rho\nu}(x) y^\rho y^\nu + O(\xi^3)$:

$$d_\mu = \frac{\partial}{\partial \xi^\mu} + a_\mu(x, \xi). \tag{A.18}$$

where $\gamma^\mu_{\rho\nu}$ is the affine connection, and the full connection $a_\mu(x,\xi)$ includes terms that depend on the affine connection and its derivatives. The expansion of (A.13) for this case is determined in [8]. We then obtain the expression (3.4) with

$$\begin{aligned}
T(p,x) &= -\frac{1}{2}\mathrm{Tr}\ln\left[1 + 2\Delta(x,p)p^2\mathcal{R}_5(x,p)\right], \\
\Delta^{-1} &= -T^{\mu\nu}\Delta_\mu\Delta_\nu + \hat{h} + X + (p^\nu + G^\nu)\, P_{\mu\nu}\widehat{M}^\mu, \\
\Delta_\mu &= p_\mu + G_\mu + \delta_\mu, \qquad -p^2\mathcal{R}_5 = (p^\nu + G^\nu)\, P_{\mu\nu}\hat{N}^\mu, \qquad h = -\frac{i}{2}\sigma_{\mu\nu}G^{\mu\nu} \\
G_\mu &= \sum_{m=0} \frac{m+1}{(m+2)!}\left(-iD\cdot\frac{\partial}{\partial p}\right)^m G_{\nu\mu}\frac{\partial}{\partial p_\nu}, \quad G_{\mu\nu} = [D_\mu, D_\nu], \\
\hat{F} &= \sum_0^\infty \frac{(-i)^n}{n!}\left(D\cdot\frac{\partial}{\partial p}\right)^n F, \quad F = h, M^\mu, N^\mu, \quad D\cdot\frac{\partial}{\partial p}X \equiv [D_\mu, X]\frac{\partial}{\partial p_\mu}, \\
P^{\mu\nu}\gamma_\nu &= P^\mu = \gamma^\mu - \frac{1}{6}r^{\mu\rho\sigma\nu}\gamma_\nu\frac{\partial^2}{\partial p^\rho\partial p^\sigma} + O\left(\frac{\partial^3}{\partial p^3}\right), \\
T^{\mu\nu} &= g^{\mu\nu} - \frac{1}{3}r^{\mu\rho\sigma\nu}\frac{\partial^2}{\partial p^\rho\partial p^\sigma} - \frac{i}{6}\nabla^\lambda r^{\mu\rho\sigma\nu}\frac{\partial^3}{\partial p^\rho\partial p^\sigma\partial p^\lambda} + O\left(\frac{\partial^4}{\partial p^4}\right), \\
X &= -\frac{r}{3} - \frac{i}{3}\nabla^\mu r\frac{\partial}{\partial p^\mu} + O\left(\frac{\partial^2}{\partial p}\right), \\
\delta_\mu &= \frac{i}{9}\left(\nabla_\mu r_{\rho\nu} - \nabla_\nu r_{\rho\mu}\right)\frac{\partial^2}{\partial p_\nu\partial p_\rho} + O\left(\frac{\partial^3}{\partial p^3}\right),
\end{aligned} \tag{A.19}$$

Finally we write $\Delta^{-1} = -p^2(1+\mathcal{R})$ and expand

$$\Delta = (1+\mathcal{R})^{-1}(-p^{-2}) = \sum_{n=0}(-\mathcal{R})^n(-p^{-2}) \tag{A.20}$$

to obtain the expression (3.8), where we have set $\mu_0 = 0$.

Once all these manipulations have been performed we can simplify the expression for the fermion connection by using simply

$$D^{\pm}_{\mu} = \tilde{D}^{\pm}_{\mu} + i\gamma_5 L_{\mu}. \tag{A.21}$$

The point is that the part of the gaugino connection arising from the dilaton has been included in the "vector" ($\mathcal{J}^V_\mu$):

$$\partial_\mu + \mathcal{J}^V_\mu = \frac{1}{2}\left(D^+_\mu + D^-_\mu\right) = \frac{1}{2}\left(\tilde{D}^+_\mu + \tilde{D}^-_\mu\right) + i\gamma_5 L_\mu, \tag{A.22}$$

rather than in the "axial vector" ($\mathcal{J}_\mu$) part of the connection.

We conclude this appendix by listing some Dirac traces that are useful in the evaluation of T_- and of the ghostino and fermion determinants:

$$\mathrm{Tr}\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -4i\epsilon^{\mu\nu\rho\sigma}, \quad \mathrm{Tr}\gamma_5\sigma^{\alpha\beta}\sigma^{\mu\nu} = 4i\epsilon^{\alpha\beta\mu\nu}, \quad \epsilon^{0123} = -g^{-1}\epsilon_{0123} = g^{-\frac{1}{2}},$$

$$\begin{aligned}
\mathrm{Tr}(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\epsilon\gamma^\zeta) = -4i[&\epsilon^{\gamma\delta\epsilon\zeta}g^{\alpha\beta} + \epsilon^{\alpha\beta\gamma\delta}g^{\epsilon\zeta} \\
&+\epsilon^{\alpha\beta\epsilon\gamma}g^{\delta\zeta} + \epsilon^{\alpha\beta\gamma\zeta}g^{\delta\epsilon} + \epsilon^{\alpha\beta\delta\epsilon}g^{\gamma\zeta} + \epsilon^{\alpha\beta\zeta\delta}g^{\gamma\epsilon} + \epsilon^{\alpha\beta\epsilon\zeta}g^{\gamma\delta}],\\
\mathrm{Tr}(\gamma_5\sigma^{\alpha\beta}\gamma^\gamma\sigma^{\delta\epsilon}\gamma^\zeta) &= 4i[\epsilon^{\alpha\beta\gamma\delta}g^{\epsilon\zeta} + \epsilon^{\alpha\beta\epsilon\gamma}g^{\delta\zeta} + \epsilon^{\alpha\beta\delta\epsilon}g^{\gamma\zeta} + \epsilon^{\alpha\beta\zeta\delta}g^{\gamma\epsilon} + \epsilon^{\alpha\beta\epsilon\zeta}g^{\gamma\delta}],\\
\mathrm{Tr}(\gamma_5\sigma^{\alpha\beta}\sigma^{\gamma\delta}\gamma^\epsilon\gamma^\zeta) &= 4i[\epsilon^{\alpha\beta\gamma\delta}g^{\epsilon\zeta} + \epsilon^{\alpha\beta\epsilon\gamma}g^{\delta\zeta} + \epsilon^{\alpha\beta\gamma\zeta}g^{\delta\epsilon} + \epsilon^{\alpha\beta\delta\epsilon}g^{\gamma\zeta} + \epsilon^{\alpha\beta\zeta\delta}g^{\gamma\epsilon}],\\
\mathrm{Tr}\sigma_{\rho\sigma}\sigma^{\mu\nu}F^{\rho\sigma}_aF^b_{\mu\nu} &= 8F^{\mu\nu}_aF^b_{\mu\nu}, \quad \mathrm{Tr}\sigma_{\rho\sigma}\sigma_{\mu\nu}\sigma_{\lambda\tau}F^{\rho\sigma}_aF^{\mu\nu}_bF^{\lambda\tau}_c = 32iF^{\mu\nu}_aF_{b\mu\rho}F^{\rho}_{c\ \ \nu},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}\left(\sigma\cdot A\sigma\cdot B\sigma\cdot C\sigma\cdot D\right) = 16\left[A^{\mu\nu}B^{\rho\sigma}C_{\mu\nu}D_{\rho\sigma} + (A\cdot B)(C\cdot D) + A^{\mu\nu}(B\cdot C)D_{\mu\nu}\right]\\
+64\left(A^{\mu\nu}B_{\mu\rho}C^{\rho\sigma}D_{\nu\sigma} - A^{\mu\nu}B_{\mu\rho}C_{\nu\sigma}D^{\rho\sigma} - A^{\mu\nu}B^{\rho\sigma}C_{\mu\rho}D_{\nu\sigma}\right),
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Tr}\left(\gamma^{\mu}\sigma\cdot A\gamma^{\nu}\sigma\cdot B\right) &= 8\left(g^{\mu\nu}A_{\rho\sigma}B^{\rho\sigma}+2A^{\mu\rho}B_{\rho}{}^{\nu}+2A_{\rho}{}^{\nu}B^{\mu\rho}\right),\\
\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\sigma\cdot A\sigma\cdot B\right) &= 8\left(g^{\mu\nu}A_{\rho\sigma}B^{\rho\sigma}+2A^{\mu\rho}B_{\rho}{}^{\nu}-2A_{\rho}{}^{\nu}B^{\mu\rho}\right),\\
\mathrm{Tr}\left(Z_{\mu\nu}\gamma^{\mu}\sigma\cdot A\gamma^{\nu}\sigma\cdot B\gamma_5\right) &= 8ir^{\mu}_{\nu}\left(\tilde{A}^{\nu\rho}B_{\mu\rho}-A^{\nu\rho}\tilde{B}_{\mu\rho}\right), \qquad \text{(A.23)}
\end{aligned}
$$

where $\sigma\cdot A=\sigma_{\mu\nu}A^{\mu\nu}$, *etc.*, and $Z_{\mu\nu}=\frac{1}{4}\gamma^{\rho}\gamma^{\sigma}r_{\rho\sigma\nu\mu}$ is the field strength arising from the spin connection (note that $\gamma_{\mu}\gamma_{\nu}Z^{\mu\nu}=\frac{1}{2}r$). To evaluate the last trace in (A.23) we used the relations (B.14) and (C.25).

B. Relations among operators

In this appendix we derive relations among the various operators that appear in the traces needed to evaluate the one-loop effective action. We adopt the gauge sign conventions of [10], [29]:

$$
\begin{aligned}
\mathcal{D}_{\mu} &= \nabla_{\mu}+iA_{\mu}, \quad A_{\mu}=T_aA^a_{\mu}, \quad T^{\bar{\imath}}_{a\bar{\jmath}}=\left(T^i_{aj}\right)^*,\\
\mathcal{D}_{\mu}z^i &= \partial_{\mu}z^i+iA^a_{\mu}(T_az)^i, \quad \mathcal{D}_{\mu}\bar{z}^{\bar{m}}=\partial_{\mu}\bar{z}^{\bar{m}}-iA^a_{\mu}(T_a\bar{z})^{\bar{m}},\\
F_{\mu\nu} &= \frac{1}{i}[\mathcal{D}_{\mu},\mathcal{D}_{\nu}]=\nabla_{\mu}A_{\nu}-\nabla_{\nu}A_{\mu}+i[A_{\mu},A_{\nu}],\\
F^a_{\mu\nu} &= \nabla_{\mu}A^a_{\nu}-\nabla_{\nu}A^a_{\mu}-c^a{}_{bc}A^b_{\mu}A^c_{\nu}. \qquad \text{(B.1)}
\end{aligned}
$$

Our other conventions and notations are given in Appendix A of I.

We first consider constraints on covariant scalar derivatives that follow from gauge invariance. We define

$$
\begin{aligned}
\mathcal{K}_{ab} &= \frac{1}{x}K_{\bar{m}j}(T_a\bar{z})^{\bar{m}}(T_bz)^j, \quad \mathcal{D}_a=K_i(T_az)^i, \quad \mathcal{D}=\frac{1}{2x}\mathcal{D}_a\mathcal{D}^a,\\
f_{ab}(z) &= \delta_{ab}f(z), \quad f=x+iy. \qquad \text{(B.2)}
\end{aligned}
$$

The classical scalar potential is $\hat{V}+\mathcal{D}$, where $\hat{V}$ has been defined in I. It follows from the gauge invariance of the Kähler potential K that:

$$
\begin{aligned}
\delta_aK &= K_i(T_az)^i-K_{\bar{m}}(T_a\bar{z})^{\bar{m}}=0, \quad D_iD_j\mathcal{D}_a=D_{\bar{m}}D_{\bar{n}}\mathcal{D}_a=0,\\
K_{i\bar{m}}D_{\bar{n}}(T_a\bar{z})^{\bar{m}} &= K_{j\bar{n}}D_i(T_az)^j, \quad D^i(T_a\bar{z})^{\bar{m}}=D^{\bar{m}}(T_az)^i,
\end{aligned}
$$

$$K_{ij}(T_az)^j + K_j(T_a)^j_i = K_{i\bar{m}}(T_a\bar{z})^{\bar{m}}, \quad D_kD_j(T_az)^i = -R_{j\bar{m}k}{}^i(T_a\bar{z})^{\bar{m}}, \quad \text{(B.3)}$$

where $K_{ij} = \partial_i\partial_j K = \partial_i K_j$, and the second and third lines follow from the first by taking successive scalar derivatives. Here $\partial_I = \partial/\partial z^I$, $I = i, \bar{\imath}$, D_I is the reparameterization covariant scalar derivative, and $R_{i\bar{m}j\bar{n}}$ is the Kähler curvature tensor. Indices are lowered and raised, respectively, with the Kähler metric $K_{i\bar{m}}$ and its inverse $K^{i\bar{m}}$. Similarly, it follows from the gauge invariance of f that

$$\begin{aligned}
\delta_a f &= f_i(T_az)^i = 0, \\
f_{ij}(T_az)^j\bar{f}^i &= -f_iD_j(T_az)^i\bar{f}^j, \quad f_{ij}(T_az)^j(T_bz)^i = -f_i(T_az)^jD_j(T_bz)^i, \\
f_i\bar{f}_{\bar{m}}D^i(T_a\bar{z})^{\bar{m}} &= -f^{\bar{n}}\bar{f}_{\bar{n}\bar{m}}(T_a\bar{z})^{\bar{m}} = -\bar{f}^i f_{ij}(T_az)^j, \quad f_{ij} = D_iD_jf, \quad \text{(B.4)}
\end{aligned}$$

and from the gauge invariance of the superpotential W that

$$\begin{aligned}
A_i(T_az)^i &= A_{\bar{m}}(T_a\bar{z})^{\bar{m}} = \mathcal{D}_aA, \\
A_{ij}(T_az)^i + A_iD_j(T_az)^i &= \mathcal{D}_aA_j + K_{j\bar{m}}(T_a\bar{z})^{\bar{m}}A,
\end{aligned}$$

$$A_{ijk}(T_az)^i + A_{ij}D_k(T_az)^i + A_{ik}D_j(T_az)^i + A_iD_kD_j(T_az)^i$$
$$= \mathcal{D}_aA_{jk} + K_{j\bar{m}}(T_a\bar{z})^{\bar{m}}A_k + K_{k\bar{m}}(T_a\bar{z})^{\bar{m}}A_j. \quad \text{(B.5)}$$

The tensors $A_{i_1\cdots i_n}$ are reparameterization invariant covariant derivatives [4] of $A = e^KW$. Using (B.3) and the definitions (B.2) we obtain

$$\mathcal{K}_{ab} - \mathcal{K}_{ba} = \frac{i}{x}c_{abc}\mathcal{D}^c, \quad \mathcal{K}^{ab}(\mathcal{K}_{ab} - \mathcal{K}_{ba}) = -\frac{1}{2x^2}C_G^{(a)}\mathcal{D}_a\mathcal{D}^a, \quad \text{(B.6)}$$

where $C_G^{(a)}$ is the Casimir in the adjoint representation, c_{abc} are the structure constants of the gauge group, and

$$(T_bz)^iD_i(T_az)^j = (T_az)^iD_i(T_bz)^j + ic_{abc}(T^cz)^j,$$

$$\mathcal{D}_bK_{\bar{m}j}(T_a\bar{z})^{\bar{m}}(T^az)^iD_i(T^bz)^j = \mathcal{D}_bK_{\bar{m}j}(T_a\bar{z})^{\bar{m}}(T^bz)^iD_i(T^az)^j - \frac{1}{2}C_G^{(a)}\mathcal{D}_a\mathcal{D}^a. \quad \text{(B.7)}$$

Combining (B.3) and (B.5) we obtain

$$
\begin{aligned}
A_i D^i (T_a \bar{z})^{\bar{m}} &= A_i D^{\bar{m}} (T_a z)^i = -A_i^{\bar{m}} (T_a z)^i + A^{\bar{m}} \mathcal{D}_a + A (T_a \bar{z})^{\bar{m}}, \\
\bar{A}^i D_i (T_a z)^k &= \bar{A}_{\bar{n}} D^{\bar{n}} (T_a z)^k = \bar{A}_{\bar{n}} D^k (T_a \bar{z})^{\bar{n}} = -\bar{A}^k_{\bar{n}} (T_a \bar{z})^{\bar{n}} + \bar{A}^k \mathcal{D}_a + \bar{A} (T_a z)^k,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}^a \bar{A}^{jk} A_{ij} D_k (T_a z)^i &= -\frac{1}{2} \bar{A}^{jk} A_{ijk} (T_a z)^i \mathcal{D}^a \\
&+ \frac{1}{2} R_{j\bar{m}k}{}^i \bar{A}^{jk} A_i \mathcal{D}^a (T_a \bar{z})^{\bar{m}} + x \mathcal{D} A_{ij} \bar{A}^{ij} + \mathcal{D}^a (T_a \bar{z})^{\bar{m}} \bar{A}^i_{\bar{m}} A_i. \quad \text{(B.8)}
\end{aligned}
$$

To evaluate the one-loop effective action, we find it convenient to introduce the scalar field reparameterization covariant derivatives of the variable ρ, defined as the squared gauge coupling:

$$
\begin{aligned}
\rho = \frac{1}{x} &= g^2, \quad \rho_i = D_i \rho = -\frac{f_i}{2x^2}, \quad \rho^i = K^{i\bar{m}} D_{\bar{m}} \rho = K^{i\bar{m}} \rho_{\bar{m}}, \\
\rho_{ij} &= D_i D_j \rho = -\frac{1}{2x^2} \left(f_{ij} - \frac{f_i f_j}{x} \right), \\
D_{\bar{m}} \rho_i = \rho_{\bar{m}i} &= -\frac{1}{x} \bar{f}_{\bar{m}} \rho_i = 2x \rho_{\bar{m}} \rho_i, \\
D_j \left(x^2 \rho_i \rho^i \right) &= x^2 \rho^i \rho_{ij}, \quad D_{\bar{m}} \left(x^2 \rho_i \rho^i \right) = x^2 \rho^i_{\bar{m}} \rho_i, \\
D_j D_k \left(x^2 \rho_i \rho^i \right) &= x^2 \rho^i \rho_{ijk}, \quad etc., \\
f_{\bar{m}ij} = R^k_{i\bar{m}j} f_k &= -2x^2 \rho_{\bar{m}ij} - 2x \bar{f}_{\bar{m}} \rho_{ij} - \frac{f_i f_j \bar{f}_{\bar{m}}}{2x^2}. \qquad \text{(B.9)}
\end{aligned}
$$

It follows from $[D_{\bar{m}}, D_i](x^2 \rho_i \rho^i) = 0$ that

$$
\bar{f}^k \rho^j{}_{ki} + \frac{1}{x} \bar{f}^k \bar{f}^j \rho_{ki} = f_k \rho_i{}^{kj} + \frac{1}{x} f_k f_i \rho^{kj}. \qquad \text{(B.10)}
$$

In addition we introduce the variable

$$
a = A + \frac{\bar{f}^i}{2x} A_i = e^{K/2} \left(\bar{m}_\psi - \bar{m}_\lambda \right), \quad a_{i_1 \cdots i_n} = D_{i_1} \cdots D_{i_n} a. \qquad \text{(B.11)}
$$

The variables a, ρ_{ij} and $1 - x^2 \rho^i \rho_i$, and all covariant derivatives thereof, vanish for effective supergravity theories obtained from superstrings in the classical limit: $f(z) = s,\ K = -\ln(s + \bar{s}) + G(z, \bar{z} \neq s, \bar{s}),\ W_s = 0.$

We will also need the following identities involving the Yang-Mills field strength and the space-time curvature. It follows from manipulating products of the antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ that

$$\begin{aligned}
\widetilde{M}_1^{\mu\nu} M^2_{\mu\rho} &= \frac{1}{2} g^{\nu}_{\rho} M_1^{\mu\sigma} \widetilde{M}^2_{\mu\sigma} - M^1_{\mu\rho} \widetilde{M}_2^{\mu\nu}, \quad \widetilde{M}^i_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M_i^{\rho\sigma}, \\
\tilde{F}^a_{\mu\nu} F_b^{\mu\nu} \tilde{F}^b_{\rho\sigma} F_a^{\rho\sigma} &= -(F^a_{\mu\nu} F_b^{\mu\nu})^2 - \left(F^a_{\mu\nu} F_a^{\mu\nu}\right)^2 + 4 F^a_{\rho\mu} F_a^{\rho\nu} F_b^{\sigma\mu} F^b_{\sigma\nu}, \\
(\tilde{F}^a_{\mu\nu} F_a^{\mu\nu})^2 &= -2(F^a_{\mu\nu} F_b^{\mu\nu})^2 + 4 F^a_{\mu\nu} F_{a\rho\sigma} F^{b\mu\rho} F_b^{\nu\sigma}, \qquad \text{(B.12)}
\end{aligned}$$

where $M^i_{\mu\nu}$ is any antisymmetric tensor-valued operator. Using the first of these gives

$$\begin{aligned}
\mathrm{Tr} A^{\mu\nu} B_{\mu\rho} \tilde{C}_{\nu\sigma} D^{\sigma\rho} &= \frac{1}{4}\mathrm{Tr}\left[(\tilde{D}\cdot A)(B\cdot C) - (A\cdot B)(\tilde{C}\cdot D) - \tilde{A}^{\mu\nu} B_{\rho\sigma} C_{\mu\nu} D^{\rho\sigma}\right], \\
\mathrm{Tr} A^{\mu\nu} B_{\mu\rho} \tilde{C}^{\sigma\rho} D_{\sigma\nu} &= \frac{1}{4}\mathrm{Tr}\left[(A\cdot B)(\tilde{C}\cdot D) + (D\cdot A)(\tilde{B}\cdot C) - A^{\mu\nu} \tilde{B}_{\rho\sigma} C_{\mu\nu} D^{\rho\sigma}\right], \\
\mathrm{Tr} A^{\nu\sigma} B^{\rho\mu} \tilde{C}_{\mu\nu} D_{\rho\sigma} &= \frac{1}{4}\mathrm{Tr}\left[(\tilde{A}\cdot B)(C\cdot D) - (D\cdot A)(\tilde{B}\cdot C) - \tilde{A}^{\mu\nu} B_{\rho\sigma} C_{\mu\nu} D^{\rho\sigma}\right], \\
\tilde{F}^a_{\mu\nu} F_a^{\mu\rho} &= \frac{1}{4} g^{\rho}_{\nu} \tilde{F}^a_{\mu\sigma} F_a^{\mu\sigma}. \qquad \text{(B.13)}
\end{aligned}$$

It follows from the the symmetry properties of the space-time Riemann tensor that

$$r_{\rho\sigma\mu\nu} F_a^{\nu\sigma} F^{a\mu\rho} = \frac{1}{2} r_{\mu\nu}{}^{\rho\sigma} F_a^{\mu\nu} F^a_{\rho\sigma}, \qquad \text{(B.14)}$$

and, using (B.12) with $M_1 = F,\ M_2 = \tilde{F},\ \widetilde{M}_2 = -F$,

$$\begin{aligned}
r_{\rho\sigma\mu\nu} \tilde{F}_a^{\nu\sigma} \tilde{F}^{a\mu\rho} &= \frac{1}{2} r_{\mu\nu}{}^{\rho\sigma} \tilde{F}_a^{\mu\nu} \tilde{F}^a_{\rho\sigma} \\
&= 2 r^{\mu}_{\nu} F^a_{\mu\rho} F_a^{\nu\rho} - \frac{1}{2} r F^a_{\mu\rho} F_a^{\mu\rho} - \frac{1}{2} r_{\mu\nu}{}^{\rho\sigma} F_a^{\mu\nu} F^a_{\rho\sigma}. \qquad \text{(B.15)}
\end{aligned}$$

In addition:

$$\begin{aligned}
F^a_{\mu\nu} [\mathcal{D}^{\mu}, \mathcal{D}_{\rho}] F_a^{\rho\nu} &= c_{abc} F^a_{\mu\nu} F^{b\mu\rho} F^{c\nu}{}_{\rho} \\
&\quad + r^{\mu}_{\nu} F^a_{\mu\rho} F_a^{\nu\rho} - \frac{1}{2} r_{\mu\nu}{}^{\rho\sigma} F_a^{\mu\nu} F^a_{\rho\sigma}. \qquad \text{(B.16)}
\end{aligned}$$

It is convenient to isolate terms that do not contribute to the S-matrix, using the classical equations of motion:

$$
\begin{aligned}
g^{-\frac{1}{2}}\mathcal{L}_I &= -K_{IJ}D_\mu\mathcal{D}^\mu z^J - \hat{V}_I - \frac{1}{x}\mathcal{D}_a(T^a z)^J K_{IJ} - \frac{1}{2}f_I\begin{cases}\mathcal{W} \\ \overline{\mathcal{W}}\end{cases}, \quad I,J = \begin{cases} i,\bar{\jmath} \\ \bar{\imath},j\end{cases}, \\
(xg)^{-\frac{1}{2}}\mathcal{L}_{a\mu} &= (xg)^{-\frac{1}{2}}g_{\mu\nu}\frac{\partial\mathcal{L}}{\partial A^a_\nu} = \mathcal{D}'^\nu\mathcal{F}_{a\nu\mu} + \tilde{\mathcal{F}}_{a\nu\mu}\frac{\partial^\nu y}{x} \\
&\quad + \frac{i}{\sqrt{x}}K_{i\bar{m}}\left(\mathcal{D}_\mu\bar{z}^{\bar{m}}(T_a z)^i - \mathcal{D}_\mu z^i(T_a\bar{z})^{\bar{m}}\right). \qquad \text{(B.17)}
\end{aligned}
$$

