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Touching the transcendentals: tractional motion from the birth of calculus to future perspectives

Pietro Milici

When the rigorous foundation of calculus was developed, it marked an epochal change in the approach of mathematicians to geometry. Tools from geometry had been one of the foundations of mathematics until the 17th century but today, mainstream conception relegates geometry to be merely a tool of visualization. In this snapshot, however, we consider geometric and constructive components of calculus. We reinterpret “tractional motion”, a late 17th century method to draw transcendental curves, in order to reintroduce “ideal machines” in math foundation for a constructive approach to calculus that avoids the concept of infinity.

1 Foundational role of ideal machines

Machines, interpreted here to mean any mechanical tools, play various roles in mathematics. They can transfer mathematical concepts to real-world applications and foster deeper understanding, particularly as used for visualization. But they have also played a foundational role since antiquity, particularly with regards to geometry. The fundamental question we want to address in this snapshot is the following: Can machines also constitute a foundation for advanced mathematics, avoiding abstract concepts such as infinite objects or processes?

The Greek mathematician Euclid (about 300 BCE) wrote the book *The Elements*, which is regarded to be the beginning of modern geometry. He considered the straightedge and compass to be foundational tools in mathematics. Indeed, the mathematical notions of straight lines and circles can be seen as an idealization of these tools. Their constructive power was captured by the following axioms:

- Given two distinct points, it is possible to construct the straight line through them.
- Given two distinct points A and B, it is possible to construct the circle with center A and radius AB.

Many geometrical problems can be solved using only straightedge-and-compass constructions (for instance, dividing any given angle into two halves, or constructing a square whose area is twice that of a given square). However, some are not solvable in this way. Consider the three classical problems of antiquity: constructing an angle which is exactly a third of a given angle (trisecting the angle); constructing a cube whose volume is twice the volume of a given cube (doubling the cube); and, most famously, constructing a square whose area is equal to the area of a given circle (squaring the circle). All of these problems were proved in the 19th century to be unsolvable using only straightedge and compass. Doubling the cube was known to the ancient Greeks to be possible by finding the intersection of a parabola and a hyperbola (which are not constructable using a straightedge and compass). So, more powerful tools were needed, in particular for analyzing curves.

Let us fast forward to the 17th century and to the work of the French philosopher and mathematician René Descartes (1596–1650). His latinized name Cartesius led to the word "Cartesian", as in Cartesian coordinates or the Cartesian product. With his book *La Géométrie* [6], Descartes is often described as the father of "analytic geometry", that is, using algebraic equations to represent curves. But nowhere in his work did he ever graph an equation; curves were constructed from geometrical actions, often pictured as idealized machines, thus generalizing the classical constructions of Euclid and others. Only after the curves, whose construction had to be "clear and distinct", had been drawn, did he introduce notation and analyze the curve to arrive at an equation. This is an important point to keep in mind, that the 17th century perspective was essentially the opposite of ours. For Descartes, a knowledge of geometric problems and constructions, and the intrinsic value of geometry, was taken for granted. He justified algebra by showing that it could faithfully represent geometry. Nowadays, the link to geometry is much less visible and we take completely for granted the algebraic manipulation of equations.

We will now consider logarithmic spirals of which straight lines and circles are particular cases. They were first studied by Descartes and the Italian physicist and mathematician Evangelista Torricelli (1608–1647) in 1638, and later by the Swiss mathematician Jacob Bernoulli (1655–1705). These curves are easier to define using polar coordinates, (r, θ), where r is the distance from the origin and θ is the angle from the x–axis. *Logarithmic spirals* are given by the equation

$$r = ae^{k\theta},$$

for two real constants $a \neq 0$ and k.[1] The geometric property shared by these curves is that given the center O of the spiral and a point P on the spiral, the angle between the line OP (that is, the radial direction), and the tangent line to the spiral at P is constant for every P. Notice that if $k = 0$, we obtain a circle, and thinking of a straight line as a "degenerate" circle, we can see that if we introduce a machine to draw logarithmic spirals, we will have a machine that can solve any problem solvable with straightedge and compass, but potentially problems beyond these.

Consider the *equiangular compass* that is shown in Figure 1. For a fixed point O, the wheel A defines a constant angle ϕ with the radial direction, thus this device traces logarithmic spirals. The constructive power of the equiangular compass is represented by the following axiom:

> Given three distinct points O, P, and Q, it is possible to construct a logarithmic spiral with center O, passing through P, and with inclination $\phi = \angle OPQ$.