The first of these gives, in particular ($M^2_\psi = m_\psi\bar{m}_\psi$, $M^2_\lambda = m_\lambda\bar{m}_\lambda$):

$$
\begin{aligned}
\frac{f_i}{\sqrt{g}}\mathcal{L}^i = \left(\frac{\bar{f}^i}{\sqrt{g}}\mathcal{L}_i\right)^* &= -\nabla^2 x - i\nabla^2 y - 2x^4\rho^i\rho_i\overline{\mathcal{W}} + \frac{1}{x}\left(\partial_\nu x + i\partial_\nu y\right)\left(\partial^\nu x + i\partial^\nu y\right) \\
&\quad -2x^2\rho_{ij}\mathcal{D}_\mu z^j\mathcal{D}^\mu z^i + 2xe^{-K}\left(2\bar{a}A - \bar{a}^i A_i\right) + 2x\left(\hat{V} + M^2_\psi - M^2_\lambda\right), \\
-\frac{\partial_\mu x}{\sqrt{g}x}\mathcal{D}^\mu z^i\mathcal{L}_i + \text{h.c.} &= \left(\frac{\partial_\mu x}{x}\mathcal{D}^\mu z^i K_{i\bar{m}}D^\nu\mathcal{D}_\nu\bar{z}^{\bar{m}} + \text{h.c.}\right) - \frac{\nabla^2 x}{x}V \\
&\quad + \frac{\partial_\mu x\partial^\mu x}{x^2}\left(V + \frac{x}{4}F^2\right) + \frac{\partial_\mu y\partial^\mu x}{4x}F\tilde{F} + \text{total derivative}, \\
\frac{a + bx^2\rho^i\rho_i}{x\sqrt{g}}\mathcal{D}^a(T_a z)^I\mathcal{L}_I &= \frac{a + bx^2\rho^i\rho_i}{x}\Big(2K_{j\bar{m}}\mathcal{D}^\mu z^i\mathcal{D}_\mu\bar{z}^{\bar{m}}\mathcal{D}^a D_i(T_a z)^j + 8x\mathcal{D}M^2_\psi \\
&\quad -2\mathcal{D}^a\mathcal{D}^b\mathcal{K}_{ab} - e^{-K}\left[\mathcal{D}^a(T_a z)^i A_{ij}\bar{A}^j + \text{h.c.}\right] \\
&\quad + \left\{K_{i\bar{n}}K_{j\bar{m}}\mathcal{D}^\mu z^j(T_a\bar{z})^{\bar{m}}\left[(T_a z)^i\mathcal{D}_\mu\bar{z}^{\bar{n}} + (T_a\bar{z})^{\bar{n}}\mathcal{D}_\mu z^i\right] + \text{h.c.}\right\} \\
&\quad - \frac{\partial_\mu x}{x}\mathcal{D}^a K_{j\bar{m}}\left[\mathcal{D}^\mu z^j(T_a\bar{z})^{\bar{m}} + (T_a z)^j\mathcal{D}^\mu\bar{z}^{\bar{m}}\right]\Big) \\
&\quad + bx\mathcal{D}^a\left[\mathcal{D}_\mu z^k\rho^j\rho_{kj}K_{i\bar{m}}\left(\mathcal{D}_\mu\bar{z}^{\bar{m}}(T_a z)^i + \mathcal{D}_\mu z^i(T_a\bar{z})^{\bar{m}}\right) + \text{h.c.}\right] \\
&\quad + \text{total derivative}. \qquad \text{(B.18)}
\end{aligned}
$$

We absorb a part of the one loop correction into the Kähler potential; a shift δK in the Kähler potential gives a shift $\Delta_{\delta K}\mathcal{L}$ in the Lagrangian:

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\Delta_{\delta K}\mathcal{L} &= -\delta K\hat{V} + \delta K_{i\bar{m}}\left(e^{-K}\bar{A}^i A^{\bar{m}} + \mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\right) \\
&\quad - \left\{\delta K_i\left[e^{-K}\bar{A}^i A + \frac{1}{2x}\mathcal{D}_a(T^a z)^i\right] + \text{h.c.}\right\}. \qquad \text{(B.19)}
\end{aligned}
$$

Taking $\delta K = \mathcal{D}$, the last equation in (B.18) can be written as

$$\begin{aligned}
\frac{a+bx^2\rho^i\rho_i}{x\sqrt{g}}\mathcal{D}^a(T_az)^I\mathcal{L}_I &= \left(a+bx^2\rho^i\rho_i\right)\left(\frac{2}{\sqrt{g}}\Delta_{\mathcal{D}}\mathcal{L}+2\mathcal{D}\left[e^{-K}a\bar{a}-3M_\psi^2-3M_\lambda^2-\hat{V}\right]\right. \\
&\quad +\left[K_{i\bar{n}}K_{j\bar{m}}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j(T_a\bar{z})^{\bar{m}}(T_a\bar{z})^{\bar{n}}+\text{h.c.}\right] \\
&\quad +i\frac{\partial_\mu y}{x^2}\mathcal{D}^a\left[K_{i\bar{m}}(T_az)^i\mathcal{D}_\mu\bar{z}^{\bar{m}}-\text{h.c.}\right] \\
&\quad \left.-\frac{1}{x^2}\mathcal{D}\left[\partial_\mu x\partial^\mu x+\partial_\mu y\partial^\mu y\right]\right) \\
&\quad +bx\mathcal{D}^a\left[\mathcal{D}_\mu z^k\rho^j\rho_{kj}K_{i\bar{m}}\left(\mathcal{D}_\mu\bar{z}^{\bar{m}}(T_az)^i+\mathcal{D}_\mu z^i(T_a\bar{z})^{\bar{m}}\right)+\text{h.c.}\right] \\
&\quad +\text{total derivative.} \qquad \text{(B.20)}
\end{aligned}$$

C. Matrix Elements and Supertraces

In this Appendix we list matrix elements of operators appearing in Eqs. (4.2–4.5) and traces needed to evaluate the divergent contributions to the one-loop effective action (4.6). Notation and conventions are defined in Appendix A of I, and the relevant part of the tree Lagrangian [10], [20] is[12]

$$\begin{aligned}
\frac{1}{\sqrt{g}}\mathcal{L}(g,K,f) &= \frac{1}{2}r+K_{i\bar{m}}\mathcal{D}^\mu z^i\mathcal{D}_\mu\bar{z}^{\bar{m}}-\frac{x}{4}F_{\mu\nu}F^{\mu\nu}-\frac{y}{4}\tilde{F}_{\mu\nu}F^{\mu\nu}-V \\
&\quad +\frac{ix}{2}\bar{\lambda}\not{D}\lambda+iK_{i\bar{m}}\left(\bar{\chi}_L^{\bar{m}}\not{D}\chi_L^i+\bar{\chi}_R^i\not{D}\chi_R^{\bar{m}}\right) \\
&\quad +e^{-K/2}\left(\frac{1}{4}f_i\bar{A}^i\bar{\lambda}_R\lambda_L-A_{ij}\bar{\chi}_R^i\chi_L^j+\text{h.c.}\right) \\
&\quad +\left(i\bar{\lambda}_R^a\left[2K_{i\bar{m}}(T_a\bar{z})^{\bar{m}}-\frac{1}{2x}f_i\mathcal{D}_a-\frac{1}{4}\sigma_{\mu\nu}F_a^{\mu\nu}f_i\right]\chi_L^i+\text{h.c.}\right) \\
&\quad +\mathcal{L}_\psi+\text{four}-\text{fermion terms}, \\
\frac{1}{\sqrt{g}}\mathcal{L}_\psi &= \frac{1}{4}\bar{\psi}_\mu\gamma^\nu(i\not{D}+M)\gamma^\mu\psi_\nu-\frac{1}{4}\bar{\psi}_\mu\gamma^\mu(i\not{D}+M)\gamma^\nu\psi_\nu-\left[\frac{x}{8}\bar{\psi}_\mu\sigma^{\nu\rho}\gamma^\mu\lambda_aF_{\nu\rho}^a\right.
\end{aligned}$$

[12] In I we defined $\epsilon^{0123}=1$; here we denote by $\epsilon^{\mu\nu\rho\sigma}$ the covariantly constant tensor– see (A.23). With this definition there is no factor $g^{-\frac{1}{2}}$ multiplying the $F\tilde{F}$ term in the Lagrangian. See also footnote 1.

$$+\bar{\psi}_\mu \not{\mathcal{D}}\bar{z}^{\bar{m}} K_{i\bar{m}}\gamma^\mu L\chi^i - \frac{1}{4}\bar{\psi}_\mu\gamma^\mu\gamma_5\lambda^a\mathcal{D}_a + i\bar{\psi}_\mu\gamma^\mu L\chi^i m_i + \text{h.c.}\Big],$$

$$\bar{M} \;=\; (M)^\dagger = e^{K/2}\left(WR + \overline{W}L\right), \quad m_i = e^{-K/2}A_i. \tag{C.1}$$

If we define

$$\text{STr}F = \text{Tr}F_\Phi - \frac{1}{4}\mathbf{Tr}F_\Theta - 2\text{Tr}F_{gh} + 2\text{Tr}F_{Gh}, \quad -\frac{i}{2}T_- = \sqrt{g}\frac{\ln\Lambda^2}{32\pi^2}T, \tag{C.2}$$

where $\mathbf{Tr}F_\Theta$ is defined below [see (C.24)], the effective Lagrangian (4.2) is

$$\frac{1}{\sqrt{g}}\mathcal{L}_1 = -\frac{\Lambda^2}{32\pi^2}\text{STr}H + \frac{\ln\Lambda^2}{32\pi^2}\left[\text{STr}\left(\frac{1}{2}H^2 - \frac{1}{6}rH + \frac{1}{12}\hat{G}_{\mu\nu}\hat{G}^{\mu\nu}\right) + T\right], \tag{C.3}$$

In the following subsections we list the matrix elements that were not included in I; the subscript 0 refers to the contributions without the Yang-Mills sector that are given in Appendix B of I, except that ordinary derivatives are replaced by gauge covariant derivatives.[13]

The contributions to STrH from each supermultiplet have been given in [13]; below we list the analogous contributions to STrH^2 and STrG^2; we drop all total derivatives.

1. Boson matrix elements

As in [9] we rescale the quantum gauge fields: $A_\mu = \sqrt{x}\mathcal{A}_\mu$. Then the operator H_Φ can be expressed as

$$Z_\Phi H_\Phi \;=\; H + X + Y - N - S - K,$$

[13]In (B.21) of I $\frac{1}{2}$STrH^2 should be modified as follows: the last term in the first line should be multiplied by e^K, the term $-\frac{1}{2}re^{-K}A_{ij}\bar{A}^{ij}$ should be added, and the third and forth lines from the bottom should read:

$$+\frac{N-47}{4}\mathcal{D}_\mu z^j\mathcal{D}^\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{m}}\mathcal{D}^\nu\bar{z}^{\bar{n}}K_{i\bar{n}}K_{j\bar{m}} - \frac{N+17}{4}\mathcal{D}_\mu\bar{z}^{\bar{m}}\mathcal{D}^\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{n}}\mathcal{D}^\nu z^j K_{i\bar{n}}K_{j\bar{m}}$$
$$+\frac{1}{2}\mathcal{D}_\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{m}}K_{i\bar{m}}R_{j\bar{n}}\left(\mathcal{D}^\mu z^j\mathcal{D}^\nu\bar{z}^{\bar{n}} - \mathcal{D}^\nu z^j\mathcal{D}^\mu\bar{z}^{\bar{n}}\right).$$

In addition, the term $-\frac{1}{6}\mathcal{D}_\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{m}}K_{i\bar{m}}R_{j\bar{n}}\left(\mathcal{D}^\mu z^j\mathcal{D}^\nu\bar{z}^{\bar{n}} - \mathcal{D}^\nu z^j\mathcal{D}^\mu\bar{z}^{\bar{n}}\right)$ should be added to the right hand side of $\frac{1}{12}$STr$G_{\mu\nu}G^{\mu\nu}$ in the same equation.

$$\begin{aligned}\Phi^T Z_\Phi H_\Phi \Phi &= z^I H_{IJ} z^J + h^{\mu\nu} X_{\mu\nu,\rho\sigma} h^{\rho\sigma} + 2h^{\mu\nu} Y_{\mu\nu I} z^I - \hat{A}^\mu N_{\mu\nu} \hat{A}^\nu \\ &\quad -2\hat{A}^\mu S_{\mu I} z^I - 2h^{\mu\nu} K_{\mu\nu,\rho} \hat{A}^\rho, \end{aligned} \tag{C.4}$$

with, in addition to the matrix elements of Z_Φ given in I,

$$Z_{i,a\mu} = Z_{\mu\nu,a\rho} = 0, \quad Z_{a\mu,b\nu} = -g_{\mu\nu}\delta_{ab}. \tag{C.5}$$

Using the results of [9] and Section 2 above, the elements of H, X, Y are modified with respect to those given in (B.3) of I by[14]

$$\begin{aligned} H_{IJ} &= (H_0)_{IJ} + \mathcal{D}_{IJ} + q_I^a q_{aJ} + v_{IJ} - (V_\mu V^\mu)_{IJ}, \quad q_a^i = -\frac{i}{\sqrt{x}}(T_a z)^i, \\ q_i^a &= \frac{i}{\sqrt{x}}(T^a \bar{z})^{\bar{m}} K_{i\bar{m}}, \quad v_{i\bar{m}} = v_{\bar{m}i} = (V_\mu V^\mu)_{i\bar{m}} = (V_\mu V^\mu)_{\bar{m}i} = 0, \\ (V_\mu V^\mu)_{IJ} &= \frac{f_I f_J}{x f_{IJ}} v_{IJ} = \frac{1}{8x^2} f_I f_J \left(\mathcal{F}_a^{\mu\nu} \mathcal{F}_{\mu\nu}^a \mp i\tilde{\mathcal{F}}_a^{\mu\nu} \mathcal{F}_{\mu\nu}^a\right), \end{aligned}$$

$$v_{IJ} - (V_\mu V^\mu)_{IJ} = -\frac{x}{4}\rho_{IJ}\left(\mathcal{F}_a^{\mu\nu}\mathcal{F}_{\mu\nu}^a \mp i\tilde{\mathcal{F}}_a^{\mu\nu}\mathcal{F}_{\mu\nu}^a\right), \quad I,J = \begin{cases} i,j \\ \bar{\imath},\bar{\jmath} \end{cases},$$

$$\begin{aligned} Y_{\mu\nu I} &= -\frac{1}{2}\left(D_\mu \mathcal{D}_\nu + D_\nu \mathcal{D}_\mu\right) K_{IJ} z^J - \frac{1}{8} f_I F_{\mu\rho}^a F_{a\nu}{}^\rho \pm \frac{i}{32} g_{\mu\nu} f_I F_a^{\sigma\rho} \tilde{F}_{\sigma\rho}^a, \quad I,J = \begin{cases} i,\bar{\jmath} \\ \bar{\imath},j \end{cases}, \\ X_{\mu\nu,\rho\sigma} &= (X_0)_{\mu\nu,\rho\sigma} - 2P_{\mu\nu,\rho\sigma}\mathcal{D} + \frac{1}{2}P_{\mu\nu,\rho\sigma}\mathcal{F}_{\lambda\tau}^a \mathcal{F}_a^{\lambda\tau} + \frac{1}{4}\left(\mathcal{F}_{\mu\rho}^a \mathcal{F}_{a\nu\sigma} + \mathcal{F}_{\nu\rho}^a \mathcal{F}_{a\mu\sigma}\right) \\ &\quad -\frac{1}{16}\left(\mathcal{F}_{\mu\lambda}^a \mathcal{F}_{a\rho}{}^\lambda g_{\nu\sigma} + \mathcal{F}_{\nu\lambda}^a \mathcal{F}_{a\rho}{}^\lambda g_{\mu\sigma} + \mathcal{F}_{\mu\lambda}^a \mathcal{F}_{a\sigma}{}^\lambda g_{\nu\rho} + \mathcal{F}_{\nu\lambda}^a \mathcal{F}_{a\sigma}{}^\lambda g_{\mu\rho}\right), \end{aligned} \tag{C.6}$$

where $f_I \equiv f_i(f_{\bar{\imath}})$ for $I = i(\bar{\imath})$, *etc.*. The potential $V = \hat{V} + \mathcal{D}$ now includes the D-term $\mathcal{D}$ defined in (B.2) above:

$$\begin{aligned} \mathcal{D}_i &= -\frac{1}{2x} f_i \mathcal{D} + \frac{1}{x}\mathcal{D}_a K_{i\bar{m}} (T^a \bar{z})^{\bar{m}}, \\ \mathcal{D}_i^j &= \frac{1}{2x^2} f_i f^j \mathcal{D} - \frac{1}{2x^2} f_i \mathcal{D}_a (T^a z)^j - \frac{1}{2x^2} f^j \mathcal{D}_a K_{i\bar{n}} (T^a \bar{z})^{\bar{n}} \end{aligned}$$

[14]The Lorentz indices in U_{IJ} and $\mathcal{R}_{IJ}$ in Eq.(B.3) of I should be contracted.

$$
\begin{aligned}
&+\frac{1}{x}(T_a z)^j K_{i\bar n}(T^a\bar z)^{\bar n}+\frac{1}{x}\mathcal{D}_a D_i(T^a z)^j,\\
\mathcal{D}_{ij} &= x\rho_{ij}\mathcal{D}-\frac{1}{2x^2}\mathcal{D}_a(f_iK_{j\bar m}+f_jK_{i\bar m})(T^a\bar z)^{\bar m}\\
&+\frac{1}{x}K_{j\bar m}(T^a\bar z)^{\bar m}K_{i\bar n}(T_a\bar z)^{\bar n}. \qquad \text{(C.7)}
\end{aligned}
$$

The additional nonvanishing elements of $Z_\Phi H_\Phi$: are $-N_{a\mu,b\nu}$, $S_{a\mu,I}$, and $K_{\mu\nu,a\rho}$, with[15]

$$
\begin{aligned}
N_{a\mu,b\nu} &= g_{\mu\nu}\left(\mathcal{K}_{ab}+\mathcal{K}_{ba}-\frac{1}{2}\mathcal{F}_{a\rho\sigma}\mathcal{F}_b^{\rho\sigma}\right)+2c_{abc}F^c_{\mu\nu}+\frac{1}{2}\left(5\mathcal{F}_{a\mu\rho}\mathcal{F}_{b\nu}{}^{\rho}-\mathcal{F}_{a\nu\rho}\mathcal{F}_{b\mu}{}^{\rho}\right)\\
&-\frac{x^2}{2}\rho_i\rho^i\left(\mathcal{F}_{a\mu\rho}\mathcal{F}_{b\nu}{}^{\rho}+\mathcal{F}_{a\nu\rho}\mathcal{F}_{b\mu}{}^{\rho}-\frac{1}{2}g_{\mu\nu}\mathcal{F}_{a\rho\sigma}\mathcal{F}_b^{\rho\sigma}\right)\\
&-g_{\mu\nu}\delta_{ab}\left(\frac{\nabla^2 x}{2x}-\frac{\partial_\rho x\partial^\rho x}{4x^2}+\frac{\partial_\rho y\partial^\rho y}{2x^2}\right)+\delta_{ab}\left(\frac{\nabla_\mu\partial_\nu x}{x}-\frac{\partial_\mu x\partial_\nu x}{x^2}+\frac{\partial_\mu y\partial_\nu y}{2x^2}+r_{\mu\nu}\right),\\
S_{a\mu,I} &= \pm i\frac{2}{\sqrt{x}}K_{IK}\left[D_\mu(T_a z)^K-\frac{\partial_\mu x}{2x}(T_a z)^K\right]-\frac{x}{2}\rho_{IJ}\left(\mathcal{F}_{a\nu\mu}\mp i\tilde{\mathcal{F}}_{a\nu\mu}\right)\mathcal{D}^\nu z^J\\
&+\frac{1}{4x}f_I\left[\mathcal{D}'^\nu\mathcal{F}_{a\nu\mu}+\frac{3\partial^\nu x}{2x}\left(\mathcal{F}_{a\nu\mu}\mp i\tilde{\mathcal{F}}_{a\nu\mu}\right)\right]\\
&+2\mathcal{D}^\nu z^K K_{IK}\mathcal{F}_{a\mu\nu}, \quad I,J,K=\begin{cases} i,j,\bar k\\ \bar\imath,\bar\jmath,k\end{cases},\\
K^a_{\mu\nu,\rho} &= -\frac{1}{2}\left(\mathcal{D}''_\mu\mathcal{F}^a_{\nu\rho}+\mathcal{D}''_\nu\mathcal{F}^a_{\mu\rho}\right)+\frac{1}{4}\left(g_{\mu\rho}\mathcal{D}''^\sigma\mathcal{F}^a_{\sigma\nu}+g_{\nu\rho}\mathcal{D}''^\sigma\mathcal{F}^a_{\sigma\mu}\right)\\
&-\frac{\partial^\sigma x}{8x}\left(g_{\mu\rho}\mathcal{F}^a_{\sigma\nu}+g_{\nu\rho}\mathcal{F}^a_{\sigma\mu}\right)+\frac{3\partial^\sigma y}{8x}\left(g_{\mu\rho}\tilde{\mathcal{F}}^a_{\sigma\nu}+g_{\nu\rho}\tilde{\mathcal{F}}^a_{\sigma\mu}\right)\\
&-\frac{1}{8x}\left(\partial_\mu y\tilde{\mathcal{F}}^a_{\nu\rho}+\partial_\nu y\tilde{\mathcal{F}}^a_{\mu\rho}\right)-g_{\mu\nu}\tilde{\mathcal{F}}^a_{\sigma\rho}\frac{\partial^\sigma y}{4x}. \qquad \text{(C.8)}
\end{aligned}
$$

In writing the above expressions we used the notation in (2.2–3) and the first identity in (B.12) with $M_1=\mathcal{F}_a$, $M_2=\tilde{\mathcal{F}}_b$, $\widetilde{M}_2=-\mathcal{F}_b$. The inverse metric Z^{-1} must be included in evaluating the traces of these operators, which are defined such that

$$
\mathrm{Tr}H_\Phi = \mathrm{Tr}H+\mathrm{Tr}X+\mathrm{Tr}N,
$$

[15]In [5], [9], there is an additional graviton-gauge mass term $Q_{\mu\nu,a\rho}$; this term drops out when the prescription (2.10) is adopted.

$$\mathrm{Tr}H_\Phi^2 = \mathrm{Tr}H^2 + \mathrm{Tr}X^2 + \mathrm{Tr}N^2 + 2\mathrm{Tr}Y^2 - 2\mathrm{Tr}K^2 - 2\mathrm{Tr}S^2. \quad (C.9)$$

In the expressions for the traces[16] to be given below, space-time indices are raised with $g^{\mu\nu}$ and scalar indices are raised with $K^{i\bar{m}}$.

Finally we need [17]

$$\begin{aligned}
\hat{G}_{\mu\nu} &= \left(G_z + G_G + G_g + G_{gz} + G_{Gz} + G_{gG}\right)_{\mu\nu}, \\
\left(G^z_{\mu\nu}\right)^I_J &= \left(G^z_{0\mu\nu}\right)^I_J \pm iF^a_{\mu\nu} D_J (T_a z)^I, \quad I,J = \begin{cases} i,j \\ \bar{\imath},\bar{\jmath} \end{cases}, \\
\left(G^z_{\mu\nu}\right)^I_J &= \left(G^z_{0\mu\nu}\right)^I_J, \quad I,J = \begin{cases} i,\bar{\jmath} \\ \bar{\imath},j \end{cases}, \\
\left(G^G_{\mu\nu}\right)_{\alpha\beta,\gamma\delta} &= \left(G^G_{0\mu\nu}\right)_{\alpha\beta,\gamma\delta} + \frac{1}{4}\Big[\mathcal{F}^a_{\alpha\mu}\mathcal{F}_{a\gamma\nu} g_{\beta\delta} \\
&\quad + \mathcal{F}^a_{\alpha\mu}\mathcal{F}_{a\delta\nu} g_{\beta\gamma} + \mathcal{F}^a_{\beta\mu}\mathcal{F}_{a\gamma\nu} g_{\alpha\delta} + \mathcal{F}^a_{\beta\mu}\mathcal{F}_{a\delta\nu} g_{\alpha\gamma} - (\mu \leftrightarrow \nu)\Big], \\
\left(G^g_{\mu\nu}\right)_{a\rho,b\sigma} &= g_{\rho\sigma}\left(c_{abc}F^c_{\mu\nu} + \frac{1}{2}\left[\mathcal{F}_{a\lambda\mu}\mathcal{F}^{\lambda}_{b\ \nu} - \mathcal{F}_{a\lambda\nu}\mathcal{F}^{\lambda}_{b\ \mu}\right]\right) + \delta_{ab} r_{\sigma\rho\mu\nu} \\
&\quad - \delta_{ab}\left(\epsilon_{\rho\nu\sigma\lambda}\left[\frac{\nabla_\mu \partial^\lambda y}{2x} - \frac{\partial^\lambda y \partial_\mu x}{2x^2}\right] - (\mu \leftrightarrow \nu)\right) \\
&\quad - \delta_{ab}\frac{1}{4x^2}\left(\partial_\lambda y \partial^\lambda y g_{\rho\nu} g_{\mu\sigma} + \partial_\sigma y \partial_\nu y g_{\rho\mu} + \partial_\rho y \partial_\mu y g_{\nu\sigma} - (\mu \leftrightarrow \nu)\right) \\
&\quad + \frac{1}{2}\left[\mathcal{F}_{a\sigma\mu}\mathcal{F}_{b\rho\nu} - \mathcal{F}_{a\rho\mu}\mathcal{F}_{b\sigma\nu} + x^2 \rho_i \rho^i \left(\mathcal{F}_{a\mu\rho}\mathcal{F}_{b\nu\sigma} + \tilde{\mathcal{F}}_{a\mu\rho}\tilde{\mathcal{F}}_{b\nu\sigma}\right) - (\mu \leftrightarrow \nu)\right] \\
\left(G^{gz}_{\mu\nu}\right)_{a\rho,I} &= \left(G^{gz}_{\mu\nu}\right)_{I,a\rho} = -\mathcal{D}_\mu\left[\frac{x}{2}\rho_I\left(\mathcal{F}_{a\nu\rho} \mp i\tilde{\mathcal{F}}_{a\nu\rho}\right)\right] \\
&\quad - \epsilon_{\rho\mu\sigma\lambda}\frac{\partial^\lambda y}{8x^2} f_I \left(\mathcal{F}_{a\nu}{}^{\sigma} \mp i\tilde{\mathcal{F}}_{a\nu}{}^{\sigma}\right) - (\mu \leftrightarrow \nu), \quad I = \begin{cases} i \\ \bar{\imath} \end{cases}, \\
\left(G^{Gz}_{\mu\nu}\right)_{\alpha\beta,I} &= -4\left(G^{Gz}_{\mu\nu}\right)_{I,\alpha\beta} = \pm\frac{ix\rho_I}{2}\left[\tilde{\mathcal{F}}^a_{\alpha\mu}\mathcal{F}_{a\beta\nu} + \tilde{\mathcal{F}}^a_{\beta\mu}\mathcal{F}_{a\alpha\nu} - (\mu \leftrightarrow \nu)\right], \quad I = \begin{cases} i \\ \bar{\imath} \end{cases},
\end{aligned}$$

[16]There is a term missing from $\mathrm{Tr}Y^2$ in I, namely:

$$-4\mathcal{D}_\mu \bar{z}^{\bar{m}} \mathcal{D}^\mu \bar{z}^{\bar{n}} \mathcal{D}_\nu z^j \mathcal{D}^\nu z^i R_{\bar{n}j\bar{m}i} + 4\mathcal{D}_\mu \bar{z}^{\bar{m}} \mathcal{D}^\mu z^i \mathcal{D}_\nu z^j \mathcal{D}^\nu \bar{z}^{\bar{n}} R_{\bar{m}j\bar{n}i}.$$

[17]In (B8) of I the expression for $\mathrm{Tr}R_{\mu\nu}R^{\mu\nu}$ should be multiplied by 2 and the fourth line of (B8) of I should read: $\left(G^G_{\mu\nu}\right)_{\gamma\delta,\alpha\beta} = \delta_{\alpha\beta,\rho\sigma}\left(r^\rho_{\gamma\mu\nu} g^\sigma_\delta + r^\rho_{\delta\mu\nu} g^\sigma_\gamma\right)$.

$$
\begin{aligned}
\left(G^{gG}_{\mu\nu}\right)_{a\rho,\alpha\beta} &= \frac{1}{4}\left[\left(g_{\beta\rho}\mathcal{D}_\mu - \frac{\partial^\lambda y}{2x}\epsilon_{\rho\mu\beta\lambda}\right)\mathcal{F}_{a\alpha\nu} + \left(g_{\alpha\rho}\mathcal{D}_\mu - \frac{\partial^\lambda y}{2x}\epsilon_{\rho\mu\alpha\lambda}\right)\mathcal{F}_{a\beta\nu} - (\mu\leftrightarrow\nu)\right],\\
\left(G^{gG}_{\mu\nu}\right)_{\alpha\beta,a\rho} &= \left(g_{\beta\rho}\mathcal{D}_\mu - \frac{\partial^\lambda y}{2x}\epsilon_{\rho\mu\beta\lambda}\right)\mathcal{F}_{a\alpha\nu} + \left(g_{\alpha\rho}\mathcal{D}_\mu - \frac{\partial^\lambda y}{2x}\epsilon_{\rho\mu\alpha\lambda}\right)\mathcal{F}_{a\beta\nu}\\
&\quad -g_{\alpha\beta}\left(\mathcal{D}_\mu\mathcal{F}_{a\rho\nu} - \frac{\partial^\lambda y}{2x}\epsilon_{\rho\mu\sigma\lambda}\mathcal{F}^{\sigma}_{a\ \nu}\right) - (\mu\leftrightarrow\nu). \qquad \text{(C.10)}
\end{aligned}
$$

2. Fermion matrix elements

As described in I, we take the Landau gauge condition $G=0$, where

$$
\begin{aligned}
G &= -\gamma^\nu(i\not{D} - \bar{M})\psi_\nu - 2(\not{\mathcal{D}}z^i K_{i\bar{m}}R\chi^{\bar{m}} + \not{\mathcal{D}}\bar{z}^{\bar{m}}K_{i\bar{m}}L\chi^i)\\
&\quad +\frac{x}{2}\sigma^{\nu\rho}\lambda_a F^a_{\nu\rho} + 2im_I\chi^I - \gamma_5\mathcal{D}_a\lambda^a, \qquad \text{(C.11)}
\end{aligned}
$$

which we implement by introducing an auxiliary field α. After an appropriate shift in the gravitino field ψ_μ, we obtain for the bilinear fermion couplings of the gravity sector:

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\mathcal{L}_{\psi+\alpha} &= -\frac{1}{2}\bar{\psi}^\mu(i\not{D} - \bar{M})\psi_\mu - \bar{\alpha}(i\not{D} + 2M)\alpha\\
&\quad +ix\bar{\psi}_\mu \not{F}^\mu_a\lambda^a - 2\bar{\psi}_\mu(\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}L\chi^i + \mathcal{D}^\mu z^i K_{i\bar{m}}R\chi^{\bar{m}})\\
&\quad -\bar{\alpha}\left(\frac{x}{2}\sigma^{\nu\rho}\lambda_a F^a_{\nu\rho} - 2im_I\chi^I + \gamma_5\mathcal{D}_a\lambda^a\right). \qquad \text{(C.12)}
\end{aligned}
$$

To obtain the ghostino determinant we use the supersymmetry transformations [10]

$$
\begin{aligned}
i\delta\chi^i &= \frac{1}{2}(\not{\mathcal{D}}z^i R - i\bar{m}^i L)\epsilon, \quad i\delta\chi^{\bar{m}} = \left[\frac{1}{2}(\not{\mathcal{D}}\bar{z}^{\bar{m}}L - im^{\bar{m}}R)\right]\epsilon,\\
i\delta\psi_\mu &= (iD_\mu - \frac{1}{2}\gamma_\mu M)\epsilon, \quad i\delta\lambda^a = \left[\frac{i}{4}\gamma^\mu\gamma^\nu F^a_{\mu\nu} - \frac{1}{2x}\gamma_5\mathcal{D}^a\right]\epsilon, \qquad \text{(C.13)}
\end{aligned}
$$

yielding

$$
\begin{aligned}
D^2 + H_{Gh} = \frac{\partial\delta G}{\partial\epsilon} &= D^\mu D_\mu - \frac{1}{2}\gamma^\mu\gamma^\nu[D_\mu, D_\nu] - i[\not{D}, M] - 2M\bar{M} + \bar{m}^i m_i + \mathcal{D}\\
&\quad +2i\bar{m}_{\bar{m}}\not{\mathcal{D}}\bar{z}^{\bar{m}}L + 2im_i\not{\mathcal{D}}z^i R + \frac{x}{2}\sigma_{\sigma\rho}F^{\sigma\rho}_a[\frac{1}{4}\sigma^{\mu\nu}F^a_{\mu\nu} - \frac{1}{x}\gamma_5\mathcal{D}^a]\\
&\quad -\mathcal{D}_\mu z^i K_{i\bar{m}}\mathcal{D}^\mu\bar{z}^{\bar{m}} + \frac{1}{2}\gamma_5[\gamma^\mu,\gamma^\nu]\mathcal{D}_\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\mathcal{D}_\nu z^i. \qquad \text{(C.14)}
\end{aligned}
$$

The metric for the gaugino field, as obtained from the classical supergravity Lagrangian given in (A.9) of I, is $Z_{ab} = \delta_{ab}x$. Following [5] we rescale the gaugino field $\lambda = \sqrt{x}\lambda'$, so for the rescaled field λ', $Z_{ab} = \delta_{ab}$. The matrix elements of M_Θ are given by (2.17), (A.11) and (B.9-10) of I and by[18]

$$\begin{aligned}
M^a_b &= \left(\bar{M}^a_b\right)^* = \delta^a_b m_\lambda, \quad m_\lambda = -\frac{e^{-K/2}}{2x} f_k \bar{A}^k, \\
M^a_I &= \delta^{ab}\left(m_{bI} + M^{\mu\nu}_{bI}\sigma_{\mu\nu}\right), \quad M^I_a = \frac{1}{2}K^{IJ}\left(m_{Ja} + M^{\mu\nu}_{Ja}\sigma_{\mu\nu}\right), \\
m_{ai} &= m_{ia} = \frac{i}{\sqrt{x}}\left(\frac{1}{2x}f_i\mathcal{D}_a - 2K_{i\bar{m}}(T_a\bar{z})^{\bar{m}}\right) = m^*_{a\bar{\imath}}, \\
M^{\mu\nu}_{aI} &= -M^{\mu\nu}_{Ia} = -\frac{ix}{4}\rho_I\left(\mathcal{F}^{\mu\nu}_a \mp i\tilde{\mathcal{F}}^{\mu\nu}_a\right), \quad I = \begin{cases} i \\ \bar{\imath} \end{cases}, \\
2M^\alpha_a &= -\bar{M}^a_\alpha = m_{\alpha a} + M^{\mu\nu}_{\alpha a}\sigma_{\mu\nu}, \quad 2\bar{M}^\alpha_a = -M^a_\alpha = \bar{m}_{\alpha a} + \bar{M}^{\mu\nu}_{\alpha a}\sigma_{\mu\nu}, \\
m_{\alpha a} &= -\bar{m}_{\alpha a} = \frac{1}{\sqrt{x}}\mathcal{D}_a, \quad M^{\mu\nu}_{\alpha a} = \bar{M}^{\mu\nu}_{\alpha a} = -\frac{1}{2}\mathcal{F}^{\mu\nu}_a,
\end{aligned} \tag{C.15}$$

with covariant derivatives as defined in (A.21) [see also (B.11) of I]

$$\begin{aligned}
D_\mu m^\lambda &= -e^{-K/2}\left(\mathcal{D}_\mu\bar{z}^{\bar{m}}\left[\bar{a}_{\bar{m}} - \bar{A}_{\bar{m}}\right] + \mathcal{D}_\mu z^i\left[\frac{f_i}{2x}\bar{a} - x\rho_{ik}\bar{A}^k\right]\right), \\
D_\rho M_{aA} &= \tilde{D}_\rho M_{aA} - i\frac{\partial_\rho y}{2x}M_{aA}\gamma_5, \quad D_\rho M_{Aa} = \tilde{D}_\rho M_{Aa} + i\frac{\partial_\rho y}{2x}M_{Aa}\gamma_5, \quad A = i, \bar{m}, \alpha, \\
\tilde{D}_\rho M^{\mu\nu}_{ai} &= -\tilde{D}_\rho M^{\mu\nu}_{ia} = -\left(\tilde{D}_\rho \bar{M}^{\mu\nu}_{a\bar{\imath}}\right)^* = \left(\tilde{D}_\rho \bar{M}^{\mu\nu}_{\bar{\imath}a}\right)^* \\
&= -\frac{ix}{4}\left[\rho_i\left(\mathcal{D}_\rho + i\frac{\partial_\rho y}{x}\right) + \mathcal{D}_\rho z^j\rho_{ij}\right]\left(\mathcal{F}_{a\mu\nu} - i\tilde{\mathcal{F}}_{a\mu\nu}\right), \\
\tilde{D}_\rho m_{ai} &= \tilde{D}_\rho m_{ia} = \left(\tilde{D}_\rho m_{a\bar{\imath}}\right)^* \\
&= \frac{i}{\sqrt{x}}\left[\mathcal{D}_a\left(\frac{f_i}{4x^2}\left[2i\partial_\rho y - \partial_\rho x\right] - x\rho_{ij}\mathcal{D}_\rho z^j\right) + \frac{\partial_\rho x}{x}K_{i\bar{m}}(T_a\bar{z})^{\bar{m}}\right. \\
&\quad \left. + \frac{1}{2x}f_i(K_{j\bar{m}}(T_a\bar{z})^{\bar{m}}\mathcal{D}_\rho z^j + \text{h.c.}) - 2K_{i\bar{m}}D_{\bar{n}}(T_a\bar{z})^{\bar{m}}\mathcal{D}_\rho\bar{z}^{\bar{n}}\right], \\
\tilde{D}_\mu m_{\alpha a} &= -\left(\tilde{D}_\mu\bar{m}_{\alpha a}\right)^* = \frac{1}{\sqrt{x}}\left(K_{i\bar{m}}\left[\mathcal{D}_\mu z^i(T_a\bar{z})^{\bar{m}} + \mathcal{D}_\mu\bar{z}^{\bar{m}}(T_a z)^i\right] - \frac{\partial_\mu x}{2x}\mathcal{D}_a\right),
\end{aligned}$$

[18](B.10) of I should read $M^\mu_I = -2Z_{IJ}\mathcal{D}^\mu z^J$, $M^I_\mu = \mathcal{D}_\mu z^I$. The equation before (2.16) should read $A = e^K W = e^{K/2}\bar{M}$.

$$\tilde{D}_\rho M^{\mu\nu}_{\alpha a} = \left(\tilde{D}_\rho \bar{M}^{\mu\nu}_{\alpha a}\right)^* = -\frac{1}{2}\mathcal{D}_\rho \mathcal{F}^{\mu\nu}_a. \tag{C.16}$$

Here α is the auxiliary field introduced in I to implement the gravitino gauge fixing; its couplings to chiral and Yang-Mills matter are given in (3.10) of I. In addition, there is a λ-ψ connection [5], $(D_\mu)_{a\nu} = (D_\mu)_{\nu a} = -\mathcal{F}_{a\nu\mu}$, that contributes as follows to the covariant derivatives of the fermion mass matrix:

$$\begin{aligned}
(D_\rho M)_{a\mu} &= -(D_\rho M)_{\mu a} = -e^{-K/2}\bar{a}\mathcal{F}_{a\mu\rho}, \\
(D^\rho M)^\mu_I &= -2K_{IJ}D^\rho \mathcal{D}^\mu z^J - M^a_I \mathcal{F}^{\mu\rho}_a, \quad (D_\rho M)^I_\mu = D_\rho \mathcal{D}_\mu z^I + M^I_a \mathcal{F}^a_{\mu\rho}, \\
(D_\rho M)^a_I &= D_\rho M^a_I + 2K_{IJ}\mathcal{D}^\mu z^J \mathcal{F}^a_{\mu\rho}, \quad (D_\rho M)^I_a = D_\rho M^I_a + \mathcal{D}^\mu z^I \mathcal{F}^a_{\mu\rho}, \\
(D^\rho M)^\mu_\alpha &= -M^a_\alpha \mathcal{F}^{\mu\rho}_a, \quad (D_\rho M)^\alpha_\mu = M^\alpha_a \mathcal{F}^a_{\mu\rho},
\end{aligned} \tag{C.17}$$

The nonvanishing matrix elements of $G_{\mu\nu}$ involving the gaugino field are

$$\begin{aligned}
\left(G^\pm_{\mu\nu}\right)_{ab} &= c_{abc}F^c_{\mu\nu} + \delta_{ab}\left(\pm\Gamma_{\mu\nu} + i\gamma_5 L_{\mu\nu} + Z_{\mu\nu}\right) + \left(\mathcal{F}_{a\rho\mu}\mathcal{F}^\rho_{b\ \nu} - \mu \leftrightarrow \nu\right), \\
\left(G^\pm_{\mu\nu}\right)_{a\rho} &= -\left[(\mathcal{D}_\mu + i\gamma_5 L_\mu)\mathcal{F}_{a\rho\nu} - (\mu \leftrightarrow \nu)\right], \\
\left(G^\pm_{\mu\nu}\right)_{\rho a} &= -\left[(\mathcal{D}_\mu - i\gamma_5 L_\mu)\mathcal{F}_{a\rho\nu} - (\mu \leftrightarrow \nu)\right].
\end{aligned} \tag{C.18}$$

As in I, $\mathcal{D}_\mu$ is the gauge and general coordinate covariant derivative, $\Gamma_{\mu\nu}$ and $Z_{\mu\nu}$ are given in (B.13) of I, and[19]

$$\mathcal{F}_{\mu\nu} = \sqrt{x}F_{\mu\nu}, \quad L_\mu = -\frac{\partial_\mu y}{2x}, \quad L_{\mu\nu} = \frac{1}{2x^2}\left(\partial_\mu x \partial_\nu y - \partial_\nu x \partial_\mu y\right). \tag{C.19}$$

The other matrix elements of $G_{\mu\nu}$ are as given in Appendix (B.12) of I, except that now the chiral matter connection includes the gauge field:

$$(G_{\mu\nu})^I_J = (R_{\mu\nu})^I_J \pm iF^a_{\mu\nu}D_J(T_a z)^I + \delta^I_J\left(Z_{\mu\nu} \pm \Gamma_{\mu\nu}\right), \quad I, J = \begin{cases} i, j \\ \bar{\imath}, \bar{\jmath} \end{cases}, \tag{C.20}$$

[19] We use the notation $L_\mu, L_{\mu\nu}$, to denote the field operators defined in (C.19), and also the matrices defined by these fields multiplying the unit projection operator in the space of gauginos, as in (3.9–11), (A.22–23), (C.22), *etc.*

where $(R_{\mu\nu})^I_J$ is defined in (B.8) of I, and the ψ-λ connection gives an additional contribution to the gravitino matrix element[20] of $G_{\mu\nu}$:

$$\left(G^{\pm}_{\mu\nu}\right)_{\rho\sigma} = g_{\rho\sigma}\left(\pm\Gamma_{\mu\nu} + Z_{\mu\nu}\right) - r_{\rho\sigma\mu\nu} + \left(\mathcal{F}^a_{\rho\mu}\mathcal{F}_{a\sigma\nu} - \mu \leftrightarrow \nu\right). \qquad \text{(C.21)}$$

Finally, in the 8×8 matrix notation of (2.14–17), setting $G_{\mu\nu} = \tilde{G}_{\mu\nu} + i\gamma_5 L_{\mu\nu}$,

$$\begin{aligned}
H_\Theta &= M_\Theta M_\Theta + \frac{1}{4}[\gamma^\mu,\gamma^\nu]\tilde{G}_{\mu\nu} - i\,\not{D}M_\Theta - 2D^\mu M^\Theta_{\mu\nu}\gamma^\nu - 4\gamma^\rho\gamma_\sigma M^\Theta_{\mu\rho}M^{\mu\sigma}_\Theta \\
&\qquad -2L_\mu L^\mu + i\tilde{D}^\mu L_\mu\gamma_5 + 2i\gamma_\mu\gamma_\rho\gamma_\nu\gamma_5[L^\rho, M^{\mu\nu}_\Theta], \\
\hat{D}^\Theta_\mu &= \tilde{D}_\mu + 2\gamma^\nu M^\Theta_{\mu\nu} + \sigma_{\mu\nu}\gamma_5 L^\nu, \\
\hat{G}^\Theta_{\mu\nu} &= \tilde{G}_{\mu\nu} + 2\gamma^\rho\left(\tilde{D}_\mu M^\Theta_{\nu\rho} - \tilde{D}_\nu M^\Theta_{\mu\rho}\right) + 4\gamma^\rho\gamma^\sigma\left(M^\Theta_{\mu\rho}M^\Theta_{\nu\sigma} - M^\Theta_{\nu\rho}M^\Theta_{\mu\sigma}\right) \\
&\quad + \left[\sigma_{\rho\mu}\left(\gamma_5\tilde{D}_\nu L^\rho - 2iL_\nu L^\rho\right) - (\mu\leftrightarrow\nu)\right] - 2iL_\rho L^\rho\sigma_{\mu\nu} - 4i[\not{L}, M^\Theta_{\mu\nu}]\gamma_5 \\
&\quad -2\left[\gamma_\mu\left(\{L^\rho, \widetilde{M}^\Theta_{\rho\nu}\} - i[L^\rho, M^\Theta_{\rho\nu}]\gamma_5\right) + \{L_\mu, \widetilde{M}^\Theta_{\nu\rho}\}\gamma^\rho - (\mu\leftrightarrow\nu)\right], \\
M_\Theta &= m_\Theta + M^{\mu\nu}_\Theta\sigma_{\mu\nu} = m_\Theta + M_\sigma.
\end{aligned} \qquad \text{(C.22)}$$

Then, defining $H_\Theta = H_1 + H_2 + H_3$, with

$$\begin{aligned}
H_1 &= M_\Theta M_\Theta - 4\gamma^\rho\gamma_\sigma M^\Theta_{\mu\rho}M^{\mu\sigma}_\Theta, \\
H_2 &= -i\,\not{D}M_\Theta - 2\gamma^\nu D^\mu M^\Theta_{\mu\nu} + 2i\gamma_\mu\gamma_\rho\gamma_\nu\gamma_5[L^\rho, M^{\mu\nu}_\Theta], \\
H_3 &= \frac{1}{4}[\gamma^\mu,\gamma^\nu]\tilde{G}_{\mu\nu} - 2L_\mu L^\mu + i\tilde{D}^\mu L_\mu\gamma_5, \\
G'_{\mu\nu} &= \tilde{G}_{\mu\nu} - Z_{\mu\nu},
\end{aligned} \qquad \text{(C.23)}$$

we find the following traces (**Tr** includes the Dirac trace): $\mathbf{Tr}1 \equiv 8\mathrm{Tr}1$, where Tr is over internal symmetry indices only):

$$\begin{aligned}
\frac{1}{8}\mathbf{Tr}H_1 &= \frac{1}{8}\mathbf{Tr}\left[m_\Theta m_\Theta - 2M^\Theta_{\mu\nu}M^{\mu\nu}_\Theta\right] = \mathrm{Tr}\left[\bar{m}m - 2\bar{M}_{\mu\nu}M^{\mu\nu}\right] \\
&= \frac{1}{8}\mathbf{Tr}\left(M^2_\Theta\right)_0 + 4\mathcal{K}^a_a - 2\mathcal{D}\left(1 - x^2\rho^i\rho_i\right) + N_G M^2_\lambda + \frac{x}{2}F^a_{\mu\nu}F^{\mu\nu}_a,
\end{aligned}$$

[20]The last line of Eq. (B12) of I should read $(G_{\mu\nu})^\rho_\sigma = \delta^\rho_\sigma(\gamma_5\Gamma_{\mu\nu} + Z_{\mu\nu}) - r^\rho_{\sigma\nu\mu}$.

$$
\begin{aligned}
\frac{1}{8}\mathbf{Tr}H_1^2 &= \frac{1}{8}\mathbf{Tr}\Big[(m_\Theta m_\Theta)^2 + (\sigma_{\mu\nu}\sigma_{\rho\sigma}M_\Theta^{\mu\nu}M_\Theta^{\rho\sigma})^2 + 4m_\Theta M_\Theta^{\mu\nu}m_\Theta M_{\mu\nu}^\Theta \\
&\quad +16M_\Theta^{\mu\nu}M_{\rho\nu}^\Theta\left(M_{\mu\sigma}^\Theta M_\Theta^{\rho\sigma} - M_\Theta^{\rho\sigma}M_{\mu\sigma}^\Theta\right)\Big] \\
&= \mathrm{Tr}\Big[(\bar m m)^2 + 2\bar m M^{\mu\nu}\bar m M_{\mu\nu} + 2\bar M^{\mu\nu}m\bar M_{\mu\nu}m + 4\bar M_{\mu\nu}M_{\rho\sigma}\bar M^{\mu\nu}M^{\rho\sigma} \\
&\quad +8\left(\bar M^{\mu\nu}M_{\mu\nu}\right)^2 - 16\bar M^{\mu\nu}M^{\rho\sigma}\bar M_{\mu\rho}M_{\nu\sigma}\Big], \qquad \text{(C.24)}
\end{aligned}
$$

and using (B.12), partial integration and the relation

$$
\not{D}\gamma^\mu\gamma^\nu M_{\mu\nu} = 2\gamma^\nu D^\mu M_{\mu\nu} + 2i\gamma^\nu D^\mu \widetilde{M}_{\mu\nu}\gamma_5, \qquad \text{(C.25)}
$$

we obtain

$$
\begin{aligned}
-\frac{1}{8}\mathbf{Tr}H_2^2 &= -\frac{1}{8}\mathbf{Tr}\left\{-i\not{D}m_\Theta + 2\gamma_\nu[L_\mu, \widetilde{M}_\Theta^{\mu\nu}] + 2i\gamma_\nu\gamma_5\tilde D_\mu\widetilde{M}_\Theta^{\mu\nu}\}\right\}^2 \\
&= \mathrm{Tr}\Big\{\tilde D_\mu\bar m\tilde D^\mu m - 4\tilde D_\mu\widetilde{\bar M}^{\mu\nu}\tilde D^\rho\widetilde{M}_{\rho\nu} - 4[L_\mu\widetilde{\bar M}^{\mu\rho}][L^\nu, \widetilde{M}_{\nu\rho}] \\
&\quad +[L_\mu, \bar m][L^\mu, m] - i\left([\hat L_{\mu\nu}, \bar m]\widetilde{M}^{\mu\nu} + [\hat L_{\mu\nu}, m]\widetilde{\bar M}^{\mu\nu}\right)\Big\}, \qquad \text{(C.26)}
\end{aligned}
$$

where $\hat L_{\mu\nu}$ is defined in (3.11). The remaining traces needed to evaluate $\mathbf{Tr}H_\Theta, \mathbf{Tr}H_\Theta^2$ are:

$$
\begin{aligned}
\frac{1}{2}\mathbf{Tr}H_3 &= (N+N_G+5)r - 2N_G\frac{\partial_\mu y\partial^\mu y}{x^2}, \\
\frac{1}{2}\mathbf{Tr}H_3^2 &= N_G\mathbf{Tr}h_3^2 + (N+5)\frac{r^2}{4} - \mathbf{Tr}\left([\Gamma'_\mu, L^\mu]\right)^2 - \frac{1}{4}\mathbf{Tr}\left(G'_{\mu\nu}G'^{\mu\nu}\right) \\
\frac{1}{2}\mathbf{Tr}(H_1H_3) &= \frac{1}{2}\mathbf{Tr}\left[\left(\frac{r}{4} - 2L_\mu L^\mu\right)H_1 - 2M_{\mu\nu}^\Theta\widetilde{M}_\Theta^{\mu\nu}\tilde D_\rho L^\rho - iG'_{\mu\nu}\{M_\Theta^{\mu\nu}, m_\Theta\}\right] \\
\frac{1}{2}\mathbf{Tr}\hat G_{\mu\nu}^\Theta\hat G_\Theta^{\mu\nu} &= \frac{1}{2}\mathbf{Tr}\Big\{\tilde G_{\mu\nu}\tilde G^{\mu\nu} + 16\tilde G_{\mu\nu}M_\Theta^{\mu\rho}M_\Theta^{\nu\sigma}\gamma_\rho\gamma_\sigma + 8\tilde D_\mu M_{\nu\rho}^\Theta\left(\tilde D^\mu M_\Theta^{\nu\rho} - \tilde D^\nu M_\Theta^{\mu\rho}\right) \\
&\quad +16\left(\tilde D_\mu M_\Theta^{\mu\nu}\{L^\rho, \widetilde{M}_{\rho\nu}^\Theta\} - \tilde D_\mu\widetilde{M}_\Theta^{\mu\nu}\{L^\rho, M_{\rho\nu}^\Theta\}\right) + 4[\Gamma'_\mu, L_\nu][\Gamma'^\mu, L^\nu] \\
&\quad +2\left([\Gamma'_\mu, L^\mu]\right)^2 - 32\left(M_{\mu\nu}^\Theta M_\Theta^{\mu\nu}M_{\rho\sigma}^\Theta M_\Theta^{\rho\sigma} + M_{\nu\rho}^\Theta M_\Theta^{\mu\rho}M_{\mu\sigma}^\Theta M_\Theta^{\nu\sigma}\right) \\
&\quad -32M_{\mu\sigma}^\Theta M_{\nu\rho}^\Theta(M_\Theta^{\mu\sigma}M_\Theta^{\nu\rho} - 3M_\Theta^{\nu\sigma}M_\Theta^{\mu\rho}) + 80L_\mu L^\nu M_\Theta^{\mu\rho}M_{\nu\rho}^\Theta \\
&\quad -80L_\rho L^\rho M_\Theta^{\mu\nu}M_{\mu\nu}^\Theta + 24\tilde D_\mu L^\mu\widetilde{M}_{\nu\rho}^\Theta M_\Theta^{\nu\rho}\Big\} + N_G\mathbf{Tr}\left(\hat g^2 - \tilde g^2\right), \qquad \text{(C.27)}
\end{aligned}
$$

where $\mathbf{Tr}h_3^2, \mathbf{Tr}\hat{g}^2, \mathbf{Tr}\tilde{g}^2$ are given in (C.66), and Γ'_μ is the gaugino-gravitino connection.