It is known that this device extends straightedge-and-compass constructions. In particular, it is possible to use it to trisect any angle and double any cube as proven in [10]. Whether it can be used to solve the squaring of a circle is still an open problem.

The equiangular compass is just one example of how ideal machines can be used in geometry. From this perspective, the early 17th century was a very fertile period, when the introduction of algebra in the resolution of geometric problems required a strong geometric constructive counterpart. As hinted at above and explained further in [3], this was the *exactness issue for geometric constructions*.

[1] For more explanations and pictures of logarithmic spirals, see also https://www.mathcurve.com/courbes2d.gb/logarithmic/logarithmic.shtml.

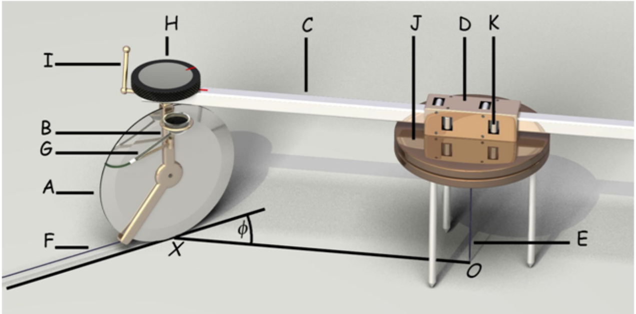

Figure 1: The *equiangular compass* for logarithmic spirals. The wheel (A) rolls on the paper and is mounted in a fork (B) locked at a fixed angle with the rod (C). The rod is constrained by the rolling of the wheel, and by the pivot (D), which allows the rod to slide and which itself rotates over the chosen center point (O). For practical reasons, there are also two pointers (E),(F), a capillary feed (G) for the ink, a knob (H) to orient the wheel, a cam (I) to lock the fork in position. To reduce friction, there are bearings in (J) and (K).

2 The introduction of tractional motion

In the 17th century, curves were generally introduced as traces of ideal machines. As already mentioned in the introduction, in *La Géométrie*, Descartes proposed a "balance" between geometric constructions and symbolic manipulation with the introduction of suitable ideal machines. In particular, he used *planar linkages* which are collections of one-dimensional straight segments of fixed lengths that are joined at their endpoints to form a graph. The segments are often called *links* or *rods*, and the shared endpoints are called *joints*. Some joints may be pinned to be fixed to specific locations (see [5]). These can clearly be seen to be a generalization of the classical straightedge and compass. So, Cartesian tools were polynomial algebra (analysis) and a class of diagrammatic constructions (synthesis) (see [3]). This setting provided a classification of curves according to which only the algebraic ones[2] were considered "purely geometrical". It was thanks to the groundbreaking method of Descartes that we developed "trust" in equations to faithfully describe the geometric properties of curves, and that

[2] An *algebraic curve* in the Euclidean plane is defined to be the set of coordinates that satisfy a polynomial equation in two variables, $P(x, y) = 0$. This is the *implicit* equation of the curve, as opposed to the *explicit* form $y = P(x)$, which can generally be graphed more easily. Curves that are not algebraic are called *transcendental*. Note that logarithmic spirals are transcendental.

analytic manipulations of equations alone, without geometric constructions to define curves, are sufficient to give solutions to problems that are fundamentally geometric.

Let us consider now the transcendental case, since after Descartes, the foundational role of ideal machines is only necessary for non-algebraic curves. The origin of a wide class of transcendental curves is the *inverse tangent problem*. As opposed to the direct tangent problem, in which we are given a curve and must calculate its tangents, in the inverse tangent case, a curve is sought given some properties that its tangent has to satisfy. In a modern setting, the problem is found in the geometrical solution of differential equations. The first documented appearance is attributed to the French architect Claude Perrault (1613–1688) in the late 17th century (see Figure 2). The role of traction, that is, of pulling something over a flat surface, in the first instrumental way of generating a curve given some tangent conditions gave rise to the name "tractional" for such constructions.

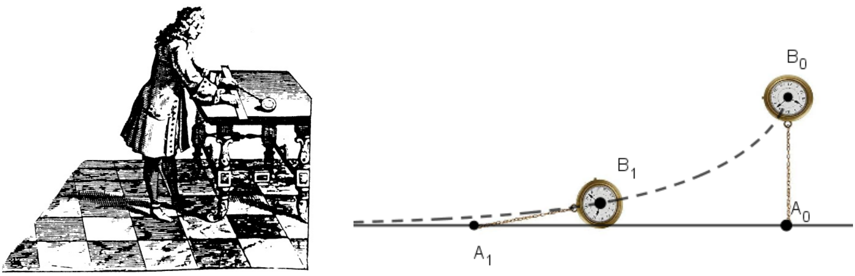

Figure 2: Moving the extremity of a chain-clock that lies flat on a table along a straight line (slowly enough to avoid inertia), the clock describes a curve called a *tractrix*.