3. Ghost matrix elements

For the gravitino ghost, H_{Gh} is defined by (C.14). For the bosonic ghosts we have

$$\begin{aligned} H^{\mu\nu}_{gh} &= \left(H^{\mu\nu}_{gh}\right)_0 + \frac{3}{2}\mathcal{F}^a_{\mu\rho}\mathcal{F}_{a\nu}{}^\rho, \\ H^{ab}_{gh} &= \mathcal{K}_{ab} + \mathcal{K}_{ba} - \frac{1}{2}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{\mu\nu}_b - \delta_{ab}\left(\frac{\nabla^2 x}{2x} - \frac{\partial_\mu x\partial^\mu x}{4x^2}\right), \\ H^{gh}_{\mu a} &= \frac{1}{\sqrt{2}}\mathcal{D}^\nu\mathcal{F}_{a\mu\nu} + \mathcal{F}_{a\mu\nu}\frac{\partial^\nu x}{\sqrt{2}x} + \sqrt{2}q_{aI}\mathcal{D}_\mu z^I, \\ (H^{gh})^a{}_\nu &= -\frac{1}{\sqrt{2}}\mathcal{D}^\mu\mathcal{F}^a_{\nu\mu} - \mathcal{F}^a_{\nu\mu}\frac{\partial^\mu x}{\sqrt{2}x} - \sqrt{2}q^a_I\mathcal{D}_\nu z^I, \end{aligned} \tag{C.28}$$

$$\begin{aligned} \left(\hat{G}^{gh}_{\mu\nu}\right)_{\rho\sigma} &= -r_{\rho\sigma\mu\nu} + \frac{1}{2}\left(\mathcal{F}^a_{\rho\mu}\mathcal{F}_{a\sigma\nu} - (\mu\leftrightarrow\nu)\right), \\ \left(\hat{G}^{gh}_{\mu\nu}\right)_{ab} &= c_{abc}F^c_{\mu\nu} + \frac{1}{2}\left(\mathcal{F}_{a\rho\mu}\mathcal{F}^\rho_b{}_\nu - (a\leftrightarrow b)\right), \\ (\hat{G}^{gh}_{\mu\nu})^a{}_\rho &= (\hat{G}^{gh}_{\mu\nu})_\rho{}^a = -\frac{1}{\sqrt{2}}\left(\mathcal{D}_\mu\mathcal{F}^a_{\rho\nu} - \mathcal{D}_\nu\mathcal{F}^a_{\rho\mu}\right). \end{aligned} \tag{C.29}$$

4. Chiral multiplet supertraces

Defining

$$\frac{1}{2}\mathrm{STr}H^2_\chi = H^i_jH^j_i + H_{ij}H^{ij} - \frac{1}{8}\mathbf{Tr}\left(H^{IJ}_\Theta H^\Theta_{IJ}\right), \quad h^\chi_{\bar{m}i} = (\bar{m}m)_{\bar{m}i}, \tag{C.30}$$

we have

$$\frac{1}{8}\mathbf{Tr}\,(H^\chi_1)^2 = \mathrm{Tr}\,h^2_\chi + \frac{x^4(\rho^i\rho_i)^2}{8}\mathcal{D}_a\mathcal{D}^bF^a_{\mu\nu}F^{\mu\nu}_b,$$

$$(h^\chi)^j_i = e^{-K}\left(A_{ki}\bar{A}^{jk} - A_i\bar{A}^j\right) - 2\mathcal{D}_\mu z^j\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}} + \frac{1}{4x^2}f_i\bar{f}^j\mathcal{D}$$

$$
\begin{aligned}
& -\frac{1}{2x^2}f_i\mathcal{D}_a(T^az)^j - \frac{1}{2x^2}\bar{f}^j\mathcal{D}_aK_{i\bar{n}}(T^a\bar{z})^{\bar{n}} + \frac{2}{x}(T_az)^jK_{i\bar{n}}(T^a\bar{z})^{\bar{n}}, \\
H_i^j &= (h^\chi)_i^j + \delta_i^j\left(\hat{V} + M_\psi^2\right) + R^j_{k\bar{m}i}\left(e^{-K}\bar{A}^kA^{\bar{m}} + \mathcal{D}_\mu z^k\mathcal{D}^\mu\bar{z}^{\bar{m}}\right) \\
& +\frac{1}{4x^2}f_i\bar{f}^j\mathcal{D} + \frac{1}{x}\mathcal{D}_aD_i(T^az)^j, \\
H_{ij} &= e^{-K}\left(A_{jik}\bar{A}^k - A_{ij}\bar{A}\right) - \mathcal{D}_\mu\bar{z}^n\mathcal{D}^\mu\bar{z}^{\bar{m}}\left(2K_{j\bar{n}}K_{i\bar{m}} + R_{i\bar{m}j\bar{n}}\right) \\
& -\frac{1}{2x^2}\mathcal{D}_a(f_iK_{j\bar{m}} + f_jK_{i\bar{m}})(T^a\bar{z})^{\bar{m}} - x^2\rho_{ij}\mathcal{W}, \qquad \text{(C.31)}
\end{aligned}
$$

where

$$
\mathcal{W} = \mathcal{W}_a^a, \quad \mathcal{W}_b^a = \frac{1}{4}\left(F^a_{\mu\nu}F_b^{\mu\nu} - iF^a_{\mu\nu}\tilde{F}_b^{\mu\nu}\right) - \frac{1}{2x^2}\mathcal{D}^a\mathcal{D}_b \qquad \text{(C.32)}
$$

is the bosonic part of the F-component of the chiral superfield $W^a_\alpha W^\alpha_b$, and $W^a_\alpha = \lambda^a_\alpha + O(\theta)$ is the Yang-Mills field strength supermultiplet. Thus:

$$
\begin{aligned}
\frac{1}{8}\mathbf{Tr}\,(H_1^\chi)^2 &= \mathrm{Tr}\;h_\chi^2 + \frac{x^4(\rho^i\rho_i)^2}{4}\left[\left(\mathcal{W}^{ab} + \overline{\mathcal{W}}^{ab}\right)\mathcal{D}_a\mathcal{D}_b + 4\mathcal{D}^2\right], \\
\mathbf{Tr}\,(H_2^\chi)^2 &= \mathbf{Tr}\,(H_2^\chi)_0^2, \quad \frac{1}{8}\mathbf{Tr}H_3^\chi = \frac{1}{8}\mathbf{Tr}\,(H_3^\chi)_0 \\
\frac{1}{8}\mathbf{Tr}\,(H_3^\chi)^2 &= \frac{1}{8}\mathbf{Tr}\,(H_3^\chi)_0^2 + \frac{1}{2}D_i(T_az)^jD_j(T^bz)^iF^a_{\mu\nu}F_b^{\mu\nu} \\
& -2iF^a_{\mu\nu}\left(D_j(T_az)^iR^j_{i\bar{m}k} + D_i(T_az)^iK_{\bar{m}k}\right)\mathcal{D}^\mu z^k\mathcal{D}^\nu\bar{z}^{\bar{m}} \\
\frac{1}{4}\mathbf{Tr}H_3^\chi H_1^\chi &= \frac{1}{4}\mathbf{Tr}\,(H_3^\chi H_1^\chi)_0 - t^\chi + \frac{r}{2}\mathrm{Tr}\;h^\chi, \qquad \text{(C.33)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Tr}\;h^\chi &= e^{-K}\bar{A}^{ij}A_{ij} - \hat{V} - 3M_\psi^2 - 2\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}} + x^2\rho^i\rho_i\mathcal{D} + 2\mathcal{K}^a_a, \\
t^\chi &= \left[\left(x\mathcal{W}^{ab} + \frac{1}{2x}\mathcal{D}^a\mathcal{D}^b\right)\rho_{ij}(T_az)^i\left(2(T_bz)^j + x\rho^j\mathcal{D}_b\right) + \text{h.c.}\right] \\
& +\frac{i}{2}x^2\rho^i\rho_i\mathcal{D}_\mu z^j\mathcal{D}_\nu\bar{z}^{\bar{m}}K_{i\bar{m}}\mathcal{D}^aF_a^{\mu\nu}, \qquad \text{(C.34)}
\end{aligned}
$$

and the chiral fermion contributions to the helicity-odd operator T are

$$
T^\chi = T_3^\chi + T_4^\chi,
$$

$$
\begin{aligned}
T_3^\chi &= \left[\widetilde{X}_-^{\mu\nu}(M,\bar{M})\right]^{\bar{m}}_{\bar{n}}\left(G'^{+}_{\mu\nu}\right)^{\bar{n}}_{\bar{m}} - \left[\widetilde{X}_-^{\mu\nu}(\bar{M},M)\right]^{i}_{j}\left(G'^{-}_{\mu\nu}\right)^{j}_{i} + r^\mu_\nu \mathrm{Tr}\left(\widetilde{M}^{\nu\rho}\bar{M}_{\mu\rho} - M^{\nu\rho}\widetilde{\bar{M}}_{\mu\rho}\right)^j_i \\
&= t^\chi + \frac{1}{8}x^3\rho^i\rho_i\left(r^\mu_\nu F^{\nu\rho}_a F^a_{\mu\rho} - \frac{1}{4}rF^{\mu\nu}_a F^a_{\mu\nu}\right), \\
T_4^\chi &= \frac{8}{3}\left(\bar{M}^{\mu\nu}M^{\rho\sigma}\right)^i_j\left(\bar{M}_{\mu\nu}M_{\rho\sigma}\right)^j_i - 2\left[(\bar{m}M^{\mu\nu})^i_j(\bar{m}M_{\mu\nu})^j_i + \mathrm{h.c.}\right] \\
&= \frac{x^6(\rho^i\rho_i)^2}{96}\left[(F^a_{\mu\nu}F^{\mu\nu}_b)^2 + (F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b)^2\right] - \frac{x^4(\rho^i\rho_i)^2}{8}\mathcal{D}_a\mathcal{D}^b F^a_{\mu\nu}F^{\mu\nu}_b. \qquad \text{(C.35)}
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\mathrm{STr}H_\chi &= \mathrm{STr}\,(H_\chi)_0 + 2x^{-1}\mathcal{D}_a D_i(T^a z)^i + 2x^2\rho_i\rho^i\mathcal{D}, \\
\frac{1}{2}\mathrm{STr}H_\chi^2 &= \frac{1}{2}\mathrm{STr}\left(H^2_\chi\right)_0 - T_3^\chi + 2x\rho_i\rho^i\mathcal{D}_a\mathcal{D}_b\mathcal{K}^{ab} - \frac{1}{x}\mathcal{D}^a(T_a z)^i k_i - \frac{2}{x^2}C^a_G\mathcal{D}_a\mathcal{D}^a \\
&\quad - \left(\mathcal{W}_{ab} + \overline{\mathcal{W}}_{ab}\right)D_i(T^b z)^j D_j(T^a z)^i - \frac{x^4(\rho^i\rho_i)^2}{4}\left(\mathcal{W}^{ab} + \overline{\mathcal{W}}^{ab}\right)\mathcal{D}_a\mathcal{D}_b \\
&\quad + \frac{1}{8}x^3\rho^i\rho_i\left(r^\mu_\nu F^{\nu\rho}_a F^a_{\mu\rho} - \frac{1}{4}rF^{\mu\nu}_a F^a_{\mu\nu}\right) - r\left(\mathcal{K}^a_a + \frac{x^2\rho^i\rho_i}{2}\mathcal{D}\right) \\
&\quad + 2x^4(\rho_i\rho^i)^2\mathcal{D}^2 + 2\mathcal{D}\left(6M^2_\psi - 3M^2_\lambda - \hat{V}\right) - 6x^2\rho^i\rho_i M^2_\lambda\mathcal{D} \\
&\quad + 4\left(\hat{V} + M^2_\psi\right)\left(\mathcal{K}^a_a + x^2\rho_i\rho^i\mathcal{D}\right) - 2e^{-K}\mathcal{D}\bar{f}^i\bar{A}^j A_k\left(\rho^k{}_{ij} + \frac{\bar{f}^k}{x}\rho_{ij}\right) \\
&\quad + \frac{4e^{-K}}{x}(T_a z)^i(T^a\bar{z})^{\bar{m}}R^k_{n\bar{m}i}A_k\bar{A}^n + 2e^{-K}\mathcal{D}\left(a_i\bar{a}^i + 2a\bar{a} + A_{ij}\bar{A}^{ij}\right) \\
&\quad + \frac{2}{x}e^{-K}\left\{\left[a_j\left(2\bar{A} - \bar{a}\right) - a_{ij}\bar{A}^i + A_k\bar{A}^i\left(x\rho^k{}_{ij} + \rho_{ij}\bar{f}^k\right)\right]\mathcal{D}_a(T^a z)^j + \mathrm{h.c.}\right\} \\
&\quad + \frac{2}{x}\mathcal{D}_a\left[\left(\hat{V} + M^2_\psi\right)D_i(T^a z)^i + e^{-K}R^{k\,j}_{n\,i}A_k\bar{A}^n D_j(T^a z)^i\right] \\
&\quad + \frac{e^{-K}}{x}\left\{\mathcal{D}_a(T^a z)^i\left[4A_{ij}\bar{A}^j + R_i{}^j{}_\ell{}^k\bar{A}^\ell A_{jk}\right] - \mathcal{D}\left(2a_i\bar{A}^i - \frac{\bar{f}^i f_j}{2x^2}a_i\bar{A}^j\right) + \mathrm{h.c.}\right\} \\
&\quad + x^2\left[2\mathcal{D}_\rho z^i\mathcal{D}^\rho z^j\rho_{ij}\mathcal{W} - e^{-K}\rho^{ij}\left(A_{jik}\bar{A}^k - A_{ij}\bar{A}\right)\overline{\mathcal{W}} + \mathrm{h.c.}\right] + x^4\rho_{ij}\rho^{ij}\mathcal{W}\overline{\mathcal{W}} \\
&\quad - \frac{4}{x}\mathcal{D}^\mu z^i\mathcal{D}_\mu\bar{z}^{\bar{m}}K_{j\bar{m}}\mathcal{D}_a D_i(T^a z)^j - \frac{1}{x^2}\left(\partial_\mu x\partial^\mu x + \partial_\mu y\partial^\mu y\right)\left(1 + 2x^2\rho^i\rho_i\right)\mathcal{D} \\
&\quad + \mathcal{D}^a\left[\frac{2}{x^2}\bar{f}_{\bar{m}}\mathcal{D}_\mu\bar{z}^{\bar{m}}K_{i\bar{n}}(T_a z)^i\mathcal{D}^\mu\bar{z}^{\bar{n}} + \left(\mathcal{W}(T_a z)^j - \mathcal{W}_{ab}(T^b z)^j\right)\rho_{ij}\bar{f}^i + \mathrm{h.c.}\right] \\
&\quad + 2\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}R_{\bar{n}i\bar{m}j}\left[\frac{2}{x}(T_a z)^j(T^a\bar{z})^{\bar{n}} + \frac{1}{x}\mathcal{D}_a D^j(T^a\bar{z})^{\bar{n}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\left[2(T^a z)^j \mathcal{D}_a - \bar{f}^j \mathcal{D}\right]\left[\rho_{\bar{m}ij} + \frac{1}{x}\bar{f}_{\bar{m}}\rho_{ij}\right]\mathcal{D}^\mu \bar{z}^{\bar{m}} \mathcal{D}_\mu z^i + \text{h.c.}\right\} \\
&+\left\{\mathcal{D}_\mu z^i \mathcal{D}^\mu z^j \left[x^2 R_{\bar{n}i\bar{m}j}\rho^{\bar{m}\bar{n}}\overline{\mathcal{W}} - 2\rho_{\bar{m}ij}\mathcal{D}_a(T^a\bar{z})^{\bar{m}}\right] + \text{h.c.}\right\} \\
&+2iF^a_{\mu\nu}\left(D_j(T_a z)^i R^j_{i\bar{m}k} + D_i(T_a z)^i K_{\bar{m}k}\right)\mathcal{D}^\mu z^k \mathcal{D}^\nu \bar{z}^{\bar{m}} \\
&+ix^2\rho^i\rho_i \mathcal{D}_\mu z^j \mathcal{D}_\nu \bar{z}^{\bar{m}} K_{j\bar{m}} \mathcal{D}^a F^{\mu\nu}_a + 4x\left(\mathcal{W}^{ab}\rho_{ij}(T_a z)^i (T_b z)^j + \text{h.c.}\right), \qquad \text{(C.36)}
\end{aligned}
$$

where

$$
k_i = D_i k, \quad k = e^{-K}A_{ij}\bar{A}^{ij} - 2\hat{V} - 10M^2_\psi - 4\mathcal{K}^a_a. \qquad \text{(C.37)}
$$

Finally we have[21]

$$
\begin{aligned}
\frac{1}{12}\mathrm{STr}\hat{G}^\chi_{\mu\nu}\hat{G}^{\mu\nu}_\chi &= \frac{1}{12}\mathrm{STr}\left(\hat{G}^\chi_{\mu\nu}\hat{G}^{\mu\nu}_\chi\right)_0 - \frac{x^3\rho^i\rho_i}{12}\left(r^\mu_\nu F^a_{\mu\rho}F^{\nu\rho}_a - \frac{r}{4}F^a_{\mu\nu}F^{\mu\nu}_a\right) \\
&+\frac{x^6(\rho^i\rho_i)^2}{96}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2\right] \\
&+\frac{i}{6}F^a_{\mu\nu}K_{i\bar{m}}\mathcal{D}^\mu z^j \mathcal{D}^\nu \bar{z}^{\bar{m}} D_i(T_a z)^i. \qquad \text{(C.38)}
\end{aligned}
$$

5. Mixed chiral-gauge supertraces

For the bose sector we have $H^{\chi g}_\Phi = -S$, and

$$
\begin{aligned}
\mathrm{Tr}S^2 &= \frac{8}{x}K_{i\bar{m}}\left[D_\mu(T_a z)^i\right]\left[D^\mu(T_a\bar{z})^{\bar{m}}\right] - 4\frac{\partial^\mu x}{x^2}\left[(T_a z)^i K_{i\bar{m}} D_\mu (T^a\bar{z})^{\bar{m}} + \text{h.c.}\right] \\
&+2\mathcal{K}^a_a\frac{\partial_\mu x \partial^\mu x}{x^2} - x\left[\rho_{ij}(T^a z)^i (T_b z)^j F^{\nu\mu}_a\left(F^b_{\nu\mu} - i\tilde{F}^b_{\nu\mu}\right) + \text{h.c.}\right] \\
&-2i\left[x\rho_{\bar{m}ij}\mathcal{D}_\nu \bar{z}^{\bar{m}}(T^a z)^i \mathcal{D}_\mu z^j\left(F^{\mu\nu}_a - i\tilde{F}^{\mu\nu}_a\right) - \text{h.c.}\right] \\
&+\frac{x^2}{2}\rho_i\rho^i\left(\mathcal{D}'_\nu \mathcal{F}^{\nu\rho}_a + \frac{\partial_\nu y}{x}\tilde{\mathcal{F}}^{\nu\rho}_a\right)^2 + x\rho_i\rho^i F^{\nu\mu}_a F^a_{\rho\mu}\left(\frac{\partial_\nu y \partial^\rho y}{2} + \frac{5\partial_\nu x \partial^\rho x}{4}\right) \\
&-x\rho_i\rho^i\left[F^{\nu\mu}_a F^a_{\nu\mu}\left(\frac{9\partial_\rho x \partial^\rho x}{16} + \frac{\partial_\rho y \partial^\rho y}{4}\right) + \tilde{F}^{\nu\mu}_a F^a_{\nu\mu}\frac{\partial_\rho y \partial^\rho x}{8}\right] \\
&-\frac{\sqrt{x}}{4}\left(\mathcal{D}'_\nu \mathcal{F}^{\nu\rho}_a + \frac{\partial_\nu y}{x}\tilde{\mathcal{F}}^{\nu\rho}_a\right)\left[\bar{f}^i\rho_{ij}\mathcal{D}^\mu z^j\left(F^a_{\mu\rho} - i\tilde{F}^a_{\mu\rho}\right) + \text{h.c.}\right]
\end{aligned}
$$

[21] The term $+4\mathcal{D}_\mu z^i \mathcal{D}_\nu \bar{z}^{\bar{m}} K_{i\bar{m}} R_{j\bar{n}}\left(\mathcal{D}^\mu z^j \mathcal{D}^\nu \bar{z}^{\bar{n}} - \mathcal{D}^\nu z^j \mathcal{D}^\mu \bar{z}^{\bar{n}}\right)$ should be included in the right hand side of (B.14) of I.

$$
\begin{aligned}
&+\frac{\sqrt{x}}{2}x\rho^i\rho_i\left(\mathcal{D}''_\nu\mathcal{F}^{\nu\rho}_a+\frac{\partial_\nu y}{x}\tilde{\mathcal{F}}^{\nu\rho}_a\right)\left(\partial^\mu xF^a_{\mu\rho}-2\partial^\mu y\tilde{F}^a_{\mu\rho}\right)\\
&-F^{\nu\mu}_aF^a_{\rho\mu}\left\{\left[\left(\frac{\partial_\nu x}{2}+i\frac{\partial_\nu y}{4}\right)\mathcal{D}^\rho z^j\rho_{ij}\bar{f}^i+\text{h.c.}\right]+x^3\mathcal{D}_\nu z^i\mathcal{D}^\rho\bar{z}^{\bar{m}}\rho_{ij}\rho^j_{\bar{m}}\right\}\\
&+F^{\nu\mu}_aF^a_{\nu\mu}\left\{\left[\left(\frac{3\partial_\rho x}{16}+i\frac{\partial_\rho y}{8}\right)\mathcal{D}^\rho z^j\rho_{ij}\bar{f}^i+\text{h.c.}\right]+\frac{x^3}{4}\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^{\bar{m}}\rho_{ij}\rho^j_{\bar{m}}\right\}\\
&-\frac{1}{16}\tilde{F}^{\nu\mu}_aF^a_{\nu\mu}\left[i\left(\partial_\rho x+i\partial_\rho y\right)\mathcal{D}^\rho z^j\rho_{ij}\bar{f}^i+\text{h.c.}\right]\\
&+8xF^{\nu\mu}_aF^b_{\nu\mu}\mathcal{K}^a_b+4i\left(\frac{2}{\sqrt{x}}\mathcal{D}''_\mu\mathcal{F}^{\mu\nu}-\frac{\partial_\mu x}{x}F^{\mu\nu}_a\right)\left[(T^az)^iK_{i\bar{m}}\mathcal{D}_\nu\bar{z}^{\bar{m}}-\text{h.c.}\right]\\
&-2\frac{\partial^\mu x}{\sqrt{x}}F^a_{\mu\rho}\mathcal{D}''_\nu\mathcal{F}^{\nu\rho}_a-\frac{3\partial_\nu y\partial^\nu x}{4x}F^{\mu\rho}_a\tilde{F}^a_{\mu\rho}-F^{\nu\mu}_aF^a_{\rho\mu}\frac{\partial_\nu x\partial^\rho x}{x}\\
&+2xF^{\nu\mu}_aF^a_{\rho\mu}\left[4\mathcal{D}^\rho z^i\mathcal{D}_\nu\bar{z}^{\bar{m}}K_{i\bar{m}}+x\left(\mathcal{D}^\rho z^i\mathcal{D}^\nu z^j\rho_{ij}+\text{h.c.}\right)\right]\\
&-\frac{i}{2}x^2\tilde{F}^{\nu\mu}_aF^a_{\nu\mu}\left(\mathcal{D}_\rho z^j\mathcal{D}^\rho z^i\rho_{ij}-\text{h.c.}\right). \qquad\qquad (\text{C.39})
\end{aligned}
$$

In writing this expression we dropped total derivatives and used (B.10) and (B.12–B.14), as well as the Yang-Mills Bianchi identity. In addition we used (B.3–5) and (B.8) and

$$
\begin{aligned}
&\sqrt{x}\rho_{ij}\left(\mathcal{F}^{\nu\mu}_a-i\tilde{\mathcal{F}}^{\nu\mu}_a\right)\mathcal{D}_\nu z^iD_\mu(T^az)^j=-\mathcal{D}_\nu z^i\rho_{ij}(T^az)^j\left(\sqrt{x}\mathcal{D}''_\mu\mathcal{F}^{\nu\mu}_a-i\tilde{F}^{\nu\mu}_a\partial_\mu x\right)\\
&\quad-x(T^az)^j\left(F^{\nu\mu}_a-i\tilde{F}^{\nu\mu}_a\right)\left(\rho_{ij}D_\mu\mathcal{D}_\nu z^i+\rho_{\bar{m}ij}\mathcal{D}_\mu\bar{z}^{\bar{m}}\mathcal{D}_\nu z^i\right)+\text{total derivative},\\
&-iF^a_{\mu\nu}\left[\mathcal{D}^\mu z^i\mathcal{D}^\nu\bar{z}^{\bar{m}}K_{j\bar{m}}D_i(T_az)^j-\text{h.c.}\right]=\text{total deriv.}\\
&\quad+i\mathcal{D}^\mu F^a_{\mu\nu}K_{i\bar{m}}\left[\mathcal{D}^\nu\bar{z}^{\bar{m}}(T_az)^i-\mathcal{D}^\nu z^i(T_a\bar{z})^{\bar{m}}\right]+xF^a_{\mu\nu}F^{\mu\nu}_b\mathcal{K}^b_a,\\
&\quad F^a_{\mu\nu}D^\mu\mathcal{D}^\nu z^I=\pm\frac{i}{2}F^a_{\mu\nu}F^{\mu\nu}_b(T^bz)^I,\quad I=\left\{\begin{matrix}i\\ \bar{\imath}\end{matrix}\right.. \qquad\qquad (\text{C.40})
\end{aligned}
$$

To evaluate the fermion matrix elements we use (3.34); we have

$$
\begin{aligned}
\frac{1}{8}\mathbf{Tr}\,(H_1^{\chi g})^2 &= \mathrm{Tr}h^2_{\chi g}+2\left[(\bar{m}M^{\mu\nu})^i_a(\bar{m}M_{\mu\nu})^a_i+\left(\bar{M}^{\mu\nu}m\right)^i_a\left(\bar{M}_{\mu\nu}m\right)^a_i+\text{h.c.}\right]\\
&= -T^{\chi g}_4+e^{-K}\mathcal{D}\left(2a_i\bar{a}^i+8a\bar{a}\right)+2\mathcal{D}\left(\hat{V}-M^2_\psi\right)\\
&\quad+\frac{e^{-K}}{x}\left[4(T_az)^iA_{ij}(T^az)^j\left(\bar{a}-\bar{A}\right)-2\left((T_az)^i\mathcal{D}^aA_{ij}\bar{a}^j+\text{h.c.}\right)\right]
\end{aligned}
$$

$$+4\frac{e^{-K}}{x}(T_a z)^i(T^a\bar{z})^{\bar{n}}A_{ki}\bar{A}^k_{\bar{n}} + 2M_\lambda^2\left[2\mathcal{K}^a_a + \left(3x^2\rho^i\rho_i - 4\right)\mathcal{D}\right]$$
$$+2\frac{e^{-K}}{x}\left\{a_i\left(\bar{a}-\bar{A}\right)\left[\bar{f}^i\mathcal{D} - (T^a z)^i\mathcal{D}_a\right] + \text{h.c.}\right\},$$

$$\begin{aligned}-\frac{1}{8}\mathbf{Tr}\,(H_2^{\chi g})^2 &= 2\left(\tilde{D}_\mu\bar{m}\right)^i_a\left(\tilde{D}^\mu m\right)^a_i - 8\left(\tilde{D}_\mu\bar{M}^{\mu\nu}\right)^i_a\left(\tilde{D}^\rho M_{\rho\nu}\right)^a_i \\ &+\left\{[\hat{L}_{\mu\nu},\bar{m}]^i_a\,(M^{\mu\nu})^a_i + [\hat{L}_{\mu\nu},\bar{m}]^a_{\bar{m}}\,(M^{\mu\nu})^{\bar{m}}_a + \text{h.c.}\right\} \\ &+2[L_\mu,\bar{m}]^i_a[L^\mu,m]^a_i + 8L^\rho L_\mu\left(\bar{M}_{\rho\nu}\right)^i_a(M^{\mu\nu})^a_i \end{aligned} \quad \text{(C.41)}$$

with

$$\begin{aligned}-8\left(\tilde{D}_\mu\bar{M}^{\mu\nu}\right)^i_a\left(\tilde{D}^\rho M_{\rho\nu}\right)^a_i &= -\frac{x^2\rho_i\rho^i}{4}\left(\mathcal{D}''_\nu\mathcal{F}^{\nu\mu} + \frac{\partial_\nu y}{\sqrt{x}}\tilde{F}^{\nu\mu}\right)^2 \\ &+\frac{x\rho_i\rho^i}{4}\left(\sqrt{x}\mathcal{D}''_\nu\mathcal{F}^{\nu\mu} + \partial_\nu y\tilde{F}^{\nu\mu} - \frac{1}{2}\partial_\nu xF^{\nu\mu}\right)\partial^\rho xF_{\rho\mu} \\ &+\frac{x\rho_i\rho^i}{4}\left(\frac{1}{8}\partial_\rho x\partial^\rho xF^{\nu\mu}F_{\nu\mu} + \frac{1}{4}\partial_\rho x\partial^\rho yF^{\nu\mu}\tilde{F}_{\nu\mu} - \partial_\nu y\partial^\rho yF^{\nu\mu}F_{\rho\mu}\right) \\ &+\frac{1}{8}\left\{\mathcal{D}^\rho z^i\rho_{ij}\bar{f}^j\left(F_{\rho\mu} - i\tilde{F}_{\rho\mu}\right)\left[\sqrt{x}\mathcal{D}''_\nu\mathcal{F}^{\nu\mu} - i\partial_\nu y\left(F^{\nu\mu} + i\tilde{F}^{\nu\mu}\right)\right] + \text{h.c.}\right\} \\ &-\frac{1}{32}\left[\mathcal{D}^\rho z^i\rho_{ij}\bar{f}^j\partial_\rho x\left(F^{\nu\mu}F_{\nu\mu} - iF^{\nu\mu}\tilde{F}_{\nu\mu}\right) + \text{h.c.}\right] \\ &-\frac{x^3}{2}\mathcal{D}^\rho z^i\mathcal{D}_\nu\bar{z}^{\bar{m}}\rho_{ij}\rho^j_{\bar{m}}F^{\nu\mu}F_{\rho\mu} + \frac{x^3}{8}\mathcal{D}^\rho z^i\mathcal{D}_\rho\bar{z}^{\bar{m}}\rho_{ij}\rho^j_{\bar{m}}F^{\nu\mu}F_{\nu\mu}, \\ 8L^\rho L_\mu\left(\bar{M}_{\rho\nu}\right)^i_a(M^{\mu\nu})^a_i &= \frac{x\rho^i\rho_i}{32}\left(4\partial_\nu y\partial^\rho yF_{\rho\mu}F^{\nu\mu} - \partial_\rho y\partial^\rho yF_{\mu\nu}F^{\mu\nu}\right), \\ \left\{[\hat{L}_{\mu\nu},\bar{m}]^i_a\,(M^{\mu\nu})^a_i + [\hat{L}_{\mu\nu},\bar{m}]^a_{\bar{m}}\,(M^{\mu\nu})^{\bar{m}}_a + \text{h.c.}\right\} &= \tau_3^{\chi g}, \end{aligned} \quad \text{(C.42)}$$

and, using (C.40),

$$\begin{aligned}2\left(\tilde{D}_\mu\bar{m}\right)^i_a\left(\tilde{D}^\mu m\right)^a_i + 2[L_\mu,\bar{m}]^i_a[L^\mu,m]^a_i &= -2\frac{\partial^\mu x}{x^2}\left[(T_a z)^iK_{i\bar{m}}D_\mu(T^a\bar{z})^{\bar{m}} + \text{h.c.}\right] \\ &+\frac{4}{x}K_{i\bar{m}}D_{\bar{n}}(T_a\bar{z})^{\bar{m}}\mathcal{D}_\mu\bar{z}^{\bar{n}}D_j(T^a z)^i\mathcal{D}^\mu z^j + 4x\mathcal{K}^b_aF^a_{\mu\nu}F^{\mu\nu}_b \\ &+x\rho_i\rho^i\left\{K_{i\bar{n}}K_{j\bar{m}}\mathcal{D}^\mu z^j(T_a\bar{z})^{\bar{m}}\left[(T^a z)^i\mathcal{D}_\mu\bar{z}^{\bar{n}} + (T^a\bar{z})^{\bar{n}}\mathcal{D}_\mu z^i\right] + \text{h.c.}\right\}\end{aligned}$$

$$
\begin{aligned}
&-\partial^\mu x\rho_i\rho^i\mathcal{D}^a K_{j\bar m}\left[(T_a z)^j\mathcal{D}_\mu\bar z^{\bar m}+(T_a\bar z)^{\bar m}\mathcal{D}_\mu z^j\right]\\
&-2\left\{K_{j\bar m}(T_a\bar z)^{\bar m}\mathcal{D}_\mu z^j\left[\rho_{ik}(T^a z)^i\mathcal{D}^\mu z^k+\rho_{\bar m\bar n}(T^a\bar z)^{\bar m}\mathcal{D}^\mu\bar z^{\bar n}\right]+\text{h.c.}\right\}\\
&-\frac{1}{2x}\mathcal{D}_a\left\{\rho_{ij}\mathcal{D}_\mu z^i\bar f^j K_{k\bar m}\left[(T^a z)^k\mathcal{D}^\mu\bar z^{\bar m}+(T^a\bar z)^{\bar m}\mathcal{D}^\mu z^k\right]+\text{h.c.}\right\}\\
&+2\mathcal{D}_a\left[\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^k\mathcal{D}_k(T^a z)^j+\text{h.c.}\right]-\frac{2i\partial_\mu y}{x}\mathcal{D}_a\left[\rho_{ik}(T^a z)^i\mathcal{D}^\mu z^k-\text{h.c.}\right]\\
&+\left(\partial_\mu x\partial^\mu x+3\partial_\mu y\partial^\mu y\right)\frac{\rho_i\rho^i}{2}\mathcal{D}+2x^2\mathcal{D}\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar z^{\bar m}\rho^j_{\bar m}\\
&+\left[\frac{2}{x}F^{\mu\nu}_a\partial_\nu y K_{i\bar m}\mathcal{D}_\mu z^i(T^a\bar z)^{\bar m}-x\mathcal{D}\rho^i\rho_{ij}\mathcal{D}_\mu z^i\left(\partial_\mu x+2i\partial_\mu y\right)+\text{h.c.}\right]\\
&+4xF^a_{\mu\nu}F^{\mu\rho}_a\mathcal{D}_\rho z^i\mathcal{D}^\nu\bar z^{\bar m}K_{i\bar m}+\mathcal{K}^a_a\left(\frac{\partial_\mu x\partial^\mu x}{x^2}-\frac{\partial_\mu y\partial^\mu y}{x^2}\right)\\
&+2i\left(\frac{2}{\sqrt{x}}\mathcal{D}''_\mu\mathcal{F}^{\mu\nu}_a-\frac{\partial_\mu x}{x}F^{\mu\nu}_a\right)K_{i\bar m}\left[\mathcal{D}_\nu\bar z^{\bar m}(T_a z)^i-\mathcal{D}_\nu z^i(T_a\bar z)^{\bar m}\right].
\end{aligned}
\tag{C.43}
$$

We write the χ-λ contribution to T as

$$
\begin{aligned}
T^{\chi g} &= T_4^{\chi g}+T_3^{\chi g}+\tau_3^{\chi g}+t_3^{\chi g}=T'_{\chi g}+t_3^{\chi g},\\
T_4^{\chi g} &= -4\left(\bar m M^{\mu\nu}\right)^i_a\left(\bar m M_{\mu\nu}\right)^a_i+\text{h.c.}\\
&= \left(x\mathcal{W}+\mathcal{D}\right)\left(x^2\rho^i\rho_i M_\lambda^2+(a-A)\bar a^i\frac{f_i}{2x}\right)+\text{h.c.},\\
t_3^{\chi g} &= -\frac{16}{3}\left[\left(\tilde D^\sigma\bar M_{\sigma\mu}\right)^i_a\left(\tilde D_\rho M^{\rho\mu}\right)^a_i+L^\sigma L_\rho\left(\bar M_{\sigma\mu}\right)^i_a\left(M^{\rho\mu}\right)^a_i\right]\\
&\quad+\frac{16i}{3}L^\sigma\left[\left(\bar M_{\sigma\mu}\right)^i_a\left(\tilde D_\rho M^{\rho\mu}\right)^a_i-\text{h.c.}\right],\\
\tau_3^{\chi g} &= -\frac{\rho^i\rho_i}{2}\partial_\mu y\partial_\nu x F^{\mu\nu}_a\mathcal{D}^a+\frac{\partial_\mu y}{2x}F^a_{\rho\nu}\left(\partial^\rho x\tilde F^{\mu\nu}_a-\partial^\rho y F^{\mu\nu}_a\right)\\
T_3^{\chi g} &= 2iL^\rho m^a_i\tilde D_\rho\bar m^i_a+\text{h.c.}\\
&= -2\partial_\mu y\partial^\mu y\rho_i\rho^i\mathcal{D}+2\frac{\partial_\mu y}{x}F^{\mu\nu}_a\left[K_{i\bar m}\mathcal{D}_\nu z^i(T^a\bar z)^{\bar m}+\text{h.c.}\right]+\frac{\partial_\mu x\partial_\nu y}{x^2}F^{\mu\nu}_a\mathcal{D}^a\\
&\quad-\frac{i\partial_\mu y}{2x}\left\{\mathcal{D}^\mu z^i\left[\mathcal{D}\bar f^j\rho_{ij}+\frac{4}{x}K_{j\bar m}(T^a\bar z)^{\bar m}D_i(T^a z)^j\right]-\text{h.c.}\right\}\\
&\quad+\frac{2i}{x}\mathcal{D}^a\partial^\mu y\left[\rho_{ij}(T_a z)^i\mathcal{D}_\mu z^j-\text{h.c.}\right],
\end{aligned}
\tag{C.44}
$$

where

$$8iL^{\sigma}\left(\bar{M}_{\sigma\mu}\right)^{i}_{a}\left(\tilde{D}_{\rho}M^{\rho\mu}\right)^{a}_{i}+\text{h.c.}=\frac{i}{32}\left[\mathcal{D}^{\rho}z^{i}\rho_{ij}\bar{f}^{j}\left(4\partial_{\nu}yF^{a}_{\mu\rho}F^{\mu\nu}_{a}-\partial_{\rho}yF^{\mu\nu}_{a}F^{a}_{\mu\nu}\right)-\text{h.c.}\right]$$
$$+\frac{x\rho_{i}\rho^{i}}{4}\partial^{\rho}y\left[\tilde{F}^{a}_{\rho\nu}\left(\sqrt{x}\mathcal{D}''_{\mu}\mathcal{F}^{\mu\nu}_{a}+\partial_{\mu}y\tilde{F}^{\mu\nu}_{a}-\partial_{\mu}xF^{\mu\nu}_{a}\right)+\partial_{\nu}yF^{a}_{\mu\rho}F^{\mu\nu}_{a}\right]. \qquad \text{(C.45)}$$

In addition we have

$$\begin{aligned}\text{Tr}\left(\hat{G}^{\chi g}_{\Phi}\right)^{2} &= 4\left(G^{gz}_{\mu\nu}\right)_{a\rho,i}\left(G^{gz}_{\mu\nu}\right)^{i,a\rho}=\mathbf{Tr}\left(\hat{G}^{\chi g}_{\Theta}\right)^{2}\\ &= 64\left(\tilde{D}_{\mu}\bar{M}_{\nu\rho}\right)^{i}_{a}\left(\tilde{D}^{\mu}M^{\nu\rho}-\tilde{D}^{\nu}M^{\mu\rho}\right)^{a}_{i}-128i\left[L^{\nu}\left(\bar{M}_{\nu\rho}\right)^{i}_{a}\left(\tilde{D}_{\mu}M^{\mu\rho}\right)^{a}_{i}-\text{h.c.}\right],\end{aligned}$$

$$\begin{aligned}64\left(\tilde{D}_{\mu}\bar{M}_{\nu\rho}\right)^{i}_{a}\left(\tilde{D}^{\mu}M^{\nu\rho}-\tilde{D}^{\nu}M^{\mu\rho}\right)^{a}_{i} &= -2x^{2}\rho_{i}\rho^{i}\left(\mathcal{D}''_{\mu}\mathcal{F}^{\mu\nu}_{a}+\frac{\partial_{\mu}y}{\sqrt{x}}\tilde{F}^{\mu\nu}_{a}\right)^{2}\\ &-\frac{1}{4}\left\{\mathcal{D}^{\rho}z^{i}\rho_{ij}\bar{f}^{j}\left[F^{\mu\nu}_{a}F^{a}_{\mu\nu}\partial_{\rho}x-i\tilde{F}^{\mu\nu}_{a}F^{a}_{\mu\nu}\left(\partial_{\rho}x+i\partial_{\rho}y\right)+4i\partial_{\nu}yF^{a}_{\mu\rho}F^{\mu\nu}_{a}\right]+\text{h.c.}\right\}\\ &+\left(\sqrt{x}\mathcal{D}''_{\mu}\mathcal{F}^{\mu\nu}_{a}+\partial_{\mu}y\tilde{F}^{\mu\nu}_{a}\right)\left\{2x\rho^{i}\rho_{i}\partial^{\rho}xF^{a}_{\rho\nu}+\left[\left(F^{a}_{\rho\nu}-i\tilde{F}^{a}_{\rho\nu}\right)\mathcal{D}^{\rho}z^{i}\rho_{ij}\bar{f}^{j}+\text{h.c.}\right]\right\}\\ &+F^{\mu\nu}_{a}F^{a}_{\mu\nu}\left[\frac{x}{4}\rho_{i}\rho^{i}\partial_{\rho}x\partial^{\rho}x+x^{3}\rho_{ij}\rho^{j}_{\bar{m}}\mathcal{D}_{\rho}z^{i}\mathcal{D}^{\rho}\bar{z}^{\bar{m}}\right]+\frac{x\rho_{i}\rho^{i}}{2}\tilde{F}^{\mu\nu}_{a}F^{a}_{\mu\nu}\partial^{\rho}y\partial_{\rho}x\\ &-F^{\mu\rho}_{a}F^{a}_{\nu\rho}\left[x\rho_{i}\rho^{i}\left(\partial_{\mu}x\partial^{\nu}x+2\partial_{\mu}y\partial^{\nu}y\right)+4x^{3}\rho_{ij}\rho^{j}_{\bar{m}}\mathcal{D}_{\mu}z^{i}\mathcal{D}^{\nu}\bar{z}^{\bar{m}}\right], \qquad \text{(C.46)}\end{aligned}$$

Using the classical equations of motion (B.17–20), we obtain, with $k^{1}=-4\mathcal{K}^{a}_{a}$,

$$\begin{aligned}\frac{1}{2}\text{STr}H^{2}_{\chi g} &= -T'_{\chi g}+\left(\mathcal{D}_{\mu}z^{i}\mathcal{D}^{\mu}\bar{z}^{\bar{m}}+\bar{A}^{i}A^{\bar{m}}e^{-K}\right)\left(k^{1}_{i\bar{m}}-\frac{4}{x}(T^{a}z)^{j}(T_{a}\bar{z})^{\bar{n}}R_{i\bar{m}j\bar{n}}\right)\\ &-e^{-K}\left(k^{1}_{i}\bar{A}^{i}A+\text{h.c.}\right)-\frac{4x^{2}\rho^{i}\rho_{i}}{\sqrt{g}}\Delta_{\mathcal{D}}\mathcal{L}-\frac{x^{2}\rho^{i}\rho_{i}}{gx}\mathcal{L}_{a\mu}\mathcal{L}^{a\mu}\\ &+\frac{2x\rho^{i}\rho_{i}}{\sqrt{g}}\left[i\mathcal{L}^{a\mu}\left(K_{i\bar{m}}\mathcal{D}_{\mu}\bar{z}^{\bar{m}}(T_{a}z)^{i}-\text{h.c.}\right)+\mathcal{D}^{a}(T_{a}z)^{I}\mathcal{L}_{I}\right]\\ &+\frac{1}{\sqrt{g}}\mathcal{L}^{\rho}_{a}\left\{x\rho^{i}\rho_{i}\partial^{\mu}y\tilde{F}^{a}_{\mu\rho}+\left[\frac{\bar{f}^{i}}{2}\rho_{ij}\mathcal{D}^{\nu}z^{j}\left(F^{a}_{\nu\rho}-i\tilde{F}^{a}_{\nu\rho}\right)+\text{h.c.}\right]\right\}\\ &-\frac{1}{12}\text{STr}\hat{G}^{2}_{\chi g}-t^{\chi g}_{3}+4x^{2}\rho^{i}\rho_{i}\mathcal{D}\left[3M^{2}_{\psi}+\hat{V}-e^{-K}a\bar{a}\right]+10x^{2}\rho^{i}\rho_{i}\mathcal{D}M^{2}_{\lambda}\end{aligned}$$

$$
\begin{aligned}
&+4x\left[\rho_{ij}(T^az)^i(T^bz)^j\left(\mathcal{W}_{ab}+\frac{1}{2x^2}\mathcal{D}_a\mathcal{D}_b\right)+\text{h.c.}\right]-4M_\psi^2\mathcal{K}_a^a\\
&+2\left[ix\rho_{\bar m ij}\mathcal{D}_\nu\bar z^{\bar m}(T^az)^i\mathcal{D}_\mu z^j\left(F_a^{\mu\nu}-i\tilde F_a^{\mu\nu}\right)+\text{h.c.}\right]\\
&-\frac{3x\rho_i\rho^i}{4}\left(F^{\nu\mu}-i\tilde F^{\nu\mu}\right)\left(F_{\rho\mu}+i\tilde F_{\rho\mu}\right)\left(\partial_\nu x\partial^\rho x+\partial_\nu y\partial^\rho y\right)\\
&-\left(\frac{1}{x^2}+\rho^i\rho_i\right)\partial_\mu y\partial_\nu x\mathcal{D}^aF_a^{\mu\nu}+\frac{\rho_i\rho^i}{2}\mathcal{D}\left(5\partial_\mu x\partial^\mu x+3\partial_\mu y\partial^\mu y\right)\\
&-ix\rho^i\rho_iK_{i\bar m}\left[\mathcal{D}^\rho\bar z^{\bar m}(T_az)^i-\mathcal{D}^\rho z^i(T_a\bar z)^{\bar m}\right]\partial^\mu y\tilde F_{\mu\rho}^a\\
&+\left\{\mathcal{W}\left[2x^3\rho^i\rho_iM_\lambda^2+(a-A)f_i\bar a^ie^{-K}-\left(\partial_\rho x+i\partial_\rho y\right)\mathcal{D}^\rho z^j\rho_{ij}\frac{\bar f^i}{2}\right]+\text{h.c.}\right\}\\
&+2x^2\rho_{ij}\rho_{\bar m}^j\mathcal{D}\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar z^{\bar m}-2x^2\left[\left(F_a^{\nu\mu}F_{\rho\mu}^a-iF_a^{\nu\mu}\tilde F_{\rho\mu}^a\right)\mathcal{D}^\rho z^i\mathcal{D}^\nu z^j\rho_{ij}+\text{h.c.}\right]\\
&+2\frac{\partial^\mu x}{\sqrt{x}}F_{\mu\rho}^a\mathcal{D}_\nu''\mathcal{F}_a^{\nu\rho}+\frac{\partial_\nu y\partial^\nu x}{x}F_a^{\mu\rho}\tilde F_{\mu\rho}^a+F_a^{\nu\mu}F_{\rho\mu}^a\left(\frac{\partial_\nu x\partial^\rho x}{x}-\frac{\partial_\nu y\partial^\rho y}{x}\right)\\
&-e^{-K}\mathcal{D}\left(2a_i\bar a^i+16a\bar a\right)+2\mathcal{D}\left(3\hat V+17M_\psi^2\right)+4M_\lambda^2\left(\mathcal{D}+\mathcal{K}_a^a\right)\\
&+\frac{e^{-K}}{x}\left\{2\mathcal{D}^a(T_az)^iA_{ij}\left(\bar a^j-2\bar A^j\right)-a_i\left(\bar a-\bar A\right)\left[\bar f^i\mathcal{D}-2(T^az)^i\mathcal{D}_a\right]+\text{h.c.}\right\}\\
&-4x\mathcal{K}_a^bF_{\mu\nu}^aF_b^{\mu\nu}-\rho_i\rho^i\left[\left(\partial^\mu x+2i\partial^\mu y\right)K_{j\bar m}(T_az)^j\mathcal{D}_\mu\bar z^{\bar m}+\text{h.c.}\right]\mathcal{D}^a\\
&-\frac{i}{2}K_{i\bar m}\left(\mathcal{D}^\nu\bar z^{\bar m}(T_az)^i-\mathcal{D}^\nu z^i(T_a\bar z)^{\bar m}\right)\left[\bar f^i\rho_{ij}\mathcal{D}^\rho z^j\left(F_{\rho\nu}^a-i\tilde F_{\rho\nu}^a\right)+\text{h.c.}\right]\\
&-2\left\{K_{j\bar m}(T_a\bar z)^{\bar m}\mathcal{D}_\mu z^j\left[\rho_{ik}(T^az)^i\mathcal{D}^\mu z^k+\rho_{\bar\ell\bar n}(T^a\bar z)^{\bar\ell}\mathcal{D}^\mu\bar z^{\bar n}\right]+\text{h.c.}\right\}\\
&+\frac{1}{2x}\mathcal{D}_a\left\{\rho_{ij}\mathcal{D}_\mu z^i\bar f^jK_{k\bar m}\left[(T^az)^k\mathcal{D}^\mu\bar z^{\bar m}+(T^a\bar z)^{\bar m}\mathcal{D}^\mu z^k\right]+\text{h.c.}\right\}\\
&+\left\{\rho_{ij}\mathcal{D}^\nu z^i\left[2\mathcal{D}_a\mathcal{D}_k(T^az)^j\mathcal{D}_\nu z^k+\bar f^jF_a^{\mu\rho}\left(\frac{\partial_\mu x}{2}F_{\nu\rho}^a-\frac{\partial_\nu x}{8}F_{\mu\rho}^a\right)\right]+\text{h.c.}\right\}\\
&-4xF_{\mu\nu}^aF_a^{\mu\rho}\mathcal{D}_\rho z^i\mathcal{D}^\nu\bar z^{\bar m}K_{i\bar m}+\mathcal{K}_a^a\left(\frac{\partial_\mu x\partial^\mu x}{x^2}+\frac{\partial_\mu y\partial^\mu y}{x^2}\right)\\
&-2i\left(2\frac{\mathcal{D}_\mu''\mathcal{F}_a^{\mu\nu}}{\sqrt{x}}-\frac{\partial_\mu x}{x}F_a^{\mu\nu}\right)K_{i\bar m}\left[\mathcal{D}_\nu\bar z^{\bar m}(T_az)^i-\mathcal{D}_\nu z^i(T_a\bar z)^{\bar m}\right]\\
&+\frac{x\rho_i\rho^i}{4}\left(F_a^{\mu\nu}F_{\mu\nu}^a\partial_\rho x\partial^\rho x+F_a^{\mu\nu}\tilde F_{\mu\nu}^a\partial_\rho x\partial^\rho y\right)\\
&-\frac{13x\rho_i\rho^i}{96}\left(F_a^{\mu\nu}F_{\mu\nu}^a\partial_\rho y\partial^\rho y-4F_a^{\mu\rho}F_{\nu\rho}^a\partial_\mu y\partial^\nu y\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{12}\mathrm{STr}\hat{G}^2_{\chi g} &= -t_3^{\chi g} - \frac{x^2\rho_i\rho^i}{4}\left(\mathcal{D}''_\mu\mathcal{F}^{\mu\nu}_a + \frac{\partial_\mu y}{\sqrt{x}}\tilde{F}^{\mu\nu}_a\right)^2 \\
&+x\rho_i\rho^i\left[\frac{\partial^\rho x}{4}F^a_{\rho\nu}\left(\sqrt{x}\mathcal{D}''_\mu\mathcal{F}^{\mu\nu}_a + \partial_\mu y\tilde{F}^{\mu\nu}_a\right) + \frac{\partial^\rho y\partial_\rho x}{16}\tilde{F}^{\mu\nu}_a F^a_{\mu\nu}\right] \\
&+\frac{1}{8}\left[\left(F^a_{\rho\nu} - i\tilde{F}^a_{\rho\nu}\right)\mathcal{D}^\rho z^i\rho_{ij}\bar{f}^j + \mathrm{h.c.}\right]\left(\sqrt{x}\mathcal{D}''_\mu\mathcal{F}^{\mu\nu}_a + \partial_\mu y\tilde{F}^{\mu\nu}_a\right) \\
&+\frac{1}{32}F^{\mu\nu}_a\tilde{F}^a_{\mu\nu}\left[i\left(\partial_\rho x + i\partial_\rho y\right)\mathcal{D}^\rho z^i\rho_{ij}\bar{f}^j + \mathrm{h.c.}\right] \\
&-\frac{x^3}{2}\rho_{ij}\rho^j_{\bar{m}}\left(F^{\mu\rho}_a F^a_{\nu\rho}\mathcal{D}_\mu z^i\mathcal{D}^\nu\bar{z}^m - \frac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu}\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^m\right) \\
&+\frac{x\rho_i\rho^i}{32}\left[F^{\mu\nu}_a F^a_{\mu\nu}\partial_\rho x\partial^\rho x - 4F^{\mu\rho}_a F^a_{\nu\rho}\left(\partial_\mu x\partial^\nu x + 2\partial_\mu y\partial^\nu y\right)\right] \\
&-\frac{1}{32}\left[\left(F^a_{\mu\nu}\partial_\rho x + 4iF^a_{\mu\rho}\partial_\nu y\right)F^{\mu\nu}_a\rho_{ij}\bar{f}^i\mathcal{D}^\rho z^j + \mathrm{h.c.}\right] \\
&+\frac{x\rho_i\rho^i}{48}\left(F^{\mu\nu}_a F^a_{\mu\nu}\partial_\rho y\partial^\rho y - 4F^{\mu\rho}_a F^a_{\nu\rho}\partial_\mu y\partial^\nu y\right). \qquad \text{(C.47)}
\end{aligned}
$$