Many mathematicians worked on clarification and definition of tractional motion from both a practical and purely mathematical perspective (see [12]). Physically, the machine solving an inverse tangent problem had to avoid the lateral motion of a point with respect to a given direction. This can be accomplished by something that, like the blade of a pizza-cutter or the front wheel of a bicycle, guides the direction of the motion (as with the equiangular compass of Figure 1).

The German polymath Gottfried Wilhelm Leibniz (1646–1716) was particularly interested in these constructions. As Leibniz is considered a cofounder of modern calculus (alongside the English scientist Isaac Newton (1643–1727)), we can infer that tractional motion may also have played an important role in the development of calculus. However, the analytic tools that Leibniz used to solve calculus problems involved the concept of infinity (although we would have to wait until the 19th century for this to be put on a solid formal foundation). As happened with the Cartesian machines used in *La Géométrie*, tractional constructions gradually fell out of favour, eventually becoming obsolete.

Even though almost forgotten [3], I would assert that tractional constructions can provide an alternative foundation to derivatives in calculus, making it possible to construct a large class of transcendental objects without the need for infinity. From a cognitive perspective, such an approach is no longer based on the concept of infinity (as suggested in [7]), but on something more concrete, namely, "the wheel direction defines the tangent to a curve" (see also Figure 3). This concept is present in everyday experience (to turn when riding a bicycle, we turn the direction of the handlebar).

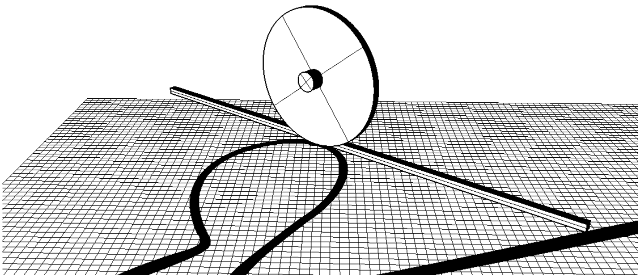

Figure 3: Considering a wheel rolling on a curve, the direction of the wheel (in the image represented by a bar) is the tangent to the curve.

3 Definition and limitations of tractional constructions

Tractional machines historically constituted an inhomogeneous class of devices with some similar ideas (the guidance of the tangent). That means that there was no well-defined notion of tractional machines until the recent work [8]. With little variations, the main idea behind tractional machines is that they are linkages, as used by Descartes, extended with the possible introduction of ideal wheels (compare Figure 3) to be posed on rods. As (partially) proved by the British mathematician Alfred Kempe (1849–1922) in 1876, for any finite algebraic part of an algebraic curve, we can construct a linkage tracing it. But

[3] Tractional constructions first appeared with foundational purposes in the late 17th century, were forgotten after the middle 18th century, and were later autonomously rediscovered in the late 19th century for engineering applications.

what about the constructive limitations of tractional constructions? To solve this problem (compare [9]), that historically remained unsolved, we can use 20th century differential algebra (see [4] for a clear introduction).

In the 19th century, the interest in solving equations with polynomials (thought of as expressions made up of variables such as x and y and coefficients such as real numbers) led to the development of *abstract algebra*, defining algebraic structures such as groups, rings, and fields.[4] These structures consist of sets equipped with one or two binary operations that generalize the arithmetic operations of addition and multiplication of real numbers.

Differential algebra extends these algebraic structures with the introduction of a unary operation, the *derivation* $a \mapsto a'$, which is defined by the conditions $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$. A ring with derivation is then called a *differential ring* and so on. Thus, instead of polynomials, differential algebra deals with differential polynomials (for example $x^2 x'^3 y + xy''^2 - x'y'$), and there are algorithms to eliminate some variables in systems of differential polynomials. Roughly speaking, the difference between differential and non-differential polynomials is that the variables $x, y, \ldots$ in polynomials represent elements in a ring such as real numbers, while in differential polynomials, the variables $x, y, x', \ldots$ represent functions of these elements. Considering a point of a tractional machine as a pair of functions $P = (x, y)$ that both vary in time, the tangent condition that P cannot move out of the direction given by the pair of real numbers (Δ_x, Δ_y) can be translated into the equation $y'\Delta_x = x'\Delta_y$. Thus, inverse tangent conditions can be converted to differential polynomials.