6. Mixed chiral-gravity supertraces

For the bosonic sector $H^{\chi G}_\Phi = S$; using (C.39) we obtain

$$
\begin{aligned}
\mathrm{Tr}Y^2 &= \mathrm{Tr}Y_0^2 + 2x\mathcal{K}^a_b F^b_{\mu\nu}F^{\mu\nu}_a \\
&+\frac{x^4\rho_i\rho^i}{8}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2 + \left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right] \\
&+2i\left(\frac{\mathcal{D}''_\mu\mathcal{F}^{\mu\rho}_a}{\sqrt{x}} - \frac{\partial_\mu x}{x}F^{\mu\rho}_a\right)K_{i\bar{m}}\left[\mathcal{D}_\rho\bar{z}^{\bar{m}}(T^a z)^i - \mathcal{D}_\rho z^i(T^a\bar{z})^{\bar{m}}\right] \\
&-x^2\left[\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j F^a_{\nu\rho}\left(F^{\nu\rho}_a - \frac{i}{2}\tilde{F}^{\nu\rho}_a\right) - 2\rho_{ij}\mathcal{D}_\nu z^i\mathcal{D}^\mu z^j F^a_{\mu\rho}F^{\nu\rho}_a + \mathrm{h.c.}\right] \\
&-F^a_{\mu\nu}F^{\mu\nu}_a\left(\frac{\nabla^2 x}{2} - \frac{\partial_\rho x\partial^\rho x - \partial_\rho y\partial^\rho y}{x}\right) + \frac{2\partial_\mu y\partial^\nu y}{x}F^{\mu\rho}_a F^a_{\nu\rho} \\
&-2F^a_{\mu\rho}\mathcal{D}''_\nu\mathcal{F}^{\nu\rho}_a\frac{\partial^\mu x}{\sqrt{x}} - F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\left(\frac{\nabla^2 y}{2} - \frac{\partial_\rho x\partial^\rho y}{x}\right), \qquad \text{(C.48)}
\end{aligned}
$$

For the fermions, we have

$$
\frac{1}{8}\mathbf{Tr}\left(H^{\chi G}_1\right)^2 = \mathrm{Tr}h^2_{\chi G} + 4\left[(\bar{m}M^{\mu\nu})^i_{\ \alpha}(\bar{m}M_{\mu\nu})^\alpha_{\ i} + \mathrm{h.c.}\right] + 8\left(\bar{M}_{\mu\nu}M_{\rho\sigma}\right)^i_{\ \alpha}\left(\bar{M}^{\mu\nu}M^{\rho\sigma}\right)^\alpha_{\ i}
$$

$$
\begin{aligned}
& +16\left(\bar{M}^{\mu\nu}M_{\mu\nu}\right)^i_{\ \alpha}\left(\bar{M}^{\rho\sigma}M_{\rho\sigma}\right)^\alpha_{\ i} - 32\left(\bar{M}^{\mu\nu}M^{\rho\sigma}\right)^i_{\ \alpha}\left(\bar{M}_{\mu\rho}M_{\nu\sigma}\right)^\alpha_{\ i} \\
= \ & \frac{1}{8}\mathbf{Tr}\left(H_1^{\chi G}\right)^2_0 + 2e^{-K}\mathcal{D}\left(A_i\bar{a}^i + \text{h.c.}\right) + 4\mathcal{D}\left(2M_\lambda^2 - \hat{V} - e^{-K}a\bar{a}\right) \\
& -2x^2\rho^i\rho_i\mathcal{D}^2 - \frac{2}{x}\mathcal{D}_a\mathcal{D}_b\mathcal{K}^{ab} - 2\frac{e^{-K}}{x}\mathcal{D}^a\left[(T_a z)^i A_{ij}\bar{A}^j + \text{h.c.}\right] \\
& -\frac{x^2}{4}\rho^i\rho_i\mathcal{D}_a\mathcal{D}^b F^a_{\mu\nu}F^{\mu\nu}_b - \frac{x^4\rho_i\rho^i}{32}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right],
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{8}\mathbf{Tr}\left(H_2^{\chi G}\right)^2 = 2\sum_{A=\alpha,\nu}\left(\tilde{D}_\mu\bar{m}\right)^i_{\ A}\left(\tilde{D}^\mu m\right)^A_{\ i} - 8\left(\tilde{D}_\sigma\bar{M}^{\sigma\nu}\right)^i_{\ \mu}\left(\tilde{D}^\rho M_{\rho\nu}\right)^\mu_{\ i} = \\
& -\frac{1}{8}\mathbf{Tr}\left(H_2^{\chi G}\right)^2_0 - x^2F^a_{\mu\nu}F^{\mu\nu}_b\rho^i\rho_i\mathcal{D}_a\mathcal{D}^b - 8\left(\tilde{D}_\sigma\bar{M}^{\sigma\nu}\right)^i_{\ \mu}\left(\tilde{D}^\rho M_{\rho\nu}\right)^\mu_{\ i}, \quad \text{(C.49)}
\end{aligned}
$$

where

$$
\left(\tilde{D}_\sigma\bar{M}^{\sigma\nu}\right)^i_{\ \mu}\left(\tilde{D}^\rho M_{\rho\nu}\right)^\mu_{\ i} = -\frac{x^4\rho_i\rho^i}{128}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2 + \left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2\right], \quad \text{(C.50)}
$$

giving

$$
\begin{aligned}
\frac{1}{2}\text{STr}H^2_{\chi G} = \ & \frac{1}{2}\left(\text{STr}H^2_{\chi G}\right)_0 - 2e^{-K}\mathcal{D}\left(A_i\bar{a}^i + \text{h.c.}\right) - 4\mathcal{D}\left(2M_\lambda^2 - \hat{V} - e^{-K}a\bar{a}\right) \\
& +2x^2\rho^i\rho_i\mathcal{D}^2 + \frac{2}{x}\mathcal{D}_a\mathcal{D}_b\mathcal{K}^{ab} + 2\frac{e^{-K}}{x}\mathcal{D}^a\left[(T_a z)^i A_{ij}\bar{A}^j + \text{h.c.}\right] + 2xF^a_{\mu\nu}F^{\mu\nu}_b\mathcal{K}^a_b \\
& +2i\left(\frac{\mathcal{D}''_\mu\mathcal{F}^{\mu\rho}_a}{\sqrt{x}} - \frac{\partial_\mu x}{x}F^{\mu\rho}_a\right)K_{i\bar{m}}\left[\mathcal{D}_\rho\bar{z}^{\bar{m}}(T^a z)^i - \mathcal{D}_\rho z^i(T^a\bar{z})^{\bar{m}}\right] \\
& -2F^a_{\mu\rho}\mathcal{D}''_\nu\mathcal{F}^{\nu\rho}_a\frac{\partial^\mu x}{\sqrt{x}} - F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\left(\frac{\nabla^2 y}{2} - \frac{\partial_\rho x\partial^\rho y}{x}\right) + \frac{2\partial_\mu y\partial^\nu y}{x}F^{\mu\rho}_a F^a_{\nu\rho} \\
& -F^a_{\mu\nu}F^{\mu\nu}_a\left(\frac{\nabla^2 x}{2} - \frac{\partial_\rho x\partial^\rho x - \partial_\rho y\partial^\rho y}{x}\right) - \frac{3}{4}x^2\rho^i\rho_i\mathcal{D}_a\mathcal{D}^b F^a_{\mu\nu}F^{\mu\nu}_b \\
& +\frac{x^4\rho^i\rho_i}{32}\left[7\left(F^{\mu\nu}_a F^a_{\mu\nu}\right)^2 + 7\left(F^{\mu\nu}_a\tilde{F}^a_{\mu\nu}\right)^2 + 6\left(F^{\mu\nu}_a F^b_{\mu\nu}\right)^2 + 6\left(F^{\mu\nu}_a\tilde{F}^b_{\mu\nu}\right)^2\right] \\
& -x^2\left\{\rho_{ij}\mathcal{D}_\mu z^i\left[\mathcal{D}^\mu z^j F^a_{\nu\rho}\left(F^{\nu\rho}_a - \frac{i}{2}\tilde{F}^{\nu\rho}_a\right) - 2\mathcal{D}^\nu z^j F^a_{\nu\rho}F^{\mu\rho}_a\right] + \text{h.c.}\right\}. \quad \text{(C.51)}
\end{aligned}
$$

The contribution to T is

$$
T^{\chi G} = T_4^{\chi G} + T_3^{\chi G},
$$

$$
\begin{aligned}
T_3^{\chi G} &= -\frac{16}{3}\left(\tilde{D}_\sigma \bar{M}^{\sigma\nu}\right)^i_{\ \mu}\left(\tilde{D}^\rho M_{\rho\nu}\right)^\mu_{\ i},\\
T_4^{\chi G} &= -4\left[(\bar{m}M^{\mu\nu})^i_{\ \alpha}(\bar{m}M_{\mu\nu})^\alpha_{\ i} + \text{h.c.}\right] + \frac{16}{3}\left(\bar{M}^{\rho\sigma}M^{\mu\nu}\right)^i_{\ \alpha}\left(\bar{M}_{\rho\sigma}M_{\mu\nu}\right)^\alpha_{\ i}\\
&= \frac{x^2\rho_i\rho^i}{48}\left[12F^{\mu\nu}_a F^b_{\mu\nu}\mathcal{D}^a\mathcal{D}_b - \left(F^{\mu\nu}_a F^b_{\mu\nu}\right)^2 - \left(F^{\mu\nu}_a \tilde{F}^b_{\mu\nu}\right)^2\right], \qquad \text{(C.52)}
\end{aligned}
$$

which for future reference [see (C.59,62)] we write as

$$
2T_4^{\chi G} + T_3^{\chi G} = x^2\rho_i\rho^i\left\{\frac{1}{24}\left[\left(F^{\mu\nu}_a F^a_{\mu\nu}\right)^2 + \left(F^{\mu\nu}_a \tilde{F}^a_{\mu\nu}\right)^2\right] + \frac{1}{2}F^{\mu\nu}_a F^b_{\mu\nu}\mathcal{D}^a\mathcal{D}_b\right\}. \qquad \text{(C.53)}
$$

The contribution to $\mathrm{STr}\hat{G}^2$ is

$$
\begin{aligned}
\mathrm{Tr}\left(\hat{G}^\Phi_{zG}\right)^2 &= x^4\rho_i\rho^i\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2\right],\\
\frac{1}{2}\mathbf{Tr}\left(\hat{G}^\Theta_{\chi G}\right)^2 &= x^4\rho_i\rho^i\left[\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2 + \left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2\right],
\end{aligned}
$$

$$
\frac{1}{12}\mathrm{STr}\hat{G}^2_{\chi G} = \frac{x^4\rho_i\rho^i}{24}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2 - \left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2 - \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right]. \qquad \text{(C.54)}
$$

7. Yang-Mills and gravity supertraces

For the remaining bosonic contributions, we have $H^{g+G}_\Phi = X - N - K$; we write $N_{ab} = N'_{ab} + \delta_{ab}n$, and evaluate separately in the next subsection the terms that depend only on n and are proportional to N_G, the number of gauge degrees of freedom. Then:

$$
\begin{aligned}
\mathrm{Tr}X &= \mathrm{Tr}X_0 - 20\mathcal{D} + xF^a_{\mu\nu}F^{\mu\nu}_a,\\
\mathrm{Tr}X^2 &= \mathrm{Tr}X_0^2 + 40\mathcal{D}^2 + 80\mathcal{D}\hat{V} - 8r\mathcal{D} + xF^a_{\mu\nu}F^{\mu\nu}_a(r - 4V)\\
&\quad + 2r^\mu_\nu xF^a_{\mu\rho}F^{\nu\rho}_a - 6xr_{\mu\nu}{}^{\rho\sigma}F^{\mu\nu}_a F^a_{\rho\sigma} - \frac{3x^2}{8}\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2\\
&\quad + \frac{x^2}{2}\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2 + \frac{29x^2}{8}\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2 + \frac{5x^2}{8}\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2,\\
\mathrm{Tr}N &= 8\mathcal{K}^a_a + N_G n,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}N^2 \;=\;& 8(\mathcal{K}_{ab}\mathcal{K}^{ba}+\mathcal{K}_{ab}\mathcal{K}^{ab})+4C^a_G F^a_{\mu\nu}F^{\mu\nu}_a+N_G n^2+4r\mathcal{K}^a_a\\
&+4x\left(1-\frac{x^2\rho_i\rho^i}{2}\right)\left(r^\mu_\nu F^a_{\mu\rho}F^{\nu\rho}_a-\frac{1}{4}rF^a_{\mu\nu}F^{\mu\nu}_a\right)+12xc_{abc}F^a_{\mu\nu}F^{b}_{\rho}{}^{\mu}F^{c\rho\nu}\\
&-4\mathcal{K}^a_a\left(\frac{\nabla^2x}{x}+\frac{3\partial^\mu y\partial_\mu y}{2x^2}\right)-2\left(2-x^2\rho_i\rho^i\right)\frac{\partial^\mu x}{\sqrt{x}}F^a_{\mu\rho}\mathcal{D}''_\nu\mathcal{F}^{\nu\rho}_a\\
&+\left(1-\frac{x^2\rho_i\rho^i}{2}\right)\left[F^a_{\mu\nu}F^{\mu\nu}_a\left(\frac{\partial_\rho x\partial^\rho x}{x}-\frac{\partial_\rho y\partial^\rho y}{2x}\right)+2F^a_{\mu\rho}F^{\nu\rho}_a\frac{\partial_\nu y\partial^\mu y}{x}\right]\\
&-\left(\mathcal{D}^\nu z^i\rho_{ij}\bar{f}^j+\text{h.c.}\right)F^{\mu\rho}_a\left(\partial_\mu xF^a_{\nu\rho}-\frac{\partial_\nu x}{4}F^a_{\mu\rho}\right)\\
&+\frac{13x^2}{8}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2\right]-\frac{5x^2}{8}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right]\\
&-\frac{x^4\rho_i\rho^i}{2}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right]\\
&+\frac{x^6\left(\rho_i\rho^i\right)^2}{8}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right],\\
\mathrm{Tr}K^2 \;=\;& 10\mathcal{D}''^\mu\mathcal{F}^a_{\mu\rho}\mathcal{D}''_\nu\mathcal{F}^{\nu\rho}_a+6r^\nu_\mu xF^{\mu\rho}_aF^a_{\nu\rho}-3r_{\mu\nu}{}^{\rho\sigma}xF^{\mu\nu}_aF^a_{\rho\sigma}+6xc_{abc}F^a_{\mu\nu}F^b_\rho{}^\mu F^{c\rho\nu}\\
&-F^{\mu\nu}_aF^a_{\mu\nu}\left(\frac{3\nabla^2x}{2}-\frac{2\partial_\rho x\partial^\rho x}{x}+\frac{25\partial_\rho y\partial^\rho y}{8x}\right)+F^{\lambda\nu}_aF^a_{\rho\nu}\left(\frac{5\partial_\lambda x\partial^\rho x}{2x}+\frac{6\partial_\lambda y\partial^\rho y}{x}\right)\\
&-\frac{1}{\sqrt{x}}\mathcal{D}''^\mu\mathcal{F}^a_{\mu\nu}\left(10F^{\lambda\nu}_a\partial_\lambda x-11\tilde{F}^{\lambda\nu}_a\partial_\lambda y\right)-\frac{1}{4}F^{\mu\nu}_a\tilde{F}^a_{\mu\nu}\left(3\nabla^2y-\frac{\partial_\rho y\partial^\rho x}{x}\right).
\end{aligned}
\tag{C.55}
$$

In writing these expressions we dropped total derivatives and used (B.10) and (B.12–B.14), as well as the Yang-Mills Bianchi identity.

Finally, writing $\left(\hat{G}^g_{\mu\nu}\right)^a_b=\left(\hat{G}'_{\mu\nu}\right)^a_b+\hat{g}_{\mu\nu}\delta^a_b$, we have

$$
\begin{aligned}
\mathrm{Tr}\left(\hat{G}^{g+G}_\Phi\right)^2 \;=\;& \mathrm{Tr}\left(G^G_\Phi\right)^2_0-4C^a_GF^a_{\mu\nu}F^{\mu\nu}_a+N_G\hat{g}^2+\tilde{F}^a_{\mu\nu}F^{\mu\nu}_a\left(3\nabla^2y-\frac{4\partial_\nu x\partial^\nu y}{x}\right)\\
&+x^3\rho_i\rho^i\left(4r^\nu_\mu F^{\mu\rho}_aF^a_{\nu\rho}-rF^a_{\mu\nu}F^{\mu\nu}_a\right)+8\left(\mathcal{D}''_\mu\mathcal{F}^{\mu\nu}_a\right)^2+8xr^\nu_\mu F^a_{\nu\rho}F^{\mu\rho}_a\\
&+\frac{1}{2x}F_{\mu\nu}F^{\mu\nu}\left(4\partial^\rho x\partial_\rho x-\partial^\rho y\partial_\rho y\right)-\frac{2}{x}F_{\mu\rho}F^{\nu\rho}\left(\partial^\mu x\partial_\nu x+2\partial^\mu y\partial_\nu y\right)\\
&+\frac{x^6\left(\rho_i\rho^i\right)^2}{4}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2-3\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2-3\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{x^4\rho_i\rho^i}{2}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right]\\
&-\frac{3x^2}{2}\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2-x^2\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2+\frac{5x^2}{4}\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\\
&-\frac{x\rho_i\rho^i}{2}\left(F^{\mu\nu}_aF^a_{\mu\nu}\partial_\rho y\partial^\rho y-4F^{\mu\rho}_aF^a_{\nu\rho}\partial_\mu y\partial^\nu y\right)\\
&-\left(8\partial^\mu xF_{\mu\rho}-4\partial^\mu y\tilde{F}_{\mu\rho}\right)\frac{\mathcal{D}''_\nu\mathcal{F}^{\nu\rho}}{\sqrt{x}},
\end{aligned}
\tag{C.56}
$$

where C^a_G is the Casimir in the adjoint representation, and we used (C.10) and (B.12–14).

For the fermions we define $H^{g+G}=H^g+H^G+H^{gG}$, with

$$
\begin{aligned}
H^g_{ab} &= H'_{ab}+\delta_{ab}\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\left(h^g_1+h^g_2+h^g_3\right),\\
\left(G^g_{\mu\nu}\right)^a_b &= \left(G'_{\mu\nu}\right)^a_b+\delta^a_b\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\left(\tilde{g}_{\mu\nu}+i\gamma_5L_{\mu\nu}\right),\\
\left(\hat{G}^g_{\mu\nu}\right)^a_b &= \left(\hat{G}'^g_{\mu\nu}\right)^a_b+\delta^a_b\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\hat{g}^g_{\mu\nu},
\end{aligned}
\tag{C.57}
$$

where $h_i,\tilde{g}_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ are 4×4 Dirac matrices. Then we obtain:

$$
\begin{aligned}
\frac{1}{8}\mathbf{Tr}H^{g+G}_1 &= \frac{1}{8}\mathbf{Tr}\left(H^G_1\right)_0+\frac{N_G}{4}\mathbf{Tr}h_1+2\mathcal{K}^a_a-\mathcal{D}\left(2-x^2\rho^i\rho_i\right)+\frac{x}{2}F^a_{\mu\nu}F^{\mu\nu}_a,\\
\frac{1}{8}\mathbf{Tr}\left(H^{g+G}_1\right)^2 &= \frac{1}{8}\mathbf{Tr}\left(H^G_1\right)^2_0+\frac{N_G}{4}\mathbf{Tr}h^2_1+2M^2_\lambda\left[2\mathcal{K}^a_a+\mathcal{D}\left(3+x^2\rho^i\rho_i\right)\right]\\
&\quad+\mathcal{D}^2\left[2-2x^2\rho_i\rho^i+\left(x^2\rho_i\rho^i\right)^2\right]+4\mathcal{K}^{ab}\mathcal{K}_{ba}-\frac{2}{x}\left(1-x^2\rho^i\rho_i\right)\mathcal{K}^{ab}\mathcal{D}_a\mathcal{D}_b\\
&\quad-\frac{1}{x^2}\left(\partial^\mu x\partial_\mu x+\partial^\mu y\partial_\mu y\right)\mathcal{D}-\frac{8}{x}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}(T_az)^j(T^a\bar{z})^{\bar{n}}K_{i\bar{n}}K_{j\bar{m}}\\
&\quad+\frac{2}{x^2}\mathcal{D}_a\left[\left(\partial_\mu x+i\partial_\mu y\right)\mathcal{D}^\mu\bar{z}^{\bar{m}}(T^az)^iK_{i\bar{m}}+\text{h.c.}\right]\\
&\quad+2\mathcal{D}\left(M^2_\psi+\hat{V}-4e^{-K}\bar{a}a\right)+xF^a_{\mu\nu}F^{\mu\nu}_a\left(\frac{1}{2}M^2_\lambda+M^2_\psi-e^{-K}\bar{a}a\right)\\
&\quad+\frac{1}{2}F^a_{\mu\nu}F^{\mu\nu}_b\mathcal{D}_a\mathcal{D}^b\left(1-\frac{x^2\rho^i\rho_i}{2}+\frac{(x^2\rho^i\rho_i)^2}{4}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{x^2}{16}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2-\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2-\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right] \\
& +\frac{x^4\rho^i\rho_i}{32}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right], \\
-\frac{1}{8}\mathbf{Tr}\left(H_2^{g+G}\right)^2 &= -\frac{1}{8}\mathbf{Tr}\left(H_2^G\right)_0^2-\frac{N_G}{4}\mathbf{Tr}h_2^2+\mathcal{D}_a\mathcal{D}^bF^a_{\mu\nu}F^{\mu\nu}_b-2xe^{-K}\bar{a}aF^a_{\mu\nu}F^{\mu\nu}_a \\
& -\left(\frac{\partial^\mu x\partial_\mu x}{2x^2}-\frac{\partial^\mu y\partial_\mu y}{2x^2}\right)\mathcal{D}+\frac{\partial^\mu x}{x^2}\mathcal{D}_a\left[\mathcal{D}_\mu z^iK_{i\bar{m}}(T^a\bar{z})^{\bar{m}}+\text{h.c.}\right] \\
& -\frac{1}{x}\left\{K_{i\bar{n}}K_{j\bar{m}}\mathcal{D}^\mu z^j(T_a\bar{z})^{\bar{m}}\left[(T^az)^i\mathcal{D}_\mu\bar{z}^{\bar{n}}+(T^a\bar{z})^{\bar{n}}\mathcal{D}_\mu z^i\right]+\text{h.c.}\right\} \\
& -\frac{x^2}{4}\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2+\frac{1}{4x}\left(\partial^\mu x\partial_\nu x-\partial^\mu y\partial_\nu y\right)F^a_{\mu\rho}F^{\nu\rho}_a \\
& -\frac{1}{8x}\left(\partial^\mu x\partial_\mu x-\partial^\mu y\partial_\mu y\right)F^a_{\mu\nu}F^{\mu\nu}_a, \\
\frac{1}{8}\mathbf{Tr}H_3^{g+G} &= \frac{1}{8}\mathbf{Tr}\left(H_3^G\right)_0+\frac{N_G}{4}\mathbf{Tr}h_3, \\
\frac{1}{8}\mathbf{Tr}\left(H_3^{g+G}\right)^2 &= \frac{N+5}{16}r^2+\frac{N_G}{4}\mathbf{Tr}h_3^2+\frac{\partial_\mu y\partial^\nu y}{2x^2}F^{\mu\rho}_aF^a_{\mu\rho}-\frac{1}{16}\mathbf{Tr}\tilde{G}'^2_{g+G}, \\
\frac{1}{4}\mathbf{Tr}\left(H_1H_3\right)^{g+G} &= \frac{1}{4}\mathbf{Tr}\left(H_1H_3\right)_0^G+\frac{N_G}{2}\mathbf{Tr}\left(h_1h_3\right)-\frac{1}{4}F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\left(\nabla^2y-\frac{\partial^\rho x\partial_\rho y}{x}\right) \\
& +\left(r-\frac{2\partial_\rho y\partial^\rho y}{x^2}\right)\left(\mathcal{K}^a_a+\frac{x^2}{2}\rho^i\rho_i\mathcal{D}\right)+\left(r-\frac{\partial_\rho y\partial^\rho y}{x^2}\right)\left(\frac{x}{4}F^a_{\mu\nu}F^{\mu\nu}_a-\mathcal{D}\right) \\
& -T_3^{gG}+i\left(2-\frac{x^2\rho^i\rho_i}{2}\right)\mathcal{D}_\mu z^j\mathcal{D}_\nu\bar{z}^{\bar{m}}K_{i\bar{m}}\mathcal{D}^aF^{\mu\nu}_a, \\
\frac{1}{2}\mathbf{Tr}\hat{G}^2_{g+G} &= \frac{1}{2}\mathbf{Tr}\tilde{G}'^2_{g+G}+N_G\mathbf{Tr}\hat{g}^2+\frac{5}{2}\left(r^2-2r^{\mu\nu}r_{\mu\nu}\right)+\tilde{F}^a_{\mu\nu}F^{\mu\nu}_a\left(6\nabla^2y-8\frac{\partial_\nu x\partial^\nu y}{x}\right) \\
& +x^3\rho^i\rho_i\left(2r^\mu_\nu F^a_{\mu\rho}F^{\nu\rho}_a-\frac{r}{2}F^a_{\mu\nu}F^{\mu\nu}_a\right)-8\left(\mathcal{D}''_\mu\mathcal{F}^{\mu\nu}_a\right)^2-8xr^\nu_\mu F^a_{\nu\rho}F^{\mu\rho}_a \\
& -\frac{2}{x}F_{\mu\nu}F^{\mu\nu}\left(\partial^\rho x\partial_\rho x-\partial^\rho y\partial_\rho y\right)+\frac{2}{x}F_{\mu\rho}F^{\nu\rho}\left(\partial^\mu x\partial_\nu x-7\partial^\mu y\partial_\nu y\right) \\
& -\frac{5x\rho^i\rho_i}{4}\left(F_{\mu\nu}F^{\mu\nu}\partial^\rho y\partial_\rho y-4F_{\mu\rho}F^{\nu\rho}\partial^\mu y\partial_\nu y\right) \\
& +8\left(\partial^\mu xF_{\mu\rho}+\partial^\mu y\tilde{F}_{\mu\rho}\right)\frac{\mathcal{D}''_\nu\mathcal{F}^{\nu\rho}}{\sqrt{x}}-\frac{x^6}{4}\left(\rho^i\rho_i\right)^2\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{x^4\rho^i\rho_i}{2}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}\tilde F^{\mu\nu}_b\right)^2-\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2-\left(F^a_{\mu\nu}\tilde F^{\mu\nu}_a\right)^2\right]\\
&+\frac{x^2}{2}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2-\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+5\left(F^a_{\mu\nu}\tilde F^{\mu\nu}_b\right)^2-5\left(F^a_{\mu\nu}\tilde F^{\mu\nu}_a\right)^2\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\mathbf{Tr}\tilde G'^2_{g+G} &= 4\left(r^2-4r^{\mu\nu}r_{\mu\nu}\right)+20\Gamma_{\mu\nu}\Gamma^{\mu\nu}-4C^a_GF^a_{\mu\nu}F^{\mu\nu}_a\\
&+16\mathcal{D}''^\nu\mathcal{F}^a_{\rho\nu}\mathcal{D}''_\mu\mathcal{F}^{\rho\mu}_a-16\frac{\partial^\nu x}{\sqrt{x}}F^a_{\rho\nu}\mathcal{D}''_\mu\mathcal{F}^{\rho\mu}_a+16xr^\mu_\nu F^{\rho\nu}_aF^a_{\rho\mu}\\
&+\frac{4}{x}\left(\partial_\rho x\partial^\rho xF^a_{\mu\nu}F^{\mu\nu}_a-\partial_\mu x\partial^\nu xF^{\mu\rho}_aF^a_{\nu\rho}\right)+4x^2\left(F^a_{\mu\nu}\tilde F^{\mu\nu}_a\right)^2\\
&-2x^2\left[\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}\tilde F^{\mu\nu}_b\right)^2+\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2\right].
\end{aligned}
\tag{C.58}
$$

The nonvanishing contributions to T^{g+G} are:

$$
\begin{aligned}
T^{gG}_3 &= 2\left[\widetilde X^{\mu\nu}_-(M,\bar M)\right]^a_\rho\left(G'^+_{\mu\nu}\right)^\rho_a-2\left[\widetilde X^{\mu\nu}_-(\bar M,M)\right]^a_\rho\left(G'^-_{\mu\nu}\right)^\rho_a\\
&= -\frac{1}{4}\left[F^a_{\mu\nu}F^{\mu\nu}_a\nabla^2x+\tilde F^a_{\mu\nu}F^{\mu\nu}_a\left(\nabla^2y-\frac{\partial_\rho x\partial^\rho y}{2x}\right)\right]+\frac{\partial_\mu x\partial^\nu x}{2x}F^a_{\nu\rho}F^{\mu\rho}_a,\\
T^g_3 &= \left[\widetilde X^{\mu\nu}_-(M,\bar M)\right]^a_b\left(G'^+_{\mu\nu}\right)^b_a-\left[\widetilde X^{\mu\nu}_-(\bar M,M)\right]^a_b\left(G'^-_{\mu\nu}\right)^b_a\\
&+iN_G\frac{\partial^\mu y}{x}\left(\bar m_\lambda\tilde D_\mu m_\lambda-\text{h.c}\right)+r^\mu_\nu\mathrm{Tr}\left(\widetilde M^{\nu\rho}\bar M_{\mu\rho}-M^{\nu\rho}\widetilde{\bar M}_{\mu\rho}\right)^j_i\\
&= \frac{i}{2}x^2\rho^i\rho_i\mathcal{D}_\mu z^j\mathcal{D}_\nu\bar z^{\bar m}K_{i\bar m}\mathcal{D}^aF^{\mu\nu}_a+\frac{1}{8}x^3\rho^i\rho_i\left(r^\mu_\nu F^{\nu\rho}_aF^a_{\mu\rho}-\frac{1}{4}rF^{\mu\nu}_aF^a_{\mu\nu}\right)+N_Gt_3,\\
T^{gG}_4 &= -4i\left(m\bar M_{\mu\nu}\right)^a_\alpha\left(m\widetilde{\bar M}^{\mu\nu}\right)^\alpha_a+\text{h.c.}=\frac{x}{2}F^a_{\mu\nu}F^{\mu\nu}_aM^2_\lambda+ixF^a_{\mu\nu}\tilde F^{\mu\nu}_ae^{-K}\left(\bar aA-\text{h.c.}\right),\\
T^g_4 &= T^\chi_4+T^{\chi G}_4.
\end{aligned}
\tag{C.59}
$$

Finally, for the ghost sector, defining $\left(H^g_{gh}\right)^a_b=\left(H'_{gh}\right)^a_b+h_{gh}\delta^a_b$, we have

$$
\begin{aligned}
\mathrm{Tr}H^g_{gh} &= 2\mathcal{K}^a_a-\frac{x}{2}F^a_{\mu\nu}F^{\mu\nu}_a+N_Gh_{gh},\quad \mathrm{Tr}H^G_{gh}=\mathrm{Tr}\left(H_{gh}\right)_0+\frac{3x}{2}F^a_{\mu\nu},\\
\mathrm{Tr}\left(H^g_{gh}\right)^2 &= N_Gh^2_{gh}+2(\mathcal{K}_{ab}\mathcal{K}^{ba}+\mathcal{K}_{ab}\mathcal{K}^{ab})-4\mathcal{K}^a_a\left(\frac{\nabla^2x}{2x}-\frac{\partial_\mu x\partial^\mu x}{4x^2}\right)\\
&+\frac{1}{2}F^a_{\mu\nu}F^{\mu\nu}_a\left(\nabla^2x-\frac{\partial_\mu x\partial^\mu x}{2x}\right)-2xF^a_{\mu\nu}F^{\mu\nu}_b\mathcal{K}_{ba}+\frac{x^2}{4}\left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\left(H^G_{gh}\right)^2 &= \left(\mathrm{Tr}H^2_{gh}\right)_0 + \frac{9x^2}{16}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2 + \left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2\right] \\
&\quad -3xF^a_{\mu\rho}F^{\nu\rho}_a\left(2\mathcal{D}^\mu z^I\mathcal{D}_\nu z^J Z_{IJ} + r^\mu_\nu\right), \\
\mathrm{Tr}\left(H^{gG}_{gh}\right)^2 &= -\left(\mathcal{D}'^\nu\mathcal{F}_{a\mu\nu}\right)^2 - \frac{1}{4x}\left(F_{a\mu\nu}\partial^\nu x\right)^2 - 4\left(q_{aI}\mathcal{D}_\mu z^I\right)^2 \\
&\quad -\mathcal{D}'^\mu\mathcal{F}^a_{\nu\mu}\left(F^{\nu\rho}_a\frac{\partial_\rho x}{\sqrt{x}} + 4q_{aI}\mathcal{D}^\nu z^I\right) - 2F^{\mu\rho}_a\frac{\partial_\rho x}{\sqrt{x}}q^a_I\mathcal{D}_\mu z^I, \\
\mathrm{Tr}\left(\hat{G}_{\mu\nu}\hat{G}^{\mu\nu}\right)^{gh} &= \left(\mathrm{Tr}G_{\mu\nu}G^{\mu\nu}\right)^{gh}_0 - \frac{\partial_\nu x\partial^\mu x}{2x}F^a_{\mu\rho}F^{\nu\rho}_a + \frac{\partial_\rho x\partial^\rho x}{2x}F^a_{\mu\nu}F^{\mu\nu}_a + 2xr^\nu_\mu F^{\mu\rho}_a F^a_{\nu\rho} \\
&\quad +2\left(\mathcal{D}'^\nu_\mu\mathcal{F}^{\mu\nu}_a\right)^2 - 2\frac{\partial^\mu x}{\sqrt{x}}F^a_{\mu\rho}\mathcal{D}''_\nu\mathcal{F}^{\nu\rho}_a - C^a_G F^a_{\mu\nu}F^{\mu\nu}_a + \frac{x^2}{4}\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2 \\
&\quad -\frac{x^2}{8}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2 + \left(F^a_{\mu\nu}F^{\mu\nu}_b\right)^2 + \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)^2\right]. \qquad \text{(C.60)}
\end{aligned}
$$

For the ghostino, $\mathrm{Tr}G_{\mu\nu}G^{\mu\nu}$ is given in (B.18) of I, and the remaining traces are modified with respect to that equation by[22]

$$
\begin{aligned}
\mathrm{Tr}H_{Gh} &= \left(\mathrm{Tr}H_{Gh}\right)_0 + 4\mathcal{D} + xF^a_{\mu\nu}F^{\mu\nu}_a, \\
\mathrm{Tr}H^2_{Gh} &= \left(\mathrm{Tr}H^2_{Gh}\right)_0 + 4\mathcal{D}^2 + 2x\mathcal{D}F^a_{\mu\nu}F^{\mu\nu}_a - 24i\mathcal{D}^a F^{\mu\nu}_a\mathcal{D}_\mu z^i K_{i\bar{m}}\mathcal{D}_\nu\bar{z}^{\bar{m}} \\
&\quad +2\left(4\mathcal{D} + xF^a_{\mu\nu}F^{\mu\nu}_a\right)\left(\hat{V} + M^2_\psi - \mathcal{D}_\rho z^i K_{i\bar{m}}\mathcal{D}^\rho\bar{z}^{\bar{m}} - \frac{r}{4}\right) \\
&\quad +2\mathcal{D}_a\mathcal{D}^b F^a_{\mu\nu}F^{\mu\nu}_b + \frac{x^2}{4}\left[\left(F^a_{\mu\nu}F^{\mu\nu}_a\right)^2 - \left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right]. \qquad \text{(C.61)}
\end{aligned}
$$

For the supertraces we obtain [see (B.17–20)]

$$
\begin{aligned}
\mathrm{STr}H^{g+G} &= \mathrm{STr}H^G_0 + N_G\mathrm{STr}h^g - 2\mathcal{D}\left(4 + x^2\rho_i\rho^i\right), \\
\frac{1}{2}\mathrm{STr}H^2_{g+G} &= \frac{1}{2}\left[\mathrm{STr}\left(H^2_G\right)_0 + \mathrm{STr}\left(H^2_{\chi G}\right)_0 - \mathrm{STr}H^2_{\chi G} + N_G\mathrm{STr}h^2_g\right] - T^{g+G}_3 - T^{gG}_4 \\
&\quad -\frac{7}{gx}\mathcal{L}_{a\mu}\mathcal{L}^{a\mu} + \frac{4}{\sqrt{gx}}\left[4i\mathcal{L}^{a\mu}\left(K_{i\bar{m}}\mathcal{D}_\mu\bar{z}^{\bar{m}}(T_a z)^i - \mathrm{h.c.}\right) + \mathcal{D}^a(T_a z)^I\mathcal{L}_I\right] \\
&\quad +\frac{\mathcal{L}^\nu_a}{\sqrt{g}}\left[\left(7 + x^2\rho^i\rho_i\right)\partial^\mu x F^a_{\mu\nu} + 3\partial^\mu y\tilde{F}^a_{\mu\nu}\right] - \frac{12}{\sqrt{g}}\Delta_{\mathcal{D}}\mathcal{L} + r\mathcal{K}^a_a
\end{aligned}
$$

---

[22] The last term in the equation for $\mathrm{Tr}H^2_{Gh}$ should be $-18\Gamma_{\mu\nu}\Gamma^{\mu\nu}$.

$$
\begin{aligned}
& +x\left(2-\frac{7x^2\rho_i\rho^i}{8}\right) r^\mu_\nu F^a_{\mu\rho} F_a^{\nu\rho} - \frac{x}{4}\left(3-\frac{7x^2\rho_i\rho^i}{8}\right) r F^a_{\mu\nu} F_a^{\mu\nu} \\
& +4\mathcal{K}^a_b F_a^{\mu\nu} F^b_{\mu\nu} + 3C^a_G\left(\mathcal{W}^a_a + \overline{\mathcal{W}}^a_a\right) + \frac{2}{x^2} C^a_G \mathcal{D}^a \mathcal{D}_a - 2x\rho^i\rho_i \mathcal{K}^{ab}\mathcal{D}_a\mathcal{D}_b \\
& -\left(5+\frac{x^2\rho_i\rho^i}{2}\right) r\mathcal{D} - \mathcal{K}^a_a\left(\frac{\partial^\mu x\partial_\mu x}{x^2} + \frac{\partial^\mu y\partial_\mu y}{x^2}\right) - 2x^2\rho^i\rho_i\mathcal{D}M^2_\lambda \\
& -4\mathcal{K}^a_a M^2_\lambda - 2e^{-K}\left(\mathcal{D}A_i\bar{a}^i + \text{h.c.}\right) + 2\mathcal{D}\left(31\hat{V} + 29M^2_\psi + 11M^2_\lambda\right) \\
& -4e^{-K}a\bar{a}\mathcal{D} + 2x\left(\mathcal{W}+\overline{\mathcal{W}}\right)\left(2M^2_\psi - M^2_\lambda\right) - 4xe^{-K}\left(\mathcal{W}A\bar{a} + \text{h.c.}\right) \\
& +2i\left(2\frac{\mathcal{D}''_\mu \mathcal{F}^{\mu\rho}_a}{\sqrt{x}} - \frac{\partial_\mu x}{x} F_a^{\mu\rho}\right) K_{i\bar{m}}\left[\mathcal{D}_\rho \bar{z}^{\bar{m}}(T^a z)^i - \mathcal{D}_\rho z^i (T^a\bar{z})^{\bar{m}}\right] \\
& -2\frac{\partial^\mu x}{x} F^a_{\mu\rho}\mathcal{D}''_\nu \mathcal{F}^{\nu\rho}_a - \frac{1}{x^2}\mathcal{D}_a\left[\left(3\partial^\mu x + 8i\partial^\mu y\right) K_{i\bar{m}}(T^a z)^i \mathcal{D}_\mu \bar{z}^{\bar{m}} + \text{h.c.}\right] \\
& +\frac{4}{x}\mathcal{D}^\mu z^i \mathcal{D}_\mu \bar{z}^{\bar{m}} K_{j\bar{m}}\mathcal{D}_a D_i (T^a z)^j - i\left(26 - x^2\rho^i\rho_i\right) K_{j\bar{m}}\mathcal{D}_\mu z^j \mathcal{D}_\nu \bar{z}^{\bar{m}}\mathcal{D}^a F_a^{\mu\nu} \\
& -\frac{i}{x}\left[\left(5 + x^2\rho^i\rho_i\right)\partial_\mu x F_a^{\mu\nu} + 5\partial_\mu y \tilde{F}_a^{\mu\nu}\right]\left(K_{j\bar{m}}\mathcal{D}_\nu \bar{z}^{\bar{m}}(T^a z)^j - \text{h.c.}\right) \\
& +\frac{13}{2x^2}\left(\partial^\mu x\partial_\mu x + \partial^\mu y\partial_\mu y\right)\mathcal{D} + \left(\frac{\partial^\rho x\partial_\rho x}{8x} + \frac{\partial^\rho y\partial_\rho y}{4x}\right) F^a_{\mu\nu} F_a^{\mu\nu} \\
& -\frac{1}{4x} F^a_{\mu\rho} F_a^{\nu\rho}\left(6\partial^\mu x\partial_\nu x - \partial^\mu y\partial_\nu y\right) - \frac{1}{x} F^a_{\mu\nu}\tilde{F}_a^{\mu\nu}\partial^\rho x\partial_\rho y \\
& -\frac{x\rho^i\rho_i}{4}\left[F^a_{\mu\nu}F_a^{\mu\nu}\left(\partial^\rho x\partial_\rho x - \frac{1}{2}\partial^\rho y\partial_\rho y\right) + F^a_{\mu\nu}\tilde{F}_a^{\mu\nu}\partial^\rho x\partial_\rho y + 2F^a_{\mu\nu}F_a^{\mu\rho}\partial^\nu y\partial_\rho y\right] \\
& +\rho^i\rho_i\partial^\mu y\partial_\mu y\mathcal{D} - F_a^{\mu\rho}\left(\frac{\partial_\mu x}{2}F^a_{\nu\rho} - \frac{\partial_\nu x}{8}F^a_{\mu\rho}\right)\left(\mathcal{D}^\nu z^i \rho_{ij}\bar{f}^j + \text{h.c.}\right) \\
& +x^2\left[\left(F^a_{\nu\rho}F_a^{\mu\rho} - iF^a_{\nu\rho}\tilde{F}_a^{\mu\rho}\right)\left(2\mathcal{D}_\mu z^i \mathcal{D}^\nu z^j - g^\nu_\mu \mathcal{D}^\sigma z^i \mathcal{D}_\sigma z^j\right)\rho_{ij} + \text{h.c.}\right] \\
& +6x^2\mathcal{W}^{ab}\overline{\mathcal{W}}_{ab} + 8\mathcal{D}_a\mathcal{D}_b\left(\mathcal{W}^{ab} + \overline{\mathcal{W}}^{ab}\right) + 2\mathcal{D}^2\left(19 - x^2\rho^i\rho_i\right) \\
& +x^3\rho^i\rho_i\left(x^2\rho^i\rho_i - 1\right)\left[x\mathcal{W}\overline{\mathcal{W}} + x\mathcal{W}^{ab}\overline{\mathcal{W}}_{ab} + \mathcal{D}\left(\mathcal{W}+\overline{\mathcal{W}}\right)\right] \\
& -x^2\rho^i\rho_i\left(\frac{3}{2} - \frac{x^2\rho^i\rho_i}{4}\right)\mathcal{D}_a\mathcal{D}_b\left(\mathcal{W}^{ab} + \overline{\mathcal{W}}^{ab}\right) \\
& -\left(8\mathcal{D} + 2xF^a_{\rho\sigma}F_a^{\rho\sigma}\right)\mathcal{D}_\mu z^i \mathcal{D}^\mu \bar{z}^{\bar{m}} K_{i\bar{m}} + 12xF^a_{\rho\mu}F_a^{\rho\nu}\mathcal{D}_\nu z^i \mathcal{D}^\mu \bar{z}^{\bar{m}} K_{i\bar{m}},
\end{aligned}
$$

$$
\frac{1}{12}\mathrm{STr}\hat{G}^2_{g+G} = \frac{1}{12}\mathrm{STr}\left(G^2_{g+G}\right)_0 + \frac{1}{12}N_G\mathrm{STr}\hat{g}^2 - \frac{1}{12}\mathrm{STr}\hat{G}^2_{\chi G} - \frac{1}{12}\mathrm{STr}\hat{G}^2_\chi
$$

$$
\begin{aligned}
&-T^{\chi G} - T_4^g - T_4^\chi + \frac{x^3\rho_i\rho^i}{4}\left(r_\mu^\nu F_a^{\mu\rho}F_{\nu\rho}^a - \frac{r}{4}F_{\mu\nu}^a F_a^{\mu\nu}\right)\\
&-\frac{1}{8x}\left(F_{\mu\nu}F^{\mu\nu}\partial^\rho y\partial_\rho y - 2F_{\mu\rho}F^{\nu\rho}\partial^\mu y\partial_\nu y\right) + \frac{i}{6}F_{\mu\nu}^a K_{j\bar{m}}\mathcal{D}^\mu z^j\mathcal{D}^\nu\bar{z}^{\bar{m}}D_i(T_a z)^i\\
&+\frac{x\rho^i\rho_i}{96}\left(F_{\mu\nu}F^{\mu\nu}\partial^\rho y\partial_\rho y - 4F_{\mu\rho}F^{\nu\rho}\partial^\mu y\partial_\nu y\right) + x^4\rho_i\rho^i\left(\mathcal{W}^{ab}\overline{\mathcal{W}}_{ab} + \mathcal{W}\overline{\mathcal{W}}\right)\\
&+x^2\rho_i\rho^i\left[\frac{3}{2}\mathcal{D}_a\mathcal{D}_b\left(\mathcal{W}^{ab} + \overline{\mathcal{W}}^{ab}\right) + x\mathcal{D}\left(\mathcal{W} + \overline{\mathcal{W}}\right) + 6\mathcal{D}^2\right]\\
&+x^4\left(\rho_i\rho^i\right)^2\left[x^2\left(\mathcal{W}^{ab}\overline{\mathcal{W}}_{ab} - \mathcal{W}\overline{\mathcal{W}}\right) - 2\mathcal{D}^2 - x\mathcal{D}\left(\mathcal{W} + \overline{\mathcal{W}}\right)\right]. \qquad \text{(C.62)}
\end{aligned}
$$

The space-time curvature dependent terms in the supertraces evaluated in sections C.4–7 give a contribution $\mathcal{L}_r$ of the form (2.23) of I with

$$
\begin{aligned}
H_{\mu\nu} &= H_{\mu\nu}^0 + H_{\mu\nu}^g + \frac{\ln\Lambda^2}{32\pi^2}\left[x\left(4 - x^2\rho_i\rho^i\right)F_{\mu\rho}^a F_{a\nu}{}^\rho - g_{\mu\nu}xF_{\rho\sigma}^a F_a^{\rho\sigma}\left(\frac{3}{2} - \frac{1}{4}x^2\rho_i\rho^i\right)\right],\\
\epsilon_0 &= (\epsilon_0)_0 + \epsilon_0^g - \frac{\ln\Lambda^2}{32\pi^2}\left\{\frac{22}{3}\mathcal{D} + 2x^2\rho_i\rho^i\mathcal{D} + \frac{2}{3x}\mathcal{D}_a D_i(T^a z)^i\right\},\\
\alpha &= \alpha_0 + \alpha^g, \qquad \beta = \beta_0 + \beta^g, \qquad \text{(C.63)}
\end{aligned}
$$

where α^g, *etc.* are evaluated in section C.8. The metric redefinition in (2.24–25) of I gives (4.11), and we get a correction[23] $\Delta_r\mathcal{L}$:

$$
\begin{aligned}
\Delta_r\mathcal{L} &= (\Delta_r\mathcal{L})_0 + \Delta_{rg}\mathcal{L} + \frac{\ln\Lambda^2}{32\pi^2}\Bigg\{\frac{N-67}{3}\mathcal{D}^2 - \frac{2N+118}{3}\mathcal{D}\hat{V} - \frac{4N+32}{3}\mathcal{D}M_\psi^2\\
&\quad + \left(\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}} - 2V\right)\left[2x^2\rho_i\rho^i\mathcal{D} + \frac{2}{3x}\mathcal{D}_a D_i(T^a z)^i\right] - 2xV\left(\mathcal{W} + \overline{\mathcal{W}}\right)
\end{aligned}
$$

[23] Eq.(B23) of I should read:

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\Delta_r\mathcal{L} &= \frac{\ln\Lambda^2}{32\pi^2}\Bigg[\left\{-2e^{-K}\left(A_{ki}\bar{A}^{ik} - \frac{2}{3}R_n^k A_k\bar{A}^n\right) - (N+17)\hat{V} - \frac{4N+32}{3}M_\psi^2\right\}\hat{V}\\
&\quad + \left[K_{i\bar{m}}\left\{\frac{N+59}{3}\hat{V} + e^{-K}\left(A_{ki}\bar{A}^{ik} - \frac{2}{3}R_n^k A_k\bar{A}^n\right) + \frac{2N+16}{3}M_\psi^2\right\} + \frac{4}{3}R_{i\bar{m}}\hat{V}\right]\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^{\bar{m}}\\
&\quad - \left\{\left(\frac{2}{3}R_{i\bar{m}} + 8K_{i\bar{m}}\right)\mathcal{D}_\rho z^i\mathcal{D}^\rho\bar{z}^{\bar{m}}g_{\mu\nu} - \frac{N+29}{6}\left(\mathcal{D}_\mu z^i\mathcal{D}_\nu\bar{z}^{\bar{m}} + \mathcal{D}_\nu z^i\mathcal{D}_\mu\bar{z}^{\bar{m}}\right)K_{i\bar{m}}\right\}\mathcal{D}^\mu z^j\mathcal{D}^\mu\bar{z}^{\bar{n}}K_{i\bar{n}}\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
&-2\mathcal{D}e^{-K}\left(A_{ij}\bar{A}^{ij}-\frac{2}{3}R^i_jA_i\bar{A}^j\right)+\frac{1}{3}\mathcal{D}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\left[4R_{i\bar{m}}-(N-61)K_{i\bar{m}}\right]\\
&+\left(\frac{N+29}{6}-x^2\rho_i\rho^i\right)\left[2x^2\mathcal{W}^{ab}\overline{\mathcal{W}}_{ab}+\left(\mathcal{W}^{ab}+\overline{\mathcal{W}}^{ab}\right)\mathcal{D}_a\mathcal{D}_b+2\mathcal{D}^2\right]\\
&+\left(\frac{N+35}{3}-x^2\rho_i\rho^i\right)\frac{x}{4}F^a_{\rho\sigma}F^{\rho\sigma}_a\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\\
&-\left(\frac{N+29}{3}-x^2\rho_i\rho^i\right)xF^a_{\rho\mu}F^{\rho\nu}_a\mathcal{D}_\nu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\Bigg\},
\end{aligned}\tag{C.64}
$$

where $\Delta_{rg}\mathcal{L}$ is given in (C.73) below.

8. Order N_G contributions

The bosonic traces are

$$
\begin{aligned}
n &= r-\frac{\nabla^2x}{x}-\frac{3\partial^\mu y\partial_\mu y}{2x^2},\\
n^2 &= r_{\mu\nu}r^{\mu\nu}-r\left[\frac{\nabla^2x}{x}-\frac{\partial_\mu x\partial^\mu x}{2x^2}+\frac{\partial_\mu y\partial^\mu y}{x^2}\right]+r^{\mu\nu}\left[\frac{2\nabla_\nu\partial_\nu x}{x}-\frac{\partial_\mu x\partial_\nu x}{x^2}+\frac{\partial_\mu y\partial_\nu y}{x^2}\right]\\
&\quad+\left(\frac{\nabla^2x}{x}\right)^2-\frac{3\partial_\mu x\partial^\mu x\nabla^2x}{2x^3}+\frac{\partial_\mu y\partial^\mu y\nabla^2x}{x^3}-\frac{\partial_\mu y\partial^\mu x\nabla^2y}{x^3}\\
&\quad+\frac{1}{4x^4}\left[3\left(\partial_\mu x\partial^\mu x\right)^2+8\left(\partial_\mu y\partial^\mu x\right)^2+3\left(\partial_\mu y\partial^\mu y\right)^2-5\partial_\mu x\partial^\mu x\partial_\nu y\partial^\nu y\right],
\end{aligned}\tag{C.65}
$$

and

$$
\begin{aligned}
\hat{g}^2 &= \frac{1}{x^2}\left[3(\nabla^2y)^2-6\frac{\partial_\mu y\partial^\mu x}{x}\nabla^2y+\frac{(\partial_\mu y\partial^\mu x)^2}{x^2}+\frac{2\partial_\mu y\partial^\mu y\partial_\nu x\partial^\nu x}{x^2}-\frac{3(\partial_\mu y\partial^\mu y)^2}{4x^2}\right]\\
&\quad+\left(r^2-4r_{\mu\nu}r^{\mu\nu}\right)-2r^{\mu\nu}\frac{\partial_\mu y\partial_\nu y}{x^2}+r\frac{\partial^\mu y\partial_\mu y}{x^2}.
\end{aligned}\tag{C.66}
$$

The fermion traces are (here $\mathbf{Tr}$ includes the ordinary Dirac trace; $\mathbf{Tr1}=4$):

$$
\begin{aligned}
\frac{1}{4}\mathbf{Tr}h_1 &= M^2_\lambda,\quad \frac{1}{4}\mathbf{Tr}h_1^2=M^4_\lambda,\\
-\frac{1}{4}\mathbf{Tr}h_2^2 &= e^{-K}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\left[\left(a_i-A_i\right)\left(\bar{a}_{\bar{m}}-\bar{A}_{\bar{m}}\right)+x^2\rho_{ik}\bar{A}^k\rho^j_{\bar{m}}A_j+\frac{\bar{f}_{\bar{m}}f_i}{4x^2}a\bar{a}\right]
\end{aligned}
$$

$$
\begin{aligned}
& +e^{-K}\left[\mathcal{D}_\mu z^i \mathcal{D}^\mu z^j (a_i - A_i)\left(\frac{f_j}{2x}\bar{a} - x\rho_{jn}\bar{A}^n\right) + \text{h.c.}\right] \\
& -\frac{1}{2}e^{-K}\mathcal{D}_\mu z^i \mathcal{D}^\mu \bar{z}^{\bar{m}}\left(\bar{f}_{\bar{m}}\rho_{ik}a\bar{A}^k + \text{h.c.}\right), \\
\mathbf{Tr}h_3 &= r - 2\frac{\partial_\mu y \partial^\mu y}{x^2}, \\
\mathbf{Tr}(h_1 h_3) &= \left(r - \frac{\partial_\mu y \partial^\mu y}{2x^2}\right) M_\lambda^2, \\
\mathbf{Tr}h_3^2 &= \frac{1}{4}r^2 - \frac{1}{2}\mathbf{Tr}\left(\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu} - Z^{\mu\nu}Z_{\mu\nu}\right) - \frac{(\nabla^2 y)^2}{x^2} \\
& +2\frac{\nabla^2 y \partial_\mu x \partial^\mu y}{x^3} - r\frac{\partial_\mu y \partial^\mu y}{x^2} + \frac{(\partial_\mu y \partial^\mu y)^2}{x^4} - \frac{(\partial_\mu y \partial^\mu x)^2}{x^4}, \\
\mathbf{Tr}\hat{g}_{\mu\nu}\hat{g}^{\mu\nu} &= \mathbf{Tr}\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu} + 6\frac{(\nabla^2 y)^2}{x^2} - 12\frac{\nabla^2 y \partial_\mu x \partial^\mu y}{x^3} + 2r\frac{\partial_\mu y \partial^\mu y}{x^2} - 4r^{\mu\nu}\frac{\partial_\mu y \partial_\nu y}{x^2} \\
& -6\frac{(\partial_\mu y \partial^\mu y)^2}{x^4} + 2\frac{(\partial_\mu x \partial^\mu y)^2}{x^4} + 4\frac{\partial_\mu y \partial^\mu y \partial_\nu x \partial^\nu x}{x^4}, \\
\mathbf{Tr}\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu} &= 4\Gamma_{\mu\nu}\Gamma^{\mu\nu} + \mathbf{Tr}Z_{\mu\nu}Z^{\mu\nu} = 4\Gamma_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{2}r^2 - 2r_{\mu\nu}r^{\mu\nu}. \qquad \text{(C.67)}
\end{aligned}
$$

To evaluate t_3, Eq. (C.58), we write it as

$$
\begin{aligned}
t_3 &= \frac{\partial^\mu x + i\partial_\mu y}{2x}\bar{m}_\lambda \tilde{D}^\mu m_\lambda - \frac{\partial^\mu x - i\partial_\mu y}{2x}\bar{m}_\lambda \tilde{D}^\mu m_\lambda + \text{h.c.} \\
&= \frac{e^{-K}}{2x}\left(\bar{f}_{\bar{m}}\mathcal{D}^\mu \bar{z}^{\bar{m}} - f_i \mathcal{D}^\mu z^i\right)\left(\bar{a} - \bar{A}\right)\left[\mathcal{D}_\mu z^j (a_j - A_j)\right. \\
& \left. -\mathcal{D}_\mu \bar{z}^{\bar{n}}\left(\frac{\bar{f}_{\bar{n}}}{2x}a - x\rho^j_{\bar{n}}A_j\right)\right] + \text{h.c.}. \qquad \text{(C.68)}
\end{aligned}
$$

The ghost traces are:

$$
\mathrm{Tr}h_{gh} = -\frac{\nabla^2 x}{2x} + \frac{\partial_\mu x \partial^\mu x}{4x^2}, \quad \mathrm{Tr}h_{gh}^2 = \left(\frac{\nabla^2 x}{2x} - \frac{\partial_\mu x \partial^\mu x}{4x^2}\right)^2. \qquad \text{(C.69)}
$$

The supertraces are

$$
-\frac{r}{6}\mathrm{STr}h = \frac{r}{3}M_\lambda^2 - \frac{r^2}{12} + r\frac{\partial_\mu x \partial^\mu x}{12x^2} + r\frac{\partial^\mu y \partial_\mu y}{12x^2},
$$

$$
\begin{aligned}
\frac{1}{2}\mathrm{STr}h^2 &= -t_3 - \frac{r^2}{16} + \frac{1}{2}r_{\mu\nu}r^{\mu\nu} - M_\lambda^2\left(\frac{r}{2} - \frac{\partial_\mu y\partial^\mu y}{4x^2}\right) - M_\lambda^4 + \frac{1}{2}\Gamma_{\mu\nu}\Gamma^{\mu\nu} \\
&\quad -r\left(\frac{\nabla^2 x}{2x} - \frac{\partial_\mu x\partial^\mu x}{4x^2} + \frac{\partial_\mu y\partial^\mu y}{4x^2}\right) + r^{\mu\nu}\left(\frac{\nabla_\nu\partial_\nu x}{x} - \frac{\partial_\mu x\partial_\nu x}{2x^2} + \frac{\partial_\mu y\partial_\nu y}{2x^2}\right) \\
&\quad +\frac{\left(\nabla^2 x\right)^2}{4x^2} + \frac{\left(\nabla^2 y\right)^2}{4x^2} - \frac{1}{2x^3}\left[\left(\partial_\mu x\partial^\mu x - \partial_\mu y\partial^\mu y\right)\nabla^2 x + 2\partial_\mu x\partial^\mu y\nabla^2 y\right] \\
&\quad +\frac{5}{16x^4}\left|\left(\partial_\mu x + i\partial_\mu y\right)\left(\partial^\mu x + i\partial^\mu y\right)\right|^2 - \frac{3}{16x^4}\left(\partial_\mu y\partial^\mu y\right)^2 \\
&\quad +e^{-K}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\left[\left(a_i - A_i\right)\left(\bar{a}_{\bar{m}} - \bar{A}_{\bar{m}}\right) + x^2\rho_{ik}\bar{A}^k\rho^j_{\bar{m}}A_j - \frac{\bar{f}_{\bar{m}}f_i}{4x^2}a\bar{a}\right] \\
&\quad +e^{-K}\left\{\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\left[\left(a_i - A_i\right)\left(\frac{f_j}{2x}\bar{A} - x\rho_{jn}\bar{A}^n\right) + \frac{f_i f_j}{4x^2}a\bar{a}\right.\right. \\
&\qquad \left.\left. -\frac{f_j}{2}\left(a - A\right)\rho_{ik}\bar{A}^k\right] + \text{h.c.}\right\} \\
&\quad +\frac{e^{-K}}{2x}\left\{\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\bar{f}_{\bar{m}}\left[\left(\bar{a} - \bar{A}\right)\left(a_i - A_i\right) - x\rho_{ik}A\bar{A}^k\right] + \text{h.c.}\right\}, \\
\frac{1}{12}\mathrm{STr}\hat{g}^2 &= \frac{1}{16}\left(r^2 - 4r_{\mu\nu}r^{\mu\nu}\right) + \frac{3}{16x^4}\left(\partial_\mu y\partial^\mu y\right)^2 - \frac{1}{6}\Gamma_{\mu\nu}\Gamma^{\mu\nu}.
\end{aligned} \tag{C.70}
$$

Dropping the total derivative

$$
\partial_\mu\left(\frac{\partial^\mu x}{x}M_\lambda^2\right) = \frac{\nabla^2 x}{x}M_\lambda^2 - \frac{\partial^\mu x\partial_\mu x}{x}M_\lambda^2 + \frac{\partial^\mu x}{x}\mathcal{D}_\mu M_\lambda^2
$$

and using the equations of motion (B.18), we can write

$$
\begin{aligned}
N_G\frac{\ln\Lambda^2}{32\pi^2}\left[\mathrm{STr}\left(h^2 - \frac{r}{6}h + \frac{1}{12}\hat{g}^2\right) + t_3\right] &= \mathcal{L}_{rg} + N_G\frac{\ln\Lambda^2}{32\pi^2}\left[\frac{x^2\rho^i\rho_j}{\sqrt{g}}\mathcal{L}_i\mathcal{L}^j + \left(X^i_g\mathcal{L}_i + \text{h.c.}\right)\right] \\
&\quad +N_G\sqrt{g}\frac{\ln\Lambda^2}{32\pi^2}\left\{x^6\left(\rho^i\rho_i\right)^2\mathcal{W}\overline{\mathcal{W}} - 2M_\lambda^4 + 3M_\psi^2 - 2M_\psi^2M_\lambda^2 + \hat{V}^2\right. \\
&\quad +6e^{-K}a\bar{a}M_\psi^2 - e^{-K}\left(\bar{a}^iA_i + \text{h.c.}\right)\left(\hat{V} + M_\psi^2\right) \\
&\quad +e^{-2K}a_i\bar{A}^i\bar{a}^jA_j - 2e^{-2K}\left(\bar{a}^iA_ia\bar{A} + \text{h.c.}\right) + 2\hat{V}\left(2M_\psi^2 - 2M_\lambda^2 + e^{-K}a\bar{a}\right) \\
&\quad +e^{-K}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\left[\left(a_i - A_i\right)\left(\bar{a}_{\bar{m}} - \bar{A}_{\bar{m}}\right) + x^2\rho_{ik}\bar{A}^k\rho^j_{\bar{m}}A_j + \frac{\bar{f}_{\bar{m}}f_i}{4x^2}a\bar{a}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+e^{-K}\left\{\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\left[(a_i-A_i)\left(\frac{f_j}{2x}\bar{A}-x\rho_{jn}\bar{A}^n\right)-\frac{f_i}{2x}a_i\bar{A}-f_i\,(a-A)\,\rho_{ik}\bar{A}^k\right]+\mathrm{h.c.}\right\}\\
&+\frac{e^{-K}}{2x}\left\{\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}\bar{f}_{\bar{m}}\left[2\bar{a}a_i-x\rho_{ik}\,(a-A)\,\bar{A}^k\right]+\frac{f_if_j}{2x^2}\bar{a}\,(2a-A)\,\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j+\mathrm{h.c.}\right\}\\
&+x\left(\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j+\mathrm{h.c.}\right)\left(M_\psi^2-\hat{V}\right)+e^{-K}\left[x\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\left(a_k\bar{A}^k-2\bar{A}a\right)+\mathrm{h.c.}\right]\\
&\frac{1}{16x^4}\left|(\partial_\mu x+i\partial_\mu y)\,(\partial^\mu x+\partial^\mu y)\right|^2-x^3\rho^i\rho_i\left(\mathcal{W}+\overline{\mathcal{W}}\right)\left(M_\psi^2+\hat{V}\right)\\
&+x^3\rho^k\rho_k\left[\mathcal{W}\left(x\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j+e^{-K}A_i\bar{a}^i-2e^{-K}\bar{a}A\right)+\mathrm{h.c.}\right]\\
&+\frac{1}{3}\Gamma_{\mu\nu}\Gamma^{\mu\nu}+x^2\rho_{ij}\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\rho_{\bar{n}\bar{m}}\mathcal{D}_\nu\bar{z}^{\bar{m}}\mathcal{D}^\nu\bar{z}^n+\text{total derivative.}
\end{aligned}
\tag{C.71}
$$

where

$$
X_g^i=(X_g^{\bar{\imath}})^*=\frac{\bar{f}^i}{2x}\left[x^3\rho^j\rho_j\overline{\mathcal{W}}+x\rho_{jk}\mathcal{D}_\mu z^j\mathcal{D}^\mu z^k+e^{-K}\left(\bar{a}^jA_j-2\bar{a}A\right)-V-M_\psi^2\right],
\tag{C.72}
$$

and $\mathcal{L}_{rg}$ is of the form (2.23) of I with

$$
\begin{aligned}
\alpha^g &= -\frac{N_G}{6}\frac{\ln\Lambda^2}{32\pi^2},\quad \beta^g=\frac{N_G}{2}\frac{\ln\Lambda^2}{32\pi^2},\quad \epsilon_0^g=-\frac{\ln\Lambda^2}{32\pi^2}\frac{N_G}{3}M_\lambda^2,\\
H_{\mu\nu}^g &= N_G\frac{\ln\Lambda^2}{32\pi^2}\left\{g_{\mu\nu}\left(-\frac{\nabla^2x}{x}+\frac{2\partial_\rho x\partial^\rho x}{3x^2}-\frac{\partial_\rho y\partial^\rho y}{3x^2}\right)\right.\\
&\left.+2\frac{\nabla_\mu\partial_\nu x}{x}-\frac{\partial_\mu x\partial_\nu x}{x^2}+\frac{\partial_\mu y\partial_\nu y}{x^2}\right\}.
\end{aligned}
\tag{C.73}
$$

Finally, using the equations of motion (B.17–18) we obtain [see (C.62)]:

$$
\begin{aligned}
\Delta_{rg}\mathcal{L} &= -N_G\frac{\ln\Lambda^2}{32\pi^2}\frac{\partial_\rho x}{x}\left[F_a^{\rho\mu}\mathcal{L}_\mu^a+\left(\mathcal{D}^\rho z^i\mathcal{L}_i+\mathrm{h.c.}\right)\right]\\
&+\sqrt{g}N_G\frac{\ln\Lambda^2}{32\pi^2}\left\{-\frac{1}{3}V^2+\frac{1}{3}M_\lambda^2\left(\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}-2V\right)+\frac{1}{3}V\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\right.\\
&+\left(\frac{\partial_\nu x\partial^\nu x}{x^2}+\frac{\partial_\nu y\partial^\nu y}{x^2}\right)\left(\frac{1}{3}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}-\frac{1}{6}V-\frac{x}{8}F_a^{\mu\rho}F_{\mu\rho}^a\right)\\
&+\left(\frac{x}{2}F_a^{\mu\rho}F_{\nu\rho}^a-\mathcal{D}_\nu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\right)\left(\frac{\partial_\mu x\partial^\nu x}{x}+\frac{\partial_\mu y\partial^\nu y}{x}\right)-\frac{1}{3}\left(\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}\right)^2\\
&+\frac{1}{2}K_{i\bar{m}}K_{j\bar{n}}\left(\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\mathcal{D}_\nu\bar{z}^{\bar{m}}\mathcal{D}^\nu\bar{z}^{\bar{n}}+\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{n}}\mathcal{D}_\nu\bar{z}^{\bar{m}}\mathcal{D}^\nu z^j\right)
\end{aligned}
$$

$$+\frac{1}{4}xF^a_{\rho\sigma}F_a^{\rho\sigma}\mathcal{D}_\mu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}} - xF^a_{\rho\mu}F_a^{\rho\nu}\mathcal{D}_\nu z^i\mathcal{D}^\mu\bar{z}^{\bar{m}}K_{i\bar{m}}$$
$$+\frac{x^2}{16}\left[\left(F^a_{\rho\sigma}F_b^{\rho\sigma}\right)^2+\left(F^a_{\rho\sigma}\tilde{F}_b^{\rho\sigma}\right)^2\right]\Bigg\}. \tag{C.74}$$

Combining the results of sections C.4–8, and evaluating $\mathcal{L}_1-\mathcal{L}_r+\Delta_r\mathcal{L}-\Delta_K\mathcal{L}-\Delta_x\mathcal{L}-\mathcal{L}_AX^A-\mathcal{L}_A\mathcal{L}_BX^{AB}$, with

$$-4e^{-K}\bar{A}^iA\mathcal{L}_i+\text{h.c.} = -\left(\mathcal{L}_IX^I\right)_0-4\sqrt{g}\left[x\left(\mathcal{W}+\overline{\mathcal{W}}\right)M^2_\psi-4\mathcal{D}M^2_\psi-xe^{-K}\left(\mathcal{W}\bar{a}A+\text{h.c.}\right)\right],$$

yields the result given in (4.6–12), where we used gauge invariance to write

$$\begin{aligned}
0 &= \delta_a\left(\mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\rho_{ij}\right) = \mathcal{D}_\mu z^i\mathcal{D}^\mu z^j\left[(T_az)^k\rho_{ijk}+2\rho_{ik}D_j(T_az)^k-(T_a\bar{z})^{\bar{m}}\rho_{\bar{m}ij}\right]\\
&= \mathcal{D}_\mu z^i\left(\mathcal{D}^\mu\left[(T_az)^j\rho_{ij}\right]+\left[\rho_{ik}D_j(T_az)^k-(T_a\bar{z})^{\bar{m}}\rho_{\bar{m}ij}\right]\mathcal{D}_\mu z^j-(T_az)^j\rho_{\bar{m}ij}\mathcal{D}^\mu\bar{z}^{\bar{m}}\right),\\
0 &= \delta_a\left(\rho_i\bar{A}^i\right) = \rho_{ij}(T_az)^j\bar{A}^i-\rho_i(T_a\bar{z})^{\bar{m}}\bar{A}^i_{\bar{m}}+\mathcal{D}_a\rho_i\bar{A}^i,\\
0 &= \delta_a\left(f_i\bar{f}^i\right) = 2x\left[\rho^i_{\bar{m}}(T_a\bar{z})^{\bar{m}}f_i-\rho_{ij}(T_az)^j\bar{f}^i\right]\\
0 &= \delta_a\left(\rho_i\hat{V}^i\right) = \rho_{ij}(T_az)^j\hat{V}^i-\frac{e^{-K}}{x}\left(a_j\bar{A}-a_{ij}\bar{A}^i+A_{ij}\bar{a}^i-x\rho_{\bar{m}ij}A^{\bar{m}}\bar{A}^i\right)(T^az)^j\\
&\quad-\frac{e^{-K}}{x}\left[\mathcal{D}_a(a\bar{a}-A\bar{a}-a\bar{A})+2x\rho^i_{\bar{m}}(T_a\bar{z})^{\bar{m}}A_i(\bar{a}-\bar{A})\right].
\end{aligned} \tag{C.75}$$

We also used the following identities, that hold up to a total derivative:

$$\begin{aligned}
0 &= \mathcal{D}_\mu z^i\mathcal{D}^\mu\left(\mathcal{D}_a\rho_{ij}(T^az)^j\right)-\rho_{ij}\mathcal{D}_a(T^az)^j\left[g^{-\frac{1}{2}}\mathcal{L}^i+\hat{V}^i+\frac{1}{x}\mathcal{D}_b(T^bz)^i+\frac{1}{2}\bar{f}^i\overline{\mathcal{W}}\right],\\
0 &= -\frac{\partial^\mu y}{x}K_{i\bar{m}}\left[(T_az)^i\mathcal{D}^\nu\bar{z}^{\bar{m}}+(T_a\bar{z})^{\bar{m}}\mathcal{D}^\nu z^i\right]F^a_{\mu\nu}\\
&\quad+\mathcal{D}_a\frac{\partial^\mu y}{x^2}\left[\frac{1}{\sqrt{g}}\mathcal{L}^a_\mu+2\partial^\nu xF^a_{\mu\nu}-iK_{i\bar{m}}\left(\mathcal{D}_\mu\bar{z}^{\bar{m}}(T^az)^i-\mathcal{D}_\mu z^i(T^a\bar{z})^{\bar{m}}\right)\right],\\
0 &= -\frac{\partial^\mu x}{x}K_{i\bar{m}}\left[(T_az)^i\mathcal{D}^\nu\bar{z}^{\bar{m}}+(T_a\bar{z})^{\bar{m}}\mathcal{D}^\nu z^i\right]\tilde{F}^a_{\mu\nu},\\
0 &= -\frac{\partial_\mu x}{x^2}\mathcal{D}^aK_{i\bar{m}}\left[(T_az)^i\mathcal{D}^\mu\bar{z}^{\bar{m}}+(T_a\bar{z})^{\bar{m}}\mathcal{D}^\mu z^i\right]\\
&\quad+\mathcal{D}\left[\frac{1}{x^2}\left(\partial_\mu x\partial^\mu x+\partial_\mu y\partial^\mu y\right)-4\left(M^2_\psi-M^2_\lambda\right)-2\left(\hat{V}+e^{-K}a\bar{a}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& + \left(\frac{\bar{f}^i \mathcal{L}_i}{2x\sqrt{g}} + e^{-K} a_i \bar{A}^i + x\rho_{ij} \mathcal{D}_\mu z^i \mathcal{D}^\mu z^j + x^3 \rho^i \rho_i \mathcal{W} + \text{h.c.} \right) \Bigg], \\
0 \;\; = \;\; & -2x^2 \rho^i \rho_i \mathcal{D}_\nu A^\nu + \left(\bar{f}^i \rho_{ij} \mathcal{D}_\nu z^j + \text{h.c.} \right) A^\nu, \qquad \text{(C.76)}
\end{aligned}
$$

where $-\mathcal{D}_\nu A^\nu$ is given by the right hand sides of the second and third equations in (C.76), with $A_\nu = (\partial^\mu y/x)\mathcal{D}_a F^a_{\mu\nu}$, $(\partial^\mu x/x)\mathcal{D}_a \tilde{F}^a_{\mu\nu}$, respectively.

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Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098