With differential algebra (specifically, with elimination algorithms), we obtain that the behaviour of any point of a tractional machine locally can be parametrized by *differentially algebraic* functions[5]. That means that with tractional constructions we can trace many transcendental curves, but there are still curves out of reach as not all functions are differentially algebraic. However, elementary functions (trigonometric, exponential, logarithmic ones) and even most of the transcendental functions of analysis handbooks are differentially algebraic. Historically, the first example of a function that is not differentially algebraic was Euler's gamma function[6], as proven by the German mathematician Otto Hölder (1859–1937) in 1886. Curiously, differentially algebraic functions form the same class as the functions obtainable by Shannon's *General*

[4] For an introduction to the concept of fields, see Snapshot 4/2014 *What does ">" really mean?* by Bruce Reznick.

[5] A function y is called *differentially algebraic* if it satisfies a differential equation of the form $P(t, y, y', \ldots, y^{(n)}) = 0$ where P is a nontrivial polynomial in $n + 2$ variables.

[6] Euler's gamma function is a continuous extension of the factorial function to all complex numbers which was introduced by the Swiss mathematician and physicist Leonhard Euler (1707–1783).

Purpose Analog Computer (GPAC) from [11], written many centuries after the introduction of tractional constructions. As visible in the title of Shannon's 1941 paper, the GPAC was a theoretical model for the analog computer called *differential analyzer* [7] and still today it seems that differentially algebraic functions constitute a limit for analog computation.

From a foundational perspective, it is interesting that the analysis of tractional machines does not need infinity: tractional machines can be investigated in a purely symbolic way with the 20th century differential algebra without the need for infinitary objects or processes. That can be considered as an extension of Descartes' foundational balance between geometric constructions and symbolic manipulation but far beyond polynomial algebraic boundaries. This time the dualism is no longer between curves that are algebraic or not, but between functions that are differentially algebraic or not.

4 Further perspectives

Apart from the theoretical model, tractional machines can be useful for didactical purposes, in particular to foster a deep and conscious understanding of calculus and differential equations. These topics pose several difficulties because they involve the manipulation of infinitary objects. Research in math education, which focused on these difficulties for a long time, has highlighted obstacles and proposed different approaches. Indeed, the actual manipulation of an artifact can help students to experience and internalize the underlying mathematical contents, if suitably introduced into educational pathways (see [1]).

The adoption of tractional tools in laboratories to improve the learning of students has already been present in the Italian tradition: we may recall Giovanni Poleni from Padua (1683–1761) who invented the "calculating clock" and Ernesto Pascal from Naples (1865–1940) who reorganized the teaching of mathematics at his university. An interdisciplinary commitment would consist in developing suitable didactical activities concerning tractional motion with the aid of both concrete machines and digital tools, as already realized for algebraic machines (see [2]).

To conclude with future perspectives, we note that the geometric "legitimation" of analytic results (as described for the Cartesian approach in the first section) is somehow present also today in some fields of advanced mathematics. An example of how mathematical machines and theory worked together in the 20th century is given in the Snapshot 13/2019 *Analogue mathematical instru-*

[7] The differential analyzer, completed by the American engineer Vannevar Bush (1890–1974) in 1931 at MIT, was one of the first machines operationally used to solve general differential equations. Such equations were solved using mechanical integrations by wheel-and-disk mechanisms (see the images and videos at http://www.mit.edu/~klund/analyzer/).

ments: Examples from the "theoretical dynamics" group (France, 1948–1964) by Loïc Petitgirard.

On the contrary, for other fields of advanced mathematics, we still are looking for a geometric interpretation. Let us describe one example here: Differentially algebraic functions are solutions of differential polynomials, that is, of polynomials with non-negative, integer order derivatives. Negative integer-order derivatives could be considered indefinite integrals, but what could it mean to consider derivatives of non-integer order? This question, first posed by Leibniz, is at the core of "fractional calculus". While fractional calculus finds use in many fields of science and engineering, it still lacks a widely accepted geometric interpretation. Fractional calculus analytically involves the use of Euler's gamma function, which is not a differentially algebraic function, so this class of problems cannot be coped by tractional motion. An exciting problem is thus to generalize tractional constructions in order to include this class of problems, extending the power of analog computation while still avoiding the introduction of infinity or approximation. After the exactness issue for geometric constructions from the 17th century that was described in the first section, we have now a new quest for exactness: Are you ready?

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